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ABSTRACT

This book is intended to make common seventh-grade mathematical concepts both interesting and easy to understand. The text is designed to meet the particular needs of those children who have "accumulated discouragements" in learning mathematics. The reading level required of pupils has been reduced. Individual chapter titles are: The Basic Operations; A Different Look; Geometry; and Factors and Primes. (MP)

Southwest Educational Development Laboratory

MATHEMATICS Book S

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PREFACE

The exercises included in this book were prepared to make mathematics both interesting and easy to understand.

Teachers and mathematicians with the Southwest Educational Development Laboratory adapted these materials. They were guided by the following beliefs:

- Children are interested in mathematics.
- Learning is enhanced by emphasis on understanding of concepts rather than on memorization of rules, and understanding results from being actively involved in experiences from which concepts are to be abstracted.
- Alternative sequences of mathematical concepts can be followed, and yet the structure of mathematics can be preserved.
- Children can learn more mathematics than they are now learning.

Edwin Hindsman
Executive Director

ACKNOWLEDGMENTS

These materials were prepared by the Southwest Educational Development Laboratory's Mathematics Education Program during two summer writing conferences. The 1968 Summer Mathematics Writing Conference participated in the initial adaptation of these materials, and the 1969 Summer Mathematics Writing Conference participated in their revision.

The 1969 Summer Mathematics Writing Conference, held in Austin, Texas, was coordinated by Floyd Vest, Professor of Mathematics Education, North Texas State University, Denton, Texas. He was assisted by James Hodge, Professor of Mathematics, North Texas State University, and Palma Lynn Ross, Department of Mathematics, University of Texas at El Paso.

Participants for the 1969 writing conference included: Carmen Montes, Santiago Peregrino, Rebecca Rankin, Rudolph Sanchez, and Flora Ann Sanford, El Paso Independent School District, El Paso, Texas; Jimmye Blackmon, J. Leslie Fauntleroy, Barbara Graham, and Sophie Louise White, East Baton Rouge Parish Schools, Baton Rouge, Louisiana; Lawrence A. Couvillon and James Keisler, Louisiana State University, Baton Rouge, Louisiana; and Socorro Lujan, Mathematics Education, Southwest Educational Development Laboratory, Austin, Texas.

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Chapter 5:

The Basic Operations: A Different Look

Section 5-1 Addition in Expanded Form

Look at the simple addition problem 4 + 3 = ? Of course, right away you know the answer is 7. If you did not know the answer you could think of a set A that has 4 members and a set B that has 3 other members, and form the union of the two sets. If you then count the number of members in the union you would have the answer to the addition problem.

This way of doing addition is fine as long as you have problems like the one above. What about problems like 892 + 367? You could find the sum of these two numbers using sets, just as you did with 4 + 3; but it would be a long, slow process. It is our base ten system that makes addition so easy. Look at the example below:

	Expanded Form		
892	800 + 90 + 2		
+ 367	300 + 60 + 7		
1,259	1100 + 150 + 9	9	sum of ones
		150	sum of tens
		1100	sum of hundreds
		1259	final sum

Exercise 5-1

Use expanded form to find the following sums. Follow the example above.

1. 246 =
 + 139 =

2. 784 =
 + 926 =

3. 777 =
 + 964 =

4. 123 =
 + 987 =

5. 486 =
 + 766 =

6. 949 =
 + 892 =

BRAINBUSTER:

7. 2,345,678 =
 + 9,876,543 =

8. Write the answer to the BRAINBUSTER in words. ________________

____

____

Section 5-2 Addition in Short Form

Now that you have done some addition using expanded form, let us see if we can shorten the method and do an addition problem more quickly.

Example:

Follow the steps as we work the addition problem

563 + 787 + 384 = 1734

```
  563
  787
+ 384
```

Step 1. Add the digits in the ones place, that is,

3 + 7 + 4 = 14. Place the partial sum here. ⟶ 14

Step 2. Add the digits in the tens place, that is,

60 + 80 + 80 = 220. Place partial sum here. ⟶ 220

Step 3. Add the digits in the hundreds place, that is,

500 + 700 + 300 = 1500. Place partial sum here. ⟶ 1500

Step 4. Add the partial sums to find the final sum. ⟶ 1734

Notice that if you use this method it is just as easy to add from left to right as from right to left.

Exercise 5-2

Work the following problems as is done in the example above.

1. 46 + 17   2. 25 + 32   3. 22 + 57   4. 727 + 324

5.
```
   476
   398
  7256
+   89
```

6.
```
  403
  213
  414
+ 898
```

7.
```
  1777
+  598
```

8.
```
  3456
  4634
+ 7279
```

9.
```
  23456
  98765
+ 45678
```

10.
```
  9999
+ 1111
```

Section 5-2(a) Addition in Short Form

This is the final short form of addition which you will want to be able to do well. See if this isn't something like the way you think when you are doing an addition problem.

```
 (1)
  68
 257
   5
```

(a) 8 + 7 = 15, "put down the 5, 'carry' the 1."

```
(1)(1)
  68
 257
  25
```

(b) (1) + 6 + 5 = 12, "put down the 2, 'carry' the 1."

```
(1)(1)
  68
 257
 325
```

(c) (1) + 2 = 3, "put down the 3."

Exercise 5-2(a)

1. In part (a), when you "carry the 1", what does this '1' stand for?

____

2. In part (b), when you "carry the 1", what does this '1' stand for?

____

3. In part (c), when you added (1) + 2 = 3, the '3' stands for three

____

Work the following problems using the short form as in the example on the preceding page. Circle your "carry" if it will help.

4. 35 + 59

5. 69 + 79

6. 75 + 68

7. 465 + 654

8. 345 + 567 + 789

9. 3579 + 2468 + 2345 + 9876

10. 98765432 + 23456789 + 98765432 + 23456789

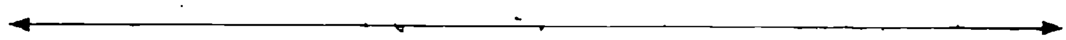
11. BRAINBUSTER. Write the answer to problem 10 in words. __________

____

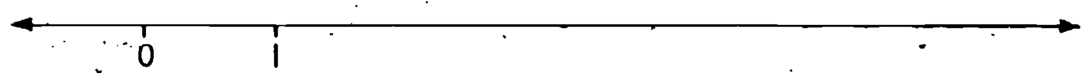
____

Section 5-3 The Number Line

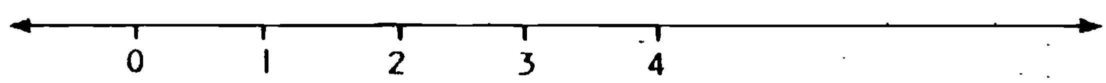
A very useful way of picturing numbers is to look at what is called a number line. First we draw a line, as shown below, with arrows on both ends. The arrows show that the line goes on and on in both directions.



Next, we pick a point (any point) on the line and let it correspond to zero. To the right of zero we mark a point corresponding to 1. (This part of the number line, between 0 and 1, is called the unit interval.)



Now, we mark 2 to the right of 1, 3 to the right of 2, 4 to the right of 3, etc. The distance between 1 and 2, 2 and 3, 3 and 4, etc., is the same distance as between 0 and 1.



The part of the number line between any two points is called a line segment. (Segment means "part of", so you can think of a line segment as "part of" a line.)

Exercise 5-3

Use the number line above to answer the following questions:

1. What is the smallest whole number represented on the number line?

____

2. What is the smallest counting number represented on the number line?

____

3. The distance between 3 and 4 is the __________ as the distance between 0 and 1.

4. The distance between 4 and 5 is the same as the distance between ______ and ______.

5. What do the arrows on either end of the number line mean?

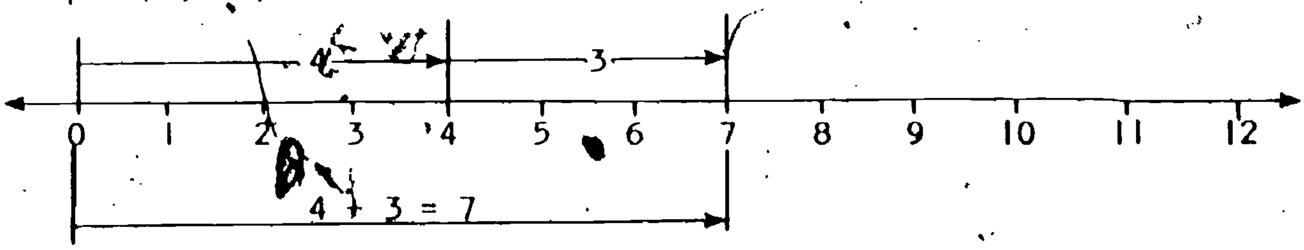
____

6. Is there a largest number on the number line? __________ Why?

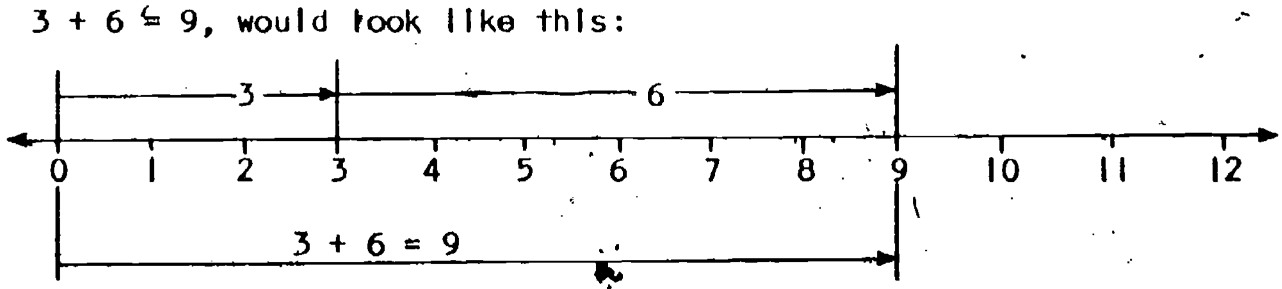
____

Section 5-4 Addition on Number Line

To add the number 3 to the number 4, start from 0, move four units to the right to the number 4, then move three more units to the right. We stop at 7; so, 4 + 3 = 7.

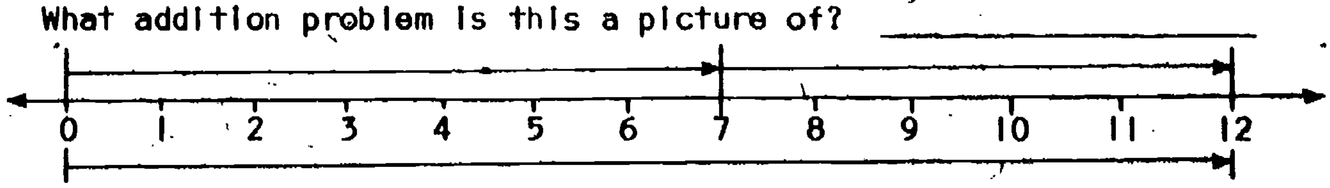


3 + 6 = 9, would look like this:

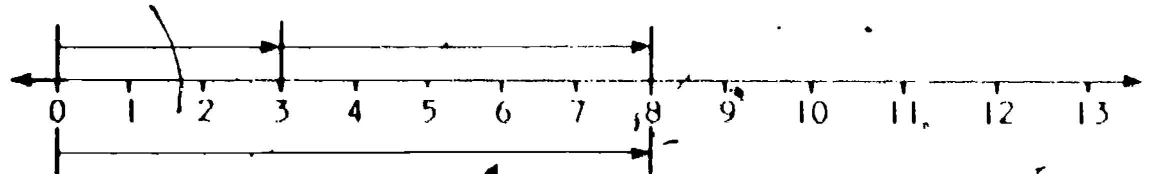


Exercise 5-4

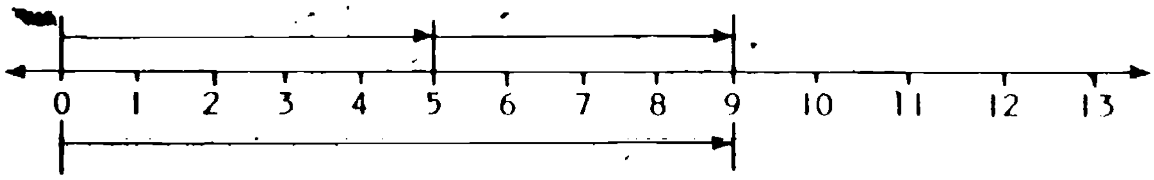
1. What addition problem is this a picture of? __________



2. What addition problem is this a picture of? ________________



3. Is this a picture of 4 + 5 = 9 or 5 + 4 = 9? ____________ Why?



4. Draw number lines and show the addition of the following:

a. 5 + 9 = 14

b. 6 + 3 = 9

c. 8 + 9 = 17

d. 9 + 8 = 17

e. 7 + 9 = 16

5. BRAINBUSTER. Can you show $(3 + 5) + 4 = 12$ on the number line? Try it!

Section 5-5 Regrouping in Subtraction

Let us now review the operation of subtraction. You should understand exactly what happens when two numbers are subtracted, especially what you are doing when you "borrow."

Example: $68 - 49 = 19$

	68	6 tens + 8 ones
−	49	4 tens + 9 ones

Looking ahead, you can see that 8 − 9 cannot be done with whole numbers. Therefore it will be necessary to regroup the 6 tens into 5 tens + 10 ones. Then we "borrow" the 10 ones and add them to the 8 ones. Now we can complete the subtraction as shown below:

	68	6 tens + 8 ones	5 tens + 18 ones
−	49	4 tens + 9 ones	4 tens + 9 ones
	19		1 ten + 9 ones

Exercise 5-5

Work the following problems as is done in the example above. Do not use the numerals 1, 10, 100, or 1,000. Write these numerals in words as is done in the example, i.e. ones, tens, etc.

1. 53
   - 36

2. 75
   - 37

3. 764
   - 199

4. 402
   - 139

5. 710
   - 287

6. 3,456
   - 1,567

Section 5-6 Subtraction in Expanded Form

The example below shows the subtraction 68 - 49 using expanded form. Study it carefully.

Step 1. 68 and 49 are written in expanded form:

68	60 + 8
- 49	40 + 9

Step 2. Looking ahead we see that 8 - 9 cannot be done with whole numbers. Therefore, we regroup 68 as 50 + 18.

68	50 + 18
- 49	40 + 9

Step 3. Now we are ready to subtract 9 from 18 and 40 from 50.

68	50 + 18
- 49	40 + 9
19	10 + 9

Exercise 5-6

Work the following problems as done in the example above. Show all the regrouping.

1. 58
 - 39

2. 73
 - 56

3. 125
   - 37

4. 452
   - 168

5. 503
   - 247

6. 3,532
   - 1,654

Section 5-7 Subtraction in Short Form

Study carefully the forms of subtraction below. This should make clear the meaning of "borrowing" in subtraction.

Short Form		Expanded Form	
342		342	300 + 40 + 2
- 187		- 187	100 + 80 + 7

Short Form		Expanded Form	
3 3~~4~~2 - 187	2 - 7 cannot be done with whole numbers, so we regroup the tens.	342 - 187	300 + 30 + 12 100 + 80 + 7
23 ~~3~~~~4~~2 - 187	3 - 8 cannot be done with whole numbers, so we regroup the hundreds.	342 - 187	200 + 130 + 12 100 + 80 + 7
23 ~~3~~~~4~~2 - 187 155	Now subtract.	342 - 187 155	200 + 130 + 12 100 + 80 + 7 100 + 50 + 5

Exercise 5-7

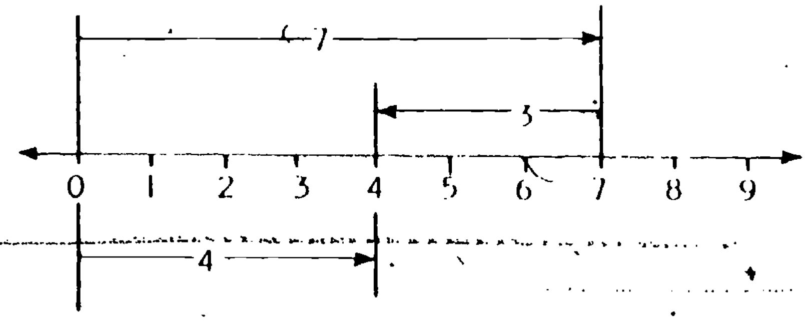
Work the following problems using the short form shown above?

1. 246 − 139
2. 926 − 784
3. 964 − 777
4. 40 − 13
5. 2323 − 987
6. 766 − 486
7. 949 − 892
8. 310 − 178
9. 6055 − 4723
10. 2003 − 876

Section 5-8 Subtraction on Number Line

To subtract the number 3 from the number 7, start from 0, move 7 units to the right to the number 7, now move 3 units to the left from the number 7. Where did you stop? ans. ________________

Below is a picture of the subtraction 7 - 3 = 4.



Exercise 5-8

Draw number lines and show the following subtractions:

1. 12 - 5 = 7

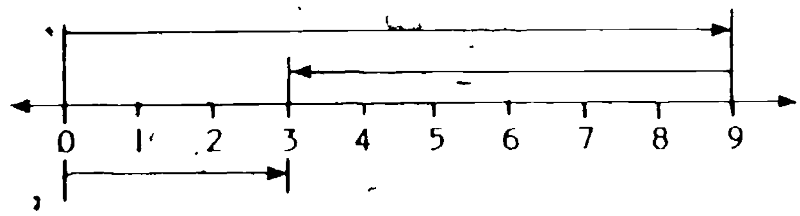
2. 9 - 6 = 3

3. 10 - 1 = 9

4. 15 - 12 = 3

5. 9 - 9 = 0

6. What subtraction is pictured on the number line below? __________



Section 5-9 Multiplication with Numbers Ending in Zero

The numbers 10, 100, 1000, 10,000, etc., are easy to work with in multiplication problems. There is a pattern to the answers (products) when these numbers are part of the problem. Study the following examples:

3 · 10 = 30

12 · 100 = 1200

37 · 100 = 3,700

6 · 1000 = 6000

125 · 10,000 = 1,250,000

Do you see the pattern? If we multiply some number by 1 followed by, say 3 zeros, the product will be that number followed by 3 zeros.

Exercise 5-9

1. 7 · 10 = ________________.

2. 5 · 100 = ________________.

3. 8 · 10,000 = ________________.

4. 32 · 1,000 = ________________.

5. 69 · 100,000 = ________________.

6. 507 · 10 = ________________.

7. 1,728 · 10,000 = ______________________.

8. 4 · 1,000,000 = ______________________.

9. 100 · 98 = ______________________.

10. 10 · 32 = ______________.

Express each of the following numbers as a product of 2 numbers. One of the numbers must be a power of 10, that is, 10, or 100, or 1000, or 10,000, etc., and the other number must not have a zero in the ones place.

Examples:

570 = 57 · 10; 107,000 = 107 · 1,000; 3,700 = 37 · 100

11. 360 = ______________.

12. 5,800 = ______________.

13. 90 = ______________.

14. 397,000 = ______________.

15. 250,000 = ______________.

16. 9,700 = ______________.

17. 546,000,000 = ______________.

18. 70,500 = ______________.

19. 8,700 = ______________.

20. 1,010 = ______________.

Now you know how to multiply, easily, in problems like 10 · 5, 32 · 100, etc., and also know that 50 = 5 · 10, 3200 = 32 · 100, etc. Let us see if you can make another discovery about multiplication. Look at the following multiplication problem and then answer the questions that follow.

30 · 70 = 2100

Exercise 5-9(a) [Note: problems continue on next page.]

1. Does 30 = 3 · 10? ______________.

2. Does 70 = 7 · 10? ______________.

3. Then, does $30 \cdot 70 = (3 \cdot 10) \cdot (7 \cdot 10)$? ______________________.

4. Does it make any difference in the answer (product) how we arrange the numbers to be multiplied? For example, does $2 \cdot 3 \cdot 5 = 3 \cdot 5 \cdot 2$? ______________________.

5. Then does $30 \cdot 70 = (3 \cdot 10) \cdot (7 \cdot 10) = 3 \cdot 7 \cdot 10 \cdot 10$?

6. $3 \cdot 7 =$ ______________________.

7. $10 \cdot 10 =$ ______________________.

8. Then, does $30 \cdot 70 = (3 \cdot 7) \cdot (10 \cdot 10) = 21 \cdot 100$? ____________.

9. Then, the product of $30 \cdot 70 = 21 \cdot 100 =$ ______________________.

Examples: (Note: 'p' stands for product.)

(a) $p = 20 \cdot 30$

$p = (2 \cdot 10) \cdot (3 \cdot 10)$

$p = 2 \cdot 3 \cdot 10 \cdot 10$

$p = 6 \cdot 100$

$p = 600$

(b) $p = 300 \cdot 50$

$p = (3 \cdot 100) \cdot (5 \cdot 10)$

$p = 3 \cdot 5 \cdot 100 \cdot 10$

$p = 15 \cdot 1000$

$p = 15{,}000$

Do the exercises below and on the next page in the same manner as the examples above. Do not leave out any steps. [Note: problems continue on next page.]

10. $p = 30 \cdot 40$

$p =$ ______________________

$p =$ ______________________

$p =$ ______________________

$p =$ ______________________

11. p = 70 · 800 ______________.
p = ______________.
p = ______________.
p = ______________.
p = ______________.

12. p = 900 · 60 ______________.
p = ______________.
p = ______________.
p = ______________.
p = ______________.

13. p = 9 · 400 ______________.
p = ______________.
p = ______________.
p = ______________.
p = ______________.

14. p = 600 · 800 ______________.
p = ______________.
p = ______________.
p = ______________.
p = ______________.

15. p = 7,000 · 500 ______________.
p = ______________.
p = ______________.
p = ______________.
p = ______________.

Let us see if you have arrived at the quick and easy method of multiplying numbers that end in zeros. Look at the problem below and the questions that follow.

600 · 30 = 18,000

1. How many zeros are there in the numeral 600? ans. ____________.
2. How many zeros are there in the numeral 30? ans. ____________.
3. How many zeros altogether? ans. ____________.
4. How many zeros are there in the product 18,000? ans. ____________.
5. Now let us take a look at the above problem again. Observe the pattern:

$$\underbrace{600}_{2\text{ zeros}} \cdot \underbrace{30}_{1\text{ zero}} = \underbrace{18,000}_{6 \cdot 3,\ 3\text{ zeros}}$$

Exercise 5-9(b)

Study these examples:

400 · 700 = (4 · 7) followed by 4 zeros = 280,000

60 · 30 = (6 · 3) followed by 2 zeros = 1,800

8000 · 9000 = (8 · 9) followed by 6 zeros = 72,000,000

Then complete the following:

1. 30 · 500 = 15,000
2. 500 · 70 = ____________
3. 300 · 9,000 = ____________
4. 8,000 · 9 = ____________
5. 600 · 900 = ____________
6. 80 · 7,000 = ____________

7. 700 · 700 = ______________.

8. 40 · 9,000 = ______________.

9. 9,000 · 9,000 = ______________.

10. 4,000 · 700 = ______________.

Section 5-10 Multiplication in Expanded Form

Let us now see if we can make use of what you have just learned to help you understand everyday multiplication problems. Look at the examples below:

$$\begin{array}{r} 24 \\ \times\ 7 \\ \hline \end{array} \quad \begin{array}{r} 20 + 4 \\ 7 \\ \hline \end{array} \quad \begin{array}{r} 20 \\ 7 \\ \hline \end{array} + \begin{array}{r} 4 \\ 7 \\ \hline \end{array} \quad \begin{array}{r} 140 \\ 28 \\ \hline \end{array} \begin{array}{l} = (7 \cdot 20) \\ = (7 \cdot 4) \end{array}$$

$$140 + 28 = 168$$

Let us try another one.

$$\begin{array}{r} 235 \\ \times\ 3 \\ \hline \end{array} \quad \begin{array}{r} 200 + 30 + 5 \\ 3 \\ \hline \end{array} \quad \begin{array}{r} 200 \\ 3 \\ \hline \end{array} + \begin{array}{r} 30 \\ 3 \\ \hline \end{array} + \begin{array}{r} 5 \\ 3 \\ \hline \end{array} \quad \begin{array}{r} 600 \\ 90 \\ 15 \\ \hline \end{array} \begin{array}{l} = (3 \cdot 200) \\ = (3 \cdot 30) \\ = (3 \cdot 5) \end{array}$$

$$600 + 90 + 15 = 705$$

Exercise 5-10

Work the following problems in the same manner as the examples above:

1. $\begin{array}{r} 47 \\ \times\ 4 \\ \hline \end{array}$

2. $\begin{array}{r} 68 \\ \times\ 9 \\ \hline \end{array}$

3. 63
   × 7

4. 41
   × 5

5. 98
   × 8

6. 264
   × 6

7. 473
   × 7

8. 987
   × 3

Section 5-10(a) Multiplication in Expanded Form

In Section 5-10 you learned to do multiplication in expanded form. Let us now see if we can shorten the process a little. Using the same examples as in Section 5-10 we can shorten the procedure

from:

24	20 + 4	20 + 4		140 = (7 · 20)	
× 7	7	7	7	28 = (7 · 4)	
		140	28	168	

to:

$$\begin{array}{r l} 24 & \\ \times\ 7 & \\ \hline 28 & = (7 \cdot 4) \\ 140 & = (7 \cdot 20) \end{array}$$

Using the shorter form on the right, 3 · 235 would look like this.

$$\begin{array}{r l} 235 & \\ \times\ 3 & \\ \hline 15 & = (3 \cdot 5) \\ 90 & = (3 \cdot 30) \\ 600 & = (3 \cdot 200) \\ \hline 705 & \end{array}$$

Exercise 5-10(a)

Work the following problems using the shorter form as in the examples above:

1. $\begin{array}{r} 47 \\ \times\ 4 \\ \hline \end{array}$ 2. $\begin{array}{r} 68 \\ \times\ 9 \\ \hline \end{array}$ 3. $\begin{array}{r} 63 \\ \times\ 7 \\ \hline \end{array}$ 4. $\begin{array}{r} 41 \\ \times\ 5 \\ \hline \end{array}$

5. $\begin{array}{r} 368 \\ \times\ 7 \\ \hline \end{array}$ 6. $\begin{array}{r} 264 \\ \times\ 6 \\ \hline \end{array}$

7. $\begin{array}{r} 473 \\ \times\ 8 \\ \hline \end{array}$ 8. $\begin{array}{r} 987 \\ \times\ 5 \\ \hline \end{array}$

Section 5-10(b) Multiplication in Expanded Form

In the last two sections we multiplied two-digit numbers by one digit numbers. Let us see what happens if we use pairs of larger numbers.

Example 1:

46	40 + 6	40 + 6	40 + 6	40	6	40	6
× 34	30 + 4	30	4	30	30	4	4

1200 + 180 + 160 + 24 = 1564

As you can see, this is a rather long procedure. Let us see if we can shorten it a little.

46	40 + 6	
× 34	30 + 4	
	24	= (4 · 6)
	160	= (4 · 40)
	180	= (30 · 6)
	1200	= (30 · 40)
	1564	

Exercise 5-10(b)

Work the following problems as in example 2 above: [Note: Study problem (5) before working problems (6) - (9).]

1. 67 × 24
2. 45 × 63
3. 97 × 69
4. 32 × 78

5. 346 300 + 40 + 6

× 67 60 + 7

42 = (7 · 6)

280 = (7 · 40)

2100 = (7 · 300)

360 = (60 · 6)

2400 = (60 · 40)

18000 = (60 · 300)

23182

6. 473

× 24

7. 839

× 48

8. 735

× 93

9. BRAINBUSTER

648

× 375

Section 5-11 Multiplication in Short Form

Study the steps below and see if this isn't something like the way you think when you are doing a multiplication problem.

Example: 46 · 34 = 1564

Step 1.

```
 (2)
  46
x 34
   4
```

4 times 6 is 24, put down the 4, "carry" the 2.

Step 2.

```
 (2)
  46
x 34
 184
```

4 times 4 is 16, plus 2 "carry" is 18, put down the 18.

Step 3.

```
 (1)
 (2)
  46
x 34
 184
  8
```

3 times 6 is 18, put down the 8, "carry" the 1.

```
 (1)
 (2)
  46
x 34
 184
138
```

3 times 4 is 12, plus 1 "carry" is 13, put down the 13.

```
 (1)
 (2)
  46
x 34
 184  ——> First partial product
138   ——> Second partial product

1564  ——> Final product
```

now add the partial products

Exercise 5-11

The following questions refer to the steps taken on page 25. You may wish to refer to Section 5-10(b), page 23, also.

1. In step 1., when you "carry the 2", the '2' stands for 2 _______ or _______.

2. In step 2., where it says "4 times 4 is 16", it really means 4 times _______ is _______.

3. In step 2., when you "put down the 18", the '18' stands for 18 _______ or _______.

4. In step 3., where it says "3 times 6 is 18", you are actually multiplying by 3 _______ or _______.

5. In step 3., when you "put down the 8", the '8' stands for 8 _______ or _______.

6. In step 4., where it says "3 times 4 is 12", the '3' stands for _______, the '4' stands for _______, and the '12' stands for _______.

7. In step 5., you are actually adding 184 to _______ in order to get the final product 1564.

8. The following multiplication problem is done for you. Write the expanded form within the parentheses. [Hint: See problem (5), Exercises 5-10(b).]

```
   678
   439
    72 = (              )
   630 = (9  ·  70      )
  5400 = (              )
   240 = (              )
  2100 = (              )
 18000 = (              )
  3200 = (              )
 28000 = (              )
240000 = (              )
297642
```

Exercise 5-11(a)

Work the following problems using any method you wish.

1. 81 × 6

2. 64 × 8

3. 478 × 7

4. 835 × 8

5. 7498 × 6

6. 48 × 91

7. 92 × 56

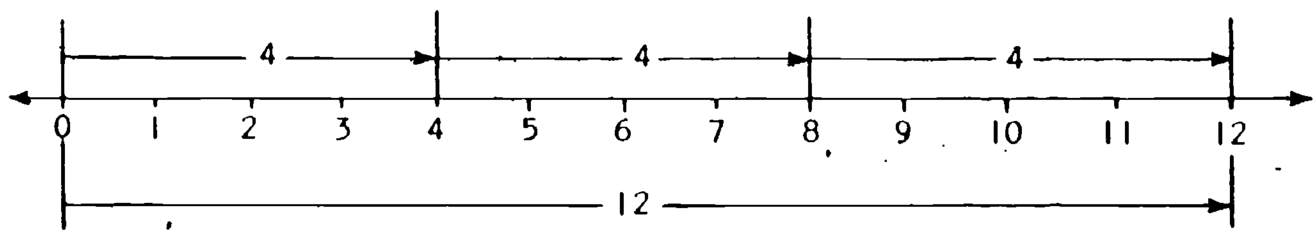
8. 87 × 34

9. 538 × 74

10. 846 × 372

Section 5-12 Multiplication on Number Line

Just as addition and subtraction may be shown on the number line, multiplication may also be shown. For example, to show $3 \cdot 4 = 12$, consider an arrow for 4. Three such arrows laid end to end (tail to head) indicate $3 \cdot 4$.



Exercise 5-12

Draw number lines for each of the following:

1. $2 \cdot 9$

2. $3 \cdot 5$

3. $5 \cdot 3$

4. $2 \cdot 6$

5. 6 · 2

6. 6 · 3

7. 3 · 6

Section 5-13 Division as Repeated Subtraction

Let us now take a look at two different division problems.

Example 1: $\frac{35}{7}$

$$\begin{array}{r} 35 \\ -\ \underline{7} \\ 28 \\ -\ \underline{7} \\ 21 \\ -\ \underline{7} \\ 14 \\ -\ \underline{7} \\ 7 \\ -\ \underline{7} \\ 0 \end{array}$$

Example 2: $\frac{29}{9}$

$$\begin{array}{r} 29 \\ -\ \underline{9} \\ 20 \\ -\ \underline{9} \\ 11 \\ -\ \underline{9} \\ 2 \end{array} \longrightarrow \text{remainder}$$

Notice, in Example 1, we kept subtracting 7's until the difference was less than 7. In this case, the remainder is zero. How many times did we subtract 7? ans. ____________ This tells you that there are 5 sevens in 35. Therefore $\frac{35}{7} = 5$, remainder 0; or $\frac{35}{7} = 5$ r 0.

Notice, in Example 2, we subtracted 9's until the difference was less than 9; in this case, 2. How many times did we subtract 9? ans. ____________ What was the remainder? ans. ____________ This tells you that there are 3 nines in 29 and 2 left over. Therefore, $\frac{29}{9} = 3 \text{ r } 2$.

Exercise 5-13

Work the following problems by repeated subtraction as shown in the preceding examples:

1. $\frac{63}{7}$ 2. $\frac{125}{25}$ 3. $\frac{72}{12}$ 4. $\frac{66}{13}$ 5. $\frac{38}{3}$

Section 5-14 Division in Short Form

After doing the exercises in Section 5-13 (especially the problem $\frac{38}{3}$), you should realize that although the method works, the process can become very long and tiresome. Try dividing $\frac{1,728}{3}$ by repeated subtraction if you are not convinced, and see how long it takes.

Let us use the idea of repeated subtraction, but let us shorten the process some. Look at the examples below:

	$\frac{69}{3} = 3 \overline{)\,69}$		Add
Subtract ten 3's, $(10 \cdot 3)$ →	$-\ 30$	10	↓
	39		
Subtract ten 3's, $(10 \cdot 3)$ →	$-\ 30$	10	↓
	9		
Subtract three 3's, $(3 \cdot 3)$ →	$-\ 9$	3	↓
remainder →	0	23	→ answer

With a little more practice, you will be able to shorten the process even more by taking fewer steps.

	$\frac{69}{3} = 3 \overline{)\,69}$	Add
Subtract twenty 3's (20 · 3) ⟶	− 60	20
	9	
Subtract three 3's (3 · 3) ⟶	− 9	3
remainder ⟶	0	23 ⟶ answer (Quotient)

Exercise 5-14

Work the following problems using the method explained above:

[Note: problems continue on next page.]

1. $\frac{95}{5}$

2. $\frac{97}{4}$

3. $\frac{123}{8}$

4. $\frac{920}{7}$ = 7 ⟌ 920

	920	
	− 700	100
	220	
	− 210	30
	10	
	− 7	1
remainder →	3	131

5. $\frac{1334}{6}$

6. $\frac{1417}{9}$

7. $\frac{9250}{7}$

8. $\frac{8427}{6}$

9. $\frac{9437}{4}$

Section 5-14(a) Division in Short Form

The following is an example of division where the divisor is a 2 digit number.

	$\frac{675}{25} = 25\overline{)675}$	Add
Subtract ten 25's (10 · 25) →	− 250	10
	425	
Subtract ten 25's (10 · 25) →	− 250	10
	175	
Subtract four 25's (4 · 25) →	− 100	4
	75	
Subtract three 25's (3 ·25) →	− 75	3
remainder →	0	27 → answer (Quotient)

Exercise 5-14(a)

Work the following problems as shown in the example above:

1. $\frac{914}{44}$

2. $\frac{1498}{21}$

3. $\frac{823}{55}$

4. $\frac{1828}{78}$

5. $\frac{7094}{38}$

6. $\frac{27345}{23}$

Section 5-14(b) Division in Short Form

The method of division shown below is only slightly different from the method you have been using the past few days. We will now place the partial quotients above the dividend.

```
                                          1359 ←—— quotient
                                             9     ↑
                                            50     |
                                           300     |
                                          1000    Add
                                        4)5439
Subtract one thousand 4's               - 4000
                                          1439
Subtract three hundred 4's              - 1200
                                           239
Subtract fifty 4's                      -  200
                                            39
Subtract nine 4's                       -   36
                                             3  remainder
```

Exercise 5-14(b)

Work the following examples using the method shown above: [Note: problems continue on the next page.]

1. $9\overline{)1233}$

2. $8\overline{)9683}$

3. $4\overline{)26547}$

4. $7\overline{)7983}$

5. $30\overline{)1628}$

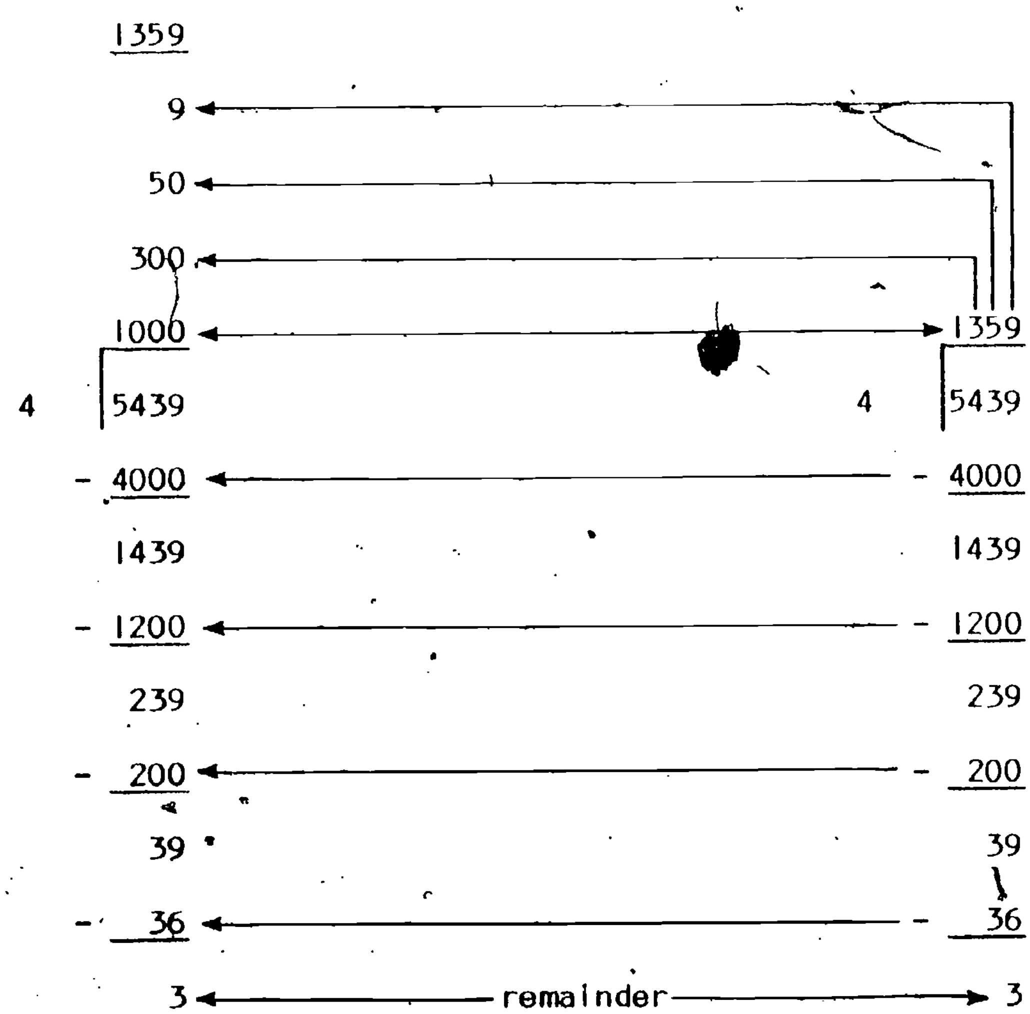
6. $44\overline{)914}$

Section 5-14(c) Division in Short Form

There is a shorter way to write your quotient in division. It will allow you to do your work more quickly.

Study the example below.

(a) Shorter Method (b) Shortest Method



In (b), to show the partial quotient 1000, we can write 1 in the thousands place. Instead of writing 300, we can write 3 in the hundreds place. Instead of writing 50, we can write 5 in the tens place. Then we can write 9 in the ones place.

Exercise 5-14(c)

Work the following as shown in example (b) on the preceding page.

1. $3\overline{)963}$ 2. $4\overline{)848}$ 3. $5\overline{)499}$

4. $4\overline{)648}$ 5. $6\overline{)4882}$ 6. $8\overline{)6896}$

7. $6\overline{)4928}$ 8. $9\overline{)6524}$ 9. $8\overline{)7932}$

Exercise 5-14(d)

Divide, using any method you wish:

1. $\frac{604}{82}$ 2. $\frac{914}{44}$ 3. $\frac{1498}{21}$

4. $\frac{8446}{65}$ 5. $\frac{9687}{21}$ 6. $\frac{7840}{32}$

7. $\frac{2376}{18}$ 8. $\frac{8767}{72}$ 9. $\frac{19780}{20}$

Review Exercise 5-15

1. Vocabulary -- Describe in your own words:

(a) digits
(b) partial sum
(c) number line
(d) regrouping
(e) quotient
(f) partial quotient
(g) product
(h) difference
(i) multiplier
(j) partial product
(k) divisor

2. Work the following using any method you wish, but show all your work:

Add:

(a)
```
    578
  4,549
    496
 27,083
```

(b)
```
  6,324
    796
 39,137
  4,034
```

Subtract:

(c)
```
 58,931
  6,336
```

(d)
```
 6,719
 2,480
```

Multiply:

(e)
```
 354
  26
```

(f)
```
 709
  61
```

Divide:

(g) $8\overline{)2472}$ (h) $20\overline{)8160}$

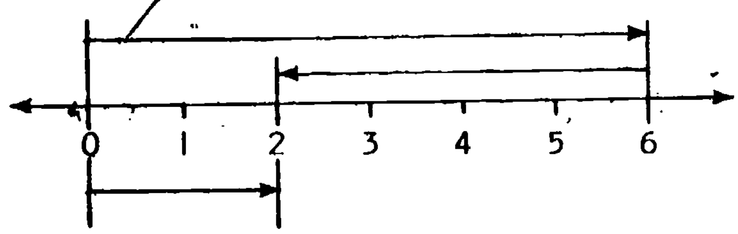
3. Draw number lines showing each of the following:

(a) 3 + 8 = 11

(b) 10 - 4 = 6

(c) 3 · 3 = 9

4. What problem is this a picture of? ______________________



Chapter 6:

Geometry

Section 6-1 Introduction

The world is full of physical objects. Touching or handling such objects helps us get an idea of their shapes and sizes. We can tell the differences between smooth and rough, small and large objects. We see that some objects have ends, corners, edges, sides. Some objects are straight, some are flat, and some are round.

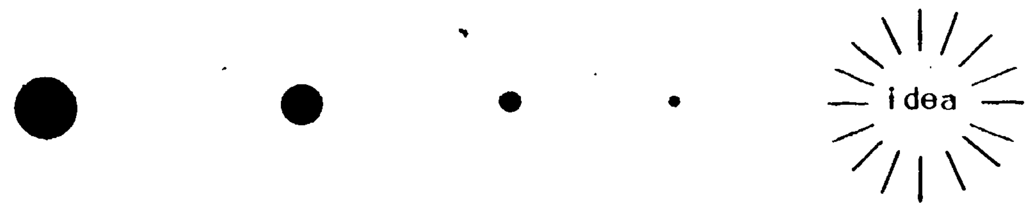
Touching and seeing help us tell the differences between the shapes, sizes and forms of the objects around us. These differences help us when we want to group these objects into special groups. Geometry has developed from a study of shapes of objects in the world around us.

The idea of number in arithmetic is a mathematical idea which grew from the need to know how many members are in certain sets. In geometry, the ideas of point, line, plane, and space are mathematical ideas that grew when people wanted to group certain sets of figures and to measure their sides or edges. Just as in arithmetic you studied numbers and the operations on them, in geometry you will study points, lines, and planes and how they relate to each other.

We shall study about a part of geometry that has to do with how such things as points, lines and planes are related. You will notice that numbers are used very little in this chapter.

Section 6-2 - Points

Let us begin with a simple figure in geometry, a point. Think of the following: the tip of a pin or a needle; the end of a sharpened stick or pencil; the corner of a box or a piece of paper; a grain of sand. All of these show what we mean by a point.



Which of these is the best picture of a point? The smaller the dot, the better the picture.

Is the dot you make on a paper with a sharpened pencil a point? Is the hole you make in a paper with the sharp tip of a pin a point? If you answered yes to the last two questions, you have an idea of what we imagine a point to be. But it is incorrect to say that a period, a dot, or a pin hole are points. Points, like numbers in arithmetic, are a creation of the mind -- they are an idea. The smallest dot you can make with your pencil is not a point. It is simply a picture of what we, in our minds, imagine a point to be.

We will think of a point as having position, but not size. The pictures of points that we draw on our paper and on the board will help us to see and remember the position of points we want to talk about. When we say, "draw a point" we will mean "draw a picture of a point".

We have studied sets before. We said that sets were important in mathematics. In our study of geometry we will look at sets again, but this time we will talk about sets of points.

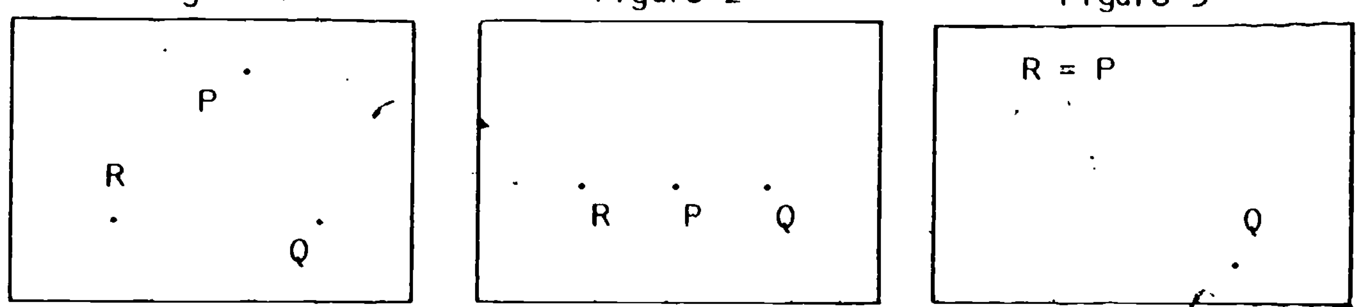
We will now think about some important sets of points.

Section 6-3 Line Segments

Draw two points on your paper. Label them A and B. Mark two more points on your paper labeled C and D so that your drawing looks like this:

Can we say that C is between A and B? D is also between A and B. What can you say about B? About A? Look at the drawings below:

Figure 1 Figure 2 Figure 3

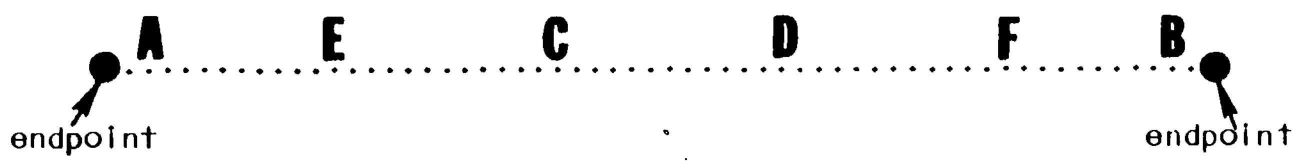


In Figure 1, none of the three points is between the other two. In Figure 2, P is between R and Q. In Figure 3, P is not between R and Q because there are only two points indicated. R and P are two different names for the same point.

Draw points E and F on your paper between points A and B. Your drawing should look something like this.

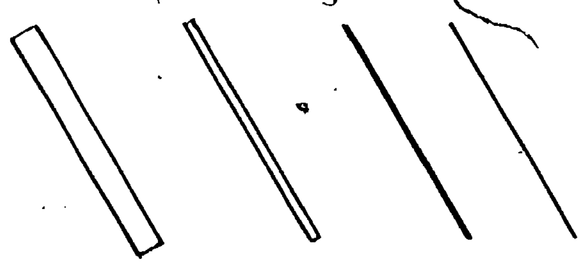
Continue to add points that are between A and B to your drawing. If you had a very sharp point on your pencil, how many points could you draw? Could you continue forever drawing points on your paper if your pencil kept getting sharper and sharper?

Your drawing may look like this now.



As you continue to draw points between A and B, notice that there are many points between every two (different) points. The set of all points between A and B, together with the endpoints A and B is the line segment between A and B. A line segment is a set of points made up of two endpoints and all the points between them. It is hard to draw a line segment by showing many points.

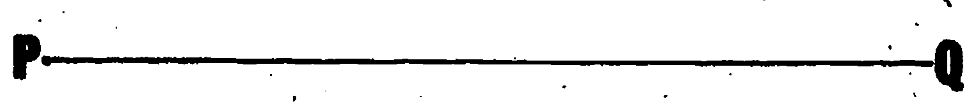
It is easier if we draw a line segment like this. ____________.



Which of these is the best picture of a line segment? The thinner the drawing, the better the picture.

Are there some objects in the classroom that suggest a line segment? The edge of a box or a pencil might suggest a line segment. A piece of string stretched between two poles will also suggest a line segment.

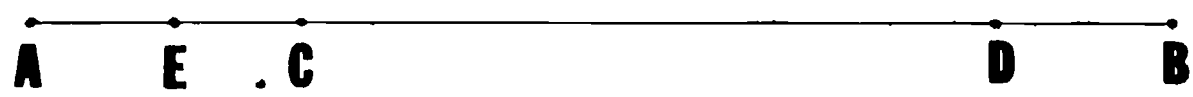
To name a line segment we use the names of the two endpoints. If the picture of a line segment looked like this



then the line segment could be named $\overline{PQ}$ or $\overline{QP}$. We use the bar ($\overline{\ \ }$) over the letters to remind us that we are talking about a line segment.

Exercise 6-3

1. Draw a line segment $\overline{AB}$ and mark points C, D, and E on it as shown.



Write as many true statements about your drawing as possible.

For example: E is between A and C.

2. In the figure for Exercise 1, which of the following statements are true and which ones are not true?

(a) A is between B and E. ______________

(b) C is not between A and D. ______________

(c) E is between A and D. ______________

(d) E is between A and C. ______________

(e) C is not between E and B. ______________

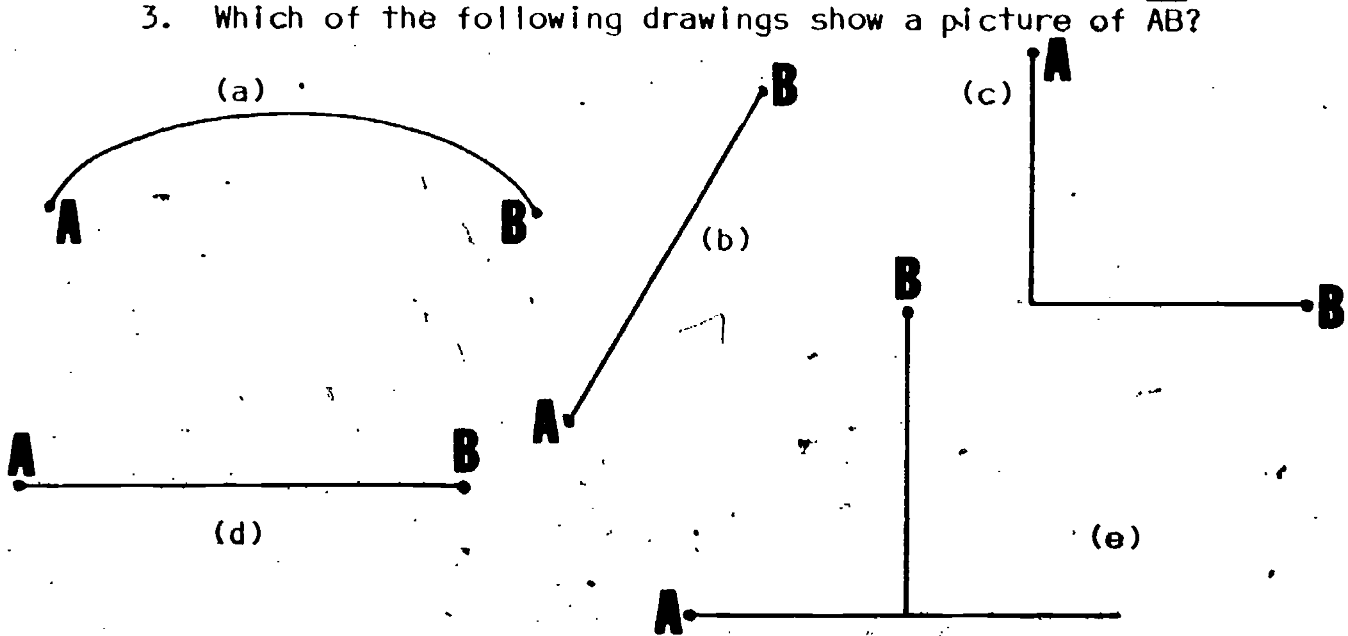
(f) B is between D and E. ______________

(g) D is between E and B. ______________

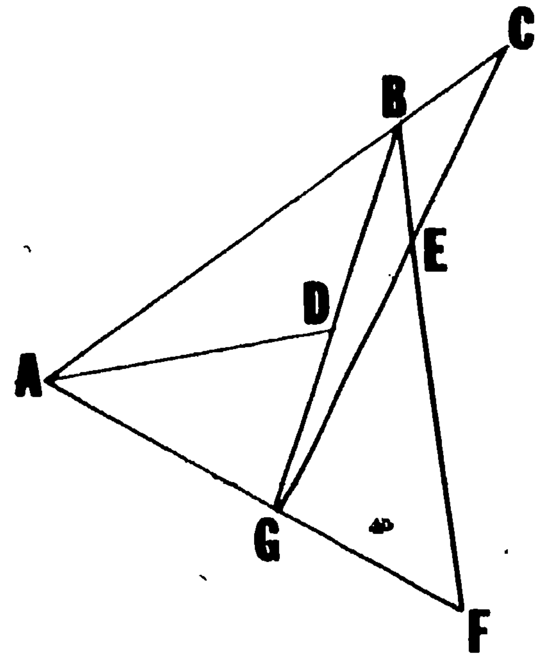
(h) E is between D and B. ______________

(i) C is between E and D. ______________

3. Which of the following drawings show a picture of $\overline{AB}$?



4. In the figure on the right, B is on $\overline{AC}$, D is on $\overline{BG}$, E is on $\overline{BF}$ and on $\overline{CG}$, and G is on $\overline{AF}$, as shown.



(a) How many line segments shown have the endpoint A? D? E? ________

(b) Which point is an endpoint of the least number of segments? ______

(c) Which points are the endpoints of the greatest number of line segments? ____________________

(d) How many more line segments can be drawn using only the given points as endpoints? ______________

5. How many line segments can be drawn in the following manner?

(a) When three points are used, if they are not in the same line segment. ________________

(b) When four points are used, no three in the same line segment.

____

(c) When five points are used, no three in the same line segment.

____

(d) When six points are used, no three in the same line segment.

____

(e) Can you guess, from your answers above, how many line segments can be drawn when seven points are used, no three in the same line segment, without drawing a picture? ______________

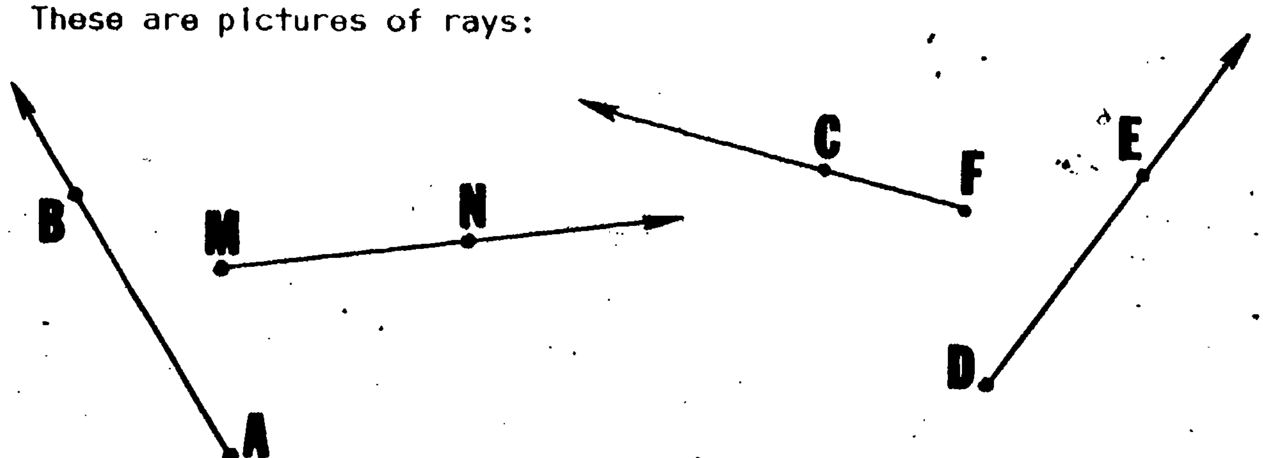
Section 6-4 Rays and Lines

There are other important sets of points that have special properties. Some of these new sets of points are very much like line segments. Recall that <u>a line segment is a set that contains two points and all the points between them</u>. Draw a line segment on your paper and label it $\overline{PQ}$. Now find a new point named R so that Q is between P and R. Your picture might look like this:

Now find a new point named S so that R is between Q and S. Now find other points T, U, V in the same way so that R is between P and each of them. If you continue in this way you will soon come to the edge of the paper. If your paper were very wide could you keep on finding new points? If your paper did not stop could you find the last point?

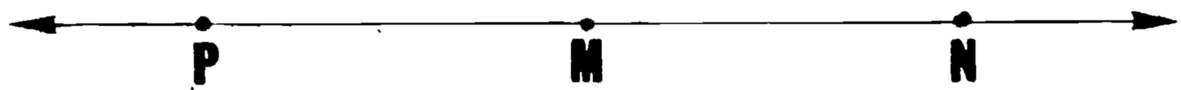
Connect the points as you did in drawing a line segment. The figure you have shown is called a <u>ray</u>. The ray you have drawn has only one endpoint. The line segments we studied before had two endpoints. A ray is formed by extending a line segment infinitely in one direction. The endpoint of the ray you have drawn is named P. We name a ray by using the name of the endpoint first and any other point of the ray second.

These are pictures of rays:



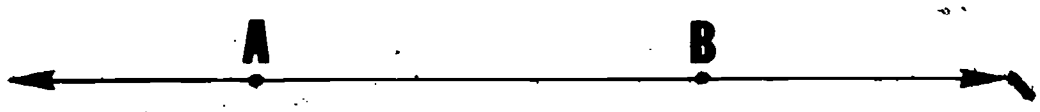
What are the endpoints of the rays? What are the names of the rays? The arrow shows that there is no end of the set of points in a ray. The rays shown on the previous page are $\overrightarrow{AB}$, $\overrightarrow{MN}$, $\overrightarrow{TC}$ and $\overrightarrow{DE}$. The bar with the arrow above the letters reminds us that we are talking about a ray. Remember, the first letter always names the endpoint.

In the figure below we have three points P, M and N, with M between P and N. We draw the two rays MP and MN.

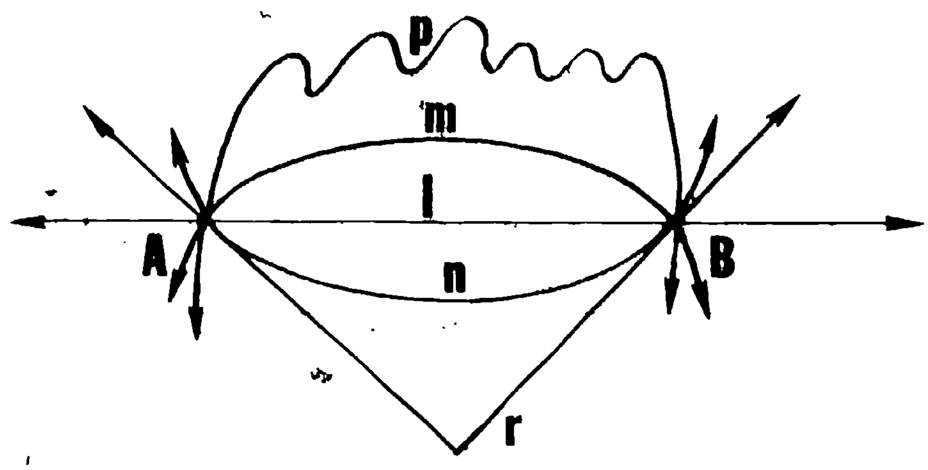


We have a picture of two rays that are opposite each other. The ray MN goes on and on to the right without end and the ray MP goes on and on to the left without end; the two arrows remind us of this fact. We say that two rays are <u>opposite</u> each other if they have <u>only</u> their endpoints in common and the endpoint is between the other points of the two rays. The line shown above is the line through the two points P and N (or P and M, or M and P, etc.). We write it as $\overleftrightarrow{PN}$ (or $\overleftrightarrow{PM}$ or $\overleftrightarrow{MP}$, etc.).

Let A and B be any two points. Then the line AB, written $\overleftrightarrow{AB}$, can be thought of as the union of the two rays, $\overrightarrow{AB}$ and $\overrightarrow{BA}$. Why?



Remembering that a point has no size, how many <u>lines</u> do you think two points show? The picture that follows shows several "lines" through the points A and B. These "lines" have been named with small letters, so that we can talk about them.



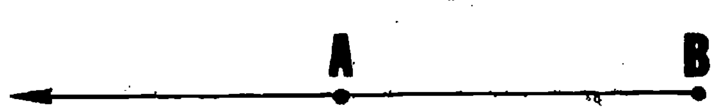
Which of the above shows a line? If your answer was line AB, then you already know what is meant by a line. We think of a line as being "straight." From now on when we say "line AB" we shall mean the one and only line through points A and B. The bar with the two arrows above the letters will remind us that we are talking about a line e.g.($\overleftrightarrow{AB}$)

To Review:

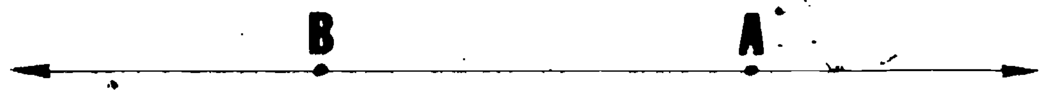
(1) This is the <u>line segment</u>, $\overline{AB}$ or $\overline{BA}$, having the endpoints A and B.

(2) This is the <u>ray</u> from A through B, or $\overrightarrow{AB}$.

(3) This is the <u>ray</u> from B through A, or $\overrightarrow{BA}$.



(4) This is the line, $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$, containing the points A and B.



(5) Given any two points A and B,

(a) There is exactly one line segment from A to B or from B to A.

($\overline{AB}$ or $\overline{BA}$)

(b) There is exactly one ray through B having A as an endpoint.

($\overrightarrow{AB}$)

(c) There is exactly one ray through A having B as endpoint.

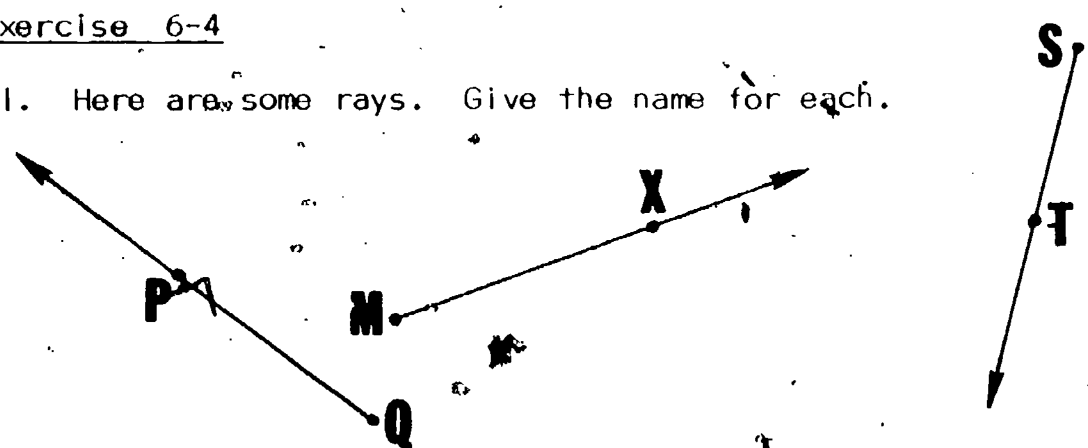
($\overrightarrow{BA}$)

(d) There is exactly one line containing the points A and B.

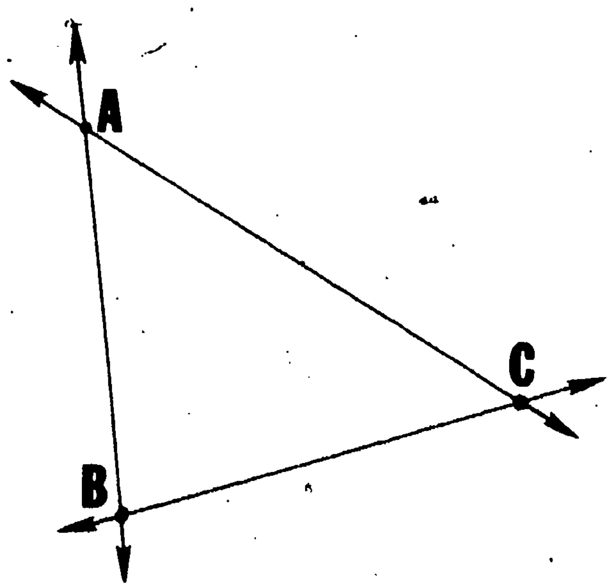
($\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$)

Exercise 6-4

1. Here are some rays. Give the name for each.



2. In the figure on the right,



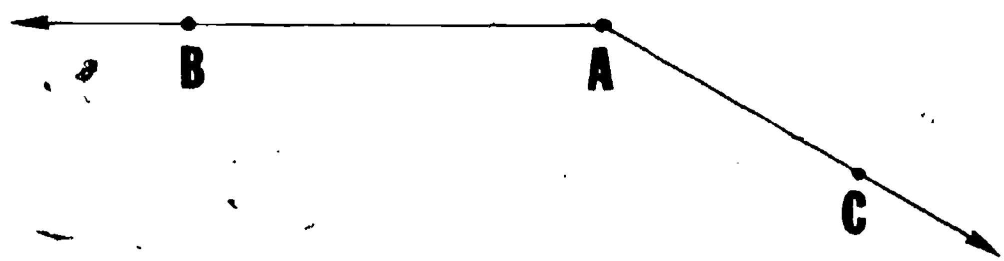
(a) Name three lines.

(b) Name three line segments.

(c) Name six rays.

3. We said that $\overleftrightarrow{AB}$ could be thought of as the union of the two rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$. What is the intersection of these two rays?

4. The following picture shows two rays, $\overrightarrow{AB}$ and $\overrightarrow{AC}$, having the same endpoint. Do these rays form a line? Why?



5. Are the two rays pictured in Exercise 4 opposite rays? Why?

6. A ray could be defined as the union of points A and B, all points between A and B, and all points beyond A from B on line $\overleftrightarrow{AB}$. Does this set define $\overrightarrow{AB}$ or does it define $\overrightarrow{BA}$? __________

7. Which of the following statements are true?

A ray has (a) one endpoint.
(b) two endpoints.
(c) many endpoints.
(d) no endpoint.

A line has (a) one endpoint.
(b) two endpoints.
(c) many endpoints.
(d) no endpoint.

A line segment has (a) one endpoint
(b) two endpoints.
(c) many endpoints.
(d) no endpoints.

Section 6-5. Flatness and Planes.

Which of the following surfaces do you think are flat?

(a) A table top

(b) The surface of the earth

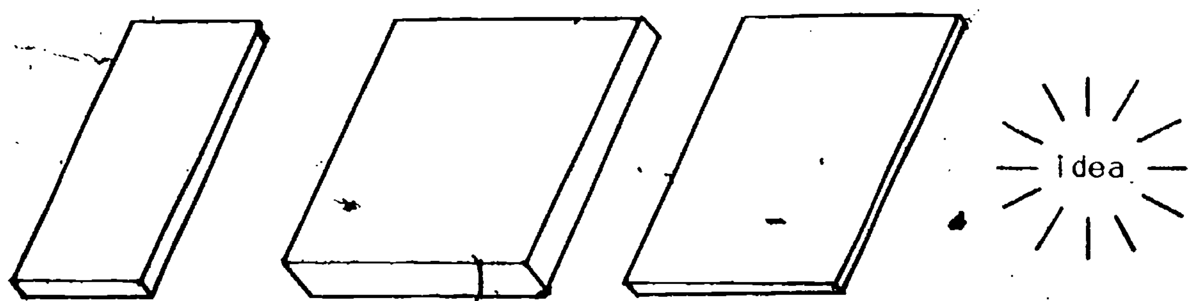
(c) A pane of glass

(d) The blackboard

(e) The surface of a ball

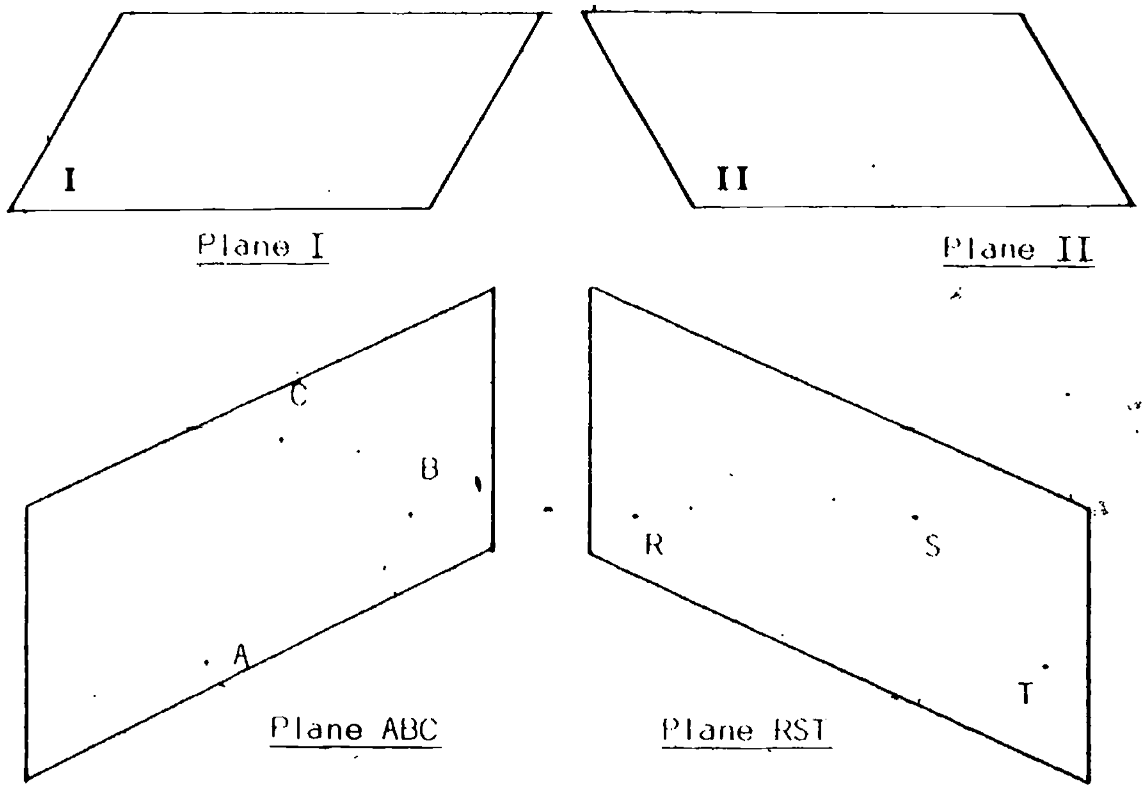
If your answers were (a), (c), and (d), then you already have some idea of what is meant by flatness. Look around the classroom and see if there are other objects that show flatness.

Now imagine a flat surface such as a table top extending indefinitely far in every direction. If you start at any point on this flat surface, you can walk in any direction without reaching an edge. This is the idea that we wish to have when we talk about a plane. We think of a plane as a special set of points that has infinite length and width, but no thickness.



Which is the best picture to show a plane? The flatter the picture, the better the drawing.

We sometimes draw pictures like the ones below to show planes.



As the figures above suggest, we sometimes name a plane with a Roman numeral, such as I, II, III, etc., or name three points (not all on the same line) that are in the plane. We say that three points not all on the same line show <u>exactly one plane</u>.

Exercise 6-5

1. Mark two points, A and B, on the plane of your paper and draw the line AB.

   (a) Do you think that every point of the line segment AB also lies in the plane of the paper? ______________

   (b) Do you think that every point of the line AB lies in the plane of the paper? ______________

   (c) Could there be another plane, different from the plane of your paper, that also contains every point of line AB? ______

2. Look at the corner of your classroom where the plane of the side wall meets (or intersects) the plane of the front wall. The edge where the two walls meet is what we think of as a line segment. The intersection of the planes, however, is a line.

   Now look at the plane of the ceiling where it meets the planes of the front and side walls. How many points do all three of these planes have in common? ____________

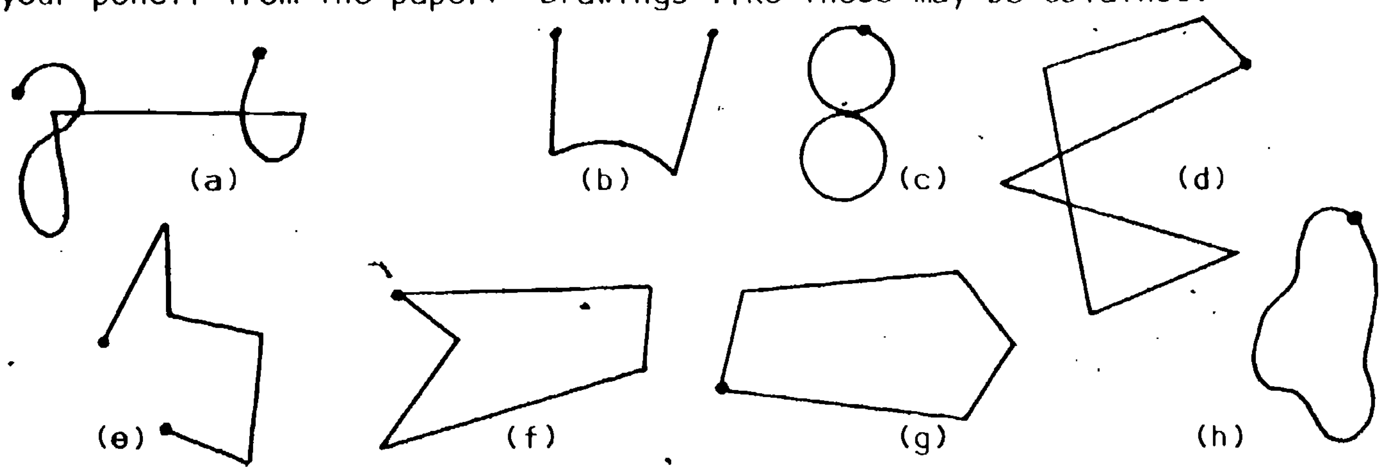
3. If two points of a line lie in a given plane, do you think all the points of the line lie in that plane? ____________

4. Could a line have only one point in common with a given plane?

   ____

Section 6-6 Paths

Mark a point, A, anywhere on your paper and place the tip of your pencil at that point. Trace out any drawing you like without lifting your pencil from the paper. Drawings like these may be obtained:



All such drawings are called paths. Note that (a), (c), and (d) cross themselves at least once. Paths (a), (b), and (e) do not end where they started. Paths (d), (e), (f), and (g) consist entirely of line segments. Paths (f), (g), and (h) do not cross themselves and go back to the starting point.

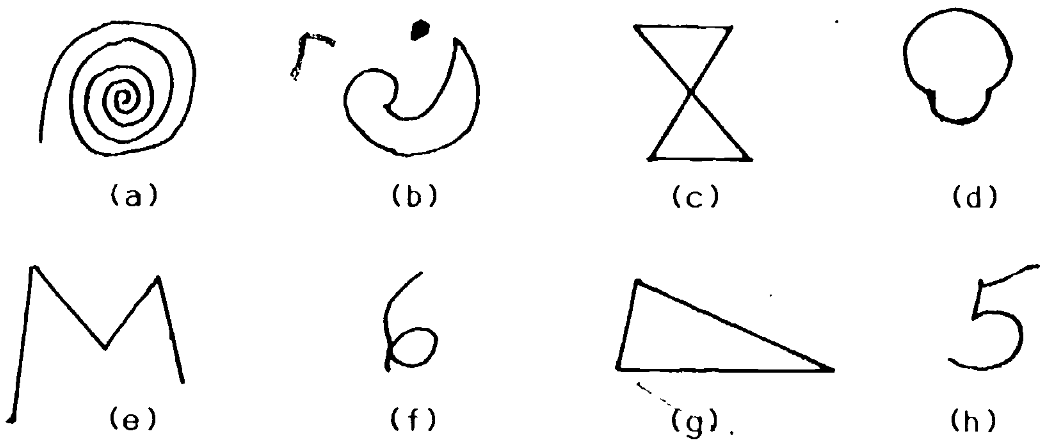
A path that never crosses itself is a simple path. (b), (e), (f), (g), (h)

A path that does not cross itself and does not go back to the starting point is a simple open path. (b), (e)

A path that never crosses itself and goes back to the starting point is called a simple closed path. (f), (g), (h)

Exercise 6-6

1. Which of the following are (i) simple paths, (ii) simple open paths, and (iii) simple closed paths?

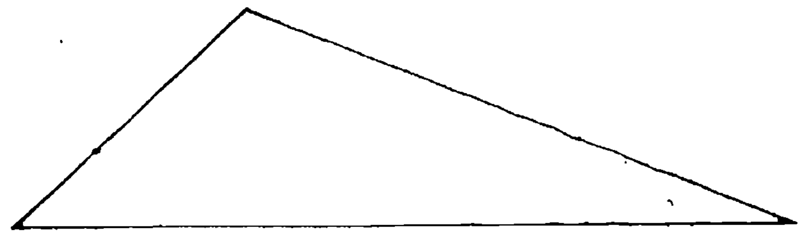


2. Draw three different simple open paths.

3. Draw two paths which are not simple paths.

Section 6-7 Regions

Mark on a piece of paper three points which do not lie on the same line and draw the three line segments joining them. Your drawing may look like the one on the following page.



We call this figure a triangle. We say that the triangle is made up of the union of the three line segments. We can also say that a triangle is a simple closed path.

Notice as you look at the drawing of the triangle that the line segments that make up the triangle divide the plane of your paper into three sets of points. The three sets of points are (i) the points on the inside of the triangle, (ii) the points on the triangle, and (iii) the points on the outside of the triangle.

Study the figures below and see if you can identify the points which are on the inside, the points which are on the outside and the points which are on each of the simple closed paths.

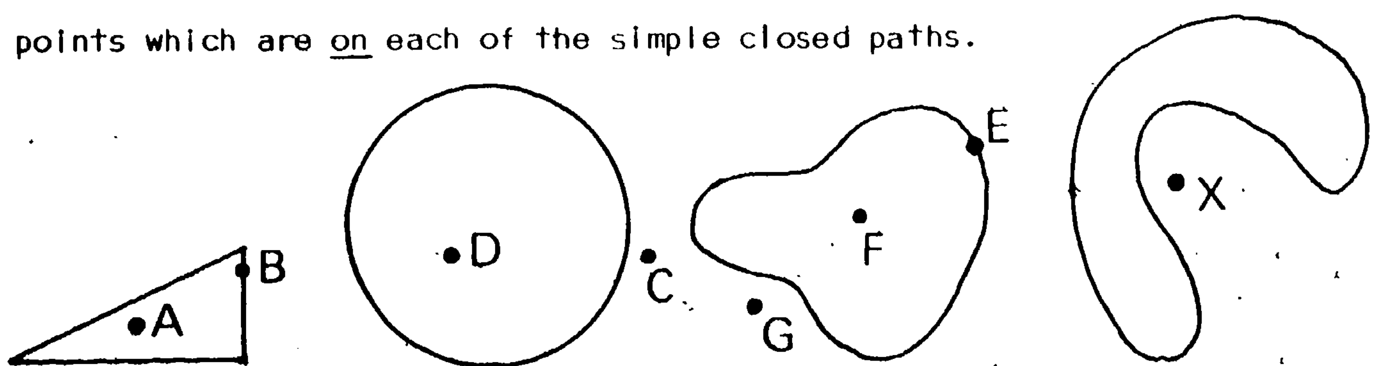


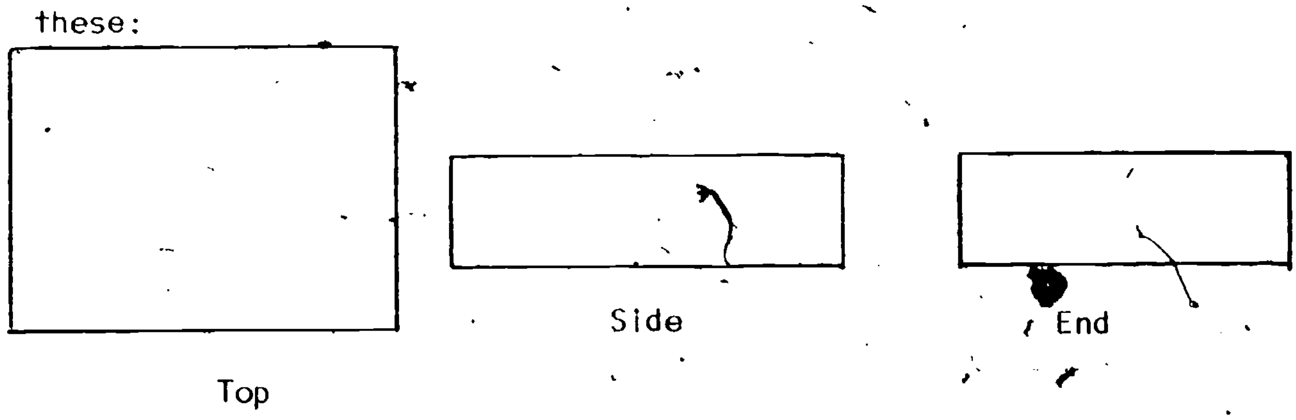
Figure I Figure II Figure III Figure IV

Point A is inside figure I, and point B is on figure I. Point D is inside figure II while point C is outside figure II. In figure III point G is on the outside, point F is on the inside while point E is on the figure. In figure IV, point X is on the outside of the simple closed path.

Look at figure I. We said that figure I divides the plane of the paper into three sets -- the set of all points inside the triangle, the set of all points on the triangle, and the set of all points outside the triangle. Let us think about two of these sets together. Think about the set of all points <u>on</u> the triangle together with the set of points <u>inside</u> the triangle. <u>This new set of points is called a region</u>. Since the simple closed path is a triangle we call the figure a <u>triangular region</u>.

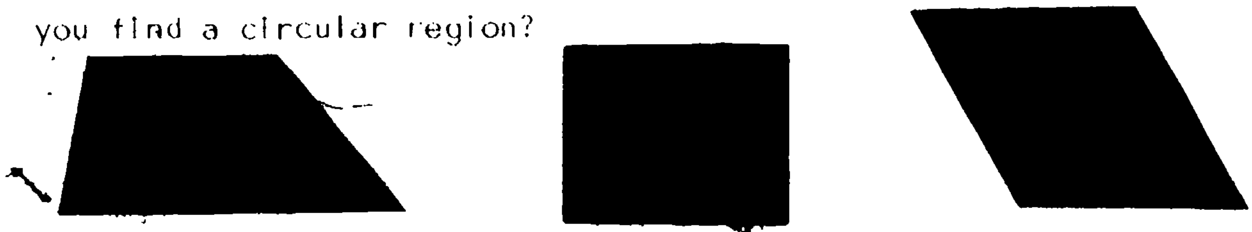
We may show that we are talking about the triangular region by drawing a figure like this one:

If you place a matchbox on a piece of paper, you can trace three different kinds of figures. Your tracings should look something like these:

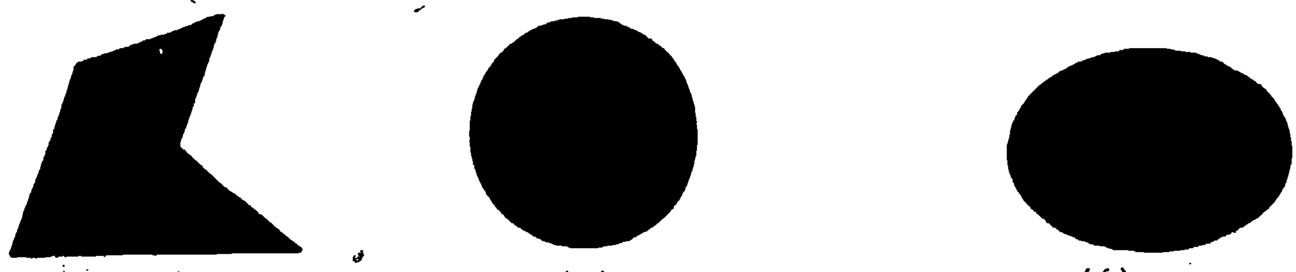


You could use a book to get the same kind of drawings on a large piece of paper. Each drawing will be made up of four line segments and four angles. If all the angles are right angles (square corners) we call the figure a rectangle. <u>A rectangle is a simple closed path that is made up of four line segments and four square corners.</u>

Which of the pictures below represents a rectangular region? Can you find a circular region?



(a) (b) (c)



(d) (e) (f)

<u>Section 6-8 Polygons and Polygonal Regions</u>

Cut a triangular region out of paper or cardboard. Your figure should look like Figure (1).

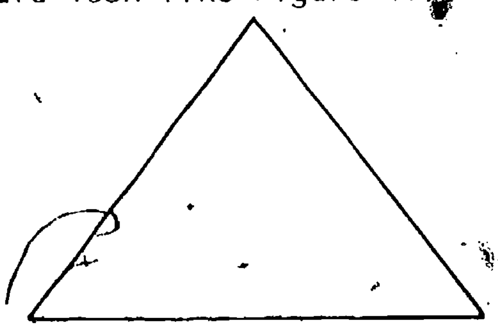


Figure (1)

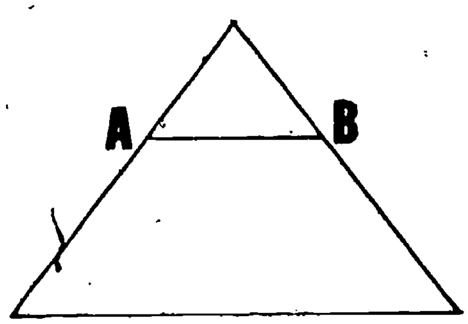


Figure (2)

Now cut along a line segment AB as in Figure (2). The region you now have has four edges as in Figure (3). Now cut along a line segment CD as in Figure (4). The region you now have has five edges as in Figure (5). (Figures 3, 4, and 5 are shown on the following page.)

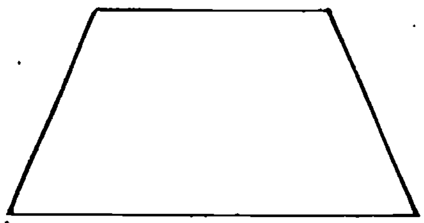


Figure (3)

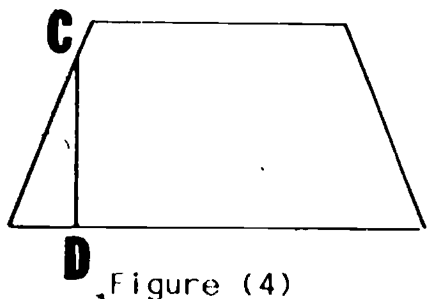


Figure (4)

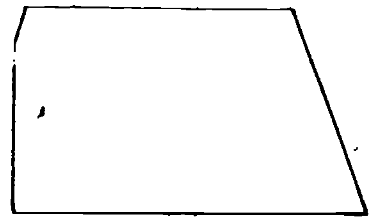


Figure (5)

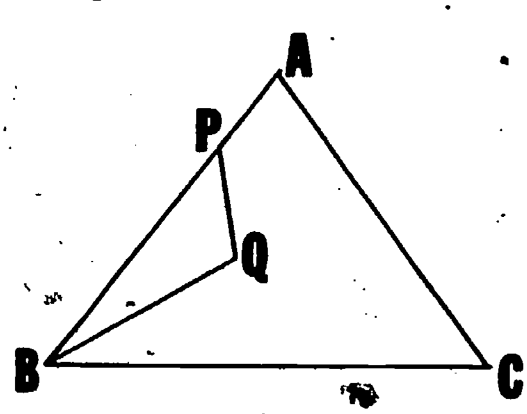
This process of cutting the cardboard or paper along line segments could be repeated many times and you would get figures similar to the ones shown above.

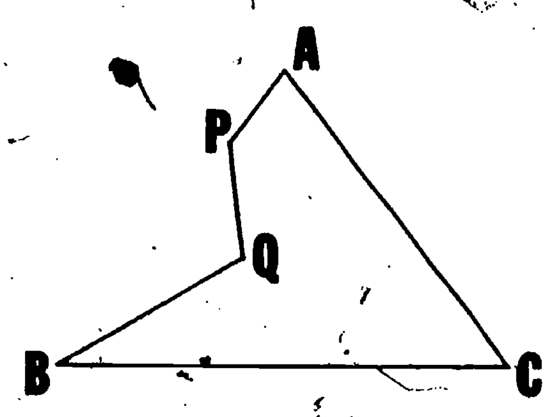
You can see that by cutting along line segments we can get plane figures with three, four, or five sides. In fact, we could get figures with as many sides as we might choose. Such figures represent polygons. A polygon is a simple closed path made up of line segments.

The common endpoints are called vertices. (A single common endpoint is called a vertex.) The line segments are called sides.

Some polygons have names according to the number of sides they have. We know already that a 3-sided polygon is called a triangle. A 4-sided polygon is called a quadrilateral. A 5-sided polygon is a pentagon.

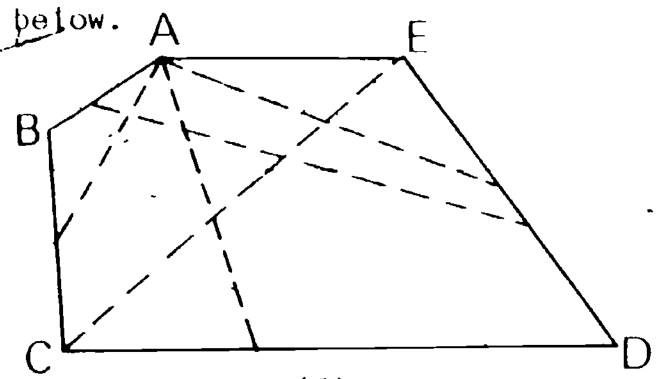
Mark a triangular piece of cardboard and cut it along line segments PQ and BQ as shown below on the left. You should get a figure like the one shown on the right, a five-sided polygon different from the polygon in figure (5) above.



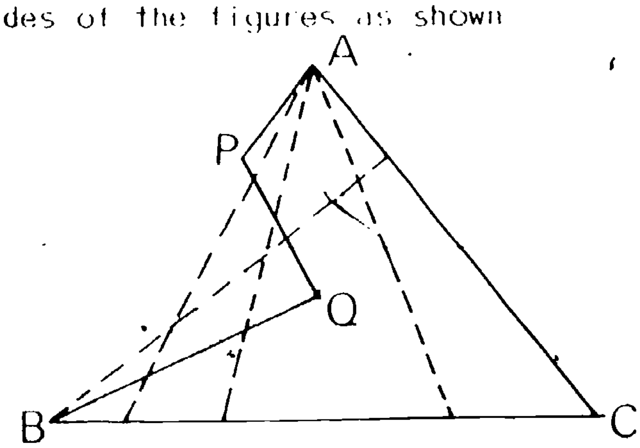


Now, let us see if we can discover the ways in which the two figures are different.

Label the vertices of the first figure A, B, C, D, E. Draw line segments whose endpoints are on the sides of the figures as shown below.



(6)



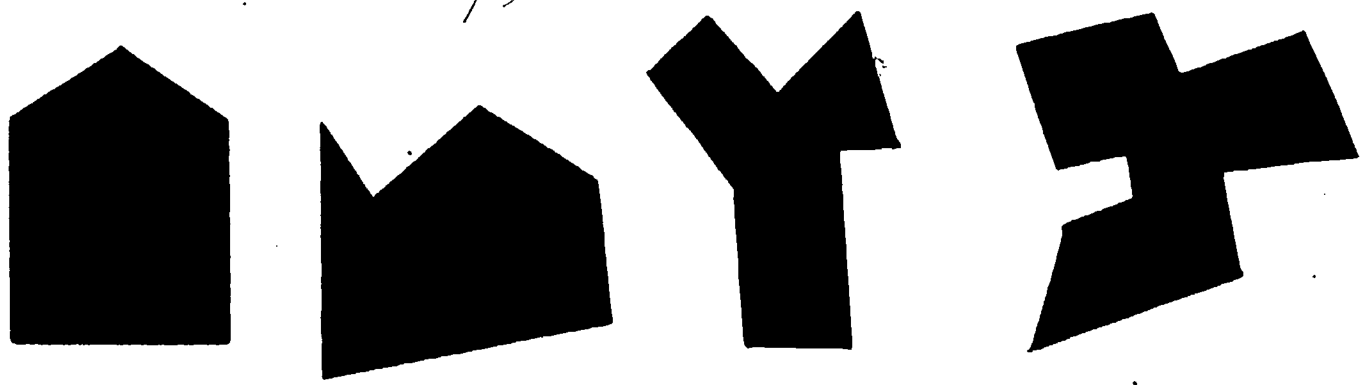
(7)

Observe that all the line segments of the first figure lie inside the polygon. In the second figure some of the line segments have points which are outside the polygon. Polygons like the one shown in figure (6) are called convex polygons. They have the property that all line segments whose endpoints are on the sides of the polygon have no points outside the polygon.

If any line segment whose endpoints are on the sides of the polygon does have points on the outside of the polygon, the polygon is called a concave polygon. Figure (7) would be an example of a concave polygon.

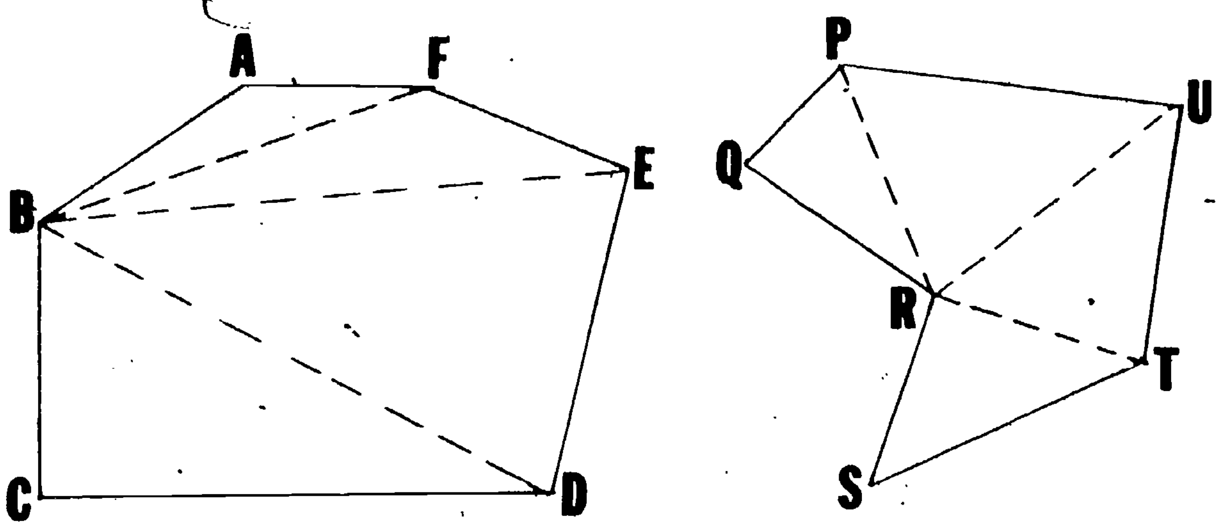
You will remember that when we talked about regions we said that we were thinking about the union of two sets of points -- the set of points on the inside of the simple closed path that formed the figure and the set of points on the figure. Can you guess what is meant by polygonal region? It is the set of all points bounded by the simple closed path (all the points inside the figure) and all the points on the boundary (all the points on the figure).

The figures below are pictures of polygonal regions.



Exercise 6-8

(1) Draw two hexagons (six-sided figures) ABCDEF and PQRSTU, the first one being convex and the second one concave.



Draw the line segments $\overline{BF}$, $\overline{BE}$, $\overline{BD}$, $\overline{RU}$, $\overline{RP}$, and $\overline{RT}$ (these are called diagonals of the polygons).

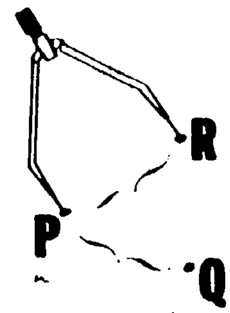
In this way the polygonal regions are subdivided into triangular regions whose interiors do not overlap.

(2) Draw any polygonal region, convex or concave, and show how it can be subdivided into triangular regions which do not overlap.

Section 6-9 Circles

You will need a compass for this part of your work. Mark two points, P and Q, on your paper. Using your compass, can you find a point R that is just as far from point P as point Q is?

Point Q is as far from point P as point R is.



Can you find three other points that are as far from P as Q and R are? Call them S, T, U. Can you find ten more? Can you find many others?

With your compass connect all the points in order. The figure you have drawn is called a circle. In drawing the circle we found that we had,

(1) a point, P, called the center of the circle, and

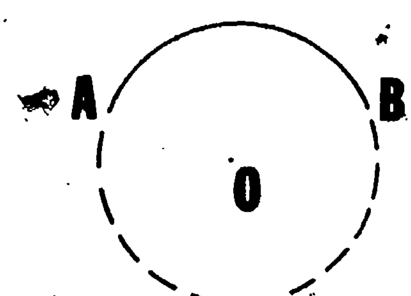
(2) a fixed distance between the center, P, and each of the points Q, R, S, T, U, that you found.

Now we can say that a circle is the set of all points in a plane that are a fixed distance from a given point in the same plane.

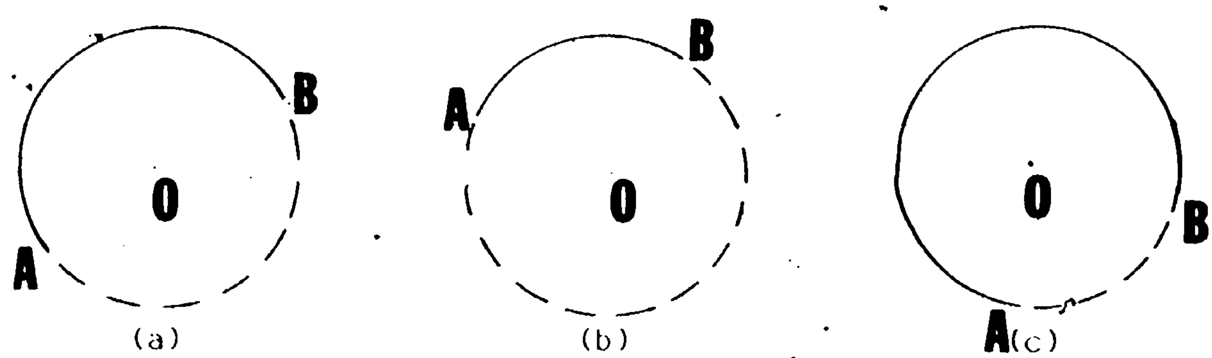
A line segment that has one endpoint as the center of the circle and the other endpoint on the circle is called a radius of the circle.

A compass is a convenient instrument for drawing a circle because the opening can be adjusted.

In the following work you are sometimes required to use a compass to draw only a part of a circle as in the figure below.

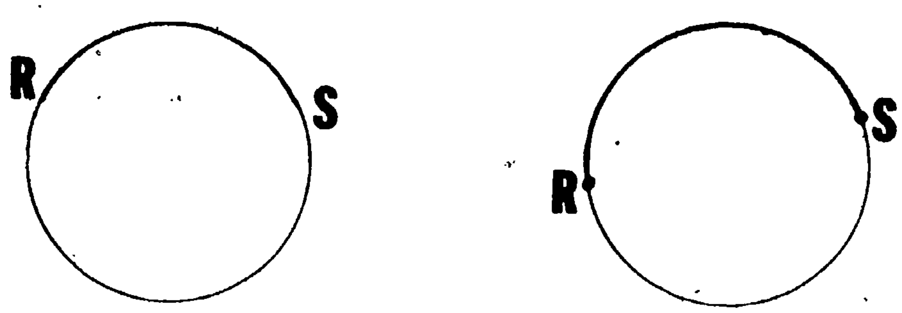


We call that part of the circle shown as a solid line in the figure a <u>circular arc</u> or <u>arc AB</u>. We sometimes use the symbol $\widehat{AB}$ to show a circular arc, and this is read "arc AB". The <u>center of an arc</u> is the center of the circle which contains the arc. The figure below shows three arcs of different sizes and their centers.



(a) (b) (c)

In Figure (a) the arc AB consists of half the circle and is called a semi-circular arc or <u>semi-circle</u>. In Figure (b) arc AB is smaller than a semi-circle and Figure (c) shows an arc that is larger than a semi-circle. If we take two points on a circle, such as R and S in the Figure (d) below, they determine two arcs of the circle. In Figure (d), the arc shown in heavy line is smaller than a semi-circle. The arc shown in lighter line is larger than a semi-circle.



(d) (e)

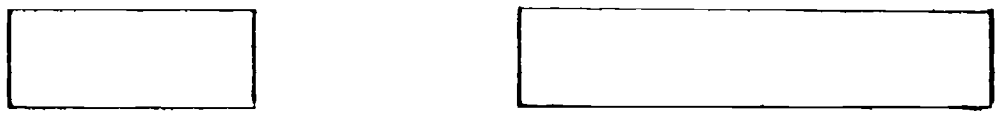
The two points R and S may also be chosen so that the two arcs formed are both semi-circles as shown in figure (e).

Exercise 6-9

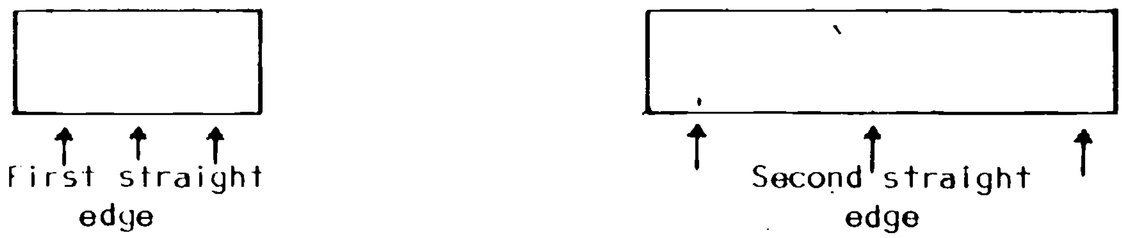
1. Use your compass to draw three circles of different sizes. Are these circles paths? Are they simple paths? Are they closed paths or open paths? Draw a radius of each circle.
2. Use your compass to draw a semi-circular arc; an arc smaller than a semi-circle; and an arc larger than a semi-circle. Are these arcs simple paths? Are they closed paths or open paths?
3. With your ruler, mark two points A and B on your paper two inches apart. With an opening that is a little more than one inch, use your compass to draw two circles, one with center at A and the other with center at B. Do these two circles intersect or cross each other? If so, how many times? Now take an opening that is a little less than one inch and draw two circles with centers at A and B as before. Do these circles cross each other?
4. With your ruler mark two points R and S that are three inches apart on your paper. Tell what opening you would need to set on your compass so that circles with centers at R and S would:
   (a) Intersect in two points.
   (b) Touch each other in just one point.
   (c) Not intersect each other at all.

Section 6-10 Pairs of Line Segments

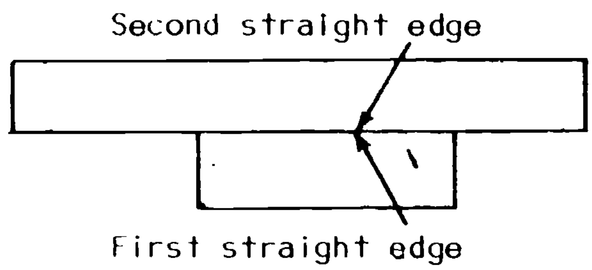
To see what pairs of line segments are like, let us look at two straight edges. This is a good place to start because we draw line segments with straight edges. If we have two pieces of wood or heavy cardboard that look like this,



then we have many straight edges. <u>Here are two examples of straight edges</u>:

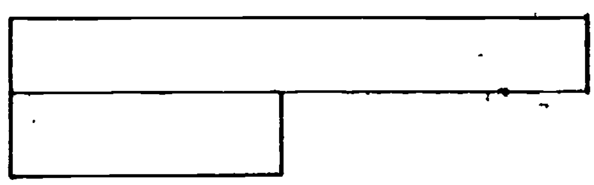


If we fit the two edges together, we could get something like this:



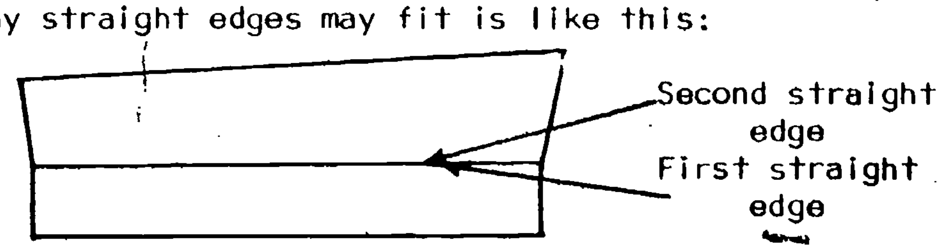
We notice that there are no holes where the edges meet. The two edges are both straight. Try this yourself with two straight edges.

If we slide one of the edges along the other until their ends just meet, they will look something like this:



We see that the edges do not fit exactly. The second edge is <u>longer</u> than the first and the first is <u>shorter</u> than the second.

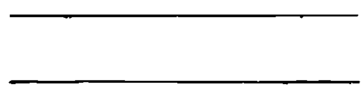
Another way straight edges may fit is like this:



Can you find two straight edges that fit this way? We say that straight edges such as these fit exactly. Their endpoints fit together.

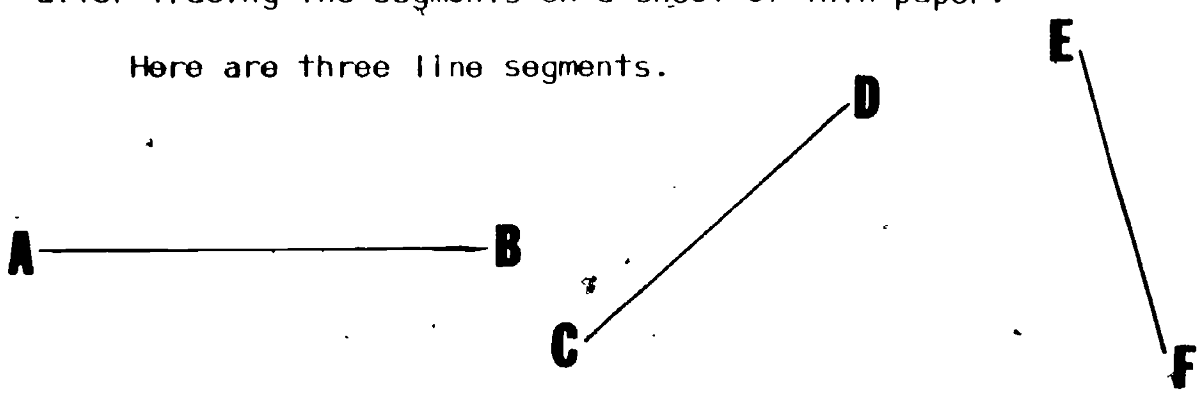
> Whenever two straight edges fit together so that endpoints fit endpoints, we say <u>they fit exactly</u>. We also say <u>they are congruent</u>. Congruent is another word for <u>fit exactly</u>.

Here is a pair of congruent line segments.



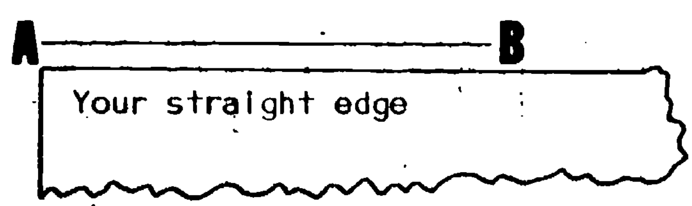
The page can be folded over so that the segments fit exactly. Try it after tracing the segments on a sheet of thin paper.

Here are three line segments.



We cannot make any two of these line segments fit together by folding the page without tearing the page. How can we tell if two of them are congruent? For example, how can we tell if either $\overline{CD}$ or $\overline{EF}$ is congruent to $\overline{AB}$? One way is to cut $\overline{AB}$ out of the page and try to fit it to $\overline{CD}$ and to $\overline{EF}$. But isn't there some other way? Yes, there are several.

One other way, and it is a good way, is to make a straight edge that fits $\overline{AB}$, and then try to fit the edge to $\overline{CD}$ and $\overline{EF}$. For example, take the edge of a piece of paper or the edge of a folded piece of paper. Now fit it to $\overline{AB}$. Fit one endpoint of the edge at A.



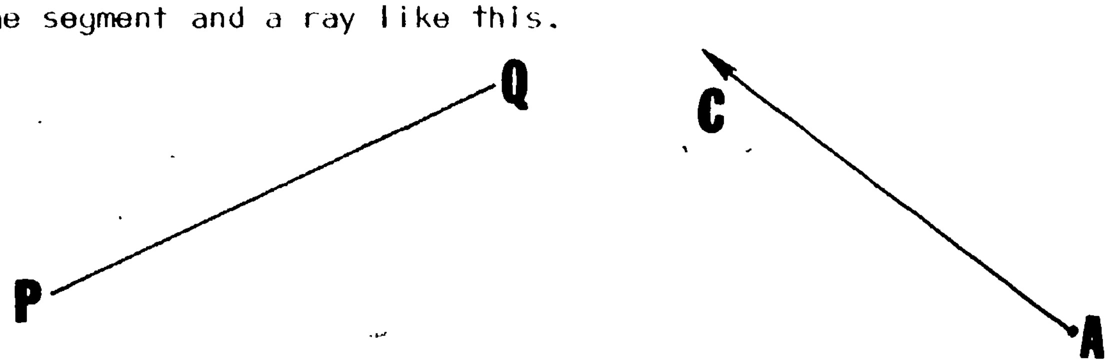
Then mark the point on the edge that fits B. Call the point you mark B', and call the endpoint that fits A, A', like this:

The part of the straight edge that starts at A' and ends at B' is a copy of $\overline{AB}$. If this copy fits $\overline{CD}$ then $\overline{AB}$ and $\overline{CD}$ are congruent. If it does not, then $\overline{AB}$ and $\overline{CD}$ are not congruent.

Try now to fit your edge to $\overline{CD}$. Can you fit it so that A' fits C at the same time that B' fits D? What do you conclude about $\overline{AB}$ and $\overline{CD}$?

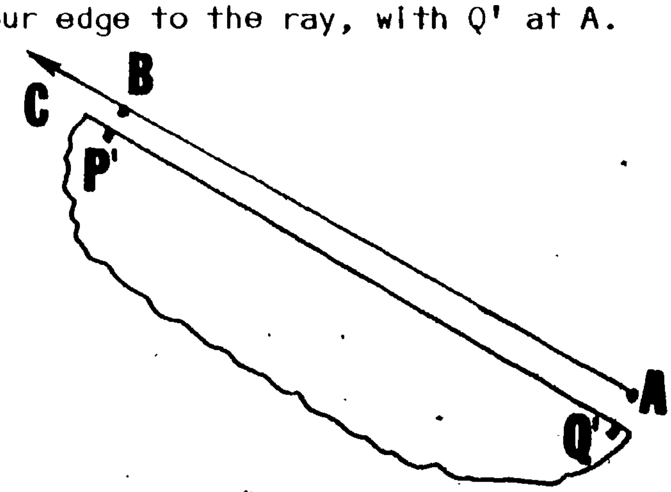
Now try to fit your edge to $\overline{EF}$. Can you fit it so that A' fits E at the same time that B' fits F? What do you conclude?

Here is another question. On a separate piece of paper, draw a line segment and a ray like this.



How would you find a point B on the ray AC so that $\overline{AB}$ is congruent to $\overline{PQ}$? We want to transfer the segment $\overline{PQ}$ to the ray AC.

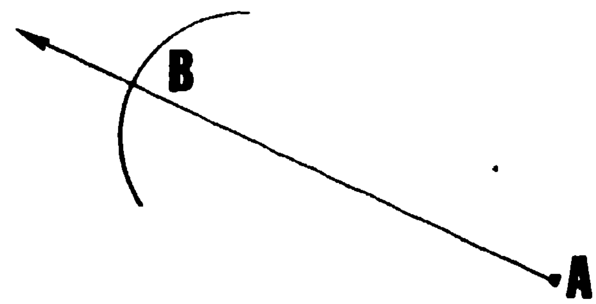
Make a copy of $\overline{PQ}$ on a straight edge, calling the endpoints of the copy P' and Q'. Then fit your edge to the ray, with Q' at A.



P' is then matched with some point on the ray. If you mark this point and call it B you will have made a line segment $\overline{AB}$ congruent to $\overline{PQ}$. Do you agree? Why?

We can also use a compass to transfer $\overline{PQ}$ to $\overrightarrow{AC}$.

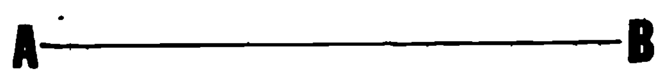
Fit the compass points to P and Q. This sets the compass. Without changing this setting, put the pin point at A and draw an arc that intersects the ray. The arc intersects the ray at B. (See the following diagram.)



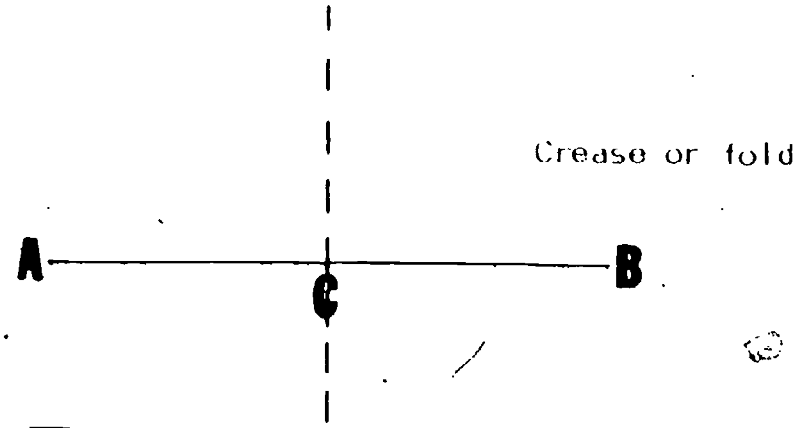
Carry out the construction with your compass, and compare $\overline{P'Q'}$ with your new $\overline{AB}$.

Section 6-11 Perpendicular Bisector

In this section we are going to show you with pictures and with words how to construct a very special kind of line segment. To begin with, draw a line segment $\overline{AB}$ in the middle of a clean piece of paper.



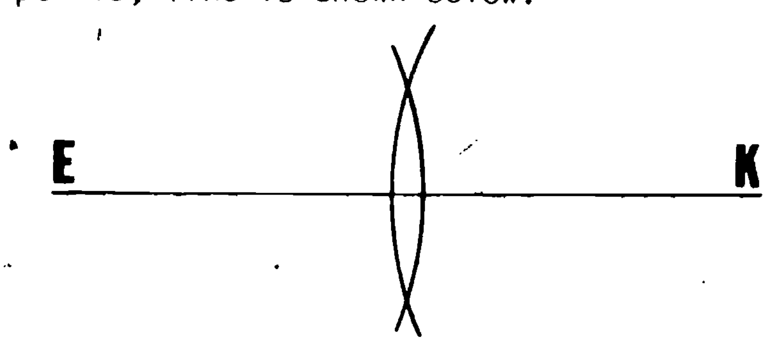
Then fold the paper so that the two endpoints are on top of each other. Press the paper flat to make a fold. Unfold the paper and mark the point where the fold cuts $\overline{AB}$. Call this point C. What can you say about the line segments $\overline{AC}$ and $\overline{CB}$? They are congruent because they match each other in folding. They fit exactly!



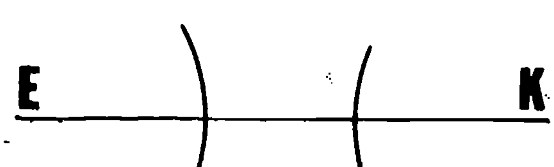
Since $\overline{AC}$ and $\overline{CB}$ are congruent, C is half way between A and B. We call C the midpoint of $\overline{AB}$.

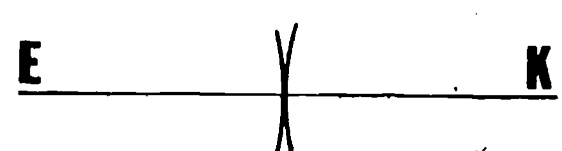
Let's look at another way of finding the midpoint of a line segment. In the middle of another clean sheet of paper draw a line segment.

Set your compass, and with E and K as centers draw arcs that meet each other in two points, like is shown below.



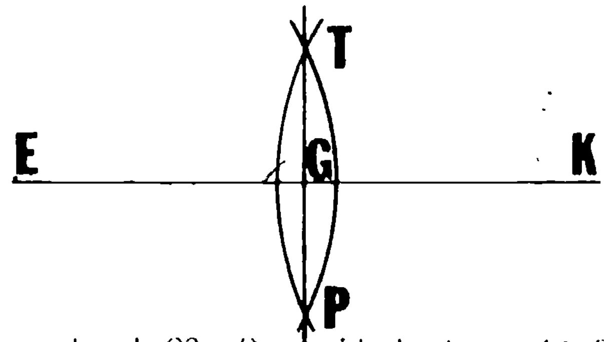
Be sure to use the same setting of your compass for both arcs. If your arcs do not meet, or if they meet at exactly one point,





then change your compass to a wider setting, and draw two arcs again.

Name the points where the arcs meet T and P. Then draw $\overline{TP}$. The point where $\overline{TP}$ crosses $\overline{EK}$, we will call G.



What can you say about G? Does it look as if $\overline{EG}$ and $\overline{GK}$ are congruent? Fold your paper along $\overline{TP}$. Hold your paper up to the light. Do you see that $\overline{EG}$ and $\overline{GK}$ fit exactly? They will fit exactly if you have done your drawing and folding carefully. G is the midpoint of $\overline{EK}$ because $\overline{EG}$ and $\overline{GK}$ are congruent. We say that $\overline{TP}$ is a <u>bisector</u> of $\overline{EK}$ because it intersects $\overline{EK}$ at its midpoint.

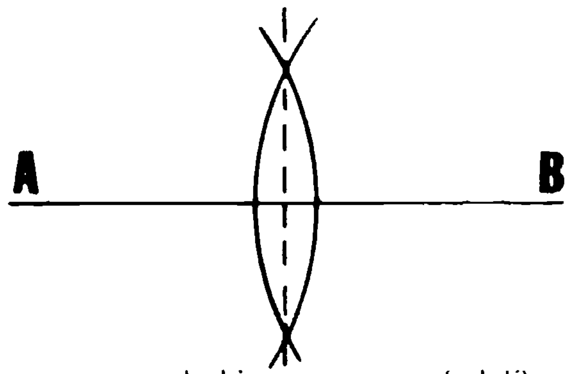
Keeping your paper folded along $\overline{TP}$, make another fold, this time along $\overline{EG}$. This fold makes a square corner or a <u>right angle</u> at G. Unfold your paper and look at the creases. Do you see that $\overline{EG}$ and $\overline{TG}$ make a right angle? What other right angles do you see?

If you have any doubt about where the right angles are in your drawing, take a moment to check your observations with a square corner made from another piece of paper.

Whenever two line segments cross and form right angles, we say that the line segments are <u>perpendicular</u>.

We see that the bisector $\overline{TP}$ is perpendicular to $\overline{EK}$. For this reason, we call $\overline{TP}$ a <u>perpendicular bisector</u> of $\overline{EK}$.

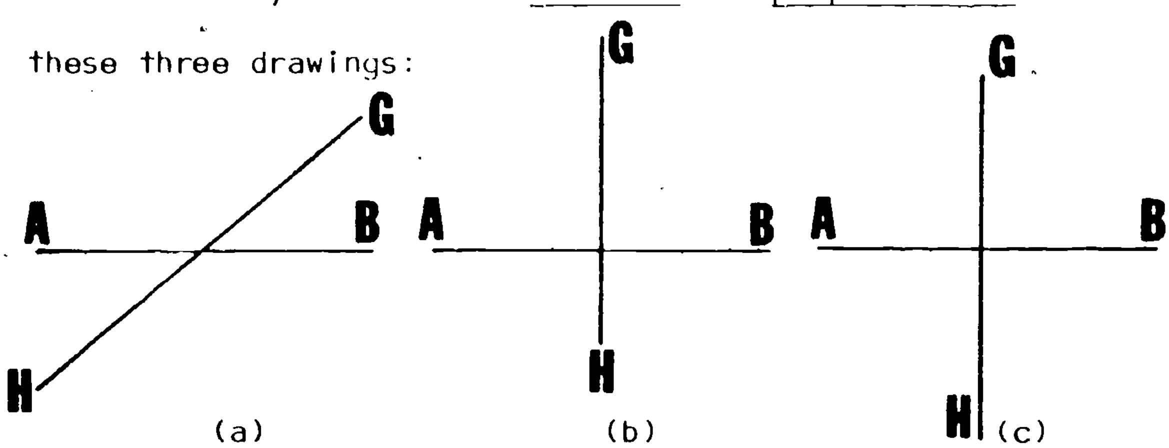
Now go back to the piece of paper on which you drew $\overline{AB}$. Remember how you folded this paper to find the midpoint C of $\overline{AB}$. With A and B as centers, use your compass once more to draw arcs that meet each other in two points. As before, make sure that you use the same setting for drawing both arcs.



Do the points where the arcs meet lie on your fold? If you have drawn carefully, they will.

Draw a line segment that joins the two points. Do you see that it lies right in the fold? The fold you made is the perpendicular bisector of $\overline{AB}$.

To make your ideas of <u>bisector</u> and <u>perpendicular</u> clearer, study these three drawings:



(a) (b) (c)

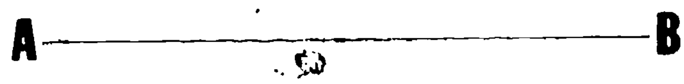
(1) In figure (a) $\overline{GH}$ is a bisector of $\overline{AB}$ but is not a perpendicular bisector of $\overline{AB}$. Also, $\overline{AB}$ is a bisector of $\overline{GH}$ but is not a perpendicular bisector of $\overline{GH}$.

(2) In figure (b), $\overline{GH}$ is a perpendicular bisector of $\overline{AB}$. $\overline{AB}$ does not bisect $\overline{GH}$, but does make square corners with $\overline{GH}$.

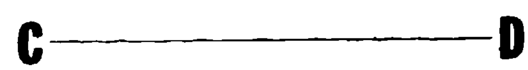
(3) In (c) each of the two line segments is a perpendicular bisector of the other.

Exercise 6-11

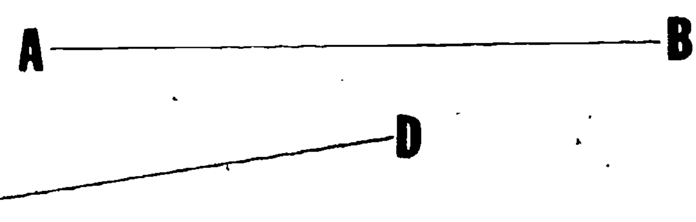
1. Suppose that we have traced a straightedge and made a line segment like this:



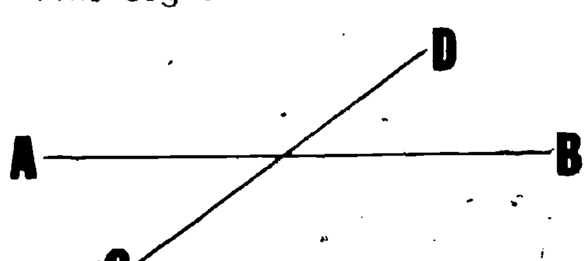
If we choose another straightedge and trace it on the same paper, we shall have a second line segment like that below.



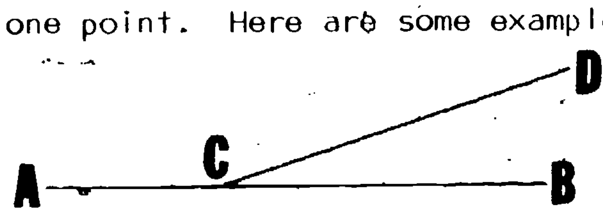
There are many positions in which $\overline{AB}$ and $\overline{CD}$ might be drawn. We can draw the two line segments in such a way that they have no point in common; that is, they do not intersect.



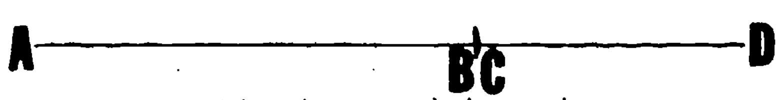
If two line segments have one or more points in common, they are said to intersect. There are many positions in which the two line segments intersect in exactly one point. Here are some examples:



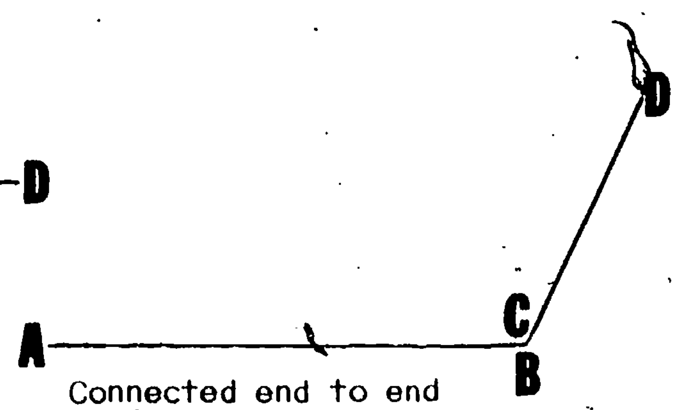
Crossing



Touching



Lined up end to end



Connected end to end

Finally there are positions in which they intersect in more than one point.

Use two paper straightedges to show each of the above positions.

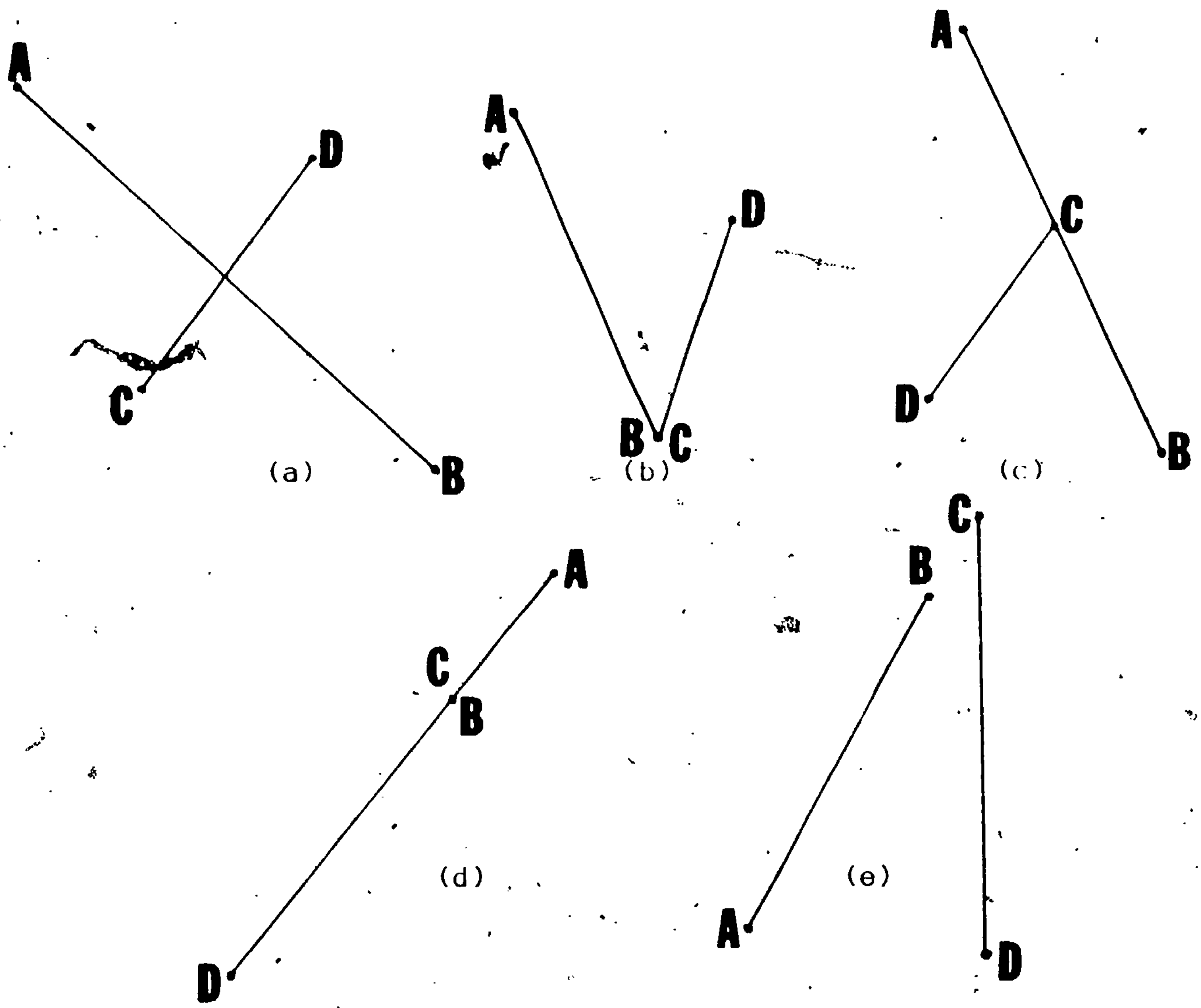
2. Earlier we learned that when line segments cross and are also perpendicular they form right angles. Line segments can also make right angles like this.

We now extend our definition of perpendicular to include any two line segments that make a right angle.

> Line segments are perpendicular if they make one or more right angles.

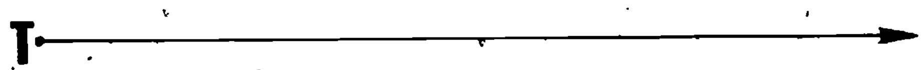
How many different right angles do you think two line segments can make?

3. Here are some pairs of line segments. Use one of the following phrases to describe the relationship of each pair: "crossing", "touching", "connected end to end", "lined up end to end", "making a right angle", "non-intersection". Example: Figure (a) shows two lines crossing.

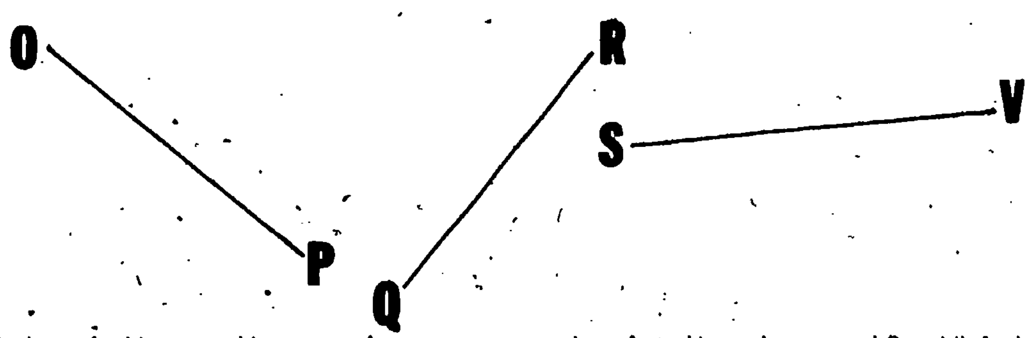


4. Test each pair of line segments in Exercise 3 for congruence.

5. Draw a ray on a piece of paper, and call its endpoint T.

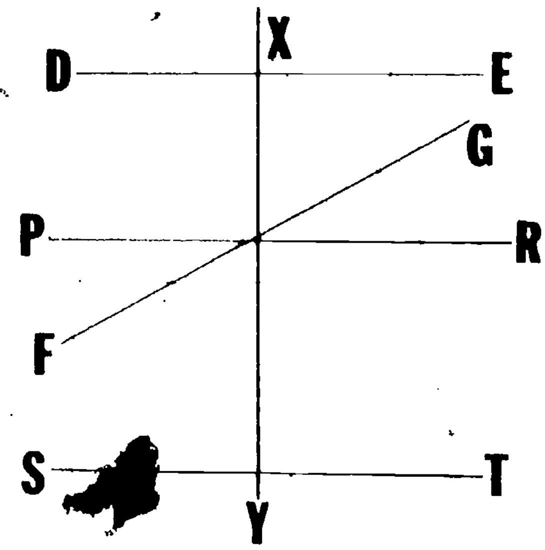


On your ray draw line segments, with endpoints at T, that are congruent to these three line segments?



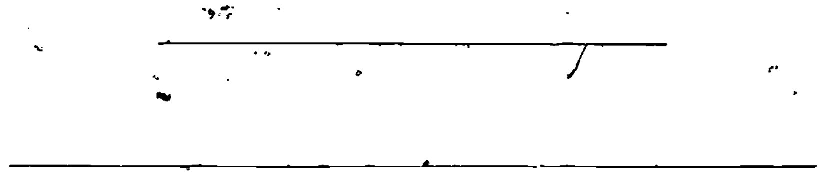
Which of these three given segments is the longest? Which is the shortest? (Do not use a ruler.)

6. In the picture which follows, try to find a line segment that is a perpendicular bisector of $\overline{XY}$; of $\overline{DE}$; of $\overline{FG}$.

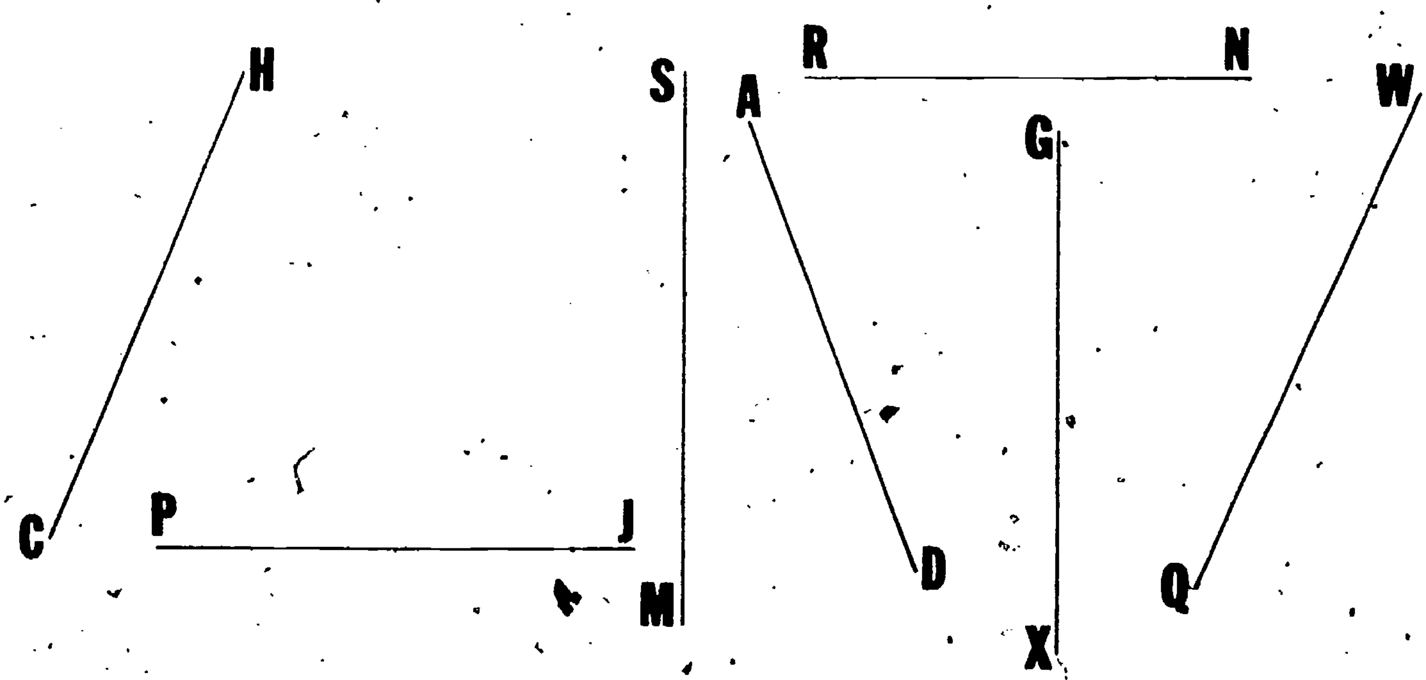


7. Make a paper straightedge and bisect it by folding. Bisect each resulting half of the straightedge in the same way.

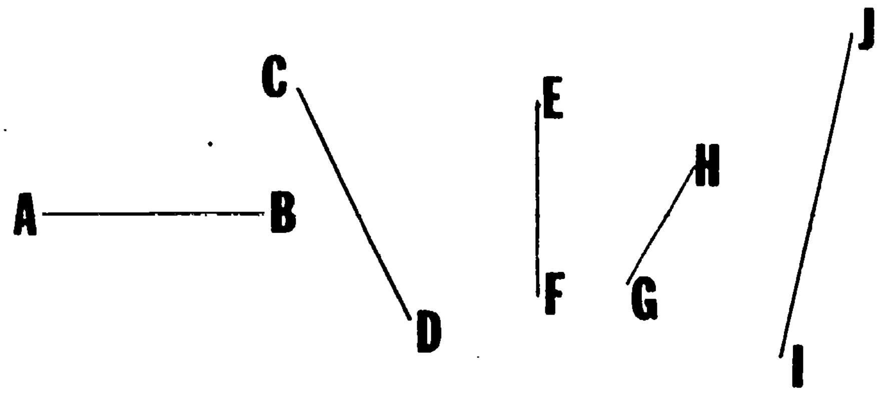
8. Copy each of these line segments and use a straightedge and compass to construct a perpendicular bisector of each copy.



9. Among these line segments, which pairs are congruent pairs?

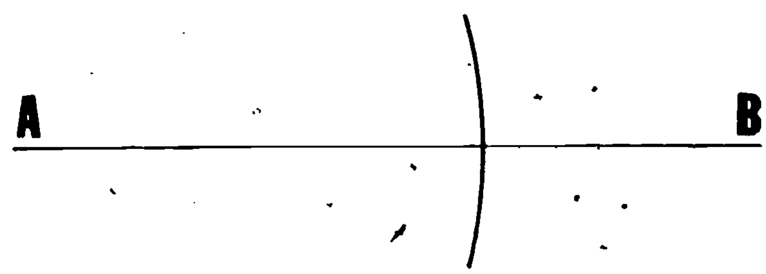


10. By using the compass, compare these line segments for length. Tell which is the longest, the next longest, the next longest, and so forth.



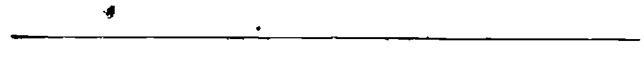
11. When we constructed perpendicular bisectors with a compass we made sure that we drew the two arcs with the same setting of the compass. Something interesting will happen if you use different settings for the two arcs. Let's discover what it is.

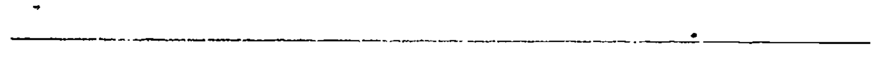
Start with a line segment $\overline{AB}$ drawn on a piece of paper. With A as center, draw an arc as shown below.



Then change your compass to a smaller setting and draw an arc with center B that intersects the first arc in two points. Connect these two points with a line segment. What do you see? Test your answer with a square corner. Where is the midpoint of $\overline{AB}$?

12. Copy each of these line segments and find the midpoint of your copy:

(a) 

(b) 

(c) 

Section 6-12 Pairs of Lines

We have just studied how line segments can intersect in many different ways. In this section we shall see how lines intersect.

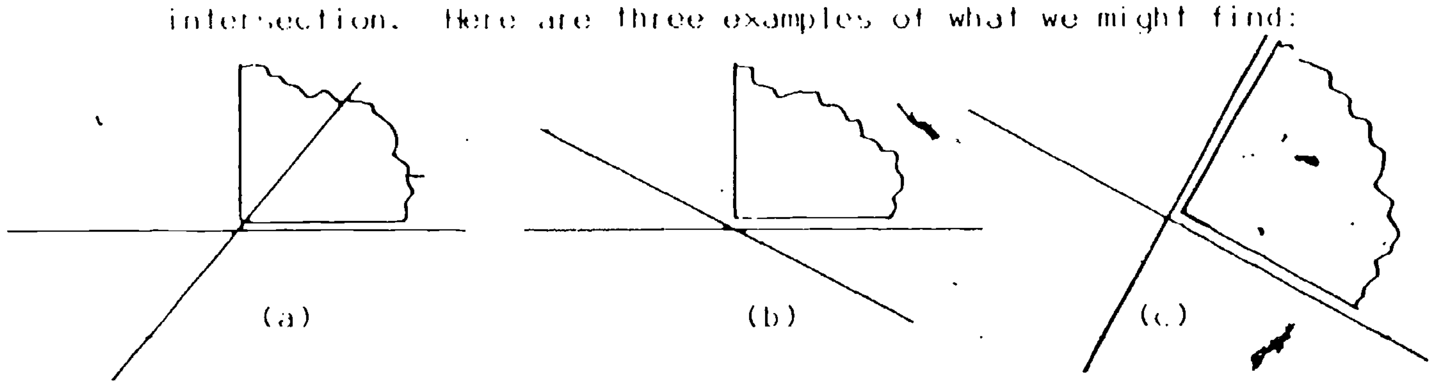
If there are two lines in a plane, then (1) they have no point in common, or (2) have one point in common, or else (3) are the same line. To see this, suppose that two lines have more than one point in common. Choose two of these common points. Then each of our lines goes through these two points. But through two points there is just one line. So the two lines are the same.

After this, when we say "two lines" we shall mean two that are not the same line. We can now say that:

Given any two lines in a plane, either they do not intersect or they intersect at just one point.

If two lines intersect, we call the point where they intersect the point of intersection.

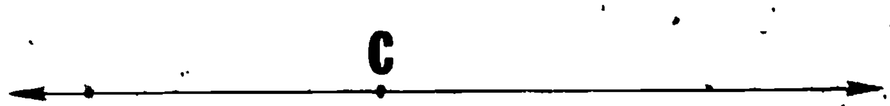
If two lines intersect, we can try to fit a right angle at the intersection. Here are three examples of what we might find:



In each case, we put the tip of the square corner at the point of intersection. Then we fit one straight edge of the corner to one of the straight lines. Now, see whether the other straight edge fits the other straight line. If it does, we call the two straight lines perpendicular (just as we did for line segments). In example (c) above, the two lines are perpendicular. In the other two examples the lines are not perpendicular.

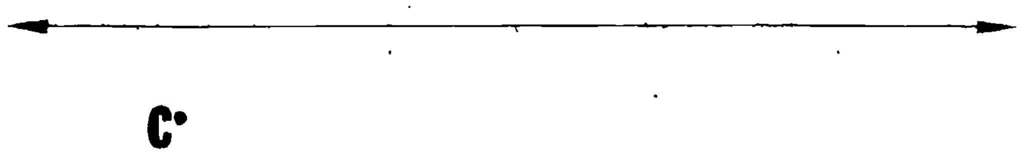
> Two lines are perpendicular if:
> (a) they intersect and
> (b) a square corner fits exactly at the intersection.

On a piece of paper, draw a line and a point on the line:

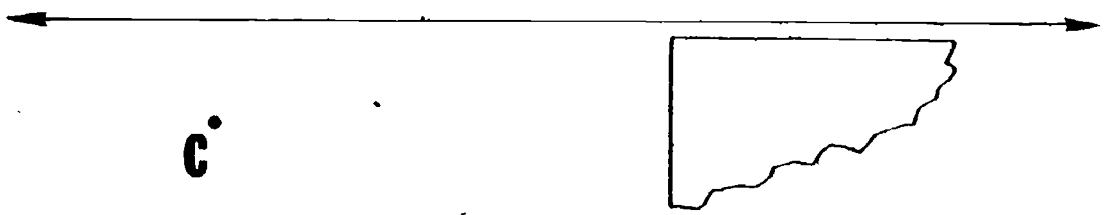


How would you draw a line that is perpendicular to your line at C? You may do this by placing a square corner in this position.

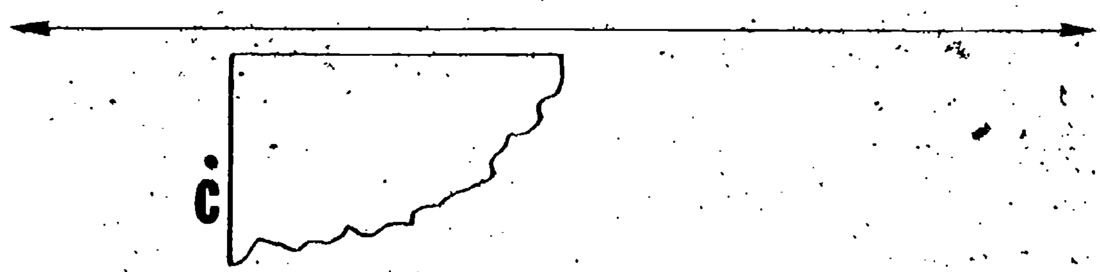
Now suppose that we are given a line, and a point C not on the line:



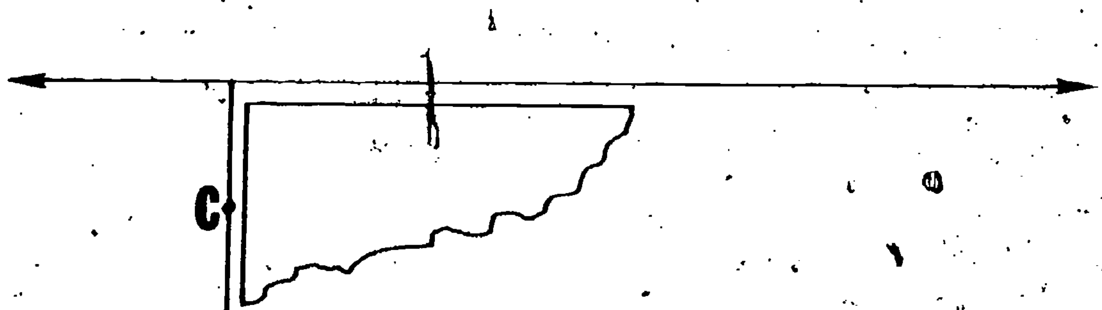
How could we draw another line that is perpendicular to the given line and goes through C? If we use a square corner it is easy. We fit one edge of the square corner to the given line, like this,



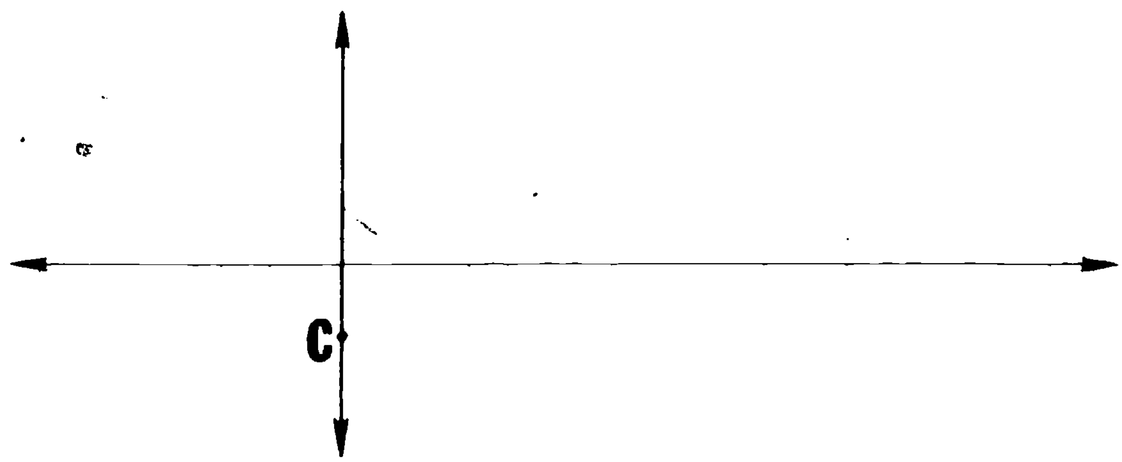
and then slide the square corner along the line, (keeping it fitted to the line) until the other straight edge of the square corner comes to the given point.



Here we stop and trace the edge that is at the point. This makes a line segment,

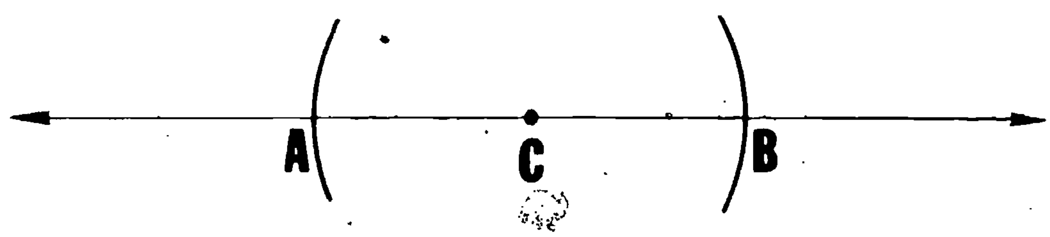


and the line segment determines a straight line which passes through C and is perpendicular to the line with which we started.

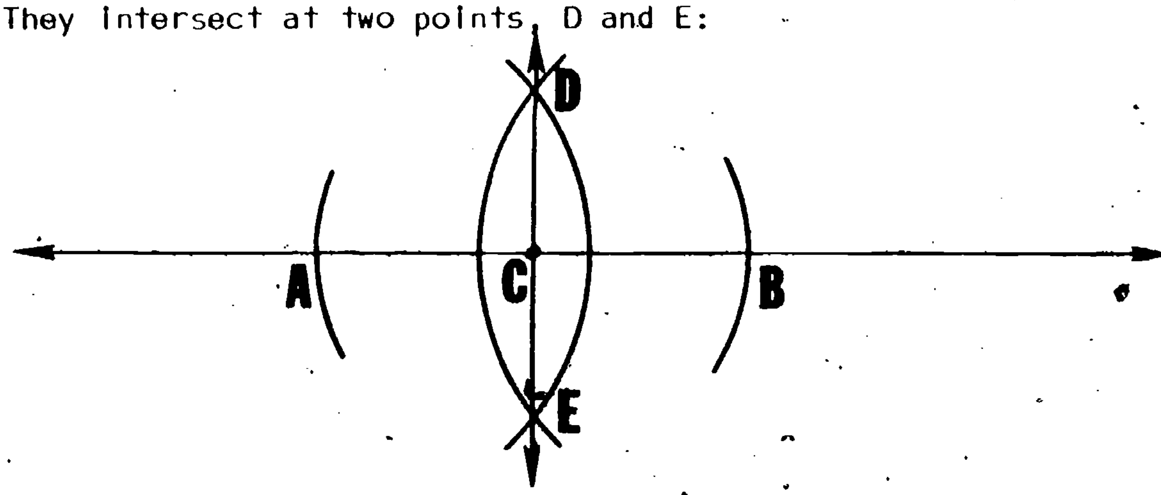


There is another way to draw perpendicular lines. All you need is a straight edge and a compass.

Start once more with a line and a point C on it. Put the point of your compass at C and cut the line with an arc on each side of C.



(We have let A and B be the names of the points at which the arcs cut the line.) Now open your compass some more, and with A and B as centers, draw two more arcs, the way we did when we bisected line segments. They intersect at two points, D and E:

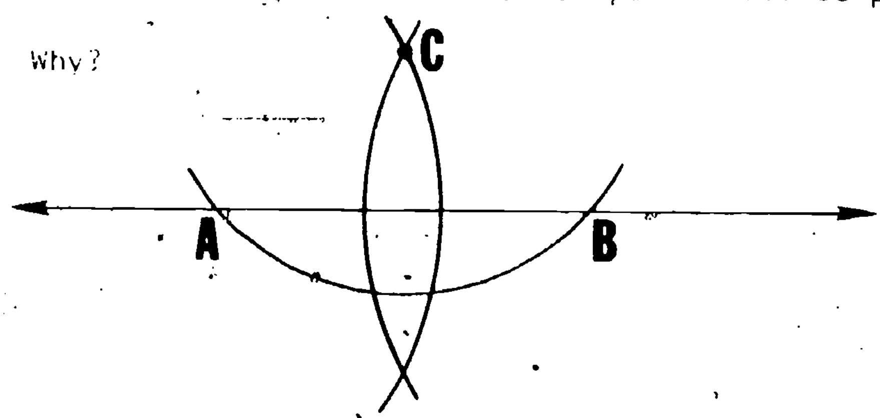


The straight line through D and E is perpendicular to the given line, $\overleftrightarrow{AB}$, at C. Why?

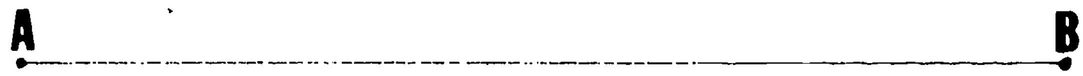
There are several things you can do here to convince yourself that $\overleftrightarrow{DE}$ is perpendicular to $\overleftrightarrow{AB}$. One is to remember the way you learned to construct perpendicular line segments. The line segment $\overline{DE}$ is perpendicular to $\overline{AB}$. Since $\overline{DE}$ and $\overline{AB}$ make right angles, so do $\overleftrightarrow{DE}$ and $\overleftrightarrow{AB}$.

Another thing you can do to convince yourself that $\overleftrightarrow{DE}$ and $\overleftrightarrow{AB}$ are perpendicular is to fold your paper along $\overleftrightarrow{DE}$. If you then fold your paper along $\overleftrightarrow{CA}$ you will have made a paper square corner whose edges lie along $\overleftrightarrow{CA}$ and $\overleftrightarrow{CD}$.

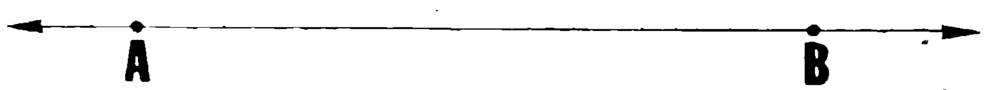
We have used a straight edge and compass to draw a line perpendicular to a given line through a given point, C, on the given line. Now start with a line and a point, C, not on the line. Open your compass enough so that when you put the point at C you can draw an arc that cuts the line in two points. Call these two points A and B. Then, without changing the setting on your compass, make big arcs with A and B as centers. These arcs intersect at two points, one of which is C itself. The line through C and the other point will be perpendicular to $\overleftrightarrow{AB}$. Why?



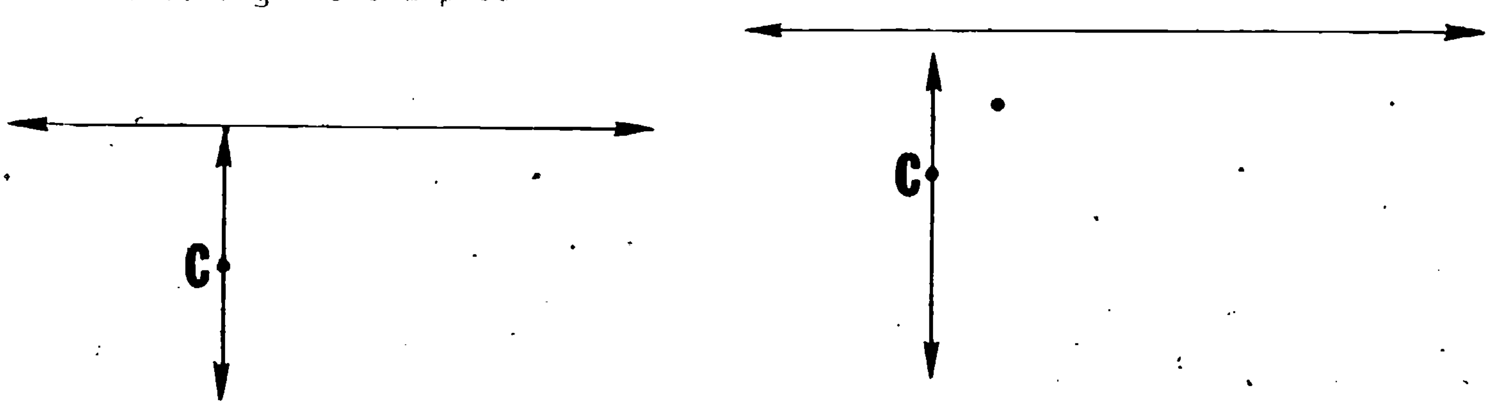
You have now seen enough examples to understand the difference between the line segments we draw and the pictures we draw when we think of lines. Line segments are shown by drawings that we make with straight edges. Line segments have endpoints. If we draw a line segment $\overline{AB}$ joining two points A and B,



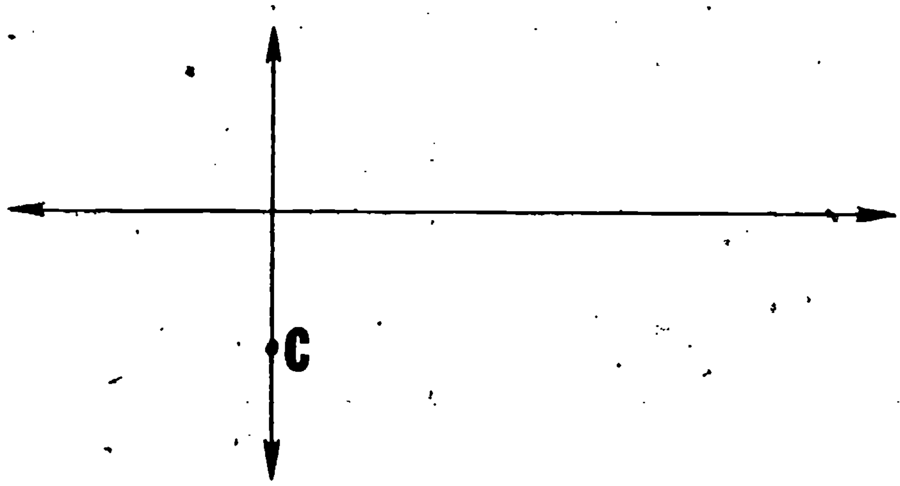
We can think of the line segment AB extended to line $\overleftrightarrow{AB}$. $\overline{AB}$ is just a part of this line. <u>We can not draw all of the line.</u> When we draw a picture for someone, then, how do we show him that we are thinking of the line $\overleftrightarrow{AB}$ and not just the segment $\overline{AB}$? We make the segment a little longer, so that it extends past A and B, and then put arrow heads on it to show that we are thinking about the entire line such as we show here:



Remember also how we made the drawing for the line through <u>C</u> perpendicular to the given line. We did <u>not</u> draw pictures like the following two examples:

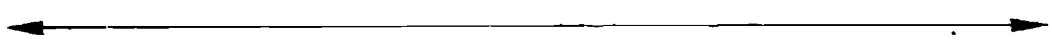


What we drew was like the following figure which clearly shows the intersection of the two lines.

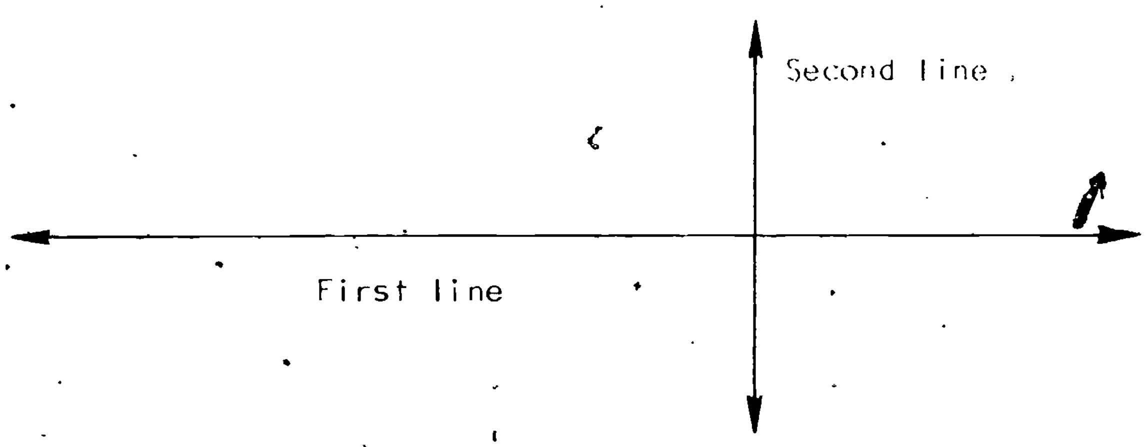


When we draw part of a line in a geometric figure, we draw only enough to show clearly the relationship of the line to the other parts of the figure. When we have drawn enough of the line, we say that we have drawn the line.

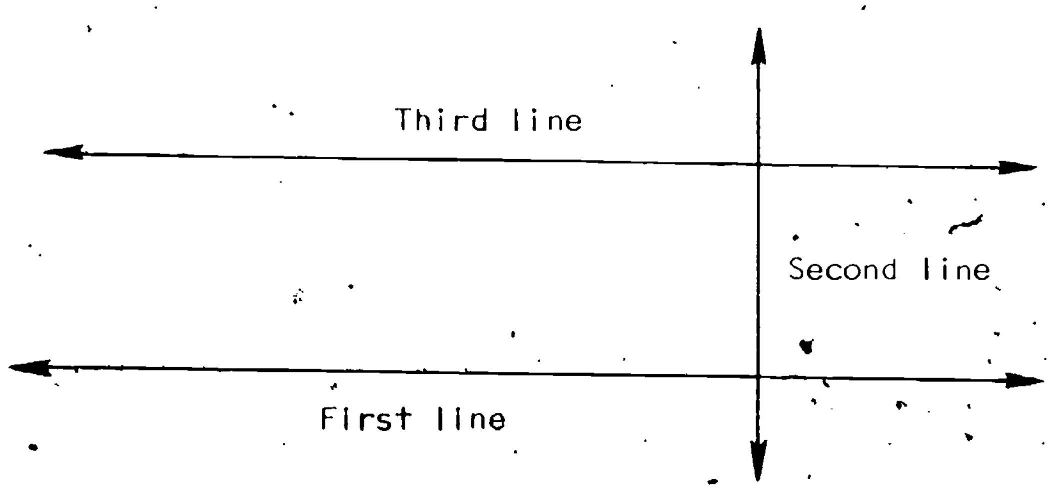
Now, let us explore another important relationship concerning lines. Two lines in a plane are parallel if there is another straight line that is perpendicular to each of them. Since you know how to draw perpendicular lines, it will be easy for you to draw parallel lines. For example, start with a line like this on a piece of paper:



Draw a second line that is perpendicular to it (this is easily done with a square corner).

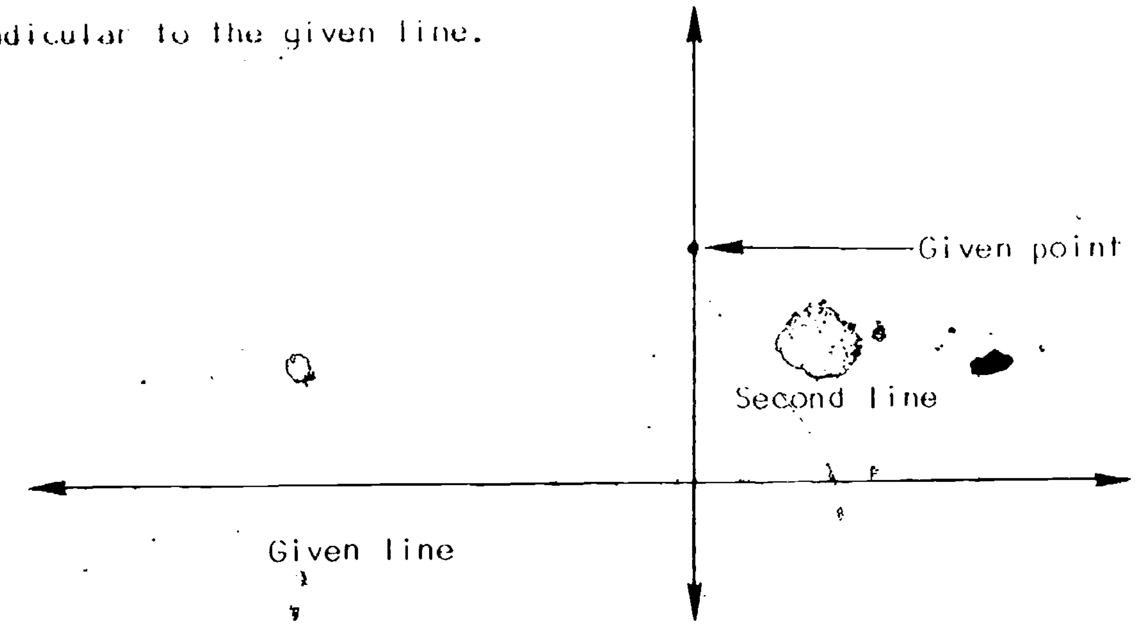


Then draw a third line that is perpendicular to the second line:

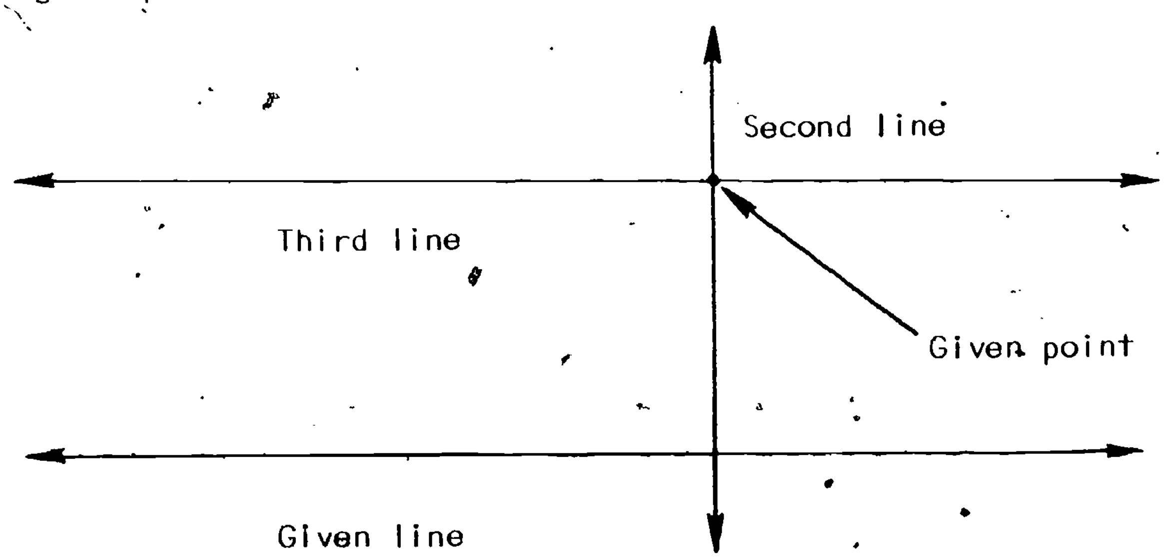


The first and third lines are parallel since the second is perpendicular to each of them.

You can do even more. Given any line and any point not on it you can find another line that is parallel to the given line and goes through the given point. Here is how: Through the given point construct a line perpendicular to the given line.

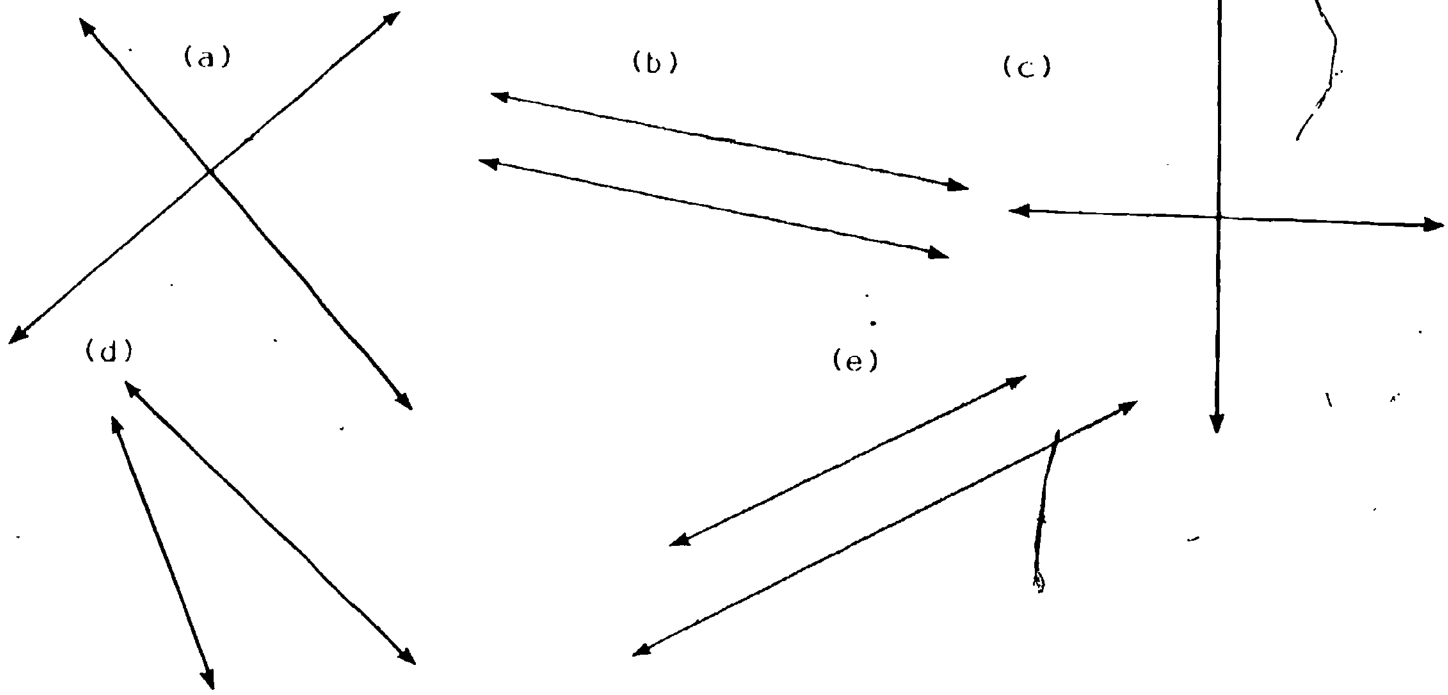


Then construct a third line that is perpendicular to the second line at the given point. Your result will look like this:



Exercise 6-12

1. First, by sight, then by using a square corner, test each of the following pairs of straight lines. Which are parallel? Which are perpendicular?



2. In Exercise 1, which pairs of lines do you think are intersecting? Remember that we can never draw all of any line; two straight lines may intersect although the parts of them shown in a drawing may not intersect.

3. It is a fact that two lines in a plane that are parallel cannot intersect; therefore:

   (a) Can two parallel lines in a plane be perpendicular?

   (b) Can two perpendicular lines in a plane be parallel?

4. This exercise will lead to a discovery. In order to make the discovery you will have to draw a line that is perpendicular to one of two parallel lines. This will mean drawing a perpendicular line a number of times. It is more convenient to use your square corner instead of your compass for drawing perpendiculars. If you have a

good square corner you can draw perpendicular lines faster and with more accuracy than you can with a compass and a straight edge. Now for the discovery.

Step 1:

On a clean piece of paper, draw two parallel lines. Mark two points on each line, and give them names, so that one of the lines is $\overleftrightarrow{TP}$ and the other is $\overleftrightarrow{TH}$. Through T draw a line perpendicular to $\overleftrightarrow{TP}$. Be sure to draw enough of this line to show its intersection with $\overleftrightarrow{TH}$. Call the point of intersection B. What do you think is the relation between $\overline{BT}$ and $\overline{TH}$? What do the corners at B look like?

Step 2:

Now through P draw a line perpendicular to $\overleftrightarrow{TP}$. Let F be the name of the point where this line intersects $\overleftrightarrow{TH}$. What do the corners at F look like?

Step 3:

Now fold your paper so that $\overleftrightarrow{TP}$ and $\overleftrightarrow{TH}$ are right on top of each other. This puts B on top of T and F on top of P. Hold your paper up to the light. What do you see?

Do you see that the right angles that you constructed at T fit exactly the corners at B? Do you see that the corners at F fit exactly with the right angles that you made at P?

Step 4:

Now unfold your paper, and pick any point you like on $\overleftrightarrow{TH}$. Call it Y, for "your point". Through Y draw a line perpendicular to

$\overleftrightarrow{IH}$. What can you say about the relationship between [illegible] and $\overleftrightarrow{IH}$? What is the relationship between this new line [illegible] $\overleftrightarrow{BT}$?

By now you should have some very definite thoughts about what happens when you draw a line perpendicular to one of two parallel lines.

> A line that is perpendicular to one of two parallel lines is also perpendicular to the other.

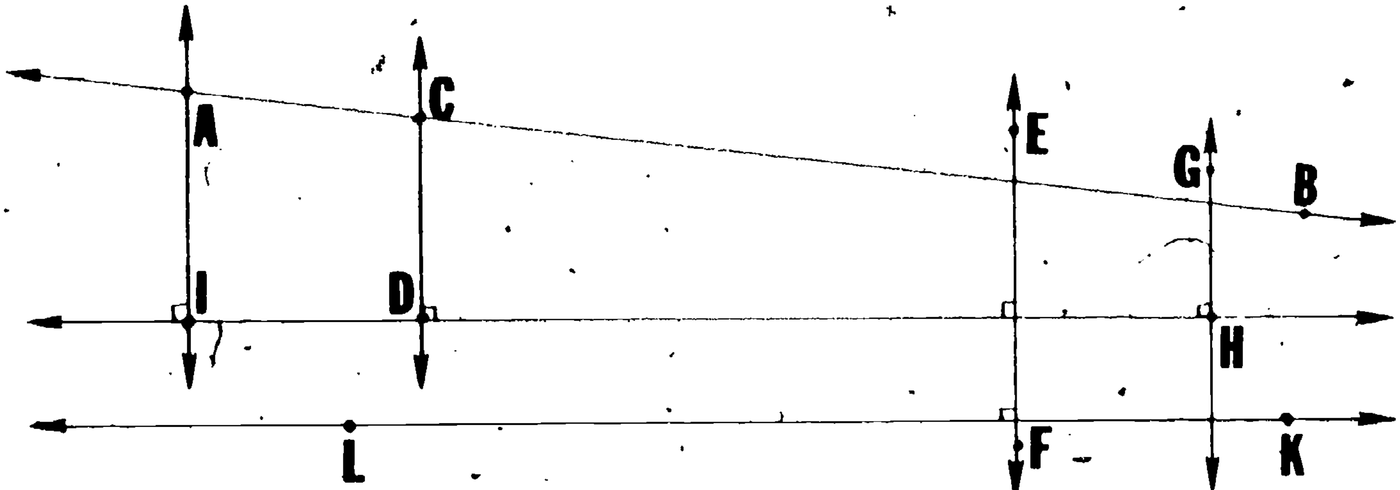
5. As you worked your way through Exercise 4, you may have wished for a way to keep track of angles that you knew were right angles. In the picture below, you will see that we have drawn a little mark, (⌈ or ⌉), in some of the corners. This little square corner is the symbol which we use to show that two intersecting lines are perpendicular. For example, we have put this mark at I, D, H, and at two other points to indicate that the lines crossing there are perpendicular.

As you study the picture on the following page, see if you can answer these two questions:

(a) Which pairs of lines in the following figure are pairs of perpendicular lines?

(b) Which are pairs of parallel lines?

Be careful here! We haven't marked <u>all</u> the right angles, and we haven't drawn all of the intersections.



6. Do you think that the following statement is a true statement?

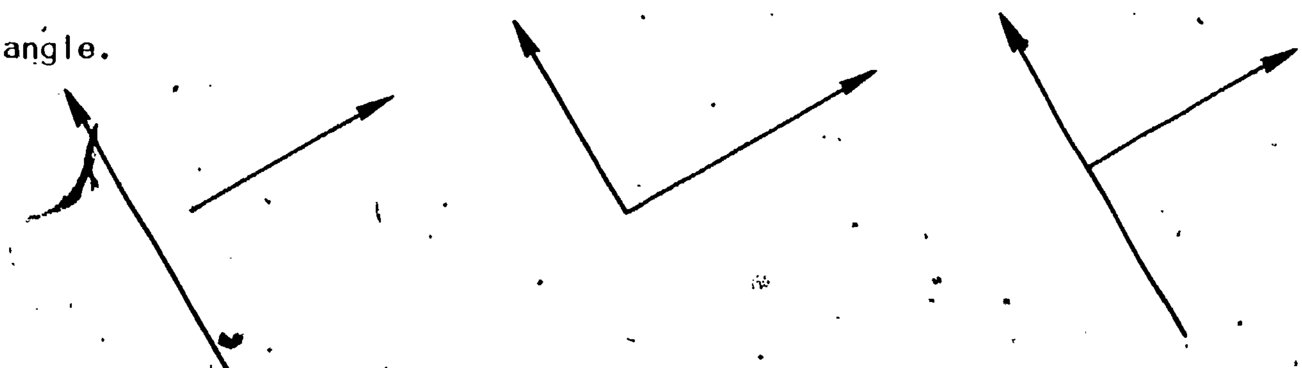
> A line that is parallel to one of two parallel lines is parallel to the other one also.

Section 6-13 Rays and Angles

Now let us investigate pairs of rays in a plane. There are several ways in which a pair of rays may intersect, but one is especially important. When two rays have the same endpoint, but no other point in common, we say that the two rays make an angle. That is:

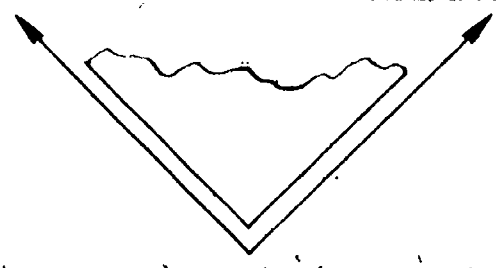
> An angle is a union of two rays that have the same endpoint, but no other point in common and are not opposite rays.

Of the three pairs of rays pictured below, only the middle pair is an angle.

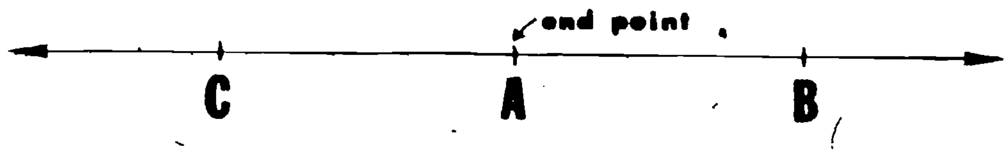


If two rays make an angle, we shall also say that they are connected end to end. The common endpoint is called the vertex of the angle, and the two rays are the sides or edges of the angle.

There is a special angle -- the one in which a square corner fits the sides and the vertex. We call it a right angle. Here is an example:

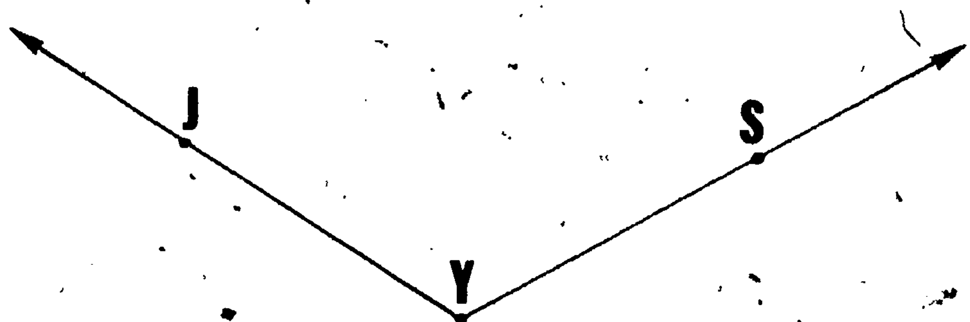


Remember, when two rays have their endpoints in common so that the endpoint is between the other points of the two rays ($\overrightarrow{AB}$ and $\overrightarrow{AC}$)



the rays are called opposite rays. In this book, when we speak of an angle, we mean one for which the two rays are not opposite rays.

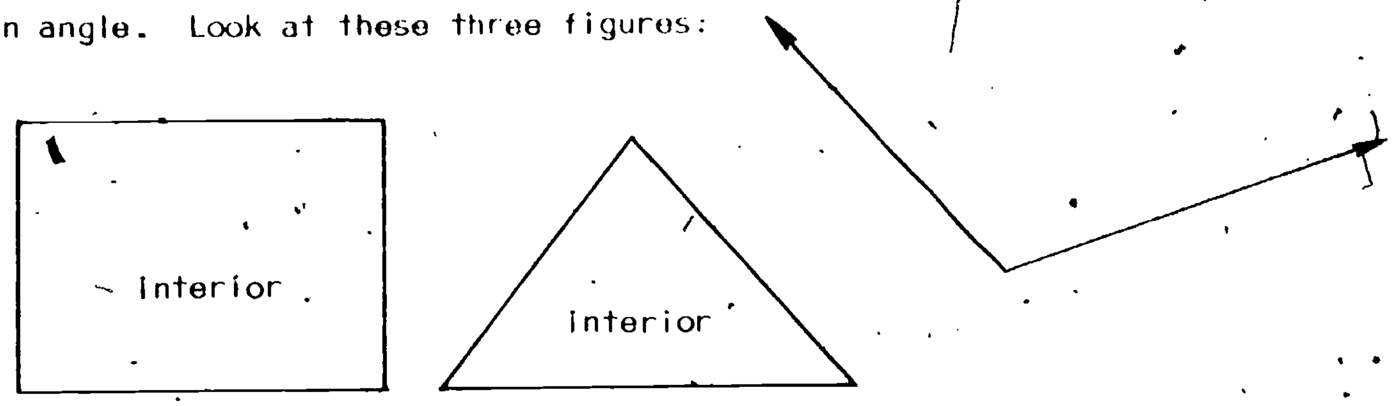
Now, let's see how we name angles. Suppose we have two rays that make an angle, and the rays are already named (if they are not, we can name them). The vertex has a letter name, perhaps Y, and the sides bear names like $\overrightarrow{YJ}$ and $\overrightarrow{YS}$. We could say "the angle $\overrightarrow{YJ}$ and $\overrightarrow{YS}$". This is quite all right, but we can be more brief.



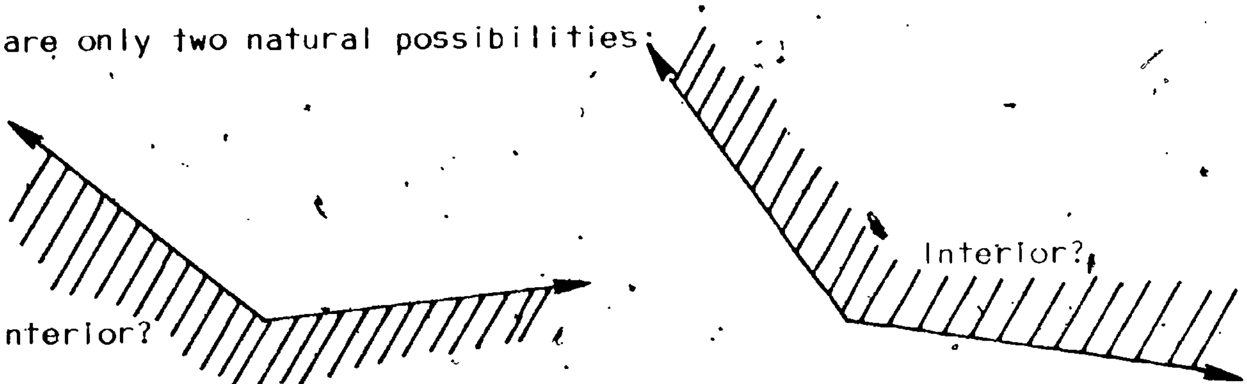
We can also say angle JYS. To be even more brief we can use a little mark, ($\angle$), to remind us that JYS is an angle. We can then say, "$\angle$JYS" or "$\angle$SYJ". At times we may wish to be more brief and call the angle by a numeral rather than by letters. For example, we may wish to call our angle, $\angle 1$, rather than $\angle$JYS.

The order in which the letters are placed is not important except that the vertex letter is named in the middle. If you see the symbols "$\angle ZXQ$" or "$\angle QXZ$", you will know that they name an angle which consists of the two rays $\overrightarrow{XZ}$ and $\overrightarrow{XQ}$.

Just as a triangle and a rectangle have interior regions, so does an angle. Look at these three figures:

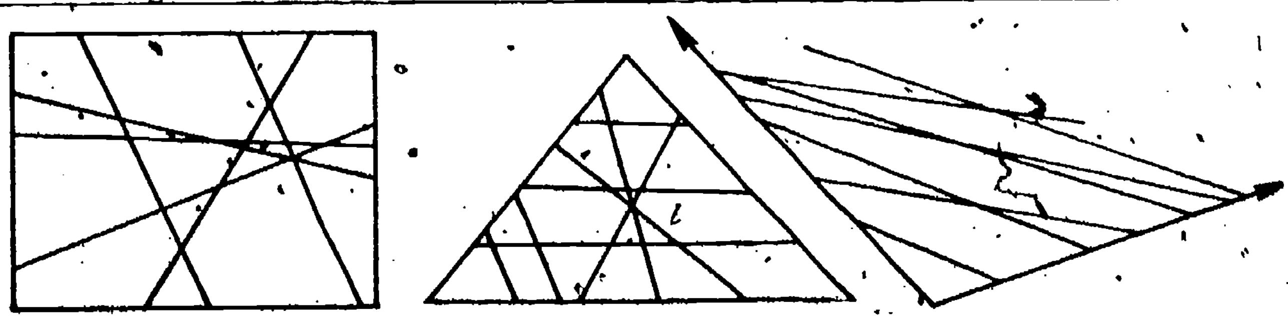


What should we regard as the interior of the one that is an angle? There are only two natural possibilities:



(a) (b)

As you may have guessed, example (b) correctly illustrates the interior of an angle. Does this choice have anything in common with the interior of a triangle and the interior of a rectangle? (Remember, the rectangle and the triangle are "<u>closed</u>" and the angle is "<u>open</u>".) All three have sides, and this gives us a common way of forming the interior: <u>draw all line segments that have their endpoints on the sides of the figure.</u>



Exercise 6-13

1. Draw, if possible, a pair of rays that:

   (a) do not intersect.

   (b) do not intersect and are not parallel. (Note: two rays are parallel if the lines of which they are part are parallel.)

   (c) intersect at a single point.

   (d) intersect at a single point and are not an angle.

   (e) intersect at exactly two points.

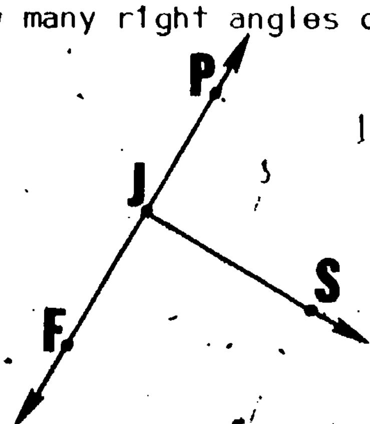
   (f) have in common a line segment.

   (g) make a right angle.

   (h) make an angle but do not intersect.

   If, in any of these cases you think that there is no such pair, give your reasons for thinking so.

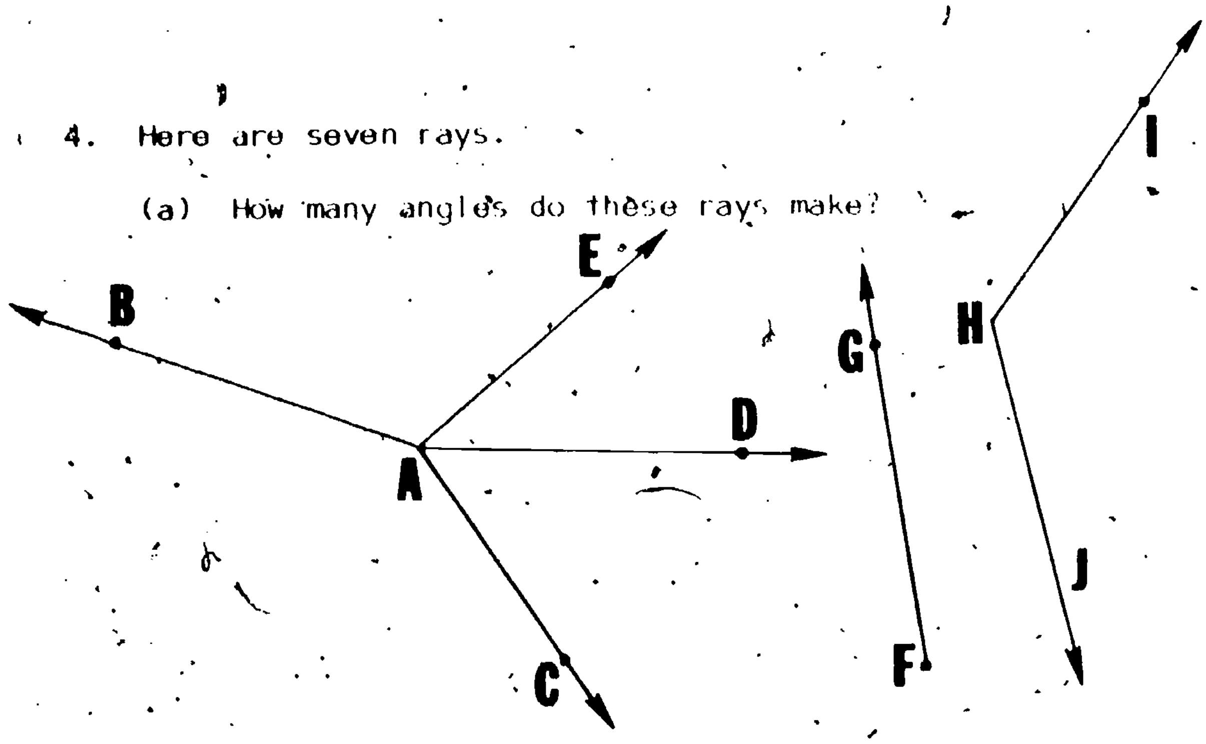
2. Here are three rays, $\overrightarrow{JP}$, $\overrightarrow{JS}$, and $\overrightarrow{JF}$. How many angles can you name from them? How many right angles do they make?



3. If, in Exercise 2, you added a ray opposite $\overrightarrow{JS}$, how many right angles could you make?

4. Here are seven rays.

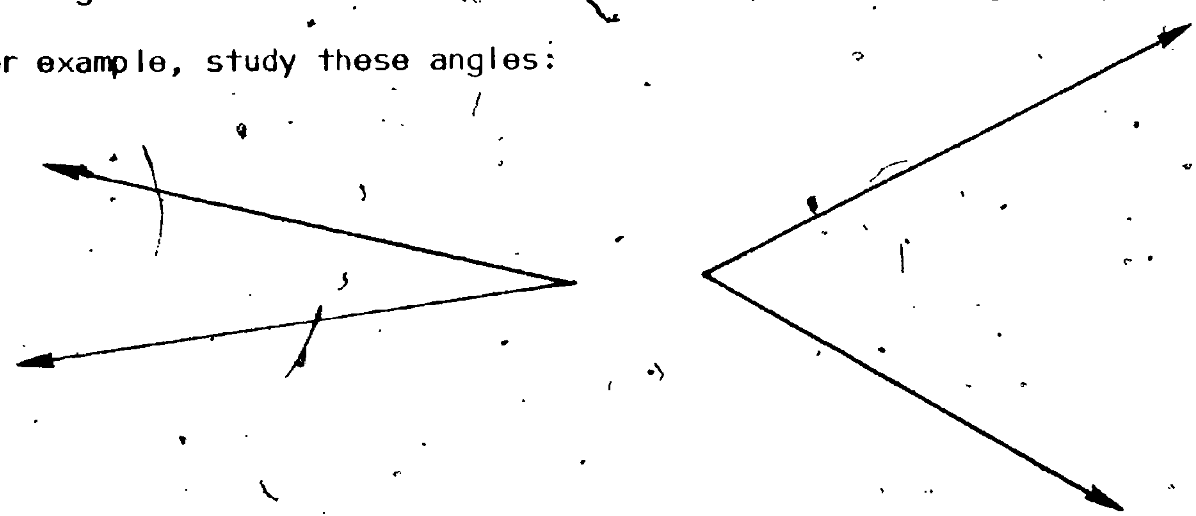
(a) How many angles do these rays make?



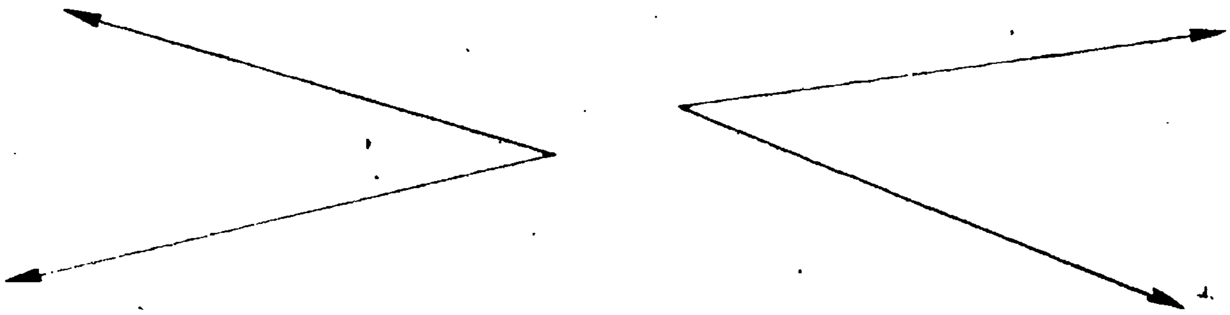
5. On a piece of tracing paper, copy the rays that start at A. Then shade the interiors of ∠BAC and of ∠DAE.

Section 6-14 Comparing Angles

How can you tell when one angle is larger than another, or when two angles have the same size? Sometimes you can tell just by looking. For example, study these angles:

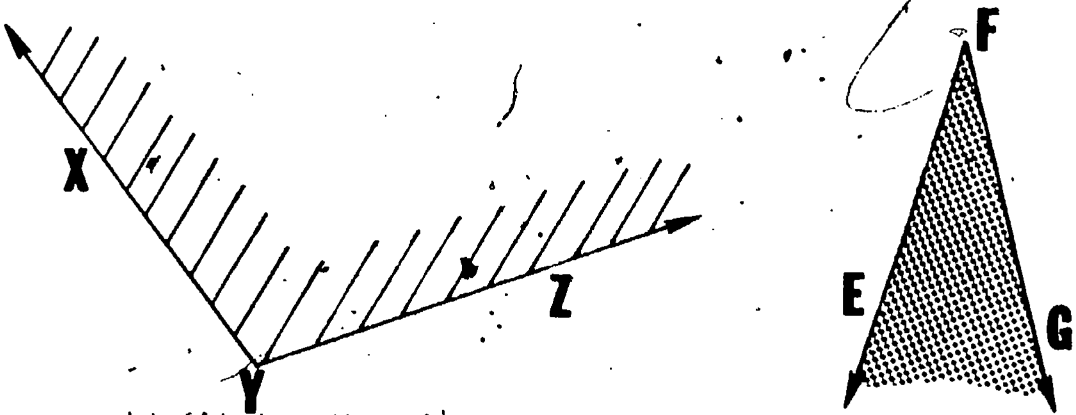


You would not have any trouble saying which is larger, would you? But now look at these:

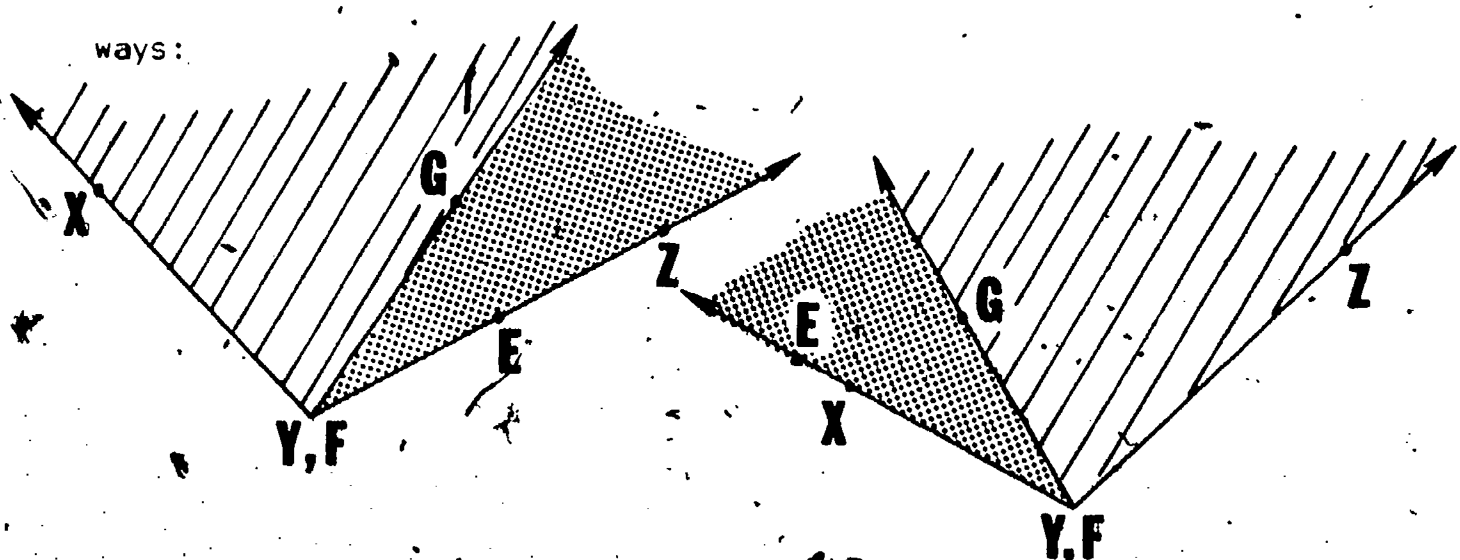


(a) (b)

Do you think one of them is larger than the other, or would you say that they have the same size? If we could bring them closer together it would be easier to compare them. In fact, a good way to compare angles (when you can do it) is to try to put one right on top of the other. Move the angles together so that the interiors overlap and a side of one angle fits exactly on a side of the other angle. If we did this with the two angles, ∠XYZ and ∠EFG, as shown below,



they would fit together in one of four ways. Here we see two of the ways:

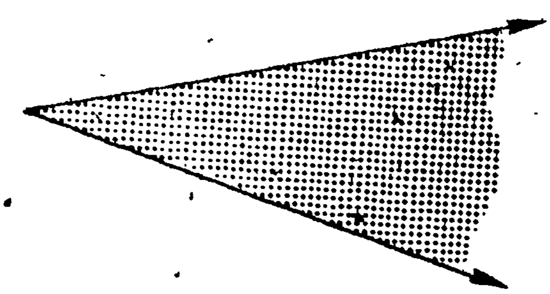


Can you determine the other two ways?

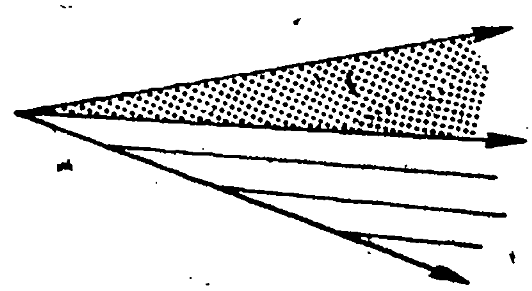
In each of the ways to compare angles it is clear that the interior of ∠EFG is contained in the interior of ∠XYZ, and that the interior of ∠XYZ is not contained in the interior of ∠EFG. It is also clear that we want to consider ∠EFG as being smaller than ∠XYZ, and ∠XYZ as larger than ∠EFG.

Whenever we put two angles together with their interiors overlapping, and with a ray of one angle fitted exactly to a ray of the other angle, one of two things happens:

(a) The other two rays fit exactly, so that the angles fit exactly. In this case the angles have the same size and we call them congruent.

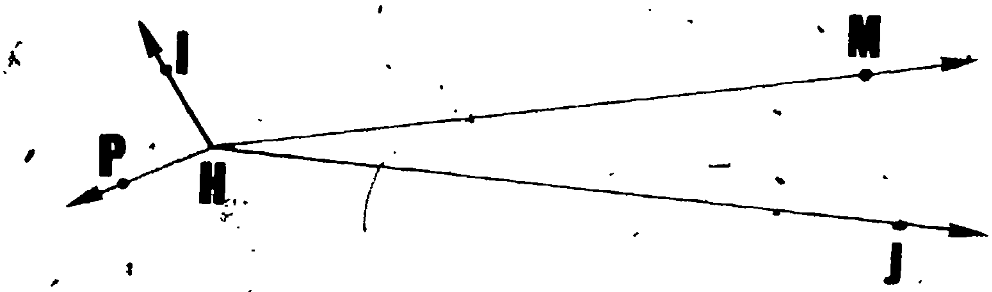


(b) A ray of one angle lies in the interior of the other angle. In this case the angles are not congruent. One angle is smaller, and the other is larger.



Exercise 6-14

1. Copy the drawing below by tracing it on thin paper.



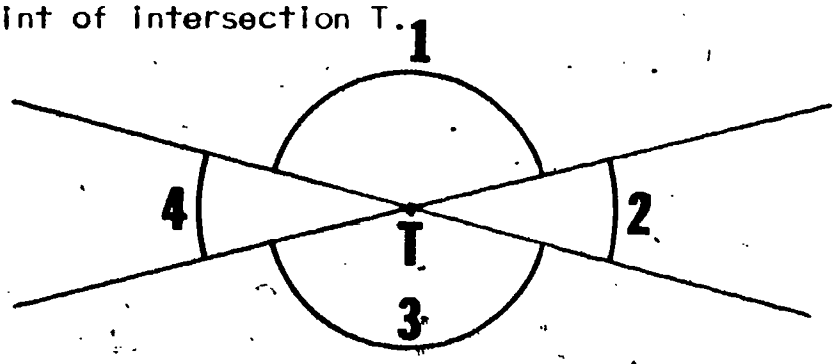
Before folding your paper, decide which of your angles you think is larger, your copy of ∠IHP or your copy of ∠MHJ. Now check your statement by folding. Were you right?

2. Make a drawing similar to the one above. Be sure to make straight lines and clearly name the points. Now, let a fellow-student take your drawing and you take his and make the same comparisons you made in example 1. You might like to do several examples like this.

Section 6-15 Angles Made by Lines

After you read this section you will see that there are countless congruent angles all about us.

On a clean piece of paper, draw two intersecting lines, and call their point of intersection T.



The lines make four rays, all starting at T. The rays, taken in pairs one from each line, make four different angles. Do you see them in your drawing? We have numbered ours 1, 2, 3, and 4. These four angles are called the angles made by the two lines. A side of each of these angles comes from each line.

Angles 1 and 3 are called vertically opposite angles because they have a common vertex and because their rays are opposite pairs.

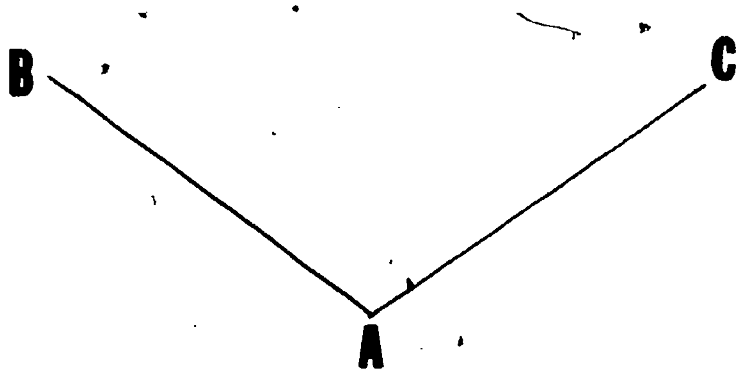
Do you think that 2 and 4 are vertically opposite? Why?

Do you think that 1 and 3 are congruent? Do they look congruent? Try to fit them together by folding your paper. What do you conclude?

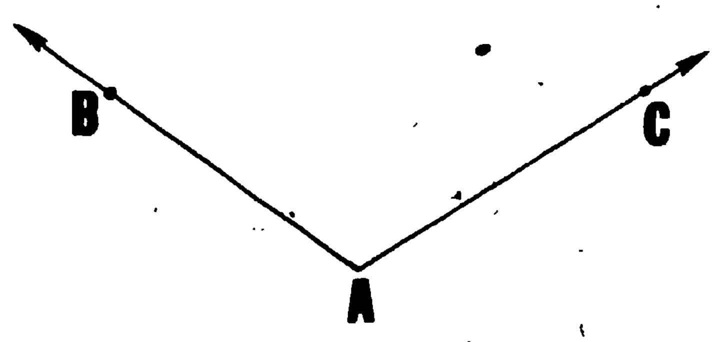
Is 2 congruent to 4? Test them by folding. What do you say?

Vertically opposite angles are congruent.

Before working the next set of exercises, let us briefly review line segments. If two line segments have a common endpoint, the pair is not an angle, because an angle is a pair of rays that have a common endpoint. But given two line segments $\overline{AB}$ and $\overline{AC}$ with a common endpoint,

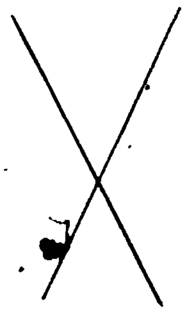


we can make an angle from them by forming the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

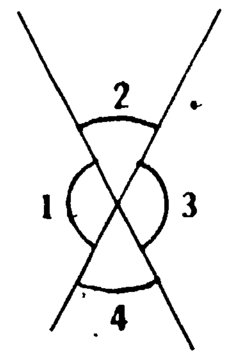


This angle, $\angle BAC$, is called the angle made by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Likewise, when two line segments cross we say that they make four angles. This is shown below:

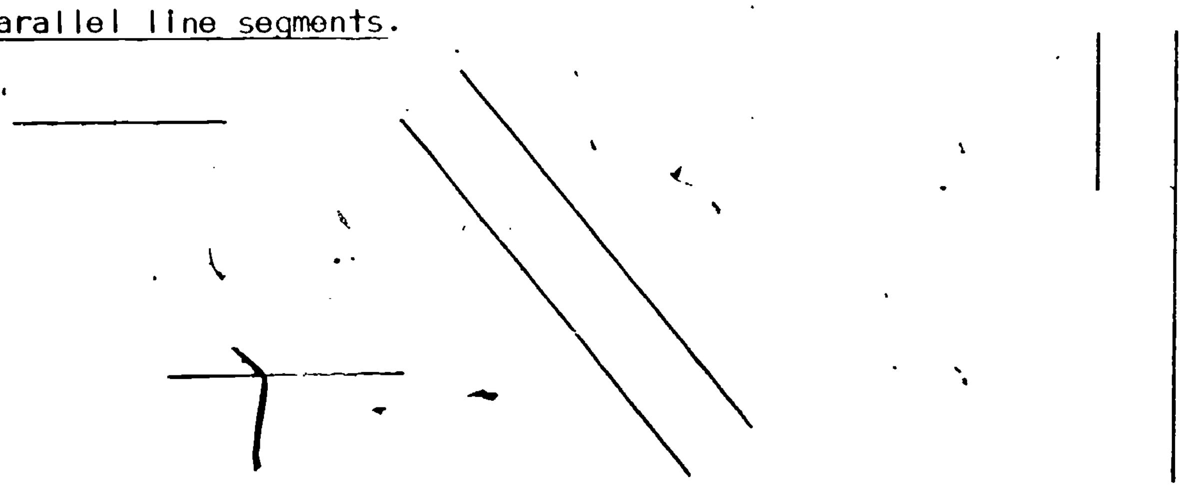


Crossing line segments



The four angles made by the crossing line segments

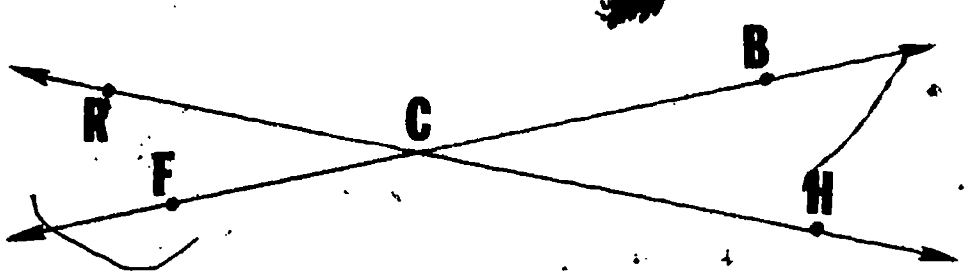
If two line segments lie on lines that are parallel, we call the line segments themselves parallel. These figures are examples of parallel line segments.



Exercise 6-15

1. What does it mean when we say that two rays are parallel? When a ray and a line segment are parallel? When a line and a line segment are parallel?

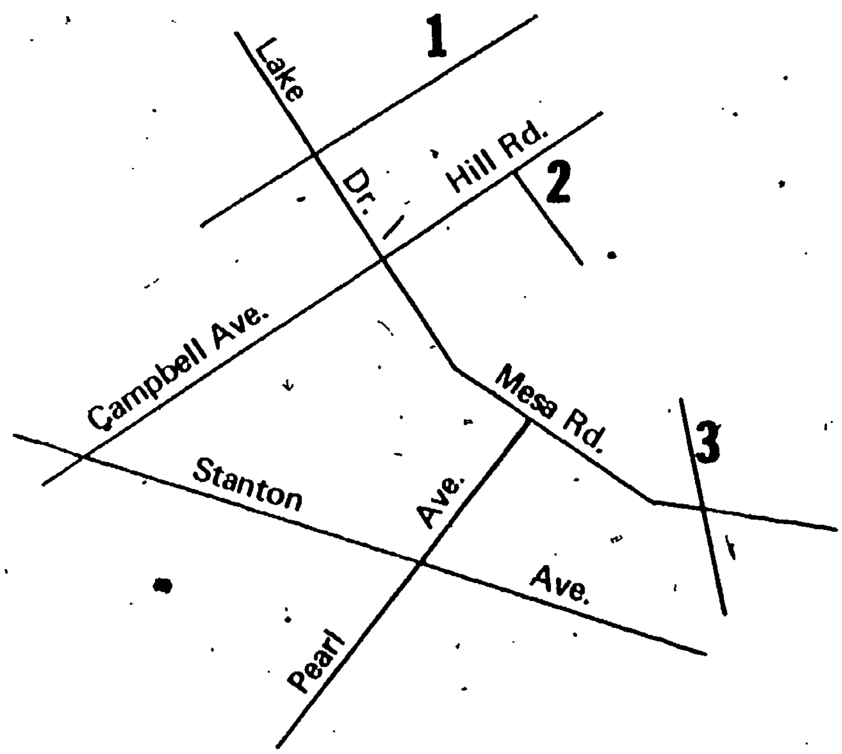
2. In the picture below, name the vertically opposite angles.



3. The line segments below are part of a map of Bordersville. Some of the streets are named, and others are numbered so that it will be easier for you to refer to them when you think about the questions.

   (a) How many pairs of perpendicular line segments can you find? How many parallel ones?

   (b) At each intersection, what angles are congruent?



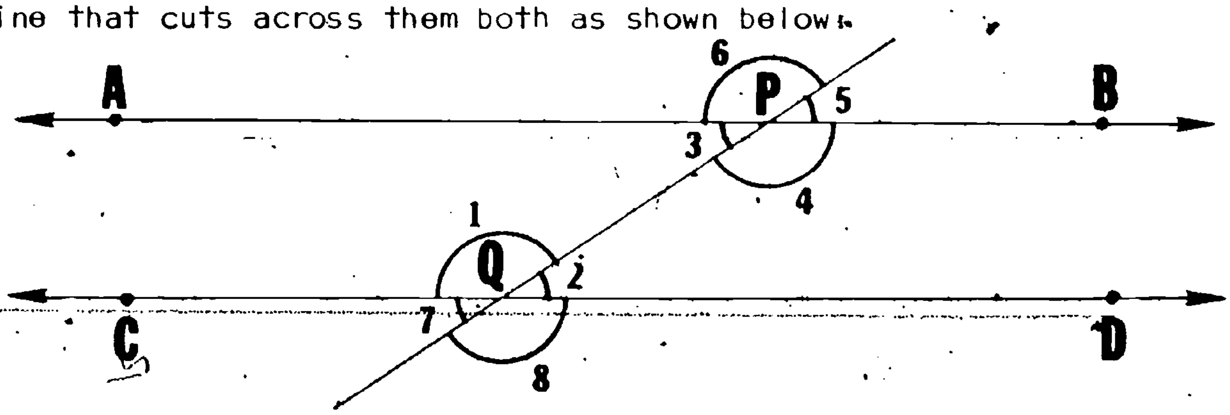
4. Think about where you have seen things that suggest vertically opposite angles. Do you know two streets that are straight where they cross? If you were to draw a map of your school playground, would you have made any vertically opposite angles? Can you find any line segments in your classroom that cross each other?

5. Look around your classroom for parallel and perpendicular line segments. For example, look at the windows. Are the sides parallel? Do you see any perpendicular lines?

Section 6-16 Alternate Interior Angles

There is a famous geometric figure in which congruent angles always appear. Let us see if you can discover them.

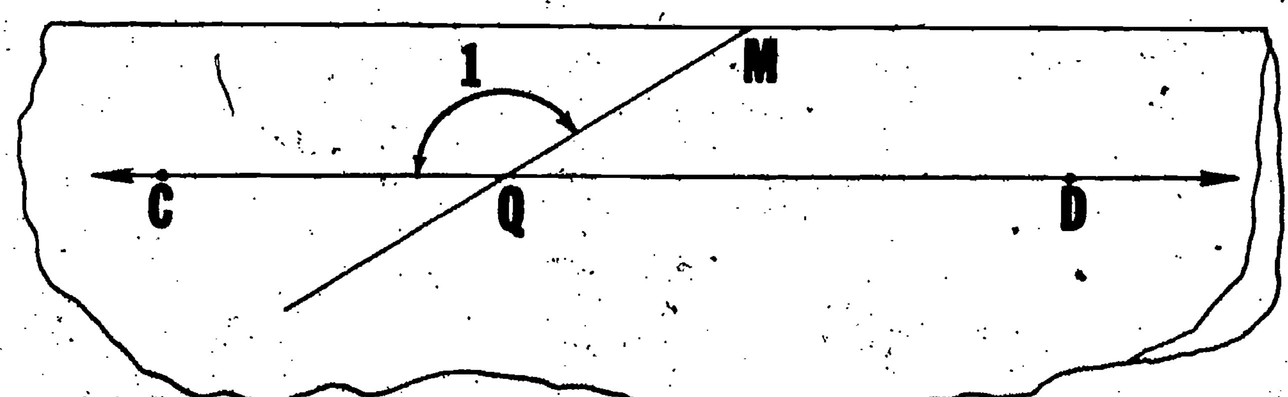
On a clean, thin sheet of paper draw two parallel lines. Now, draw a line that cuts across them both as shown below:



A line like this is a transversal of the parallel lines. The transversal makes four angles with each of the parallel lines. Can you name them? Each angle at P is congruent to another angle at P, and each angle at Q is congruent to another angle at Q. For example, ∠1 is congruent to ∠8 and ∠7 is congruent to ∠2. Why? (∠1 is vertically opposite ∠8 and ∠7 is vertically opposite ∠2.) Now, do you see any angle at P that seems to be congruent to an angle at Q? Check yourself by fitting them together by folding.

Here is one way. Fold your paper so that $\overleftrightarrow{CD}$ lies on top of $\overleftrightarrow{AB}$. Then your paper, on one side, looks like this:

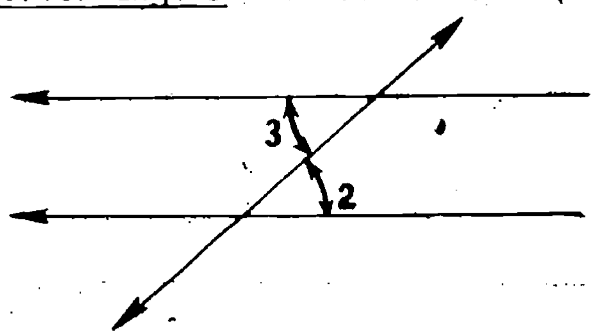
fold fold



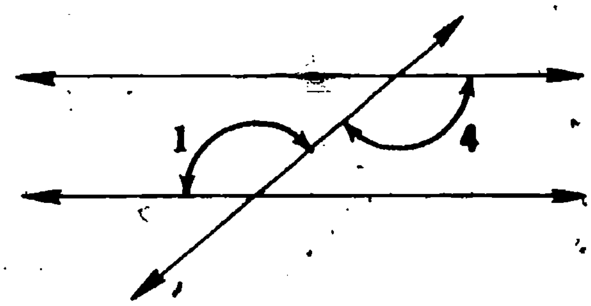
We have called the point where $\overline{PQ}$ crosses the fold, M. Fold the paper again to make a square corner at M. What seems to appear when you hold your paper to the light? If you cannot see clearly enough through all that paper, unfold your paper and draw the lines more heavily. Then fold and have another look. What do you conclude about $\angle 1$ and $\angle 4$? Do they appear to be congruent?

What seems to be true about $\angle 3$ and $\angle 2$? When you fold your paper, do they fit exactly? Are $\angle 3$ and $\angle 2$ congruent?

Do you see that the four angles we have been talking about, $\angle 2$, $\angle 3$, $\angle 4$, and $\angle 1$, are the only angles in the picture that have an edge containing $\overline{PQ}$? These four angles are called the <u>interior angles</u> formed by the transversal and the parallel lines. $\angle 3$ and $\angle 2$ are called <u>alternate interior angles</u>. $\angle 1$ and $\angle 4$ are also <u>alternate interior angles</u>. These are shown in the following examples:



Alternate interior angles



Alternate interior angles

Do you see that alternate interior angles have different vertices? Notice also that the <u>interiors of alternate interior angles</u> lie on <u>opposite</u> sides of the transversal.

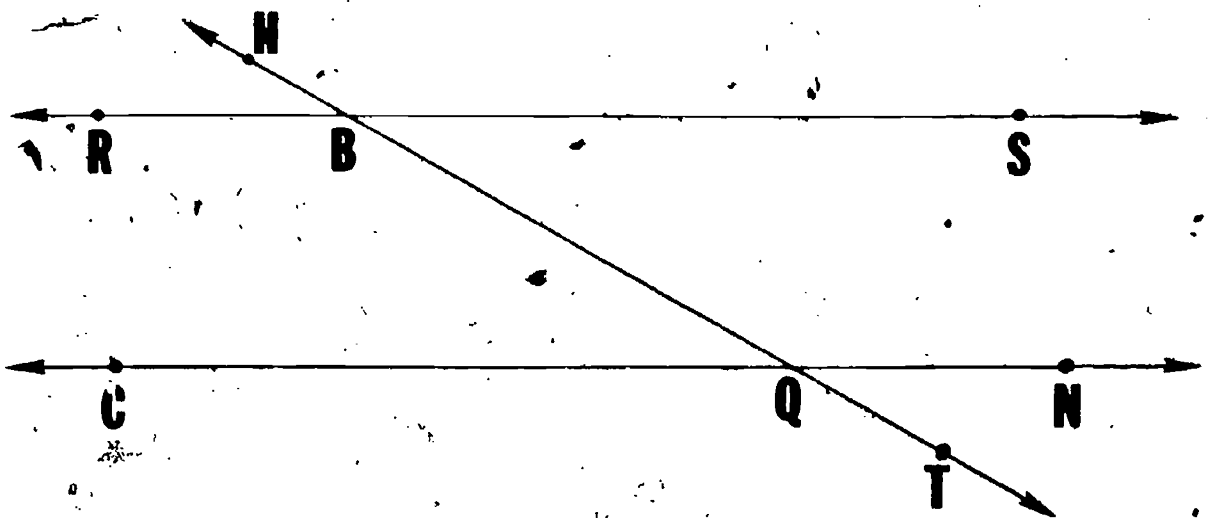
Alternate interior angles, formed by a transversal crossing two parallel lines, are congruent.

Exercise 6-16

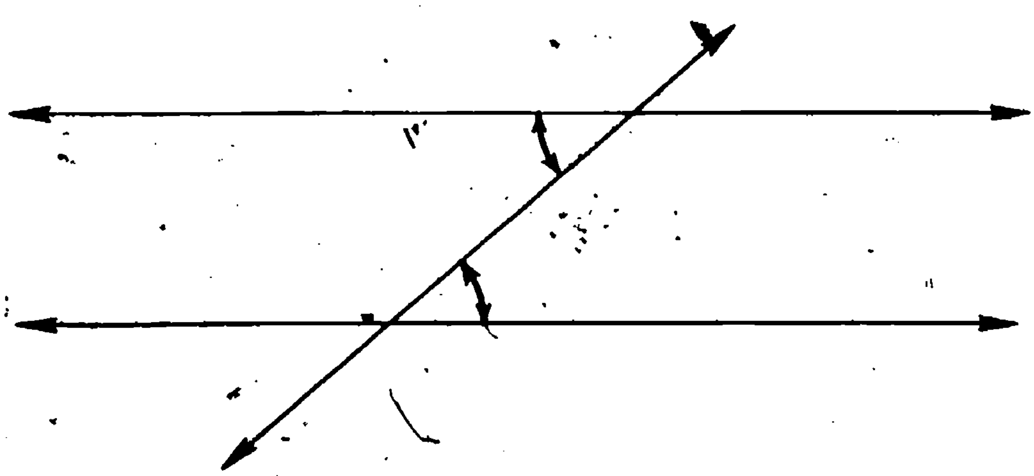
1. Here is a picture of a transversal crossing two parallel lines.

   (a) Use your knowledge of vertically opposite angles, and of alternate interior angles, to name all the angles congruent to $\angle HBR$.

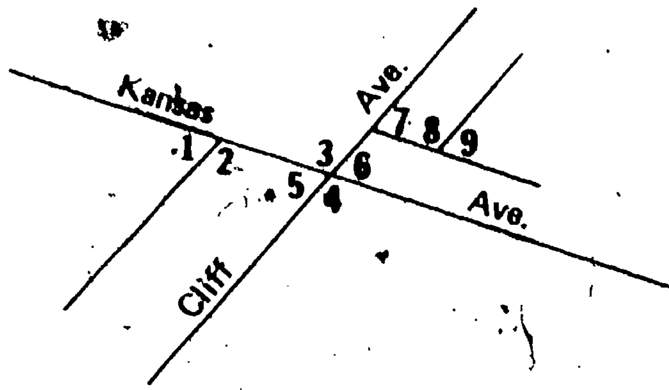
   (b) What angles are congruent to $\angle BQN$?



2. On tracing paper make a copy of the figure which follows. Shade the interiors of the alternate interior angles marked with arcs.



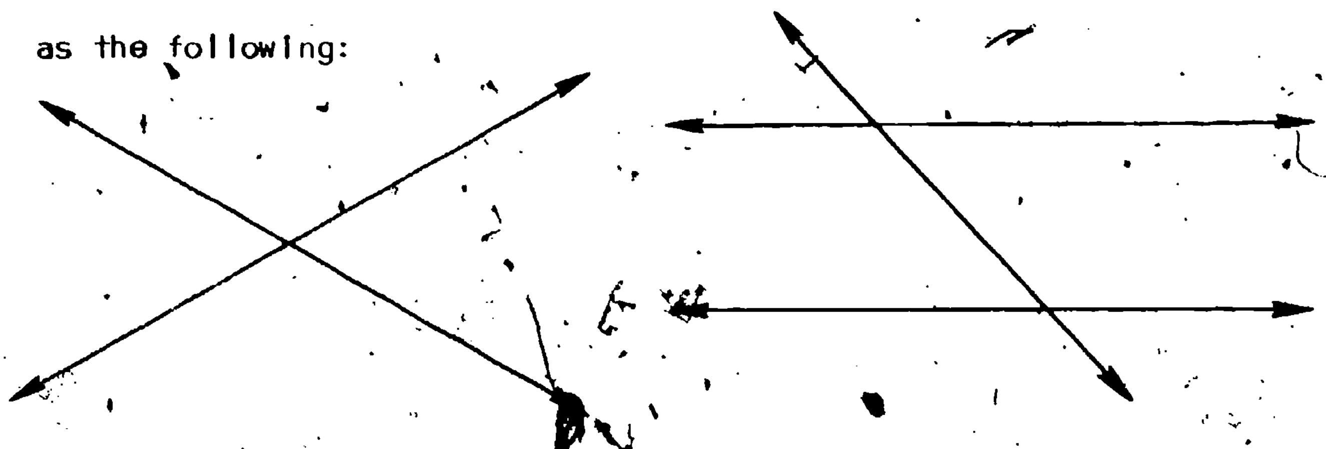
3. Look at the diagram of certain street intersections in the city of El Paso. The angles formed by the streets have been numbered.



Make a list of the congruent angles formed in the diagram.

Section 6-17 Using a Compass to Compare Angles

By now you know what angles are, and how lines and line segments make angles. You also know how to compare angles on paper whenever you can fold one on top of the other. You have used this method of comparison to look at angles made by crossing lines in figures such as the following:



Your study of figures such as those above led you to identify congruent angles made by objects in the world around you.

But what about angles which cannot be moved about? How are you going to compare them?

Do you remember how we compared line segments? We made a copy of one and compared the copy with the other. If we could make copies of angles, then we could compare one angle with another by comparing a copy of it with the other. Then, instead of saying that two angles are congruent if they fit together exactly, we could say,

> Two angles are congruent if a copy of either one fits the other exactly.

We could also say,

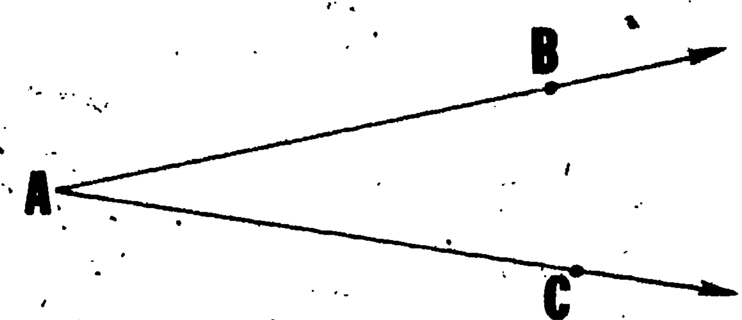
> One angle is larger (or smaller) than another if a copy of it is larger (or smaller) than the other.

How, then, might we copy angles?

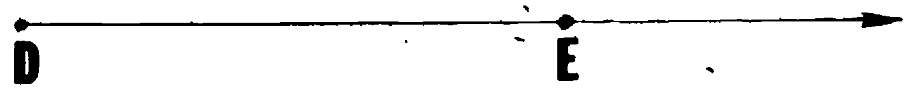
Some angles are very easy to copy. Right angles are, for example. Any two right angles are congruent. Do you agree?

One way is to put thin paper over the angle and copy the angle onto the paper by tracing. Another way is to cut a model of the angle from a sheet of paper. Here is another way which you might like even better.

Let us each start with an angle drawn on a piece of paper. Ours looks like this:

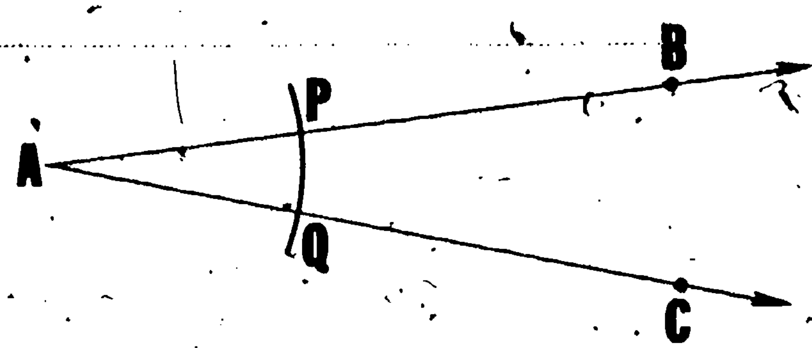


Yours may look different, but that does not matter, because we are going to show you with pictures and with words how to copy your angle. Start with your angle BAC on a piece of paper, and on another piece of paper draw a ray:

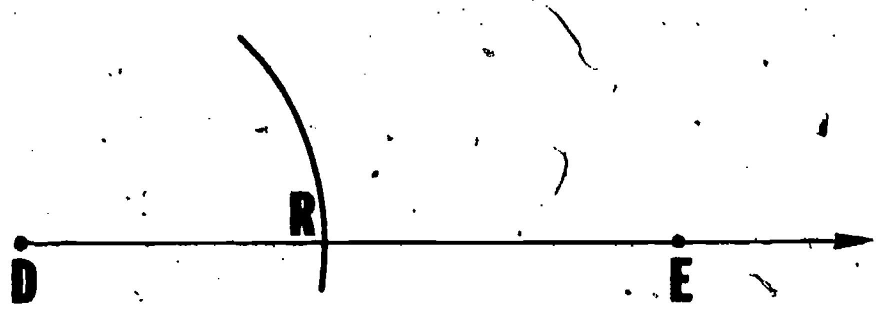


Now, let's see how we construct an angle which is congruent to ∠BAC and has $\overrightarrow{DE}$ as one side.

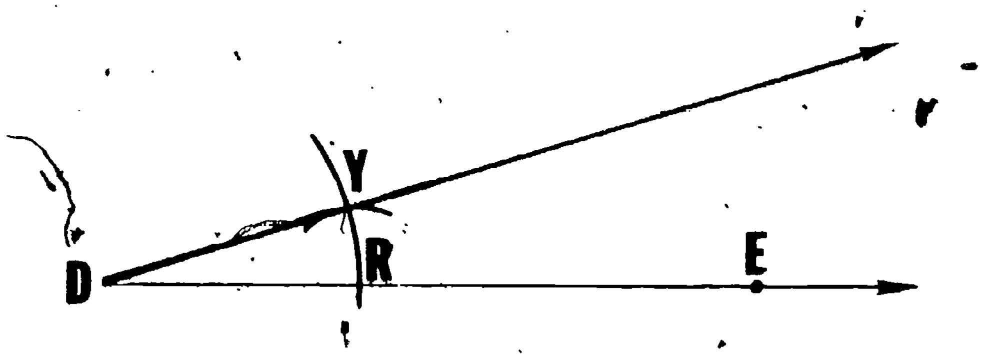
Take your compass and with vertex A as center draw an arc that cuts both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



(We have called the points where the arc cuts the rays P and Q.) Leaving your compass unchanged, put the point at D and draw an arc that cuts $\overrightarrow{DE}$. You should draw this arc longer than the one that cuts ∠BAC.



Returning to $\angle$BAC, set your compass so that one point fits on P at the same time that the other fits on Q. Without changing this setting, put the pin point at R and draw an arc that cuts the first arc you drew there. Call the point of intersection Y, if you like, and draw the ray $\overrightarrow{DY}$.



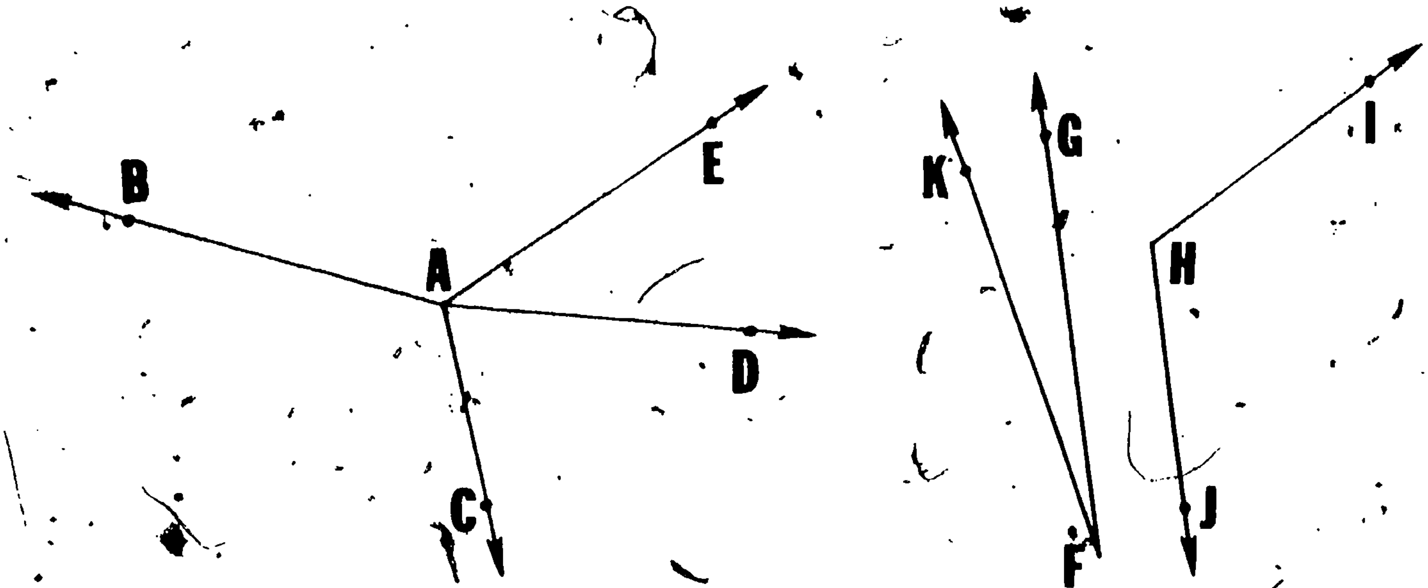
If you have done your work correctly your new angle, YDR, is congruent to $\angle$BAC.

Now that we can copy angles, the following observation can be made about comparing their sizes:

> The size of two angles can be compared by matching one with a copy of the other. Since the copy can be moved about we simply observe what happens when we try to fit it to the other angle.

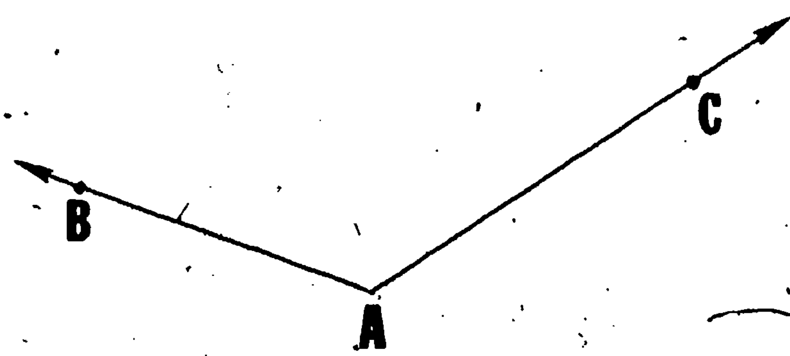
Exercise 6-17

1. At the beginning of Section 6-14 we looked at two angles that are difficult to compare by sight. Trace one of them now on thin paper and compare it with the other. What do you conclude?
2. By sight only, list the angles named on the next page in the order of their sizes, beginning with the largest.

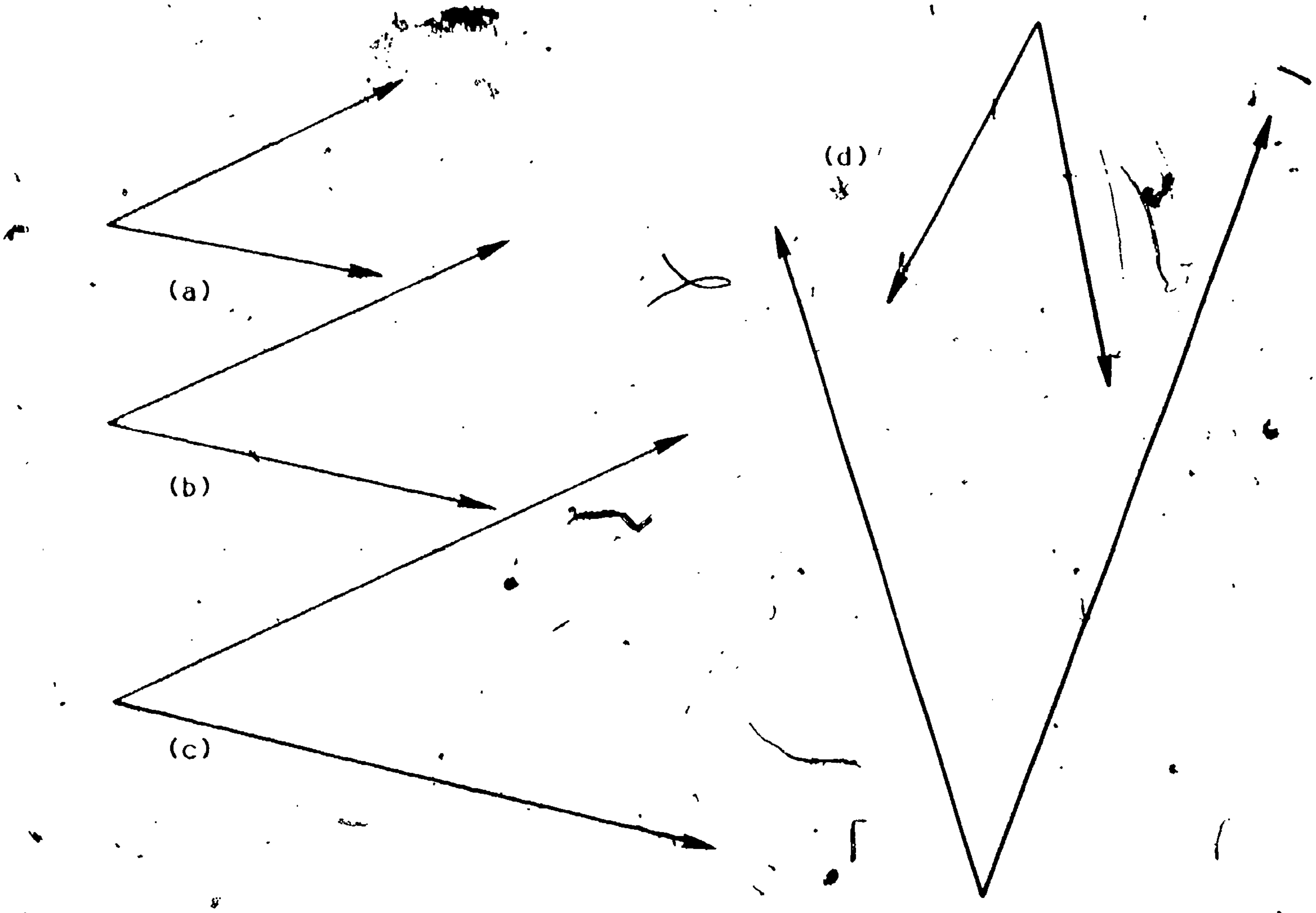


3. Draw a ray, and on it construct an angle congruent to ∠BAC of Exercise 2, above. On the same ray, construct another angle, this one congruent to ∠IHJ of Exercise 2. Which of these two angles is larger?

4. Copy the following angle, ∠BAC, on a piece of thin paper. On another piece of paper, draw a ray with endpoint D and use your compass to construct an angle congruent to ∠BAC and having your ray as one side. <u>Put your thin paper copy over the compass copy,</u> and compare them. Does the thin paper copy give you a way to show that the compass copy and ∠BAC are congruent?



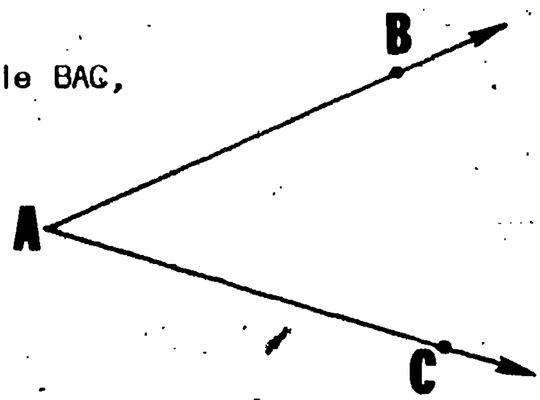
5. Compare the following angles using any of the methods we have discussed. What are your conclusions?



Section 6-18 Bisecting an Angle

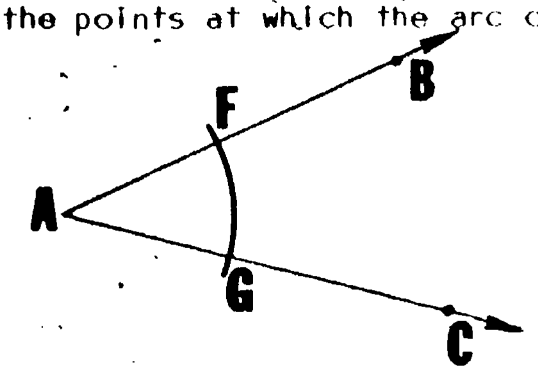
Another important construction with angles is called bisection. We bisect an angle by dividing it into two congruent angles. Remember, we bisected a line segment by dividing it into two segments that were congruent.

To bisect the angle BAC,

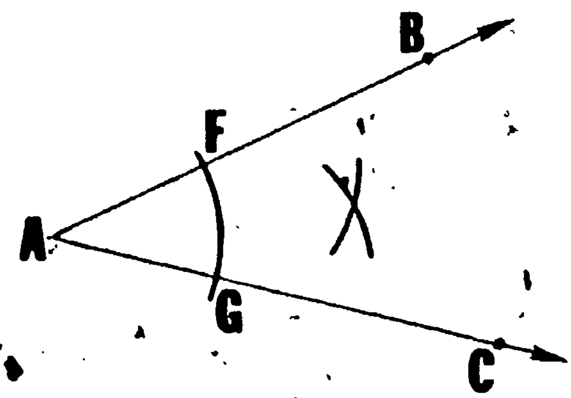


we find a ray, $\overrightarrow{AM}$, that lies in the interior of $\angle$BAC in such a way that $\angle$BAM and $\angle$CAM are congruent.

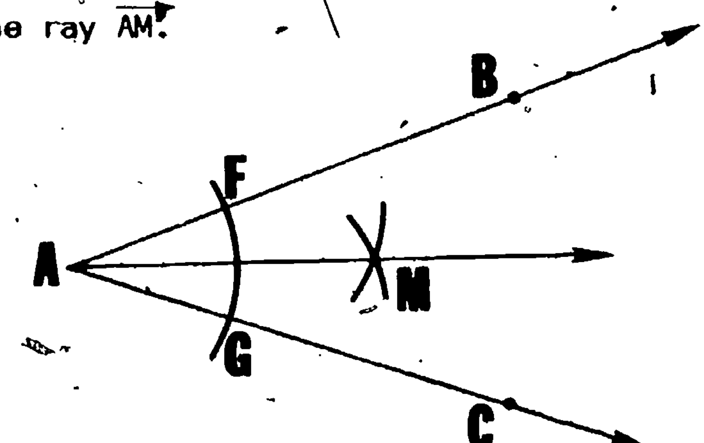
This is how we do it. With the point A as center, draw an arc that cuts both $\overrightarrow{AB}$ and $\overrightarrow{AC}$. In the picture below, F and G are the names of the points at which the arc cuts the rays.



With the same setting, and with F and G as centers, make two more arcs that intersect away from A, as shown in the figure that follows.



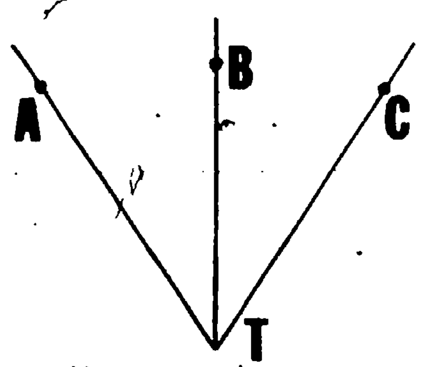
Now, let <u>M</u> be the point of intersection of these two arcs, and draw the ray $\overrightarrow{AM}$.

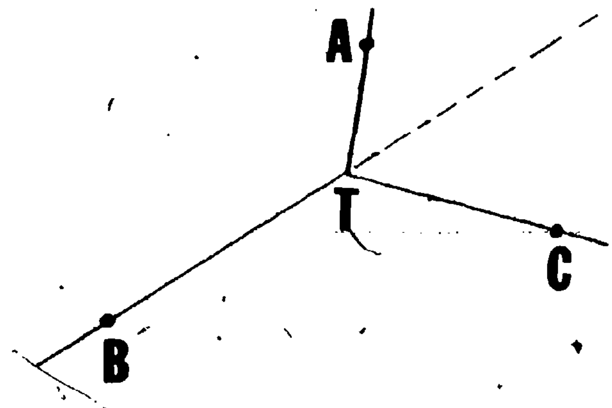


If you fold your paper along $\overrightarrow{AM}$, you will find that $\overrightarrow{AC}$ fits exactly on $\overrightarrow{AB}$. Try it. Do you see that ∠BAM and ∠CAM are congruent? $\overrightarrow{AM}$ <u>is the bisector of</u> ∠BAC. We have bisected ∠BAC.

Exercise 6-18

1. In the two figures below, tell how you would use paper folding to compare $\angle ATB$ and $\angle CTB$. If they are congruent, how would you know? If they are not congruent, how would you identify the larger one?

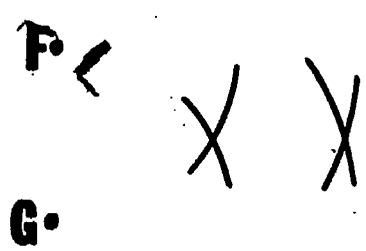




2. Draw three angles on your paper and bisect each. Test each bisection by paper folding.

3. Earlier, when we discussed angle bisection, we used the same setting of the compass for the two intersecting arcs. Would our drawing have been affected had we changed the setting before we drew the two intersectiong arcs? Let's see what happens.

On a piece of paper, draw two points, labeling one F and the other G. Set your compass, and draw two intersecting arcs, one with center F and one with center G. Change the setting, and on the same side of F and G draw two more intersecting arcs. So far, you have something like this:



Now, by changing the setting several more times, add some more pairs of arcs to your picture. What pattern do you see? Draw some pairs of arcs on the other side of F and G. What happens? Can you make any conclusions about the way we made our bisector? Did we need the same setting for all three arcs?

Review Exercise 6-19

1. How many different lines may contain:

   (a) One certain point?

   (b) A certain pair of points?

2. How many different planes may contain:

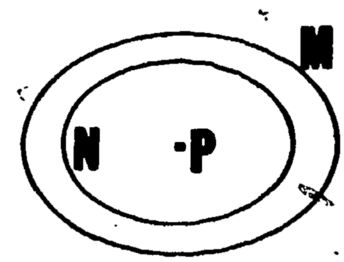
   (a) One certain point?

   (b) A certain pair of points?

   (c) A certain set of three points?

3. Draw a picture of two simple closed paths whose intersection is exactly two points. How many simple closed paths are shown in your figure?

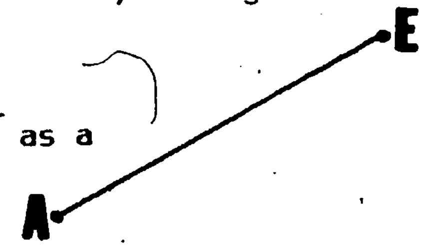
4. Describe the region between curve M and curve N in terms of intersection, interior, and exterior.



5. Draw two triangles whose intersection is a side of each. Is the union of the other sides of both triangles a simple closed path? How many simple closed paths are represented in your figure?

6. (a) Draw $\overline{AE}$ on your paper.

   (b) Draw a circle with center at A and $\overline{AE}$ as a radius. Call the circle C.



(c) Is a radius of a circle part of the circle? Why?

(d) Draw a diameter of your circle. Is a diameter of a circle part of the circle?

7. Draw a circle with a center marked A and a radius $\overline{AE}$. Can you imagine another circle with center E and radius $\overline{AE}$? Is there more than one such circle?

8. Imagine two circles.

(a) Do they have to be in the same plane?

(b) Could their intersection be the empty set?

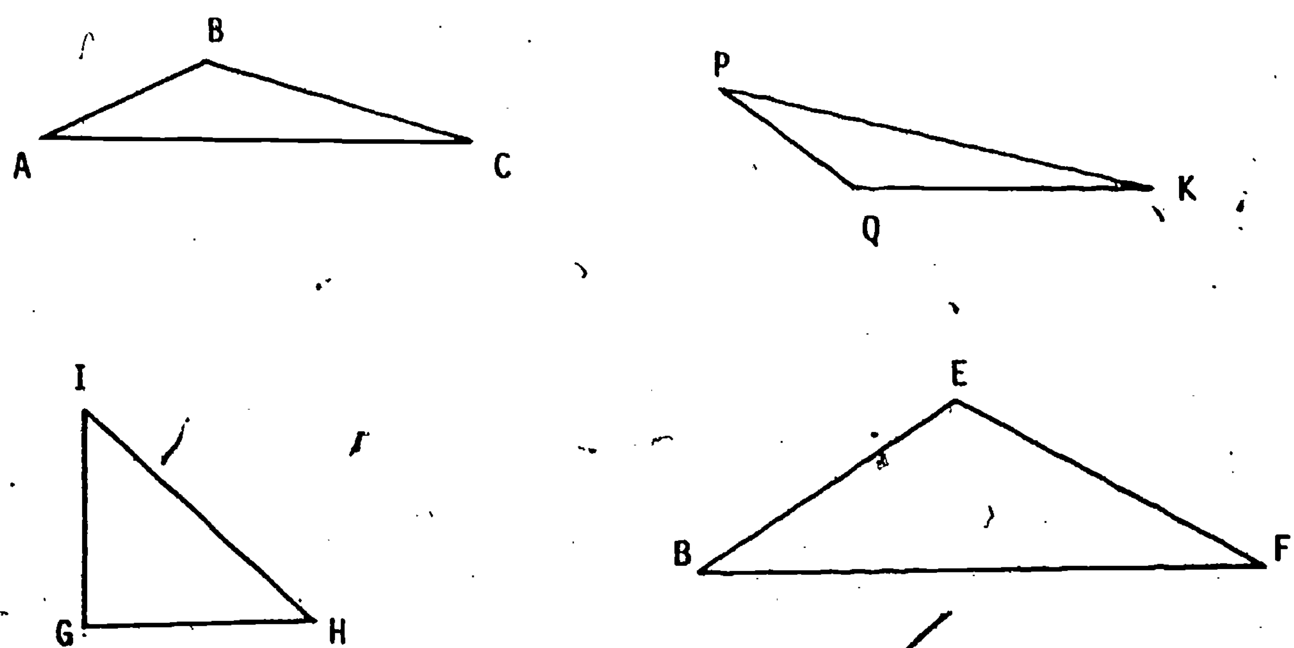
(c) Could they intersect in exactly one point? Show how.

(d) Could they intersect in exactly two points? Show how.

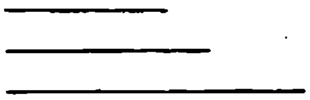
(e) Could they intersect in exactly three points? Show how.

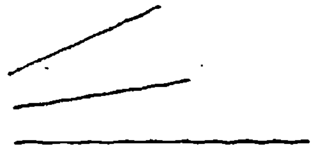
(f) Could they intersect in more than three points?

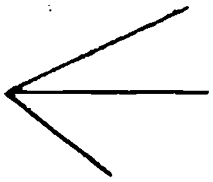
9. Copy each of the following triangles, using a compass and straightedge.

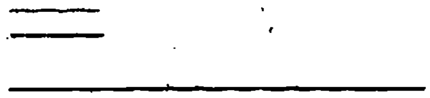


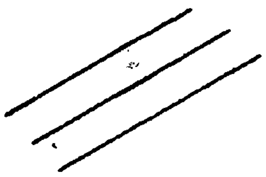
10. Construct a triangle using the lengths of the given line segments for the lengths of the sides of the triangle. Are all the constructions possible?

(a) 

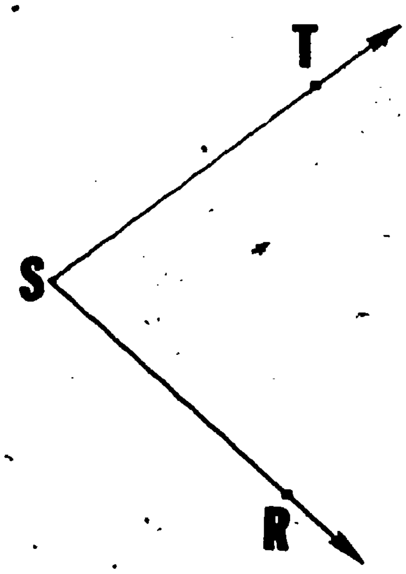
(b) 

(c) 

(d) 

(e) 

(f) 



11. (a) Trace angle RST. Choose a point in the interior of angle RST. Call this point, W. Draw $\overrightarrow{SW}$.

(b) Compare the size of $\angle$RST with the size of $\angle$RSW.

(c) Compare the size of $\angle$RST with the size of $\angle$WST.

12. In the interior of $\angle$ZYX, place a point N near Z and draw $\overrightarrow{YN}$. Fold along $\overrightarrow{YN}$. Which has the larger size, $\angle$XYN or $\angle$NYZ?

Chapter 7:

Factors and Primes

Section 7-1 Natural Numbers and Whole Numbers

In Chapter 3 you learned that the set of counting numbers {1, 2, 3, ...} is also called the set of natural numbers. If we include the number zero in this set, we have another set of numbers which we have called the set of whole numbers. You have learned that the set of whole numbers may be written as W = {0, 1, 2, 3, ...}.

Set of Natural Numbers	N = {1, 2, 3, ...}
Set of Whole Numbers	W = {0, 1, 2, 3, ...}

In this chapter we shall study only the natural numbers. Therefore, every time we use the word "number" in this chapter we shall mean natural number.

Exercise 7-1

1. How does the set of natural numbers differ from the set of whole numbers?
2. What is the intersection of N and W?
3. What is the union of N and W?
4. What is the smallest natural number?
5. What is the smallest whole number?
6. If 1 is added to any natural number, the sum is another natural number. With this fact could you convince a friend that there is no largest natural number?

7. Think about these two sets: (a) {pages in every book ever printed} and (b) {natural numbers}. Which set has more elements?

8. Let <u>w</u> represent a whole number. Can we always say that <u>w</u> represents a natural number? Why?

Section 7-2 Factors and Divisors

We need to review two words in our mathematics vocabulary. The words are <u>factor</u> and <u>divisor</u>. Look carefully at the following examples.

$6 = 3 \cdot 2$	3 and 2 are <u>factors</u> of 6.
$15 = 5 \cdot 3$	5 and 3 are <u>factors</u> of 15.
$10 = 10 \cdot 1$	10 and 1 are <u>factors</u> of 10.

We look again at the factors of 6, 10, and 15 just shown:

$$\begin{array}{r} 3 \\ 2\overline{)6} \\ \underline{6} \\ \text{remainder } 0 \end{array} \qquad \begin{array}{r} 5 \\ 3\overline{)15} \\ \underline{15} \\ \text{remainder } 0 \end{array} \qquad \begin{array}{r} 1 \\ 10\overline{)10} \\ \underline{10} \\ \text{remainder } 0 \end{array}$$

$$\begin{array}{r} 2 \\ 3\overline{)6} \\ \underline{6} \\ \text{remainder } 0 \end{array} \qquad \begin{array}{r} 3 \\ 5\overline{)15} \\ \underline{15} \\ \text{remainder } 0 \end{array} \qquad \begin{array}{r} 10 \\ 1\overline{)10} \\ \underline{10} \\ \text{remainder } 0 \end{array}$$

When 6 is divided by 2 or 3, the remainder is zero.
<u>2</u> and <u>3</u> are called <u>divisors</u> (exact divisors) of 6.

When 15 is divided by 3 or 5, the remainder is zero.
<u>3</u> and <u>5</u> are <u>divisors</u> (exact divisors) of 15.

When 10 is divided by 1 or 10, the remainder is zero.
<u>1</u> and <u>10</u> are <u>divisors</u> (exact divisors) of 10.

In mathematics the words <u>factor</u> and <u>divisor</u> mean the same thing.

$56 = 8 \cdot 7$ — 8 and 7 are <u>factors</u> of 56.

8 and 7 are also <u>divisors</u> of 56 since

$$7\overline{)56} = 8,\quad 56,\quad r = 0 \qquad \text{and} \qquad 8\overline{)56} = 7,\quad 56,\quad r = 0$$

We know that 4 is a factor of 12 since $12 = 4 \cdot 3$. Also, 4 is a divisor of 12 since 12 divided by 4 equals 3 with remainder 0.

> If <u>a</u>, <u>b</u>, and <u>n</u> are natural numbers such that $n = a \cdot b$, then <u>a</u> and <u>b</u> are called <u>factors</u> or <u>divisors</u> of <u>n</u>.
>
> Example: If $21 = 3 \cdot 7$, then 3 and 7 are <u>factors</u> or <u>divisors</u> of 21.

<u>Exercise 7-2(a)</u>

<u>Note</u>: Exercises 1 through 5 refer to the definition shown above.

1. n = 56, a = 8. The missing factor b is ______________.
2. n = 24, b = 3. The missing divisor a is ______________.
3. n = 30. One pair of factors, a and b, is a = 15, b = 2.
4. n = 30. Another pair of divisors, a and b, is a = ________, b = ________.
5. n = 30. Another pair of factors, a and b, is a = ________, b = ________.
6. Circle the numbers from 1 to 12 which are factors of the numbers given. (a) has been done for you.

   a. 8: (1) (2) 3 (4) 5 6 7 (8) 9 10 11 12

   b. 12: 1 2 3 4 5 6 7 8 9 10 11 12

   c. 15: 1 2 3 4 5 6 7 8 9 10 11 12

   d. 21: 1 2 3 4 5 6 7 8 9 10 11 12

6. (continued from preceding page)

e. 27: 1 2 3 4 5 6 7 8 9 10 11 12

f. 23: 1 2 3 4 5 6 7 8 9 10 11 12

7. Circle the numbers to the right which have the numbers on the left as divisors. (a) has been done for you.

a.	2:	(2)	5	7	(10)	(12)	51	(44)	63
b.	3:	6	11	3	21	26	80	33	15
c.	4:	12	7	6	13	16	28	32	41
d.	5:	20	16	30	22	5	51	34	60
e.	6:	16	24	42	7	9	15	6	48
f.	7:	63	22	28	21	15	14	18	49
g.	8:	8	9	56	24	16	21	64	15
h.	9:	72	45	16	21	27	29	36	77
i.	1:	21	3	6	11	13	17	9	2

8. Write each of the following numbers as a product of 2 factors, each different from 1, if possible:

a. 12 = 4 · 3 ______

b. 36 = ______

c. 7 = 7 · 1 ______

d. 8 = ______

e. 11 = ______

f. 35 = ______

g. 39 = ______

h. 42 = 7 · 6 ______

i. 41 = ______

j. 50 = ______

It is sometimes useful to be able to find all the factors of a given number. For example, the set of all factors of 6 is {1, 2, 3, 6}.

when the numbers become larger, the complete set of factors is more difficult to find. Let us see if we can discover an easy way to find the factors of a number. For example, let us find all the factors of 36.

a. Start with 1. Is 1 a factor of 36? ans. ______. Then 1 times some number equals 36. What is that number? ans. ______

b. Try 2. Is 2 a factor of 36? ans. ______ Then 2 times some number equals 36. What is that number? ans. ______ Then 2 • ______ = 36.

c. Try 3. Is 3 a factor of 36? ans. ______ Then 3 times some number equals 36. What is that number? ans. ______ Then 3 • ______ = 36.

d. Try 4. Is 4 a factor of 36? ans. ______ Then 4 times some number equals 36. What is that number? ans. ______ Then 4 • ______ = 36.

e. Do you need to try 5? [Numbers which have 5 as a factor always end in 0 or 5.] 36 ends in a 6, so 5 is not a factor of 36.

f. Try 6. Is 6 a factor of 36? ans. ______ Then 6 times some number equals 36. What is that number?. ans. ______ Then 6 • ______ = 36.

g. We know 7 <u>is not</u> a factor of 36 because 5 • 7 = 35 and 6 • 7 = 42.

h. We know 8 <u>is not</u> a factor of 36 because 4 • 8 = 32 and 5 • 8 = 40.

i. Do you need to try 9? Why or why not? (Hint: See question d.)

You should now be ready to list the set of factors of 36.

{1 2 3 4 6 9 12 18 36}

Notice that the arrows show you every way of naming 36 as a product expression of two factors.

We see that the factors of a given number come in pairs of different factors (except for cases like 36 = 6 · 6 or other squares like 4, 9, 16, etc.). When we find <u>one</u> factor we usually find <u>two</u> factors. This fact helps us find factors quickly. We shall learn other facts later which will make the task even easier.

<u>Exercise 7-2(b)</u>

1. Explain in your own words what the following words mean: natural number, factor, whole number, divisor.

2. Complete the chart.

Number	Set of Factors	Number of Factors	Number	Set of Factors	Number of Factors
1	{1}	1	13		
2	{1, 2}	2	14	{1, 2, 7, 14}	4
3			15		
4	{1, 2, 4}	3	16		
5			17		
6			18		
7	{1, 7}	2	19		
8			20		
9			21		
10	{1, 2, 5, 10}	4	22		
11			23		
12			24		

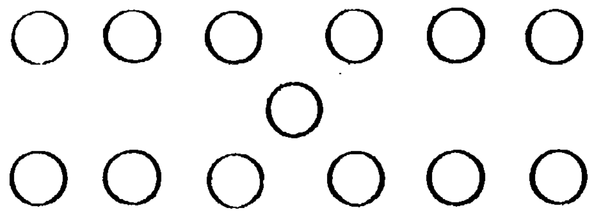
3. True or False? "If a and b are natural numbers and a is less than b, then a has fewer factors than b." The chart in Exercise 2 will help you with your answer.
4. Can you find a number which is contained in every set of factors.
5. Find the factors of the numbers listed: 27, 30, 36, 42, 48.

Section 7-3 Prime Numbers and Composite Numbers

The set of factors of 2 is {1, 2}. Also, the set of factors of 3 is {1, 3}, and the set of factors of 5 is {1, 5}. The set of factors of 4 is {1, 2, 4}, and the set of factors of 6 is {1, 2, 3, 6}. Let us make a list of sets of factors of some more numbers:

Set of factors of 7 is {1, 7}
Set of factors of 8 is {1, 2, 4, 8}
Set of factors of 9 is {1, 3, 9}
Set of factors of 10 is {1, 2, 5, 10}
Set of factors of 11 is {1, 11}
Set of factors of 1 is {1}

We see that some numbers have only two factors, the number itself and 1, while other numbers have more than two factors. A number which has exactly two different factors, the number itself and the number one, is called a prime number. Check to see that the first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. You may want to consult problem 2 in Exercises 7-2(b). Let us consider the prime number 13. Look at a group of 13 circles.



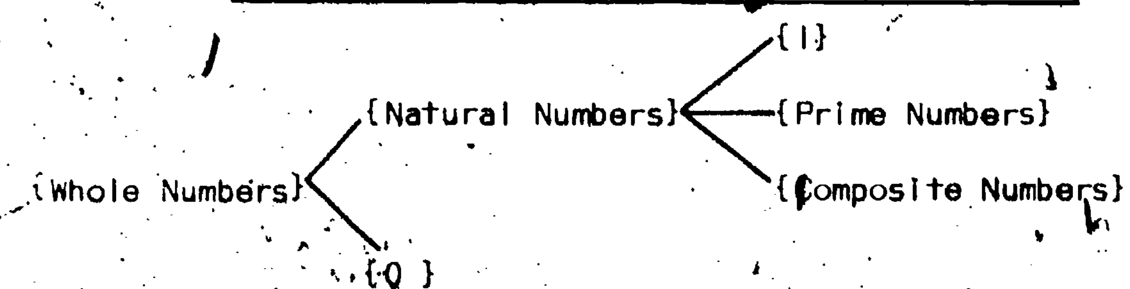
Try to see if there is any way you can divide the circles into groups of equal size. (Other than 1 group of 13 circles or 13 groups with 1 circle in each group) <u>Do you see that a prime number of objects can not be "split up evenly"</u>? This is another way of saying that a prime number has no factors other than itself and one.

Those numbers that have more than two different factors are called <u>composite numbers</u>. The set of the first five composite numbers is {4, 6, 8, 9, 10}. Can you list the next five composite numbers?

What can we say about the number 1? We notice that 1 is neither in the set of prime numbers, nor in the set of composite numbers. The set of factors of 1 is simply {1}. Hence the number 1 is <u>not</u> prime, and it is <u>not</u> composite.

In our discussion above we have put the natural numbers into three different sets: the set of <u>prime</u> numbers, the set of <u>composite</u> numbers, and the set consisting of the <u>number 1</u>.

> 1. A natural number which has exactly two different factors, itself and 1, is called a <u>prime number</u>.
> 2. A natural number which has more than two different factors is called a <u>composite number</u>.
> 3. The number 1 is neither composite nor prime.



A method for finding the prime numbers less than any given number was suggested by a Greek mathematician named Eratosthenes (276? - 195? B.C.) One way of describing the method is as follows:

1. Write all natural numbers from 2 to a given number (here, 100).

SIEVE OF ERATOSTHENES (Era-toss-the-knees)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

2. Circle 2. Now cross out all numbers with 2 as a factor which are greater than 2. That is, 4, 6, 8, ...
3. Circle 3. Now cross out all numbers with 3 as a factor which are greater than 3. That is, 6, 9, ...
4. The number 4 and all numbers with 4 as a factor have been crossed out. Why? ______________________
5. Circle 5. Now cross out all numbers greater than 5 which have 5 as a factor. That is, 10, 15, ...

6. Circle 7. Now cross out all numbers with 7 as a factor which are greater than 7. That is, 14, 21, 28, ...

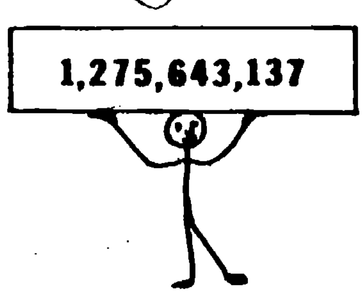
7. Circle 11. Cross out all numbers with 11 as a factor which are greater than 11. That is, 22, 33, 44, ...

   a. Did you cross out any new numbers when you considered numbers which have 11 as a factor? ans. ____________

      Why? ______________________________________

8. Continue in this manner. After a number has been circled cross out all numbers greater than the number which have it as a factor. Then circle the next number which has not been crossed out in any previous steps. Such a number has no factors smaller than itself, except 1. Hence, each of the circled numbers is a prime number.

This process is sometimes called the Sieve of Eratosthenes. Do you know what the word "sieve" means? Can you explain why it was chosen to describe this process?



Primes

{2, 3, 5, 7, 11, ...}

Composites

{4, 6, 8, 9, 10, ...}

Let us look again at the Sieve. Five of the ten columns were crossed out in the discussion of the number 2, that is, one half of the numbers here have 2 as a factor. It is useful to remember that except for 2, prime numbers must end in 1, 3, 5, 7, or 9.

> A natural number which has 2 as a factor is called an even number. One which does not have 2 as a factor is called an odd number.

Five of the ten columns are made up entirely of even numbers. Five of the ten columns are made up entirely of odd numbers. When we list the numbers in natural order, 1, 2, 3, 4, 5, 6, 7, 8, etc., we see that after each odd number comes an even number. And after each even number comes an odd number.

Exercise 7-3

1. Give 5 examples of each of the following types of numbers:

   a. prime ________________

   b. composite ________________

   c. even ________________

   d. odd ________________

2. List the first 15 prime numbers.

3. A = {even numbers}, B = {odd numbers}, C = {prime numbers}, D = {composite numbers}, E = {numbers with 3 as a factor}. Find the following sets and describe them as simply as you can.

   a. $A \cap B$ __________ e. $D \cup C$ __________

   b. $A \cup B$ __________ f. $A \cap C$ __________

   c. $C \cap E$ __________ g. $B \cap E$ __________

   d. $D \cap C$ __________ h. $B \cap C$ __________

4. Cross out the word which <u>does not</u> apply.

   a. It seems that the product of an odd number and even number is always an <u>odd</u>, <u>even</u> number.

   b. The product of two primes is never a <u>prime</u>, <u>composite</u> number.

   c. It seems that the sum of two even numbers is always an <u>odd</u>, <u>even</u> number.

   d. It seems that the sum of two odd numbers is always an <u>odd</u>, <u>even</u> number.

   e. It seems that the product of two odd numbers is always an <u>odd</u>, <u>even</u> number.

   f. If <u>n</u> is a prime number larger than 2, then <u>n</u> is an <u>odd</u>, <u>even</u> number.

5. We notice that 4 = 2 + 2   10 = 3 + 7

   6 = 3 + 3   12 = 5 + 7

   8 = 3 + 5   14 = 3 + 11

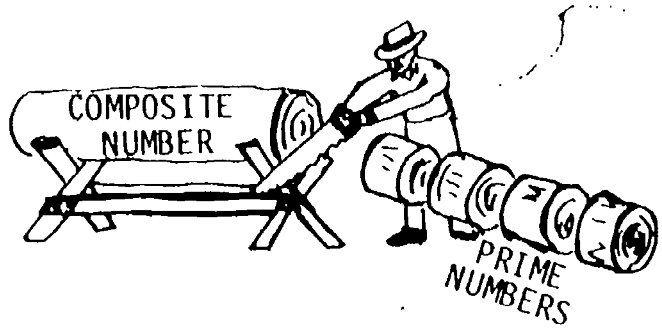
   We see that all the even numbers greater than 2 and less than or equal to 14 can be written as the sum of two prime numbers. Extend this list to include all even numbers less than or equal to 32.

16 = ________	22 = ________	28 = ________
18 = ________	24 = ________	30 = ________
20 = ________	26 = ________	32 = ________

   Do you think that every even number greater than 2 can be expressed as the sum of two primes? ____________________

6. Find two numbers which have 5 as a factor. Add the numbers together. Does the sum have 5 as a factor? Repeat this using 4, 7, 11, 12 instead of 5. Put into your own words a rule which seems to be present.

Section 7-4 Prime Factorization of Natural Numbers



Look at the following products of prime factors. Check the multiplication.

$12 = 2 \cdot 2 \cdot 3$ $\qquad$ $154 = 2 \cdot 7 \cdot 11$

$30 = 2 \cdot 3 \cdot 5$ $\qquad$ $80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

When a number has been written as a product of prime numbers, we say that we have found the prime factorization of that number, or that the number has been written as a product of prime factors. What numbers have been factored into prime factors below?

______ $= 2 \cdot 11 \cdot 13$ $\qquad$ ______ $= 3 \cdot 5 \cdot 5 \cdot 19$

______ $= 41 \cdot 43$ $\qquad$ ______ $= 17 \cdot 29 \cdot 5$

Can you find the prime factorizations of these numbers?

80 = ____________ $\qquad$ 63 = ____________

84 = ____________ $\qquad$ 29 = ____________

Did you have trouble factoring 29 into a product of primes? Of course, you found that 29 is already a prime number. When we talk about the prime factorization of a number, we shall mean a composite number.

Prime factorization can be used to find all the factors of a number quickly. The set of factors of 30 is {1, 2, 3, 5, 6, 10, 15, 30}

Do you agree? Write 30 as a product of primes and then check your result with the explanation which follows.

$30 = 2 \cdot 3 \cdot 5$ Factors with one prime, 2, 3, 5

6 15

$30 = 2 \cdot 3 \cdot 5$ Factors with 2 primes, 6, 10, 15

10

All factors have been found except 1 and 30. But any number always has itself and one as factors. We have found all of the factors of 30.

The factors of 36 are {1, 2, 3, 4, 6, 9, 12, 18, 36}. We shall find all but 1 and 36 by the method above. $36 = 2 \cdot 2 \cdot 3 \cdot 3$

$36 = 2 \cdot 2 \cdot 3 \cdot 3$ Factors with one prime, 2, 3

4 9

$36 = 2 \cdot 2 \cdot 3 \cdot 3$ Factors with 2 primes, 4, 6, 9

6

12

$36 = 2 \cdot 2 \cdot 3 \cdot 3$ Factors with 3 primes, 12, 18

18

Exercise 7-4

1. Find the prime factorization of each of the following composite numbers: 15, 16, 18, 20, 36, 48, 82, 154, 221.

2. Factor completely into a product of primes: 9, 12, 21, 24, 30, 42, 108, 125, 1,015.

3. Find the set of factors of 125, 18, 24, 108, and 1,015 by the method which uses prime factorizations. The prime factorizations were found in problems 1 and 2 of this exercise.

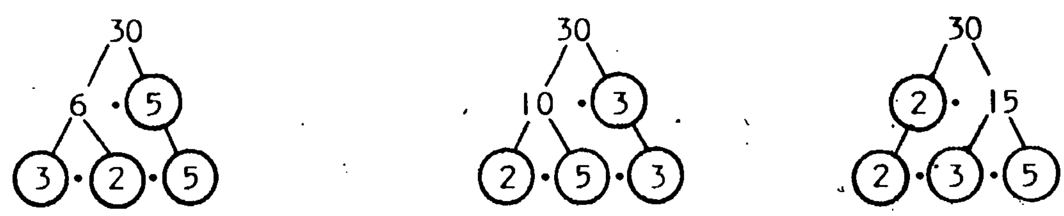
Section 7-5 Repeated Factoring: The Fundamental Theorem of Arithmetic

Every composite number is the product of smaller numbers. If one of these numbers is composite, then it also is the product of smaller numbers. If we continue this, we must come to a product expression in which no number is composite and every factor is prime. We have then found the prime factorization of the composite number.

Let us take a closer look at prime factorizations of composite numbers. Study the following:

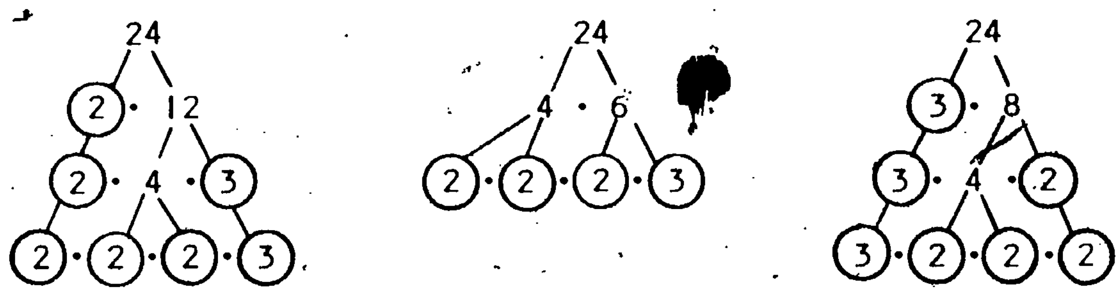
$30 = 6 \cdot 5$	$30 = 10 \cdot 3$	$30 = 2 \cdot 15$
$= (3 \cdot 2) \cdot 5$	$= (2 \cdot 5) \cdot 3$	$= 2 \cdot (3 \cdot 5)$
$= 3 \cdot 2 \cdot 5$	$= 2 \cdot 5 \cdot 3$	$= 2 \cdot 3 \cdot 5$

It seems that no matter "how" we begin to factor 30 into prime factors, we end with the same factorization. The following "factor trees" may help you understand the examples above.



Note that only prime factors are circled.

Let's use "factor trees" on 24 to factor it into prime factors.



Which primes occur in the factorization of 24?

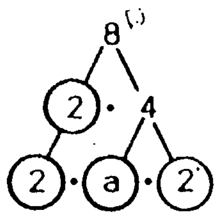
The fact that we obtain the same prime factors for 24 (or for 30) no matter how we carry out the factorization is an example of what is known as the <u>Fundamental Theorem of Arithmetic</u>. Stated simply, it says:

> Every composite number can be factored as a product of prime numbers in exactly one way, except for the order of the factors.

<u>Exercise 7-5</u>

Make copies of the following factor trees and find the missing numbers:

1.



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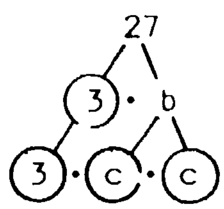

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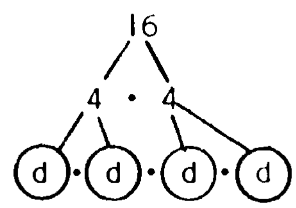

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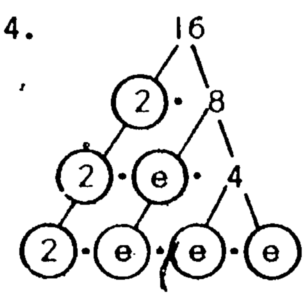
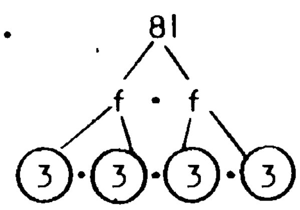
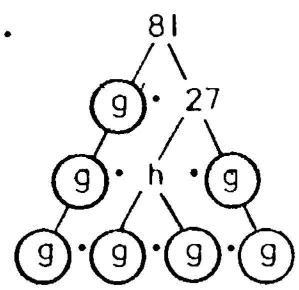


5.



6.



7. Make three different factor trees for 42.

8. Make four different factor trees for 36.

9. Make as many factor trees for 60 as you can.

10. Factor the following into primes by any method you wish.

    a. 10

    b. 15

c. 9

d. 100

e. 28

f. 16

g. 72

h. 81

i. 75

Section 7-6 Greatest Common Factor (GCF)

A number which divides two or more numbers is called a common factor of these numbers. For example, 2 divides 10 and 2 divides 12. So 2 is a common factor of 10 and 12. Generally, the greatest common factor is more useful in mathematics than other common factors. Therefore, we need to know how to find the greatest common factor.

Let's try an example.

A = {factors of 12} = {1, 2, 3, 4, 6, 12}

B = {factors of 18} = {1, 2, 3, 6, 9, 18}.

A ∩ B = {common factors of 12 and 18}, that is, factors that appear in both sets. A ∩ B = {1, 2, 3, 6}. The largest number of the set of common factors is 6. Therefore, the greatest common factor of 12 and 18 is 6, that is, the GCF = 6.

Let's try another example:

A = {factors of 24} = {1, 2, 3, 4, 6, 8, 12, 24}.

B = {factors of 60} = {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}.

A ∩ B = {common factors of 24 and 60}, that is, factors that appear in both sets. A ∩ B = {1, 2, 3, 4, 6, 12}. The largest number of the set of common factors is 12. Therefore, the greatest common factor is 12. The GCF = 12.

Exercise 7-6

1. Find the set of factors of each of the following:

   Example: 18: {1, 2, 3, 6, 9, 18}

   a. 6

   b. 8

   c. 12

   d. 15

   e. 16

   f. 21

2. Using your answers to Exercise 1, find the greatest common factor (G.C.F.) of the following sets of numbers:

   a. 6 and 8

   b. 8 and 12

   c. 12 and 15

c. 6, 8 and 12 (See problem 8 below for help.)

e. 12, 15 and 21

f. 8, 12 and 16

Find the G.C.F. of each of the following sets of numbers.

3. 15 and 25

4. 18 and 30

5. 24 and 36

6. 25 and 75

7. 32 and 48

8. 15, 30 and 36

Example: A = {factors of 15} = {1, 3, 5, 15}

B = {factors of 30} = {1, 2, 3, 5, 6, 10, 15, 30}

C = {factors of 36} = {1, 2, 3, 4, 6, 9, 12, 18, 36}

A ∩ B ∩ C = {common factors of 15, 30 and 36}

= {1, 3}

Greatest Common Factor = 3

9. 12, 24 and 48

10. 15, 30 and 45

Section 7-7 Multiples, Common Multiples and LCM

Start with {natural numbers} = {1, 2, 3, 4, 5, ...}. Let us form a new set as follows: we multiply each natural number

{1, 2, 3, 4, 5, ...} by 3 and get {3, 6, 9, 12, 15, ...}.
A = {3, 6, 9, 12, 15, ...} is called the set of multiples of 3. Each element of set A is called a multiple of 3. Each element of A has 3 as a factor. Can you tell why?

> If n, a and b are natural numbers, and n = a · b, then n is said to be a multiple of a and a multiple of b.
>
> Example: If 45 = 9 · 5, then 45 is a multiple of 9 and 5.

We thus see how the words factor and multiple are related. If a and b are factors of n, then n is a multiple of a and a multiple of b. Use this definition to fill in the following blanks:

6 = 3 · 2; therefore, 6 is a multiple of 3.

3 is a ________ of 6.

6 is a ________ of 2.

2 is a ________ of 6.

Since 2 and 3 are also called divisors of 6, it would be correct to say that 6 is divisible by both 2 and 3. However, we shall use the word multiple. So, instead of saying that 6 is divisible by 2 and 6 is divisible by 3, we shall say that 6 is a multiple of 2 and 6 is a multiple of 3. The following set is the set of multiples of a number. {6, 12, 18, 24, 30, 36, ...} Which number? Can you find a number in the set which is at the same time a multiple of 2 and a multiple of 3? You found more than one, didn't you?

You will find the following information very helpful when working with multiples of numbers. Study it carefully.

{1, 2, 3, 4, 5, 6, ...} — Multiply by 2 — B = {2, 4, 6, 8, 10, 12, ...}

{1, 2, 3, 4, 5, 6, ...} — Multiply by 3 — C = {3, 6, 9, 12, 15, 18, ...}

$$A = B \cap C = \{6, 12, 18, \ldots\}.$$

Do you see that A is the set of <u>common multiples</u> of 2 and 3? The smallest element in A is the <u>least common multiple</u> (LCM) of 2 and 3. What is it?

<u>12 x 12 Multiplication Table</u>

x	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Find the row which shows the first 12 multiples of 5.

Find the column which shows the first 12 multiples of 5.

Notice that the same set appears in each: {5, 10, 15, ..., 55, 60}.

Do you see that this table can be used to find the first 12 multiples of all numbers from 1 to 12?

Find the row which shows the first 12 multiples of 3.

Find the column which shows the first 12 multiples of 5.

15 appears in both the row and column. So it is a common multiple of 3 and 5. Is it the least common multiple?

The set of the first 12 multiples of 4 is {4, 8, 12, 16, ..., 48}.

The set of the first 12 multiples of 6 is {6, 12, 18, ..., 66, 72}.

The set of common multiples shown is {12, 24, 36, 48}. The least common multiple of 4 and 6 is 12.

Exercise 7-7

1. Use your multiplication table to list the first 12 multiples of:

   a. 2 ______________________________

   b. 3 ______________________________

   c. 5 ______________________________

   d. 6 ______________________________

   e. 8 ______________________________

   f. 7 ______________________________

   g. 9 ______________________________

   h. 12 ______________________________

2. Use the results of exercise 1 and list the common multiples of the pairs of numbers below which are less than 45.

a. 2 and 3 ____________________

b. 3 and 6 ____________________

c. 4 and 5 ____________________

d. 3 and 5 ____________________

e. 5 and 8 ____________________

f. 6 and 9 ____________________

3. If <u>n = a • b</u>, we know that <u>n</u> is a multiple of both <u>a</u> and <u>b</u>. Therefore, <u>it is a common multiple</u>. It is <u>not always</u> the <u>least</u> common multiple. For example, if n = 24, a = 4, b = 6, we see that 24 is a common multiple of 4 and 6. But it is not the least common multiple. Find the pairs below whose product, <u>a • b</u>, is the LCM (least common multiple) of the pair:

a. a = 2, b = 3, a • b = 6, LCM = ____________________

b. a = 3, b = 6, a • b = 18, LCM = ____________________

c. a = 4, b = 5, a • b = 20, LCM = ____________________

d. a = 6, b = 8, a • b = 48, LCM = ____________________

e. a = 3, b = 5, a • b = 15, LCM = ____________________

f. a = 6, b = 5, a • b = 30, LCM = ____________________

4. Find the least common multiple for each of the following:

<u>Example</u>: 3, 4 and 6

{multiples of 3} : {3, 6, 9, (12), ...}

{multiples of 4} : {4, 8, (12), 16, ...}

{multiples of 6} : {6, (12), 18, ...}

The least common multiple of 3, 4, and 6 is <u>12</u>.

4. (Continued from preceding page)

   a. 2, 3, and 4

   b. 8, 9, and 12

   c. 4, 5, and 6

   d. 4, 6, and 12

Section 7-8 LCM: Larger Numbers and An Easier Way

So far we have found LCM's only for two or three numbers where each number is less than 13. But the method used before can also be used on numbers greater than 12. Let us find the LCM of 15 and 20. We think as before:

A = {multiples of 15}

B = {multiples of 20}

A ∩ B = {common multiples of 15 and 20}.

From set A ∩ B we pick the smallest member.

This is the LCM of 15 and 20.

{1, 2, 3, 4, 5, ...} — multiply by:

- 15 → A = {15, 30, 45, 60, 75, 90, 105, 120, ...}
- 20 → B = {20, 40, 60, 80, 100, 120, 140, ...}

A ∩ B = {60, 120, ...}. The LCM of 15 and 20 = 60.

Let us see if we can find an easier and quicker method of finding the least common multiple of a set of numbers, for example, 8 and 12. We shall use the Fundamental Theorem of Arithmetic.

Consider the following:

Does 8 = 2 · 2 · 2? ans. ______

Does 12 = 2 · 2 · 3? ans. ______

Now let us consider the prime factorization of 12, that is, 2 · 2 · 3.

Is 2 · 2 · 3 a multiple of 12? ans. ______

Is 2 · 2 · 3 a multiple of 8? ans. ______

In the product of primes, 2 · 2 · 3, what other factor is needed so that it is a multiple of 8? ans. ______

Suppose we take the product of primes, 2 · 2 · 3 and place in it another factor of 2. Then the product of primes will look like this: 2 · 2 · 2 · 3.

Now consider the following:

12 will divide 2 · (2 · 2 · 3) because the product of primes within the parentheses is another name for 12. 8 will divide (2 · 2 · 2) · 3 because the product of primes within the parentheses is another name for 8. Therefore, the least common multiple of 8 and 12 is 2 · 2 · 2 · 3 = 24.

Let us carefully follow one more example of the short method.

Example: Find the least common multiple of 9 and 15.

9 = 3 · 3

15 = 3 · 5

Consider 3 · 5.

3 · 5 is a multiple of 15 because 3 · 5 is another name for 15.

3 · 5 is not a multiple of 9. To be divisible by 9 the product of primes must have two factors of 3, that is, 3 · 3.

To make 3 · 5 a multiple of 9, place one more factor of 3 in the product of primes, that is, 3 · 3 · 5.

Now, 3 · 3 and 3 · 5 will both divide 3 · 3 · 5.

Therefore, the least common multiple of 9 and 15 is 3 · 3 · 5 = 45.

When you write the problem it should look something like this:

9 = 3 · 3

15 = 3 · 5

LCM = 3 · 3 · 5 = 45

Exercise 7-8

Find the LCM of the following using prime factorizations:

1. 12 and 16 12 = 2 · 2 · 3; 16 = 2 · 2 · 2 · 2

   The LCM needs 4 factors of 2 to insure that 16 divides it and 1 factor of 3 to make certain that 12 divides it. Hence, the LCM of 12 and 16 is 2 · 2 · 2 · 2 · 3 = 48.

2. 14 and 18

3. 10 and 14

4. 16 and 18

5. 12 and 17

6. 60 and 36

7. BRAINBUSTER: 100, 250, and 200

Review Exercise 7-9

1. We have studied the following types of numbers in this chapter:

a. whole numbers b. natural numbers c. even numbers

d. odd numbers e. prime numbers f. composite numbers

Next to each of the numbers below place those letters from the above list which apply.

Example: 17 a, b, d, e

a. 2 ______ d. 29 ______

b. 15 ______ e. 24 ______

c. 1 ______ f. 0 ______

2. Name 3 factors, different from 1, of each of the following.

Problems (a) and (b) are done for you.

a. 12: 2, 3, and 6 are factors of 12.

b. 21: 3, 7, and 21 are factors of 21.

c. 14:

d. 16:

e. 24:

f. 27:

g. 32:

3. Name 3 multiples, different from the number itself, of each of the following. Problems (a) and (b) are done for you.

   a. 6: 12, 18, and 24 are multiples of 6.

   b. 11: 22, 33, and 44 are multiples of 11.

   c. 7:

   d. 12:

   e. 9:

   f. 5:

4. List all prime numbers between 1 and 50.

   ____

5. List all the multiples of 5 which are less than 61. __________

   ____

6. List the set of numbers less than 50 which are multiples of 7.

   ____

7. List the set of numbers which are less than 100 and are also multiples of both 3 and 5. ______________________

8. List the set of all common factors for each of the following:

   a. 18 and 42 ______________________________

   b. 21 and 33 ______________________________

9. Find the greatest common factor of:

   a. 18 and 42

   b. 28 and 56

10. Find the least common multiple of the following sets of numbers:

a. 8 and 10

b. 12 and 15

c. 10, 15, and 30

11. Using any method you wish, find the prime factorization of the following:

a. 105

b. 42

c. 300

d. 64

BRAINBUSTER SKILLS

The following are statements about natural numbers which mathematicians have proved to be true. Check the correctness of the given statements with examples.

12. True statement I: If a and b are natural numbers, if G stands for the greatest common faotor of a and b, and if L stands for the lest common multiple of a and b, then $a \cdot b = G \cdot L$.

<u>Example</u>: a = 6, b = 15. Then G = 3, L = 30.

a · b = 6 · 15 = 90

G · L = 3 · 30 = 90 Then <u>a · b = G · L</u>.

a. Let a = 3, b = 5. Find G and L and see if <u>a · b = G · L</u>.

b. Let a = 8, b = 20. Find G and L and show that <u>a · b = G · L</u>.

13. <u>True Statement 2</u>: If 9 is a factor of a number, then 9 divides the sum of the digits of the number; and if 9 divides the sum of the digits of a number, then 9 is a factor of the number.

<u>Examples</u>:

9 is a factor of 99; 9 + 9 = 18; 9 divides the sum of the digits, 18.

9 is a factor of 45; 4 + 5 = 9; 9 divides the sum of the digits, 9.

a. Decide without dividing if 9 is a factor of 510,211.

b. Decide without dividing if 9 is a factor of 2,115.

c. Check (a) and (b) by division.

14. <u>True Statement 3</u>: If <u>a</u> is a factor of <u>b</u>, and if <u>b</u> is a factor of <u>c</u>, then <u>a</u> is a factor of <u>c</u>.

<u>Example</u>: a = 2, b = 6. Then a is <u>a factor of b</u>.

If c = 12, 6 is a factor of 12. <u>b is a factor of c</u>.

Notice 2 is a factor of 12. So a is <u>a factor of c</u>.

a. a = 5, supply b and c to show the truth of this statement.

b. a 7, b 14. Find c such that b is a factor of c.
Does a divide c?

15. True Statement 4: The greatest common factor of two different prime numbers is 1.

Show this is true for:

a. 5 and 7

b. 11 and 13

16. True Statement 5: If 4 is a factor of a number, then 4 divides the number formed by the last 2 digits; and if 4 divides the number formed by the last two digits, then 4 is a factor of the number.

4 divides 132 since 4 divides 32, the number formed by the last 2 digits.

Does 4 divide 176,930? ________ Why? ________________

Does 4 divide 624? ________ Why? ________________