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ABSTRACT

The purpose of this document is to survey at least some of the important research literature on solving linear equations in order to identify information that might suggest ways to improve teaching effectiveness. The studies that are examined employ a wide range of research techniques, are based often on quite different perspectives, and span over half a century. It is striking, however, that there has been so much attention paid to algebra in general and equation solving in particular since 1970. The resulting recent accumulation of bits and pieces of information may leave the impression that what is known about equation solving has not been clearly synthesized. Such an impression is probably correct. The results that have been reported tend to be based on limited data and are not consistent enough to outline clearly the structure of the processes of equation solving. (Author)

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STUDENT PROCEDURES IN SOLVING EQUATIONS*

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The purpose of this paper is to survey at least some of the important research literature on solving linear equations in order to identify information that might suggest ways to improve teaching effectiveness. The studies that are examined employ a wide range of research techniques, are based often on quite different perspectives, and span over half a century. It is striking, however, that there has been so much attention paid to algebra in general and equation solving in particular since 1970. The resulting recent accumulation of bits and pieces of information may leave the impression that what is known about equation solving has not been clearly synthesized. Such an impression is probably correct. The results that have been reported tend to be based on limited data and are not consistent enough to outline clearly the structure of the processes of equation solving.

Student Performance on Equation Solving

In the period 1915-1930 there was considerable effort expended to measure student performance on many aspects of algebra learning. One of these aspects was equation solving. The data are interesting not so much for their specific values but rather for what they reveal about the expectations made of students and about the relative difficulty of solving various kinds of linear equations.

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Moore (1915a, 1915b) reported that in a one-minute test, given in March 1914, of solving equations of the form $\pm ax = \pm b$, 275 algebra students averaged 9.5 attempts and 6.2 correct solutions. In a 12-minute test of solving 13 more complex linear equations, one of which was $7x - 5\frac{6x - 11}{3} = 12$, these students averaged 7.1 attempts and 2.4 correct solutions. (A perhaps interesting aside is that on the entire battery of tests there was no pattern of differences in the number of attempts made by boys or by girls. Boys were, however, more accurate.) Rugg and Clark (1918) gave a 25-item test of simple equations (i.e., integer coefficients) to algebra students in 27 schools. (The time allowed seems to have been about 6 minutes.) On the average, 10 items were attempted and 7.6 were solved correctly. On a test of fractional equations (e.g., $\frac{4x - 2}{3} - \frac{x - 3}{4} = 0$) only 1.1 were attempted and 0.5 were correct. (The time allowed and the number of items on the test are not clearly stated, though it seems these numbers are similar to those for the simple equations test.) Reeve (1926) gave a two-minute, 22-item test of equations of the form, $x + a = b$. The median score was 15. Davis and Cooney (1977) tested 110 algebra students on 12 linear equations. The mean was 8.7 and the median was 10.

Of potentially more use are data indicating student performance levels on specific items. Hotz (1918), Reeve (1926), Küchemann (1978), and Bell, O'Brien and Shiu (1980) each provided data from special populations, while Carpenter, Coburn, Reys, and Wilson (1978) and Carpenter, Corbitt, Kepner, Lindquist, and Reys (1980) summarized data from the National Assessment of Education Progress (NAEP). Some of these data are presented in Table 1.

INSERT TABLE 1 ABOUT HERE

The diffences in samples and the gap in time among the studies permit few generalizations to be drawn. However, it is clear that the larger the number of steps required to solve an equation, the less likely students are to derive the correct solution. It is reasonable, therefore, to try to identify those aspects of the solution process that students are less able to do correctly. Possible sources of trouble might be in combining terms, in transposing terms across the equals sign, in clearing fractions, in performing arithmetic operations, or more fundamentally in understanding the concepts, "variable," "equation," or "equivalent equations." These sources will all be touched on in the remainder of the paper.

Developing Meaning for "Variable," "Equation," and "Equivalence"

Thorndike, Cobb, Orleans, Symonds, Wald, and Woodyard (1928) identified two abilities with respect to equations. The first was to solve the equation, which might mean to get a numerical answer, to solve for one variable in terms of the other, or to find the coefficients (e.g., find a and b in $y = ax + b$) given sufficiently many x, y pairs. The second was to understand the equation as an expression of a certain relationship; that is, to understand that equality is a relation and neither an operator nor an indicator that something is to be produced.

Thorndike, et al., also noted that there could easily be confusion engendered in students' minds by the ways in which letters are used for variables. For example, in the equation $5x + 7 = 4x - 1$, there is one value which when substituted for x produces a true statement. In graphing $y = 6x - 1$, on the other hand, x may assume any value at all. Finally, in the system $5x - 7 = 3y$ and $8y - 1 = 9x$, each of x and y have a unique solution. The similarity of notations to carry several meanings suggests that it is important to know how students perceive "variable." Matz (Note 1) has also identified this need.

No attempt has been made to survey the literature on "variable," but a few references do seem relevant. Davis (1975) noted that students sometimes refused to operate with terms containing a variable because they didn't know what number the variable stood for. Küchemann (1978) suggested six uses for letters used as variables. These uses with examples are as follows:

1. letter evaluated; if $a + 5 = 8$, then $a = ?$
2. letter ignored; if $a + b = 48$, then $a + b + 2 = ?$
3. letter as object; in a triangle with sides a, b, c, P(perimeter) = ?
4. letter as specific unknown; if $e + f = 8$, then $e + f + g = ?$
5. letter as generalized number; if $c + d = 10$ and $c < d$, then $d = ?$
6. letter as variable; which is larger, $2n$ or $n + 2$?

In solving linear equations, uses 1 and 2 play roles. Certainly use 1 is needed for obtaining a solution. Use 2 seems applicable to some extent if any formal procedure is used to obtain the solution.

Tonnessen (1980) has attempted to sort out students' perceptions of the concept, "variable." His work, while preliminary, suggests that there certainly is some confusion about this concept. This possibility of confusion will color all the conclusions of this paper. More needs to be learned about students' understanding of variable.

Recently attention has been paid both to understanding what "equation" means for students and to ways in which this understanding can be expanded (Davis 1975; Kieran 1979, 1980; Herscovics & Kieran 1980; Matz Note 1). Of central importance, as Thorndike, et al. (1928) pointed out, is that students understand that the equals sign is a relation. Davis (1975) also stated this, and Matz (Note 1) stated a similar view in talking about the equals sign as a constraint. Kieran (1980) pointed out that the preponderance of evidence is that elementary school students view the equals sign as a signal to write down an answer. Hence, $\square = 3 + 4$ is backwards. To expand that understanding to include the relational aspects of equality, she organized instruction in three steps.

First, students (N=6 seventh and eighth graders) were asked to write down true number sentences with more than one operation on each side of the equals sign; e.g., $3 \times 5 + 1 = 2 \times 2 + 12$. (Students tended to evaluate from left to right, without using the standard order of operations.) Second, one number was hidden (first with a finger, then with a "box," and finally with a letter) to generate equations while keeping the corresponding true number sentence always retrievable. Third, the rule, "what you do to one side you have to do to the other," was generated through work with number sentences. For example, from $2 \times 5 = 10$, the sentence $2 \times 5 + 7 = 10 + 7$ was generated.

This process is called "didactic reversal" because it builds mathematics by carefully expanding the student's cognition until the concept is attained rather than by breaking down a full blown concept into pieces that will fit into the student's cognition. The procedure suggests that the typical algebra textbook approach of presenting an equation and then substituting the solution to obtain a true number sentence may not be the best way to introduce the concept. However, the data Kieran and Herscovics used are limited, so their ideas should be viewed only as suggestions.

The other part of "equation" that seems critical is the idea of equivalence of equations. (Two equations are equivalent if the domains of the variables are identical and the solutions are also identical.) Wagner (Note 2, 1981) asked students if the equations $7 \times W + 22 = 109$ and $7 \times N + 22 = 109$ had the same solution. She inferred that "conservation of equation" existed if the response was "Yes." If the response was "No, W is larger since W is later in the alphabet," she inferred that conservation of equation was absent. Students who said that the equations had to be solved to know were classified as transitional. About 50% of 12-year-olds and about 80% of 14- and 17-year-olds conserved. There was also a significant correlation ($p < .05$) between conserving and having had algebra, though it is not clear whether there was age confounding in this analysis. Kieran (1979) also reported that students may have the misconception that the solution to an equation changes if the letter used for the variable changes, but she did not measure conservation of equation directly.

Herscovics (1979) claimed that equivalence of equation can be built up in small steps through the didactic reversal process explained earlier. Schematically this is represented in Figure 1.

INSERT FIGURE 1 ABOUT HERE

The checking of the solution is represented by the box in the lower right corner. That is, substituting the solution into the equation generates an arithmetic identity.

Herscovics and Kieran (1980) in an extension of their thinking claimed that "undoing" the equation (that is, apply inverse operations in the reverse order) "brings to the concept of equivalent equations a dynamic flavor that is lost in a formal definition" (p. 579). No data were presented in support of this. Given the extent to which this technique is used in algebra texts and the difficulties that students seem to have with generation of equivalent equations, however, one suspects that "undoing" is not as effective as Herscovics and Kieran suggest.

Kieran (1980) stated that understanding equivalent equations seems essential if the steps in equation solving are to be understood. She noted that step three in the following sequence seems to be a bookkeeping use of the equals sign.

$$2x + 3 = 5 + x$$
$$2x + 3 - 3 = 5 + x - 3$$
$$2x = 5 + x - x - 3 \qquad \text{[clearly not equivalent]}$$
$$2x - x = 5 - 3$$
$$x = 2$$

Similarly, the next sequence is also not uncommon.

$$y + 5 = 8$$
$$= 8 - 5$$
$$= 3$$

The equals sign serves as a link between steps, but equivalent equations in the mathematical sense are not generated. Even in calculus, students write

$$f(x) = 3x^2 + 5x + 7$$
$$= 6x + 5$$

in the process of computing derivatives. The use of the equals sign as a short-cut, or as an indicator of an application of an operator, or as a bookkeeping device needs to be investigated.

Bright and Harvey (Note 3) in a review of literature on equivalent equations concluded that students do not seem to know when equations are equivalent. Perhaps in light of the information on the use of the equals sign, this should be rephrased as students seem not to be concerned whether the equations they write are equivalent. Depending on the role of the equals sign, this lack of concern may be totally appropriate.

The possible confusion in students' minds about the equals sign seems to be the most obvious obstacle to overcome in generating understanding of "equation." It is clear that students do not automatically (or, developmentally) come to the same meaning for the equals sign as do mathematicians. Kieran (1979) has begun to develop ways of expanding students' understandings. Much more work seems called for, however, to explore the implications of these suggestions for students' processes for solving equations.

Equation Solving Processes and Errors in These Processes

Swain (1962) suggested two types of processes that are involved in solving linear equations. One is manipulation; for example, multiplying both sides by the same non-zero number; and the other is reduction; for example, replacing one expression in x by another (equivalent) expression. For example,

$$8x + 7 = 2x - 1$$

$$8x + 7 + (-7) = 2x + (-1) + (-7) \qquad \text{manipulation}$$

$$8x = 2x + (-8) \qquad \text{reductions}$$

$$8x + (-2x) = 2x + (-8) + (-2x) \qquad \text{manipulation}$$

$$6x = -8 \qquad \text{reductions}$$

$$\left(\frac{1}{6}\right)(6x) = \left(\frac{1}{6}\right)(-8) \qquad \text{manipulation}$$

$$x = -\frac{3}{4} \qquad \text{reductions}$$

Romberg (1975) used the phrase "sentential transformation" for these operations, and Matz (Note 1) described these operations as deductions and reductions.

Bundy (Note 4) and Bundy and Welham (Note 5) in developing a computer program to solve equations, identified three operations that are generalizations of Swain's categories. Isolation is performed if there is a single occurence of x; for example, if $3x = 12$ then the computer program divides both sides by 3 to produce $x = 4$. This is a particular instance of manipulation, at least for linear equations. Collection occurs if the number of occurences of the variable can be reduced; for example, $7x + (-3x)$ can be replaced by $4x$. This is a particular kind of reduction, but apparently in this particular program, collection can occur only when instances of x are essentially adjacent. Attraction is the procedure used to get instances of the variable closer together; for example,

$$12x + 7 = 4x - 1$$

$$12x + 7 + (-4x) = -1$$

and

$$12x + 7 + (-4x) = -1$$

$$12x + (-4x) + 7 = -1$$

would both be illustrations of attraction. This process may reflect what students think as they solve linear equations, but it also may be too formal (mathematical) to be an accurate representation. Heller and Greeno (1979) pointed out that knowledge of Bundy's three operations is not sufficient for solving equations. There must also be a higher-order strategy for choosing which operator to apply. There must be some guiding process.

Byers and Herscovics (1977) also pointed to the variety of guiding processes that students might bring to equation solving, but phrased their discussion in terms of "understanding." Four kinds of understanding were identified: (a) instrumental, in which rules are applied without knowing why, (b) relational, in which specific rules for a particular problem are derived from more general rules, (c) intuitive, in which the problem is solved based on some prior analysis, and (d) formal, in which the symbolism and notation are connected to relevant mathematical ideas to get a deductive chain. In solving the linear equation $x + 3 = 7$, students might exhibit the four levels by (a) transposing the number and changing the sign, (b) adding -3 to (or subtracting 3 from) both sides, (c) guessing, or (d) generating the string

$$x + 3 = 7$$
$$x + 3 + (-3) = 7 + (-3)$$
$$x + 0 = 4$$
$$x = 4$$

Carry, Lewis, and Bernard (Note 6) and Lewis (in press) studied the way college students solved various equations. Their work was influenced by Bundy (Note 4) and Bundy and Welham (Note 5), and although they didn't directly analyze levels of understanding, their data could be used to investigate these levels. Figure 2 shows the 14 equations that were presented to 19 introductory psychology students, 15 mathematics education and engineering students, and five research mathematicians.

INSERT FIGURE 2 ABOUT HERE

Each subject was videotaped twice. Seven equations were presented in the first session and each subject was asked to "think aloud." For the second set of seven equations presented in the second session, each subject was asked to explain the method of solution as if to a student asking for help on homework. For problem 2B, there were several differences in choices of strategies among the subjects. (The

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five mathematicians were called experts.) In particular, the experts were sometimes much more consistent in their choice of strategies. (See Table 2.)

INSERT TABLE 2 ABOUT HERE

Yet, for equation 2A, designed to be analogous to 2B, a consistent strategy was not used even among experts, and for equation 5A there was notable consistency across all four groups. (See Table 3).

INSERT TABLE 3 ABOUT HERE

Lewis suggested that the lack of similar patterns in the data, especially among the experts, may be due to the fact that equations like those in Figure 1 were so infrequently encountered that there was no need to invest the time and effort necessary to find the most concise solution process.

Of perhaps equal interest in these data were the categories of errors identified among solution attempts. Carry, Lewis, and Bernard noted a variety of categories, some of which were (a) strategy difficulties, (b) operator gaps, (c) other, and (d) arithmetical. Strategy difficulties included cancellation errors of several types (e.g., $\frac{x}{2 + x}$ becomes $\frac{1}{2}$, $x^2 - x$ becomes x); transposition errors (e.g., $7x + 8 = x + 2$ becomes $8x + 8 = 2$), combination errors (e.g., $x^2 + x + 3$ becomes $x^3 + 3$, $\frac{x}{1} + \frac{x + 1}{2}$ becomes $\frac{x + x + 1}{2}$), cross-multiplication errors (e.g., $\frac{1}{x} + \frac{1}{7}$ becomes $7 + x$), and splitting-equation errors (e.g., $\frac{5}{10} = \frac{x - 10}{x + 5}$ becomes $5 = x - 10$ and $10 = x + 5$). Some of the errors related to fractions have correspondence in work with common fractions (e.g., Bright & Harvey, Note 3). Some of the cancellation errors (e.g., $x^2 - x$ becomes x) may be language related (e.g., Davis & McKnight, Note 8), and the splitting-equation's error may be an overgeneralization of other equation solving techniques (e.g., Matz, Note 1).

Operator gaps were inferred from halts in student work. Varieties included lack of inversion (e.g., $\frac{1}{x} = \frac{a}{b}$ but not $\frac{x}{1} = \frac{b}{a}$), lack of clearing fractions (e.g., $\frac{2x + 3}{x^2} = 1$ but not $2x + 3 = 1 \cdot x^2$), lack of distributivity (e.g., $ax + bx = c$ but not $(a + b)x = c$), and dead ends (e.g., $p = A - prt$ for equation 1A). These gaps suggest that substantive information may be missing from the students' backgrounds.

The "other" category included fraction errors (e.g., $\frac{2}{x}$ becomes $2x$), grouping errors (e.g., $\frac{x + 2(x + 2)}{x + 2}$ becomes $\frac{(x + 2)(x + 2)}{(x + 2)}$), and distributivity errors (e.g., $2(x + 1)$ becomes $2x + 1$). Davis and McKnight (Note 8) also noted errors in misuse of parentheses and in using distributivity.

Carry, et al., put all the errors into three types: (a) operator, reflecting incorrect or incomplete knowledge, (b) applicability, which was mostly mishandling of parentheses, and (c) execution, which included partial executions, misreading, and miscopying. The first two types seem amenable to correction by instruction. The third may not be easily altered.

Lewis (in press) noted that the experts also made errors (e.g., transposition, confusion of numerator and denominator, incorrect cancellations) similar to those of college students, though at a lower rate. Many of these errors seemed to occur when more than one operation was done at once. Thus, some of the errors may have been careless. Yet it seems important that the errors were of the same kinds as those made by the college students.

Davis and Cooney (1977) also categorized errors made in solving linear equations, but the errors were from written records only; there were neither videotape records nor "thinking aloud" records to supplement the written work. Data were gathered from 72 regular algebra I students and 38 second-year basic algebra (algebra I in two years) students. The equations presented are given in Figure 3.

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INSERT FIGURE 3 ABOUT HERE

The categories of errors were (a) mistake in addition of real numbers either as numbers or as coefficients of x, (b) mistakes in multiplication of real numbers, (c) transposing errors (similar to strategy difficulties discussed earlier) either for addition or multiplication, (d) confusion about additive or multiplicative inverses, (e) incomplete work (similar to operation gaps discussed earlier), (f) miscopying, (g) combination errors (e.g., $-4 + 8x = 4x$), and (h) undecipherable. The similarity of these errors and those identified by Carry, et. al., seems remarkable. Too, the students made many computational errors, and there seemed to be no difference in the distribution of errors between the two kinds of algebra students. This reinforces the observation of Lewis (in press) that experts and college students made similar errors. However, the distribution of errors of those students who solved ten or eleven of the equations correctly indicated mostly (75%) computational errors rather than process errors (16%), while the errors of those students who solved two to seven equations correctly were less (50%) computational and more (38%) related to processes for solving. The difference reported between good and poor equation solvers should probably be further investigated.

DeVincenzo (1980) tested 1122 ninth-graders on both arithmetic and algebra skills and reported that for fraction skills there was consistency in errors in arithmetic and algebra work. However, more errors ($p < .05$) on the distributive principle occurred in the algebra material. This suggests that some errors in algebra might be avoided by careful building up of arithmetic skills, but there may be difficulties in algebra that improved number skills will not mitigate.

Numerous researchers have pointed out errors in equation solving. Monroe (1915a) noted arithmetic errors, copying errors, and incomplete solutions, Rugg and Clark (1918) noted arithmetic errors, combination errors (e.g., 4c - 6c

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becomes 2c, or 3x + 4 becomes 7x), incomplete solutions, transposition errors, and inverted divisions (e.g., $5x = 13$ becomes $x = \frac{5}{13}$). The last one is certainly related to errors in fractions (Bright & Harvey, Note 3). Reeve (1926) noted combination errors and mixed operation errors (e.g., $\frac{1}{3}y = 3$ becomes $y = 1$).

Matz (Note 1) noted that some procedures seem to be overgeneralized by students. For example, students learn that if $x \cdot y = 0$ then $x = 0$ or $y = 0$. This is sometimes erroneously generalized to the rule that if $x \cdot y = a$, for any a, then $x = a$ or $y = a$; that is, in solving $(x - 3)(x - 4) = 12$ students write $x - 3 = 12$ or $x - 4 = 12$. Davis and McKnight (1979, Note 7) suggested that the role of 0 as a special number in the correct rule may not be adequately emphasized when the rule is learned.

Other algebraic errors noted by Matz that may interfere with equation solving included (a,b,c,d may be numbers or algebraic expressions) (a) $\frac{a}{b} + \frac{c}{d}$ becomes $\frac{a + c}{b + d}$, (b) $\frac{a}{b + c}$ becomes $\frac{a}{b} + \frac{a}{c}$, (c) $\frac{a}{b} + \frac{c}{d}$ becomes $ad + bc$, (d) $3a^2$ is interpreted as $(3a)^2$, and (e) $\sqrt{a + b}$ becomes $\sqrt{a} + \sqrt{b}$ (also noted by Davis & McKnight, Note 7). These are clearly not unique to algebra, and at least as far as the fraction errors are concerned, may be extensions of arithmetic errors (Bright & Harvey, Note 3).

Meyerson (1976) observed that typical remediation of errors like $\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$ often takes the form of either substituting numbers for a, b, c, d to generate a false statement or deriving the true relationship ($\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$) algebraically (or some combination of these two). He claimed that this technique is based on two assumptions; first, that the pupil's belief in the mistake is not strong, and second, the error is more random than systematic, Myerson noted that if one speculates as to why students use incorrect rules, then different and perhaps more effective remediation techiques might result. For example, $\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$ may be derived from an overgeneralized multiplication of fractions rule, or it may be an overgeneralized 'baseball addition' rule. That is, if a batter has 3 hits in 5 attempts on Monday and 1 hit in 2 attempts then the cumulative record is 4 hits in 7 attempts ($\frac{3}{5} + \frac{1}{2} = \frac{4}{7}$). In ei[illegible]r case, the incorrect rule is frequently reinforced within the domain that it originated. Remediation, therefore may require careful reanalysis of the mutu[illegible] i[illegible]erference among mathematical rules and 'everyday' mathematics and may not [illegible] accomplished simply. Davis, Jockusch, and McKnight (1978) used the term, binary confusion, to denote the interference between two rules. (See Figure 4.)

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Insert Figure 4 about here.

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If the $S_1 \rightarrow P_1$ chain is learned earlier and well, and if both the stimuli S_1 and S_2 and the products P_1 and P_2 are similar, then the student may generate the incorrect chain $S_2 \rightarrow P_1$. Shevarev (1946) in discussing this same example suggested that the incorrect chain $S_2 \rightarrow P_1$ seemed to be learned at the time of instruction on $S_2 \rightarrow P_2$ because the students were already orien[illegible]ed toward the addition of exponents ($S_1 \rightarrow P_1$).

Occassionally, interference may arise from non-mathematics sources. Kieran (Note 8) observed that junior high school students seemed to perform multiple arithmetic operations from left to right; for example, 3 + 4 x 5 is 35 rather than 23. Perhaps this is interference from reading instruction, reinforced by use of simple calculators with 'left-to-right' orientation. If student do generate rules like this before beginning algebra, because of the absence of instruction to the contrary, then it may be very difficult to overcome the student's belief in the incorrect rule.

The discussion of student's equation solving procedures and errors so far roughly encompasses Byers and Herscovics' (1977) instrumental, relational, and formal levels of understanding. The undiscussed level, intuitive, also derserves some attention.

Everett (1928) suggested that for an equation like $4x + 5 = 17$, one might reason that 17 is 5 more than 4x, so $4x = 12$ and that 12 is 4 times as large as x, so $x = 3$. Davis and McKnight (Note 7) reported one instance in which this type of reasoning occurred more or less spontaneously. In solving $\frac{5}{x} = \frac{10}{3x - 7}$, the student reasoned,

> they've doubled the numerator [10 is twice as large as 5] ... [so] they must have doubled the denominator. But instead of 2x, they put 3x, so there is one x too many. [$3x - 7 = 2x + (x - 7)$]. But then they fixed up the extra x by subtracting 7. But if taking away 7 gets rid of the extra x, then x must be 7. (pp. 14-15).

The student seemed to reason that since $10 = 2 \cdot 5$ then $3x - 7 = 2 \cdot x$, so that the difference $(3x - 7) - 2x$ must be 0. Bell, O'Brien and Shiu (1980) in working with 53 algebra students noted intuitive kinds of reasoning in solving $3x + 7 = 28$ and $24 - 5x = 9$. For $29 = 14 - 5x$, however, the students could not conclude that $5x = -15$; that is, there seemed to be a blockage in being able to reason about subtraction of negative numbers. For $\frac{1}{4}(3x + 5) = 14$, most students applied the distributive principle to the left side first, rather than reason that $3x + 5$ must be $4 \cdot 14$.

Petitto (1979) interviewed nine ninth-grade students and asked them to solve several equations. (See Figure 5.)

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Insert Figure 5 About Here.

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She defined "formal processes" to mean that a linear sequence of steps was performed each of which was explicitly described as an instruction or rule applicable to a class of problems. "Informal processes" meant that the solution was organized according to perceived properties and relationships within a particular problem. She reported that intuitive techniques did not always generalize. For example, one subject solved $\frac{2}{5} = \frac{4}{x + 5}$ by noting that $4 = 2 \cdot 2$ so $x + 5 = 2 \cdot 5 = 10$, so $x = 5$. The same subject couldn't solve $\frac{14}{23} = \frac{56}{x + 2}$ and apparently did not see that $56 = 4 \cdot 14$. The subject's response to the more difficult problem was withdrawal. Petitto noted that students who used a combination of formal and informal processes were more successful than those who used only one of these processes. The switching between techniques noted by Bell, et al., seems to reinforce this conclusion.

The processes and errors presented in this section suggest several conclusions. First, errors are not random, but they also may not be effectively algorithmic. Errors may be interpretable as overgeneralizations of rules to domains which are inappropriate, but the cause of this overgeneralization may be lack of attention by the teacher to specifying the limits on rules. To assume like Davis and McKnight (1979) that students spontaneously, and perhaps unconsciously, search for 'deeper-level rules' may be a stretch of the information processing view of the world to unreasonable limits. Students may apply learned rules whenever there is not a prohibition to refrain.

Second, the possible interference among concepts or rules should be dealt with direclty. Probably this means that a teacher should identify explicitly at least some of the possible ways that the concept or rule being taught is _not_ an instance of earlier-learned concepts or rules.

Third, the apparent use of both formal and informal techniques by students suggests that instruction should include consideration of both types of processes. Flexibility in approach, suggested by the lack of consistent use of a single process for solving a given equation (Lewis, in press) may be the best goal for instruction on equation solving. Explicit attention should be given to helping students recognize what might cause a failure to reach solution. Carry, et al., (Note 6) classified such a wide variety of causes of failure that it is unreasonable to expect instruction to deal with them all. More studies need to be conducted to identify the most common causes so that instruction can be appropriately focused.

Instructional Techniques

A small amount of work has been done on improving the overall effectiveness of group instruction on equation solving. Adi (1978) investigated the relationship of the Piagetian levels of concrete and formal operations to formal and informal instruction on equation solving. Of the 75 prospective elementary school teachers that participated, 37 were at the early concrete (IIA) stage, 26 were at the late concrete (IIB) stage, and 12 were at the early formal (IIIA) stage. Cover-up and formal methods for equation solving were taught to all subjects. (Cover-up techniques were taught first.) A posttest of 12 equations was given to all subjects; instructions were that the first six were to be solved by informal methods and the last six were to be solved by formal methods. For both subscales, the IIB and IIIA subjects scored higher ($p < .01$) than the IIA subjects. This suggests that students at the early concrete stage may require substantively different instruction in equation solving.

Whitman (1976) studied the effects of intuitive equation-solving instruction on the formal techniques used by students. Seventh-graders ($N = 156$) were taught one of four ways: (a) intuitive techniques only (I), (b) intuitive followed by formal (IF), (c) formal followed by intuitive (FI), and (d) formal techniques only (F). The I students performed better than the IF students though the grade-level of the subjects calls into question their preparation to do algebra. Interviews with 31 students indicated that those who had received 'intuitive' instruction, regardless of when, used those techniques almost exclusively.

The F students had considerable difficulty. These conclusions, however, are clearly limited by the possibly inadequate or unstable preparation for algebra.

Davis (1975), however, reported that heuristic problem analysis techniques did not work well with bright seventh-grade algebra students. He found those students desirous of having well-formulated procedures to follow. Davis and McKnight (1976) distinguished between an S-algorithm (S for specific), which is a rote algorithm like "add the opposite of the constant to both sides", and a D-algorithm (D for deep), which is a more heuristic algorithm like "clear fractions first." No one seems to have investigated the instructional effects of this distinction.

Neves (Note 9) also has tried to tie down a clear picture of effective procedures for teaching equation solving. He proposed a computer program that would learn to solve equations by examining worked-out examples from textbooks. Two parts of the program are especially interesting. First, the program identifies the symbols that have been removed, transformed, or added from each step to the next. Second, the program searches for an operator (i.e., an algorithm in its library of algorithms) that will produce the identified difference. If one does not exist in its library, it asks the programmer for a new operator to put in its library that will produce the difference. The obvious analogy is that a student can ask a teacher for help. The difficulty with the analogy is that students usually assume they are supposed to know all the appropriate algorithms. Perhaps more time should be spent helping students know when their 'libraries' are incomplete.

Of the remaining instructional studies of equation solving, not much can be said. Davis (1976) taught each of two groups of eighth-grade students linear equation solving by either encoding/decoding skills or the traditional textbook approach. Although both groups learned ($p < .01$), there were no significant, between-group comparisons. Brandner (1976) tested 177 male algbra I students and observed that a guess-and-check procedure was used frequently. Although it might be possible to refine students' guessing procedures, this instructional technique does not seem to hold much long-term hope. Settle (1977) in a study of writing equations for verbal problems reported that a guess-and-test procedure in which the equation seemed to be derived from an arithmetical identity (in the sense of Kieran 1979) was more effective than the standard technique of starting by defining a variable. This may be because of confusion about the concept of variable. Comparing Settle's work to Kieran's might produce a clearer picture of appropriate instructional technqies. Along a similar vein, Stephens (1980) reported that students who equated unknown quantities before translating the relationships into symbolism were less successful in solving the resulting equations than students who first identified the unknown and then used that unknown to write equations. Why this result should be observed is not clear, but it may have to do with the meaning attached to the variable.

Finally, two microcomputer studies need to be mentioned, though it is not clear to what extent the use of microcomputers confounds the results. Boysen and Thomas (Note 10) asked 96 eighth-graders to practice solving linear equations by specifying operations for a PET microcomputer to perform; e.g., add 7 to both sides. The teacher provided the regular, in-class instruction. In one condition the students received feedback on whether the operation was correct, and if it was not the operation was not performed. In the other condition each operation was performed without any feedback on whether it would further the solution. Students used the PET for six sessions of 15 minutes each. The aptitude, field independence/dependence, was tested on all subjects. On a transfer task of simplifying complex linear equations, the explicit feedback condition was better for high field independent students and the no feed-back (or implicit feedback) condition was better for low field independent students ($p < .05$). This aptitude-by-treatment interaction may be important for structuring practice on equation solving, but it needs to be researched further.

Moore (1980) used the games POE (a computerized strategy game designed to help students learn to use the computer and to learn the rules of EQUATIONS) and EQUATIONS with 41 of 89 university entry-level algebra students. POE was played for two weeks and EQUATIONS was played for six weeks. The game-playing was apparently not a required activity, and only 12 of the 41 experimental subjects played beyond the fourth week. There was no significant game effect, but this seems at least in part due to non-participation in the experimental treatment.

The limited work on instruction is clearly not coherent. Differences in students seem to be indicative of differences in success of instructional procedures, but the patterns of differences do not seem to be clear. Perhaps careful reexamination and reinterpretation of the treatments in these studies in light of the results now available on students' equation solving processes would yield comprehensible conclusions.

Synthesis

Students have historically performed less well in solving increasingly complex linear equations. Only recently, however, has attention for this failure focused on specific aspects of the equation solving process. For example, in spite of the observation by Thorndike, et al., (1928) that one of the essentials for equation solving is understanding of the equals sign as a relation rather than an operator, Kieran (1979) seems to be the first person to investigate the extent to which this relational understanding is present in students ready to begin algebra.

ERIC

The concept "variable" would clearly seem to be critical for being able to solve equations. Yet, very little seems to be known about perception of "variable" especially in the equation solving context. Kücheman (1978) and Tonnessen (1980) both presented data that speak to an important lack of understanding.

For standard equation solving procedures taught in high school algebra I, understanding the concept "equivalence of equations" would also seem critical. Even less, however, is known about student's perceptions of this than of "variable". The best that can be said is that there is some speculation about how to teach equivalence of equations effectively (e.g., Herscovics 1979).

More information about equation solving seems to be focused on the errors that students make. The consistency of error classifications across several studies suggests that further refinement of error categories may not be important. The fairly wide range of categories suggests that instructional remediation of the errors is probably not simple. One bright spot is that DeVincenzo (1980) suggested that building arithmetic sklls, especially related to fractions, might help avoid some of the algebra errors. The literature survey of Bright and Harvey (Note 3) on equivalence of fractions tends to support the observation that algebra errors and fraction errors are similar.

Many studies point to the fact that informal processes are both available to and used by students. This suggests that instructional procedures should not totally ignore informal processes. Rather, coordination of formal and informal processes should be investigated. Perhaps, at least for above average students, informal processes might be used to lead students to formal processes. Kieran's model (1979) for teaching understanding of the equals sign may also serve as a guide for creating instruction on informal and formal equation solving processes.

Instructional Implications

For instruction the most important result from studies of equation solving may be that understanding of the equals sign is often not adequate for teaching equation solving. Kieran (1979) and Herscovics and Kieran (1980) give good guidelines for beginning to develop instruction to expand students' understanding of this concept. It should be remembered that there is no guarantee that these procedures will work; the sample of six students from which the instruction evolved is terribly inadequate for creating confidence that the instruction will in any sense be generally successful. Buth the suggestions do give teachers a point from which to start, and teachers who are confident of their abilities to use student reactions to build non-standard techniques into effective instruction should feel encouraged to dive in.

The concepts of "variable", "equation", and "equivalent equation" also may not be adequately learned by students. Unfortunately there are few indications from the research literature as to what teachers can or should do to improve understandings. One suggestion is to ask students individually what these concepts mean. Lack of clear understanding or clear misunderstanding should be dealt with individually, perhaps by work with specific examples. A second suggestion is for each teacher to review her/his own understanding of these concepts and on the basis of this review to develop discussions of these concepts for use in class. A third suggestion is that teachers should not assume that students will spontaneously develop understanding of these concepts just from doing equation solving. Explicit attention should be given to these concepts during instruction on equation solving.

There does seem to be a definite decline in student performance as the complexity of the equations increases. This is of course not surprising. But it does reinforce the fact that teachers should set realistic goals for students in terms of the kinds of equations students are expected to solve.

The most important implication for teaching related to student performance is that teachers should become aware of the kinds of errors that students make. One very important type of error is the possible overgeneralization of procedures by students to inappropriate domains. While there may be general remediation procedures to prevent this, the more obvious pedagogical approach to this problem would seem to be to specify explicitly both the limits within which a procedure applies and at least some of the regions in which the procedure does not apply. For example, cancellation is not an appropriate technique if the expression to be cancelled has any possibility of being zero; e.g., in $5(x - 7) = 3(x - 7)$, $x - 7$ should not be cancelled. Related to this approach is that algebra-specific errors; e.g., work with exponents; should become very familiar to teachers so that these errors can be at least somewhat diminished by careful instruction on the skills. In particular, the examples used by teachers should include cases like those which are known to cause trouble for students. In this way students will have appropriate models to use in their own work. Perhaps underlying all of this is that students' fraction skills should be monitored to be sure that a lack of these skils is not interfering with the development of algebra skills.

Finally, students' use of informal or intuitive skills should not be totally discouraged. Intuitive skills along with formal skills seem to make a more powerful tool than either technique alone. Acceptance of the intuitive skills that students bring into an algebra class and help in supplementing those skills with equally powerful formal procedures seems called for. Intuitive skills should not be allowed to atrophy. The failure of intuition in complex situations can be a powerful motivator for learning formal procedures.

ERIC

Equation solving is a complex task. For many students it may be the first mathematical procedure that is not completely determined by an algorithm. This alone may make it somewhat mystical for students, but to complicate it further, there are numerous chances for students to make errors that are not directly related to equation solving. Still further confusion may result from a lack of clear understanding of the objects (i.e., equations) being studied. Flexibility in approaching the task, both on the part of the teacher and on the part of the student, may be the key to instructional effectiveness. Flexibility will at least allow modifications in instruction to be made when students' performance levels are not acceptable. After all, this seems to be one of the main purposes of teaching.

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Table 1

Student Performance Data on Solving Linear Equations

Hotz (1918)[a]				Reeve (1926)[b]		Küchemann (1978)[c]	
Item	% Correct 3 months	6 months	9 months	Item	% Correct	Item	% Correct
				$x + 5 = 9$	97.5	$a + 5 = 8$	92
$2x = 4$	96.5	99.3	99.8	$2z = 10$	97.3		
$3x + 3 = 9$	86.7	97.6	98.0	$4x + 5 = 17$	95.7		
$7m = 3m + 12$	81.9	95.7	96.3	$8x = 5x + 12$	89.3		
$5a + 5 = 61 - 3a$	74.9	92.9	93.3	$6x + 3 = 2x + 35$	80.0		
$10 - 11z = 4 - 8z$	65.2	92.1	92.8				
$7n - 12 - 3n + 4 = 0$	67.5	88.7	90.1				
$c - 2(3 - 4c) = 12$	54.1	76.0	79.6				
$\frac{2}{3} z = 6$	47.6	77.6	79.7	$\frac{1}{2} x = 6$	82.8		
$\frac{2x}{3} = \frac{5}{8}$	48.3	69.4	78.4				
$\frac{1}{4} (x + 5) = 5$	6.2	68.4	70.3				
$\frac{1}{2} x + \frac{1}{4} x = 3$	39.9	47.7	68.9	$\frac{1}{3} x + \frac{1}{2} x = 30$	70.7		
$\frac{y}{3} = \frac{5}{2} - \frac{y}{4}$	17.3	57.2	63.8				
$\frac{4}{3 - x} = \frac{2}{1 + x}$	13.1	28.0	48.1				
				$0.4x - 5 = 3.8$	45.3		

[a] N = 3047 students. The number of months is the number of months the students had studied algebra.

[b] N = 1204 students.

[c] N = 3000 students in U.K.

[d] N = 53 students in U.K.

[e] Students were 17-year-olds in U.S.

Students were 17-year-olds in U.S. and had completed algebra I (or algebra II)

Bell, et. al. (1980)[d]		Carpenter, et. al. (1978)[e]		Carpenter, et. al. (1980)[f]		
Item	% Correct	Item	% Correct	Item	% Correct	
					Algebra I	Algebra
		$x - 3 = 7$	95			
$3x + 7 = 28$	100	$3x - 3 = 12$	75			
$24 - 5x = 9$	66					
$29 = 14 - 5x$	9	unreleased item requiring combining terms with variables and numbers on both sides	36	$3x + 6 - 14 = x + 2$	44	63
$8x = 16 + 16x$	8					
$17 + 6x = 2x + 9$	9					
		unreleased item, $\frac{a}{b} = \frac{x}{c}$, with $c = bn$	77			
$\frac{1}{4}(3x + 5) = 14$	28					
				$30 = \frac{2}{5c} + 10$	43	62

Table 2

Strategies for Problem 2B

Group[a]	Choice of Stategy[b]		Choice of First Step[b]	
	Transpose - Invest	Other	Transpose	Other
E	4 (80)	1 (20)	5 (100)	0 (0)
T	4 (40)	6 (60)	8 (80)	2 (20)
M	3 (21)	11 (79)	9 (64)	5 (36)
B	0 (0)	10 (100)	0 (0)	10 (100)

[a]E = experts (professional mathematicians)
T = top 10 students
M = middle 14 students
B = bottom 10 students

[b]Entries are numbers (percentages) of subjects in each group.

adapted from Lewis (in press).

Table 3

Strategies for Problems 2A and 5A

Group[a]	Strategy for Problem 2A[b]		Operation for Problem 5A[b]		
	Transpose-Invert	Other	Cross-Multiply	Clear Fractions	Other
E	0 (0)	5 (100)	5 (100)	0 (0)	0 (0)
T	2 (20)	8 (80)	6 (60)	4 (40)	0 (0)
M	1 (7)	13 (93)	11 (79)	2 (14)	1 (7)
B	0 (0)	10 (100)	5 (50)	2 (20)	3 (30)

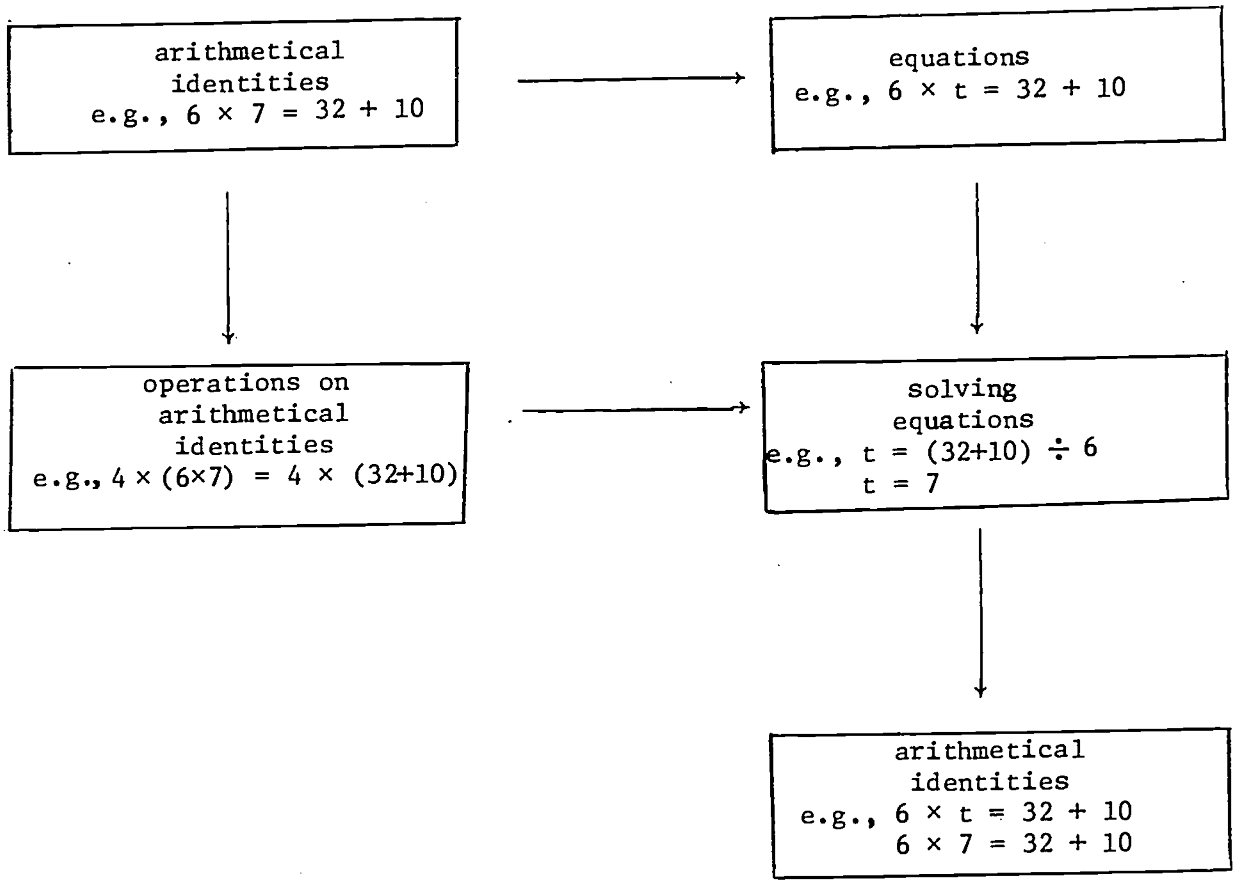
[a]E = experts (professional mathematicians)
T = top 10 students
M = middle 14 students
B = bottom 10 students

[b]Entries are numbers (percentages) of subjects in each group.

adapted from Lewis (in press).

Figure 1

Schematic Drawing of a Means for Teaching "Equation"



adapted from Herscovics (1979)

Figure 2

Equations Used by Carry, Lewis, and Bernard

1A	$A = p + prt$, solve for p	1B	$2x = x^2$
2A	$\frac{1}{3} = \frac{1}{x} + \frac{1}{7}$	2B	$\frac{1}{R} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, solve for x
3A	$9(x+40) = 5(x+40)$	3B	$7(4x-1) = 3(4x-1) + 4$
4A	$xy + yz = 2y$, solve for x	4B	$\frac{x + 3 + x}{x^2} = 1$
5A	$\frac{5}{30} = \frac{x - 10}{x + 5}$	5B	$\frac{1 - x^2}{1 - x} = 2$
6A	$x + 2(x+1) = 4$	6B	$x + 2(x+2(x+2)) = x + 2$
7A	$x - 2(x+1) = 14$	7B	$6(x-2) - 3(4-2x) = x - 12$

adapted from Carry, Lewis, and Bernard (Note 6)

Figure 3

Equations Used by Davis and Cooney

1. $5x + -4 = 8x + 8$

2. $3x - {}^{-}5 + -20 = 4$

3. $8 = -5 + x$

4. $x + -2 = -6$

5. $7x + 24 = 3x$

6. $-8 + 5 + \frac{3}{4} x = 15$

7. $\frac{3}{2} x = \frac{5}{7}$

8. $\frac{3}{2} x + 4 - 6 = -11$

9. $5x + {}^{-}7 + {}^{-}2x = -17$

10. $8 = \frac{2}{5} x + 5 - {}^{-}3$

11. $4x = 7x - 36$

12. $-21 = 10 - 4 + \frac{3}{5} x$

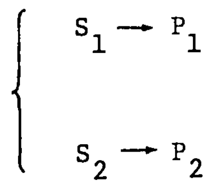
NOTE: The mixture of raised and unraised negatives signs reflects the eqations as printed in Davis and Cooney (1977).

adapted from Davis and Cooney (1977)

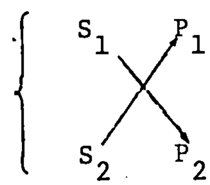
Figure 4

Binary Confusion

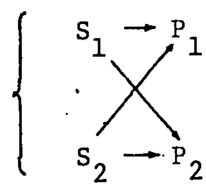
Original Learning[a]



Incorrect Associations



Binary Confusion Scheme



[a] S_i = stimulus i, P_i = product i

adapted from Davis, Jockusch, & McKnight (1978)

Figure 5

Equations Used by Petitto

Content of Equations

Difficulty of Equations	Familiar	Unfamiliar	
Easy	$\frac{1}{2} = \frac{x}{4}$	$\frac{1}{2} = \frac{2}{x + 3}$ $\frac{1}{3} = \frac{2}{x + 1}$	$\frac{2}{x} = \frac{4}{x + 5}$
Intermediate	$\frac{3}{4} = \frac{x}{8}$	$\frac{2}{5} = \frac{4}{x + 5}$ $\frac{3}{10} = \frac{9}{x + 5}$	
Difficult	$\frac{2}{9} = \frac{x}{135}$	$\frac{14}{23} = \frac{56}{x + 2}$	$\frac{13}{x + 2} = \frac{39}{x + 35}$

adapted from Petitto (1979)