DOCUMENT RESUME

ED 190 389 SE 031 434

AUTHOR Bolduc, Elroy: And Others
TITLE Mathematics Through Science, Part II: Graphing, Equations and Linear Functions. Teachers' Commentary. Preliminary Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 63
NOTE 95p.: For related document, see ED 143 539. Contains light and broken type. Best copy available.

EDRS PRICE MF01/PC04 Plus Postage.
DESCRIPTORS *Algebra: Experiments: Graphs: Interdisciplinary Approach: *Junior High Schools: Laboratories: *Mathematical Applications: Mathematics Curriculum: Mathematics Instruction: *Physical Sciences: Science Experiments: Secondary Education: *Secondary School Mathematics: Teaching Methods: Unified Studies Curriculum
IDENTIFIERS *Functions (Mathematics): *School Mathematics Study Group

ABSTRACT

These units were produced by a team exploring the possibility of developing some of the basic concepts of mathematics through simple, but significant physical science experiments. The purpose of the entire project is to see if learning and understanding of mathematics can be improved by the use of these science-related units in mathematics classes, grades 7-9. Previous knowledge of science on the part of the students or teachers is not necessary. The units supplement but do not replace, whatever mathematics textbooks are in use. The material can be inserted in a mathematics course for a two- or three-week period over the school year. The use and need for graphing sets of ordered pairs of numbers are concepts in mathematics which can be drawn from the science experiments. Other useful mathematical concepts derived from the science data are those of: (1) drawing a line of best fit from a series of graphed ordered pairs: (2) the slope concept and its physical and mathematical meaning: (3) developing the equation of the graph: (4) interpolation and extrapolation from the graph and from the equation: and (5) a mathematical generalization and its use in problem solving. (Author/MK)

SCHOOL MATHEMATICS STUDY GROUP

MATHEMATICS THROUGH SCIENCE

PART II: GRAPHING, EQUATIONS AND LINEAR FUNCTIONS

TEACHERS' COMMENTARY

(Preliminary Edition)

SMSG

2

MATHEMATICS THROUGH SCIENCE

Part II: Graphing, Equations and Linear Functions

Teachers' Commentary

(Preliminary Edition)

The following is a list of all those who participated in the preparation of this volume:

Elroy Bolduc, Norwich Free Academy, Norwich, Connecticut

Bernard Friedman, University of California, Berkeley, California

T. H. Geballe, Bell Telephone Laboratories

A. Lloyd Gelmers, Ponca City Senior High School, Ponca City, Oklahoma

Melvin Greenstadt, Fairfax High School, Los Angeles, California

E. E. Hammond, Jr., Phillips Academy, Andover, Massachusetts

Neal Haygood, Hoffman-Boston High School, Arlington, Virginia

R. R. Kadesch, University of Utah

Frank Lindsay, California State Department of Education

Eugene Meyer, S.M., St. Mary's University, San Antonio, Texas

Robert Starkey, Cubberly High School, Palo Alto, California

Bryan Thwaites, The University, Southampton, England

Anna Maria Tuske, Jordan Junior High School, Palo Alto, California

James H. Zant, Oklahoma State University

William Sandmann, Harvey Mudd College, Claremont, California

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

TEACHER'S COMMENTARY

MATHEMATICS THROUGH SCIENCE

Foreword

The units described in this Teacher's Commentary are the result of a summer's work by a writing team of fifteen competent mathematicians and scientists who spent the summer of 1963 working on this project under the direction of the School Mathematics Study Group. This team explored the possibility of developing some of the basic concepts of mathematics through simple, but significant, physical science experiments. The purpose of the entire project, both the writing of the teaching material and its experimental use in a limited number of centers over the country, is to see if learning and understanding of mathematics can be improved by the use of these units in mathematics classes, grades 7-9.

It is emphasized that the units are to be used in the mathematics classroom and that they are primarily designed to teach mathematical concepts rather than those of science. It is true that the procedures and principles of science included are sound and correct within the framework in which they are used. The experience in science which the student will gain from the study of these units will no doubt be useful in subsequent courses in the physical sciences, but the main purposes of the units are to teach mathematics.

The composition of the writing team and the procedures used in developing the units are significant. The team was composed of seven mathematicians (three college teachers and four high school teachers), five physicists (two college teachers, two high school teachers and one industrial physicist), two chemists (both high school teachers) and a coordinator of the team, from a state department of education. Some of the high school teachers have taught in more than one field, but they are listed here in their main field of interest and training.

The individual units or blocks of units were produced by smaller groups working cooperatively to perform the experiments and write them. In all cases these small writing teams consisted of representatives from both science and mathematics and of both college teachers and high school teachers. In this way it was possible to take advantage of their knowledge and experience at both levels. A college mathematician served as editor of the whole series of units, but preliminary drafts were read and criticized by members of the entire writing team.

Previous knowledge of science on the part of the students or teachers is not necessary, though some knowledge in this field will be useful. The experimental procedure is clearly described; lists of needed equipment and a kit of apparatus will be supplied. The experiments will involve basic measurements of length, mass, time and temperature. Though most of the experiments will be described in terms of the metric system of measurement, British units are not ignored. It is assumed that these experiments can be done in the mathematics classroom. Some will be done by all of the students working in small groups, and some will have to be demonstrations done by the teacher.

The use of the data collected is the important aspect of each experiment. These data will be measurements which can be graphed on a single number line or on a set of rectangular coordinates. Hence, the use and need for graphing sets of single numbers and sets of ordered pairs of numbers will be concepts in mathematics which can be drawn from the science experiments. Other useful mathematical concepts derived from the science data are those of: drawing a line of best fit from a series of graphed ordered pairs; the slope concept and its physical and mathematical meaning; developing the equation of the graph; interpolation and extrapolation from the graph and from the equation; and finally, a mathematical generalization and its use in problem solving.

Use in the Schools

When any part of this material is used in selected schools over the country during 1963-64, it will be inserted in a mathematics course for two- or three-week periods over the school year. The teachers in the system will decide on times suitable for this. Prior or current use of SMSG mathematics textbooks is not essential. The units supplement, but do not replace, whatever mathematics textbooks are in use. It seems probable that some portions of the regular textbooks may be omitted, provided teachers are satisfied that the treatment of topics developed directly from experiments clearly suffices.

EQUIPMENT LIST

Part II

Chapter 1
RELATIONS, FUNCTIONS AND GRAPHING

Students are to work in groups of two. Each group should have the following equipment.

1. Cantilever Experiment

   Science Supply

   none

   Local Supply

   5 books, identical (with dust jackets, if possible)

   5 wooden boards (1" × 8" × 10")

   1 foot ruler, with metric scale

2. Irregular Bottle Experiment

   Science Supply

   none

   Local Supply

   1 irregular-shaped bottle (12 fluid oz.)

   1 plastic pill bottle (approximately 20 cm^3)

   1 "Magic Mending" tape, Scotch brand ($\frac{3}{4}$" × 1296")

   1 foot ruler, with metric scale

   2 #10 can (water container)

Chapter 1

RELATIONS, FUNCTIONS, AND GRAPHING

In this chapter, three mathematical concepts are developed: (1) the definition of a relation, (2) the definition of a function, and (3) graphing order pairs by a coordinate system in a plane. In addition to these three concepts, it is hoped that the student will gain a strong feeling for the meaning of the domain and range of relations and functions. We also want to induce an intuitive feel, on the part of the student, for a function which is continuous as opposed to one which is not. On the basis of this intuition he should be able to see the reasoning behind graphing some functions as a "best" line which is drawn without removing the pencil from the paper as distinguished from a function whose graph is a discrete set of points.

This material is developed with the aid of two rather simple experiments: (1) the cantilever of books, and (2) the graduating of an irregular bottle. The first experiment deals with a function whose range and domain are discrete. The second experiment should convey a strong feeling that the function resulting from this experiment is continuous. Both experiments should point out to the student a need to consider very carefully all limitations placed upon the domain and range of a function or a relation.

A good reference for relations, functions, domain, range and mappings may be found in Principles of Mathematics, Allendoerfer and Oakley, McGraw-Hill, 1963, Chapter 6.

The method by which ordered pairs are graphed as arrows connecting elements of the domain to the corresponding elements of the range may at first appear to be a mapping but this is not necessarily true. A mapping of a set A into a set B is defined as a correspondence that associates with each element a of A a unique element b of B. We call b the image of a under this mapping. In some of the graphs illustrated in the text, more than one arrow had its origin at a given element in the domain and, hence, this graph was not a mapping of a set of ordered pairs.

You will notice that in the discussion of the coordinate system in a plane care was taken not to mention the x-axis or the y-axis. It was desired to give the student the feeling that the label attached to the horizontal axis and to the vertical axis would be dependent upon the sets which represent the domain and the range of the relation being graphed.

The fact that the horizontal axis is used to represent the domain is merely a convention and should be recognized as such. In some cases it may even be desirable to interchange the axis of the domain with the axis of the range. For example, if air temperature is to be plotted as a function of height, a stronger feeling for the physical situation may be gained by plotting height along the vertical axis instead of the horizontal axis.

It would be most desirable to keep the vocabulary as simple as possible. Such words as abscissa, ordinate, and variable have not been used for this reason. Also, the word variable introduces a generally obscure feeling for its definition. Bertrand Russell, who probably investigated the aspects of variables more thoroughly than anyone before him, said: "Variable is perhaps the most distinctly mathematical of all notions, it is certainly also one of the most difficult to understand. . . and in the present work [The Principles of Mathematics, 1903] a satisfactory theory as to its nature, in spite of much discussion, will hardly be found."

1.4 An Experiment: Cantilever of Books

In this experiment, a set of 5 or more books of the same size should be used (1 x 8 boards about 10" long can be substituted for the books).

For a reference which discusses the center of gravity of such a cantilever of books refer to Matter, Earth, and Sky by George Gamow, Prentice-Hall Inc., 1958, pp. 14-16.

The purpose of the experiment is to collect a set of ordered pairs and develop an understanding of the definition of a relation. The groundwork should be laid for the later development of the definition of a function. In this experiment, the student should be led to realize that both the domain and the range of this "relation" are discrete. The objects to be balanced must all be of the same size and, hence, using "half" books or "half" boards may affect the description of the objects in the domain of this relation but the numbers assigned to the domain will still be a subset of the counting numbers.

This experiment could probably be treated as a teacher demonstration. However, a strong plea is made for having each student actually perform the experiment, collect a set of data and graph the ordered pairs, first by the method developed in Section 4.5 and later by the method discussed in Section 4.10.

1.7 More Graphing

This section starts with the graduation of an irregular container. The equipment should be kept as simple as possible. What is needed is any irregularly shaped glass bottle, a 22 cc. plastic pill bottle (this can be obtained from your local drug store), and Scotch magic mending tape or masking tape. The irregular bottle was chosen in order to provide the student with a set of data which would not suggest a linear function. The student should be encouraged to develop a strong intuitive sense for having collected only a discrete sampling of the set of ordered pairs represented here. The act of displaying all ordered pairs of this function as a "mapping" from one number line to another number line gets us nowhere at this point. It is necessary, then, to develop a new method for displaying the ordered pairs of a function, or relation. The rectangular coordinate system is introduced. All four quadrants are discussed. If your class has had very little or no experience with negative numbers you may wish to pause here and introduce some elementary concepts of the set of negative numbers or you may wish to omit Section 4.9 altogether. Either approach may be used without having serious difficulties arise in the remaining Section 4.10 of the chapter.

Chapter 1
SOLUTIONS TO EXERCISES

Solutions to Exercise 1

1. (January, 31 days) (February, 29 days) (March, 31 days) (April, 30 days) (May, 31 days) (June, 30 days) (July, 31 days) (August, 31 days) (September, 30 days) (October, 31 days) (November, 30 days) (December, 31 days)

   (a) There are 12 ordered pairs in the set.

   (b) There are 12 elements in the domain of the relations. January, February, March, etc.

   (c) Set of elements in the range: {29,30,31}

   (d) There are 3 elements in the range of the relation.

2. Yes, 1-C would have been answered differently, since the set of elements in the range would be {28,30,31}. February has 28 days except during a leap year.

3. There are many relations that can be obtained from the information given in problem 3. These answers are 2 of the many.

   {(7,B), (7,C), (10,C), (22,A), (1,D)}

   {(7,B), (10,B), (7,A), (10,A), (7,C), (10,C), (7,D)}

4. The SMSG text Mathematics for Junior High School Volume 2 Part II was used to illustrate a possible solution to this exercise.

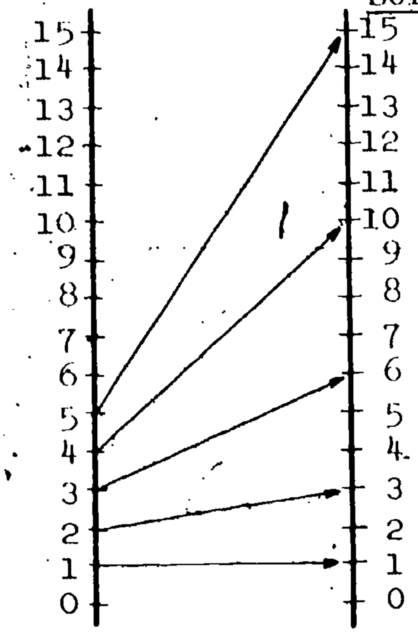
   (a) (additive inverse,237), (multiplicative inverse,237) (repeating decimal,247), (terminating decimal,247)

   (b) (exponent,121), (exponent,126), (exponent,129), (exponent,149)

11

Solutions to Exercise 2

1.

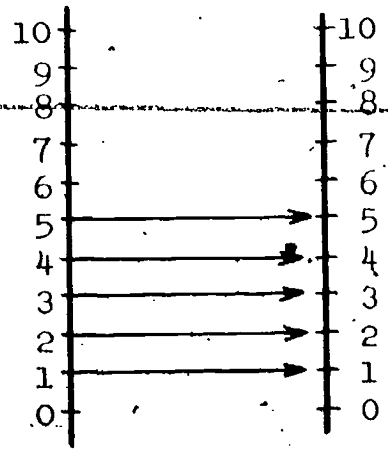


(a) domain: {1,2,3,4,5}

(b) range: {1,3,6,10,15}

(c) 21 could be a good guess since the graph shows that the difference of the elements in the range for succeeding order pairs increases by one as the element in the domain increases by a value of one.

2.

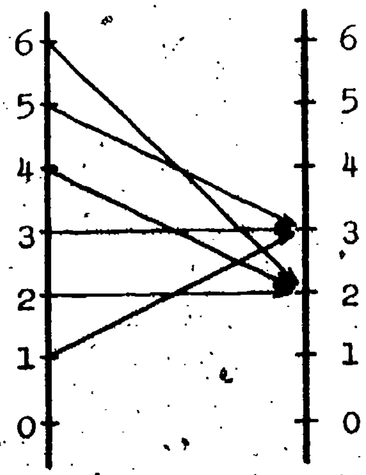


(a) domain: {1,2,3,4,5}

(b) range: {1,2,3,4,5}

(c) No. The element 6 is not contained in either the set of elements of the domain or the set of elements of the range.

3.



(a) domain: {1,2,3,4,5,6}

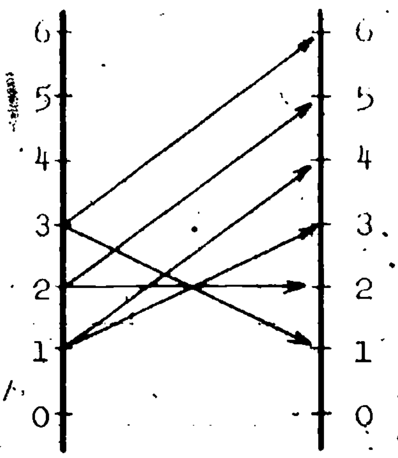
(b) range: {2,3}

(c) 3

(d) 2

(e) If the element of the domain is an even number, then the element of the range assigned to it is the number 2. If the element of the domain is an odd number, then the element of the range assigned to it is the number 3.

4.



(a) domain: {1,2,3}

(b) range: {1,2,3,4,5,6}

(c) 0 or 7

5. The ordered pairs are:

(30,85), (40,90), (50,105), (60,120), (70,140)

domain: {30,40,50,60,70}

range: {85,90,105,120,140}

6. The ordered pairs are:

$(9,2)$, $(12,\frac{3}{2})$, $(15,1)$, $(18,\frac{3}{4})$, $(21,\frac{5}{4})$, $(24,\frac{7}{8})$

domain: {9,12,15,18,21,24}

range: $\{2,\frac{3}{2},1,\frac{3}{4},\frac{5}{4},\frac{7}{8}\}$

7. The ordered pairs are:

(1,2.2), (1.1,2,2), (1.2,2.4), (1.3,2.3), (1.4,2.5)
(1.5,2.3), (1.6,2.7), (1,7,2.6)

domain: {1,1.1,1.2,1.3,1.4,1.5,1.6,1.7}

range: {2.2,2.3,2.4,2.5,2.6,2.7}

8. The ordered pairs are:

(.50,1960), (.50,1961), (.51,1962), (.52,1963), (.53,1964)
(.54,1965)

domain: {.50,.51,.52,.53,.54}

range: {1960,1961,1962,1963,1964,1965}

9. The ordered pairs are:

(3,1), (3,1.6), (4,1.2), (4,1.8)

domain: {3,4}

range: {1,1.2,1.6,1.8}

13

10. The ordered pairs are:

(2,3); (4,3), (6,7), (8,7), (10,11), (12,11)

domain: {2,4,6,8,10,12}

range: {3,7,11}

Solutions to Exercise 2

1. Set of all candidates is the domain.
   Set of all votes cast is the range.

2. Set of all triangles is the domain.
   Set of all areas is the range.

3. Set of all people is the domain.
   Set of all first names is the range.

4. Set of all counting numbers is the domain.
   Set of all counting numbers <u>except</u> <u>1</u> is the range.

5. Set of all counting numbers is the domain.
   Set of all squares of the counting numbers is the range.

6. Set of all positive integers is the domain.
   range: {1,2,3,4,0}

7. Set of all positive numbers is the domain.
   Set of all positive numbers is the range.

8. A function is described because <u>one</u> and only <u>one</u> element of the range is assigned to <u>each</u> element of the domain.

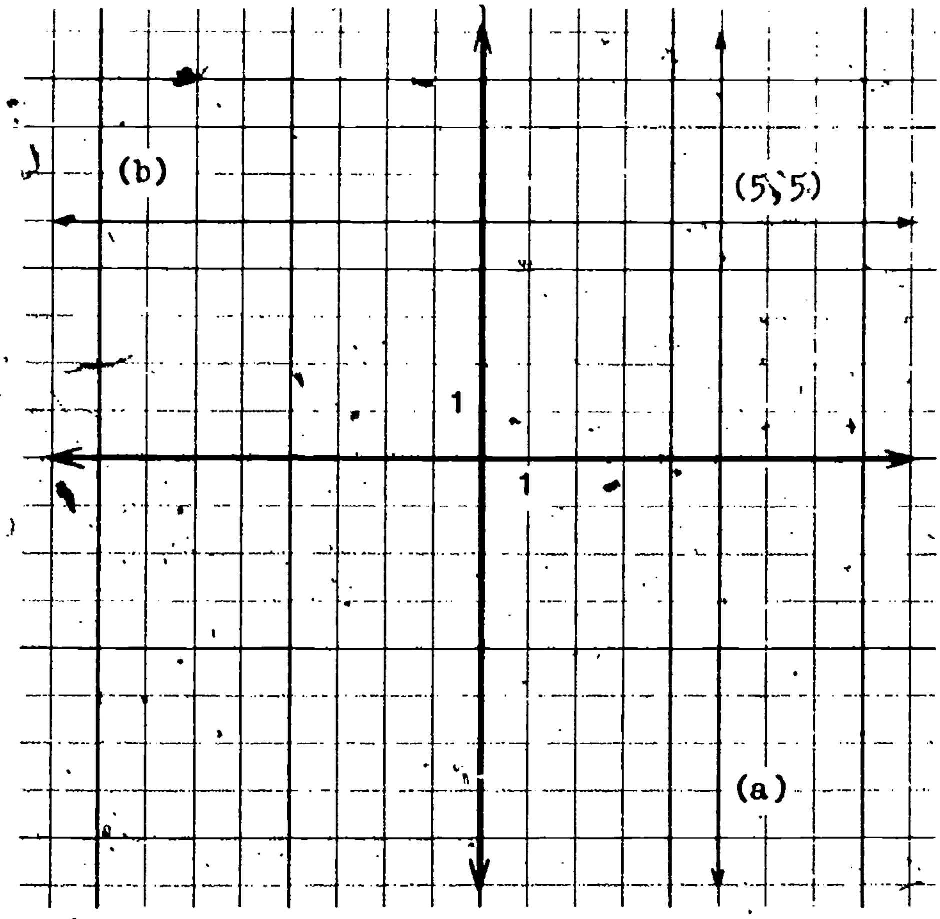
   The set of all positive numbers less than or equal to 20 is the domain.
   The set of all positive multiples of 5 is the range.

   (3.7,60); (5,75), (19.2,300)

14

Solutions to Exercise 4

1.

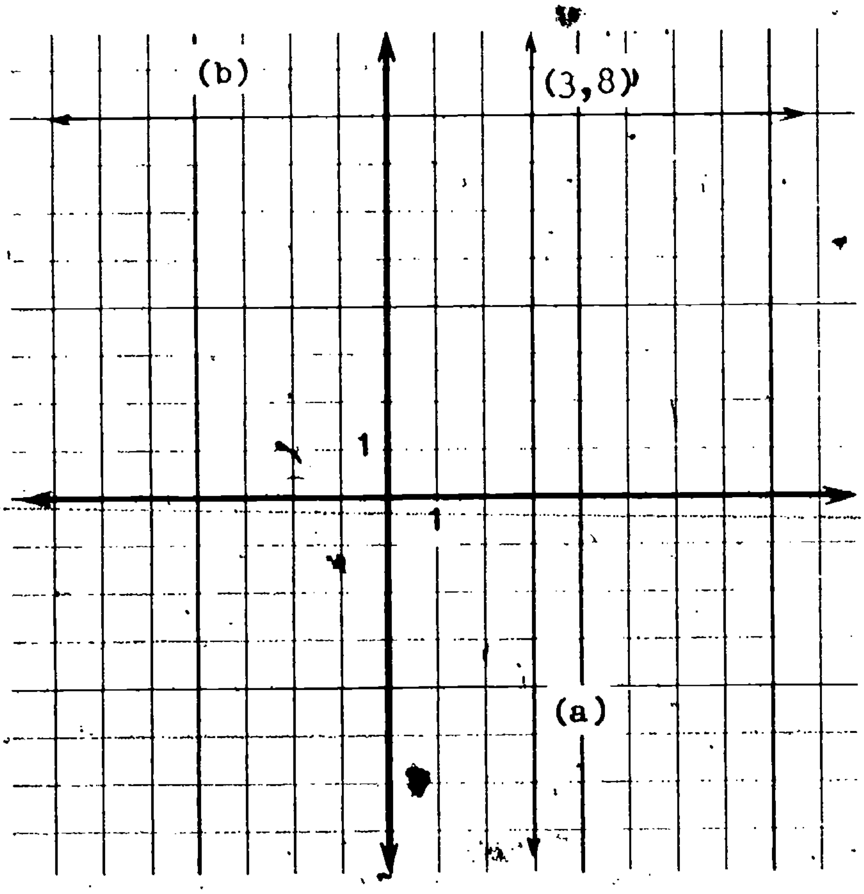


(c) The two sets have one point in common.

(d) (5,5)

15

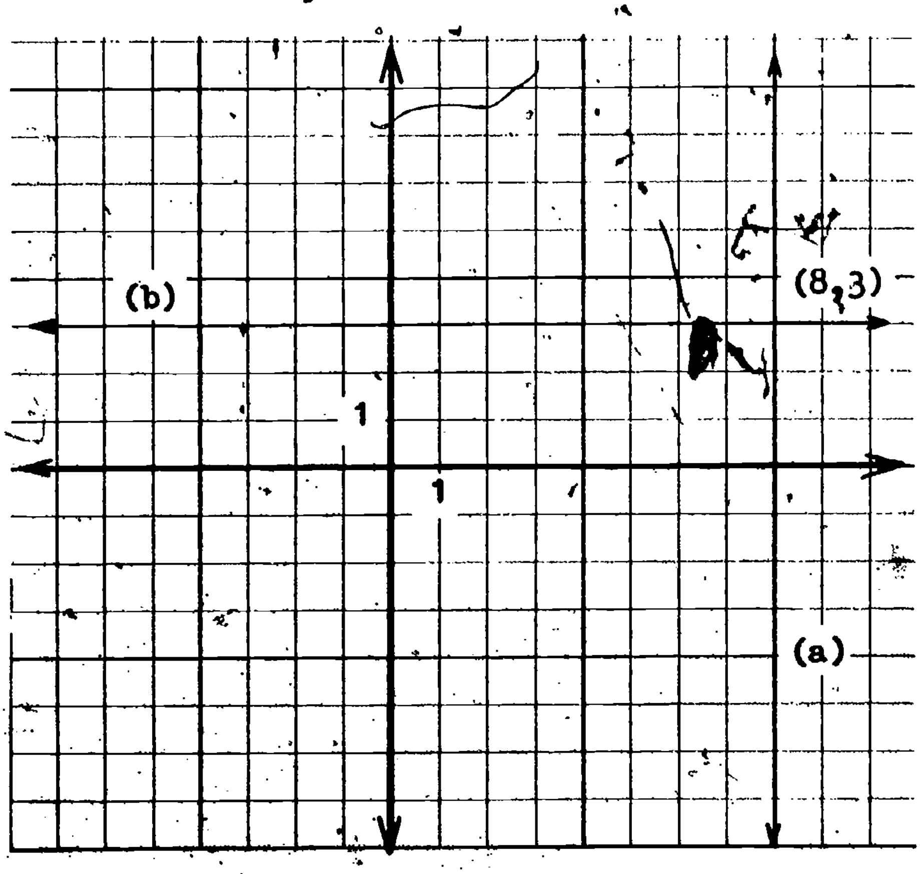
2.



(c) The two sets have <u>one point</u> in common.

(d) (3,8)

3.



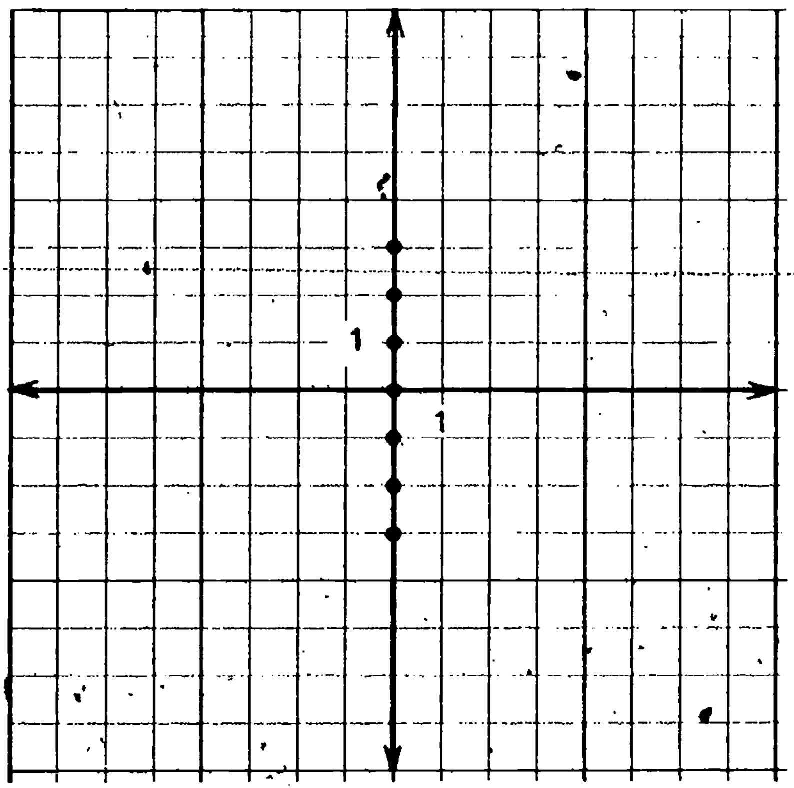
(c) The two sets have <u>one point</u> in common.

(d) (8,3)

16

4. No. The domain in Exercise 2 is 3, while the domain in exercise 3 is 8. The range in Exercise 2 is 8 while the range in Exercise 3 is 3.

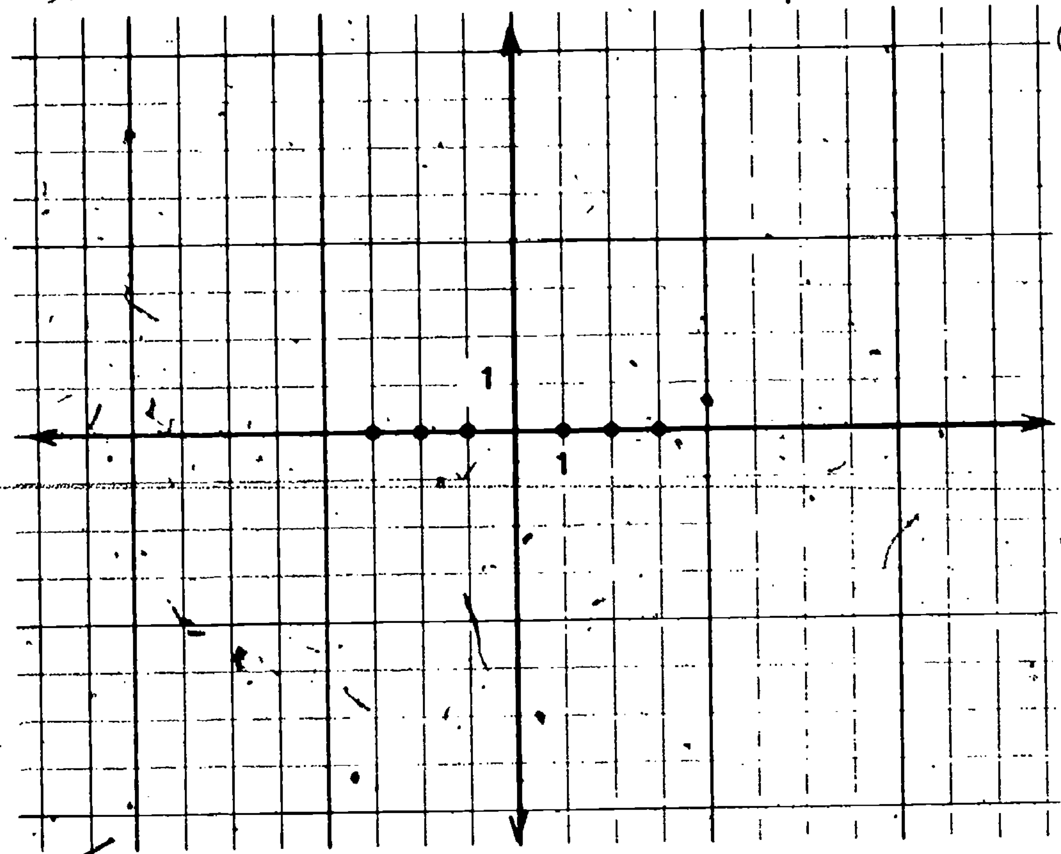
6.



(b) Yes

(c) Each point has the horizontal coordinate 0.

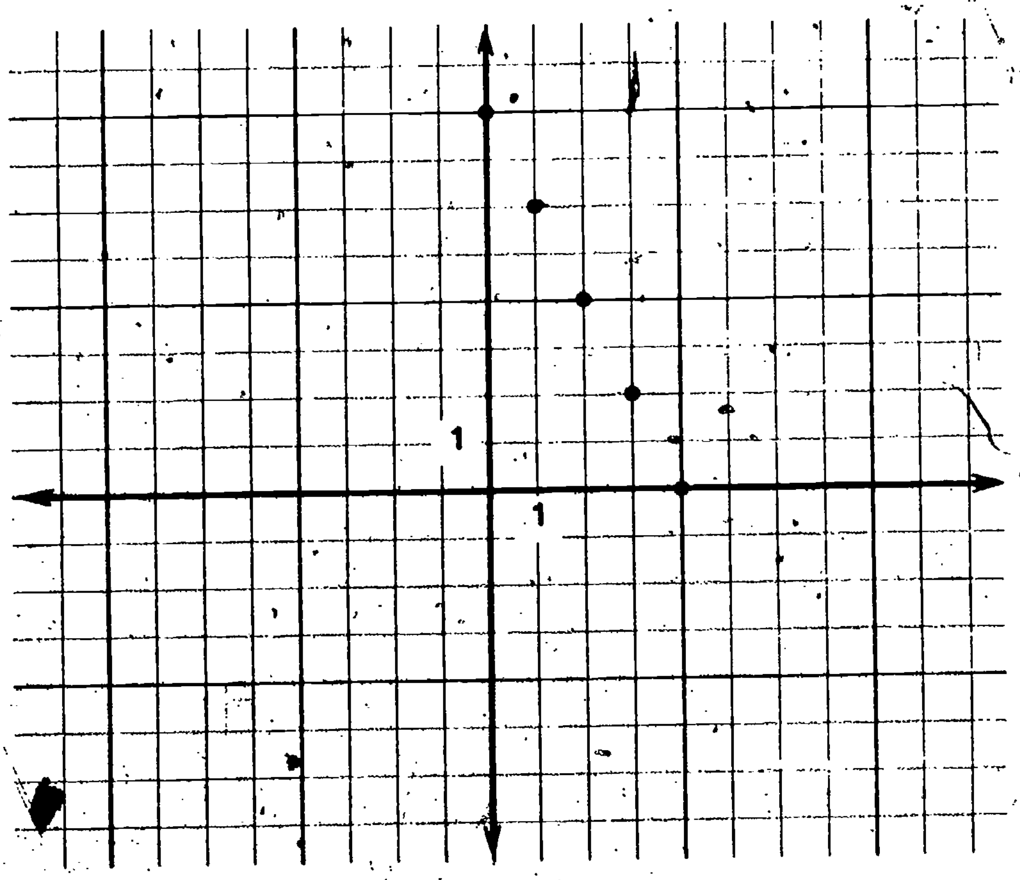
5.



(b) Yes

(c) The vertical co-ordinate for each point is 0.

7.



(b) Yes

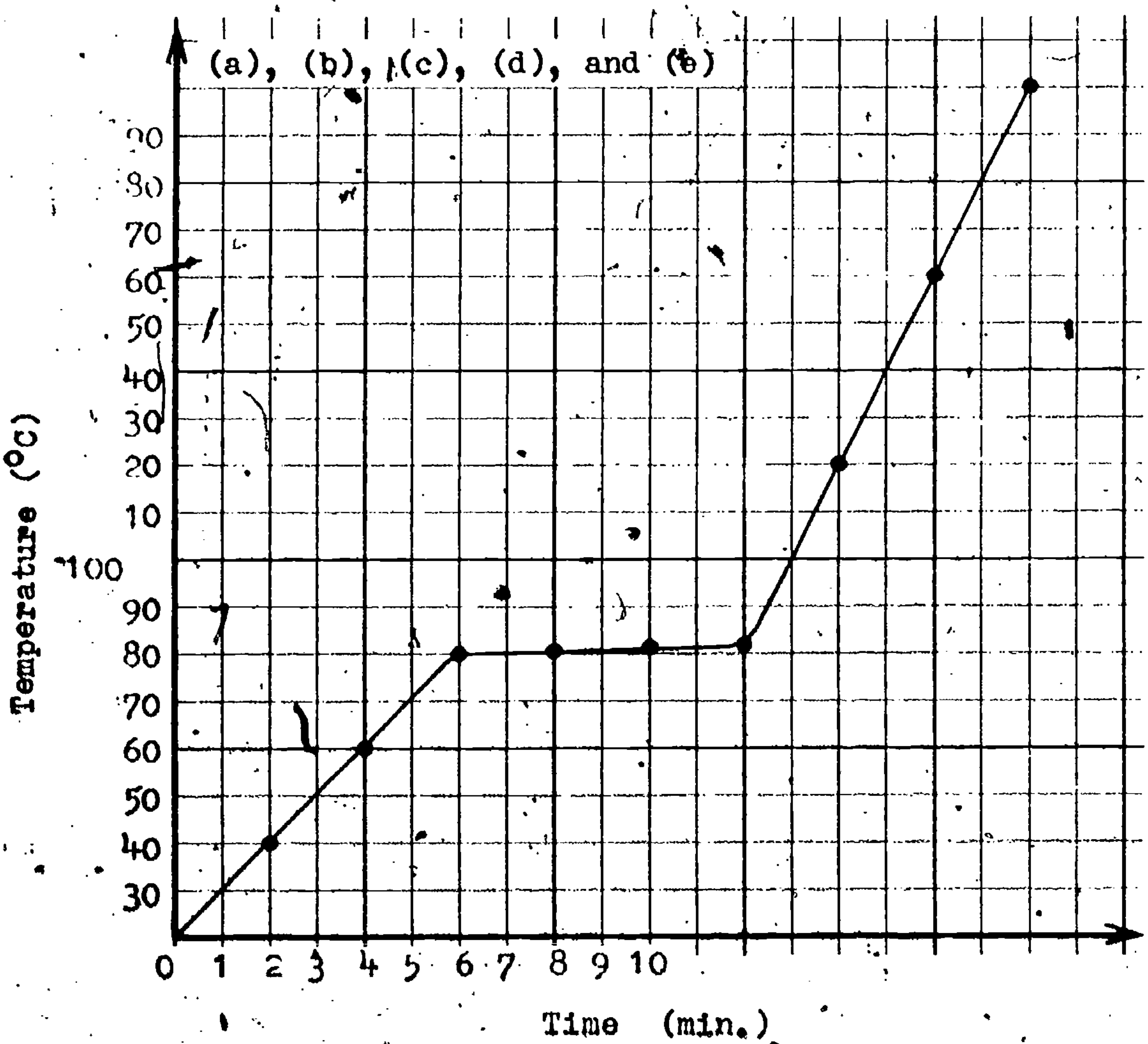
8. A(3,10) B(11,4) C(2, [illegible]) D(8,3) E(-5,[illegible]) F(5, [illegible])
G(2, 5) H(-10, -10) I(15,-4) J(-17, 3) K(0, 17)
L(-18,0)

Solutions to Exercise 5

1. (a) first quadrant
(b) negative horizontal axis
(c) fourth quadrant
(d) second quadrant
(e) origin
(f) positive vertical axis
(g) third quadrant
(h) fourth quadrant
(i) first quadrant
(j) negative vertical axis
(k) third quadrant
(l) positive vertical axis

Solutions to Exercise 6

1.



(f) $30^\circ C$ at 1 minute, $70^\circ C$ at 5 minutes, $81.5^\circ C$ at 9 minutes, $82\frac{1}{2}^\circ C$ at 11 minutes

(g) $50^\circ C$ at 3 minutes, $100^\circ C$ at 13 minutes, $150^\circ C$ at $15\frac{1}{2}$ minutes, $180^\circ C$ at 17 minutes

2. (a) set of times in seconds
   (b) set of height in feet
   (c) 44 feet
   (d) 2 seconds
   (e) 4 seconds
   (f) No, since the ball fell to earth at 6 seconds.

EQUIPMENT LIST

Part II

Chapter 2

OPEN SENTENCES AND LINEAR EQUATIONS

Students are to work in groups of four. Each group should have the following equipment.

1. See-Saw Experiment

Science Supply

- 1 meter stick
- 1 hook weight (10 gram)
- 2 hook weights (20 gram)
- 1 hook weight (50 gram)
- 1 hook weight (100 gram)
- 2 hook weights (200 gram)
- 1 triangle file (5 in.)

Local Supply

- 1 ball of string
- 1 rock
- 2 "Dixie" cups (6 oz.)
- 1 paper clip, spring-type "Everhandy" hunt clip 1
- $\frac{1}{2}$ package model clay
- 2 $\frac{3}{8}$" wooden dowels (12" length)

Chapter 2

EQUATIONS AND OPEN SENTENCES

Purpose: (a) Lead students to greater understanding of equations and open sentences in general.

(b) Show the importance of mathematics in solving problems in science.

The problems are created at the following times:

(a) At the introduction to the unit when the student makes predictions where to sit on a see-saw to make it balance.

(b) In the last part of the unit when we want to find out the mass measure of a piece of marble by using the meter stick model of the see-saw.

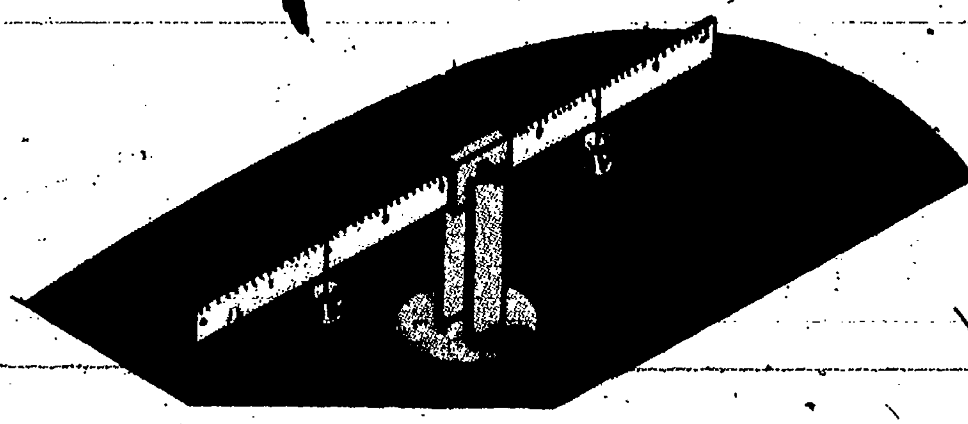
The experiments have been tried by 8th graders. We found that the students' involvement in the actual experimentation is very important. We suggest, therefore, that a teacher demonstration should be the second choice only. We want the children to develop scientific intuition and a spirit of inquiry. Let children explore. There are many ways of learning. One is by actual manipulation of objects. In the see-saw experiment, for instance, the students can feel physically the balance or imbalance of the stick.

The presentation of the experiment is fully developed in the student's text. Making the meter stick model of a see-saw would be a homework project before class work. The necessary material can be found around the house and in any hardware store.

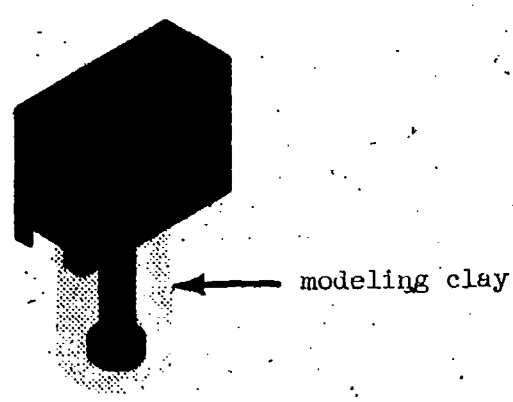
Materials:

1 - meter stick
1 - 4" x 4" x 6" block of wood
4 - 6 ounce "Dixie" cups
2 - dowels, 12 inches long
1 - lump of modeling clay
1 - set of standard weights (10, 20, 20, 50, 200, 200 grams)
1 - piece of strong string or nylon thread

In making up a meter stick model of a see-saw, the teacher should feel free to use his ingenuity and to make whatever changes are helpful, according to the composition of the class and the materials available at the time. For instance, the meter stick can be supported on a knife edge by means of a knife edge clamp, commerically made for this purpose. (See illustration.)

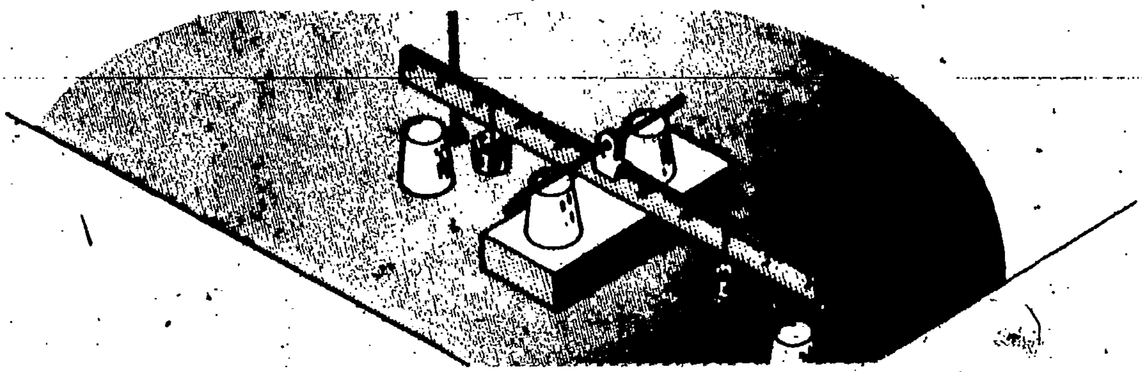


The teacher should use this commercially made knife edge with care since it is very sensitive, especially for weights greater than 150 grams. The sensitivity problem can be eliminated by sticking a piece of modeling clay at the bottom part of the clamp screw. (See illustration.)



If ~~the teacher~~ cannot obtain the standard weights, he can use pennies instead. The pennies can be suspended from the stick by small plastic bags. The pennies can be used not only in the see-saw experiments but also to find the measure of the mass of an object. Instead of grams, the mass measure would be expressed in terms of the pennies. For instance, the mass measure of the object could be equivalent to the mass measure of 25 pennies.

Before starting the actual experiment, the students should have some preliminary play on balancing, just to get the feel of it. It is very important that the stick settle exactly in a horizontal position before the distance is read off. The use of two dowels can help appreciably in this respect. The 6 ounce paper cup at each end of the stick is used to eliminate extreme tipping of the stick. (See illustration.)



The distance should be estimated to the nearest centimeter; otherwise we will not get the desired results.

Finding a general rule from the observations can be an exciting experience for the students. The teacher should use his imagination to lead the students to the discovery of the general rule: mc = 1200. Many students will easily recognize that there are two variables involved, the mass of the object and the corresponding distance from the fulcrum, and that increasing either one increases the balancing force. If the unit is being used on the 9th grade level, the teacher can develop an elementary understanding of the principle of moments. A moment is the product of a force and its distance from the fulcrum. In this experiment, whenever balance exists, two equal moments also exist. Each of these is a product of a mass and a distance. For example, when 120 grams at 10 centimeters is balanced by 60 grams at 20 centimeters, the two moments are 120×10 and 60×20.

On a more sophisticated level the teacher can even discuss that when a condition of equilibrium exists:

(a) The sum of the vertical components of the forces acting on the stick is equal to 0. $\Sigma F_v = 0$.

(b) The sum of the horizontal compnents of the forces acting on the stick is equal to 0. $\Sigma F_h = 0$.

(c) The sum of the moments is equal to 0. $\Sigma M = 0$.

Actually, on the see-saw we compare masses, not weights. Considering the fact that the concept of mass is difficult, even for college students, we avoided discussing it, but nonetheless.

always used it in the correct sense. The teacher should be careful not to use mass and weight synonymously. The distinction need not be emphasized in the classroom.

What is actually the difference between the mass and the weight of an object? The mass of an object is the same no matter where the object is. The weight of an object depends on the location of the object. For example, suppose you weight 120 pounds on your bathroom scale. If you went to the moon with your scale and weighed yourself there, the scale would read only about 20 pounds. However, your mass would still be the same.

Consider another illustration. On the earth you might have difficulty in lifting a chunk of steel, whereas on the moon you might lift it fairly easily. However, on the moon and on the earth you would have the same difficulty accelerating it with the same force.

The weight of a body is a measure of the gravitational force acting on it. Newton's Law of Universal Gravitation evaluates this force as follows:

$$F = G\frac{mM}{r^2}$$

where m is the mass of an object, M is the mass of the attracting body such as the earth, G is a constant and r is the distance between the centers of the two attracting objects. Thus, the weight is proportional ~~to~~ the mass of the object and also to the mass of the attracting body.

Every time we measure the pull of the earth on a body, we are measuring its weight. For example, a platform scale and a bathroom scale measure weight. A balance, however, compares the pull of gravity on two different objects and therefore it compares the masses of the objects.

The treatment of number sentences and number phrases depends on the teacher. The unit can be used by itself. However, the combined use of a mathematics text along with the unit may seem to be a better solution. In this section the teacher can utilize the see-saw very effectively as a physical model of equality or inequality.

The section on finding the mass measure of an object is fully developed in the student's text. In experimenting, the teacher should take time to let the student develop a feeling of approximation by comparing the mass of an object with a standard mass. Furthermore, the teacher should point out that the obtained mass measure of the object is an approximate value, since the distance that we are using to find the mass measure of the object is also an approximation to the nearest centimeter.

The following text can be used as resource material to this unit: Physics by the Physical Science Study Committee, D. C. Heath and Co., 1960, or Physics for Students of Science and Engineering, by Halliday and Resnick, John Wiley and Sons, Inc.

Exercise 1

1.

Trials	m	d
1	12	4
2	2	24
3	8	6
4	24	2
5	16	3
6	6	8

2.

m	20	40	10	20	10	50
d	15	7.5	30	15	30	6

Exercise 2

1. (a) $x + 15$
   (b) $8x$
   (c) $\frac{1}{4}x$ or $\frac{x}{4}$
   (d) $x - 4$
   (e) $\frac{18}{x}$

2. (a) 27
   (b) 96
   (c) 3
   (d) 8
   (e) $\frac{3}{2}$ or $1\frac{1}{2}$

Exercise 3

1. a, b, e, f

2. (a) $2 \times (3 + 1)$
   (b) $2 + (4 \times 3)$
   (c) $(6 \times 3) - 1$
   (d) $(12 - 1) \times 2$

3. (a) 47 (f) 44
   (b) 75 (g) 7
   (c) 70 (h) 26
   (d) 14 (i) 30
   (e) 8

4. (a) $(3 + 8) - 4$ or $3 + (8 - 4)$
   (b) $(\frac{1}{2} \times 6) + 4$
   (c) $(2 \times 3) + (4 \times 3)$

Exercise 4

1. The indicated sums are b, c and e. The indicated products are a, d and f.

2. (a) $4(47) + 4(3)$ (c) $\frac{2}{3}(8 + 4)$
   (b) $9(34) + 9(6)$ (d) $18(3.2 + .8)$

3. (a) $12(\frac{1}{3} + \frac{1}{4})$
   $12(\frac{1}{3}) + 12(\frac{1}{4})$
   $4 + 3$
   7

   (b) $\frac{1}{5}(\frac{7}{8}) + \frac{1}{5}(\frac{1}{8})$
   $\frac{1}{5}(\frac{7}{8} + \frac{1}{8})$
   $\frac{1}{5}(1)$
   $\frac{1}{5}$

   (c) $9(11 + 9)$
   $9(20)$
   180

   (d) $0(17 + 83)$
   0

   (e) $\frac{8}{9}(0 + 9)$
   $\frac{8}{9}(9)$
   8

27

4. (a) $7(22) = 7(20 + 2)$
$= 140 + 14$
$= 154$

(b) $12(33) = 12(30 + 3)$
$= 360 + 36$
$= 396$

(c) $15(36) = 15(30 + 6)$
$= 450 + 90$
$= 540$

Exercise 6

1. (a) $k + 7$
(b) $25x$
(c) $100x$
(d) $x - 3$ if x represents Sam's present age
(e) $a + 4$ if a represents John's present age
(f) $3y$
(g) $36b$

2. (a) $x + 2$
(b) $x - 8$
(c) $15 - x$
(d) $7x$
(e) $\frac{x}{3}$
(f) $x + 6$
(g) $\frac{x}{2}$
(h) $\frac{1}{3}x$

3. (a) 26
(b) 16
(c) 9
(d) 168
(e) 8
(f) 30
(g) 12
(h) 8

4. (a) $x + (x + 2)$ if x is an even number
(b) $\frac{x + 6}{2}$ or $(x + 6)\frac{1}{2}$
(c) $3x - 7$
(d) $2x + 3$
(e) $2(7 + 2)$
(f) $x + 5x$
(g) $5x + 2(10x)$ or $5x + 20x$ if x is the number of nickels Mike has

5. $2(5 + 3)$

6. $\frac{2n + 5}{3}$

7. $\frac{1}{3}(4x) + \frac{1}{4}x$

8. $\frac{y}{4}$

9. $2(a + b)$

10. 1000 k

11. 100 cm

12. 1000 d

13. 1000 p

14. 500 g

15. $100(5 + w)$ g or
500 g + 100 wg

16. 100 k + n cm

17. 10^{-2} t + s grams

18. 4 h

Exercise 7

1. $C = 32n$

2. $D = .32n$

3. $D = 2r$

4. $P = a + b + c$

Exercise 8

1. (a) T
(b) T
(c) F
(d) F

Exercise 9

1. (a) T
(b) F
(c) T
(d) T
(e) T
(f) T

4. (a) 6
(b) 8
(c) 4
(d) 3

5. 4 cm

6. $d < 4$ cm
$d > 4$ cm

Exercise 10

1. (a) $x = 2$
(b) $y > 2$
(c) $x = 3$
(d) Set of all real numbers except 3
(e) $n = 12$
(f) $b < 12$

2. (a) 8
(b) 4
(c) 6
(d) 2

Exercise 11

1. Domain of n is the set of positive integers.

   $\$1.35 \; n = \$236.25$

   $n = 175$

2. The domain of g is the set of all positive rational numbers.

   $\$.30 \; n = \$2.76$

   $n = 9.2$ gallons

3. Domain of p is the set of positive numbers

Exercise 13

1. (a) $\frac{1}{17}$ (c) $\frac{5}{4}$

   (b) $\frac{7}{8}$ (d) 1

Exercise 14

1. (a) $\frac{1}{2}$ (f) $\frac{3}{7}$

   (b) $\frac{6}{7}$ (g) 1

   (c) 49 (h) $\frac{4}{25}$

   (d) 9 (i) 2

   (e) $\frac{2}{5}$

2. (a) $5x = 30$, $x = 6$

   (b) $\frac{y}{4} = 9$, $y = 36$

   (c) $\frac{2}{7}a = 28$, $a = 98$

   (d) $3x = 60$, $x = 20$

   (e) $9x = 63$, $x = 7$

3. (a) 10 in (b) 18 lb (c) 2d cm

4. Yes, if the girl is $\frac{1000}{90}$ $(11\frac{1}{9})$ times as far as the box from the fulcrum and on the opposite side.

5. 2 feet

6. 200 lbs

3. (a) 10 in.
   (b) 18 lb.
   (c) 2d cm.

4. Yes, if the girl is $\frac{1000}{90}$ $(11\frac{1}{9})$ times as far as the box from the fulcrum and on the opposite side.

5. 2 feet

6. 200 lbs.

EQUIPMENT LIST

Part II

Chapter 3.
A GAS LAW AND LINEAR FUNCTIONS

1. Charles' Law--Teacher Demonstration

   Science Supply

   1 Charles' Law apparatus

   1 thermometer, Centigrade (-20° to 110°)

   Local Supply

   1 electrical hot plate

   2 trays of ice cubes

   salt

   1 #10 tin can

Chapter 3

A GAS LAW AND LINEAR FUNCTIONS

This unit is a survey of some of the elementary concepts included in the broad subject of linear functions. It makes no attempt to introduce all phases of the subject, but, rather, treats a few topics which arise from an analysis of the data derived from a demonstration experiment on Charles' Law. The entire unit is based on this single experiment, which is referred to throughout. In this way an attempt has been made to interweave the physical phenomenon with the related mathematical principles.

The unit is designed to be used in a ninth grade algebra class, although it is considered to be appropriate also for an eighth grade class which has received the prerequisite instruction. This presupposed knowledge consists of the following concepts and skills: the properties of the number system, algebraic operations with signed numbers, ordered pairs, functions, and graphing. For the rest the unit is a self-contained package. Therefore, there is considerable flexibility in the matter of where it will fit in existing courses. Possibly, the most convenient place to insert it is after the students have learned graphing and before introducing systems of linear equations.

<u>Scientific background to the experiment</u>. Charles' Law is a statement of one of the properties of gases, whose explanation is covered by the broad kinetic theory. The law states that the pressure exerted by a quantity of gas at constant volume varies directly as the absolute (or Kelvin) temperature, or, in mathematical terms, $P = kT$.

The pressure exerted by a gas is due to the bombardment by its molecules of the walls of the containing vessel. An increase in the number of impacts of the molecules per unit of time or in the average momentum which they bring into each impact will, of course, tend to result in an increase in the pressure. Both of these factors depend on the average velocity of the molecules. Obviously, the higher the average velocity of the molecules, the greater will be the rate of their collisions with the containing walls, and, since the momentum of a particle is equal to the product of its mass and velocity, the average momentum also increases with an increase in the average velocity. In turn the average velocity of the molecules depends on temperature. As a matter of fact, the temperature of a gas is a measure of the average kinetic energy of its molecules. Since the kinetic energy of a particle equals one-half the product of its mass and the square of its velocity ($E = 1/2\ mv^2$), it is readily seen that raising the temperature of a body of gas consists of increasing the average velocity of its molecules. Hence, it can be seen that these inter-relationships are such that increasing the temperature of a sample of gas tends to result in an increase in the pressure exerted by the gas. Furthermore, it can be shown by theoretical considerations which need not concern us here that the temperature-pressure relation should be linear, as is indicated by $P = kT$.

The above theoritical prediction is verified by experiment, but Charles' Law, like all scientific laws, is an exact expression only in ideal cases. In other words the expression $P = kT$ is rigidly correct only in the case of an ideal gas, which for our purposes can be thought of as one in which the molecules themselves are point masses, that is, have zero volume, and in which there are no inter-molecular attractive forces. This is to say that an ideal gas is one in which there are no internal factors which interfere with the behavior of the molecules toward their surroundings.

Of course, there is no such thing as an ideal gas. However, through some range of conditions, practically any gas will behave in a manner which approaches ideality. From the discussion, it can be seen that if the interfering factors are kept negligible, or nearly so, these conditions will be met. In this light, if the temperature is high enough so that the molecules are moving rapidly, the effect of their attractions for each other will be minimized. Also, if the concentration of the gas, that is, the number of molecules per unit volume, is kept low, the molecules will be far enough apart to again keep their mutual attractions low and their volumes will represent only a very small fraction of the space occupied by the body of gas as a whole.

Under the conditions of the demonstration used in this unit, the gas in the apparatus is at a high enough temperature range and low enough concentration to behave sufficiently close to ideality to meet our needs. Therefore, the data which will be collected will yield an approximately straight line. The extrapolation to the temperature axis of the corresponding line for an ideal gas results in an intersection at -273°C, the absolute zero of temperature. This point is of great importance in physical science, both theoretically and operationally. In connection with Charles' Law it is the point at which the pressure of a gas becomes zero. More fundamentally it is the temperature at which kinetic energy becomes zero and molecular motion ceases.

The line which will result from the experimental data will extrapolate to a point on the temperature axis which will be gratifyingly close to -273°C. Since we are extrapolating from a line segment which represents close to ideal behavior, naturally we miss the derivations from ideal linearity which come with lower temperatures, and, of course, we also by-pass the sharp breaks which would occur if we took the gas down to a temperature at which it would liquefy and, eventually, solidify.

If it is desired, the demonstration can be repeated using different amounts of gas (air will be used) in the constant volume of the apparatus. If it is possible to reduce the pressure in the apparatus to about 10 lb/sq in at room temperature and to increase it to about 20 lb/sq in, these would make good starting points, along with prevailing atmospheric pressure. The more air there is in the apparatus, the greater will be the slope of the graph. If all the lines are plotted on the same axes, they will show good convergence toward -273°C (or close to it) on the temperature axis.

This introductory material will take on more meaning if it is re-read after reading sections 11-1 and 11-2 of the text. In general it would probably be wise to read commentary material both before and after reading the sections of the text which that particular portion of the commentary accompanies.

Notes on Text Sections

3.1 The introduction to the unit will probably lead to a discussion of the terms underlined in the text, although at least some of the students will have encountered these terms in science courses. There is also an opportunity for the teacher to elaborate on the interrelation of science and mathematics. It can be pointed out that the investigation of problems in science has stimulated the development of new mathematics, required to enable the investigators to subject their findings to theoretical analysis. For example, Newton's invention of the calculus to help him understand the nature of motion might be discussed. On the other hand, the discovery of new mathematics makes possible and stimulates an attack on scientific problems which could not otherwise be approached. Much of our modern work on the structure and properties of the atom would be impossible without the sophisticated mathematics that has come into existence in the last century, nor would Einstein have been able to develop his theory of relativity without the work of the geometers of the Nineteenth Century. The students might also be interested in Archimedes, who, over two thousand years ago, combined in one man the mathematician, scientist, engineer, and inventor.

3.2 The Charles' Law apparatus may be made in the school shop. The metal bulb is a float from a toilet water tank. This is attached to a piece of copper tubing, which in turn, is attached to a bicycle tire valve assembly. The final attachment is a Bourdon tube pressure gauge, calibrated from 0 to 30 pounds per square inch. All connections must be made airtight and the bulb must be soldered around its equatorial seam to make it airtight also. The glass front of the gauge should be removed and the needle set to read the local atmospheric pressure when the air in the bulb is at that pressure.

The Centigrade thermometer should read from -20° to 110°. The water vessels can be 1500 ml or 2000 ml beakers, although any containers of the appropriate size may be substituted. It is convenient to have the water baths set up shortly before the experiment is to be done and, therefore, there should be at least six vessels available. Depending on what facilities are present in the classroom, a Bunsen burner (and ringstand) or electric hot plate will also be used. Therefore, at least one water vessel should be a Pyrex beaker or metal can. A complete equipment list is given at the end of this chapter.

At least six pairs of readings of temperature and pressure should be made, of which one can be at room conditions. The lowest temperature used should be below $0^{o}C$ and is obtainable by a mixture of rock salt, ice and water. It is important that this temperature be used because it is desired that the students initial graphs cross the vertical axis. The highest temperature should be that of boiling water. The remaining temperatures should be spread fairly regularly between the lowest and highest. Readings should be made in order of increasing temperature, so that the students will more readily see the relation in terms of an increase in one variable resulting in an increase in the other. For this reason, it is not advisable to start with boiling water and then produce successively lower temperatures by mixing with portions of cold water, even though it is easier to perform the demonstration in this manner. An alternative to using separate vessels, each at a discrete temperature, is to obtain some of the higher reading by heating a single vessel while the bulb and thermometer remain in it. Readings of temperature and pressure can be made at regular intervals and, in addition, the students can see the continuous and simultaneous increase in temperature and pressure. This will be of some advantage when the concept of continuity is introduced. The bulb should be entirely immersed in the liquid and, in order to get simultaneous readings of the two instruments, it may be necessary to have the assistance of one of the students.

Temperature readings should be made to the nearest whole degree and pressure readings to the nearest fifth or tenth of a scale division. Since each division represents two pounds, the pressure readings will be to the nearest 0.4 or 0.2 pound. Before reading the pressure gauge, tap it with a finger to make sure that the needle is stabilized in the correct position for that temperature. Also check to make sure that the bulb is entirely immersed in the liquid.

3.3 Make sure that the students draw the coordinate axes in such a way that they make maximum use of the space available on their graph paper. The paper should be held with its long side horizontal and the horizontal axis should be drawn about one inch from this edge. Their attention should be drawn to the fact that only the first and second quadrants of the plane are going to be used. The vertical scale should run to about one inch from the top edge of the paper, and the horizontal scale should start with the -300 point within an inch from the left edge and run to the +100 point within an inch from the right edge. If the students ask why temperature is designated by a lower case letter (t) while pressure is designated by an upper case letter (P), they should be told that the reason will be brought out later in the text.

3.4 The concept of continuity will probably be familiar to the students from previous work in mathematics and/or science. Hence, this point will not require belaboring at this time.

3.5 The students should get the idea that measurements are far from infallible. If a study of the general theory of measurement has preceded the present work, some of the ideas which are relevant to this experiment can be reviewed. The text avoids any discussion of non-ideality of the gas as a partial cause of deviation from perfect linearity. This factor can be introduced if there has been a prior discussion of the material presented earlier in this commentary under the heading of scientific background.

3.6 Every effort should be made to have available enough transparent rulers to enable each student to have the use of one for a few minutes during class time. This will be especially important if the plotted points deviate from linearity by more than a very slight amount, for it will be the only way that they will be able to see how to position the ruler for the "best line."

It is not appropriate at this level to discuss the fact that there is no one "best line" and that the choice depends on the mathematical procedure used to determine it. Rather, let the students use a bit of intuition and experimentation with the ruler to make a judgment. Point out to them that they are proceeding according to the ideas of the last few sentences of text section 3.5, that is, they are taking into account the fact that there are measurement errors and are not letting these expected errors obscure the fundamental relation that exists between the two variables.

Before the students draw their lines, stress the fact that the lines should not extend beyond the two extreme points on the paper. In the next to last sentence of the section, the statement made is not strictly correct, for it neglects the factor of non-ideality of the gas. However, it is unlikely that any of the students will pick up this subtle point, even if the nature of real gases has been discussed. If the issue does arise, it is valid to say that the effect of this factor is practically negligible under the conditions at which the experiment was performed.

Answers to Exercise 1

1. y-intercept $\approx$ 3. 2. y-intercept $\approx$ 2 3/4. 3. y-intercept $\approx$ -1/2.

3.7 From this section on there are several mathematical concepts presented. The time spent on each section will depend on the difficulty of the concepts and the ability of the students. It may be necessary to reinforce learning with additional problems. Considerable discussion of text material may also be necessary. It is recommended that sufficient time be allowed for each section, so that the average student will achieve reasonable mastery before proceeding to the next section.

The students may recognize and know by name the process of interpolation which is described in this section. The term is not mentioned by name, this being reserved for the fuller treatment presented in a later section. The only interpolations actually made in this section and the exercise are from domain to range, the reverse operation also being postponed to a more appropriate place in a later section. The chief purpose of this section is simply to establish the fact that the graph actually represents a functional relation.

Answers to Exercise 2

1. a, b, c, e, g, h represent functions. The students may need review of the fact that, in order for a functional relation to exist, each element of the domain must have related to it a unique element of the range, but that the reverse relation need not exist.

---

2. a. (-3, -7 1/2) b. (-2, -6) c. (-1/2, -3 3/4)
   d. (1/2, -2 1/4) e. (1, -1 1/2) f. (2, 0)
   g. (5, 4 1/2) h. (8, 9)

3.8 This section may well be the one that will present the most difficulties to the students. It is imperative that they learn how to calculate the value of the slope of a line, and much classroom demonstration and additional practice may be necessary to achieve this end.

It is suggested that the following method of finding slope be taught in class to supplement the text and that it be followed consistently, at least on the outset. In this method the counting process always starts with the change in x and always from left to right. Then the change in y is counted, either up or down as the case may be. The comparable subtraction process is to subtract the coordinates of the point on the left from the coordinates of the point on the right. If this method is used and emphasized to the students, the following points can be made: (1) the change in x is always positive; (2) if the change in y is counted upward it is a positive quantity, if downward it is negative--these designations will agree with the results obtained if the subtraction process is employed; (3) the sign of the slope will always be correct if it agrees with the sign of the change in y.

After the students have been thoroughly drilled in the above standard method, alternative counting processes (there are three others) can be introduced, if it is so desired, although this is not recommended. However, it might be well to show them, using specific cases, that the same value of the slope is obtained by the subtraction process wether one uses $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y_1 - y_2}{x_1 - x_2}$.

They should be strongly forewarned against the two common mistakes made by students in finding slopes, namely, "scrambling" the coordinates in the subtraction process (that is, setting up $\frac{y_2 - y_1}{x_1 - x_2}$), and inverting the fraction in either process (that is, setting up $\frac{\text{change in } x}{\text{change in } y}$).

It is suggested that a chalkboard demonstration be employed to show how a line is uniquely determined by its slope and one point through which it passes. This should be followed by several demonstrations of drawing a line, given m and b.

Answers to Exercise 3

1. The values of the slopes should be exact, since they are obtained by counting or calculation. The values that the students give for the y-intercept in the first two cases are subject to some deviation, since they will determine them by reading coordinates on graphs.

	m	b
a.	1/2	2
b.	-5/7	-6/7
c.	-6/5	0
d.	undefined	none
e.	0	4
f.	undefined	none
g.	0	0
h.	0	4
i.	undefined	none

2 and 3. It is suggested that these problems be done on the board by students and discussed in order to provide further exposure to this technique. In problem 3 the attention of the students should be drawn to the fact that the point used to help determine a line need not be the y-intercept. This is merely a very convenient point to use.

3.9 An important, if subtle, fact associated with equation (3) and the corresponding versions derived by the students is that they are all inaccurate. Equation (2), derived in general terms, is, of course, simply a modification of the slope-intercept form of the equation of a straight line. The inaccuracy is introduced when the substitutions are made for m and b in equation (2) in the process of deriving equation (3), for it is patently impossible to obtain from the graph exact coordinates of two points (used to calculate m) and the exact value of b. Even if the best line has been drawn through two of the original ordered pairs obtained from the experiment and these ordered pairs are used to calculate m, the probability of obtaining the exact value of b from the graph is extremely low. Hence, if coordinates of any point on the line (these, too, probably will not be exact!) are substituted for t and P in equation (3), the left and right members of the resulting "equation" will not reduce down to the same number. The amount of deviation will depend upon the accuracy with which all the graph readings have been made. This matter of the inaccuracy of the equation is not introduced to the student until the next section and its full development is spread out beyond that up to Exercise 6. Whenever this point is discussed, the students should be reminded that this condition will exist whenever an equation is derived from graphically-obtained data and, hence, is quite generally true in the case of equations derived to fit non-idealized physical data.

Only one form of equation of a straight line is introduced in this section. However, if it is desired to do so, the other forms may be introduced as supplementary material. It will be noted that the general form of the equation appears, without being named, in specific examples in the text and problems.

Answers to Exercise 4

1. The following answers are based on the answers to problem 1 of Exercise 3. Hence, the slopes are exact, but the y-intercepts in the first two cases are subject to some deviation in the students' answers.

a. $y = 1/2x + 2$
b. $y = -5/7x - 6/7$
c. $y = -6/5x$
d. $x = -3$ (no slope-intercept form)
e. $y = 4$
f. $x = 0$ (no slope-intercept form)
g. $y = 0$
h. $y = 4$
i. $x = 2$ (no slope-intercept form)

2. a. $m = -3/2$; $b = 9/2$
b. $m = 2/5$; $b = -7/5$
c. $m = 3/4$; $b = 9/2$
d. $m = -3/8$; $b = 0$
e. $m = -3/2$; $b = 7/4$
f. $m = 2/5$; $b = -7/5$

3. a and e are parallel (same slopes makes them parallel; different y-intercepts makes them distinct lines).

4. a and c intersect on the y-axis (same y-intercept; different slopes makes them distinct lines).

5. b and f are the same line (same slope; same y-intercept).

3.10 In this section the statement is made that interpolation in an equation permits the obtaining of more accurate results than are obtainable by interpolation in a graph. This is entirely true in the case of purely mathematical equations, but not always true in the case of an equation which is itself derived from graphically-obtained data.

When the students extend the line representing the temperature-pressure function down to, but not beyond, the horizontal axis, the significance of their using only the two upper quadrants can be pointed out, for to extend the line into the region representing negative pressures would have no physical meaning.

In this section, there can be introduced, if it is so desired, a discussion of some or all of the material concerning the deviation of a real gas at low temperatures from the ideal straight line. This will give substance to the statements made in the last two paragraphs of the section.

It will be noted that all references to interpolation and extrapolation in this section are still limited to the process of going from domain to range.

Answers to Exercise 5

1. Student discussion of limit of the range due to physical limit of low pressure.

2. Lower limit definite; upper limit indefinite.

3. The y-intercept, obtained algebraically, is $-5/3$. The students can not be sure of this value if they read their graphs honestly. This problem should be used to emphasize the superiority of a purely mathematical equation over the graph of the equation, for the purpose of interpolation.

4. The y-intercept, obtained algebraically, is 2. Ask the students if, even in this simple case, they could be certain of the exact value of the y-intercept from the graph alone.

5. The range is all the real numbers between -4 and 2.

3.11 Answers to Exercise 6

1. Neither ordered pair is a member of the solution set. Both points are very close to, but not actually on, the line represented by the equation.

2. Student review of much of the preceding work. At this time, there can be a review of the ideas presented in the commentary for section 3.9.

3.12 This section introduces for the first time, interpolation from range to domain and goes on from there to develop the point that there is nothing sacred about the designation of the horizontal axis as the x-axis and the vertical axis as the y-axis. The horizontal axis is still used to represent elements of the domain and the vertical axis elements of the range, but the domain and range themselves are interchanged, as is represented by the interchange of labels on the two axes and by a different graph. The two equations are, of course, the inverses of each other and represent the same facts as the graphs. It might be pointed out that the equivalent operation can be performed with the parabola in Figure 10 as was peformed with the linear function, simply by rotating the entire figure 90° clockwise. In this case the functional relation no longer exists. The students can be led from this to the conclusion that only in cases in which 1-1 correspondence exists will the functional relation be preserved when the domain and range are interchanged.

The remaining major points in the section are the ones which are brought out as the answers to problem 2 of Exercise 7.

Answers to Exercise 7

1.

x	y(from equation)
2	-8/3
4	-4

y	x(from equation)
1	-7/2
3	-13/2

2. a. No.
   b. Yes; the parabola in Figure 10 is an example.
   c. Yes; a horizontal line is an example $(y = b)$.
   d. Yes; a vertical line is an example $(x = a)$.

3.13 If the students have been presented with a considerable amount of the information contained in the scientific background to the experiment, it can be pointed out to them that the absolute zero of temperature has theoretical validity. This is so because, in spite of the fact that the full length of the extrapolated portion of the line does not represent the situation in a real gas, the point of absolute zero was reached by extrapolation from a segment of the line which approaches quite close to ideality. Hence, this temperature has physical significance, for it represents the zero point of kinetic energy and is the point at which the line would cross the horizontal axis, even if it were obtained experimentally, rather than by extrapolation, and even though the line would not be straight along its entire length.

Equation (5) offers the students an opportunity to think about domain and range of a physical function. They should be able to get the lower limits as -273 and 0, respectively. In graphing the (T,P) function, make sure the students use the same kind of graph paper as they did for the (t,P) function and that they use the same scale dimensions. Point out to them the use of the (0,0) point as a check on their consistency of judgment as to the best line.

If it is desired to pursue translation of axes beyond the text material, the comparison of domain and range of the (t,P) function with the domain and range of the (T,P) function should be worked through in detail and related to the two graphs and two equations. The students should be made aware that the translation of the vertical axis resulted in a change in the domain of the function but no change in the range. This change occurs only in the case of functions in which there are limitations on the domain and range.

Answers to Exercise 8

1. a. $R = F + 460$
   b. Student graphs should show $m = 1$ and $b = 1060$.
   c. Student graphs should show $m = 1$ and $b = 600$.
   d. $S = F + 1060$
   e. $S = R + 600$

2. a. $F = 9/5C + 32$
   b. Only the best students will get this one. Their answers will be $R = 9/5k + 3/5$ or $K = 9/5R - 1/3$. It should be pointed out to them that the constants used, 273 and 460, are rounded off values and that, if exact values had been used, the equations would be $R = 9/5k$ and $k = 5/9R$. They will see the logic of this if it is pointed out that at absolute zero both R and k equal 0 and that a graph of the (R,k) function will pass through (0,0).

3.14 This section presents a standard treatment of direct variation. The only subtle point in the section lies in the definition of ratio. It will be noted that the definition designates the terms of a ratio as pure numbers. In true rigor we do not say that we have a ratio of pressure to temperature, but, rather, a ratio of numbers expressing measures of those two quantities. It is not recommended that the older terminology of a rate be introduced to represent a quotient of two unlike quantities, e.g., 10 miles/hour, but, rather, that the 10 be thought of as a pure number expressing the value of the quotient of a number representing a distance in miles and a number representing a time in hours.

Answers to Exercise 9

1. a. $\frac{a_1}{b_1} = \frac{a_2}{b_2}$
   b. $a = kh$
   c. $\frac{c_1}{i_1} = \frac{c_2}{i_2}$
   d. $i = kc$
   e. $\frac{c_1}{v_1} = \frac{c_2}{v_2}$

2. 0.394, approximately. The two constants are the reciprocals (or multiplicative inverses) of each other.

3. a. $y = 2x$
   b. $y = 1/2x$

4. $y = 4\ 1/2$

5. $z = 7.5$

6. a. Student graph having $m = \frac{1}{40}$ and $b = 7$.
   b. $P = \frac{1}{40}t + 7$.
   c. $t = 48^{\circ}C$
   d. $P = 8.25$ lb/sq in.
   e. $P = \frac{1}{40}T$

7. a. $Y = 3x - 8$
   b. $X = x - 8/3$
   c. $Y = 3X$

<u>Equipment List</u>

1 Charles' Law apparatus
1 thermometer, $-20^{\circ}C$ to $110^{\circ}C$
6 (or more) vessels, at least 1500 ml capacity; beakers or metal cans preferable
1 electric hot plate or 1 Bunsen burner with 1 ringstand with large ring
1 classroom set of transparent plastic rulers

EQUIPMENT LIST

Part II

Chapter 4

AN EXPERIMENTAL APPROACH TO LINEAR FUNCTIONS

Students are to work in groups of four. Each group should have the following equipment.

1. The Bending Beam

Science Supply

1 meter stick

1 hook weight (50 gram)

1 hook weight (100 gram)

2 hook weights (200 gram)

Local Supply

1 15-inch flat-sided wooden ruler ($1\frac{3}{8}$ inch wide, $\frac{1}{8}$ inch thick)

1 3-inch "C" clamp

1 spool, button thread

2. The Falling Sphere

Science Supply

2 ball bearings, $\frac{1}{8}$-inch steel

1 magnet, alnico horseshoe (1" × $\frac{3}{4}$")

Local Supply

1 bottle, olive (or equivalent)

syrup, light "Karo" (enough to fill each bottle)

1 ruler (30.5-centimeter)

3. The Trampoline Experiment

Local Supply

- 1 9-inch aluminum pie plate
- 6 15-cent balloons, spherical
- 1 10 x 24-inch sheet bristol board
- 1 pound of plastolene
- 2 glass marbles
- 1 desk lamp (or slide projector)
- 2 $\frac{5}{8}$ - inch nylon bearings

4. Charles' Law--Teacher Demonstration

Science Supply

- 1 Charles' Law apparatus
- 2 thermometers, Centigrade (-20° to 110°)

Local Supply

- 1 electrical hot plate
- 2 trays of ice cubes
- 32 sheets frosted acetate ($8\frac{1}{2}$" x 11") (1 sheet per student)

Chapter 4

AN EXPERIMENTAL APPROACH TO LINEAR FUNCTIONS

Introduction

The purpose of this section of the booklet which your students will study for the next few weeks is to teach them some of the fundamental concepts of mathematics regarding linear functions. As you know, these are expressions which involve variables to the first degree; actually first degree polynomials in a single variable.

The need for such knowledge can be presented to the students through the physical sciences. As we tell the students, "The mathematics will be developed to meet the particular needs of a set of experimental situations." However, after the mathematics is introduced in this manner a number of logical extensions of particular mathematical structures will be made.

This approach to mathematics must not be made, as we have done so often in the past, by telling the student what will take place. The student himself must encounter first-hand the experimental situations from which the mathematics will arise. That is, he must do the experiments himself, measure the things which change, record the data in an orderly fashion, and examine it critically for whatever general relations it shows.

The experiments presented here for your students to do have been designed and performed by scientists and mathematicians, working as a team, college teachers and high school teachers, who feel that this approach to the study of mathematics has real merit in our attempt to teach the subject with understanding and interest.

You, as teachers, will be given the results of these experiments, the data we have collected, but the student should be expected to find his own data and make his own analysis of it. Thus the results of each team of students as experimenters may be different, but if their work is done carefully the differences should not be too great. Even though differences exist, the real need for graphing the data collected, examining the graphs for definite relations, functions and equations should be apparent.

In this chapter four experiments are used. Each results in a linear function, but illustrates different aspects of this topic. The first experiment on The Loaded Beam investigates what happens when a beam (a 15-inch flexible ruler) is clamped to a desk at one end and loaded with a series

51

of increasing masses at the free end. This gives a series of ordered pairs of numbers which are graphed on a coordinate system. Through discussion, the student is led to the conclusion that, if other masses had been used, many other ordered pairs could have been found and graphed within the interval of the experiment. This leads finally to a "best straight line" which will, within the errors of measurement, be a graphical representation between the two variables: load on the end of the beam and the resulting deflection of the end of the beam. At this point the mathematics of the slope and intercept of the line is introduced, followed with a general discussion and practice in graphing linear functions and an introduction to relations and functions. A similar procedure will be used in each of the experiments.

The other experiments on The Falling Sphere, The Trampoline and Charles's Law will also deal with linear functions, but will gradually introduce additional ideas on this topic.

An outline of the topics included in the various experiments appears below:

A. The Loaded Beam

Graphing the Experimental Points

- setting scales
- filling in the line
- "best straight line"

Exercise

Slope and Intercepts

- slope
- vertical axis intercept
- $y = mx + b$ $(b = m1 + b)$
- interpolation and extrapolation

Graphing Linear Equations

- positive and negative slopes
- drawing the line from an equation

Relations and Functions

- domain and range
- "generator"
- definition of a function
- geometrical conditions of a function
- converse of a relation

geometric condition for the converse of a relation

to be a function

definition of 1-1 functions

B. <u>The Falling Sphere</u>

The Graph and the Equation

new "braid"

best line

slope

equation, $d = mt$

Exercise

The Point-Slope Form

slope-intercept form, $y = mx + b$

through origin, $y = mx$

slope-intercept on x-axis, $y = m(x - a)$

point-slope easily gives these three as special cases

Physical Units

standards of mass, length, time

undefined quantities

MKS and British systems

Mathematical Approach to Units

form for elements of set

multiplication and division

addition

units of various illustrations

Exercise

Replacement of Units

three kinds of replacement

tables for MKS and British

illustrative problems

Exercise

C. <u>The Trampoline</u>

Function of Integers

no best line, not a straight line

other functions of integers only

Exercise

Mathematical Trampoline Model

relation of height of bounce to previous bounce

53

ordered pairs

graph of ordered pairs (h_n, h_{n+1})

slope of line

meaning of slope

length of bounce from slope m

equation as function of (n, h)

calculating values of bounce

limit of domain, extension?

Experimental Extension

change of constants of experiment

nylon to glass ball

slope, change of m

Exercise

D. Charles's Law

Extending the Temperature Domain

definition of extrapolation

prediction of temperature for 0 pressure

extend graph to intersect horizontal axis

Graphical Translation of Coordinate Axes

shift axes, not graph, linear shift

new axes- X, Y

slope remains same

horizontal and vertical shifts

shift for Charles's Law, new temperature scale

both axes shifted

scales

Algebraic Translation of Coordinate Axes

point-slope form, to $(-c, -d)$

shifts h and k from $y - d = m(x - c)$

4.1 The Loaded Beam Experiment

In performing this experiment and all others, you should have the students follow directions carefully. They should clamp the ruler onto the desks firmly and with the same distance extending beyond the desk each time they do the experiment. Different groups may use different distances. The holes bored in the rulers should all be at the same place (1 inch from the end). You may have to do this in the school shop, $\frac{1}{8}$ inch holes are about the right size. Since the thread must support up to 300 grams, it should be strong. Small fishing line would be excellent. If you use sewing thread, No. 24 or button thread would probably be sufficient. Make a loop from a 5 inch length, place it around the ruler and draw it down through the hole. The weights may be hung directly on the projecting loop.

All of these things are important because any measurement is subject to error and we must be careful not to introduce others.

This experiment should be done by groups of 4 or 5 students.

The equipment needed is listed below:

- 8 15-inch flexible wooden rulers
- 8 3-inch C-clamps
- 1 spool button thread
- 8 meter sticks
- 8 kits of hooked weights, each to contain 1 50-gram, 1 100-gram and 2 200-gram weights

The following data were collected on four trails for this experiment. These data are listed here on a suggested standard data sheet which we will use for all experiments. If possible, this should be dittoed for the use of your students during this course.

THE LOADED BEAM EXPERIMENT

Load l (grams)	Trial 1 Position p (centimeters)	Trial 2 Position p (centimeters)	Trial 3 Position p (centimeters)	Trial 4 Position p (centimeters)		
0	20.0	20.1	20.1	20.2		
30	20.4	20.5	20.4	20.6		
60	20.8	20.9	20.9	21.0		
90	21.3	21.4	21.4	21.5		
120	21.7	21.9	21.8	21.9		
150	22.1	22.3	22.3	22.3		
180	22.5	22.7	22.8	22.7		
210	22.9	23.2	23.2	23.1		
240	23.4	23.6	23.7	23.6		
270	23.8	24.0	24.2	24.0		
300	24.2	24.4	24.6	24.4		

4.2 Graphing the Experimental Points

There are a number of items included in the graphing of these experimental points. These are all discussed in the student's text and need not be repeated. However, be sure the students understand these. The exercise at the end of the unit will help in seeing whether the ideas have been understood.

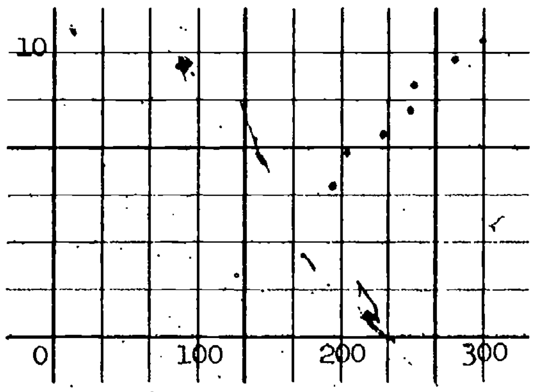
Getting the "best straight line" involves assuming it can be drawn where it ought to be. Often it will not be too hard to get a line which will satisfy the student. Don't put too much stress on this now. There will be more practice and the students should learn quickly.

The accompanying graph of the data shows the four trials and has a "best line" dashed in. While this will probably not be the same as many of the lines obtained by members of the class, it will be used to solve some of the problems for the teacher which may come up in the exercises.

Answers to Exercises, Section 2.2

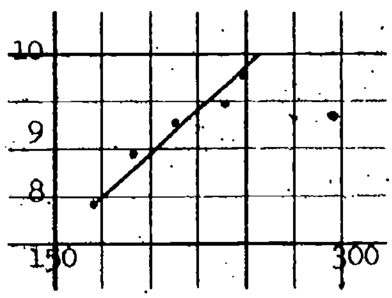
1. no, no

2. The graph would not "fill" the paper and hence the divisions on the axes would have to be shorter.



3. This makes the graph appear on a small portion of the page instead of filling the page.

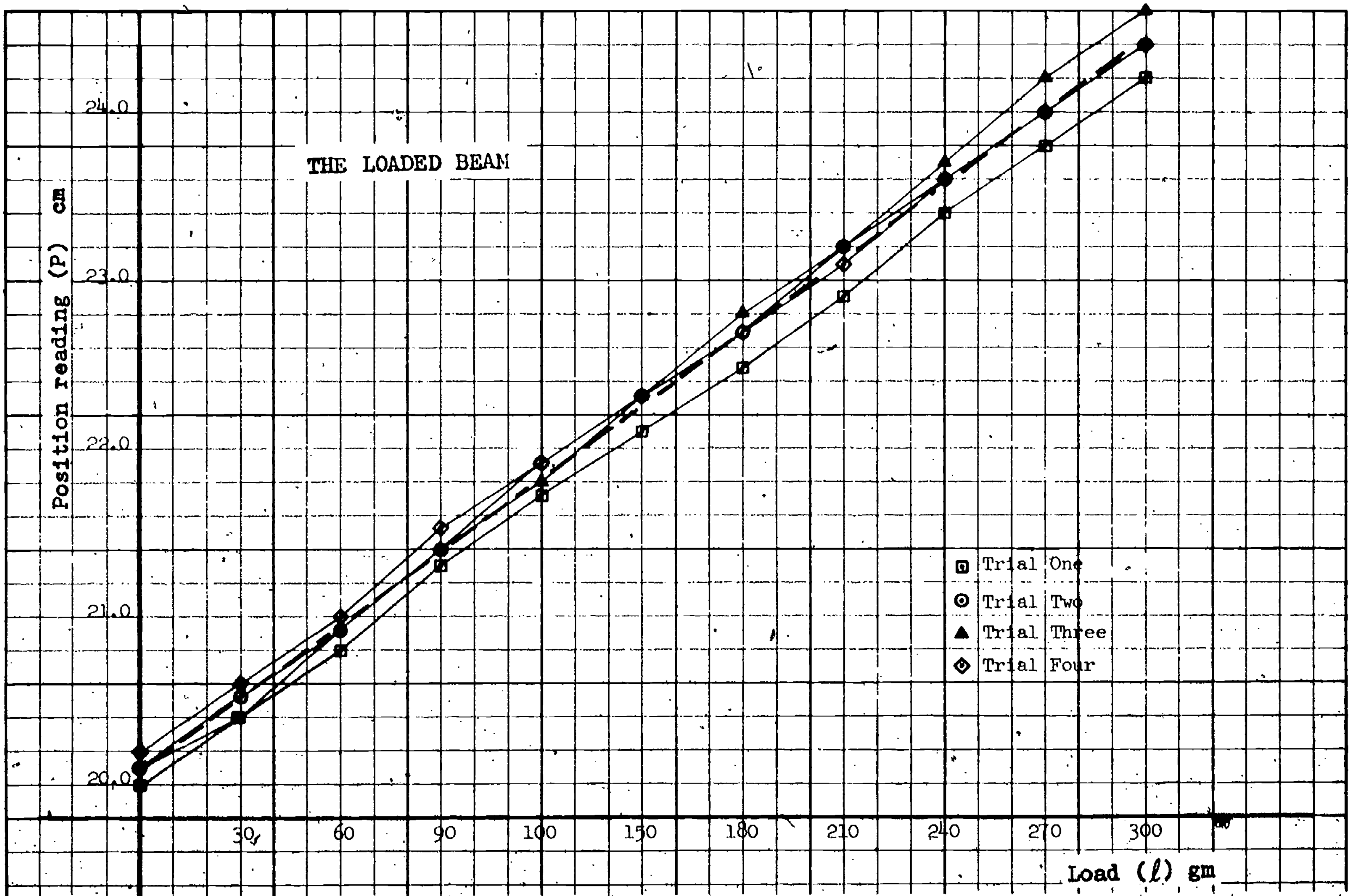
4. and 5.



In heating the iron rod, any reasonable length of time may be used.

6. Have students find this on their graph. From the graph in No. 5 above, 165° will correspond approximately to 8.29; 9.3 will correspond approximately to 186°.

7. Probably due to inaccuracy of measurements or of graphing the data.

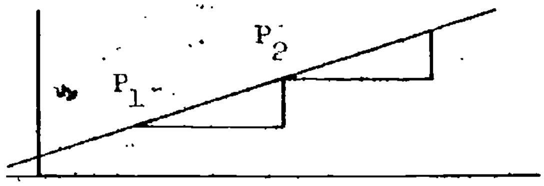

THE LOADED BEAM
Position reading (P) cm
24.0
23.0
22.0
21.0
20.0
30
60
90
100
150
180
210
240
270
300
Load (ℓ) gm
Trial One
Trial Two
Trial Three
Trial Four

4.3 Slope and Intercepts

This will perhaps be the student's first introduction to the concept of slope as a precise concept. Defining the slope as

$$\frac{\text{vertical change}}{\text{horizontal change}}$$

will, we think, make sense to the student and he will probably be willing to accept the statement that the slope of a straight line is a constant. The reason, of course, is that for any two points the vertical change and horizontal change are sides of similar right triangles and their ratios are equal. To find the slope of a particular line, the student must choose two specific points on the line, find their coordinates by measurement or counting squares at this level. Then by subtracting the vertical values he will have the vertical change, and the horizontal change may also be found by subtraction. If you apply this to the "best straight line" for the Loaded Beam graph above, you may choose the points A and B on the line. A is chosen on the load line at 60 and you can determine that the vertical distance to the point is about 20.9. Hence, the horizontal distance is 60 and the vertical distance is 20.9. The second point B has a horizontal distance of 270 and a vertical distance of 24. The difference in the vertical distances is $24 - 20.9 = 3.1$ and the difference in the horizontal distances, taken in the same order, is $270 - 60 = 210$. Hence the slope is



$$\frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{3.1}{210} = 0.015$$

The reason that this is so small is that the horizontal distance increases so much more rapidly than the vertical distance.

The y-intercept, as stated in the student's text, is the point where the line crosses the y-axis. In this section the student will usually read this from the graph and must see that the coordinates of the point are (0, b). To find the equation of the line, he must be able to see that if this point (0, b) is used with an arbitrary point (ℓ, p), which may be anywhere on the line, he can write an equality which will be the equation of the line in terms of the two variables ℓ and p and the two known quantities m and b. This gives

$$p = m\ell + b.$$

On the loaded beam graph the value of b is 20.1. Hence, the equation of the "best straight line" is

$$p = 0.015\ell + 20.1$$

Have your students find the equation of their "best straight line."

In order to apply this to an actual case where he knows the slope is $m = \frac{5}{4}$, as in the student's text, he can read from the graph that $b = 6$, and know that the coordinates of the y-intercept are $(0, 6)$. This gives the equation

$$\frac{p - 6}{\ell - 0} = \frac{5}{4}$$

To get the final equation which appears in the student's text, he must be able to

multiply by ℓ $$\frac{\ell(p - 6)}{\ell} = \ell\left(\frac{5}{4}\right)$$

$$p - 6 = \frac{5\ell}{4}$$

add 6 $$p - 6 + 6 = \frac{5\ell}{4} + 6$$

$$p = \frac{5\ell}{4} + 6.$$

If the student has not learned these skills with understanding, then you will need to teach them before you can go further. You will recognize that this involves the use of both the additive inverse and the multiplicative inverse and that the student will have to understand these aspects of the real number system before he can go on.

The use of "interpolation" here may be confusing to you. At this point it is straight line interpolation and it is first done graphically by finding the value of y <u>on the graph</u> which corresponds to some value of x. For the scientists these values of the first variable x must always be between those for which he has taken experimental data. He must also assure himself that these values have meaning and that he could have made measurements on them if he had wanted to. (This is not always true. See Experiment C of this chapter.) For example, he could have placed a load of 50 grams on the loaded beam (ruler) and measured the position of the end. After the equation is available, he may let $\ell = 50$ and solve for p by a substitution process. This is familiar to you as a mathematics teacher.

You and your students should realize by now that the relation expressed in the ordered pairs, by their graph, and by the "best" line and its equation, have definite meaning only for the interval of measurements made and recorded. We suspect that heavier loads could have been added, but we also know that at some point the ruler will break, or a metal beam might bend permanently. We know that it does not make sense to add negative loads, though you could contrive to introduce a force in a negative direction, that is, <u>up</u> instead of down. However, there are limits in all cases and you never <u>know</u> what data beyond the actual measurements taken will exhibit the same relation as that taken between the two variables. Hence, scientists are always cautious about going beyond the limits of the data taken. This is called <u>extrapolation</u>. The procedures for predicting values of the other variable are the same as with interpolation, but use it cautiously. More will be said about this in other experiments.

Answers to Exercises, Section 2.3

These problems are to be solved by reading the coordinates of the points and intercepts from the figures.

1. $m_{(p \text{ and } q)} = \frac{1}{2}$, $m_{(p \text{ and } r)} = \frac{1}{2}$, $m_{(q \text{ and } r)} = \frac{1}{2}$

2. For ℓ_2, $m = 1$; for ℓ_3, $m = 2$; for ℓ_4, $m = 3$
   intercept $b = 0$ $\quad b = 0 \quad b = 0$

3. For ℓ_5, $m = \frac{1}{3}$ $\quad$ For ℓ_6, $m = \frac{1}{3}$
   $b = 0$ $\quad b = 3$

4. Its slope <u>and</u> y-intercept.

5. $y = \frac{1}{2}x$, $\quad y = x$, $\quad y = 2x$
   $y = 3x$, $\quad y = \frac{1}{3}x$, $\quad y = \frac{1}{3}x + 3$

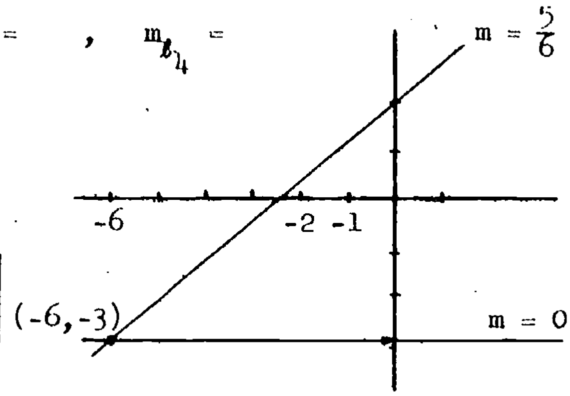
6. It was the <u>position</u> on the meter stick from which the amount of deflection was measured. No.

4.4 Graphing Linear Functions

This is a mathematical extension of graphing various linear functions. We have followed the practice in mathematics and used x and y and coordinates of a point as (x, y) when needed. Points have been used in various orders to find slopes and the idea of a negative slope is introduced here for the first time. We have also introduced lines where the calculated slope is 0, (horizontal lines), and lines for which the slope is undefined (vertical lines). The other ideas are explained in the text.

Answers to Exercises, Section 2.4

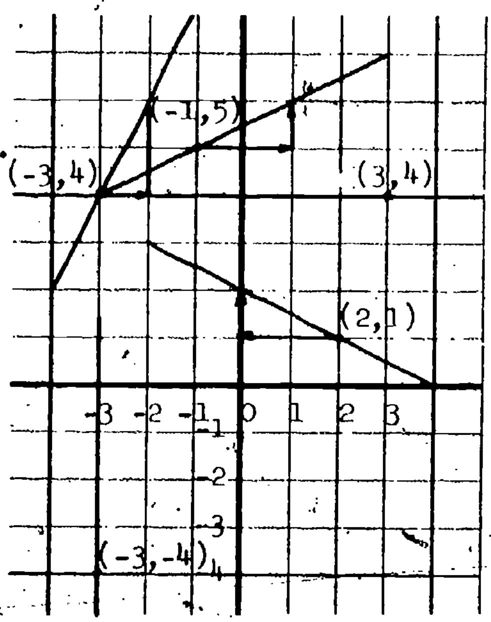
1. $m_{\ell_2} =$, $m_{\ell_2} =$, $m_{\ell_3} =$, $m_{\ell_4} =$



2. 0, slope undefined

3. (a) and (b)

4.



5. $m_1 = \frac{3 - (-1)}{3 - 1} = \frac{3 + 1}{2} = \frac{4}{2} = 2$

$m_2 = \frac{-9 - (-1)}{-3 - 1} = \frac{-8}{-4} = 2$ $\qquad m_1 = m_2$, $(-3, -9)$ is on the line

6. (a) $y = \frac{2}{3}x - 2$; (b) $y = \frac{3}{4}x$; (c) $y = -2x + \frac{4}{3}$; (d) $y = -7x - 5$

7. $m = \frac{4 - 0}{3 - 0} = \frac{4}{3}$. 0, $\quad y = \frac{4}{3}x + 0 = \frac{4}{3}x$

8. The slope would have been negative.

9. $m = \frac{2 - (-4)}{-3 - 3} = \frac{6}{-6} = -1$

$\frac{y - 2}{x - (-3)} = -1$ $\qquad$ $\frac{y - (-4)}{x - 3} = -1$

$y - 2 = -x - 3$ $\qquad$ $y + 4 = -x + 3$

$y = -x - 1$ $\qquad$ $y = -x - 1$

10. (a) $y = \frac{1}{5}x + 3$ (b) $y = \frac{12}{5}x - 4$ (c) $y = \frac{2}{3}x - 2$

(d) $y = -\frac{8}{5}x + 6$ (e) $y = -\frac{1}{3}x + 2$ (f) $y = 3$

(g) $x = -3$ (h) $y = \frac{1}{7}x + \frac{10}{7}$

4.5 Relations and Functions

The last section of this chapter is a careful attempt to show the students what is meant by the concepts of Relation and Function. Since these ideas will be used many times in later work in this chapter and in more advanced work in both science and mathematics, every effort should be made to see that this section is clearly understood.

The definition of a function given in the text tells when a given relation is a function, but the graphical methods given for telling this may prove most useful at this point. The answers to part (d) - (k) of the example in Figure 20 are:

(d) function, (e) not a function, (f) function,
(g) function, (h) function, (i) not a function,
(j) not a function, (k) not a function.

This exercise is not hard and can be done in a few minutes during class.

Discuss carefully the meaning of the converse of a relation given in the paragraphs following this example. After they have solved the problems in the exercise, especially number 5, discuss the meaning of inverse functions.

Answers for Exercises, Section 2.5

1. (Solved in text)
2. domain {2}, range {3,4,5} relation is in function
   For converse: domain {3,4,5}, range {2}, converse not a function
3. domain {3,6,7}, range {3}, relation is a function
   For converse: domain is {3}, range {3,6,7}, not a function
4. domain {3,4}, range {6, -2}, not a function
   For converse: domain {6,-2}, range {3,4}, not a function
5. domain {-1,-2,-3}, range {-3,-5,-7}, a function
   For converse: domain {-3,-5,-7}, range {-1,-2,-3}, a function

Most of the points regarding slope, intercept, equation, function, relation, etc., will be developed further later in this chapter and in the others. Hence, while you should not attempt at this time to develop these concepts completely, do be careful not to over-simplify them and give the students erroneous or incorrect ideas.

Answers for Exercises, Section 2.5

1. a. {0,30,60,90,...,300}, {20,20.4,20.8,...,24.6}
   b. Yes
2. Yes
3. No. They include all of the values of d from 30 to 300, inclusive and from 20 to 24.6 inclusive.
4. The domain is the same, the range varies slightly
5. $l = -\frac{p}{m} - \frac{b}{m}$ We could have hung enough mass on the beam to give ~~the~~ desired amount of deflection.
6. Yes. Since a vertical line will not cut the graph of $\ell = -\frac{p}{m} - \frac{b}{m}$ in more than one point, the converse is a function.

4.6 <u>The Falling Sphere Experiment</u>

Though this is an investigation of <u>terminal velocity</u> and the measurements made are of time and distance, only after the data has been graphed and analyzed and the slope of the "best straight line" found at the end of B.1 is the idea of velocity introduced.

The "falling sphere" is actually a small steel bearing ($\frac{1}{8}$ inch in diameter). It is used because it reaches its terminal velocity quickly in Karo syrup and because it is attracted to a small magnet. It can, therefore, be brought to the surface and positioned properly for the repeated trials used in this experiment.

This experiment will be done by the class working in small groups of 4 to 5 each. The equipment needed for each group is:

- small upright glass cylinder (8 inches by $1\frac{1}{4}$ inches)
- steel ball bearing, $\frac{1}{8}$ inch
- small magnet
- centimeter ruler (19 cm.)
- metronome from music department
- Karo syrup (white)
- 10 paper strips, 1 × 10 inches
- cellophane tape

Data for the table of values (these are also ordered pairs of numbers) is found by measuring the marked tape and by finding time after the sphere has fallen 0 seconds, 2 seconds, 4 seconds, etc. The table of values found for this experiment is given in the table on the next page.

THE FALLING SPHERE EXPERIMENT

Time t (seconds)	Trial 1 Distance d (millimeters)	Trial 2 Distance d (millimeters)	Trial 3 Distance d (millimeters)	Trial 4 Distance d (millimeters)		
0	0	0	0	0		
2	5.5	6.5	6.5	6.5		
4	11.5	13.5	12.0	15.5		
6	20.0	21.0	19.0	23.0		
8	27.5	27.5	27.0	30.0		
10	35.0	35.5	34.0	37.5		
12	43.5	42.0	40.5	45.0		
14	50.5	49.5	47.0	52.5		
16	56.0	57.0	54.0	59.0		
18	62.5	65.5	61.5	67.0		

4.7 The Graph and the Equation

Graphing the data and the drawing of the "best straight line" follows the same procedure as in the Loaded Beam experiment. A difference is that the origin, whose coordinates are (0, 0), is a point on all the graphs.

When the line has been drawn by each student, he can find its slope by finding the coordinates of one point (t_1, d_1) on the line, since he already has the point (0, 0). The coordinates of the point (t_1, d_1) must be found by measurement or by counting on the student's graph. The slope m which the student finds will be a number, usually a fraction. It will be a constant for his graph, but will differ from his classmates'.

To find the equation of the line we use again the same procedure as in the Loaded Beam experiment. That is, we select an arbitrary point on the line with coordinates (t, d). By also using the point (0, 0) the equation comes out easily as

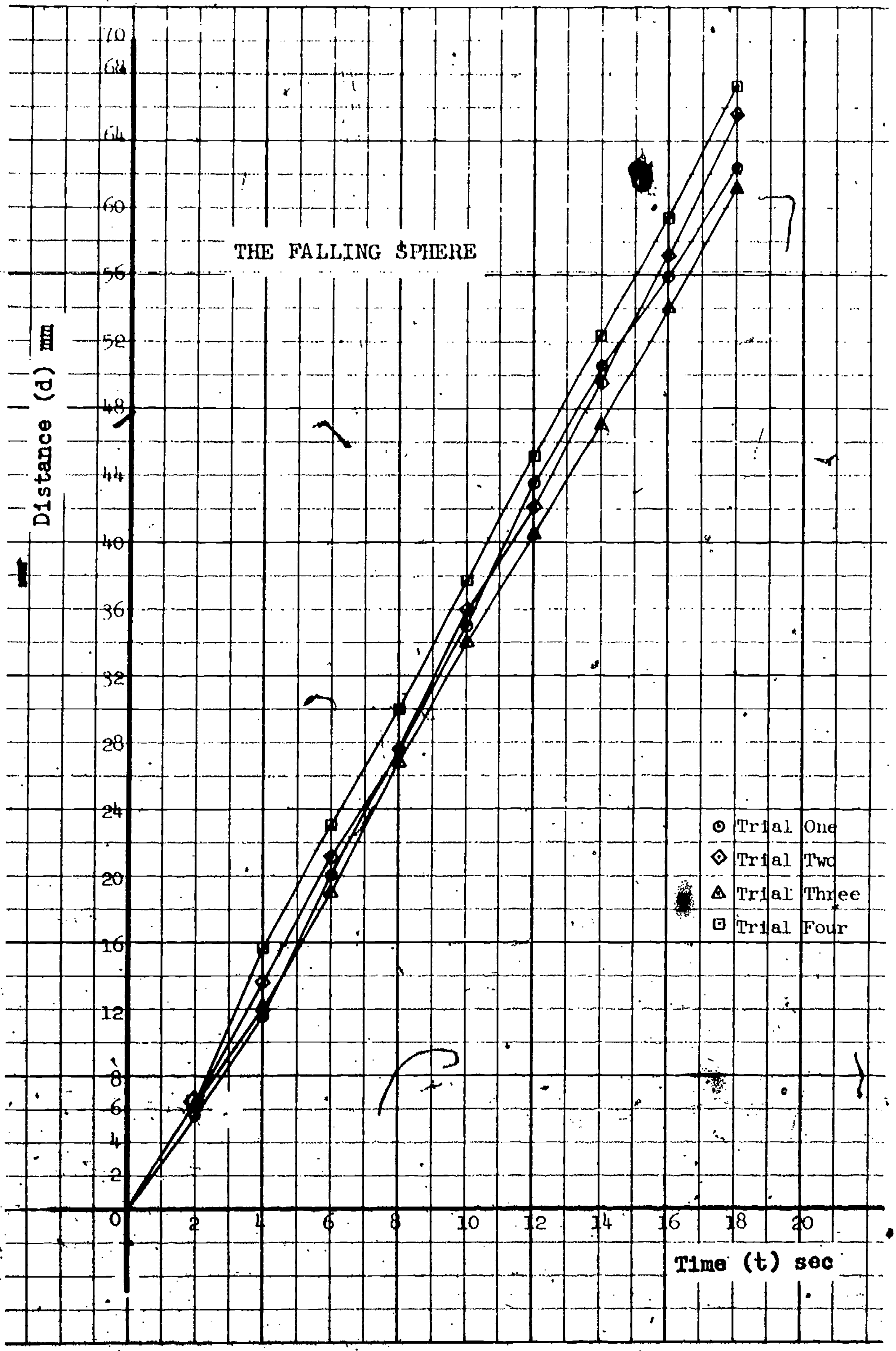
$$d = mt$$

where m is the student's constant slope. The equation of the best line on the graph here is:

$$d = \frac{7}{2}x .$$

Only after this point in the analysis of the data of the experiment is the velocity mentioned. The discovery that the slope of the line represents the terminal velocity of the falling ball bearing should emphasize a meaning of the slope not often encountered in the mathematics classroom. As indicated in the student's text, this will be discussed more fully in Section B.3 on Physical Units.

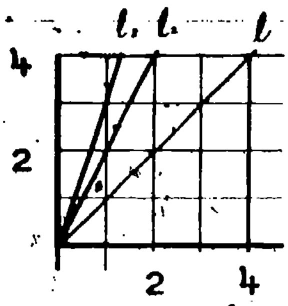
Special Note: Have each student save both his data sheet and his graph for all experiments done. These will be necessary to solve exercises in this chapter and in later chapters. (You may want to take them up and store them.)

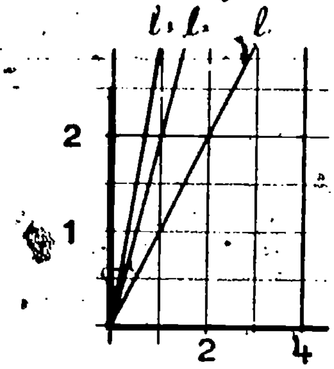

THE FALLING SPHERE
Distance (d) mm
Time (t) sec
70
68
64
60
56
52
48
44
40
36
32
28
24
20
16
12
8
6
4
2
0
2
4
6
8
10
12
14
16
18
20
Trial One
Trial Two
Trial Three
Trial Four

68

Exercise

The purpose of the first two problems in this exercise is to give alternate ways to find the best straight line from the data already collected.





The slopes are the same because slope is defined by

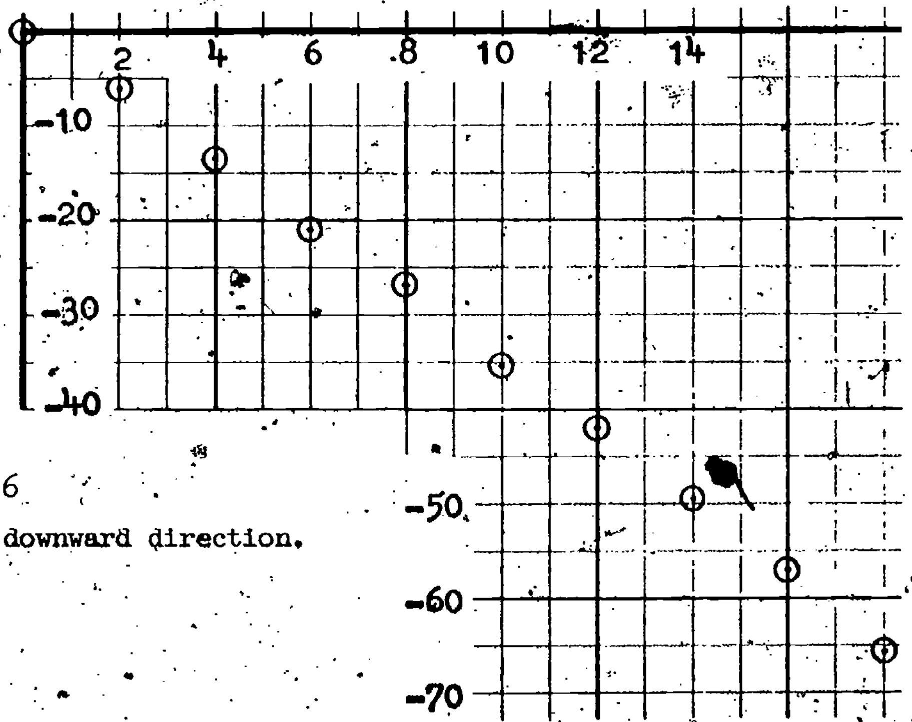
$$m = \frac{\text{Vertical distance}}{\text{Horizontal distance}}.$$

Hence the slope of ℓ_1 in both cases is $m = \frac{1}{1} = \frac{4}{4} = 1.$

5. Problem 5 introduces for the first time negative coordinates. This may not be clear to the students, but let them think about it for a while. Actually the distances were measured in a downward direction. By using a pulley arrangement in the Loaded Beam experiment, the beam could have been pulled upwards, which is opposite to the action in the experiment. You might want to suggest this variation to the experiment to some of your better students. The graph of this problem appears below.

Table for No. 5.

time	distance
0	0
2	-6.5
4	-13.5
6	-21.0
8	-27.5
10	-35.5
12	-42.0
14	-49.5
16	-57.0
18	-65.5



The slope is $m = -3.6$

The velocity is in a downward direction.

4.8 The Point-Slope Form

This section is a mathematical development of the various forms for the equation of a straight line culminating with the point-slope form. A brief development shows how the special forms can be quickly derived from the point-slope form. For example, if one of the points is (0, 0), then

$$y - d = m(x - c)$$

becomes

$$y - d = m(x - 0)$$

or

$$y = mx,$$

the line which passes through the origin.

Exercises for 2.8

1 and 2. In these problems the students will have to calculate the slope first using the coordinates of the two points read from the graph. Then by the use of this slope and one of the given points together with an arbitrary point (x, y) the equation may be written in the point-slope form or in one of the special forms when it applies to the particular line. For example, for line ℓ_2 the given points R and S have the coordinates (3, 2) and (12, 7). Hence the slope is

$$m = \frac{7 - 2}{12 - 3} = \frac{5}{9}$$

Using the point (3, 2) with $m = \frac{5}{9}$, the point-slope form gives

$$\frac{y - 2}{x - 3} = \frac{5}{9}$$

$$9y - 18 = 5x - 15$$

$$5x - 9y = -3$$

Using (12, 7)

$$\frac{y - 7}{x - 12} = \frac{5}{9}$$

or

$$5x - 9y = -3$$

3. The reason for this problem is to show that the intercept may be found from the equation and that we do not have to depend on reading these from the graph. Since this graph has scales instead of axes, it is not possible to read the intercepts on the graph. The procedure, of course, is to substitute

0 for x in the equation and solve for y, then substitute 0 for y and solve for x. For ℓ_6 the equation is $y = -\frac{10}{7}x + \frac{276}{7}$, the x-intercept is $(27.6, 0)$ and the y-intercept is $(0, \frac{276}{7})$.

4. The student might have to measure the coordinates for the point selected. This may be done by counting squares if the line crosses at an intersection.

5. $y = 3x - 18$, $m = 3$, y-intercept number 18

$y = -2x + 4$, $m = -2$, y-intercept number 4

$y = \frac{5}{2}x - 12$, $m = \frac{5}{2}$, y-intercept number -12

$y = -4x - 12.5$, $m = -4$, y-intercept number -12.5

4.9 Physical Units

This suggestion of introducing physical units in the mathematics classroom is against recent trends in the teaching of mathematics. Elementary arithmetic and algebra have been developed by the use of the real number system with the basic operations of addition and subtraction. It is true that the other systems and other operations are introduced briefly, but this has been done so that the real number system will be understood.

There is good reason for writing open sentences and equations in terms of the real numbers. The properties of this system are clear and concise and not too difficult for children to understand and use. This enables us to be sure that our results are reasonable and mathematically sound. Recent writers of mathematics text books for the schools have kept the real number system and its properties closely in mind and have succeeded well in conforming to this pattern. When they have needed physical units in the comparatively few problems which have been used to illustrate open sentences and equations, it has been suggested that children think of the number of such units rather than the units themselves. When the solution of such an equation has been found, then the student is expected to recall the various units involved with these numbers and assign a single or a compound unit to the answer. The answer to the problem is then such units as 30 pounds per square inch, 50 feet per second, and so forth.

This is not considered adequate for the physical sciences. Hence, traditionally mathematicians and scientists have solved problems differently. The scientists have considered the use of units necessary for a proper understanding of the physical situations which can be stated in mathematical form. The mathematicians have not considered this a serious problem and hence students have been confused. At least at the elementary level, no simple set of basic principles of defining a system of units and the operations needed for them has been available to the students and teachers.

Yet the solution of this teaching problem is simple. It is possible to define physical units as a set of elements with three (or five) undefined terms. Under an operation of "multiplication," signified by the symbol $\otimes$, and not unlike multiplication with real numbers, this forms an abelian group. This is described more carefully below.

4.10 A Mathematical Approach to Units and Dimensions.

It will now be shown that all physical measurements in a single system of units are the elements of a single set which forms a group, under a given operation of multiplication. This set not only includes the standards for length, mass and time, but all the physical quantities that are derivable from these three undefined quantities. A few of the derived quantities are force, velocity, acceleration, density, momentum and energy. The set could be expanded to include the standards of temperature and electric charge, but this will not be done in the treatment that follows.

Physical quantities expressed in the meter-kilogram-second (MKS) system of units are the elements of one group, while the British system of units constitutes yet another group.

Although many of the mathematical details of the following development will not concern the student directly, it is thought best to present the notion of physical quantities as members of a group in a rather formal way:

Let the elements of a set U be

$$nL^{\frac{p}{q}} M^{\frac{r}{s}} T^{\frac{t}{u}}$$

where n is a real number $\neq 0$, p, q, r, s, t, u are integers, and q, s, u $\neq 0$.

The symbols L, M and T refer to the standards of length, mass and time. In the MKS system of units, L ~ meter, M ~ kilogram and T ~ second. In the British system of units, L ~ foot, M ~ slug, and T ~ second. Using this notation, a length such as 5.4 meters would be written $5.4\ L^1M^0T^0$. An acceleration such as $0.6\frac{\text{meter}}{\text{sec}^2}$ would be written $0.6L^1M^0T^{-2}$. Also, a force (F = ma) of 12 newtons would be written $12\ L^1M^1T^{-2}$. The real numbers are also elements of the set; $1\ L^0M^0T^0$ is 1, $5.3\ L^0M^0T^0 = 5.3$, $-3\frac{\text{Kg}}{\text{Kg}} = -3$.

Let a binary operation of "multiplication," symbolized by "$\otimes$," be defined as follows:

$$nL^{\frac{p}{q}} M^{\frac{r}{s}} T^{\frac{t}{u}} \times n'L^{\frac{p'}{q'}} M^{\frac{r'}{s'}} T^{\frac{t'}{u'}} \quad nn'L^{\frac{p}{q}\frac{p'}{q'}} M^{\frac{r}{s}\frac{r'}{s'}} T^{\frac{t}{u}\frac{t'}{u'}}$$

Clearly, the operation exhibits <u>closure</u> and yields a <u>unique</u> product.

The set is a group U ("units" group) with respect to the multiplication operation indicated above because the following properties are satisfied.

(i) If a, b, c are any elements of U, then

$(a \otimes b) \otimes c = a \otimes (b \otimes c)$ (associative law)

(ii) There exists an element e of U such that

$e \otimes a = a \otimes e = a$

for every element a of U. (existence of an identity)

We have $e = 1\ L^0 M^0 T^0$.

(iii) For every element a of U, there exists an element x of U such that

$a \otimes x = x \otimes a = e$ (existence of inverses)

We have, for $a = nL^{\frac{p}{q}} M^{\frac{r}{s}} T^{\frac{t}{u}}$

$x = \frac{1}{n} L^{-\frac{p}{q}} M^{-\frac{r}{s}} T^{-\frac{t}{u}}$

Further, for every a and b in U

$a \otimes b = b \otimes a$ (commutative property)

U is therefore an abelian group.

One final property of physical quantities is required. It is contained in the following definition:

<u>Definition</u>: For any two elements in the group

$$nL^{\frac{p}{q}} M^{\frac{r}{s}} T^{\frac{t}{u}} \oplus n'L^{\frac{p}{q}} M^{\frac{r}{s}} T^{\frac{t}{u}} = (n'+n) L^{\frac{p}{q}} M^{\frac{r}{s}} T^{\frac{t}{u}}$$

The above definition serves to define the operation "$\oplus$" which can be applied if and only if the values of $\frac{p}{q}$, $\frac{r}{s}$ and $\frac{t}{u}$ are the same for both elements. Notice that in the special case where $n + n' = 0$, the sum is not an element of the group. Otherwise, the operation "$\oplus$" gives closure, and in all cases yields a unique "sum."

<u>Theorem.</u> If a, b, c are elements of U, then

$$a \otimes (b \oplus c) \quad (a \otimes b) \oplus (a \otimes c) \text{ (distributive law)}$$

Here the operation " $\oplus$ " is defined by the definition. As a consequence, the " $\oplus$ " operation indicated on the right above is proper whenever the " $\oplus$ " operation on the left is proper. In other language, if b and c have the same units, the units of $a \otimes b$ will be the same as those of $a \otimes c$.

The mathematical basis for the treatment of physical properties is important in the following ways:

1. The definition of the elements of the group focuses attention upon a single system of units.

2. The operation " $\otimes$ " is one which always yields another element of the same group. If, for example, MKS units are strictly adhered to, all new quantities encountered must be MKS quantities.

3. The operation " $\otimes$ " is the one actually performed in science when "hybrid" units such as ft. lb., meter $\sec^{-2}$ and slug ft. $\sec^{-2}$ are obtained.

4. The operation " $\oplus$ " allows one to add feet to feet, kg to kg, and newton meter to newton meter. This operation permits one to obtain 0, a situation often encountered in physical situations, but 0 is not an element of the group and " $\otimes$ " has no mathematical or physical meaning for it.

5. Division in physical situations can be considered in the same manner as in mathematics. Division is multiplication by an inverse. That is, meter^{-2} sec is the same as $\frac{1}{\text{meter}^2}$ sec.

6. The distributive law is not really necessary for the analysis of physical situations. In all cases the addition operation " $\oplus$ " could be performed prior to the one of multiplication.

7. Conversion of units may be performed without considering millimeters as an element in the MKS group, nor ounces in the British group. "Millimeters" and "ounces" may be taken as different names for elements that <u>are</u> in these groups. Identity is then expressed as mm $= \frac{1}{1000}$ meter, cm $= \frac{1}{100}$ meter and km $=$ 1000 meter. A conversion table of this form is used simply to <u>replace</u> mm, cm and km whenever they are encountered, by its equivalent--an element <u>in the group</u>. To get <u>out of the group</u>, one

would require identies in the form $1000 \text{ mm} = 1$ meter, $100 \text{ cm} = 1$ meter and $\frac{1}{1000}$ km $= 1$ meter. Notice that no algebraic manipulations are required in either procedure.

This topic will almost certainly appear strange to the mathematics student. He may even think that it is not sound mathematically. However, it is sound and perhaps you can convince him of this by presenting to him, or having him recall, certain systems which use only a subset of the real numbers, like the whole numbers, the even numbers, numbers modulo 12, or "clock" numbers. Also other definitions of addition or multiplication can be used as effectively as the ordinary ones.

Answers for Exercises, Section 2.10

1. 12 kg meter; $13 \frac{\text{Kg}}{\text{sec}}$; 15 meters; $10 \frac{\text{meter}}{\text{sec}^2}$; $147 \frac{\text{Kg meter}}{\text{sec}^2}$; 5.4 sec

2. $5 \frac{\text{meter}}{\text{sec}^2}$; 1.8 meter Kg; no ans.; $7.6 \frac{\text{meter}}{\text{Kg}^2}$; no answer

3. The unit of C is Kg; the unit of d is meter; the unit of m is $\frac{\text{m}}{\text{Kg}}$

4. The unit of C is sec; the unit of d is inch, the unit of m is $\frac{\text{inch}}{\text{sec}}$.

5. The unit of slope is $\text{m} \frac{\text{meter}}{\text{meter}}$ or m, a real number.

6. The unit of x is $\frac{\text{sl}^2}{\text{ft}}$; the unit of b is sl.

4.11 <u>Replacement of Units</u>

The purpose of this section is to teach the student how to replace units written with various units within the MKS system into the standard units of meter, kilogram and second. This is done by the use of tables. A similar table is included for the British system. Hence you will want your students to change millimeters into meters and kilometers into meters, etc. The answers to the problems of Exercise B.4 are:

1. 86 mm = 0.086 meter; 1700 m; 1.5 Kg. $0.0073 \frac{\text{meter}}{\text{sec}}$; $2600 \frac{\text{meter}}{\text{Kg}}$; $0.0000057 \frac{\text{Kg}^2}{\text{sec}}$; $219 \frac{\text{meter}}{\text{Kg}}$

2. $88 \frac{\text{ft}}{\text{sec}}$; 72 ft^2 ; 2 ft^2 ; $0.38 \frac{\text{ft}}{\text{sec}}$; $0.0015 \frac{\text{slug}}{\text{sec}}$; $0.0067 \frac{\text{ft slug}}{\text{sec}^2}$; $4 \frac{\text{slug}}{\text{ft}}$

3. $\frac{Kg}{cm}$

4. 1.9 meter, 0.0522 Kg, no answer (can't add)

5. $176 \frac{ft}{sec}$; $\frac{19}{129} \frac{ft}{sec}$

4.12 The Trampoline Experiment

This experiment deals with data which is not linear in character, but which can be can be analysed by the use of derived data which is linear. This will present problems to both the student and the teacher. However, the explanation given here with that in the student's text book should be sufficient.

This experiment will be performed by the teacher as a class demonstration. You will need to use student help to do the experiment. The equipment needed will be:

- 1 9-inch aluminum pie plate
- 6 15-cent balloons, spherical
- 1 10 x 24-inch sheet bristol board
- 1 pound of plastolene
- 2 glass marbles
- 1 desk lamp (or slide projector)
- 2 $\frac{5}{8}$ - inch nylon bearings

4,13 Functions of Integers

The results of this experiment, when graphed on coordinate paper in Figure 3 indicate clearly that the points do not lie ~~in a~~ straight line. While this is a perfectly good graph and has an equation, the ordered pairs for the graph are single points and the use of the equation, if the students could find it, is beyond this present level of mathematical knowledge. Certainly it is not a linear function as the chapter heading indicates. The data obtained for this experiment and the graph of the data is included in Figure 1 and Figure 2.

THE TRAMPOLINE

Bounce number n	Height in cm (h) trial 1	trial 2	trial 3	trial 4	Average height h	Corrected height h_n	h_{n+1}	Calculated heights h_{n+1}
0	50.1	(obtained from shadow)			50.1	46.0	40.3	39.6
1	44.5	44.5	44.4	44.3	44.4	40.3	34.2	34.0
2	38.3	38.2	38.2	38.4	38.3	34.2	27.7	29.3
3	31.9	31.8	31.8	31.8	31.8	27.7	23.0	25.2
4	27.3	26.8	27.0	27.2	27.1	23.0	19.7	21.6
5	23.8	23.7	23.9	23.6	23.8	19.7	17.0	18.6
6	21.3	21.1	21.0	21.0	21.1	17.0	15.5	16.0
7	19.2	19.3	19.5	20.3	19.6	15.5	13.0	13.8
8	17.3	16.8	17.3	16.9	17.1	13.0	11.0	11.9
9	15.1	14.9	14.8	15.5	15.1	11.0	9.0	10.1
10	13.8	11.2	13.7	13.8	13.1	9.0		

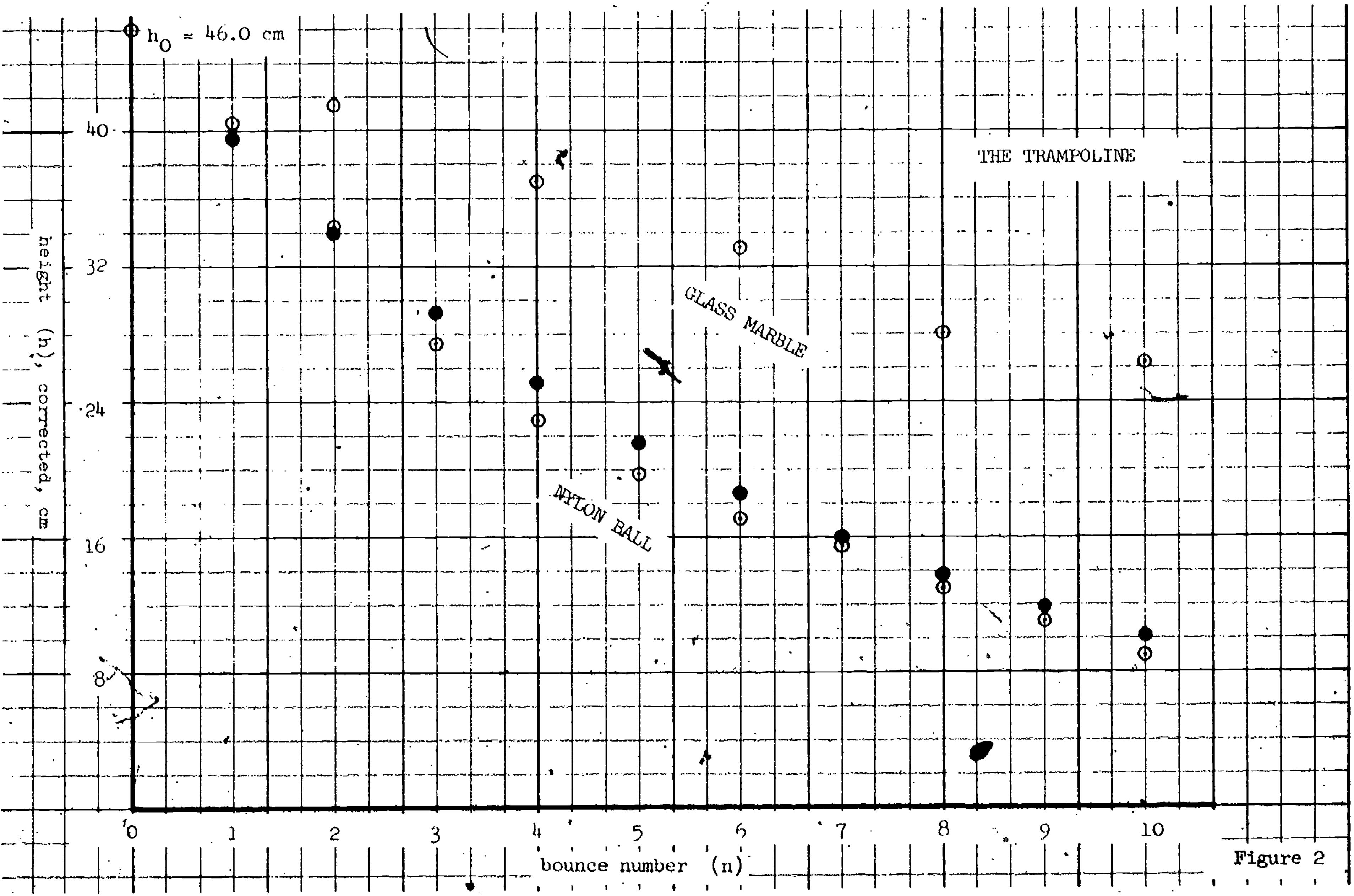
$\frac{5}{8}$-inch nylon bearing (diam. 1.4 cm)

light source ≈ 4 meter distant

obtained by subtracting 4.1 cm

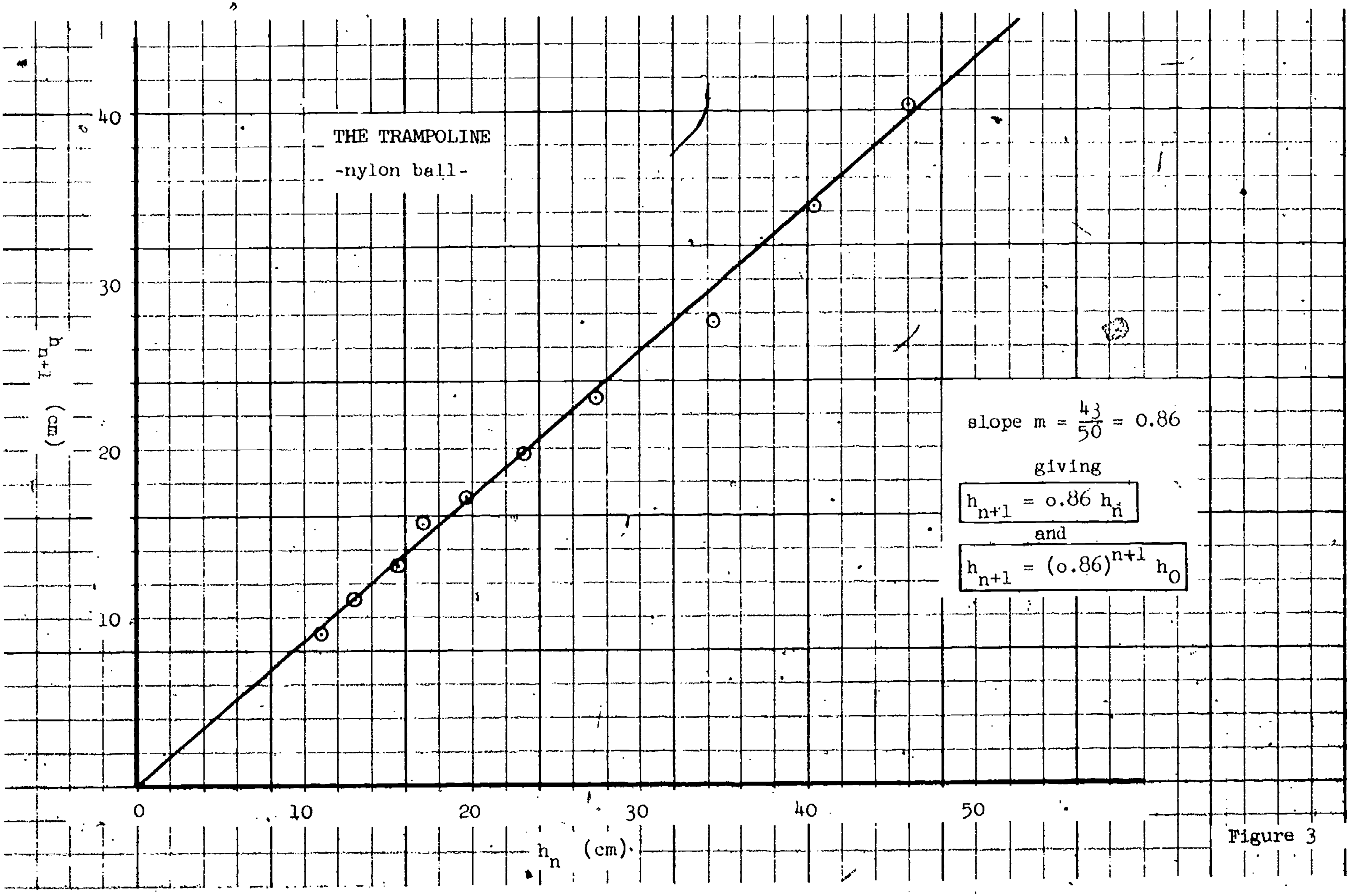
NYLON BALL

Figure 1



$h_0 = 46.0$ cm
THE TRAMPOLINE
GLASS MARBLE
NYLON BALL
height (h), corrected, cm
40
32
24
16
8
0
1
2
3
4
5
6
7
8
9
10
bounce number (n)

Figure 2



THE TRAMPOLINE
-nylon ball-
slope m = 43/50 = 0.86
giving
$h_{n+1} = 0.86\ h_n$
and
$h_{n+1} = (0.86)^{n+1}\ h_0$
h_{n+1} (cm)
h_n (cm)
0
10
20
30
40
50

Figure 3

Answers for Exercises after 2.13

1. No. The bounce height occurs once, and once only, for each integral number 0, 1, 2, 3 etc. There is not such thing as a bounce number $3\frac{1}{3}$.

2. Yes. Each number n (0, 1, 2, 3, etc.) gives a unique corresponding height h.

3. Yes, for greater than 10, but the height will be harder to measure. As n increases, the bounce heights become more difficult to measure. Furthermore the bounces soon stop!

4. Data for this could be best collected from local newspapers. Almanacs and calendars include general information for some time zones and these times are not usually true locally. Such times for special localities are usually given in T.V. weather broadcasts.

4.14 Mathematical Trampoline Model

Hence at this point we look more closely at the data and try to find some representation which does yield a straight line graph. Fortunately in this case we succeed quickly, but in later attempts it will not be as easy or as successful.

We suspect that there is a relation between the height to which the ball bounces and the height to which it bounced the time before. With the data already collected we are able to make a table of values for each height and the succeeding bounce, which is actually the next bounce. The table, which is written in the student's text as the ordered pairs

$$(h_o, h_1)$$
$$(h_1, h_2)$$
$$(h_2, h_3)$$
$$\vdots$$
etc.

has the following ordered pairs from the data presented earlier from the experiment done by the writing team. (See Figure 1.)

If these ordered pairs are arranged in a table we can consider the first elements as being represented by h_n and the second by h_{n+1} and the table would appear again as in Figure 1.

These were used in plotting the points for the graph in Figure 34 of the student's text. When they use their own data this may be slightly different.

In this case it is also possible to fill in the points on the line by varying the height from which the ball is dropped. The slope must be obtained by measuring distances for the coordinates of two points on the line. For example, when $h_n = 30$, $h_{n+1} = 22.5$ and when $h_n = 20$, $h_{n+1} = 15.5$, hence the slope is

$$m = \frac{43}{50} = 0.86$$

and the equation of the line is

$$h_{n+1} = 0.86\ h_n .$$

If the equation is written in the form

$$h_{n+1} = mh_n$$

then

$$m = \frac{h_{n+1}}{h_n} .$$

This will enable us to determine all of the bounce heights in terms of the bounce number and the height of the previous bounce--or more simply, in terms of the number n and the original height from which the ball was dropped. The data table in C.1 could have been written in the form

n	h
0	$m \cdot 50$
1	$m(m \cdot 50) = m^2 \cdot 50$
2	$m^3 50$
3	$m^4 50$
⋮	⋮
n	$m^{n+1} 50$

The result of this is the equation given in the student's text

$$h_{n+1} = m^{n+1} h_o .$$

Since m is constant (as shown by the straight line graph, Figure 34) for a given trampoline system, a nylon ball, kind of balloon, etc., the variables are n and h_o, and the equation is one with a variable in the exponent. It is certainly not linear and is restricted to a particular domain, but we have been able to find the equation by the use of our knowledge of linear functions.

This development and use of the linear function will certainly be strange to the secondary mathematics teacher. The procedure is not familiar to most mathematics students, even at the college level, though applied mathematicians and scientists have found it very useful. However, there seems to be no reason why secondary students should not be able to understand what has been done and recognize its usefulness. The procedure will be used again in succeeding chapters with varying degrees of success. It is considered that this knowledge will be extremely useful to students as they go into science courses at higher levels.

4.15 Experimental Extension

This section is self-explanatory and will give the students the idea that the results of a scientific experiment change when the conditions under which data is collected vary.

Exercises

1. This refers to the student's counterpart of Figure 34 in the text. The domain h_n will be approximately $6 \le h_n \le 50$ and the range h_{n+1} $5 \le h_{n+1} \le 40$. (The students' measurements may vary considerably.) The domain and range of the mathematical equation representing this data cannot safely be extended too far from the experimental data.

2. For $h_o = 10$

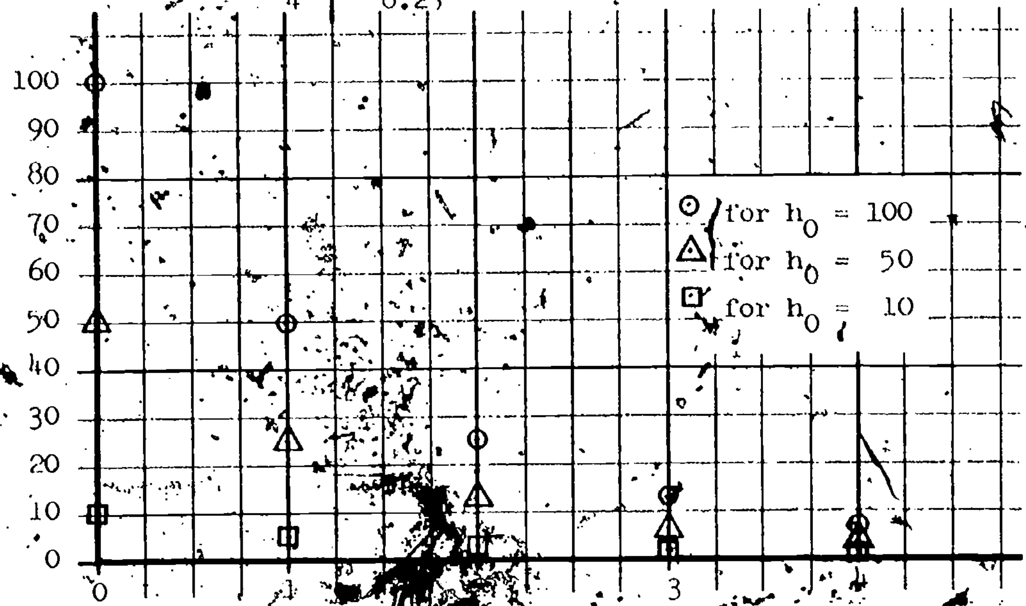
n	h
0	10
1	5
2	$2\frac{1}{2}$
3	$1\frac{1}{4}$
4	$\frac{5}{8}$

For $h_o = 50$

n	h
0	50
1	25
2	12.5
3	6.25
4	$3\frac{1}{8}$

For $h_0 = 100$

n	h
0	100
1	50
2	25
3	12.5
4	6.25



3. For $m = 0.3$

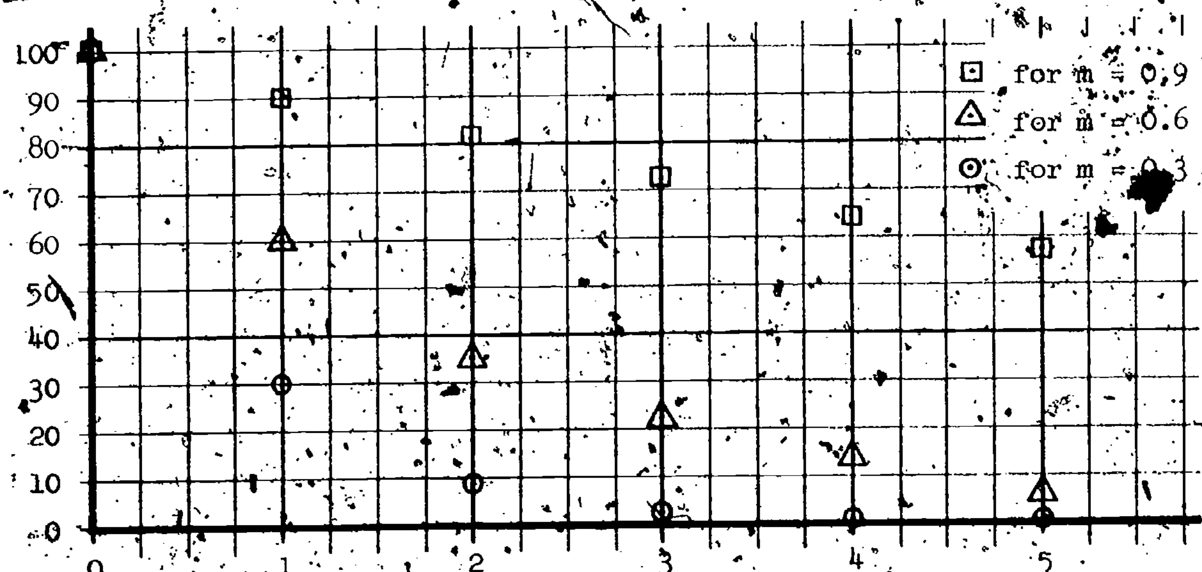
n	h
0	100
1	30
2	9
3	2.7
4	.81
5	.24

For $m = 0.$[illegible]

n	h
0	100
1	60
2	36
3	21.6
4	13
5	7

For $m = 0.9$

n	h
0	100
1	90
2	81
3	72.9
4	64.6
5	58.1



THE TRAMPOLINE

Bounce number n	Height in cm (h) Trial 1	Trial 2	Trial 3	Average height h	Corrected height hn
0	50.2			50.2	46.1
2	45.3	45.4	45.5	45.4	41.3
4	41.3	41.3	41.2	41.3	37.2
6	37.2	37.4	37.0	37.2	33.1
8	32.4	32.4	32.4	32.4	28.3
10	[illegible]	31.1	30.6	30.8	26.7

Calculations not carried farther because $\frac{h_{n+2}}{h_n}$ yields the square of the slope m obtained with nylong ball.

--DATA DISPLAYED ON (n,h) PLOT--

GLASS MARBLE

FIGURE 4

4.16 Charles' Law Experiment

This experiment will deal with the change in pressure of an enclosed gas (ordinary air) vs. the temperature. However, we shall be interested in this experiment because there is some physical meaning to the extension of the domain (the temperature) beyond the experimental data collected and the accompanying mathematical concepts of graphical and algebraic translation of the coordinate axes.

Because of the expense of the equipment and the fact that hot water is used, this experiment will be done by the teacher as a class demonstration. You will need to have a few students assist you in the experiment. The equipment needed is:

A Charles Law apparatus
Centigrade thermometer
Vessel for water
A source of hot water
Ice
Heater to raise water to 100 degrees C (optional)
Ring stand to hold apparatus (optional)

Data for the table of values (temperature, pressure) will be read by the experimenters and should be copied on the chalkboard and by each student in the class. The table of values found in this experiment is given on the following page.

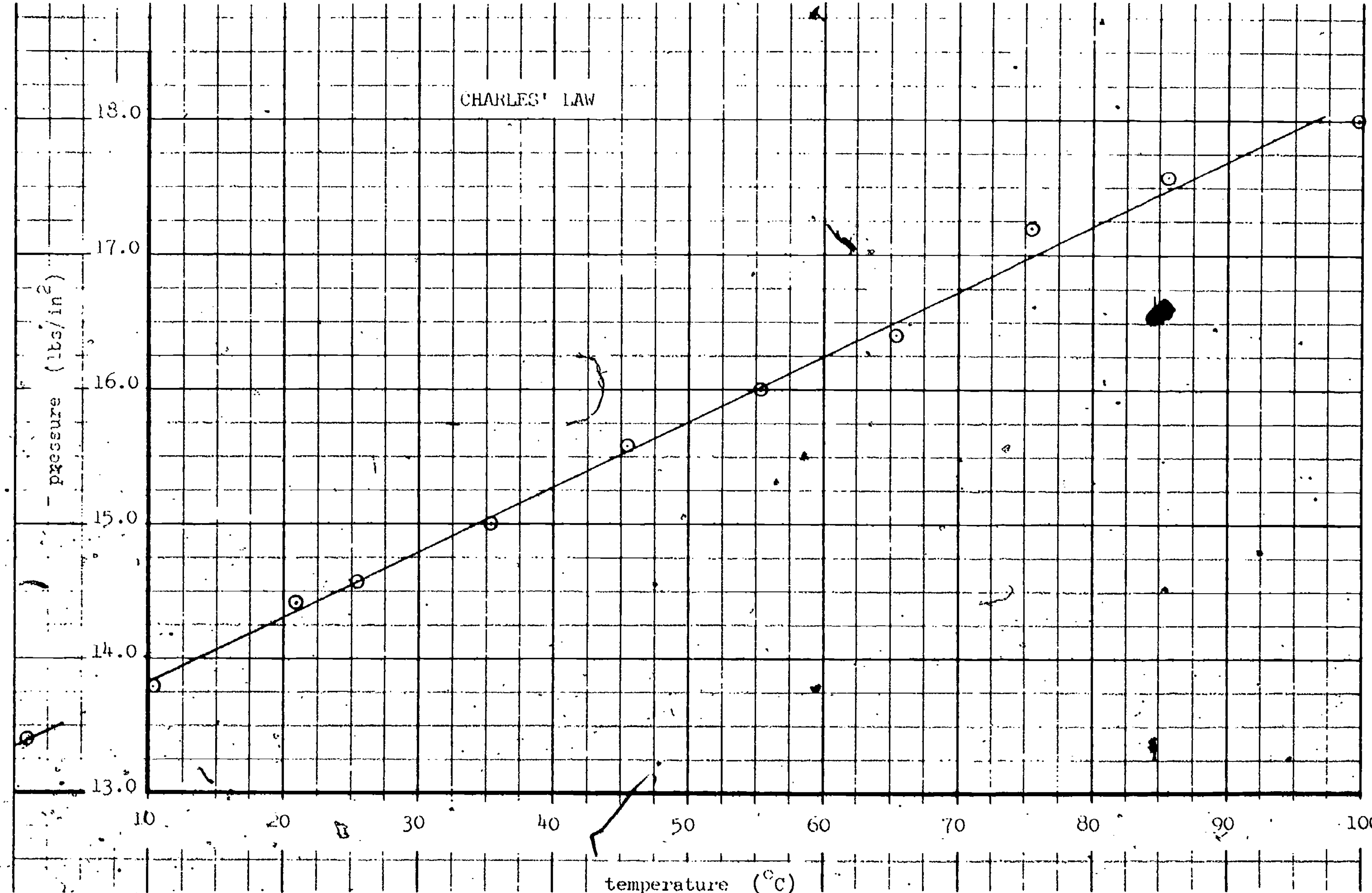
C H A R L E S' L A W

Temp. (°C)	Pressure (lbs/in^2)								
99.4	18.0	← boiling water							
85.5	17.6								
75.5	17.2								
65.5	16.4								
55.5	16.0								
45.5	15.6								
35.5	15.0								
25.5	14.6								
21.0	14.4								
10.5	13.8								
1.2	13.4	← ice mixture							
guage tapped gently before each reading.									

Each student will then graph this data on a coordinate system using the appropriate scales. He should then be able to draw his best straight line as has been done in the other experiments.

The students should also be able to write the equation of the line quickly by the use of the point-slope form or perhaps by the slope-intercept form if you were able to reduce the temperature to 0 degrees C in the experiment.

For the data given above, the graph appears on the following page.


CHARLES' LAW
18.0
17.0
16.0
15.0
14.0
13.0
pressure (lbs/in^2)
10
20
30
40
50
60
70
80
90
100
temperature (°C)

A best straight line is drawn and its slope is found by the use of the points M and N to be

$$m = \frac{16.50 - 15.25}{65 - 40} = \frac{1.25}{25} = 0.05$$

If (C, P) represents any arbitrary point on the line, we may use the point M (40, 15.25) to write the equation of the line

$$P - 15.25 = 0.05\ (C - 40)$$

$$P = 0.05C - 24.75$$

No exercise is given at this point because we are more interested in the possibility of extending the range of the domain and range of the function.

4.17 <u>Extending the Temperature Domain</u>

The idea of interpolation is brought up again here and some problems at the end of the section on this experiment as well as on preceeding ones are included. However, the idea of <u>extrapolation</u> is emphasized more, though still with some caution. There are apparent physical limits in the other experiments which are more easily recognized. These limits must be considered by the students in solving the exercises.

Since it makes more sense to extend the domain and hence the graph of Charles's Law to the left, we do so and are able to arrive at a temperature for zero pressure which is approximately the <u>absolute zero</u> which scientists know is slightly lower than -273 degrees C. Perhaps the students have also heard of this "magic" temperature.

Answers to Exercises after 2.17

1. No predictions are possible, because this value of ℓ is not in the domain of the function.

2. 35.1

3. Some are: a stronger beam, the same beam turned on its edge, a shorter beam.

4. None, this is not in the domain of the function.

5. $d = \frac{7}{2}t, \longrightarrow d = \frac{7}{2}(120) = 420$

6. Some are: extending the length of the glass vessel, decreasing the relative weight of the sphere, increasing the viscosity of the liquid.

4.15 Graphical Translation of Coordinate Axes

Translating axes by the use of the sheet of frosted acetate, one of which will be furnished for each student, is not difficult and will often help the student to simplify the equation of the line and the orientation of the graph. We will also use translation of axes in later experiments which will yield quadratic functions. Before the acetate sheets are passed out the teacher should draw a set of coordinate axes in the middle of each sheet. Use a heavy ball-point pen. Do not write any number or scales on the sheet. Students will write these with a pencil since they can then be easily erased.

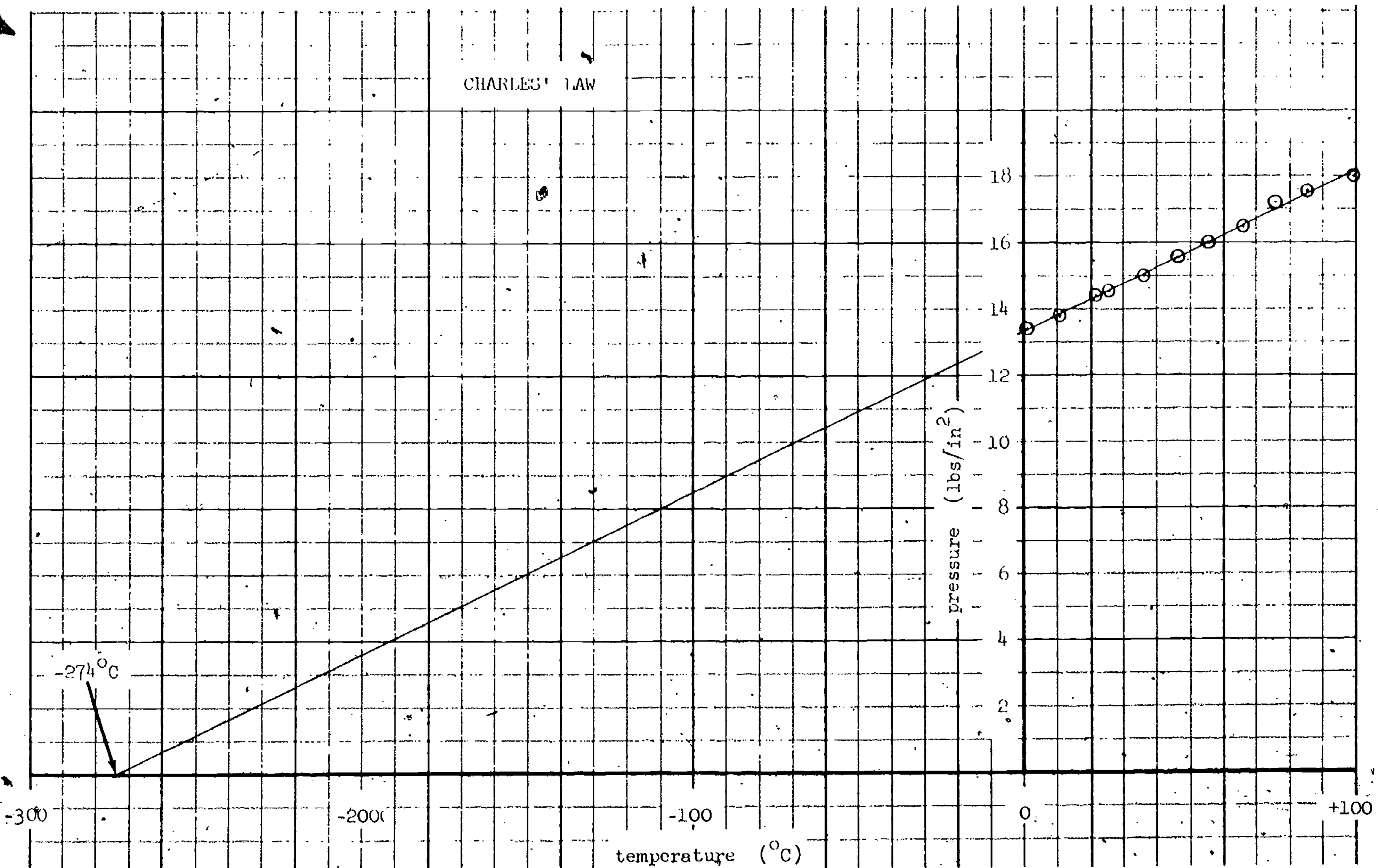
While it is possible to position the new axes in any manner whatsoever, (i.e., they can be translated and rotated) translation horizontally and vertically will be sufficient for our purposes.

The amount and direction of translation may be purely arbitrary but when we use this on graphs based on science experiments the physical situation usually suggests a "natural" location for the new axes. For example, when dealing with Charles's Law it will be the point at which the pressure is zero. The only way the student will be able to approximate this point will be to extend his graph to the left until it crosses the temperature-axis (not included in his original graph). If this point of zero pressure is near to -273 degrees on the C-axis (between -260 degrees C and -285 degrees C) it will be close enough. The equation in the new coordinate system will obviously be in the form

$$y = mx .$$

Answers to Exercises at the end of 2.18

1. $y = 0.015\ell + 20.1$
2. Have the meter stick placed so the end of the beam coincides with 0 on the stick.
3. (a) $y = \frac{1}{2}x$ (b) $y = \frac{1}{2}x + 3.5$ (c) $y = \frac{1}{2}x + 2$
4. (a) $y = -\frac{1}{3}x$ (b) $y = -\frac{1}{3}x$ (c) $y = -\frac{1}{3}x$


CHARLES' LAW
18
16
14
12
10
8
6
4
2
pressure (lbs/in^2)
-274°C
-300
-200
-100
0
+100
temperature ($^{\circ}$C)

Exercises

1. Using the equation $P - 15.25 = 0.05\ (C - 40)$ we translate the axes to $(-273, 0)$

$$Y + 0 = 0.05\ (X - 233).$$

The new units of temperature are degrees C, of pressure pounds per square inch. The slope of the line is the same, since this is not affected by translation of the axes.

2. $y - 3 = \frac{1}{2}\ (x - 2)$

(a) $Y = \frac{1}{2}\ X$

(b) $Y - 3 = \frac{1}{2}\ (X - 5)$

(c) $Y = \frac{1}{2}\ (X + 2)$

Compare

4. (a) $Y + \frac{1}{3} = \frac{1}{3}\ (X - 1)$

(b) $Y - 7 = -\frac{1}{3}\ (X + 21)$

(c) $Y - 1 = -\frac{1}{3}\ (X + 3)$

Compare

94

4.10 Algebraic Translation of Coordinate Axes

This section is purely a mathematical discussion of translation of the axes. Though we did write the equations of the line representing Charles's Law at the end of the preceding paragraph, this cannot always be done as easily, especially when the graph represents curved lines. (See Chapter 2 and 3 on quadratic functions.)

The explanations of the algebraic translation of axes in the student's text is clear and needs no elaboration here. You may feel, however, that more drill similar to Exercise 2 will be needed. All you will need to provide this drill is to give the class equations of the sort $3x - 2y = 7$ and ask them to translate the origin to various points like the y intercept, the point $(5, 4)$ or $(0, 6)$, and so on.

To translate the axes of this equation to the point $(5, 4)$ write it in the form

$$y = \frac{3}{2}x - \frac{7}{2}$$

or

$$y - 0 = \frac{3}{2}\left(x + \left(-\frac{7}{3}\right)\right).$$

It is now in the point-slope form and you may use equation (2)

$$Y + (0) + K = m\left(X + \left(-\frac{7}{3}\right) + h\right)$$

where h and K are 5 and 4 respectively. This becomes

$$Y + 0 + 4 = \frac{3}{2}\left(X + \left(-\frac{7}{3}\right) + 5\right)$$

$$Y + 4 = \frac{3}{2}\left(X + \frac{8}{3}\right)$$

or

$$Y = \frac{3}{2}X$$

Since the point $(5, 4)$ is on the line you would expect to get equations in the form

$$Y = mX .$$

If, on the other hand, you want to translate to the point $(5, -4)$, it is done in this manner:

$$Y + (0) + (-4) = \frac{3}{2}\left(X + \left(-\frac{7}{3}\right) + 5\right)$$

$$Y - 4 = \frac{3}{2}\left(X + \frac{8}{3}\right)$$

$$Y - 4 = \frac{3}{2}(X + 4)$$

$$Y = \frac{3}{2}X + 8$$