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ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises concerned with simultaneous equations and points and lines in a plane, a summary of the problems in the film "Graphing Absolute Value Equations," and the supplement. (MK)

THE ARITHMETIC PROJECT COURSE FOR TEACHERS

TOPICS: Simultaneous Equations. Points and Lines in a Plane.

FILM: Graphing Absolute Value Equations, Grade 2

SUPPLEMENT: Graphing Simultaneous Equations

17

NAME: ....

This booklet is part of a course for teachers produced by The Arithmetic Project in association with Education Development Center. Principal financial support has come from the Carnegie Corporation of New York, the University of Illinois, and the National Science Foundation.

The course is available from:

THE ARITHMETIC PROJECT
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Newton, Massachusetts 02160

I. SIMULTANEOUS EQUATIONS

Consider the equation

$$\square + \triangle = 5$$

Here are some pairs of numbers that work:

$\square$	$\triangle$
0	5
5	0
3	2
4	1
$2\frac{1}{2}$	$2\frac{1}{2}$
7	-2
$6\frac{15}{17}$	$-1\frac{15}{17}$

(Notice that $(2\frac{1}{2}, 2\frac{1}{2})$ is included. In an expression containing frames of different shapes, you <u>are</u> allowed to put the same number in different-shaped frames. In $\square + \square = 5$ you <u>must</u> use $2\frac{1}{2}$ in both boxes; for $\square + \triangle = 5$ you <u>may</u> use $2\frac{1}{2}$ in both box and wedge, or you may use any other pair of numbers that work.)

1. Complete the blanks in the table below to give six different pairs of numbers that work in

$$\square + \triangle + \triangle = 7$$

$\square$	$\triangle$
	1
7	
	-2

Now consider both equations together:

$$\begin{cases} \square + \triangle = 5 \\ \square + \triangle + \triangle = 7 \end{cases}$$

We can ask, "Is there a pair of numbers that will work in both equations at the same time?" In this example the answer is yes. The pair (3, 2) will work. (Is it the only pair?)

When a bracket is used to tie two or more equations together, the "same shape, same number" rule applies to all the frames in the equations grouped by the bracket.

Another example:

$$\begin{cases} \boxed{8} + \triangle_{4\frac{1}{2}} + \triangle_{4\frac{1}{2}} = 17 \\ \boxed{8} + \triangle_{4\frac{1}{2}} = 12\frac{1}{2} \end{cases}$$

Find the numbers that work in the following pairs of equations. (In one of the problems, no pair of numbers will work.)

2. $$\begin{cases} \square + \triangle + \triangle = 12 \\ \square + \triangle = 10 \end{cases}$$

3. $$\begin{cases} \square + \triangle = 20 \\ \square + \triangle + \triangle = 28 \end{cases}$$

4. $$\begin{cases} \square + \triangle = 20 \\ \square + \triangle + \triangle + \triangle = 30 \end{cases}$$

5. $\begin{cases} \triangle + \square = 40 \\ \square + \triangle + \triangle + \triangle = 60 \end{cases}$

6. $\begin{cases} \square + \triangle = 14 \\ \square + \triangle + \triangle + \triangle + \triangle + \triangle = 16 \end{cases}$

7. $\begin{cases} \square + \triangle = 14 \\ \triangle + \square + \square + \square + \square + \square = 16 \end{cases}$

8. $\begin{cases} \square + 3 \times \triangle = 680 \\ \square + 2 \times \triangle = 500 \end{cases}$

9. $$\begin{cases} \square + \triangle = 118 \\ \square - \triangle = 0 \end{cases}$$

10. $$\begin{cases} \triangle - \square = 0 \\ \square + \square + \triangle = 12 \end{cases}$$

11. $$\begin{cases} \square + \triangle = 14 \\ \square + \triangle = 8 \end{cases}$$

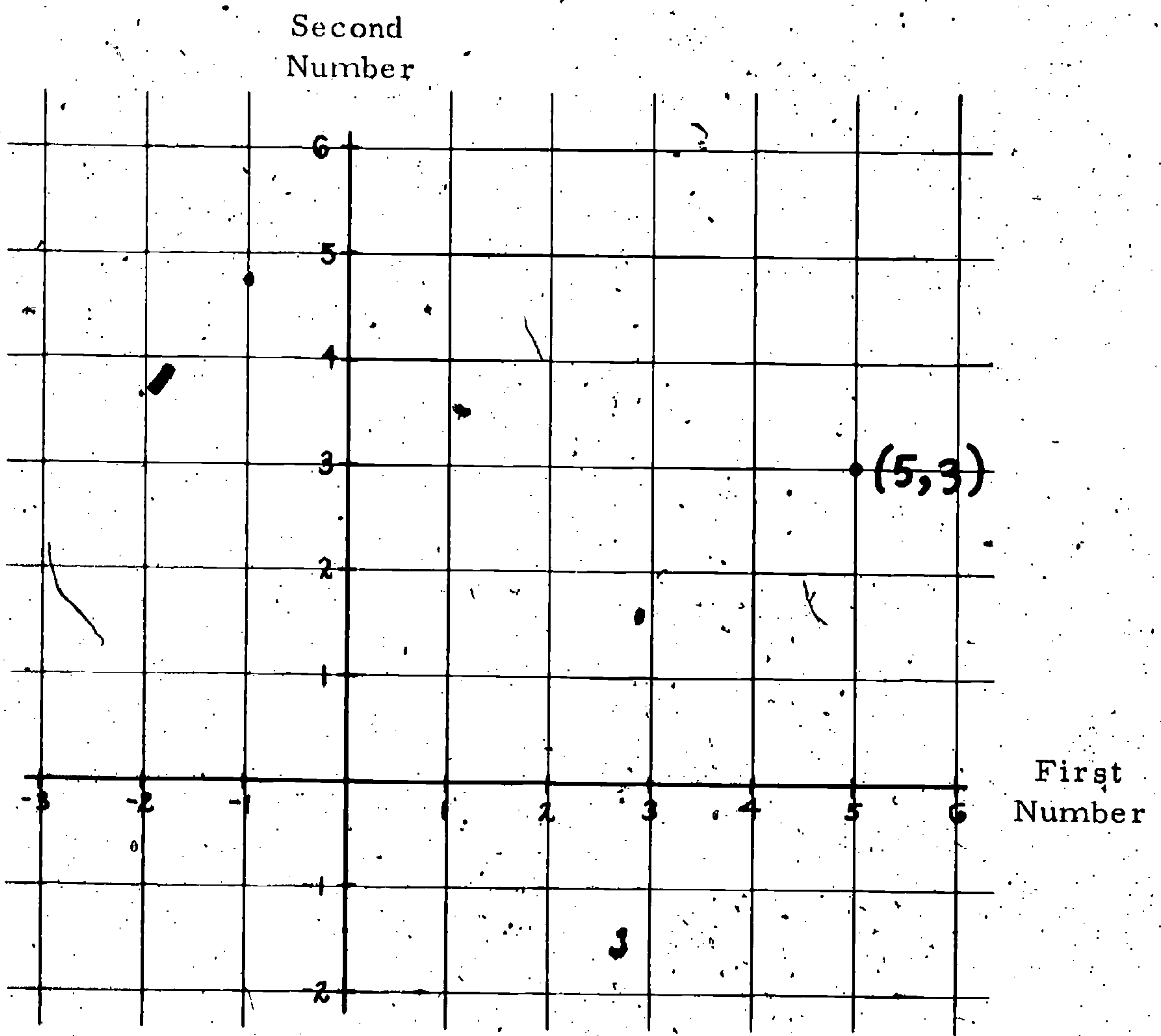
☆12. $$\begin{cases} \square + \triangle = 36 \\ \square + \square + \triangle + \Diamond = 96 \\ \square + \Diamond + \Diamond + \square + \triangle = 139 \end{cases}$$

☆13. $$\begin{cases} \square + \square + \triangle = 42 \\ \square + \triangle + \triangle = 33 \end{cases}$$

☆14. $$\begin{cases} \square + 3 \times \triangle = 20 \\ \triangle + \square + 2 \times \square = 40 \end{cases}$$

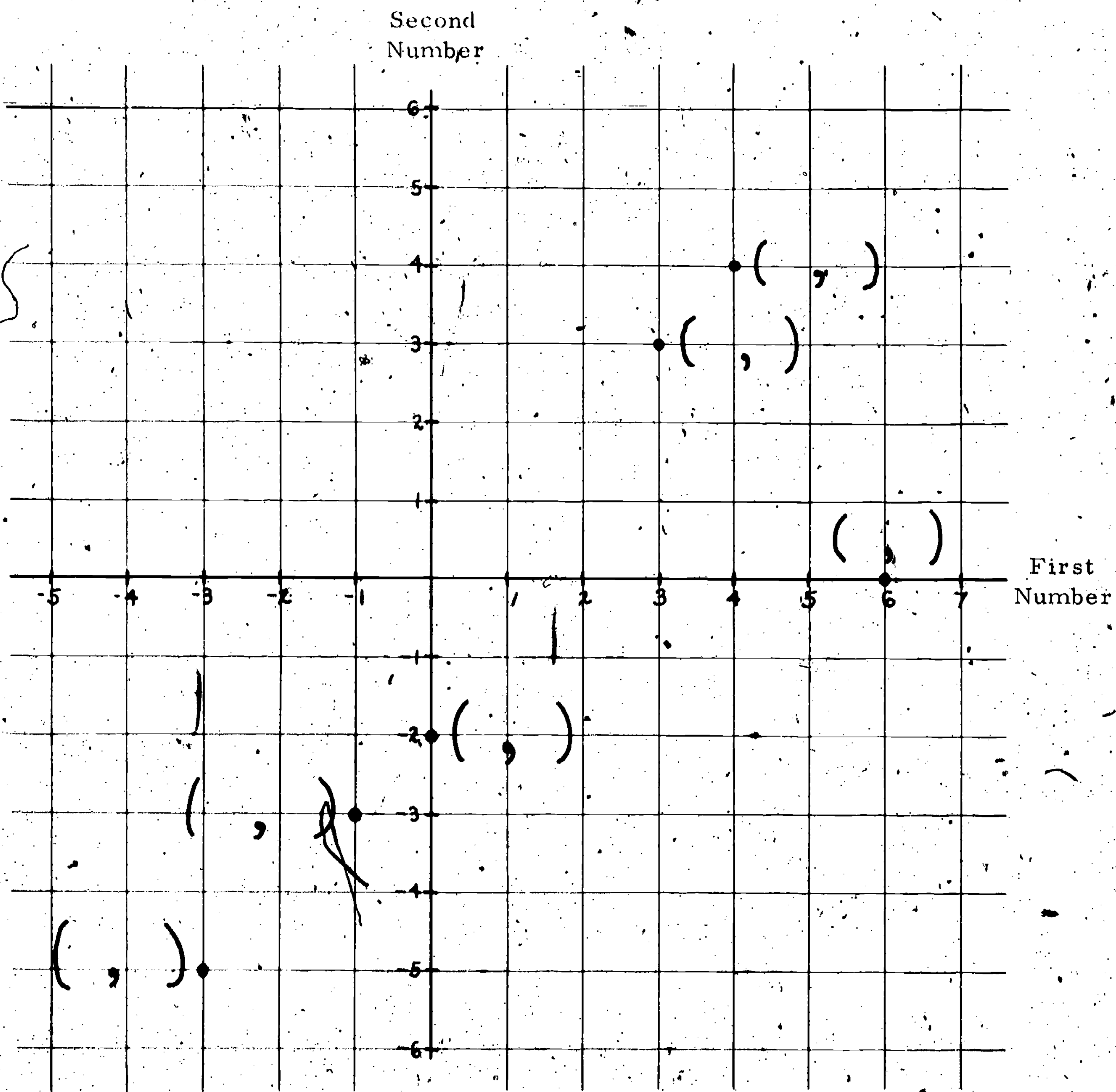
II. POINTS AND LINES IN A PLANE

Much of your work in this course has been related to number lines. We will now extend the idea of a number line to a number plane. This extension is customarily done by having two number lines perpendicular to each other. This allows you to name points in a plane by giving two numbers—a first number and a second number. By convention, if the first number is 5 and the second number is 3, the point is:



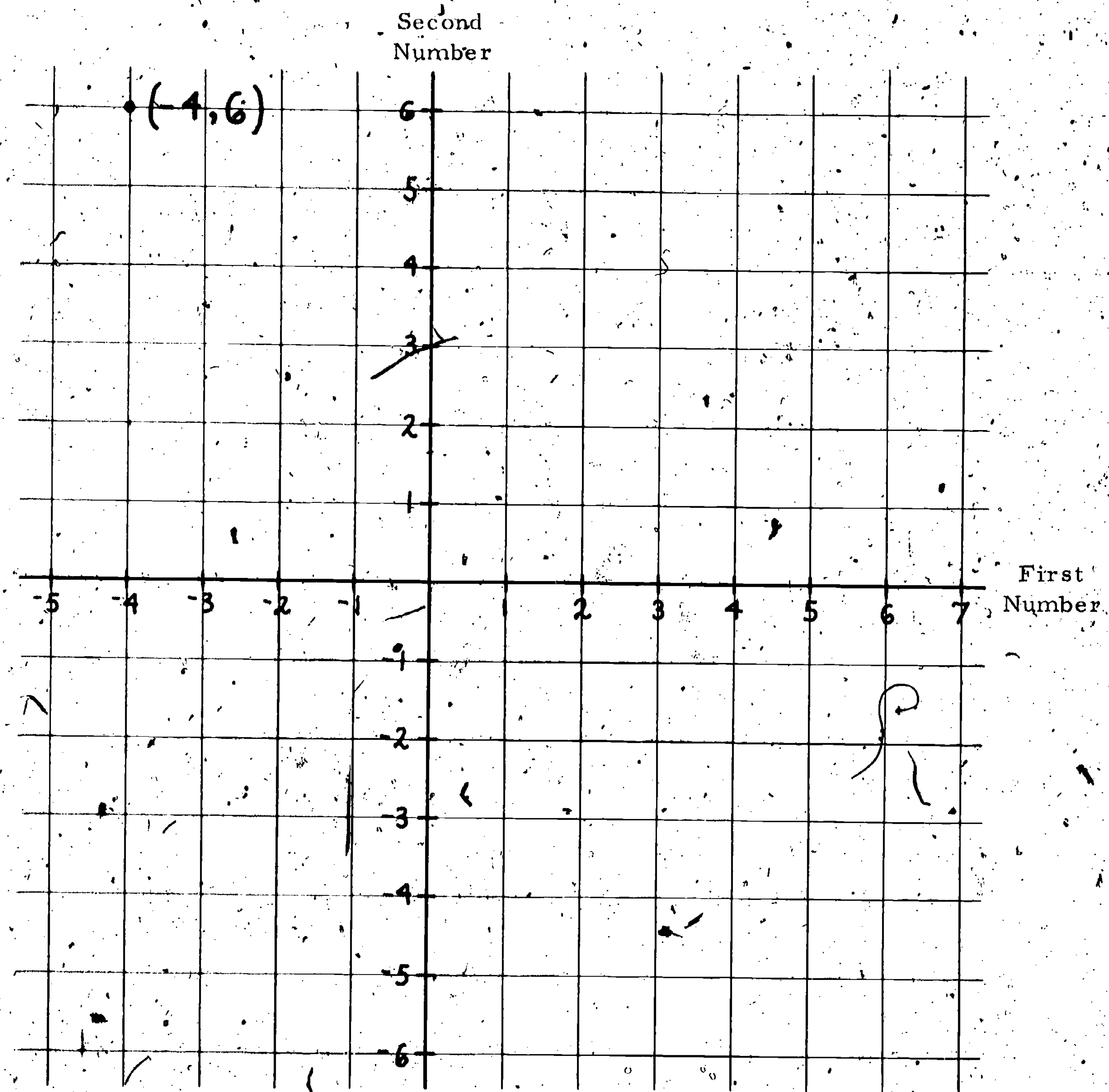
It is customary to write (5,3) instead of mentioning first number and second number.

1. Here are six points. Beside each point write its numbers.



2. On the following diagram, the point (-4, 6) has been labelled. In a similar way, locate with dots and label these points:

(3, 5) , (-3, 5) , (-3, -5) , (-5, 3)



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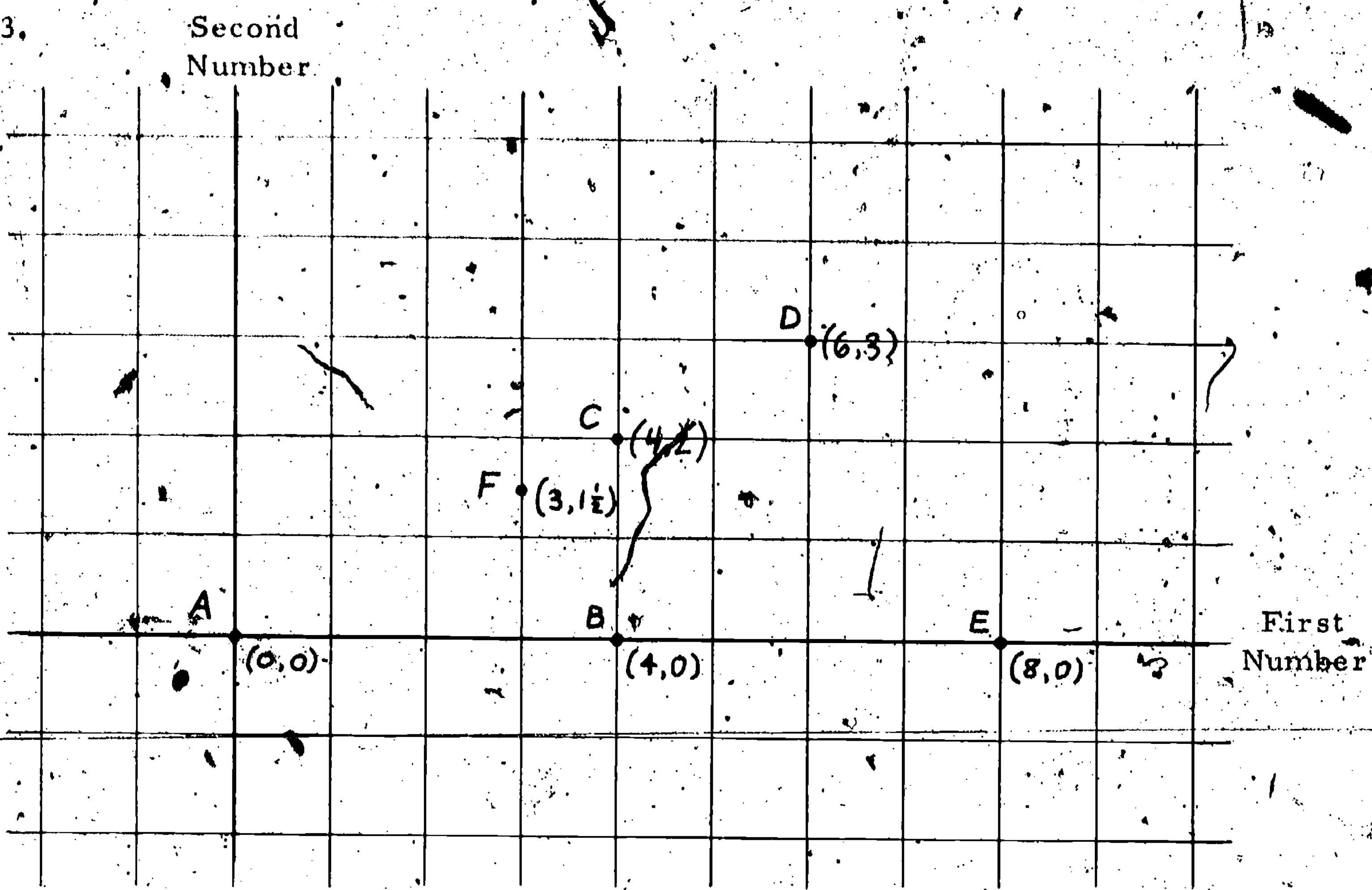
3.



In order to make our terminology clear, we shall say that the point B above is halfway between A and E. Point C is not halfway between A and E because it is not on the line through A and E. Only one of the statements below is false. Find it.

(a) C is between A and D.

(b) The distance between A and C is the same as the distance between E and C.

(c) Point E is not halfway between A and B.

(d) F is between A and D.

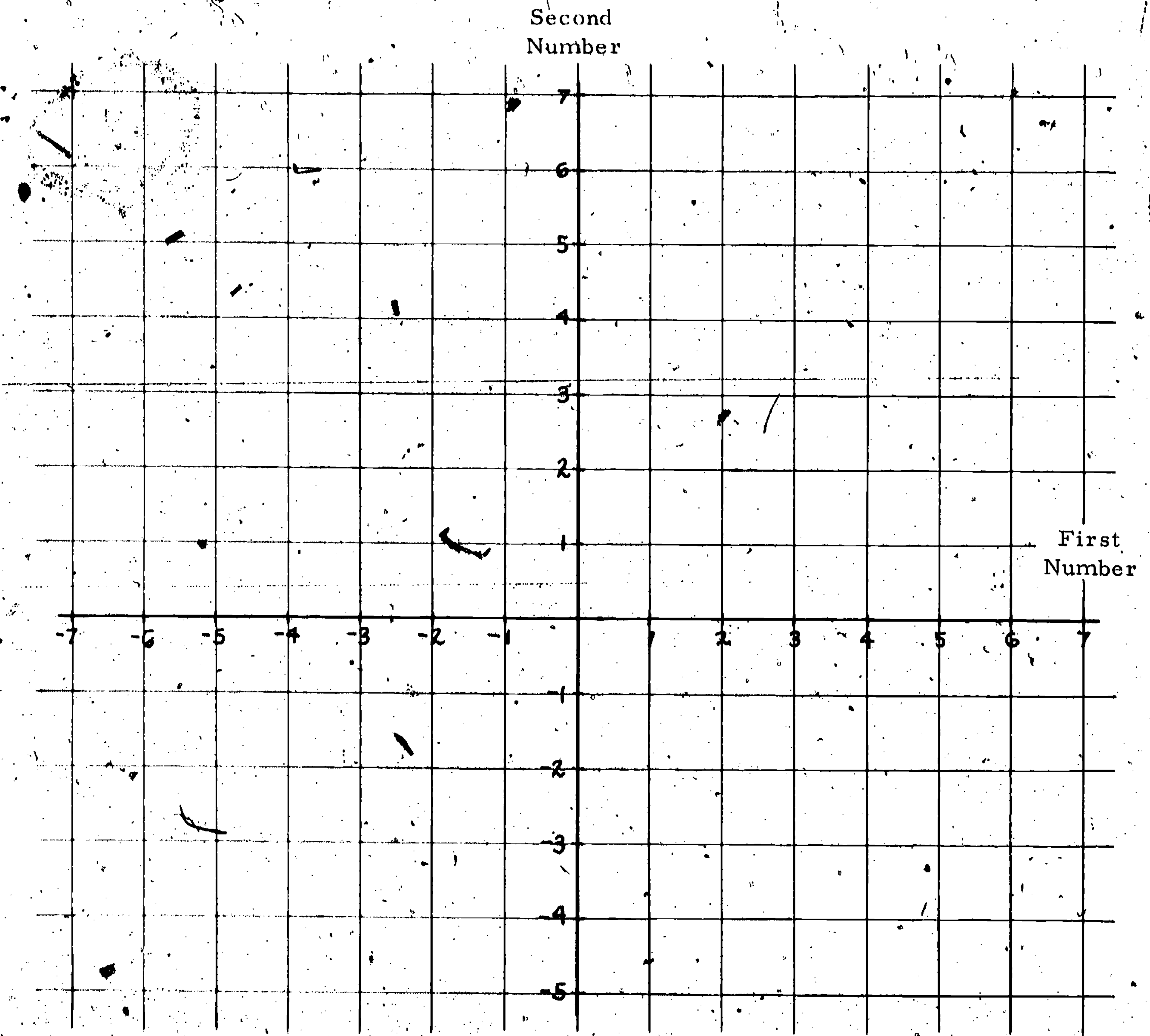
(e) F is halfway between A and D.

(f) C is halfway between A and D.

4. Is point B halfway between A and C ? ________

At the bottom of the page comment on your answer.

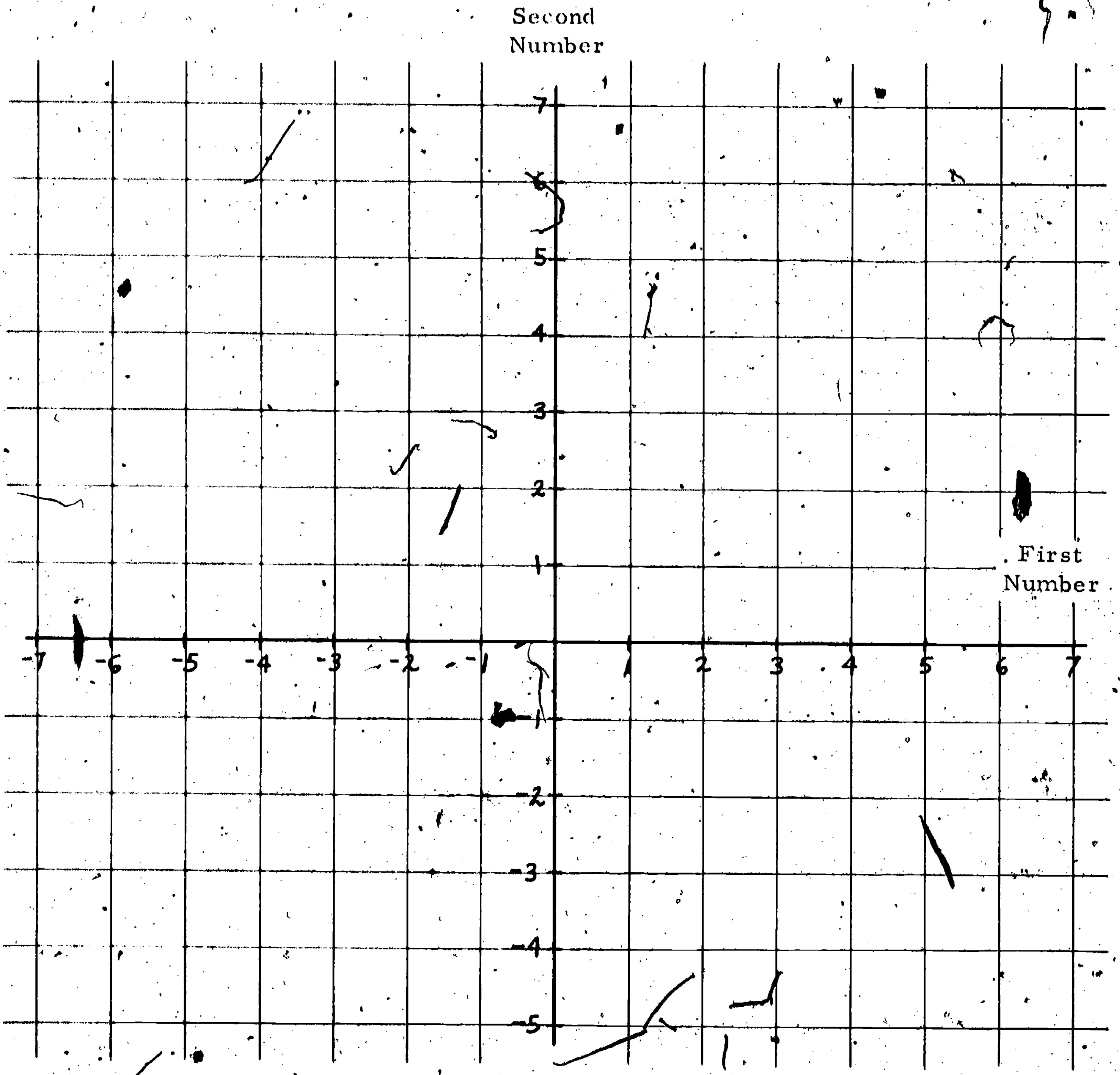
Point A : (-3 , -4)
B : (1 , 0)
C : (4 , 3)



Comment on your answer:

5. Is point D halfway between E and F. ? ________ At the bottom of the page, comment on your answer.

Point D : (0, 0)
E : (-1, 6)
F : (2, -5)

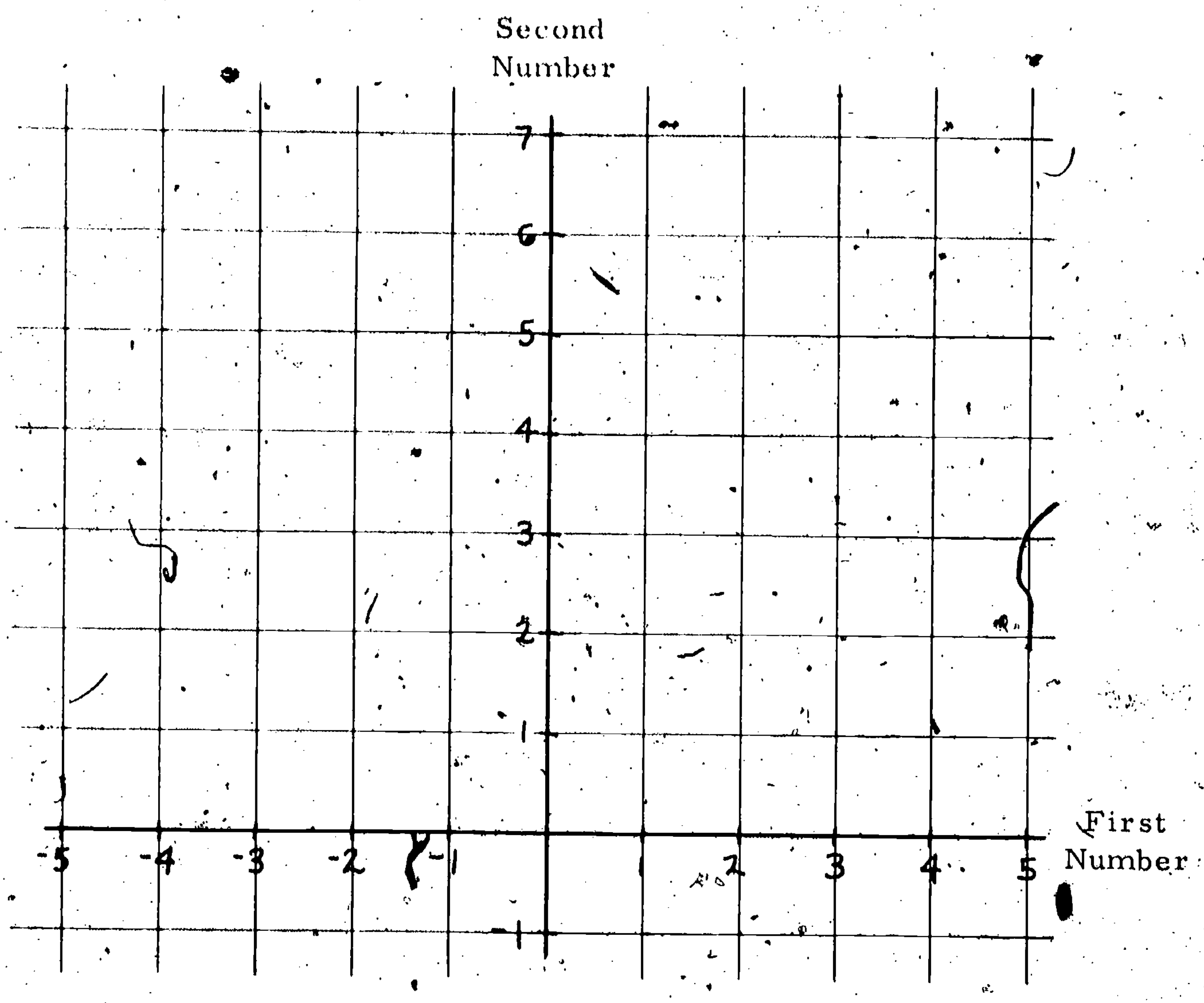


Comment on your answer:

6. Point M : $\left(-4\frac{1}{7},\ 4\frac{2}{3}\right)$

Point N : $\left(4\frac{1}{7},\ 5\frac{1}{3}\right)$

Locate with a dot, and label, the point halfway between M and N.

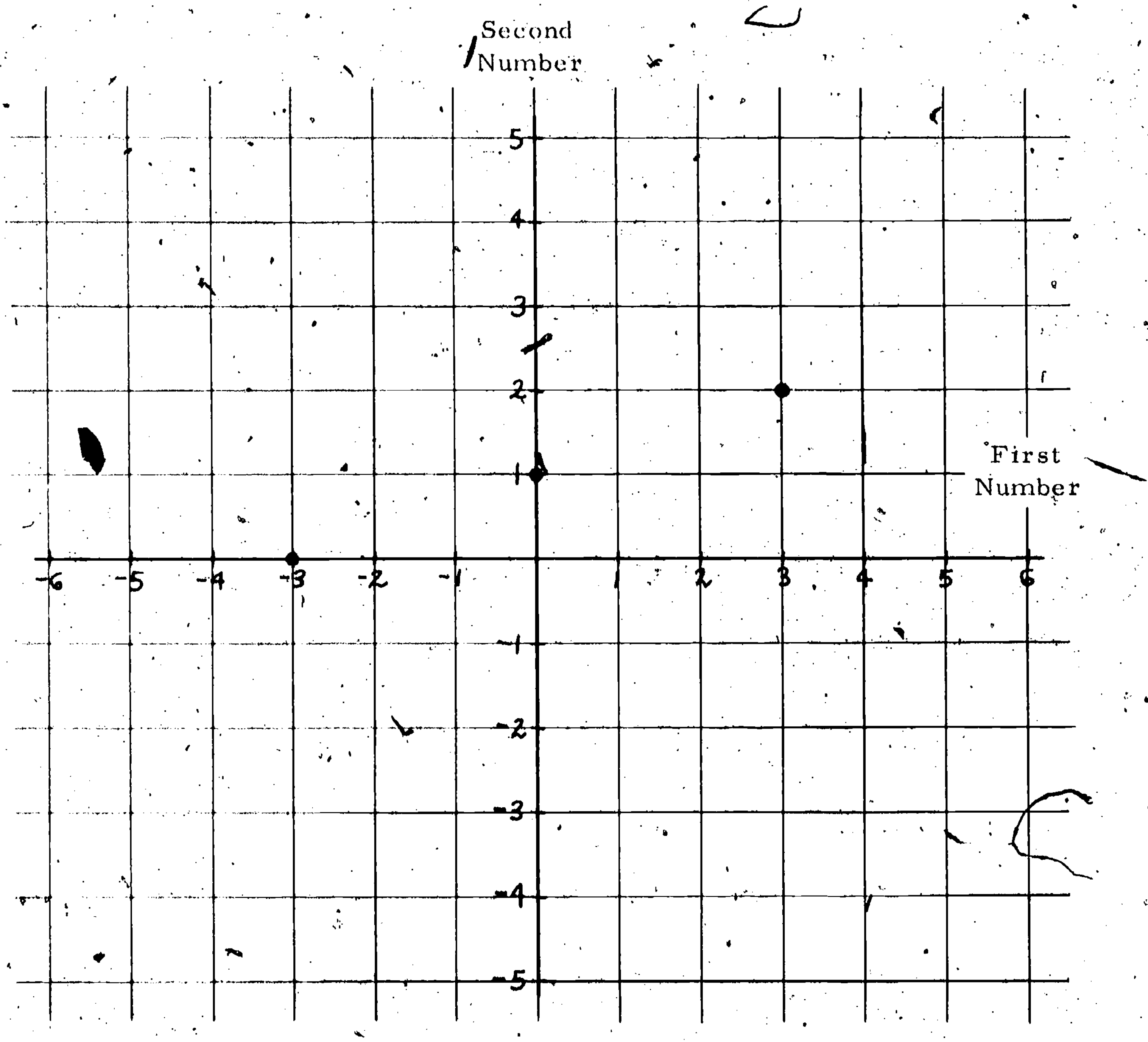


Tell how you can be sure of your answer without drawing a line or using a straightedge.

7. Here are three points on a line. Find two other points on the same line which are not between any of the given points.

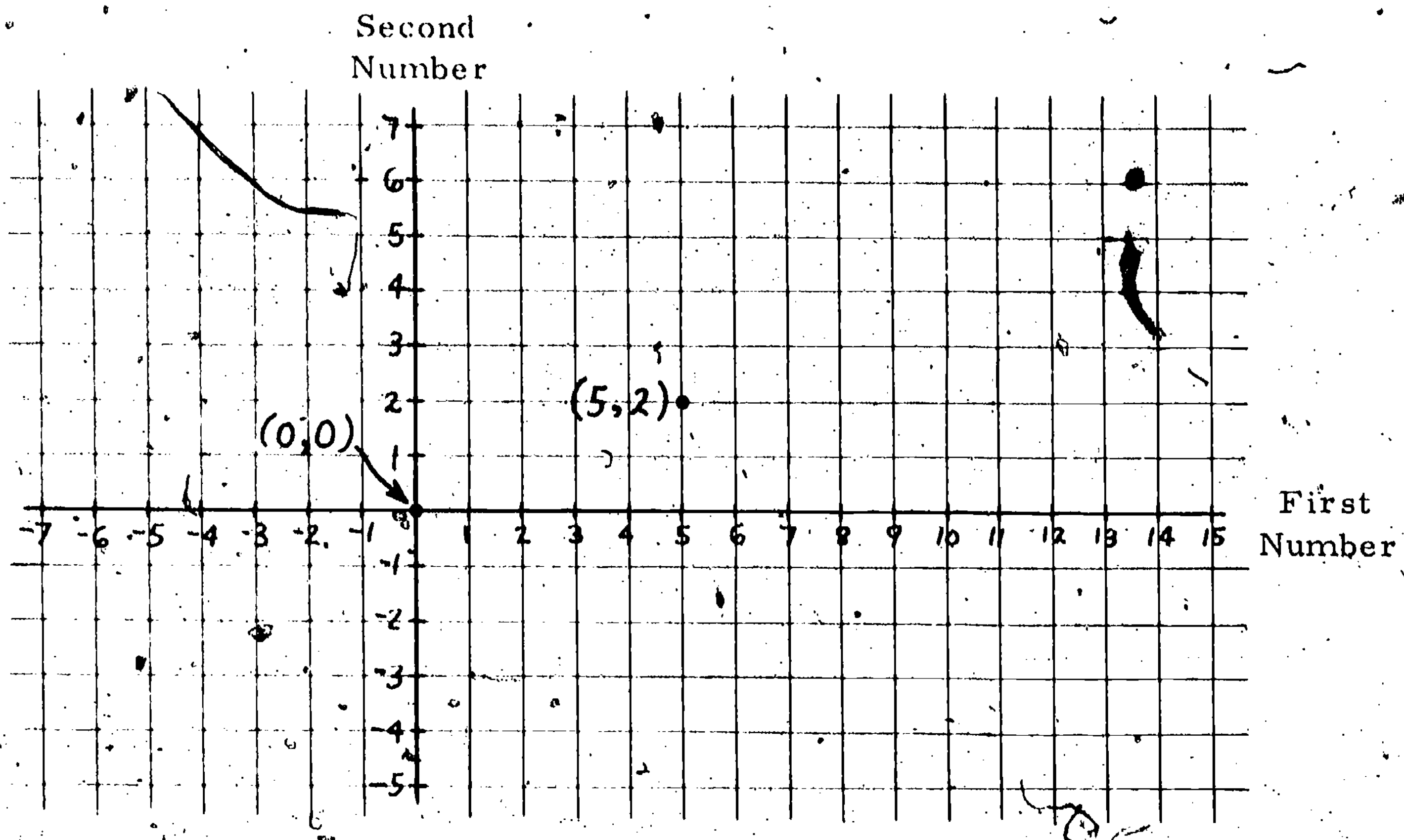
(____ , ____)

(____ , ____)



8. Here are two points. There is exactly one line that goes through both of them. Find the point on this line with second number -2.

(, -2)

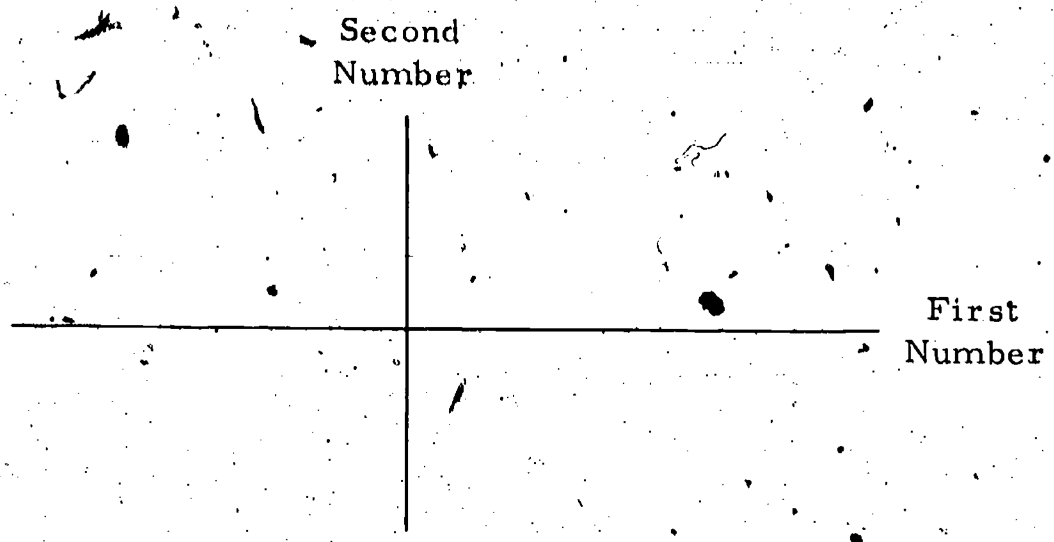


With the first number 15 : (15 ,)

With the first number $17\frac{1}{2}$: ($17\frac{1}{2}$,)

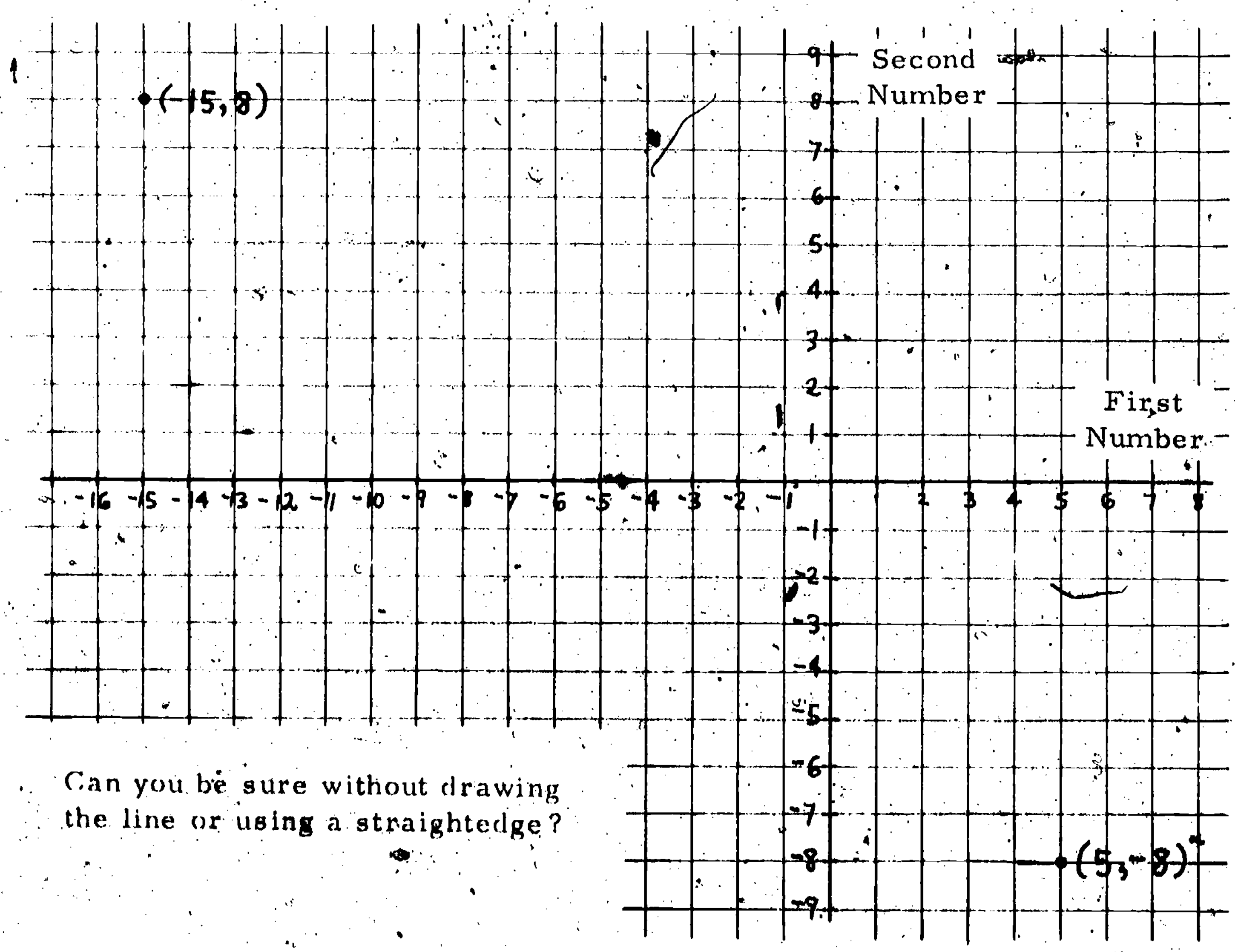
With the first number $31\frac{1}{4}$: ($31\frac{1}{4}$,)

9. These two lines are called <u>axes</u> :



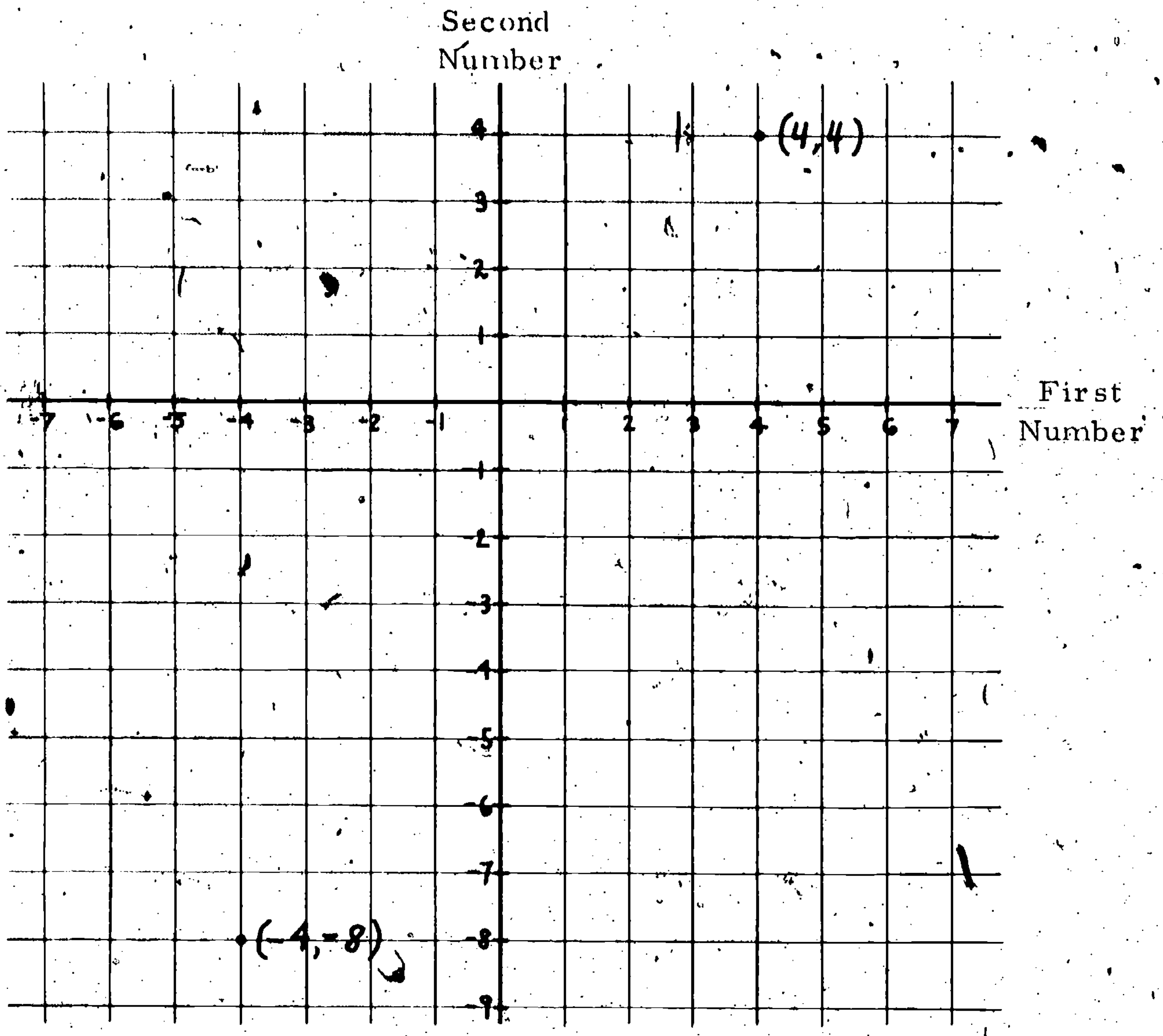
Every point on either axis has 0 for one of its two numbers. For example, these points are on the first-number axis: (3 , 0) ; (−23 , 0) ; ($415\frac{1}{2}$, 0). (0 , 7) is on the second-number axis. The point (0 , 0) is called the origin.

Find where the line going through the two points given below intersects each axis: (______ , ______) ; (______ , ______)



Can you be sure without drawing the line or using a straightedge?

10. Find where the line going through the two points intersects each axis. (Plot the points of intersection and give their numbers.)



If you used a straightedge to draw the line through (4, 4) and ($^-4$, $^-8$), one of the points of intersection looks something like ($1\frac{1}{3}$, 0) or ($1\frac{1}{4}$, 0). Tell how you can be sure exactly what this point is.

11. Point M is (−3, −5). The line through point M and the origin also contains some of the following points. Which ones?

(1, 1) _____

(5, 3) _____

(3, 5) _____

(5, 8) _____

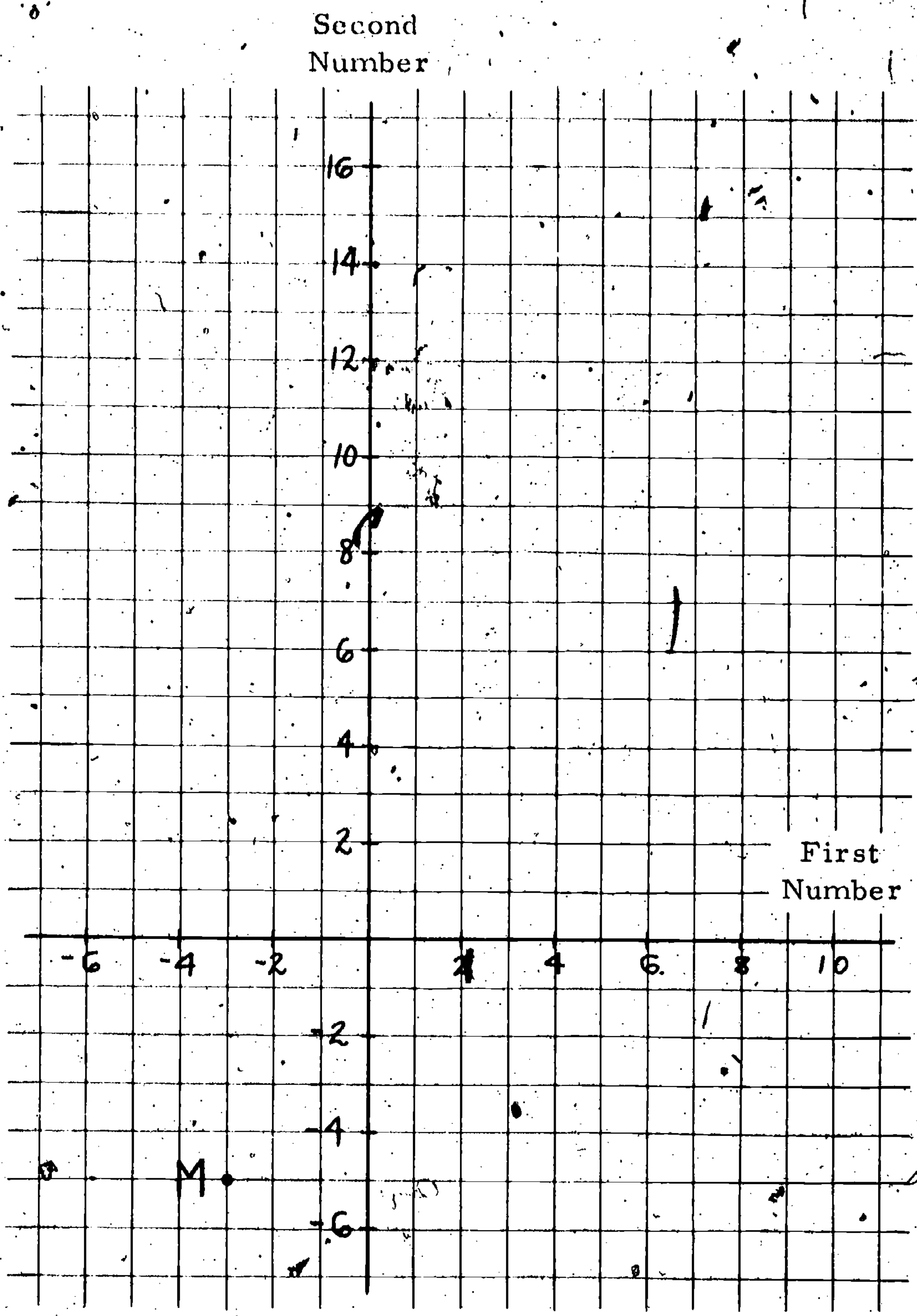
(6, 10) _____

(7, 12) _____

(9, 15) _____

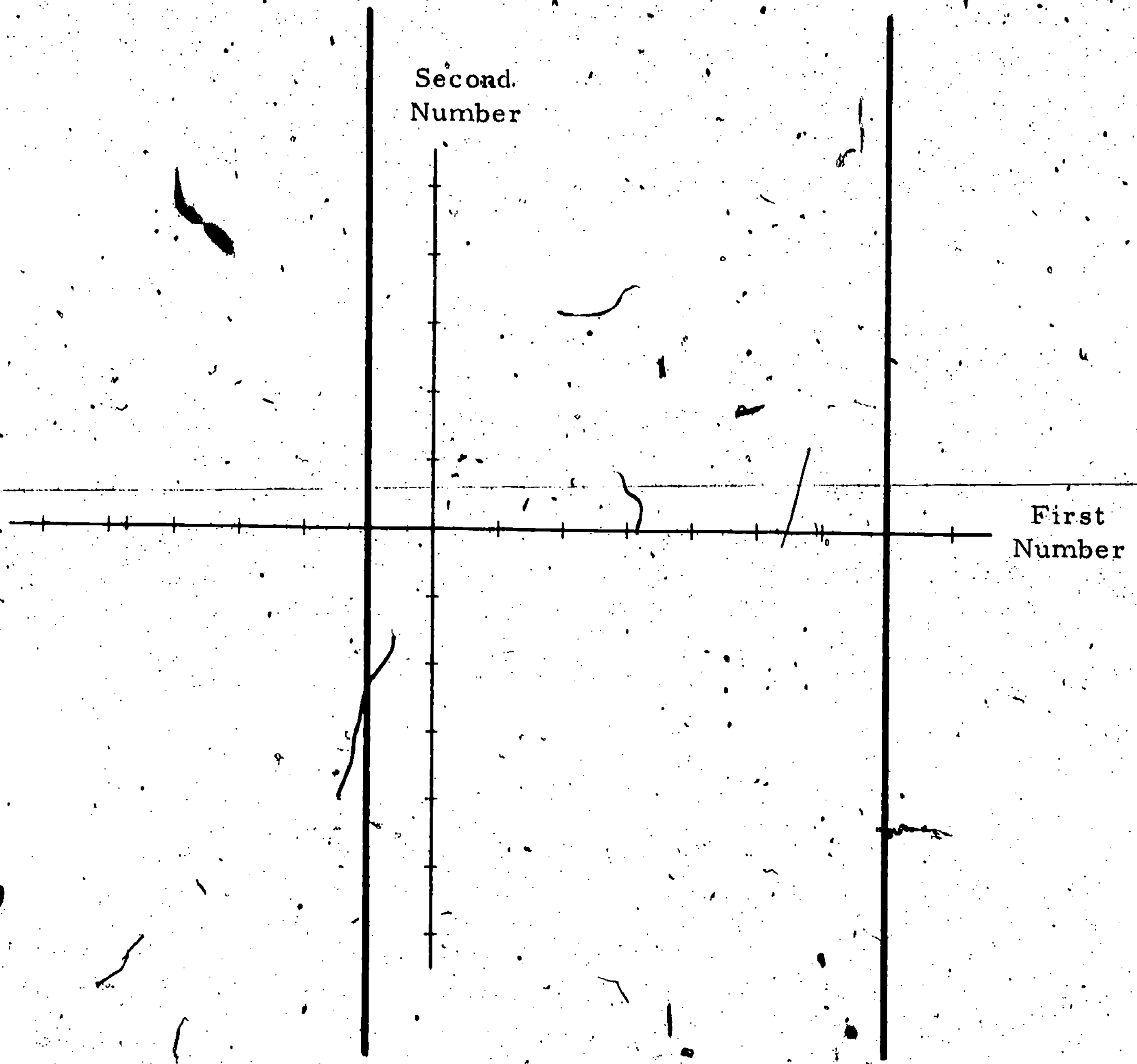
(10, 16) _____

(12, 20) _____



(Hint: Four points should be checked.)

☆12.



Here are some clues about a point in the plane. Find the point and show it on the graph above.

(1) The point lies on the line halfway between the two dark lines.

(2) The first number and second number of the point are both integers.

(3) One of the numbers is $^{-}2$ times the other number.

Summary of Problems in the Film
"Graphing Absolute Value Equations"

2nd Grade, James Russell Lowell School, Watertown, Massachusetts
Teacher: Mrs. Marie Hermann

"Let's see, I think you all know what these bars do."	
$\left\| -5 \right\| =$	
"Who knows what the answer to that is?"	(0 is given.)
	(5)
$\left\| -10 \right\| =$	(10)
$\left\| 18\frac{1}{2} \right\| =$	(18 is given.)
"I think you're thinking of these bars. $\lfloor 18\frac{1}{2} \rfloor$ Then the answer would be 18. But these are just straight bars."	
$\left\| 1\frac{3}{4} \right\| =$	($1\frac{3}{4}$)
$\left\| 10 - 3 \right\| =$	(7)
$\left\| 6 + -11 \right\| =$	(5)
$\left\| \square \right\| = \frac{2}{3}$	($\frac{2}{3}$)
"$\frac{2}{3}$ works. Anybody know something else that works?"	($\frac{2}{3}$ below)
$\left\| \square \right\| + 6 = 26$	(20, 20 below)
$\left\| \square - 3 \right\| = 5$	(Answers given: 8, 8 below, 6 below, $7\frac{3}{3}$, 2 below)
$\left\| \square - 5 \right\| = 100$	(105, 95 below)

$|\square| + \triangle = 5$

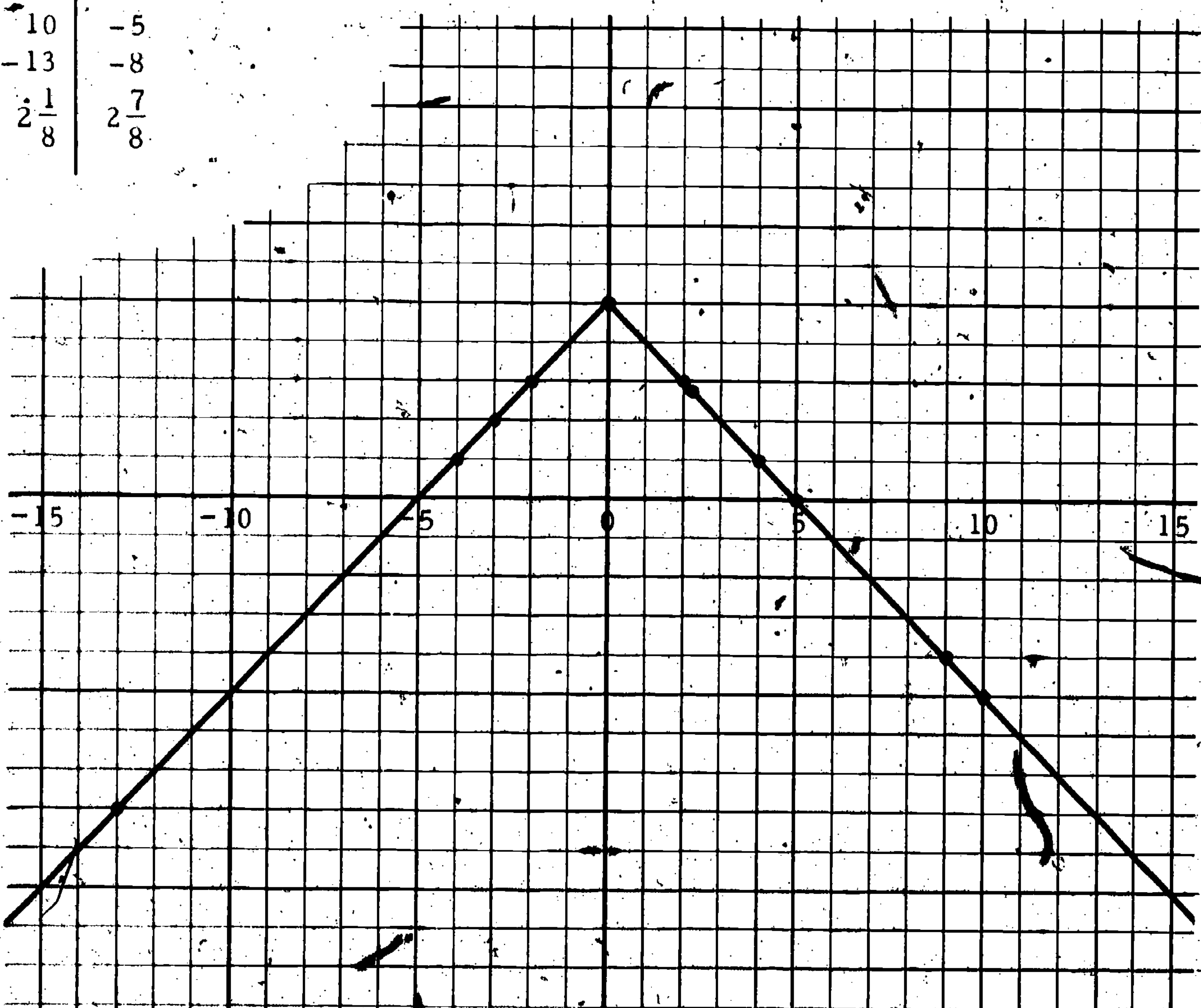
"Now we have to find two numbers, a number for the box and a number for the wedge so that when you get finished you get 5."

(The teacher chose a student to plot the points and a student to fill in a table. The table and the graph are shown below.)

$\square$	$\triangle$
2	3
−2	3
−4	1
5	0
4	1
−3	2
0	5
9	−4
10	−5
−13	−8
$2\frac{1}{8}$	$2\frac{7}{8}$

(Wrong answers given)

$\square$	$\triangle$
−2	−3
−6	1
1	−4



"Can anybody give me an equation so that we'll get something that looks like that, an upside-down V, but move it up or move it down?"

"Take away the 5 and put 6."
"Put 15 there instead of 5."
"Put 15 below and it'll go all the way down to the bottom."
"I know one that can make it so high that you couldn't even fit it on the graph board."

$$|\square| + \triangle = 100$$

"Let's see, Gayl, what was the one you gave me?"

$$|\square| + \triangle = 15$$

"Do you think you could come up and draw it without doing any work at all?"

"Can anybody figure out how to flip it over so that instead of having an upside-down V, you'd get a right-side-up V?"

Suggestions:

$$\lfloor\square\rfloor + \triangle = 15$$

$$|\square| = 5$$

$$\square + \triangle = -10$$

$$|\square| - \triangle = 5$$

$\square + \triangle = -10$ was tried, but it did not work.

$|\square| - \triangle = 5$ was tried. The graph of this equation did form a right side up V.

Supplement

Graphing Simultaneous Equations

by Sally Agro

I.

In the written lesson in this booklet you were solving simultaneous equations such as these:

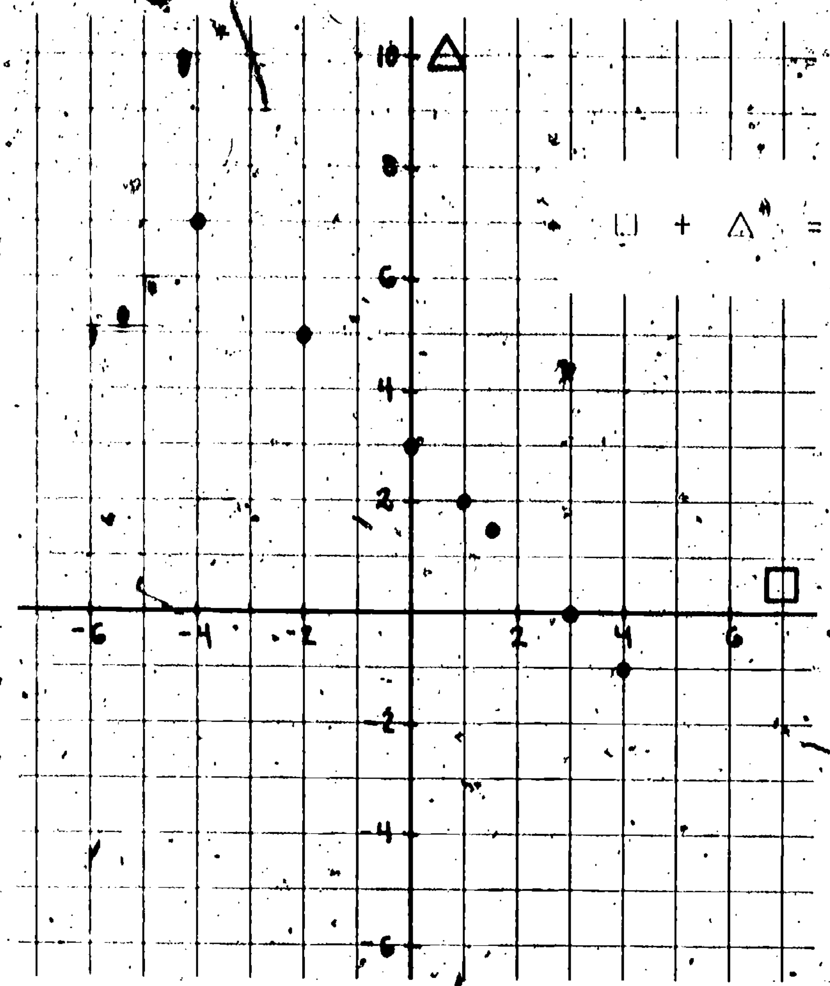
$$\begin{cases} \Box + \triangle = 3 \\ \Box + \Box + \triangle = 8 \end{cases}$$

Perhaps you devised a method for solving any such problems by comparing the frames and the numbers in the two equations. Here is an alternative approach.

Consider the top equation, $\Box + \triangle = 3$. By making a table, you can find many pairs of numbers that work in the equation (that is, whose sum is 3).

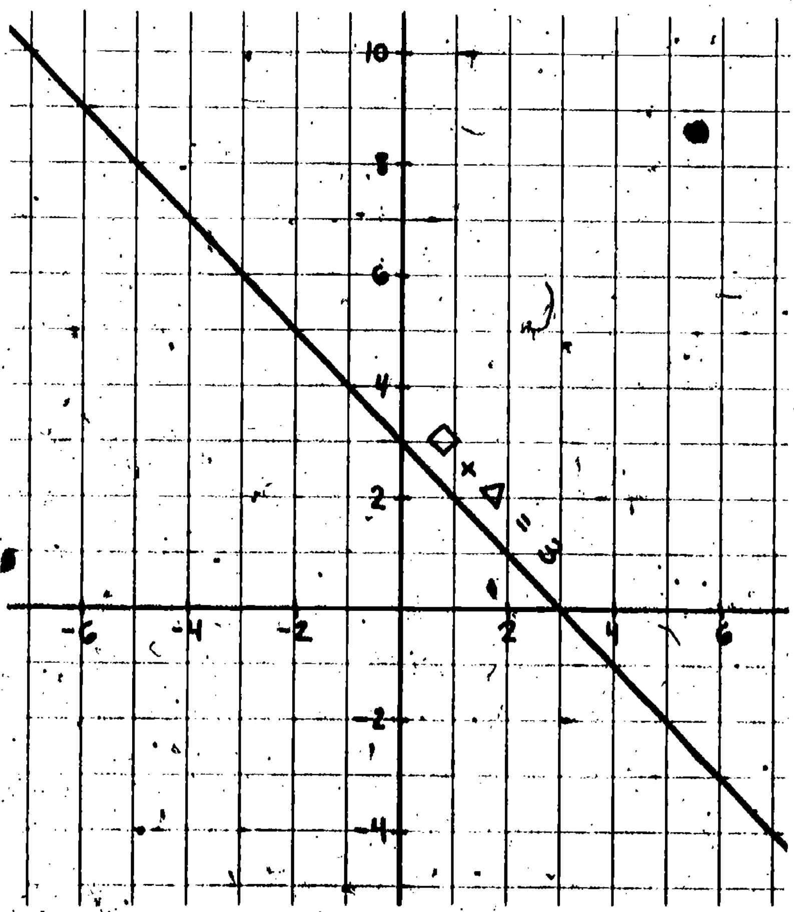
$\Box$	$\triangle$
3	0
0	3
1	2
−4	7
$1\frac{1}{2}$	$1\frac{1}{2}$
−2	5
4	−1
⋮	

You can put a dot on a graph to represent each pair of numbers in your table. For convenience, we will use the first number axis for the box ($\Box$) numbers and the second number axis for the wedge ($\triangle$) numbers.

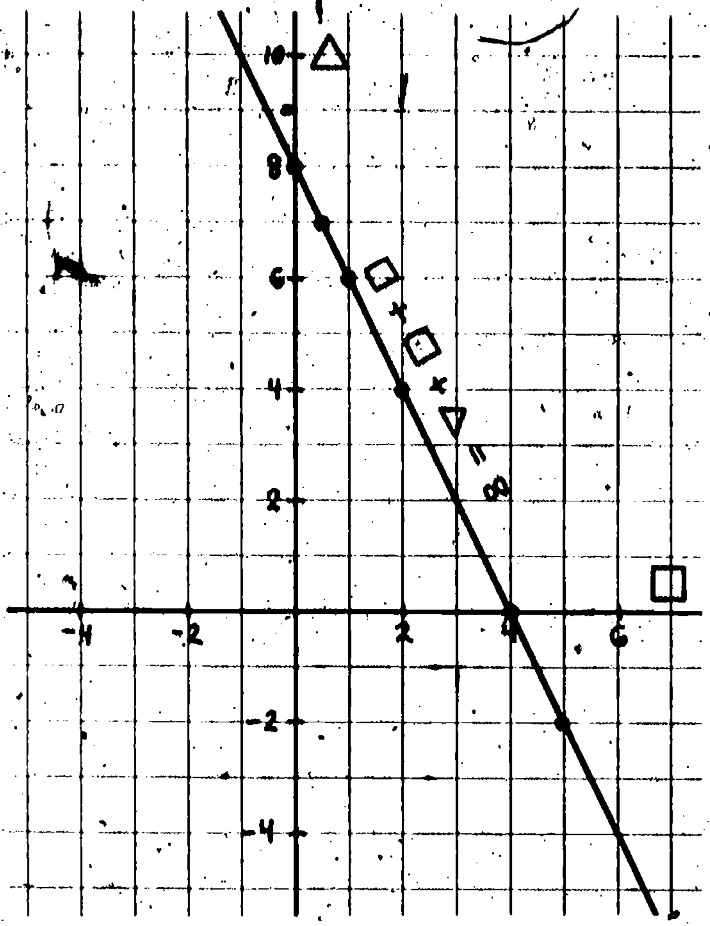


□	△
3	0
0	3
1	2
−4	7
$1\frac{1}{2}$	$1\frac{1}{2}$
−2	5
4	−1

Of course, you could continue to put more number pairs in your table, and put appropriate dots on your graph. Eventually, all the numbers that work in the equation □ + △ = 3 would be represented by the multitude of dots making up the straight line:

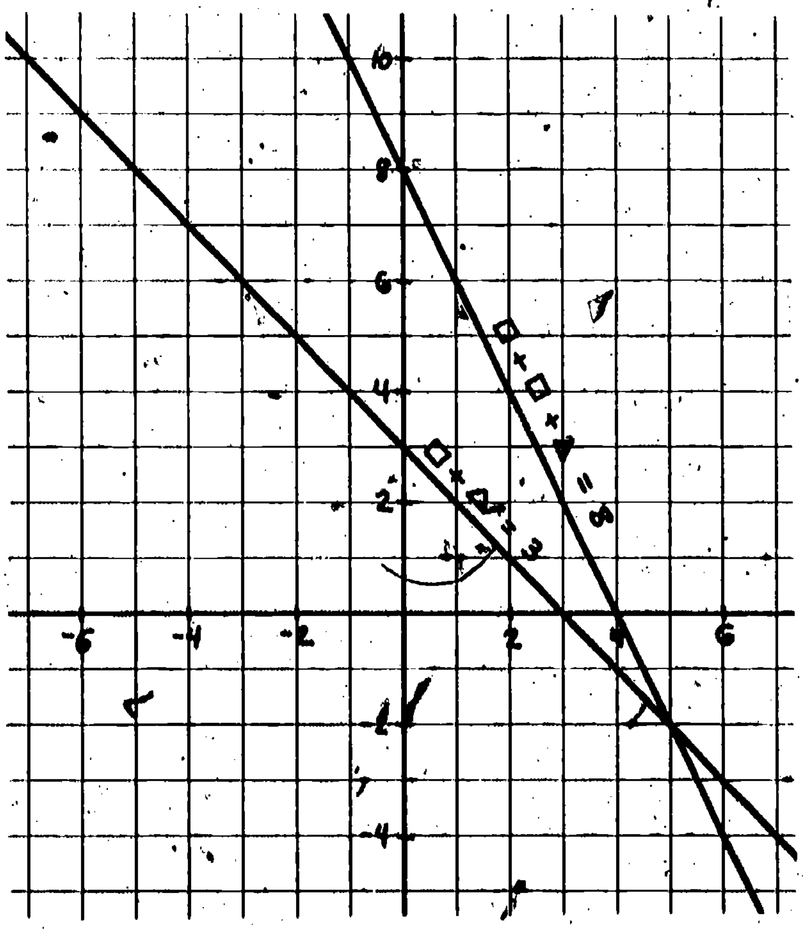


If you follow a similar procedure for the equation $\square + \square + \triangle = 8$, you get this picture:



□	△
0	8
4	0
2	4
1	6
5	−2
$\frac{1}{2}$	7

Since you are looking for pairs of numbers that work in both equations simultaneously, place the lines for each equation on the same graph.



The point at which the lines cross is (5 , −2), which gives the solution to this problem:

$$\begin{cases} \square + \triangle = 3 \\ \square + \square + \triangle = 8 \end{cases}$$

Namely, putting 5 in the boxes and −2 in the wedges satisfies both equations.

On the next few pages, you will find several pairs of simultaneous equations with their graphs and tables of values. Looking at these graphs, the following conclusions can be drawn:

(a) If the lines depicting the two equations cross, there is one solution. Since they are straight lines, they can only cross once.

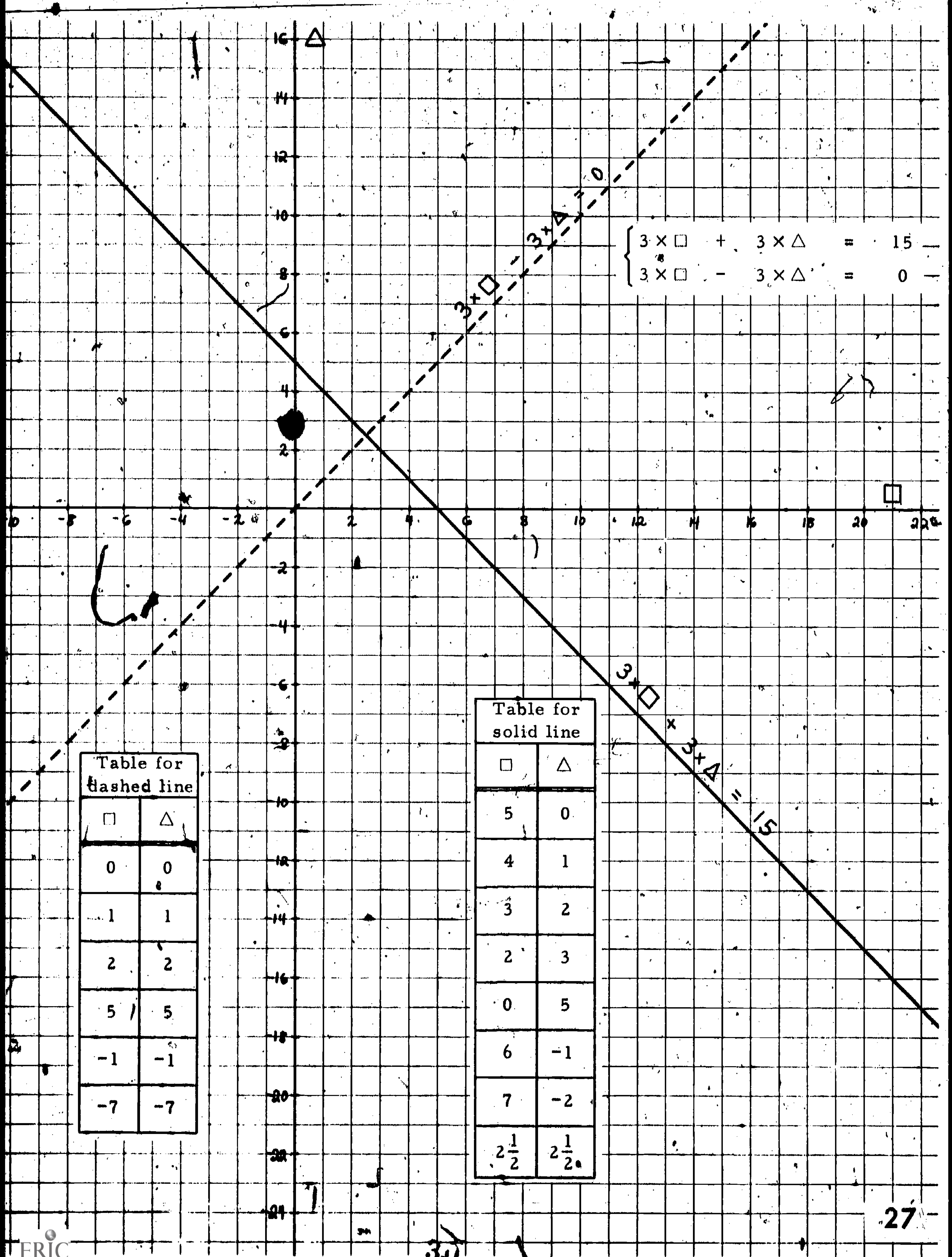
(b) If each of the two equations involved is represented by the very same line, they have many common solutions (infinitely many).

(c) If the lines are parallel, no solution exists.*

You can find the solution to simultaneous equations of type (a) by looking for the place where the lines meet and reading off the point. Make sure you are correct by trying the first number of the point in the boxes and the second number in the wedges.

---

* This discussion is limited to simultaneous equations containing only two unknown quantities (or variables) such as □ and △. One should avoid using more than two different shaped frames in this context since the possibilities for different types of solutions become quite complex for equations of three variables. An example of such a system of simultaneous equations is problem ☆12 on page 5 of this written lesson, where each of the three equations determines a plane. The three planes mutually intersect at (17, 19, 43). This solution of (17, 19, 43) represents one point in three-dimensional space.



$$\begin{cases} 3 \times \square + 3 \times \triangle = 15 \\ 3 \times \square - 3 \times \triangle = 0 \end{cases}$$

Table for dashed line

$\square$	$\triangle$
0	0
1	1
2	2
5	5
−1	−1
−7	−7

Table for solid line

$\square$	$\triangle$
5	0
4	1
3	2
2	3
0	5
6	−1
7	−2
$2\frac{1}{2}$	$2\frac{1}{2}$

30

$$\begin{cases} 2\times\square + \triangle = 10 \\ 7\times\square - 3\times\triangle = 9 \end{cases}$$

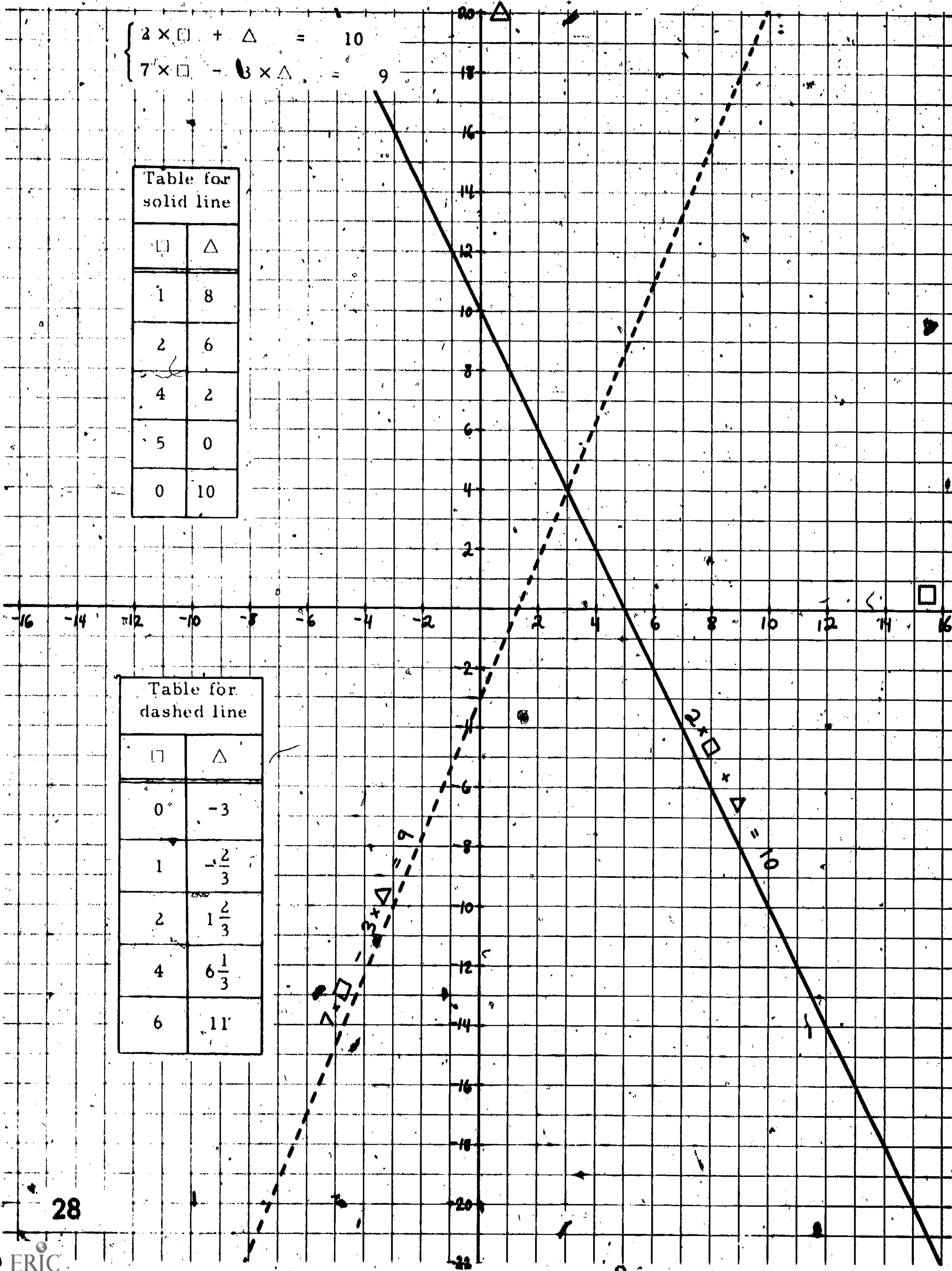
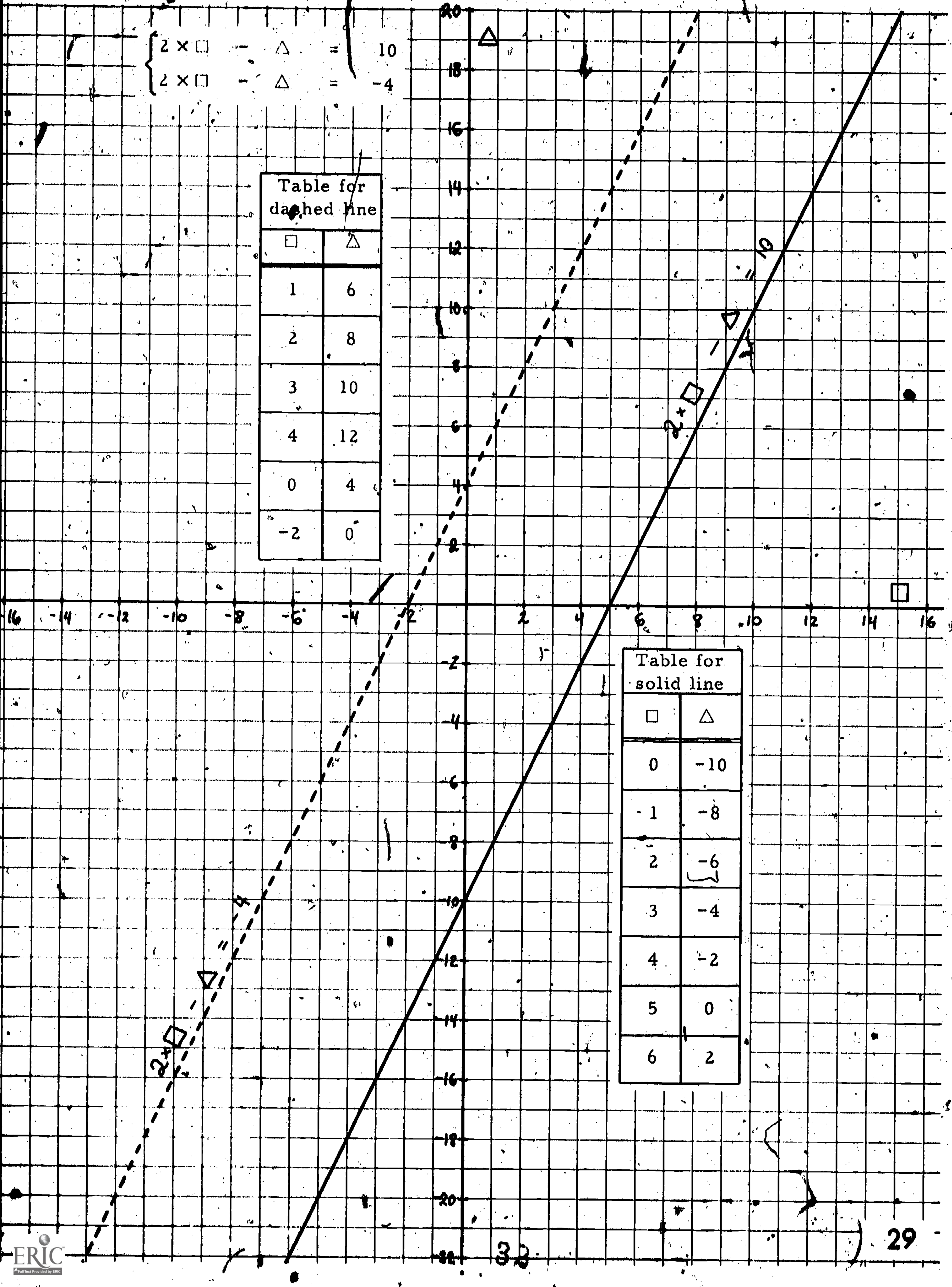


Table for solid line	
□	△
1	8
2	6
4	2
5	0
0	10

Table for dashed line	
□	△
0	−3
1	$-\frac{2}{3}$
2	$1\frac{2}{3}$
4	$6\frac{1}{3}$
6	11



$$\begin{cases} 2 \times \square - \triangle = 10 \\ 2 \times \square - \triangle = -4 \end{cases}$$

Table for dashed line

$\square$	$\triangle$
1	6
2	8
3	10
4	12
0	4
−2	0

Table for solid line

$\square$	$\triangle$
0	−10
1	−8
2	−6
3	−4
4	−2
5	0
6	2

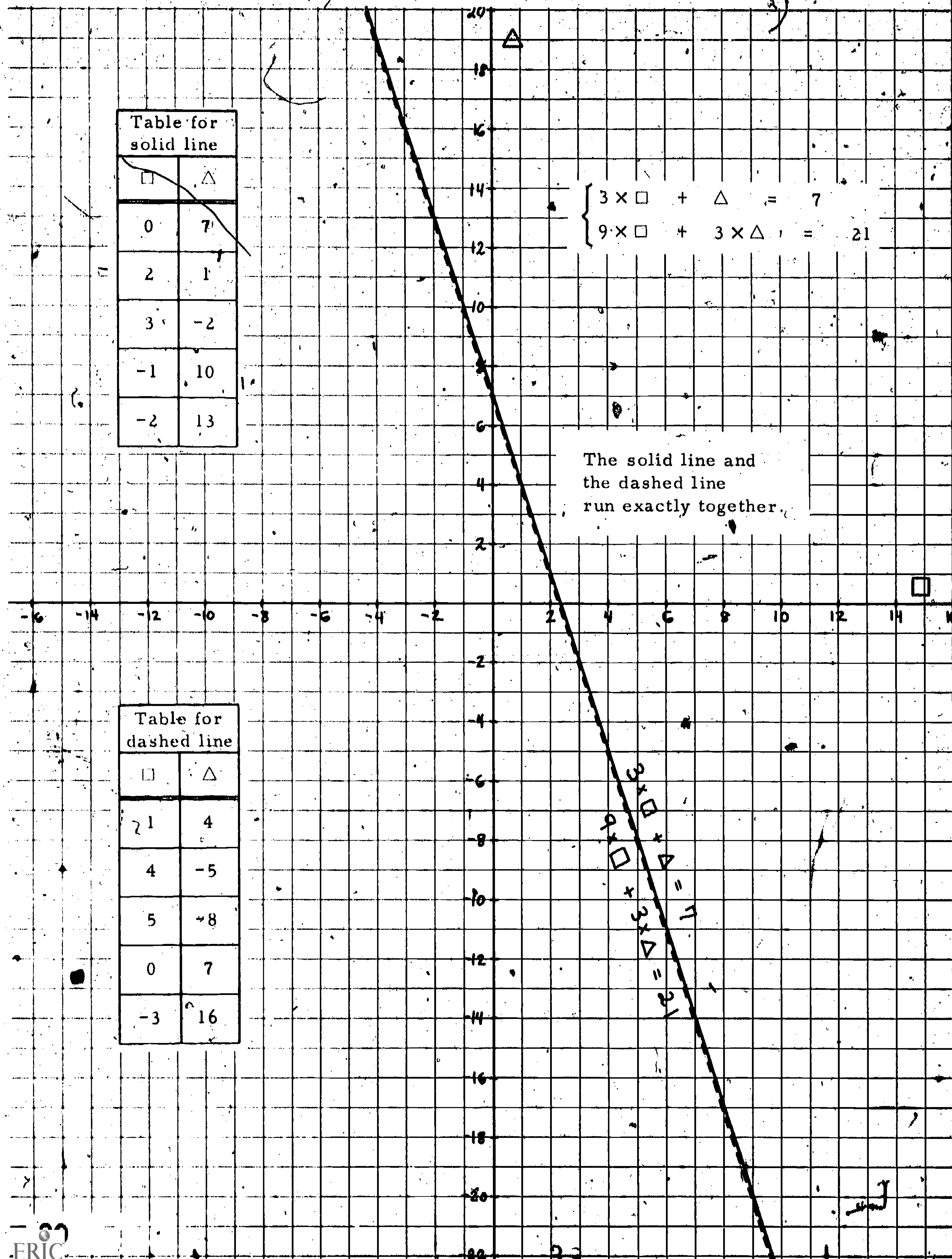


Table for solid line	
□	△
0	7
2	1
3	−2
−1	10
−2	13

$$\begin{cases} 3 \times \square + \triangle = 7 \\ 9 \times \square + 3 \times \triangle = 21 \end{cases}$$

The solid line and the dashed line run exactly together.

Table for dashed line	
□	△
1	4
4	−5
5	−8
0	7
−3	16

On the graph paper following this page, graph these pairs of simultaneous equations and find their solutions:

1. $$\begin{cases} 5 \times \square + 5 \times \triangle = 20 \\ 3 \times \square - \triangle = 4 \end{cases}$$

2. $$\begin{cases} 2 \times \square + \triangle = -16 \\ -\square + \triangle = -10 \end{cases}$$

3. $$\begin{cases} 3 \times \square + 2 \times \triangle = 20 \\ 3 \times \square + 2 \times \triangle = 16 \end{cases}$$

Comment: ______________________________

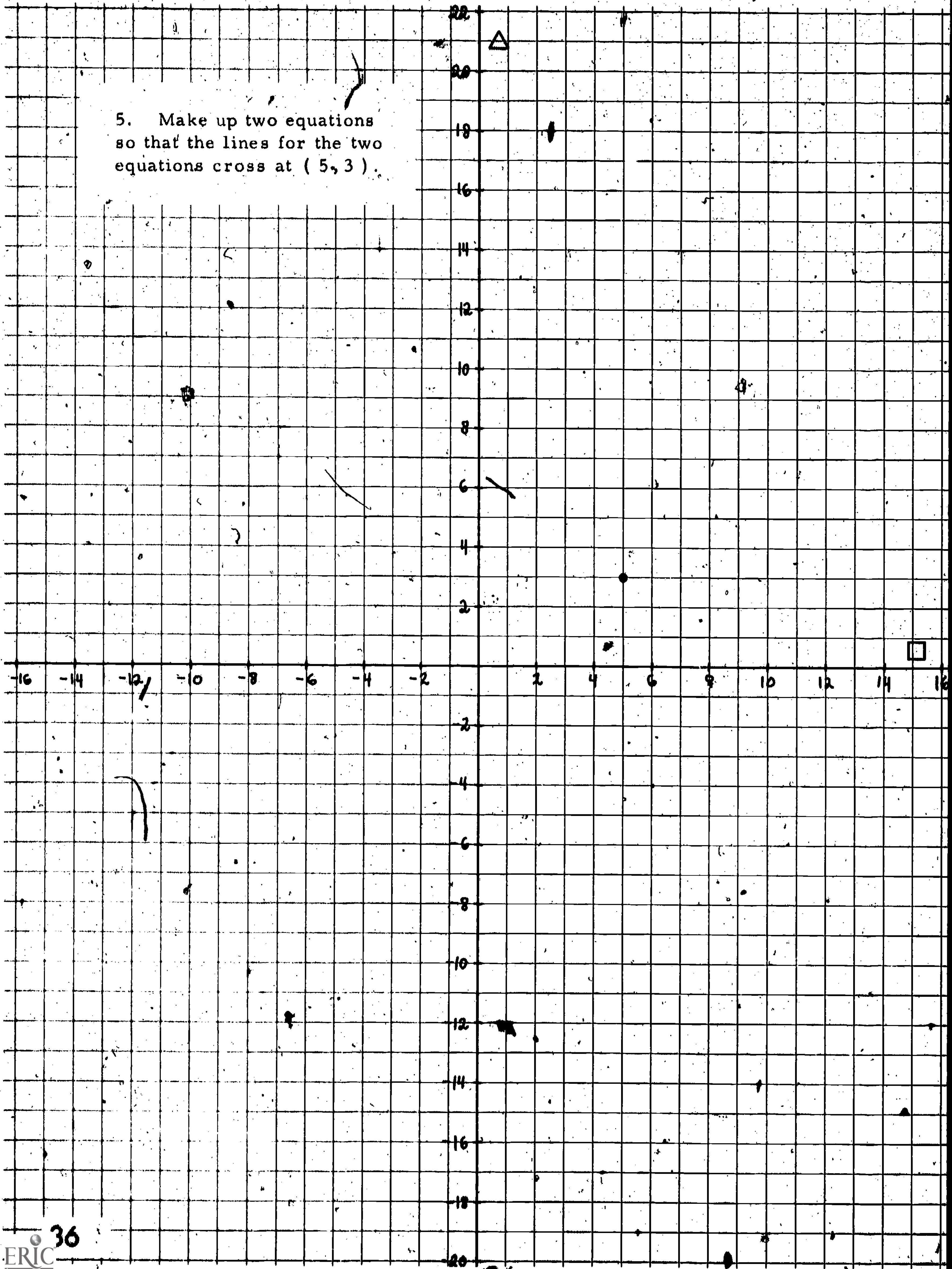
4. $$\begin{cases} \square - \triangle = 7 \\ 9 \times \square - 9 \times \triangle = 63 \end{cases}$$

What does the graph tell you about this pair of simultaneous equations? Why should this be so?

(Answers to all problems are on pages 47 through 49.)

32 - 35

5. Make up two equations so that the lines for the two equations cross at (5, 3).



6.

The tables of values for two equations are given at the right. Where do the lines cross? __________ Write the two equations in the spaces provided below. (The equations are to be of the same form as the previous equations in this section, namely, of the form

$$n \times \square + m \times \triangle = p$$

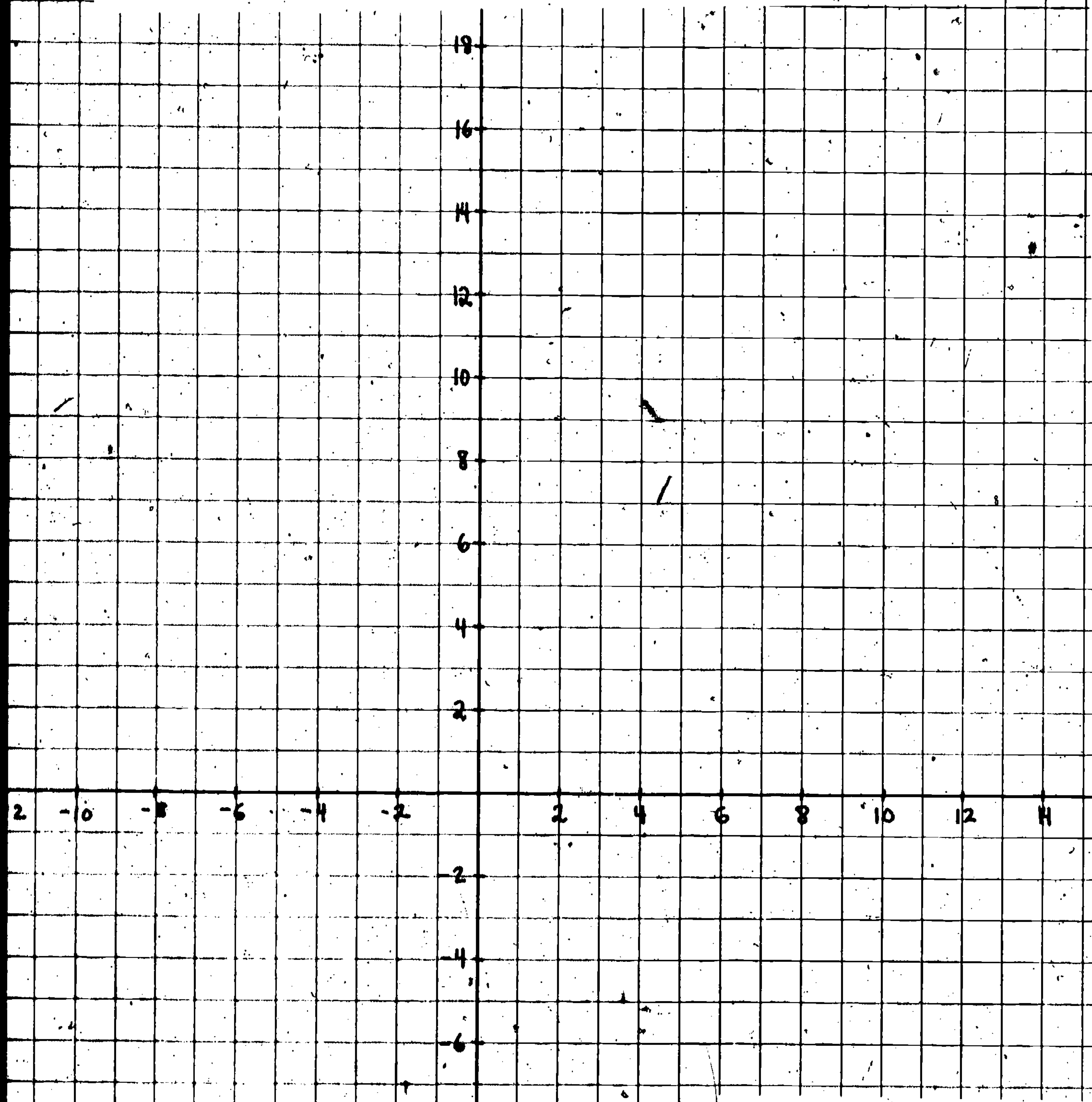
or

$$n \times \square - m \times \triangle = q$$

where n, m, p, and q are numbers you choose so that they work for the table.)

Table c	
$\square$	$\triangle$
3	4
0	7
−1	8
$5\frac{1}{2}$	$1\frac{1}{2}$
−10	17

Table d	
$\square$	$\triangle$
5	6
0	−9
3	0
2	−3
−1	−12

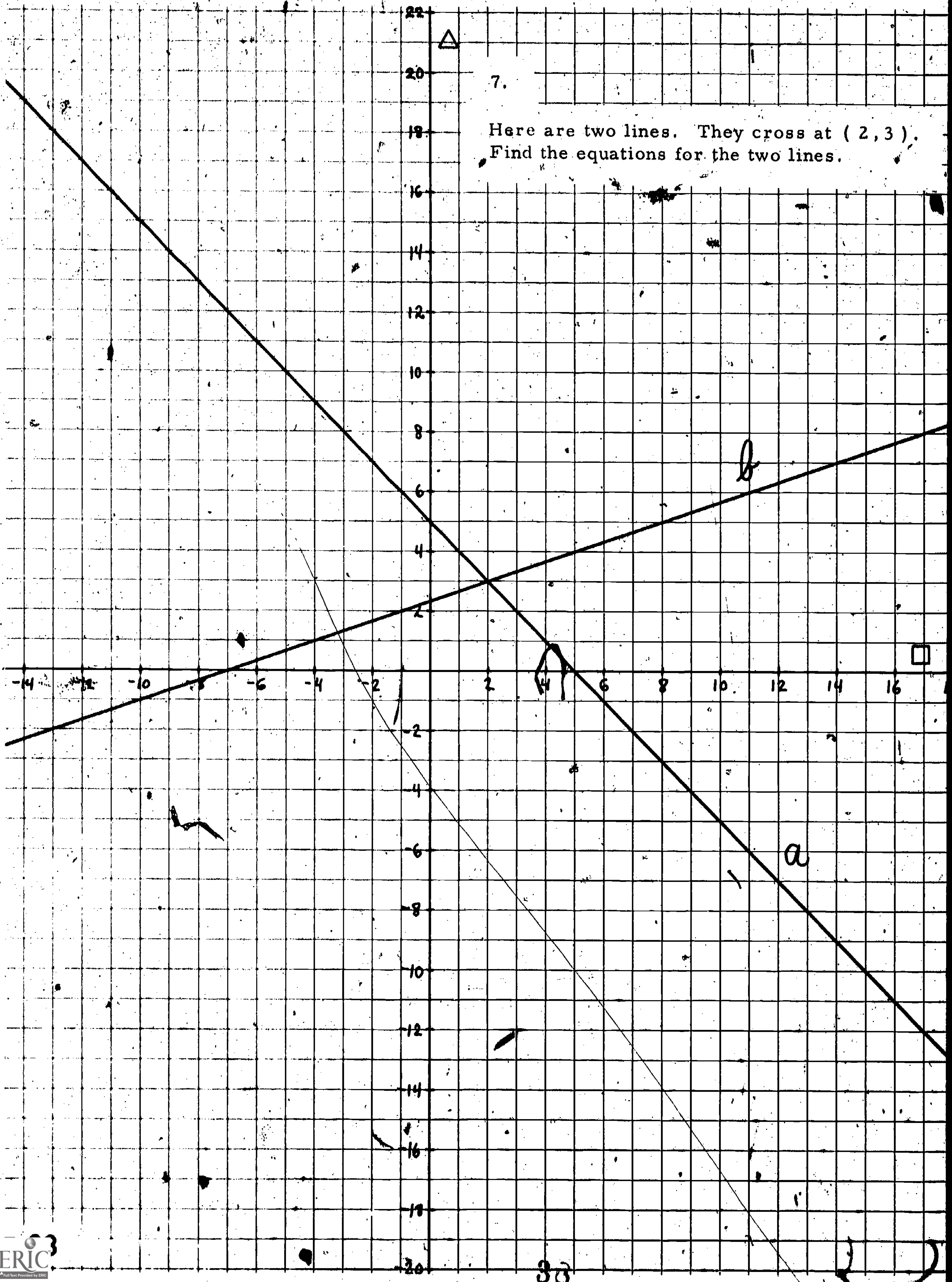


Equation c : __________

Equation d : __________

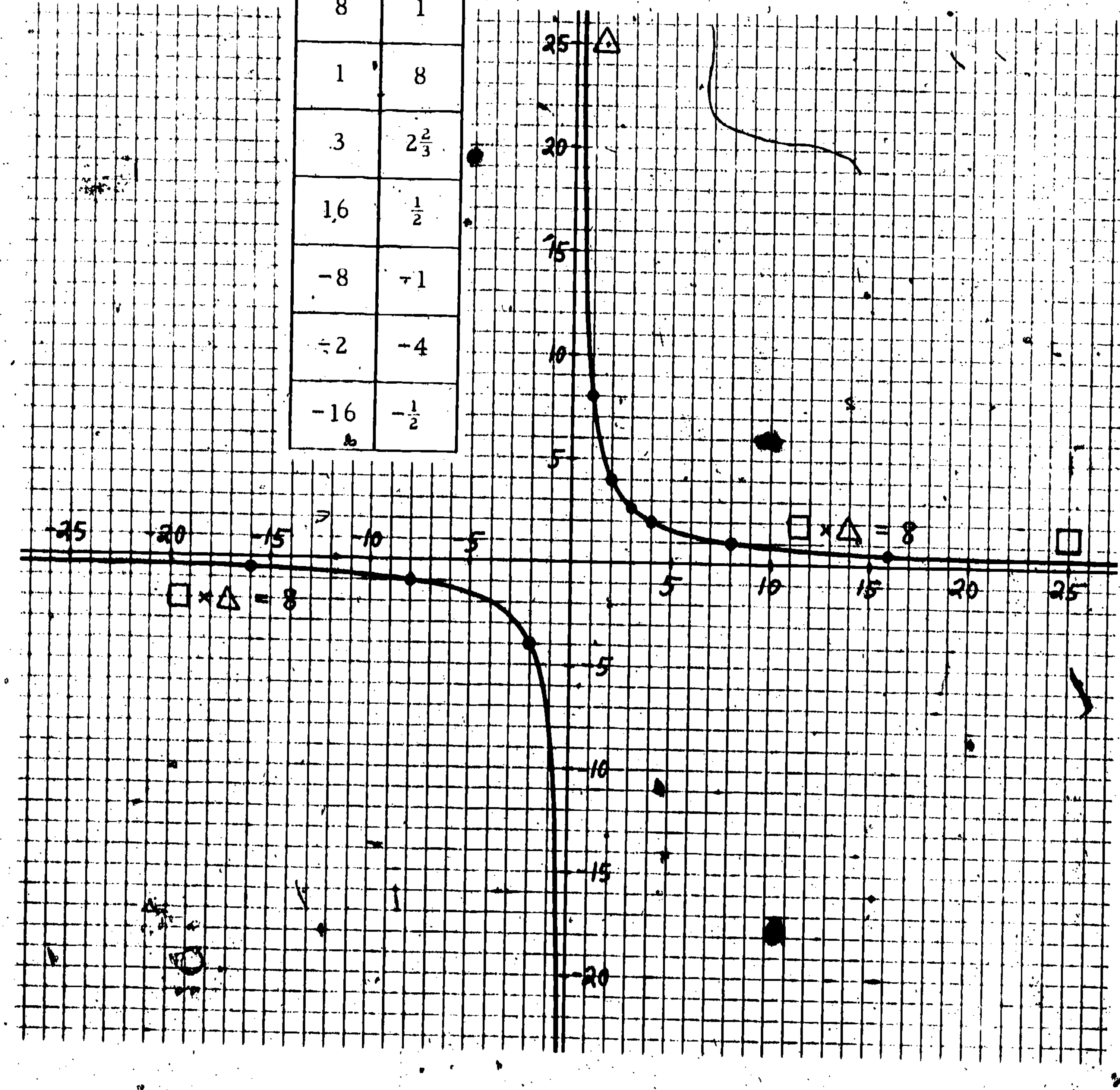
7.

Here are two lines. They cross at (2 , 3). Find the equations for the two lines.



II. Until now, the number of solutions to a system of simultaneous equations has been zero, one, or many. A question that might occur at this point is how to find a system of simultaneous equations with exactly two solutions. Since two straight lines can never cross twice, one way to get two solutions is to find an equation that gives a curved line when it is graphed. Then a straight line could pass through it twice. Surprisingly, an equation as simple as $\square \times \triangle = 8$ will give a curve with two branches when graphed. Below is a graph of this equation with a table of values.

$\square$	$\triangle$
4	2
2	4
8	1
1	8
3	$2\frac{2}{3}$
16	$\frac{1}{2}$
−8	−1
−2	−4
−16	$-\frac{1}{2}$



Once the equation giving the curved line has been chosen, the linear equation must be chosen so that its straight line passes through the curve twice. A simple type of equation that will do this is $\square + \triangle = n$; where n is some number that you choose to do the job. In order to make the solution relatively easy to find, look for pairs of numbers that work for the $\square \times \triangle$ equation. For example, using $\square \times \triangle = 8$ as one of the two equations, two pairs of numbers that work are $(2, 4)$ and $(4, 2)$. $2 + 4 = 6$ and $4 + 2 = 6$, so $\square + \triangle = 6$ is a linear equation whose graph would also pass through the points $(2, 4)$ and $(4, 2)$. Thus, the simultaneous equations

$$\begin{cases} \square + \triangle = 6 \\ \square \times \triangle = 8 \end{cases}$$

would have solutions of $(2, 4)$ and $(4, 2)$.

1. Another choice for the linear equation could be $\square + \triangle = 9$. What would the two solutions be for

$$\begin{cases} \square + \triangle = 9 \\ \square \times \triangle = 8 \end{cases}$$

Put them here: __________, __________

2. Using equations of the $\square + \triangle = n$ type only, fill in the following blanks so that the resulting simultaneous equations will have two solutions. (The table of values on the preceding page will help.)

(a) $\begin{cases} \square \times \triangle = 8 \\ \_\_\_\_\_\_\_\_\_\_ \end{cases}$

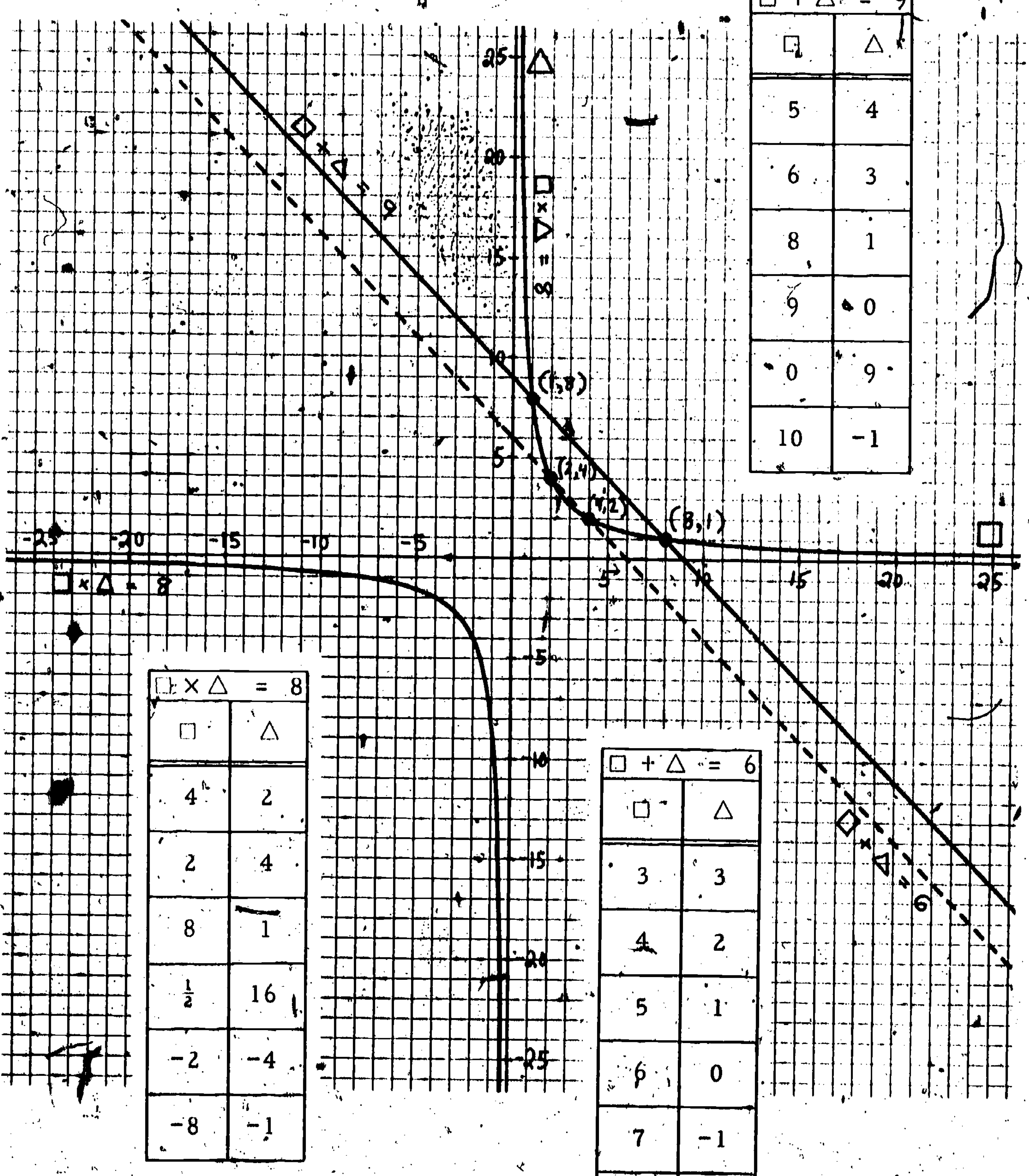
(b) $\begin{cases} \square \times \triangle = 8 \\ \_\_\_\_\_\_\_\_\_\_ \end{cases}$

(c) $\begin{cases} \square \times \triangle = 8 \\ \_\_\_\_\_\_\_\_\_\_ \end{cases}$

2. (cont.) (d) Where would the straight line have to be so that

$$\begin{cases} \square \times \triangle = 8 \\ \square + \triangle = n \quad (n \text{ is some number}) \end{cases}$$

will have exactly one solution? (The graph given below will help.) Is it possible for there to be no solution?



□ + △ = 9	
□	△
5	4
6	3
8	1
9	0
0	9
10	−1

□ × △ = 8	
□	△
4	2
2	4
8	1
$\frac{1}{2}$	16
−2	−4
−8	−1

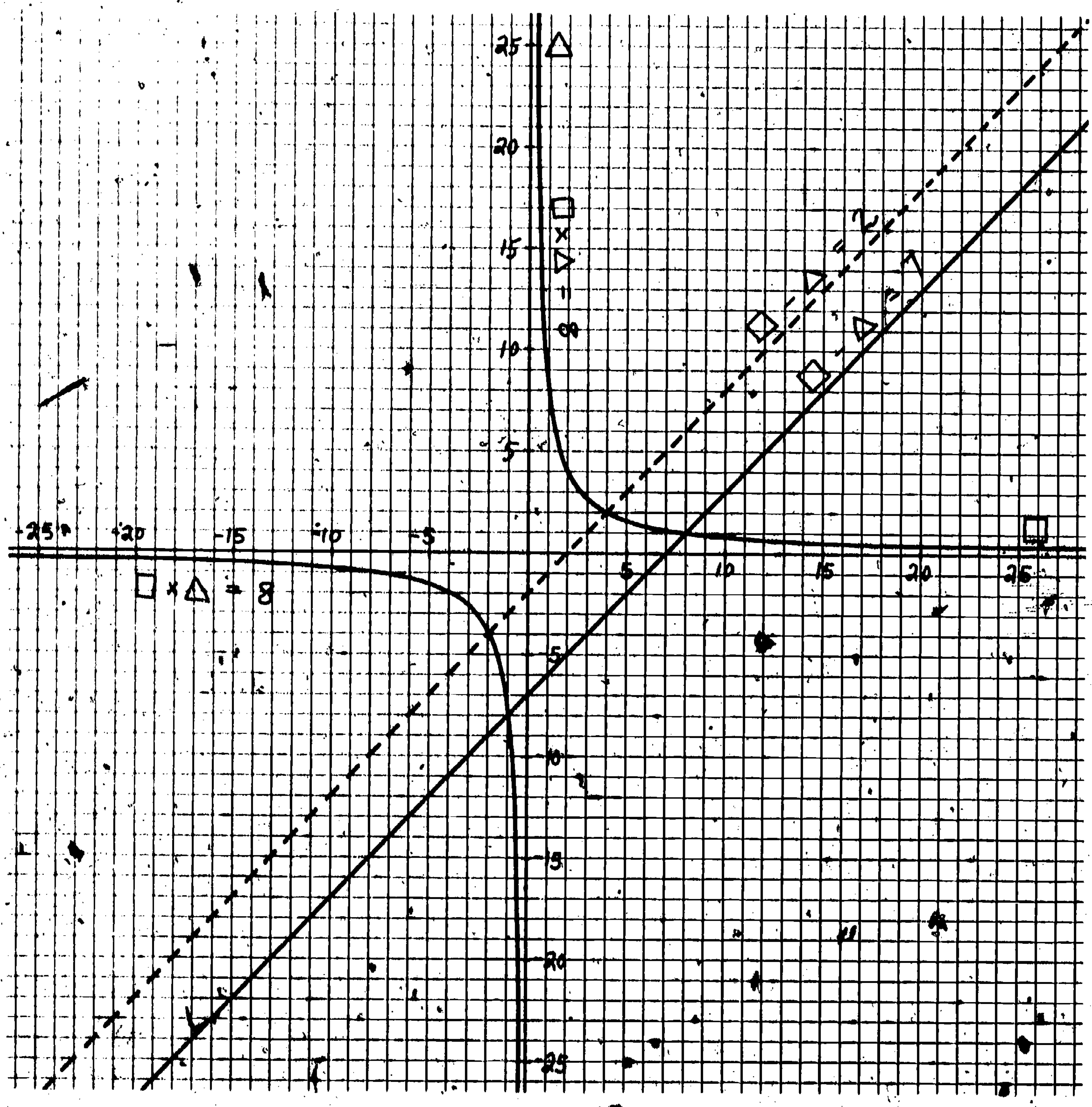
□ + △ = 6	
□	△
3	3
4	2
5	1
6	0
7	−1
0	6

III. If you feel confident working with negative numbers, an equation of the form $\square - \triangle = n$ (n is some number, as before) when paired with $\square \times \triangle = 8$, will also give two solutions, one of them involving negative numbers. Notice that the pairs of numbers (4, 2) and (⁻2, ⁻4) which satisfy $\square \times \triangle = 8$, also satisfy the equation. $\square - \triangle = 2$, since $4 - 2 = 2$ and $(^-2) - (^-4) = 2$. The graphs of $\square \times \triangle = 8$, $\square - \triangle = 2$ and $\square - \triangle = 7$ are given below.

1. Which pairs of numbers will work for the following simultaneous equations?

$$\begin{cases} \square - \triangle = 7 \\ \square \times \triangle = 8 \end{cases}$$

__________, __________



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IV. Finding other equations that will cross the curved lines in two places is for you now largely a matter of experimentation. Looking at the table of values for $\square \times \triangle = 8$ and trying various pairs of numbers in the expression $2 \times \square + \triangle$ (which was arbitrarily chosen) led to this pair of simultaneous equations:

$$\begin{cases} \square \times \triangle = 8 \\ 2 \times \square + \triangle = 10 \end{cases}$$

$\square$	$\triangle$	$2 \times \square + \triangle$
2	4	8
4	2	10
1	8	10
8	1	17
16	$\frac{1}{2}$	$32\frac{1}{2}$
$\frac{1}{2}$	16	17
−2	−4	−8
−4	−2	−10
−8	−1	−17
−1	−8	−10

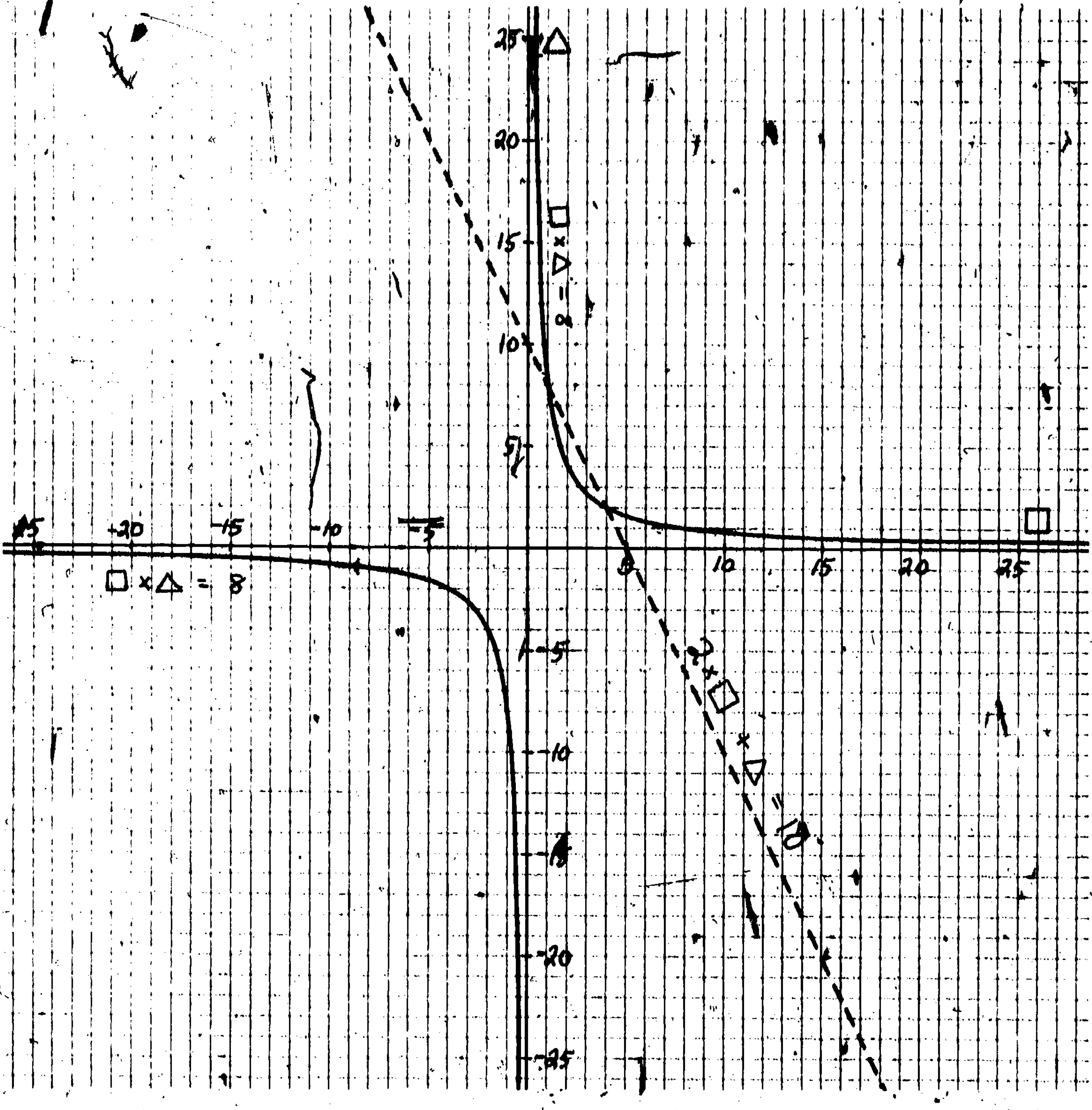
There are two other pairs of simultaneous equations with two solutions that can be made up from this table. Can you find them? Write them here:

1. $\begin{cases} \square \times \triangle = 8 \\ 2 \times \square + \triangle = \_\_\_\_ \end{cases}$

2. $\begin{cases} \square \times \triangle = 8 \\ 2 \times \square + \triangle = \_\_\_\_ \end{cases}$

The graphs of $\square \times \triangle = 8$ and $2 \times \square + \triangle = 10$ are on the following page.

$$\begin{cases} \square \times \triangle = 8 \\ 2 \times \square + \triangle = 10 \end{cases}$$



V. Each of these pairs of simultaneous equations has two whole number solutions. Graph each pair and list the number pairs that work in the spaces provided. (Graph paper is attached at the end of the supplement.)

1. $$\begin{cases} \square \times \triangle = 24 \\ 3 \times \square + 2 \times \triangle = 30 \end{cases}$$

Number pairs that work are __________ and __________

2. $\begin{cases} \square \times \triangle = 24 \\ 4 \times \square + \triangle = 20 \end{cases}$ ________ , ________

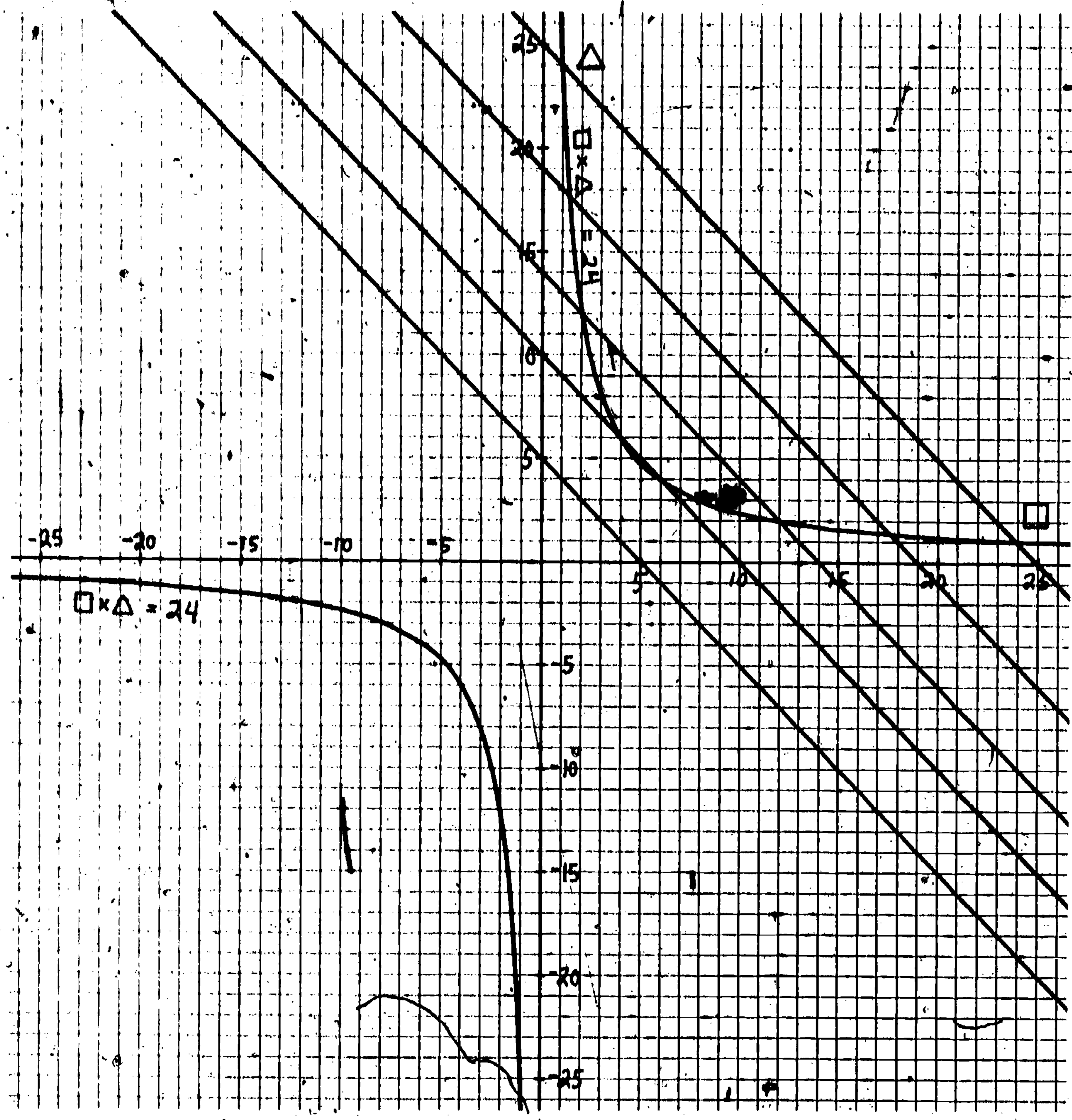
3. $\begin{cases} \square \times \triangle = 24 \\ 4 \times \square + \triangle = 28 \end{cases}$ ________ , ________

4. $\begin{cases} \square \times \triangle = -12 \\ 2 \times \square + \triangle = -10 \end{cases}$ ________ , ________

5. $\begin{cases} \square \times \triangle = 5{,}000 \\ 10 \times \square + \triangle = 600 \end{cases}$ ________ , ________

One way to make up problems of this type is as follows: Choose any equation of the $\square \times \triangle = n$ type. (We chose $\square \times \triangle = 24$ in the problems above because 24 has lots of factors. Other equations with many whole number solutions, such as $\square \times \triangle = 30$ or $\square \times \triangle = 12$, would also be fruitful.) Then, take a linear expression such as $\square + \triangle$, $\triangle + 2 \times \square$, $\square + 4 \times \triangle$, etc. Now construct a table of solutions for $\square \times \triangle = n$ as was done earlier (on page 43), computing values for the linear expression as well. If, out of all the pairs that work for the $\square \times \triangle$ equation, you find two that give the same answer when used in the linear expression that you picked, you are in business. If not, you then should discard that particular linear expression and try another one in its place. Usually, however, it is possible to find two solutions for the first linear expression that you try. After your students have spent some time making up tables and choosing equations, they may find other and faster ways to make up new pairs of simultaneous equations.

In particular, students may wish to graph the $\square \times \triangle = n$ equation and then move the graph of the linear equation up or down to intersect the curve at whole-number points. For example, suppose that $\square \times \triangle = 24$ is chosen for the curve and $\square + \triangle = N$ (N is any positive number) as the linear equation. Any two equations of the form $\square + \triangle = N$ will have graphs that are parallel. By changing N, we will get a whole family of parallel lines. On the next page is a graph of the equation $\square \times \triangle = 24$ and some graphs representing equations of the form $\square + \triangle = N$. Can you find N for each line on the graph?

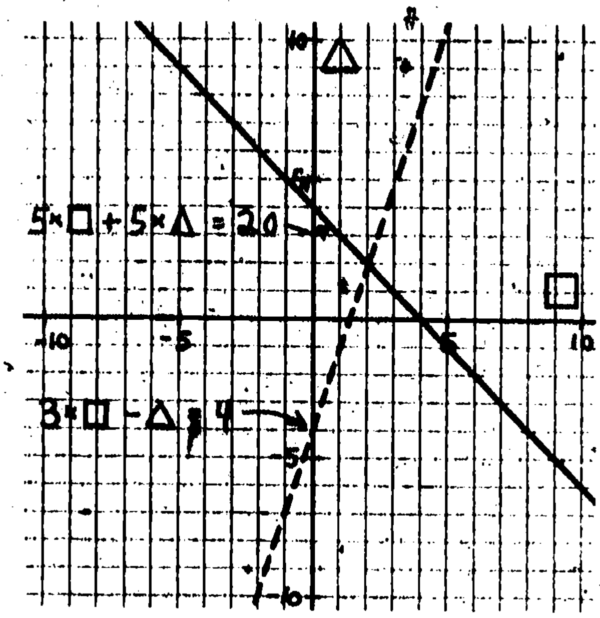


This technique will also work for other linear equations.

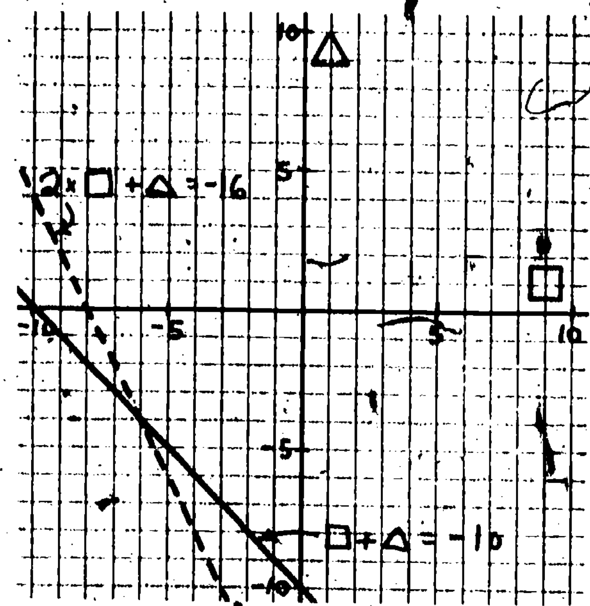
Notice that □ × △ = 24 and □ + △ = 19 <u>have</u> two solutions in common. But the numbers at the places where they cross would be too nasty to work with in elementary classrooms.

Answers

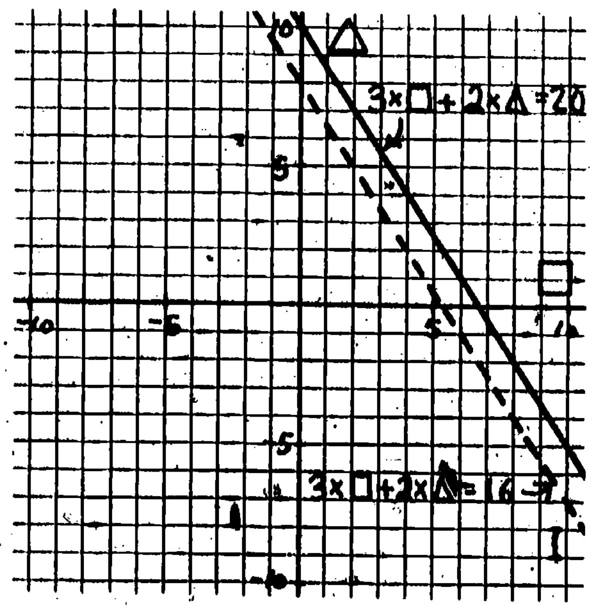
I. 1. (page 31) Solution: (2 , 2)



2. (page 31) Solution: (−6 , −4)



3. (page 31) Solution: none

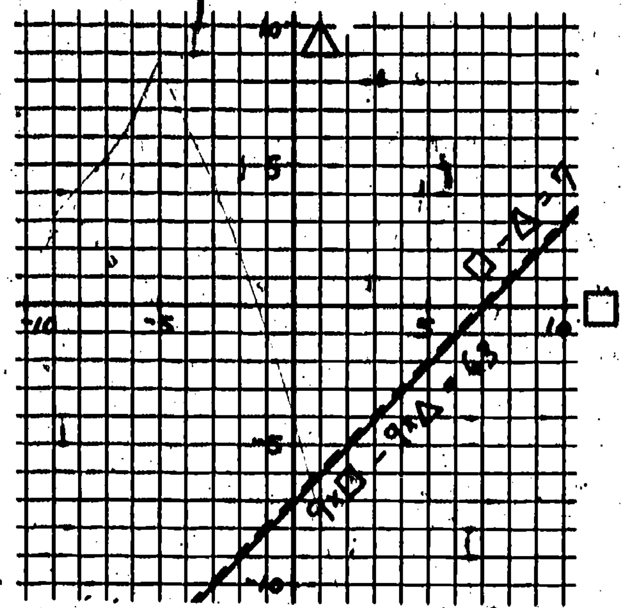


Comment for this problem should convey the idea that these two equations are exactly the same on the left of the equal sign, but are different on the right of the equal sign. Their graphs are two parallel lines, and hence the equations have no common solution.

4. (page 31)

Solution: Any pair that works for

$$\square - \triangle = 7$$



The graphs of these two equations are exactly the same. Thus, the equations are really different forms of the same equation. Multiplying both sides of the first equation by nine gives the second equation.

5. (Page 36)

There are an infinite number of equations whose graphs go through the point (5 , 3). Thus, it is likely that no two people will give identical answers. However, here are a few pairs of equations that would work:

$$\begin{cases} \square - \triangle = 2 \\ 3 \times \square + \triangle = 18 \end{cases} \qquad \begin{cases} \square + \triangle = 8 \\ 2 \times \square - \triangle = 7 \end{cases}$$

$$\begin{cases} 2 \times \square + \triangle = 13 \\ 3 \times \square - 2 \times \triangle = 9 \end{cases} \qquad \begin{cases} \square - 2 \times \triangle = -1 \\ 3 \times \square - \triangle = 12 \end{cases}$$

$$\begin{cases} 3 \times \square - 5 \times \triangle = 0 \\ 3 \times \square + 5 \times \triangle = 30 \end{cases} \qquad \begin{cases} 3 \times \square + 2 \times \triangle = 21 \\ 2 \times \square + 3 \times \triangle = 19 \end{cases}$$

$$\begin{cases} \square = 5 \\ \triangle = 3 \end{cases}$$

6. (Page 37). The lines cross at the point (4 , 3).

Equation c : $\square + \triangle = 7$

Equation d : $3 \times \square - \triangle = 9$

7. (Page 38)

Equation a : $\square + \triangle = 5$

Equation b : $\square - 3 \times \triangle = -7$

II.

1. (page 40) The two solutions would be (8 , 1) and (1 , 8).

2. a, b, c (page 40)

Possible solutions are $\square + \triangle = 16\frac{1}{2}$, $\square + \triangle = 5\frac{2}{3}$, $\square + \triangle = -9$, $\square + \triangle = -6$, $\square + \triangle = -16\frac{1}{2}$.

There are many others.

d. (page 41)

The straight line would have to hit the "corner" of either of the two branches of the curve, for a $\square$ value of somewhere between 3 and $2\frac{2}{3}$ or somewhere between -3 and $-2\frac{2}{3}$. To get an equation that does this, $\square$ must equal $\triangle$. Solving $\square \times \square = 8$ gives irrational answers; namely, the square root of eight ($\sqrt{8}$), and $-\sqrt{8}$. So each of the pairs of equations

$$\begin{cases} \square \times \triangle = 8 \\ \square + \triangle = \sqrt{8} \end{cases} \quad \text{and} \quad \begin{cases} \square \times \triangle = 8 \\ \square + \triangle = -\sqrt{8} \end{cases}$$

has exactly one solution.

It is possible for there to be no solution. These two pairs of simultaneous equations have no solutions:

$$\begin{cases} \square \times \triangle = 8 \\ \square + \triangle = 2 \end{cases} \qquad \begin{cases} \square \times \triangle = 8 \\ \square + \triangle = 1 \end{cases}$$

III. 1. (page 42) Solutions are (8 , 1) and (−1 , −8).

IV. 1. and 2. (page 43) Equations that can be made up from the table:

$$2 \times \square + \triangle = 17$$

$$\text{and} \quad 2 \times \square + \triangle = -10.$$

V. 1. (page 44) Number pairs that work are (8 , 3) and (2 , 12)

2. (page 45) (3 , 8) and (2 , 12)
3. " (6 , 4) and (1 , 24)
4. " (−6 , 2) and (1 , −12)
5. " (50 , 100) and (10 , 500)

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