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AUTHOR Carpenter, Thomas P.; Moser, James M.
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ABSTRACT

This paper describes the research program of the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling. The major interest is in the development of children's abilities to solve verbal addition and subtraction problems and particularly in the processes and strategies used by children. Three factors are considered: (1) problem structure, (2) student characteristics, and (3) the nature of instruction. An analysis of verbal problems is presented. This analysis includes a discussion of various types of problem entities: discrete sets, continuous attributes, and actions or transformations. Problem structure is also analyzed along three dimensions: action vs. static relationships, set inclusion, and order (larger vs. smaller). Examples of various problem types are given. Results from the first year of an ongoing (1978-1981) longitudinal study of primary age students are included. These results are mainly gathered from individual problem-solving interviews with about 150 subjects. Various strategies employed and their change over time are examined. The paper concludes with a discussion of the implications of the research, both present and contemplated, for instruction. (Author/MK)

Theoretical Paper No. 79

An Investigation of the Learning of Addition and Subtraction

by Thomas P. Carpenter
and James M. Moser

November 1979

Wisconsin Research and Development
Center for Individualized Schooling

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AN INVESTIGATION OF THE LEARNING OF ADDITION AND SUBTRACTION

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Thomas P. Carpenter and James M. Moser

Conceptual Paper No. 1 of
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Report from the Project on
Studies in Mathematics

Thomas A. Romberg and Thomas P. Carpenter
Faculty Associates

James M. Moser
Senior Scientist

Wisconsin Research and Development Center
for Individualized Schooling
The University of Wisconsin
Madison, Wisconsin

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- conducting and synthesizing research to clarify the processes of school-age children's learning and development
- conducting and synthesizing research to clarify effective approaches to teaching students basic skills and concepts
- developing and demonstrating improved instructional strategies, processes, and materials for students, teachers, and school administrators
- providing assistance to educators which helps transfer the outcomes of research and development to improved practice in local schools and teacher education institutions

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WISCONSIN RESEARCH AND DEVELOPMENT
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Abstract

This paper describes the research program of the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling. The major interest is in the development of children's abilities to solve verbal addition and subtraction problems and particularly in the processes and strategies used by children. Three factors are considered: 1) problem structure, 2) student characteristics, and 3) the nature of instruction.

An analyses of verbal problems is presented. This analysis includes a discussion of various types of problem entities: discrete sets, continuous attributes, and actions or transformations. Problem structure is also analyzed along three dimensions: action vs. static relationships, set inclusion, and order (larger vs. smaller). Examples of various problem types are given.

Results from the first year of an ongoing (1978-1981) longitudinal study of primary age students are included. These results are mainly gathered from individual problem-solving interviews with about 150 subjects. Various strategies employed and their change over time are examined.

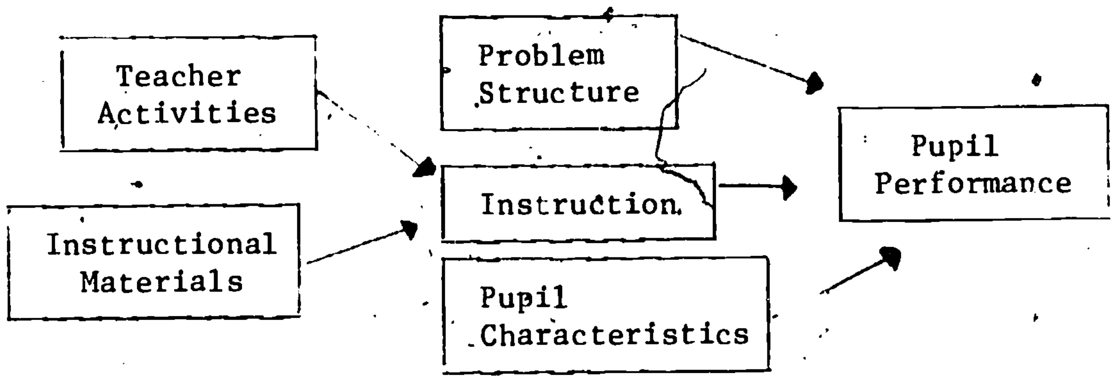
The paper concludes with a discussion of the implications of the research, both present and contemplated, for instruction.

A major aim of mathematical instruction is to enable students to acquire concepts and skills requisite for solving problems of many types. A principal goal of mathematical education research is to understand how children acquire those concepts and skills and to understand how selected pedagogical and psychological factors are related to that acquisition. The Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling is presently conducting a program of research focused on a small set of those concepts and skills. Its interest lies in arithmetical learning, and in particular, in the acquisition of concepts and skills related to addition and subtraction of whole numbers.

A primary focus of the mathematics project of the Wisconsin Research and Development Center for Individualized Schooling is to study the processes that children use to solve simple verbal addition and subtraction problems and to identify how these processes evolve over time. We believe that this investigation will not only help us to better understand children's problem solving skills but will also help us to understand how children acquire basic addition and subtraction concepts and skills.

The types of problems that we are concerned with are the simple story problems or word problems commonly found in elementary mathematics textbooks that can be solved by a single operation of addition or subtraction. We are not suggesting that children necessarily solve these problems by adding or subtracting. In fact we have found that young children generally do not solve them by applying an arithmetic operation. It is convenient, however, to define the problem domain in terms of these operations.

Our research is investigating various factors that influence children's problem solving behavior. These factors are the structure of the problem, characteristics of the children and in particular certain cognitive processes, instructional materials, and teacher's classroom behaviors. The interrelationship of these factors is depicted below.



Our research has progressed the furthest regarding the effect of problem structure, and this aspect will be the primary focus of this paper. In order to understand the effect of problem structure it is necessary to characterize the major structural differences between different addition and subtraction problems.

An Analysis of Verbal Problems

There are several approaches that previous research has taken to characterize verbal problems. One is to classify problems in terms of syntax, vocabulary level, number of words in a problem, etc. (Jerman, 1973; Suppes, Loftus & Jerman, 1969). A second approach differentiates between problems in terms of the open sentences they represent (Grouws, 1972; Rosenthal & Resnick, 1974; Lindvall & Ibarra, 1978). We have chosen a third alternative that considers the semantic characteristics of the problem. Our analysis is generally consistent with other analyses based on problem structure (Gibb, 1956; Greeno, 1978; Nesher & Katriel, 1978; Vergnaud & Durand, 1976), but we have introduced certain distinctions not included in previous analyses of problem types. In our research we have been primarily concerned with structural characteristics involving the action or relationships described in the problem. In order to fully characterize verbal problems, however, it is also necessary to consider the nature of the entities in the problem.

Nature of the Entities in the Problem

We have identified three distinct types of entities in addition and subtraction problems. What all three have in common is that they are measurable or can be represented by a number.

The first type we consider is a discrete collection of objects. In this case, it is possible to represent the elements in a set given in a particular problem by counting out an appropriate number of physical objects to make an equivalent set. There is a one-to-one correspondence between the problem set and a constructed set or between the problem set and a set of counting words so that one can actually think of each element in the constructed or spoken set as representing an element in the problem set.

In contrast to discrete sets, we can consider entities characterized by an attribute which is continuous in nature, such as length, age or temperature. For continuous measures, however, any set that might be constructed would represent the quantity in a very different sense. There would not be a one-to-one correspondence, because there are no identifiable elements in this type of problem set. Thus, for continuous quantity problems, the constructed set would represent the number assigned to the attribute in the problem, but would not represent the attribute directly. Needless to say, continuous quantities present potentially more complex problem situations. It is also possible that continuous measures that are not directly observable such as age or weight are more complex than measures such as length or area that tend to be more easily discerned on a visual basis.

Both of the previous categories involve a measurable entity that is acted upon or transformed to yield another measurable entity. In the third category the entities are actions or transformations. The following examples illustrate this distinction. In the first example the entities are sets and in the second they are transformations.

> John had 8 pennies. He spent 5 pennies. How many pennies did he have left?
>
> Mary had some pennies. Her father gave her 8 more pennies. Then she spent 5 pennies. How many more pennies did she have than she started with?

In the first problem the entities are sets of pennies that could be directly represented by sets of objects. In the second problem the entities are a change in the total number of pennies, not a set of pennies; the problem deals with the magnitude of the composite change, not with a set. In the first problem there is a set of 8 objects, and 5 objects are removed from it. In the second problem there is an initial relative set of unspecified magnitude. Eight objects are joined to it, and 5 are removed. The 5 objects are not removed from the set of 8 objects that were added but from the larger set. The following problem illustrates why this is a critical point.

> Mary had some pennies. Her father gave her 5 more pennies. Then she spent 8 pennies. How many fewer pennies did she have than she started with?

This problem also illustrates another characteristic of transformations; they have both a magnitude and a direction. In other words they may be represented by both positive and negative integers.

In the research we have completed, we have only used problems involving discrete sets. Consequently all of the examples in the next section dealing with problem structure involve discrete sets. However, continuous quantities could easily be substituted for the sets in the examples. It is somewhat more difficult to fit problems involving compositions of transformations into the model, and they may in fact represent somewhat distinct problem types (cf. Vergnaud & Durand, 1976).

Problem Structure

We have identified three orthogonal dimensions that characterize the different action or relationships involved in verbal addition and subtraction problems.

The first dimension is based upon whether an active or static relationship between sets or objects is implied in the problem. Some problems may contain an explicit reference to a completed or contemplated action causing a change in the size of problem entities. For example, "Sue had 8 apples in a basket. Then she put 6 more apples in that basket. How many apples did she have altogether?" Contrasted to such situations are those in which no action is implied; that is, there is a static relationship. As an example, consider, "There are 7 apples in a basket. Four are red and the rest are green. How many of the apples are green?"

The presence or absence of action carries with it a temporal aspect. When an action is performed, there is usually an initial state which is changed or transformed as a result of the action. Thus, a before-after relationship is part of the situation. This is not the case when there is no action. We do not suggest that temporal considerations are different from the action/static dimension but rather that they are simply a potentially different manner of considering such situations.

The second dimension involves a set inclusion or set-subset relationship. In certain problems two of the entities involved in the problem are necessarily a subset of the third. In other words, either the unknown quantity is made up of the two given quantities, or one of the given quantities is made of the other given quantity and the unknown. For example, consider the following problem: "There are seven children on the playground. Three are boys and the rest are girls. How many are girls?" The set of boys and the set of girls are subsets of the set of children. The alternative is that one of the quantities is disjoint from the other two. For another example, consider the following problem: "There are seven girls and three boys on the playground. How many more girls than boys are there?" In this problem, removing a set of three girls and counting the number of girls in the remaining set of four girls is one way of determining the answer. The distinction between this problem and the preceding one is that the set of boys is disjoint from all of the sets of girls involved.

The third dimension is best described as an order relationship. In the static relationship among the entities, there may be the notion that one entity is larger or smaller than another. Where action is described in the problem, that action may result in something being made larger (increased) or being made smaller (decreased).

It is helpful to visualize these three dimensions by means of a three dimensional solid that represents a two-by-two-by-two matrix (Figure 1).

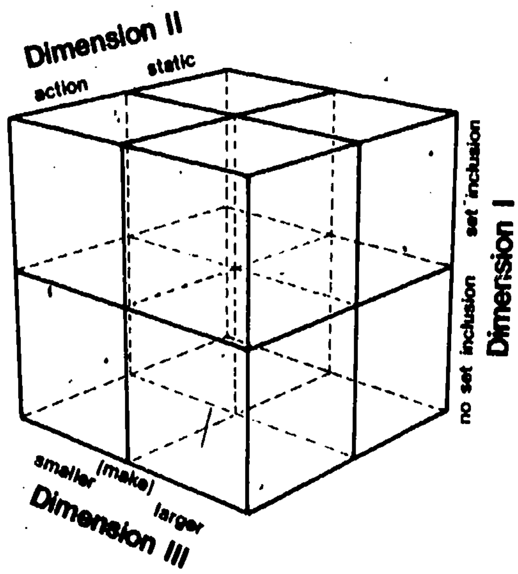


Figure 1. Schematic diagram of characteristics of addition/subtraction problems.

For each cell of the matrix there are three distinct problem types depending upon which quantities are given and which is the unknown. Although the action or relationship involved in each problem is essentially the same, the problems are very different and potentially involve different methods of solution. In fact each cell of the matrix contains both addition and subtraction problems. Furthermore, there are significant differences in difficulty between problems within a single cell of the matrix that are a function of which quantities are given and which is the unknown. (Grouws, 1972; Lindvall & Ibarra, 1978). The distinction between different problems within a cell in the matrix are illustrated by the examples given of Joining and Separating problems. In general the same sort of variation is possible for problems in the other cells of the matrix. A characterization of problems corresponding to each cell of the matrix follows.

Joining. Joining situations often arise in early mathematics instruction because they are generally easy to understand and tend to be familiar to young children. Joining is the process of actively putting together an entity B with an entity A to form a new single entity C. A is made larger or increased by B so that the union has measure c. Both A and B are subsets of C. Figure 2 indicates where Joining occurs in the three dimensional matrix.

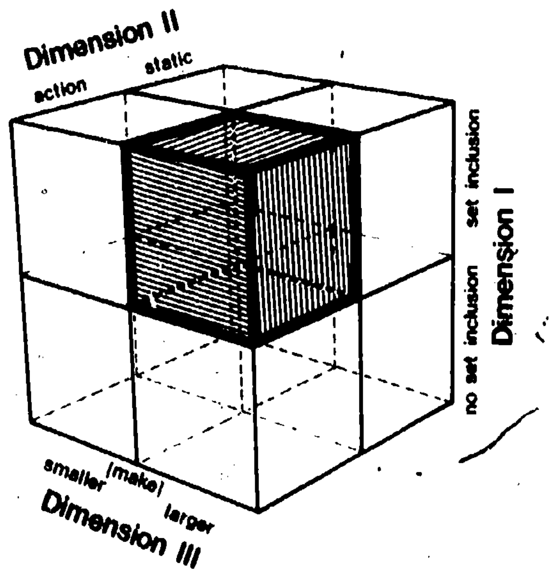


Figure 2. Schematic representation of "Joining" problems.

The following examples illustrate the three basic types of Joining problems.

> Wally had 3 pennies. His father gave him 6 more pennies. How many pennies did he have altogether?

Wally had 3 pennies. His father gave him some more pennies. Then he had 9 pennies altogether. How many pennies did his father give him?

Wally had some pennies. His father gave him 6 more pennies. Then he had 9 pennies altogether. How many pennies did Wally have to begin with?

Separating. Separating problems have the same characteristics as Joining problems except that the action involves a decrease rather than an increase (See Figure 3).

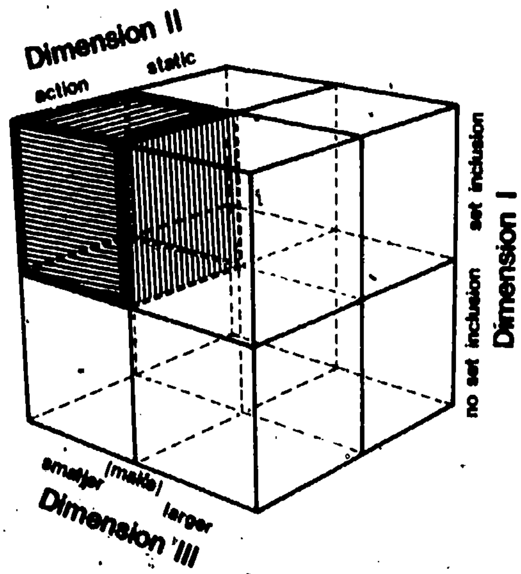


Figure 3. Schematic representation of "Separating" problems.

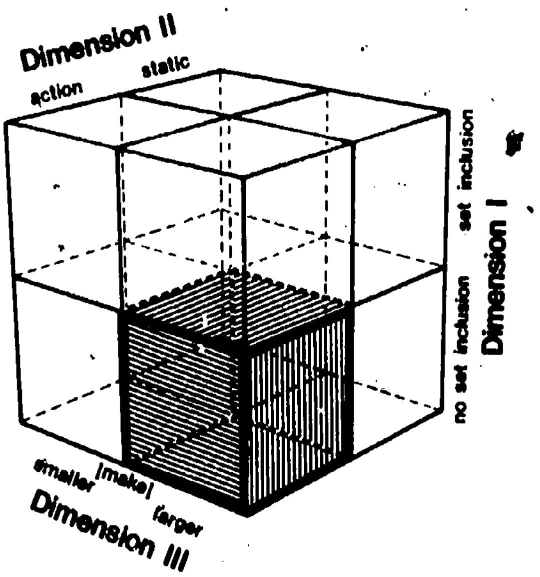
In Separating problems a subset is removed from a given set. The three basic Separating problems are illustrated below.

Fred had 8 pieces of candy. He gave 3 pieces to Jane. How many pieces did he have left?

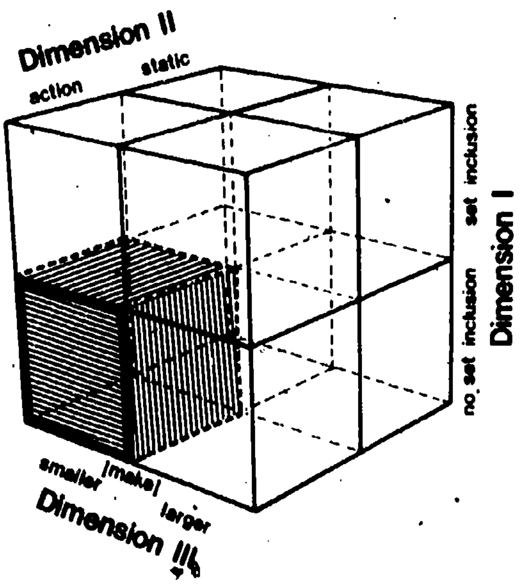
Fred had 8 pieces of candy. He gave some to Jane. He had 5 pieces left. How many did he give to Jane?

Fred had some candy. He gave 3 pieces to Jane. He had 5 pieces left. How many pieces did he have to start with?

Equalizing. Equalizing problems are not as well-known as Joining and Separating. They are used extensively in the earlier sections of the Developing Mathematical Processes (DMP) program that was developed at the Wisconsin Research and Development Center (Romberg, Harvey, Moser, Montgomery, 1974). There are two types of Equalizing problems; one involves an increase and one involves a decrease (See Figure 4).



Equalize-add on



Equalize-take away

Figure 4. Schematic representation of "Equalize" problems.

Equalizing problems involve the same sort of action that is found in Joining and Separating problems but there is also a comparison involved. Basically equalizing is a process of changing one of two entities so that the two are then equal on some particular attribute. The following is an example of an Equalizing-add on problem.

> There are 6 boys and 9 girls in the dancing class. How many more boys have to be put in the class in order for there to be the same number of boys and girls?

Equalizing-take away problems are virtually identical except that the action involves a decrease in the larger set.

> There are 6 boys and 9 girls in the dancing class. How many girls have to leave the class in order for there to be the same number of boys and girls?

Part-Part-Whole. Part-Part-Whole problems involve a static relationship existing between an entity having a particular attribute and its two disjoint, but complementary, parts. The Part-Part-Whole problem type -- a static situation in which the set-inclusion relationship is present -- does not appear to divide itself naturally along the third dimension of order. It is tempting to try to dichotomize the problem types on some logical basis. However, such attempts have proven fruitless. Thus, we choose to show Part-Part-Whole problems as occupying two cells of the matrix (Figure 5). Some examples follow.

> There are 2 boys and 6 girls in the dancing class. How many children are there altogther?

> There are 8 children in the dancing class. Six of them are girls. How many boys are in the class?

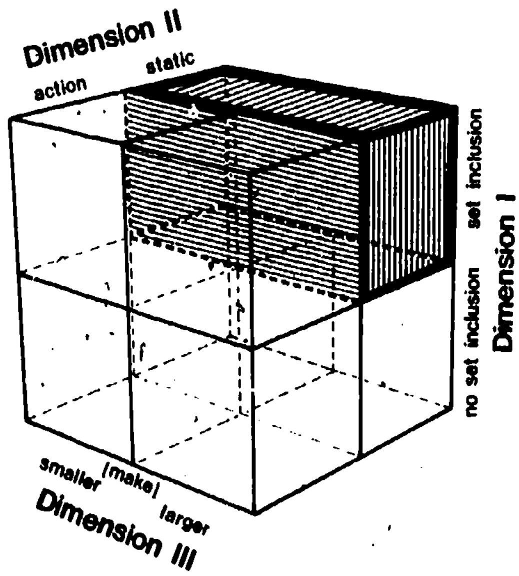
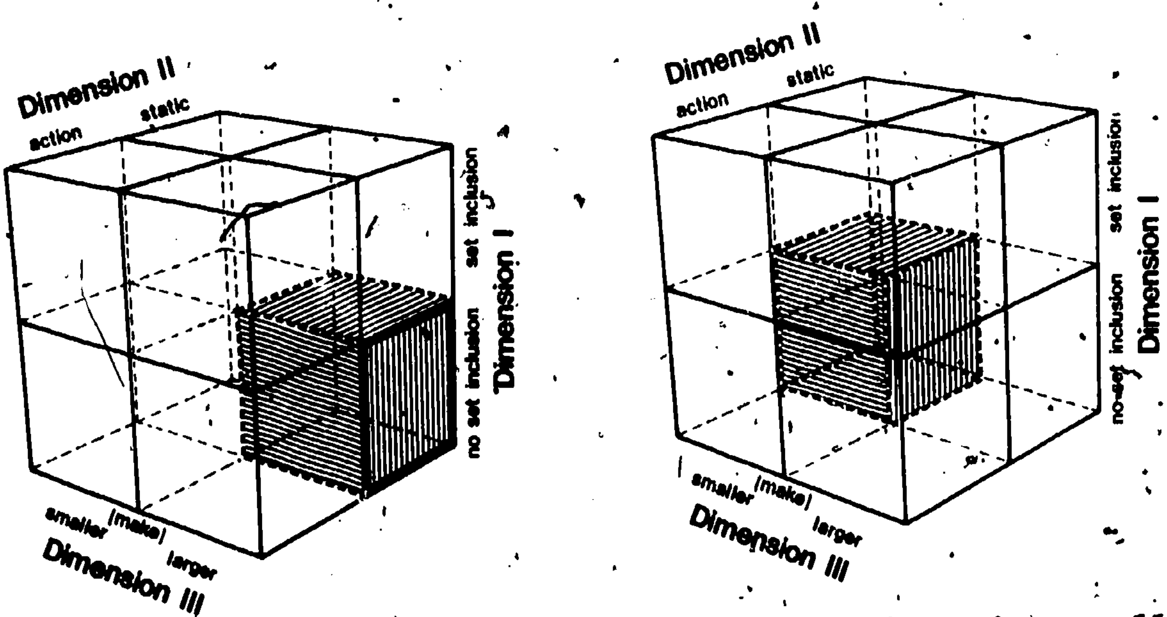


Figure 5. Schematic representation of "Part-Part-Whole" problems.

Comparison. In Comparison problems there is a static relationship of order existing between two disjoint entities. As the name implies the Comparison problems involve a comparison of two quantities. These problems are similar to the Equalizing problems except that there is no implied action to increase or decrease one of the quantities. As with the Equalizing problems there are two types of comparison problems (See Figure 6). A Comparison-larger problem situation involves the amount by which the larger of two compared entities exceeds the smaller on a stipulated attribute. In this situation, the student is directed by the problem language to focus on the fact that some entity A is larger than a second entity B. Several examples follow.



Comparison-larger

Comparison-smaller

Figure 6. Schematic representation of "Comparison" problems.

> Joe has 5 records. Mike has 13 records. Mike has how many more records than Joe?
>
> Joe has 5 records. Mike has 8 more records than Joe. How many records does Mike have?

A second, similar type of problem is called the Comparison-smaller problem. As with the Comparison-larger problem, it involves a static relationship of order between two disjoint entities. However, the Comparison-smaller problem focuses attention on the smaller of the two entities and on the amount by which it is less than the second entity. The problems are identical to the Comparison-larger problems except that the question is who has fewer.

> Joe has 5 records. Mike has 13 records. Joe has how many fewer records than Mike?

Our model does not unambiguously characterize all problems, and there are some problems that are difficult to place in a single cell in the model. For most types of addition and subtraction problems, we have evidence that children's solutions reflect the distinctions between problem types characterized by the model. The one contrast we have not systematically investigated is the one between the two types of comparison problems. It is possible that this distinction is not useful in characterizing children's performance. This would imply that the increase-decrease dimension of our model would only apply to the action dimension of the matrix.

So far we have only investigated 9 of the 21 problems characterized by the model. These problems represent six of the seven cells of the matrix; the one exception is the Comparison-smaller cell. In our initial studies we have been concerned with the processes that young children use to solve addition and subtraction problems. Consequently we have begun with problems that logical analysis or empirical evidence would suggest are most likely to be solved by young children. In general these problems are ones in which the action or relationships described in the problem can be directly modeled without trial and error.

Research Program

In the studies conducted or presently in progress, the principal focus is on the strategies children use to solve basic verbal addition and subtraction problems. In spring 1977, a pilot study involving 43 first grade children was carried out (Carpenter, Hiebert & Moser, 1979). A following study involving the same subjects was conducted in May of the same year

(Carpenter, Moser and Hiebert, in press). Both studies provided interesting findings in their own right. In addition, they helped identify suitable problem tasks, refine data gathering and data reporting techniques, and clarify many of the problem solving strategies children use. These findings provided the basis for the design of our major research effort, a three year longitudinal study of 150 children that is currently in progress.

Procedures

The longitudinal study is investigating three major sets of variables. The basic dependent variable is children's performance on simple addition and subtraction verbal problems. The primary method of assessing children's performance is through individual interviews administered at the beginning, middle, and end of each school year. These results are supplemented with paper and pencil achievement monitoring tests administered approximately every six weeks and unit tests given after each major unit of arithmetic instruction.

The second set of variables deals with measures of specific cognitive abilities that potentially are related to performance on addition and subtraction problems. Included in this set of variables are measures of conservation, class inclusion, and information processing capacity. The third major set of variables that we are investigating involves the kind and amount of instruction that pupils receive on individual topics. A key component of these measures, which are gathered through direct observation in the classroom, is the recording of pupil allocated and engaged time, using techniques developed in the Beginning Teacher Evaluation Study (Jones & Romberg, 1979). In following sections, results from the first year of the study will be discussed.

Subjects

Subjects for the study consist of 150 children from eight first grade classrooms. The classes were in three elementary schools that all draw from predominantly white middle to upper-middle class neighborhoods. The schools were selected in an attempt to control for instruction. All three used a modified version of Developing Mathematical Processes (DMP), the activity-oriented problem solving program developed at the Wisconsin Research and Development Center (Romberg, et al., 1974). The modifications occur primarily with instructional units whose objectives deal with writing and solving open sentences that represent addition and subtraction problems. These modifications are described in detail in Kouba and Moser (1979).

At the time of the first interview in October 1978, subjects had generally experienced only the readiness activities typical of a kindergarten mathematics curriculum. Some had begun work on writing numerals. By the time of the second interview in January 1979, the instruction had covered only two purely arithmetic units, Writing Numbers and Comparison Sentences. The other topics dealt with measurement and geometry. Comparison Sentences introduces the notion of a mathematical sentence, though it only deals with representing a static relation (equality) between two numbers. Thus, at the time of the second interview the children had received no formal instruction in symbolic representation of addition and subtraction. On the other hand, several lessons which included problem situations involving joining, separating, part-part-whole and comparison had been presented. In those instances, modeling with objects to determine the solutions had been suggested.

By the time of the third interview in May 1979, several instructional units on addition and subtraction were presented. The units required approximately two months of instruction and focused on the following objectives: writing number sentences of the form $a + b = \square$ or $a - b = \square$ to represent concrete and verbal problem situations, and solving number sentences of the form $a + b = \square$ and $a - b = \square$ for sums between 0 and 10. The problem situations were of the joining, separating, comparison and part-part-whole types. Several key features highlight these units. First, the children are strongly encouraged to use modeling behaviors by representing numbers with sets of physical objects. Second, various forms of counting are suggested. Finally, analysis of verbal problems is taught using a device that has the part-part-whole relationship as its basis. This device tends to highlight the inverse relationship of addition and subtraction.

Problems

Each interview included six problem types, two having an additive structure and four having a subtractive structure. Representative problems and the order in which they were given in the interview are presented in Table 1. The specific problems were selected because 1) they were representative of problems commonly included in elementary mathematics texts, 2) they include the three basic, but different, types associated with subtraction, 3) they were problems that the younger subjects were most likely to be able to solve, and 4) the earlier pilot study (Carpenter, et al., 1979) had indicated that they would elicit different patterns of solution. Each problem type was presented under four different conditions, resulting from the crossing of two variables, number size and the availability of manipulative

Table 1

Representative Addition and Subtraction Problems

1. Joining (Addition)	Wally had 3 pennies. His father gave him 5 more pennies. How many pennies did Wally have altogether?
2. Separating (Subtraction)	Tim had 11 candies. He gave 7 candies to Martha. How many candies did Tim have left?
3. Part-Part-Whole (Subtraction)	There are 6 children on the playground. 4 are boys and the rest are girls. How many girls are on the playground?
4. Part-Part-Whole (Addition)	Sara has 6 sugar donuts. She also has 9 plain donuts. How many donuts does Sara have altogether?
5. Comparison (Subtraction)	Joe has 3 balloons. His sister Connie has 5 balloons. How many more balloons does Connie have than Joe?
6. Joining (Subtraction)	Kathy has 5 pencils. How many more pencils does she have to put with them so she has 7 pencils altogether?

aids. The manipulative dimension involved the presence or absence of physical objects that could be used to represent the action or relationships described in the problems. Number size included a set of smaller number triples, the sum of whose addends was between 5 and 9, and a larger set for which the sum was between 11 and 16.

The assignment of number triples to problem types involved a six-by-six Latin square design resulting in six sets of six problem tasks each of which were uniformly and randomly distributed across subjects.

Individual Interviews

The interviews were broken into two parts, with the 12 problems involving smaller numbers given on one day and the remaining 12 with larger numbers given on a succeeding day. Interviews were cut short at any time it became apparent a subject was floundering. The interview procedures were not clinical in the sense described by Opper (1977). Rather, they could be considered as an attempt at naturalistic observation. If a student's strategies could be directly observed, no follow-up questions were posed. If not, the interviewer attempted to determine the strategy by asking the child further questions. All interviewers followed a standardized routine for questioning children and coding responses. All interviewers were trained to the point that intra and intercoder reliability coefficients were greater than .90 (Martin, in press).

We now turn to a presentation and discussion of results obtained to date. Because the longitudinal study is still in progress, some data have been only partially analyzed. The analysis of the data will be reported on a cross-sectioned basis. Tracing of the development of individual children over time has not yet been completed. However, the longitudinal study results together with the results from the pilot studies cited earlier (Carpenter, et al., 1979; Carpenter, et al., in press) provide a reasonably consistent picture of children's initial solution processes for simple verbal problems. Addition will be discussed first, followed by subtraction.

Addition

Our basic interest is in the strategies children use, both before they receive formal (i.e., school) instruction and during and after they receive initial instruction in the operations of addition and subtraction. Because these children have, at best, limited exposure to the formal operations of arithmetic, the strategies they do exhibit are a result of their intuition, their invention, and of informal instruction and experiences (e.g., parents, older siblings, kindergarten). Most of the strategies are based on counting and are similar to the strategies for solution of numerical addition problems identified by Suppes and Groen (1967) and Groen and Parkman (1972). Other strategies exhibited by our subjects are not based strictly on counting.

Addition Strategies

In all problem contexts reported, the measurable entities were discrete sets. In two of the four settings, plastic cubes were made available for modeling. In a third, the numbers were sufficiently small that the subjects' ten fingers could be easily used as representations of the two sets given in the problem. Only in the fourth setting, larger numbers without the cubes present, was it true that physical representations of the sets described in the problem were relatively inaccessible. As we shall report, this fourth setting induced different behaviors on the part of some children. The manipulatives, cubes or fingers, were used in two distinct ways. In one case, they stood as direct representatives of the problem entities. In the second, they served as a marker or tracking aid to help the child remember some counting sequence.

The various types of strategies used to solve addition verbal problems would seem to occur in a logical order of difficulty, or degree of sophistication. This is a suggested ordering made on a logical analysis of the levels of abstraction. Empirical evidence supporting or contradicting this order has not yet been analyzed. These strategies are summarized below. In the following discussion, m is the smaller addend in the problem, n the larger, and t is the sum. In other words $m + n = t$.

Counting All with Models. Cubes or fingers are used to independently count out and represent both sets. Then the union of the two sets is counted. Three distinct counting sequences are used, each associated with a direct one-to-one count of a set in the problem. One is 1, 2, ..., m; another is 1, 2, ..., n; the third is 1, 2, ..., m + n. The answer is the number of objects in the union set. We make no distinction if the smaller or larger set is modeled first. In actual practice, most children modeled the sets in the order in which they were given, which was always the smaller one first.

Counting All without Models. This is essentially the SUM strategy as identified in the response latency studies by Suppes and Groen (1967) and Groen and Parkman (1972). Neither set is modeled. The counting sequence begins with "one" and a simple counting procedure is executed until either m or n is reached. At that point, a double-count is initiated as the child continues until the final word, m + n, is reached in the sequence. The first count at the intermediate point continues as m + 1, m + 2, etc. (or n + 1, n + 2, ...) while the second and presumably simultaneous count is 1, 2, 3, ... n (or 1, 2, 3, ... m). Keeping track of the second count may be done by objects (rarely observed by us), by fingers, or mentally. The answer is the final number in the counting sequence.

Counting On from First (smaller) Number. Exactly like the previous strategy with the major exception that the counting sequence begins with m or m + 1. Again, tracking of the second count may be done with objects, fingers, or mentally.

Counting On from Larger Number. This is the MIN strategy identified in the response latency studies. Here the counting sequence begins with n or n + 1. Tracking is done as in the other strategies.

Number Fact. Although the children we are working with had not been taught number facts until the latter part of the school year, some of them learned a great deal about addition outside of school, including a wide range of number facts. These children were generally able to apply their knowledge of addition facts to solve simple verbal problems.

Heuristic. Heuristic strategies are employed to generate solutions from a small set of known basic facts. These strategies usually are based on doubles or numbers whose sum is 10. For example, to solve a problem representing 6 + 8 = ? a child responded that 6 + 6 = 12 and 6 + 8 is just 2 more than 12. In another example involving the operation 4 + 7 = ? another child responded that 4 + 6 = 10 and 4 + 7 is just 1 more than 10.

The latter three counting strategies that involve the double, or simultaneous counting deserve further discussion. Although we did not observe any instances of this behavior, it is theoretically possible to use a clearly different process to track the second set involved in the double count. In the three strategies described, the second set is being constructed, either by cubes, fingers or a string of counting words, as the double counting is being carried out. In contrast, a child could conceivably construct that second set (probably on a physical basis) prior to beginning the double count. Then the tracking would be carried out by successively removing objects from the constructed set and would end when that set was exhausted.

The information processing demands on the child would seem to be much less for the latter situation in which the second, tracked set was constructed prior to the simultaneous count than in the former where the second set is being constructed as the child is counting. It would appear that the child would have to continuously check whether the second count had yet reached the desired target number. When carried out mentally, it was difficult to determine how the child knew when to stop. Some children appeared to use some sort of rhythmical or cadenced counting. Others explicitly described a double count. But children generally had difficulty describing this process. When fingers were used to construct the second set, it seems to us that children have a special kinesthetic, quasi-subitizable sense about knowing when a particular number of fingers have been raised (or lowered).

Addition Results

A summary of results for the two addition problems is presented in Tables 2 and 3. (Wording of the problems is given in Table 1.) Although both interviews 1 and 2 were conducted before children received formal instruction in addition, most children were able to solve both addition problems. In fact the overall pattern of responses for both problems is almost identical both in terms of number correct and strategy. This suggests that there is very little difference in the way that children approach these two types of problem.

It is not the case, however, that all addition problems are equivalent. In an earlier pilot study (Carpenter, et al., 1979), the following Comparison-larger problem was found to be significantly more difficult than Joining and Part-Part-Whole problems:

> Ralph has 8 pieces of gum. Jeff has 5 more pieces than Ralph. How many pieces of gum does Jeff have?

Although over 80 percent of the first graders in that study could solve the other two addition problems, fewer than 25 percent correctly solved this Comparison problem. Over 50 percent gave one of the numbers in the problem as their answer. They did not seem to be able to understand that "Jeff had 5 more pieces of gum than Ralph" and interpreted it as "Jeff had 5 more pieces of gum." Although not performing at a high level, children were still better able to deal with the "more than" relation in the Comparison problem with subtractive structure. It seems to be this particular addition comparison context that gave them difficulty.

Table 2

Results for Joining (Addition) Problems [Interview Task #1]

Condition	Interview	Number Correct*	STRATEGY: Counting All, with models	Counting All, without models	Counting on, from first	Counting on, from larger	Numerical, number fact	Numerical, heuristic
Smaller numbers Physical objects	1	96	63	7	13	9	14	0
	2	126	83	0	7	13	25	4
	3	138	50	0	8	23	48	5
Smaller numbers No physical objects	1	90	49	11	11	8	10	8
	2	113	50	2	15	14	26	4
	3	137	25	1	21	21	63	2
Larger numbers Physical objects	1	68	67	0	5	11	1	3
	2	99	80	0	12	18	2	6
	3	125	70	0	17	33	17	3
Larger numbers No physical objects	1	44	29	0	11	12	0	2
	2	68	25	1	25	12	1	7
	3	106	21	1	34	42	18	3

* n = 144 for Interview 1; n = 150 for Interview 2 and 3

Table 3

Results for Part-Part-Whole (Addition) Problems [Interview Task #4]

			STRATEGY					
			Counting All		Counting On		Numerical	
Condition	Interview	Number Correct*	with models	without models	from first	from larger	number fact	heuristic
Smaller numbers Physical objects	1	108	77	7	7	8	8	4
	2	123	78	1	7	12	20	9
	3	137	49	1	6	29	45	2
Smaller numbers No physical objects	1	92	54	8	8	13	11	5
	2	108	47	5	13	14	27	8
	3	137	21	1	19	24	54	0
Larger numbers Physical objects	1	72	73	1	4	5	2	2
	2	107	73	0	15	19	3	5
	3	126	68	2	13	37	13	6
Larger numbers No physical objects	1	41	23	2	12	11	1	1
	2	68	25	3	12	25	4	6
	3	99	19	0	31	49	8	5

* n = 144 for Interview 1; n = 150 for Interview 2 and 3

There is a difference in the structure of the addition problems that may account for this difference in difficulty. As noted in the earlier discussion of problem types both the Joining and Part-Part-Whole problems have the set-inclusion dimension. Thus, when a child constructs sets representing both entities and takes their union, that child is actually modeling the problem. This is not the case for the addition Comparison problem for which the set-inclusion property does not hold. The union of sets representing the quantities described in the Comparison problem does not quite model the relationship of the problem.

For the Joining and Part-Part-Whole problems, it appears that some children are able to represent and solve problems involving small numbers before they can solve similar problems involving larger numbers. In theory, the process of solving problems with small numbers or large numbers are the same when physical objects are available. But the problems with smaller numbers were significantly easier.

Of particular interest was the fourth interview condition where larger numbers were used but no physical aids were available. Since it is more difficult to represent numbers larger than 10 with fingers, many children opted to use the Counting On strategies rather than the less advanced Counting All strategy that they would use when physical aids were present.

There was also a marked increase in the Counting On strategies over time. Although we cannot completely rule out the possibility of informal instruction or some formal instruction by some of the classroom teachers involved in the study, we would still propose that it is strong evidence in support of the theory put forth by Groen and Resnick (1977) that children invent these strategies for themselves.

Subtraction

It is generally acknowledged that subtraction is harder for children than addition. Although a number of reasons could be advanced for this difference, we propose that a possible cause of this difficulty is the fact that there are several distinct representations possible for subtraction problems while addition is generally defined as the union of two sets. We have identified three basic types of subtraction strategies which represent the distinctly different actions of separating, joining, and comparing. Some of these strategies operate at different levels of abstraction in much the same way as the addition strategies. There is the low level modeling of sets and actions with physical objects accompanied by simple counting and the more sophisticated counting strategies that involve the double-count and tracking procedures. While not directly associated with a specific problem or strategy type, use of number facts and heuristics are also used to solve subtraction problems. The different strategies are described below. Some have been identified in the response latency study of numerical subtraction problems carried out by Woods, Resnick and Groen (1975).

Subtraction Strategies

Rather than describe the strategies in the order of level of sophistication, we have listed them by the various types that correspond to the problem types. The number sentences $m - n = d$ or $n + d = m$ represent the mathematical operation characterized by the problem.

Separating From, with models. The child uses concrete objects or fingers to construct the larger given set m and then takes away or separates, one at a time, a number of cubes or fingers equal to the smaller given number n in the problem. Counting the set of remaining cubes yields the answer. Three distinct counting sequences are used. The first is 1, 2, ..., m; the second is 1, 2, ..., n; and the third is 1, 2, ..., m - n (or d).

Counting Down From. In a more abstract representation of the separating from strategy, a child initiates a backwards counting sequence beginning with the given larger number m. It is conceivable that a child could precede that by counting 1, 2, ..., m; but we never observed it. The backwards counting sequence contains as many counting number words as the given smaller number. The last number uttered in the counting sequence is the answer. Here a double-count is necessary to keep track that the correct number of counting words has been uttered. As with the counting on strategies for addition, the tracking may be accomplished by a constructed set of cubes (rarely seen) or fingers, or mentally. This is the method number 2 identified in the Woods et al. (1975) study.

Separating To, with models. The Separating To strategy is similar to the Separating From strategy except that the separating continues until the smaller quantity is attained rather than until it has been removed. In the concrete case, after the larger set m is counted out, the child removes cubes one at a time until the remainder n is equal to the second given number of the problem. Counting the number of cubes (d) removed gives the answer. Again, three distinct counting sequences are used.

Counting Down To. A child initiates a backwards counting sequence beginning with the larger given number. The sequence ends with the smaller number. By keeping track of the number of counting words uttered in this sequence, either mentally or by using fingers or cubes, the child determines the answer to be the number of counting words uttered in the sequence. It is interesting to observe that Woods, et al. (1975) did not identify this strategy. From a response latency perspective, it would involve the same number of steps as a Counting Up From Given strategy.

Adding On, with models. With concrete objects the child sets out a number of cubes equal to the smaller given number (n). The child then adds cubes to that set one at a time until the new collection is equal to the larger given number (m). Counting the number of cubes added on (d) gives the answer. Here too, three counting sequences are used. The first is 1, 2, ..., n. The second is n + 1, n + 2, ..., m. No tracking is needed because the child knows to stop whenever the word "m" is uttered. The third count is 1, 2, ..., m - n (d).

Counting Up from Given. A child initiates a forward counting sequence beginning with the smaller given number n. The sequence ends with the larger given number m. Again, by using any of the available devices, the child keeps track of the number of counting words uttered in the sequence, and thereby determines the answer. This is method number 3 of the Woods', et al., (1975) study.

Matching. Matching is only feasible when concrete objects are available. The child puts out two sets of cubes, each set standing for one of the given numbers. The sets are then matched one-to-one. Counting the unmatched cubes gives the answer.

Greeno (1978) has hypothesized that children may use a single strategy to solve all subtraction problems. He suggests, for example, that certain problems are associated directly with a subtraction operation. Others are first transformed to one of the representations that is directly associated with an operation. This analysis would seem to imply that all of the problems that are initially transformed into the same basic representation would generate the same solution strategy.

An alternative hypothesis is that different strategies would be used, depending on the structure of the problem. As we have just seen, certain of the strategies naturally model the action described in specific problems. The Separating problem is most clearly modeled by the separating strategies. On the other hand, the implied joining action of the Joining (missing addend) problems is most closely modeled by the Adding On or Counting Up strategy. Comparison problems deal with static relationships between sets rather than action. In this case the Matching strategy appears to provide the best model.

For the Part-Part-Whole subtraction problem the situation is more ambiguous. Since Part-Part-Whole problems have no implied action, neither the Separating nor Adding On strategies (or their counting analogues), which involve action, exactly model the given relationship between quantities. And since one of the given entities is a subset of the other, there are not two distinct sets that can be matched. In the next section we shall present evidence that the second hypothesis best characterizes children's solution strategies. In other words, children tend to model the action or relationship described in the problem rather than attempting to relate the problem to a single operation of subtraction.

Subtraction Results

The data for each of the four subtraction problems are presented in Table 4, 5, 6, and 7. The incidence of the Separating To and Counting Down To strategies was so small that that category is not included. For the sake of readability, uncodable responses as well as incorrect responses such as guessing, repeating one of the given numbers, or adding instead of subtracting are also not included in the tables. We would observe, however, that there were relatively few instances of these types of errors. Most often children who were unable to solve a problem because they were unable to represent the action or relationship in the problem. They very seldom, however, represented it in an incorrect or inappropriate way.

The results indicate that the dominant factor in determining children's strategy was the structure of the problem. The strategy used by the great majority of children modeled the action or relationship described in the problem. This was true through all three interviews and under all problem conditions. For the Separating probem (Table 4), almost all children used a subtractive strategy (Separating, or Counting Down). For the Joining-Missing Addend problem (Table 5), almost all children used a strategy (Adding On or Counting Up). The results were not quite as overwhelming for the Comparison problem (Table 6), but the Matching strategy was the most frequently used strategy when physical objects were available. In general, this strategy is

Table 4

Results for Separating (Subtraction) Problems [Interview Task #2]

Condition	Interview	Number Correct*	STRATEGY						
			SUBTRACTIVE		ADDITIVE		COMPARATIVE	NUMERICAL	
			Separate	Count down from	Add on	Count up from given	Match	Number fact	Heuristic
Smaller numbers	1	92	72	6	3	3	0	7	1
Physical objects	2	108	93	5	0	3	0	17	5
	3	134	85	13	0	0	0	32	7
Smaller numbers	1	74	53	7	1	3	0	9	5
No physical objects	2	92	60	5	0	3	0	15	4
	3	129	65	12	1	9	0	37	3
Larger numbers	1	60	81	2	1	0	0	1	1
Physical objects	2	92	102	1	1	4	0	2	3
	3	111	96	13	1	9	0	6	8
Larger numbers	1	24	33	6	0	1	1	1	1
No physical objects	2	31	26	9	0	6	0	2	2
	3	68	35	21	2	12	0	15	8

* n = 144 for Interview 1; n = 150 for Interviews 2 and 3

Table 5

Results for Joining (Subtraction) Problems [Interview Task #6]

			STRATEGY						
			SUBTRACTIVE		ADDITIVE		COMPARATIVE	NUMERICAL	
Condition	Interview	Number Correct*	Separate	Count down from	Add on	Count up from given	Match	Number fact	Heuristic
Smaller numbers Physical objects	1	90	6	0	60	15	2	7	2
	2	120	2	0	68	24	0	23	4
	3	133	2	0	47	30	0	51	3
Smaller numbers No physical objects	1	86	0	0	53	28	0	9	2
	2	110	0	0	43	24	0	25	3
	3	132	1	0	30	39	0	49	3
Larger numbers Physical objects	1	56	7	0	45	13	5	2	2
	2	86	3	0	65	18	1	3	6
	3	114	5	0	70	27	3	12	8
Larger numbers No physical objects	1	36	0	0	10	26	0	1	2
	2	59	0	0	16	39	0	2	3
	3	92	1	0	21	65	0	11	8

* n = 144 for Interview ; n = 150 for Interviews 2 and 3

Table 6

Results for Comparison (Subtraction) Problems [Interview Task #5]

Condition	Interview	Number Correct*	STRATEGY: SUBTRACTIVE Separate	SUBTRACTIVE Count down from	ADDITIVE Add on	ADDITIVE Count up from given	COMPARATIVE Match	NUMERICAL Number fact	NUMERICAL Heuristic
Smaller numbers Physical objects	1	75	21	1	10	9	18	4	1
	2	84	18	1	4	13	36	7	3
	3	108	16	4	3	9	31	30	3
Smaller numbers No physical objects	1	61	10	0	16	17	2	6	2
	2	76	9	2	16	22	4	13	1
	3	97	9	2	8	27	5	30	5
Larger numbers Physical objects	1	55	18	0	6	9	26	1	1
	2	61	12	0	5	13	45	2	1
	3	85	21	2	7	17	50	4	7
Larger numbers No physical objects	1	29	2	1	5	18	0	1	3
	2	42	1	1	9	26	0	3	6
	3	58	3	0	7	34	0	10	5

* n = 144 for Interview 1 and n = 150 for Interviews 2 and 3

Table 7

Results for Part-Part-Whole (Subtraction) Problems [Interview Task #3]

Condition	Interview	Number Correct*	STRATEGY						
			SUBTRACTIVE		ADDITIVE		COMPARATIVE	NUMERICAL	
			Separate	Count down from	Add on	Count up from given	Match	Number fact	Heuristic
Smaller numbers Physical objects	1	67	41	1	10	7	1	5	2
	2	82	45	1	3	7	0	12	5
	3	119	72	2	6	5	0	27	4
Smaller numbers No physical objects	1	50	32	1	6	5	0	3	4
	2	71	28	2	6	7	0	19	4
	3	104	47	4	0	11	0	31	2
Larger numbers Physical objects	1	51	55	2	7	3	0	0	3
	2	68	67	0	6	5	0	3	3
	3	105	83	8	3	11	1	9	7
Larger numbers No physical objects	1	19	19	0	4	11	0	0	2
	2	34	12	2	2	16	0	1	4
	3	64	26	14	3	18	0	11	4

* n = 144 for Interview 1 and n = 150 for Interviews 2 and 3

not possible when there are not objects available to construct the two sets to put in one-to-one correspondence. Interestingly, we did have several instances where children tried to match their fingers on one hand with those on the other hand. This occurred when the number triplet 2 - 3 - 5 was used.

The ambiguity of the Part-Part-Whole problem (Table 7) is reflected in children's selection of strategies. Although a majority tended to use a subtractive strategy, the additive strategies were used by a significant minority, especially in the fourth condition where manipulative objects and sufficient fingers were not available to model the separating process.

In the first two interviews the Counting Down strategy was used relatively infrequently. Although a subtractive strategy was almost universally used to solve the Separating problem, children tended to use the Separating strategy with physical objects or fingers. Over three times as many children used the Counting Up strategy to solve the Joining-Missing Addend problem as used the Counting Down strategy to solve the Separating problem. Counting Down is a difficult process. And when explicitly asked to count backwards a given number of steps, only about 50 percent of the first-graders in our sample could do so. Although our data are not conclusive in this regard and others have identified Counting Down as a basic subtraction strategy (Woods et al., 1975), we would conjecture that some children never use a Counting Down strategy prior to learning basic subtraction facts.

General Discussion of Strategies

It has been clearly established that children enter school with reasonably well developed counting procedures, and that they invent strategies based on counting procedures to solve simple addition and subtraction problems (cf. Ginsburg, 1977; Resnick 1978). In fact, the investigations carried out by a number of other researchers support this conclusion. Ours is no exception. Our research also offers some support to the conclusion that children first apply these strategies to small numbers and subsequently extend them to larger number domains (Gelman & Gallistel, 1978).

Our research indicates that in solving simple verbal problems children use various counting techniques to directly represent the action or relationships described in the problem. Our current results do not offer a complete picture of the evolution of these representation processes. However, based on the data we have, we would make an educated guess that at the earliest stages children solve problems directly by representing the quantities described in the problem and then performing the indicated action on these representations.

In our current reseatch we have focused on problems in which the action or relationships described in the problems can be directly modeled. In other problems like the following missing minuend problem this is not the case.

> Mary had some marbles. After she lost
> 5 of them, she had 8 marbles left. How
> many marbles did Mary have to start with?

In this problem the initial state is the unknown quantity. To directly model this action would require some sort of trial and error strategy in which children guess at the size of the initial set and check their guess by removing 5 elements to see if there are 8 elements left. It is possible that this sort of problem will generate trial and error variations of the strategies that we have identified. Rosenthal and Resnick (1974) also suggest that trial and error strategies might be employed for this type of problem. However, we are aware of very little empirical evidence that indicates that children systematically use trial and error strategies to solve these problems rather than transforming them so that they can be solved directly.

This analysis might help explain differences in difficulty between different problem types. It would imply that problems in which the quantities given in the problem are operated on directly would be easier than problems in which they were not. In the analysis of the Comparison addition problem in the addition results section above this was indeed the case. This analysis may also explain why Separating problems like:

> You have some stamps. You give
> 7 stamps to Judy. You now have
> 4 stamps. How many stamps did
> you have to begin with?

are significantly more difficult than related Separating or Missing Addend problems (Grouws, 1972; Lindvall & Ibarra, 1978). In the action described in the problem, 7 stamps are being removed from an unknown quantity. The only way to directly model this action is to already know the answer to the problem or to use trial and error.

The difference in difficulty between action and static problems (Nesher & Katriel, 1978; Steffe, 1970) may also reflect how clearly the action or relationships are specified in the problem. In the Comparison and Part-Part-Whole subtraction problems children were less consistent in their choice of strategy than they were for the Separating or Joining subtraction problems. This may reflect the fact that children had more difficulty figuring out how to model the relationships in the static problems, which was ultimately reflected in their ability to solve the problem correctly. In this regard it is noteworthy that in our study there was no difference in performance on the action and static state addition problems. Since there is only one general model of addition the exact representation of the action or relationship was not an issue.

Although they may have difficulty applying them to all problem situations, it appears that early in their development of subtraction concepts children have a variety of strategies for solving different subtraction problems. There may be a general overriding strategy of modeling the action or relationship described in a problem. But it manifests itself in several very different ways that provide different interpretations of subtraction. We would hypothesize that at first children do not recognize the interchangeability of their strategies. This would account for the fact that there is such a close match between problem structure and strategy. Even though a Counting Up strategy is much easier and with the numbers in our problems more efficient than a Counting Down strategy, most children in our sample continued to attempt to use some form of Separating strategy for the Separating problem. Woods et al. (1975) hypothesized that older children are able to choose the most efficient of the counting strategies to solve numerical problems. So far we have no data to support this conclusion with regard to children's solutions of verbal problems. They would suggest, however, that younger children have independent conceptions of subtraction. A completely developed concept of subtraction invovles an integration of all these interpretations.

Our data do provides some insights into how that development may take place. Apparently, the first step might involve a shift to more abstract counting strategies from concrete strategies that completely model the problem. Although at the time of the third interview most of the children in our sample continued to use a strategy that represented the action or relationship described in the problem, almost half of them were using the more abstract Counting Up and Counting Down strategies rather than the more concrete Adding On and Separating strategies. Thus, the ability to choose between strategies representing different interpretations of subtraction seems to come after the ability to use more abstract versions of a given strategy in a particular problem.

So far we have said very little about the relationship between the formal mathematics that children learn as part of the mathematics curriculum and the informal strategies they invent independently. We have completed one study that examines the relationship between children's symbolic representation of addition and subtraction problems and their strategies for solving them (Carpenter, et al., in press). After several months of instruction, most children could write addition and subtraction sentences of the form $a + b = \square$ or $a - b = \square$ to represent Joining and Separating problems but had more difficulty representing the other types of problems. At this stage very few children recognized that the arithmetic sentence was a mechanism that they might use to help them solve the problem. Most of them continued to use the verbal problem as the basis for deciding how to solve the problem. In fact, in spite of instructions to the contrary, about 25 percent of the subjects would solve a problem before writing a sentence. In general few children clearly understood the relationship between the number sentence and the problem.

This pilot study also provided some evidence that children's strategies were less influenced by problem structure after several months of instruction on addition and subtraction. So far this trend is not quite as evident in the data from the longitudinal study.

As a final comment to this section it is interesting to contrast the performance of the children we have studied and the problem solving abilities of older students. We have found that young children very carefully analyze problems and base their solutions on the structure and content of the problem. This analytic ability is precisely what older children lack. Although they are generally successful in solving simple addition, subtraction, multiplication, and division word problems, they have a great deal of difficulty with even simple nonroutine problems that involve anything more than a straightforward application of a single arithmetic operation (Carpenter, Corbitt, Kepner, Lindquist, & Reys, in press a, b).

Other Variables

In the discussion so far we have attempted to characterize the processes children use to solve simple verbal addition and subtraction problems and how they may evolve over time. In this regard we have focused on the effect of problem structure on children's solution processes. In addition to problem structure, there are two other important variables that we are investigating: characteristics of the child solving the problem and the nature of instruction the child has received. So far we have not made as much progress in examining these variables as we have in identifying the effect of problem structure, bu we would like to briefly characterize some of the factors we have investigated and those we are continuing to investigate.

Individual Differences

There clearly are differences in the rate at which children acquire basic addition and subtraction problem solving skills. At the time of each interview, a great deal of variability in performance was observed. What is not yet clear is whether children are simply at different stages of development with respect to a given skill or whether different children go through different patterns of development in the acquisition of addition and subtraction concepts and skills. For example, are children who use a Counting On strategy simply further along in their acquisition of addition concepts than children who use a Counting All strategy, or do some children need to rely on Counting All strategies up until the time they develop formal addition concepts? The fact that there is a steady increase over time in the number of children using a Counting On strategy would argue for the fact that children using a Counting All strategy were simply at a lower level in their acquisition of addition concepts. Other studies have also examined this issue and have generally concluded that older children increasingly use more advanced coutning strategies. (cf. Woods et al., 1975).

So far we have only examined our data on a cross sectional basis. As we examine our data on a longitudinal basis and trace the change of performance of individual children over time, we should begin to get a better idea of whether there is a well defined sequence in which children acquire addition and subtraction problem solving concepts and skills.

One of the factors that we have examined to attempt to account for individual differences in the acquisition of addition and subtraction problem solving strategies is the relation of these strategies to measures of more general cognitive abilities that might be prerequisites. (Carpenter & Hiebert, in press a, b). From an instructional point of view, the question of whether the ability to solve problems or apply strategies is tied to the development of certain basic cognitive abilities is an important one. There are potentially different instructional implications if the ability to solve certain problems or use certain strategies is closely linked to fundamental cognitive abilities whose development is difficult to accelerate than if this is not the case.

The variables that we have explored are several of the logical abilities that Piaget (1952) proposes represent the foundation of number concepts, and a measure of information processing as characterized by Case (1978) and Pascual-Leone (1970, 1976). In addition to the argument that these variables represent fundamental concepts that underlie the development of the most basic number concepts, there were several other reasons for their selection. A strictly logical analysis of the addition and subtraction problems and strategies themselves would suggest that these concepts are directly involved in certain of the problems and strategies. For example, the concept of class inclusion is a basic dimension of certain addition and subtraction problems. The Part-Part-Whole problems deal with subordinate relations very similar to those found in classical class inclusion tasks. Similarly many of the addition and subtraction strategies involve transformations that presuppose conservation. Although we have not yet been able to identify the specific information processing requirements of individual strategies, different strategies seem to place very different demands on children's information processing capacity, and it is reasonable that more advanced strategies may require more advanced information processing capacities.

There are not only logical reasons for considering these variables; there are empirical reasons as well. Previous studies have found that measures of basic Piagetian variables are highly correlated with performance on arithmetic achievement tests (Carpenter, 1979a; Carpenter, Hiebert, Blume, Martin & Pimm, in press). Although these studies have done little to uncover explicit relationships between these variables and specific arithmetic skills, Steffe, Spikes, & Hirstein (1976) conclude that certain of these abilities are required to learn to apply some of the more advanced counting strategies.

In spite of the reasons that one might put forth to expect these variables to be productive in helping account for children's performance on addition and subtraction problems, we have found that this is not the case. Not only have we found that these basic abilities are not prerequisites for solving certain problems or applying certain strategies, but the correlations are modest at best. Based on some other research we have done on the learning of measurement concepts (Hiebert, 1979), we have concluded that these variables are useful in explaining performance on tasks whose logical structure is similar to the task measuring the basic cognitive ability. For tasks that are based on the application of a skill like counting, performance is not as closely related.

The Influence of Instruction

Although children clearly invent strategies to solve problems that they are not explicitly taught, it is unlikely that the construction of these strategies is unaffected by instruction. Psychologists studying the development of early number concepts have generally assumed that specific instruction plays relatively little part in the development of these basic concepts. Although this assumption may be appropriate for early number concepts on which children receive comparatively little formal instruction in school, it is less valid, for describing children's acquisition of addition and subtraction concepts. It may be that within the limits imposed by the range of common practices within the elementary curriculum, variations in instruction have relatively little effect on the strategies that children employ. But this is an unwarranted assumption on an a priori basis. It is important to at least monitor instruction in order to have some idea of the match between formal instruction and the informal mathematics that children construct themselves.

Since we are ultimately concerned with applying our knowledge about the development of addition and subtraction concepts in children to the design of instruction, we are especially interested in the effect of instruction on the development of these concepts. Furthermore, since we are also concerned with the implications of our results for the mathematics curriculum in schools, we have chosen to conduct our studies in natural school settings rather than artificially controlled laboratory experiments.

In the studies that are completed or in progress, we have attempted to control mathematics instruction. All of the children in the major studies we have completed have been studying from the same mathematics program, a modified version of Developing Mathematical Processes. To further take into account variations in the kind or amount of instruction that are introduced by the teacher or the individual child, we are examining specific classroom experiences of individual children. A major element in this dimension of our research is the observation of allocated and engaged time, using techniques developed in the Beginning Teacher Evaluation Study (Jones & Romberg, 1979). Classroom observations of participating teachers and students will provide evidence as to what types of activities students have engaged in, the amount of time they have engaged in them, and the teaching behaviors that affect engagement.

The observation data are still incomplete, but the available data clearly indicate that even with the same mathematics program there are significant differences in instruction and pupil engagement. It is not yet clear whether these differences have any effect on the strategies that children use to solve addition and subtraction problems. But there is evidence that these differences are related to differences in learning certain content like knowledge of basic facts. This would seem to have implications for the time at which children would switch from invented strategies to formal arithmetic operations.

It is reasonable that instruction would have an increasingly greater effect in the later stages of acquisition of addition and subtraction concepts. The very early concepts, like those studied by Brush (1978) are probably relatively independent of specific instruction. Even the construction of counting strategies does not appear to be greatly influenced by instruction. Whether the transition to more sophisticated strategies could be accelerated by instruction is an open question. But even without specific instruction, children construct the strategies themselves. The shift to solving problems by using basic facts and formal algorithms certainly is related to instruction. It appears likely that, depending on instruction, this shift may take place earlier in some children than in others. Whether it is productive or in the long range beneficial to attempt to accelerate this transition to formal mathematical operations is also an open question that requires further research.

Implications for Instruction

Our long range goal for studying children's learning of basic mathematics concepts is to provide a basis for designing more effective instruction. It is not our objective to generate increasingly fine ground analyses of children's behavior but to provide a description of children's learning, and do so at a level that it may potentially impact instruction. Consequently we have selected a content area that is a central focus of the mathematics curriculum, we have selected relevant variables, and we have chosen to study the acquisition of this content in real school environments rather than controlled laboratory settings.

Applying knowledge about children's learning to instructional decision making is not trivial. Although instruction should be consistent with the ways children learn, the most effective instruction cannot be deduced directly from an examination of children's spontaneous learning. This issue has been discussed at greater length in another paper (Carpenter, 1979b; see also Glaser, 1976; & Resnick, 1976). The point is that we are not proposing that our research in its current state clearly specifies an appropriate sequence of instruction. A great deal of intermediate research is still required that specifically attempts to establish how instruction can be designed to effectively build upon the spontaneous acquisition of addition and subtraction concepts that we have observed and facilitate the transition to formal addition and subtraction operations.

On a long range basis we see our research having implications for instruction in two general areas: the selection and sequencing of content and the individualization of instruction. There is ample evidence that children enter school with well developed counting processes and that they naturally depend upon these processes to deal with problems involving numbers. The typical mathematics program, however, fails to build upon the richness and growing sophistication of these strategies. This is one area in which research involving the design of instructional alternatives might build upon our research. A second involves the integration of verbal problems into the mathematics curriculum. It has typically been assumed that children must first master computational skills before they can apply them to solve problems. We have demonstrated, however, that children can solve basic verbal problems before they learn formal addition and subtraction skills. Rather than requiring computational skills for their solution, basic problems give meaning to addition and subtraction operations. This suggests that verbal problems might provide a basis for introducing addition and subtraction concepts and that verbal problems may be effectively integrated into the instructional sequence a great deal more extensively than is now the case.

An effective program for individualizing instruction must be based on some measure of how children are different. If we can establish a clear picture of how addition and subtraction concepts are acquired, this knowledge could provide one basis for individualizing instruction. Presumably different content and different types of instruction would be appropriate for children at different stages in the acquisition of addition and subtraction concepts. The analysis of the acquisition of these concepts potentially provides a basis for evaluating children's concepts and skills and designing instruction that is appropriate for children at different stages.

More detailed knowledge of children's addition and subtraction processes should provide a more substantial basis for making instructional decisions. But as the above examples indicate, there is already a reasonable base to support a great deal of instructional research.

Brush, L.R. Preschool children's knowledge of addition and subtraction. Journal for Research in Mathematics Education, 1978, 9, 44-45.

Carpenter, T.P. Cognitive development research and mathematics education. (Theoretical Paper No. 73). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979. (a)

Carpenter, T.P. Research on children's thinking and the design of mathematics instruction. In R. Lesh & D. Mierkiewicz, (Eds.), Applied mathematics problem solving. Columbus, OH: ERIC, 1979. (b)

Carpenter, T.P., Corbitt, M.K., Kepner, H., Lindquist, M.M., & Reys, R.E. Results and implications of the second NAEP mathematics assessment: Elementary school. Arithmetic Teacher, in press. (a)

Carpenter, T.P., Corbitt, M.K., Kepner, H., Lindquist, M.M. & Reys, R.E. Results and implications of the second NAEP mathematics assessment: Secondary school. Mathematics Teacher, in press. (b)

Carpenter, T.P. & Hiebert, J. Information processing capacity and relation skills. Madison: Wisconsin Research and Development Center for Individualized Schooling, in press. (a)

Carpenter, T.P. & Hiebert, J. Information processing capacity and addition skills. Madison: Wisconsin Research and Development Center for Individualized Schooling, in press. (b)

Carpenter, T.P., Hiebert, J., Blume, G.W., Martin, C., & Pimm, D. Analysis and synthesis of research on children's number concepts and skills. (Conceptual Paper No. 3). Madison: Wisconsin Research and Development Center for Individualized Schooling, in press.

Carpenter, T.P. & Hiebert, J., & Moser, J.M. The effect of problem structure on first-graders' initial solution processes for simple addition and subtraction problems. (Technical Report No. 516). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Carpenter, T.P., Moser, J.M., & Hiebert, J. A study of the effect of instruction on children's representation of selected addition and subtraction problems. Madison: Wisconsin Research and Development Center for Individualized Schooling, in press.

Case, R. Piaget and beyond: Toward a developmentally based theory and technology of instruction. In R. Glaser (Ed.), Advances is instructional psychology (Vol. 1). Hillsdale, NJ: Lawrence Erlbaum, 1978.

Gelman, R. & Gallistel, C.R. The child's understanding of number. Cambridge, MA: Harvard University Press, 1978.

Gibb, E.G. Children's thinking in the process of subtraction. Journal of Experimental Education, 1956, 25, 71-80.

Ginsburg, H. Children's arithmetic: The learning process. New York: D. Van Nostrand, 1977.

Glaser, R. Components of a psychology of instruction: Toward a science of design. Review of Educational Research, 1976, 46, 1-24.

Greeno, J.G. Some examples of cognitive task analysis with instructional implications. Paper presented at the ONR/NPRDC Conference, San Diego, CA, March 1978.

Groen, G.J. & Parkman, J.M. A chronometric analysis of simple addition. Psychological Review, 1972, 79, 329-343.

Groen, G. & Resnick, L.B. Can preschool children invent addition algorithms? Journal of Educational Psychology, 1977, 69, 645-652

Grouws, D.A. Differential performance of third-grade children in solving open sentences of four types. (Doctoral dissertation, University of Wisconsin, 1971). Dissertation Abstracts International, 1972, 32, 3860A. (University Microfilms No. 72-1028)

Hiebert, J. The effect of cognitive development on first grade children's ability to learn linear measurement concepts. (Technical Report No. 506). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Jerman, M. Problem length as a structural variable in verbal arithmetic problems. Educational Studies in Mathematics, 1973, 5, 109-123.

Jones, G.A. & Romberg, T.A. Three "Time on task" studies and their implications for teaching and teacher education. (Project Paper 79-6). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Kouba, V. & Moser, J.M. Development and validation of curriculum units related to initial sentence writing. Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Lindvall, C.M. & Ibarra, C.G. An analysis of incorrect procedures used by primary grade pupils in solving open addition and subtraction sentences. Paper presented at the Annual Meeting of the American Educational Research Association, Toronto, March 1978. (ERIC Document Reproduction Service No. ED 155 049).

Martin, C. Intra and intercoder reliability for addition and subtraction interviews. Madison: Wisconsin Research and Development Center for Individualized Schooling, in press.

Nesher, P.S. & Katriel, T. Two cognitive modes in arithmetic word problem solving. Paper presented at the second annual meeting of the International Group for the Psychology of Mathematics Education, Osnabruck, West Germany, September 1978.

Opper, S. Piaget's clinical method. The Journal of Children's Mathematical Behavior, 1977, 1 (4), 90-107.

Pascual-Leone, J. A mathematical model for the transition rule in Piaget's developmental stages. Acta Psychologica, 1970, 32, 301-345.

Pascual-Leone, J. A view of cognition from a formalist's perspective. In K.F. Reigel & J. Meacham (Eds.), The developing individual in a changing world, The Hague: Mouton, 1976.

Piaget, J. The child's conception of number. New York: Humanities Press, 1952.

Resnick, L.B. Task analysis in instructional design: Some cases from mathematics. In D. Klahr (Ed.), Cognition and instruction, Hillsdale, NJ: Lawrence Erlbaum Associates, 1976.

Resnick, L.B. The role of invention in the development of mathematical competence. Paper presented at the Workshop on Children's Mathematical Learning, Learning Research and Development Center, Pittsburgh, September 1978.

Romberg, T.A., Harvey, J.G., Moser, J.M., and Montgomery, M.E. Developing Mathematical Processes. Chicago: Rand McNally, 1974.

Rosenthal, D.J.A. & Resnick, L.B. Children's solution processes in arithmetic word problems. Journal of Educational Psychology, 1974, 66, 817-825.

Steffe, L.P. Differential performance of first-grade children when solving arithmetic addition problems. Journal for Research in Mathematics Education, 1970, 1, 144-161.

Steffe, L.P., Spikes, W.C., & Hirstein, J.J. Quantitative comparisons and class inclusion as readiness variables for learning first grade arithmetical content. Athens, GA: The Georgia Center for the Study of Learning and Teaching Mathematics, University of Georgia, 1976.

Suppes, P. & Groen, G. Some counting models for first grade performance data on simple facts. In J.M. Scandura (Ed.), Research in Mathematics Education, Washington, D.C.: National Council of Teachers of Mathematics, 1967.

Suppes, P., Loftus, E.F., & Jerman, M. Problem solving on a computer-based teletype. Educational Studies in Mathematics, 1969, 2, 1-15.

Vergnaud, G. & Durand, C. Structures additives et complexité psychogénétique. La Revue Francaise de Pedagogie, 1976, 36, 28-43.

Woods, S.S., Resnick, L.B., & Groen, G.J. An experimental test of five process models for subtraction. Journal of Educational Psychology, 1975, 67, 17-21.

Center Planning and Policy Committee

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Wayne Otto
Center Co-Directors

Dale D. Johnson
Area Chairperson
Studies in Language:
Reading and Communication

Marvin J. Fruth
Area Chairperson
Studies in Implementation
of Individualized Schooling

Penelope L. Peterson
Area Chairperson
Studies of Instructional Programming
for the Individual Student

James M. Lipham
Area Chairperson
Studies of Administration and
Organization for Instruction

Thomas A. Romberg
Area Chairperson
Studies in Mathematics and Evaluation
of Practices in Individualized Schooling

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Psychology

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Professor
Educational Administration

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Associate Professor
Curriculum and Instruction

W. Patrick Dickson
Assistant Professor
Child and Family Studies

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Associate Professor
Educational Administration

Marvin J. Fruth
Professor
Educational Administration

Dale D. Johnson
Professor
Curriculum and Instruction

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Educational Psychology

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Professor
Educational Administration

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Psychology

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Educational Administration

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English and Linguistics

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Curriculum and Instruction

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Curriculum and Instruction

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Educational Psychology

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Professor
Educational Psychology

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