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ABSTRACT

This student text presents instructional materials for a unit of mathematics within the Biomedical Interdisciplinary Curriculum Project (BICP), a two-year interdisciplinary precollege curriculum aimed at preparing high school students for entry into college and vocational programs leading to a career in the health field. Lessons concentrate on biomedical applications in the mathematics areas of propagation of error, vectors, and linear programing. Reading materials, graphs, illustrations, and problems accompany each lesson. (CS)

BIOMEDICAL MATHEMATICS

UNIT II

PROPAGATION OF ERROR, VECTORS AND LINEAR PROGRAMMING

STUDENT TEXT
REVISED VERSION, 1975

THE BIOMEDICAL INTERDISCIPLINARY CURRICULUM PROJECT

SUPPORTED BY THE NATIONAL SCIENCE FOUNDATION

Any opinions, findings, conclusions or recommendations expressed herein are those of the author and do not necessarily reflect the views of the National Science Foundation

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SECTION 1:

1-1 A Different Type of Question

Suppose that you are diving in the ocean. You know that the volume of your air tank is V liters and that the pressure when you began was P atmospheres. Your breathing rate is x liters per minute, and you are at a depth of d meters. Before diving you applied your knowledge of the gas laws. You calculated that you had enough oxygen to stay at a depth of d meters for exactly one hour.

How long would you stay under water? Exactly one hour? 59 minutes and 59 seconds? 55 minutes? 50 minutes? What do you need to know before you can decide how long you may safely be submerged?

It is our purpose in the succeeding sections to help you answer this type of question. But before we discuss how to answer such questions let us consider several more situations.

Knowing the vital capacity of an individual's lungs helps to diagnose respiratory diseases. Vital capacity information from a large group of young people allows us to predict a normal vital capacity for a person of any particular height. Suppose a young person is 170 centimeters tall. The predicted vital capacity for young people 170 cm tall is 4.6 liters. However, this person's vital capacity is only 4.0 liters.

Does this indicate that his lungs are not functioning properly? Is the dysfunction serious?

Consider another problem. Our equations indicate that a person with a mass of 73 kilograms (about 160 pounds) can drink no more than five 12-ounce cans of beer without the concentration of alcohol in his blood reaching .10 per cent. If the legal limit in his state is .10 per cent, how much beer can he drink and then drive legally?

In Biomedical Science you will analyze a food for its content of certain nutrients. Federal law requires meat sold as "hamburger" to be no more than 30 per cent fat. If you analyze a sample of hamburger and find that it is 31 per cent fat, are you entitled to complain?

You have been asked a large number of questions, none of which you can answer. You have been given equations relating diving time to oxygen pressure and volume, relating vital capacity to height, and relating blood-alcohol concentration to quantity of alcohol consumed. Yer these questions have not allowed you to answer any of the questions we have asked. Why?

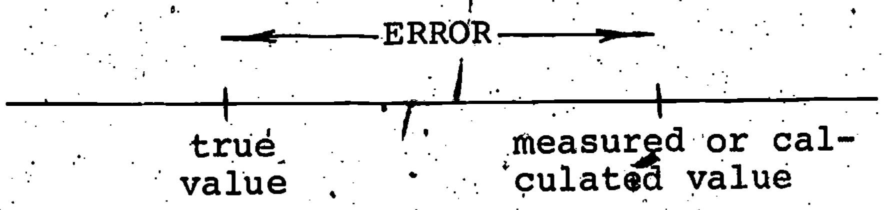
The questions all relate to matters of certainty and uncertainty. To answer the questions we need to know how accurate our predictions are. And the accuracy of the predictions depends on the accuracy of other things: the equations we use, the numbers we substitute into the equations and the procedures we use to obtain the numbers. The more we know about the accuracy of our measurements, our procedures and our equations, the better we are able to state the range of uncertainty of our predictions. This brings us to the subject of error.

1-2 Error

Error often means doing something wrong. Errors are mistakes, the things that Elmo makes. He pours the wrong chemical into the wrong solution. He adds 3 and 5 and gets 9.

However, the word error has a more general meaning than this, and a meaning that is far more important to scientific investigations. Not only Elmo's experiments contain error; the experiments of everyone have error in every procedure. Error is an unavoidable part of the experimental process.

In mathematics the word error has a very specific meaning. Error is the difference between the measured or calculated value of a particular quantity and its true value. This idea is illustrated below.



Unfortunately, the exact value of many errors can never be determined. Instead we are forced (when we are being "fussy") to estimate a range around a measured value in which we expect the true value to lie.

Errors come from many sources. For example, in Unit I Math we concentrated on a particular source of error, imprecision. Recall that precision for length measurements is closely related to the limits of careful scale reading. It was in this context that we introduced the term range of imprecision. There is another type of measurement error which was touched upon briefly in Biomedical Science, inaccuracy. This source of error is related to variability in the measuring instrument. For example, a ruler is longer when it is hot than when it is cold. This variation in length is not noticeable in an ordinary metric ruler but it is very noticeable in a micrometer, an instrument that can measure very small changes in length. Both imprecision and inaccuracy contribute to measurement error.

On the following graph, the vital capacity of a large number of female children and adolescents is plotted against the cube of height. We may use the data on the graph to predict the vital capacity of a specific girl whose height we know. Yet our predicted vital capacity will almost certainly not be her actual vital capacity, even if she is in perfect health. This error in our prediction is caused by the fact that people are different. We call this form of error statistical variation.

A third type of error occurs in experimental procedures. One investigation of Boyle's Law involved a plunger with weights and a syringe (see diagram on following page). Friction between the plunger and the syringe was a source of error. (You may recall that you obtained different data when the plunger was moving down and when it was moving up. We tried to minimize the error caused by friction by averaging the two numbers.) We call this type of error procedural error.

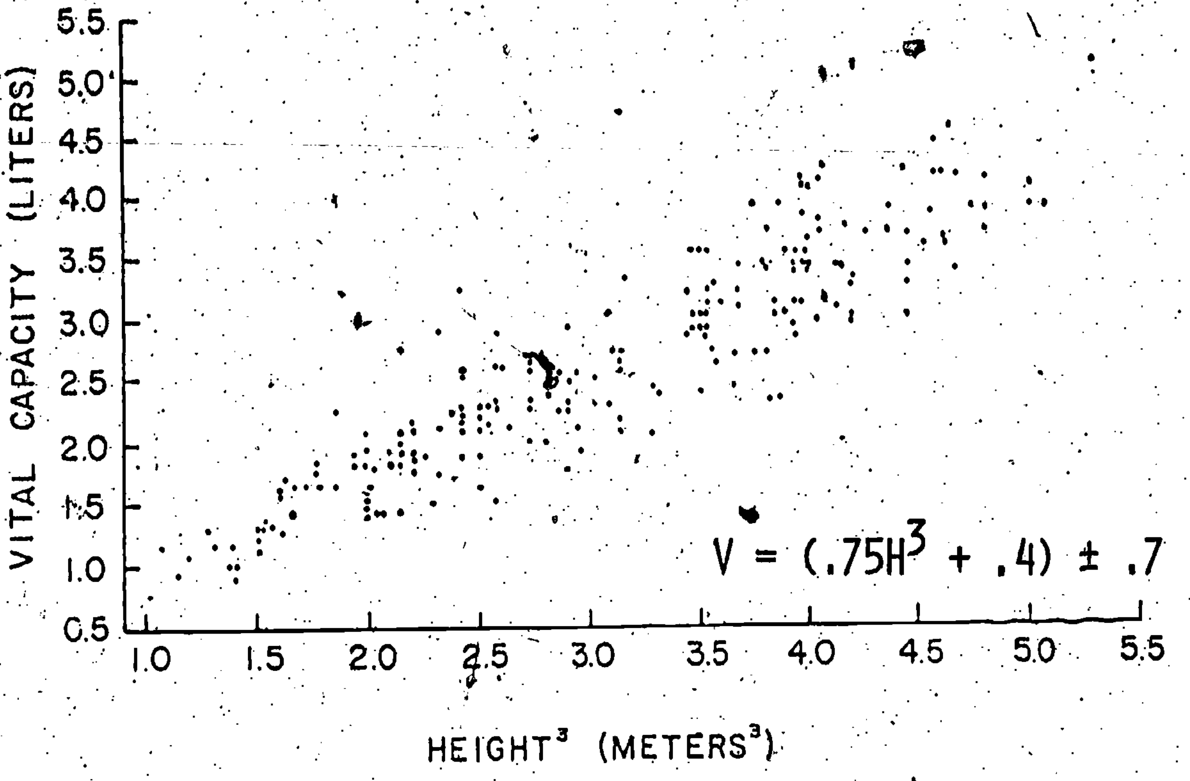
A fourth source of error is the equations we use. For example, the gas law equation we have used is

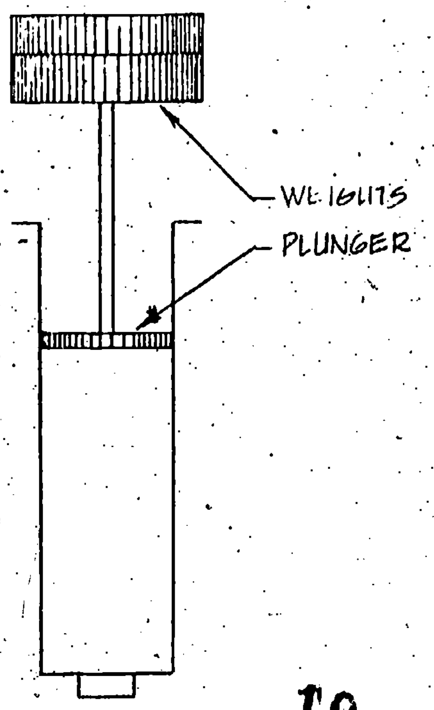
$$\frac{PV}{T} = \text{constant.}$$

This equation, although very useful, is only an approximation. A closer approximation is Van der Waals' equation.

$$\frac{\left(P + \frac{a}{V^2}\right)(V - b)}{T} = \text{constant}$$

VITAL CAPACITY VS. HEIGHT^3
IN FEMALE CHILDREN AND ADOLESCENTS





The terms a and b are constants that are different for every gas. Van der Waals' equation is more complicated than our simpler gas law, and more difficult to use. However, it is generally true that the greater the number of factors that are considered in a theory the smaller the error and the better the prediction. Often a more sophisticated mathematical model must be used to obtain sufficient accuracy.

An equation is a mathematical model of a theory, and when an equation is not accurate, it is a reflection that the theory is not accurate. Thus we call this type of inaccuracy theoretical error.

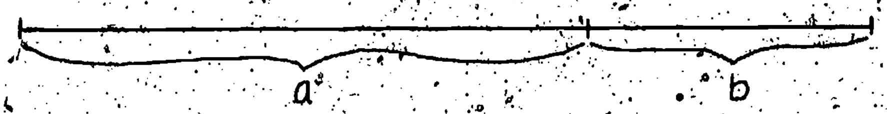
We have identified four types of error: measurement error, statistical vaiation, procedural error and theoretical error. In future sections we will not be interested in assigning errors to one category or another. We only listed the types to illustrate our concept of error. What is important is that you be able to recognize sources of error and, when possible, to estimate the maximum size of the error. We will call this estimate the uncertainty. The midpoint together with the uncertainty will be called a range of uncertainty. For example,

$$\overbrace{\underbrace{47.04}_{\text{midpoint}} \pm \underbrace{.01}_{\text{uncertainty}} \text{ grams}}^{\text{range of uncertainty}}$$

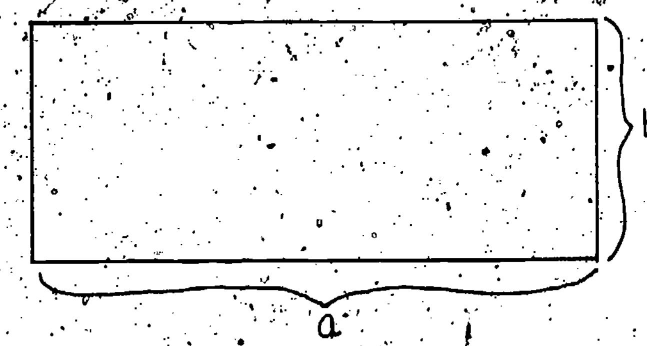
Uncertainty is a much more general term than imprecision and refers to estimates of the maximum possible error from any conceivable source.

When the maximum size of an error may be estimated, it is possible to determine what effect the error may have on the final result. Each time an operation is performed on a number that is not absolutely accurate, the error is passed on to the number that results from the operation. This is called propagation of error.

In this sequence of lessons concerning error we will develop methods to calculate the effect of error on results. For example, we will learn how to answer questions of the following types:



The measured length of a is expressed by the range of uncertainty 100 ± 3 units and the measured length of b by the range of uncertainty 50 ± 1 units. How is the sum $a + b$ expressed as a range of uncertainty?



The measured lengths of a and b are 100 ± 3 units and 50 ± 1 units. The area of the rectangle is the product ab square units. How is this expressed as a range of uncertainty?

The procedures to which we will introduce you cannot always be used. Many errors cannot be numerically estimated. However, these procedures are one set of tools that can help you to obtain a better idea of the accuracy of computed results.

1-3 Propagation of Error Under Addition

In the previous section we presented two measured quantities as ranges of uncertainty. We then asked what their sum is, expressed as a range of uncertainty. The ranges of uncertainty of the two measurements are $a = 100 \pm 3$ and $b = 50 \pm 1$. (For the time being we will ignore their units.)

The maximum value that a could be is $100 + 3 = 103$. The maximum value of b is $50 + 1 = 51$. Therefore, the maximum value of the sum of a and b is $103 + 51 = 154$.

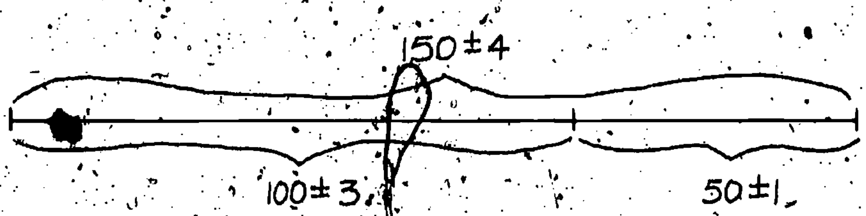
The minimum value of a is $100 - 3 = 97$, while the minimum value of b is $50 - 1 = 49$. The minimum value of the sum is $97 + 49 = 146$.

The sum of a and b may be as great as 154 or as small as 146. This may be expressed as a range of uncertainty by first determining the midpoint. The midpoint is the average of 154 and 146, or

$$\frac{154 + 146}{2} = 150$$

150 is 4 less than 154, the maximum value, and 4 more than 146, the minimum value. Thus the range of uncertainty of the sum of a and b is

$$150 \pm 4$$



Observe what we have done. The midpoint (150) of the sum is the average of the maximum possible sum and the minimum possible sum. The uncertainty of the sum (4) is half the difference between the maximum sum and the minimum sum.

Note also that the midpoint of the sum is equal to the sum of the midpoints of the numbers being added. That is to say,

$$150 = 100 + 50$$

Note also that the uncertainty of the sum is equal to the sum of the uncertainties of the addends

$$4 = 1 + 3$$

The observations made in the previous paragraph are true not only for the two ranges of uncertainty that we added. They are true for the addition of any two ranges of uncertainty.

We will demonstrate the general case by adding two measurements expressed as ranges of uncertainty $x \pm \Delta x$ and $y \pm \Delta y$. Recall that Δ is the Greek letter "delta." Delta as a prefix means roughly "difference in," "change in" or "variation in." For example, Δx is the difference between two values of x before and after some change has occurred. Remember, Δx is a single quantity; it is not the product of Δ and x.

We may find the sum of $x \pm \Delta x$ and $y \pm \Delta y$ in the same way that we found the sum in the numerical example. The maximum value of the sum is

$$x + \Delta x + y + \Delta y = (x + y) + (\Delta x + \Delta y)$$

The minimum value of the sum is

$$x - \Delta x + y - \Delta y = x + y - \Delta x - \Delta y$$
$$= (x + y) - (\Delta x + \Delta y)$$

The midpoint of the range of uncertainty of the sum is the average of the maximum sum and the minimum sum.

$$\frac{(x + \Delta x) + (y + \Delta y) + (x - \Delta x) + (y - \Delta y)}{2}$$
$$= \frac{x + \Delta x + y + \Delta y + x - \Delta x + y - \Delta y}{2}$$

This expression may be simplified to

$$\frac{2x + 2y}{2} = x + y.$$

The midpoint of the sum is $x + y$.

The difference between the midpoint and the maximum value is

$$\underbrace{[(x + \Delta x) + (y + \Delta y)]}_{\text{maximum}} - \underbrace{(x + y)}_{\text{midpoint}} = \Delta x + \Delta y$$

The difference between the midpoint and the minimum value is the same.

$$\underbrace{(x + y)}_{\text{midpoint}} - \underbrace{[(x - \Delta x) + (y - \Delta y)]}_{\text{minimum}} = \Delta x + \Delta y$$

The uncertainty of the sum is thus $\Delta x + \Delta y$. The range of uncertainty of the sum is given by the following formula.

$$\boxed{(x \pm \Delta x) + (y \pm \Delta y) = (x + y) \pm (\Delta x + \Delta y)}$$

We may summarize the general case by the following formal statement. Under the operation of addition, uncertainty is additive.

1-4 Polynomials

The expressions $x + \Delta x$ and $y + \Delta y$ are examples of binomials. Binomials are expressions in which two terms are connected by plus or minus signs. Biomials are one type of a more general class, polynomials. Polynomials are expressions in which any number of terms are connected by + or - signs.

Since operations with polynomials will be used in our study of error, this is a good place to review the algebra of polynomials.

We hope that a rigorous treatment of the polynomials is not needed, and that a few examples will refresh your memory sufficiently to enable you to go on to the problem set.

EXAMPLE:

Add $x^3 + 2x^2 + 2x + 1$ and $x^3 + 3x^2 - 4x + 5$

SOLUTION:

We simply add like terms.

$$\begin{array}{r} x^3 + 2x^2 + 2x + 1 \\ +(x^3 + 3x^2 - 4x + 5) \\ \hline 2x^3 + 5x^2 - 2x + 6 \end{array}$$

EXAMPLE:

Subtract $x^2 + 4xy - 6$ from $2x^2 + 3xy + 4$

SOLUTION:

We simply subtract like terms.

$$\begin{array}{r} 2x^2 + 3xy + 4 \\ -(x^2 + 4xy - 6) \\ \hline x^2 - xy + 10 \end{array}$$

We will also be faced with problems which require the multiplication of two polynomials. For these problems you will need to recall the distributive principle.

$$(a + b)h = ah + bh$$

This principle can also be stated in other forms depending on order and number of quantities involved.

$$(a + b + c)h = ah + bh + bc$$

$$h(a + b) = ha + hb$$

EXAMPLE:

Find the product of $(2x + 5y - 3)$ and $(4x - 3y)$

SOLUTION:

We use the distributive principle.

$$\begin{aligned}(2x + 5y - 3) \cdot (4x - 3y) &= 2x(4x - 3y) + 5y(4x - 3y) - 3(4x - 3y) \\ &= 8x^2 - 6xy + 20xy - 15y^2 - 12x + 9y \\ &= 8x^2 + 14xy - 15y^2 - 12x + 9y\end{aligned}$$

PROBLEM SET 1:

1. What four kinds of error are discussed in the text?

2. You are a carpenter's apprentice for the summer. The carpenter asks you to measure the length of board A and board B which he is going to nail together to make board C (A + B = C). Suppose board A measures 3 ± .01 m and board B measures 4 ± .01 m.

a. What is the maximum possible length of board C?

b. What is the minimum possible length of board C?

c. The midpoint of the range of uncertainty may be found by averaging the maximum and the minimum possible values. Determine the midpoint.

d. Write the range of uncertainty in the form (midpoint) ± (uncertainty).

3. One beef pot pie contains 43 ± .5 grams of carbohydrate and one coke contains 37 ± .5 grams of carbohydrate. Suppose that you have just finished a meal consisting of one beef pot pie and one coke.

a. What is the maximum amount of carbohydrate you could have consumed?

b. What is the minimum amount of carbohydrate consumed?

c. Write the amount consumed as a range of uncertainty.

4. As a result of your investigation in the upcoming Nutrient Analysis of Foods Laboratory Activity, you find that your sample of hamburger is

Water	(50 ± 2)%
Protein	(25 ± 1)%
Fat	(20 ± 1)%

What is the range of uncertainty of the sum?

Perform the additions indicated in Problems 4 through 9.

5. (73 ± .4) + (.7 ± .05)

6. (1.437 ± .012) + (.304 ± .009)

7. 6,789,000 ± 4,500
 + 14,380,000 ± 7,000

16

8. $(.04906 \pm .00072) + (.13987 \pm .00007) + (.000963 \pm .000004)$

9. Let $937 \pm 5 = x \pm \Delta x$
and $463 \pm 21 = y \pm \Delta y$

a. $x =$ c. $y =$

b. $\Delta x =$ d. $\Delta y =$

For Problems 10 through 14, find the sum of the polynomials.

10.
$$\begin{array}{r} 2x^2 - 3xy + y^2 \\ -4x^2 - 6xy - y^2 \\ \underline{x^2 + xy - y^2} \end{array}$$

11.
$$\begin{array}{r} z - 3y - 4\Delta y \\ -7z \qquad - 6\Delta y \\ \underline{5y - \Delta y} \end{array}$$

12. $x^3 - 4x^2 - 7x + 6$, $2x^3 - 5x + 2$ and $-5x^3 + 6x^2 - 8$

13. $m^2 - mn + n^2$, $4n^2 + m^2 - 3mn$ and $5m^2 + n^2$

14. $\frac{1}{3}x + \frac{3}{4}\Delta x - \frac{2}{5}z$, $\frac{1}{4}x + \Delta x - \frac{1}{4}z$ and $\frac{1}{2}x - \frac{3}{8}\Delta x - \frac{1}{10}z$

15.
$$\begin{array}{r} 2x - 3y + 6z \\ \underline{-(4x - 5y + 2z)} \end{array}$$

16.
$$\begin{array}{r} x^3 \qquad\qquad\qquad - y^3 \\ \underline{-(-2x^3 + x^2y - xy^2 + 2y^3)} \end{array}$$

17. Subtract $x^2 - 6x + 5$ from $2x^2 - 4x + 3$

18. What do you add to $2x - 6$ to make $x^2 + 3x + 2$?

19. From the sum of $2x + \Delta x$ and $3x - 4\Delta x$ subtract $5x + 2\Delta x$.

For Problems 20 through 26, perform the indicated multiplications (use distributive property).

20.
$$\begin{aligned} (2x + 1)(x + 3) &= 2x(x + 3) + 1(x + 3) \\ &= 2x^2 + 6x + x + 3 \\ &= ? \end{aligned}$$

21. $(ab - 3)(ab + 3)$

22. $(x^2 - 2y)(x^2 - 5y)$

23. $(a^2 + 2a + 1)(a + 1)$

24. $(x - y)^3 = (x - y)(x - y)(x - y)$

SECTION 2:

2-1 Implied Uncertainty

In your study of nutrition you will use food tables. Food tables give the amounts of nutrients contained in given portions of foods.

TABLE 1

Food/ Measure Weight (grams)		Water	Food Energy	Pro-tein	Fat (Total Lipid)	Car-bohy-drate	Iron	Thia-min	Ascor-bic Acid
	g	%	cal.	g	g	g	mg	mg	mg
Macaroni, 1 cup cooked, enriched	130	64	190	6	1	39	1.4	0.23	0
Muffins, 1 muffin enriched white flour, 2-3/4 in diameter	48	38	140	4	5	20	.8	.08	Trace
Noodles, 1 cup (egg noodles) enriched	160	70	200	7	2	37	1.4	.23	0
Oatmeal, 1 cup cooked	236	86	130	5	2	23	1.4	.19	0
Apple, 1 sector	135	48	345	3	15	51	.4	.03	1
Cherry, 1 sector	135	47	355	4	15	52	.4	.03	1
Lemon Meringue, 1 sector	120	47	305	4	12	45	.6	.04	4
Pumpkin, 1 sector	130	59	275	5	15	32	.6	.04	Trace
Pizza, cheese, 1 sector	75	45	185	7	6	27	.7	.04	4
Popcorn, 1 cup popped, with oil, salt	14	3	65	1	3	8	.3	--	0
Pretzels, 5 sticks	5	8	20	Trace	Trace	4	0	Trace	0
Rice, white, 1 cup all commercial varieties, enriched, cooked	168	73	185	3	Trace	41	1.5	.19	0
Rice, puffed, 1 cup	14	4	55	1	Trace	13	.3	.06	0

Since the numbers in the table were determined by experiment, they are somewhat uncertain. For example, the nutritive value of macaroni would be expected to vary between different brands; this is an example of statistical variation.

However, the data are not expressed as ranges of uncertainty. This is because the uncertainty is implied in the numbers themselves. For example, note the numbers in the column giving the percentage of water: 64, 38, 70, 86, etc. Apparently the data have been rounded to the nearest integer. The actual experimental value for the percentage of water in macaroni might have been 64.3 or 63.92; we only know that it was closer to 64 than to any other integer. In other words the 64 represents a value somewhere within the shaded region below.

Expressing the shaded portion as a range of uncertainty, we obtain 64 ± .5. The uncertainty, .5 in this case, is called the implied uncertainty. The other values in the column can also be written as ranges of uncertainty, becoming 38 ± .5, 70 ± .5 and so on.

The data in the protein, fat and carbohydrate tables have the same implied uncertainty as the water percentages. The numbers in the protein column, for example, would be expressed as ranges of uncertainty as 6 ± .5, 4 ± .5, 7 ± .5, etc.

Observe, however, that the milligrams of iron in each food is given to the nearest tenth of a milligram. Since the data are rounded to the nearest tenth of a milligram, it is implied that one cup of macaroni may contain any amount of iron between 1.35 mg and 1.45 mg.

Thus the implied uncertainty is .05 mg, and the iron in a cup of macaroni may be expressed as 1.4 ± .05 mg.

The amount of thiamin in each food is given to the nearest hundredth of a milligram. This implies that the error is at most .005 mg. The quantity of thiamin in a cup of macaroni may be written .23 ± .005 mg.

The amount of ascorbic acid in each food is apparently given to the nearest milligram, with an implied uncertainty of .5 mg. The presence of the word "trace" in the column, however, implies that 0 means no ascorbic acid was found.

The relation between rounding numbers and implied uncertainty can now be stated. When data are rounded to the nearest 1 g, then the implied uncertainty is .5 x 1 = .5 g. If rounding is to the nearest .01 mg then the implied uncertainty is .5 x .01 = .005 mg. In general, when data have been rounded to the nearest x, the implied uncertainty is .5x.

Note that every number in the food energy column ends in either 0 or 5, and it is safe to conclude that all data have been rounded to the nearest 5 calories. Since the data are rounded to the nearest 5 calories, the implied uncertainty is half of 5, or 2.5 calories. The actual number of calories in a cup of macaroni lies somewhere between 187.5 or 192.5 and may be expressed as 190 ± 2.5 calories.

The first column of numbers, which shows the weight in grams of one portion of each food is more complex than the others. The presence of such numbers as 48 and 236 implies an uncertainty of .5 grams. But the fact that 8 of the 13 numbers end in 0 or 5 seems to be more than coincidence (which it might be, of course). Could it be that many but not all of the weights were rounded to the nearest 5 grams?

The four pies and pizza are especially suspicious because although "one cup" is a specific unit of volume, "one sector" is not. We may reasonably guess that the implied uncertainty in many of the weights is 2.5 grams.

When you are dealing with a large set of data, such as a complete food table, it is usually not too difficult to determine the implied uncertainty. But often you encounter just a single measurement, and in some cases the implied uncertainty of a single number cannot be determined.

Consider the numbers 12 and 10, for instance. When the number 12 appears as a single measurement, the uncertainty is implied to be .5. When 10 appears alone, however, the implied uncertainty may be .5 or it may be 5; we do not know.

Or take 101 and 100. The number 101 implies an uncertainty of .5. But 100 could imply an uncertainty of .5, or 5 or even of 50. When a number ends in a zero, as 10 and 100 do, the implied uncertainty is ambiguous.

This ambiguity may be avoided by using scientific notation. For example, a single measurement of 130 could be either 130 ± .5 or 130 ± 5. However, 130 may be expressed in scientific notation as 1.30×10^2 or 1.3×10^2. The final zero in 1.30 is not needed to express the size of the number; its only function is to express accuracy. 1.30 tells us that the measurement was not 1.29 or 1.31. The number 1.3, however could represent a rounded value of 1.29 or 1.31 or even 1.34. The implied uncertainty of 1.30×10^2 is $.005 \times 10^2$, while the implied uncertainty of 1.3×10^2 is $.05 \times 10^2$.

You have a choice of two ways of avoiding confusion caused by a final zero in measurement. For example, consider the number 130. Is the uncertainty 5 or .5? The confusion may be cleared up by stating an explicit range of uncertainty, either 130 ± .5 or 130 ± .5. It may also be cleared up by stating 130 in scientific notation, either 1.3×10^2 (uncertainty = 5) or 13.0×10^1 (uncertainty = .5).

You may occasionally encounter other conventions. If the final zero is significant, a measurement of 130 may be written 130. (130 followed by a decimal point) or 13<u>0</u>. Both 130. and 13<u>0</u> imply an uncertainty of ±.5. Often, however, you will not be able to tell what uncertainty is intended.

When measurements contain digits to the right of the decimal point, no confusion can occur. 373.94 has an implied uncertainty of .005. The implied uncertainty of .73926 is .000005, while .739260 implies an uncertainty of .0000005.

2-2 Rounding Numbers

Throughout the Biomedical Mathematics course you will have occasion to round numbers. For example, suppose that you need to round the number 12.68 to the nearest integer. 12.68 lies between 12 and 13 and clearly is closer to 13. Therefore, the rounded value is 13.

Or consider the problem of rounding 6.213 g to the nearest tenth of a gram. We merely note that 6.213 is closer to 6.2 than 6.3. Therefore the correct rounded value is 6.2 g.

One additional problem remains. How should we round 6.25 g to the nearest tenth of a gram? Since 6.25 lies equally close to 6.2 and 6.3, we must make an arbitrary decision. <u>In this situation we will adopt the convention of rounding so that the final digit of the rounded value is even. The answer is therefore 6.2.</u>

According to the above convention, we round 5.35 to 5.4 and 8.85 to 8.8.

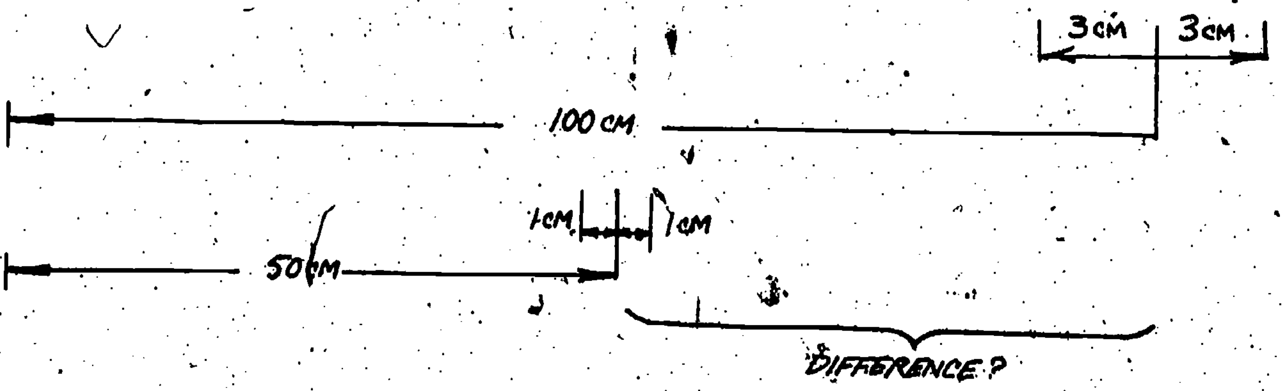
2-3 Propagation of Error Under Subtraction

In Section 1-3 we found that under the operation of addition, the uncertainty is additive. The sum of two ranges of uncertainty is the sum of the midpoints plus or minus the sum of the uncertainties.

$$(x \pm \Delta x) + (y \pm \Delta y) = (x + y) \pm (\Delta x + \Delta y)$$

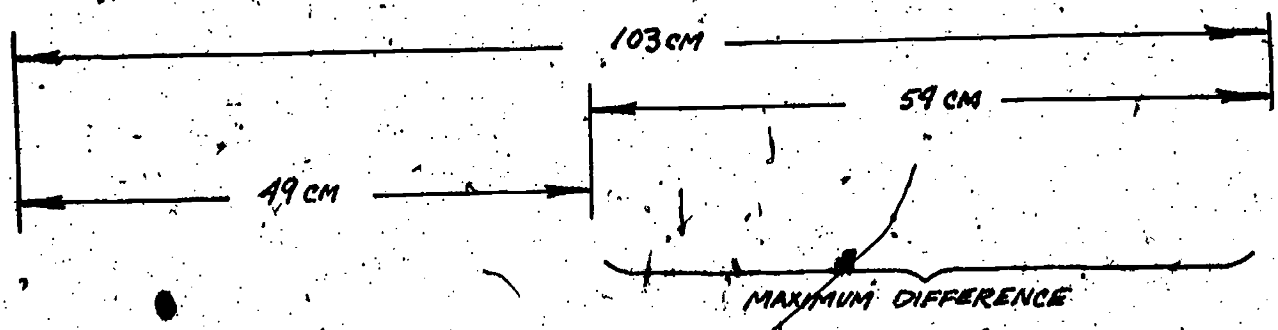
$$(100 \pm 3) + (50 \pm 1) = 150 \pm 4$$

In this section we wish to determine the uncertainty when one range of uncertainty is subtracted from another. We wish to find, for example, the uncertainty of the difference when 50 ± 1 is subtracted from 100 ± 3.

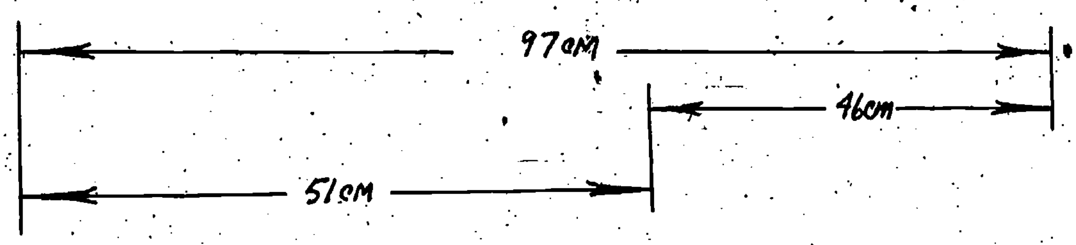


We determine the uncertainty by first finding the greatest possible difference between the two measurements and the smallest possible difference between the measurements. The greatest difference is found by starting with as much as possible and subtracting as little as possible. This is

$$(100 + 3) - (50 - 1) = 103 - 49$$
$$= 54$$



The smallest value the difference may have is found by starting with as little as possible and subtracting as much as possible.



The difference between 100 ± 3 and 50 ± 1 may be as great as 54, or as small as 46.

The midpoint of the range of uncertainty is the average of 54 and 46.

$$\frac{54 + 46}{2} = \frac{100}{2} = 50$$

The midpoint 50 is 4 less than the maximum possible difference 54, and 4 greater than the minimum possible difference, which is 46. The uncertainty of the difference is therefore 4. The difference between 100 ± 3 and 50 ± 1 is expressed by the range of uncertainty.

$$50 \pm 4$$

We have determined the difference between two specific ranges of uncertainty. Our next task is to derive a general formula for the difference between two ranges of uncertainty. We will do this by repeating the procedure we followed in the example above, but using algebraic expressions rather than specific numbers.

In finding the difference between $x \pm \Delta x$ and $y \pm \Delta y$, you subtract $x \pm \Delta x$ from $y \pm \Delta y$. The result is greatest when $y \pm \Delta y$ is as large as possible and $x \pm \Delta x$ is as small as possible. This occurs when $x - \Delta x$ is subtracted from $y + \Delta y$.

$$(y + \Delta y) - (x - \Delta x) = y + \Delta y - x + \Delta x$$

The difference between $x \pm \Delta x$ and $y \pm \Delta y$ is least when $x \pm \Delta x$ is as large as possible and $y \pm \Delta y$ is as small as possible.

The minimum difference is therefore,

$$(y - \Delta y) - (x + \Delta x) = y - \Delta y - x - \Delta x$$

The midpoint is the average of the maximum and minimum differences. It is

$$\frac{(y + \Delta y - x + \Delta x) + (y - \Delta y - x - \Delta x)}{2} = \frac{2y - 2x}{2}$$

$$= y - x.$$

The midpoint is simply the differences between x and y.

The uncertainty is the difference between the midpoint and the maximum or between the minimum and the midpoint, or what is equivalent, half the difference between the minimum and the maximum. The uncertainty is

$$\frac{(y + \Delta y - x + \Delta x) - (y - \Delta y - x - \Delta x)}{2}$$

$$= \frac{y + \Delta y - x + \Delta x - y + \Delta y + x + \Delta x}{2}$$

$$= \frac{2\Delta y + 2\Delta x}{2}$$

$$= \Delta y + \Delta x$$

The uncertainty of the difference between two ranges of uncertainty is thus the sum of their uncertainties. Uncertainty is additive under the operation of subtraction.

The difference between two ranges of uncertainty $x \pm \Delta x$ and $y \pm \Delta y$ is given by the following formula.

$$\boxed{(y \pm \Delta y) - (x \pm \Delta x) = (y - x) \pm (\Delta x + \Delta y)}$$

We have considered the addition of one range of uncertainty to another and the subtraction of one range of uncertainty from another. But what if we combine the operations of addition and subtraction? What is the range of uncertainty, for example, of the following expression?

$$(6 \pm 1) + (83 \pm 2) - (74 \pm .5)$$

The problem is not difficult if it is divided into steps. First we find the range of uncertainty of

$$(6 \pm 1) + (83 \pm 2)$$

Using the formula for the addition of two ranges of uncertainty

$$(x \pm \Delta x) + (y \pm \Delta y) = (x + y) \pm (\Delta x + \Delta y),$$

we obtain

$$(6 \pm 1) + (83 \pm 2) = (6 + 83) \pm (1 + 2)$$
$$= 89 \pm 3$$

The second step is to find the range of uncertainty of

$$(89 \pm 3) - (74 \pm .5)$$

We apply the formula for the difference between two ranges of uncertainty.

$$(y \pm \Delta y) - (x \pm \Delta x) = (y - x) \pm (\Delta x + \Delta y)$$

The result is

$$(89 \pm 3) - (74 \pm .5) = (89 - 74) \pm (3 + .5)$$
$$= 15 \pm 3.5$$

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PROBLEM SET 2:

The table below lists some of the nutritional needs of people of various ages.

TABLE 2.—RECOMMENDED DAILY DIETARY ALLOWANCES (ABRIDGED)[1]

[Designed for the maintenance of good nutrition of practically all healthy persons in the U.S.A.]

Persons	Age in years[2] From up to	Weight in pounds	Height in inches	Food energy	Protein	Calcium	Iron	Vitamin A	Thiamin	Riboflavin	Niacin equivalent[3]	Ascorbic acid
				Calories	Grams	Grams	Milligrams	International units	Milligrams	Milligrams	Milligrams	Milligrams
Infants	0-1/6	9	22	lb. x 54.5	lb. x 1.0	0.4	6	1,500	0.2	0.4	5	35
	1/6-1/2	15	25	lb. x 50.0	lb. x .9	0.5	10	1,500	0.4	0.5	7	35
	1/2 1	20	28	lb. x 45.5	lb. x .8	0.6	15	1,500	0.5	0.6	8	35
Children	1 2	26	32	1,100	25	0.7	15	2,000	0.6	0.6	8	40
	2 3	31	36	1,250	25	0.8	15	2,000	0.6	0.7	8	40
	3 4	35	39	1,400	30	0.8	10	2,500	0.7	0.8	9	10
	4 6	42	43	1,600	30	0.8	10	2,500	0.8	0.9	11	40
	6 8	51	48	2,000	35	0.9	10	3,500	1.0	1.1	13	40
	8 10	62	52	2,200	40	1.0	10	3,500	1.1	1.2	15	40
Boys	10 12	77	55	2,500	45	1.2	10	4,500	1.3	1.3	17	40
	12 14	95	59	2,700	50	1.4	18	5,000	1.4	1.4	18	45
	14 18	130	67	3,000	60	1.4	18	5,000	1.5	1.5	20	55
Men	18 22	147	69	2,800	60	0.8	10	5,000	1.4	1.6	18	60
	22 35	154	69	2,800	65	0.8	10	5,000	1.4	1.7	18	60
	35 55	154	68	2,600	65	0.8	10	5,000	1.3	1.7	17	60
	55 75+	154	67	2,400	65	0.8	10	5,000	1.2	1.7	14	60
Girls	10 12	77	56	2,250	50	1.2	18	4,500	1.1	1.3	15	40
	12 14	97	61	2,300	50	1.3	18	5,000	1.2	1.4	15	45
	14 16	114	62	2,400	55	1.3	18	5,000	1.2	1.4	16	50
	16 18	119	63	2,300	55	1.3	18	5,000	1.2	1.5	15	50
Women	18 22	128	64	2,000	55	0.8	18	5,000	1.0	1.5	13	55
	22 35	128	64	2,000	55	0.8	18	5,000	1.0	1.5	13	55
	35 55	128	63	1,850	55	0.8	18	5,000	1.0	1.5	13	55
	55 75+	128	62	1,700	55	0.8	10	5,000	1.0	1.5	13	55
Pregnant				+200	65	+0.4	18	6,000	+0.1	1.8	[illegible]	60
Lactating				+1,000	75	+0.5	18	8,000	+0.5	2.0	20	60

Give the implied uncertainties of the set of numbers in each of the following columns. Include units.

1. iron
2. thiamin
3. niacin
4. vitamin A

Give the implied uncertainty of each of the numbers below. If more than one answer is possible list all the possibilities.

5. 27
6. .322
7. 5.82
8. 10
9. 6000

In each case perform the indicated rounding operation.

10. Round 6.3 g to the nearest whole gram.

11. Round 8.48 m to the nearest whole meter.

12. Round 7.35 mg to the nearest .1 mg.

13. Round 6.205 cm to the nearest .01 cm

14. Suppose that you wish to perform the following subtraction.

$$(62 \pm 1) - (30 \pm 2)$$

a. What is the largest possible value for the difference?

b. What is the smallest possible value?

c. The midpoint of the range of uncertainty may be found by averaging the largest and smallest possible values. Find the midpoint.

d. What is the uncertainty in this problem?

e. Write the difference as a range of uncertainty.

(Multiple Choice)

15. $(80 \pm .5) - (20 \pm .3) =$

a. $(80 - .5) \pm (20 - .3)$
b. $(80 - .3) \pm (20 - .5)$
c. $(80 - 20) \pm (.5 + .3)$
d. $(80 + 20) \pm (.5 - .3)$

(Multiple Choice)

16. $(s \pm \Delta s) - (t \pm \Delta t) =$

a. $(s - \Delta s) \pm (t - \Delta t)$

b. $(s - \Delta t) \pm (t - \Delta s)$

c. $(s - t) \pm (\Delta s + \Delta t)$

d. $(s - t) \pm (\Delta s - \Delta t)$

Perform the following operations, indicating the resulting range of uncertainty.

17. $(9.3 \pm .4) - (6.6 \pm .3)$

18. $(8.5 \pm .2) - (12.4 \pm .7)$

19. $(9.4 \pm .6) + (2.1 \pm .15) - (3.1 \pm .15)$

20. $(8.0 \pm .7) - (2.0 \pm .8) + (-4.1 \pm .2)$

21. $(6.9 \pm 1.3) + (8.2 \pm 2.3) - (9.0 \pm 1)$

22. $(3.7 \pm .4) - (2.1 \pm .2)$

23. A student is trying to determine the quantity of water in a griddle cake. If the cake weighs 27.500 ± .005 grams before drying and 12.150 ± .005 grams after drying, determine the range of uncertainty of water in the cake.

24. A recipe calls for flour, water, and sugar. If you used 5.330 ± .002 grams of flour, 8.220 ± .003 grams of sugar and there were 15.000 ± .010 grams of the final mixture, how much water did you use?

SECTION 3:

3-1 Propagation of Error Under Multiplication

We have been investigating the propagation of error under mathematical operations. In previous lessons we considered the operations of addition and subtraction. We will now investigate the propagation of error under multiplication by considering a specific example, as we did with addition and subtraction.

Our example concerns Boyle's Law. Boyle's Law states that the product of pressure and volume is constant for any gas at constant temperature. The problem is as follows. The pressure of a gas is $10 \pm .1$ atmospheres and the volume is $3 \pm .2$ liters. What is the range of uncertainty of the product of pressure and volume?

In other words, what is

$$(10 \pm .1)(3 \pm .2)$$

We first determine the maximum value that the product may have. The maximum possible product is found by multiplying the maximum possible pressure and maximum possible volume. This situation resembles that of addition, in which we also combined two maximum values.

The maximum possible pressure is $10 + .1 = 10.1$ atmospheres, while the maximum possible volume is $3 + .2 = 3.2$ liters. The maximum possible product of these two factors is then

$$(10.1)(3.2) = 32.32 \text{ liter-atmospheres.}$$

The minimum possible product is found by multiplying the minimum possible pressure by the minimum possible volume.

$$(10 - .1)(3 - .2) = (9.9)(2.8)$$

$$= 27.72 \text{ liter-atmospheres}$$

The product may have any value from 27.72 to 32.32. The midpoint of this range is the average of the extremes, or

$$\frac{27.72 + 32.32}{2} = \frac{60.04}{2}$$

$$= 30.02 \text{ liter-atmospheres}$$

The uncertainty of the product is the difference between the maximum and the midpoint (or between the midpoint and the minimum).

$$32.32 - 30.02 = 2.30 \text{ liter-atmospheres},$$

or, equivalently, half the difference of the extremes

$$\frac{32.32 - 27.72}{2} = 2.30 \text{ liter-atmospheres}$$

The midpoint of the product is 30.02; the absolute error is 2.30. The range of uncertainty of the product is thus 30.02 ± 2.30 liter-atmospheres.

$$(10 \pm .1)(3 \pm .2) = 30.02 \pm 2.30$$

When two ranges of uncertainty are added, the midpoint of the sum is the sum of the midpoints of the addends. However, the result of our multiplication is not analogous to the result of addition. The product of the midpoints of the factors is not the midpoint of the product. The product of the midpoints of the factors is $10 \cdot 3 = 30$, while the midpoint of the product is 30.02.

Multiplication of two ranges of uncertainty is not analogous to addition in another way. When two ranges of uncertainty are added, the uncertainty of the sum is the sum of the uncertainties of the addends. However, the uncertainty of a product is not the product of the uncertainties of the factors. The product of the uncertainties of the factors is $(.1)(.2) = .02$, while the actual uncertainty of the product is 2.30. Clearly, the propagation of error in multiplication is more complex than it is in addition.

In order to determine the rules of obtaining the product of two ranges of uncertainty we will perform a multiplication algebraically. We will find the range of uncertainty of the product of $x \pm \Delta x$ and $y \pm \Delta y$.

The maximum possible value of this product is

$$(x + \Delta x)(y + \Delta y) = xy + x\Delta y + y\Delta x + \Delta x\Delta y$$

The minimum possible value of the product is

$$(x - \Delta x)(y - \Delta y) = xy - x\Delta y - y\Delta x + \Delta x\Delta y$$

The midpoint of the product is the average of the maximum possible value and the minimum possible value. It is

$$\frac{(xy + x\Delta y + y\Delta x + \Delta x\Delta y) + (xy - x\Delta y - y\Delta x + \Delta x\Delta y)}{2}$$

$$= \frac{2xy + 2\Delta x\Delta y}{2}$$

$$= xy + \Delta x\Delta y$$

The uncertainty is the difference between the maximum possible value and the midpoint. This difference is

$$(xy + x\Delta y + y\Delta x + \Delta x\Delta y) - (xy + \Delta x\Delta y)$$

$$= x\Delta y + y\Delta x$$

The midpoint of the product is $xy + \Delta x\Delta y$, and the uncertainty of the product is $x\Delta y + y\Delta x$. The range of uncertainty of the product is therefore given by the following formula.

$$(x \pm \Delta x)(y \pm \Delta y) = (xy + \Delta x\Delta y) \pm (x\Delta y + y\Delta x)$$

Often an uncertainty is very small compared to the midpoint. When Δx is very small relative to x, and Δy is very small relative to y, the product $\Delta x\Delta y$ is extremely small compared to the product xy. In these cases little accuracy is lost by ignoring the term $\Delta x\Delta y$ in the midpoint expression $(xy + \Delta x\Delta y)$.

$$xy + \Delta x\Delta y \simeq xy$$

This leads to an approximate formula for the product of two ranges of uncertainty

$$(x \pm \Delta x)(y \pm \Delta y) \simeq xy \pm (x\Delta y + y\Delta x)$$

For example, consider the product of the two ranges of uncertainty (10 ± 1) and (100 ± 1). The product xy is

$$10 \cdot 100 = 1000$$

The product $\Delta x\Delta y$ is

$$1 \cdot 1 = 1$$

The exact midpoint $xy + \Delta x\Delta y$ is thus

$$1000 + 1 = 1001$$

The approximate midpoint xy is 1000, which is very close to the exact midpoint.

The uncertainty of this product is $x\Delta y + y\Delta x$.

$$x\Delta y + y\Delta x = 10 \cdot 1 + 100 \cdot 1$$
$$= 110$$

The exact range of uncertainty of the product is thus

$$1001 \pm 110$$

The approximate range of uncertainty is

$$1000 \pm 110$$

The approximation is not as good when the uncertainties of the factors are larger relative to the midpoints. For example, the exact range of uncertainty of the product $(8 \pm 4)(10 \pm 3)$ is

$$(8 \cdot 10 + 4 \cdot 3) \pm (8 \cdot 3 + 10 \cdot 4)$$
$$= (80 + 12) \pm (24 + 40)$$
$$= 92 \pm 64$$

The approximate range of uncertainty, however, obtained by ignoring the $\Delta x \Delta y$ term, is

$$(8 \cdot 10) \pm (8 \cdot 3 + 10 \cdot 4) = 80 \pm 64$$

This approximate range of uncertainty of the product is not very close to the exact range of uncertainty.

Using the approximate formula for the product of two ranges of uncertainty sacrifices accuracy. It is in fact a form of computational error. However, the loss of accuracy is compensated by the ease of computation. Because computations are made easier, the $\Delta x \Delta y$ term is commonly ignored when finding the range of uncertainty of a product.

3-2 Implied Error and Multiplication

An interesting problem occurs when measurements having implied uncertainty are multiplied. Suppose the measurements are 9.99 and 10.01. If the implied errors of these numbers are neglected, their product is

$$9.99 \cdot 10.01 = 99.9999$$

The implied uncertainty of 99.9999 is .00005. But is such a small implied uncertainty justified? Do we really know the product that accurately?

Let us repeat the multiplication taking into account the implied uncertainties of the two factors. The implied uncertainties of both 9.99 and 10.01 are .005. These factors may then be expressed as ranges of uncertainty.

$$9.99 \pm .005 \quad \text{and} \quad 10.01 \pm .005$$

The product of these ranges of uncertainty (using the approximate formula) is

$$\begin{aligned}(9.99 \pm .005) \cdot (10.01 \pm .005) &= (9.99 \cdot 10.01) \pm [9.99\,(.005) + 10.01\,(.005)] \\ &= 99.9999 \pm .005(9.99 + 10.01) \\ &= 99.9999 \pm .005 \cdot 20 \\ &= 99.9999 \pm .1\end{aligned}$$

The uncertainty of the product is actually .1. The product is not known with the accuracy than an implied uncertainty of .00005 indicates.

The point is not that implied uncertainties are not valid, but that stating the product as 99.9999 is misleading. The accuracy with which the factors are known does not justify implying that the product is known with such great accuracy.

Notice that very little accuracy is lost if we round the midpoint of the product to 100. If we state the product as 1.00×10^2, the implied uncertainty is $.005 \times 10^2$ or .5. If the product is stated as 1.000×10^2, the implied uncertainty is $.0005 \times 10^2$ or .05. In the first case, we are implying a greater uncertainty than is necessary. In the second case, we are implying less uncertainty than there actually is.

The only way out of this dilemma is to state the product as a range of uncertainty. The most exact statement of the product is

$$99.9999 \pm .1$$

when the approximate formula is used. Another very reasonable statement would be

$$100 \pm .1$$

Keep in mind that this discussion applies only to numbers which have an implied uncertainty, such as measurements. In the realm of pure mathematics, the statement

$$9.99 \cdot 10.01 = 99.9999$$

is a true statement.

PROBLEM SET 3:

1. Given the expression 6 ± 3.2, answer the following.

   a. What is the largest possible value of the expression?

   b. What is the smallest possible value of the expression?

   c. What is the midpoint of the expression?

   d. What are the uncertainties of the expression?

2. Given the expression $g \pm \Delta t$, answer the following.

   a. What is the largest possible value of the expression?

   b. What is the smallest possible value of the expression?

   c. What is the midpoint of the expression?

   d. What is the uncertainty of the expression?

3. Given the product $(6 \pm 1)(7 \pm 2)$,

   a. What is the maximum product?

   b. What is the minimum product?

   c. Find the average of the maximum and minimum products. This value is the midpoint of the range of uncertainty of the product.

   d. Subtract the minimum product found in Part b from the maximum product found in Part a, and divide the difference by 2 to obtain the uncertainty.

e. Express the range of uncertainty of the product in the form (midpoint) ± (uncertainty)

4. Given the product $(6 \pm 1)(7 \pm 2)$,

a. What is the uncertainty of the first factor?

b. What is the uncertainty of the second factor?

c. Find the product of the uncertainties.

d. Is the answer to Part 4c the same as the answer to Part 3d? That is, did the uncertainty of the product turn out to be the product of the uncertainty of the factors?

e. If your answer to 4d is "no" which is the correct way to solve problems of this type?

5. An arithmetic example of the algebraic expression $(a \pm \Delta a)(b \pm \Delta b)$ is $(10 \pm 1)(15 \pm 2)$.

a. What are a, Δa, b and Δb in the example?

b. Given that $(a \pm \Delta a)(b \pm b) = (ab + \Delta a\Delta b) \pm (a\Delta b + b\Delta a)$, the expression $(ab + \Delta a\Delta b)$ is the _______ of the range of uncertainty of the product, and $(a\Delta b + b\Delta a)$ is the ________ of the range of uncertainty of the product.

c. Use the expression in Part b to calculate the range of uncertainty of the numerical example.

6. Given that the range of uncertainty of the product $(x \pm \Delta x)(y \pm \Delta y)$ may be stated approximately by the expression $xy \pm (x\Delta y + y\Delta x)$ and exactly by the expression $(xy + \Delta x\Delta y) \pm (x\Delta y + y\Delta x)$, find both the approximate and exact products for each of the following.

a. $(11 \pm 1)(12 \pm 2)$

b. $(18 \pm .1)(10 \pm .01)$

c. $(15 \pm .01)(15 \pm .05)$

d. $(h \pm \Delta h)(s \pm \Delta s)$

7. Elmo wants to find the area of his classroom by using a ruler to measure the length and width. After several hours of wearisome work, he finds the width to be 10 meters, give or take $\frac{1}{10}$ meter, and he finds the length to be 7 meters, give or take $\frac{1}{15}$ meter. What is the exact range of uncertainty of the area? The approximate range of uncertainty?

8. a. What is the implied uncertainty of the number 9.98?

b. What is the implied uncertainty of the number 10.02?

c. Ignore error considerations to determine the product (9.98)(10.02). State the implied uncertainty of the result.

d. Calculate the true uncertainty of the product by use of the formula $(x\Delta y + y\Delta x)$.

The true uncertainty is 2000 times larger than the implied uncertainty of the incorrectly stated product in Part c. This is one of the most common errors made by students. Beware!

SECTION 4:

4-1 Relative Uncertainty

By now it should be apparent that every measurement has some degree of uncertainty. One method of reducing uncertainty is through careful attention to the measurement procedure. But ultimately the accuracy of a measurement is limited by the quality of the measuring instrument used. If you use a metric ruler which is poorly constructed and calibrated, there is little hope of an accurate measurement.

When required, scientific measurements are made with high precision, high accuracy instruments. In Biomedical Science, the flasks, graduated cylinders and balances which you use have all been designed with accuracy in mind. Yet they are still imperfectly made and manufacturers often include an estimate of accuracy in equipment catalogues.

A description of a 250 ml Erlenmeyer Flask states that the graduations are accurate to ±5%. What does this statement mean? It means that the true value of the volume is within five percent of what the flask reads. In other words, the uncertainty of a reading of 200 ml is 5% of 200 ml or

$$.05 \times 200 = 10 \text{ ml}$$

Therefore, we can write the reading as a range of uncertainty

$$200 \pm 10 \text{ ml}$$

Often, however, the uncertainty is left as a percentage. In that case, the above range of uncertainty would be written as follows.

$$200 \pm 5\% \text{ ml}$$

When an uncertainty is stated as a percentage of the midpoint, it is called a relative uncertainty. The term "relative uncertainty" is used because the size of the uncertainty is related to the size of the midpoint. The greater the midpoint, the greater is the uncertainty.

We have two ways to express uncertainty: absolute uncertainty and relative uncertainty. When the size of an uncertainty remains the same regardless of the size of the midpoint, the range of uncertainty is usually stated in terms of an absolute uncertainty. For example, if a thermometer can be read only to the nearest degree, the scale reading related uncertainty is .5 degrees for any temperature. The absolute uncertainty in scale reading is the same for a measurement of 80° C as it is for 20° C. The ranges of uncertainty which express this scale reading uncertainty are 80 ± .5° C and 20 ± .5° C.

When the size of an uncertainty is directly proportional to the size of the midpoint, the range of uncertainty is usually stated in terms of a relative uncertainty. In the example of the Erlenmeyer Flask, the size of the uncertainty was directly proportional to the volume reading. Therefore we would usually give readings in terms of relative uncertainty, for example 75 ± 5% or 200 ± 5%.

4-2 Interconverting Relative Uncertainty and Absolute Uncertainty

Converting relative uncertainty to absolute uncertainty, and the reverse, is not difficult, as the following examples show.

EXAMPLE:

Express 10 ± 1% g in terms of absolute uncertainty.

SOLUTION:

The absolute uncertainty is 1 per cent of 10 grams, which is

$$\frac{1}{100} \cdot 10 = \frac{1}{10}$$

$$= .1 \text{ g}$$

The absolute uncertainty is .1 grams. The range of uncertainty using absolute uncertainty is

$$10 \pm .1 \text{ g}$$

EXAMPLE:

Express 50 ± 1 cm in terms of relative uncertainty.

SOLUTION:

The absolute uncertainty is 1, and the midpoint is 50. 1 is $\frac{1}{50}$ of 50. To convert $\frac{1}{50}$ to a percentage, we multiply by 100%.

$$\frac{1}{50} \cdot 100\% = 2\%$$

Since the absolute uncertainty is 2 per cent of the midpoint, the relative uncertainty is 2 per cent.

The range of uncertainty is expressed as

$$50 \pm 2\% \text{ cm}$$

EXAMPLE:

Suppose that quarter-mile tracks are laid out with an absolute uncertainty of 5 feet.

a. Express the range of uncertainty in terms of relative uncertainty.

b. When a person has run 8 laps, what is the range of uncertainty of the distance he has run, using relative uncertainty?

c. What is the range of uncertainty of the distance he has run, in terms of absolute uncertainty?

SOLUTION:

a. $\frac{1}{4}$ mile = 1,320 feet. The absolute uncertainty is $\frac{5}{1,320}$ of the midpoint. We convert $\frac{5}{1,320}$ to a percentage by multiplying by 100.

$$\frac{5}{1,320} \cdot 100\% = \frac{500}{1,320}\%$$
$$\approx .38\%$$

The length of one lap is thus

$$1,320 \pm .38\% \text{ ft.}$$

b. For one lap the absolute uncertainty is 5 feet; for two laps it is 10 feet. The absolute uncertainty is proportional to the midpoint. Therefore, the relative uncertainty is the same for every distance. The midpoint is

$$8 \cdot 1,320 = 10,560 \text{ ft}$$

The range of uncertainty is

$$10,560 \pm .38\% \text{ ft.}$$

c. .38% of 10,560 is

$$\frac{.38}{100} \cdot 10,560 \approx 40 \text{ ft}$$

The length of 8 laps is

$$10,560 \pm 40 \text{ ft.}$$

When a runner runs 8 laps he may run as much as 40 feet more than two miles, or as much as 40 feet less than two miles.

We conclude this section by converting absolute uncertainty to relative uncertainty and the reverse for ranges of uncertainty involving algebraic quantities.

The expression $x \pm \Delta x$ is a range of uncertainty in terms of absolute uncertainty. The absolute uncertainty Δx is $\frac{\Delta x}{x}$ of the midpoint x. The fraction $\frac{\Delta x}{x}$ is converted to a percentage by multiplying by 100%. The relative uncertainty is

$$100 \frac{\Delta x}{x} \%$$

The range of uncertainty in terms of relative uncertainty is consequently

$$x \pm 100 \frac{\Delta x}{x} \%$$

We may convert a range of uncertainty with algebraic quantities in the opposite direction also. The expression $x \pm p\%$ is a range of uncertainty expressed in terms of relative uncertainty. The relative uncertainty is p per cent, so the absolute uncertainty is $\frac{p}{100}$ times the midpoint, which is x.

$$\text{Absolute uncertainty} = \frac{p}{100} \cdot x$$

Therefore the range of uncertainty in terms of absolute uncertainty is

$$x \pm \frac{px}{100}$$

PROBLEM SET 4:

1. Given the expression $10 \pm .1$ calories
   a. What is the absolute uncertainty?
   b. Compute the relative uncertainty.
   c. Write the expression using relative uncertainty.

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2 Given the expression 150 ± 2% meters

a. What is the relative uncertainty?

b. Compute the absolute uncertainty.

c. Write the expression using absolute uncertainty.

Write the following expressions using relative uncertainty.

3. 5 ± .1 seconds

4. 20 ± .1 calories

5. 100 ± .5 meters

Write the following expressions using absolute uncertainy

6. 8.8 ± 1% calories

7. 5.67 ± 10% grams

8. 25.25 ± .5% calories

Perform the indicated operations and express the resulting range of uncertainty first in terms of (a) absolute uncertainty and then in terms of (b) relative uncertainty.

9. (5 ± .1) + (5 ± .2)

10. (10.3 ± .2) - (.3 ± .01)

11. Given the expression (20 ± 1%) - (15 ± 2%)

a. Convert the relative uncertainties to absolute uncertainties.

b. Write the expression using absolute uncertainties.

c. Determine the range of uncertainy of the difference.

d. Express your answer to Part c in terms of relative uncertainty. (Note that your answer is not 5 ± 3%, which indicates that relative uncertainty is not additive under the operation of subtraction.)

12. 30 ± 1% = x ± p%

a. x =

b. p =

c. $\Delta x = \frac{px}{100}$

$\Delta x =$

13. 28 ± 7 = x ± Δx

a. x =

b. $\Delta x =$

c. $p = \frac{\Delta x}{x} \cdot 100\%$

p =

12

14. According to an equipment catalog, a 100 ml graduated cylinder has an error of .6%. This indicates that any measurement made with the cylinder may be in error by as much as .6%, which is equivalent to .006 of the midpoint. For the following problems we will ignore scale reading error in order to keep the problem simple.

a. What is the maximum amount of liquid that could be in the cylinder when it is filled to the 80 ml mark?

b. What is the maximum amount of liquid when it is filled to the 45 ml mark?

c. The cylinder is used to pour first 80 ml and then 45 ml of water into a beaker. What is the maximum amount of water in the beaker? (Assume that all of the water can be poured out of the cylinder each time.)

d. What would be the maximum amount of water in the beaker if the same procedures were used, except that 70 ml were transferred first and then 55 ml?

15. Suppose that a 50 ml buret has an uncertainty of .1%. Suppose that the buret is filled to the 40 ml mark.

a. What is the absolute uncertainty of the 40 ml reading.

b. Express the reading in the form (midpoint) ± (absolute uncertainty).

16. Elmo and Alfred are in a class that is studying the metric system. Alfred is the best student in the class; Elmo is not.

The students are learning to convert inches to centimeters by measuring distances in inches on a yardstick using a meter stick. They measure the length in centimeters of 5 inches, 10 inches, 15 inches, 20 inches and 30 inches.

Alfred does his measuring by placing the meter stick next to the yardstick. He finds that 5 inches corresponds to 12.70 centimeters. The meter stick has divisions of one millimeter, and Alfred figures his measurement may be in error by as much as .2 mm, or .02 cm. Therefore, he expresses his measurement as a range of uncertainty,

12.70 ± .02 cm

Alfred then measures the "length" of 10 inches in the same way--by placing the yardstick next to the meter stick and noting that 10 inches is 25.40 centimeters long.

Elmo, however, uses a different method. He measures the length of each inch, one at a time. He finds that the first inch is 2.54 centimeters long. Error is Elmo's specialty; although he is just plain incompetent in most things, he is good in error analysis. He knows that his measurement of 2.54 cm has an absolute uncertainty of .02 cm.

Elmo measures the second inch and finds that its length too is 2.54 ± .02 cm. The length of 2 inches is the sum of the lengths of the first two inches.

$$(2.54 \pm .02) + (2.54 \pm .02) = 5.08 \pm .04 \text{ cm}$$

Since absolute uncertainty is additive under the operation of addition, Elmo adds .02 cm to his uncertainty with each measurement. By the time he has measured 5 inches, his uncertainty is 5 x .02 = .10 centimeters. Elmo expresses the length of 5 inches as

$$12.70 \pm .10 \text{ cm}$$

Elmo continues to measure the other lengths in the same way, long after Alfred has gone home.

a. Which student had constant <u>absolute</u> uncertainty?

b. Which student had constant <u>relative</u> uncertainty?

c. What was his <u>relative</u> uncertainty?

d. What was the absolute uncertainty of his measurement of the length in centimeters of 30 inches?

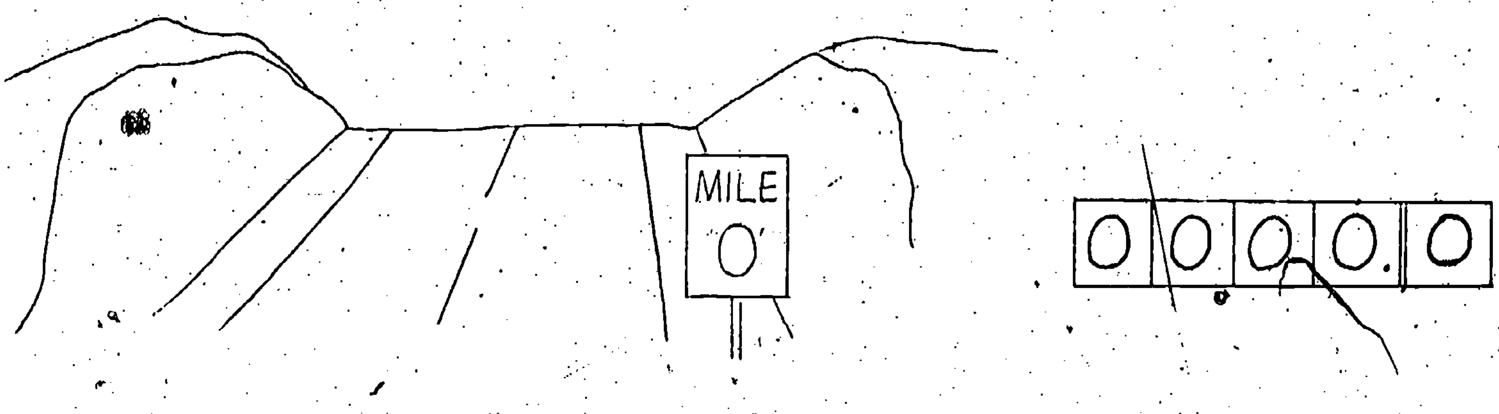
SECTION 5:

5-1 Errors That Can Be Corrected

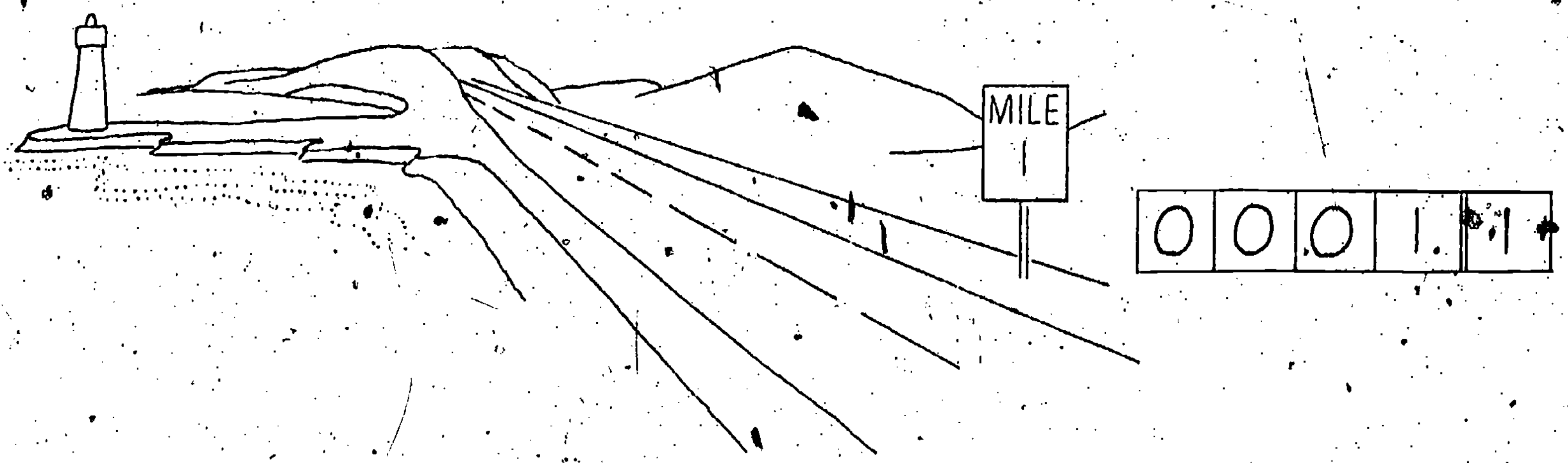
Occasionally you see signs along highways that read "Speedometer Check" or "Speedometer Test Section." These signs are followed by signs indicating one-mile intervals.

Speedometer check signs may be used to determine the accuracy of a speedometer. They may also be used to check the odometer, the instrument that indicates the number of miles traveled by a car.

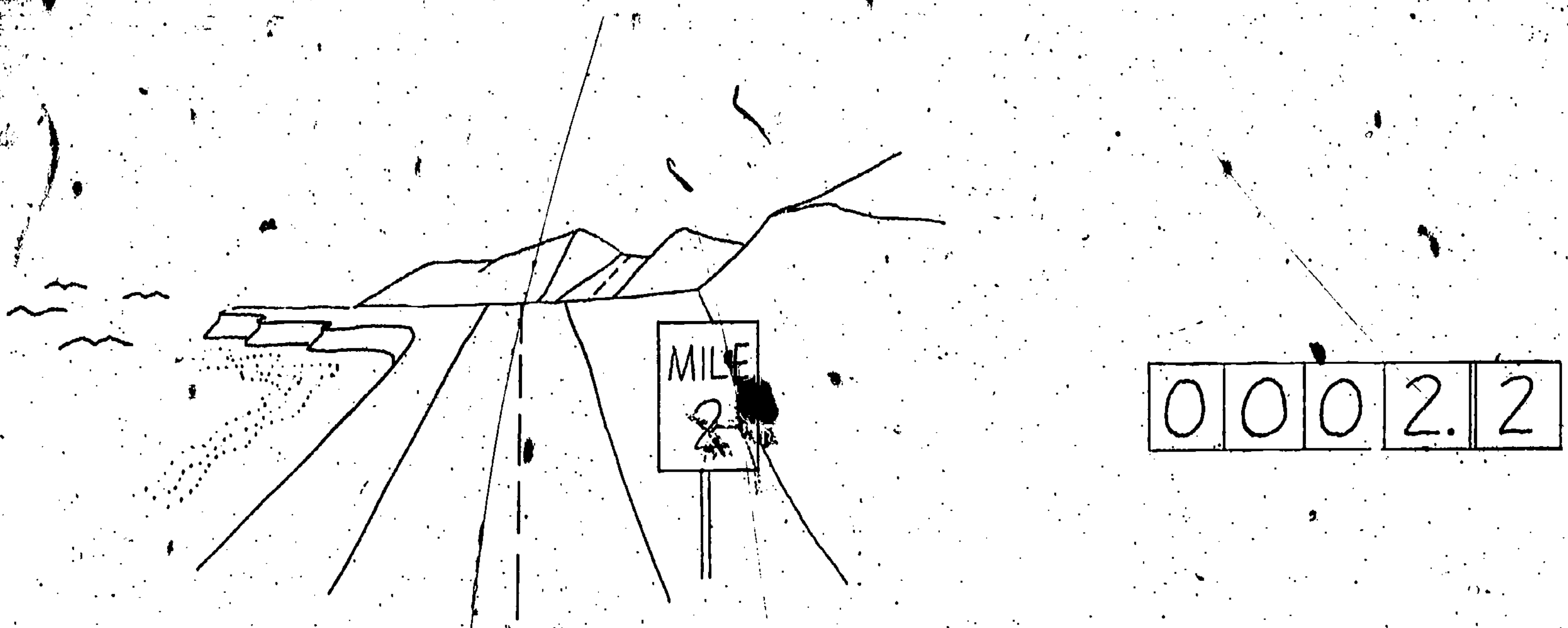
Assume that our car not only has an odometer that records the total mileage traveled by the car, but also has a trip odometer, which may be set to zero at any time. We approach a speedometer test section and set the trip odometer at zero as we pass the sign indicating the beginning of the first test mile.



As we pass the end of the first mile, the odometer reads 1.1.



As we pass the end of the second test mile, the odometer reads 2.2.



We continue our trip, and the odometer eventually reads 132.0. But how many miles have we actually driven since the start of the speedometer check?

Measurement error is the difference between the true measurement and the reading of the measuring instrument. Up until now we could only approximate the maximum size of the error. We call this approximation uncertainty. In the case of speedometer correction we will be able to do more. We will be able to estimate both the direction and the size of a major error. Other errors will remain, such as error in the placement of the sign posts and error in reading the odometer exactly when the signs are passed. But we are ignoring uncertainties in our readings and concentrating on a major error which can be corrected.

Notice that the odometer reading is always too large. At the end of one mile the odometer reading is too large by .1 miles. At the end of two miles the reading is too large by .2 miles. To put it another way, the actual distance traveled is always less than the odometer reading and the discrepancy grows with the length of the trip. We may correct the odometer trip reading of 132.0 miles by using the information obtained from the speedometer check. The question is, what must we subtract from the odometer reading to obtain the actual mileage of the trip?

First note that when the odometer reading was 1.1 the error was .1 miles. The relative error was

$$\frac{.1}{1.1} \cdot 100\% = \frac{100}{11}\%$$

or approximately 9.1%.

When the odometer read 2.2, the relative error was

$$\frac{.2}{2.2} \cdot 100\% = \frac{100}{11}\%$$

the same as for the first mile.

The reason why odometer readings have constant relative error is not the subject of this mathematics section. But we may mention that tire size is one factor. An odometer counts the number of revolutions made by the wheels, and converts this to miles by multiplying by a factor assumed to be the distance traveled per revolution. If the tires have a greater circumference than that upon which the assumption was based, the car travels a greater distance per revolution. If the car travels a greater distance per revolution than was assumed in designing the odometer, the odometer indicates fewer miles than actually were traveled. If the tires are of smaller circumference (due to being badly worn, for instance), the odometer indicates more miles than the miles actually traveled. The theoretical error caused by assuming an incorrect tire size introduces a constant relative error into the odometer reading (constant, at least, as long as the tire size remains constant).

The relative error of our odometer is approximately 9.1 per cent. When the odometer reads 132.0, the absolute error of the reading is

$$\frac{9.1}{100} \cdot 132.0 \approx 12.0$$

The absolute error is 12.0, and since the odometer reading is greater than the actual number of miles traveled, we subtract 12.0 from 132.0.

$$132.0 - 12.0 = 120.0$$

When the odometer reads 132.0, the actual number of miles traveled is 120.0.

The constant relative error that occurs in the odometer reading may be corrected. We corrected the reading by checking the odometer against true measurements, the mile markers. This type of error does not involve uncertainty. Correctible error is not of the "plus-or-minus" kind, which involves maximum or minimum values. Rather we may state the error by the following expression.

$$\text{Actual miles} \approx \text{Odometer reading} - 9.1\%$$

PROBLEM SET 5:

1. A bathroom scale reads 2 lb when no one is standing on it. It reads 102 lb when a standard 100-lb weight is placed on it.

a. The scale seems to have a(n) (relative, absolute) error.

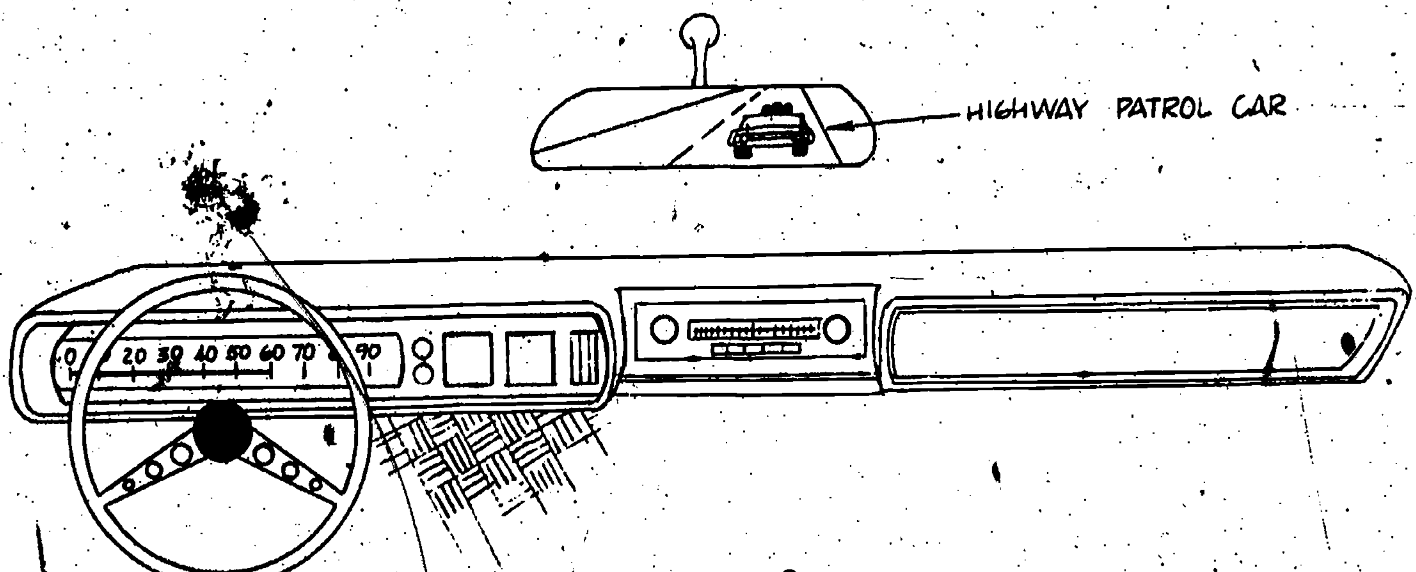
b. If it reads 113 lb after Belinda stands on it, what is Belinda's probable weight?

2. Another bathroom scale reads zero lb with no weight on it and 102 with a 100-lb weight on it.

a. This scale probably has a(n) (absolute, relative) error.

b. If it reads 357 lb when Errorbella steps on, what is Errorbella's probable weight?

3. Speedometers and odometers are generally in agreement. For example, if you travel for one minute at a constant 60 mph, the odometer will register one mile, as it should. This means that the speedometer will have the same relative error as the odometer. The same correction factor may be used for speed as for odometer correction. It is easy to imagine situations in which the ability to correct for speedometer error might be really useful, such as when a Highway Patrol car appears in your rearview mirror and your indicated speed is a little above the speed limit.



TRUE SPEED?

Assume that the relative error of the speedometer is the same as that of the odometer. The odometer has a $\frac{1}{11}$ relative error. It indicates more miles than actual miles. Show that the true speed is about 54.5 mph.

4. Odometer Reading True Miles

Odometer Reading	True Miles
0000.0	0
0005.2	5

a. Show that the absolute error in the odometer reading after 5 true miles is .2 miles.

b. Show that the relative error of the odometer is in the neighborhood of 3.8% (not 4% exactly).

c. By use of the formula

$$\frac{.2}{5.2} \text{(Odometer Reading)} = \text{Absolute Error}$$

show that the absolute error in an odometer reading of 130 miles is 5 miles.

d. Since the odometer reading is (less, more) than the actual number of miles traveled, the absolute error should be (added, subtracted) from the odometer reading.

e. Show that the corrected value for distance traveled is 125 miles.

f. Show that if the speedometer correction factor is the same as the odometer correction factor, then a speedometer reading of 65 mph has an absolute error of 2.5 mph.

g. Show that the corrected speed is 62.5 mph.

5.

Odometer Reading	True Miles
35026.2	0
35029.9	4

a. Show that the odometer indicated a distance of 3.7 miles traveled in a distance of 4 true miles.

b. Show that the absolute error of the odometer after 3 true miles is .3 miles.

c. Show that the relative error of the odometer is in the neighborhood of 8.1%.

d. Show that an indicated distance of 125 miles is approximately an actual distance of 135 miles.

e. State why the correction was added to the indicated distance.

f. Show that an indicated 55 mph is approximately an actual speed of 59.5 mph. (NOTE: This situation can arise when oversized tires are put on the rear. You may think you're legal, but you aren't.)

6. Odometer Reading True Miles

Odometer Reading	True Miles
25626.8	0
25630.0	3

a. After an indicated 210 miles, how many actual miles have you gone? State your answer to the nearest mile.

b. What is your actual speed when your indicated speed is 60 mph? State your answer to the nearest mph.

7. Elmo was cruising along one day in his Locomobile. He had just purchased some oversized tires from Reet Red's Square Deals and Round Wheels Warehouse. The salesman told him that it would change his speedometer reading. Accordingly he drove straight to a speedometer check zone where he obtained this information. His odometer showed only 1.7 miles in 2 actual miles. Elmo did some lightning calculations, ".3 miles in 2 miles is a 15% relative error. If my speedometer shows 65 mph then 15% of 65 is 9.75 or about 10. Since the odometer reads less than true distance my true speed must be less than my indicated speed by 10 mph. I must be going about 55 mph." Elmo continued to cruise along at an indicated 65 mph. After an extremely short time poor Elmo was pulled over for speeding. The officer claimed he was going 76 mph.

a. Identify two mistakes Elmo made in his calculations.

b. Do you think the judge will have mercy on Elmo when he tells him his story?

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SECTION 6:

6-1 Propagation of Error Under Division

We have developed procedures for determining the propagation of error when ranges of uncertainty are added, subtracted or multiplied. But we have not yet determined the result of dividing one range of uncertainty by another. For example, what happens when we divide 2999 ± 70 by 100 ± 1.

$$\frac{2999 \pm 70}{100 \pm 1} = ?$$

The quotient has its maximum possible value when 2999 ± 70 is as great as possible and 100 ± 1 is as small as possible. Therefore, we obtain the maximum possible quotient by dividing 2999 + 70 by 100 - 1.

$$\frac{2999 + 70}{100 - 1} = \frac{3069}{99}$$

$$= 31$$

The quotient has its minimum possible value when the smallest possible value of 2999 ± 70 is divided by the largest possible value of 100 ± 1.

$$\frac{2999 - 70}{100 + 1} = \frac{2929}{101}$$

$$= 29$$

The quotient may be as great as 31 or as small as 29. The midpoint of this range is

$$\frac{31 + 29}{2} = \frac{60}{2}$$

$$= 30$$

The absolute uncertainty is the difference between the maximum and the midpoint, or half the difference between the maximum and the minimum.

$$\frac{31 - 29}{2} = \frac{2}{2}$$

$$= 1$$

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The absolute uncertainty is 1, so the range of uncertainty of the quotient is

$$30 \pm 1$$

Writers in the field of mathematics are not usually required to use data from actual experiments. They are free to use in examples whatever numbers they wish, and consequently they tend to choose numbers that are easy to work with. In the example above, for instance, numbers were chosen so that the maximum and minimum values of the quotient are whole numbers.

In actual scientific work we are usually faced with examples in which numbers are not so easy to work with. It is therefore useful to have an algebraic formula for the quotient of two ranges of uncertainty. Then you need only substitute the numbers into the formula. Let us suppose that we want to divide the range $x \pm \Delta x$ by the range $y \pm \Delta y$.

$$\frac{x \pm \Delta x}{y \pm \Delta y} = ?$$

The problem that we have just posed is nothing more than an algebra problem, but the computations are quite lengthy. Therefore, we will just present the result and then show how it can be applied to specific problems. The result is

$$\boxed{\frac{x \pm \Delta x}{y \pm \Delta y} = \frac{xy + \Delta x \Delta y}{y^2 - \Delta y^2} \pm \frac{x\Delta y + y\Delta x}{y^2 - \Delta y^2}}$$

In order to illustrate the use of this formula, let us return to our numerical example. The example is the division of 2999 ± 70 by 100 ± 1. We will substitute these numbers into our algebraic expression. We set $x = 2999$ and $\Delta x = 70$; we set $y = 100$ and $\Delta y = 1$.

$$\frac{x \pm \Delta x}{y \pm \Delta y} = \frac{xy + \Delta x \Delta y}{y^2 - \Delta y^2} \pm \frac{x\Delta y + y\Delta x}{y^2 - \Delta y^2}$$

$$\frac{2999 \pm 70}{100 \pm 1} = \frac{2999 \cdot 100 + 70 \cdot 1}{100^2 - 1^2} \pm \frac{2999 \cdot 1 + 100 \cdot 70}{100^2 - 1^2}$$

$$= \frac{299900 + 70}{10000 - 1} \pm \frac{2999 + 7000}{10000 - 1}$$

$$= \frac{299970}{9999} \pm \frac{9999}{9999} = 30 \pm 1$$

This result is in agreement with our previous answer.

When $\Delta x \Delta y$ is a small fraction of xy and Δy^2 is very small compared to y^2, little accuracy is lost by ignoring the terms Δy^2 and $\Delta x \Delta y$. When $\Delta x \Delta y$ and Δy^2 are ignored the formula for the range of uncertainty of the quotient simplifies to

$$\frac{x \pm \Delta x}{y \pm \Delta y} \approx \frac{xy}{y^2} \pm \frac{x\Delta y + y\Delta x}{y^2}$$

$$\approx \frac{x}{y} \pm \frac{x\Delta y + y\Delta x}{y^2}$$

Therefore,

$$\boxed{\frac{x \pm \Delta x}{y \pm \Delta y} \approx \frac{x}{y} \pm \frac{x\Delta y + y\Delta x}{y^2}}$$

This approximate formula may be applied to the example above. 70 is much smaller than 299,900 and 1 is much smaller than 10,000. Let us repreat the computation, ignoring the terms $\Delta x \Delta y$ and Δy^2.

$$\frac{x \pm \Delta x}{y \pm \Delta y} \approx \frac{x}{y} \pm \frac{x\Delta y + y\Delta x}{y^2}$$

$$\frac{2999 \pm 70}{100 \pm 1} \approx \frac{2999}{100} \pm \frac{2999 \cdot 1 + 100 \cdot 70}{100^2}$$

$$\approx \frac{2999}{100} \pm \frac{2999 + 7000}{10,000}$$

$$\approx \frac{2999}{100} \pm \frac{9999}{10,000}$$

We simplify this expression to obtain

$$\frac{2999 \pm 70}{100 \pm 1} \approx 29.99 \pm .9999$$

The actual quotient is 30 ± 1, so very little accuracy has been lost in the approximation. In general, the smaller Δx is with respect to x, and the smaller Δy is in relation to y, the better is the approximation. Use of the approximate formula causes a slight loss in accuracy, but it simplifies the computation tremendously.

We will show one more example to illustrate the use of the approximate formula. The example has been chosen to make a point concerning implied uncertainty.

Students often take two measured pieces of data, such as 1 and 3, dividing 3 into 1, and reporting the quotient as .33333333. Or what is worse, since it involves more work, dividing 97 into 103 and reporting the result as 1.06185567. The numbers are excellent approximations and suggest that the student is trying to do well, but what about implied uncertainty? The implied uncertainty of a measurement of 97 is .5, and the implied uncertainty of 103 is also .5. The implied uncertainty of 1.06185567 is .000000005. Is such a small implied uncertainty justified?

<u>EXAMPLE:</u>

Divide 103 ± .5 by 97 ± .5.

<u>SOLUTION:</u>

.5 is much less than 103 and 97, so we use the approximate formula.

$$\frac{103 \pm .5}{97 \pm .5} \simeq \frac{103}{97} \pm \frac{(.5)\,103 + (.5)\,97}{97^2}$$

$$\simeq \frac{103}{97} \pm \frac{.5(103 + 97)}{97^2}$$

$$\simeq \frac{103}{97} \pm \frac{100}{9409}$$

Before computing the midpoint $(\frac{103}{97})$ it is a good idea to estimate the uncertainty $(\frac{100}{9409})$. This estimate of the uncertainty will tell us how many decimal places to calculate in the midpoint. First we round 9409 to the nearest hundred to allow us to cancel some zeros in the numerator.

$$9409 \simeq 9400$$

Consequently

$$\frac{100}{9409} \simeq \frac{100}{9400}$$

$$\simeq \frac{1}{94}$$

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Next, we make the further approximation

$$\frac{1}{94} \simeq \frac{1}{90}$$

$$\simeq .011$$

Notice that this estimate of uncertainty has only two nonzero digits in it. This is the maximum number that you will ever need. Most of the time a single digit is adequate. Since we aren't interested in doing any more work than necessary, we round the uncertainty back to one nonzero digit.

$$.011 \simeq .01$$

This estimate of the uncertainty (.01) tells us that the midpoint should be rounded to the nearest hundredth, i.e., so that both the uncertainty and the midpoint of the product have the same implied uncertainty.

PROBLEM SET 6:

Suppose that we wish to carry out the following division.

$$\frac{1998 \pm 180}{100 \pm 1}$$

1. The largest possible value for this expression is $\frac{1998 + 180}{100 - 1}$. Compute this value.

2. Write a quotient for the smallest possible value of the expression.

3. Compute the value in Question 2.

4. Find the midpoint of the range of uncertainty by using the formula

$$\text{midpoint} = \frac{\text{largest value + smallest value}}{2}$$

5. Find the absolute uncertainty by using the formula

$$\text{absolute uncertainty} = \frac{\text{largest value - smallest value}}{2}$$

6. State the quotient as a range of uncertainty.

7. a. Use the approximate formula

$$\frac{x \pm \Delta x}{y \pm \Delta y} \simeq \frac{x}{y} \pm \frac{x\Delta y + y\Delta x}{y^2}$$

to compute the same quotient,

$$\frac{1998 \pm 180}{100 \pm 1}$$

b. Is the answer in part a close to that of problem 6?

8. From a food table with an implied absolute uncertainty of .5 grams it is found that 50 grams of egg contain about 6 grams of protein.

a. Express the ratio of protein to total food mass as a quotient of ranges of uncertainty, i.e., in the form $\frac{x \pm \Delta x}{y \pm \Delta y}$.

b. Substitute into the relation

$$\frac{y \pm \Delta x}{y \pm \Delta y} = \frac{xy + \Delta x\Delta y}{y^2 - \Delta y^2} \pm \frac{x\Delta y + y\Delta x}{y^2 - \Delta y^2}$$

to obtain a numerical statement of the range of uncertainty of the quotient. Do not simplify.

c. Note that in this case

$$\Delta x\Delta y = .25$$

$$\Delta y^2 = .25$$

Rewrite the numerical expression found in Part b, ignoring the .25 value term.

Use the relation

$$\frac{x \pm \Delta x}{y \pm \Delta y} \simeq \frac{x}{y} \pm \frac{x\Delta y + y\Delta x}{y^2}$$

to calculate the approximate quotients of the following expressions:

9. $\frac{40 \pm 1}{2 \pm .1} \simeq$

10. $\frac{6 \pm .1}{10.0 \pm .2} \simeq$

11. $\frac{6.0 \pm .02}{10.0 \pm .05} \simeq$

12. $\frac{4.02 \pm .01}{10.00 \pm .01} \simeq$

13. $\frac{15.6 \pm .04}{20.0 \pm .01} \simeq$

14. $\frac{3.63 \pm .02}{10.00 \pm .02} \simeq$

15. $\frac{16 \pm .5}{50 \pm .5} \simeq$

16. $\frac{4.25 \pm .03}{6.80 \pm .03} \simeq$

PROBLEM SET 7:

Take your time in answering these questions. There should be plenty of time to finish in one class period.

1. Nurse Nathan wishes to determine how many units of tetanus antitoxin are in

$$100 \pm 1 \text{ ml}$$

of a solution labeled

$$1500 \pm 30 \text{ units antitoxin/ml of solution.}$$

a. Supply the missing <u>units</u> in the following dimensional algebra problem.

(100 ± 1) _____ x (1500 ± 30) _____ = (Range of Uncertainty of product) _____.

b. Use the approximate formula

$$(x \pm \Delta x)\ (y \pm \Delta y) \approx xy \pm (x\Delta y + y\Delta x)$$

to show that

$$(100 \pm 1)\ (1500 \pm 30) \approx 150{,}000 \pm 4500.$$

c. Use the relation

$$\text{relative uncertainty } (\%) = \frac{\Delta x}{x} \cdot 100\%$$

to show that the relative uncertainty of 100 ± 1 is 1%.

d. Show calculations that confirm that the relative uncertainty of 1500 ± 30 is 2%.

e. Show calculations that confirm that the relative uncertainty of 150,000 ± 4500 is 3%.

f. Show calculations which confirm that the sum of the relative uncertainties of the factors is also 3%.

2. Nurse Naomi wishes to determine how many mg of the antibiotic terramycin are in

$$1.5 \pm .015 \text{ ml}$$

of a solution labeled

$$200 \pm 1 \text{ mg terramycin/ml solution.}$$

a. Supply the missing units in the following dimensional algebra problem.

(1.5 ± .015) _____ x (200 ± 1) _____ = (Range of Uncertainty) _____.

b. Use the approximate formula

$$(x \pm \Delta x)(y \pm \Delta y) \approx xy \pm (x\Delta y \pm y\Delta x)$$

to show that

$$(1.5 \pm .015)(200 \pm 1) \approx 300 \pm 4.5$$

c. Use the relation

$$\text{relative uncertainty } (\%) = \frac{\Delta x}{x} \cdot 100\%$$

to show that the relative uncertainty of 1.5 ± .015 is 1%.

d. Show calculations that confirm that the relative uncertainty of 200 ± 1 is .5%.

e. Show calculations which confirm that the relative uncertainty of 300 ± 4.5 is 1.5%.

f. Show calculations which confirm that the sum of the relative uncertainties of the factors is also 1.5%.

3. On your own paper write the information missing in the following statements.

a. In problem 1 the sum of the relative uncertainties of the factors is _____. (numerical answer)

b. In problem 1 the relative uncertainty of the approximate product is _____. (numerical answer)

c. In problem 2 the sum of the relative uncertainties of the factors is ________________. (numerical answer)

d. In problem 2 the relative uncertainty of the approximate product is _____. (numerical answer)

e. For both of the previous examples the sum of the relative uncertainties of the __________ is equal to the relative uncertainty of the __________.

4. Relative uncertainty may be determined from the following equation.

$$\frac{\text{(absolute uncertainty)}}{\text{(midpoint of the range of uncertainty)}} \times 100\%$$

The product of two ranges of uncertainty may be approximately expressed by the following equation.

$$\underbrace{(x \pm \Delta x)}_{\text{Factor 1}}\ \underbrace{(y \pm \Delta y)}_{\text{Factor 2}} \approx \overbrace{\underbrace{xy}_{\text{Midpoint}} \pm \underbrace{(x\Delta y + y\Delta x)}_{\text{Absolute Uncertainty}}}^{\text{Product}}$$

a. Express the relative uncertainty of each factor as an algebraic quotient.

b. Express the relative uncertainty of the product as an algebraic quotient.

c. Determine the numerator of the right-hand side of the following equation.

$$\left(\frac{\Delta x}{x} + \frac{\Delta y}{y}\right) 100\% = \left(\frac{?}{xy}\right) \cdot 100\%$$

d. The left side of the equation in Part c represents the sum of the __________ uncertainties of the factors.

e. The right side of the equation in Part c represents the __________ uncertainty of the approximate product.

5. Consider the following division problem.

$$\frac{2499 \pm 75}{100 \pm 1}$$

$$\text{where } x \pm \Delta x = 2499 \pm 75$$

$$y \pm \Delta y = 100 \pm 1$$

a. Show calculations which confirm the truth of the equation

$$\frac{2499 + 75}{100 - 1} = 26$$

Note that this is the <u>maximum</u> possible value for the quotient.

b. Show calculations which confirm the truth of the equation

$$\frac{2499 - 75}{100 + 1} = 24$$

Note that this is the <u>minimum</u> possible value for the quotient.

c. From the maximum and minimum possible values for the quotient, confirm that the true range of uncertainty for the quotient will be 25 ± 1.

d. Show calculations which confirm that the ratio of the midpoints of the numerator and denominator is 24.99.

e. Show calculations which confirm that the relative uncertainty of the numerator is approximately 3%. Use the approximation $2499 \approx 2500$.

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f. Show calculations that the relative uncertainty of the denominator is 1%.

g. Show claculations that confirm that the relative uncertainty of the quotient is 4%.

h. On your own paper write words that will accurately complete this sentence.

For the foregoing division problem the __________ of the __________ uncertainties of the numerator and denominator closely approximated the relative uncertainty of the quotient.

6. An algebraic problem for the quotient of two ranges of uncertainty similar to the preceding Problem 4 (on products) will not be presented here because it is much more complicated than the product of ranges of uncertainty. If you are happy about this, write yes on your paper. Perhaps this means that man was meant to multiply, not divide.

REVIEW PROBLEM SET 8:

The first four problems are intended to help you review how ranges of uncertainty behave under the operations of addition, subtraction, multiplication and division.

1. <u>Addition</u>

$(20 \pm .04) + (43 \pm .07) =$ a. __________ range of uncertainty (sum)

b. __________ absolute uncertainty (first term) c. __________ absolute uncertainty (second term) d. __________ absolute uncertainty (of sum)

e. The absolute uncertainty in Part d is the __________ of the absolute uncertainties in Parts b and c.

2. <u>Subtraction</u>

$(801 \pm 4) - (168 \pm 2) =$ a. __________ range of uncertainty (difference)

b. __________ absolute uncertainty (first term) c. __________ absolute uncertainty (second term) d. __________ absolute uncertainty (of difference)

e. The absolute uncertainty in Part d is the __________ of the absolute uncertainties in Parts b and c.

3. <u>Multiplication</u> Use the approximate formula

$$(x \pm \Delta x)(y \pm \Delta y) \approx xy \pm (x\Delta y + y\Delta x)$$

to compute the range of uncertainty in Part a. Then complete Parts b through h.

$(100 \pm 2) \cdot (500 \pm 5)$ a. __________ range of uncertainty (product)

b. __________ absolute uncertainty (first factor) c. __________ absolute uncertainty (second factor) d. __________ absolute uncertainty (of product)

e. __________ relative uncertainty (first factor) f. __________ relative uncertainty (second factor) g. __________ relative uncertainty (of product)

h. The relative uncertainty in Part g is the __________ of the relative uncertainties in Parts e and f.

4. <u>Division</u> Use the approximate formula

$$(x \pm \Delta x) \div (y \pm \Delta y) \approx \frac{x}{y} \pm \left(\frac{x\Delta y + y\Delta x}{y^2}\right)$$

to compute the range of uncertainty in Part a. Then complete Parts b through h.

(300 ± 3) ÷ (10 ± 1) a. ______ range of uncertainty (quotient)

b. ______ absolute uncertainty (dividend)

c. ______ absolute uncertainty (divisor)

d. ______ absolute uncertainty (of quotient)

e. ______ relative uncertainty (dividend)

f. ______ relative uncertainty (divisor)

g. ______ relative uncertainty (of quotient)

h. The relative uncertainty in Part g is the ______ of the relative uncertainties in Parts e and f.

5. Convert the relative uncertainty to absolute uncertainty and also rewrite the range of uncertainty using the absolute uncertainty.

a. 1500 2% meters

b. 6.02 10% grams

6. In each case find the implied absolute uncertainty of the given number.

a. 13 d. 33.4215

b. .04 e. 675.320

c. 1,000

7. In the last month Hortense has driven round trip to Nurdsburg between 8 and 10 times. In the process she has driven between 840 and 960 miles.

a. Express the distance driven as a range of uncertainty.

b. Express the number of trips as a range of uncertainty.

c. Show calculations verifying that the length of one round trip to Nurdsburg cannot be more than 120 miles.

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d. Show calculations verifying that the length of one round trip to Nurdsburg is at least 84 miles.

e. Express the length of one round trip to Nurdsburg as a range of uncertainty.

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SECTION 9:

9-1 Introduction to Vectors

Shortly you will be starting the study of nutrition in Biomedical Science class. You will investigate the requirements for a healthful diet, how the digestive system functions to meet the nutritional needs of the body and some medical problems associated with poor nutrition and malfunctions of the digestive system.

The mathematics material which you are beginning now is intended to lay some groundwork for your study of nutrition. You will be solving problems involving the basic substances necessary in a healthy diet. All of us must have a certain minimum amount of protein, fat and carbohydrate. In addition, everyone needs certain amounts of vitamins and minerals.

You will be able to answer questions of the following sort when you have finished this sequence of math lessons.

1. For what length of time can you swim on the energy obtained by eating one hot dog?

2. How many spoonfuls of peanut butter and how many spoonfuls of jelly should be used to make a sandwich containing 12 grams of protein and 72 grams of carbohydrate?

3. What is the least expensive mixture of peanuts and cashew nuts containing at least as much protein, fat and carbohydrate as a pound of liver?

9-2 Vectors

Mathematicians frequently find it convenient to devise special notations to make their work easier. A mathematical notation is often a kind of shorthand intended to minimize the amount of writing (and reading). Vectors are a type of notation that make certain types of mathematical operations simpler to perform.

Consider the following table, which gives the total mass of four breakfast foods in the first column. The quantities of protein, fat and carbohydrate are given in succeeding columns.

	mass	protein	fat	carbo-hydrate
one cup of orange juice	249 g	2 g	1 g	26 g
one scrambled egg	64 g	7 g	8 g	1 g
one slice of bread	23 g	2 g	1 g	12 g
one glass of milk	244 g	9 g	9 g	12 g

A mathematician interested not in nutrition but only in pure mathematics could call such an array of numbers a matrix. He could write the array in matrix notation:

$$\begin{bmatrix} 249 & 2 & 1 & 26 \\ 64 & 7 & 8 & 1 \\ 23 & 2 & 1 & 12 \\ 244 & 9 & 9 & 12 \end{bmatrix}$$

Matrix notation reduces the writing required, and allows the mathematician to concentrate on the numbers.

A single row of numbers is called a row vector. For example, a row vector can be written for one cup of orange juice:

$$[249, 2, 1, 26]$$

A single column of numbers is a column vector. A column vector for the quantities of fat in all four foods listed in the table is:

$$\begin{bmatrix} 1 \\ 8 \\ 1 \\ 9 \end{bmatrix}$$

Vectors are enclosed in square brackets, and the number making up a row vector are separated by commas. If a letter is used to represent a vector, an arrow is written above the letter. For example, a mathematician would know that the statement

$$\vec{V} = [4, 5, 6]$$

identifies a vector for two reasons. He would know this because of the arrow over the $\vec{V}$, and because of the square brackets enclosing numbers which are separated by the commas.

The numbers within the brackets are the components of a vector. The numbers 4, 5 and 6 in the previous example are the components of the vector $\vec{V}$. The components of a vector may be pure or abstract numbers. They may be numbers associated with units, such as 4 cm, 5 cm and 6 cm, or 4 kg, 5 kg and 6 kg, or even 4 cm, 5 kg and 6 sec. In our orange juice row vector, the components are associated with units of grams. Thus

$$[249, 2, 1, 26]$$

represents 249 g, 2 g, 1 g and 26 g, respectively.

In our study of nutrition we will work mostly with row vectors. And we will adopt a convention for row vectors of three components. Our convention will be that the first component is the grams of protein in a food, the second component is the grams of fat, while the third component is the grams of carbohydrate. Thus the vector

$$\vec{V} = [p, f, c]$$

represents p grams of protein, f grams of fat and c grams of carbohydrate.

We may represent the amounts of protein, fat and carbohydrate in one cup of orange juice by the vector expression

$$\vec{J} = [2, 1, 26].$$

We interpret the statement to mean that the food (orange juice) represented by $\vec{J}$ contains 2 grams of protein, 1 gram of fat and 25 grams of carbohydrate.

Observe that the order in which the components are written is important.

$$[26, 1, 2]$$

is not the same as

$$[2, 1, 26]$$

The first vector indicates 26 grams of protein, 1 gram of fat and 2 grams of carbohydrate, but the second indicates 2 grams of protein, 1 gram of fat and 26 grams of carbohydrate.

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9-3 Scalar Multiplication

One cup of orange juice contains about 2 grams of protein, 1 gram of fat and 26 grams of carbohydrate. How much of each type of nutrient does <u>two</u> cups of orange juice contain? Two cups of orange juice contain twice as much as one cup, or $2 \cdot 2 = 4$ grams of protein, $2 \cdot 1 = 2$ grams of fat and $2 \cdot 26 = 52$ grams of carbohydrate.

If vectors are to be useful, we should be able to perform these multiplications using vector notation, and in fact we can. We may calculate the nutrients in two cups of orange juice by writing

$$2 \cdot \vec{J} = 2 \cdot [2, 1, 26].$$

The multiplication on the right side is done by writing

$$2 \cdot \vec{J} = [2 \cdot 2, 2 \cdot 1, 2 \cdot 26]$$

and finally

$$2\vec{J} = [4, 2, 52].$$

In the same manner we may determine the quantities of protein, fat and carbohydrate in <u>three</u> cups of orange juice. We may write

$$3\vec{J} = 3 \cdot [2, 1, 26]$$

and solve this expression to obtain

$$3\vec{J} = [3 \cdot 2, 3 \cdot 1, 3 \cdot 26]$$

$$3\vec{J} = [6, 3, 78].$$

Three cups of orange juice contain 6 grams of protein, 3 grams of fat and 78 grams of carbohydrate.

We may represent any row vector as

$$\vec{V} = [v_1, v_2, \ldots, v_n].$$

The dots indicate that the vector may have any number of components. The general expression for the multiplication of a vector by a number is

$$x\vec{V} = [x \cdot v_1, x \cdot v_2, \ldots, x \cdot v_n].$$

The number multiplying the vector is called a <u>scalar</u> or <u>scalar number</u>. The operation of multiplying a vector by a scalar is

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called scalar multiplication. Scalar multiplication is one of the techniques we will use to do nutritional computations.

Scalar multiplication is a commutative operation. That is to say,

$$x\vec{V} = \vec{V}x$$

for any scalar x and any vector $\vec{V}$.

PROBLEM SET 9:

1. Which of the following are vectors?

a. $\begin{bmatrix} 3 \\ 7 \\ 4 \\ 8 \end{bmatrix}$

b. $\begin{bmatrix} 16 \\ 16 \\ 16 \end{bmatrix}$

c. $\vec{V}$

d. distance AB

e. [4, 3, 7]

f. $\begin{bmatrix} 9 & 7 & 5 & 3 \\ 8 & 6 & 4 & 2 \\ 7 & 5 & 3 & 1 \\ 6 & 4 & 2 & 0 \end{bmatrix}$

g. 17.268

2. Identify in the previous problem a row vector, a column vector, a matrix and a scalar.

3. When a mathematician sees the statement

$$\vec{R} = [10, 2, 8]$$

which two things tell him that the quantity represented is a vector?

4. We have adopted a convention for interpreting vectors. The vector [31, 8, 11] means ___ g of protein, ___ g, of fat and ___ g of carbohydrate.

5. Interpret the vector [1, 3, 14].

6. The vector for the nutritional ingredients of 245 g of yogurt is

$$\vec{Y} = [8, 4, 13]$$

How many grams of fat are there in 245 g of yogurt?

7. One scrambled egg contains 7 g protein, 8 g fat and 1 g carbohydrate.

a. Two scrambled eggs contain ___ g protein, ___ g fat and ___ g carbohydrate.

b. Write the nutritional contents of one egg as a row vector.

c. If $\vec{E} = [7, 8, 1]$, which of the following is equal to $2\vec{E}$?

(1) [49, 64, 1] (3) [9, 10, 3]

(2) [14, 8, 1] (4) [14, 16, 2]

8. A slice of bread contains 2 g protein, 1 g fat and 12 g carbohydrate.

a. How much of each do 3 slices contain?

b. Write a row vector $\vec{B}$ for the nutritional ingredients of a slice of bread.

c. Write a vector equal to $3\vec{B}$.

9. Operations such as $2\vec{E}$ and $3\vec{B}$ are called:

a. vector multiplication

b. scalar multiplication

10. Perform the indicated scalar multiplications, using vector notation.

a. One glass of milk contains 9 g protein, 9 g fat and 12 g carbohydrate. How much of each do 2 glasses contain?

b. A piece of Swiss cheese contains 8 g protein, 8 g fat, and 1 g carbohydrate. How much of each do 8 pieces contain?

11. The vector for the nutritional ingredients of one cup of cooked spinach is

$$\vec{S} = [5, 1, 6]$$

a. During an unusually taxing day, Popeye ate 12 cups of spinach. Write a row vector for the nutritional ingredients of the spinach he ate.

b. If Popeye needs 70 g of protein a day, did he eat enough spinach to meet his requirement?

12. The vector for the nutritional ingredients of one cup of raw oysters is [20, 4, 8]. How many cups of oysters are represented by the vector [5, 1, 2]?

13. The nutrient vector $\vec{P}$ of one sweet potato is

$$\vec{P} = [2, 1, 36]$$

The vector for one cup of roasted cashew nuts is

$$\vec{C} = [24, 64, 41]$$

a. Find the nutrient vector for 20 sweet potatoes.

b. Twenty sweet potatoes contain more __________ and __________ than one cup of cashews.

c. Twenty sweet potatoes contain less __________ than one cup of cashews.

d. How many sweet potatoes must one eat to get the same amount of fat as that in one cup of cashews?

14. Suppose that the nutrient vector for 195 g of mashed potatoes is

$$\vec{M} = [4, 8, 24]$$

How many grams of mashed potatoes are represented by the vector

$$[.5, 1, 3]?$$

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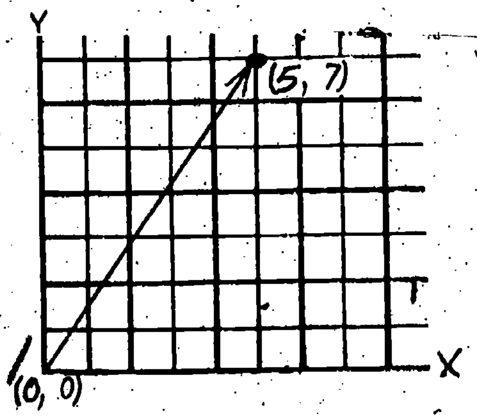
SECTION 10:

10-1 Equivalent Vectors

The previous section concentrated on vectors with three components, since these vectors will be used in your study of nutrition. This section deals with two component vectors. The components of these vectors will be interpreted as pure numbers; that is, numbers without units.

Vectors with two components may be represented graphically in two-space. A vector is represented on a graph as an arrow. For example, the vector [5, 7] may be represented as an arrow whose "tail" is located at the point (0, 0) and whose "head" is at the point (5, 7).



Note the difference in notation between a vector and the coordinates of a point. The coordinates of a point are enclosed in parentheses, while a vector is enclosed in square brackets. The numbers within the brackets are the components of the vector. We may refer to the x-component and y-component of a vector, but these components are different from x-coordinates and y-coordinates.

Coordinates designate points on a graph. Components of a vector, however, are the differences between the coordinates of the head of an arrow and the coordinates of the tail of the arrow. The components of a vector may be found by subtracting the coordinates of the tail from the coordinates of the head. The tail of the vector shown in the graph above has the coordinates (0, 0). The head has the coordinates (5, 7). The x-component of the vector is consequently 5 - 0 = 5. The y-component of the vector is 7 - 0 = 7. Thus we represent the vector by the expression [5, 7].

The vector represented in the graph above is not the only vector with an x-component of 5 and a y-component of 7. Consider the

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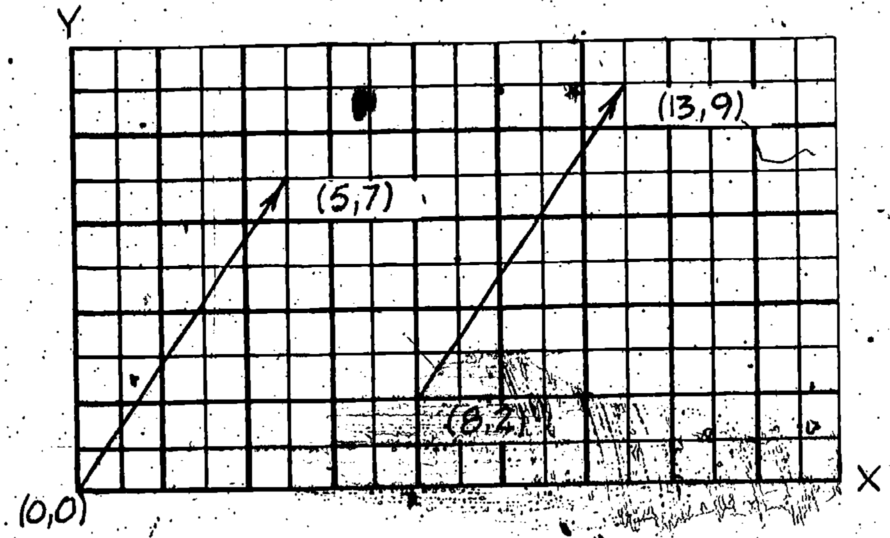
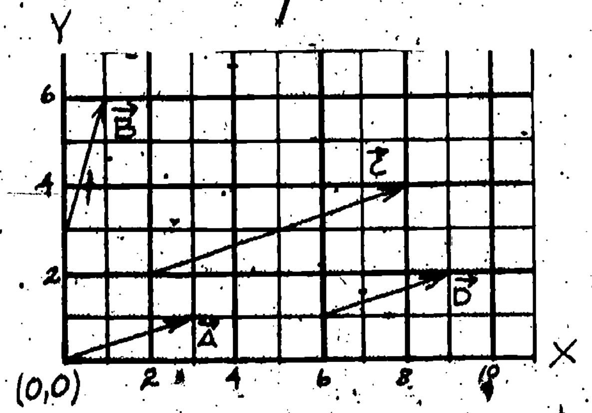
vector on the right in the following graph.



The coordinates of the head of the right-hand vector are (13, 9), while the tail is located at the point (8, 2). Therefore, the x-component of the vector is 13 - 8 = 5, and the y-component is 9 - 2 = 7. The right-hand vector is thus represented in vector notation as [5, 7]. The right-hand vector has the same components as the left-hand vector and is represented by the same notation, although the heads and tails have different coordinates.

The right-hand vector and the left-hand vector are said to be equivalent. In vector theory two vectors may be equivalent although they do not occupy the same position when graphed.

Let us further illustrate the idea of equivalence by another example. Consider the four vectors represented in the following graph. Which are equivalent?



To determine the equivalence of the vectors we must find the components of each. Recall that the components of a vector are found by subtracting the coordinates of the tail from the coordinates of the head.

Thus the components of vector $\vec{A}$ expressed in vector notation are [3, 1]. The x-component of vector $\vec{B}$ is 1 - 0 = 1, while the y-component is 6 - 3 = 3. In vector notation $\vec{B}$ is [1, 3]. $\vec{B}$ is not equivalent to $\vec{A}$, because its x-component is not equal to the x-component of $\vec{A}$, nor are the y-components equal.

The x-component of vector $\vec{C}$ is 8 - 2 = 6; the y-component is 4 - 2 = 2. $\vec{C}$ in vector notation is consequently [6, 2]. Although $\vec{C}$ points in the same direction as $\vec{A}$, it is not equivalent to $\vec{A}$, because the two vectors do not have the same components. Nor is $\vec{C}$ equivalent to $\vec{B}$.

The components of vector $\vec{D}$ are 9 - 6 = 3 and 2 - 1 = 1. In vector notation, $\vec{D}$ is [3, 1]. $\vec{D}$ is equivalent to $\vec{A}$, because its components are equal to the components of $\vec{A}$.

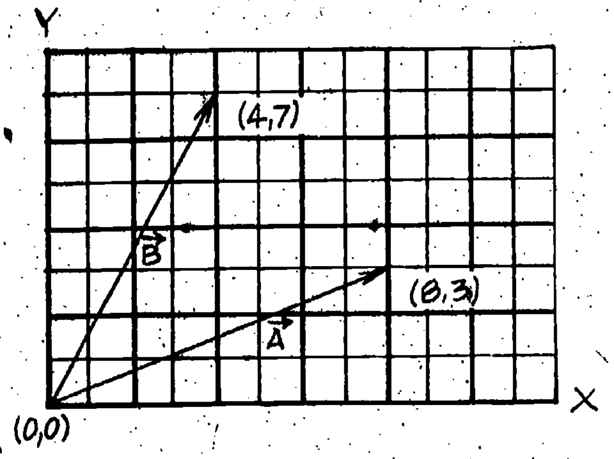
Let us summarize this section with a formal definition of equivalence. Two vectors are said to be equivalent if and only if their respective components are equal.

10-2 Adding Vectors

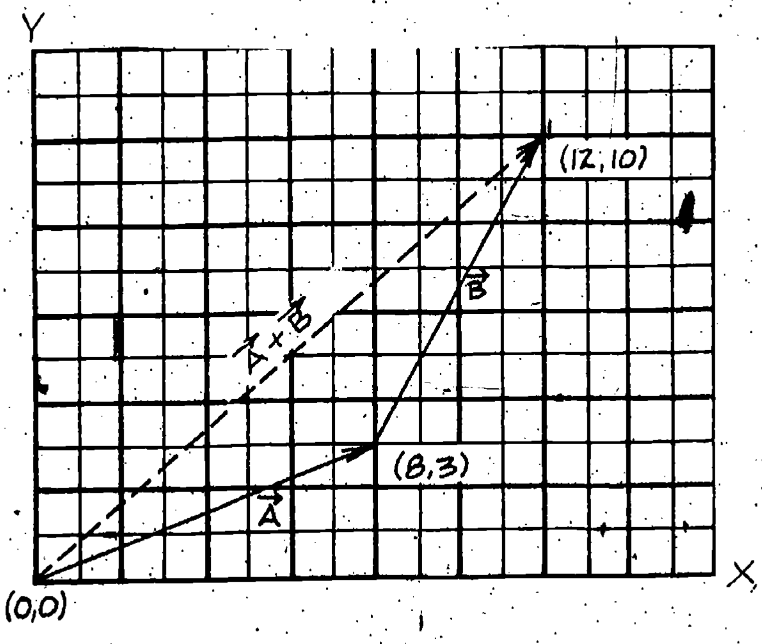
In Section 9 we adopted a convention for expressing the quantities of protein, fat and carbohydrate in a food as a vector. The first component represents grams of protein, the second component grams of fat and the third grams of carbohydrate. One scrambled egg contains 7 grams of protein, 8 grams of fat and 1 gram of carbohydrate, so we write the vector for a scrambled egg [7, 8, 1]. One slice of bread contains 2 grams of protein, 1 gram of fat and 12 grams of carbohydrate; its vector is [2, 1, 12].

In Section 9 we discussed how to determine, using vector notation, the amount of each nutrient in two eggs or three slices of bread. We did not discuss, however, how to use vectors to calculate the combined nutritional value of one egg and one slice of bread. In this section you will learn how to solve problems of this type by adding vectors.

We will start by adding vectors on a graph on the following page. Consider vectors $\vec{A}$ and $\vec{B}$ on the graph.



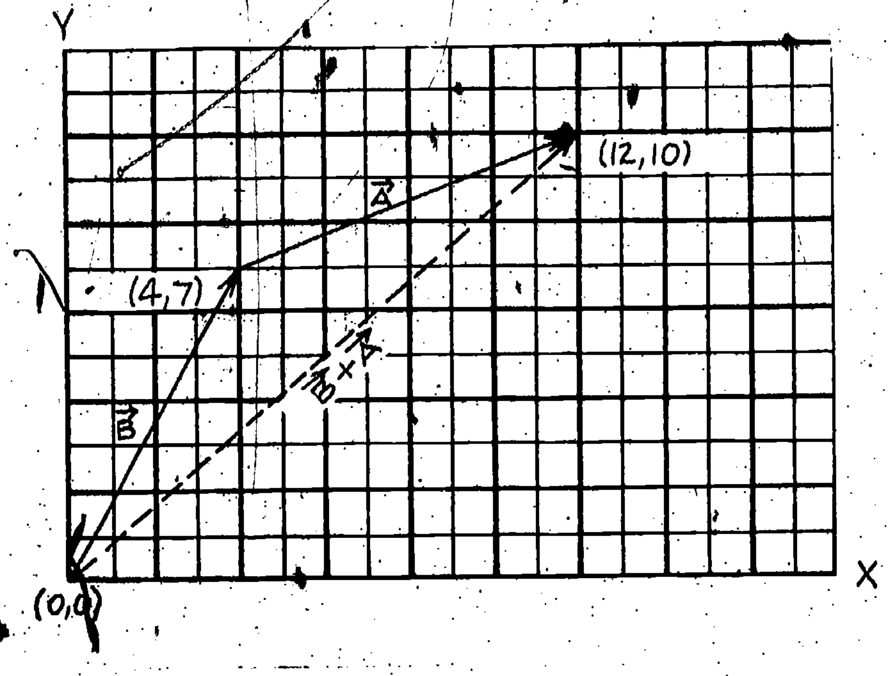
Vectors may be added graphically by placing them "head-to-tail." This is done by drawing an equivalent vector with its tail located at the head of the other vector. In our example, we draw a vector equivalent to $\vec{B}$ with its tail at the point (8, 3). The head of this equivalent vector is then at the point (12, 10).



The sum of vectors $\vec{A}$ and $\vec{B}$ is a vector with its tail at (0, 0) and its head at (12, 10). That is to say

$$\vec{A} + \vec{B} = [12, 10]$$

We may also perform the addition by placing a vector equivalent to $\vec{A}$ with its tail at the same point as the head of $\vec{B}$.



This sum is again the vector [12, 10], so we may say that

$$\vec{B} + \vec{A} = [12, 10].$$

Thus,

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

and vector addition is <u>commutative</u>.

It is not necessary to add vectors graphically. Vectors may be added by simply adding components. The vectors in our example may be added by adding the first component of $\vec{A}$ to the first component of $\vec{B}$, and the second component of $\vec{A}$ to the second component of $\vec{B}$.

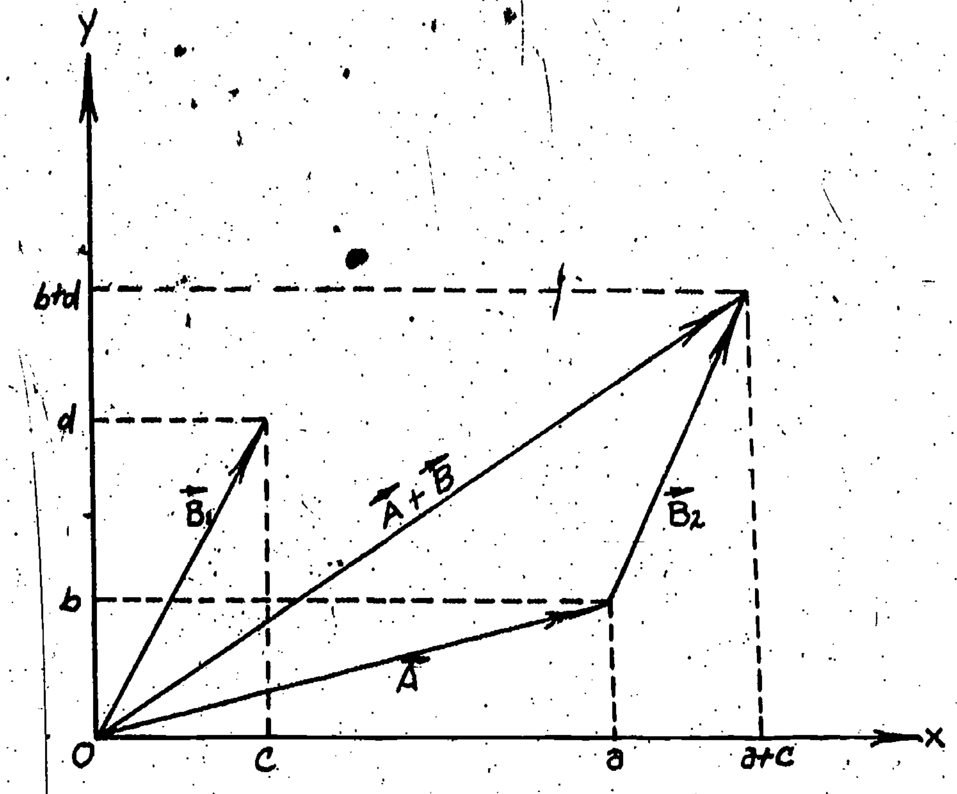
$$[4, 7] + [8, 3] = [4 + 8, 7 + 3]$$
$$= [12, 10]$$

The result is the same as that obtained by graphical addition.

We have demonstrated the addition of vectors for one particular case. We may also state the general case. The addition of vector [a, b] to vector [c, d] is given by the expression

$$[a, b] + [c, d] = [a + c, b + d]$$

The general case may also be shown graphically. The next graph represents the addition of vector $\vec{A} = [a, b]$ and vector $\vec{B} = [c, d]$.



Note that $\vec{B}_1$ and $\vec{B}_2$ are equivalent vectors. The components of a vector are the differences in coordinates between the head and the tail. The differences in coordinates of $\vec{B}_1$ are $c - 0 = c$ and $d - 0 = d$. The differences in coordinates of $\vec{B}_2$ are given in vector notation as

$$[a + c - a, b + d - b] = [c, d]$$

We conclude this section by returning to the problem we posed at the beginning of the section. How much protein, fat and carbohydrate is contained in one scrambled egg and one slice of bread? The scrambled egg vector is [7, 8, 1]; the bread vector is [2, 1, 12]. We may determine the total quantity of each nutrient by vector addition:

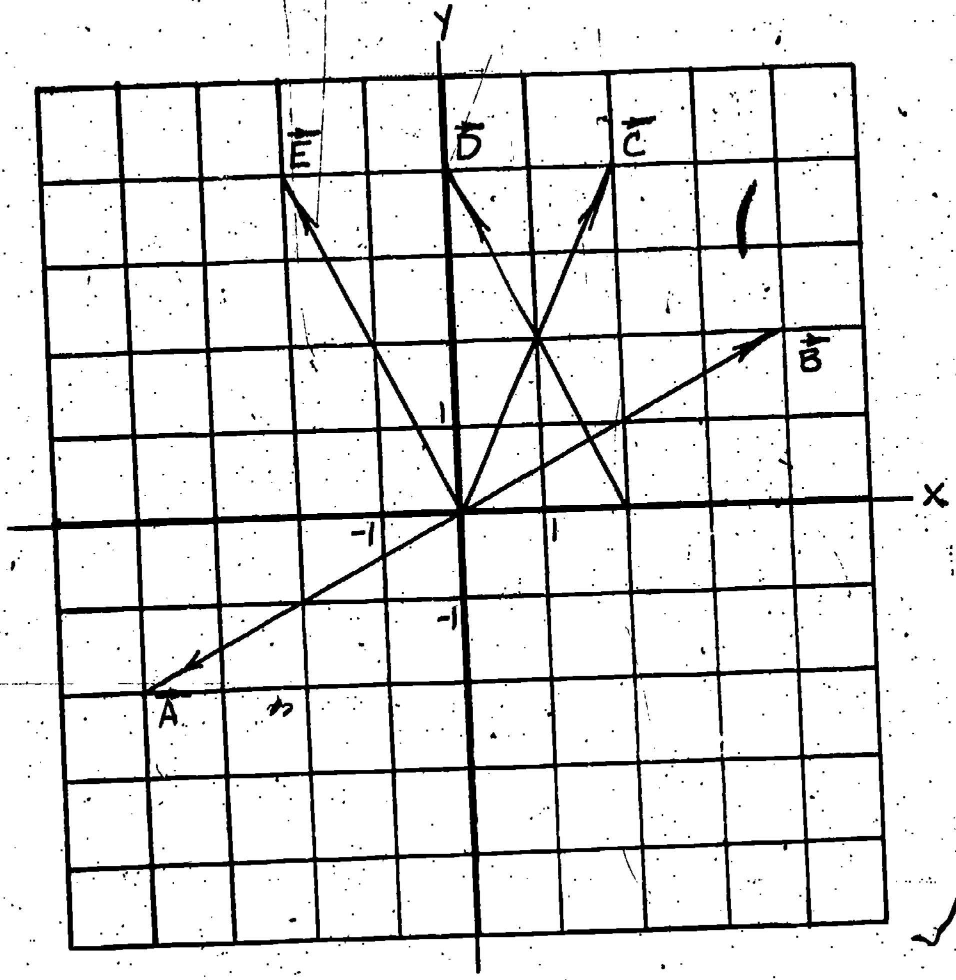
$$[7, 8, 1] + [2, 1, 12] = [7 + 2, 8 + 1, 1 + 12]$$
$$= [9, 9, 13]$$

A scrambled egg and slice of bread together contain 9 grams of protein, 9 grams of fat and 13 grams of carbohydrate.

This simple example illustrates the use of vector addition in doing dietary calculations.

PROBLEM SET 10:

1.



a. Which is vector [4, 2]?

b. The head of vector [4, 2] is at what point?

c. The tail of vector [4, 2] is at what point?

d. Which vector on the graph is vector [2, 4]?

e. Which is vector [-4, -2]?

f. Vector $\vec{D}$ is said to be ______ to vector $\vec{E}$.

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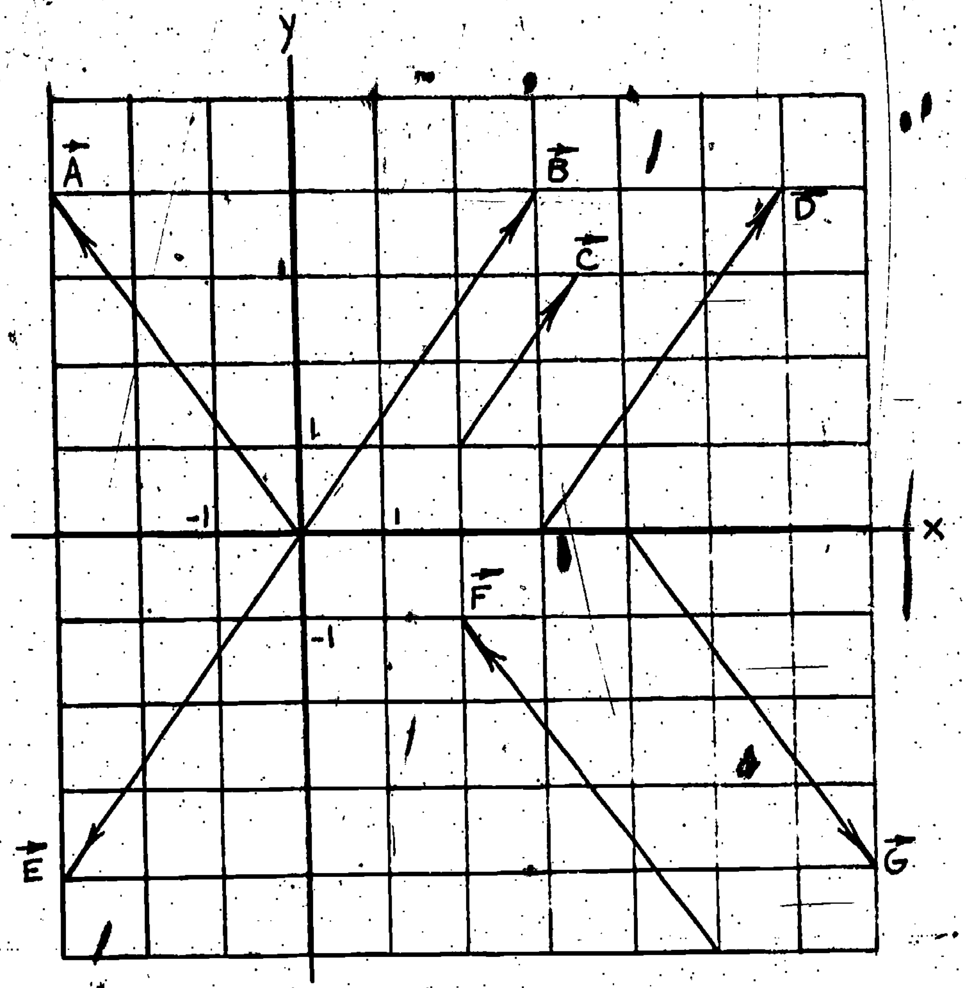


a. Vector $\vec{B}$ is designated

vector [?, ?].

b. Designate vector $\vec{A}$ and vector $\vec{E}$.

c. Which vector is equivalent to vector $\vec{A}$?

d. Which vector vectors are equivalent to vector $\vec{B}$?

e. Which vector is designated [1.5, 2]?

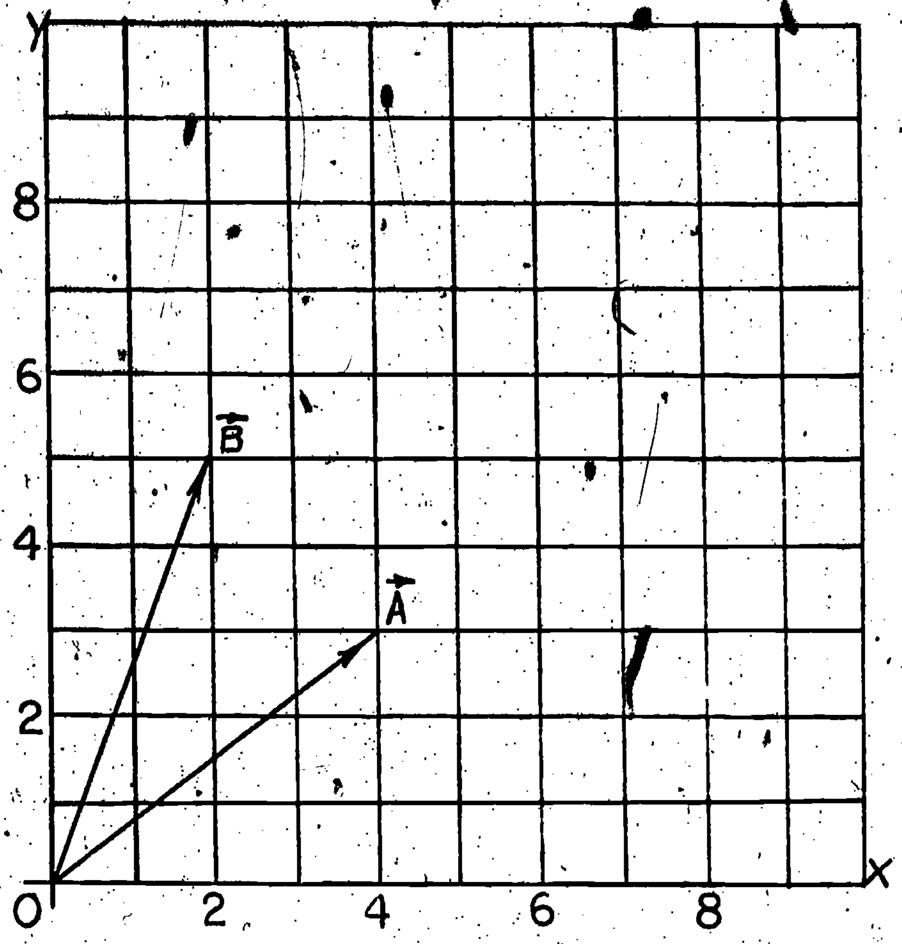
3. Draw the following vectors on a graph.

a. [1, 3]

b. [5, 2]

c. [4, -2]

d. [6, 0]



We wish to add vectors $\vec{A}$ and $\vec{B}$.

a. Draw a vector equivalent to $\vec{B}$ with its tail at the same point as the head of $\vec{A}$.

b. Draw a vector from the origin to the head of the vector you drew equivalent to $\vec{B}$.

c. This vector is the vector sum $\vec{A} + \vec{B}$. Which of the following vectors is it?

(1) [6, 8]   (3) [9, 5]

(2) [8, 6]   (4) [-2, 2]

d. Repeat the procedure but this time draw a vector equivalent to $\vec{A}$ with its tail at the same point as the head of $\vec{B}$.

e. Is the resultant vector sum of part d the same as part c?

f. Does $\vec{A} + \vec{B} = \vec{B} + \vec{A}$?

g. Is the addition of vectors commutative?

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5. Vectors may be added simply by adding their components. Which of the following is a correct method of adding the two vectors in Problem 4?

a. [4, 3] + [2, 5] = [4 + 3, 2 + 5] = [7, 7]

b. [4, 3] + [2, 5] = [4 + 2, 3 + 5] = [6, 8]

c. [4, 3] + [2, 5] = 4 · 2 + 3 · 5 = 23

d. [4, 3] + [2, 5] = 4 · 3 + 2 · 5 = 22

6. In general, [a, b] + [c, d] = ? (choose one)

a. [a + b, c + d]

b. [ab + cd]

c. [a + c, b + d]

7. Add the following vectors:

a. [3, 9] + [7, 4]

b. [16, 0] + [1, 3]

c. [-3, 2] + [7, -1]

d. [1, 3] + [-4, -5]

e. [a, b] + [2, 3]

f. [3, 1] + [6, 9] + [7, 4]

g. [5, 0] + [4, 7] + [-3, 6]

h. [3, 6, 2] + [4, 7, 1]

i. [5, 0, 3] + [-2, 6, 5]

j. [6, 3, 1] + [6, 4, 2] + [6, 5, 3]

8. Let two tablespoons of peanut butter = $\vec{P}$.

Let two slices of white bread = $\vec{B}$.

$\vec{P}$ = [8, 16]

$\vec{B}$ = [4, 2]

The order of the vector components is protein, fat.

a. Graph $\vec{P} + \vec{B}$.

b. Graph $\vec{B} + \vec{P}$.

c. Does $\vec{B} + \vec{P} = \vec{P} + \vec{B}$?

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9. Let two tablespoons of chocolate syrup = $\vec{S}$.

Let one piece of cherry pie = $\vec{C}$.

Let one scoop of vanilla ice cream = $\vec{V}$.

$\vec{C} = [4, 15]$

$\vec{V} = [2, 8]$

$\vec{S} = [0, 1]$

The order of the vector components is protein fat.

a. Graph $\vec{C} + \vec{V} + \vec{S}$.

b. Graph $\vec{S} + \vec{C} + \vec{V}$.

c. Calculate $\vec{C} + \vec{V} + \vec{S}$ and write the result using vector notation.

d. Do the graphical and calculated values agrees?

10. Let one cup of whole milk = $\vec{M}$.

Let one ounce of corn flakes = $\vec{C}$.

Let two tablespoons of sugar = $2\vec{S}$

$\vec{M} = [9, 9, 12]$

$\vec{C} = [2, 0, 24]$

$2\vec{S} = [0, 0, 24]$

Calculate $\vec{M} + \vec{C} + 2\vec{S}$

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Refer to the following table in Problems 11 through 18.

Let one slice of ham = $\vec{H}$.

Let one slice of swiss cheese = $\vec{C}$.

Let one slice of white bread = $\vec{W}$.

Let one slice of French bread = $\vec{F}$.

Let one slice of rye bread = $\vec{R}$.

	gram Protein	gram fat	gram carb.	mg calcium	mg iron	I.U. vit. A	mg thiamin	mg niacin	I.U. vit. C
$\vec{H}$ =	[11,	10,	0,	6,	1.6,	0	.25,	1.5,	0]
$\vec{C}$ =	[8,	8,	1,	262,	.3,	320,	.00,	.0,	0]
$\vec{W}$ =	[2,	1,	12,	16,	.6,	0,	.06,	.5,	0]
$\vec{F}$ =	[2,	1,	13,	10,	.5,	0,	.06,	.6,	0]
$\vec{R}$ =	[2,	0,	12,	17,	.4,	0,	.04,	.3,	0]

11. How many milligrams of iron would a ham and cheese on rye sandwich have? (We are ignoring mustard, mayonnaise, etc. for the present.)

12. Which kind of bread is richest in iron?

13. Which kind of bread has the least niacin?

14. Write a column vector giving the mg thiamin in each food.

15. Write the amount of protein, fat and carbohydrate for one slice of French bread in vector notation.

16. Write a vector for one slice of cheese showing the amount of protein, fat, carbohydrate, calcium, iron, vitamin A, thiamin, niacin, and vitamin C, in that order.

17. Write in vector notation the sum $2\vec{H} + \vec{C} + 2\vec{R}$, using protein, fat, carbohydrate, calcium, iron, and vitamin A, in that order, as the components.

18. In this problem use protein, fat and carbohydrate, in that order, as the components of all vectors.

    a. Describe the makeup of the sandwich described by the vector

$$\vec{H} + 2\vec{C} + 2\vec{F}$$

b. Compute the sum in Part a.

c. This sandwich has the same amount of ______ as the sandwich in Problem 17.

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SECTION 11:

11-1 Subtracting Vectors

If zero is added to a number, the sum is the number itself.

$$n + 0 = n$$

Zero is said to be the additive identity for the system of scalar numbers. Zero is the additive identity because when it is added to a number, the sum is the same number.

When the vector [0, 0, 0] is added to a three-component vector, the sum is the vector itself.

$$[a, b, c] + [0, 0, 0] = [a, b, c]$$

Therefore [0, 0, 0] is the additive identity for vectors with three components. This vector is called the zero vector.

Two scalar numbers may add together to give the additive identity, zero. We then say that one number is the additive inverse of the other. The additive inverse of 5 is -5, because

$$5 + (-5) = 0.$$

We may also speak of the additive inverse of a vector. The additive inverse of vector [5, 6, 7] is [-5, -6, -7] because

$$[5, 6, 7] + [-5, -6, -7] = [0, 0, 0].$$

Recall that [0, 0, 0] is the additive identity of three-component vectors.

In Section 9-3 you were introduced to scalar multiplication of vectors. If vector $\vec{V} = [a, b, c]$ is multiplied by scalar x, the product is

$$x\vec{V} = [x\cdot a, x\cdot b, x\cdot c].$$

One way to obtain the additive inverse of a vector is scalar multiplication by -1. Vector $\vec{C}$ multiplied by -1 gives the additive inverse of $\vec{C}$.

$$(-1)\vec{C} = (-\vec{C})$$

In general then,

$$\vec{C} + (-1)\vec{C} = \vec{C} + (-\vec{C})$$
$$= \vec{0}$$

$\vec{0}$ is the additive identity vector.

We have briefly discussed additive identity and additive inverse as an introduction to subtraction of vectors. Recall that subtracting a number is equivalent to adding the number's additive inverse. For example, subtracting 7 from 5 is equivalent to adding -7 to 5.

$$5 - 7 = 5 + (-7) = -2$$

Subtracting -8 from 4 is equivalent to adding 8 and 4.

$$4 - (-8) = 4 + (+8) = 12$$

The same principle is true for subtracting vectors. Subtracting a vector is the same as adding its additive inverse. As an example let $\vec{A} = [3, 7, 5]$ and $\vec{B} = [1, 4, 2]$, and let us calculate $\vec{A} - \vec{B}$.

$$\vec{A} - \vec{B} = \vec{A} + (-1)\vec{B}$$

$$= [3, 7, 5] + (-1)[1, 4, 2]$$

The right-hand vector is multiplied by the scalar -1.

$$\vec{A} - \vec{B} = [3, 7, 5] + [-1, -4, -2]$$

We then add the two vectors.

$$\vec{A} - \vec{B} = [3 - 1, 7 - 4, 5 - 2]$$

$$= [2, 3, 3]$$

The example above illustrates subtraction of vectors. In general for vectors of three components,

$$[a, b, c] - [d, e, f] = [a - d, b - e, c - f].$$

If this formula is used to solve a problem, the first three steps in the solution above need not be done. We may simply write

$$\vec{A} - \vec{B} = [3, 7, 5] - [1, 4, 2]$$

$$= [3 - 1, 7 - 4, 5 - 2]$$

$$= [2, 3, 3]$$

Scalar multiplication and vector addition are commutative operations. What about vector subtraction? If subtraction is commutative, then A - B = B - A. Let us find out by using the vectors A and B of the previous paragraph.

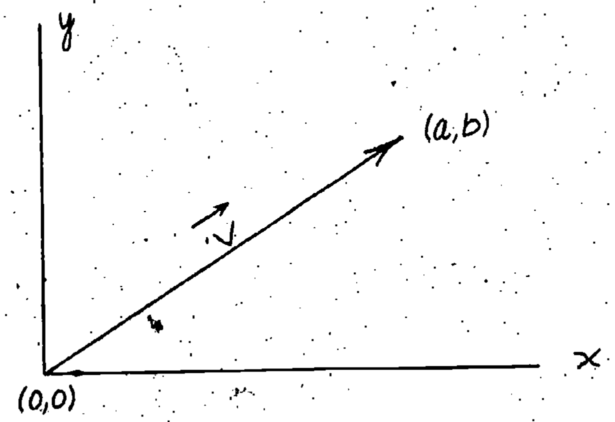
$$B - A = [1, 4, 2] - [3, 7, 5]$$

$$= [-2, -3, -3]$$

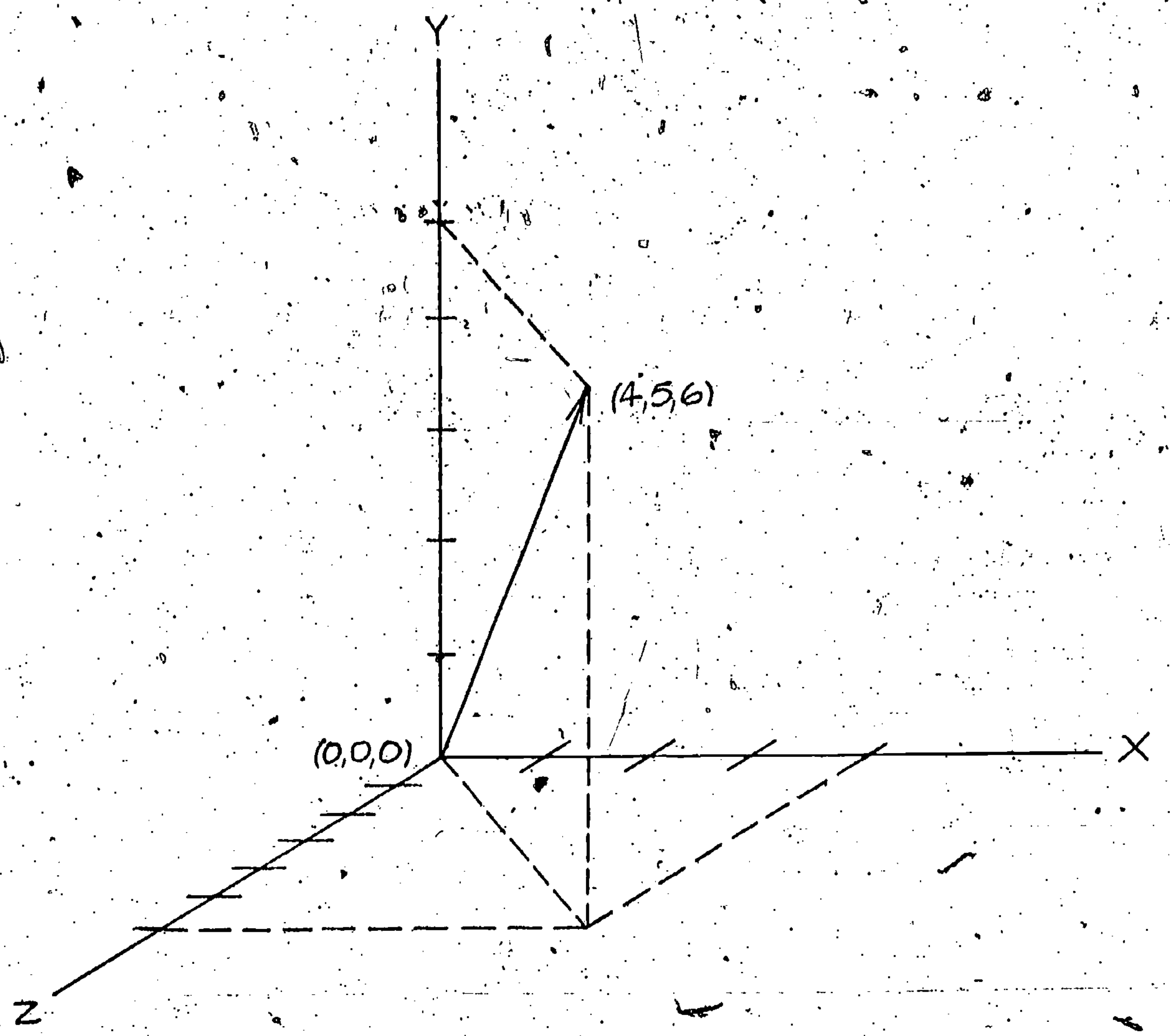
B - A is not equal to A - B. Therefore vector subtraction is not commutative.

11-2 Dimensions of Vector Spaces

In Section 10-1 we represented vectors on graphs. The vectors we represented had two components. A two-component vector can be represented in two-dimensional space, or two-space, that is, on a plane.



Vectors of three components may be represented in three-dimensional space, or three-space. There is a third coordinate in three-space; this coordinate is conventionally called the z-coordinate. It is conventional to state the three coordinates in the order (x, y, z). Likewise the components of a vector in three-space are in the order [x, y, z].



The vector illustrated above is designated [4, 5, 6]. Visualize the x and y axes as being in the plane of the paper and the z axis coming out of the paper at a right angle.

Vectors are not restricted to two components or three components. A vector in four-space is represented by four components. A vector in n-space is represented by n components. However, we have no satisfatory way to represent vectors with four or more components graphically.

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PROBLEM SET 11:

1. a. If $\vec{V} = [6, 4]$, $2\vec{V} = ?$

   b. If $\vec{V} = [2, -7, 1]$, $(-1)\vec{V} = ?$

(Multiple Choice)

2. The additive identity of the real number 7 is: a. -7, b. 1, c. 0, d. 24

(Multiple Choice)

3. The additive identity of a vector is the vector which when added to that vector gives the same vector. The additive identity of vector [3, 1, 2] is: a. [-3, -1, -2], b. [1, 1, 1], c. [2, 1, 3], d. [0, 0, 0]

4. What is the additive identity of [0, -2, 4]?

(Multiple Choice)

5. Suppose we add a real number and another number (n) and get a sum of zero. The name mathematicians have given to this other number (n) is: a. additive identity, b. additive inverse, c. identity inverse, d. scalar sum

6. When a vector and its additive inverse are summed, the result is the __________ vector.

In each of Problem 7 through 10 find the additive inverse of the given vector.

7. [5, 6, 7]

8. [1, 0, 10]

9. [-2, 5, -6]

10. [-100, -50, -1]

(Multiple Choice)

11. Which scalar multiplication gives the additive inverse of the vector [x, y, z]? a. (1)[x, y, z], b. (0)[x, y, z], c. (-1)[x, y, z], d. [-x, -y, -z] · [x, y, z]

Which is correct?

a. $\vec{A} - \vec{B} = \vec{A} + (-1)(-\vec{B})$

b. $\vec{A} - \vec{B} = \vec{A} + (1)\vec{B}$

c. $\vec{A} - \vec{B} = \vec{A} + (-1)\vec{B}$

13. Which is correct?

a. [6, 9] - [4, 2] = [6, 9] + [-4, -2] = [2, 7]

b. [6, 9] - [4, 2] = [6, 9] - [-4, -2] = [10, 11]

c. [6, 9] - [4, 2] = [6 - 9] + [4 - 2] = 6

14. Which is correct?

a. [a, b] - [c, d] = [a - c, b - d]

b. [a, b] - [c, d] = [a - b, c - d]

c. [a, b] - [c, d] = [c - a, d - b]

15. Is subtraction of vectors commutative?

16. Perform the following vector subtractions.

a. [12, 16] - [9, 5]

b. [3, 8] - [2, 0]

c. [7, 4] - [18, 11]

d. [3, 5] - [6, -8]

e. [6, 8, 10] - [3, 5, 7]

f. [a, b, c,] - [x, y, z]

g. [8, 3] + [19, 1] - [7, 2]

h. [14, 10] - [1, 0] - [2, 1]

In Problems 17 through 21 suppose that

$$\vec{R} = [3, -1, 2]$$

$$\vec{S} = [-2, 5, -3]$$

$$\vec{T} = [0, -1, 2]$$

17. Carry out the following steps in order to compute $2\vec{R} - 3\vec{S}$

a. compute $2\vec{R}$

b. compute $3\vec{S}$

c. Subtract the vector in part b from the vector in part a.

18. Compute $5\vec{S} - 6\vec{T}$

19. Compute $\vec{T} - 10\vec{R}$

20. Compute $\vec{R} - \vec{S} - \vec{T}$

21. Compute $\vec{S} - 3\vec{R} + 4\vec{T}$

In Problems 22 through 27, refer to the following table which gives [protein, fat, carbohydrate] vectors for various foods.

steak (1 serving)	$\vec{A} = [20, 27, 0]$
fried perch (1 serving)	$\vec{B} = [16, 11, 6]$
chicken pot pie (one)	$\vec{C} = [23, 31, 3]$
peas (1 cup)	$\vec{D} = [9, 1, 19]$
corn (1 ear)	$\vec{E} = [3, 1, 16]$
broccoli (1 stalk)	$\vec{F} = [6, 1, 8]$
mashed potatoes (1 cup)	$\vec{G} = [4, 1, 25]$
french bread (1 slice)	$\vec{H} = [2, 1, 13]$

22. Find the nutrient vector for a meal consisting of one serving of steak, half a cup of peas and two slices of french bread. That is, compute

$$\vec{A} + \frac{1}{2}\vec{D} + 2\vec{H}$$

23. One evening Elmo had a meal consisting of one chicken pot pie, one cup of mashed potatoes and a certain unknown vegetable. The nutrient vector $\vec{M}$ for the entire meal was [35, 33, 38]. Carry out the following steps to find the nutrient vector $\vec{X}$ for the unknown vegetable.

a. Find the nutrient vector for the pot pie and mashed potatoes combined (add $\vec{C}$ and $\vec{G}$).

b. Subtract the vector in part a from $\vec{M}$ to find $\vec{X}$.

24. Norbert Numnitz had a meal of roast veal, one ear of corn, and one cup of mashed potatoes. The vector $\vec{M}$ for the meal was [30, 16, 41]. Find the vector $\vec{X}$ for the roast veal.

25. If one serving of fried perch, one serving of baked squash and two slices of french bread give a nutrient vector

$$\vec{M} = [24, 14\ 64]$$

find the vector $\vec{X}$ for a helping of baked squash.

26. A meal with $\vec{M} = [53, 25.5, 62.5]$ consisted of two helpings of perch, three stalks of broccoli, half a cup of mashed potatoes and one beer. Find the nutrient vector for one beer.

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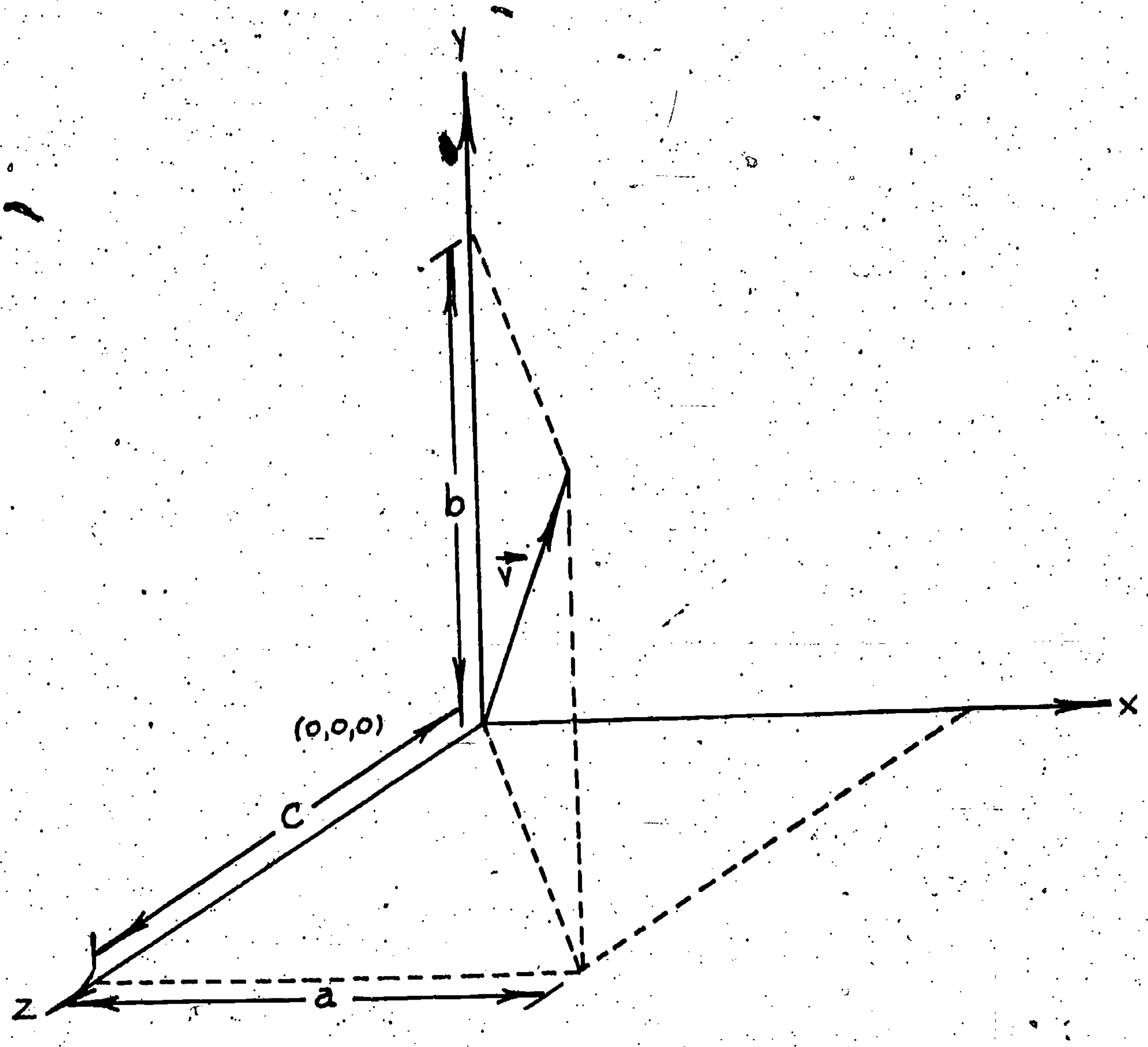
27. Ima Goodwin had a meal with $\vec{M}$ = [41, 35, 51] consisting of one pot pie, two ears of corn and some broccoli. How much broccoli did she eat?

28.



a. The minimum number of components necessary to specify $\vec{V}$ in this figure is ______.

(Multiple Choice)

b. Vector $\vec{V}$ is designated

(1) [a, b] (3) [a, b, c]

(2) [a, b] + [c]

29. Is the vector [1, 6, 9, 3] possible? In what kind of space?

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SECTION 12:

12-1 Dot Products of Vectors

The quantities of protein, fat and carbohydrate in a food may be expressed by a three-component vector.

$$\vec{F} = [f_1, f_2, f_3]$$

Food $\vec{F}$ contains f_1 grams of protein, f_2 grams of fat and f_3 grams of carbohydrate.

Often you will be given the amounts of protein, fat and carbohydrate in a food and asked to determine the calories of energy provided by that food. Nutritional research has shown that a gram of protein provides roughly 4 calories, a gram of fat about 9 calories, while a gram of carbohydrate yields approximately 4 calories. f_1 grams of protein therefore provide $4f_1$ calories, f_2 grams of fat provide $9f_2$ calories and f_3 grams of carbohydrate provide $4f_3$ calories. The total calories provided by food $\vec{F}$ is

$$\text{Total Calories} = f_1 \text{ g prot} \cdot \frac{4 \text{ cal}}{\text{g prot}} + f_2 \text{ g fat} \cdot \frac{9 \text{ cal}}{\text{g fat}} + f_3 \text{ g carb} \cdot \frac{4 \text{ cal}}{\text{g carb}}$$

$$= f_1 \cancel{\text{g prot}} \cdot \frac{4 \text{ cal}}{\cancel{\text{g prot}}} + f_2 \cancel{\text{g fat}} \cdot \frac{9 \text{ cal}}{\cancel{\text{g fat}}} + f_3 \cancel{\text{g carb}} \cdot \frac{4 \text{ cal}}{\cancel{\text{g carb}}}$$

$$= 4f_1 + 9f_2 + 4f_3 \text{ calories}$$

The calculation above is an example of what is called a dot product or inner product. A dot product is the product of two vectors.

The numbers of calories in each of the three types of nutrients may be written as a vector of three components. Call this vector $\vec{K}$.

$$\vec{K} = [k_1, k_2, k_3]$$

where

$$k_1 = 4 \frac{\text{calories}}{\text{g protein}}$$

$$k_2 = 9 \frac{\text{calories}}{\text{g fat}}$$

$$k_3 = 4 \frac{\text{calories}}{\text{g carbohydrate}}$$

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Thus

$$\vec{K} = [4, 9, 4]$$

The dot product of $\vec{K}$ and $\vec{F}$ is written

$$\vec{K} \cdot \vec{F} = [4, 9, 4] \cdot [f_1, f_2, f_3] \text{ calories.}$$

A dot product is found by multiplying the first component of one vector by the first component of the other, the second component of one by the second component of the other and the third component of one by the third component of the other. These three products are then added. The sum is the dot product.

The dot product of $\vec{K}$ and $\vec{F}$ may be determined in this way.

$$\vec{K} \cdot \vec{F} = [4, 9, 4] \cdot [f_1, f_2, f_3] \text{ calories}$$

$$= 4f_1 + 9f_2 + 4f_3 \text{ calories}$$

This expression gives the total calories provided by food $\vec{F}$. Note that the dot product of two vectors is not itself a vector but rather a scalar number.

We may write an expression that defines the dot product for the general case, the multiplication of two vectors of n components. Suppose we have the two vectors

$$\vec{A} = [a_1, a_2, \ldots, a_i, \ldots, a_n]$$

$$\vec{B} = [b_1, b_2, \ldots, b_i, \ldots, b_n]$$

Then

$$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + \ldots + a_ib_i + \ldots + a_nb_n$$

Observe that both vectors must have the same number of components, because each term of the product contains one component of each vector.

The right side of the previous expression is rather long and unwieldy. Mathematicians do not like clumsy notation or expressions requiring large amounts of writing. As we have said before, new notation is often created to reduce the amount of writing. There is a problem though, and the problem is that such "shorthand" can frighten those not familiar with it. There is a tendency to avoid learning and using it. However, learning new notation is generally worth the effort, because it simplifies handling numbers. Simplifying the handling of numbers is a large part of what mathematics is about.

Expressions such as that on the right side of the equation above may be written in more compact form by using summation notation. In order to see how summation notation works, suppose we start with a simple example. Suppose that we want to find a quick way of writing the following sum.

$$a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9$$

We begin by noting a few things about this sum. All the terms are the same except for the subscripts. The subscripts begin at two and run up to nine. In summation notation, we would write the sum as follows.

$$\sum_{i=2}^{9} a_i$$

The symbol "$\sum$" is the capital sigma of the Greek alphabet. Sigma is the counterpart of our "S", the first letter of the word "sum." The notation "i=2" beneath the $\sum$ indicates that the summation begins with the term in which i is equal to 2. The 9 above the $\sum$ means that the sum ends with the term in which i = 9. In other words, the sum is from a_2 to a_9.

Let us illustrate summation notation with another simple problem.

EXAMPLE:

Calculate $\sum_{i=1}^{4} a_1$, given that

$$a_1 = 1$$

$$a_2 = 2$$

$$a_3 = 3$$

$$a_4 = 4$$

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SOLUTION:

$$\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4$$

$$= 1 + 2 + 3 + 4$$

$$= 10$$

The sum of a_i for $i = 1$ to 4 is thus 10.

The dot product of two vectors may be expressed in terms of summation notation. Let us use the vectors used previously,

$$\vec{A} = [a_1, a_2, \ldots, a_i, \ldots, a_n]$$

$$\text{and } \vec{B} = [b_1, b_2, \ldots, b_i, \ldots, b_n]$$

The dot product of $\vec{A}$ and $\vec{B}$ is $\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + \ldots + a_ib_i + \ldots + a_nb_n$. This dot product may be written more easily using summation notation.

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{n} a_i b_i$$

The following problem illustrates the use of the dot product to calculate the number of calories provided by a specific food.

EXAMPLE:

A beef pot pie is 10% protein, 15% fat and 19% carbohydrate. How many calories are provided by a 200 g serving of beef pot pie?

SOLUTION:

The first step is to find the number of grams of protein, fat and carbohydrate in the 200 g serving. This is easily done, using the percentages given.

.10 X 200 = 20 g protein
.15 X 200 = 30 g fat
.19 X 200 = 38 g carbohydrate

Therefore the protein, fat, carbohydrate vector $\vec{B}$ for the serving is

$$\vec{B} = [20, 30, 38]$$

The calories per gram of protein, fat and carbohydrate are given by the vector $\vec{K}$.

$$\vec{K} = [4, 9, 4]$$

The total calories in the serving of beef pot pie is the dot product of $\vec{K}$ and $\vec{B}$.

$$\vec{K} \cdot \vec{B} = [4, 9, 4] \cdot [20, 30, 38] \text{ calories}$$
$$= 4 \cdot 20 + 9 \cdot 30 + 4 \cdot 38 \text{ calories}$$

Performing the indicated multiplications and addition, we obtain

$$\vec{K} \cdot \vec{B} = 80 + 270 + 152 \text{ calories}$$
$$= 502 \text{ calories}$$

Is dot product multiplication commutative? Does it matter whether we write $\vec{K} \cdot \vec{B}$ or $\vec{B} \cdot \vec{K}$? Let us determine $\vec{B} \cdot \vec{K}$ and compare the result to the result of the example above.

$$\vec{B} \cdot \vec{K} = [20, 30, 38] \cdot [4, 9, 4] \text{ calories}$$
$$= 20 \cdot 4 + 30 \cdot 9 + 38 \cdot 4 \text{ calories}$$
$$= 80 + 270 + 152 \text{ calories}$$
$$= 502 \text{ calories}$$

Thus $\vec{B} \cdot \vec{K} = \vec{K} \cdot \vec{B}$. In fact the dot product operation is always commutative.

PROBLEM SET 12:

Find the dot product of the following:

(Multiple Choice)

1. [a, b] • [c, d] =

   a. (a + b)(c + d)   c. ab + cd

   b. (a + c)(b + d)   d. ac + bd

2. [3, 7] • [9, 4] =

   a. 75   c. -1

   b. 55   d. 35

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3. $[-1, 3]\cdot[8, 2] =$

4. $[5, 0]\cdot[1, 18] =$

5. $[x, y]\cdot[1, 1] =$

6. $[x, y]\cdot[1, 0] =$

7. $[x, y]\cdot[0, 1] =$

(Multiple Choice)

8. $[a, b, c]\cdot[x, y, z] =$

   a. $(a + b + c)(x + y + z)$ c. $ax + by + cz$

   b. $(abc)(xyz)$ d. $(a + x)(b + y)(c + z)$

9. $[1, 2, 3]\cdot[3, 2, 1] =$

10. $[1, 2, 3]\cdot[1, 1, 1] =$

11. $[x, y, z]\cdot[0, 0, 0] =$

12. $[x, y, z]\cdot[0, 0, 1] =$

13. $[x, y, z]\cdot[-1, -1, -1] =$

14. $[x_1, x_2, x_3, x_4, x_5, x_6]\cdot[y_1, y_2, y_3, y_4, y_5, y_6] =$

For Problems 15 through 17, find the dot products and give the proper units.

15. [6 cal/g, 9 cal/g]·[4 g, 3 g]

16. [3 acorns/acre, 7 acorns/acre]·[2 acres, 1 acre]

17. [300 people/sq mile, 600 people/sq mile, 200 people/sq mile] · [4 sq mile, 70 sq mile, 100 sq mile]

18. Is it possible to have a dot product $[a, b, c]\cdot[x, y]$?

19. True or False? $[a, b, c]\cdot[x, y, z] = [ax, by, cz]$

20. Discuss the difference between $2[x, y, z]$ and $[2, 2, 2]\cdot[x, y, z]$

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(Multiple Choice)

21. $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$

may be written in summation notation as

a. $\sum_{i=1}^{7} a_i$ b. $\sum_{i=1}^{7} a_7$ c. $\sum_{i=1}^{7} 7$ d. $\sum_{i=1}^{7} i$

22. Write the sum

$$x_2 + x_3 + x_4 + x_5$$

in summation notation.

23. Write the sum

$$s_1 + s_2 + s_3 + \ldots + s_{100}$$

in summation notation.

24. Vector $\vec{A} = [a_1, a_2, \ldots a_n]$ and vector $\vec{B} = [b_1, b_2, \ldots, b_n]$

Write $\vec{A} \cdot \vec{B}$ using the summation sign.

25. Write your answer to Problem 14 using the summation sign.

(Multiple Choice)

26. $1 + 2 + 3 + \ldots + n$ can be written as:

a. $\sum_{i=1}^{n} n$ b. $\sum_{i=1}^{n} i$ c. $\sum_{n=1}^{n} i$ d. $\sum_{i=1}^{3} i$

27. We know from investigation that a gram of protein contains 4 calories, a gram of fat 9 calories, and a gram of carbohydrate 4 calories. These numbers are called conversion factors. Write a 3-component vector from these conversion factors and call it $\vec{K}$. Remember that the order is important.

28. A sirloin steak is 24% protein, 32% fat and 0% carbohydrate.

a. How many grams each of protein, fat, and carbohydrate are in one gram of steak?

b. How many grams of each are in 100 grams of steak?

c. Write a vector $\vec{S}$ showing the grams of each in a 100-gram steak.

29. A side order of French fries has a mass of 50 g; 4% is protein, 12% fat and 35% carbohydrate.

a. Find the number of grams of protein, fat, and carbohydrate in the side order and write a vector $\vec{F}$.

b. Use the dot product operation to find the number of calories in a side order of French fries.

30. In a lettuce and tomato salad, the tomato vector $\vec{T}$ is [1, 0, 4] and the lettuce vector $\vec{L}$ is [2, 0, 3]. A tablespoon of blue cheese dressing contains 1 g protein, 8 g fat and 1 g carbohydrate.

a. Write a vector for the salad with a tablespoon of dressing.

b. Use the dot product to find the number of calories in the salad.

31. A glass of milk has a mass of 244 g, of which 9 g are protein, 9 g are fat and 12 g are carbohydrate. A piece of chocolate cake has a mass of 120 g, of which 5 g are protein, 20 g are fat and 67 g are carbohydrate.

a. Using vector notation, find the number of calories in a glass of milk.

b. Do the same for a piece of chocolate cake.

c. Add the milk vector and the cake vector, and use the sum to find the total calories in the two foods.

d. Add the calories obtained in a and b. Does the sum equal the answer to c?

32. A 17-year-old student requires 3000 calories a day and has already consumed 2000 calories. Will a dinner of the salad, steak, french fries, milk and chocolate cake described in Problems 28 through 31 give him sufficient calories? How many calories more or less than he requires will the dinner provide?

*33. Food is converted to energy in the body by a chemical process in which oxygen reacts with the protein, fats, and carbohydrates to form water, carbon dioxide, wastes and energy. The water formed is called metabolic water and is available to the body to perform the normal functions of water. The amount of metabolic water formed is .4 g/g of protein, 1.0 g/g of fat and .6 g/g of carbohydrate.

a. Write g of metabolic water per g of protein, fat and carbohydrate, as a vector, $\vec{W}$.

b. A quart of milk has a mass of 976 g and contains 36 g protein, 36 g fat and 48 g carbohydrate. Write the last three numbers as vector $\vec{M}$.

c. Use the dot product $\vec{W} \cdot \vec{M}$ to find the amount of metabolic water formed by the oxidation of a quart of milk.

d. Assume that the remainder of the 976 g of milk is water. How much water does the quart of milk contain?

e. Use the results of c and d to find the total amount of water gained by drinking a quart of milk.

f. Use the results of e and the following data to determine whether consumption of a quart of milk causes a net increase or decrease of water in the body.

Energy content of 1 quart of milk: 640 cal.
Water lost in heat dissipation: .6 g/cal.
Water lost in wastes: .8 g/g of milk

g. Would a diet of nothing but milk (no other food, no water) make a person thirstier or less thirsty?

*34. To convert a gram of protein to energy .9 liter of oxygen is used; to convert a gram of fat 1.9 liters are used; and .8 liter is used to convert a gram of carbohydrate.

a. Write the amount of oxygen as a vector $\vec{O}_2$ and use the vector to determine how much oxygen is used to convert to energy food containing 10 g protein, 20 g fat and 55 g carbohydrate.

b. How much oxygen is consumed metabolizing a quart of milk? (Use data from Problem 33b.)

PROBLEM SET 13:

1. One frankfurter has a mass of 51 grams, of which 6 grams are protein, 14 grams are fat, and 1 gram is carbohydrate. A hot dog bun has a protein, fat, carbohydrate vector of [4, 2, 24].

a. Use vector notation to find the number of calories in one frankfurter. There are approximately 4 cal/g of protein, 9 cal/g of fat, and 4 cal/g of carbohydrate.

b. Find how many calories the bun has, using vector operations.

c. How many calories does a hot dog and bun have?

2. The following table gives the number of calories per kilogram per minute a person uses in performing various physical activities. In order to find the number of calories you use in performing such activities, you have to multiply the figure given for an activity by your mass in kilograms.

Activity	cal/kg-min
Washing, dressing	.032
Sitting; eating	.025
Sitting; watching TV	.023
Standing	.028
Walking	.055
Running	.147
Typing; driving a car	.035
Dancing	.087
Swimming	.163

a. Find your mass to the nearest kilogram. 1 lb ≈ .454 kg

b. Find the number of calories per minute you use in performing any three of the activities listed.

3. How many calories would you gain if you ate two hot dogs including buns and it took you 8 minutes to eat them?

4. Choose any three activities from the table in Problem 2 and find out how long you would have to spend at each of the activities to use up the net calories supplied by two hot dogs.

* 5. Effie the heffalump spends 24 hours a day eating. She eats nothing but hot dogs night and day, consuming them at a rate of one every four minutes. Effie's mass never changes. What is her mass? Assume that Effie uses .025 cal/kg-min as she eats.

6. Suppose you had the following dinner and it took you 40 minutes to finish the meal.

$\frac{1}{3}$ head lettuce	$= \vec{L} = [1, 0, 2]$
1 tbsp. blue cheese dressing	$= \vec{D} = [1, 8, 1]$
3-oz. steak	$= \vec{S} = [20, 27, 0]$
10 French fries	$= \vec{F} = [2, 7, 20]$
1 cup milk	$= \vec{M} = [9, 9, 12]$
1 piece chocolate cake	$= \vec{C} = [5, 20, 67]$

a. How many calories does this meal provide?

b. How many calories would you use while eating the meal?

c. Choose any three activities from the table in Problem 2, and find out how long you could perform each activity with the net calories you obtained from the meal.

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SECTION 14:

14-1 Introduction to Vector Equations

We have now studied some operations that may be done on vectors: scalar multiplication, addition, subtraction and dot product multiplication. These operations may be used to solve equations that contain vectors.

The principle of equivalence is important to solving vector equations. (We discussed equivalence in Section 2-1.) Two vectors are equivalent if and only if their respective components are equal.

Thus if two vectors are equivalent and we know the components of one, we know the components of the other. Suppose that vectors $\vec{A}$ and $\vec{B}$ are equivalent.

$$\vec{A} = \vec{B}$$

We know the components of $\vec{A}$.

$$\vec{A} = [5, 7, 9]$$

The components of $\vec{B}$, however are the unknowns x, y and z.

$$\vec{B} = [x, y, z]$$

We wish to find these three unknowns.

Since $\vec{B}$ is equivalent to $\vec{A}$, each component of $\vec{B}$ is equal to the corresponding component of $\vec{A}$.

$$[x, y, z] = [5, 7, 9]$$

Therefore $x = 5$, $y = 7$ and $z = 9$.

The equation we just solved is a very simple example of a vector equation. We will turn our attention to more complex vector equations, but our methods of solution will rely on the same principle of equivalence.

As an example of a diet problem that may be solved by means of a vector equation, consider the following. A certain person's diet should include 30 grams of protein and 24 grams of carbohydrate per meal. One gram of salmon contains .20 grams of protein and no carbohydrate, while a gram of apple contains no protein and .12 grams of carbohydrate. How much of each should the person eat to obtain 30 grams of protein and 24 grams of carbohydrate?

We write the contents of a gram of salmon and of a gram of apple as vectors, $\vec{S}$ and $\vec{A}$. These vectors have only two components, since we can ignore the fat component in this problem.

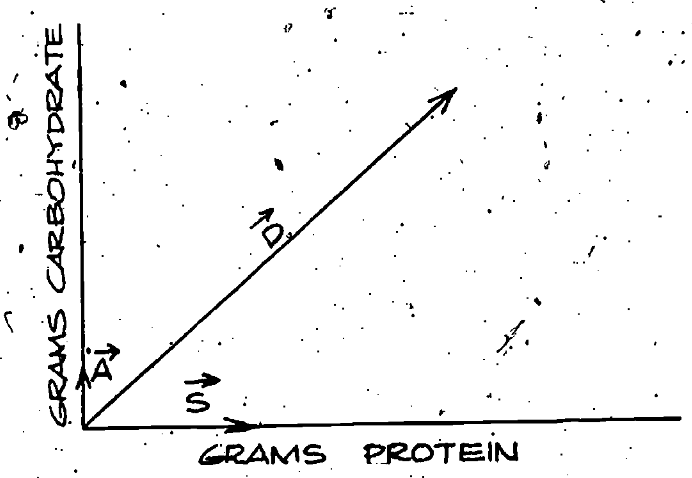
$$\vec{S} = [.20, 0]$$

$$\vec{A} = [0, .12]$$

We also express the requirement of the diet as a two-component vector, $\vec{D}$.

$$\vec{D} = [30, 24]$$

These three vectors are shown on the graph below. (The scaling is approximate.)



Let x be the number of grams of salmon this person should eat and y be the grams of apple. The quantity of protein and carbohydrate provided by x grams of salmon is

$$x\vec{S}$$

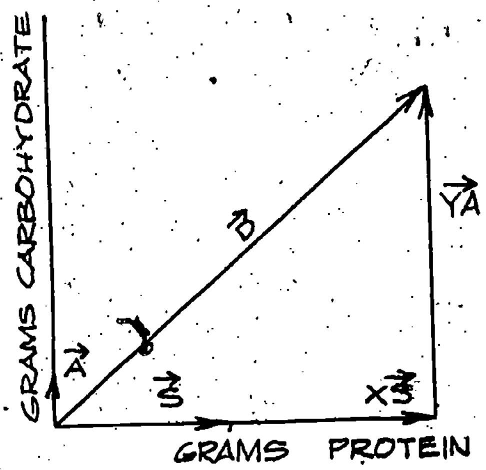
The amount of protein and carbohydrate in y grams of apple is

$$y\vec{A}$$

The total quantities of protein and carbohydrate in x grams of salmon and y grams of apple must equal the amount of protein and carbohydrate in the diet. In other words, the sum of the two scalar multiplications above must be equivalent to the diet vector $\vec{D}$.

$$x\vec{S} + y\vec{A} = \vec{D}$$

This problem may be interpreted in terms of the graph above. It is obvious that the sum of $\vec{A}$ and $\vec{S}$ is not $\vec{D}$. However, if $\vec{A}$ and $\vec{S}$ could be "stretched" to certain lengths, their sum would be equal to $\vec{D}$. This "stretching" is done by scalar multiplication. Vector $\vec{S}$ is "stretched" by scalar x, while $\vec{A}$ is "stretched" by scalar y.



Thinking in another way, it is obvious that one gram of salmon and one gram of apple is not sufficient to provide 30 grams of protein and 24 grams of carbohydrate. We need many times one gram of salmon and many times one gram of apple. The "many times" are represented by the multiples x and y.

The values of x and y may be found by substituting the numerical values of $\vec{S}$, $\vec{A}$ and $\vec{D}$ into the vector equation.

$$x\vec{S} + y\vec{A} = \vec{D}$$

The result of the substitution is

$$x[.20, 0] + y[0, .12] = [30, 24].$$

The first step in solving this equation is to perform the scalar multiplications.

$$[.20x, 0] + [0, .12y] = [30, 24]$$

Next we add the two vectors on the left.

$$[.20x, .12y] = [30, 24]$$

The resulting vector on the left is equivalent to the vector on the right. Therefore, the respective components of the vectors are equal.

$$.20x = 30 \text{ and } .12y = 24$$

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We solve the first equation for x.

$$x = \frac{30}{.20}$$

$$\doteq 150 \text{ g}$$

We solve the second equation for y.

$$y = \frac{24}{.12}$$

$$= 200 \text{ g}$$

The diet should include 150 grams of salmon and 200 grams of apple.

We have solved a vector equation in which the vectors have two components. One component is zero in one vector and the other component is zero in another vector. The problem we solved represents a general type of problem. We are given a diet $\vec{D} = [d_1, d_2]$ and two foods $\vec{A} = [a_1, 0]$ and $\vec{B} = [0, b_2]$. We are asked to find a combination of the two foods that provide d_1 grams of one nutrient and d_2 grams of a second nutrient. In other words, we must find numbers x and y such that

$$x\vec{A} + y\vec{B} = \vec{D}$$

In terms of the graphical representation, the sum of vector $\vec{A}$ "stretched" x times and vector $\vec{B}$ "stretched" y times must be equivalent to vector $\vec{D}$.

The solution of this vector equation is accomplished as in the salmon and apple problem. We substitute for $\vec{A}$, $\vec{B}$ and $\vec{D}$.

$$x[a_1, 0] + y[0, b_2] = [d_1, d_2]$$

The scalar multiplications are performed.

$$[xa_1, 0] + [0, yb_2] = [d_1, d_2]$$

We add the two vectors on the left.

$$[xa_1, yb_2] = [d_1, d_2]$$

The first component on the left is set equal to the first component on the right, and the second component on the left is set equal to the second component on the right.

$$xa_1 = d_1 \quad \text{and} \quad yb_2 = d_2$$

We solve for x and y.

$$x = \frac{d_1}{a_1} \quad \text{and} \quad y = \frac{d_2}{b_2}$$

This general solution may be used for any case with two food vectors in which one component of one food vector is zero and the other component of the second food vector is zero.

PROBLEM SET 14:

1. Two vectors are equivalent if and only if:

   a. their components are scalar multiples.

   b. their components are the same.

   c. their components are even numbers.

2. Are vectors $\vec{A}$ and $\vec{B}$ equivalent? (Answer yes or no for each)

   a. $\vec{A} = [4, 6]$, $\vec{B} = [6, 4]$

   b. $\vec{A} = [2, 3]$, $\vec{B} = [2, 3]$

   c. $\vec{A} = [0, 1]$, $\vec{B} = [1, 0]$

   d. $\vec{A} = [2, 1, 3]$, $\vec{B} = [2, 1, 3]$

   e. $\vec{A} = [46, 29]$, $\vec{B} = [46, 29]$

3. What values must x and y have to make the following statement true?

$$[x, 10] = [16, y]$$

4. $\vec{v} = [x, y, z]$, $\vec{w} = [r, s, t]$

   If $\vec{v} = \vec{w}$, x = ?

   y = ?

   z = ?

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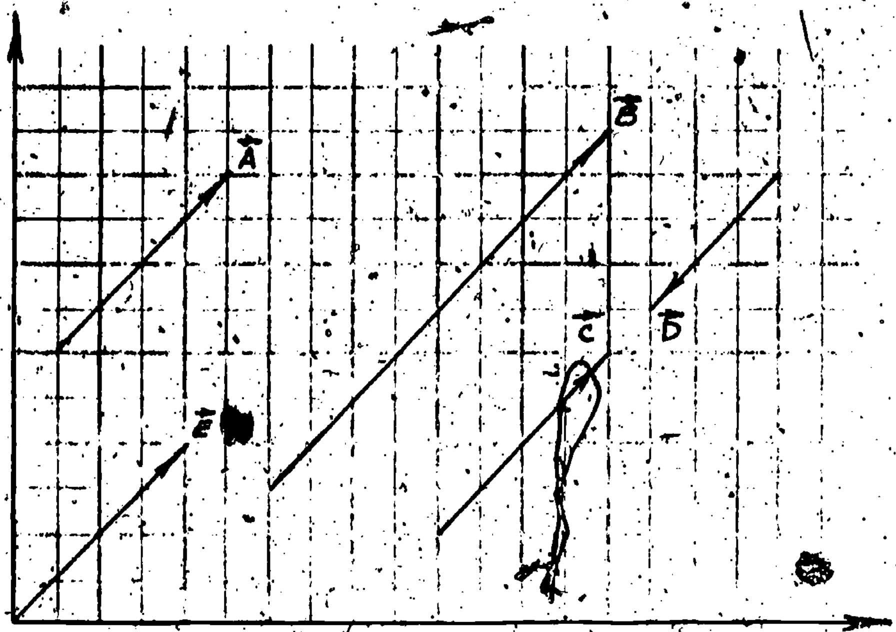
5. 

Choose the equivalent vector pairs.

a. $\vec{A}$ and $\vec{B}$ d. $\vec{D}$ and $\vec{E}$

b. $\vec{C}$ and $\vec{E}$ e. $\vec{C}$ and $\vec{D}$

c. $\vec{A}$ and $\vec{C}$ f. $\vec{A}$ and $\vec{E}$

6. Which is the two-food diet equation?

a. $x\vec{A} + y\vec{B} = \vec{D}$

b. $(x + y) \cdot (\vec{A} + \vec{B}) = \vec{D}$

c. $x \cdot \frac{\vec{A}}{\vec{B}} + y \cdot \frac{\vec{B}}{\vec{A}} = \vec{D}$

7. Suppose food $\vec{A}$ contains no carbohydrate and food $\vec{B}$ no protein. Then $\vec{A} = [a, 0]$ and $\vec{B} = ?$

a. [b, 0]

b. [0, b]

c. [a, b]

Find x and y in each of the following problems.

8. $x[1, 0] + y[0, 1] = [3, 4]$

9. $x[3, 0] + y[0, 4] = [3, 12]$

10. $x[1.5, 0] + y[0, .3] = [45, 1.5]$

11. The menu of a quaint little seaside restaurant lists "Apples and Salmon for Two" for lunch. The meal contains 34 g protein and 36 g carbohydrate. Apples are 0% protein and 12% carbohydrate, while salmon is 20% protein and 0% carbohydrate. How many grams of apples and of salmon does the meal contain?

12. Phil is restricted to a diet of 37 g fat and 67.5 g carbohydrate per meal. He is having ham and bananas for lunch today. A slice of ham (84 g) is 22% fat and has no carbohydrate. A banana (150 g) has no fat and is 15% carbohydrate.

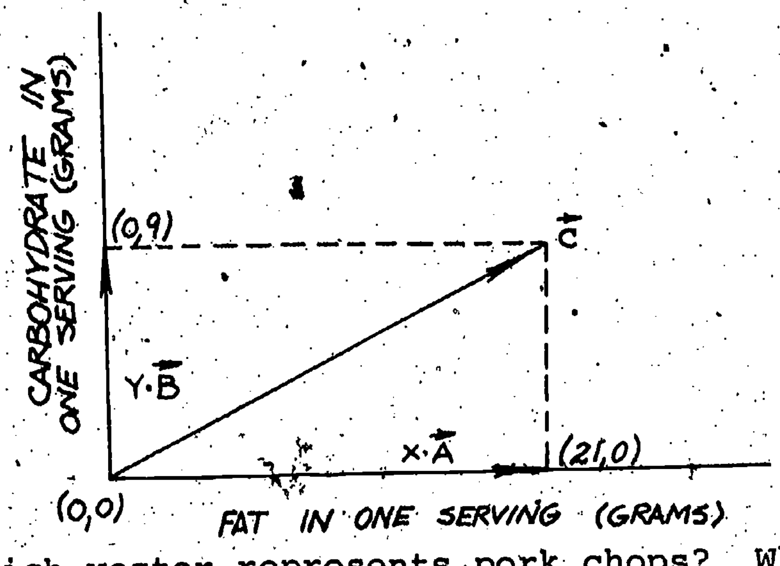
a. Write a vector $\vec{H}$ for the grams of fat and carbohydrate content of one slice of ham.

b. Write a vector $\vec{B}$ for the content of one banana.

c. Write a diet vector $\vec{D}$ for Phil's meal.

d. Solve a vector equation to find out how many bananas and slices of ham Phil should eat.

13. For winning the Intramural Frisbee Championship the victors, called the Frizzer Freaks, were treated to a dinner by the losers with the main dish being pork chops topped with cherries. One pork chop (98 g) is 21% fat and 0% carbohydrate, and one cup of cherries (130 g) is 0% fat and 15% carbohydrate.



a. Which vector represents pork chops? Which represents cherries? Which is the diet vector?

b. How many grams of fat are contained in one serving? How many grams of carbohydrate?

c. How many grams of pork chops and grams of cherries are in one serving?

SECTION 15:

15-1 A Second Kind of Vector Equation

In Section 14 we demonstrated the solution of vector equations of the type

$$x[a_1, 0] + y[0, b_2] = [d_1, d_2]$$

In this section we will add one degree of complexity. One of the vectors on the left side will not have a zero component. The vector equations will be of the type

$$x[a_1, a_2] + y[0, b_2] = [d_1, d_2]$$

Note that neither component of the first vector is zero.

We will demonstrate the solution of this type of problem by solving a specific example.

Assume that Fred's lunchtime diet should include 34 grams of protein and 62 grams of carbohydrate. In his refrigerator are only fishsticks and apples. Fred is understandably disgusted, but is too lazy to go to the store. Fishsticks are 17 per cent protein and 7 per cent carbohydrate. Apples are 12 per cent carbohydrate but contain no protein. How many grams of fishsticks and how many grams of apples should Fred eat?

We let $\vec{F}$ be the nutritional vector for one gram of fishstick. $\vec{A}$ represents the protein and carbohydrate in one gram of apple. If 17 per cent of fishsticks is protein, one gram contains .17 gram of protein. One gram of fishstick also contains .07 gram of carbohydrate. A gram of apple contains 0 grams of protein and .12 gram of carbohydrate. The vectors $\vec{F}$ and $\vec{A}$ are thus

$$\vec{F} = [.17, .07]$$

$$\vec{A} = [0, .12]$$

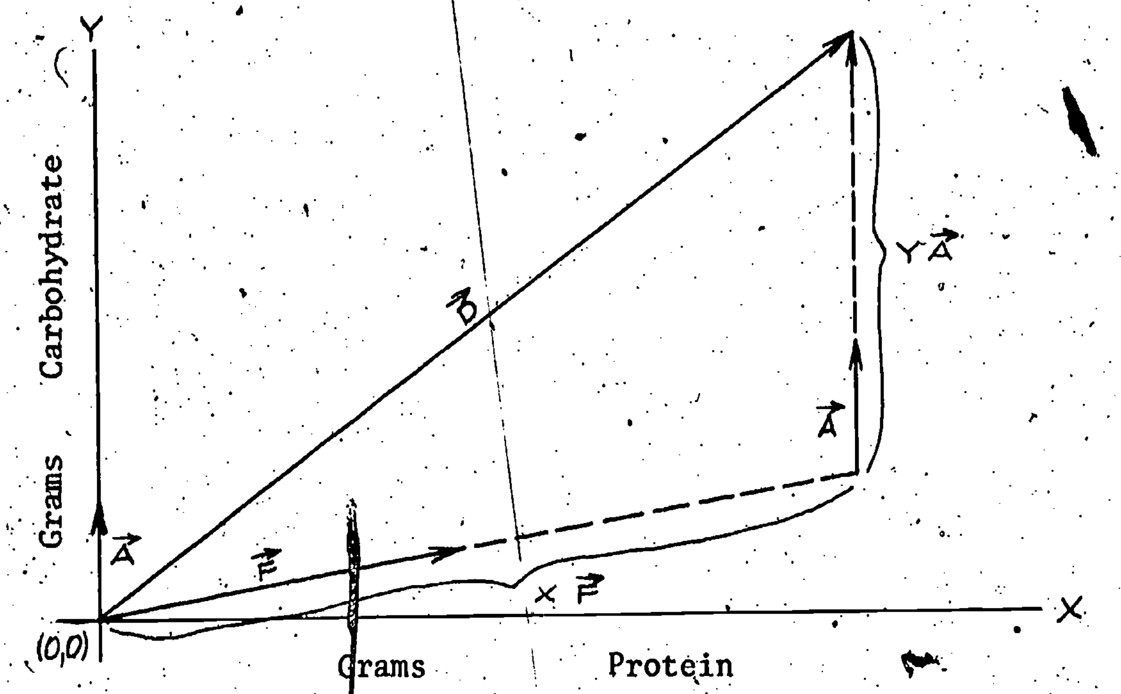
We let $\vec{D}$ be the vector expressing Fred's diet. Then

$$\vec{D} = [34, 62]$$

The number of grams of fishsticks to be eaten by Fred is x; the number of grams of apples is y. Wr write a vector equation as we did in Section 14.

$$x\vec{F} + y\vec{A} = \vec{D}$$

This type of problem is represented graphically below.



We must expand $\vec{F}$ and expand $\vec{A}$ so that the sum of the resulting vectors is equivalent to $\vec{D}$. We expand $\vec{F}$ by a factor x and expand $\vec{A}$ by a factor y.

We must solve the vector equation for x and y. We begin by substituting numerical values for $\vec{F}$, $\vec{A}$ and $\vec{D}$.

$$x[.17, .07] + y[0, .12] = [34, 62]$$

We perform the scalar multiplications of the first two vectors.

$$[.17x, .07x] + [0, .12y] = [34, 62]$$

We then add the two vectors on the left.

$$[.17x + 0, .07x + .12y] = [34, 62]$$

$$[.17x, .07x + .12y] = [34, 62]$$

Since the vector on the left is equivalent to the vector on the right, the first components of each are equal, and the second components of each are equal.

$$.17x = 34$$

$$.07x + .12y = 62$$

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The first equation may be solved for x.

$$x = \frac{34}{.17}$$

$$x = 200$$

In order to solve the second equation for y we substitute the value of x obtained from the first equation.

$$.07(200) + .12y = 62$$

$$14 + .12y = 62$$

We solve this equation for y.

$$.12y = 62 - 14$$

$$.12y = 48$$

$$y = \frac{48}{.12}$$

$$y = 400$$

The answers indicated that Fred should eat 200 grams of fishsticks and 400 grams of apples.

Our method of solving this problem differed only in the final step from the solution of the vector equations in Section 14. In Section 14 each of the final equations could be solved directly for an unknown. In the type of problem that concerns us here, one equation contains both unknowns. Therefore, the equation containing one unknown is solved directly, and then the numerical value of the unknown is substituted into the other equation.

PROBLEM SET 15:

Vector $\vec{A} = [3, 2]$ and vector $\vec{B} = [0, 1]$. Vector $\vec{D} = [6, 7]$. Problems 1 through 3 concern is finding x and y such that $x\vec{A} + y\vec{B} = \vec{D}$.

1. Are $x\vec{A}$ and $y\vec{B}$ vectors or scalars?

   $\vec{A}$, $\vec{B}$ and $\vec{D}$ are shown on the graph on the following page.

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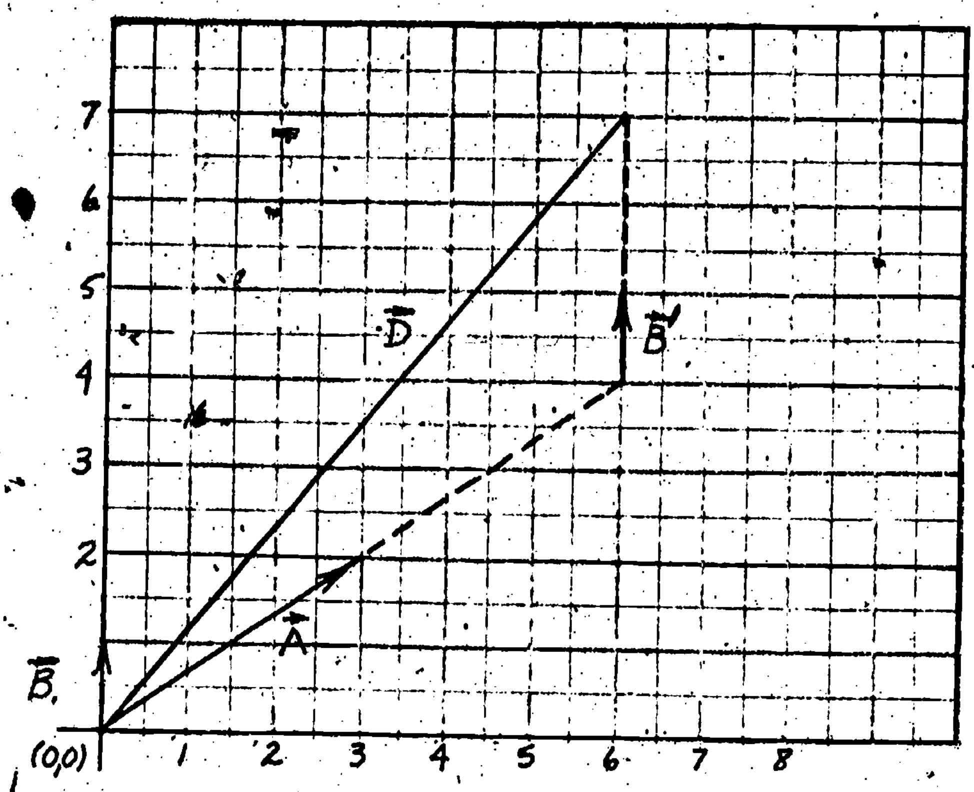

2. Vector $\vec{B}'$ is said to be ________ to vector $\vec{B}$.

3. When the vectors $\vec{A}$ and $\vec{B}'$ are extended along the dashed lines we obtain the vectors $x\vec{A}$ and $y\vec{B}$.

a. The vector $x\vec{A}$ which satisfies $x\vec{A} + y\vec{B} = \vec{D}$ is:

(1) [3, 2] (3) [6, 4]

(2) [6, 2]

b. Similarly, $y\vec{B}$ is:

(1) [0, 7] (3) [3, 0]

(2) [0, 3]

c. What does x equal and what does y equal?

In Problems 4 through 8 provide the expression which belongs in the position of the question mark.

4. x[4, 2] = [?, 2x]

5. y[5, 6] = [5y, ?]

6. x[8, 2] + y[3, 0] = [8x, 2x] + [?, 0]

7. [x, 10x] + [0, 7y] = [x, ?]

8. x[21, 5] + y[16, 0] = [21x + 16y, ?]

In each of Problems 9 through 11 a vector equation is given. Write the associated system of linear equations. Do not solve the system.

9. $[6x + y, 5y] = [20, 2]$

10. $[7x, 6x + 2y] = [5, 15]$

11. $[100x + 18y, 12x] = [18, 0]$

12. Solve the following system of equations.

$$2x + 5y = 100$$

$$8x + 0y = 200$$

Solve the following vector equations for x and y.

13. $x[4, 2] + y[0, 3] = [16, 14]$

14. $x[0, 7] + y[1, 2] = [5, 17]$

15. $x[5, 0] + y[6, 4] + [7, 3] = [25, 15]$

16. $x[3, 1] + x[7, 3] + y[9, 0] = [56, 8]$

17. A glass of orange juice contains 61 mg vitamin C and 14 mg calcium. A bowl of corn flakes contains no vitamin C and 5 mg calcium. A certain diet requires 122 mg vitamin C and 33 mg calcium.

a. Write two-dimensional vectors [mg vitamin C, mg calcium] for orange juice, for corn flakes and for the dietary requirements.

b. Let x = the number of glasses of orange juice to be drunk and y = the number of bowls of corn flakes to be eaten to achieve the diet mentioned above. This information may be written as a vector equation. Which equation describes this statement?

(1) $x[61, 0] + y[14, 5] = [122, 33]$

(2) $x + y = 122 + 33$

(3) $x[61, 14] + y[0, 5] = [122, 33]$

(4) $x[122, 33] = y[61 + 0, 14 + 5]$

c. Solve the vector equation.

d. How many glasses of orange juice and how many bowls of corn flakes are needed to fulfill the diet?

18. A slice of bread contains 2 g protein and 12 g carbohydrate. A spoonful of peanut butter contains 4 g protein and 3 g carbohydrate. A spoonful of jelly contains no protein and 14 g carbohydrate. How many spoonfuls of peanut butter and how many spoonfuls of jelly would make a sandwich (with 2 slices of bread) containing 12 g protein and 72 g carbohydrate?

19. An average strawberry contains 9 units of vitamin A and 9 mg of vitamin C. A scoop of ice cream contains 120 units of vitamin A and no vitamin C. How many scoops of ice cream and how many strawberries give 465 units of vitamin A and 225 mg of vitamin C?

20. A slice of white bread contains .015 mg vitamin B_1 and .020 mg vitamin B_2. A tablespoon of honey contains no vitamin B_1 and .012 mg vitamin B_2. To fulfill a requirement of .045 mg vitamin B_1 and .096 mg vitamin B_2 how many slices of bread should you eat, and how many tablespoons of honey should you use on each slice of bread?

21. One egg contains 6 g of protein and 6 g of fat. Non-fat milk contains no fat and 9 g of protein per cupful. A girl has read a magazine that she should eat 24 g of protein and 6 g of fat during breakfast. How many eggs and cups of milk should she have if she follows these directions? Use vector notation.

22. Martha's monkey loves bananas. To make him happy and to make the vet happy, too, Martha should give her monkey a daily diet that has at least 59.4 g of protein and 230 g of carbohydrate. She decides on bananas and hamburger. One banana contains one gram of protein and 23 grams of carbohydrate. A gram of hamburger has .247 gram of protein and no carbohydrate. How many grams of hamburger and how many bananas should the monkey eat a day?

23. Mother Hubbard went to her cupboard, but it wasn't quite bare. It contained a jar of peanut butter and a sack of sugar. Mother Hubbard and her poor dog need at least 100 grams of protein and 150 grams of carbohydrate to survive for a day.

	grams protein	grams carbohydrate
One tablespoon peanut butter = $\vec{P}$ =	[4 ,	3]
One cup sugar = $\vec{S}$ =	[0 ,	200]
Diet = $\vec{D}$ =	[100 ,	150]

How many tablespoons of peanut butter and how many cups of sugar do Mother Hubbard and her dog need to eat in one day?

24. Claude ate a meal at a pizza parlor. How many glasses of cola and how many sections of pizza would he have to eat to fulfill a diet $\vec{D}$?

	grams protein	grams carbohydrate
Diet = $\vec{D}$ =	[21 ,	105]
One section pizza = $\vec{P}$ =	[7 ,	27]
One glass of cola = $\vec{C}$ =	[0 ,	24]

How many calories does this meal have from protein and carobhydrate alone?

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SECTION 16:

16-1 The Substitution Method for Solving Vector Equations

All of the vector equations so far have included at least one vector with a component of zero. We will now consider cases in which none of the components of any food vector is zero.

In the previous two sections we demonstrated the solution of vector equations by specific examples. In this section we will again use a specific problem, in this case to demonstrate the solution of vector equations in which no food vectors have any zero components.

We are given vectors $\vec{F}_1$ and $\vec{F}_2$ representing the nutrient content of two foods.

$$\vec{F}_1 = [2, 4]$$

$$\vec{F}_2 = [3, 2]$$

We are asked to determine the quantities of these foods needed to fulfill a diet given by vector $\vec{D}$.

$$\vec{D} = [12, 16]$$

As we have done previously, we multiply the food vectors by scalar multipliers and write a vector equation.

$$x\vec{F}_1 + y\vec{F}_2 = \vec{D}$$

The scaler multipliers are x and y. The equation expresses the fact that x grams of the first food and y grams of the second food fulfill the requirements of the diet.

You should by now be familiar with the first several steps of the solution. We substitute the numerical values of the vectors into the equation and perform the scalar multiplications:

$$x[2, 4] + y[3, 2] = [12, 16]$$

$$[2x, 4x] + [3y, 2y] = [12, 16]$$

The two vectors on the left side of the equation are added. Then the first component on the left is set equal to the first component on the right, and the second components are likewise set equal to each other.

$$[2x + 3y, 4x + 2y] = [12, 16]$$

$$2x + 3y = 12 \quad \text{(Equation 1)}$$

$$4x + 2y = 16 \quad \text{(Equation 2)}$$

The result is a pair of equations, each of which contains both unknowns x and y.

We have not previously encountered this situation. Neither equation can be solved directly for one unknown. However, we can solve one equation for one unknown in terms of the other unknown. We can transfer x to the right side of the equation, for example, and isolate y on the left side. This is solving for y in terms of x. The following steps show how to solve for y in terms of x in Equation 2.

$$4x + 2y = 16 \quad \text{(Equation 2)}$$

$$2y = 16 - 4x$$

$$y = 8 - 2x$$

We now substitute the expression $8 - 2x$ for y in Equation 1.

$$2x + 3y = 12 \quad \text{(Equation 1)}$$

$$2x + 3(8 - 2x) = 12$$

$$2x + 24 - 6x = 12$$

$$24 - 4x = 12$$

This equation may be solved for x.

$$-4x = -12$$

$$x = 3$$

We have now solved for x but have not yet found y. We may find y by substituting the value of x into either Equation 1 or Equation 2. Since we have already solved Equation 2 for y in terms of x we will substitute into that result.

$$y = 8 - 2x$$

$$y = 8 - 2(3)$$

$$y = 2$$

The method of solution in the above example is called the <u>substitution method</u>. Let us review the steps involved.

1. We solved one of the equations for one of the unknowns in terms of the other.

2. We substituted the expression we obtained in step 1 into the other equation. We solved this equation for the second unknown.

3. We substituted the value of the second unknown, obtained in step 2, into the expression obtained in step 1. The resulting equation was solved for the first unknown.

When we solved the pair of equations in the example, we did step 1 by solving Equation 2 for y. We could just as well have solved Equation 2 for x, or solved Equation 1 for x or y.

<u>EXAMPLE</u>:

Solve for x and y.

$$.20x + .25y = 10 \quad \text{(Equation 1)}$$

$$.10x + .75y = 15 \quad \text{(Equation 2)}$$

<u>SOLUTION</u>:

Many people prefer to work with linear equations having coefficients which are integers. In order to put Equation 1 in this form we first note that the coefficients are expressed in hundredths. Therefore we multiply the entire equation by 100, obtaining

$$20x + 25y = 1000$$

Next we notice that 20, 25 and 1000 are all divisible by 5. Therefore we divide the entire equation by 5, obtaining

$$4x + 5y = 200 \quad \text{(Equation 1)}$$

The equation is now <u>reduced to lowest terms</u>. This is important in simplifying subsequent calculations. Equation 2 can also be converted, reducing to

$$2x + 15y = 300 \quad \text{(Equation 2)}$$

Suppose that we now solve Equation 1 for x.

$$4x = 200 - 5y$$

$$x = 50 - \frac{5}{4}y$$

We substitute this expression into Equation 2 and solve for y.

$$2(50 - \frac{5}{4}y) + 15y = 300$$

$$100 - \frac{5}{2}y + 15y = 300$$

$$100 + \frac{25}{2}y = 300$$

$$\frac{25}{2}y = 200$$

$$y = 16$$

The value of x is now easily computed.

$$x = 50 - \frac{5}{4}y$$

$$= 50 - \frac{5}{4}(16)$$

$$= 30$$

We conclude that $x = 30$ and $y = 16$.

PROBLEM SET 16:

1. Solve the following vector equations for x and y.

   a. $x[6, 4] + y[7, 3] = [75, 35]$

   b. $x[2, 1] + y[4, 5] = [28, 23]$

   c. $x[.003, .006] + y[.002, .005] + [.090, .080] = [.310, .600]$

   d. $x[7, 3] + y[9, 4] + x[2, 0] = [99, 39]$

   e. $x[15, 9] + y[12, 3] = 3[14, 7]$

2. A carrot contains 5500 units of Vitamin A and 4 mg of Vitamin C. A stalk of celery contains 100 units of Vitamin A and 4 mg of Vitamin C. John's diet requires 5900 units of Vitamin A and 20 mg of Vitamin C. Write a vector equation. Solve it.

3. An apple contains .04 mg of Vitamin B_1 and .02 mg of Vitamin B_2. A pear contains .04 mg of Vitamin B_1 and .07 mg of Vitamin B_2. How many apples and how many pears would you have to eat to get 1.00 mg of Vitamin B_1 and 1.20 mg of Vitamin B_2, the minimum daily requirements?

4. One cup of macaroni contains 1 g of fat and 39 g carbohydrate. A cup of cheese has 36 g fat and 2 g carbohydrate. If you are on a diet restricting you to 20 g fat and 79 g carbohydrate, how many cups each of macaroni and cheese can you eat, and what is the ratio of macaroni to cheese?

5. A sandwich consists of two slices of bread, a spoonful of peanut butter and a spoonful of jelly. A spoonful of peanut butter contains 4 g protein and 3 g carbohydrate. A spoonful of jelly contains no protein and 14 g carbohydrate. A slice of bread contains 2 g protein and 12 g carbohydrate. A glass of milk contains 9 g protein and 12 g carbohydrate. If you desire to have a lunch of 35 g protein and 77 g carbohydrate, how many sandwiches and how many glasses of milk should you have?

6. Master Chef Bandersnatch likes to make salads that have a protein-to-carbohydrate ratio of 1 to 3. Note that any scalar multiple of the diet vector $\vec{D} = [1, 3]$ will have such a ratio.

$$\text{one head lettuce} = \vec{L} = [3, 6]$$

$$\text{one carrot} = \vec{C} = [1, 5]$$

Make a Chef Bandersnatch salad from lettuce and carrots. What would be the proportions of the smallest salad that made use of an integral number of carrots and heads of lettuce?

SECTION 17:

17-1 The Addition-Subtraction Method for Solving Vector Equations

In Section 16 we solved a vector equation in which no component of any vector was zero. The equation was

$$x[2, 4] + y[3, 2] = [12, 16]$$

We converted this vector equation to the two equations

$$2x + 3y = 12, \quad \text{(Equation 1)}$$

$$4x + 2y = 16 \quad \text{(Equation 2)}$$

We obtained the values for x and y by solving the second equation for y in terms of x, substituting this expression for y into the first equation and solving this equation for x. Finally we substituted the numerical value of x into an earlier equation and solved it for y. The solution we obtained was $x = 3$ and $y = 2$. This procedure is called the substitution method.

We will now discuss an alternate procedure, called the <u>addition-subtraction method</u>, which may be used to solve the same pair of equations.

$$2x + 3y = 12 \quad \text{(Equation 1)}$$

$$4x + 2y = 16 \quad \text{(Equation 2)}$$

Consider the result of multiplying the first equation by 2. We designate this result by 2(Equation 1).

$$4x + 6y = 24 \quad 2\text{(Equation 1)}$$

Suppose now that we subtract Equation 2 from this new form of Equation 1.

$$\begin{array}{rl} 4x + 6y = 24 & 2\text{(Equation 1)} \\ -(4x + 2y = 16) & -\text{(Equation 2)} \\ \hline 4y = 8 & \end{array}$$

Observe that we have eliminated terms containing x. The resulting equation can be solved for y.

$$y = \frac{8}{4}$$

$$y = 2$$

The value of x is now easily found by substituting 2 for y in either of the original equations.

$$2x + 3y = 12 \quad \text{(Equation 1)}$$
$$2x + 3(2) = 12$$
$$2x + 6 = 12$$
$$2x = 6$$
$$x = 3$$

The values of x and y, not surprisingly, are the same that we obtained by the substitution method. The addition-subtraction method produces a new pair of equations with the same solution set as the original pair. When two systems of equations have the same solution set, they are said to be <u>equivalent systems</u>.

The crucial step in the addition-subtraction method is the adjustment of the equations so that one of the variables has the same coefficient in both equations. Usually this involves multiplying one or both equations by appropriate numbers. Another example follows.

<u>EXAMPLE</u>:

Solve for x and y.

$$.5x + .2y = 9 \quad \text{(Equation 1)}$$
$$.4x + .3y = 10 \quad \text{(Equation 2)}$$

<u>SOLUTION</u>:

Neither variable has the same coefficient in both equations. Therefore we must adjust the equations by multiplication. We can make the coefficient of y equal to .6 in both equations if we multiply Equation 1 by 3 and Equation 2 by 2.

$$1.5x + .6y = 27 \quad 3\text{(Equation 1)}$$
$$.8x + .6y = 20 \quad 2\text{(Equation 2)}$$

We now subtract the second equation from the first.

$$\begin{array}{rl} 1.5x + .6y = 27 & 3\text{(Equation 1)} \\ -(.8x + .6y = 20) & -2\text{(Equation 2)} \\ \hline .7x \qquad = 7 & \\ x = 10 & \end{array}$$

We now substitute this value into the original Equation 1 to find the value of y.

$$.5x + .2y = 9 \qquad \text{(Equation 1)}$$

$$(.5)(10) + .2y = 9$$

$$5 + .2y = 9$$

$$.2y = 4$$

$$y = 20$$

Let us now consider a more practical question about the addition-subtraction method. When should you use this method and when should you use the substitution method? In many problems, personal preference is the best guide. In general, the addition-subtraction method is most convenient when the coefficient of an unknown in one equation is a simple multiple of the coeffienct in the other equation. The substitution method is most convenient when one of the coefficients is a small integer. Both methods will always work. Consider the following problem which we will solve by both methods.

<u>EXAMPLE</u>:

Solve for x and y.

$$x + 14y = 76 \qquad \text{(Equation 1)}$$

$$11x - 4y = 46 \qquad \text{(Equation 2)}$$

<u>SOLUTION</u>: (SUBSTITUTION METHOD)

Solve the first equation for x.

$$x = 76 - 14y$$

Substitute this expression into the second equation.

$$11(76 - 14y) - 4y = 46$$

$$836 - 154y - 4y = 46$$

$$836 - 158y = 46$$

Solve for y.

$$-158y = -790$$

$$y = 5$$

Substitute y = 5 into the first equation.

$$x + 14(5) = 76$$

$$x + 70 = 76$$

$$x = 76 - 70$$

$$x = 6$$

SOLUTION: (ADDITION-SUBTRACTION METHOD)

Multiply the first equation by 11.

$$11x + 154y = 836 \qquad 11(\text{Equation } 1)$$

Subtract from this the second equation.

$$\begin{array}{rl} 11x + 154y = 836 & 11(\text{Equation } 1) \\ -(11x - 4y = 46) & -(\text{Equation } 2) \\ \hline 158y = 790 & \\ y = 5 & \end{array}$$

Substitute y = 5 and solve for x.

$$x + 14(5) = 76$$

$$x + 70 = 76$$

$$x = 6$$

Which method do you prefer?

PROBLEM SET 17:

Solve the vector equations in Problems 1 through 6 for x and y by the addition-subtraction method.

1. x[3, 6] + y[4, 5] = [29, 43]
2. x[9, 9] + y[3, 7] = [570, 850]
3. x[8, 24] + y[6, 24] = [2600, 9600]
4. x[.20, 1.00] + y[.18, .36] = [2, 64]
5. x[.13, .17] + y[.09, .06] = [38.40, 38.10]
6. x[.20, .25] + y[.14, .35] = [110, 225]

7. Which of the following systems of equations are equivalent to the system

$$3x + 7y = 55$$
$$7x - 9y = -49?$$

There may be more than one correct answer in this problem. Recall that equivalent systems have the same solution set.

a. $x + 3 = 5$

$y - 9 = -2$

b. $6x + 14y = 110$

$7x - 9y = -49$

c. $13x + 5y = 61$

$5y = 35$

d. $3x + 8y = 59$

$7x - 8y = -40$

8. John, a football player, loves fried chicken drumsticks and milk. He was told that during training he should eat 767 g of protein, and 169 g of carbohydrate each day. John decided to eat drumsticks and milk three meals a day. Fried chicken drumsticks are 32.5% protein and 1.3% carbohydrate. Milk is 3.9% protein and 5.2% carbohydrate.

a. How many grams of protein and of carbohydrate are in one gram of chicken? In one gram of milk?

b. Write protein-carbohydrate food vectors for chicken and for milk.

c. How many grams of chicken and of milk will John need per day to fulfill his diet?

d. If John fulfills his diet by eating only chicken and milk, he will consume 239 g of fat per day. Use the conversion vector to determine his daily caloric intake.

9. For dessert, Jane likes apple pie a la mode (with ice cream on top). How many pieces of pie and how many grams of ice cream can she eat if she can have only 39 g of fat and 93 g of carbohydrate?

There are 15 g of fat and 51 g of carbohydrate in a piece of apple pie. Ice cream is 12% fat and 21% carbohydrate. Use vector notation and be careful with your units (pieces and grams).

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REVIEW PROBLEM SET 18:

If $\vec{A} = [3, 2, -1]$, $\vec{B} = [-2, 5, 7]$ and $\vec{C} = [1, -4, 8]$ compute the following vectors.

1. $\vec{A} - \vec{B}$
2. $3\vec{C} - 2\vec{A}$
3. $2\vec{A} + \vec{B} - 3\vec{C}$

The following are nutrition vectors for the ingredients of a meal.

one hamburger patty	$\vec{H} = [21, 7, 0]$
one cup wax beans	$\vec{B} = [2, 1, 10]$
one cup rice casserole	$\vec{R} = [4, 2, 38]$

Suppose that a certain meal consists of two hamburger patties, one cup of beans and half a cup of rice casserole.

4. Express the nutrition vector for the meal in terms of the vectors $\vec{H}$, $\vec{B}$ and $\vec{R}$.

5. Compute the components of the vector in Problem 4 and write the vector in terms of the components.

6. Compute the number of calories in the meal.

7. One evening Norbert Numnitz had a meal consisting of one terrapin steak, 2 ears of corn and half a cup of mixed vegetables. Use the information below to compute the nutrient vector for <u>one cup</u> of the mixed vegetable dish.

total meal	$\vec{M} = [27, 7, 36]$
one terrapin steak	$\vec{T} = [19, 4, 0]$
one ear of corn	$\vec{C} = [3, 1, 16]$

The following table lists the calories used during certain activities.

Activity	cal/kg-min
Washing, dressing	.032
Sitting; eating	.025
Sitting; watching TV	.023
Standing	.028
Walking	.055
Running	.147
Typing; driving a car	.035
Dancing	.087
Swimming	.163

Suppose that a 70-kg person consumes a meal with nutrient vector $\vec{M} = [25, 12, 21]$.

8. Find the number of calories in the meal.

9. The meal took 30 minutes to eat. How many calories were used in eating?

10. The meal was followed by a 60-minute walk. How many calories were used on the walk?

11. At the end of the walk, how many calories remained from the meal?

12. The nutrient vector for one chicken drumstick is $\vec{D} = [12, 4, 0]$. Salmonella, a 60-kg woman, ate a drumstick so slowly that she just broke even on calories (neither gained nor lost calories). How long did she spend eating the drumstick?

13. Find x and y in each vector equation.

a. $[x, 4] = [3, y]$

b. $[x + 2, 5] = [-2, y - 1]$

c. $[17, 11] = [2x - 1, y + 5]$

14. Solve for x and y.

$$x[3, 5] + y[2, 4] = [34, 61]$$

15. A snack of raw tuna and tangerines contains 3.6 g fat and 7.2 g carbohydrate. Tuna is 4% fat and contains no carbohydrate. Tangerines contain no fat and are 9% carbohydrate. Give the amounts of tuna and tangerines in the snack.

16. A lunch with 33 g protein and 21 g fat consists of chicken drumsticks and avocado salad. The nutrient vectors are

one drumstick [12 g protein, 0 g fat]
one helping salad [6, 14]

How many drumsticks and how many helpings of salad are in the lunch?

17. A 90-g steak contains 21 g of protein and 27 g fat. A 50-g order of French fries contains 2 g protein and 6 g fat. How many grams of steak and how many grams of French fries must be consumed to provide 31 g of protein and 45 g of fat?

SECTION 19:

19-1 Inequalities

Certain nutritional problems involve determining how sufficient quantities of nutrients may be supplied at the least cost. The solution to such problems could affect the lives of many people in poor countries trying to cope with starvation. An example of such a problem is the following.

> One pound of whole wheat flour contains 60 grams of protein, 20 mg of niacin and no vitamin A. A pound of soy flour contains 168 grams of protein, 10 mg of niacin and 500 IU (international units) of vitamin A. Suppose whole wheat flour costs 15 cents per pound, while soy flour costs 30 cents per pound. What is the least expensive combination of the two types of flour that meets or exceeds a requirement of 252 grams of protein, 30 mg of niacin and 500 IU of vitamin A?

This problem differs from previous problems in one important respect. The difference is the phrase "meets or exceeds." The combination of whole wheat flour and soy flour need not contain the exact amounts specified in the dietary requirements; it may contain more than is required. The combination may contain 252 grams of protein, for example, or it may contain more.

Since the amount of protein in the flours must be either equal to or greater than the amount specified, the algebra of inequalities is involved. The algebra of inequalities has its own special notation. We will now study this notation in order to better understand the nutritional problem which we have posed.

The statement $x > y$ means that x is greater than y. For example, $6 > 4$. $x \geq y$ means that x is either greater than or equal to y. It is true that $6 \geq 4$, and it is also true that $6 \geq 6$.

$x < y$ means that x is less than y. For example, $6 < 8$. $x \leq y$ indicates that x is less than or equal to y.

The statement $x \neq y$ means that x is not equal to y. For example, $6 \neq 4$ and $6 \neq 8$.

You may remember which sign indicates "greater than" and which sign "less than" by noting that the wider side of the sign is toward the larger number. The wide side in $6 > 4$ is toward 6, and in $6 < 8$ it is toward 8.

Inequalities in many respects are like equalities. An inequality is a statement that may be true, false or open. (Recall that an open statement may be either true or false.) The following inequalities are true.

$4 > 3$ $9 \geq 3$

$3 < 5$ $5 \geq 5$

$5 \neq 7$

However, the following inequalities are false.

$4 < 2$ $4 \neq 4$

$2 > 4$ $7 \geq 9$

The following are open inequalities, because they may be either true or false.

$x > 4$ $y \leq x + 2$

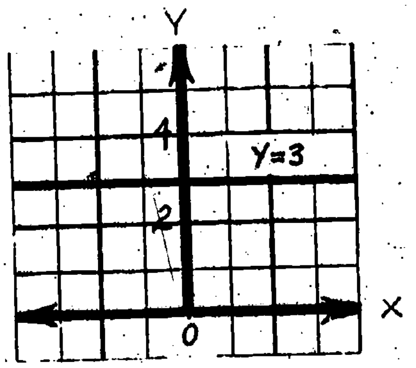
$y \neq 7$ $y \leq mx + b$

The solution set of an open statement is the set of numbers that makes the statement true.

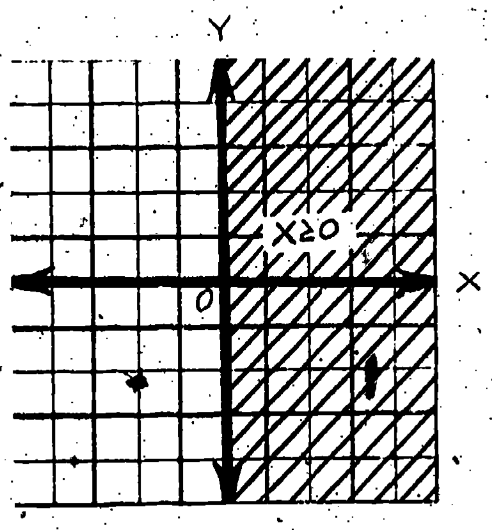
19-2 Graphing Inequalities

One of the best ways to study inequalities is through graphing. Graphing inequalities will be essential in our solution of the nutritional problem we posed earlier. Therefore we will give this subject some attention now.

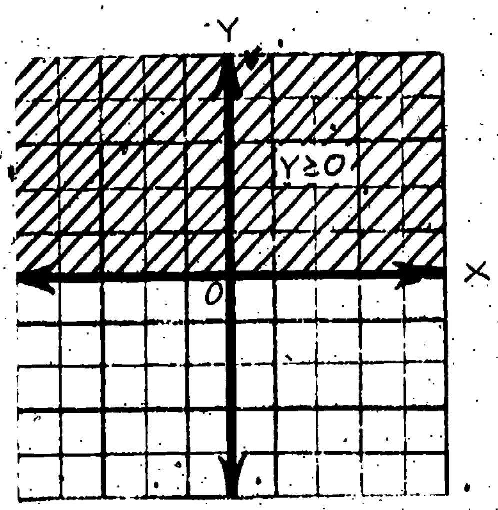
Equalities may be represented graphically. For example, $y = 3$ may be represented by a line, as shown below.



Inequalities may also be represented on a graph. The inequality $x \geq 0$ is shown by shading the entire region in which x is greater than or equal to zero.



The inequality $y \geq 0$ may be shown by shading the entire region in which y is greater than or equal to zero.



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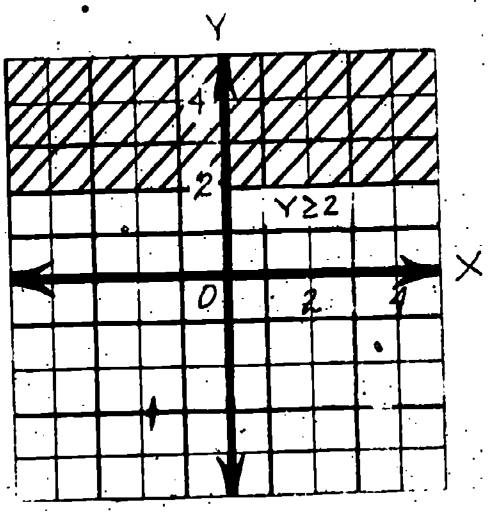
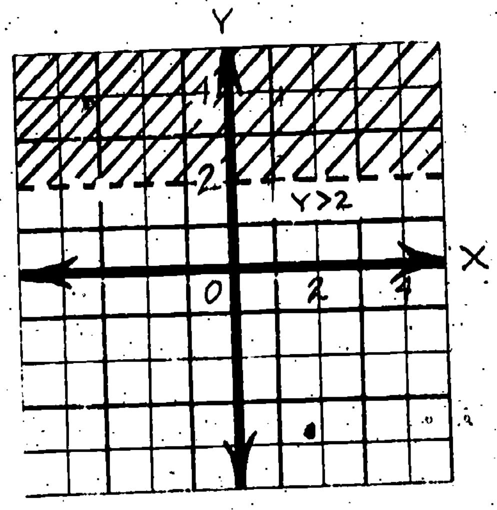
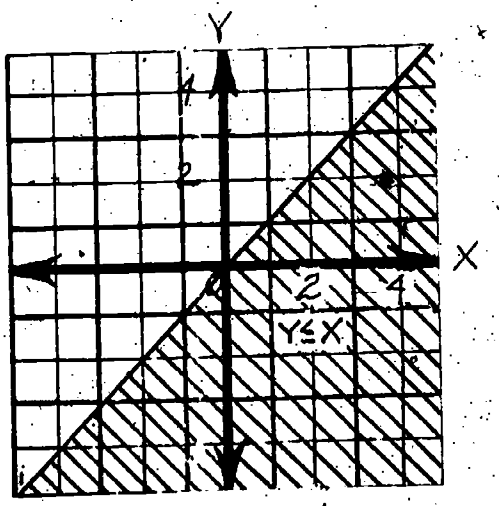
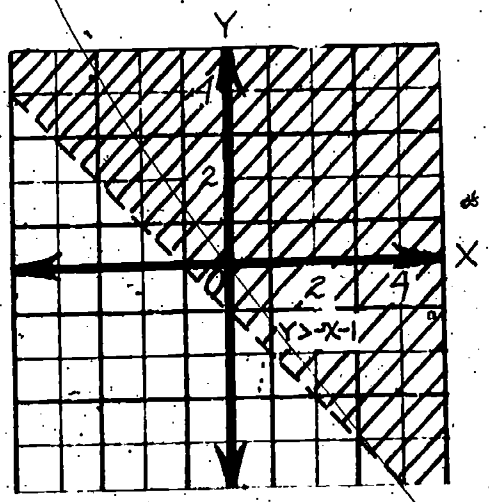
When one side of an inequality is greater than the other side ($a > b$), as opposed to being greater than or equal to ($a \geq b$), a dashed line bounds the shaded region. For example, the graph on the left below is $y \geq 2$, while the graph on the right is $y > 2$. The dashed line indicates that the line itself is not included in the region, in other words that $y = 2$ is not part of the solution set.





Other examples of inequalities represented graphically are shown below and on the next page.





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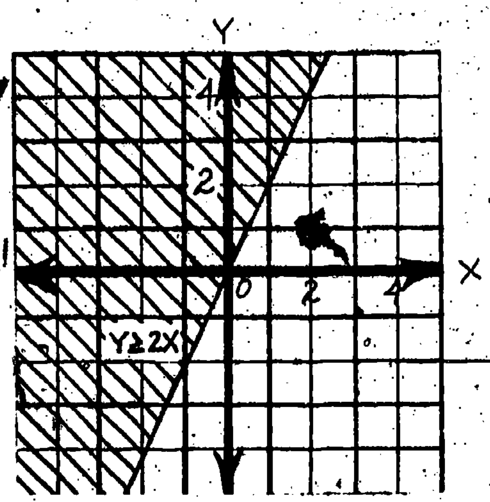
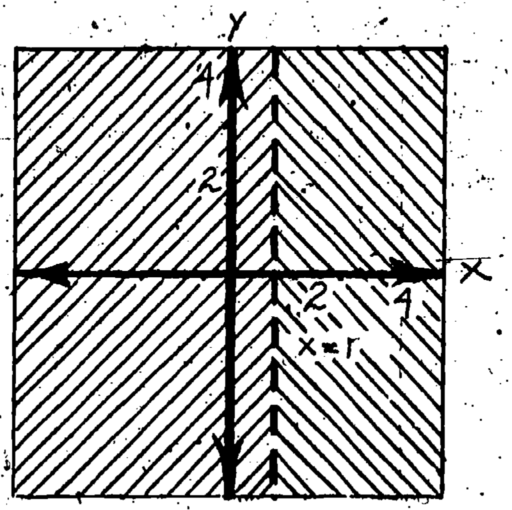




Often a problem contains more than one constraint. For example, in our flour problem the amount of whole wheat flour can not be negative. We cannot buy a negative amount of whole wheat flour. The same goes for soy flour. If x is the quantity of whole wheat flour purchased and y is the quantity of soy flour purchased then

$$x \geq 0 \quad \text{and} \quad y \geq 0$$

How can we graph both these inequalities at once? The solution is developed on the following graphs. Where the two shaded regions overlap is the region where both constraints are true (Figure 4).

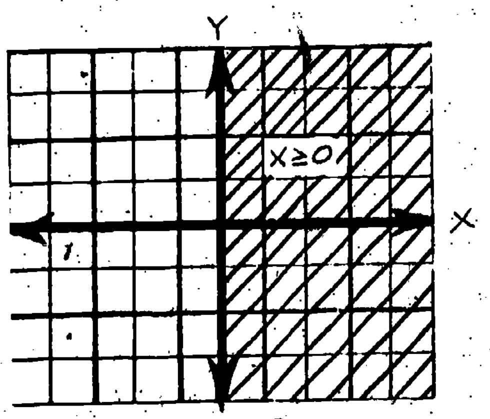


FIGURE 1

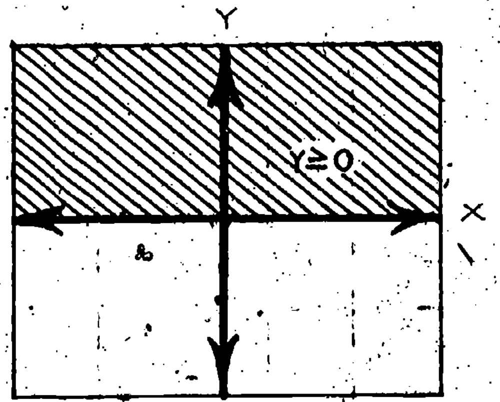


FIGURE 2

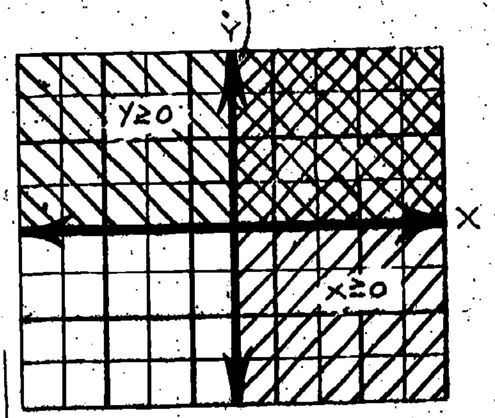


FIGURE 3

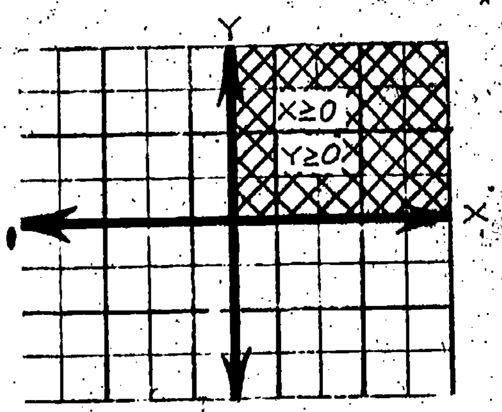
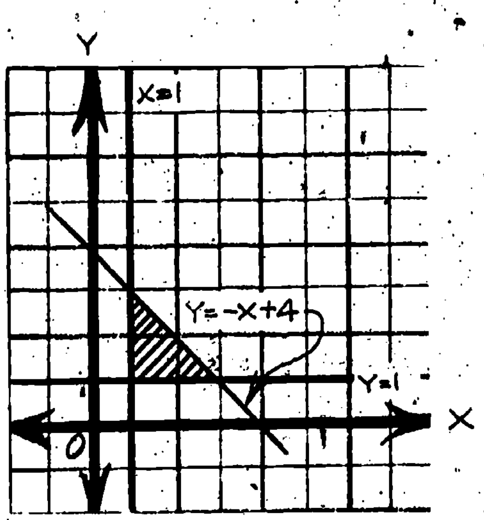


FIGURE 4

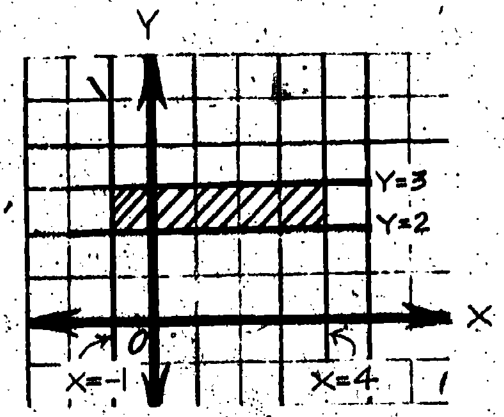
More than two inequalities may be represented on the same graph, and sometimes the resulting shaded region is entirely bounded. For example, the three inequalities $x \geq 1$, $y \geq 1$ and $y \leq -x + 4$ define the bounded shaded region in the following graph. By contrast, the shaded regions in the previous graphs are unbounded; that is, they extend indefinitely in one or more directions.



As a final example, consider the region defined by the constraints

$$x \geq -1$$
$$x \leq 4$$
$$y \geq 2$$
$$y \leq 3$$

These four constraints are represented on the graph below.



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PROBLEM SET 19:

1. State whether the following inequalities are <u>true</u>, <u>false</u> or <u>open</u>.

   a. $10 > 7$

   b. $15 \leq 6$

   c. $-9 > 3$

   d. $-15 < -4$

   e. $p \leq 3$

   f. $2 \leq s$ <u>and</u> $s \leq 10$

   g. $-10 > -3 + 5$

   h. $2x + y \geq 7$

   i. $-3 \geq -10$

2. Express the following statements symbolically:

   a. Five is less than six.

   b. Negative three is greater than or equal to negative 10.

   c. Three is greater than two and 3 is less than 5.

   d. s is a negative number.

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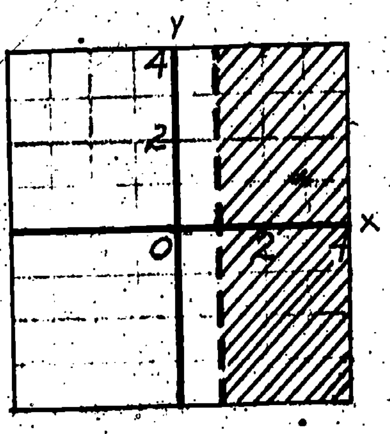
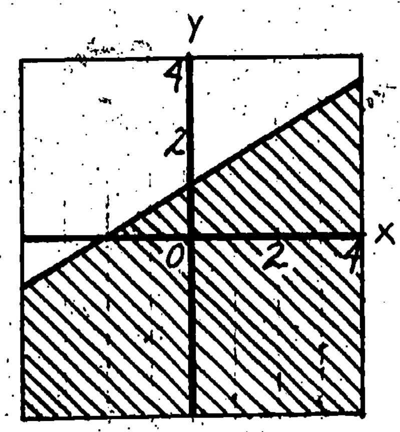
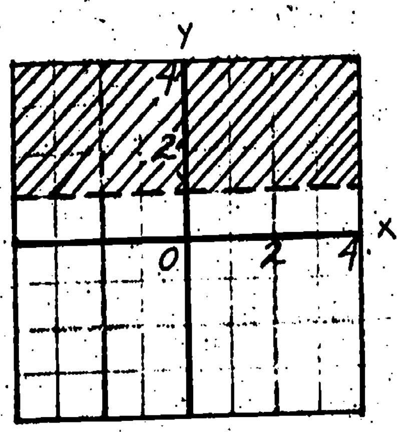
3. Match the following inequalities with the graphs below.

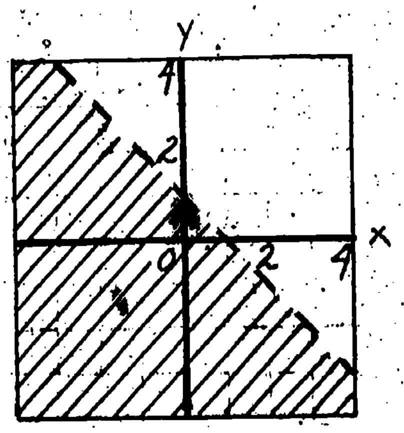
a. $x \geq 0$ c. $x \neq 3$ e. $y < -1$

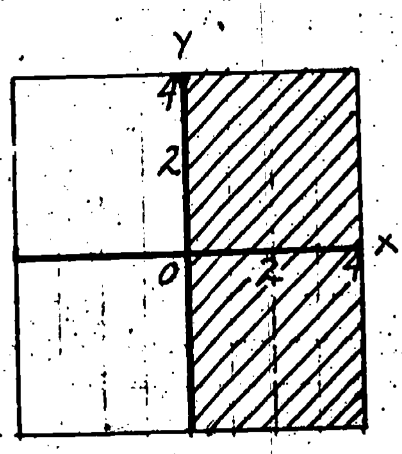
b. $y > 1$ d. $y \geq 2x + 1$ f. $y < -x + 1$

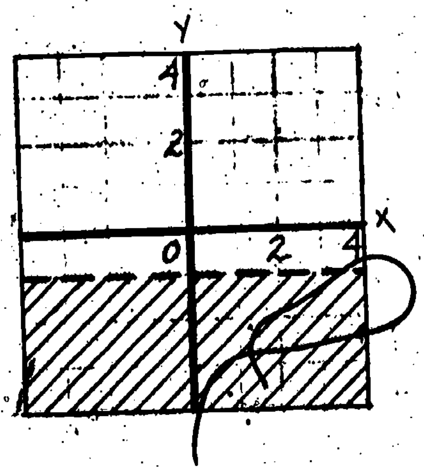
S. 

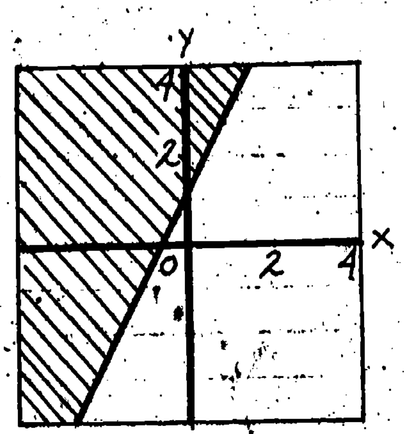
T. 

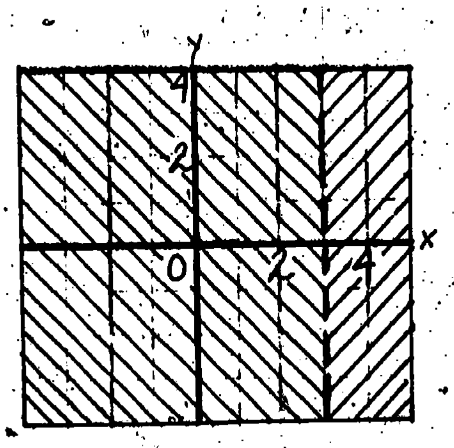
U. 

V. 

W. 

X. 

Y. 

Z. 

4. Match the following sets of inequalities with their graphs at the right.

a. $x > 0$

$x \leq 4$

$y \leq 0$

$y > -\frac{3}{2}$

b. $x \geq 0$

$y \geq 0$

$y \leq 2x + 1$

$x \leq 2$

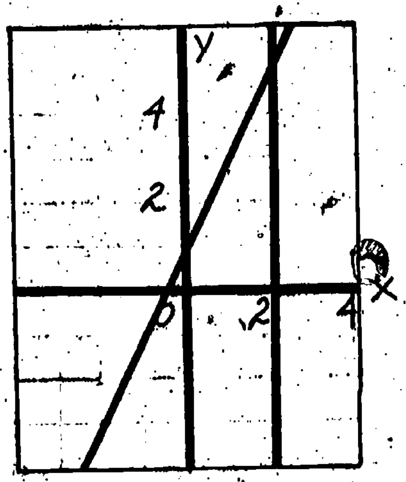
c. $x \leq 0$

$x > -\frac{3}{2}$

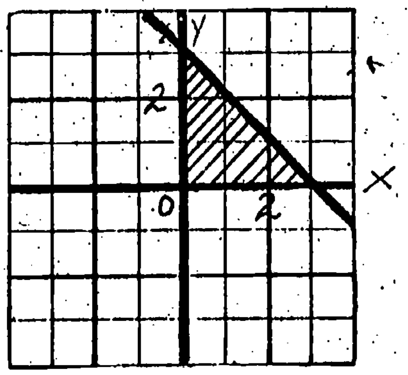
$y \geq 0$

$y \leq -x + 1$

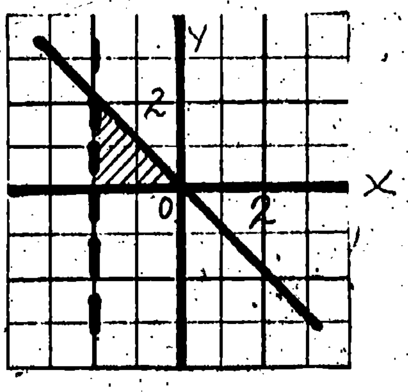
v.



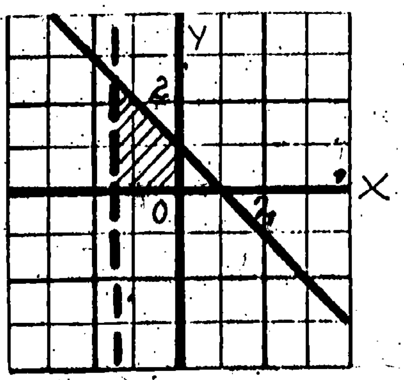
w.



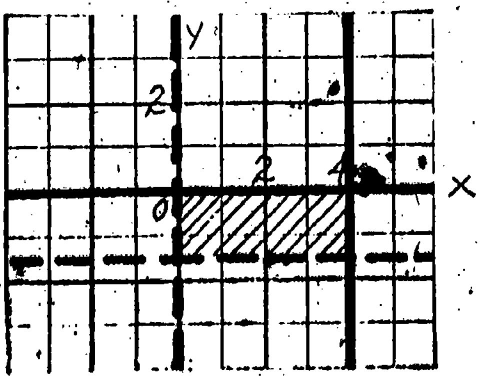
x.



y.



z.



In each of problems 5 through 7, graph the set of inequalities on one graph and shade the bounded region.

5. $y \leq 1$

$x \leq 2$

$y \geq -1$

$x \geq 0$

6. $x \leq 0$

$x > -2$

$y \leq -x$

$y \geq x$

7. $x \geq 1$

$x \leq 5$

$y \geq -x + 2$

$y \leq -x + 5$

$y \geq 0$

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SECTION 20:

20-1 Vector Inequalities

Let us return to the problem presented in Section 19. The problem is to find the least expensive combination of whole wheat flour and soy flour that provides at least 252 grams of protein, 30 mg of niacin and 500 IU of vitamin A. We may represent these quantities by the diet vector $\vec{D}$.

$$\vec{D} = [252, 30, 500]$$

The grams of protein, milligrams of niacin and international units of vitamin A in one pound of whole wheat flour and in one pound of soy flour are given, respectively, by the vectors

$$\vec{W} = [60, 20, 0]$$

$$\text{and} \quad \vec{S} = [168, 10, 500]$$

We let x be the number of pounds of whole wheat flour and y the number of pounds of soy flour. We may express the fact that the whole wheat flour and soy flour must supply at least the specified quantities of each nutrient by writing

$$x\vec{W} + y\vec{S} \geq \vec{D}$$

$$\text{or } x[60, 20, 0] + y[168, 10, 500] \geq [252, 30, 500]$$

Both expressions are vector inequalities.

Recall our interpretation of a vector equality. Suppose that two vectors, $[a_1, a_2]$ and $[b_1, b_2]$ are equal.

$$[a_1, a_2] = [b_1, b_2]$$

Then the respective components are equal.

$$a_1 = b_1 \text{ and } a_2 = b_2$$

A vector inequality is interpreted in a similar way. The statement

$$[a_1, a_2] \geq [b_1, b_2]$$

means that each component on the left side is greater than or equal to the comparable component on the right side.

$$a_1 \geq b_1 \text{ and } a_2 \geq b_2$$

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Let us return to the vector inequality written for the whole wheat and soy flour problem.

$$x[60, 20, 0] + y[168, 10, 500] \geq [252, 30, 500]$$

We convert the left side of the expression into a single vector by first performing the scalar multiplication and then adding the two vectors.

$$[60x, 20x, 0] + [168y, 10y, 500y] \geq [252, 30, 500]$$

$$[60x + 168y, 20x + 10y, 500y] \geq [252, 30, 500]$$

Each of the three components on the left side is greater than or equal to its counterpart on the right side. We may express this fact by writing an inequality for each component.

$$60x + 168y \geq 252$$

$$20x + 10y \geq 30$$

$$500y \geq 500$$

20-2 Algebraic Manipulation of Inequalities

You have learned to graph linear functions when they are in the form $y = mx + b$. Therefore, we will convert each of the three inequalities to either

$$y \geq mx + b \quad \text{or} \quad y \leq mx + b$$

in order to represent them graphically.

However, we have not yet seen how inequalities may be manipulated. Therefore we will once again leave our flour problem briefly to discuss how to handle inequalities.

Equal quantities may be added to or subtracted from both sides of an inequality. For example, start with the inequality

$$9 > 5$$

If 3 is added to both sides, the result is

$$12 > 8$$

If 3 is subtracted from both sides, the result is

$$6 > 2$$

Try other examples if you wish, and you will see that the resulting expressions are always true.

Both sides of an inequality may also be multiplied or divided by the same positive number. For example, if 9 and 5 are both multiplied by 2, the resulting inequality is true.

$$18 > 10$$

And if 9 and 5 are both divided by 2, the resulting expression is also true.

$$4\frac{1}{2} > 2\frac{1}{2}$$

However, if both sides of an inequality are multiplied or divided by a negative number, the direction of the inequality must be reversed. For example,

$$9 > 5$$

is true, but

$$-2(9) > -2(5)$$

is not true. Rather

$$-18 < -10$$

9 is greater than 5, but -2 times 9 is less than -2 times 5. Likewise

$$(9 \div -2) > (5 \div -2)$$

is false, but

$$(9 \div -2) < (5 \div -2)$$

is true.

The first inequality of our flour problem expresses the constraint that the protein content of the two flours be equal to or exceed the amount specified.

$$60x + 168y \geq 252$$

In order to graph this inequality it is convenient to put it in the form

$$y \geq mx + b \quad \text{or} \quad y \leq mx + b$$

Since we may subtract the same quantity from both sides of an inequality, let us subtract 60x from both sides of the protein inequality

$$60x + 168y - 60x \geq 252 - 60x$$

$$168y \geq -60x + 252$$

We may also divide both sides by the same positive number, so we divide by 168.

$$\frac{168y}{168} \geq \frac{-60x + 252}{168}$$

$$y \geq \frac{-60x}{168} + \frac{252}{168}$$

$$y \geq -\frac{5}{14}x + \frac{3}{2}$$

Our second inequality represents the constraint that the flours contain at least 30 milligrams of niacin.

$$20x + 10y \geq 30$$

We convert this to the standard form in the same way that we converted the protein inequality. We first subtract 20x from both sides.

$$20x + 10y - 20x \geq 30 - 20x$$

$$10y \geq -20x + 30$$

We then divide each side by 10.

$$\frac{10y}{10} \geq \frac{-20x + 30}{10}$$

$$y \geq -\frac{20x}{10} + \frac{30}{10}$$

$$y \geq -2x + 3$$

The third inequality is

$$500y \geq 500$$

This inequality expresses the constraint that the flour that is purchased contains at least 500 IU of vitamin A. It is simplified to the standard form by dividing both sides by 500.

$$\frac{500y}{500} \geq \frac{500}{500}$$

$$y \geq 1$$

20-3 Region of Feasible Solutions

We have been developing a procedure for solving a problem. The problem is to find the least expensive combination of x pounds of whole wheat flour and y pounds of soy flour that contains at least certain specified amounts of protein, niacin and vitamin A.

We will be concerned with finding the least expensive combination in Section 21. In this section we are concerned only with determining the possible values that x and y may have while meeting the nutritional requirements.

These values are represented by a region of a graph of x versus y. The region is determined by the constraints the problem imposes.

In Section 19-2 we pointed out that x and y cannot be negative numbers. This constraint is expressed by the inequalities

$$x \geq 0$$

$$y \geq 0$$

In Section 20-2 we derived expressions for three other constraints.

$$y \geq -\frac{5}{14}x + \frac{3}{2}$$

$$y \geq -2x + 3$$

$$y \geq 1$$

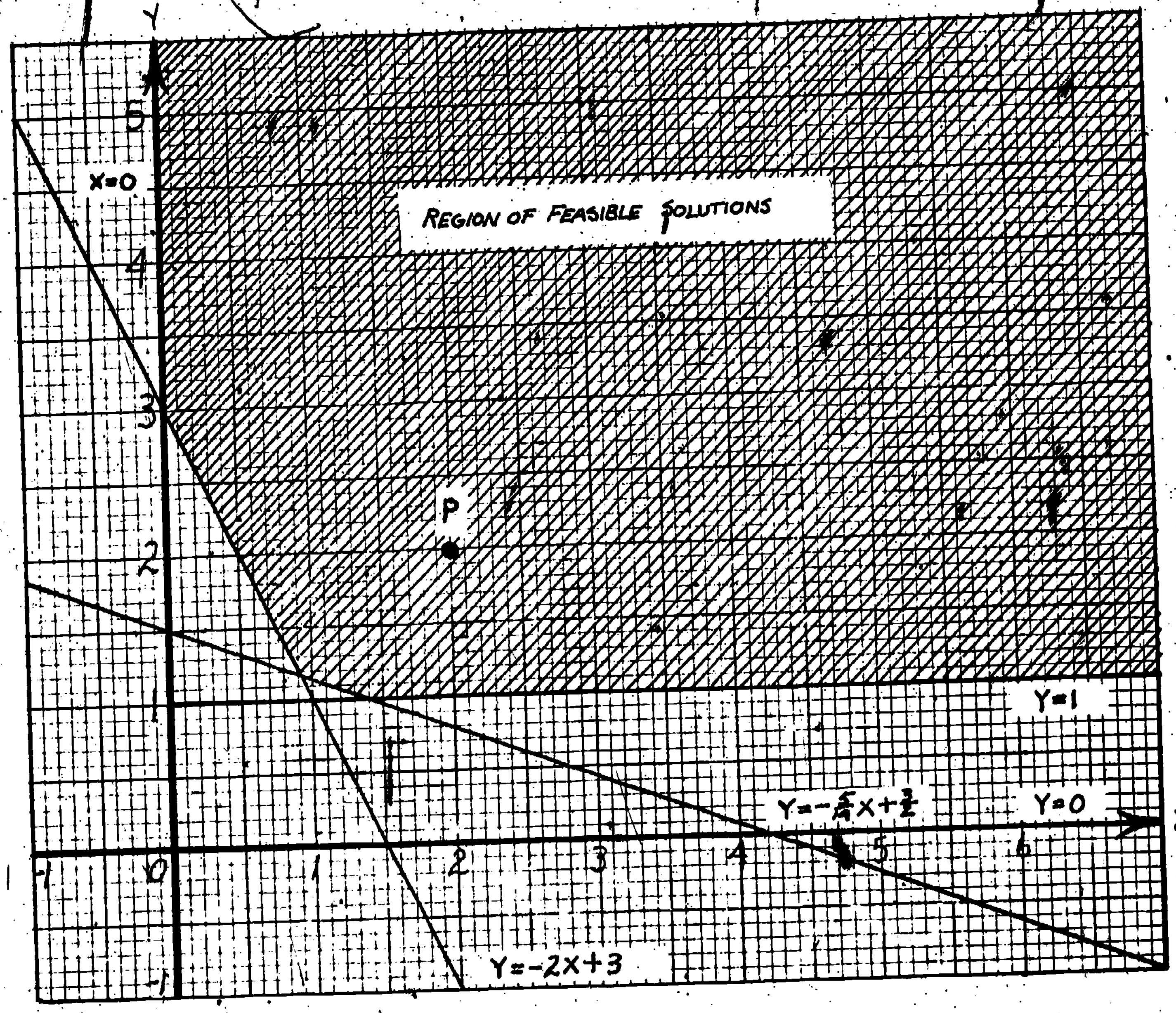
The solution to the problem must be within the region where all five inequalities are true. This region is called the <u>region of feasible solutions</u>, because it represents all combinations of x and y that are feasible solutions to the problem. The region of feasible solutions is represented by the shaded region in the graph on the next page (and it extends indefinitely upward and to the right).

The region of feasible solutions is bounded by the lines representing the four equations

$$x = 0 \qquad y = -2x + 3$$

$$y = 1 \qquad y = -\frac{5}{14}x + \frac{3}{2}$$

(The constraint that y be greater than or equal to zero is not involved, because another constraint requires that y be greater than or equal to one.)



Any pair of values of x and y within the region of feasible solutions is a solution to all five inequalities. For example, the point x = 2, y = 2 (P on the graph) is within the region of feasible solutions. This point represents 2 pounds of whole wheat flour and 2 pounds of soy flour.

If we substitute these values of x and y into the five inequalities, all five expressions are true.

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$$x \geq 0 \qquad 2 \geq 0$$

$$y \geq 0 \qquad 2 \geq 0$$

$$y \geq -\frac{5}{14}x + \frac{3}{2} \qquad 2 \geq -\frac{5}{14}(2) + \frac{3}{2}$$

$$y \geq -2x + 3 \qquad 2 \geq -2(2) + 3$$

$$y \geq 1 \qquad 2 \geq 1$$

The last three inequalities were obtained from the vector inequality

$$x\,[60,\ 20,\ 0] + y\,[168,\ 10,\ 500] \geq [252,\ 30,\ 500]$$

If we substitute x = 2 and y = 2 into this vector inequality, we obtain

$$2\,[60,\ 20,\ 0] + 2\,[168,\ 10,\ 500] \geq [252,\ 30,\ 500]$$

$$[120,\ 40,\ 0] + [336,\ 20,\ 1000] \geq [252,\ 30,\ 500]$$

We add the vectors on the left side and obtain

$$[456,\ 60,\ 1000] \geq [252,\ 30,\ 500]$$

This expression is true, because each component of the vector on the left is greater than or equal to the corresponding component of the vector on the right. Two pounds of whole wheat flour and two pounds of soy flour contain 456 grams of protein, 60 mg of niacin and 1000 IU of vitamin A. All of these quantities exceed the specified amounts.

Of course (2, 2) is only one of an infinite number of points lying in the region of feasible solutions. We could just as well have chosen (6, 4) or even (80, 1000), which represents 80 lb of wheat flour and 1000 lb of soy flour.

Our task in the next section will be to find the values of x and y within the region of feasible solutions that meet the specification for the lowest cost.

PROBLEM SET 20:

Perform algebraic manipulations that will isolate the variable on one side or the other of the inequality sign.

1. $y + 6 > 10$

2. $-8 \leq 2x$

3. $-2s \geq 18$

4. $-2q - 8 \leq 4$

Transform the following inequalities into the standard form $y \leq mx + b$ or $y \geq mx + b$.

5. $5y \geq x - 7$

6. $2y + x \geq 15$

7. $-3x - 6y \leq 12$

8. $-2x + 6 \leq y - 8$

9. Given that $\vec{A} = [2, 3, 4]$, $\vec{B} = [1, 8, 7]$ and $\vec{C} = [5, 10, 15]$ and that $x\vec{A} + y\vec{B} \geq \vec{C}$

a. Substitute the component form of the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ into the equation and write the new equation.

b. Perform the scalar multiplication.

c. Perform the indicated addition.

d. Use the inequality convention for vectors to transform the vector inequality into a set of algebraic inequalities.

10. Given the vector inequality

$$x\vec{A} + y\vec{B} \geq \vec{C}$$

and that $\vec{A} = [2, 4]$, $\vec{B} = [3, 8]$, $\vec{C} = [15, 10]$

a. Substitute the component form of the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ into the vector equation and write the new equation.

b. Perform the indicated scalar multiplication.

c. Perform the indicated vector addition.

d. Use the inequality convention for vectors to transform the vector inequality into a system of algebraic inequalities.

e. Rewrite the inequalities in the form $y \geq mx + b$.

f. The inequalities of Part e and the inequalities $x \geq 0$ and $y \geq 0$ define a region of feasible solutions. Graph and label this region.

11. Given the vector inequality

$$x\vec{R} + y\vec{P} \geq \vec{D}$$

and given that

$$\vec{R} = [2, 1, -1], \vec{P} = [4, -1, 1], \vec{D} = [8, -3, -5]$$

a. Following the steps of Problem 10, transform the vector inequality into a set of algebraic inequalities.

b. Given that $x \geq 0$ and $y \geq 0$, graph the five inequalities and label the resulting region of feasible solutions.

SECTION 21:

21-1 The Optimal Solution

In the previous two sections we were developing a procedure for solving a problem. The problem is to find the combination of x pounds of whole wheat flour and y pounds of soy flour that provides 252 grams of protein, 30 milligrams of niacin and 500 IU of vitamin A for the least cost. We assume that whole wheat flour costs 15 cents per pound, and soy flour costs 30 cents per pound.

In Section 20, we graphed the region of feasible solutions. This region is the part of the graph in which the values of x and y satisfy all the constraints of the problem. Two of the constraints are that x and y have non-negative values since we cannot have less than zero pounds of either flour.

$$x \geq 0$$
$$y \geq 0$$

Three other constraints were obtained from the vector inquality

$$x[60, 20, 0] + y[168, 10, 500] \geq [252, 30, 500]$$

We expressed the vector inequality as three inequalities, and we converted these inequalities to the form

$$y \geq mx + b \quad \text{or} \quad y \leq mx + b$$

The results were

$$y \geq -\frac{5}{14}x + \frac{3}{2}$$

$$y \geq -2x + 3$$

$$y \geq 1$$

The region of feasible solutions, the region in which the five constraints are true, is shown on the graph in Section 20-3.

Now we must find a solution within the region of feasible solutions that involves the least cost. The values of x and y resulting in the lowest cost are the optimal solution to the problem.

We begin by writing an expression for cost as a function of x and y. Whole wheat flour costs 15 cents per pound, and soy flour costs 30 cents pound, so if we let c be the total cost in cents

$$c = 15x + 30y$$

We can find the cost corresponding to any point (x, y) on the plane by simply substituting into the equation above. Our objective is to find the point in the region of feasible solutions which gives the smallest possible value for c.

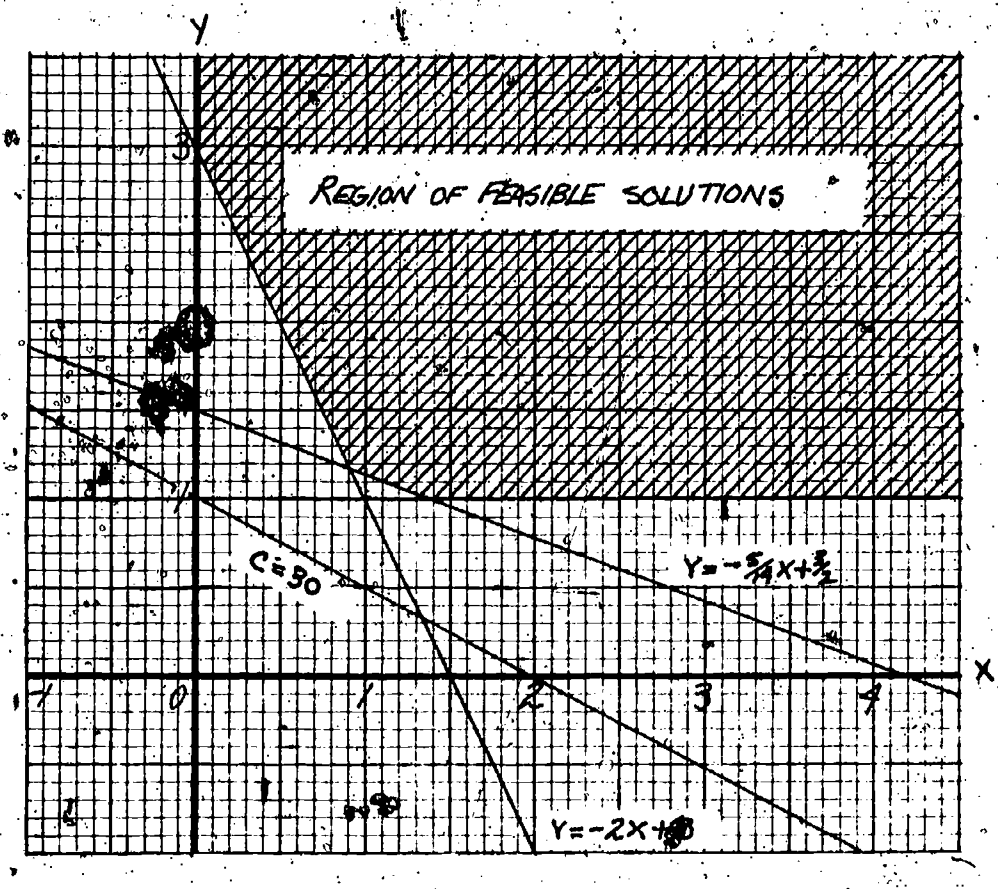
We can begin by asking, for example, which points on the plane correspond to a cost of 30 cents? These will be the points (x, y) satisfying the equation

$$30 = 15x + 30y$$

This is a linear equation which can easily be put in the familiar form y = mx + b. The result is

$$y = -\frac{1}{2}x + 1 .$$

The graph of this equation is shown below as the line labeled c = 30.



The line $y = -\frac{1}{2}x + 1$ does not pass through the region of feasible solutions. Therefore there is no feasible combination of flours costing 30 cents. As you will see shortly, the cost must be greater than 30 cents.

Before we try other values of c, we put the cost equation $c = 15x + 30y$ in the form $y = mx + b$.

$$15x + 30y = c$$

$$30y = -15x + c$$

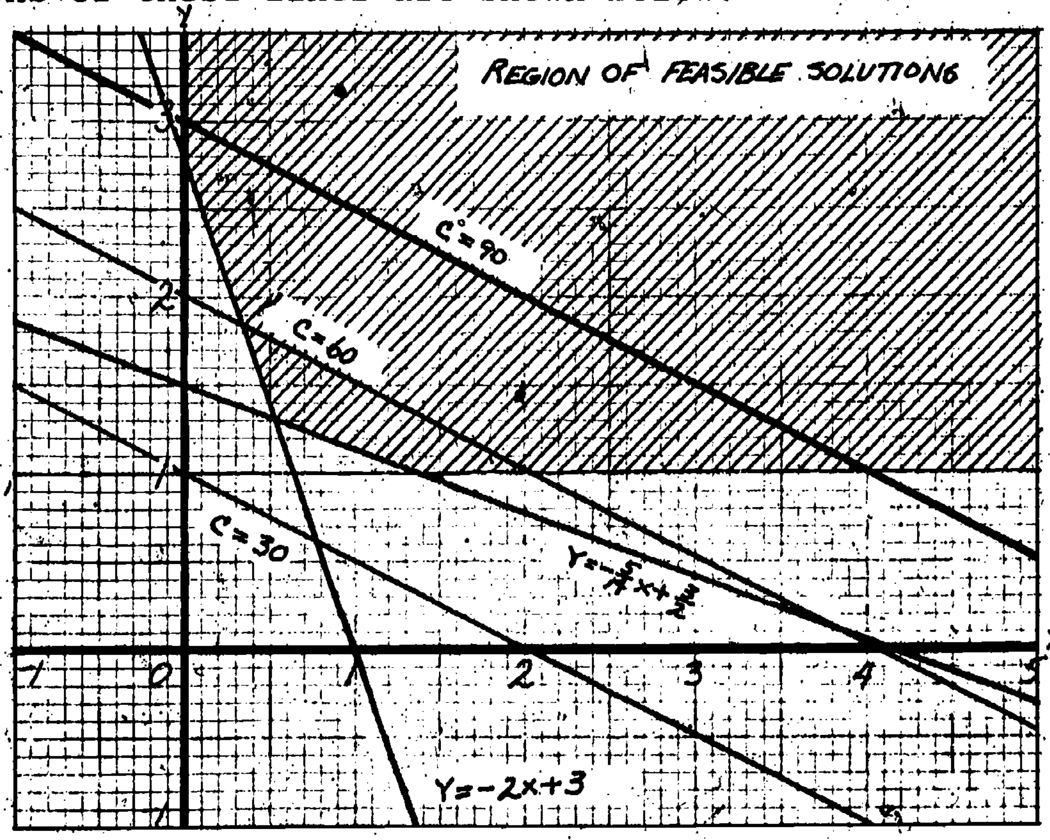
$$y = -\frac{1}{2}x + \frac{1}{30}c$$

We now check cost values of 60 and 90 cents. When these values are substituted into the equation above we get the two equations.

$$y = -\frac{1}{2}x + 2$$

$$y = -\frac{1}{2}x + 3$$

The graphs of these lines are shown below.



There are two important things to note about the cost lines.

1. The 60-cent and 90-cent lines pass through the region of feasible solutions. Therefore, the nutrient requirements may be met by combinations of flours with these costs.

2. The cost lines are all parallel because they have the same slope.

What about the cheapest combination? We reason as follows. There must be a cost line parallel to the ones drawn on the graph

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and just touching a corner of the shaded region. The line will correspond to a cost somewhere between 30 cents and 60 cents. To get an idea of what the line looks like, think of sliding the c = 30 line upward toward the c = 60 line until you touch the shaded region. The first point touched will be the intersection of the lines $y = -2x + 3$ and $y = -\frac{5}{14}x + \frac{3}{2}$.

The coordinates of the intersection of the lines $y = -\frac{5}{14}x + \frac{3}{2}$ and $y = -2x + 3$ is the solution of the two equations. We may find the values of x and y at this point by solving the pair of equations.

We have encountered two methods for solving a pair of linear equations. One is the substitution method; the other is the addition-subtraction method. The results, which you may verify for yourself, are

$$x = \frac{21}{23}$$

$$y = \frac{27}{23}$$

The least expensive solution to our nutrient problem is $\frac{21}{23}$ pounds of whole wheat flour and $\frac{27}{23}$ pounds of soy flour.

The cost of these quantities of flour is calculated by the equation

$$\begin{aligned} c &= 15x + 30y \\ &= 15\left(\frac{21}{23}\right) + 30\left(\frac{27}{23}\right) \\ &= \frac{15}{23}(21 + 2 \cdot 27) \\ &= \frac{15}{23}(21 + 54) \\ &= \frac{15 \cdot 75}{23} \\ c &\simeq 49 \text{ cents} \end{aligned}$$

21-2 Linear Programming

The procedure we used to solve the flour problem in Sections 19, 20 and 21-1 is called <u>linear programming</u>. We digressed from solving the problem in several instances to discuss related matters. Since you may wish to see the solution to the problem in a form that is easier to refer to, it is reviewed below as an example.

EXAMPLE:

Suppose that whole wheat flour costs 15 cents per pound and contains 60 g of protein, 20 mg of niacin and no vitamin A. Soy flour costs 30 cents per pound and contains 168 g of protein, 10 mg niacin and 500 IU of vitamin A. How much of each type of flour should you buy to obtain at least 252 g of protein, 30 g of niacin and 500 IU of vitamin A at the least cost?

SOLUTION:

1. Let $\vec{W}$ represent the nutrient content of whole wheat flour.

$$\vec{W} = [60, 20, 0]$$

Let $\vec{S}$ represent the nutrient content of soy flour.

$$\vec{S} = [168, 10, 500]$$

Let $\vec{D}$ represent the dietary requirements.

$$\vec{D} = [252, 30, 500]$$

Let x be the number of pounds of whole wheat flour and y be the pounds of soy flour.

2. Write inequalities to express the constraint that x and y cannot be negative numbers.

$$x \geq 0$$
$$y \geq 0$$

3. Write a vector inequality to express the constraint that the flours must contain at least as much of each nutrient as the amounts specified by the diet vector.

$$x[60, 20, 0] + y[168, 10, 500] \geq [252, 30, 500]$$

4. Perform the indicated scalar multiplications.

$$[60x, 20x, 0] + [168y, 10y, 500y] \geq [252, 30, 500]$$

Add the vectors on the left.

$$[60x + 168y, 20x + 10y, 500y] \geq [252, 30, 500]$$

Write this vector inequality as three separate inequalities.

(a) $60x + 168y \geq 252$

(b) $20x + 10y \geq 30$

c. $500y \geq 500$

5. Convert each of these inequalities to the form $y \geq mx + b$ or $y \leq mx + b$.

(a) $168y \geq -60x + 252$

$$y \geq -\frac{5}{14}x + \frac{3}{2}$$

(b) $10y \geq -20x + 30$

$$y \geq -2x + 3$$

(c) $y \geq 1$

6. Represent the two constraints of Step 2 and the three constraints of Step 3 graphically. This is done by graphing the lines representing the following five equations.

$$x = 0$$

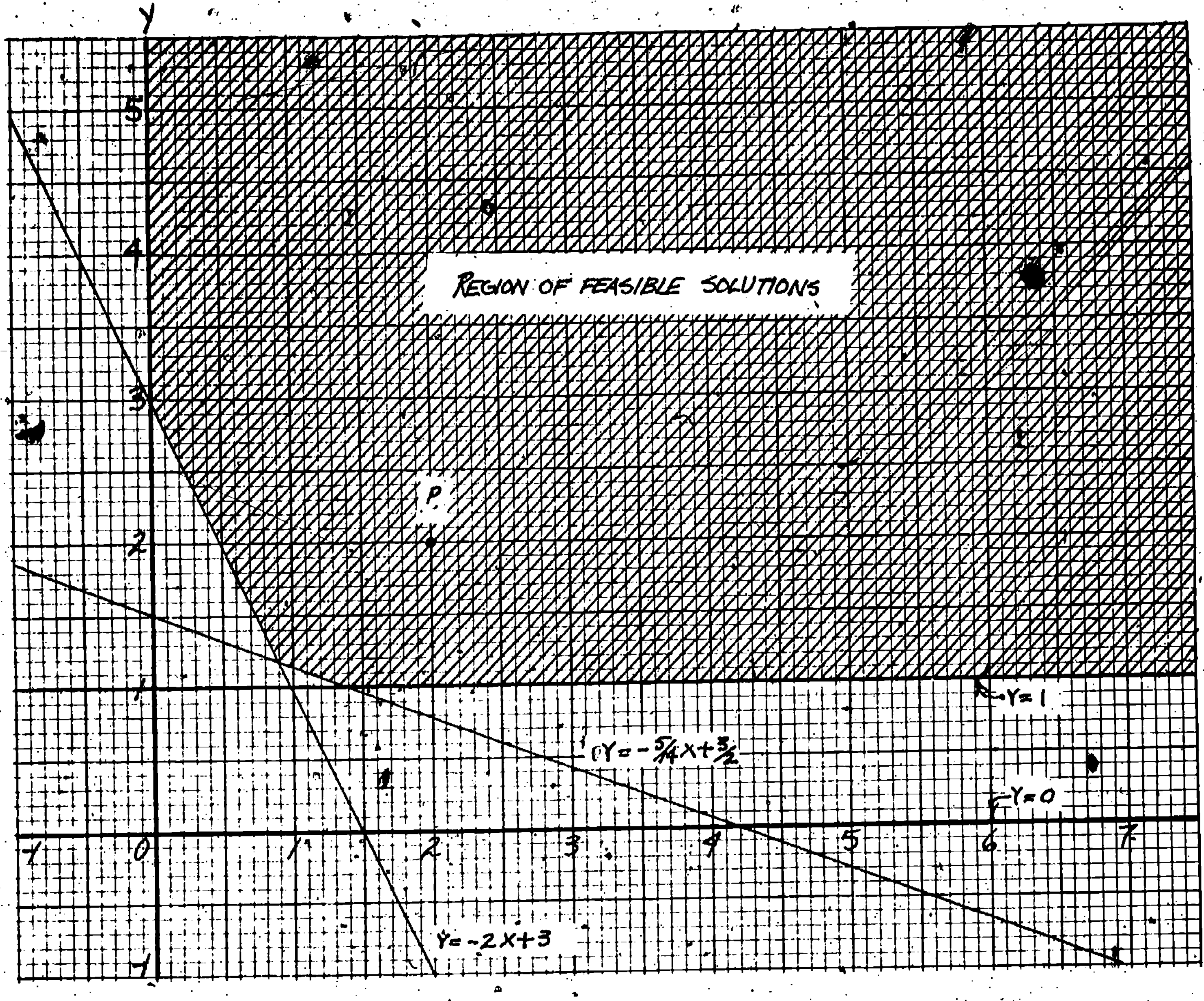
$$y = 0$$

$$y = -\frac{5}{14}x + \frac{3}{2}$$

$$y = -2x + 3$$

$$y = 1$$

Indicate the region of feasible solutions.



7. Write an equation for cost (c) in terms of x and y.

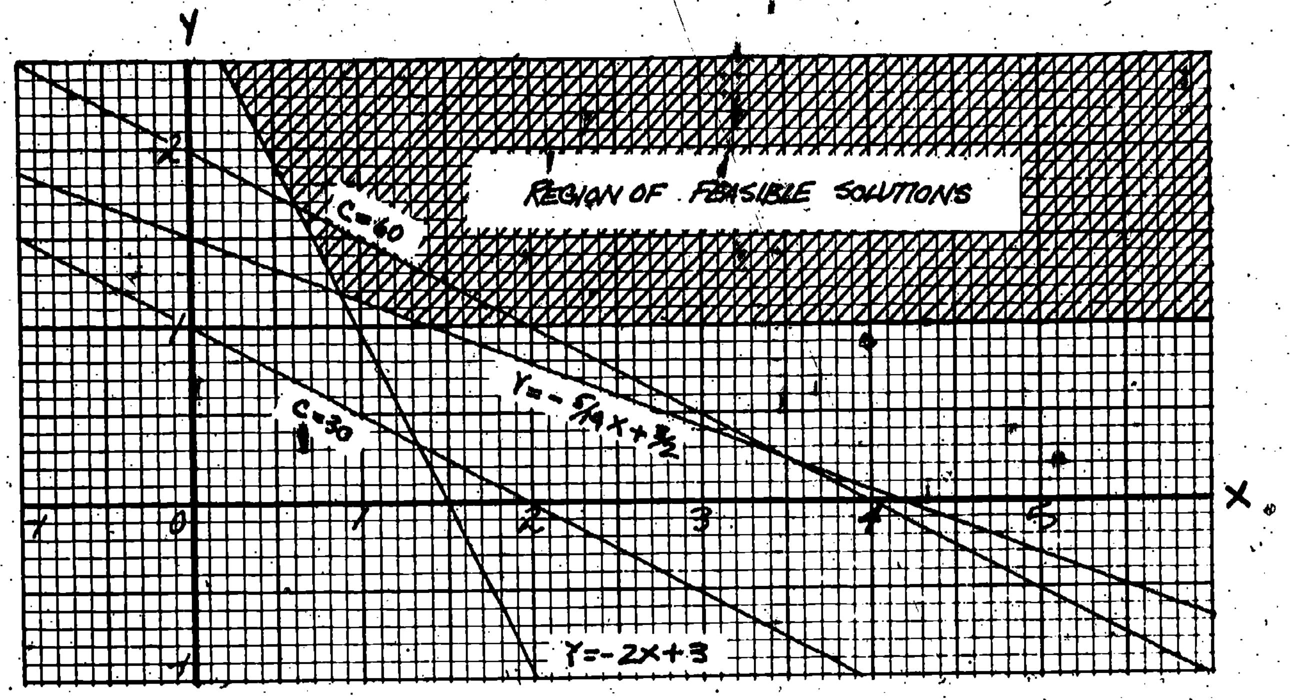
$$c = 15x + 30y$$

Convert this equation to one that gives y in terms of x and c.

$$30y = -15x + c$$

$$y = -\frac{1}{2}x + \frac{1}{30}c$$

8. Assume values for c and substitute those values in the equation of Step 7. Draw the lines representing these equations on the graph. The equations when c = 30 and c = 60 are represented on the following graph.



9. Locate on the graph the point at which c has its least value within the region of feasible solutions. This point is the intersection of the lines $y = -\frac{5}{14}x + \frac{3}{2}$ and $y = 2x + 3$. Determine the coordinates of the intersection by solving the two equations simultaneously. In this case it is convenient to use the substitution method.

$$-\frac{5}{14}x + \frac{3}{2} = -2x + 3$$

$$\frac{23}{14}x = \frac{3}{2}$$

$$x = \frac{21}{23}$$

Substitute $x = \frac{21}{23}$ into either expression for y.

$$y = -\frac{5}{14}\left(\frac{21}{23}\right) + \frac{3}{2}$$

$$y = \frac{27}{23}$$

10. Determine the cost of this solution by substituting $x = \frac{21}{23}$ and $y = \frac{27}{23}$ into the expression for cost.

$$c = 15\left(\frac{21}{23}\right) + 30\left(\frac{27}{23}\right)$$

$$c \approx 49 \text{ cents}$$

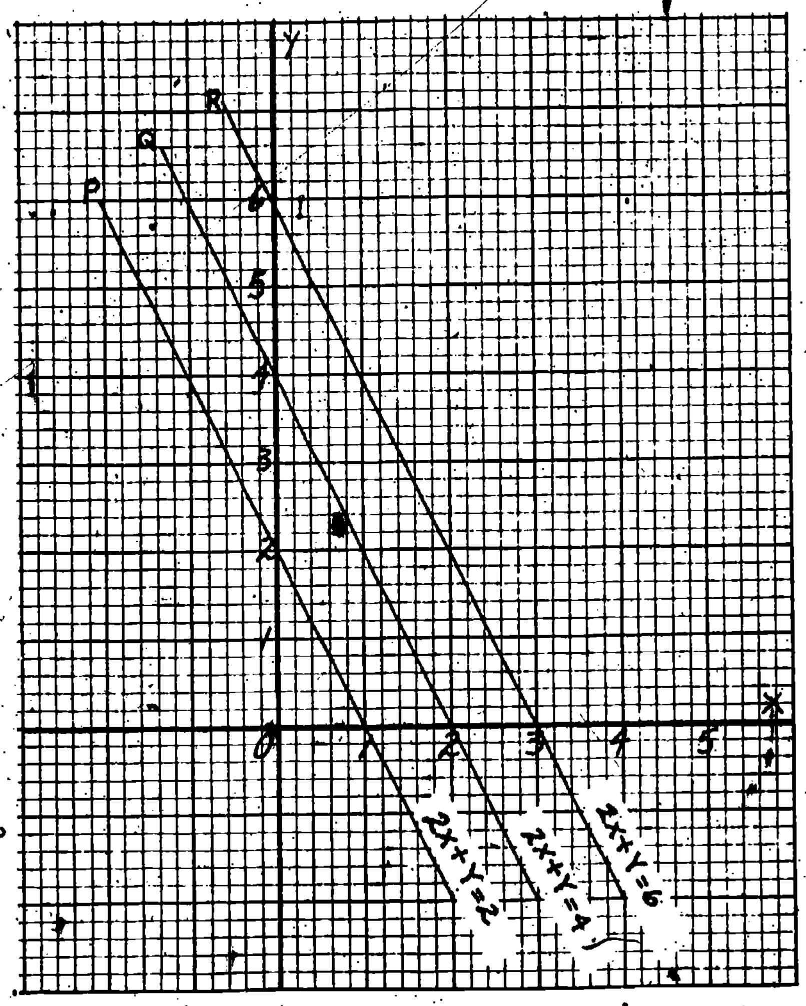
The least expensive solution to the problem is to purchase $\frac{21}{23}$ pounds of whole wheat flour and $\frac{27}{23}$ pounds of soy flour at a cost of approximately 49 cents.

PROBLEM SET 21:

1. Suppose that in a certain linear programming problem the cost function is

$$c = 2x + y \quad \text{(c in dollars)}$$

The graph below shows three cost lines based on this equation.



a. Line Q corresponds to a cost of _______ dollars.

(Multiple Choice)

b. If the line $2x + y = 3$ were added to the graph, it would lie

(1) between lines P and Q.

(2) between lines Q and R.

(3) above line R.

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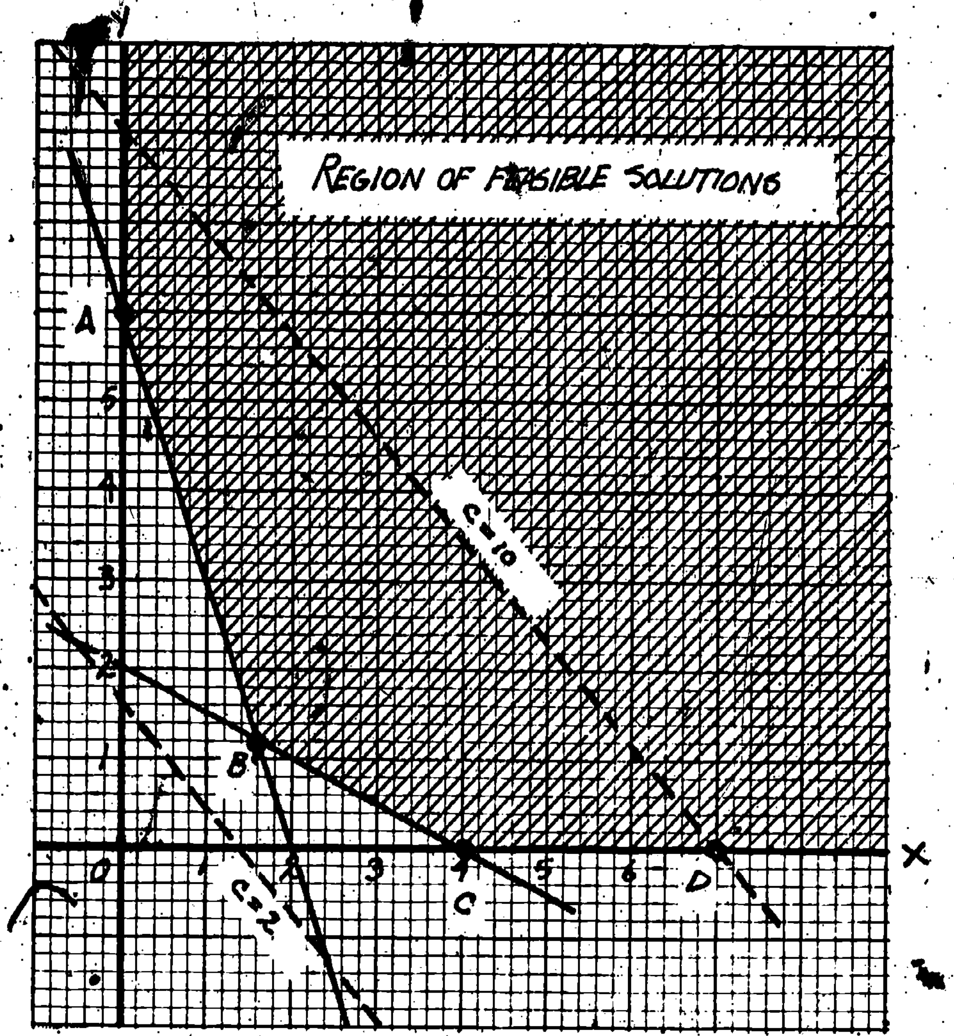
c. Lines which are closer to the origin correspond to ______ (higher, lower) values of c.

2. The graph below shows a shaded region of feasible solutions and dashed lines corresponding to costs of 2 and 10.



a. The point D (7, 0) corresponds to a cost of ______.

b. Give the letter of the point in the region of feasible solutions which will yield the minimum cost.

(Multiple Choice)

c. The minimum cost will

(1) be at least 10.

(2) be no bigger than 2.

(3) lie between 2 and 10.

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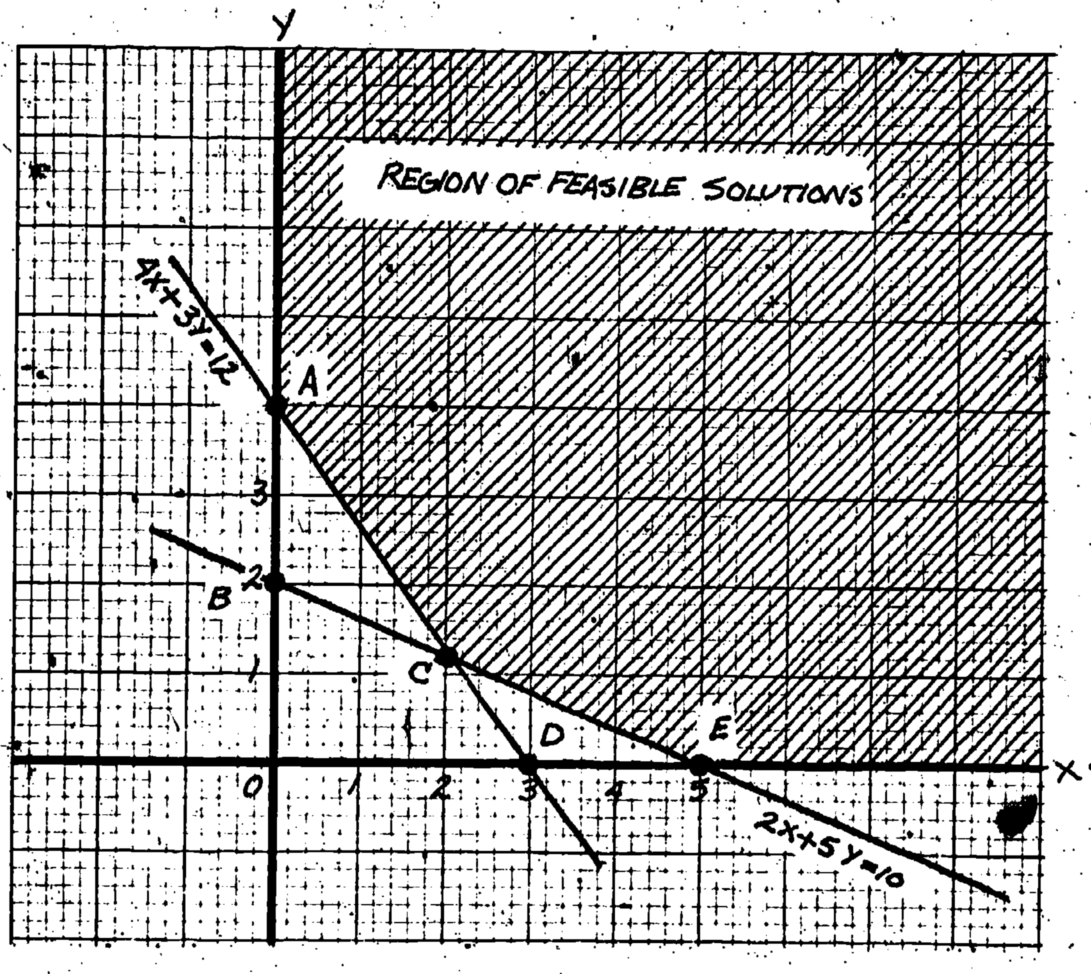
3. The graph below shows the region of feasible solutions determined by the inequalities

$$x \geq 0 \qquad 4x + 3y \geq 12$$
$$y \geq 0 \qquad 2x + 5y \geq 10$$



Suppose we want to find the minimum value of the function

$$S = 2x + y$$

on the shaded region.

a. Put the equation $S = 2x + y$ in the form $y = mx + b$.

b. Reproduce the region of feasible solutions on your own graph paper. Then add graphs of the equation in part a when $s = 3$ and $s = 5$.

c. Give the letter of the point in the region of feasible solutions which corresponds to a minimum value of s.

d. What are the coordinates of the point?

e. Substitute into the equation $s = 2x + y$ to find the minimum value of s.

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4. In this problem you will be working with the region of feasible solutions defined by the following inequalities

$$x \geq 0 \quad y \geq 0$$

$$y \geq -\frac{3}{2}x + 3$$

$$y \geq -\frac{1}{2}x + 2$$

a. Graph the region of feasible solutions defined by the inequalities.

b. Suppose that you want to find the minimum value of the function $w = x + y$ on the region of feasibility. Put this equation in the form $y = mx + b$.

c. On your graph, draw in the lines corresponding to the values $w = 2$ and $w = 4$.

d. Find the coordinates of the point in the region where w will be a minimum (solve a system of linear equations).

e. What is the minimum value of w?

Consider the following problem:

5. Elmo has decided to try living on wild foods. He is planning a modest dinner partly consisting of swamp cabbage and raw snails. He figures he will need to collect enough cabbage and snails to meet or exceed a diet vector of $\vec{D}$ = [60 g protein, 40 g carbohydrate]. One pound of swamp cabbage has a nutrient vector of $\vec{C}$ = 15 g protein, 25 g carbohydrate and one pound of snails a vector of $\vec{S}$ = [30 g protein, 10 g carbohydrate]. It takes Elmo 30 minutes to collect a pound of swamp cabbage and 20 minutes to collect a pound of snails. What is the least time he can spend in collecting dinner?

a. Let x = number of pounds of swamp cabbage, and y = number of pounds of snails. Explain why it must be true that

$$x\vec{C} + y\vec{S} \geq \vec{D}$$

b. Given that

$$x\vec{C} + y\vec{S} \geq \vec{D}$$

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Show intermediate steps between this vector and the following inequalities.

(1) $x + 2y \geq 4$ (protein)

(2) $5x + 2y \geq 8$ (carbohydrate)

c. For each of the above inequalities show steps between them and the following inequalities.

(1) $y \geq -\frac{1}{2}x + 2$

(2) $y \geq -\frac{5}{2}x + 4$

d. Why is it also true that $x \geq 0$ and $y \geq 0$?

e. Carefully graph the inequalities of parts c and d and label the region of feasible solutions. Label the horizontal axis "pounds of cabbage" and the vertical axis "pounds of snails." Scale the graph so that 2 cm = 1 lb.

f. Let T represent the number of minutes required to collect the food. Explain the equation

$$T = 30x + 20y$$

g. Derive the equation

$$y = -\frac{3}{2}x + \frac{1}{20}T$$

h. Graph two time equations on the graph for Part e, one for T = 40 minutes and one for T = 80 minutes.

i. Locate the point in the region of feasible solutions where the time T will be a minimum. Mark the point with a star (*).

j. Find the coordinates of the point in Part i. (Solve a system of linear equations.)

k. What is the minimum time T?

Consider the following problem.

6. One pound of hamburger contains 90 g fat and 100 IU of vitamin A. One pound of chicken contains 15 g fat and 400 IU of vitamin A.

If hamburger costs \$1.00/lb and chicken costs \$.50/lb how many pounds of each should you buy to meet or exceed a diet vector of $\vec{D}$ = [180 g fat, 800 IU vitamin A] for the least amount of money?

a. If $\vec{H}$ represents one pound of hamburger and $\vec{C}$ represents one pound of chicken, is it true that

$\vec{H} = [90, 100]$ and
$\vec{C} = [15, 400]$?

If it is incorrect write the correct vectors.

b. If x = number of pounds of hamburger needed, and y = number of pounds of chicken needed, explain why it must be true that

$$x\vec{H} + y\vec{C} \geq \vec{D}$$

c. Given that

$$x\vec{H} + y\vec{C} \geq \vec{D}$$

must be true, show intermediate steps between this vector equation and the following set of inequalities.

(1) $90x + 15y \geq 180$ (fat)

(2) $100x + 400y \geq 800$ (vitamin A)

d. For each of the above inequalities show intermediate equivalent inequalities between them and the following inequalities.

(1) $y \geq -6x + 12$ (fat)

(2) $y \geq -\frac{1}{4}x + 2$ (vitamin A)

e. Why is is also true that $x \geq 0$ and $y \geq 0$?

f. Carefully graph the inequalities of Parts e and f and label the region of feasible solutions. Label the horizontal axis "Pounds in Hamburger" and the vertical axis "Pounds in Chicken." Let 1 cm = 1 lb.

g. Use dimensional algebra arguments and the information

1 lb hamburger = 1 dollar

1 lb chicken = $\frac{1}{2}$ dollar

to derive the expression

$$\text{cost in dollars} = x + \frac{1}{2}y$$

for x lb of hamburger and y lb of chicken.

h. Derive the equation

$$y = -2x + 2c$$

from the equation

$$c = x + \frac{1}{2}y$$

i. Graph two cost equations on the graph for Part f, one for c = \$2. and one for c = \$4.

j. The minimum cost is that point where a cost line will first intersect the region of feasible solutions. Locate this point on the graph and mark it with a star.

k. What are the exact coordinates of the point? How many pounds of hamburger and how many pounds of chicken should be purchased?

l. Find the minimum cost.

7. Prof. L. Mobucs claims that he can cure colds with his triple-formula pills. The vector that represents his blue pills is

$\vec{B}$ = [1 grain aspirin, 8 grains sugar, 6 mg cinnamon]

The vector representing his red pills is

$\vec{R}$ = [2 grains aspirin, 5 grains sugar, 1 mg cinnamon]

Prof. Mobucs' research has convinced him that the vector

$\vec{C}$ = [12 grains aspirin, 74 grains sugar, 24 mg cinnamon]

must be attained or exceeded to effect his truly unbelievable cure. Determine the minimum number of pills that Prof. Mobucs should prescribe.

a. If x = the number of blue pills
and y = the number of red pills
h = total number of pills

write an interpretation for the constraints

$$x \geq 0 \text{ and } y \geq 0$$

b. Write one sentence explaining the constraint

$$x\vec{B} + y\vec{R} \geq \vec{C}$$

c. From the vector inequality

$$x[1, 8, 6] + y[2, 5, 1] \geq [12, 74, 24]$$

derive the following system of linear inequalities.

$$x + 2y \geq 12 \quad \text{(aspirin)}$$

$$8x + 5y \geq 74 \quad \text{(sugar)}$$

$$6x + y \geq 24 \quad \text{(cinnamon)}$$

Show all work.

d. From the inequalities in Part c derive the following set of inequalities. Show all work.

$$y \geq -\frac{1}{2}x + 6 \quad \text{(aspirin)}$$

$$y \geq -\frac{8}{5}x + \frac{74}{5} \quad \text{(sugar)}$$

$$y \geq -6x + 24 \quad \text{(cinnamon)}$$

e. Graph the region of feasible solutions. Remember, there are five constraints. Label each axis accurately.

f. Label the region of feasible solutions on your graph.

g. Write a one sentence interpretation of the algebraic statement

$$n = x + y$$

h. From the equation in Part g derive the equation

$$y = -x + n$$

i. (1) Graph the equation in Part h for $n = 4$ on the graph for Part e. Write $n = 4$ on the line.

(2) Repeat part (1) for $n = 8$ and $n = 10$.

j. Refer to the graph you have been constructing. Select the appropriate symbols in order to make the following statement true.

(1) $s = 8$ implies that $x\vec{B} + y\vec{R} \; ? \; \vec{C}$

(2) $s < 10$ implies that $x\vec{B} + y\vec{R} \; ? \; \vec{C}$

(3) $s = 10$ implies that $x\vec{B} + y\vec{R} \; ? \; \vec{C}$

k. The minimum number of pills necessary to meet or exceed Prof. L. Mobucs' requirements is achieved when x = _____ and y = _____. The number of pills is the sum of x and y, which is _____.

8. Of the twenty-odd amino acids making up protein, eight cannot be synthesized by humans and are known as the essential amino acids. A diet which provides adequate amounts of the three amino acids tryptophan (TRP), lysine (LYS) and methionine (MET) will generally provide enough of the other essential amino acids. Meat is an excellent source of the essential amino acids, but they may also be obtained from several other sources. One of the sources is nuts. In the table below is listed the approximate number of grams of the amino acids TRP, LYS and MET in a pound of beef, a pound of peanuts and a pound of cashews.

	TRP	LYS	MET
Beef	1.2	8.0	2.4
Peanuts	1.5	5.0	1.2
Cashews	2.0	3.0	1.5

Peanuts cost \$.50 a pound and cashews sell for \$1.50 a pound. Our problem is to find the cheapest combination of peanuts and cashews that contain an amount of each of the three amino acids equal to or greater than the amount in a pound of beef.

a. Express the quantity of the amino acids in beef as a three-dimensional vector $\vec{B}$. Also write three-dimensional vectors for $\vec{P}$ and $\vec{C}$, the amino acids in peanuts and cashews, respectively.

b. Let x = pounds of peanuts and y = pounds of cashews. Identify a vector inequality expressing the constraint that the amount of each amino acid in the nuts be equal to or greater than the amount in a pound of beef.

(1) $x[2.0, 3.0, 1.5] + y[1.5, 5.0, 1.2] \geq [1.2, 8.0, 2.4]$

(2) $x[1.5, 5.0, 1.2] + y[2.0, 3.0, 1.5] \geq [1.2, 8.0, 2.4]$

(3) $x[1.5, 5.0, 1.2] + y[2.0, 3.0, 1.5] \geq [1.2, 8.0, 5.0, 2.4]$

(4) $x[1.5, 5.0, 1.2] - y[2.0, 3.0, 1.5] \geq [1.2, 8.0, 5.0, 2.4]$

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c. The vector inequality may be written as three separate inequalities, one for each amino acid. The inequality of tryptophan, for example, is

$$1.5x + 2.0y \geq 1.2$$

Write corresponding inequalities for lysine and methionine.

d. The three inequalities in Part c may be solved for y. For tryptophan, for example, we obtain $y \geq -\frac{3}{4}x + \frac{3}{5}$. Solve each of the two other inequalities for y.

e. Which choice below expresses the impossibility of a negative amount of either nut?

(1) $x = 0$ and $y = 0$ (4) $x \geq 0$ and $y \geq 0$

(2) $x = y$ (5) $x \leq 0$ and $y \leq 0$

(3) $x \geq \vec{P}$ and $y \geq \vec{C}$

f. Graph the inequalities of Part d along with the two you chose in Part e. Label the region of feasible solutions. Label the axes.

g. The cost, c, of x pounds of peanuts and y pounds of cashews is given by which equation?

(1) $c = .50x + 1.50y$ (3) $c = 1.50x - .50y$

(2) $c = 1.50x + .50y$ (4) $c = x + y$

h. The equation of Part g may be rewritten as

(1) $y = -.50x + 1.50c$ (3) $y = -\frac{2}{3}x + \frac{1}{3}c$

(2) $y = -\frac{1}{3}x + \frac{2}{3}c$ (4) $y = \frac{1}{3}x + \frac{2}{3}c$

i. Plot your answer to Part h on your graph for c = \$.50, c = \$1.00 and c = \$1.50.

j. What is the lowest value of c for which the cost line intersects the region of feasible solutions? This is the minimum cost of nuts that will provide enough of each amino acid.

At the point of minimum cost, x = _____ and y = _____.

Therefore, the cheapest combination of nuts that contains at least as much of the three amino acids as a pound of beef is _____ pounds of peanuts and _____ pounds of cashews.

9. In Problem 8, a diet of peanuts and cashews was determined which contains as much of three of the eight essential amino acids as a pound of beef. In case a diet of nuts does not appeal to you, another satisfactory vegetarian diet may be devised with egg and pinto beans. The table below gives the approximate grams of each of the three amino acids in a pound of pinto beans and in one egg, as well as giving again approximately the grams of each in a pound of beef.

	TRP	LYS	MET
1 lb pinto beans	1.0	7.5	1.0
1 egg	0.1	0.4	0.2
1 lb beef	1.2	8.0	2.4

Assume that eggs sell for 60¢ a dozen (5¢ apiece) and that pinto beans cost 30¢ a pound. Find the least expensive combination of eggs and beans that provides at least as much of each essential amino acid as a pound of beef does.

10. An outfitter in charge of setting up the menu for group backpacking trips plans to serve scrambled eggs and pancakes as part of the breakfast menu. To minimize weight, both are purchased in the form of dried preparations intended for backpacking. The protein, fat and carbohydrate contents of 100 grams of each dried preparation is as follows.

	grams protein	grams fat	grams carbohydrate
scrambled eggs	52	42	0
pancake mix	15	15	60

On the basis of dietary consideration the outfitter wants to serve enough of the two dishes each morning to provide a total of at least 20 grams of protein, 18 grams of fat and 30 grams of carbohydrate per person.

a. Suppose that the outfitter is interested in minimizing the weight of the ingredients to be carried. How many grams of each dried product should he allot per person per day?

b. Suppose that the scrambled egg mixture costs $1.00 per 100 grams and the pancake mix costs $0.15 per 100 grams. If the outfitter is interested in minimizing the combined cost of the products, how many grams of each dried product should he allot per person per day?

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PROBLEM SET 22:

In Problems 1 through 4 compute the range of uncertainty.

1. $(80 \pm .10) + (17 \pm .05)$

2. $(.037 \pm .002) + (.162 \pm .001) + (1.06 \pm .01)$

3. $[(5.09 \pm .01) \times 10^6] + [(6.34 \pm .01) \times 10^5]$

4.
$$\begin{array}{r} 8,000,000 \pm 1,000 \\ \underline{+19,000,000 \pm 500} \end{array}$$

5. The following is part of a table which appeared in the Biomedical Science Text.

ABSORBANCE-TRANSMITTANCE TABLE

Trans. (%)	Absorbance	Trans. (%)	Absorbance	Trans. (%)	Absorbance	Trans. (%)	Absorbance
0.0	----	25.0	.600	50.0	.300	75.0	.125
0.5	2,300	25.0	.595	50.5	.295	75.5	.120
1.0	2,000	26.0	.585	51.0	.290	76.0	.120
1.5	1,825	26.5	.575	51.5	.290	76.5	.115
2.0	1,700	27.0	.570	52.0	.285	77.0	.115
2.5	1,600	27.5	.560	52.5	.280	77.5	.110

a. What is the implied uncertainty of the transmittance values in the table?

b. What is the implied uncertainty in the absorbance values?

6. The dry mass of a flask is 75.03 ± .02 g. After the addition of a portion of hamburger the flask and hamburger have a combined mass of 94.87 ± .02 g. What is the range of uncertainty of the mass of hamburger?

7. $(8.9 \pm .1) - (3.1 \pm .1)$

8. $(8.32 \pm .01) - (5.10 \pm .02) - (1.01 \pm .01)$

9. $[(3.75 \pm .01) \times 10^2] - [(2.01 \pm .02) \times 10^3]$

10. $(17.01 \pm .02) - (1.31 \pm .01) + (1.52 \pm .03)$

Use the approximate formula for the product of two ranges of uncertainty $(x \pm \Delta x)(y \pm \Delta y) \approx xy \pm (x\Delta y + y\Delta x)$ to solve Problems 11 through 14.

11. $(20 \pm 1)(15 \pm 1) =$

12. a. $(10 \pm .1)^2$

b. $(10 \pm .1)^3$

13. A square has a side 20 ± .02 cm in length. Determine the range of uncertainty of the area.

14. A cube has an edge of 1 ± .1 meters. Determine the range of uncertainty of the volume.

State the relative uncertainty of each of the ranges of uncertainty in Problems 15 through 17.

15. 50 ± .5

16. 250 ± .5

17. 50.0 ± .03

State the absolute uncertainty of the ranges of uncertainty in Problems 18 through 20.

18. 87 ± 5%

19. 92 ± .5%

20. 146 ± 1%

21. Determine the range of uncertainty of (187.1 ± .5) + (22.9 ± .2) in terms of

a. absolute uncertainty

b. relative uncertainty

22. Riley wanted to calculate the miles per gallon that his Burpmobile was getting. When he filled up his tank his odometer read 40,356.2 miles. When he ran out of gas it read 40,612.4 miles. Burpmobile gas tanks hold 12 gallons. His odometer has a correctable uncertainty of 4%. The odometer errs on the high side. In other words, the indicated number of miles traveled minus 4% gives the corrected number of miles traveled. How many miles per gallon did Riley's Burpmobile get? Round your answer to the nearest .1 mile per gallon.

23. Riley readjusted the ignition timing on the old Burpmobile. He went 190.0 ± .05 actual miles on 10 ± .05 gallons of gas.

a. What is the range of uncertainty of his miles-per-gallon calculation. Use the formula $\frac{x \pm \Delta x}{y \pm \Delta y} \simeq \frac{x}{y} \pm \frac{x\Delta y + y\Delta x}{y^2}$.

b. Do you think Riley ought to learn more about ignition timing?

24. A solution of 125 ± 1 ml contains 6.25 ± .03 grams of eggwhite.

a. What is the concentration of the solution in grams eggwhite per ml of solution? State your answer in the form (midpoint ± absolute uncertainty).

b. What is the concentration (as a range of uncertainty) in grams eggwhite per 100 ml of solution?

25. The protein concentration of an eggwhite solution may be obtained by this dimensional algebra problem.

$$\frac{\text{(E) g eggwhite}}{\text{(M) 100 ml solution}} \cdot \frac{\text{(P) g protein}}{\text{(F) g eggwhite}} \cdot \frac{\text{(C) g protein}}{\text{100 ml solution}}$$

If the relative uncertainty of each of the factors is

factor	relative uncertainty
E	.5%
M	2.0%
P	1.1%
F	.2%

what is the relative uncertainty of the protein concentration (C)?

26. A slice of bread contains 2 grams protein, 1 gram fat, and 12 grams carbohydrate.

a. Write a row vector $\vec{B}$ for the nutritional ingredients of one slice of bread.

b. If a loaf of bread has 20 slices, write a vector $\vec{L}$ which represents the nutritional ingredients of the loaf.

27. Given that $\vec{A} = [-1, 0, 3]$, $\vec{B} = [0, 0, 1]$, $\vec{C} = [-2, 3, 5]$

a. Find $-1\vec{B}$

b. Find $-2\vec{A} + 1\vec{B} - 3\vec{C}$

28. If $\vec{X} = [1, 2, -1]$ and $\vec{L} = [0, 1, -10]$

a. $\vec{X} - \vec{L} =$

b. $\vec{X} + \vec{L} =$

c. $2\vec{X} - 3\vec{L} =$

29. If $\vec{A} = [2, 1, -2]$, $\vec{B} = [0, 1, -1]$

a. $\vec{A} \cdot \vec{B} =$

b. $\vec{B} \cdot \vec{A} =$

c. Does $\vec{B} \cdot \vec{A} = \vec{A} \cdot \vec{B}$?

30. $\sum_{i=2}^{6} i =$

31. The vector $\vec{K} = [4, 9, 4]$ represents the number of calories contained respectively in 1 gram of protein, 1 gram of fat and 1 gram of carbohydrate. Given that a lettuce and tomato salad vector is $\vec{S} = [3, 0, 7]$, use the device of dot product to determine the number of calories in the salad. Show your work.

32. From the vector expression $[8, 11] = [2x + y, -x - y]$

a. Derive, by application of the principle of vector equivalence, two open sentences.

b. Solve the open sentences for x and y.

33. A group of Girl Scouts eat Swiss cheese and apples for lunch. If their total diet intake was [64 g protein, 80 g carbohydrate], determine the number of apples and the number of pieces of Swiss cheese that were eaten (1 apple = [0, 18], 1 piece of Swiss cheese = [8, 1]).

34. In order to achieve a diet of [800 units vitamin A, 200 mg vitamin C] how many California and Florida oranges must you eat? (1 California orange ≈ [250 units vitamin A, 80 mg vitamin C], and 1 Florida orange ≈ [300, 60].)

35. Solve the equations by the substitution method.

$$3x + 3y = 15$$
$$6 = 5x + y$$

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36. State whether the following inequalities are true, false, or open.

a. $15 \leq 15$ c. $3 \leq a$ and $10 \geq a$

b. $x \geq y$ d. $-9 \geq -3$

37. Translate the following into mathematical statements choosing letters when needed to represent unknown quantities.

a. Five is greater than negative two, but less than 8.

b. Twice a certain number is three more than one half the number.

c. The total number of nut trucks and bolt trucks is at least seven but no more than fifteen.

d. T is a positive number.

38. Graph the region determined by the following set of inequalities.

$x \leq 0$ $y \geq -3$

$y \leq x + 1$

39. Graph the region determined by the following set of inequalities.

$x \geq -4$ and $x \leq 3$

$y > -4$ and $y \leq -2$

40. Graph the region determined by the following set of inequalities.

$y \leq -x$ $y > -4$

$x > 1$

41. Transform the following inequalities into the standard form $y \leq mx + b$ or $y \geq mx + b$.

a. $3y \leq 6x + 15$ c. $-x - y \leq 0$

b. $15 - x \geq 2y$ d. $2x \leq -y - 3$

42. Given that $a\vec{X} + b\vec{Y} \geq \vec{Z}$, where $\vec{X} = [2, -1, 3]$, $\vec{Y} = [0, -2, 1]$ and $\vec{Z} = [-1, 0, -1]$, transform the vector inequality into a set of algebraic inequalities. Show your work.

43. Given that $\vec{A} = [1, 3]$, $\vec{B} = [0, -2]$ and $\vec{C} = [-1, -1]$,

a. Transform the inequality $s\vec{A} + t\vec{B} \leq \vec{C}$ into a set of algebraic inequalities.

43. b. Graph the region determined by the inequalities.

44. Given the vector inequality $x\vec{R} + y\vec{P} \geq \vec{Q}$ where $\vec{R} = [0, 1, 1]$, $\vec{P} = [4, -2, 0]$, $\vec{Q} = [-3, 2, 1]$, derive algebraic inequalities and shade the region of simultaneous solution.

45. The following table shows the protein and carbohydrate contents of lentils and soybeans.

	Protein	Carbohydrate
Lentils (100 g portion)	25 g	60 g
Soybeans (100 g portion)	35 g	35 g

Suppose that we want a combination of lentils and soybeans which will contain at least as much protein and carbohydrate as the diet vector.

$$\vec{D} = [210 \text{ g protein}, 455 \text{ g carbohydrate}]$$

Find the combination satisfying this requirement such that the combined mass of lentils and soybeans is as small as possible. What is the smallest possible mass? (Show all work!)

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SECTION 23:

23-1 Introduction to Statistics

During the next few weeks you will be studying the mathematical field called statistics. When you have mastered some of the basic concepts, you will perform several activities in which you will collect data and apply statistical techniques to the results.

Roughly speaking, the field of statistics involves the study of certain trends or patterns which arise in large populations or groups. In medicine, patterns of disease are studied using statistical methods. An example is the investigation of the relationship between smoking and emphysema.

What is involved in testing whether smoking and emphysema are related? If you have known a smoker or two who developed emphysema it proves nothing at all; it could have just happened by chance. On the other hand, suppose you work in a hospital and you notice that most of the patients with emphysema are smokers. Can you now conclude that smokers are more likely to develop emphysema than nonsmokers? The answer is still no. Perhaps most people in the general population are smokers. Then of course most emphysema victims will be smokers too.

In order to show that smoking and emphysema are statistically related we must do two things:

1. We must show that a higher percentage of smokers than nonsmokers develop emphysema.

2. We must have data on a large number of people so we can be confident of our results.

We want data on a large number of people so we can be reasonably sure that our results aren't just an accident--just a freak occurrence because we happened to investigate an unusual group of people. But how do we know when we have collected data on enough people? How can we know how likely it is that our results are just an "accident" and prove nothing at all? In the sections to come you will learn how to use mathematics to help answer such questions.

Before we go on, however, a word of caution is needed. Medical researchers are always trying to find the cause of particular medical problems. Therefore they do statistical research to identify variables which are commonly associated with the ailment. However not all things which are associated with an ailment are causes of the ailment. A misunderstanding of this idea could lead to disastrous results. For example, suppose Dr. Elmo's research has shown that all emphysema victims have heads. The distinguished Dr. Elmo concludes that heads are the cause of emphysema, and recommends the headectomy (removal of the head) of everyone. Perhaps Dr. Elmo should be the first to try out his revolutionary new procedure.

23-2 Randomness

In everyday life we are faced with common events which we hardly think about at all. But once in a while something happens which is so unusual that we take special notice of it. For example, suppose you went for a walk and the first five people you passed were all naked. This would certainly be an unusual event, but it could occur by chance alone. Now and then some people get the notion to go out and walk around the block naked. If five such people happened to do so at the same time, that could account for the event. But the combination is so unlikely that you would probably look for some other explanation besides chance. Perhaps you walked into a nudist colony by mistake or were having an hallucination.

An event which occurs by chance alone is called a random event. Some random events occur frequently. Others are very unusual. An unusual event like the one above is variously described as a rare event, an event with a low relative frequency, an event with a low probability.

In our study of statistics we will be particularly interested in rare random events. You will begin by performing a random events activity with a set of 16 coins. You shake them up and then dump them out on your desk top. You probably already have some correct expectations about what will happen. For example, you would probably be surprised if you got 16 heads or 16 tails. This isn't impossible, but it is a rare event. You probably would expect 15 heads and one tail to happen more often than 16 heads and so forth.

What number of heads do you think would be seen most often? The answer has something to do with our expectations about the way "fair" coins behave. In the long run we expect to see just as many heads as tails. This has implications for our set of 16 coins. It suggests that we might expect to see 8 heads and 8 tails more often than any other combination. We can summarize our intuitive feelings by saying that we expect 16 heads very arely, 8 heads most often and zero heads very rarely. Would you expect 10 heads or 9 heads to occur more often? Since 10 heads is farther away from 8 we would expect 10 heads to happen less often than 9 heads. This then is the general trend of things for 16 coins. The farther away from 8 heads the less often we expect to see it. How often we see an expected pattern of events is somewhat dependent upon the number of trials. In an experiment with a million trials we will produce a pattern that agrees very closely with the expected pattern. When the number of trials is small we will more often see unexpected patterns of events.

23-3 A Coin-Dumping Tally

So far we have put forth some ideas on the outcomes when 16 coins are shaken up and dumped on a desk. How will these expectations bear out in practice? In order to find out, you will spend a class period dumping a set of 16 coins. To help you record the experience and apply it, we will define some words to describe the activity. You will be repeatedly dumping 16 coins and counting the number of heads. You will be doing this again and again. It would seem reasonable to call one dumping a "dump." However, we won't because there is a lack of dignity in the word "dump." You may wish to agrue this point. We will call each dump of the coins a trial. This is the language of statisticians. For example, a statistician would say "The result of Trial 14 was 11 heads."

When 16 coins are dumped there are 17 possible outcomes, from 16 heads all the way down to zero heads. We will call each of these possibilities an event. For example, 16 heads is an event. In general, the set of all possible outcomes of a trial is the set of possible events. For example, when a single die is thrown, there are 6 possible events.

Since you dump out the 16 coins again and again, it is likely that a particular event will happen more than once. We will call the number of times a particular event occurs the <u>frequency</u> of the event. For example, Wilbur Caliente performed 50 trials. The frequency of 9 heads was 8. In other words, the event 9 heads occurred 8 times.

When you do your coin dumping, your results should be recorded in a manner similar to the way we have recorded the results of an actual set of 50 trials.

Event (Number of Heads)	Observed Frequency		Total Number of Heads for Each Event
0		0	
1		0	
2		0	
3		0	
4		0	
5	~~////~~	5	5 x 5 = 25
6	~~////~~ ///	8	6 x 8 = 48
7	~~////~~ ///	8	etc. 56
8	~~////~~ ////	9	72
9	~~////~~ ///	8	72
10	~~////~~ /	6	60
11	////	4	44
12	//	2	24
13		0	
14		0	
15		0	
16		0	
	total number of trials	= 50	total number of heads = 401

$$\text{Mean} = \frac{401}{50}$$

$$= 8.02 \text{ heads per trial}$$

When you have completed all of your trials you will report your frequencies to your teacher, who will record the grand totals for the entire class. Then you should calculate the mean (average)

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number of heads per trial for your data. We have demonstrated how to do this in the example shown on the previous page. Your teacher will calculate the mean number of heads per trial for the entire class. You are doing this to test a statistical prediction. The prediction is that the mean for the whole class will be closer to 8 than most of the students' individual means.

PROBLEM SET 23

1. (True or false) If you have known a couple of smokers who developed emphysema then it proves that smoking causes emphysema.

2. Suppose that you work in a hospital and you notice that most of your emphysema patients are smokers. One possible explanation is that smoking causes emphysema. What are <u>other</u> possible explanations?

3. What two things must be done to show that smoking causes emphysema?

4. Listed below are pairs of variables which are associated with each other. Which pairs are related as cause and effect?

a. It has been found that countries with <u>high per capita incomes</u> tend to have <u>high heart disease rates</u>.

b. It has been found that 82.4% of all people who <u>break their hips</u> were wearing <u>shoes</u> at the time of their accident.

c. An <u>increase in temperature</u> is associated with an <u>increase in volume</u> for a quantity of trapped gas at constant pressure.

d. An <u>increase in pressure</u> is associated with a <u>decrease in volume</u> for a quantity of trapped gas at constant temperature.

e. People <u>go to lunch in California</u> when the <u>clock strikes 3:00 PM in New York</u>.

5. List two variables that are commonly associated with one another that are <u>not</u> related as cause and effect. Do not use any example listed above.

6. Identify the random events in the following list. All decks, dice and coins are "fair."

a. A coin is tossed. It turns up heads.

b. A pair of dice is rolled. A pair of sixes comes up.

c. A card is dealt. It is a jack.

d. A key is inserted into a lock. It opens the lock.

e. A car runs out of gas. The car stops.

f. A person rolls a pair of dice. He gets ten 7's before he rolls a 2.

7. List two random events not listed in this text.

8. List two nonrandom events not listed in this text.

9. Which event in Problem 6 is the rarest random event?

10. Sixteen coins are shaken up and repeatedly dumped on a table top.

a. Which two numbers of heads would we expect to see <u>least</u> often?

b. What number of heads would we expect to see <u>most</u> often?

c. In the long run, the farther away from 8 heads, the (more, less) often we expect to see it.

11. Six coins are shaken up and repeatedly dumped on a table top. Below we have listed all possible events which might occur (h = "heads"). The set of all possible events = {0h, 1h, 2h, 3h, 4h, 5h, 6h}.

a. Which two events would you expect to see <u>least</u> often?

b. Which single event would you expect to see <u>most</u> often?

c. Which event would you expect to see more often, 3h or 2h?

d. Which event would you expect to see more often, 4h or 5h?

12. Ten coins are shaken up and repeatedly dumped on a desk top. Below, we have a table that orders our expectations about the number of heads. It goes from most often to least often. Insert 0h, 1h, 2h, etc., in the appropriate box in the table.

Most Often							Least Often

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13. a. When 10 coins are dumped there are ___ possible events.

b. When "n" coins are dumped there are ___ possible events.

In class you dumped 16 coins at a time in each trial. In the next problems suppose you are dumping 100 coins at a time and counting the number of heads. Therefore, on any given trial you can get anything between zero and 100 heads. Suppose the number of trials is very large.

14. We expect that the most frequent event will be ____ heads.

15. Which event do we expect to be rarer, zero heads or 3 heads?

16. Which event do we expect to be rarer, 3 heads or 52 heads?

17. Which event do we expect to be rarer, 40 heads or 62 heads?

After four trials of a four-coins-at-a-time coin dumping experiment the tally sheet looked like the one below.

Event (Number of Heads)	Tally	Observed Frequency
0	1	1
1		0
2		0
3	1	1
4	11	2

18. Does the tally sheet prove that the coins are behaving in a nonrandom fashion? (Yes or No)

19. How do the results in the table differ from our expectations about randomness?

20. What might explain the pattern in the table?

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Below is the tally sheet for the same experiment after <u>20</u> trials.

Event (Number of Heads)	Tally	Observed Frequency
0	/	1
1	~~////~~ /	6
2	~~////~~ //	7
3	////	4
4	//	2

21. How many possible events are there?

22. How many trials (or dumps) were there in the above activity?

23. The event "4 heads" occurred with frequency _____.

24. The most frequent event was _____ heads.

25. The rarest event was _____ heads.

26. What was the <u>total</u> number of heads appearing in all 20 trials?

27. What was the <u>mean</u> number of heads per trial?

The following table shows the results of measuring the systolic blood pressures of 102 men aged 75 and above.

Systolic Pressure (mm Hg)	Observed Frequency
110-129	18
130-149	31
150-169	23
170-189	20
190-209	7
210-229	1
230-249	2
Total	1[illegible]

28. a. How many possible events are there in this survey?

b. How many trials were there?

29. The event of a blood pressure in the range 190-209 mm Hg occurred with a frequency of _____.

30. a. The most frequent event was a blood pressure in the range of ____ mm Hg.

b. The rarest observed event was a blood pressure in the range of ____ mm Hg.

Some of the best medical surveys available are based on military personnel and on inmates of mental hospitals. This is no surprise since such people are given systematic medical examinations. The following table gives the head heights in mm of 68 male mental patients. The source of the table did not explain exactly what "head height" means, but this is unimportant for our purposes.

DATA ON HEAD HEIGHT OF MALE INMATES OF HADDINGTON DISTRICT ASYLUM

Head Height (mm)	Frequency
115-119	1
120-124	0
125-129	10
130-134	15
135-139	17
140-144	16
145-149	5
150-154	3
155-159	1

31. The most frequent event is a head height in the range ____ mm.

32. The rarest event is a head height in the range ____ mm.

33. This survey was based on only 68 males. If the survey were done on 1,000,000 males, which event is likely to be more unusual, 115-119 mm or 120-124 mm?

34. How many trials were there?

Create tally sheets and record the data for the experiments described below.

35. 153 trials of an experiment with five possible events is run.

The frequency of the zeroth event was 6.

The frequency of the first event was 27.

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The frequency of the second event was 80.

The frequency of the third event was 33.

The frequency of the fourth event was ?

Be sure to include the numerical value for the frequency of the fourth event.

36. 127 trials of an experiment with seven possible events is run.

Heights in the range <u>150</u> cm to <u>154</u> cm are observed <u>7</u> times.

Heights in the range <u>155</u> cm to <u>159</u> cm are observed <u>17</u> times.

Heights in the range <u>160</u> cm to <u>164</u> cm are observed <u>41</u> times.

Heights in the range <u>165</u> cm to <u>169</u> cm are observed <u>36</u> times.

Heights in the range <u>170</u> cm to <u>174</u> cm are observed <u>15</u> times.

Heights in the range <u>175</u> cm to <u>179</u> cm are observed <u>9</u> times.

Heights in the range <u>180</u> cm to <u>184</u> cm are observed <u>?</u> times.

SECTION 24:

24-1 Tally Ho!

A tally similar to the one you made in the previous section is one way to display a frequency distribution. It shows the frequencies of different events. We will later examine more elaborate ways of displaying a frequency distribution. But there is much more that can be learned from the simple tally in Section 23-3.

For example, were our expectations confirmed for that particular set of trials? In other words, did 8 heads occur most often? Were frequencies lower for events farther away from 8 heads? Certainly 8 heads was most often seen and, in general, the farther away from 8 heads the less frequent the event.

There is an exception, though. Can you spot it? Six and 7 heads both occurred 8 times. But 6 is farther away from 8 than 7, therefore we expect that its frequency should be less than the frequency of 7. This result underlines something important to keep in mind about random events. We can only predict what is most likely to happen. We can never be certain exactly what will happen.

The unexpected pattern of 6 heads and 7 heads happens more often with small numbers of trials. (The grand tally for the whole class probably will not show this kind of behavior.) There is an underlying principle here. It is that frequency distributions will more closely fit our expectations (if they are correct) as the number of trials increases. The exercise in computing the mean number of heads per trial was designed to test this hypothesis. If the mean for the entire class was closer to 8 than a majority of the means for individuals, then our expectations proved accurate for your class. However, if the grand mean wasn't closer, this does not disprove our hypothesis. Since we are dealing with random events, there is always the possibility that a rare event may be observed.

24-2 Relative Frequency: A Zero-to-One Scale for Never to Always

Is an event that happens only once a rare event? The answer depends on the number of trials. If there were only one trial, then the answer would be, "Not necessarily." If there were a million trials, the event would indeed be considered a rare one. The ratio of the observed frequency to the total number of trials is called the relative frequency of an event. This is summarized in the formula below.

$$R = \frac{O}{N}$$

where R = relative frequency

O = observed frequency

N = total number of trials

For example, the relative frequency of 8 heads in our earlier example is the observed frequency (9) divided by the total number of trials (50).

$$R = \frac{9}{50}$$

$$= .18$$

What this figure means is that the event 8 heads occurred in 18% of the trials. You will calculate relative frequencies for your own and the entire class tally.

Remember that a relative frequency of one means that an event always happened, while an R of zero means that it never happened. Also, relative frequency is a descriptive statistic. It describes how often a particular event happened.

24-3 Probability--An Expected Relative Frequency

Probability is a predictive statistic. It predicts how often a particular event can be expected to happen on the average.

Probability is related to expected frequency and the number of trials by the formula

$$p = \frac{E}{N}$$

where p = probability

E = expected frequency

N = number of trials.

This formula looks a lot like the previous one and it is. p and R are related quantities. A probability of one means that an event is certain to happen. A probability of zero means that an event cannot happen. Relative frequency deals with history while probability deals with future expectations.

In the relative frequency formula, R is generally the unknown. N and O are known. However, the p is generally a known quantity in the probability formula. The unknown is usually E or expected frequency. We can use this relationship to predict the expected number of heads for a particular number of trials.

The first thing we must do is to determine the probability of getting a head on a single throw of a coin. We shall assume that the coin is "fair" (heads and tails are equally likely) and that it cannot land on edge. Since a head is one of two equally likely events, the chance or probability of getting a head on a single throw is one-half or .5.

Suppose we flip a single coin 16 times. What is the expected frequency of heads? We know p (it is .5) and N (it is 16). Note that we are calling the flip of a single coin a trial. What we will calculate is the expected frequency of heads in 16 flips.

$$p = \frac{E}{N}$$

where $p = .5$

$E = ?$

$N = 16$

We substitute these numbers into the formula.

$$.5 = \frac{E}{16}$$

$$(.5)(16) = E$$

$$8 = E$$

When a single coin is flipped 16 times the expected frequency of heads is 8. What does "expected" mean in this statement? Two things. It means that 8 heads is more likely than any other outcome when we flip a single coin 16 times. It also means that if we repeat the series of 16 flips many, many times, we expect the average number of heads per series of 16 flips to be very close to 8.

You may have already realized that flipping one coin 16 times is equivalent to dumping 16 coins at once, as you did in class. This equivalence permits us to state that 8 heads is the most likely event when we dump 16 coins at once. Notice that this does not mean that 8 heads will come up in a majority of the trials. Eight heads did in fact appear more often in our example than any other number, but still only 18% of the time.

For the same reason we expect the average to be very close to 8 heads per trial (of 16 coins at a time) after a very large number of trials. In Section 23 the average number of heads per trial after 50 trials was 8.02. This is indeed very close to 8.

For the sake of variety we consider another example of the use of the probability formula.

EXAMPLE:

What is the expected frequency of a 3 in 60 rolls of a single die?

SOLUTION:

Since we expect that each side of the die will come up equally often, the probability of a 3 coming up is one out of six or $\frac{1}{6}$. The number of trials is 60.

$$p = \frac{E}{N}$$

$$\text{where} \quad p = \frac{1}{6}$$

$$N = 60$$

$$E = ?$$

We substitute this information into the formula.

$$\frac{1}{6} = \frac{E}{60}$$

$$10 = E$$

We see that the expected frequency is 10 for a 3 coming up in 60 rolls of a single die.

PROBLEM SET 24:

1. A simple tally is one way to display a ________.

2. (T or F) Statistics allow us to make exact predictions about the future.

3. Relative frequency sets up a zero-to-one scale for ________ to ________.

4. In the formula

$$R = \frac{O}{N},$$

a. R =

b. O =

c. N =

d. Which variable is most commonly the unknown?

5. Relative frequency is a descriptive statistic. It describes how often an event ________.

6. Probability is a ________ statistic.

7. (Probability, Relative frequency) predicts the pattern of future events while (probability, relative frequency) describes the pattern of past events.

8. In the formula

$$p = \frac{E}{N}$$

a. p =

b. E =

c. N =

d. Which variable is most commonly the unknown?

(Multiple Choice)

9. When 16 coins are dumped simultaneously we expect to see 8 heads (half the time, every time, more than half the time, more often than any other event).

10. A single die is rolled 90 times.

a. On a single roll what is the probability that a 5 will come up?

b. What is N in the formula $p = \frac{E}{N}$?

c. Calculate the expected number of times that a 5 will come up.

d. If 5 came up 17 times, would you be suspicious about the fairness of the die?

e. If 5 came up 90 times, would you be suspicious?

11. In order to compute the relative frequency of an event we divide the observed frequency by _____.

12. If the relative frequency of an event was .62, it means that the event occurred in _____% of the trials.

The following table appeared in Problem Set 23. It shows the outcomes of 20 dumps of 4 coins.

Event (Number of Heads)	Observed Frequency
0	1
1	6
2	7
3	4
4	2

13. Show calculations verifying that the relative frequency of the event 4 heads was .10.

14. Show calculations verifying that the relative frequency of the event 2 heads was .35.

The following is the same table as above, except that the observed frequencies have been replaced with relative frequencies.

Event (Number of Heads)	Relative Frequency
0	.05
1	.30
2	.35
3	.20
4	.10

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15. What event occurred in 5% of the trials?

16. One of the following statements is true. Which one is it?

a. The event 1 head occurred with the same relative frequency as the event 2 heads.

b. The event 1 head occurred with the same relative frequency as the combined event 3 heads and 4 heads.

c. The event 0 heads was more frequent than the event 4 heads.

The following table is based on a survey of hemoglobin values for 54 female medical students.

Event (Hemoglobin value ranges in grams per 100 milliliters blood) "V"	Relative Frequency
11.5<V≤12.0	.037
12.0<V≤12.5	.056
12.5<V≤13.0	.166
13.0<V≤13.5	.222
13.5<V≤14.0	.204
14.0<V≤14.5	.222
14.5<V≤15.0	.037
15.0<V≤15.5	.056

17. How many possible events are there for this set of data? Remember that an event is a possible outcome of a trial. A trial in this case is a measurement of hemoglobin value.

18. How many trials were there?

19. 16.6% of the students had hemoglobin values falling in the range _____ g/ml.

20. A hemoglobin value in the range greater than 14.5 to 15.0 g/ml was observed in _____% of the cases.

21. Notice that the relative frequency of a value in the range greater than 13.5 to 14.0 g/ml is .204. Show that the corresponding observed frequency must be 11. (You will need to use the formula $R = \frac{O}{N}$ and solve for 0.)

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22. What was the observed frequency of values in the range greater than 14.5 to 15.0?

23. What is the sum of the relative frequency column? (Don't add--think about it.)

24. Recall that the probability p is 0.5 that a coin will come up heads on a given toss. Compute the expected number of heads when a coin is tossed 20 times. (You will need to use the formula $p = \frac{E}{N}$ and solve for E.)

25. The probability p that a certain experimental drug will be effective is 0.6. If 100 patients are treated with the drug, what is the expected number of patients upon whom the drug will be effective?

26. Calculate the relative frequency of each of the following observed frequencies. N = 250

a. 117

b. 220

c. 25

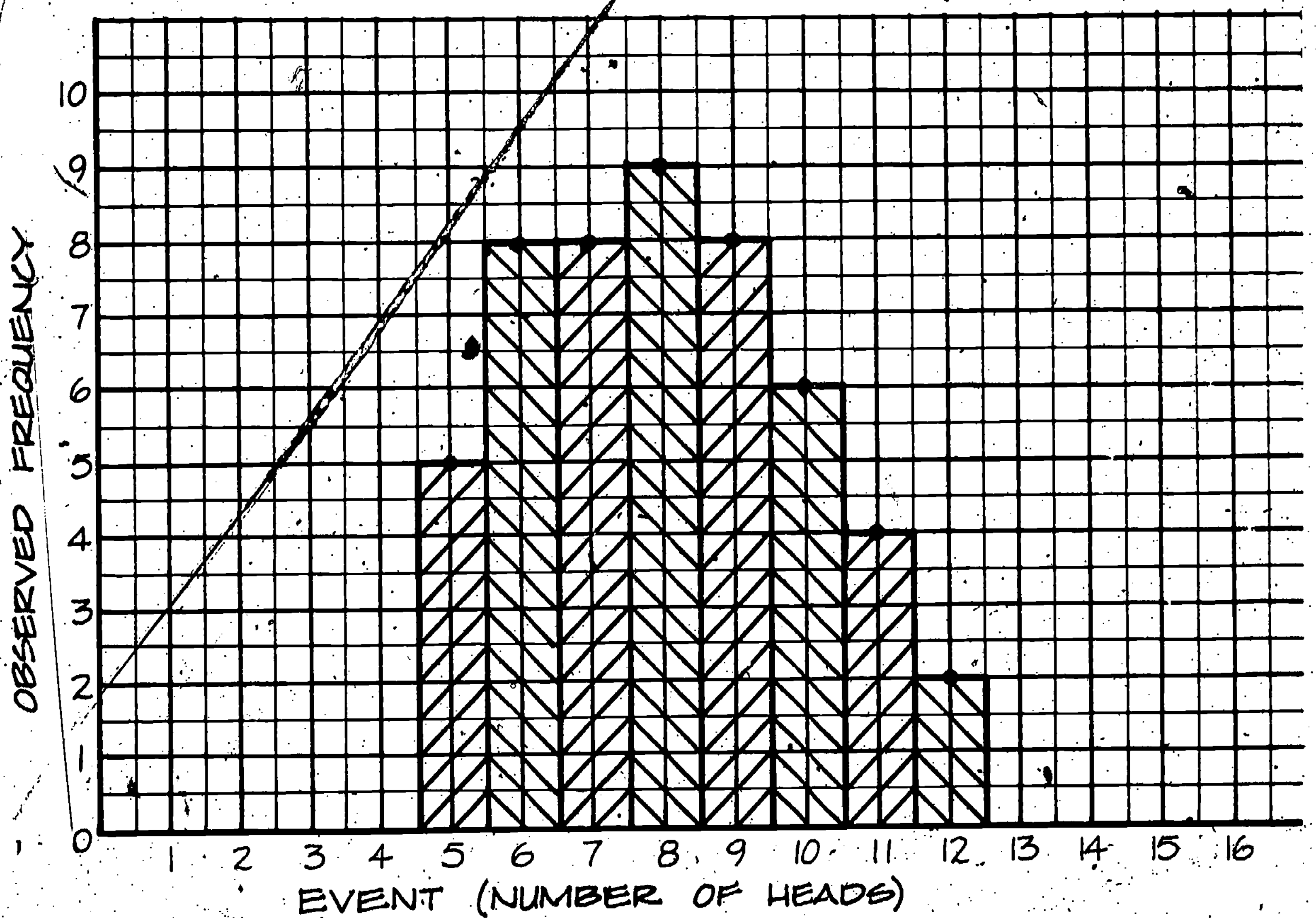
d. 93

27. Calculate the relative frequency of each event for the total class data from the coin dumping activity of Section 23.

SECTION 25:

25-1 A Descriptive Frequency Histogram

We mentioned frequency distributions in the previous sections. We will develop a few more details in our description of them. We can make a graph of the tally of Section 23. We will put observed frequency on the vertical axis and event (number of heads) on the horizontal axis.



A graph like the one above is called a frequency histogram. It is a good way of "picturing" a frequency distribution. Notice that the vertical axis is scaled in terms of observed frequency. For example, it says that 12 heads were observed 2 times.

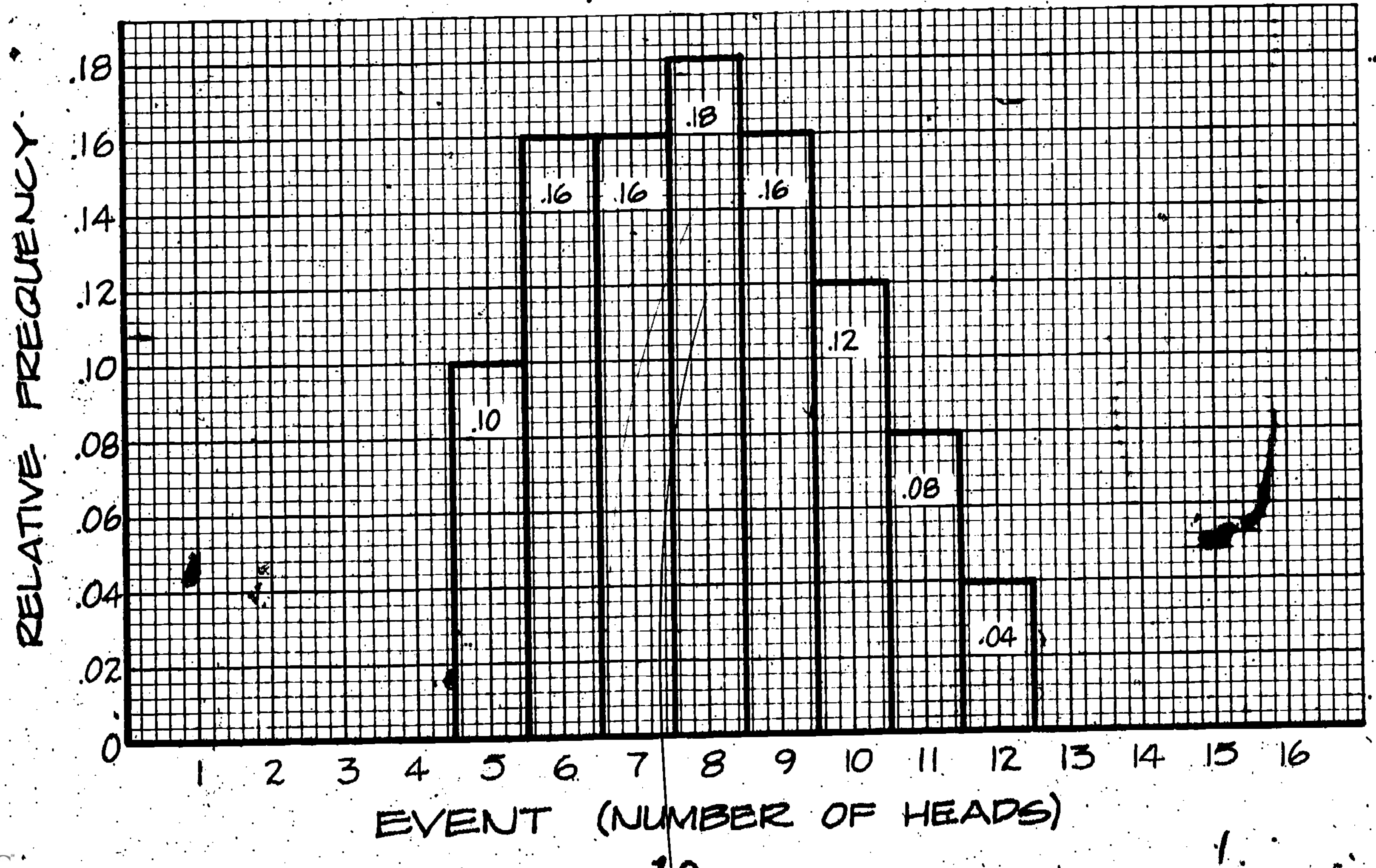
A histogram is a convenient method of displaying biomedical data also. For example, we will construct a histogram for the hemoglobin data of Problem Set 24.

25-2 A Relative Frequency Histogram

In the future we will often want to compare observed frequencies with probabilities. It turns out that relative frequency is more comparable with probability than observed frequency. For this reason, it is convenient to scale the vertical axis of our histogram in terms of relative frequency instead of observed frequency. The relative frequency of each event is summarized below.

EVENT Number of Heads	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Relative Frequency	0	0	0	0	0	.10	.16	.16	.18	.16	.12	.08	.04	0	0	0	0

The graph of this frequency distribution is similar to the previous graph. The only difference is in the vertical axis. It is scaled in terms of relative frequency. As you can see, the relative frequency of each event is written on the corresponding histogram bar. A graph of this type is called a <u>relative frequency histogram</u>. Note that the shape is identical to the observed frequency histogram.



The two graphs and the tally are all different ways of representing frequency distributions. In math class you will construct a relative frequency histogram. It will show the frequency distribution of the results of the class coin-dumping exercise.

25-3 Combined Relative Frequencies

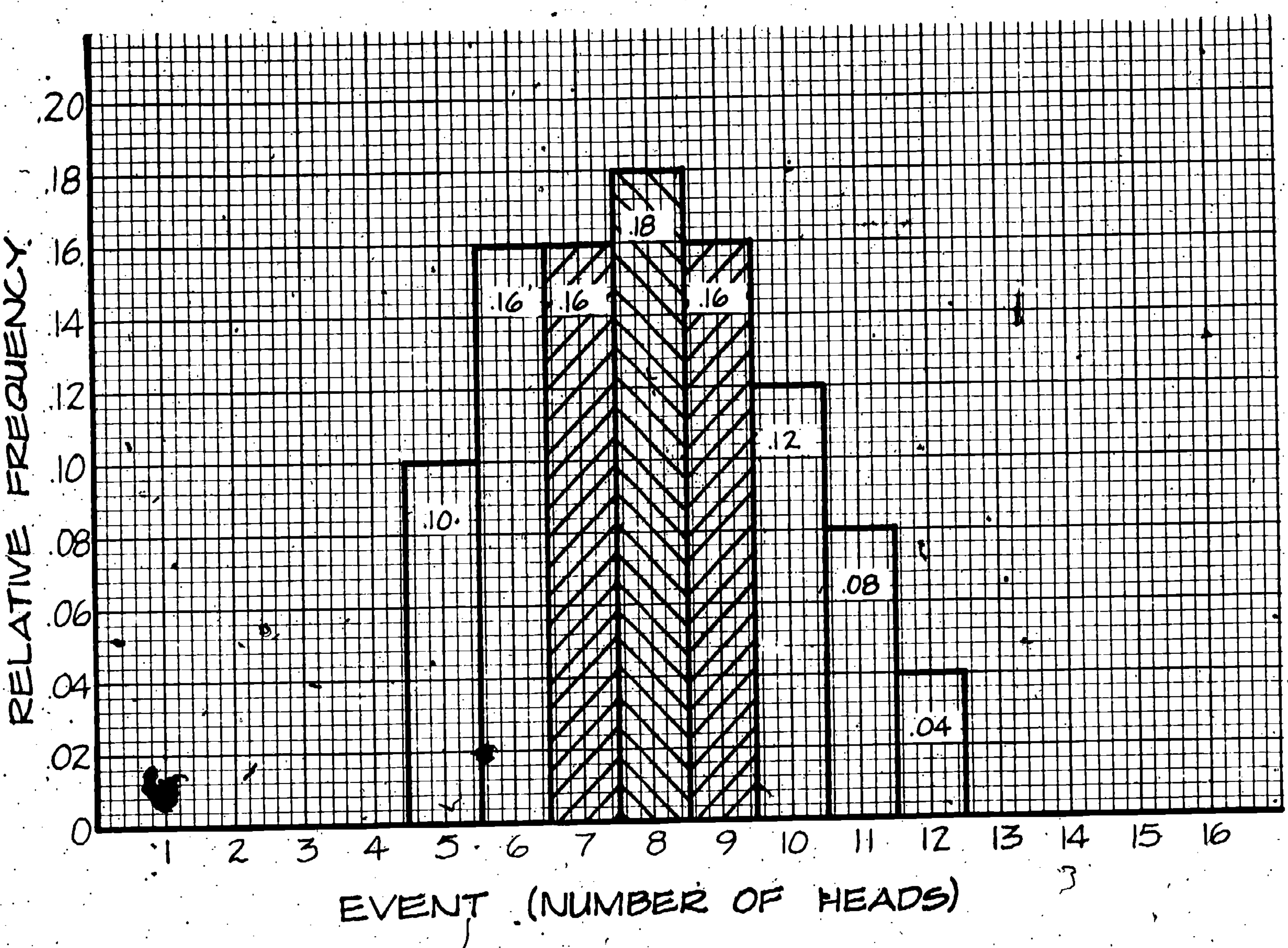
This sequence of lessons is leading up to the interpretation of a statistic called chi square. The chi-square procedure produces a probability for a combination of events. To understand the meaning of this probability we are going to start out by finding relative frequencies for combinations of events. For example, how often did either 7, 8 or 9 heads come up? Half of the time. We can determine this by adding the relative frequencies of each of the three events.

$$.16 + .18 + .16 = .50$$

An important point to keep in mind is that the sum of the relative frequencies of all the events must always be one. This is true because the sum accounts for every event that occurred, and therefore the sum represents an event that always happened. An event that always happens has a relative frequency of 1. Try adding up all the relative frequencies on the histogram bars of our example. The sum should be 1.

25-4 Area and Histograms

A very important feature of histograms is the relation between the area of a bar and relative frequency. All of the bars have a base of 1. The height of each bar is the relative frequency of the particular event. The area of each bar is the product of the base and the height. Since all of the bases have a width of 1, the area of each bar is numerically identical to the relative frequency of the particular event. This idea is very useful when describing combined relative frequencies. The shaded region of the graph on the next page represents the combined relative frequencies of 7, 8 and 9 heads. The shaded area is .5 and the unshaded area is .5.



The relationship between area and relative frequency is not always as obvious as it is with our coin-dumping histogram. Consider the histogram on the next page of the hemoglobin values of the 54 female medical students.

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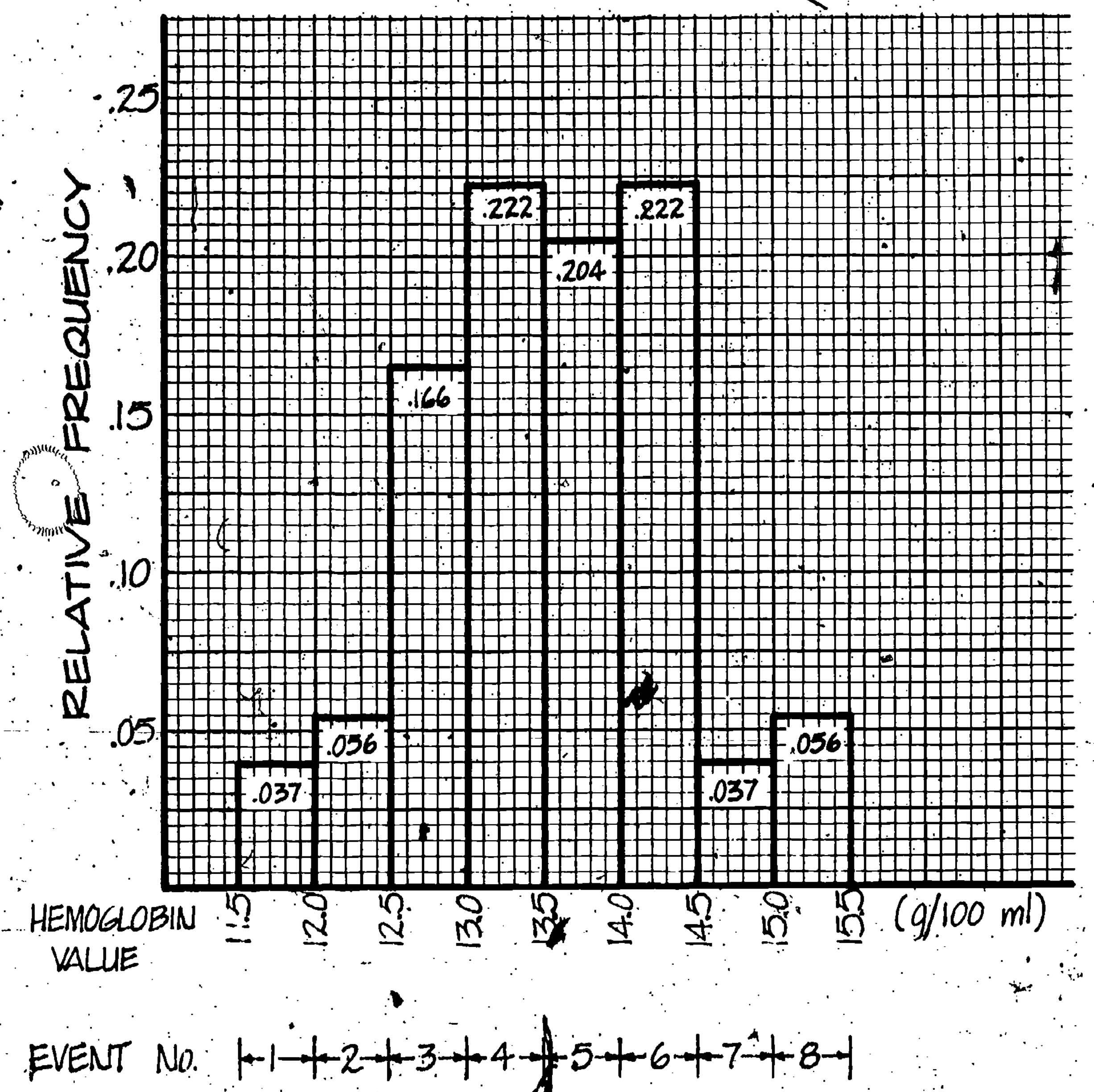

Notice that the base of each histogram bar is .5 g hemoglobin per 100 ml. If the height of each bar (relative frequency) were multiplied by this .5, then the area of each bar would not be numerically equal to the relative frequency. However, this is an artificial problem which is related to the scaling of the horizontal axis. We can consider each hemoglobin value range to be one event wide. Now, if we multiply the height of each bar by its width in terms of events, then the height of each bar will again be numerically equal to the area. Consequently the total area will be 1, just like the total relative frequency. When we look at the histogram in this light, then it is similar to our coin-dumping example.

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PROBLEM SET 25:

The following table summarizes the outcome of dumping four coins 20 times. (This example appeared in the previous two problem sets.)

EVENT (Number of heads)	OBSERVED FREQUENCY	RELATIVE FREQUENCY
0	1	.05
1	6	.30
2	7	.35
3	4	.20
4	2	.10

1. Draw a frequency histogram of the above results with the vertical axis scaled in <u>observed frequency</u>.

2. Draw a relative frequency histogram of the above results.

In Problems 3-5, give the relative frequency of each combined event. Refer to the table above or to your histogram.

3. Zero heads or four heads.

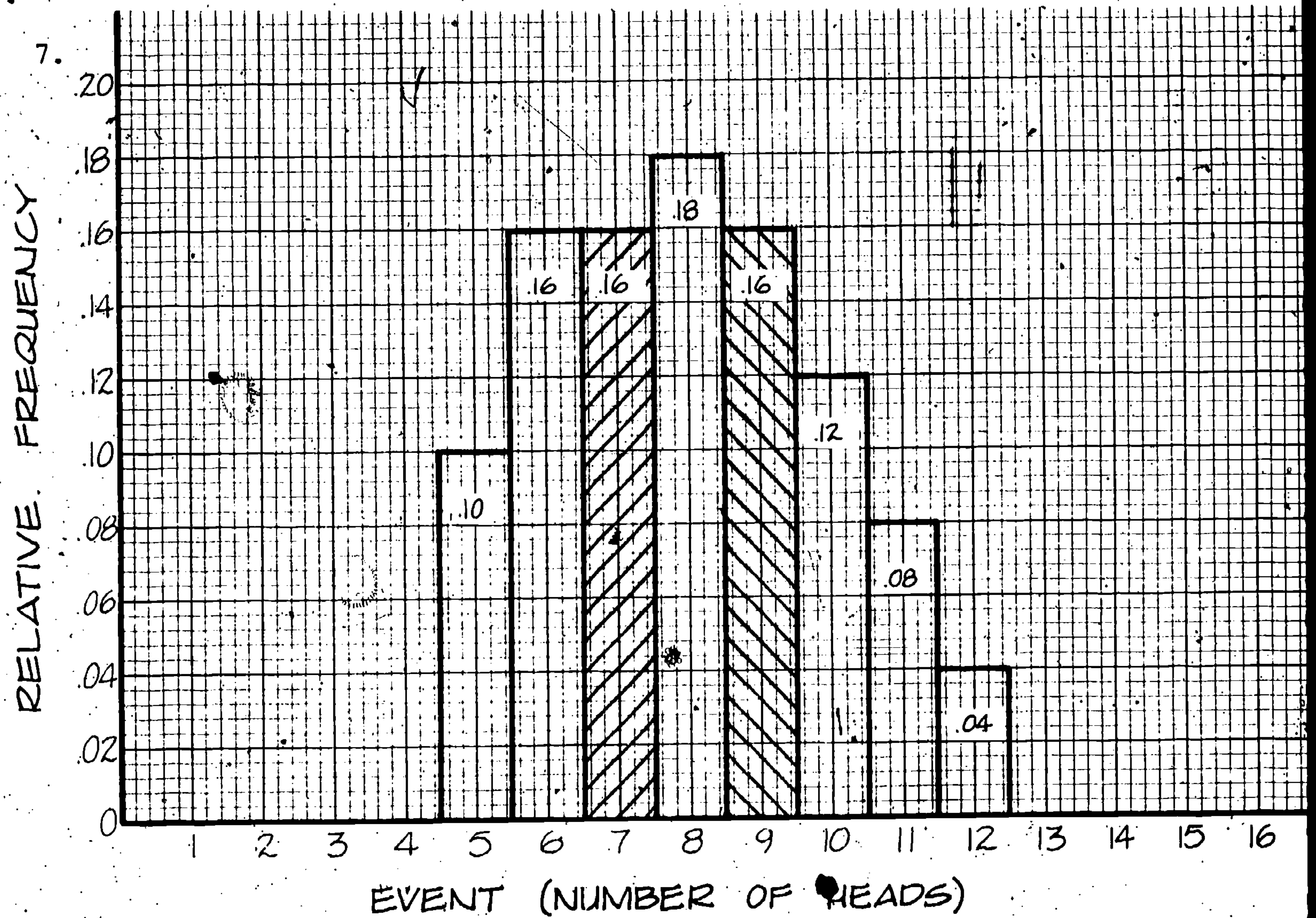
4. One head or two heads.

5. More than one head.

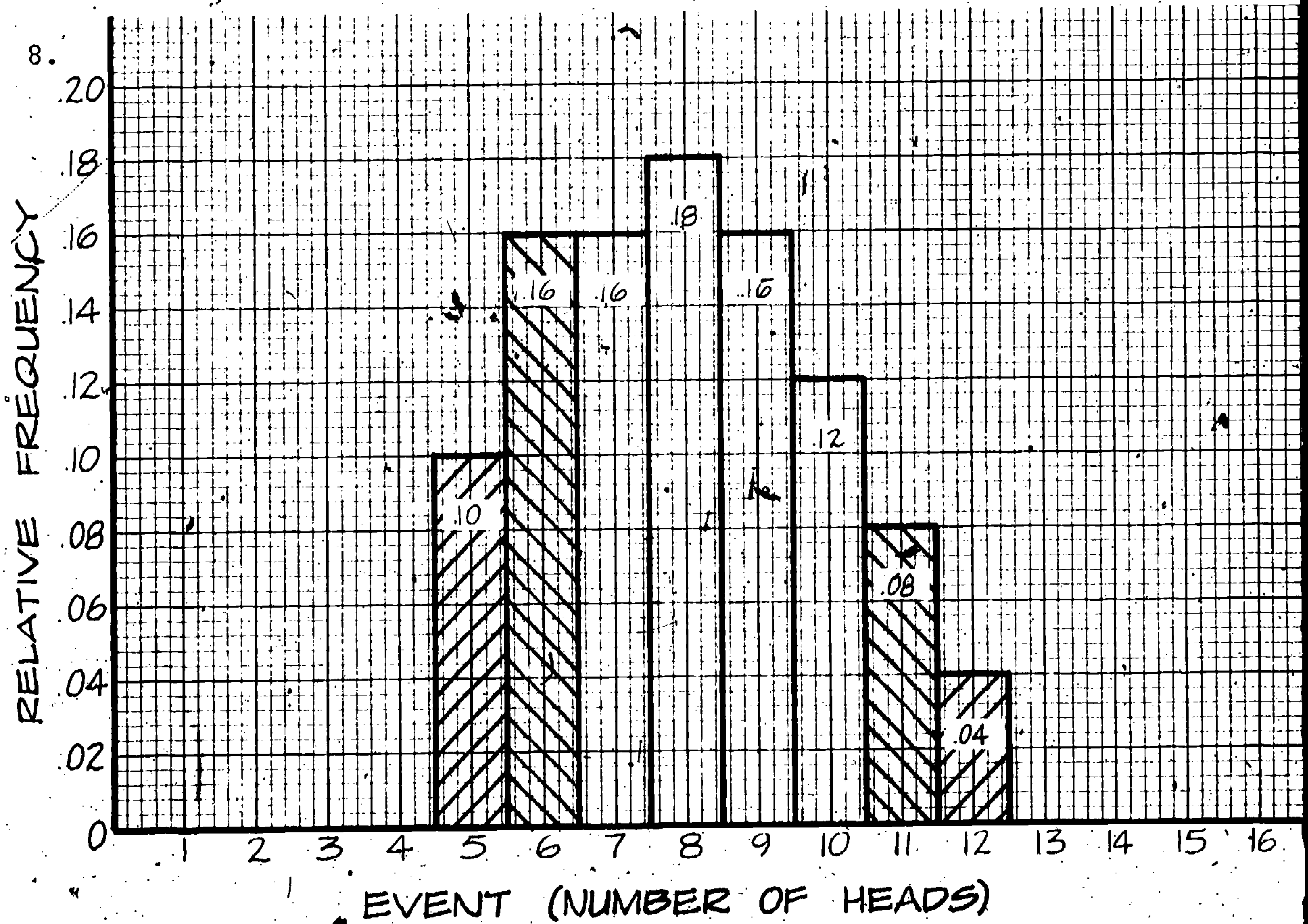
6. What two events may be combined to give a relative frequency of exactly 0.50?

The following relative frequency histograms are the same as in the text example. In each case, give the area of the shaded portion of the histogram.

7.

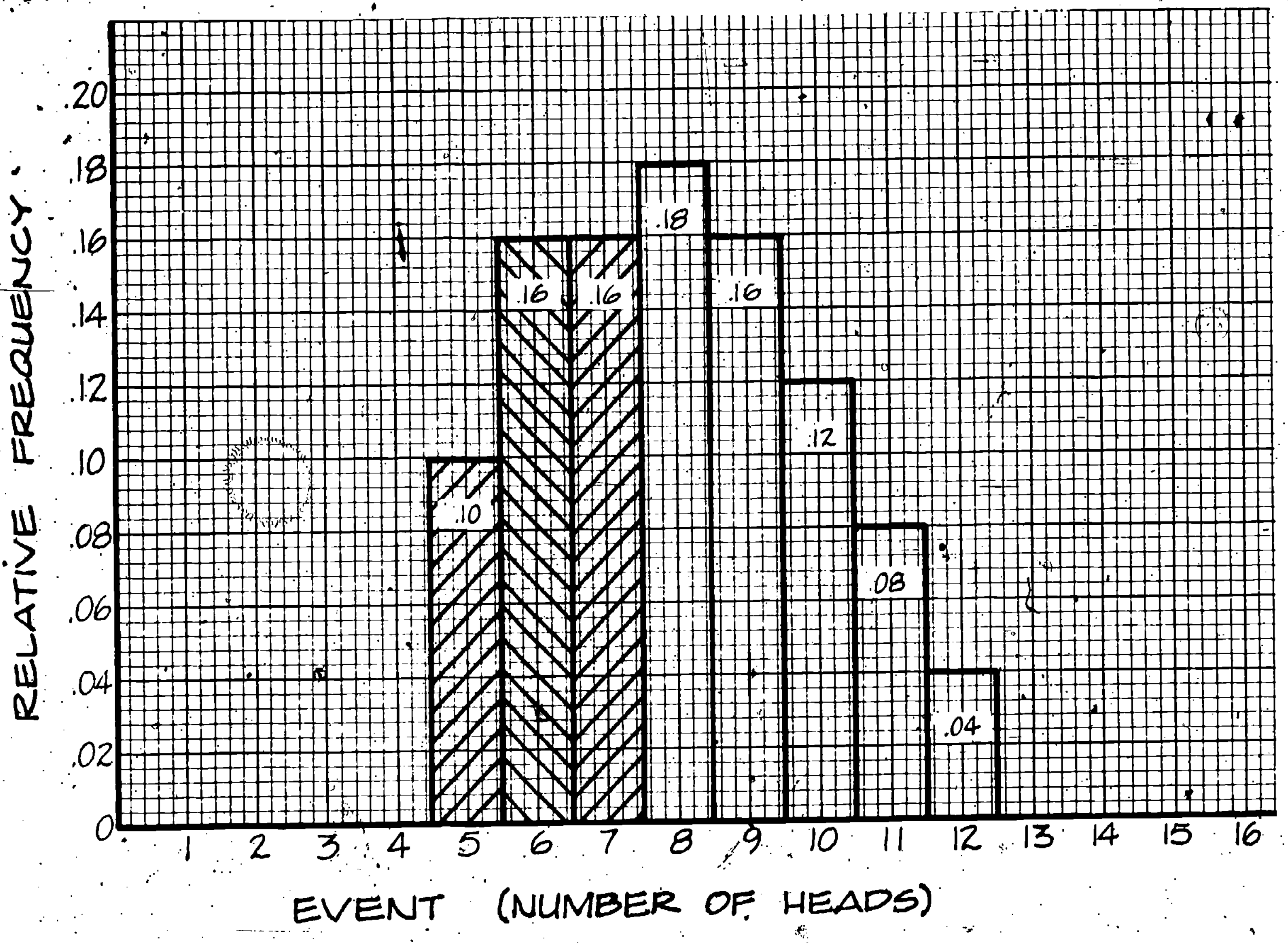

.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
0
RELATIVE FREQUENCY
.10
.16
.16
.18
.16
.12
.08
.04
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
EVENT (NUMBER OF HEADS)

8.


.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
0
RELATIVE FREQUENCY
.10
.16
.16
.18
.16
.12
.08
.04
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
EVENT (NUMBER OF HEADS)

19

9.



10. How far away from event 8 heads are the events which correspond to the shaded bars in Problem 7?

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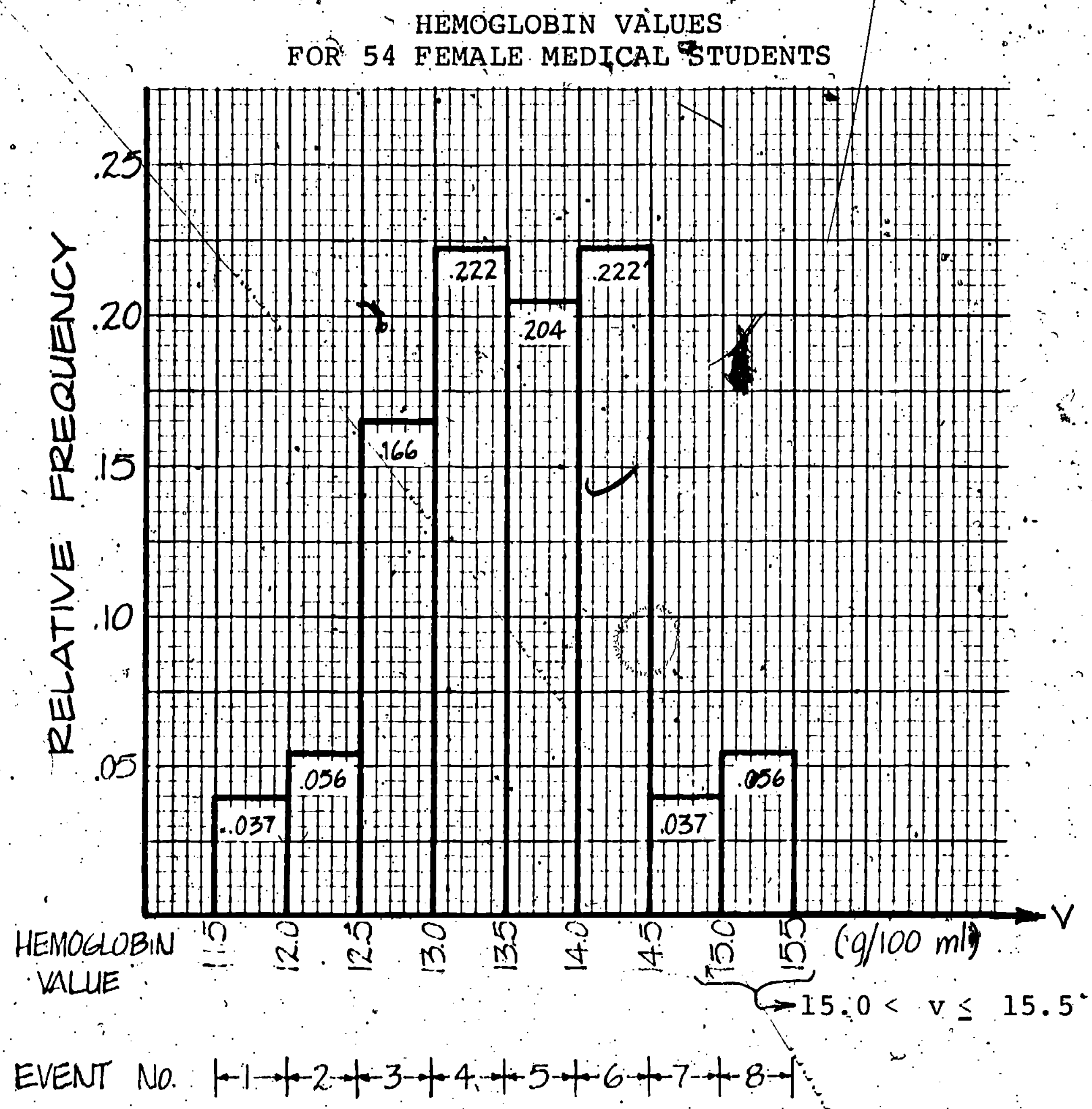
The relative frequency histogram below is the same as the table on page 194. The relative frequencies are written on the bars.



For Problems 11 through 16 determine the relative frequency of the combined events.

11. A hemoglobin value (v) in the range greater than 12.5 to 13.0 ($12.5 < v \leq 13.0$).

12. The combined events, 1, 2 and 3.

13. A value (v) in the range greater than 13.5 to 15.5 ($13.5 < v \leq 15.5$).

14. The combined events 1, 2, 7 and 8.

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15. A value (v) in the range greater than 12.5 to 14.0 ($12.5 < v \leq 14.0$).

16. What is the relative frequency of a value (v) in the range greater than 11.5 to 15.5 ($11.5 < v \leq 15.5$)?

17. Calculate the observed frequency of each event in the hemoglobin histogram. Keep in mind that each observed frequency must be a whole number.

18. The combination of events 4, 5 and 6 accounted for what fraction of all observed events?

19. Construct a relative frequency histogram for the total class data from the coin dumping activity of Section 23.

SECTION 26:

26-1 Yet Another Frequency Histogram

How often will the number of males in a randomly selected class deviate by 4 or more from the average number for all classes?

This is an example of the kind of question which we will be answering with the chi square procedure.

In this section we will develop a histogram that focuses on "deviations...from the average" and a table that focuses on the "... or more" phrase in the question above.

To develop these new tools we return (as usual) to the familiar coin-dumping data.

Recall from Section 24 that the expected frequency of heads in a toss of 16 coins is 8. In other words, 8 is the expected number of heads per trial. Generally, the farther from 8 heads the more rare the event. We can focus on this pattern by drawing a histogram based on deviation from the expected number of heads. Eight heads has a zero deviation from the expected value. Both 7 and 9 have a deviation of 1 from the expected value. The frequency histogram will picture the relative frequencies of particular deviations from the expected value. The information needed to construct such a histogram is the table below.

EVENT (number of heads)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
RELATIVE FREQUENCY	0	0	0	0	0	.10	.16	.16	.18	.16	.12	.08	.04	0	0	0	0

Your table will have the form shown below. Note that we have changed the event row to record deviations from the expected event (8 heads). For example, the relative frequency of a deviation of 1 is the sum of the relative frequencies of 7 and 9 heads, namely, .16 + .16, or .32.

EVENT (Deviation from expected value) Δh	0	1	2	3	4	5	6	7	8
RELATIVE FREQUENCY	.18	.32	.28	.18	.04	0	0	0	0

We are using Δh ("delta h") for deviation from the expected number of heads. Recall that deltas are supposed to remind you of differences since both delta and difference start with the letter "d". Δh will always be the positive value of the difference between a given event and the expected event.

EXAMPLE:

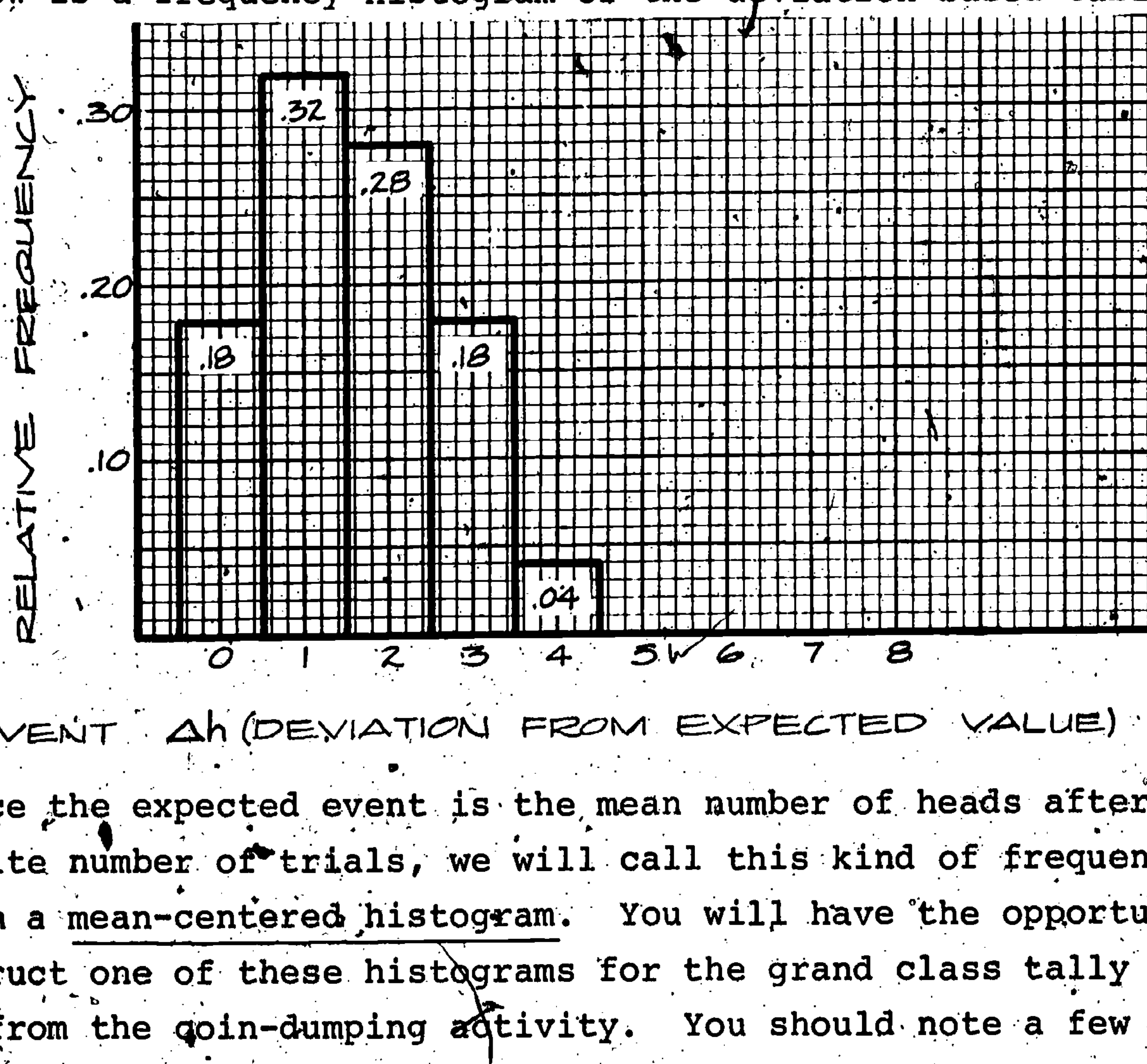
Expected event = 13 heads
Given event = 20 heads
Δh = ?

SOLUTION:

Δh = 20 - 13
= 7

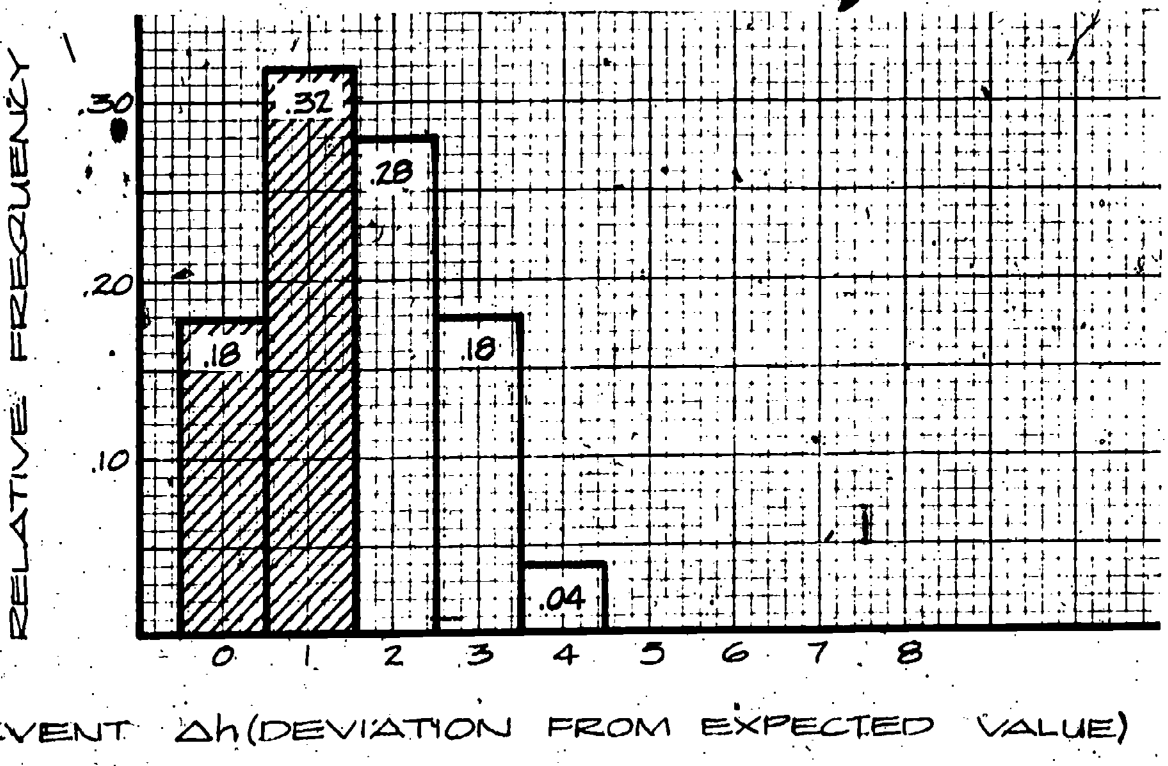
In other words, the event 20 heads is 7 events away from the event 13 heads.

Below is a frequency histogram of the deviation based table.



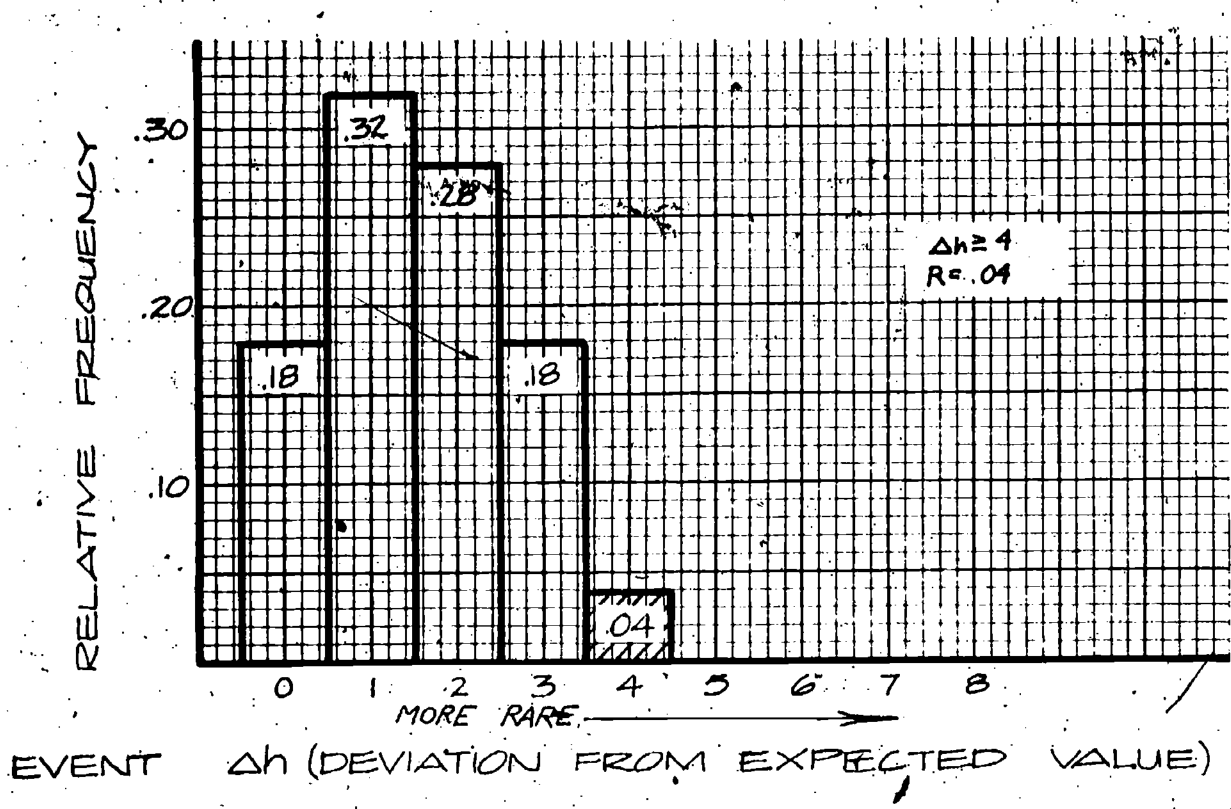
Since the expected event is the mean number of heads after an infinite number of trials, we will call this kind of frequency histogram a mean-centered histogram. You will have the opportunity to construct one of these histograms for the grand class tally resulting from the coin-dumping activity. You should note a few features of this particular frequency histogram. First of all, movement to the right is movement away from the expected value. Second, the highest relative frequency is not at zero or the expected value. It is just to the right of it. From there on, the farther away from the expected value, the shorter the bars.

This kind of frequency histogram has the same relation between relative frequency and area as our earlier histograms. For example, the area of the zero bar is (.18) x (1) = (.18). Area is equivalent to relative frequency. Look at the graph on the next page. The first two bars have been shaded in. The shaded region represents a combined relative frequency of .18 + .32 = .50 for how often 7, 8 or 9 heads were seen.



26-2 Rare Events and the Phrase "... Or More"

We mentioned earlier our desire to measure the rarity of random events. This kind of histogram makes our task easier. Events farther to the right on the horizontal scale were seen less often. They are farther away from the "expected" event. We can use this histogram to answer such questions as, "How often did the number of heads deviate by 4 or more from the expected value?". The shaded region in the graph on the following page represents the answer.



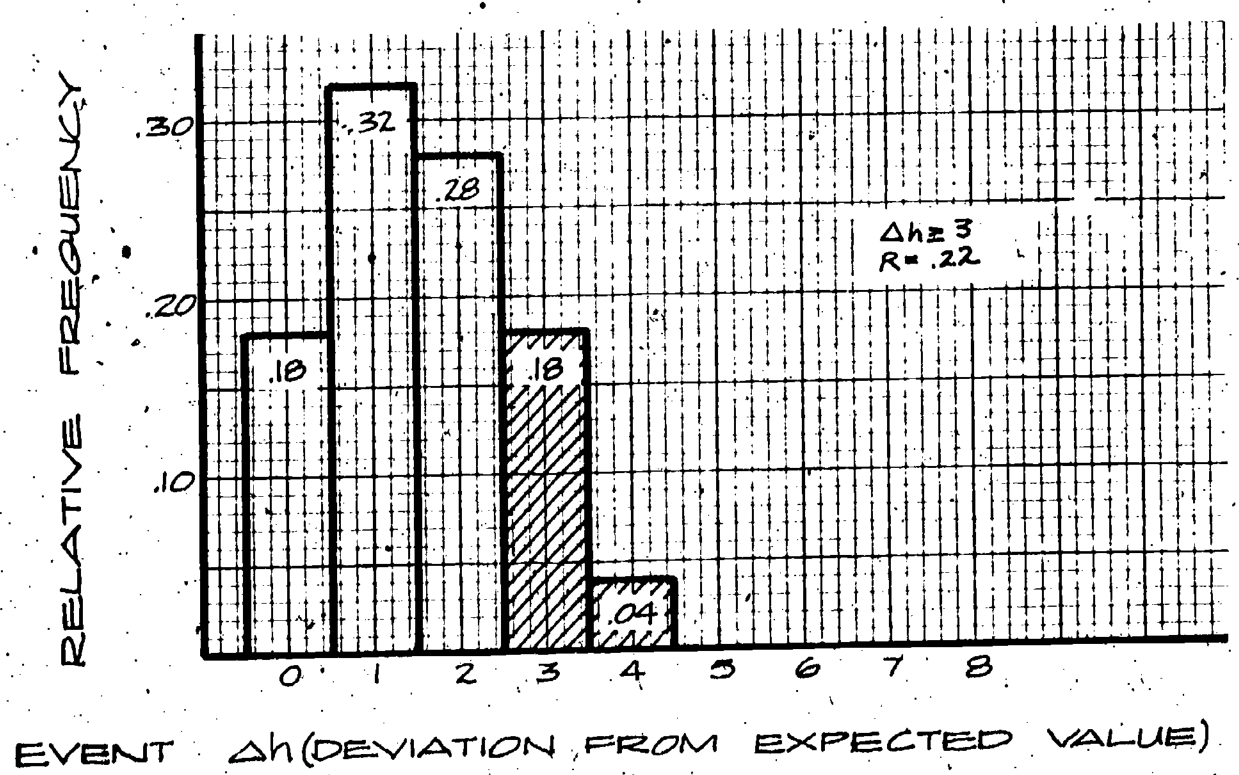
In other words, a deviation of 4 or more from the expected value occurred in only 4% of the trials.

This kind of question is very similar to the one we opened the section with. But for a few phrases they are identical.

How often did the number of heads deviate by 4 or more from the expected value?

Now often will the number of males...deviate by 4 or more from the average?

The graph below represents the answer to another question about the rarity of a certain set of events. Can you give the question?



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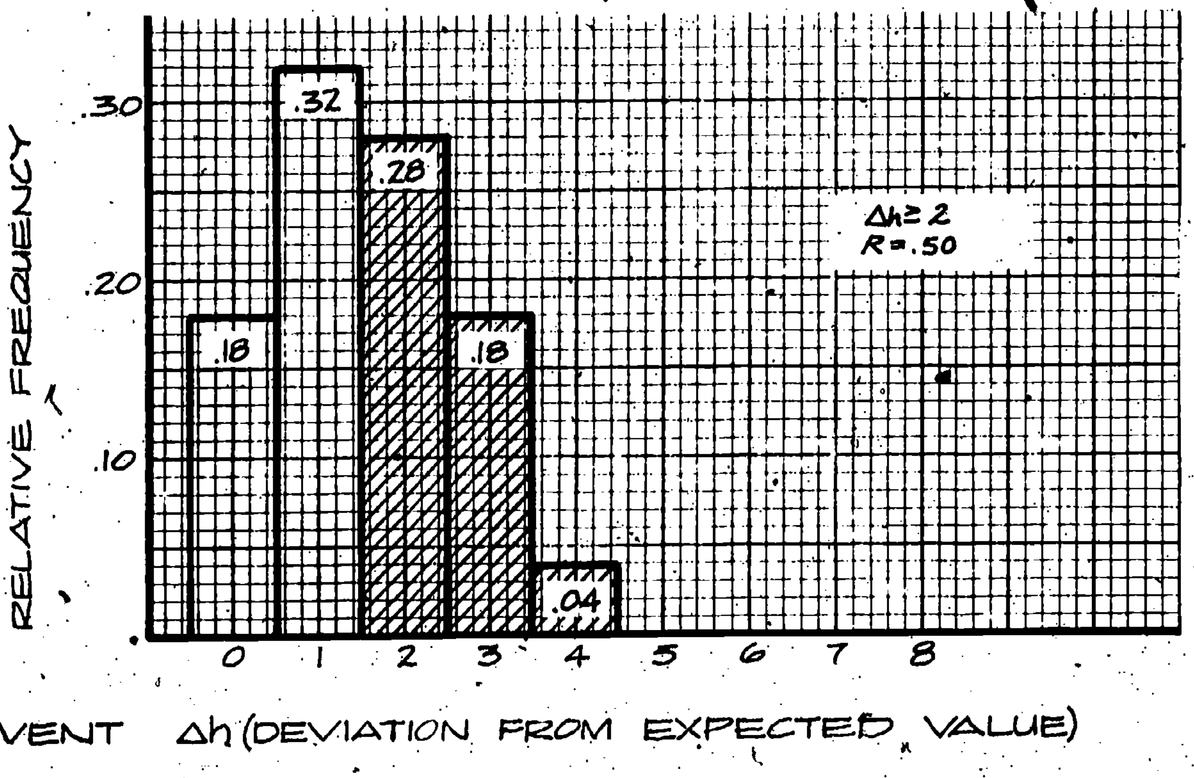
The question is, "How often did the number of heads deviate by 3 or more from the expected value?" The area of the shaded region is (.18) + (.04) = .22.

The shaded area below represents the combined realtive frequency of a deviation of 2 or more.



Look back now at the last three graphs. Notice that we have moved from right to left, shading in an additional bar in each step. In the next section we will demonstrate how to construct a table which records the combinated relative frequencies which correspond to the process we have visually produced here.

26-3 A Table of Combined Events

A table that shows the relative frequencies of different combinations of events will have this form.

TABLE OF COMBINED EVENTS

EVENT (Deviation from expected value) Δh	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more	6 or more	7 or more	8 or more
RELATIVE FREQUENCY	?	?	?	?	?	?	?	?	?

Notice that the events in the table all have an "or more" tag. We can find the relative frequencies for these combined events by following the reasoning of the previous discussion and using the information in the table below. This is the same table that we developed in Section 26-1. Just below is the table of combined events.

TABLE 1

EVENT (Deviation from expected value) $\Delta h =$	0	1	2	3	4	5	6	7	8
RELATIVE FREQUENCY	.18	.32	.28	.18	.04	0	0	0	0

TABLE 2
COMBINED EVENTS

EVENT (Deviation from expected value) $\Delta h \geq$	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more	6 or more	7 or more	8 or more
RELATIVE FREQUENCY	1.00	.82	.50	.22	.04	0	0	0	0

Table 2 may be most efficiently developed by starting from the right and working left, zig-zagging between tables. For example the relative frequency for deviations of 3 or more may be found by adding R for 4 or more (.04, Table 2) to the R for a deviation of exactly 3 (.18, Table 1) to get R = .22 (Table 2) for deviations of 3 or more.

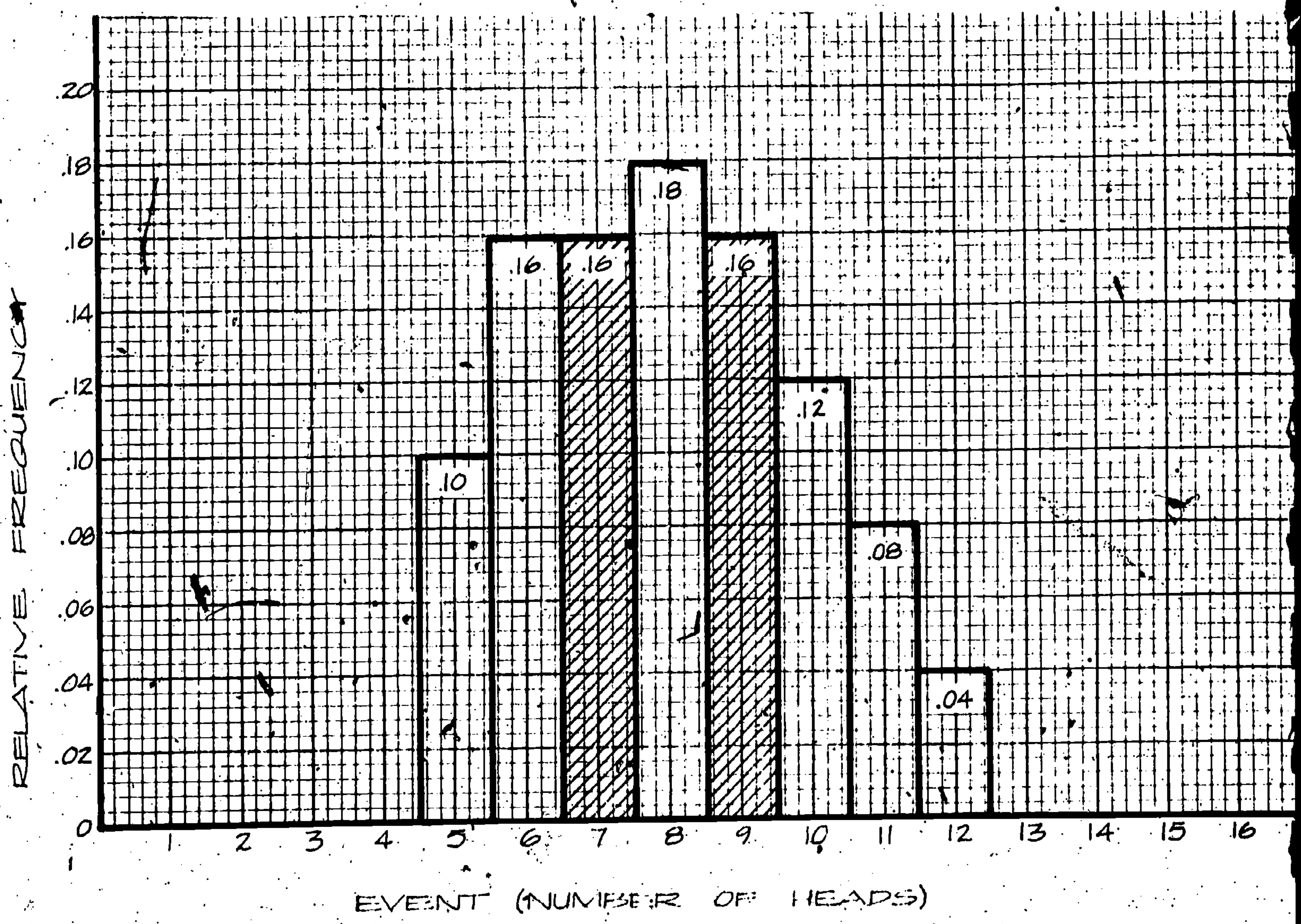
Now it is time to take a break from calculations and look at the table we have constructed. Remember that a relative frequency of 1 means that an event always happened. We see by looking at the Table of Combined Events that 0 or more heads happened all the time. This is very reasonable. From a mathematical point of view, we also know that we got the relative frequency for $\Delta h \geq 0$ by adding up all of the relative frequencies. And, we know that the sum of all of the relative frequencies must be 1.

Now notice that the relative frequencies decrease from 1 to 0 as you move from left to right (→) in the table. In other words, if an event is to the right of another event, the right-most event is more rare.

Finally, remember that the relative frequency which is entered in the combined event table represents an area. It is the combined area of the histogram bars for Δh greater than or equal to the stated Δh.

PROBLEM SET 26:

The following relative frequency histograms are the same as the one developed in Section 25 based on the tossing of 16 coins.

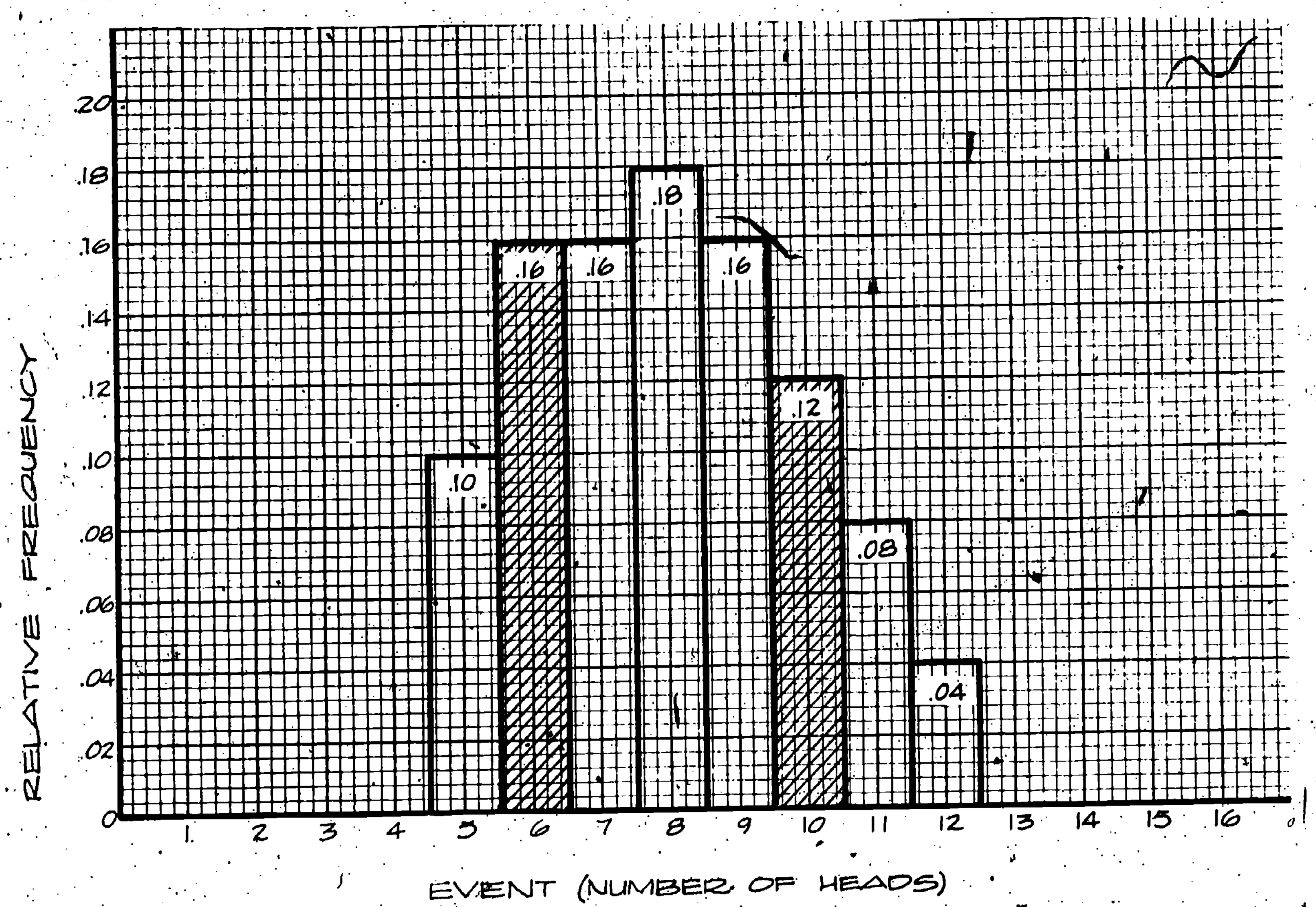


1. The shaded area in the above histogram represents the relative frequency of the combined event, "____ heads or ____ heads."

2. Therefore, the shaded area represents the relative frequency of a deviation (Δh) of ____ from the expected value of 8.

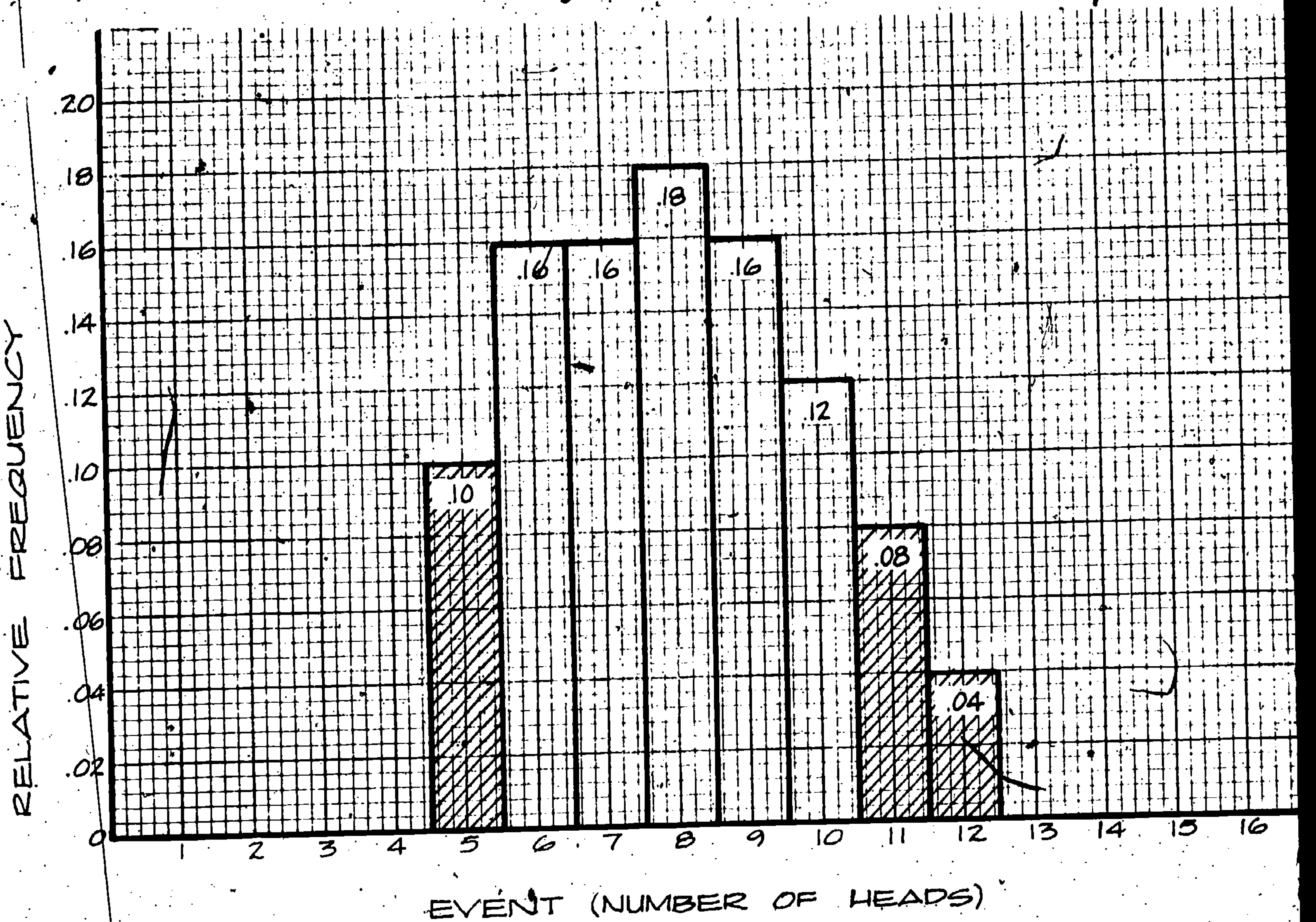
3. Therefore, the relative frequency of a deviation of 1 from the expected value is ____.

20?



4. The shaded area in the above histogram represents the relative frequency of a deviation of ____ from the expected value.

5. The relative frequency of this deviation from the expected value is ____.



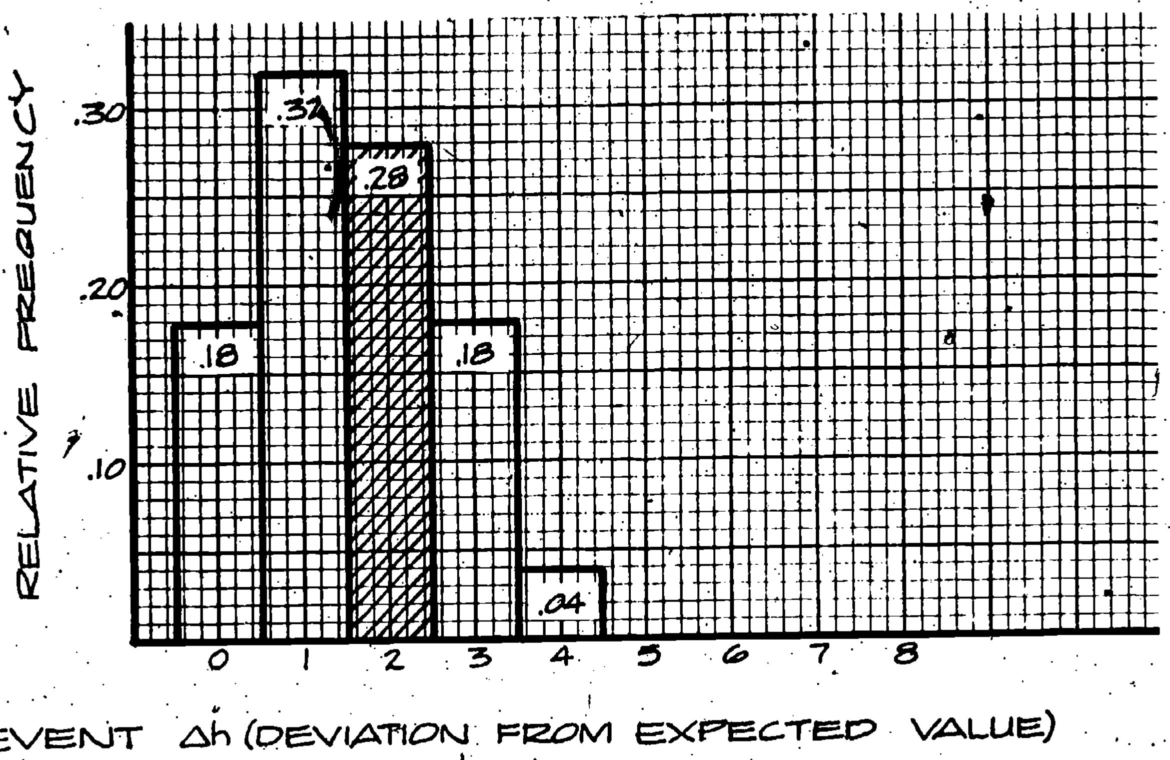
6. Give the letter of the correct completion of the sentence: "The shaded area in the histogram above represents the relative frequency of:

a. a deviation of 2 from the expected value."

b. a deviation of 3 or more from the expected value."

c. a deviation of 3 from the expected value."

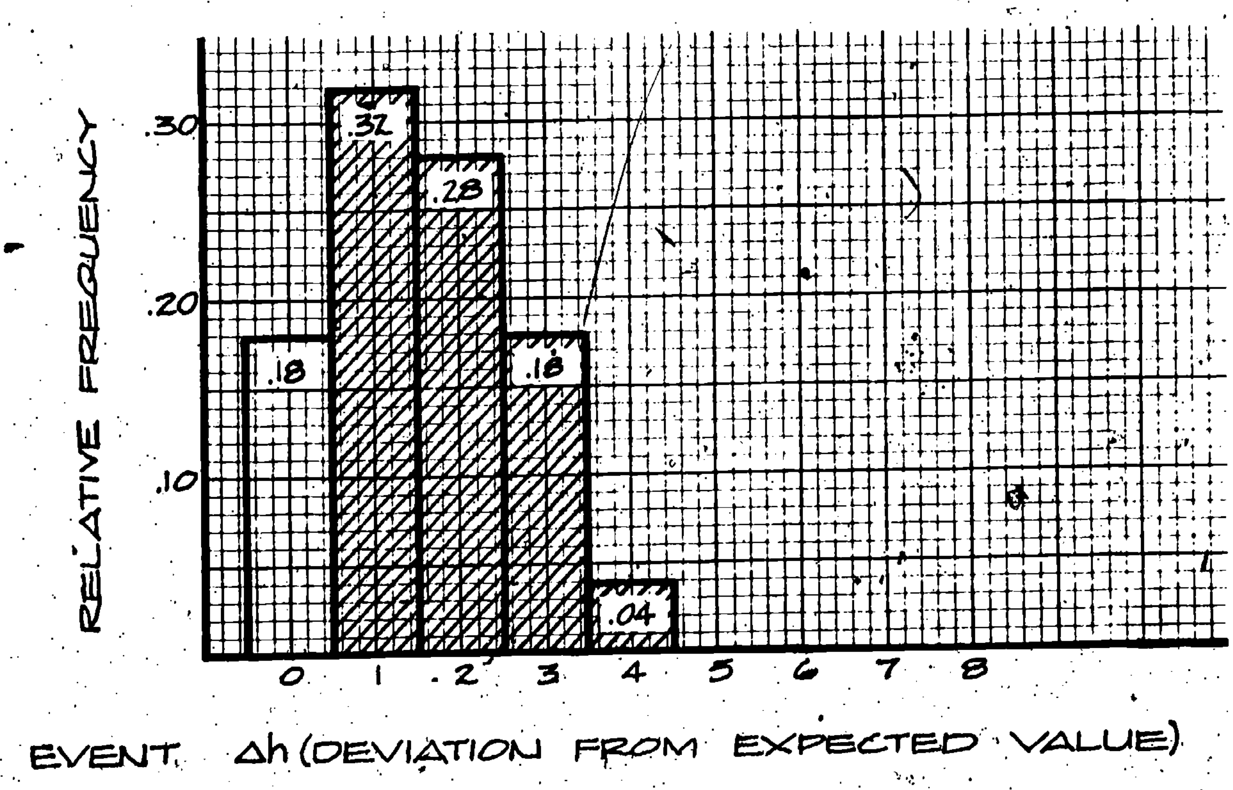
d. a deviation of 4 or more from the expected value."



The above mean-centered histogram is the same as the one developed in Section 26. It is another way of displaying the results of tossing 16 coins.

7. The shaded area represents the relative frequency of a deviation of ____ from the expected value.

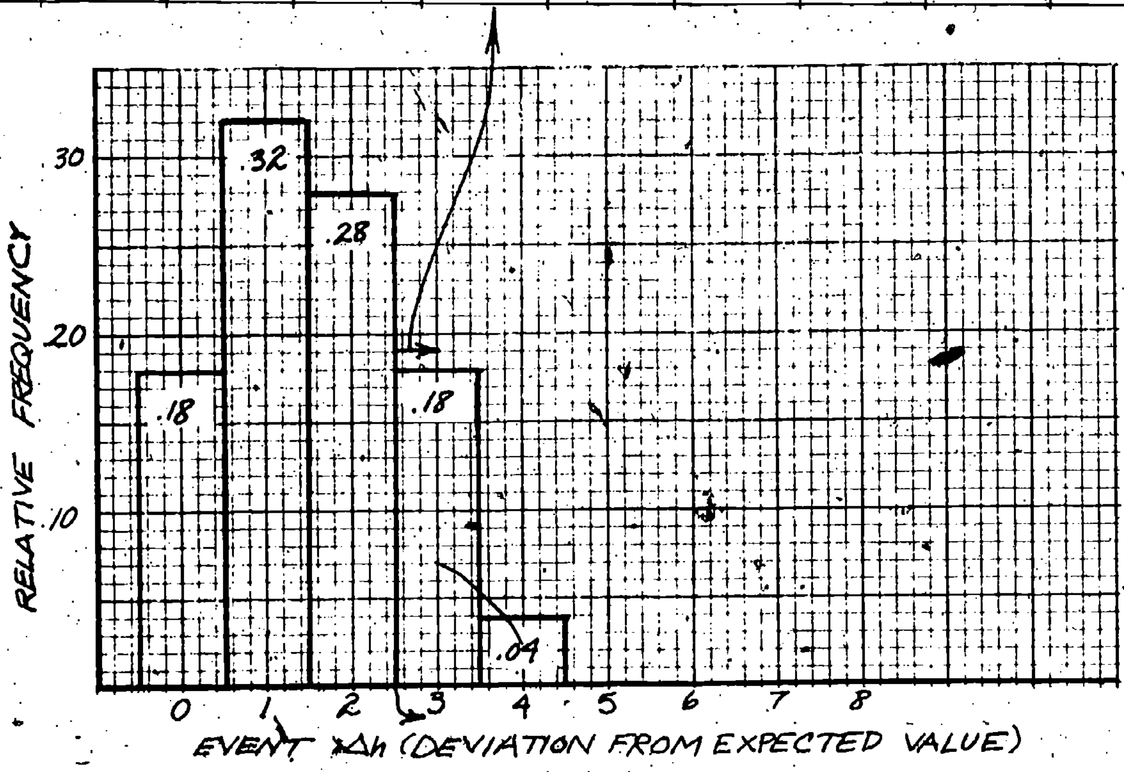
8. Does the relative frequency .28 agree with your answer to Problem 5?



9. The shaded area above represents the relative frequency of a deviation of ____ or more from the expected value.

10. Here we have copied the Table of Combined Events and the Mean-Centered Histogram developed in this section. We have drawn in lines to show how a particular table entry corresponds to the area of a combination of histogram bars.

Δh	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more	6 or more	7 or more	8 or more
R	1.00	.82	.50	.22	.04	0	0	0	0



a. Copy both the table and the graph on your own paper.

b. Draw in lines similar to the ones above for each relative frequency in the R row of the table.

The following table summarizes the outcome of tossing 4 coins 20 times (you have seen this example in previous problem sets).

EVENT (Number of Heads)	RELATIVE FREQUENCY
0	.05
1	.30
2	.35
3	.20
4	.10

11. The expected value for the table above is 2 heads. The relative frequency of a deviation of 1 from the expected value is ____.

12. Use the table above to complete the following table.

EVENT (Deviation from expected value)	Δh	0	1	2
RELATIVE FREQUENCY				

13. Construct a mean-centered histogram based on the table of Problem 11.

14. Using the data of Problem 11, complete the following table. Notice the addition of the "or more" phrase to the column headings.

EVENT (Deviation from expected value)	Δh	0 or more	1 or more	2 or more
RELATIVE FREQUENCY				

15. A deviation of ________ or more was observed in 65% of the trials.

16. 14 coins are repeatedly dumped.

a. What is the expected number of heads for a single trial?

b. List all events for which $\Delta h = 5$.

c. When $h = 9$, $\Delta h = ?$

d. When $h = 5$, $\Delta h = ?$

e. List all events for which $\Delta h \geq 5$.

f. List all events for which $\Delta h \geq 3$.

g. List all events for which $\Delta h \leq 3$
(Remember that 0 is less than 3.)

17. 12 coins are repeatedly dumped.

a. What is the expected number of heads for a single trial?

b. List all events for which $\Delta h = 0$.

c. List all events for which $\Delta h = 1$.

d. List all events for which $\Delta h \leq 1$.

e. List all events for which $\Delta h = 5$.

f. List all events for which $\Delta h \geq 2$.

g. List all events for which $\Delta h \leq 2$.

18. Calculate the observed (O) frequency of each combined event in the table of combined events in Problem 10.

N = 50.

19. Construct a mean centered histogram for the total class data for the coin dumping activity of Section 23.

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SECTION 27:

27-1 Frequency Distributions from a Theoretical Point of View

So far we have been concerned only with the description of frequency distributions that we have obtained experimentally. You now have quite a good idea of how 16 coins behave when they are tossed a large number of times. But this information did not come particularly easily. Tossing coins and tabulating results took quite a bit of time. And to make things more complicated, maybe some of the coins you used were lopsided, and maybe you didn't throw them in a random way. Maybe the frequency distribution you developed in class is a "freak" and doesn't look the way it should.

Because of the above considerations it is very useful to be able to predict the appearance of a given frequency distribution. Since a frequency distribution is based on random events, we can never predict it exactly, but mathematicians have developed methods for producing an "ideal" model. Their predictions for 16 coins are summarized in the table that follows. Remember that a probability is a theoretical relative frequency. It is related to expected frequency and the number of trials by the equation

$$p = \frac{E}{N}$$

where p = probability

E = expected frequency

N = number of trials.

NUMBER OF HEADS	APPROXIMATE PROBABILITY	RELATIVE FREQUENCY	NUMBER OF HEADS	APPROXIMATE PROBABILITY	RELATIVE FREQUENCY
0	.000015	0	9	.174561	.16
1	.000244	0	10	.122192	.12
2	.001831	0	11	.066650	.08
3	.008545	0	12	.027771	.04
4	.027771	0	13	.008545	0
5	.066650	.10	14	.001831	0
6	.122192	.16	15	.000244	0
7	.174561	.16	16	.000015	0
8	.196381	.18			

In acquainting yourself with the information in the table, notice that there is rough agreement between the theoretical relative frequencies (probabilities) and the observed relative frequencies. A statistician would tell us that the larger the number of trials, the more closely we should expect the two to agree. Notice also that some of the probabilities are very small. Perhaps it is a little difficult to appreciate what they mean. We can illustrate their meaning in a little more depth by using the formula

$$p = \frac{E}{N}$$

EXAMPLE:

How often do we expect to see 14 heads out of 16 coins in 50 trials?

SOLUTION:

By referring to the table we find that the probability of 14 heads is $\approx .001831$; therefore, $p \approx .001831$. The problem says that $N = 50$. We substitute this information into the equation

$$p = \frac{E}{N}$$

to get

$$.001831 \approx \frac{E}{50}$$

$$.09 \approx E.$$

So we see that the expected frequency of 14 heads is much much less than 1. This means that we would very rarely see 14 heads in 50 dumps of 16 coins.

Since we found the expected frequency for 14 heads in 50 dumps to be much less than 1, it would be interesting to find out how many trials it would take to expect to see 14 heads once.

EXAMPLE:

How many trials are necessary to have an expected frequency of 1 for 14 heads out of 16 coins?

SOLUTION:

We know that

$$p \approx .001831$$

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In the problem it states that

$$E = 1$$

The problem is to find N. We substitute our known information into the formula

$$p = \frac{E}{N}$$

to get

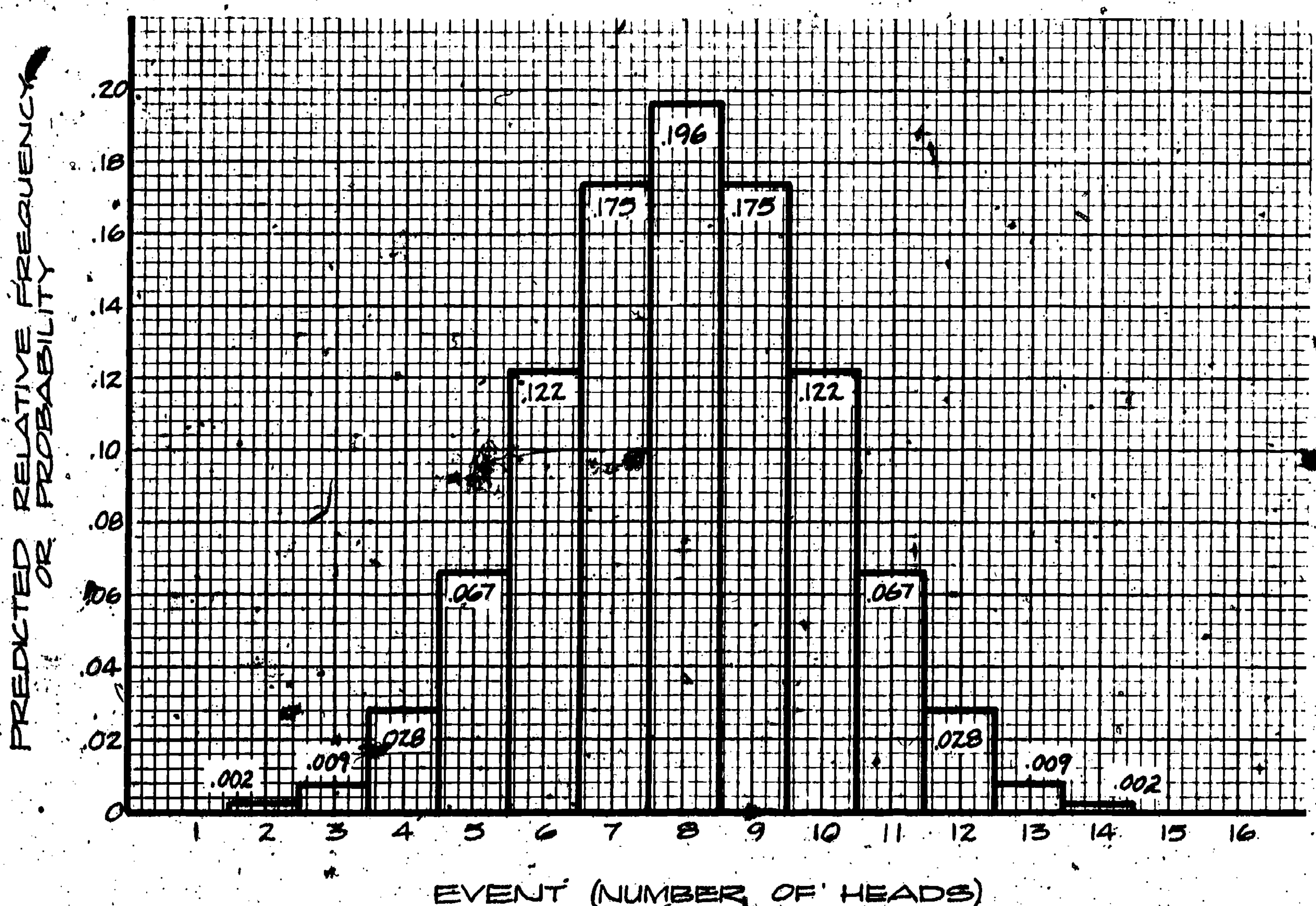
$$.001831 \approx \frac{1}{N}$$

$$N \approx \frac{1}{.001831}$$

$$N \approx 546$$

In other words, we expect to see 14 heads out of 16 coins about once out of every 546 trials.

Below we have made a histogram that represents the probabilities for 16 coins.



Again, it is important to notice some features of the above graph. First, it is symmetrical to the left and right around 8 heads. Second, it has a shape similar to a bell. In statistics there are a variety of bell-shaped histograms. We will not now pursue the mathematics involved in the derivation of these probabilities. Later, when we deal with the subject of genetics, we will explore this topic in greater detail. For our present purposes we will assume that the mathematical derivations are accurate and merely use the results.

Interestingly, the same histogram describes rather well our expectations about the number of males (or females) in a family of 16 children. For example, it predicts the .122 (roughly one eighth) of all 16-children families would have exactly 10 male children.

27-2 A Theoretical Mean-Centered Histogram

Just as in Section 26, we are interested in a histogram that shows the relative frequencies of the different deviations from the expected value (Δh's). The table is constructed just as before.

EVENT (Deviation from expected value) or Δh	Predicted Relative Frequency or PROBABILITY
0	.19638
1	.34912
2	.24438
3	.13330
4	.05554
5	.01709
6	.00366
7	.00049
8	.00003

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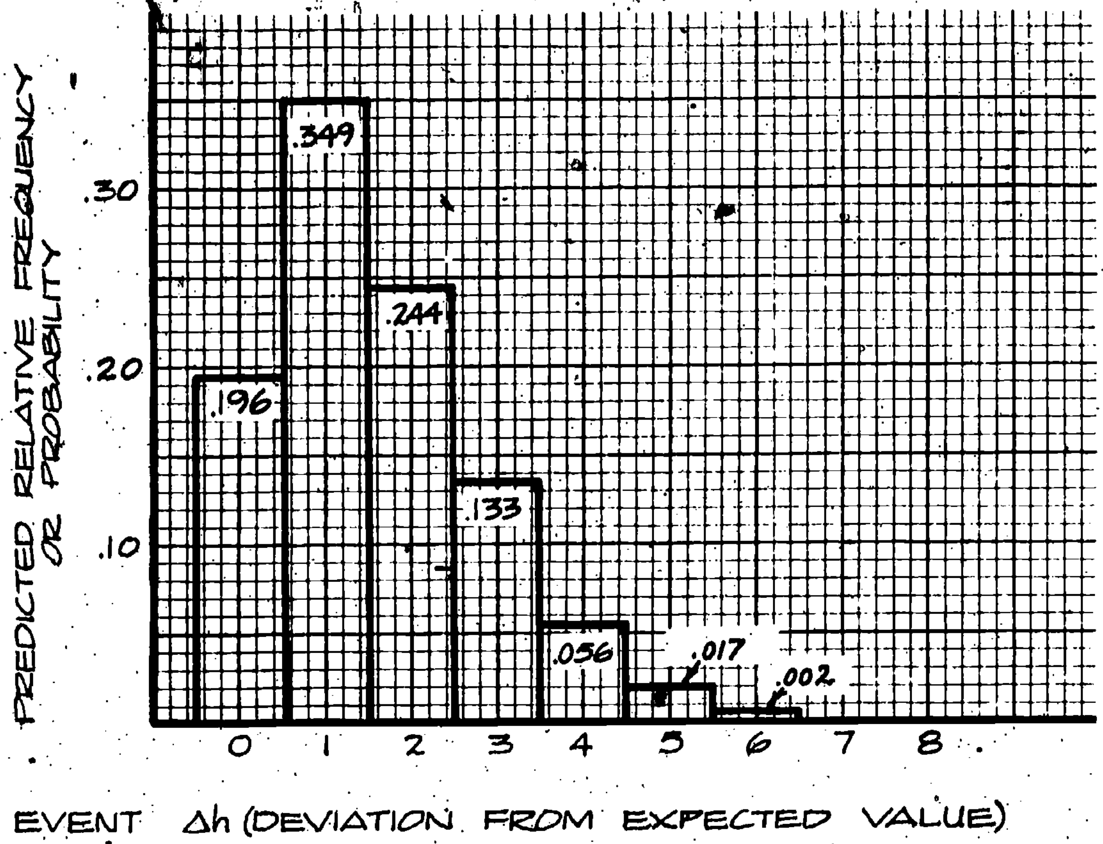
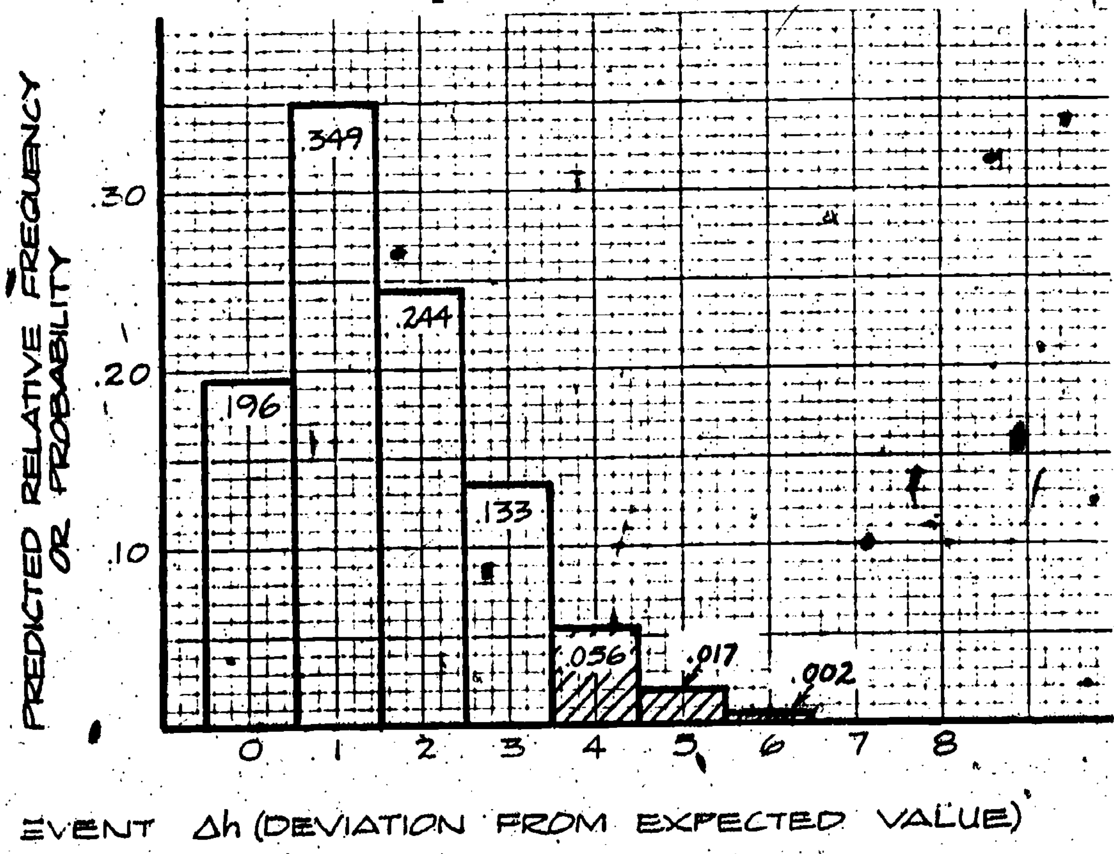
A histogram of this theoretical frequency distribution appears below.



Just as before, there is a relation between the area of histograms and the theoretical relative frequencies. The shaded region in the histogram below represents the probability that the number of heads will deviate from 8 by 4 or more.

In the context of a 16-child family it predicts that only .00366 (about (1/275)th) of such families will the number of males or female children be exactly 14 (Δh = 6).



The table below shows the combined probability of different deviations from the mean. Notice that events become rarer as you move from left to right.

EVENT (Deviation from expected value) Δh	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more	6 or more	7 or more	8 or more
COMBINED PROBABILITY	1.00	.804	.454	.210	.077	.021	.004	5×10^{-4}	3×10^{-5}

Remember that the above table is based on theoretical predictions. It tells what relative frequencies we expect to find after an infinite number of trials. It is illuminating to compare the above theoretical table with the one we got from our run of 50 trials. This table follows.

EVENT (Deviation from expected value) Δh	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more	6 or more	7 or more	8 or more
COMBINED RELATIVE FREQUENCY	1.00	.82	.50	.22	.04	0	0	0	0

PROBLEM SET 27:

1. List two reasons why it is useful to be able to predict the appearance of coin-produced frequency histograms.

2. How does a probability differ from a relative frequency?

3. a. Complete the table below. All entries in the third column should be positive.

Event (number of heads)	P	R	(p - R)
5	.07	.10	.03
6	.12	.16	?
7	.17	.16	?
8	.20	.18	?
9	.17	.16	?
10	.12	.12	?
11	.07	.08	?
12	.03	.04	?

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b. Construct a table similar to the one on the preceding page which compares probabilities with the observed relative frequencies of the total class coin-dumping data. Include all events with non-zero observed relative frequencies.

In Problems 4 through 11 refer to the table on Page 209 of the text.

4. The event which is expected in about 19.6% of the trials is _______ heads.

5. Thirteen heads and ______ heads are expected to occur with equal frequency.

6. In about 2.78% of the trials we expect the event

a. 1 head

b. 8 heads

c. 12 heads

7. If the number of trials were 1000 then the expected frequency of 7 heads would be ______.

8. If the number of trials were 100,000 then the expected frequency of 15 heads would be _______.

9. How many trials are necessary to have an E of 1 for 16 heads? (E = expected frequency)

10. How many trials are necessary to have an E of 1 for 8 heads? (Use p ≈ .20)

11. a. How many trials are necessary to have an E of 1 for 15 heads? (Use p ≈ .00025)

b. How many trials are necessary to have an E of 1 for 1 head?. (Hint: No calculation is necessary).

12. List two important features of the histogram in Section 23-1 of the text.

In Problems 13 through 15 refer to the table on Page 212 of the text.

13. The probability of getting either 10 heads or ____heads is about .244.

14. A deviation of 7 from the expected value is expected to occur about 49 times in every _______trials.

15. The rarest event is a deviation of ______ from the expected value.

16. Construct a mean-centered histogram for the theoretical frequency distribution described in the table below. It is the distribution for repeated dumpings of 8 fair coins.

EVENT (number of heads)	P
0	.004
1	.031
2	.109
3	.219
4	.274
5	.219
6	.109
7	.031
8	.004

17. a. Construct a table of combined probabilities for the frequency distribution in Problem 16.

b. Let N = 50. Calculate an E for each p of part a.

18. Construct a mean-centered histogram for the theoretical frequency distribution described in the table below. It is the distribution for repeated dumpings of 6 coins.

EVENT (number of heads)	P
0	.016
1	.094
2	.234
3	.312
4	.234
5	.094
6	.016

19. a. Construct a table of combined probabilities for the frequency distribution in Problem 18.

b. Let N = 50. Calculate E for each p of part a.

The following table shows the predicted and observed relative frequencies of the various deviations from the expected value for repeated dumpings of 16 coins. (See p.214 of the text.)

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EVENT (Deviations from expected value)	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more	6 or more	7 or more	8 or more
COMBINED PROBABILITY	1.00	.804	.454	.210	.077	.021	.004	$5x10^{-4}$	$3x10^{-5}$
COMBINED RELATIVE FREQUENCY FOR SAMPLE DATA	1.00	.82	.50	.22	.04	0	0	0	0

20. Both the combined probability and the observed relative frequency are the same in the "0 or more" column. Why must they agree?

21. Look at the last four columns of the table. The probabilities are not 0; but the observed relative frequencies are all 0. How can this discrepancy be explained?

22. Copy and complete this table. Use the relative frequencies from your class's coin-dumping activity.

EVENT (Deviations from expected value)	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more	6 or more	7 or more	8 or more
COMBINED PROBABILITY	1.00	.804	.454	.210	.077	.021	.004	$5x10^{-4}$	$3x10^{-5}$
COMBINED RELATIVE FREQUENCY									

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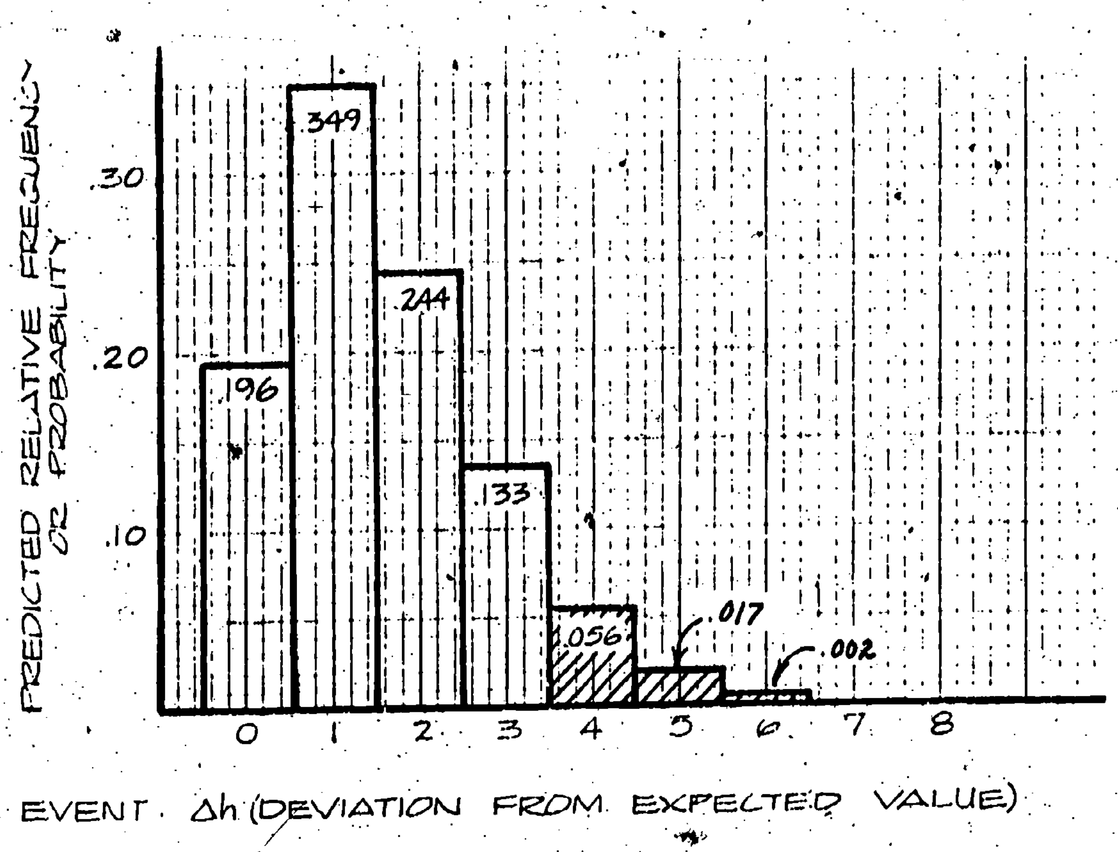
SECTION 28:

28-1 A Quick Review

How often do we expect the number of heads to differ from the expected outcome by 4 or more when 16 coins are tossed? In Section 16 we found that this question could be answered by combining the probabilities for Δh = 4, 5, 6, 7 and 8. Graphically this is represented by the shaded area in the histogram below.



The area for the entire shaded region is about .08. Therefore we expect a deviation of 4 or more in about 8% of all cases.

28-2 The Chi-Square Procedure

The question posed in the last paragraph was easy to answer using the mean-centered histogram which we have developed. However, it is not very efficient to construct a table or histogram each time we run into such a question. It is natural to look for a shortcut.

Fortunately there is a formula which we can use to find the approximate area of part of a histogram. Instead of drawing the histogram and adding the areas of the bars, we substitute into the formula and then look up the area in a table. In order to illustrate this process we will look at a specific example.

Suppose you toss 16 coins and you get 12 heads and 4 tails. You can display the event in a table with two boxes.

HEADS	12
TAILS	4

The observed numbers are 12 heads and 4 tails. What are the expected numbers of heads and tails? By now you can probably answer this question easily. We expect 8 heads and 8 tails. The expected values are entered in the table as follows.

HEADS	12 (8)
TAILS	4 (8)

A table of this form is called a contingency table. It is a convenient way of displaying observed and expected frequencies. Notice that each observed value deviates by 4 from the expected value.

Since you will be working with contingency tables, remember which frequencies go inside the parentheses and which do not. One way to remember is to think of the frequencies <u>without</u> parentheses as representing actual experience; these are the <u>observed</u> frequencies. We can think of the parentheses as indicating that a frequency did not actually occur, in other words, an <u>expected</u> frequency.

We are now going to answer the following question. "With what relative frequency do we expect a deviation as great or greater than that in the table above?" This is the same as asking how often we expect to observe a deviation of 4 or more.

The first thing we do is calculate the quantity

$$\frac{(|O - E| - \frac{1}{2})^2}{E}$$

for each of the two "boxes" of the contingency table. Here

O = observed frequency
E = expected frequency

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The expression $|0 - E|$ means "absolute value". Absolute values are always possible. In other words,

$$|12 - 8| = |4 - 8|$$
$$4 = 4$$

Substituting the values from the first box, we get

$$\frac{(|12 - 8| - \frac{1}{2})^2}{8} = \frac{(4 - \frac{1}{2})^2}{8}$$
$$= \frac{12.25}{8}$$
$$\approx 1.53$$

Substituting the values from the second box, we get

$$\frac{(|4 - 8| - \frac{1}{2})^2}{8} = \frac{(4 - \frac{1}{2})^2}{8}$$
$$\approx 1.53$$

You have probably noticed that this result is the same as for the first box. This happens whenever $|0 - E|$ is the same for each box and E is the same for each box. Knowing this can save us from repeating calculations in some cases. But WATCH OUT! These values are not always the same for the different boxes.

We now compute the quantity χ^2, called chi square, because χ is the Greek letter chi. χ^2 is just the sum of the two numbers obtained above, one for each box of the table. Remembering that $\sum$ stands for "sum," we can write the formula for χ^2 as follows.

$$\chi^2 = \sum \frac{(|0 - E| - \frac{1}{2})^2}{E}$$

In our particular example,

$$\chi^2 \approx 1.53 + 1.53$$
$$\approx 3.1$$

Now we look up this value of χ^2 in a χ^2-probability table. Notice that 3.1 appears in the χ^2 column opposite a p of .08.

χ^2 - p TABLE

χ^2	p	χ^2	p	χ^2	p	χ^2	p
.00016	.99	.045	.84	.46	.50	2.0	.16
.00063	.98	.054	.82	.56	.46	2.3	.14
.0017	.97	.064	.80	.72	.42	2.5	.12
.0028	.96	.081	.78	.84	.38	2.7	.10
.0039	.95	.098	.76	.97	.34	2.9	.09
.0063	.94	.12	.74	1.1	.30	3.1	.08
.0087	.93	.13	.72	1.2	.28	3.4	.07
.011	.92	.15	.70	1.3	.26	3.6	.06
.014	.91	.21	.66	1.4	.24	3.8	.05
.016	.90	.27	.62	1.5	.22	4.3	.04
.026	.88	.34	.58	1.6	.20	4.9	.03
.035	.86	.40	.54	1.8	.18	5.4	.02
						6.6	.01

We conclude that a deviation of 4 or more from 8 heads can be expected to occur about 8% of the time. Notice that this agrees with the value obtained in Section 28-1. The χ^2 method does not always give such good results, but it is more than sufficient for most purposes.

Recall that we were able to find exactly 3.1 in the χ^2 column. Often this will not be possible. For example, suppose that we want to find p for $\chi^2 = .28$. When we look in the χ^2 column, we see that .28 falls between the table entries of .27 and .34.

χ^2	p
.27	.62
.28 ⟷	?
.34	.58

What p should be reported for a χ^2 of .28? Since .28 is closer to .27 than .34, report that $p \approx .62$. In similar situations follow this rule. <u>Report the p which corresponds to the nearest χ^2 table entry</u>. If your χ^2 value falls exactly in the middle of two χ^2 table entires, report the smallest p.

To summarize, there are three steps in using χ^2 to estimate expected relative frequency.

1. Construct a contingency table displaying observed and expected frequencies.

2. Compute χ^2 by substituting into the formula.

3. Look up the probability or expected relative frequency in the χ^2 table.

28-3 A Handy, Dandy Shortcut in Chi-Square Calculations

$$(x + .5)^2 = x^2 + x + .25$$
$$= x(x + 1) + .25$$

"So what?" you might ask. Well it just so happens that whenever the observed and expected frequencies are integers, the chi-square formula requires you to square numbers that end in .5. If you enjoy multiplying out such numbers as $(4.5)^2$, $(9.5)^2$ and $(14.5)^2$ stop reading right here. Go directly to the problem set.

Still with us? The equation above tells us that you can find the square of any number that ends in .5 by the following procedure.

1. Take the integral part of the number and multiply it by one more than itself.

2. Add .25 to the result of step 1.

A few examples should give you the hang of it.

$$(4.5)^2 = 4(5) + .25 = 20.25$$
$$(9.5)^2 = 9(10) + .25 = 90.25$$
$$(14.5)^2 = 14(15) + .25 = 210.25$$

If this procedure confuses you, then don't use it.

PROBLEM SET 28:

The following contingency table shows the outcome of a toss of 16 coins.

HEADS	$11^{(8)}$
TAILS	$5^{(8)}$

1. The 8's in parentheses represent the __________ (expected, observed) frequencies of heads and tails.

2. The other number not in parentheses represent the __________ (expected, observed) frequencies of heads and tails.

3. Show calculations verifying that

$$\frac{(|0 - E| - \frac{1}{2})^2}{E} \approx 0.8$$

for the left-hand box of the contingency table.

4. Compute $\frac{(|0 - E| - \frac{1}{2})^2}{E}$ for the right box of the contingency table.

5. $\chi^2 \approx$? (Add the answers to Problems 3 and 4.) Express your value of χ^2 with an implied uncertainty of $\approx .05$. (i.e., round to nearest tenth.)

6. What is the probability corresponding to your value of χ^2? (Use the χ^2 table on p. 221.)

7. This is the probability that the number of heads will deviate by ______ or more from 8.

8. Refer to the table on p.214 and write down the probability of a deviation of 3 or more using that method. (If you followed all the above steps carefully, this value should differ from the one in Problem 6 by only .01.)

Refer to the χ^2-probability table on p. 221 for Problems 9 through 11.

9. As χ^2 gets larger, the probability of the event gets __________ (larger, smaller).

10. As χ^2 of .00016 corresponds to an event which we expect to happen _____ % of the time.

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11. If χ^2 > ________, than p < .5. Refer to the χ^2-p table.

Suppose that you toss 16 coins and get 3 heads and 13 tails.

12. Set up a contingency table showing these results.

13. Compute χ^2. Round your value to the nearest tenth.

14. Refer to your χ^2 table. What probability corresponds to your value of χ^2?

15. Therefore we expect a deviation of 5 or more from 8 heads about _____% of the time.

Now suppose you simultaneously toss 100 coins and get 57 heads and 43 heads.

16. Set up a contingency table showing these results.

17. Compute χ^2 (with an implied error of .05).

18. What is the corresponding probability?

19. Do we expect a deviation as great or greater than the one observed to have a p > .50? (Yes or no.)

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SECTION 29:

29-1 Chi Square and Your Biomed Class

At this point you have seen how to use chi square to estimate the probabilities of various outcomes when tossing coins. Fortunately the chi-square procedure can be applied to many other situations as well. The coin-flipping approach was used to introduce the concept because it is easy to visualize. In the next two sections you will see how chi square can be applied to an entirely different type of problem.

Let us turn our attention now to your Biomed class. Your class represents some small portion or <u>sample</u> of all the people in your grade at school. There are many possible ways to choose such a sample. For example, a random process could be used. The names of all the students in your grade could be placed in a hat, and then your class could be chosen by drawing names. In this case we would expect that your class would be much like a scaled-down version of your entire grade. That is, we would expect about the same porportions of males, females, brown-eyed people, football stars, cheerleaders, bongo drum players and so on, as there are in your entire grade.

In summary, your class was formed by a random process, than we <u>expect</u> that its composition will be about the same as the composition of your entire grade. But what should we conclude if the composition of your class is quite different from that of your entire grade? There are two possible conclusions. First, we might conclude that your Biomed class was not formed by a random process. Second, we may have observed a rare random event, analogous to getting almost all heads when 16 coins are tossed.

In order to tackle a bite-sized problem, we will restrict our attention to the sex composition of your class. We will use chi square to estimate the probability of randomly selecting from your entire grade a class with the same sex composition as your Biomed class. The lower the probability, the more likely it is that we need to look for a non-random explanation for the observed sex ratio.

In the math class we are merely going to estimate the probability. It is the business of the social scientist to decide when and how to look for non-random explanations in a problem of this type.

29-2 The Proportions of Random Samples

You might expect that the probability of selecting a male, for example, at random from your grade would be .50. This is not necessarily so. Suppose you were in an all-male or all-female school. Then the probability of picking one sex would be 1 while, the probability of picking the other would be 0. Your school probably lies somewhere between the extremes of all-female or all-male. Can you think of a way to find the probability of picking a male or female at random? One place to start is with the school records. Certainly if you knew the numbers of males and females in your grade, then you would be one step closer to getting the information you need. For the sake of having some numbers to crank through our statistical mill, we sent Elmo down to the office of Pudworthy High School. He badgered the secretaries until they took pity on him.

Elmo came back with these numbers: males--230, females--270. You already know how to proceed from here. Remember how we calculated the relative frequency for each number of heads? We will calculate the relative frequency for the males and females in the same way.

$$R = \frac{O}{N}$$

where R = relative frequency

O = observed frequency

N = total number of students

The number of students in Elmo's grade at Pudworthy High School is 270 + 230 = 500.

First, we calculate the relative frequency of the number of males.

$$R = \frac{230}{500}$$

$$= \frac{46}{100}$$

$$= .46$$

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Similarly, the relative frequency for the number of females is

$$R = \frac{270}{500}$$

$$= .54$$

Of course, we could just as easily have obtained this number by substracting .46 from 1.

We conclude that Elmo's entire grade is 46% male and 54% female. For the sake of argument, we are considering the Biomed class at Pudworthy to be a random sample of the entire grade. Therefore we expect the Biomed class to be 46% male and 54% female, too. The Biomed class has 25 students. We can now calculate the expected frequencies of the males and females in the Biomed calss.

MALES: $E = (.46)(25)$

$= 11.5$

FEMALES: $E = (.54)(25)$

$= 13.5$

Do not be concerned that the expected numbers of males and females are not whole numbers. Expected frequencies often come out this way. The numbers do not imply a person who is half male and half female!

At this point we have calculated the expected frequencies of males and females. What about the observed frequencies? A head count in Elmo's class yields the results, 17 males and 8 females.

We can summarize our results in a <u>contingency table</u>.

MALES	17 (11.5)
FEMALES	8 (13.5)
TOTAL	25

Notice that each expected frequency (the number in parentheses) is the total (bottom box) times the relative frequency of males and females respectively.

In the next section we will apply the chi square procedure to these data.

PROBLEM SET 29:

At Nurdsburg High School there are 200 students in grade 10, 110 are females and 90 are males.

1. What is the relative frequency of males in grade 10?

2. What is the relative frequency of females in grade 10?

3. The grade 10 algebra class contains 20 students. If this class is a random sample of the tenth grade, what is the expected number of males in the algebra class?

4. What is the expected number of females in the algebra class?

5. Suppose the algebra class is actually made up of 13 females and 7 males. Construct a contingency table displaying the observed and expected frequencies of males and females.

The following table gives the relative frequencies of blood types in the U.S. white population.

BLOOD TYPE	RELATIVE FREQUENCY
0	.45
A	.40
B	.11
AB	.04

6. In a randomly selected group of 100 white Americans, what is the expected frequency of blood type AB?

7. In a randomly selected group of 1100 white Americans, what is the expected frequency of blood type 0?

8. In a randomly selected group of 12 white Americans, how many do we expect to have either blood type A or blood type B?

The following table gives the relative frequencies of blood types in the U.S. black population.

BLOOD TYPE	RELATIVE FREQUENCY
0	.47
A	.28
B	.20
AB	.05

9. In a randomly selected group of 80 black Americans, how many do we expect to have blood type AB?

10. In a randomly selected group of 450 black Americans what is the expected frequency of blood type A?

11. In a randomly selected group of 12 black Americans, how many do we expect to have either blood type A or B? Is this answer the same as in Problem 8?

The table below gives the relative frequencies of blood types in the West African population.

BLOOD TYPE	RELATIVE FREQUENCY
0	.46
A	.23
B	.26
AB	.05

12. Calculate the expected frequency of blood type B in a randomly selected group of 150 West Africans.

13. Calculate the expected frequency of blood type A in a randomly selected group of 30 West Africans.

14. A survey of 400 randomly selected people of one ethnic group gave the following results.

BLOOD TYPE	OBSERVED FREQUENCY
0	180
A	96
B	100
AB	24

These data most closely fit which ethnic group, U.S. white, U.S. black or West African? (Hint: calculate relative frequencies.)

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I. TONGUE ROLLING

II. TONGUE FOLDING

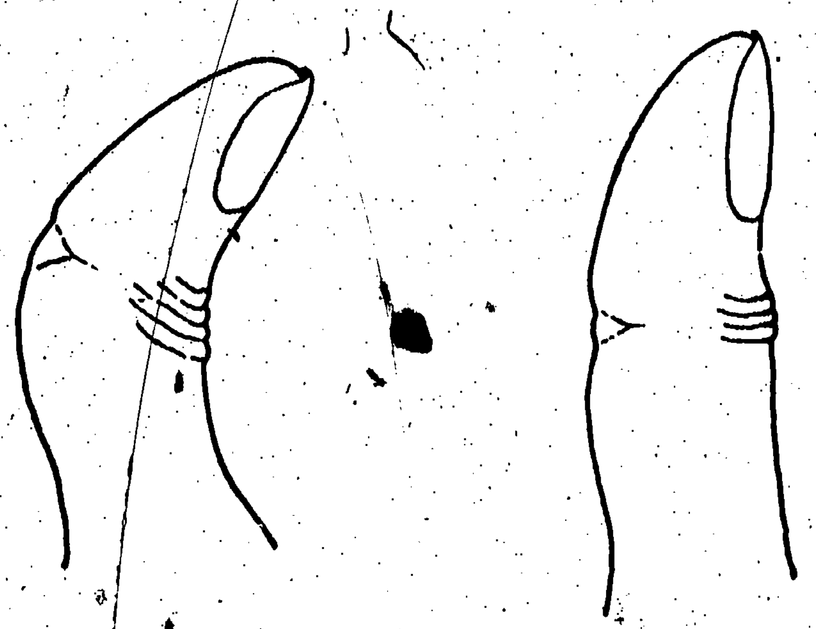
15. Many people inherit the ability to curl up the edges of their tongue as in picture I above. Can you? The relative frequency of tongue rollers in the U.S. population is about .67 and non-rollers .33.

In a group of 500, randomly closen from the U.S. population, what are the expected numbers of tongue rollers and on-rollers?

16. Suppose that the sample of 500 actually contains 320 tongue rollers and 180 non-rollers. Construct a contingency table displaying the expected and observed frequencies.

17. A trait which is quite rare in the U.S. population is the ability to fold the tip of the tongue back when the tongue is extended (see picture II above). The relative frequency of people with this ability is about .002. In a high school with 1000 students, how many tongue folders would you expect?

18. Taking the population of the U.S. to be 200 million, approximately how many tongue folders are there in the U.S.?



I. HITCH-HIKER'S THUMB   II. STRAIGHT THUMB

Some people have the ability to hyperextend the last joint of the thumb forming the "hitch-hiker's thumb" shown in picture I. There is some difficulty in deciding who has this trait because people can hyperextend to different degrees. We will disregard this detail and summarize the relative frequencies of "hitch=hiker's thumb."

U.S. white population: .25
U.S. black population: .36

19. In a sample of 25 randomly chosen from the U.S. black population, what is the expected frequency of hitch-hiker's thumb?

20. Suppose that the observed frequency of hitch-hiker's thumb in the above sample is 12. Construct a contingency table for this sample.

21. In a random sample containing 80 white people and 20 black people, what is the expected number of people with hitch-hiker's thumb? (The answer is not a whole number.)

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SECTION 30:

30-1 How Odd (Statistically) Is Your Class?

We are now prepared to apply the chi-square procedure to the sex ratio of Elmo's Biomed class at Pudworthy High School. By following the same procedure you can calculate the chi-square probability for your class.

Let us recall the results obtained in the last section. The Biomed class at Pudworthy contains 25 students. If the class were a random sample of Elmo's entire grade, we would expect the class to contain 11.5 males and 13.5 females. In reality the class contains 17 males and 8 females. We summarized these facts in a contingency table.

MALES	17 (11.5)
FEMALES	8 (13.5)
TOTAL	25

In applying chi square to these data, the technique will be exactly like that of Section 28. You can refer to that section as an aid in tracing through the development.

The formula for chi square is

$$\chi^2 = \sum \frac{(|0 - E| - \frac{1}{2})^2}{E}$$

In order to apply this formula we substitute the values from each box or <u>cell</u> of the contingency table into the expression

$$\frac{(0 - E - \frac{1}{2})^2}{E}$$

and then add the two resulting expressions. The computations follow.

$$\chi^2 = \frac{(|17 - 11.5| - \frac{1}{2})^2}{11.5} + \frac{(|8 - 13.5| - \frac{1}{2})^2}{13.5}$$

$$= \frac{5^2}{11.5} + \frac{5^2}{13.5}$$

$$\simeq 4.0$$

Now we refer to the χ^2-probability table on p. 221. It will tell us the probability that a random sample will deviate from the expected values by as much or more than in Elmo's class. In the table we find that a χ^2 of 3.8 corresponds to a probability of .05 and a χ^2 or 4.3 corresponds to a probability of .04. Since 4.0 is closer to 3.8 than 4.3 we report a probability of .05. In other words, we would expect only about 5 out of 100 randomly selected classes to have as large a deviation in sex composition as Elmo's class. In 95% of randomly selected classes, the sex distribution will be closer to the predicted values.

This is the point at which social scientists have to make a decision. They ask themselves, "Am I observing a rare random event? Or is there some other explanation?" Typically, when the probability is as low as .05, social scientists will start looking for some other explanation. For Elmo's class they would look for factors which would cause more males than females to end up in the Biomed class.

In math class you will calculate χ^2 for your own class. From this value of χ^2 the probability of random selection may be found. If the probability is low ($p < .05$), then you will discuss possible factors which might have caused your class to deviate markedly from the expected sex composition.

PROBLEM SET 30:

The following contingency table shows the expected and observed frequencies of males and females in a high school class.

MALES	7 (9)
FEMALES	13 (11)
TOTAL	20

1. Compute χ^2 for this table. Round your answer to the nearest hundredth.

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2. Refer to your χ^2-probability table and find the probability corresponding to your value of χ^2.

3. Complete the following sentence.

"Therefore we expect a random deviation of _________ or more from the expected values about ________% of the time."

In Problems 15 of Problem Set 29 we listed the relative frequencies of tongue rollers (.67) and non-rollers (.33).

4. Suppose that a group of 100 people consists of 56 tongue rollers and 44 non-rollers. Construct a contingency table displaying the expected and observed frequencies.

5. Compute χ^2, rounding your answer to the nearest tenth.

6. What is the corresponding probability?

7. Does this probability indicate that the group of tongue rollers and non-rollers were randomly selected or not? Explain.

8. A certain drug is known to be effective in 60% of cases treated. A doctor who has used the drug on 100 patients finds that the drug was effective in 50 cases. How often would we expect a deviation this great or greater due to chance alone?

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REVIEW PROBLEM SET 31:

The following table shows the results of 368 separate measurements of the vital capacity of a single individual. The measurements were taken to the nearest .01 liter.

VITAL CAPACITY (liters)	OBSERVED FREQUENCY	RELATIVE FREQUENCY
6.10-6.14	1	.003
6.15-6.19	3	.008
6.20-6.24	13	.035
6.25-6.29	29	.079
6.30-6.34	78	.212
6.35-6.39	91	.247
6.40-6.44	61	.166
6.45-6.49	51	.139
6.50-6.54	31	.084
6.55-6.59	7	.019
6.60-6.64	3	.008
TOTAL	368	

1. How many different events are there?

2. The event of a vital capacity in the range 6.50-6.54 liters occurred with a frequency of ________.

3. The most frequent event was a vital capacity in the range of ________ liters.

4. The rarest observed event was a vital capacity in the range of ________ liters.

5. To find the relative frequency of a given event we divide the observed frequency of that event by ________.

6. The relative frequency of which event is .247?

7. This means that that event happened in ______% of the trials.

8. Which event happened in 8.4% of the trials?

9. The event "vital capacity in the range 6.60-6.64 liters" occurred with the same relative frequency as the event ________.

10. Draw a frequency histogram of the vital capacity measurement results given in the preceding table. Label the vertical axis "Observed Frequency."

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11. a. Draw a relative frequency histogram of the data in the preceding table.

b. How does the shape of this histogram compare with that in Problem 10?

The following table records the results of 20 trials of dumping 10 coins.

EVENT (Number of Heads)	OBSERVED FREQUENCY	TOTAL NUMBER OF HEADS for each event
0	0	0
1	0	0
2	2	4
3	1	3
4	3	12
5	3	15
6	6	36
7	3	21
8	1	8
9	1	9
10	0	0
	TOTAL # OF TRIALS 20	TOTAL # OF HEADS 108

12. Draw a frequency histogram of the above data. Label the horizontal axis "Event (number of heads)" and the vertical axis "Observed Frequency."

13. Complete the following relative frequency table.

EVENT (number of heads)	0	1	2	3	4	5	6	7	8	9	10
RELATIVE FREQUENCY	0		.10	.05	.15						

14. Draw a relative frequency histogram of these data.

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In Problems 15 through 17, give the relative frequency of each combined event. Refer to your completed table (Problem 13) or to your histogram (Problem 14).

15. Zero heads or 7 heads.

16. Four heads or 6 heads.

17. More than 3 heads.

18. The combination of ____heads or ____heads occurs with a relative frequency of exactly 0.40.

19. What is the relative frequency of the combined event, 9 heads or less?

20. Recall that we are dealing with 20 dumps and 10 coins each. What is the expected number of heads per trial?

21. The relative frequency of a deviation of 1 from the expected value is ________.

22. Complete the following table.

EVENT Δh (Deviation from expected value)	0	1	2	3	4	5
RELATIVE FREQUENCY						

23. Construct a mean-centered histogram based on the table of Problem .

24. Using the data of Problem 26, complete the following table.

EVENT (Deviation from expected value)	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more
RELATIVE FREQUENCY						

25. A deviation of_____or more was observed in 40% of the trials.

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The following table gives the probabilities (the theoretical relative frequencies or the mathematical predictions) for 20 trials of dumping 10 coins.

EVENT (# of heads)	0	1	2	3	4	5	6	7	8	9	10
PROBABILITY	.001	.010	.044	.117	.205	.246	.205	.117	.044	.010	.001

26. The event which is expected in 24.6% of the trials is ________ heads.

27. Two heads and ________ heads are expected to occur with equal frequency.

28. If the number of trials were 1000, then the expected frequency of 2 heads would be ____________.

29. Complete the following table using the probability data just preceding Problem Set 26.

EVENT Δh (deviation from expected value)	0	1	2	3	4	5
PREDICTED RELATIVE FREQUENCY, OR PROBABILITY						

30. The probability of getting either 3 heads or ________ heads is .234.

31. A deviation of 4 from the expected value is expected to occur 200 times in every ________ trials.

32. Complete the following table using the table you completed in Problem 29.

EVENT Δh (deviation from expected value)	0 or more	1 or more	2 or more	3 or more	4 or more	5 or more
COMBINED PROBABILITY						

33. What is the probability of an event deviating by 3 or more from the expected value?

34. In what percent of the trials would we expect a deviation of 3 or more?

The following contingency table represents the outcome of a toss of 10 coins.

HEADS	8 (5)
TAILS	2 (5)
TOTAL	10

35. The numeral 5 in parentheses represents the ________ (expected, observed) frequencies of heads and tails.

36. The numeral 8 represents the ________ (expected, observed) frequency of heads.

37. Evaluate $\frac{(|0 - E| - \frac{1}{2})^2}{E}$ for

a. The box in the contingency table labeled "HEADS."

b. The box in the table labeled "TAILS."

38. $\chi^2 = ?$ Round your answer to the nearest tenth.

39. Use the χ^2 table on page 221 of your text to find the probability corresponding to your value of χ^2. What is that probability (to the nearest hundredth.)

40. This is the probability that the number of heads will deviate by ________ or more from 5.

41. Refer to the table you constructed to answer Problem 36 and write the probability of a deviation of 3 or more determined by that method.

42. Referring to the table preceding Problem 12, we can set up a contingency table to determine whether the total number of heads which occurred was a rare event, given the total number of coins tossed.

a. How many trials were there in all, taking 1 trial to be 1 toss of 1 coin?

b. What is the expected number of heads in that many trials?

c. What is the expected number of tails?

d. What were the observed frequencies for head and tails, respectively?

e. Complete the contingency table below.

HEADS	___ ( )
TAILS	___ ( )
TOTAL	

f. Evaluate χ^2, round your answer to the nearest tenth.

g. What is the probability associated with your value for χ^2?

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SECTION 32

32-1 Drugs and the Placebo Effect

At this point we are ready to look at a slightly more complicated situation in which chi square can provide useful information. Specifically, we are going to tackle the problem of measuring the effectiveness of a new drug before it is released for medical use. After a drug is developed, and tested on laboratory animals for toxicity, it must be tested on a group of people who have the medical problem which it is intended to treat. Moreover, the effect of the drug must be compared with that of a placebo given under comparable conditions.

A placebo is an inactive medication. Therefore it should have no physical effect on the human body. Nevertheless, a placebo may change a patient's symptoms through psychological avenues. Among patients who are not told they have taken placebos, as many as 60% may experience an improvement in their condition.

Because of this placebo effect, we do not know whether a drug is really effective, even if 60% or so of patients show improvement. Therefore it is standard procedure to give some patients the real drug and other patients a placebo. The results can then be compared. We can also state that the investigation is properly controlled. The following example illustrates the method.

200 patients suffering from pain were selected, of which 100 were given a placebo and 100 were given a newly developed drug for pain relief. 60% of the patients receiving the placebo improved and 80% of the patients receiving the new drug improved.

You will notice right away that the patients on the new drug did better than those on the placebo. But is this because the new drug is better or is it just a chance phenomenon? This is where chi square is useful; it can tell us the probability that the difference is due to chance alone.

We begin by exhibiting the observed frequencies in a contingency table.

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	DRUG	PLACEBO	TOTAL
IMPROVED	80	60	140
DID NOT IMPROVE	20	40	60
TOTAL	100	100	200

This table is a little more complicated than the ones in previous lessons. There are four boxes, or cells, in the table instead of two.

At this point, we have no expected frequencies, since we don't know what to expect. First we must make an assumption. The assumption is that there is no difference between the drug and the placebo in terms of improving the condition of patients. This would mean that the relative frequency of improvement would be the same for both the drug and the placebo.

To obtain the expected frequencies, based on our assumption, we reason this way. In the total group of 200, 140 improved and 60 didn't. Therefore the relative frequencies are as follows.

<u>IMPROVED:</u> $\frac{140}{200} = .70$

<u>DID NOT IMPROVE:</u> $\frac{60}{200} = .30$

If the relative frequency of improvement is the same for both groups of patients, it must be .70 for each group. Similarly, the relative frequency of no improvement must be .30 for each group. Of the 100 patients given the new drug, we would expect .70 x 100 = 70 to improve and .30 x 100 = 30 to show no improvement. The same goes for the group given the placebo. We enter the expected frequencies in the contingency table.

	DRUG	PLACEBO	TOTAL
IMPROVED	80 (70)	60 (70)	140
DID NOT IMPROVE	20 (30)	40 (30)	60
TOTAL	100	100	200

Now we can apply the χ^2 formula to the data of the table. The formula is

$$\chi^2 = \sum \frac{(|O-E| - \frac{1}{2})^2}{E}$$

In all our previous examples there were only two cells in the contingency table, and therefore only two terms to add in the χ^2 formula. But in the present case there are four cells in the contigency table. Therefore, there will be four terms in the χ^2 sum, one for each cell.

$$\chi^2 = \frac{(|80-70| - \frac{1}{2})^2}{70} + \frac{(|60-70| - \frac{1}{2})^2}{70} + \frac{(|20-30| - \frac{1}{2})^2}{30} + \frac{(|40-30| - \frac{1}{2})^2}{30}$$

$$\simeq 1.3 + 1.3 + 3.0 + 3.0$$

$$\simeq 8.6$$

Now refer to the χ^2-probability table on p. 221. It turns out that 6.6 is the largest value of χ^2 listed. Therefore our value of 8.6 is "off the scale." We can conclude that the corresponding probability is less than .01.

There is less than one chance in a hundred that the new drug is no more effective than the placebo. In other words, it is highly likely that the drug is effective. This kind of information, along with data on any side-effects caused by the drug, would be used to make a decision about whether to release the drug for general use.

PROBLEM SET 32:

Dramamine is a drug used in the prevention of motion sickness. In an actual study on the effectiveness of Dramamine, 108 people were given Dramamine and 108 people were given a placebo. The entire group of 216 was then subjected to turbulent flight conditions. A record was kept of those who became sick (i.e., vomited) and those who did not. The table below shows the following results.

	DRAMAMINE	PLACEBO	TOTAL
SICK	31	60	91
NOT SICK	77	48	125
TOTAL	108	108	216

1. In order to determine the expected frequencies, we assume that ____________________________________.

2. The total number of people who became sick was _____.

3. The total number of people who did not become sick was _____.

4. The fraction $\frac{91}{216}$ represents the relative frequency of those who ________(did, did not) become sick.

5. Therefore, out of 108 people, we expect _____ to have become sick. (Your answer should end in .5.)

6. The fraction $\frac{125}{216}$ represents the relative frequency of those who ________(did, did not) become sick.

7. Out of 108 people, we expect _____ to have not become sick. (Your answer should end in .5.)

8. Copy the contingency table and add to it the expected frequency for each of the four cells.

9. Compute χ^2 for your table. If you do not have a calculator, round the denominators to the nearest 10 before dividing.

10. The probability that Dramamine is no better than the placebo in preventing vomiting is less than _____. This might be considered strong evidence that the drug is effective.

SECTION 33:

33-1 Introduction to the Cola Activity

Everybody is a consumer; we all buy things. When we buy products such as breakfast cereal, rice, bread and so forth, we must choose a single brand from a great variety of brands. What factors influence the choices we make? The people who sell us things are very interested in answers to this question. They spend millions for "motivational research."

We will spend pennies. By means of a cola-tasting activity we will demonstrate how information about motivation can be obtained. We will ask, "Is taste related to brand preference for Coca-Cola and Pepsi-Cola?" Taste is only one among many factors which might influence brand preferences. Some, like taste, are related to information which our senses can gather about the cola itself. Others are not.

SENSE FACTORS : fizziness, temperature, ...

OTHER FACTORS : attitudes of friends, advertising, ...

In the Cola Activity we will use chi square to investigate whether the taste of a cola (either Coke or Pepsi) is related to the preference for a particular brand.

Here is what will happen. You will be asked to state your preference for Coke or Pepsi. This will be recorded. Your recorded preference for Coke or Pepsi will be called your "brand preference." Then you will taste unidentified samples of Coke and Pepsi. They will be called X and Y, or Y and X, or something else. Your choice on this test will be called your "taste preference." When the taste and brand preferences for the entire class have been recorded, the true identities of Brands X and Y will be revealed.

Notice that we have chosen two drinks which are very similar as far as human senses are concerned. Both colas are the same color and will be served to you at the same temperature. Every effort will be made to make sure that you base your choice only on the taste. This is an effort to make taste the only sense-related factor available for decision making.

33-2 The Unerring Tasters

The students of Oracle High School performed the Cola Activity. The results are summarized in the contigency table below.

		BRAND PREFERENCE		
		COKE	PEPSI	TOTAL
TASTE PREFERENCE	COKE	30 (18)	0 (12)	30
	PEPSI	0 (12)	20 (8)	20
	TOTAL	30	20	50

First, let's get familiar with the table.

QUESTION 1:

How many students said they preferred the Coke brand?

ANSWER:

30. Brand preference information is found by reading columns. The "30" which answers the question is found at the bottom of the "Coke Brand Preference" column.

QUESTION 2:

How many students preferred the taste of Pepsi?

ANSWER:

20. The "20" which answers the question is found on the right of the "Pepsi Taste Preference" row.

Notice that all students who said they preferred Coke actually chose the taste of Coke. The same is true for Pepsi.

An inspection of the table reveals a strong pattern of association between Coke brand preference and Coke taste preference and also between Pepsi brand preference and Pepsi taste preference. From a statistical point of view this is a very strong correlation.

There are four possible relations which might account for such a strong correlation.

Cause and Effect Relationships	1. Taste may cause brand preference. 2. Brand preference may cause taste preference. 3. Some third factor may be causing both. 4. The pattern may be just a very rare chance occurrence.

Notice that if we can eliminate possibility 4, then all other possibilities are cause-and-effect explanations of some sort. The chi square procedure allows us to calculate how often a chance occurrence would produce a pattern like this.

The expected frequencies in the table are calculated in the standard way. We assume (for the sake of comparison) that <u>each BRAND PREFERENCE subgroup is a random sample of the total group with respect to TASTE PREFERENCE</u>. For example, COKE taste preference had a relative frequency of .60 for the entire group. This is found by dividing the number in the upper right box (30) by the total number of subjects (50). The expected frequency of Coke taste preference in a random subgroup of 30 would be (.6)(30) or 18 (upper left box). Other expected frequencies are calculated similarly.

Without even calculating χ^2 we can see that it would be quite large, corresponding to a $p < .01$. In other words, the pattern of observed frequencies in this table would be a very rare random event. In fact, the probability is so small that we will neglect the possibility that the pattern is a chance occurrence and assume that some sort of a cause-and-effect explanation is responsible for it.

33-3 The Coin Flippers

Elmo's Pudworthy High School Biomed class attempted to administer the Cola Activity to the Home Economics class. Unfortunately they ran into some problems. Every student in the class claimed that he or she liked Coke or Pepsi equally well. Not a single student could decide whether he liked Coke or Pepsi better. This stopped progress. Nobody was allowed to taste the drinks until he stated a brand preference. At last, Elmo had a brilliant (?) idea. He suggested that each student flip a coin to decide on a brand preference. All agreed that this was an acceptable solution. The table below is a record of the Home Economics class' data.

		BRAND PREFERENCE		
		BRAND COKE (HEADS)	PEPSI (TAILS)	
TASTE PREFERENCE	COKE	16 (15)	9 (10)	25
	PEPSI	14 (15)	11 (10)	25
		30	20	50

There is more information in this table and it will probably take a little study to interpret it correctly.

QUESTION 1:

How many students said they preferred the COKE BRAND? (Or, equivalently, How many students got "HEADS" on the flip of the coin?)

ANSWER:

30, which is found at the bottom of the COKE-HEADS column. Of the 30 in this group not all students chose a single taste. In fact 16 preferred the COKE TASTE and 14 preferred the PEPSI TASTE.

QUESTION 2:

How many students preferred the taste of Pepsi?

ANSWER:

25, which is found at the right of the PEPSI TASTE row. Also notice that 14 of these were in the heads group and 11 in the tails group.

It is important to note that Elmo's solution produced randomly selected BRAND PREFERENCE subgroups. Therefore, the pattern of observed frequencies in the table is a true random event. Now notice that the observed and expected frequencies do not agree exactly, although they are close. This is a common random pattern. (Recall our earlier coin-dumping activity. Even though 8 heads was the expected number of heads for a trial involving 16 coins, we actually got exactly 8 heads less than a fifth of the time.)

When we determine p for this table, we will be calculating how often randomly produced tables will have χ^2 values as large or larger than the χ^2 for this table.

Without going through the details of χ^2 calculation you should by now expect χ^2 to be small because the differences between observed and expected frequencies are small. Small χ^2's correspond to large probabilities. In fact $\chi^2 \approx .083$ and $p \approx .93$. This means that 93 out of 100 tables produced in exactly the same way would have χ^2 values this large or larger. In other words, it is very common for randomly produced tables to have χ^2 values which are larger.

33-4 Looking Back--Looking Forward

We have just examined two extreme χ^2 tables. The first one was for a group of students who knew definitely which brand they preferred and who later were able to choose, without a single error, the taste of the brand they had previously stated. We also found that this table had a larger χ^2 and $p < .01$. Such a small "p" implies that it is very unlikely that this table could have occurred as a result of a random process. Therefore, if we examined the table and then claimed, "The students of Oracle High School prefer the taste of their previously stated brand preference," then no one could turn around and agree that the results could just as easily have happened by chance.

The second contingency table was for a group of students who had no predisposition to prefer either brand. When forced to make a brand-preference choice, they flipped a coin. This produced a table in which there was little disagreement between observed and expected frequencies and, consequently, a low χ^2. For this table p was .93 which implies that the observed pattern is also a common random event.

When you calculate χ^2 and find p for the Cola Activity results for your Biomed class, you will be finding out how often deviations from expected frequencies as large or larger than those observed would happen as a result of a random method of brand-preference selection.

In other words, you will be calculating how often deviations as large or larger than those actually observed could have happened had brand preferences been chosen by a random method such as a coin flip or the roll of a die. Stated still another way, you will presume a random method of brand-preference selection and then see whether p is low or high. If $p < .05$, as it was for the Oracle High School contingency table, then we say that the pattern in the table departs significantly from common random patterns. If $p \geq .05$, as it was for the Home Economics table, then we say that chance alone could account for the observed pattern.

33-5 What Can Be Concluded if $p \geq .05$?

Suppose that you determine a $p \geq .05$ for the table which results from your own Cola Activity. The meaning is straightforward. The departure from common random patterns is not significant. You must conclude that your class as a whole could just as easily have gotten the results by some random method of BRAND-PREFERENCE selection, similar to the method the Home Economics class used. Therefore, you must conclude that taste is not a major factor in brand preference for your test group. This conclusion quite naturally raises the question, "If brand preferences aren't related to taste, what (if anything) is?

Recall that the colas are nearly identical in their sense-related characteristics. Therefore, it is reasonable to ask whether brand preference is related to other influences like peer pressure, advertising and so forth.

33-6 What Can Be Concluded if $p < .05$

Suppose $p < .05$; can it then be concluded that there is a cause-and-effect relation between TASTE PREFERENCE and BRAND PREFERENCE? In other words, can we conclude that a pattern approaching the Oracle High School pattern is being observed? It would certainly be convenient if this were true. Unfortunately, $p < .05$ does not necessarily imply a cause-and-effect type of explanation because of another unlikely (but possible) pattern which could also result in low probabilities.

33-7 Static

Consider the following table.

		BRAND PREFERENCE		
		COKE	PEPSI	
TASTE PREFERENCE	COKE	0	25 (12.5)	25
	PEPSI	25 (12.5)	0 (12.5)	25
		25	25	50

Examine the table. What is the pattern? It is a weird one. Everyone who said they preferred Coke chose the Pepsi taste and the reverse. This is another mathematically possible pattern which would result in high χ^2 and low p. We will call this pattern <u>negative correlation</u>. We will call the Oracle High School pattern, the one in which brand and taste preferences agreed perfectly, <u>positive correlation</u>.

Fortunately there is a simple mathematical test to determine which kind of pattern is responsible for your $p < .05$. For example,

	COKE	PEPSI	
COKE	20 (15)	5 (10)	25
PEPSI	10 (15)	15 (10)	25
	30	20	50

$\chi^2 = 6.75$

$p < .01$

The diagonal from upper left to lower right (↘) represents positive responses i.e., COKE-COKE, PEPSI-PEPSI. If the sum of the observed frequencies in this diagonal (20 + 15 = 35) is greater in the other diagonal (10 + 5 = 15), then a positive pattern is being observed. If not, then it is a negative pattern.

In the given example it is positive because 35 > 15.

For other χ^2 tables the diagonal rule may not work. In such cases you'll have to examine the tables to discover which boxes represent positive responses and which represent negative. Then if the sum of the positive responses is greater than the negative, a positive pattern is being observed.

33-8 $p < .05$ (Reconsidered)

If you determine a $p < .05$ and the pattern is positive, then you may conclude (finally) that for the most part the people in your test group form BRAND preferences on the basis of TASTE. A word about the significance of this conclusion is in order. It is a finding for the test group. It really says nothing about the ability of a given individual to recognize the taste of their favorite cola. In order to answer this question for a single person a different kind of experiment is required. For example, a given person could be repeatedly tested and the data from this activity analyzed.

One final caution. You may be tempted to think that the conclusions drawn from your own Cola Activity investigation would apply to all possible test groups. Unfortunately, this is not so. This is because your test subjects (high school students) are not a random sample of the entire U.S. population. In fact, they may not even be a random sample of the entire U.S. high school population. However, there is a good chance that your results may apply to the entire population of your own high school. How accurately they describe your school depends on how representative of the whole school your test groups were.

33-9 Ending on a Positive Note

The results of chi square analysis must be qualified so much that it is easy to forget what chi square can actually tell us.

1. If $p \geq .05$, the assumption of any sort of cause-and-effect relationship is unjustified.

2. $p < .05$ suggests some kind of a cause-and-effect explanation.

PROBLEM SET 33:

Questions 1 through 5 refer to the following table.

		BRAND PREFERENCE		
		COKE	PEPSI	TOTAL
TASTE PREFERENCE	COKE	26 ( )	4 ( )	30
	PEPSI	9 ( )	11 ( )	20
	TOTAL	35	15	50

1. How many students preferred the TASTE of COKE?

2. How many students said they preferred COKE by BRAND?

3. Of the PEPSI BRAND-PREFERENCE group, how many actually chose the TASTE of PEPSI?

4. Of the COKE BRAND-PREFERENCE group, how many preferred the TASTE of PEPSI?

5. a. Copy the table.

   b. Calculate the expected frequencies and fill them in.

   c. Calculate χ^2 for the table. (Hint: $\sum \frac{1}{E} \approx .4$)

   d. Determine p

   e. What can be concluded?

6. a. Calculate χ^2 for this table. (Hint: $\sum \frac{1}{E} \approx .18$)

		BRAND PREFERENCE		
		COKE	PEPSI	TOTAL
TASTE PREFERENCE	COKE	19 (14)	21 (26)	40
	PEPSI	16 (21)	44 (39)	60
	TOTAL	35	65	100

   b. Determine p

   c. What can be concluded?

7. a. Which table below shows a <u>negative correlation</u>?

A.

	COKE	PEPSI	TOTAL
COKE	8	2	10
PEPSI	12	18	30
TOTAL	20	20	40

B.

	COKE	PEPSI	TOTAL
COKE	10	10	20
PEPSI	25	15	40
TOTAL	35	25	60

b. Justify your answer to Part a mathematically.

8. Suppose taste is not related to brand preference ($p \geq .05$). What other factors (if any) might be?

9. The Cola Activity is designed so that only one factor is available as the basis for a TASTE PREFERENCE choice? What is it?

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PROBLEM SET 34:

Following a charity luncheon there was an outbreak of nausea. Consequently an investigation was conducted to find out which food was responsible. The results are summarized in the table below.

EGG SALAD

	ATE	DID NOT EAT	TOTAL
ILL	38 (31)	3 (10)	41
NOT ILL	27 (34)	18 (11)	45
TOTAL	65	21	86

$\chi^2 \approx 10.7$

$p < .01$

MACARONI AND CHEESE

	ATE	DID NOT EAT	TOTAL
ILL	20 (16.2)	21 (24.8)	41
NOT ILL	14 (17.8)	31 (27.2)	45
TOTAL	34	52	86

$\chi^2 \approx 2.12$

$p \approx .14$

1. How many people ate egg salad and then became ill?

2. How many people ate egg salad and then did not become ill?

3. How many people did not eat macaroni and cheese and did not become ill?

4. How many people ate macaroni and cheese and did not become ill?

5. What assumption is used as the basis for calculating the expected frequencies?

6. a. For the egg salad table, what is the probability of producing a table with deviations as large or larger as a result of a random subgroup selection procedure?

   b. For the macaroni and cheese table, interpret "p ≈ .14."

7. Which food shows the stronger positive association with becoming ill?

8. For which food is the association with becoming ill more likely to be attributable to chance?

9. a. If the nausea was caused by eating egg salad, how can you account for the 20 people who ate the macaroni and cheese and became ill?

b. On the other hand, notice that 3 people who experienced nausea also claimed that they ate no egg salad. What might explain this?

Genetics is the study of transmission of certain characteristics from parents to offspring. Many genetics experiments use fruit flies because they reproduce rapidly and therefore many generations may be observed in a short time. They also happen to be easy to work with.

One proposed mechanism for the transmission of eye color in fruit flies predicts an approximately equal split between red-and white-eyed females for the first generation after a cross between males and females of different-colored eyes. We will treat the results of a particular fruit fly experiment just as if they had occurred as a result of a coin-dumping exercise. Expected frequencies will be calculated just as they were for the coin-dumping data.

Problems 10 through 14 use these data.

Red-eyed females	86
White-eyed females	91
Total	177

10. The theory predicts a relative frequency of .5 for each color.

a. Calculate the expected frequency of each color.

b. Copy the contigency table and insert the expected frequencies.

11. Calculate χ^2 for the data.

12. Determine p.

13. Do the observed frequencies represent significant deviations from common random patterns (i.e., $p < .05$)?

14. What does the probability for this table imply about the original assumption that each eye color is equally likely?

Problems 15 and 16 refer to these data.

Red-eyed females	74
White-eyed females	103
Total	177

15. a. Determine p for the table.

b. Do the observed deviations represent a significiant departure from common random patterns (i.e., $p < .05$)?

16. What does the probability for this table imply about the original assumption that red- and white-eyed females are equally likely?

Elmo's class at Pudworthy High School conducted an experiment to determine whether positive feedback influences subjects' opinions of their own performance. The subjects were divided into two groups, an experimental group and a control group. The experiment was conducted in two phases.

In the first phase, all subjects were given a list of ten nonsense words, were told that these words were actually biomedical terms with which they were not familiar, and were asked to guess the meaning of each term. After guessing the meanings of the terms, each subject was asked to rate his own performance on a scale of 1 to 10.

In the second phase, all subjects were given another list of ten nonsense words with the same instructions. During this phase, confederates of the experimenter circulated among the subjects, looking at their work. The confederates gave positive feedback to experimental subjects (i.e., "You're doing pretty well." "It looks like you've got a knack for this kind of work.") They said nothing to the control subjects; if the control subjects asked them any questions, the confederates made neutral remarks (e.g., "Keep working."). At the end of the second phase, each subject was asked to rate his performance on the second phase, using the same scale of 1 to 10.

After the experiment was conducted, the experimenters examined all subjects' answer sheets from both phases of the experiment, and determined for each subject whether the subject's self-rating was higher for the second phase than for the first phase. The results for 100 subjects (50 experimental and 50 control) were as follows.

TREATMENT

		experimental (positive feedback)	control (no feedback)	Total
change in self-rating from first to second phase	higher	35	15	50
	lower or no change	15	35	50
	total	50	50	100

17. What assumption is used as the basis for the calculations of expected frequencies?

18. a. Determine p for the data.

b. Do the observed frequencies represent a significant departure from common random patterns?

c. Is it likely that chance alone could account for the pattern in the table?

d. What does p imply for the assumption used to generate the expected frequencies?

e. What appears to be the relationship between positive feedback and self-evaluation rating?

TABLE OF CONTENTS

FOR

SUPPLEMENTARY SECTIONS

SECTION X:

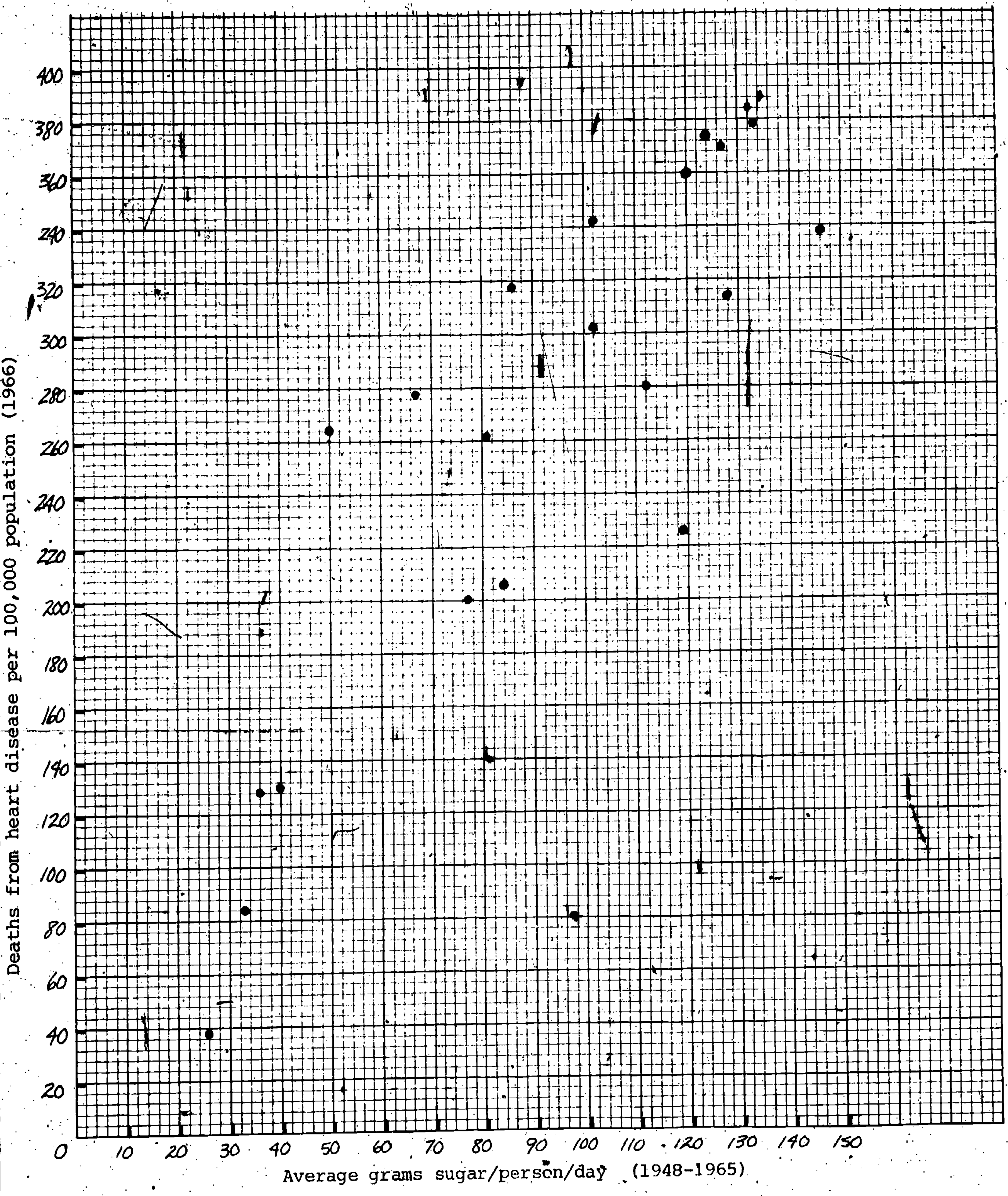
X-1 Correlation: Sugar and Heart Disease:

Since the development of antibiotics and immunizations, diseases caused by bacteria and virusus have declined as causes of death in the United States. Instead we are faced with diseases for which it is difficult to find a single cause, such as cancer and heart disease. There are many factors involved in these diseases, such as heredity, dietary habits, cigarette smoking and the psychological state of the individual. One of the major projects of medical research is deciding which factors are most important in causing a particular illness. In order to do this, researchers look at large groups of people and try to find trends or correlations between certain factors and a given disease. The Framingham Study, mentioned in the Biomedical Science Text, is an example of such a statistical study.

In order to get an idea of how a statistical study works, look at the table on the following page which lists the death rates per 100,000 people from heart disease for twenty countries in 1966. They are listed in order from highest to lowest death rate. Also listed is the average number of grams of sugar consumed per person per day over the years 1948-1965, with the exception of Hungary for which there were insufficient data before 1960. The value listed for Hungary is an educated guess.

Country	Average grams sugar/ person /day (1948-1965)	Number of deaths/100,000 caused by heart disease (1966)
United Kingdom	130.3	378.5
Denmark	123.8	373.1
U.S.A.	127.5	371.2
Sweden	119.7	359.4
Finland	102.3	342.2
Australia	146.5	338.6
Austria	86.2	316.8
New Zealand	128.2	314.9
Norway	102.2	301.2
Switzerland	111.8	278.9
Hungary	67	277.4
Italy	50.0	263.3
West Germany	81.3	262.1
Netherlands	119.2	227.7
Israel	84.5	206.3
France	77.5	199.6
Spain	40.3	130.5
Greece	35.8	128.0
Japan	33.5	83.2
Taiwan	26.0	39.4

When the data are represented by a graph, the relationship between sugar intake and heart disease is easier to see. This graph is shown on the following page.



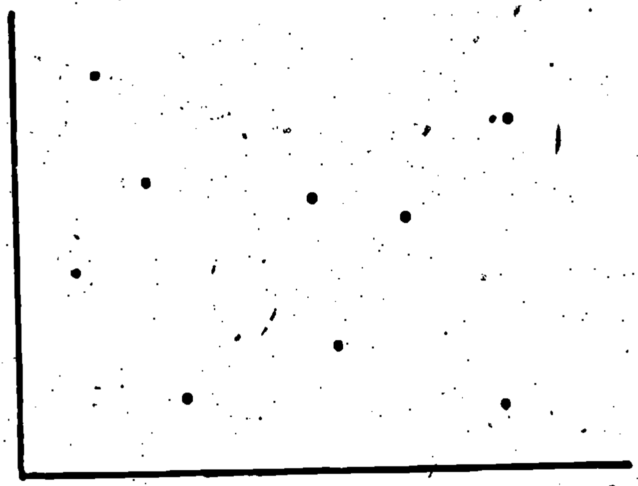
This graph seems to show an association between sugar intake and heart disease. The points are somewhat scattered but there is a tendency for high sugar intake to be associated with high death rate. But how strong is the association?

You may remember from Unit I that one way of measuring the strength of a relationship is to see how close the plotted points come to forming a straight line. This can be done visually by drawing a "best" straight line through the points and examining the degree to which the points tend to cluster around this "best" line.

Statisticians have a more precise way of doing much the same thing. They speak in terms of the <u>correlation</u> between two variables. They have developed a scale for measuring correlation, and have defined a term called the <u>correlation coefficient</u>. If a graph of the data is a set of points that lie on a perfectly straight line, the correlation coefficient is 1 (Figure 1). If the set of points is scattered completely at random over the graph, the correlation coefficient is 0 (Figure 2).

CORRELATION COEFFICIENT = 1

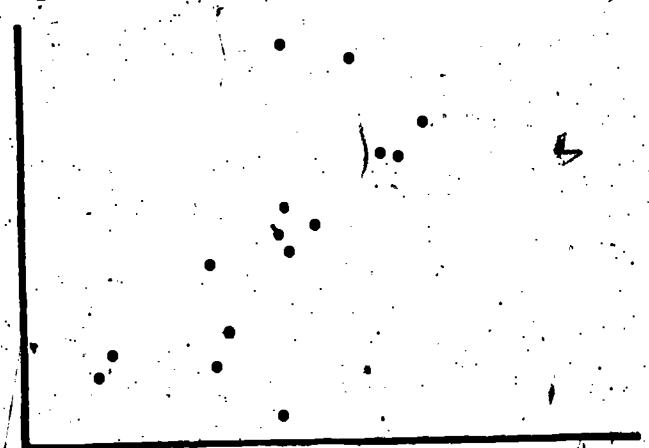
FIGURE 1



CORRELATION COEFFICIENT = 0

FIGURE 2

Relationships which lie somewhere between the linear and random cases will have correlation coefficients somewhere between 0 and 1. Figures 3 and 4 show examples.



CORRELATION COEFFICIENT = .66

FIGURE 3

CORRELATION COEFFICIENT = .95

FIGURE 4

If the correlation coefficient is close to 1 we say there is a high correlation between the variables. If it is close to 0 we say there is a low correlation. Now look back at the graph of heart disease as a function of sugar consumption. The correlation coefficient for that graph is .85. This indicates quite a strong relationship.

Can we now conclude that sugar consumption is a cause of heart disease? The answer is no. We could just as easily conclude that heart disease causes people to eat sugar. Or a third factor might cause both increased sugar consumption and heart attacks. To use a far-fetched example, we could argue that money is such a third factor. We could say that money allows people to buy more sugar and that worrying about money causes heart diseases. Based only on our graph, this is a possible explanation. The correlation demonstrates only that sugar consumption and heart disease are related; we cannot automatically conclude that one causes the other.

To use another example, the relationship between annual income and the death rate from heart disease turns out to have a correlation coefficient of .78, which is quite a high value. So both sugar and annual income are correlated with heart disease. We have already mentioned one far-fetched explanation, namely that money allows people to buy more sugar and worrying about money causes heart disease. We could also put forth the absurd explanation that people with heart disease develop a craving for sugar and that sugar gives people more energy to earn more money.

There are many possible explanations, some absurd and some plausible. On the basis of statistics alone we cannot choose between them. To prove a cause-and-effect relation between, say sugar and heart disease, controlled experiments are necessary. The mechanism by which foods cause heart disease must be specified. It so happens that a mechanism has been hypothesized to explain the correlations between sugar (and fat) consumption and heart disease. Experimental evidence has been collected to support the hypothesis. This is the topic of a later science section.

PROBLEM SET X:

In science class you discussed the relationship between fat intake and heart disease. In order to focus on this relationship, look at the table below which lists the death rates per 100,000 people from heart disease for twenty countries in 1966. They are listed in order from highest to lowest death rate. Also listed is the average number of grams of fat in milk, eggs and meat consumed per person per day from 1948 to 1965, with the exception of Hungary for which there were insufficient data before 1960. The value listed for Hungary is an educated guess.

Country	Average grams fat in M,E,M/ person/day (1948-1965)	Number of deaths/100,000 caused by heart disease (1966)
United Kingdom	68.0	378.5
Denmark	67.5	373.1
U.S.A.	77.3	371.2
Sweden	61.2	359.4
Finland	52.0	342.2
Australia	78.6	338.6
Austria	44.5	316.8
New Zealand	90.0	314.9
Norway	52.9	301.2
Switzerland	58.6	278.9
Hungary	32	277.4
Italy	20.1	263.3
West Germany	43.3	262.1
Netherlands	45.0	227.7
Israel	25.3	206.3
France	51.0	199.6
Spain	16.7	130.5
Greece	19.6	128.0
Japan	4.6	83.2
Taiwan	15.3	39.4

1. Draw a graph displaying the data in the table.

2. Based on your graph alone, does there seem to be an association between fat intake and death from heart disease?

3. Look at all the graphs in the text and make an educated guess on the correlation coefficient for the graph of Question 1.

4. Based on your graph, can you assume that fat intake causes heart disease?

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SECTION Y:

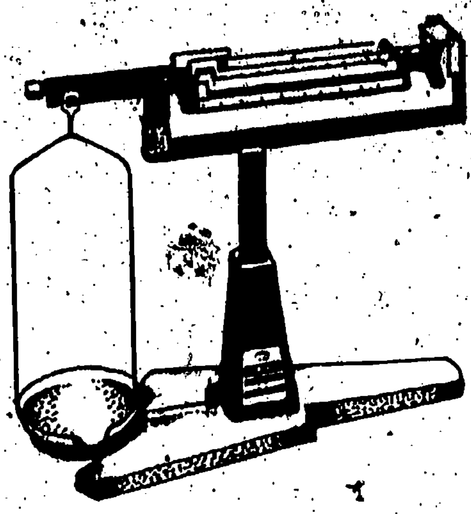
Y-1 Uncertainties in the Vitamin C Analysis

You will shortly be involved in a laboratory activity to measure the ascorbic acid concentration of orange juice. Measurement is involved in every phase of this activity. Involved in every measurement is some uncertainty. We will discuss how the uncertainty associated with each measurement translates into an uncertainty in the final result. Also, we will discuss factors which need to be considered when deciding on an estimated uncertainty.

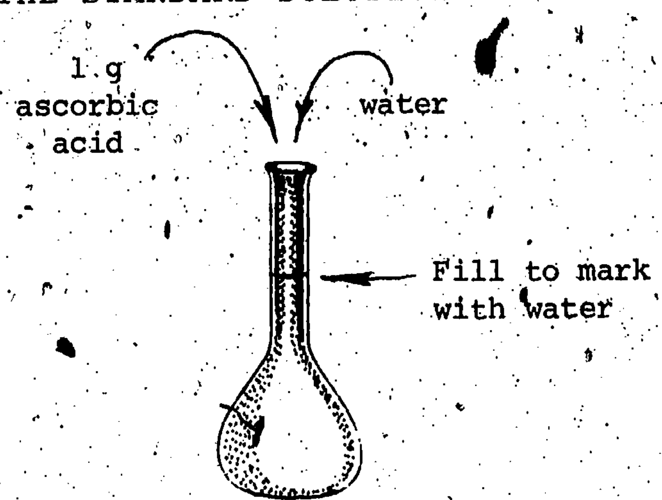
Y-2 The Standard Ascorbic Acid Solution

Igor, your friendly laboratory assistant, will prepare a standard ascorbic acid solution. A balance will be used to measure 1 ± .01 g of ascorbic acid which will be transferred to a 1-liter volumetric flask. Then enough water will be added to make 1 liter.

PART I: MAKING THE STANDARD SOLUTION



1. Weigh 1 g ascorbic acid



2. Add ascorbic acid to flask and add water to mark.

DIAGRAM 1

Both the balance and the volumetric flask contribute some uncertainty to the concentration of vitamin C in the solution.

$$\text{Concentration of Ascorbic Acid} = \overbrace{1\text{ g}}^{\text{Balance}} \Big/ \underbrace{1000\text{ ml}}_{\text{Volumetric Flask}}$$

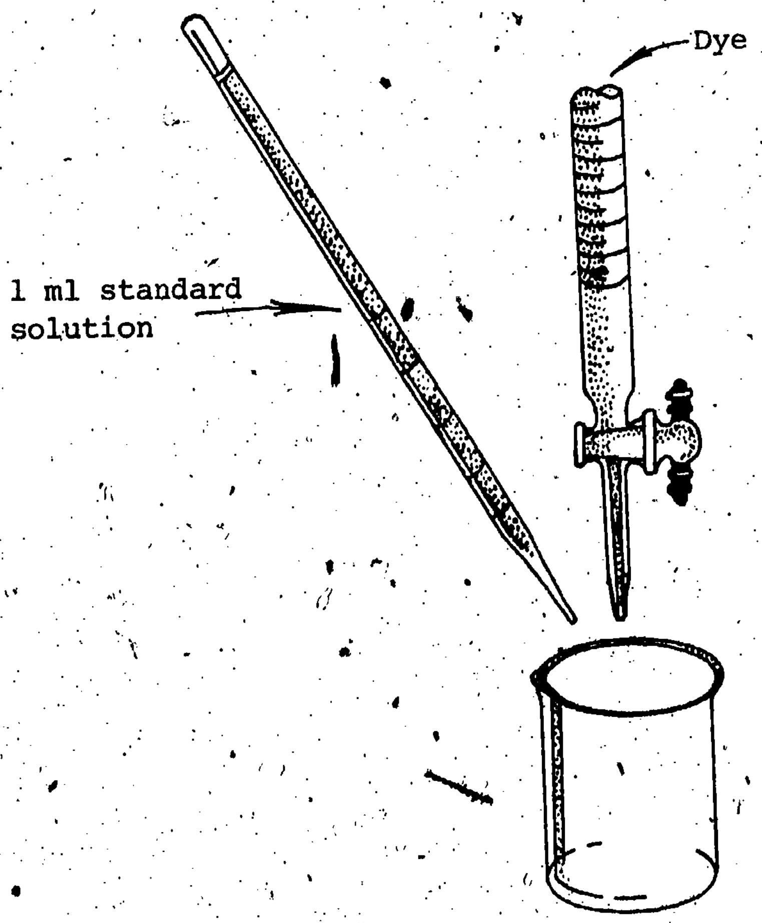
Since the concentration is a quotient, the relative uncertainties of the two measurements are added together to obtain the uncertainty of the concentration.

Relative Uncertainty of Standard Solution = Relative Uncertainty of Mass Measurement + Relative Uncertainty of Volume Measurement

Y-3 Standardizing the Dye

In the next part of the activity you will use the standard ascorbic acid solution to standardize the dye.

PART II: STANDARDIZING THE DYE



1. Add 1 ml standard solution to beaker
2. Titrate with x ml dye

DIAGRAM 27

A pipet is used to measure 1 ml of standard solution and a buret is used to measure the amount of dye that will react with the amount of ascorbic acid in the test solution. The objective is to find the conversion factor between ml dye and mg ascorbic acid. In addition, we want ascorbic acid in the numerator when we are through, because we want the answer to indicate how much vitamin C is in a given amount of orange juice. Therefore, we begin with the concentration of the standard ascorbic acid solution and then link up the remaining measurements so that the "ml st'd sol'n" units cancel.

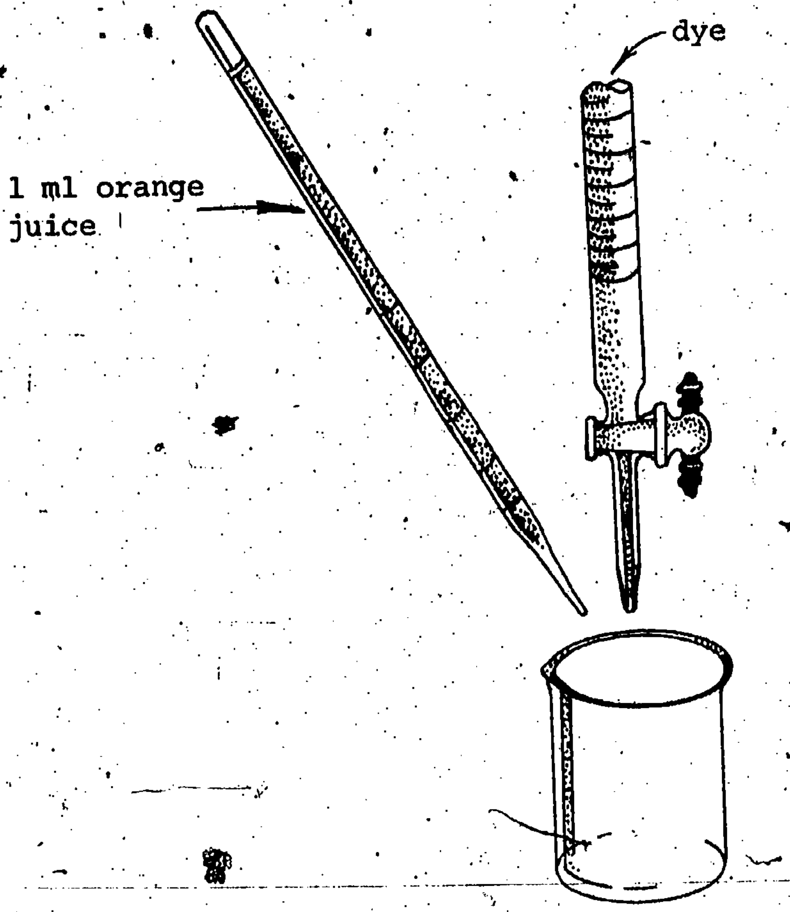
$$\underbrace{\overbrace{\frac{\text{1 g ascorbic acid}}{\text{1000 ml st'd sol'n}}}^{\text{balance}}}_{\text{volumetric flask}} \cdot \underbrace{\frac{\text{1000 mg ascorbic acid}}{\text{1 g ascorbic acid}}}_{\text{conversion factor-exact}} \cdot \underbrace{\overbrace{\frac{\text{1 ml st'd sol'n}}{\text{x ml dye}}}^{\text{pipet}}}_{\text{buret}} = \frac{\text{1 mg ascorbic acid}}{\text{x ml dye}}$$

Since all of the arithmetic operations are either multiplications or divisions, the relative uncertainty of the result is the sum of the relative uncertainties of all of the measurements involved.

Y-4 Determination of the Vitamin C Concentration of an Orange Juice Sample

In this part of the activity you will determine the volume of dye that reacts with the ascorbic acid in a 1-ml sample of orange juice. The procedure is the same as described in Section Y-3.

PART III: TITRATING ORANGE JUICE SAMPLES



1. Add 1 ml orange juice to beaker

2. Titrate with y ml dye

DIAGRAM 3

The amount of dye used (y ml dye) divided by the amount of orange juice (1 ml OJ) is a conversion factor which can be multiplied by the conversion factor of Section Y-3 to find the mg of ascorbic acid in 1 ml of orange juice.

$$\underbrace{\frac{1 \text{ mg ascorbic acid}}{x \text{ ml dye}}}_{\text{The uncertainty is constant and comes from the previous section.}} \; \frac{\overbrace{y \text{ ml dye}}^{\text{Buret}}}{\underbrace{1 \text{ ml OJ}}_{\text{Pipet}}} = \frac{y \text{ mg ascorbic acid}}{x \text{ ml OJ}}$$

Once again, since all of the arithmetic operations are either multiplications or divisions, we can find the relative uncertainty of the result by adding the relative uncertainties of the factors.

Y-5 Finding The Relative Uncertainties

The relative uncertainty of the ascorbic acid concentration of orange juice samples depends upon the relative uncertainties of all the measurements involved. Four instruments were needed. They were

a. A balance for measuring 1 g of ascorbic acid.

b. A volumetric flask for measuring the volume of the standard solution.

c. A pipet for measuring 1-ml samples of ascorbic acid and orange juice samples.

d. A buret for measuring the amount of dye used (twice).

Our task is to estimate the relative uncertainties associated with measurements made by each of these instruments. Occasionally the accuracy of an instrument will already be stated in relative terms. Often, however, we will know the absolute uncertainty and need to convert it to a relative uncertainty. Recall that relative uncertainty is the ratio of the absolute uncertainty to the size of measurement, i.e.,

$$\text{relative uncertainty} = \frac{\text{absolute uncertainty}}{\text{size of measurement}} (100\%)$$

You should use this formula to convert absolute uncertainties to relative ones.

Before any uncertainty calculations can be made, it is necessary to know the uncertainties of the instruments used. Therefore, the class will scrutinize the instruments and arrive at a common decision for the uncertainties associated with each instrument. To arrive at your decision for the estimate of absolute uncertainty, you will have to consider

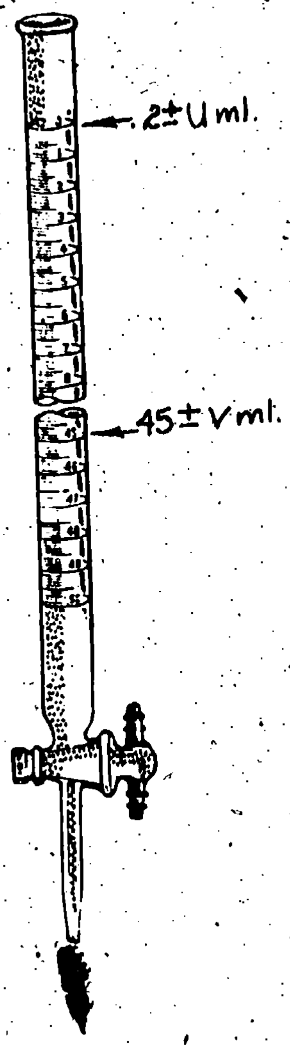
a. Imprecision or scale-reading uncertainty--roughly one-tenth of the smallest division.

b. Inaccuracy--information about the inaccuracy of your instruments may be obtained from the catalog from which they were ordered. Accuracy may be referred to as "tolerance" in these catalogs. These two considerations will be adequate for the flask and pipet.

An estimation of the uncertainty for the balance is more complicated, but has been considered earlier. Presumably, you have already arrived at a decision for the balance.

Y-6 <u>The Buret</u>

A buret measurement involves two readings, one at the top at the beginning of the titration and one lower down at the finish. To find the amount of fluid added, the difference between the two readings is determined. Recall that the absolute uncertainty of a difference is the sum of the two absolute uncertainties. For the situation depicted in the diagram the measurement would be 44.8 ± (u + v) ml.



To estimate the uncertainty of the initial reading (u), we need only consider inaccuracy and imprecision. To estimate the uncertainty of the endpoint we also need to consider the problem of seeing the endpoint. When excess dye is added to the ascorbic acid solution, the solution changes from colorless to faint pink, a color change that is difficult to see when it first happens. This factor will have to be considered in the estimation of the uncertainty (v) of the final reading.

PROBLEM SET Y:

Determine the range of uncertainty of each of the following products and quotients. State uncertainty in relative terms.

1. $(100 \pm 1\%)(.250 \pm 2\%)$

2. $(4.50 \pm 1.5\%)(.400 \pm 1.2\%)$

3. $(.702 \pm .8\%)(100 \pm .4\%)$

4. $\frac{280 \pm 1.1\%}{70 \pm 3.2\%}$

5. $\frac{560 \pm .03\%}{80 \pm .13\%}$

6. $\frac{.04 \pm .03\%}{1 \pm .01\%}$

Determine the range of uncertainty of each of the following sums and differences. State uncertainty in absolute terms.

7. $(50.0 \pm .2\text{ ml}) - (30.0 \pm .2\text{ ml})$

8. $(30.2 \pm .15\text{ ml}) - (.2 \pm .2\text{ ml})$

9. $(28.7 \pm .10\text{ ml}) - (0 \pm .12\text{ ml})$

10. $(17.41 \pm .01\text{ g}) - (3.20 \pm .01\text{ g})$

11. $(14.01 \pm .02\text{ g}) + (123.24 \pm .05\text{ g})$

12. $(26.497 \pm .001\text{ g}) + (18.203 \pm .001\text{ g})$

13. The first step in the vitamin C activity is making a standard ascorbic acid solution.

a. What two measuring instruments are used in this step?

b. The ratio of two measurement is the concentration of the standard solution. The (relative, absolute) uncertainty of the concentration is the (sum, product) of the (relative, absolute) uncertainties of the two measurements.

14. In the second part of this activity you determine the amount of dye that reacts with 1 ml of standard solution.

a. Name the two measuring instruments connected with this part of the activity.

b. Which instrument requires you to find the difference of two readings?

c. For which instrument will the absolute uncertainty be the sum of two absolute uncertainties?

15. Suppose the balance has an absolute uncertainty of .01 g and the volumetric flask has an uncertainty of 5 ml.

a. What is the relative uncertainty of a measurement of 1 g?

b. What is the relative uncertainty of a measurement of 1000 ml?

c. What is the relative uncertainty of the quotient of the two measurements?

16. Before the start of a titration a buret read .3 ml. At the endpoint it read 40.3 ml.

a. If each reading has an uncertainty of .1 ml, find the range of uncertainty (in absolute terms) of the volume of fluid delivered.

b. Find the relative uncertainty (in percent) of the amount of fluid delivered.

17. The dimensional algebra problem below describes how to find the number of mg of ascorbic acid in 1 ml of orange juice (AA = Ascorbic Acid). List the measuring instrument corresponding to each letter.

$$\underbrace{\frac{\overbrace{1\text{ g AA}}^{a}}{\underbrace{1000\text{ ml st'd sol'n}}_{b}}} \cdot \frac{10^3\text{ mg AA}}{1\text{ gAA}} \cdot \frac{\overbrace{1\text{ ml st'd sol'n}}^{c}}{\underbrace{x\text{ ml dye}}_{d}} \cdot \frac{\overbrace{y\text{ ml dye}}^{e}}{\underbrace{1\text{ ml OJ}}_{f}} = \frac{y}{x}\ \frac{\text{mg AA}}{\text{ml OJ}}$$

18. Assume the following relative uncertainties for measurements made by each instrument and calculate the relative uncertainty of the result of the dimensional algebra problem of Problem 17.

balance - 1%
volumetric flask - .2%
pipet - .3%
buret - .4% (combined relative uncertainty of initial reading and endpoint)

SECTION Z:

Z-1 A Sample Ascorbic Acid Uncertainty Calculation

For the purpose of demonstrating a typical ~~uncertainty~~ uncertainty calculation, we will assume the following uncertainties for each of the measuring instruments involved.

- balance - .01 g
- volumetric flask - .3 ml
- pipet - .01 ml
- buret - .1 ml for the initial reading
  .2 ml for the endpoint

The dimensional algebra problem which links up all of the measurements will tell us how to use the uncertainty information to find the uncertainty of the result.

LET AA = ASCORBIC ACID

SS = STANDARD SOLUTION

$$\frac{1\ g\ AA}{10^3\ ml\ SS} \cdot \frac{10^3\ mg\ AA}{g\ AA} \cdot \frac{1\ ml\ SS}{X\ ml\ dye} \cdot \frac{y\ ml\ dye}{1\ ml\ OJ} \cdot \frac{29.6\ ml\ OJ^*}{1\ fl\ oz\ OJ} \cdot 6\ fl\ oz\ OJ$$

	PART I	PART II		PART III		
Factor	$\frac{1\ g\ AA}{10^3\ ml\ SS}$	$\frac{10^3\ mg\ AA}{g\ AA}$	$\frac{1\ ml\ SS}{X\ ml\ dye}$	$\frac{y\ ml\ dye}{1\ ml\ OJ}$	$\frac{29.6\ ml\ OJ^*}{1\ fl\ oz\ OJ}$	6 fl oz OJ
Numerator measured by	balance		pipet	buret		
Denominator measured by	volumetric flask		buret	pipet		

The result of the calculation described above will be the number of mg of ascorbic acid in a 6-oz glass of orange juice. We are currently interested in the uncertainty of this result. Since all of the calculations in the formula are either multiplications or divisions, it will be most convenient if the uncertainties are first stated in relative terms. Then we can add all the uncertainties involved.

Examples of the calculations of the relative uncertainties for the balance, pipet and volumetric flask follow.

*The use of this rounded conversion factor introduces a constant error of (+.09%). For simplicity, we will neglect this error in our calculations.

BALANCE:

$$\frac{\text{absolute uncertainty}}{\text{size of measurement}}(100\%) = \frac{.01}{1}(100\%)$$

$$= 1\%$$

PIPET:

$$\text{relative uncertainty} = \frac{.01}{1}(100\%)$$

$$= 1\%$$

VOLUMETRIC FLASK:

$$\text{relative uncertainty} = \frac{.3}{1000}(100\%)$$

$$= .03\%$$

This leaves only the buret uncertainties to convert to relative terms. To do this we must know something about x and y. For example, let

x = 40 ml (The volume of dye that reacts with 1 ml of standard solution.

y = 15 ml (The volume of dye that reacts with 1 ml of orange juice)

First we will consider the buret uncertainty in Part II, the standardization of the dye. The total absolute uncertainty of the buret measurement is the sum of the uncertainties of the two readings, or .1 ml + .2 ml = .3 ml. This translates into a relative uncertainty of .75% for x.

$$\frac{\text{absolute uncertainty}}{\text{size of measurement}}(100\%) = \frac{.3\text{ ml}}{40\text{ ml}}(100\%)$$

$$= .75\%$$

The relative uncertainty for y is found in a like fashion.

$$\frac{.3\text{ ml}}{15\text{ ml}}(100\%) = 2\%$$

Now we have all the information needed to calculate the uncertainty of the result. We simply add the relative uncertainties of the terms in the dimensional algebra formula above.

PART I	balance	1.0%
	volumetric flask	.03%
PART II	pipet	1.0%
	buret (x)	.75%
PART III	buret (y)	2.0%
	pipet	+ 1.0%
		5.78%

This result implies that the true number of mg of ascorbic acid in a 6-oz glass of orange juice could lie at most 5.78% to either side of the calculated result, i.e., the midpoint of the range of uncertainty.

If you are required to calculate the uncertainty for the amount of vitamin C in different kinds of orange juice, you will be pleased to learn that all error calculations are identical except for the relative uncertainty of y. This means that for each separate uncertainty calculation, only the relative uncertainty of y needs to be recalculated; then this number can simply be added to the constant sum of the uncertainties of the other terms.