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ABSTRACT

This module is part of a series designed to be used by life science students for instruction in the application of physical theory to ecosystem operation. Most modules contain computer programs which are built around a particular application of a physical process. This module is used to introduce the biology student to differential calculus, a branch of mathematics which is being used more often in recent ecological and physiological models. Since biological systems are dynamic, their mathematical models must describe rates of change of relevant variables, a process which requires calculus. The module introduces modeling in biology by reviewing differential calculus and using only examples from life sciences. A problem set reviews information in the text proper and presents additional information not included in the text. An associated computer program, DIFF, uses graphics to check the user's own calculations and to demonstrate the validity of general solutions. (Author/CS)

PHYSICAL PROCESSES IN TERRESTRIAL AND AQUATIC ECOSYSTEMS

APPLIED MATHEMATICS

CALCULUS-DIFFERENTIATION

by

Richard C. Hertzberg

CENTER FOR QUANTITATIVE SCIENCE IN
FORESTRY, FISHERIES AND WILDLIFE
University of Washington

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PREFACE

Calculus has quietly invaded several areas of biology in the last few years, reflecting a greater desire for precise explanations of biological phenomena. Mathematical modeling of rates of dynamic processes requires calculus. This module introduces modeling in biology by reviewing differential calculus using only examples from life sciences. The problem set should be worked since several ideas are presented which are not in the text proper. An associated computer program, DIFF, uses graphics to check the user's own calculations and demonstrate the validity of general solutions. Previous exposure to calculus is recommended but not required.

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INTRODUCTION

Differential calculus is being used more often in recent ecological and physiological models as data become more precise and the processes become better understood. Since most biological systems are dynamic, their mathematical models must describe rates of change, not just current values, of the relevant variables. Although most models consider changes over time, the techniques of calculus depend only on the mathematical function involved and thus any independent variable may be used, such as spatial dimensions, organism weight, temperature, etc. As a result, the implications from a model of one process may be applied to the model of a different process as long as the mathematical functions involved are the same.

FUNCTIONS OF ONE VARIABLE

Rates of Change

The simplest graph of a dynamic relationship is a straight line. The equation for a straight line is

$$y = mx + b \qquad (1)$$

where y and x are variables, m and b are constants. An example of this relation is the oxygen uptake by the lobster. The oxygen consumption (y) depends on the oxygen concentration (x) in the surrounding environment, so that y is a function of x. A typical graph of this function in shown in Figure 1.

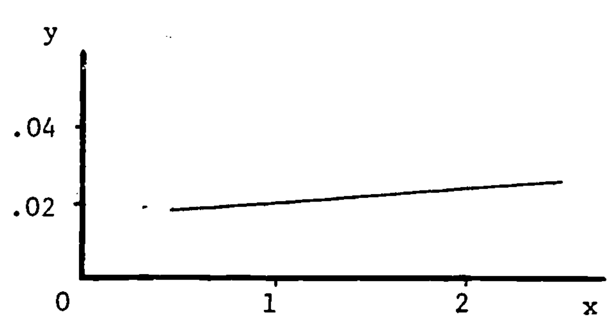

Figure 1. Oxygen consumption.

The equation for this function is

$$y(x) = .004x + .016$$

The number .004 represents the slope of the line; that is, the ratio of the change in y to the change in x. When x changes by 1 unit, y changes by .004 units. In this particular application, when the water O_2 concentration increases by one ml/l, the lobster O_2 consumption increases by .004 ml per hour-gm body weight. The slope (m in Equation (1)) then represents the rate of change of y with x.

When the graph is not a straight line, the function it represents is more complex than above and the rate of change cannot be expressed so easily. Note that for each unit change in x in Fig. 1, y changes by .004, regardless of the value of x. The rate of change is then constant. In the graph of Fig. 2, the rate of change is not constant. To see this, approximate Fig. 2 by two connected tangent lines (Fig. 3a) and note that the slope differs with each line. As the approximation improves (Fig. 3b), it uses more lines and thus presents more slopes. Using an infinite number of lines, we would duplicate the curve (in Fig. 2) and have a slope that changes with each value of x. The slope then depends on x and clearly is not constant. In fact, one definition of the slope of a curve at a point is the slope of the tangent line at that point.

Note that the slope is ambiguous at the points where two straight lines meet (Figure 3a). We say the slope is "undefined" at such "corner" points.

In general, the rate of change of a function is also a function of x and possesses its own equation. The rate of change is denoted $\frac{dy}{dx}$ to reflect the ratio of the change in y to the change in x. When the graph

Figure 2. Example of a function with a changing slope.

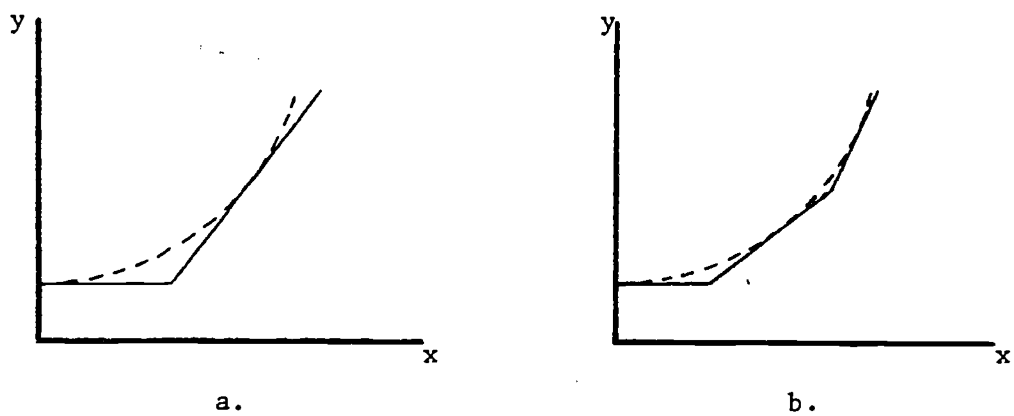

Figure 3. Approximations to curve in Figure 2.

is a straight line, the equation for y is

$$y(x) = mx + b$$

and the rate of change is

$$\frac{dy}{dx} = m$$

The rate of change is called the derivative. Its functional form depends on the equation for y. For example, an empirical relation between oxygen consumption (Q) and body weight (W) is

$$Q = 3W^2$$

The derivative of this function is

$$\frac{dQ}{dW} = (3)(2W) = 6W$$

The graph of $Q = 3W^2$ is given in Fig. 4a. The derivative at W = 1 represents the slope of the line tangent to the curve at the point where W = 1, as shown in Figure 4b. At W = 1, the slope is calculated to be

$$\frac{dQ}{dW} = 6(1) = 6$$

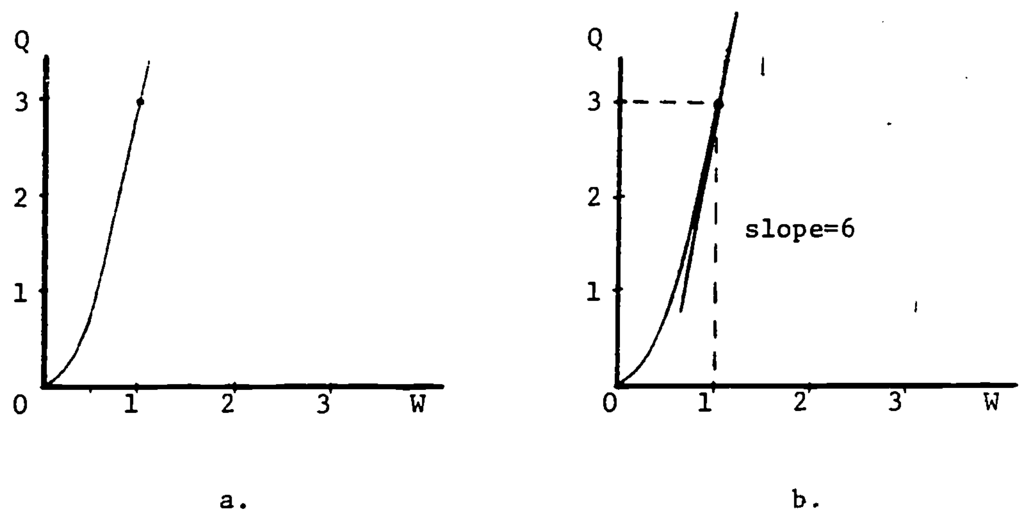

Figure 4. Derivative as the slope of the tangent line.

The number "3" in the formula $Q = 3W^2$, as well as the "2" in the exponent, are empirically determined and differ with body size and species. The general formula is

$$Q = aW^b \tag{2}$$

The derivative of this power function is

$$\frac{dQ}{dW} = a \cdot b \cdot W^{b-1}$$

Table 1 gives the more common functions and their derivatives,

Table 1. Derivatives of Elementary Functions

$y(x)$	dy/dx
x^n	nx^{n-1}
e^x	e^x
$\ell n\ x$	$1/x$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

For a more complete table, see any calculus text, or any math handbook (see Bibliography).

Composite Functions

When a function is composed of several simple functions its derivative can be evaluated in stages. In the simple cases where y equals the sum or product of two functions, f(x), g(x), the rules for differentiation (finding the derivative) are:

$$y(x) = f(x) + g(x), \quad \frac{dy}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

$$y(x) = f(x)\ g(x), \quad \frac{dy}{dx} = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

For example, if $y(x) = x^2(x-1) + 2x$, then

$$\frac{dy}{dx} = x^2\frac{d(x-1)}{dx} + (x-1)\frac{d(x^2)}{dx} + 2$$

$$= x^2(1) + (x-1)(2x) + 2$$

When y is a function of a function, the <u>chain</u> <u>rule</u> provides the differentiation method:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

For the composite exponential function $y = e^{2x}$, we have

$$u(x) = 2x, \qquad y(u) = e^{u}$$

$$\frac{dy}{du} = e^{u}, \qquad \frac{du}{dx} = 2$$

Therefore

$$\frac{dy}{dx} = 2e^{u} = 2e^{2x}$$

The Gompertz growth curve is used occasionally to describe the population size (N) of some species as a function of time (t), and is given by the equation

$$N(t) = ae^{-be^{-kt}}$$

where a,b,k are constants. The derivative dN/dt then represents the rate of growth of the population. Here the chain rule is applied twice:

$$N(t) = ae^{u(t)}, \qquad u(t) = -be^{-kt}$$

$$\frac{dN}{du} = ae^{u}$$

Write u(t) as $u = -be^{v(t)}$ where $v = -kt$. Then

$$\frac{dv}{dt} = -k, \qquad \frac{du}{dt} = \frac{du}{dv}\frac{dv}{dt} = -be^{v}(-k)$$

$$\frac{dN}{dt} = \frac{dN}{du}\frac{du}{dv}\frac{dv}{dt} = ae^{u}(-be^{v})(-k) = abk\ e^{-be^{-kt}} \cdot e^{-kt}$$

Higher Derivatives

The derivative dy/dx of a function y(x) is called the first derivative of y(x). If we write

$$\frac{dy}{dx} = g(x)$$

then differentiating g(x) produces

$$\frac{dg}{dx} = h(x)$$

which is called the second derivative of y(x), and is written

$$\frac{d^2y}{dx^2} .$$

Other notation used for the first derivative includes y' and $\dot{y}$, for the second derivative, y" and $\ddot{y}$. Since the second derivative is also a function, it too can be differentiated to give the third derivative, and so on. The n^{th} derivative is written (there is no general dot notation)

$$\frac{d^ny}{dx^n} , \qquad y^{[n]}$$

Critical Points

The first and second derivatives can be used to determine three special points on the graph of the function, namely, the relative maxima, relative minima and the inflection points. The relative maximum is easily visualized: the curve rises, reaches a peak, and then falls. The peak is the relative maximum. It is relative because the curve may rise even higher in a different place on the graph. Similarly, the relative minimum constitutes a low point on the curve. An inflection point is best illustrated by an example.

The logistic growth function describing population size is

$$N = N_0 \frac{(1+b)}{1+be^{-kt}} = N_0(1+b)(1+be^{-kt})^{-1}$$

where k is a growth coefficient, N_0 is the population size at t = 0 and $N_0(1+b)$ represents the carrying capacity of the environment. The growth rate is then

$$\frac{dN}{dt} = N_0(1+b)(-1)(1+be^{-kt})^{-2}(be^{-kt})(-k)$$

$$= N_0(1+b)bke^{-kt}(1+be^{-kt})^{-2}$$

The coefficient k is always positive. In this example, we restrict b to be greater than 1.

At a relative maximum, the tangent line is horizontal so the slope is zero. Then the maximum growth rate occurs when the derivative of the growth rate equals zero.

$$\frac{d\left(\frac{dN}{dt}\right)}{dt} = 0$$

or equivalently,

$$\frac{d^2N}{dt^2} = 0$$

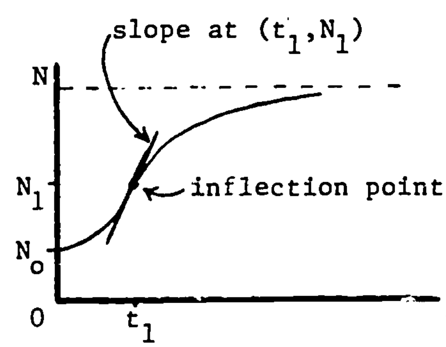

Figure 5. Maximum growth rate at the inflection point.

The second derivative of a function equals zero at the point of inflection, where the curvature changes from curving upward to curving downward, or vice versa. Then the maximum growth rate occurs when the population function N(t) is at its inflection point (see Fig. 5). Since the slope is decreasing (leveling off) following the inflection point, and increasing before the inflection point, it is certainly maximal (steepest) at that point. This point can also be viewed as the relative maximum on the graph of the growth rate, dN/dt (see Fig. 6).

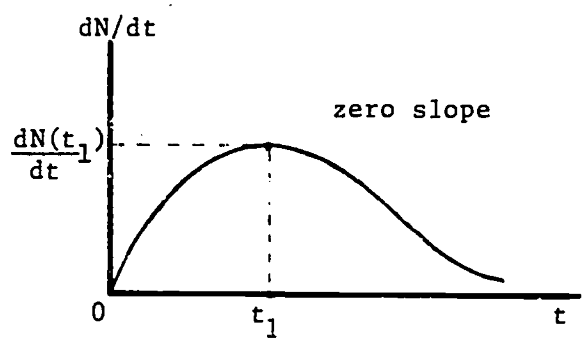

Figure 6. Growth rate as a function of time.

We now locate this point of maximum growth:

$$0 = \frac{d^2N}{dt^2} \qquad \text{at } (t_1, N_1)$$

$$0 = \frac{d}{dt}[N_o(1+b)bke^{-kt}(1+be^{-kt})^{-2}] \quad \text{at} \quad (t_1,N_1)$$

$$0 = N_o\ (1+b)bk^2e^{-kt_1}(be^{-kt_1}-1)(1+be^{-kt_1})^{-3}\ .$$

Since all factors are positive except $(be^{-kt_1}-1)$, then

$$0 = be^{-kt_1}-1$$

Solving for t_1 gives

$$t_1 = (\frac{1}{k})\ln b$$

and then substituting into the original expression for N,

$$N_1 = N_o(1+b)(1+be^{-k(\frac{1}{k}\ell nb)})^{-1} = N_o(1+b)/2\ .$$

this says that the growth rate is highest when the population is one-half of the carrying capacity.

FUNCTIONS OF SEVERAL VARIABLES

Partial Derivatives

The models treated thus far involve functions of one variable. Oxygen uptake is given as a function of just the surrounding oxygen concentration. The population size depends only on time. A more complicated model, however, may involve many variables. Growth certainly depends on available food supply in addition to time. Oxygen consumption also depends on more factors than ambient oxygen concentration. One such model is discussed by Bayne, Thompson and Widdows (1973).

The model begins with the equation

$$\frac{dC}{dt} = aW^b \tag{3}$$

where C is the amount of oxygen consumed up to time t, and a, b and W are constants.* The notation dC/dt demands that we may be able to consider C <u>only</u> as a function of t. This is not always the case. Bayne, et al., studied mussels (Mytilus) with regard to the effects of food and temperature on oxygen consumption. One of their data sets gives values for the coefficients a and b for winter vs. summer at two activity levels:

Table 2. Oxygen consumption for Mytilus edulis

Parameter	Season	Activity standard	Activity routine
a	Winter	1.76	2.64
	Summer	1.87	2.64
b	Winter	0.724	0.774
	Summer	0.670	0.702

*Comparing (3) with (2) gives $\frac{dC}{dt} = Q$.

Standard activity represents the resting state. Routine refers to the post-feeding time period where some filtration (i.e. muscle activity) is occurring. From table 2, we recognize significant dependence of "a" on the level of activity and dependence of "b" on both activity and season. Let season and activity be denoted s and m, respectively. Then, in place of "a" and "b" we write a(m) and b(s,m) to show the dependence on the variables m and s. Now C, a and b are dependent variables and W, s, m and t are the independent variables. We must now write (3) as

$$\frac{\partial C}{\partial t} = a(m)W^{b(s,m)} \tag{4}$$

The derivative notation is different from that in (3) to indicate more than one independent variable. This derivative is called a partial derivative and represents the rate of change of C with time while all other variables are held constant. Note that by holding all of the independent variables (except t) constant, we also hold a and b constant. So this partial derivative is obtained by differentiating the function C with respect to t and treating all of the remaining independent variables as constants.

As a simple example, consider the function

$$y = xt^2 \tag{5}$$

Then

$$\frac{\partial y}{\partial x} = t^2 \quad \text{(t held constant)}$$

and

$$\frac{\partial y}{\partial t} = x \cdot 2t \quad \text{(x held constant)}$$

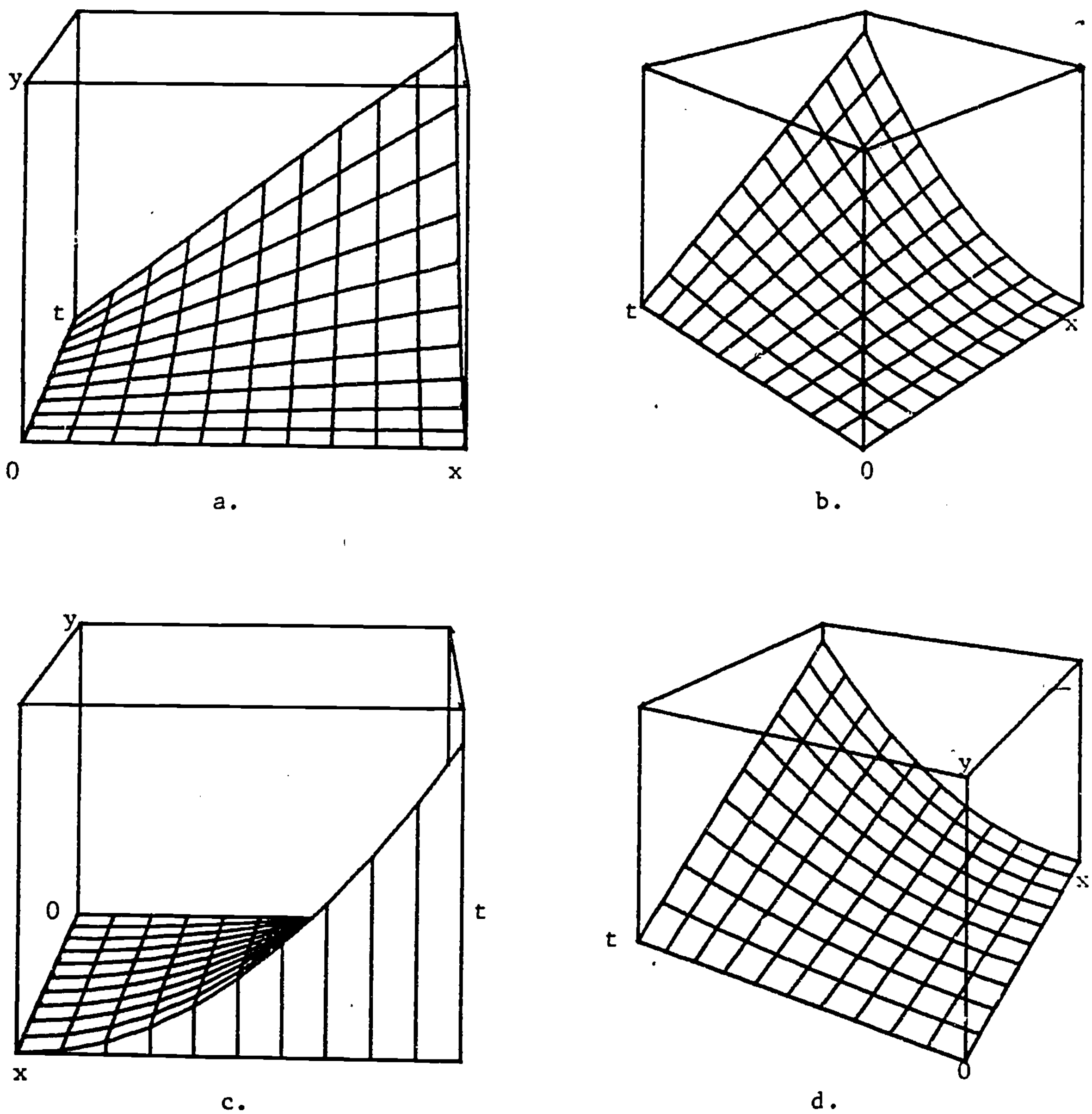

Fig. 7. The function $y = xt^2$.

A partial derivative is by nature merely one simple relation (of many) extracted from a complicated function. In (5), when both x and t vary, the graph of y vs. x vs. t is a three-dimensional surface (figure 7). When t is held constant, the graph (y vs. x) is a straight line (with slope t^2); when x is held constant, the graph (y vs. t) is a parabola, as shown in Figures 8 and 9. These latter two graphs are much simpler than the surface of figure 7. Note that the straight line (y vs. x) is the far edge of figure 7a, and the parabola (y vs. t) is the near edge of figure 7c.

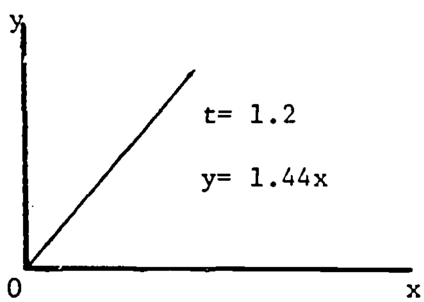

Figure 8. $\partial y/\partial x = t^2$.

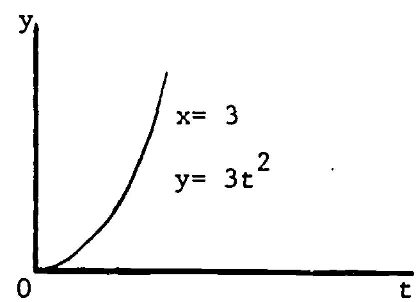

Figure 9. $\partial y/\partial t = 2xt$.

A more complicated example is the complete expression for (4). The level of activity is based on the fraction of the maximal filtration rate. Then m=0 represents the "standard" state, m=1 gives the "active" state, and m=.4 is the "routine" state. Consider oxygen consumption during summer. Then

$$a(m) = 1.87(m+1)$$

and (3) becomes

$$\frac{\partial C}{\partial t} = 1.87(m+1)W^{(.7)} \qquad (6)$$

Changes in "b" are not significant so that an average value, .7, can be used. Since (6) involves four variables, and thus cannot be plotted, we revert to the notation of (2), i.e., $Q = \frac{\partial C}{\partial t}$. Then

$$Q = 1.87(m+1)W^{(.7)} \tag{7}$$

This last expression is similar in form to (5) and its graph has a similar shape. Problem 3 discusses (7) in more detail.

Critical Points in Three Dimensions

The extension of a critical point to functions of two variables is quite natural. A point $P = (x_o, y_o, z_o)$ is a critical point for the function

$$z = f(x,y)$$

if

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

at the point P. The classification of the critical point is, however, more complicated. Since there are <u>three</u> second derivatives, many cases could be considered:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial(\partial f/\partial x)}{\partial x} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial(\partial f/\partial y)}{\partial y}$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial(\partial f/\partial y)}{\partial x} = \frac{\partial(\partial f/\partial x)}{\partial y} \equiv \frac{\partial^2 f}{\partial y \partial x}$$

This last "mixed" second derivative can be evaluated in either order <u>only if</u> the function f(x,y) is continuous in x and y. The functions used in the examples which follow are continuous so that the order of differentiation is arbitrary. Rather than considering all combinations of sign (+,0,-) in the second derivatives, we treat only three, which classify a relative maximum, relative minimum, and a saddle point. Define

$$L = f_{xx}(x_o,y_o)\cdot f_{yy}(x_o,y_o) - f_{xy}^2(x_o,y_o)$$

where

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad , \text{ etc.}$$

P is a relative maximum if $L > 0$ and $f_{xx}(x_o,y_o) < 0$.

P is a relative minimum if $L > 0$ and $f_{xx}(x_o,y_o) > 0$.

P is a saddle point if $L < 0$.

When L=0, the situation is "undetermined" since its resolution is beyond the scope of this module.

An interesting example is the function

$$z = x^3 + y^3 - 3xy + 15$$

which seems to describe some of the properties of water falling across a rock face (Clow and Urquhart, 1974). Some of these properties are well known. The water will often dig potholes in the rock, especially if it falls onto a ledge. The corners and edges of the ledge eventually become rounded. We would then use a function which drops rapidly, levels off then drops steeply again. Figure 10 shows a three-dimensional computer plot of this function, looking across the origin into the positive octant (x>0, y>0, z>0). The required shape is evident, with the pothole just beginning to form. In fact, the function does possess a relative minimum at the point (x, y, z) = (1, 1, 14). This model is discussed further in problem 6.

The saddle point in the waterfall model is located at the point (0, 0, 15). The region around the saddle point represents the front part of the ledge. It is displayed in the computer-drawn graph of figure 11, expanded vertically to highlight the saddle shape. Some basic features of derivatives are shown here:

a) The slope changes from point to point.

b) The slope depends on the orientation of the tangent line. Thus, at a given point, the slope found by $\partial z/\partial x$ may be different from the slope found using $\partial z/\partial y$.

c) The slope at a relative maximum or relative minimum is zero, i.e. horizontal.

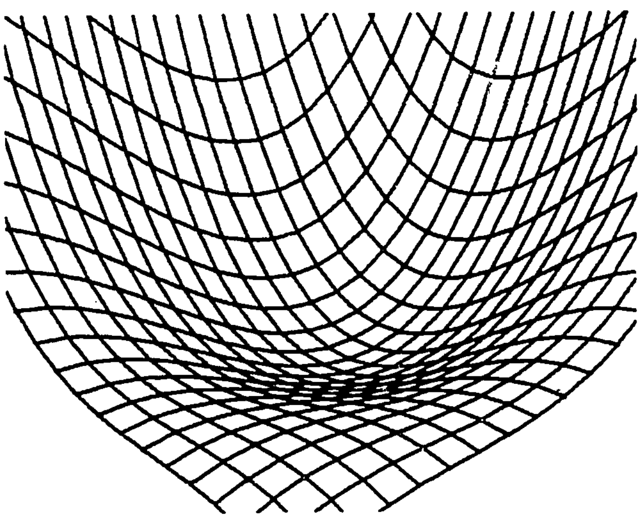

Figure 10. Waterfall function.

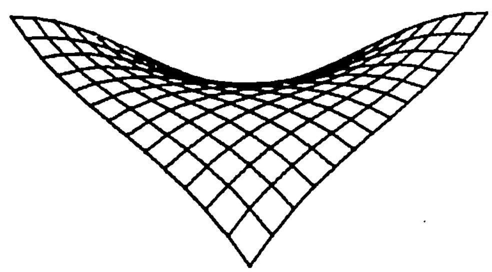

Figure 11. Saddle point of waterfall function.

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PROBLEM SET

1. a. The logistic population growth function satisfies the differential equation

$$\frac{dx}{dt} = Ax(N - x)$$

where $x=x(t)$ is the population size and N is the carrying capacity of the environment. Use this equation to show that the maximum growth rate (max dx/dt) occurs at the inflection point $x=N/2$. Assume A is positive. (HINT: write the equation with $v = dx/dt$, and set $dv/dx = 0$).

b. The "relative growth rate" is defined by

$$R = \frac{1}{x} \left(\frac{dx}{dt} \right)$$

where dx/dt is as given in part (a). Let x be given in "numbers of animals" and evaluate the units of R. The logistic curve is often used to describe "crowding effects" including intraspecies competition. Use the equation of part (a) to determine the value of x which maximizes R, and explain this result in terms of crowding.

2. Leaves usually have small openings called stomata to allow passage of gases between their interior and exterior. Through them carbon dioxide passes in for capture by photosynthesis and the resulting oxygen passes out. Water vapor also escapes through the stomata, sometimes leading to dehydration. Thus plants have guard cells around the stomata to regulate their size. Action of the guard cells varies the shape of stomatal openings from a long narrow slit to nearly a circle. Throughout most of this variation the opening has approximately the shape of an ellipse with a constant length perimeter, typically about 35 μ.

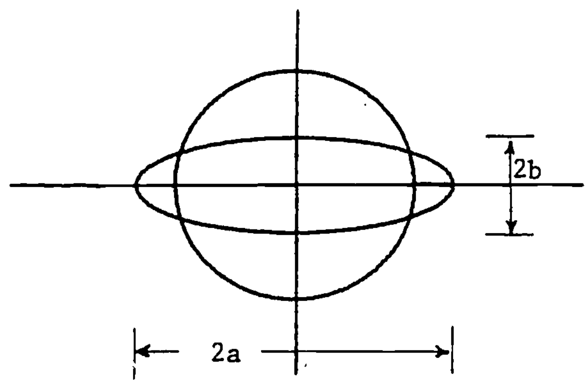
2.(cont.) A good approximation to the perimeter of an ellipse is

$$P = 2\pi\sqrt{(a^2+b^2)/2}$$

The area is given by

$$A = \pi ab$$

Set P=35 and evaluate the area, A, in terms of just the width, b. Show that the area reaches a maximum when a=b, i.e., when the stomatal opening is a circle.

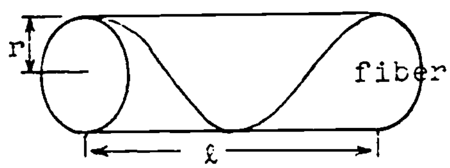


Constant-perimeter ellipses.

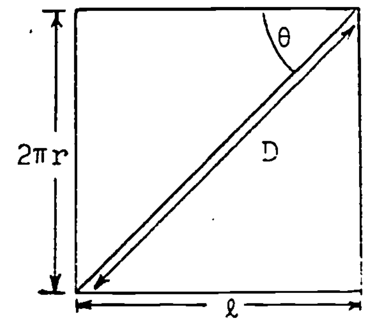
3. In the O_2 consumption model represented by equation (7), the variable Q has units of μl/hr. Write Q in ml/hr and find $\frac{\partial Q}{\partial m}$ for W=1000 mg. What are the units of $\frac{\partial Q}{\partial m}$? What might this derivative represent biologically? That is, why would a biologist be interested in this derivative?

4. A study of shape changes in nemerteans and flatworms (Alexander, 1968) theorizes that a basement membrane encloses the body and contains fibers which run in helices around the body. From Figure b, if the length D of the fiber is fixed, then $\ell = D\cos\theta$ and when the shape is cylindrical, the

4.(cont.) circumference and volume are $2\pi r = D\sin\theta$, $v = \pi r^2 \ell$.



a. Worm membrane



b. "unrolled" membrane

a. Express v as a function of θ (with D a parameter) and show v=0 when θ=0 or θ=π/2 radians. What do these two cases mean physically?

b. Find θ which gives a maximum for v. Prove it is a relative maximum and not a relative minimum. Note that laboratory dissections show that in the relaxed worm the fibers run at about 55° to the axis of the body. Why would the relaxed worm have the maximum volume?

5. Studies of insect flight (Alexander, 1968) use the theory of "forced vibrations" to explain the muscle action responding to nervous stimuli. If the "forcing function" is assumed to be

$$F\sin(2\pi nt), \quad t = \text{time}, \quad F = \text{constant},$$

then the steady amplitude A (magnitude of the vibrations) is given by

$$A = F[(s-4\pi^2n^2m)^2 + (2\pi nK)^2]^{-\frac{1}{2}}$$

where K = viscous damping coefficient

m = wing mass

n = frequency

s = stiffness of the vibrating medium

Find the frequency (n), called the resonant frequency, which gives the maximum amplitude.

6. This exercise verifies the location of the relative minimum in the example of "water falling across a rock face". The function is

$$z = x^3 + y^3 - 3xy + 15$$

Find all the critical points and determine which one is the relative minimum.

7. Heat transfer in soils depends on many factors; among them is the variation in the soil itself (de Vries, 1975).

Since most of the variation is in the vertical direction, a simple mathematical model is the one dimensional diffusion equation,

$$C \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} (\lambda \frac{\partial T}{\partial z})$$

where we define

T = temperature

t = time

z = vertical space coordinate

C = volumetric heat capacity

λ = thermal conductivity

When C and λ are uniform in depth and constant in time, we have the simple diffusion equation:

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial z^2}$$

where $a = \lambda/C$ is called the thermal diffusivity of the soil. The temperature at the surface gives boundary conditions for the model. For sinusoidal variation of surface temperature, we can write the boundary conditions as

$$T(t,0) = T_a + \theta_o \cos\omega t$$

$$T(t,\infty) = T_a = \text{constant}$$

7.(cont.) Show that the solution to this model is given by the following function:

$$T(t,z) = T_a + \theta_o e^{-z/d} \cos(\omega t - z/d)$$

with

$$d = (2a/\omega)^{\frac{1}{2}}$$

Be sure to verify that this function satisfies the diffusion equation <u>and</u> the boundary conditions.

ANSWERS TO THE PROBLEM SET

1. a. Find the x which gives max(dx/dt). Differentiate dx/dt with respect to x, equate to zero and solve for x:

$$\frac{d}{dx}\left(\frac{dx}{dt}\right) = A(N - 2x)$$

$$0 = A(N - 2x)$$

$$x = N/2$$

Since the second derivative is negative, i.e.

$$\frac{d^2}{dx^2}(dx/dt) = -2A < 0 \quad ,$$

then at x=N/2, dx/dt is maximal.

b. Find dR/dx, set equal to zero, solve for x:

$$R = \frac{1}{x}[Ax(N-x)] = A(N-x)$$

$$\frac{dR}{dx} = -A < 0$$

So no relative max exists, and the maximum must be at the lower end of the domain of x: since $0 \leq x \leq N$, then max (R) occurs at x=0. This model then implies that crowding effects are present whenever any animals exist.

2.

$$35 = 2\pi\sqrt{(a^2+b^2)/2}$$

$$\left(\frac{35}{2\pi}\right)^2 = \frac{a^2+b^2}{2}$$

$$a = \left(\frac{35^2}{2\pi^2} - b^2\right)^{\frac{1}{2}}$$

2. (cont.) The area A in terms of b is then

$$A = \pi\left(\frac{35^2}{2\pi^2} - b^2\right)^{\frac{1}{2}} b = \pi\left(\frac{35^2 b^2}{2\pi^2} - b^4\right)^{\frac{1}{2}}$$

Maximize A(b):

$$\frac{dA}{db} = \frac{\pi}{2}\left(\frac{35^2 b^2}{2\pi^2} - b^4\right)^{-\frac{1}{2}} \left(\frac{35^2 b}{\pi^2} - 4b^3\right)$$

$$0 = \frac{35^2 b}{\pi^2} - 4b^3 \quad , \quad \text{assume } b \neq 0.$$

$$= \frac{35^2}{\pi^2} - 4b^2$$

$$b = \frac{35}{2\pi} \quad , \quad \text{since } b > 0.$$

$$a = \left(\frac{35^2}{2\pi^2} - \frac{35^2}{4\pi^2}\right)^{\frac{1}{2}} = \frac{35}{2\pi} \quad , \text{ since } a > 0.$$

Thus the maximum area occurs when a=b, i.e. a circle.

3. To convert (7) so Q is in ml/hr, we divide by 1000:

$$Q = 1.87\times10^{-3}(m+1)W^{.7}$$

$$\frac{\partial Q}{\partial m} = 1.87\times10^{-3}W^{.7}$$

$$= .234 \qquad \text{for } W = 1000$$

The units are then ml O_2 per hour per unit of activity.

4. a. First solve for $r(\theta)$:

$$r = \frac{D}{2\pi} \sin\theta$$

$$v = \left(\frac{D}{2\pi} \sin\theta\right)^2 (D\cos\theta)$$

$$v(\theta) = \frac{D^3}{4\pi} \sin^2\theta\cos\theta$$

4. a. (cont.)

$$v(0) = 0 \text{ since } \sin(0) = 0$$

$$v(90^{\circ}) = 0 \quad \text{since} \quad \cos(90^{\circ}) = 0$$

b.

$$\frac{dv}{d\theta} = \frac{D^3}{4\pi}\{2\sin\theta\cos^2\theta - \sin^3\theta\}$$

$$0 = 2\sin\theta\cos^2\theta - \sin^3\theta \qquad \text{Assume } \theta>0$$

$$0 = 2\cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = 2\cos^2\theta$$

$$\tan\theta = \sqrt{2}$$

$$\theta = 54.74^{\circ}$$

5. Maximize A by minimizing the denominator:

$$0 = \frac{d}{dn}[(s-4\pi^2n^2m)^2+(2\pi nK)^2] = 2(s-4\pi^2n^2m)(-8\pi^2mn)+8\pi^2K^2n$$

$$0 = -2(ms-4\pi^2m^2n^2) + K^2$$

$$2ms-K^2 = 8\pi^2m^2n^2$$

$$n = \frac{2ms-K^2}{8\pi^2m^2}$$

6. First calculate the required partial derivatives:

$$\frac{\partial z}{\partial x} = 3x^2 - 3y \quad , \quad \frac{\partial z}{\partial y} = 3y^2 - 3x$$

$$\frac{\partial^2 z}{\partial x^2} = 6x \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = -3 \quad , \quad \frac{\partial^2 z}{\partial y^2} = 6y$$

Now find the critical point(s) by simultaneously solving

$$\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = 0$$

Thus

$$3x^2-3y=0 \qquad 3y^2-3x=0$$

6.(cont.) From the first equation we obtain:

$$y=x^2$$

Substitute into the second equation and solve for x:

$$3(x^2)^2-3x = 0$$

$$3x(x^3-1) = 0$$

Thus x=0, 1 . The minimum is said to be at (1,1) .

With x=1 we evaluate y:

$$y = (1)^2 = 1$$

To classify this critical point, we evaluate

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)-\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \quad \text{at} \quad (x,y) = (1,1):$$

$$(6\cdot 1)(6\cdot 1)-(-3)^2 = 36 - 9 > 0$$

Since

$$\frac{\partial^2 z}{\partial x^2} = 6 > 0 \quad \text{at} \quad (x,y) = (1,1)$$

then (1,1) is indeed a relative minimum.

7. First show the boundary conditions to be satisfied:

$$T(t,0) = T_a+\theta_o e^{-0/d} \cos(\omega t-0/d)$$

$$= T_a+\theta_o \cos\omega t$$

$$T(t,\infty) = T_a + \theta_o e^{-\infty/d} \cos(\omega t-\infty/d)$$

$$= T_a + 0$$

$$= T_a$$

7.(cont'd.) Now show that the diffusion equation is satisfied:

$$\frac{\partial T}{\partial t} = -\theta_o e^{-z/d} \omega \sin(\omega t - z/d)$$

$$\frac{\partial T}{\partial z} = -\frac{1}{d}\theta_o e^{-z/d} \cos(\omega t - z/d) - \theta_o e^{-z/d}(-\frac{1}{d})\sin(\omega t - z/d)$$

$$= \frac{\theta_o e^{-z/d}}{d} [\sin(\omega t - z/d) - \cos(\omega t - z/d)]$$

$$\frac{\partial^2 T}{\partial z^2} = (-\frac{1}{d}) \frac{\theta_o}{d} e^{-z/d} [\sin(\omega t - z/d) - \cos(\omega t - z/d)]$$

$$- (\frac{1}{d}) \frac{\theta_o}{d} e^{-z/d} [\cos(\omega t - z/d) + \sin(\omega t - z/d)]$$

$$= -\frac{2\theta_o}{d^2} e^{-z/d} \sin(\omega t - z/d)$$

Substituting into the diffusion equation:

$$-\theta_o e^{-z/d} \omega \sin(\omega t - z/d) = -a \frac{2\theta_o}{d^2} e^{-z/d} \sin(\omega t - z/d)$$

The equation is certainly true when $\sin(\omega t - z/d) = 0$. Now assume that $\sin(\omega t - z/d) \neq 0$ and divide both sides by $-\theta_o e^{-z/d} \sin(\omega t - z/d)$:

$$\omega = 2a/d^2$$

Substituting for d, we obtain

$$\omega = 2a/[(2a/\omega)^{\frac{1}{2}}]^2$$

$$= \omega$$

Thus the equation is indeed satisfied.

COMPUTER EXERCISES

Program DIFF is basically a plotting routine where the graph of a function demonstrates some property or use of derivatives. The program is designed to motivate the following concepts: the derivative as a slope, the zero slope at a critical point, partial derivatives, and the distinction between continuous and discrete rates of change.

The specific user options and program features are detailed in the User's Guide for Program DIFF. The program is designed for easy conversion to be compatible with a continuous off-line plotter (such as Calcomp). Thus one option is setting XSLICE, YSLICE to "slice" and remove part of the three-dimensional graph to expose a hidden profile. This option is of limited benefit with a line printer, but is of great advantage with an off-line plotter for displaying critical points.

In each exercise, the user must input certain parameter values. By repeating an exercise with different parameter values, the user can gain a better intuitive understanding of how the behavior of functions and their derivatives depends on the chosen parameter values.

1. In the example in the text treating oxygen consumption (see also problem 3 above), the oxygen consumption (Q) depended on both the activity level (m) and the dry weight (W):

$$Q = .00187(m + 1)W^{.7}$$

where the units of Q are $m\ell O_2$/hr. For the computer exercise, we use the correspondence

$$Q \rightarrow z, \; m \rightarrow x, \; W \rightarrow y$$

so that the equation is

$$z = .00187(x + 1)y^{.7}$$

Choose a value for y with $0 < y \leq 3000$ mg and evaluate the first partial derivative $\partial z/\partial x$ at that value of y. Now use program DIFF:

1. Choose function 1.

2a. If a line printer is used, obtain the plot and check that the curvature with increasing x agrees with what your partial derivative suggests. Is the shape supposed to be a straight line, concave upward, concave downward?

2b. If an off-line printer is used, set XSLICE = 0 and YSLICE = your choice of y. Obtain the plot. Does the exposed profile agree with your partial derivative? Repeat the exercise with XSLICE = 0, YSLICE = 0. Does the graph have the same general shape as Fig. 7 in the text? What is different? Now reverse the roles of x, y: pick x so that $0 \leq x \leq 1$, evaluate $\partial z/\partial y$ at that value, set YSLICE = 0 and XSLICE = your value for x and obtain your plot. Does the exposed profile agree with your function for $\partial z/\partial y$? That is, is the shape supposed to be a straight line, circle, parabola ... ?

2. This exercise illustrates how different parameter values can affect the properties of a function and its critical points. Function 2 is used in this exercise and is a simple polynomial in two independent variables:

$$z = (P1)x^2 + (P2)y^2$$

First calculate all the partial derivatives needed to locate and classify a critical point for general values of P1, P2. Now use program DIFF:

1. Choose function 2.

2a. If a line printer is used, set values for P1, P2 (from -1.0 to 1.0) and obtain the plot. Locate and classify the critical point for these parameter values. Now check that the plot does show the same type of critical point at the same location given by your calculations. Repeat for several values of P1, P2 and note the dramatic change in type of critical point. How does the location change?

2b. If an off-line plotter is used, choose values for P1, P2, XSLICE and YSLICE and obtain the plot. Repeat the exercise with different values for XSLICE, YSLICE only, and obtain the plot. Classify the critical point as described in the text and compare with the plot. By repeating the exercise with different slicing values (XSLICE, YSLICE), you can search for the critical point by observing the changes in the exposed profile. For example, run the program four times using the following values:

(XSLICE,YSLICE) = (-2,-2)(-2,-1),(-2,0),(-2,1)

You can also keep YSLICE = -2 and vary XSLICE. Check that the exposed profile agrees with your first derivative. For example, if XSLICE = -2, YSLICE = -1, then the sliced edge should represent $\partial z/\partial x$ at y = -1.

3. When a beam of rectangular cross-section is cut lengthwise from a log, its strength can sometimes be well described by the following function:

$$S = k\,W\,D^2$$

where

S = strength of beam

W = width of beam

D = depth of beam

as shown in Figure A, and where k is a constant which depends on the type of tree used. Assume that the log is circular in cross-section and that the beam is cut so each corner reaches the outside of the log, as shown. Assume k = 0.1.

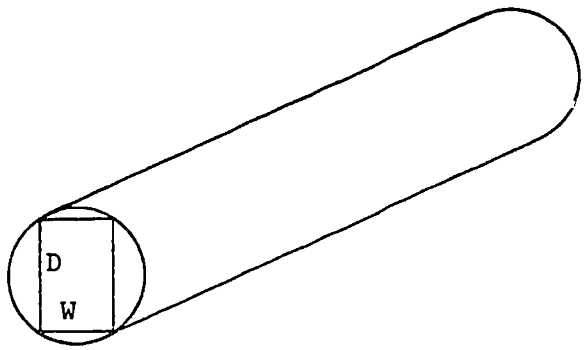


Figure A. Rectangular beam cut from a log.

a. Find the width and depth of the beam which give the maximum strength. Assume the radius of the log is r.

b. Now use program DIFF:

1. Choose function 3.

2. Let P2 = radius of the log. Choose P2 so that $0 < P2 \leq 10$.

3. Plot the function. From the graph with y representing S and x representing W, estimate the width W which gives the maximum strength.

4. Now calculate the depth D. Check that your estimates for W, D agree with your general formula of part a.

4. Animal populations newly introduced into a region have been observed to increase rapidly in number and soon thereafter to fall drastically (Caughley 1970a,b) as indicated in Figure B. One theory is that the population at first "senses" an infinite food supply and then reproduces rapidly to the

point of overgrazing the area. The birth rate remains the same but the death rate (perhaps of young) dramatically increases until the population is low enough to match the food supply. The population then increases more slowly and seems to stabilize. Migration is also involved but is poorly understood. The second population rise to a "steady-state" suggests some adaptation or social "learning" by the population concerning their new environment.

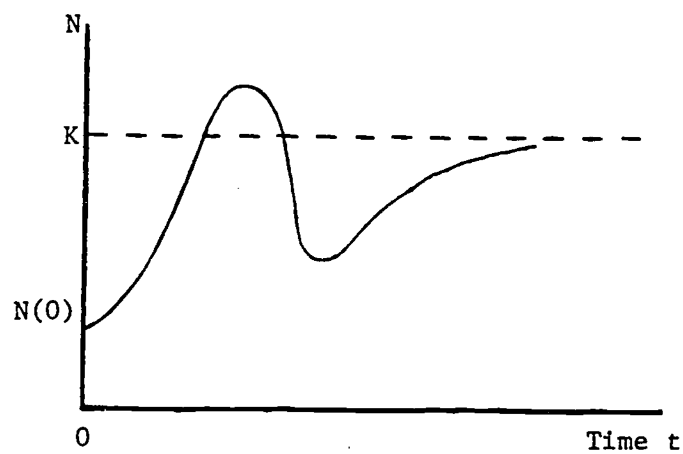


Figure B. Population dynamics of introduced species.

a. Consider the logistic equation written as follows:

$$N(t) = K(1 + be^{-rt})^{-1}$$

where K, b, r are positive constants, and N(0) < K. In an attempt to describe the behavior in Figure B, one could modify the above logistic function by allowing r to vary over time, i.e. r = r(t). Can the population ever exceed K? If not, then this modified logistic function is not a good model. Answer this for a general r(t) in any fashion: by examining the function N(t) itself, by evaluating the maximum of N(t), etc.

b. An alternate model is developed if the discreteness of the population is taken into account. The logistic function in part (a) assumes that the population size, N(t), will change <u>continuously</u> as time changes. In certain species, populations do not change size smoothly (see Sladen and Bang 1969). The breeding time is a short period, once a year so that many off-spring are born at the same time. The population size then increases in large jumps. This discrete change is not modeled well by differential equations. A closely related field is the study of "difference equations" (see Goldberg 1958) where discrete changes are allowed. The derivative of the logistic function satisfies the differential equation

$$\frac{dN(t)}{dt} = rN(t)\left(\frac{K-N(t)}{K}\right)$$

From this formulation, we see that the logistic model assumes that the population is always aware of how far from the carrying capacity is the current population size and so continuously adjusts for this difference. (Note that the derivative depends on this difference, K-N(t)). Actually, plentiful food one season may cause too many births the following season, with a definite time lag between abundant food and severe overpopulation and with few adjustments in between.

The computer is used here to illustrate this inadequacy of differential calculus. The derivative above is replaced by a difference:

$$\frac{N(t+1) - N(t)}{(t+1) - t} = rN(t)\left(\frac{K-N(t)}{K}\right)$$

This model assumes the population changes only once per time period (e.g., annually) and is unable to adjust in between. Is this more or less realistic than the original logistic model?

The computer program for this problem plots the population size (N) as a function of time (t) for a given set of parameters (constants). You should note in each plot whether N fluctuates at all, whether the initial drop is steep, and whether the population size eventually stabilizes at the carrying capacity (K).

The input parameters are identified as:

P1 = initial size

P2 = growth rate (r)

However, the computer program restricts P1 to [-10,10]. The range of N_o (initial population size) is chosen to be [50,250]. Thus we use P1 to represent, but not equal, the initial size as follows:

$$P1 = \frac{N_o - 150}{10} .$$

The carrying capacity is fixed at K = 500. Investigate the different kinds of behavior of the solution by using program DIFF:

1. Choose function 4.
2. Choose P1, P2 so that $-10 \le P1 \le 10$ and $1 \le P2 \le 4$.
3. Obtain the plot.

Check that the initial size on the graph is correctly represented by your choice of P1. Repeat the exercise if you wish, but be sure two of your plots have P1 = -10, P2 = 3.5 and P1 = -10, P2 = 1.5. Can you estimate the value for P2 above which the oscillations are no longer regular (or periodic), i.e. May's "chaos" (1978)?

Do you think this "difference" model could be a fairly good description of the introduced population? How would you include social or genetic "learning" into the model? That is, once the population size rises and then falls, what parameters might change in the model so that the next rise would be more gradual?

5. The reaction (Y) of the body to a dose (X) of drug can be represented by the function:

$$Y(X) = X^2(P1/2 - P2 \cdot X/3)$$

$$= \frac{P1}{2} X^2 - \frac{P2}{3} X^3$$

where P1 and P2 depend on certain body characteristics and on the maximum dosage which can be administered. Y indicates the strength of the reaction, measured in millimeters of mercury if blood pressure is being tested, or perhaps degrees Celsius if change in body temperature is being measured.

Find the dose that has "maximum sensitivity," i.e. where the rate of increase of Y is greatest. Is this "critical point" a maximum or minimum for Y'? What is this point called on the graph for Y?

Now use program DIFF:

1. Choose function 5.
2. Choose P1, P2 so that $0 \le P1 \le 5$, $.1 \le P2 \le 1.0$.
3. Plot the function and calculate the correct value for X at the "critical point." Check that the graph of Y(X) has the behavior you predicted at this point.

ANSWERS TO THE COMPUTER EXERCISES

1. $\partial z/\partial x = (.00187y^{.7})$ = constant for given y.

   Exposed profile should be a straight line.

   $\partial z/\partial y = (.00187)(x+1)(.7)y^{-.3}$

   Exposed profile curves upward with decreasing slope.

2. $\partial z/\partial x = 2(P1)x$, $\partial z/\partial y = 2(P2)y$

$\partial^2 z/\partial x^2 = 2(P1)$, $\partial^2 z/\partial x\partial y = 0$, $\partial^2 z/\partial y^2 = 2(P2)$

$\partial z/\partial x = 0$ if $x = 0$

$\partial z/\partial y = 0$ if $y = 0$

Thus $(x,y) = (0,0)$ is the only critical point.

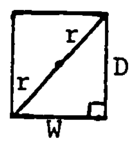
To classify the critical point, evaluate:

$$\frac{\partial^2 x}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 4(P1)(P2)$$

Pick $P1 = .6$, $P2 = 1.0$

Then $4(P1)(P2) > 0$, $\frac{\partial^2 z}{\partial x^2} = 2(P1) > 0$ and the point is a relative minimum.

3. From the diagram, we use the Pythagorean theorem to obtain



$$(2r)^2 = W^2 + D^2$$

Thus $$D^2 = 4r^2 - W^2$$

$$S = (.1)(4)r^2W - (.1)W^3$$

$$\frac{dS}{dW} = (.4)r^2 - (.3)W^2$$

$$0 = (.4)r^2 - (.3)W^2$$

The critical point is then at $W = \sqrt{4r^2/3} = 2r/\sqrt{3}$.

We have

$$\frac{d^2S}{dW^2} = -.6W < 0$$

so that when $W = 2r/\sqrt{3}$, S is indeed at a maximum.

The depth is then

$$D = \sqrt{4r^2 - W^2}$$

$$= \sqrt{4r^2 - 4r^2/3}$$

$$= 2\sqrt{2/3}\, r$$

4.a. Even assuming $r = r(t)$, N cannot exceed K if $N(0) < K$. We see this by writing N as a fraction.

$$N(t) = \frac{K}{1 + be^{-r(t)t}}$$

Since $b > 0$, the denominator exceeds 1, and thus

$$N(t) < K$$

b. Treat r as r(t) in the difference equation model, with

$$r(0) = r_o$$

$$r(1) = r_o/2$$

$$r(2) = r_o/3$$

$$r(3) = r(2)$$

or some such scheme to decrease r as time increases.

5. Evaluate the derivative.

$$Y' = (P1)X - (P2)X^2$$

The "rate of increase" of Y is greatest when Y' is maximal. Find the max(Y') by differentiating Y' and setting $Y'' = 0$.

$$Y'' = \frac{d(Y')}{dx} = (P1) - (2)(P2)X$$

$$0 = (P1) - (2)(P2)X$$

$$X = (P1)/(2(P2))$$

The value for Y is then

$$Y = \left[\frac{P1}{2(P2)}\right]^2 \left[\frac{P1}{2} - \frac{(P1)(P2)}{6(P2)}\right] = \frac{(P1)^3}{12(P2)^2}$$

This point is an inflection point on the graph Y versus X.

USER'S GUIDE FOR PROGRAM DIFF

Identification

DIFF - A program which displays properties of derivatives of mathematical functions

Authors - Richard Hertzberg, Mark Bailey, Center for Quantitative Science in Forestry, Fisheries and Wildlife, University of Washington, Seattle. February 1979.

Purpose

Program DIFF is the computer supplement to the instructional module "Calculus-Differentiation," by Richard Hertzberg, which reviews the basic principles and uses of differential calculus, with special emphasis on ecological and physiological applications. The computer program displays graphs of selected functions so that certain derivatives are visible. Most plots serve as checks to the user's own calculations.

Operation

The user controls program DIFF through certain input variables. The principal one, NFC, selects the function to be displayed and also enables or disables other inputs listed in the INPUT TABLE below. Other inputs control the parameters in a function, the portions of a function to be displayed, the number of plots, and the structure of the plots themselves.

Setting NFC = 1 selects the function

$$F(x,y) = 0.00187(x + 1)y^{0.7}$$

and disables the input variables P1, P2, and NPLOTS. The user may alter the variables XSLICE and YSLICE which act as "slicing" variables that cut through the z axis of the function along certain planes in order to reveal hidden profiles. The slicing effect merely sets the z values of the function to ZMIN (the smallest z value in the plot, which is usually displayed as a blank) whenever $x < XSLICE$ and $y < YSLICE$.

Setting NFC = 2 selects the function

$$F(x,y) = p_1x^2 + p_2y^2 \quad .$$

This option enables all of the inputs listed in the INPUT TABLE. The parameters of the function, p_1 and p_2, are represented by the arrays P1 and P2 which can hold up to six different sets of values and generate up to six different plots, where the number of plots is controlled by NPLOTS. For example, if

P1 = 1, 2, 3, 4, 5, 6,

P2 = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6,

NPLOTS = 4 ,

then DIFF would produce four plots as follows:

plot 1: $1x^2 + 0.1y^2$

plot 2: $2x^2 + 0.2y^2$

plot 3: $3x^2 + 0.3y^2$

plot 4: $4x^2 + 0.4y^2$.

Also, by repeating NFC = 2 with different values for XSLICE and YSLICE, the user can "search" for three-dimensional critical points by observing various function profiles.

Setting NFC = 3 selects the function

$$F(x) = 0.1x(4p_2^2 - x^2)$$

which can be written as

$$F(x) = 0.1xD^2(x)$$

in terms of the notation used in computer exercise no. 3. This option disables the input variables XSLICE, YSLICE, NPLOTS, P1 and the last five elements of P2. P2(1) represents the function parameter p_2.

Setting NFC = 4 selects the function

$$F(y) = p_2 y(1-y/500) + y.$$

This function represents the step-wise solution to the difference equation

$$y_{k+1} = p_2 y_k(1-y_k/500) + y_k \;, \quad \text{where } y_k = y(x_k) \;,$$

which is the discrete form of the logistic differential equation

$$dy/dx = p_2 y(1-y/500) \;,$$

as given in computer exercise no. 4. This initial value for y (y(0)) is determined by the function

$$y(0) = 10.0\, p_1 + 150.0 \;.$$

This option disables the input variables XSLICE, YSLICE, NPLOTS, and the last five elements of P1 and P2. P1(1) represents the parameter p_1 in the initial value equation, and P2(1) represents the function parameter p_2.

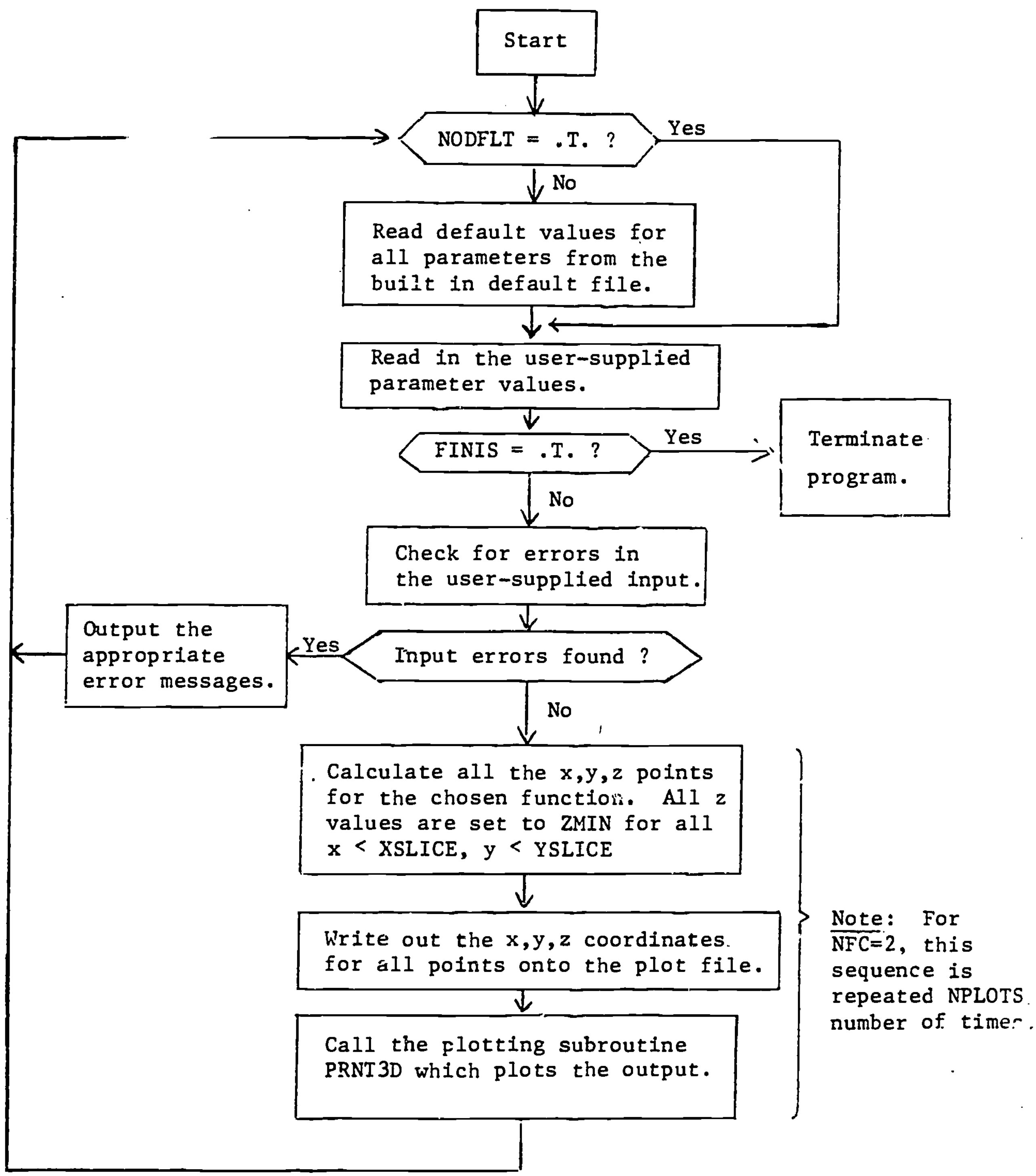
Setting NFC = 5 selects the function

$$F(x) = (p_1/2)x^2 - (p_2/3)x^3 \quad ,$$

This option disables the input variables XSLICE, YSLICE, NPLOTS, and the last five elements of the arrays P1 and P2. The function parameters p_1 and p_2 are represented by P1(1) and P2(1).

Program Organization

The program is organized according to the following flow chart:



Input

All input is handled by the format free input package (Gales and Anderson, 1978) which permits a user to assign values to variables by a "name=value" convention. Not all variables need be explicitly assigned by the user, however, as unassigned variables automatically assume default values. The input consists of any number of data sets, each of which is terminated by a dollar sign ($). Each data set generates a separate printer plot.

The input for DIFF is divided into three classes: (a) variables having mathematical significance: NFC, XSLICE, YSLICE, P1, P2 and NPLOTS; (b) variables which control certain program operations, such as program termination or the handling of default input: IPRINT, ECHO, NODFLT, and FINIS; and (c) variables which control the printer plots (default values are in parentheses): XMIN (0), XMAX (0), YMIN (0), YMAX (0), ZMIN (0), ZMAX (0), XRICH (0), YRICH (0), DFAULT (0), OVPRNT (.F.), AVE (.T.), INT2D (.F.), NX (60), NY (45), and ZMAP (0,1,2,3,4,5,6,7,8,9). The variables in the first two classes are explained in the following INPUT TABLE, whereas the printer plot variables are explained in the user's guide for PRNT3D (Gales, 1978). The user normally may ignore the PRNT3D variables since DIFF controls them internally. However, if he chooses to change any of them, he should do so with great care. In particular, the variables XRICH and YRICH, if made too small, will cause DIFF to generate a very large number of enrichment points, and consume far too much computer time.

INPUT TABLE

Name	Type and Dimensions	Range Limits	Description
NFC	Integer	1,5	Identifies function to be plotted. Default value: NFC = 1
P1 P2	Real (6) Real (6)	-10,10 -10,10	Function parameters. Their physical significance and, to some extent, their range limits, depend on the particular function specified. Default values: P1 = 1,1,1,1,1,1 P2 = 1,1,1,1,1,1
XSLICE YSLICE	Real Real	XMIN,XMAX YMIN,YMAX	Deletes part of the function by setting Z=ZMIN for x < XSLICE, y < YSLICE. Default values: XSLICE = 0 YSLICE = 0
NPLOTS	Integer	1,6	The number of plots to be drawn. NPLOTS is used only when NFC=2. Default value: NPLOTS = 1
IPRINT	Logical	.F., .T.	A logical value which causes the current values for all input variables (default as well as current user input) to be printed. Default value: IPRINT = .F.
ECHO	Logical	.F., .T.	A logical value which causes the user's input to be echoed if ECHO=.T., or suppresses echoing if ECHO=.F. Default value is: ECHO=.T.

Input Table (continued)

Name	Type and Dimensions	Range Limits	Description
NODFLT	Logical	.F., .T.	A logical value which suppresses the input of default values if NODFLT=.T. Default value: NODFLT = .F.
FINIS	Logical	.F., .T.	A logical value which causes program termination if and only if FINIS=.T. Default value: FINIS=.F.

Note: XMIN, XMAX, YMIN, YMAX determine the range of points to be plotted and are set internally for each function.

The last four variables deserve special mention.

1. The logical variable IPRINT controls the output of all input variables which are currently in effect (default values as well as those specified in the current input set). Setting IPRINT=TRUE (or T or .T.) displays the input variables; setting IPRINT=FALSE (or F or .F.) suppresses the display.
2. The logical variable ECHO controls the echoing of the input cards. Setting ECHO=TRUE causes the subsequent input set to the echoed; setting ECHO=FALSE suppresses the echo for the subsequent input set.
3. The logical variable NODFLT can be used to inhibit the automatic assignment of default values to input variables. If NODFLT is set TRUE in the current input set, then the current input set is assigned default values as usual, but all subsequent input sets merely accumulate more input values. In effect, the input values which exist after th i-th input set is read, become the default values for the (i+1)-th input set. The standard default values may then be restored by setting NODFLT=FALSE, but, again, the effects of this change are delayed until the next input set is read. To a limited extend, NODFLT permits a user to set up his own default values and can be very useful for executing a number of input sets which differ only in a few parameters. Consider the following example in which a user wishes to slice the same function by using several different values for XSLICE and YSLICE:

```
/INPUT SET 1:  THE FOLLOWING VALUES BECOME THE DE FACTO/
/DEFAULTS FOR ALL SUBSEQUENT INPUT SETS:               /
 NODFLT = TRUE, NFC = 2, P1 = .1, .4, .5, 2.3,
```

```
P2 = 2.2, 2.6, 2.5, 2.8, NPLOTS = 4, XSLICE = 0, YSLICE = 0, $
/INPUT SET 2:  SLICE THE ABOVE FUNCTION/
XSLICE = -1, YSLICE = 0, $
/INPUT SET 3:  SLICE IT ANOTHER WAY/
XSLICE = 0, YSLICE = -1, $
/INPUT SET 4:  SLICE IT YET ANOTHER WAY/
XSLICE = 1.2, YSLICE = -1.2, $
/INPUT SET 5:  NOW STOP/
FINIS = TRUE, $
```

4. The logical variable FINIS controls program termination. The user should add the card:

   FINIS = TRUE, $

   as the very last input set. If FINIS is not set, the program will terminate abnormally.

Output

DIFF produces sets of plots, via subroutine PRNT3D, which display function values. Each plot contains a title, legend, x and y axis annotation, and printer plot lines or surfaces. The title displays some of the values, and ranges of values, for the variables used to generate the plot.

The plot legend, in conjunction with the numbers along the x and y axes, allows users to interpret the plot numerically. The x and y axis numbers are of the form ±N.NNN and differ from their true values by powers of 10 which are specified by the scale factors in the legend. For example, the first line of the plot legend for RUN 3 reads.

SCALE FACTORS = X-AXIS:E+00 Y-AXIS:E-01 Z-AXIS:E+00

hence the point, indicated by a "1" character near the top right of the plot, is near (x=1.246, y=3.100) = ($x=1.246 \times 10^{0}$, $y=3.100 \times 10^{-1}$) = (x=1.246, y=0.31). The remaining two lines of the legend specify the number of points mapped to each z-level (-9 means > 99).

Restrictions

The input values are restricted to the ranges given in the INPUT TABLE. Due to the physical interpretation of the function parameters, further restrictions are indicated in the description of each computer exercise.

Error Messages

Three types of errors may occur in program execution:

1. Syntax errors in the user's input.
2. Parameter range exceeded.
3. Plot parameter errors.

Input errors 1 and 2 generate an appropriate error message, the calculations are skipped, and the next input set is read. For type 3 errors, the program suppresses the plot, outputs the error message and reads the next input set. If the plot file is empty, an error message is printed but the plot proceeds. Type 1 and type 3 error messages are listed in the user's guides for subroutines FFORM and PRNT3D.

Sample Runs

The control cards, input cards, and line printer output for five sample runs appear on the next few pages.

```
DIFF,CM70000,T10,P2.
ACCOUNT,*******,*******.
COMMENT.
COMMENT.*****************************************
COMMENT.* THE ABOVE CARDS IDENTIFY THE JOB      *
COMMENT.* (DIFF), SPECIFY THE CENTRAL MEMORY    *
COMMENT.* REQUIREMENTS (70000 OCTAL), THE       *
COMMENT.* CENTRAL PROCESSOR TIME (10 SECONDS)   *
COMMENT.* THE JOB PRIORITY (P2), AND THE        *
COMMENT.* ACCOUNT NUMBER AND PASSWORD           *
COMMENT.*****************************************
COMMENT.
ATTACH,BDIFF,ID=BDIFF.
ATTACH,BPR3D,ID=BPR3D.
ATTACH,BFF,ID=BFF.
COMMENT.
COMMENT.*****************************************
COMMENT.* THE ABOVE CARDS ATTACH THE MAIN       *
COMMENT.* PROGRAM (BDIFF), AND THE SUPPORT      *
COMMENT.* ROUTINES PRINT3D (BPR3D) AND THE      *
COMMENT.* FREE FORM INPUT ROUTINE (BFF). THEY   *
COMMENT.* ARE ALL IN BINARY                     *
COMMENT.*****************************************
COMMENT.
LOAD,BDIFF,BPR3D,BFF.
EXECUTE,DIFF.
COMMENT.
COMMENT.*****************************************
COMMENT.* THE ABOVE CARDS LOAD THE ROUTINES     *
COMMENT.* INTO CENTRAL MEMORY AND PASS CONTROL  *
COMMENT.* TO DIFF FOR EXECUTION                 *
COMMENT.*****************************************
COMMENT.
*EOR
/**********************************************************************/
/                                                                      /
/   THE FOLLOWING DEFAULT VALUES ARE USED                              /
/                                                                      /
/   NFC=1,     NPLOTS=1,                                               /
/   P1 = 1.,1.,1.,1.,1.,1.,   P2 = 1.,1.,1.,1.,1.,1.,                  /
/   XSLICE = 0., YSLICE = 0.,                                          /
/   ECHO=.T., NODFLT=.F.,            FINIS=.F.,                        /
/   NX=60,     NY=45,     ZMAP=0,1,2,3,4,5,6,7,8,9,                    /
/   XMIN=0.0, XMAX=0.0, YMIN=0.0, YMAX=0.0, ZMIN=0.0, ZMAX=0.0,        /
/   XRICH=0.0,            YRICH=0.0,            DFAULT=0.0,            /
/   OVPRNT = .F.,   AVE = .T.,   INT2D = .F.,   IPRINT = .F.           /
/                                                                      /
/                                                                      /
/********************************RUN 1*********************************/
/                                                                      /
    NFC=1, $
/                                                                      /
/                                                                      /
/********************************RUN 2*********************************/
/                                                                      /
    NFC=2, $
/                                                                      /
/                                                                      /
/********************************RUN 3*********************************/
/                                                                      /
```

```
   NFC=3, $
/                                                                            /
/                                                                            /
/*********************************RUN 4**************************************/
/                                                                            /
   NFC=4, P1=-10, P2=1.2, $
/                                                                            /
/                                                                            /
/*********************************RUN 5**************************************/
/                                                                            /
   NFC=5, XRICH=0.05, YRICH=0.005, IPRINT = .T., $
/                                                                            /
/                                                                            /
/**********************************STOP**************************************/
/                                                                            /
   FINIS=.T., $
/                                                                            /
/                                                                            /
/****************************************************************************/
*EOP
```

```
OGRAM -DIFF- READY FOR INPUT
/**********************************************************************/
/                                                                      /
/   THE FOLLOWING DEFAULT VALUES ARE USED                              /
/                                                                      /
/   NFC=1,     NPLOTS=1,                                               /
/   P1 = 1.,1.,1.,1.,1.,1.,  P2 = 1.,1.,1.,1.,1.,1.,                   /
/   XSLICE = 0., YSLICE = 0.,                                          /
/   ECHO=.T., NODFLT=.F.,          FINIS=.F.,                          /
/   NX=60,     NY=45,     ZMAP=0,1,2,3,4,5,6,7,8,9,                    /
/   XMIN=0.0, XMAX=0.0, YMIN=0.0, YMAX=0.0, ZMIN=0.0, ZMAX=0.0,        /
/   XRICH=0.0,            YRICH=0.0,            DFAULT=0.0,            /
/   OVPRNT = .F.,   AVE = .T.,   INT2D = .F.,  IPRINT = .F.            /
/                                                                      /
/                                                                      /
/********************************RUN 1*********************************/
/                                                                      /
    NFC=1, $
```

```
                          OXYGEN CONSUMPTION
                  FOR F = 0.00187 * (X+1.0) * Y**0.7
       XSLICE =             0          YSLICE =          0

         0        .153       .322       .492       .661       .831      1.000
         X---------X----------X----------X----------X----------X----------X
  3.000Y
        I3  3  4  4  4  4  4  5  5  5  5 5  6  6  6  6  6  6  7  7
        I
        I3  3  3  4  4  4  4  4  5  5  5 5  5  5  6  6  6  6  6  7
        I
  2.659Y3  3  3  4  4  4  4  4  4  5  5 5  5  5  5  6  6  6  6  6
        I
        I3  3  3  3  4  4  4  4  4  4  5 5  5  5  5  5  6  6  6  6
        I
        I3  3  3  3  3  4  4  4  4  4  4 5  5  5  5  5  5  6  6  6
  2.318Y
        I3  3  3  3  3  3  4  4  4  4  4 4  5  5  5  5  5  5  5  6
        I
        I3  3  3  3  3  3  3  4  4  4  4 4  4  4  5  5  5  5  5  5
        I
  1.977Y2  3  3  3  3  3  3  3  4  4  4 4  4  4  4  5  5  5  5  5
        I
        I2  2  3  3  3  3  3  3  3  3  4 4  4  4  4  4  4  5  5  5
        I
        I2  2  2  2  3  3  3  3  3  3  3 4  4  4  4  4  4  4  4  5
  1.636Y
        I2  2  2  2  2  3  3  3  3  3  3 3  3  4  4  4  4  4  4  4
        I2  2  2  2  2  2  3  3  3  3  3 3  3  3  3  4  4  4  4  4
        I
        I2  2  2  2  2  2  2  2  3  3  3 3  3  3  3  3  3  3  4  4
  1.295Y
        I2  2  2  2  2  2  2  2  2  2  3 3  3  3  3  3  3  3  3  3
        I
        I1  1  2  2  2  2  2  2  2  2  2 2  2  3  3  3  3  3  3  3
        I
   .955Y1  1  1  1  2  2  2  2  2  2  2 2  2  2  2  2  3  3  3  3
        I
        I1  1  1  1  1  1  1  2  2  2  2 2  2  2  2  2  2  2  2  2
        I
        I1  1  1  1  1  1  1  1  1  1  1 2  2  2  2  2  2  2  2  2
   .614Y
        I1  1  1  1  1  1  1  1  1  1  1 1  1  1  1  1  2  2  2  2
        I
        I      1  1  1  1  1  1  1  1  1 1  1  1  1  1  1  1  1  1
        I
   .273Y                              1  1 1  1  1  1  1  1  1  1  1
        I
        I
        I
      0Y
         X---------X----------X----------X----------X----------X----------X
         0        .153       .322       .492       .661       .831      1.000
                                     ACTIVITY LEVEL

ALE FACTORS = X-AXIS: E+00    Y-AXIS: E+03    Z-AXIS: E+00
)-74=        0(-9),      .133(69),     .267(85),     .400(-9),     .533(77)
)-79=     .667(51),      .800(24),     .933( 3),    1.067( 0),    1.200( 0)
```

```
ROGRAM -DIFF- READY FOR INPUT
    /                                                                              /
    /                                                                              /
    /**********************************PUN 2**********************************/
    /                                                                              /
        NFC=2, $
```

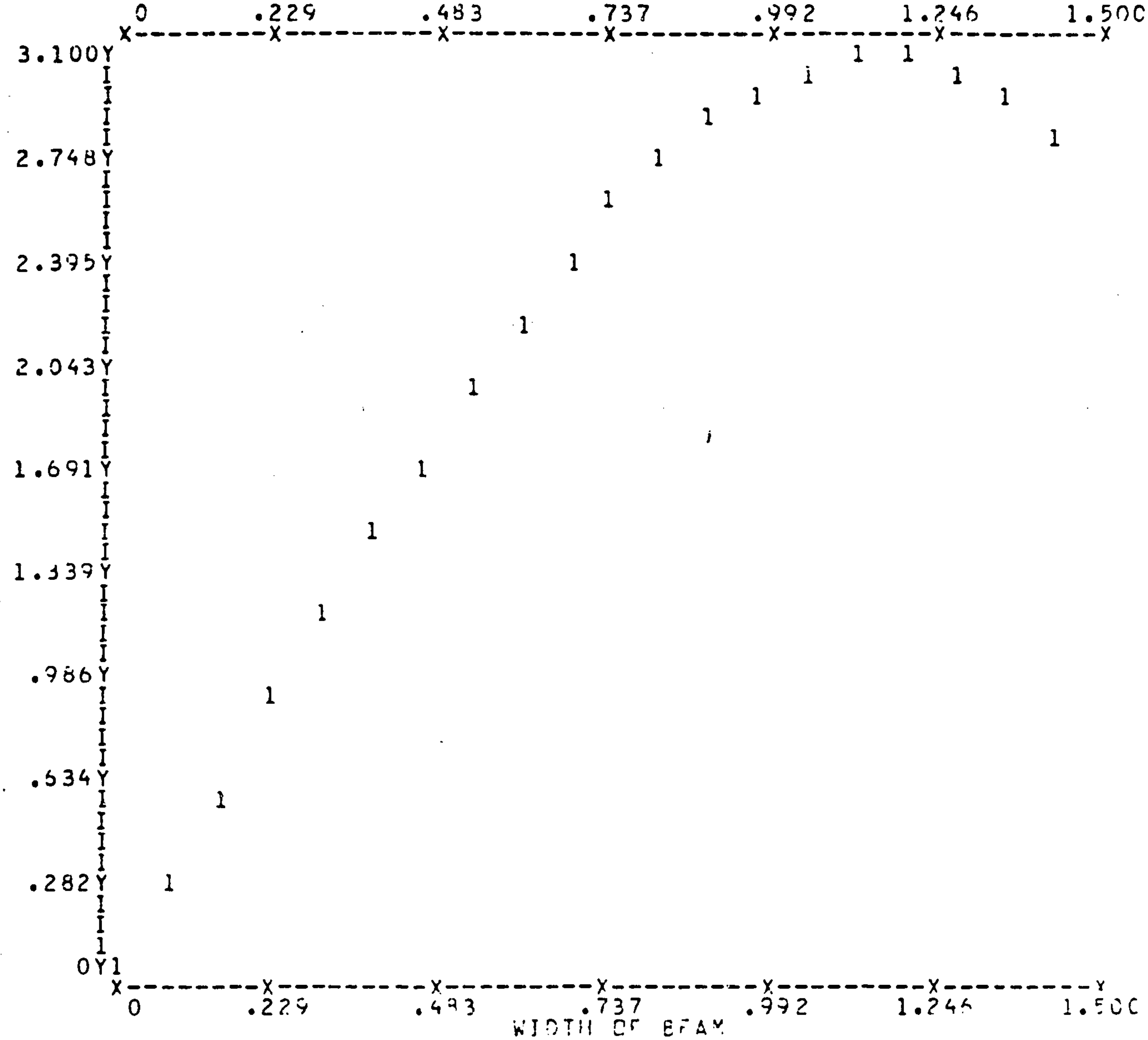
```
                      PIT, PEAK, OR PASS OR ...
                      FOR F = P1*X**2 + P2*Y**2
        XSLICE =       -2.0000          YSLICE =      -2.0000
            P1 =        1.0000              P2 =       1.0000

   -2.000     -1.390       -.712       -.034       .644       1.322      2.000
       X---------X----------X----------X----------X----------Y----------Y
 2.000Y
      I8   7  6  6  5  5  4  4  4  4  4 4  4  4  4  5  5  6  6  7
      I
      I7   7  6  5  4  4  4  3  3  3  3 3  3  3  4  4  4  5  6  7
      I
 1.545Y7   6  5  4  4  3  3  3  2  2  2 2  2  3  3  3  4  4  5  6
      I
      I6   5  5  4  3  3  2  2  2  2  2 2  2  2  2  3  3  4  5  5
      I
      I6   5  4  3  3  2  2  2  1  1  1 1  1  2  2  2  3  3  4  5
 1.091Y
      I5   4  4  3  2  2  1  1  1  1  1 1  1  1  1  2  2  3  4  4
      I
      I5   4  3  3  2  1  1  1                 1  1  1  2  3  3  4
      I
  .636Y4   4  3  2  2  1  1                       1  1  2  2  3  4
      I
      I4   3  3  2  1  1                             1  1  2  3  3
      I
      I4   3  2  2  1  1                             1  1  2  2  3
  .182Y
      I4   3  2  2  1  1                             1  1  2  2  3
      I4   3  2  2  1  1                             1  1  2  2  3
      I
      I4   3  2  2  1  1                             1  1  2  2  3
 -.273Y
      I4   3  3  2  1  1                             1  1  2  3  3
      I
      I4   4  3  2  1  1  1                       1  1  1  2  3  4
      I
 -.727Y5   4  3  2  2  1  1  1                 1  1  1  2  2  3  4
      I
      I5   4  3  3  2  2  1  1  1  1    1  1  1  1  2  2  3  3  4
      I
      I5   5  4  3  3  2  2  1  1  1  1 1  1  1  2  2  3  3  4  5
-1.182Y
      I6   5  4  4  3  2  2  2  2  1  1 1  2  2  2  2  3  4  4  5
      I
      I6   6  5  4  4  3  3  2  2  2  2 2  2  2  3  3  4  4  5  6
      I
-1.636Y7   6  5  5  4  4  3  3  3  3  3 3  3  3  3  4  4  5  5  6
      I
      I8   7  6  5  5  4  4  4  3  3  3 3  3  4  4  4  5  5  6  7
      I
-2.000Y9   8  7  6  6  5  5  4  4  4  4 4  4  4  5  5  6  6  7  8
       X---------X----------X----------X----------X----------Y----------Y
   -2.000     -1.390       -.712       -.034       .644       1.322      2.000

CALF FACTORS = X-AXIS: E+00    Y-AXIS: E+00    Z-AXIS: E+00
0-Z4=     .002(-9),     .891(76),    1.779(79),    2.668(77),    3.557(72)
5-Z9=    4.445(37),    5.334(22),    6.223(11),    7.111( 4),    8.000( 1)
```

```
ROGRAM -DIFF- READY FOR INPUT
     /                                                                              /
     /                                                                              /
     /**********************************RUN 3***********************************/
     /                                                                              /
        NFC=3, $
```

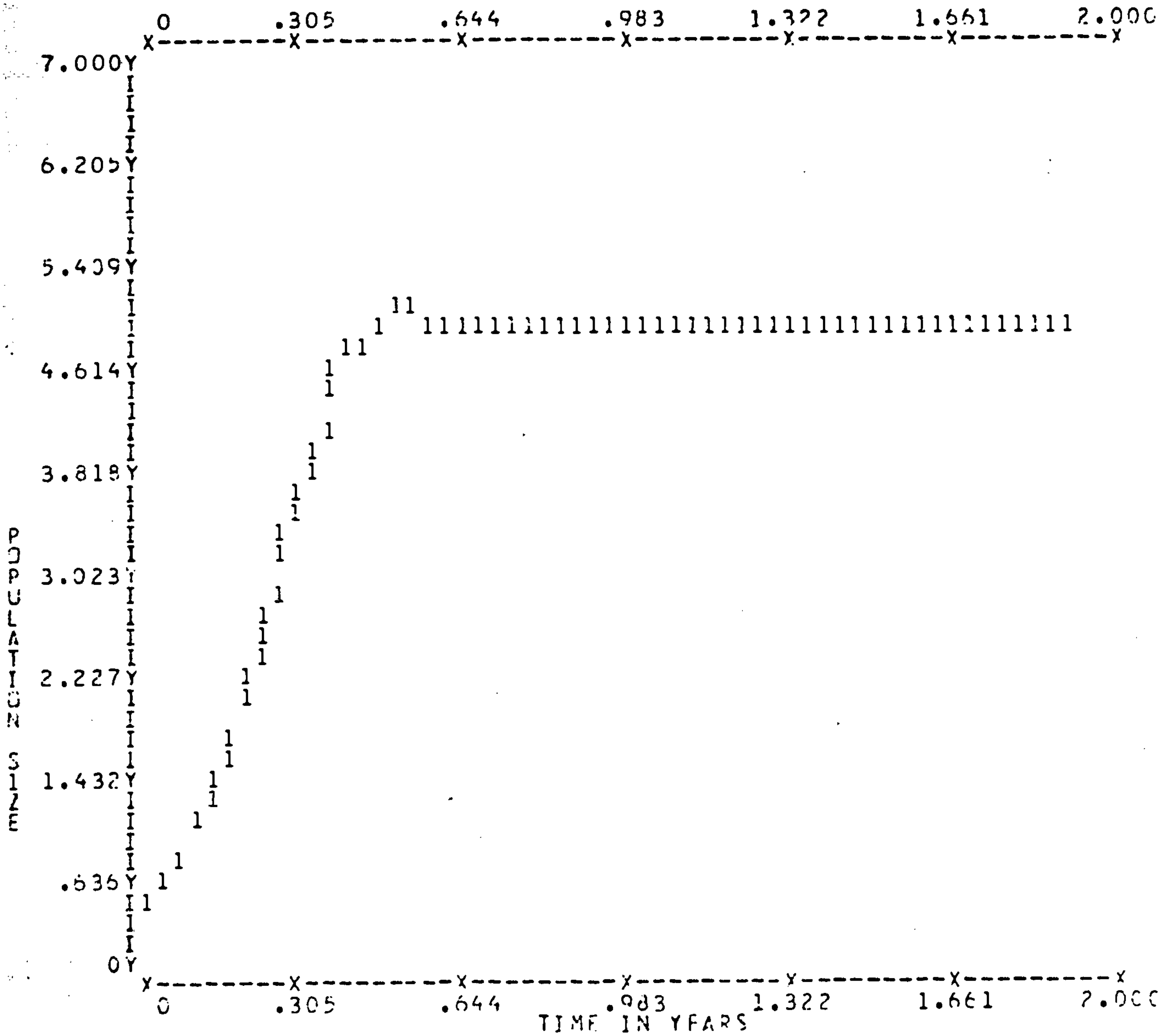
STRENGTH OF WOOD BEAM
FOR Y(X,D) = 0.1*X * D(X)**2
P1 = 1.0000 P2 = 1.0000



CALE FACTORS = X-AXIS: E+00 Y-AXIS: E-01 Z-AXIS: E+00
0-Z4= 0(-9), 1.000(20), 2.000(0), 3.000(0), 4.000(0)
5-Z9= 5.000(0), 6.000(0), 7.000(0), 8.000(0), 9.000(0)

```
PROGRAM -DIFF- READY FOR INPUT
   /                                                                      , /
   /                                                                        /
   /*****************************RUN 4**********************************/
   /                                                                        /
       NFC=4, P1=-10, P2=1.2, S
```

LOGISTIC GROWTH - DIFFERENCE EQUATION
FOR DY = (P2*Y*(1.-Y/500.))*DX
P1 = -10.0000 P2 = 1.2000



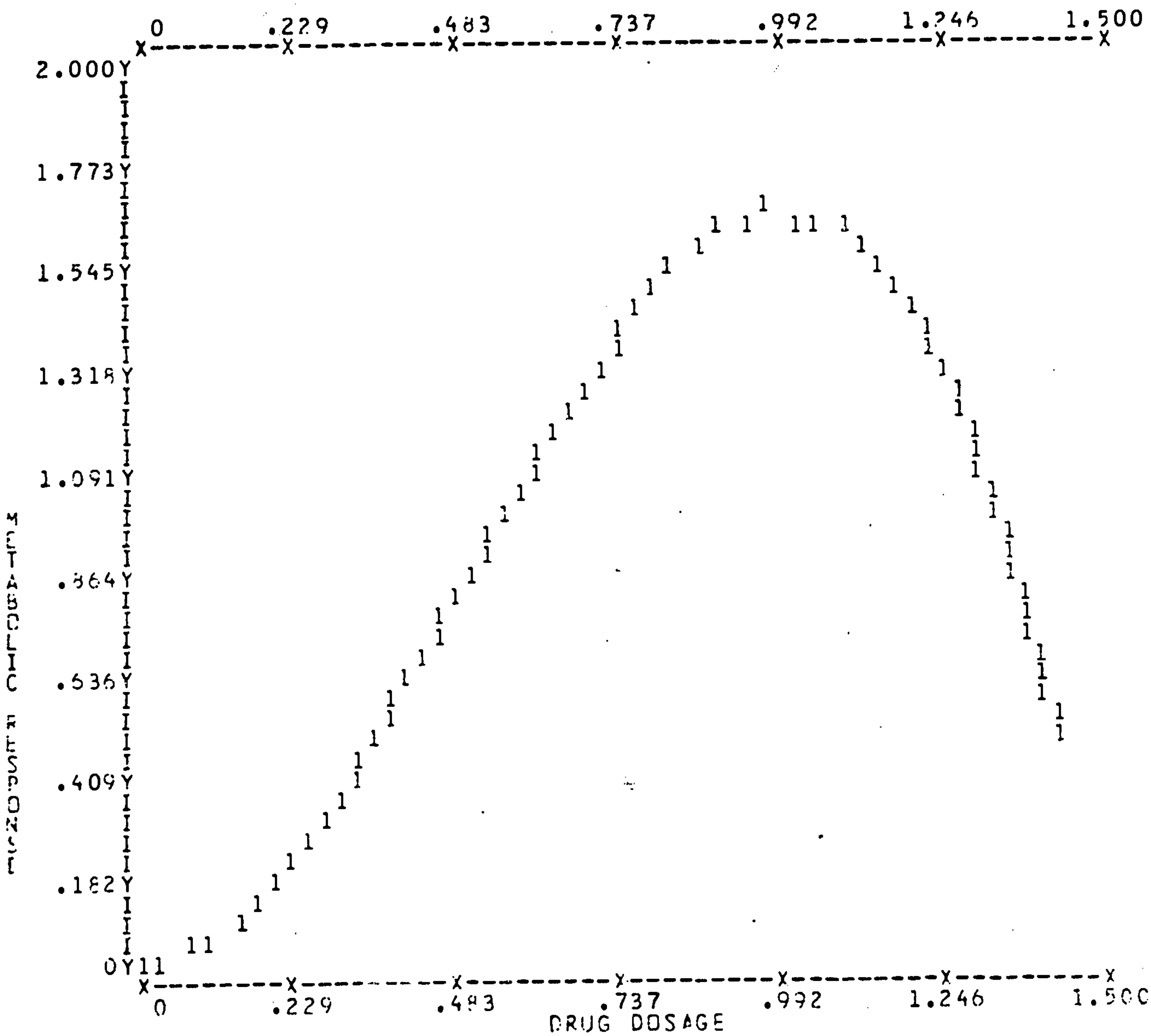
SCALE FACTORS = X-AXIS: E+01 Y-AXIS: E+02 Z-AXIS: E+00
Z0-Z4= 0(-9), 1.000(68), 2.000(0), 3.000(0), 4.000(0)
Z5-Z9= 5.000(0), 6.000(0), 7.000(0), 8.000(0), 9.000(0)

```
PROGRAM -DIFF- READY FOR INPUT
 /                                                                      /
 /                                                                      /
 /*******************************RUN 5*********************************/
 /                                                                      /
    NFC=5, XRICH=0.05, YRICH=0.005, IPRINT = .T., $
```

RENT VALUES FOR ALL INPUT VARIABLES:

```
     =            5,
TS   =            1,
     =  .1000000E+01,   .1000000E+01,   .1000000E+01,   .1000000E+01,
        .1000000E+01,   .1000000E+01,
     =  .1000000E+01,   .1000000E+01,   .1000000E+01,   .1000000E+01,
        .1000000E+01,   .1000000E+01,
CE   =       0      ,
CF   =       0      ,
     =  T,
S    =  F,
LT   =  F,
     =           60,
     =           45,
     =            0,             1,              2,              3,
                  4,             5,              6,              7,
                  8,             9,
     =       0      ,
     =  .1500000E+01,
     =       0      ,
     =  .2000000E+00,
     =       0      ,
     =  .9000000E+01,
H    =  .5000000E-01,
H    =  .5000000E-02,
I    =       0      ,
NT   =  F,
     =  F,
     =  F,
NT   =  T,
```

BODY RESPONSE TO FIXED DRUG DOSE
FOR Y(X) = (P1/2)*(X**2) - (P2/3)*(X**3)
P1 = 1.0000 P2 = 1.0000



SCALE FACTORS = X-AXIS: E+00 Y-AXIS: E-01 Z-AXIS: E+00
Z0-Z4= 0(-9), 1.000(69), 2.000(0), 3.000(0), 4.000(0)
Z5-Z9= 5.000(0), 6.000(0), 7.000(0), 8.000(0), 9.000(0)

```
/
/*-
/***********************************STOP***********************************/
/                                                                          /
      FINIS=.T., $
RAM -DIFF- TERMINATED
```

References

Gales, L.E. and L. Anderson. 1978. User's guide for subroutine FFORM: a format free input system. Center for Quantitative Science in Forestry, Fisheries, and Wildlife. University of Washington, Seattle, Washington.

Gales, L.E. 1978. User's guide for subroutine PRNT3D. Center for Quantitative Science in Forestry, Fisheries, and Wildlife, University of Washington, Seattle, Washington.