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ABSTRACT

This guide for the University of Illinois Arithmetic Project was designed for use by the leader of a locally conducted in-service course for teachers of elementary school mathematics. It provides extensive discussion notes and detailed guides for correcting written lessons. The guides also contain suggestions on: finding personnel to conduct the institute, planning the institute sessions, scheduling the sessions, and points to be discussed at the initial meeting. The course package also contains films showing mathematics being taught to classes of children, written lessons which teachers do between institute sessions, and supplementary materials providing further mathematical exposition and suggestions for the classroom. (MK)

THE ARITHMETIC PROJECT COURSE FOR TEACHERS

GUIDE FOR COURSE LEADERS

THE ARITHMETIC PROJECT
GUIDE FOR COURSE LEADERS

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COURSE SUPPLEMENTS

For your convenience, the following lists the supplements contained in each book.

BOOK 1	Answers to Common Questions About the Course	BOOK 11	Well-Adjusted Trapezoids
BOOK 2	Computing With Positive and Negative Numbers	BOOK 12	Functions
BOOK 3	Answers to Common Questions About the Film "Standstill Points"	BOOK 13	Graphing Number Line Jumping Rules, Part One
BOOK 4	Dividing by Zero	BOOK 14	Graphing Number Line Jumping Rules, Part Two
BOOK 5	Maneuvers on Lattices	BOOK 15	Examples of Questions Dealing With □ → □ × □
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BOOK 7	Using Centimeter Blocks to Introduce Prime Numbers to a Third Grade	BOOK 17	Graphing Simultaneous Equations
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BOOK 9	More Suggestions for Lattices	BOOK 19	More Work With Number Plane Rules
BOOK 10	Using Blocks to Introduce Other Bases of Numeration to a Fourth Grade	BOOK 20	Hybrid Rules: Jumping Rules from the Line to the Plane and from the Plane to the Line

NOTES ON USING THE COURSE

Getting Started

It is difficult to convey to people the nature of the course and how the mathematics actually relates to classroom teaching. One of the most effective ways is to invite all teachers who might possibly be interested to view one of the films and inspect sample course material.

Teachers should have a taste of the mathematics, and a taste of the materials of the course, before being asked to decide whether to take an institute. Showing an excerpt from one of the films is one possibility. You may want to have teachers do one or two pages of problems from a written lesson that relates to the film they saw. The Arithmetic Project will be happy to recommend a film and written materials that might be suitable for your group's introduction to the course. Course materials which might be distributed as samples include Maneuvers on Lattices, "Answers to Common Questions About the Course," and possibly the first written lesson.

If you wish, the initial meeting may be considered the first session of the course. (Show the first course film.) Teachers can try the institute for a few weeks before deciding finally whether to take it. They should do the written lessons as long as they are considering taking the course. ("Auditing" without doing the lessons never works; in a few weeks a person who is not working the lessons can no longer follow the discussions and almost inevitably drops out. Besides—much of the fun, as well as the substance, is in the written lessons.)

Released time or salary increment credits—whatever is available to you for in-service courses—will ensure that teachers know that the course will make substantial demands on them. School systems have given two to four in-service credits to teachers who have taken the course.

Things You Will Need

For your initial meeting, and for giving the course, you will need an attractive, quiet room with good acoustics, and a 16mm projector in good condition with a separate loudspeaker.

The room should be amply equipped with blackboards, so that discussion leaders have space to write problems and solutions, and so that participants can write their problem sequences.

For ease in viewing films it should be possible to darken the room almost totally.

Projection equipment should be of good quality, with a sound system of adequate power for your room. The loudspeaker should be near the screen.

A quiet room is important for comfortable discussions as well as for film projection. Avoid rooms with noisy air conditioners or blowers. Carpeting makes a good acoustical treatment. Tables or desks are useful for notebooks, coffee cups, and other paraphernalia, but they are not essential.

Finding Personnel to Conduct an Institute

The course is designed to be conducted by elementary teaching personnel who have a better-than-average background in mathematics and who are interested in helping others improve their mathematics teaching.

Whoever you choose to conduct the institute should actively share with participating teachers the process of learning—and learning to teach—the materials.

Thus, each discussion leader and corrector should have a class of students with whom to try out the new materials in the classroom.

To find interested teachers who might serve as leaders, before or during your initial meeting ask people to contact you if they might like to help give the institute. (Your request will generate additional interest in the institute.)

Participants in earlier institutes are ideal as leaders of an institute. However, previous experience with Project ideas is not essential, nor is any "modern" math course a prerequisite. (Most of the course is about new things to do with "old math," and the course materials provide ample guidance for anyone willing to study them.)

Planning the Institute Sessions

Each meeting of the course should last for 1 1/2 to 2 hours to give time for viewing the film, discussing it, and discussing the written work and teachers' classroom experiences.

Different orders of events are possible.

You may want to discuss the written lessons at the beginning of a session, since teachers will have questions about problems fresh in their minds. Then view the film and discuss that, Another good possibility: view the film first and discuss it; then divide into smaller groups, perhaps by grade level, to discuss the written lesson and to work on adaptations of materials.

See page vii for further suggestions.

Try to have several teachers in each school take the course at the same time so that they can share their experiences and help each other when necessary.

It also helps if correctors can work together or conveniently be in touch with one another.

Things You Can Discuss at the Initial Meeting

The course is demanding, but it should make every participant able to bring about a substantial change in his classroom teaching of mathematics.

While taking the course, most teachers should plan to spend up to two hours a week on the written lessons, occasionally a little more.

The Institute should be both work and fun. There are strictly optional starred problems in the written lessons, just for those who want an extra challenge.

Your teachers should be given every encouragement to try the ideas they are learning with their classes.

Teachers should be urged to help each other solve problems, plan lessons, observe each other in the classroom, and share successful experiences that they have had.

Here are two possible schedules for your institute sessions:

Discuss written lesson to be handed in	15 to 25 minutes
Introduce film	2 minutes
View film	25 to 45 minutes
Discuss film	10 to 15 minutes
Return corrected lesson, discuss it (Possibly sub-divide by grade level for small group discussion.)	10 to 15 minutes
Talk further about participants' experiences, lesson plans, problem sequences	0 to 20 minutes

Introduce film	2 minutes
View film	25 to 45 minutes
Discuss film	10 to 15 minutes
Discuss written lesson to be handed in	15 to 25 minutes
Return corrected lesson, discuss it (Possibly sub-divide by grade level for small group discussion.)	10 to 15 minutes
Talk further about participants' experiences, lesson plans, problem sequences	0 to 20 minutes

Introducing the film	You need say only a few words—the grade and background of the class as given in the discussion notes are usually sufficient.
Discussion of written lesson being handed in	Since this is a time when participants' interest in finding the answers is very high, discuss fully every problem anyone asks about. Let each person keep and write on his lesson. (If you wish to keep track of the answers written during the session, distribute green pencils for use during the discussion.) Full discussion not only helps people find out about the mathematics (which is what you want), it also saves work for the correctors.
Discussion about the film	This may be short or long depending on the film and how far along in the course you are. The film discussion notes frequently suggest questions about the class that was seen, as well as additional work related to the film.
Discussion about the corrected lesson being returned	Here is the chance for participants to explain questions many people missed, as found by the correctors, and for those who got unusual solutions to explain them. Questions about correctors' comments may also come up.
Further discussion of participants' experiences, lesson plans, and problem sequences	Classroom experiences, general questions about mathematics, generating and evaluating problem sequences are all relevant here. You will find many ideas in the discussion notes and Corrector's Guides, and others will occur to you as the course proceeds.

INTRODUCTION TO THE DISCUSSION NOTES

You may expect the discussions which you will be leading in this course to center around the films, the written lessons, the supplements, and questions that come up as teachers begin to try course materials in their own classrooms. Because the course is designed to be useful without expert mathematical guidance, the discussions are of particular importance. They enable the whole group to profit from the knowledge of those who have had more experience with mathematics, and from the intuition of those with special aptitude for the material. Any reasonably sized group will possess a diversity of backgrounds and talents. Exploiting this diversity for the benefit of everyone is the discussion leader's task. It is not an easy one, but it can be exciting and rewarding.

The notes in this guide deal with each film and each written lesson in the course. They are based on questions that have come up in previous institutes using these materials and on other mathematical ideas that have proved of interest to teachers.

Many of the notes are in question-and-answer form. It is not our intention that the moderator read each question to the group, call for answers, and then read "what the book says."

In many cases members of your group will ask questions similar to those given here. Of these, many will be answerable by other members of the group. The comments given in these notes will help you decide whether a question has been adequately discussed. In some cases, participants in your institute will give better answers than we have given here. In other cases, the answers are a matter of taste, and there is room for diversity of viewpoints. In still other cases, the mathematics involved definitely determines which viewpoints are suitable and which need to be corrected. You must use your own judgment in deciding if it will be helpful to read portions of this guide to a group. But bear in mind that much reading aloud will imply that these notes contain the only authoritative answers and ways to think about a problem. Not only is this false, but also it will inhibit many people in your institute from discussing the questions.

Consider the discussion notes as only one of many sources of assistance to your institute. Another important source is the Corrector's Guides and the observations of the correctors on the week-by-week work of the participants. (Yet another source is your own sense of what is worth pursuing.) Do not lean too heavily on these guides. You may disagree with parts of them. You should try to depend, instead, on the strengths of the members of your group. The institute belongs to you and to them.

<u>How to Get Started</u>

One of the first things you should do is to be sure you have a class to teach. If your normal duties do not include teaching regularly, you should arrange during the institute to have a class of children to work with on three or four times a week. The course will be immensely more effective if you are genuinely sharing with the other participants the process of learning how to teach the ideas to children.

Many of the difficulties you may encounter in achieving lively and interesting discussions will stem from such things as the composition of the group, how well its members know one another, and the general attitudes that are brought to the course. Some groups will begin quickly to explore fertile questions and to help each other probe the mathematics and how to teach it.

Many people, however, are frightened by mathematics. Others may not yet be convinced that they should use some of the curriculum innovations they have been introduced to. For whatever reasons, you may well encounter a group that, for all your efforts, continues to remain silent and unenthusiastic. If so, keep in mind two obvious but often ignored rules.

First, listen to what is being said. If this sounds too simple, recall how many discussions you have attended in which nobody in charge seemed to care whether a question had been suitably answered or whether a comment was germane. Some moderators will listen superficially for the answers in their prepared notes and move on as soon as something similar is said. Other leaders will try their best to get an argument started on any subject. Both approaches make clear a lack of interest in what is actually being said. Neither is likely to achieve a discussion that clarifies questions, solves problems people want solved, and brings out worthwhile new ideas and viewpoints.

Second, do not talk too much: the more you talk, the less the participants will. Doubtless it is being "on stage" that makes a moderator tend to talk incessantly. The urge must be resisted with all your strength. Do not fill in the long embarrassing silences with your own voice. Do not be afraid of silences. First of all, they are not as long as they seem to you. Secondly, these silences work for you: some of the people in the group may gain the courage to speak simply to relieve the pressure of the silence.

Watch carefully for little signs that someone is considering putting his hand up. Adults, unlike children, frequently are inhibited about raising their hands very high. (They may be afraid they'll be called on, or afraid they won't be—or perhaps both.) If you wait long enough, you may see a timid index finger begin to sneak up by its owner's chin.

Moving around the room will help keep you occupied while you concentrate on being silent. In some cases, if you find yourself near one or two people you know, you may find it helpful to start a local conversation. It should be related to the question or subject to be considered. You may then bring this conversation into open discussion with some remark like, "Would you say that again for everyone?" "Can you explain that to the others?" (You should not continue a local conversation long; either make the discussion public or terminate it soon.) Sometimes a reticent group will become more verbal if the moderator, while addressing a question to the whole group, concentrates on looking at two or three members at a time.

You will discover other useful techniques as you go along. Once you become acquainted with the group, you will come to know individuals who are likely to help. Consult with the correctors to find who has given interesting answers to hard problems, and (preferably before the session) seek out and ask these teachers if they would be willing to explain to the group what they did.

Many teachers who are perfectly at ease in their classrooms are painfully shy about talking to other teachers on an unfamiliar subject. Do not be discouraged if your early efforts to get people talking result only in a few nervous sentences. Eventually, after a few teachers have summoned the courage to explain something, others will be less reticent, and everyone's courage will grow.

As the Institute Proceeds

Once a discussion is really going, the task of the moderator is to help get interesting things said. Here there are no formulas. These notes will give you clues to fruitful territories for exploration, but your own sensitivity again must determine whether you should encourage a conversation to continue, cut it off, change its direction (by adding a new thought, perhaps), or postpone it for further discussion.

Of course, a nice balance is required between over-domination and laissez-faire. Any group resents being led from one predetermined question to another. A rambling discussion that goes from one unfinished topic to another is frustrating. And one which is dominated by one or two vociferous participants is not likely to be of much help to others.

Because discussions cannot resolve many of the worries which teachers have when starting the course, it is extremely valuable, if you can arrange it, to have teachers who have already begun to teach Arithmetic Project topics speak to your institute. It is comforting to a group first plunging into the course to meet someone who recently emerged from the experience and who is enthusiastic about it and working with the ideas in the classroom. If this person entered the course with little previous mathematical training, so much the better for the morale of your group. You may want to invite the correctors of written lessons for your institute to talk about their teaching experiences. Beware, however, of an overemphasis on "staff" reports to the group. As soon as the participants themselves have things to say about their own experiences, such comments can often be more useful and more convincing.

NOTES TO CORRECTORS

Thoughtful, considerate correcting is important. It helps participants profit from their mistakes. They learn more about mathematics and ways to teach it effectively. Your caring makes the whole institute experience a richer, more human relationship. (Incidentally, your own insights into teaching and mathematics will also benefit.)

Before beginning to correct papers, take the trouble to work through the written lesson yourself. Use the Corrector's Guide to correct your work afterwards. Even though you may have done the assignment previously, you will find it valuable to be reminded how you found many of the answers. Like many of the rest of us, you may be surprised to find that some of the problems again seem quite difficult and that you make some discouraging errors. Be reassured, however: an institute benefits when those who give it consider themselves partly to be taking it also, along with the other participants. Surprises lurk among the assignments, new things are to be found in the films, and new ideas and questions will be raised by participating teachers—and their students.

* * *

<u>How to help participants who are having trouble with computation:</u>

Having found an error, what does one do? Sometimes you will want simply to write the correct answer. Sometimes you will think up a question that will enable the teacher to find the answer for himself. You may be able to write down the problem the teacher probably worked. (Likely possibilities are given in the Corrector's Guides.) You may wish to restate directions if you think they have been misinterpreted. If you can guess at what stage in a solution the teacher went off the track, you may explain and correct that step, and then suggest, "Now try it again."

A wise corrector is encouraging. If a person bogs down in the middle of a lesson, tell him, "You were doing fine up to here." Then explain a few problems where the difficulty began and offer to look at his paper again. If a teacher is just on the edge of understanding, there is no need to make mistakes hurt more than they already do. If a teacher has had trouble in many places, it is unquestionably important to find a page well done and write on it, "Good."

* * *

<u>How to comment on answers that require verbal explanations:</u>

You may find verbal answers which are vague, illogical, or only partially correct. Look for the best possible interpretation. Teachers meeting these ideas for the first time often cannot express themselves well about them, just as children can't. The careful corrector emphasizes the correct ideas and expands on them, realizing that teachers who anticipate children's approaches to problems will probably be able to teach more effectively.

Many questions ask for hypothetical children's responses, such as "How might a child describe a fast method for doing this problem?" Practically any answer to this question should be considered correct, since it is hard to be sure of the capabilities of a hypothetical child. Many teachers, at least in the beginning of the course, will solve problems algebraically, while most children do not. To a teacher answering a

"How might a child..." question with an algebraic method, you might comment, "Yes, a very clever child might say this, but can you think of some method that might be used more commonly by young children?"

* * *

How to respond to teachers' sequences of problems:

It is particularly difficult to comment usefully on sequences of problems written by teachers. What yardstick can be used for grading difficulty, pacing, variety, flair, inventiveness? The notion of presenting an idea through a series of problems is a subtle one. It does not come all at once. There is no formula for doing it, and no general instructions are really of much help. Moreover, there is always the nagging chance that the corrector's suggestions might not really be as good as the work being criticized.

The real value in having teachers begin early to write problems is that writing problems, even imperfect ones, helps teachers get started in their classes. Trying these ideas with their students is crucial if teachers are to learn about them and learn effective ways to teach them.

Checklist for Correctors

* Do the assignment yourself.

* Remember that correct answers are to be found on the far right page of the guide. Answers in samples often contain errors.

* Use erasable red pencil so that you can change your mind when you decide the teacher was right after all...

* Do not write over participants' work or text; it is difficult to read afterwards. If you need more space for an explanation use a separate sheet of paper and clip it to the page in question.

* If everything on a page is correct, you might simply put a "C" or "Good" at the bottom. Make sure that there is some mark on each page attempted to indicate that you have looked it over.

* Write overall comments on the last page on which problems were answered, or inside the front cover. Do not write general comments on the front of the booklet, since they may be seen by other participants when they collect their returned work.

* Do not add parentheses to an answer unless the answer is wrong without them.

* Ignore errors in spelling and grammar. If a person consistently makes the same mathematical error throughout a written lesson comment on it only once.

* Keep equations separate. Expressions like the following are sometimes written by correctors (and others) to explain how an answer was reached:

$$6 + 7 = 13 + 9 = 22$$

Although the meaning is usually clear, the expression is incorrect, and should be written as:

$$6 + 7 = 13 \qquad 13 + 9 = 22$$

* Do not mark unattempted starred problems. Participants should feel that these problems are really optional. They may want to try them later. If a starred problem has been attempted but the answer is wrong, you may want to supply help.

* Do not overpraise easy work. As the course progresses, exercise judgment as to what was difficult for a particular person. Some participants will become increasingly sophisticated about the work.

* Interesting solutions to problems make good topics for discussions.

* It is not necessary to comment on everything. If you are in doubt about the appropriateness of a comment, refrain from making it.

* Try not to forget, as you correct papers, how it felt when the ideas were new to you. Sometimes correctors become impatient for others to learn, forgetting in their zeal how long they themselves spent becoming familiar with the material.

21

First Session

Written Lesson Discussion Notes

Here is how one moderator led a discussion of the first written lesson.

Moderator: Are there questions that you'd like to bring up about the written lesson?

Participant A: Why do you need □ ⟶ for a jumping rule? Why can't you just write 3 × □ ?

Participant B: I suppose that it's helpful to keep track of where you started a jump. Isn't this question discussed in the comments at the end of the written lesson?

Participant A: But I still don't understand.

Moderator writes □ ⟶ □ × □ + 10 − (5 × □) on board.

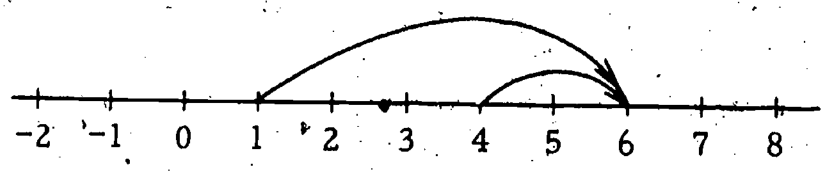
Moderator: With this jumping rule, where do you land if you start at 1 ?

Participant C: You go to 6 .

Moderator: Now start at 4 and use the rule.

Participant D: You also land at 6 .

Moderator draws both jumps :



Participant A: I see. It's the starting numbers that make the jumps different.

Moderator: Right. Now consider this rule.

Moderator (writing □ ⟶ 7): Why do you suppose you need the box and the arrow here?

Participant E: It looks as if you'll always get 7 as a landing point. Unless you have the starting points, you wouldn't be able to distinguish one jump from the other, would you?

Participant A: Yes, that's what I just said.

Participant F: Is the arrow the same as an equal sign?

Moderator (writing $\square \longrightarrow 3 \times \square - 4$): Start at 5 and make a jump with this rule.

Participant F: You land at 11.

Moderator writes:

$$\boxed{5} \longrightarrow 3 \times \boxed{5} - 4$$

$$\boxed{5} \longrightarrow 11$$

Moderator: This is read "5 goes to 11"* or "if you start at 5 you land at 11." But no one would claim that 5 = 11.

Participant F: So the arrow is not like an equal sign.

Note to moderator:

Perhaps someone in your group will recognize that jumping rules are examples of functions. The arrow is standard mathematical notation indicating some sort of jump or transformation. Further on in the course the teachers will receive supplementary material on functions. After having worked with jumping rules for some time, they will be ready to appreciate a more thorough explanation of functions as well as some interesting examples of other types of functions.

Moderator: Other questions?

Participant C: For problem 19 on page 5, I tried some numbers and they worked. But how can you be sure that all numbers work?

Moderator writes problem 19: $1 + (2 \times \square) + 2 = \square + 3 + \square$.

Participant B: Well, I figured that . . . may I come to the board?

Moderator: Yes.

Participant B: I rewrote the expression on the left of the equal sign as $2 \times \square + 1 + 2$. And the right side may be written as $\square + \square + 3$. Now each side has 2 boxes and a 3, so the expressions on either side of the equal sign, though written in a different form, are really equivalent. Therefore any number you put in the boxes will work.

---

*Of course 5 does not get up and walk to 11, but this is a common way of reading the expression.

One of the most fruitful discussion topics is question 4 on page 9 of the first written lesson which asks participants to write rules whose jumps always go to the left (to lower numbers).

Many rules suggested will be incorrect but will give some jumps to the left. It is usually interesting to pursue this in discussion. If no one raises the question, you might bring it up: "What about question 4 on page 9? Does anyone have a rule we can try?"

A rule is written on the board. For example

□ ⟶ □ - □ - □

Moderator: Does that do it?

Participant A: It goes to the left.

Moderator: Someone give me a starting place.

Participant B: 5.

Moderator (writing 5 in the boxes): Where do you land?

Participant B: Minus 5?

Moderator: Does everyone agree? OK, give another starting place.

(The moderator collects and draws several jumps, getting starting and landing points from the group. As long as no one suggests a negative number or 0, he will only have jumps going to the left.)

Moderator: Well, what about it? Have we made all the jumps we need in order to be sure?

Participant C: What about starting at 0?

Moderator writes 0 in the boxes.

Participant D: That stays on 0, doesn't it?

Moderator: 0 minus 0 minus 0 is still 0.

Participant D: Yes. So that jump doesn't go to the left. It doesn't go anywhere.

Participant E: What happens if you start on a negative number?

Moderator: O.K., give me one.

Participant E (after a pause): Negative 2 .

Moderator (writing −2 in each box): Does anyone know how to do this? (He writes (−2) − (−2) − (−2) .)

Participant E: There's something about this kind of thing in the notes at the end of the lesson.

Whether using the notes in the lesson or relying on someone in the group, the moderator should compute and put on the board the correct answer to the problem, which, in this case, is 2 . There may be some bewilderment in a typical group of teachers with respect to how one performs such a computation. The moderator should assure the group that as the course proceeds there will be more information about computing with negative numbers.

With such a rule as □ ⟶ □ − □ − □, the moderator might also ask, "Does anyone know a simpler way to write this rule?" He is looking for □ ⟶ − □ , but may get such a suggestion as □ ⟶ □ − (2 × □) , which is also the same rule.

Many of the various incorrect rules which are given for this question are like the preceding example in that they have jumps to the left only when you start at positive numbers. The moderator may say to the group that this is likely to be the case, and ask how many others may have such rules. Here are some typical examples from other institutes:

□ ⟶ □ ÷ 2

□ ⟶ −2 × □

□ ⟶ □ × 0 (This is the same rule as □ ⟶ 0 .)

Some rules will have jumps to the left if, and only if, you start above some particular number. □ ⟶ □ − □ − □ − □ − □ − 2 goes to the left providing you start above $-\frac{1}{2}$. The exact point need not always be determined (later in the course people will learn how to find it). <u>Any</u> jump to the right disqualifies the rule. If the group wants to find roughly where the transition point is, you can ask for or suggest various starting points and make a table such as the following:

Jumps with □ ⟶ □ − □ − □ − □ − □ − 2

or, in simpler form, □ ⟶ (−3 × □) − 2

Start	Land	Jump goes
10	−32	Left
5	−17	Left
0	−2	Left
−5	13	Right

By now you know that the transition is somewhere between 0 and −5. Narrowing it down further:

Start	Land	Jump goes
−1	1	Right
$-\frac{1}{2}$	$-\frac{1}{2}$	Neither right nor left

For other interesting rules that may be expected in response to this question, see the Corrector's Guide.

Film Discussion Notes

"A First Class With Number Line Rules and Lower Brackets"

Preliminary information:

You will see a fifth grade class from the Gleason School in Medford, Massachusetts. The students had never before worked with Project materials. This was their first meeting with the teacher, Lee Osburn. The film was made near the end of the school year. [Film running time: 33 min.]

Here is how one moderator led a discussion after the film.

Moderator: Near the end of the film, the teacher wrote this problem:

$$\left\lfloor \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} \right\rfloor =$$

Do you recall some of the wrong answers that the students gave?

Participant A: Six was a popular answer.

Participant B: So was $\frac{12}{5}$!

Participant C: I remember that two students gave 0.

Moderator: Where do you think these answers were coming from?

Participant D: Possibly some students saw $\frac{2}{5}$ written six times, and thought that the answer was 6.

Participant B: I suppose that you could fix the problem so the answer would be 6.

Moderator: How would you do that?

Participant B writes

$$\left\lceil \frac{2}{5} \right\rceil + \left\lceil \frac{2}{5} \right\rceil + \left\lceil \frac{2}{5} \right\rceil + \left\lceil \frac{2}{5} \right\rceil + \left\lceil \frac{2}{5} \right\rceil + \left\lceil \frac{2}{5} \right\rceil = 6$$

Participant C: In that case, if you switched each upper bracket to a lower bracket, then you would get 0 for an answer. Perhaps the students thought that since $\left\lfloor \frac{2}{5} \right\rfloor = 0$, then

$$\left\lfloor \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} \right\rfloor$$ might also equal 0.

Participant E: Of all the wrong answers, I think $\frac{12}{5}$ was closest to the correct answer, since the students just forgot to use the lower brackets.

Participant F: What else can you do with brackets?

Participant G: Are the brackets a made-up symbol, or are they used in mathematics?

Moderator: Yes, brackets are used in mathematics. We will be seeing more of this topic later in the course. If you will save your questions for a few weeks, we can discuss them then. The topic of upper and lower brackets was included in this film to give you a flavor of what lies ahead. Does anyone have questions about jumping rules?

Participant B: When the teacher wrote the first jumping rule on the board, the arrow seemed to be very long. Why did he do that?

Moderator writes $\square \longrightarrow 2 \times \square - 5$.

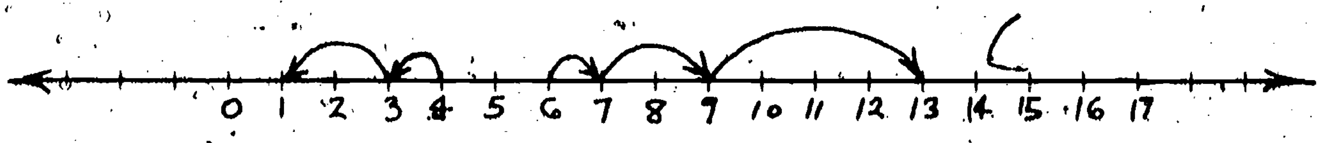
Moderator: Students first learning about jumping rules sometimes confuse the arrow with a plus, a minus, or an equal sign.

Moderator writes $\square \rightarrow \square + \square$

Moderator: When the teacher wrote this rule (using a relatively short arrow), the students did show some confusion. Tony, for example, suggested 3 for a starting number, hoping to land on 9, an odd number. Clifford thought that if you started at 1, you'd land at 3. What were they probably thinking?

Participant A: I suppose they just added all the numbers. If you make the arrow longer, it seems to separate the starting number from the landing number.

Moderator: Yes, I agree. Here's how some of the jumps looked for the rule $\square \longrightarrow 2 \times \square - 5$:



Can you find any patterns to these jumps?

Participant B: The jumps are different lengths and they go in opposite directions.

Participant D: I noticed that you have two jumps of one space, two jumps of two spaces, but then only one jump of four spaces.

Moderator: If the pattern were preserved, where should the jump from 1 land?

Participant D: Would it be four spaces to the left of 1 ?

Moderator: Yes, it would. How long do you predict the jump from 13 will be?

Participant B: Eight spaces. Then you'd have 16 spaces, 32 spaces, and so on. The distance just doubles, doesn't it?

Moderator: Let's try it. Where do you land when you put 13 into the rule?

Participant F: 21, which is eight spaces from 13.

Praticipant B: I figured out that starting at 21, you land at 37, which is a jump of 16 spaces, so I was right.

Participant A: I noticed that for a while, the teacher always took jumps from the previous landing points. Are you allowed to start on other numbers? Can you start on 10, for example?

Moderator: Yes, you certainly can. You can start on any number on the line. Do you think you can land on any number on the line? Could you land on $11\frac{1}{3}$, or even on $102\frac{5}{8}$?

Here are some other questions about the film your group may be interested in discussing. Do not feel compelled to "cover" them nor to provide all the answers.

1. Why do you suppose the teacher did not introduce negative numbers?

2. For the rule □ ⟶ □ + □, the class was looking for starting places so that they would land on an odd number. One child said, "Every number with $\frac{1}{2}$ added to it." Strictly speaking, this is not correct, since $2\frac{1}{4} + \frac{1}{2}$ would not work. What would you have done in your class?

3. Try to make [illegible] a jumping rule so that when you start on any integer, you always land on an odd number.

4. Try to make up a jumping rule so that if you start on any integer, you never land on a multiple of 3.

Second Session

Written Lesson Discussion Notes

Moderator: Are there any questions on the lesson?

Participant T: I didn't know how to go about question (j) on page 4. The rule is □ ⟶ □ − 3 and the question is where can you start so that after 5,000 jumps you land at zero?

Moderator writes, □ ⟶ □ − 3

Moderator: Where would you start to land at zero:

after 1 jump? ________ (3)

after 2 jumps? ________ (6)

after 5 jumps? ________

after 10 jumps? ________

after 100 jumps? ________

after 1,000 jumps? ________

after 5,000 jumps? ________

Participant Z: I was wondering about question 3 on page 3. The rule is still □ ⟶ □ − 3. If you start at 10 you land at 7. Since the jump has been to the left, why don't you write −3 instead of 3?

Participant L: I decided that 3 spaces is 3 spaces and that negative three spaces does not make sense. So when asked for spaces moved, I always gave a positive number.

Moderator: You are correct. Spaces moved asks only for how many spaces long the jump is, regardless of whether it is to the right or to the left.

Participant B: Does anybody have a quick way to do question (m) on page 4?

Moderator: Question (m) asks you to use the same rule and start at 150. You are to give three numbers less than 80 on which you will not land.

Participant K : I figured out that each jump subtracts 3 so after

10 jumps you are at 120

20 jumps you are at 90

30 jumps you are at 60

31 jumps you are at 57

Since you are starting at 150, which can be divided by 3, and you are subtracting 3 each time, you will always land on a number divisible by 3.

* * *

Notes to moderator:

As part of this written lesson, teachers were to write a sequence of problems on competing number line rules. Teachers generally find it helpful to see what other participants have written. Try to get a few volunteers to put their problems on the board. The group can discuss the sequencing of ideas, alternative problems, and possible pitfalls to avoid. Some teachers will be tempted to try their sequences in the classroom. You might ask the participants to report later on the results.

* *

Can you tell *a priori* the number of tie points a pair of rules will have? If one or both of the rules involves $\square \times \square$ (and not $\square \times \square \times \square$ or $\square \times \square \times \square \times \square$, etc.) there may be zero, one or two tie points. Examples:

$$\left.\begin{array}{l}\square \longrightarrow \square \times \square + 6 \\ \square \longrightarrow 10 \times \square + 6\end{array}\right\}$$ Two tie points: 0 and 10

$$\left.\begin{array}{l}\square \longrightarrow \square \times \square + 40 \\ \square \longrightarrow 10 \times \square + 15\end{array}\right\}$$ One tie point: 5

(continued next page)

$\square \rightarrow \square \times \square + 1$
$\square \rightarrow \square \times \square + 2$ } No tie points

$\square \rightarrow \square \times \square$
$\square \rightarrow 2 \times \square - 100$ } No tie points

If one or both of the rules involves $\square \times \square \times \square$, (with nothing more complicated) there are at most three tie points. Example:

$\square \rightarrow \square$
$\square \rightarrow \square \times \square \times \square$ } Three tie points: -1, 0, and 1

The connection between tie points of jumping rules and the frame equations of the first written lesson becomes clear if we observe that the rules

$\square \rightarrow \square \times \square + 6$
and $\square \rightarrow 10 \times \square + 6$

tie whenever the same starting number (same number in both boxes on the left of the arrows) yields equal landing numbers. Thus numbers that work in the equation $\square \times \square + 6 = 10 \times \square + 6$ are tie points for the pair of rules. Since the equation involves $\square \times \square$ (but not $\square \times \square \times \square$ or $\square \times \square \times \square \times \square$, etc.), at most two numbers will work—and so there are at most two tie points for the rules. In fact, $\square \times \square + 6 = 10 \times \square + 6$ has <u>exactly</u> two roots, 0 and 10, and so the rules tie at exactly two places—0 and 10.

It should be mentioned that although a pair of rules may in fact have two tie points, the tie points may be very difficult to find.

Film Discussion Notes
"Which Rule Wins?"

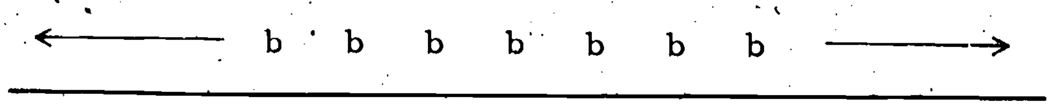
Preliminary information:

You will see a third grade class from the Cunniff School in Watertown, Massachusetts. The teacher is Phyllis Klein. The day before this class, the students had been introduced to jumping rules. The film was made in the middle of the school year. [Film running time: 30 min.]

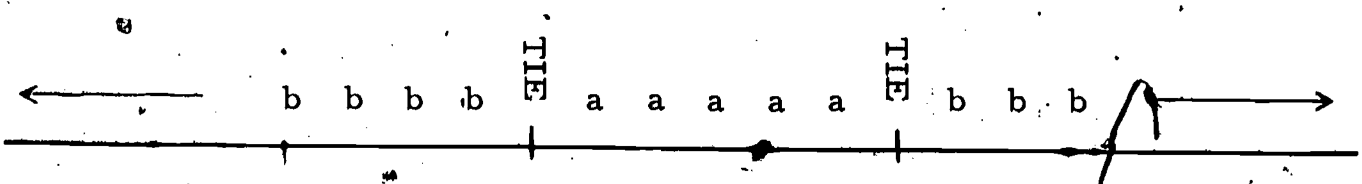
Discussion after the film:

For each pair of rules in the film there was one tie point; to the left of the tie point one of the rules won and to the right of the tie point the other rule won. It is natural to ask if any other situations are possible. Here are three which are worthwhile discussing:

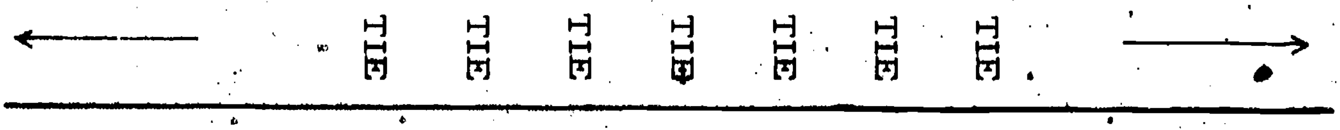
(a) Are there two rules which never tie? That is, could we have a picture like this?



(b) Are there two rules which tie exactly twice, giving a picture like this?



(c) Are there two different rules which always tie?



Probably several of the teachers will be able to find two rules which never tie.
Hint: Suppose rule a is $\square \xrightarrow{a} \square + 4$. Write rule b so that for any starting number, rule b always wins.

Some possibilities for rule b are:

$$\square \longrightarrow \square + 100$$

or

$$\square \longrightarrow \square + 5$$

or even

$$\square \longrightarrow \square + 4\frac{1}{10}$$

Situation (b) is somewhat harder. If the participants get bogged down here, you could either let them think about it for a while or suggest that they try the rule $\square \longrightarrow \square \times \square$ as one of the rules.

In regard to problem (c) there may be some teachers who feel that a pair of rules like $\square \longrightarrow \square + \square$

and

$\square \longrightarrow 2 \times \square$

are two different rules which always tie. If so, it must be emphasized that $\square \longrightarrow \square + \square$ and $\square \longrightarrow 2 \times \square$ are in fact the <u>same rule</u> simply because they do always tie. They are written differently, but since the jumps given by $\square \longrightarrow \square + \square$ are exactly the same as the jumps given by $\square \longrightarrow 2 \times \square$, the rules are not different.

Of course this is true in general: two rules are different only if one of them has a jump that the other doesn't.

Participants in your group may wonder why the teacher rarely filled in the box for the starting number when she tried numbers in the rules. The reason for this was that in the preceding class the teacher found that the children invariably added the starting number to the landing number. For instance, if 5 were put into the rule $\square \longrightarrow \square + \square$, they would say, incorrectly, that the landing number is 15.

When jumps are being drawn on the line, however, it is very important to keep track of the starting number by filling in the first box. What are some things that a teacher can do to minimize the tendency children have to add the starting number to the landing number?

Perhaps by next week some of the participants could share their experiences of introducing jumping rules in their classrooms.

The discussion notes for this written lesson have a format different from that of the first two written lessons. It is not intended that the participants should talk less this week. In fact, by the third week the participants should be taking a more active part in the discussion.

General Suggestions

The first five pages of this written lesson involve problems about parentheses. Perhaps the best way to answer questions that participants may have concerning these problems is to write the problem on the board and have another participant tell how he did it. If someone gives a wrong answer to a particular problem it is possible that he really solved a different problem. Sometimes the quickest way to explain his error is to show him the problem he actually did. For example, problem 22 on page 4 is $3 + \frac{1}{2} - 3 - \frac{1}{2} =$ ____ , and the correct answer is 0 . Assume that someone arrived at 1 as an answer. Perhaps what he did was to supply parentheses — $3 + \frac{1}{2} - (3 - \frac{1}{2}) =$ ____ , in which case the answer would be 1 . It would be worthwhile to ask a participant to explain the difference between these two problems.

The remainder of this written lesson deals with methods for finding standstill points. There are at least three general techniques for finding standstill points, and, in a typical institute, each of the three ways will be used by the participants. The moderator should encourage participants to describe their methods. Notice that problem ☆4 (a) on page 8 asks for a child's method of finding a standstill point for rules such as $\square \longrightarrow 3 \times \square - 8$ and $\square \longrightarrow 3 \times \square - 7$. Acceptable answers to this question are not likely to be as general as those described here.

One way to find standstill points is to make many jumps and notice the pattern of jumps. By looking at the symmetry of the jumps one can usually determine the standstill point(s).

Another method is to divide the number being subtracted by one less than the multiplier. If the rule is $\square \longrightarrow a \times \square - b$ (a and b are numbers with a not equal to 1) then divide b by (a − 1) . (Why couldn't we do this if a were equal to 1 ?) This is the method used by some of the students in the film "Standstill Points".

It is unlikely that students would use the following method for finding standstill points. In order for a number to be a standstill point the starting number must be equal to the landing number. If the rule is $\square \longrightarrow a \times \square - b$ (a and b are numbers), then the starting number is $\square$ and the landing number is $(a \times \square - b)$. Since these two numbers must be equal if there is a standstill point, we have $\square = a \times \square - b$. The number that works in $\square$ will be the standstill point. Notice that if a equals 1, then the equation has no solution, and there is no standstill point.

The moderator does not have to cover all these methods for finding standstill points in the discussion. They are included here to provide background for the moderator and could be used at a later date.

Specific Suggestions

Be prepared for questions about the following problems: page 6, question 1(f); page 7, question 1(j); page 8, question ☆4(a); page 9, question ☆5(d). A good approach in answering these questions is to have participants discuss their ideas and answers. The moderator should use the following information with discretion.

Page 6, question 1(f) (The rule is $\square \longrightarrow 3 \times \square - 8$).

(e)	Start at 1. One jump. Land? ______
(f)	A common wrong answer to (e) above is −4. What question might someone who makes this mistake have been answering?

Some possible answers:

1. The starting point might have been added with the landing point.

$$1 + (3 \times 1) - 8 = -4$$

2. The multiplication sign might have been read as a plus sign.

$$3 + 1 - 8 = -4$$

Page 8, question ☆4(a)

> (a) What other method besides looking at the pattern of jumps might a child suggest as a quick way to find standstill points for the rules in the foregoing problems?

This type of question actually does not have a wrong answer. The question is asking for a child's method and children say all kinds of crazy things. However, most participants (as do many students) say something like "take $\frac{1}{2}$ of the end number" or "divide the last number in the rule by 2". Members of your group who have answered this question might enjoy testing their various methods on these rules:

$$\square \longrightarrow 4 \times \square + 21$$
$$\square \longrightarrow 1 \times \square - 3$$
$$\square \longrightarrow 0 \times \square - 3$$
$$\square \longrightarrow -2 \times \square + 12$$

Page 9, question ☆5(d)

The question is to find the standstill point or points for the rule $\square \longrightarrow 0 - \square$.

Participants often think that negative numbers are standstill points for this rule. If -7 is put in the boxes, then $0 - (-7)$ has to be computed to find the landing number. Because of the two negative signs people often have trouble. Two approaches could be used to solve $0 - (-7)$.

1. A sequence of problems such as this:

$$0 - 5 = -5$$
$$0 - 4 = -4$$
$$0 - 3 = -3$$
$$0 - 2 = -2$$
$$0 - 1 = -1$$
$$0 - 0 = 0$$
$$0 - (-1) = 1$$
$$0 - (-2) = 2$$
$$0 - (-7) = 7$$

2. The use of a rule such as □ ⟶ 10 − □ :

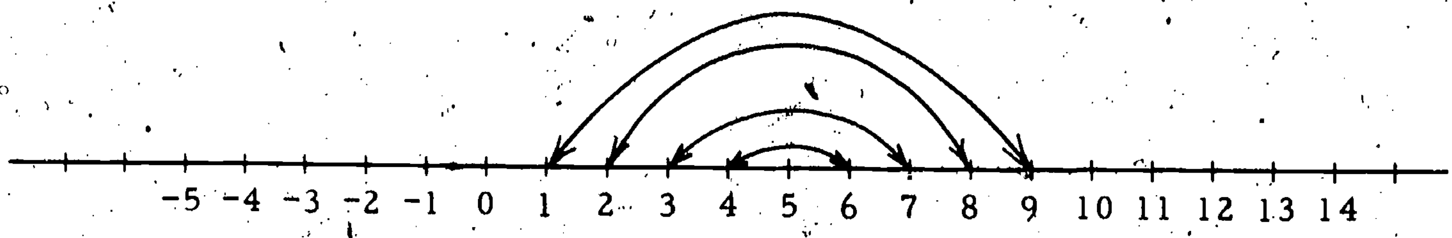
Start at 7 and use this rule. Where do you land? (3)

Start at 3 . Land? (7)

Start at 1 . Land? (9)

Start at 9 . Land? (1)

Here is a picture of some jumps taken with this rule:



Start at 11 . Where do you land? (−1)

Start at −1 .

If the pattern is preserved then you should go back to 11 from −1 , and 10 − (−1) must be equal to 11 , which it is. What is 10 − (−6) ? 5 − (−7) ? 0 − (−7) ? Where do you land with the rule □ ⟶ 0 − □ if you start at −7 ?

In one institute a participant suggested that zero must be the only standstill point for the rule □ ⟶ 0 − □ because 0 is the only number which is its own opposite. (The opposite of a number is defined as that number which, when added to the original number, yields a sum of 0.) Since □ ⟶ −□ ("Box goes to the opposite of box") can be shown to be the same rule as □ ⟶ 0 − □ ("Box goes to zero minus box"), 0 is the only standstill point.

Film Discussion Notes
"Standstill Points"

Preliminary information:

This class is a heterogeneous fifth grade from the James Russell Lowell School in Watertown, Massachusetts. The teacher is David A. Page. Before this film, he had met with the class three or four times. The filming took place in March. [Film running time: 45 min.]

The discussion that follows occurred in a previous institute. It is intended to alert the moderator to possible questions. Most of these answers were given by participants.

Q: Fairly early in the film the class was doing things like □ ⟶ 3 × □ − 19, and somebody gave as her explanation: "You just take the number on the right and cut it in half, and you stay right there." But, when the teacher was going around asking people what the standstill point was, Terry said 38. What was going on there?

A: Terry multiplied 19 by 2 instead of dividing it by 2.

Q: Nancy's answer was $6\frac{1}{3}$. Where did that come from?

A: Possibly she divided the 19 by 3.

Q: "Why is the standstill point one less than the number? How come it works?" (The person who asked this question did not state the question clearly. What she really wanted to know was why you can find the standstill point by dividing the last number in the rule by the number which is one less than the multiplier in that rule.)

A: Since we are looking for a place to start, so that we will land at the same place, we can say that for the standstill point the starting number will equal the landing number. In this particular case the rule □ ⟶ 3 × □ − 19 may be rewritten as □ ⟶ □ + 2 × □ − 19. If we can find a starting number for □ so that 2 × □ − 19 is zero, then that starting number must be a standstill point for the rule. (Why?)

Note to moderator:

Someone with more than routine insight or experience may suggest that a way to find the standstill point is simply to realize that the starting point and the landing point must be equal in that case, or

$$\square = 3 \times \square - 19$$

Now what works in the box? This method is entirely correct, of course, but it is likely to compound the confusion of those teachers who think they have just gotten it straight in their minds that the arrow in a jumping rule is not an equal sign. The moderator, if he or anyone else brings out such a method, should raise this question and try to get various explanations, making clear that the equation is a way of finding the standstill point of the jumping rule. Later in the institute he might want to come back to the question and explore whether or not one could write equations in order to find jumps of specified distances other than zero.

The moderator may want to get thoughts from those who wish to offer them, and defer further discussion of this question until the following week. This will allow the participants time to solve the problem for themselves. These notes are for whatever time he decides to pursue this topic.

Q: When the class is trying to find the standstill point for the rule $\square \longrightarrow 3 \times \square - 5$, Donald suggests starting at 0. What was he probably thinking?

A: He might have been confusing a jump starting at 0 with a jump 0 spaces long.

Q: Commenting on Carmine's jump from 9 to 8 (using the rule $\square \longrightarrow 3 \times \square - 19$), the teacher says something like, "That's a pretty short jump, which probably means you're not too far away." Can a classroom teacher and her students always rely on this principle?

A: Not always, but for most simple rules a very short jump indicates that even shorter jumps are not too far away. One has to use some common sense in applying this, though. If you start at 5 and use the rule $\square \longrightarrow 1 \times \square - \frac{1}{10}$, you get a "pretty short jump", but where is the standstill point? (There is none.) If you start at 1 and use the rule $\square \longrightarrow 1\frac{1}{100} \times \square - 1$, you get a jump from 1 to $\frac{1}{100}$, but the standstill point is not very close to 1. (It is 100.)

Q: While working on the same problem, Katherine disregards the teacher's hint completely and suggests $1\frac{1}{2}$, which is not close to 9 at all. Is she lost in a fog of her own, or is there a reason why she said $1\frac{1}{2}$?

A: Perhaps she divided the 3 by 2, getting $1\frac{1}{2}$. There's no reason to assume she even heard the teacher's hint.

Or perhaps she saw the distance from 8 to 9 was one and when the teacher suggested Carmine was not too far away, assumed this meant the answer was 'around one'.

Another possibility is that Katherine may have realized that $9\frac{1}{2}$ was the standstill point and the distance from 8 to $9\frac{1}{2}$ was $1\frac{1}{2}$.

Q: After Helen explains her method for getting standstill points ("split the number after the minus sign in half") the teacher has the class try the method on the rule $\square \longrightarrow 3 \times \square - 117$. Half of 117 is $58\frac{1}{2}$, so $58\frac{1}{2}$ is put in the boxes and the children work out the answer. Is all this arithmetic really necessary? Is there a quick way to see that $58\frac{1}{2}$ <u>must</u> be the standstill point?

A: You know that $117 = 2 \times 58\frac{1}{2}$ (that's where the $58\frac{1}{2}$ came from originally). So the landing number can be rewritten as $3 \times 58\frac{1}{2} - 2 \times 58\frac{1}{2}$. If we have three $58\frac{1}{2}$'s and subtract two of them, we have one $58\frac{1}{2}$ left.

Q: In order to check z5 as the standstill point for $\square \longrightarrow 4 \times \square -$ z15, the class has to compute z20 − z15. The teacher writes this sequence of problems:

10 − 5 = 5
10 − 4 = 6
10 − 3 = 7
10 − 2 = 8
10 − 1 = 9
10 − 0 = 10
10 − z1 = 11
10 − z2 = 12
10 − z3 = 13
10 − z4 = 14
10 − z5 = 15

He was hoping that the students would see that subtracting a z number is the same as adding a regular number. Is this a good way to handle subtraction of negative numbers? Are there other ways?

A: It sounded as if a number of children responded "9" to 10 − z1 and the teacher rejected that answer. Just by itself, the "just-follow-the-pattern" method is not necessarily convincing, but it's quick, and if you remember that the answers keep going in one direction, it is a reasonable way of remembering what happens.

There are other ways of teaching subtraction of negative numbers, and undoubtedly teachers in an institute can invent and develop many of them. For example, since 5 − 3 = □ means the same thing as □ + 3 = 5, then 10 − z1 = □ means the same as □ + z1 = 10. If the student knows how to add negative numbers, he can do this last problem.

Q: When Christine is explaining her general method she uses as an example the rule □ ⟶ 7 × □ − 22. She says that she looks at the 7, takes the "next smaller number" and divides it into 22. But there is no "next smaller number"; she means the next smaller whole number. Should the teacher have corrected her?

A: This would be purely a matter of opinion. In the context in which it was uttered, Christine's statement is perfectly clear. On the other hand, we wouldn't want children to think that 7 has a next smaller number. In this case the teacher chose to ignore Christine's lack of precision (or he didn't notice it at all), but in a later film, "Graphing with Square Brackets", you will see the same teacher handle a similar situation differently. This film is shown in the second part of the course.

Fourth Session

Written Lesson Discussion Notes

Moderator: Are there any questions on the lesson?

Participant F: I'm puzzled about question 14 on page 3. It says that the most likely wrong answer to problem 13 is 20. I can't figure out how you would get 20.

Moderator writes:

$$\square \xrightarrow{b} \square + 10$$
$$\square \xrightarrow{c} \square - 4$$
$$\square \xrightarrow{d} \square - 7$$
$$\square \xrightarrow{e} \square \times 3$$

$$\left(5 \xrightarrow{c}\right) \times \left(2 \xrightarrow{b}\right) \times \left(8 \xrightarrow{d}\right) \times \left(\tfrac{1}{3} \xrightarrow{e}\right) =$$

Participant K: I thought 15 was a more likely wrong answer. You would get that by adding the results of each step instead of multiplying them. You would have:

$$\left(5 \xrightarrow{c}\right) = 1$$
$$\left(2 \xrightarrow{b}\right) = 12$$
$$\left(8 \xrightarrow{d}\right) = 1$$
$$\left(\tfrac{1}{3} \xrightarrow{e}\right) = 1$$

and $1 + 12 + 1 + 1 = 15$.

Participant O: Maybe they added also the first starting point: $5 + 1 + 12 + 1 + 1$. That would give 20 as a wrong answer.

Participant X: You would also get 20 by multiplying instead of adding when you do $\left(2 \xrightarrow{b}\right)$. You would have $1 \times 20 \times 1 \times 1 = 20$.

Moderator: Those are all good possibilities, and with this type of question they are equally acceptable. Are there any other questions on the lesson?

Participant A: By trial and error I was able to figure out the answer to question 7 on page 4. But, frankly, trial and error can be time-consuming and boring. Can anyone explain it to me?

Participant B: After a few problems I realized that if you want to get the biggest number possible you should use your adding rule before you use the multiplying rule. Then you multiply not only your starting number but also what has been added on.

Participant C: That's right. You want to get as big a number as you can before you multiply, and when you have a dividing rule, you want to get as small as you can before you use it. For example, in problem 5 on page 7, you want to use the subtracting rule first so that you will lose as little as possible when you divide.

Moderator: Let's see, The rules on page 7 are:

$$\square \xrightarrow{f} \square + 8$$

$$\square \xrightarrow{g} \square - 12$$

$$\square \xrightarrow{h} \square \div 4$$

and question 5 asks for the largest possible number using three f's, three g's, and one h. The starting place is 116. Now what were you saying?

Participant C: Well, when you use rule h, you are going to take one-fourth of whatever is in the box. So you are going to lose three-quarters of whatever you have. In order to lose as little as possible, first use your three g's and then use h.

Moderator writes: $116 \xrightarrow{ggghfff}$

Participant D: I didn't know how to go about question 12 on page 6. (Aside: I have enough trouble getting the answer when they tell me where to start and what rule to use. How do they expect me to figure this out?)

Moderator writes:

$\square \xrightarrow{A} \square + 10$

$\square \xrightarrow{B} \square - 12$

$\square \xrightarrow{C} 3 \times \square$

Let's see. Three A's, three B's, and one C are used on a certain number, and the highest number that can be obtained is 207. Does anyone know in what order the rules would have to be used to get the highest number?

Participant B (a little exasperated): As I said before, if you add as much as you can before you multiply, you will get the biggest number. So do the three A's, then C, and then do the three B's.

Moderator (writing $\square \xrightarrow{AAACBBB}$): So we know in what order the rules were used to get 207. Does anyone have an idea about how to find the starting number?

Participant E: You can use the rules in reverse. The B's subtracted 36, so add 36 to 207. C multiplied by 3, so divide by 3. 243 ÷ 3 = 81. Then use the three A's in reverse, so you subtract 30 from 81. So 51 is the starting number.

Moderator: Now how should we use the rules to get the lowest number?

Participant A: I guess you want to get as small as possible before you multiply. So do BBBCAAA.

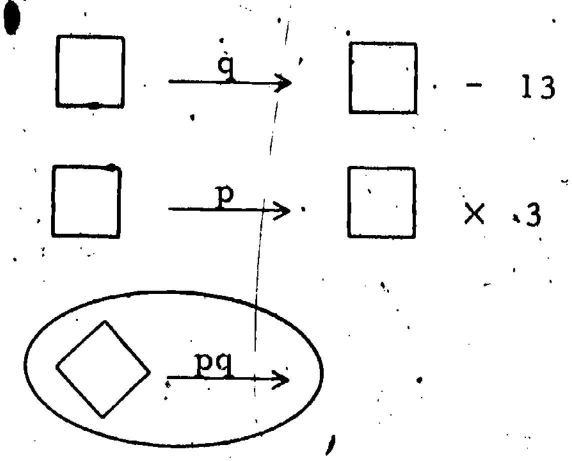
47

Moderator (writing $\left(51 \xrightarrow{BBBCAAA}\right)$): Does anyone have the answer?

Participant R: 75

Participant M: I was wondering about question 8 on page 11. I know that pq wins by 26. But does it work also if I put a negative number in the diamond?

Moderator: Let's try it with a negative number.



Someone give me a negative number.

Participant M: -4.

Moderator writes:

$$\left(-4 \xrightarrow{p}\right) = -12$$

$$\left(-12 \xrightarrow{q}\right) = -25$$

$$\text{so} \quad \left(-4 \xrightarrow{pq}\right) = -25$$

$$\left(-4 \xrightarrow{q}\right) = -17$$

$$\left(-17 \xrightarrow{p}\right) = -51$$

$$\left(-4 \xrightarrow{qp}\right) = -51$$

Which is larger, −25 or −51 ?

Participant Q: −25

Moderator: It is larger by 26 ; so, pq wins by 26 even when we start at −4. And, in fact, pq will win by 26 for all negative numbers.

The preceding is the beginning of a discussion about the fourth written lesson. It is not complete, but it is included so that you may get the flavor of how one moderator handled his group.

Following are some other ideas that the moderator could use if the participants themselves do not bring them up.

A way of determining which combination of rules is "better" is to look at what each rule does to any starting number.

For example, in problem ☆9 on page 12 where the rules are $\square \xrightarrow{a} \square - 45$ and $\square \xrightarrow{b} \square \times 17$ and the starting number is 293:

$$293 \xrightarrow{ba} = (293 \times 17) - 45$$

$$293 \xrightarrow{ab} = (293 - 45) \times 17 = (293 \times 17) - (45 \times 17)$$

Thinking of the first expression as $(293 \times 17) - (45 \times 1)$ and comparing it with $(293 \times 17) - (45 \times 17)$, we see that in the second case the 45 is being subtracted 16 more times. Therefore, ba is the "better" combination.

As a participant in an institute said, "If you use rule b first and then rule a, you will be subtracting 45 only once. If you use rule a first, then rule b, you will be subtracting 45 seventeen times. So the combination ba will win by 16×45 or 720."

For some people the answers for the problems on page 13 come as a surprise, particularly problem ☆3. It may seem that the rules should be used in a different order when the starting number is between zero and one, since jumps with the rule $\square \xrightarrow{c} \square \times \square$, starting between zero and one, will go to smaller numbers.

It would be worthwhile to discuss these problems and see if a participant could come up with a convincing argument as to why the order of the rules stays the same. A possibility would be to start at $\frac{1}{100}$ and compute the answers for the six different ways that the rules can be arranged.

$\left(\frac{1}{100} \xrightarrow{abc}\right) = \frac{4}{90,000}$ $\quad$ $\left(\frac{1}{100} \xrightarrow{acb}\right) = \frac{4}{30,000}$

$\left(\frac{1}{100} \xrightarrow{bac}\right) = \frac{4}{90,000}$ $\quad$ $\left(\frac{1}{100} \xrightarrow{bca}\right) = \frac{2}{90,000}$

$\left(\frac{1}{100} \xrightarrow{cab}\right) = \frac{2}{30,000}$ $\quad$ $\left(\frac{1}{100} \xrightarrow{cba}\right) = \frac{2}{30,000}$

This shows that acb will give the largest number when the starting number is $\frac{1}{100}$. A more general argument follows.

Rather than start at $\frac{1}{100}$ start at □ so that we have:

$$\left(\square \xrightarrow{a}\right) = 2 \times \square$$

$$\left(\left(\square \xrightarrow{a}\right) \xrightarrow{b}\right) = (2 \times \square) \div 3$$

or

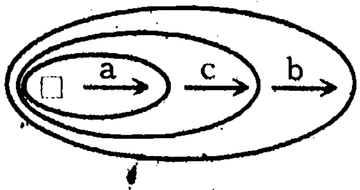
$$\frac{2}{3} \times \square$$

$$\left(\left(\left(\square \xrightarrow{a}\right) \xrightarrow{b}\right) \xrightarrow{c}\right) = \left(\frac{2}{3} \times \square\right) \times \left(\frac{2}{3} \times \square\right)$$

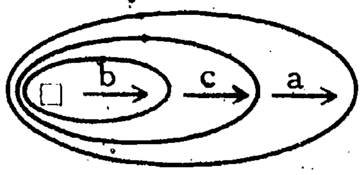
or

$$\frac{4}{9} \times \square \times \square$$

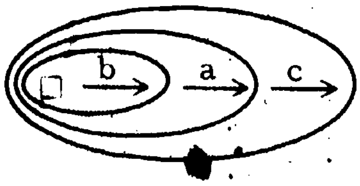
In a similar fashion we get these results for the other five arrangements of the rules.



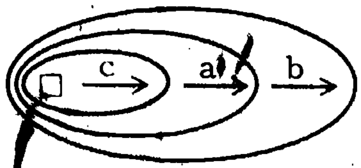
$= \frac{4}{3} \times \square \times \square$



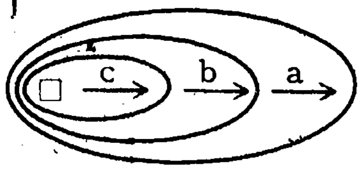
$= \frac{2}{9} \times \square \times \square$



$= \frac{4}{9} \times \square \times \square$



$= \frac{2}{3} \times \square \times \square$



$= \frac{2}{3} \times \square \times \square$

Comparing what is on the right side of the equal sign, one can see that $\frac{4}{3} \times \square \times \square$ will be the biggest number no matter what number is in the boxes (except zero, in which case they will be the same). Someone might wonder if $\frac{4}{3} \times \square \times \square$ will give the biggest number if a negative number is used. The answer is yes, since $\square \times \square$ will always be positive.

The moderator may want to return to this problem after the participants have done the written lesson in Book 1-6.

Film Discussion Notes
"Three A's, Three B's, and One C"

Preliminary Information:

In this film you will see the same class of fifth graders that you saw in the film "Standstill Points". This is the next day. The teacher is again David A. Page. Just before filming he introduced the class to the conventional notation for negative numbers. [Film running time: 48 min.]

Discussion After the Film:

Moderator:

Are there any questions or comments about the film?

Participant A:

I couldn't figure out the one with the 10 and do 6 c's.

Moderator:

Let's see. Rule c was $\square \xrightarrow{c} 2 \times \square$ and the problem was $\left(10 \xrightarrow{cccccc}\right)$. The teacher was getting mostly wrong answers. What were they?

Participant B:

120. They were going 10, 20, 40, 60, 80, 100, 120.

Participant C:

Another wrong answer was 22. They were thinking of each of the c's as worth 2. $6 \times 2 = 12$. $10 + 12 = 22$.

Moderator writes $\left(10 \xrightarrow{cccccc}\right) = 640$

Moderator:

Does anybody know how to get to 640?

Participant D:

I thought you would multiply 10×2^6; but it doesn't work out.

Moderator:

For those of us who don't remember about exponents, what is 2 to the 6th?

Participant E:

$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$

Participant D:

That's 64 so it does work. I only did 2 to the 5th before.

Moderator writes (15 cccccc→) = 960

Moderator:

When the teacher changed the 10 to 15, he commented, "The answer has to be bigger than 640 because we're starting with a bigger number." Does anyone have a fast way to get the answer which was 960?

Participant F:

Add half of 640 to 640.

Moderator:

Does that work?

Participant G:

You've increased the 10 by half of 10.

Participant H:

I see what you're doing. If you take half of your answer (640) and add it to 640 you get 960.

Moderator:

What if you had 30?

Participant J:

It would be 3 times the answer you get for 10.

Participant K:

I noticed that one child said one hundred and thirty-two, and another time a child said two-six-six for two hundred sixty-six. Isn't it important to say the numbers?

Moderator:

It is important that you do as you think best. Sometimes when children are working with big ideas and looking for generalizations, a teacher may ignore things like that for the time being.

Participant L:

In another place in the film Brian said 3 minus 10 instead of 10 minus 3, and the teacher let it go because he knew what Brian meant, and Brian certainly knew what he meant.

Participant M:

I was wondering why those loops were drawn.

Moderator writes $6 \xrightarrow{a}$ and $\left(6 \xrightarrow{a}\right)$

Moderator:

Is there a difference in meaning between these two?

Participant N:

The teacher said that $\left(6 \xrightarrow{a}\right)$ was the number you got when you made that jump.

Moderator:

Perhaps it will be helpful to remember that $6 \xrightarrow{a} = 9$ is not correct. It says that using rule a, 6 goes to equals 9. This is nonsense. It is correct to write $\left(6 \xrightarrow{a}\right) = 9$ which says that if you start at six and use rule a, the landing point is equal to 9. $6 \xrightarrow{a} 9$ is a correct way to say: using rule a, six goes to nine.

Participant N:

What if Bruce hadn't said, "Is it because you multiply at a different place?"

Participant O:

Teachers aren't always lucky enough to have a Bruce in their class.

Participant P:

Maybe the teacher would ask why $\left(53 \xrightarrow{caaabbb}\right)$ did not give the same answer.

Participant Q:

Maybe the teacher would wait until someone did catch on.

Note to moderator:

At the end of the film, after the students have found the maximum number and minimum number, the teacher asks a question. "If 115 is the largest number you can get and 88 is the smallest number you can get, what are all the numbers in between that you can hit? Can you land on all the whole numbers between 88 and 115 or only on some of them?"

If the participants would like to pursue this question, the following is one possible approach.

□ —a→ □ + 4

□ —b→ □ - 5

□ —c→ 2 × □

The position of rule c makes a difference in the landing number when 3 a's, 3 b's and one c are used. We will start with rule c on the extreme left side.

caaabbb — There are many other ways to arrange these keeping the c to the left; for example, cababab. (How many other ways are there?) However, each of these arrangements will give the same result because moving a's and b's will not change the answer. (Rules a and b commute.)

Now move the c one position to the right.

acaabbb
bcaaabb — Here we have two arrangements that will give different results.

Move the c to the right again. Can you predict how many arrangements will yield different results? If you said three, you are right.

aacabbb
abcaabb
bbcaaab

Move the c once again.

aaacbbb
aabcabb
abbcaab
bbbcaaa

Again move c.

aaabcbb
aabbcab
abbbcaa

Once more?

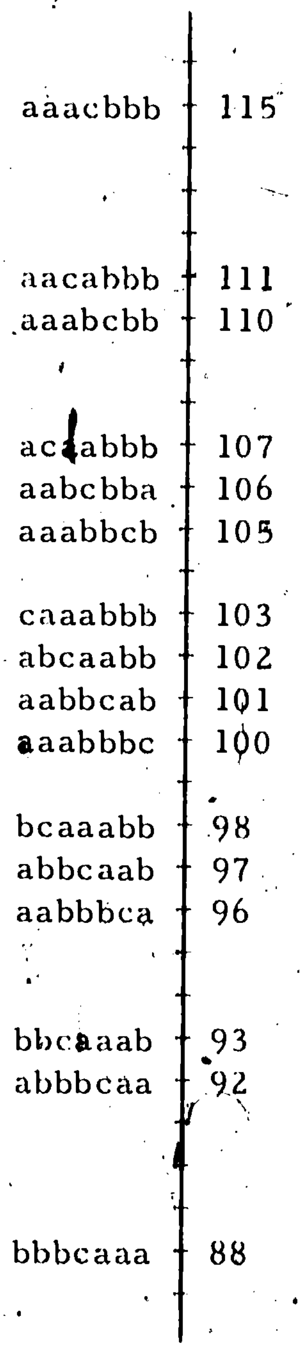
aaabbcb
aabbbca

Finally!

aaabbbc

Counting these arrangements we find there are 16 possibilities, including the maximum and minimum. The answer to the question, therefore, is: No, you cannot get all the numbers between the largest and smallest. You can get only 14 numbers between the largest and smallest.

The moderator may want to show the 16 possible landing numbers (starting at 53) on a number line and discuss such things as symmetry, pattern of numbers hit and numbers missed, and midpoints. Here is a picture with the rules that give the landing numbers:



Fifth Session
Written Lesson Discussion Notes

In this written lesson, the participants are asked to do some lattice problems with their classes. We hope that the moderator will encourage the participants to discuss the lattices and the problems that they tried. Perhaps teachers from representative grade levels could present their problems to the group. Another possibility to keep in mind is to reproduce for the entire group some of the lattices and sets of problems that participants have tried in their classrooms. This would also be a good time to discuss what children can learn from working with lattices. (See page 41 for further discussion of this question.)

Following are some of the questions that are most often asked about this written lesson. (Some possible answers are on pages 43-45.)

Page 3, problem 8 and problem 9:

14 →↑↑↓→↓←↑→↑←	=	________
36 →↑↑↓→↓←↑→↑←	=	________

Is there a reason for using the same arrows in problems 8 and 9? (Answer on page 43.)

Page 4, Sect. IV, problem 4:

5 ↗↖	=	________	**(The answer is not 5.)**

Why is 5 a likely wrong answer for 5 ↗↖ ? (Answer on page 43.)

Page 4, Sect. V, problem 1 (a):

Use the smallest possible number of arrows to make this problem true:

25 ☐ **= 41**

What is the largest number of arrows that can be used? ________

Why isn't six the largest number of arrows that can be used here? (Answer on page 43.)

Page 5, problem 4 (a):

> ☐ ↗↗ + ☐ ↗↗ = 58

How might one approach this problem in a logical fashion? (Answer on page 44.)

Page 5, problem 5 (c):

> **Explain why this problem is impossible:**
>
> **192** ⏢ − **65** ⏢ = **129**

Why is this problem impossible? (Answer on page 45.)

In problems 6 and 7, on page 5 and page 6, the participants are asked to extend the lattice. Usually problem 6 gives no trouble, but the answer to problem 7 can be the subject of disagreement. Two common answers are −3 and −13. Participants who give these answers are probably trying to keep the last digits of the numbers in each column of the lattice the same. (Thus, −3 goes under 3.) However, if −3 were to go beneath 3, then the ↓ would subtract 6 there, whereas previously it has always subtracted 10. One has to decide whether it is more important to keep the value of the arrows consistent or to preserve the pattern of the numerals in the lattice. That all the numbers in the column above 3 end in 3 is really just an accidental outcome of the fact that there are ten numbers in each row. (Try the same kind of question with a seven-fold lattice.) The moderator might use the following sequences of problems as a possible way to lead to an extension of the lattice which preserves the effect of the arrows.

(In these problems, put numbers in all the frames; no arrows allowed.)

(i)	$90\uparrow$	+	$90\downarrow$	=	____
(ii)	$26\uparrow$	+	$26\downarrow$	=	____
(iii)	$43\uparrow$	+	$43\downarrow$	=	____
(iv)	53	+	33	=	____
(v)	$7\uparrow$	+	$7\downarrow$	=	____
(vi)	17	+	□	=	14
(vii)			$7\downarrow$	=	____

(A)	$19\downarrow$	+	□	=	10
(B)	$17\downarrow$	+	□	=	10
(C)	$12\downarrow$	+	□	=	10
(D)	$11\downarrow$	+	□	=	10
(E)	$10\downarrow$	+	□	=	10
(F)	$9\downarrow$	+	□	=	10
(G)	$8\downarrow$	+	□	=	10
(H)	$7\downarrow$	+	□	=	10
(I)	△	+	13	=	10
(J)			$7\downarrow$	=	△

* * *

Following are two slightly edited discussions among participants who were taking the course.

I.

Participant A: What was the answer to problem 5(b), page 5 ?

192 ⏢	−	65 ⏢	=	127	

Moderator: Remember, we must put the same arrows in each ⏢. Would someone else like to answer this?

Participant F: I made a square with the arrows ⬚ so they get back to 192 and 65.

Participant G: I agree. No matter what arrows you put in each ⏢, the arrows must cancel themselves out, so that $\uparrow\downarrow$ in each ⏢ would work, too.

Participant M: Well then, I don't see why I have to use any arrows at all, since 192 − 65 = 127 already.

Participant Q: We don't have a no-arrow symbol. ↑↓ or →← is the closest we come to that, and the result is just as if we <u>had</u> used no arrows.

Moderator: Yes, you're right. You have to have something in the ⏢'s or you haven't done the problem.

Participant N: I disagree with all of you. I tried four arrows to the right in each ⏢, and I still got 127.

Moderator: Let's work that out:

192 →→→→ = 196

65 →→→→ = 69

and 196 − 69 = 127.

Participant N: In fact, I think that any combination of arrows would work.

Moderator: You are correct. Those of you who wish to think some more about problems of this type might see if you can find the answer (or answers) to this problem:

192 ⏢ + 65 ⏢ = 257

II.

Participant A: What are the children really learning by doing lattices?

Moderator: Would anyone like to answer that?

Participant C: Each arrow pointing up is worth 10, so that a problem like 3 ↑↑↑↑↑↑ could be done by adding

$$3 + 10 + 10 + 10 + 10 + 10 + 10$$

or by doing $3 + 6 \times 10$.

Moderator: Yes, and if you used a different lattice, such as one with multiples of 6 in the far left column, the children would get practice in the six tables.

Moderator puts three rows of this lattice on the blackboard:

12	13	14	15	16	17
6	7	8	9	10	11
0	1	2	3	4	5

Participant X: It's a good way to view division with remainders. If you had $44 \div 6$, go up 7 rows from zero on that lattice and then go over 2.

Participant M: Using arrows that go in opposite directions brings in cancellation. The children can see that ↑ cancels out ↓.

Participant L: They are learning to use numbers in base form, too.

Participant C: Is this really other number bases?

Participant D: I don't think it's the same thing because you are working with ten digits, so it couldn't be another base.

Participant L: But if you wrote all the numbers in base six (in the lattice you have on the blackboard), it would look a lot like the ten lattice.

Moderator: Do you think it would be good for introducing negative numbers? How might you do that?

Participant P: Have the arrow go down below 0 and put another row of numbers below the bottom row.

* * *

Here are some possible answers to the questions on pages 38 and 39 of these discussion notes.

Page 3, problem 8 and problem 9.

The answer to problem 8 provides a clue to the answer for problem 9, since problem 9 has the same arrows, but a different starting number. If you cross out arrows that cancel one another, such as ↑ and ↓, and → and ←, you are left with

14 ↑ ↑ → = 35

So all that those arrows did was to add 21. Adding 21 to 36 will now give the answer to problem 9.

Page 4, Sect. IV, problem 4.

Notice that 5 ↗↙ = 5 and 5 ↘↖ = 5. Students often think that ↖ and ↗ cancel out in the same way, but they do not. On this lattice, ↗ means add 11, and ↖ means add 9. Thus, 5 ↗ ↖ has the same answer as 5 ↑↑.

Another way to see that 5 ↗ ↖ cannot be 5 is to think of the geometry of the moves. The arrow ↗ moves you up and to the right; the arrow ↖ moves you up and to the left. As long as the starting number is not on the edge, you must travel vertically; you cannot stay where you are. Students sometimes decompose ↗ into → ↑ and ↖ into ← ↑. Therefore, 5 ↗ ↖ can be replaced by 5 → ↑ ← ↑. The two horizontal arrows cancel, leaving 5 ↑ ↑.

Page 4, Sect. V, problem 1(a):

There are two common ways of thinking about the correct number of arrows used. If one allows arrows that cancel, there is no largest number of arrows. There can be as many as you want. If arrows that cancel were not allowed, then 6 would be the largest number of arrows: ← ← ← ← ↑↑ (or any rearrangement of these 6 arrows.)

Page 5, problem 4(a):

There are many approaches to this problem. Here are four possible ways to do it:

(i) First rewrite the problem as

[ ] + [ ] ↗↗↗↗ = 58

Since ↗ adds 11, ↗↗↗↗ adds 44.

[ ] + [ ] + 44 = 58

[ ] + [ ] = 14

so [7] + [7] = 14

(Of course, 7↗↗ should be checked on the lattice to ensure that it does not go off the edge.)

(ii) Rewrite the problem as

[ ] ↗↗ + [ ] ↗↗ = 29 + 29

Then, each [ ] ↗↗ must equal 29. Since ↗↗ adds 22, 7 must go in the [ ].

(iii) Add arrows that will cancel out those on the left:

[ ] ↗↗↙↙ + [ ] ↗↗↙↙ = 58 ↙↙↙↙

(Since four ↙'s were added to the left of the equal sign, four ↙'s were also added to the right of the equal sign.)

58 ↙↙↙↙ = 14

So, you are left with

[ ] + [ ] = 14

(iv) Try numbers in the ☐'s and check to see if they are too big or too small.

Try 10:	10 ↗↗	+	10 ↗↗	=	64	(too big by 6)
Try 5:	5 ↗↗	+	5 ↗↗	=	54	(too small by 4)
Try 7:	7 ↗↗	+	7 ↗↗	=	58	(it works)

Page 5, problem 5(c):

Using the same arrows for the same-shaped frames, as one must, this is impossible, since 192 − 65 = 127. Both numbers will be changed by the same amount regardless of the combination of arrows used. The difference will always be 127.

Film Discussion Notes
"A Seven-Fold Lattice"

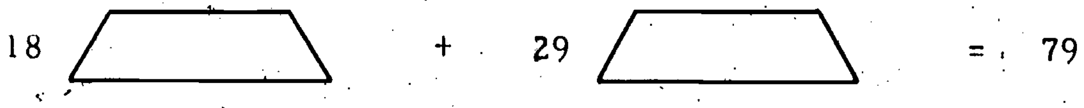
Preliminary information:

The class you will see in this film is a fifth grade heterogeneous group from the Browne School in Watertown, Massachusetts. The film teacher is Francis X. Corcoran. On two occasions prior to the filming Mr. Corcoran had taught the class using other Project materials. [Film running time: 48 min.]

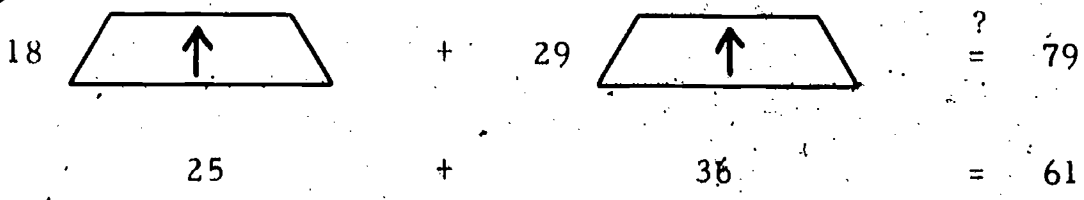
Discussion after the film:

We hope that the moderator will use these notes as a reference to help his group answer their own questions.

Participant A: The answers to the last two problems were not given in the film. What was the answer to



Moderator: In the film the teacher gave a clue by trying ↑ in the ⏢'s:



But we need to get up to 79. What other arrows do you need to put in each frame to get the necessary 18?

Participant A: An arrow up and two arrows to the right (↑ → →).

Participant X: So you would have ↑ ↑ → → in each ⏢.

Moderator: Can anyone do this problem using only three arrows in each ⏢? Only two arrows?

Participant N: For three arrows, you could have ↑ ↗ →, and for two arrows, ↗ ↗.

Participant B: What answers did the children give for this problem?

Moderator: Some popular answers were ↑↖→→ , ↑← , ↗↑→ , →↑↗↖ , ↑↑↑→ , ↗↗ , and ↑→→ .

Participant Q: And what was the answer to the last problem?

18 ⏢ + 29 ⏢ = 80

Moderator: Any ideas?

Participant R: We have just done the problem with the end number 79. Are there any arrows that get up to 80?

Participant C: You need only one more, and you'd have to split that one more into two ⏢'s , so the problem is impossible.

Moderator: Yes, on this lattice there is no arrow that adds $\frac{1}{2}$.

Participant N: Is it possible to construct a lattice so that some arrow adds $\frac{1}{2}$?

* * *

Moderator: In the problem 32 ⏢ = 58, the most common correct answer was ↑↖↖↑ . What other correct answers could you give for this?

Participants suggest the following ways: ↑↑←←↑↑ , ↗↖↖↖ , ↑↑↑↖← , ↗↗↖↖←← ,

Participant M: Of course, there are many more solutions.

Participant D: Why is it impossible to do with only three arrows?

Participant E: The problem is impossible to do using three arrows because 58 − 32 = 26, and the largest value that you can get with three arrows is 24.

Your group might find it interesting to discuss the following questions:

(a) In the beginning of the film, the teacher wrote three rows of the lattice and then had the students fill in the fourth row and some numbers in the fifth and sixth rows. Why did he spend so much time building it up? Should he have spent more time? When you introduce lattices to your class, how will you decide how much of the lattice to build?

(b) Several wrong answers were given when the teacher switched to a short way of writing arrows pointing up. For $9 \overset{10}{\uparrow}$, the answer of 90 was given. What was the child probably thinking? What are some other short ways you might use to indicate $9\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$? For $60 \overset{40}{\uparrow}$, an answer of 460 was given; how did the child probably arrive at that answer?

(c) At one point in the film the teacher gave the problem $\square \nearrow \downarrow$ and asked if the result were bigger or smaller than $\square$. He said that the number he was thinking of for $\square$ was 49. One of the students wanted to try a smaller number, namely 2. Is $2 \nearrow \downarrow$ bigger or smaller than 2? Is $49 \nearrow \downarrow$ bigger or smaller than 49? Would the result always be bigger, no matter what number is put in $\square$?

(d) When the students were trying to find where 100 was on the lattice, they noticed that 10 was in column D, 20 in column G, and 0 in column A. Do multiples of ten occur in every column? Which lattices have multiples of ten in only one column? A question to ponder: given some lattice, how could you tell in which columns multiples of any given number would fall?

Sixth Session

Written Lesson Discussion Notes

Following are the problems in sections I and II of the lesson that appear to be the most difficult for people.

Page 1

11. $52 + 2 \times (\square + \square) + 8 = 9 + 4 \times \square + 41$

Participant A: I was unable to do it. Was it a misprint?

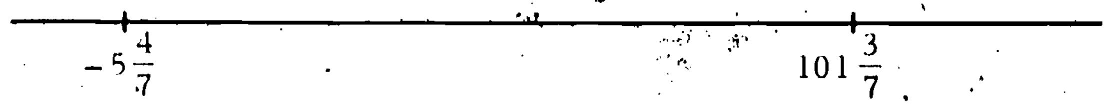
Participant B: $2 \times (\square + \square)$ is the same as $4 \times \square$.

$52 + 8 = 60$ and $41 + 9 = 50$

so you are adding 60 on the left side and 50 on the right side. No matter what number you put in the boxes, the left side will always be 10 bigger than the right side. So it's impossible.

Page 3

☆ 14.

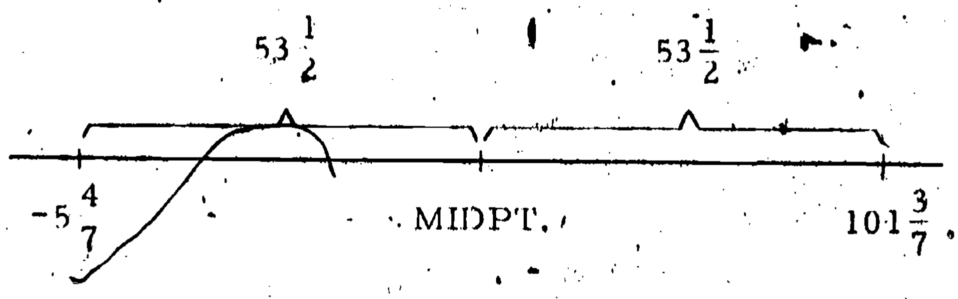


☆ 15. The most likely wrong answer to problem ☆14 is $53\frac{1}{2}$. Why might this be?

Participant C: You say that the most likely wrong answer to problem 14 is $53\frac{1}{2}$. But that is the answer I got. I added $101\frac{3}{7}$ to $5\frac{4}{7}$ and got 107. Then I divided by 2, and got $53\frac{1}{2}$. Why is it wrong?

Participant D: That $53\frac{1}{2}$ that you got is the distance from the midpoint to each of the ends. Now add $53\frac{1}{2}$ to $-5\frac{4}{7}$, or subtract $53\frac{1}{2}$ from $101\frac{3}{7}$. That will give you the midpoint.

This diagram may help:



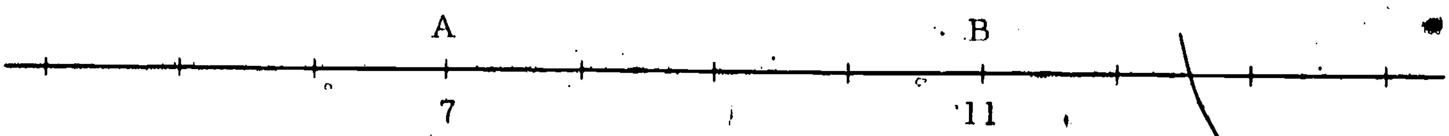
Page 4

If some participants have attempted the problems on this page and have had difficulty, it may help to label A and B with simple numbers and then ask appropriate questions. We illustrate with problem ☆16 (a).

(a) Point A goes west at 5 units per second, and at the same time point B goes west at 5 units per second. In what direction and how fast does the midpoint (halfway point) of AB move?

Midpoint moves ____________ at ____________ units per second.

Let A be 7 and B be 11 .



What is the midpoint now? (9)

Where is A after one second? (at 2)

Where is B after one second? (at 6)

Where is the midpoint? (at 4)

Has the midpoint moved east or west? (west)

In one second how many units has it moved? (5)

Your group may want to discuss whether it makes any difference what numbers you use for A and B .

Section III

Even though we are speculating as to how people arrived at the wrong answers indicated, these questions are well worth discussing. By analyzing wrong answers teachers can often correct misconceptions before they become too firmly established.

Section IV

Sometimes participants have felt that a statement such as

$$\left| -4\frac{2}{3} \right| = 4\frac{2}{3}$$

is somehow untrue because $-4\frac{2}{3}$ is not equal to $4\frac{2}{3}$. Pointing out that absolute value is an operation that makes negative numbers positive may lead to a better understanding of what the absolute value bars mean.

Another way to look at absolute value is to ask, for instance:

How many units away from 0 is 16 ?

$$|16| = 16$$

How many units away from 0 is −16 ?

$$|-16| = 16$$

The distance that a number is from 0 is the absolute value of that number.

More work with absolute value will come in the second part of the course.

Film Discussion Notes
"Counting With Dots"

Preliminary information:

The class you will see in this film is a second grade from the James Russell Lowell School in Watertown, Massachusetts. The film teacher is David A. Page. This class had worked with Project teachers for seven months prior to the filming. [Film running time: 34 min.]

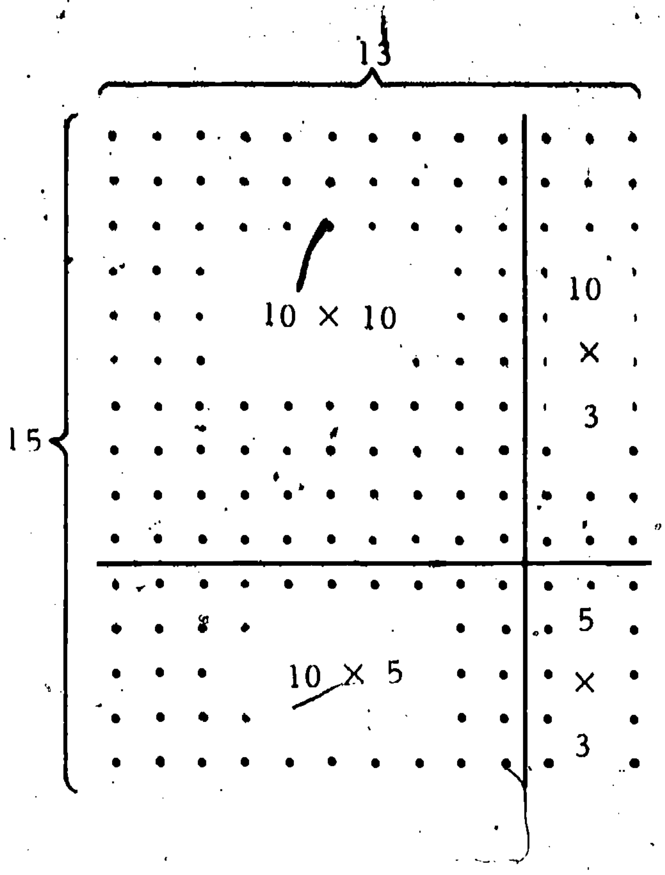
Possible discussion questions:

In the problem where the child is asked to show that there are 63 dots, do you think he really put them into groups of tens as he said he did? What other methods might he have used to obtain his answer?

To the ten-by-ten array of dots, the teacher added another row and another column of dots before asking for the total number. Why is 120 a likely wrong answer?

Draw a four-by-four array of dots on the board. How many more dots do you need to add to get a five-by-five array? Now consider the five-by-five array. How many dots will you need to get a six-by-six array? Imagine a thirteen-by-thirteen array. How many more dots will you need to obtain a fourteen-by-fourteen array? Now you have a square-shaped array which is n-by-n (where n is a number). How many more dots do you need to get to the next larger square?

Consider the problem 13 × 15 = ? How can a child use an array of dots to help him determine the answer? Here is one way to look at the problem (see diagram on next page):

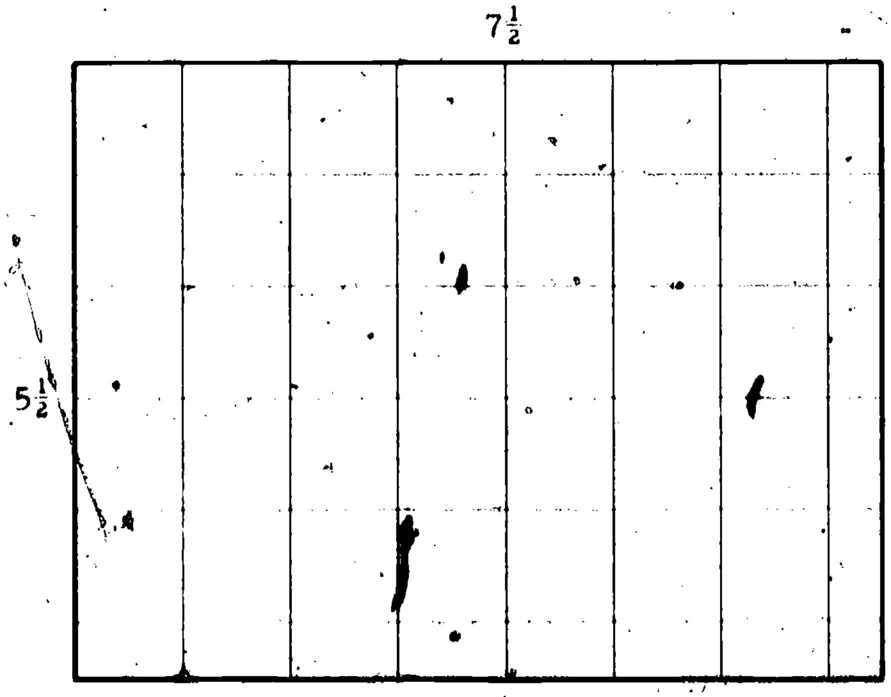


$$10 \times 10 = 100$$
$$10 \times 5 = 50$$
$$10 \times 3 = 30$$
$$5 \times 3 = 15$$

$$100 + 50 + 30 + 15 = 195 = 13 \times 15$$

Drawing dots may get to be a time-consuming job. Sheets of graph paper or a large graph board can easily be substituted. Instead of counting dots, children will now count little squares, each of which is one unit.

Another advantage of using graph paper is that you can now solve problems such as $5\frac{1}{2} \times 7\frac{1}{2} = ?$ Students and teachers can profit from finding shortcuts to this problem.



Seventh Session

Written Lesson Discussion Notes

Page 3

16. $\left\lfloor 1\frac{6}{7} + 1\frac{10}{11} + 1\frac{97}{100} \right\rfloor$ = ________

One certainly does not want to do the actual computation in this problem. You might ask your group for quick ways to get the answer.

A person might reason that each of the numbers is close to, but less than, 2. The sum will be close to, but less than, 6. The problem is reduced to $\lfloor 6$ minus a small fraction $\rfloor = 5$. A more cautious person might reason that together the fractions will certainly be more than 1. Will they be more than 2? When you combine $\frac{6}{7}$ and $\frac{10}{11}$ you have 1 and a fraction that is certainly more than $\frac{1}{2}$. This fraction combined with $\frac{97}{100}$ will give 1 and some fraction. The problem is reduced to $\lfloor 3 + 2 +$ some fraction $\rfloor$.

18. $\left\lfloor 3\frac{10}{13} + 2\frac{19}{20} + 10\frac{96}{97} + \frac{199}{200} \right\rfloor$ = ________

Reasoning similar to that suggested for problem 16 is equally applicable here since each fraction is close to but less than 1.

Page 5

☆33. $\square \div 2 > \lfloor \square \div 2 \rfloor$

Someone may be unsure about using negative numbers in this problem. If so, the group could work through a couple of examples like the following.

Use a negative integer in the box.

$\boxed{-6} \div 2 > \lfloor \boxed{-6} \div 2 \rfloor$

or $-3 > -3$, which is false.

Therefore, -6 does not work in $\square$.

Try -5.

$$\boxed{-5} \div 2 > \lfloor \boxed{-5} \div 2 \rfloor$$

or $\quad -2\frac{1}{2} > -3$

This is true, so -5 <u>does</u> work.

Now try a negative fraction.

$$\boxed{-\frac{1}{2}} \div 2 > \lfloor \boxed{-\frac{1}{2}} \div 2 \rfloor$$

or $\quad -\frac{1}{4} > \lfloor -\frac{1}{4} \rfloor$,

or $\quad -\frac{1}{4} > -1$, which is true.

Therefore, $-\frac{1}{2}$ does work in $\square$.

A few examples should convince the group that even whole numbers (positive and negative and zero) do not work and that all other numbers do work.

One approach to problems 34 through 42 is to consider how much is lost by placing the brackets in various places. For example, in problem 34 if the brackets are placed like this: $\lfloor \frac{2}{3} \rfloor + \frac{2}{3}$, then $\frac{2}{3}$ will be lost. On the other hand, if the brackets are like this: $\lfloor \frac{2}{3} + \frac{2}{3} \rfloor$, then only $\frac{1}{3}$ will be lost. Since you are looking for the largest number, you want to lose as little as possible. Therefore $\lfloor \frac{2}{3} + \frac{2}{3} \rfloor$ will give the largest number.

42. $\quad \frac{3}{4} + \frac{3}{16} + \frac{2}{16} + \frac{1}{2} + \frac{1}{4}$

Using one pair of lower brackets in this expression, the smallest amount that you can lose is $\frac{1}{16}$. There are two different ways to lose $\frac{1}{16}$:

$$\frac{3}{4} + \left\lfloor \frac{3}{16} + \frac{2}{16} + \frac{1}{2} + \frac{1}{4} \right\rfloor$$

or

$$\left\lfloor \frac{3}{4} + \frac{3}{16} + \frac{2}{16} \right\rfloor + \frac{1}{2} + \frac{1}{4}$$

* * *

Here is a problem for a group interested in a challenge. It concerns this sequence of questions in the written lesson:

28. $\left\lfloor \frac{235}{100} \right\rfloor =$ ________

29. $\lfloor 235 \div 100 \rfloor \times 100 =$ ________

30. $\lfloor 353 \div 100 \rfloor \times 100 =$ ________

31. $\lfloor 6353 \div 100 \rfloor \times 100 =$ ________

When these questions were originally written and tried in an institute for teachers, they were followed by this question:

$$\lfloor 6,353 \div \square \rfloor \times \square = 6,000$$

The plan was to extend the idea of going to the next lower multiple of 100. The author of the question assumed 1,000 to be "the" root of the equation. 1,000 does work, but it's not the only number. Some teachers in the institute gave 3,000 and 6,000, which also work. The following week more roots had been found, including 2,000, 1,200, 600, and 500. Soon, some really surprising roots had been discovered, including $545\frac{5}{11}$. The equation has, in fact, the following roots:

6,000	1,200	$666\frac{2}{3}$	$461\frac{7}{13}$
3,000	1,000	600	$428\frac{4}{7}$
2,000	$857\frac{1}{7}$	$545\frac{5}{11}$	400
1,500	750	500	375

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In your institute you might wish to review problems 28 through 31 and in this context give the equation $\lfloor 6,353 \div \square \rfloor \times \square = 6,000$. Some participants will probably give some of the more evident roots such as 1,000 and 6,000. You can then observe that there are quite a few other answers and let those who are interested search for them during the next week. Keep a tally of the roots collected for several weeks. Eventually someone may produce all the solutions, together with a system for determining them and proving that the list is complete.

Film Discussion Notes
"Lower and Upper Brackets"

Preliminary information:

You will see a class of fourth graders from the Ballard School in School District #63 in Niles, Illinois. The teacher is Mrs. Carol Daniel. While teaching a fourth grade at the Ballard School, Mrs. Daniel attended a twenty-week in-service institute using Project materials. The year after this institute Mrs. Daniel helped give an institute for the remaining teachers in District #63. It was for the second institute that this class was videotaped. This film was made from the videotape. [Film running time: 30 min.]

Questions for discussion:

1. In the film you saw the expression $4\frac{1}{2} + \frac{5}{8} + 10\frac{1}{8}$. The teacher put the brackets in two different places and asked which expression gave the smaller answer. How would you place the brackets to get the largest answer? Can you do it more than one way?

2. At one point the children suggested using numbers below zero. How do brackets work for negative numbers? If this question is raised in your group, you can mention that the use of brackets with negative numbers is explained in the Epilogue to the written lesson in Book 7. After participants have read this information the group could do a few problems with negative numbers.

3. For the equation $\lceil \square \rceil - \lfloor \square \rfloor = 0$, Ed suggested 2 below zero and then decided that it would not work. The teacher said that Mark agrees with Ed and it seems that she also agrees. Yet she summed it all up by saying that the whole numbers work. Isn't −2 a whole number? Does it work?

   To some people "whole numbers" means only the positive integers. For others, it means the integers: positive, negative and zero. (This is the meaning the Project adopts.) For this problem, −2 and all other integers work.

Eighth Session

Written Lesson Discussion Notes

A source for discussion is the collection of sample sequences together with correctors' comments that will be found in the Corrector's Guide for this lesson.

The moderator could put one or more sequences on the board and ask for comments and criticism. He might bring up the Corrector's Guide comments at the outset for discussion, or bring them out later for comparison with the comments of institute participants. It would be worthwhile to have some participants write their sequences on the board for discussion.

The following are the problems most often asked about. As with all problems brought up in the discussion, try to get the participants to tell their methods. If no one has a method, one approach is given here. Use it at your discretion.

<u>Page 1, problem 5</u>

$$\left\lfloor \frac{1}{2} + \frac{1}{2} + \frac{310}{204} + \frac{1}{2} \right\rfloor = \underline{\qquad\qquad}$$

The common wrong answer given is 2. The 2 probably comes from reading $\frac{310}{204}$ as $\frac{204}{310}$.

<u>Page 2, problem 14</u>

$$\left\lfloor \frac{\left\lfloor 2 \times 6\frac{1}{7} \right\rfloor}{\left\lfloor 1\frac{1}{2} + 1\frac{3}{4} \right\rfloor - \frac{1}{100}} \right\rfloor = \underline{\qquad\qquad}$$

Many participants reduce the problem to $\left\lfloor \frac{12}{2\frac{99}{100}} \right\rfloor$ and then compute to find an answer. A worthwhile discussion topic: are there methods that do not involve dividing 12 by $2\frac{99}{100}$?

A participant may say something like, "If you had $\frac{12}{3}$ in lower brackets the answer would be 4. Taking $\frac{1}{100}$ away from the denominator makes the answer a little bigger than 4, so that when you take lower brackets the answer is still 4."

For those participants who get 3 as an answer, here is a sequence that may help:

$\left\lfloor \frac{6}{2} \right\rfloor$ = ____ $\left\lfloor \frac{6}{3} \right\rfloor$ = ____ $\left\lfloor \frac{6}{2\frac{1}{2}} \right\rfloor$ = ____

$\left\lfloor \frac{6}{2\frac{3}{4}} \right\rfloor$ = ____ $\left\lfloor \frac{6}{2\frac{99}{100}} \right\rfloor$ = ____ $\left\lfloor \frac{12}{2\frac{99}{100}} \right\rfloor$ = ____

If necessary, before doing the last problem you may want to do these two.

$\left\lfloor \frac{12}{2} \right\rfloor$ = ____ $\left\lfloor \frac{12}{3} \right\rfloor$ = ____

Problem 15 on page 2 can be approached in a similar way.

Page 5, problems 19 and 20

(19) $\lceil 2 \times \square \rceil - \lfloor 2 \times \square \rfloor = 1$

Describe, somehow, all the numbers that work:

(20) $\lceil 2 \times \square \rceil - \lfloor 2 \times \square \rfloor = 0$

What numbers work?

For problem 19, all numbers work except integers and integers plus $\frac{1}{2}$. $7\frac{2}{3}$ will work, but neither 7 nor $7\frac{1}{2}$ will.

If negative numbers are suggested by the participants, then a few examples could be done. Again any negative number will work except integers and integers plus $\frac{1}{2}$. $-9\frac{2}{3}$, $-17\frac{5}{8}$, $-100\frac{1}{5}$ all work. -9, $-26\frac{1}{2}$, $-51\frac{1}{2}$ will not work. It may be better not to discuss negative numbers for problems 19 and 20 unless they are suggested by the participants.

For problem 20, integers and integers plus $\frac{1}{2}$ will work.

If you find that your group does not have many questions on this lesson, then here are some other things that you may want to discuss.

Which is large , $\dfrac{1 - \frac{1}{10}}{1{,}000 - \frac{1}{10}}$ or $\dfrac{1}{1{,}000}$?

Is there a fast way of determining which is larger?
(This relates to problems 9 through 12 on page 2.)

$$\lceil 5 \times \square \rceil - \lfloor 5 \times \square \rfloor = 0$$

$$\lceil 5 \times \square \rceil - \lfloor 5 \times \square \rfloor = 1$$

What numbers work?
(Compare these two problems with problems 19 and 20 on page 5 of the lesson.)

$$\left\lceil (5 \times \square) + \frac{1}{5} \right\rceil - \left\lfloor (5 \times \square) + \frac{1}{5} \right\rfloor = 0$$

How do the numbers that work in this problem compare with the numbers that work in $\lceil 5 \times \square \rceil - \lfloor 5 \times \square \rfloor = 0$?

Can you predict what would happen if the problem were

$$\left\lceil (5 \times \square) - \frac{1}{5} \right\rceil - \left\lfloor (5 \times \square) - \frac{1}{5} \right\rfloor = 0 \ ?$$

Film Discussion Notes
"Inequalities With Lower Brackets"

Preliminary information:

The class is a fifth grade from the Browne School in Waterto[illegible] Massachusetts. The teacher is Francis X. Corcoran. He had met [illegible] this class two previous times. The film was made in the spring of 1967. [Film running time: 33 min.]

Possible questions to discuss:

Why is 20 the only number tried for the problem $(5\frac{2}{3} + 5\frac{2}{3}) \times 2$ [illegible]

Discussion of this question should include ideas such as: this is a teacher's prerogative and another day he might do it differently by asking for other numbers that work. He might also come back to this proble[illegible] the next day and graph the solution. Perhaps the teacher felt that by giving one example of a number that worked the students knew all the numbers that worked and it would be a waste of time to dwell on this problem.

In the problem

$$\left\lfloor 99\frac{1}{5} + 98\frac{1}{5} + 101\frac{1}{5} + 102\frac{1}{5} + 100\frac{1}{\square} \right\rfloor = 501$$

isn't $\frac{1}{5}$ a wrong answer?

Yes, $\frac{1}{5}$ is a wrong answer for the number in the box. However, it appeared obvious in the film that the students meant the entire fraction was to be $\frac{1}{5}$ or that 5 was to go into the box. How would you handle this type of answer in your class?

When Tommy goes to the board to graph the solution to $\left\lfloor \square + \frac{1}{2} \right\rfloor > \square$ he marks only the whole numbers, which is wrong. What could he have been thinking?

Possible problems to do:

$$\left\lfloor 99\frac{1}{5} + 98\frac{1}{5} + 101\frac{1}{5} + 102\frac{1}{5} + 100\frac{1}{\square} \right\rfloor = 501$$

In the film the solutions 5 and 4 were given. What are all the numbers that work? Can someone graph the solution?

What happens if the $\frac{1}{5}$'s are changed to $\frac{2}{5}$'s and the 501 to 502?

$$\left\lfloor 99\frac{2}{5} + 98\frac{2}{5} + 101\frac{2}{5} + 102\frac{2}{5} + 100\frac{2}{\square} \right\rfloor = 502$$

Graph the solution.

In the film the solutions of the problems

$$\left\lfloor \square + \frac{1}{2} \right\rfloor > \square$$

and $\square > \left\lfloor \square + \frac{1}{2} \right\rfloor$

are graphed. The group might profit from graphing

$$\square < \left\lfloor \square + \frac{1}{4} \right\rfloor$$

and $\left\lfloor \square + \frac{1}{4} \right\rfloor < \square$

Ninth Session

Written Lesson Discussion Notes

Here are some questions that have been discussed in previous institutes.

Participant A: On page 1 I don't see why 1 (e) is wrong.

(e) 27 → → → → ↑ = 34

Participant B: 27 is in the top row because it's a multiple of 3. If you go four spaces right, you're still in the top row. If you go one arrow up, you're off the chart. That's undefined.

(The moderator may want to discuss methods for determining whether a number is divisible by 3. See page 4b of the Corrector's Guide for the second lesson.)

Moderator: Many people gave 114 as a wrong answer to problem 13 on page 2. Can someone explain why this is wrong?

13. 104 → → → → → → → = ______

Participant D: Well, I first figured out that 104 is in the bottom row, under 105, which is a multiple of 3. Since there are 7 arrows to the right and every 2 arrows adds 3, then 6 arrows adds 9. That takes you to 113 and now you have to figure out where one more arrow takes you. Since 104 is in the same relative position as 2 or 5, then one more arrow adds 2 more.

Participant E: It couldn't be 114 because that's a multiple of 3 and all multiples of 3 are in the top row.

Participant F: Oh, it would have to be 115.

Participant G: Well, I got 116 for problem 14, and if I do what was suggested for problem 13, my answer should be right.

14. 105 → → → → → → → = ______

Participant E: You're forgetting that 105 is in the top row.

Participant G: Oh, of course, I'd land on a dot!

Participant A: Is 502 the correct answer for problem 8 on page 5?

8.	501 ↑	=	______

Participant B: Every time you go up a row the number doubles, so your answer would have to be 501 doubled.

Participant A: So it's 1,002. I see.

Participant C: Does someone have a general method for finding out what row a number is in? I couldn't seem to find a pattern.

Participant F: All the odd numbers are in the bottom row. If you want to know what row 2,720 is in you keep cutting it in half.

Participant C: But how do you know when to stop splitting?

Participant F: When you hit an odd number you know you're in the bottom row. You'd get 1,360 , 680 , 340 , 170 , 85 . You stop there.

Participant C: Oh, I see! Then you climb back up. 85 is in a , 170 in b , 340 in c , 680 in d , 1,360 in e , and 2,720 in f .

Participant F: Right.

(Note to moderator: Question 23 on page 7 is well worth discussing too.)

23.	Teacher: "How can you tell what to add when you move one space to the right in some given row?" Student:

* * *

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We have included reasonably complete answers to problems ☆27 and ☆28 in the Epilogue. This will give teachers a chance, before the discussion, to mull over any differences between their responses and the arguments presented in the Epilogue. We stress that there are many ways to show that each positive integer appears exactly once in the lattice; perhaps some of the participants will have shorter, more complete or more elegant explanations, and they should be encouraged to share them with the group. In particular, some teachers may use the so-called Fundamental Theorem of Arithmetic—the theorem which says that every whole number greater than 1 can be written as a product of prime numbers and that, except for the order in which the prime numbers appear, this factorization is unique. If this theorem is used, both problem ☆27 and problem ☆28 can be explained simultaneously.

Answers like "Well, it's obvious" or "The first one hundred numbers all appear just once, so probably they all do" are unsatisfactory. Consider, for example, these two numbers:

123,299,131,511,182,514,899,124,224

and 123,299,131,511,182,514,899,124,225

The numbers are so close (in comparison to their enormous size) that for all practical purposes they are the same. (But they aren't equal, of course!) The first of these numbers is the first number in the 91 st row, which would be about 4 feet up if the numbers in the lattice are spaced as they are in the written lesson. The other number is odd and is therefore in the bottom row—about 500 quintillion miles to the right! That two numbers so widely separated on the lattice should be so close numerically suggests that some even larger number might appear twice, in two even more widely separated positions.

But, as the Epilogue shows, each positive integer appears exactly once.

Preliminary information:

In this film you will see a heterogeneously grouped class of fifth graders from the Coolidge School in Watertown, Massachusetts. The teacher is Phyllis Klein. She had met with the class 28 times prior to this film, which was made toward the end of the school year. A reproduction of the lattice in this film is on the first page of Book 9. [Film running time: 35 min.]

Why is the lattice in the film called a periodic lattice?

——Note that the word "periodic" was not used in the film and was not necessary. The children were using the properties described below to solve the problems, but neither they nor the teachers in your institute would have benefitted much by having these properties described to them at the outset. Later a teacher or discussion leader might ask if anyone wants to try to describe the "periodic" properties of the lattice.

The first thing to notice is that every number in the lattice has a position in a backward L-shaped array:

		5	•	•	10	•	•	15	•	•	20	•	
		4	•	•	9	•	•	14	•	•	19	•	. . .
1	2	3	6	7	8	11	12	13	16	17	18	21	

and that if we add 5 to any number in the lattice we get a number which has the same position as the original number. Thus, for example, 14 and 14 + 5, or 19, occupy the same position within their own backward L-shaped blocks. Since adding 5 does not change the position of the number, adding 10 or 15 or any other multiple of 5 won't change the position of the number either.

The second thing to notice is that adding 10 to a whole number does not change the last digit. Thus, 847 and 847 + 10, or 857, have the same last digit, namely 7. (This property is not a consequence of the lattice, but rather of the way we write numbers.) Of course, adding any multiple of 10, such as 20 or 40 or 1500, doesn't change the last digit either.

Combining these results we see that adding 10 (or any multiple of 10) changes neither the position of the number within its backward L-shaped configuration nor the last digit of the number. "Periodic" refers to this repetition of the last digit and position. It is this periodicity which makes questions like "What row is 13,468 in?" and "What is $23 \xrightarrow{(301)}$?" accessible to fifth graders. One can tell a lot about what happens far out in the lattice by examining the first part of it.

Probably none of the children in the film knew precisely why the methods they used worked; and there is no point in reciting lengthy reasons. The basic ideas of periodicity might become clearer to teachers if they attempt to answer a few of the same questions that were asked in the film, but using a different lattice. You might try this one:

•	•	•	7	•	•	•	14	•	•	•	21
•	•	•	6	•	•	•	13	•	•	•	20
•	•	•	5	•	•	•	12	•	•	•	19
1	2	3	4	8	9	10	11	15	16	17	18

What row is 13,468 in? What is $13{,}468 \rightarrow$? What is $23 \xrightarrow{(301)}$?

Now try to answer the same questions on a non-periodic lattice, such as this one:

					21
				15	20
			10	14	19
		6	9	13	18
	3	5	8	12	17
1	2	4	7	11	16

Periodicity can make a great difference!

Did the teacher make a mistake when she wrote 18 → → ↑ ?

——Yes. She meant to write 13 → → ↑ .

When the teacher wrote 8 → → → → → → , the writing at the top of the screen recorded the problem as 8 → → →   → → → , with a space between the two groups of arrows. Which is better to use in the classroom?

——Teachers will disagree on this. Some may feel that separating the arrows is a good idea, at least in the beginning, while others will think that it's an unnecessary crutch. There is no "right" way to do it.

George says that if you have a number followed by an arrow with a 9 over the arrow, you "triple it". To find out what "it" refers to, the teacher asks him to do $7 \xrightarrow{(9)}$, and he responds with 21 . The surprising thing here is that he's only off by 1 , even though he should have tripled 5 and added it to 7 . Is there a number in the lattice for which George's method would work? That is, is there a number which makes $\square \xrightarrow{(9)} = 3 \times \square$ true?

——Teachers should be encouraged to give reasons for the apparent non-existence of such a number. Going 9 spaces to the right on this lattice is the same as adding 15, no matter where you start. So the equation above can be rewritten as $\square + 15 = 3 \times \square$. The only way to make this true is to put $7\frac{1}{2}$ in the boxes, but $7\frac{1}{2}$ is not in the lattice.

In explaining how he got an answer of 160 for $10 \xrightarrow{(90)}$, Charlie says, in part, that "each 30 equals 90". Should the teacher have let this go by without any clarification?

——This question is debatable. How would the teachers in your institute have handled the situation?

Several children in the film gave 110 as the answer to $10 \xrightarrow{(90)}$. Where does this answer come from?

This is a wrong answer which the Arithmetic Project has never been able to explain. Of course 110 is 10 + 10 + 90, but why the students doubled the 10 is a mystery. The Project would be delighted to hear from anyone who has an explanation!

When the problems get as hard as $10 \xrightarrow{(90)}$, some children will just add the 10 and the 90. Teachers should not feel discouraged if this happens. Children have learned that sometimes the add-everything-in-sight technique works, and it's hard to break them of this habit.

Film Discussion Notes
"Surface Area With Blocks"

Preliminary information:

The class you will see in the film is a first grade from the James Russell Lowell School in Watertown, Massachusetts. It is a heterogeneous group. Prior to the filming the children had been taught arithmetic exclusively by Project teachers using Project topics. This is in contrast to the classes seen in other films in which the children had had limited or no exposure to Project topics.

Although they had worked often with blocks, this is the first time the students had considered surface area.

The film teacher is Phyllis R. Klein and this is the first time she had taught the class. [Film running time: 25 min.]

Discussion after the film:

There is no formula to insure that a lively discussion will follow any film. Many moderators, impatient for such a discussion, feel that by bringing up lots of points about the film an enthusiasm for discussion will spread throughout the group. Experience shows that this is not usually the case. Before the moderator brings up anything the participants should be given ample time to bring up their own discussion points. If they do not, then the moderator might try mentioning one or two significant points about the film. Should there still be no discussion it is probably best to give up and go on to something else.

The following is part of a discussion recorded at an institute given in a suburb of Lowell, Massachusetts.

Discussion:

Moderator	Participants
Does anybody have any comments about the film, or questions?	
	I have a few I wrote down. You said they were first graders. They know their addition pretty well. It seems to me they are still doing this in the first and second grade. They knew how to count by 10 and 6. The first one got 42 without any problems at all. The girl, Eileen, who had a problem adding 6's, is going to be lost at the end. What do you do about children like that? It seems to me these children are awfully bright. They are not an ordinary class.
	Didn't you say it was a third grade?
	This film was made last year? For that matter, one of them knew fractions.
	A lot of children know that.
	But they knew it was $\frac{1}{4}$.
This was made in May so it was pretty well along in the year.	
	They had almost completed a year of the first grade.
	It seems to me these were awfully large numbers they were adding in their heads. Eileen is really going to be lost if the rest know how to add and she doesn't.
Does anybody teach first grade in the group? What is your reaction?	

Moderator	Participants
	I think they could do it if they had the preliminary work. It is just what we do, but we don't have all those blocks to work with. They learn to count by 10's. A lot of kids know 6 and 6 is 12 and we did quarters and halves in other ways, so I think by May they would be able to do this.
	A much larger majority of the kids were working with 10's and 5's rather than 6's.
Mostly the teacher was using yellow blocks, but I think she did use a 6 block on two occasions. (Moderator got out the large blocks.) Have any of you ever used them?	
	The small ones.
	I would be interested in seeing them introduced to something like this. How do you start these first graders? Just the concept of putting across what a block is and how many stamps should be on each side I think would be something.
I would like to hear from somebody who has used them, perhaps how they introduced them. I am not usually teaching a first grade.	
	We used them in first grade. They started off with one unit and built it up so they could see the 10 block was composed of 10 small ones. That is how we introduced the blocks to the children.
	I started with the very smallest blocks.

Moderator | **Participants**

Moderator: We have the small blocks here, also. In the film you saw just the large blocks. Very often with a class it is best to let the children have some small blocks so they can do it by themselves. The teacher sometimes took the large blocks to their desks.

Participants: We used a different method. We gave the children the entire kit and let them play with the whole kit. They could stack them up and form an idea of how the blocks relate to each other and what they represented.

Moderator: You gave them some time to play with them?

Participants: Several weeks. They would be given free time to play with them and, hopefully, to work constructively with them.

Moderator: Do you really think it's necessary to give free time to play with the blocks?

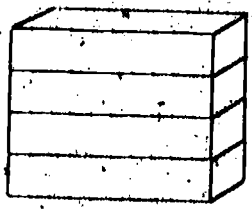
etc.

Questions the moderator might bring up:

Question:

"David had given the answers to 4 yellow blocks stacked together as 58 and when asked, "What's a good way of doing that?" he said, "I didn't get a good way of doing it. I just counted."

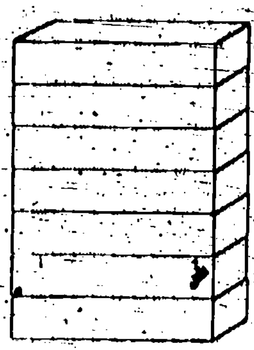
Do you think he really counted every one?



(Yellow blocks are 5 units long.)

Question:

Does anyone have a quick way of finding surface area for blocks stacked as shown in the picture so the teacher will always be ahead of the children with the answer?



Question:

When the class first discovered it took 6 stamps to cover a white block, they were then shown a red block and Dennis immediately said it took 12 stamps to cover it. What might Dennis have been thinking?

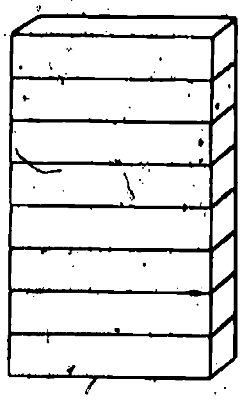
Question:

When the class was shown the orange block and asked how many stamps were needed to cover it, Stephen answered with a quick 100. What could he have been thinking?

Question:

The teacher ended the class by having the children think about 8 yellow blocks stacked this way:

What is the surface area of that figure?



Question:

(a) In the film we saw that a half white block can be covered by four stamps. Can you cover a half block with four stamps without cutting any stamps, i.e., by folding them if necessary?

How many different ways can it be done (if at all)?

Can you do it by folding only one of the stamps? Two of them?

(b) A quarter block can be covered with three stamps if you cut them up. Can you cover the same block without cutting any of the stamps?

Eleventh Session

Written Lesson Discussion Notes

When working with artificial operations, the moderator should be prepared for confusion when definitions are written using frames. By chance, all of the definitions given in the written lessons were written using first $\square$ and then $\triangle$. But $\square * \triangle = \square + \square + \triangle$ could equally well have been written $\triangle * \square = \triangle + \triangle + \square$ or $\diamond * \nabla = \diamond + \diamond + \nabla$.

Participants often ask if the artificial operations in this lesson (such as $\odot$, $*$, $\checkmark$, etc.) are of any value, or if they are just games. The moderator might comment that children often look upon them as games, but that they do have mathematical significance. The commutative and associative laws may not look very impressive to children when they are told that $13 + 5 = 5 + 13$ or that $(9 + 14) + 16 = 9 + (14 + 16)$. However, to decide whether $(2 \odot 10) \odot 20 = 2 \odot (10 \odot 20)$ is a different matter. We see that $(2 \odot 10) \odot 20$ does not equal $2 \odot (10 \odot 20)$, so we say that the operation $\odot$ is not associative. The moderator may wish to bring this out by putting problems such as the following on the board for comparison:

$3 + 4 \stackrel{?}{=} 4 + 3$	$6 + (10 + 7) \stackrel{?}{=} (6 + 10) + 7$
$3 * 4 \stackrel{?}{=} 4 * 3$	$6 * (10 * 7) \stackrel{?}{=} (6 * 10) * 7$
$3 \checkmark 4 \stackrel{?}{=} 4 \checkmark 3$	$6 \checkmark (10 \checkmark 7) \stackrel{?}{=} (6 \checkmark 10) \checkmark 7$

In the lesson $\square \odot \triangle$ is defined as "the number halfway between $\square$ and $\triangle$"; it is also defined in the following way:

$$\square \odot \triangle \stackrel{df}{=} \frac{\square + \triangle}{2}$$

The difference between these two methods is not trivial in practice, even though the results are the same; for example, it is easier to find the number halfway between $6\frac{1}{2}$ and 7 by just thinking of the point halfway between them on the number line. It would be more difficult to find their sum and divide by 2. On the other hand, for most people it is easier to find the number halfway between $1{,}000\frac{1}{2}$ and -1 by dividing their sum by 2.

Page 3, problems 16 through 23

16.	(1 ⊙ 3) ⊙ 6	=	4
17.	1 ⊙ 1 ⊙ 49	=	______
18.	20 ⊙ 20 ⊙ 2	=	______
19.	6000 ⊙ 0 20	=	______
20.	50 ⊙ 100 ⊙ 12	=	______
21.	$\frac{1}{2}$ ⊙ 400 ⊙ 400	=	______
22.	14 ⊙ 18 ⊙ 14	=	______
23.	In a student's words, what is a good strategy to follow in doing the preceding problems?		

An interesting question that might be asked about problems 16 through 23 is this: if the rules for these exercises were changed so that you could rearrange the numbers, would you get different answers?

Doing problem 20 as it stands gives

$$(50 \odot 100) \odot 12 = 43\frac{1}{2}$$

and

$$50 \odot (100 \odot 12) = 53$$

Now rearrange the numbers:

$$(12 \odot 50) \odot 100 = \_\_\_\_\_\_$$

$$12 \odot (50 \odot 100) = 43\frac{1}{2}$$

(This is the same as (50 ⊙ 100) ⊙ 12, since ⊙ is commutative.)

Predict answers to:

$$(100 \odot 50) \odot 12 = \_\_\_\_\_\_$$

$$100 \odot (50 \odot 12) = \_\_\_\_\_\_$$

$$(100 \odot 12) \odot 50 = \_\_\_\_\_\_$$

$$100 \odot (12 \odot 50) = \_\_\_\_\_\_$$

You can now find all the other ways to arrange the numbers in problem 20. In all there are fewer than 15 ways.

Did the maximum change? Which arrangement (or arrangements) gave the largest result?

* * *

Participants often have difficulty with the problems on page 5. Since there are various ways to think about these problems, by all means ask people who solved the problems to explain how they did it. Common methods involve what happens on each side of the equation as you vary what you put in the box.

Here is a somewhat formal analysis: Recall that

$$\square * \triangle \overset{df}{=} \square + \square + \triangle \quad \text{and} \quad \square \checkmark \triangle \overset{df}{=} \max(\square, \triangle)$$

A key to doing problems involving the operation $\checkmark$ is to realize that $12 \checkmark \square$, for example, is either 12 or $\square$. Each of the two possibilities leads to an equation which does not involve $\checkmark$. The roots of these two equations may or may not work in the original equation; the trial roots must be tested.

Problem 12, page 5

$$\square * 2 = 12 \checkmark \square$$

$$12 \checkmark \square \overset{df}{=} \max(12, \square)$$

$\square * 2 =$ 12 or $\square$, whichever is larger.

Try $\square * 2 = 12$.

Root: 5

See if 5 works in the original equation:

$$\boxed{5} * 2 \overset{?}{=} 12 \checkmark \boxed{5}$$

$$12 = 12$$

5 works.

Try $\square * 2 = \square$

Root: -2

See if -2 works in the original equation:

$$\boxed{-2} * 2 \overset{?}{=} 12 \checkmark \boxed{-2}$$

$$-2 \neq 12$$

-2 does not work.

In fact, 5 is the only number that works for this problem.

Problem, ☆18, page 5

$$\square * 4 = (3 \times \square) \checkmark 10$$

$$(3 \times \square) \checkmark 10 \overset{df}{=} \max(3 \times \square, 10)$$

$\square * 4 = 3 \times \square$ or 10, whichever is larger.

Try $\square * 4 = 3 \times \square$

Root: 4

See if 4 works in the original equation:

$$\boxed{4} * 4 \overset{?}{=} (3 \times \boxed{4}) \checkmark 10$$

$$12 = 12$$

4 works.

Try $\square * 4 = 10$

Root: 3

See if 3 works in the original equation:

$$\boxed{3} * 4 \overset{?}{=} (3 \times \boxed{3}) \checkmark 10$$

$$10 = 10$$

3 works.

Problem, ☆19, page 5

$$\square * 2 = (3 \times \square) \checkmark 10$$

$\square * 2 = 3 \times \square$ or 10, whichever is larger.

Try $\square * 2 = 3 \times \square$

Root: 2. See if it works in the original equation:

$$\boxed{2} * 2 \overset{?}{=} (3 \times \boxed{2}) \checkmark 10$$

$$2 \neq 10$$

2 does not work.

Try $\square * 2 = 10$

Root: 4. See if it works in the original equation:

$$\boxed{4} * 2 \overset{?}{=} (3 \times \boxed{4}) \checkmark 10$$

$$10 \neq 12$$

4 does not work.

Problem ☆20, page 5

$\square * \triangle \;=\; (3 \times \square) \checkmark 10$

In discussing problem ☆20, the moderator might find it useful to point out that by selecting different numbers for △, one can generate more problems of the same type as ☆18 and ☆19. After finding solutions to several of these, a pattern will become apparent:

number in △	number of solutions for □
7	two
6	two
5	two
4	two
3	none
2	none
1	none
0	none
−1	none

If there does exist a number for △ which will give only one solution for □, it would appear from this to be somewhere between 3 and 4. Further trial and error will quickly show that such a number is $3\frac{1}{3}$.

Another approach to problem ☆20 is to realize that whenever there were two solutions for □, it was because $(3 \times \square) \checkmark 10$ had two possible answers, according to which part of it was larger. If both parts turned out to be the same, there could be at best only one solution for □, and we make this happen by setting $3 \times \square = 10$, or $\square = 3\frac{1}{3}$. Now we have

$$3\frac{1}{3} * \triangle \;=\; 10 \checkmark 10 \;=\; 10,$$

and all we need to do is solve for △.

Participants may be curious as to why $3\frac{1}{3}$ is the only number for $\triangle$ that satisfies the conditions of this problem. In fact, we might guess from examining the table on page 80 that for $\triangle$ equal to any number larger than $3\frac{1}{3}$, two numbers will always work in $\square$. These two numbers are $\triangle$ and

$$\frac{10 - \triangle}{2}$$

The moderator may wish to check that $\triangle$ and $\frac{10 - \triangle}{2}$ will give the two numbers that work for $\square$. He can do this by making $\triangle$ equal to 100, for example, and checking that 100 and −45 both work in the equation $\square * 100 = (3 \times \square) \checkmark 10$.

In problem ☆19, page 5, $\triangle$ is equal to 2, and no number works in $\square$. The moderator may want to ask participants to predict how many numbers would work in $\square$ if $\triangle = 2\frac{1}{2}$; if $\triangle = 3$.

*   *   *

The operations that we have called circle-dot, star, and check are <u>binary</u> operations. (You have to use two numbers to get back one number.) There are also <u>singulary</u> (or <u>unary</u>) operations. In these operations, only one number is given to get back one number. The moderator may ask participants to suggest examples. Absolute value is a singulary operation. Finding lower brackets of a number ($\lfloor \ \rfloor$) and finding the positive square root of a positive number are other examples of singulary operations.

Both of the operations $\oplus$ and $\circ$ in the Epilogue to the written lesson are commutative and associative. If the participants are curious about these operations, the moderator might ask them to invent other operations with four-by-four tables that are

(a) neither commutative nor associative,

and (b) commutative but not associative.

Finding an operation that is associative but not commutative, while possible, is a much harder problem.

Page 7

At $-\frac{1}{5}$ rule a wins and at $-\frac{1}{4}$ rule b wins. Participants may become curious about the tie point between $-\frac{1}{4}$ and $-\frac{1}{5}$. There is indeed a tie point, but it is not a rational number. For your information we include the following analysis. It is not necessary that participants see it. To have a tie the landing points must be equal. This gives the equation

$$\square \times \square = 4 \times \square + 1$$

or $$\square \times \square - 4 \times \square - 1 = 0$$

This is a second degree, or quadratic, equation. Using the quadratic formula, we get

$$\square = \frac{4 \pm \sqrt{16 + 4}}{2}$$

or $\square = 2 + 2\sqrt{5}$ and $\square = 2 - 2\sqrt{5}$

The root $2 - 2\sqrt{5}$ is the one that is between $-\frac{1}{4}$ and $-\frac{1}{5}$.

Of course the elementary school teacher need not know how to find the tie points for such rules in order to use them effectively in the classroom. (Notice that the sequence of problems on pages 6 and 7 did not ask for tie points, but only for which rule won at various starting points.) Participants who want to pursue this sort of thing further can be referred to the section on quadratic equations in any high school algebra book.

103

Film Discussion Notes
"Some Artificial Operations"

Preliminary information:

The class you will see in this film is a fourth grade heterogeneous group from the Phillips School in Watertown, Massachusetts. The film teacher is Phyllis Klein. Miss Klein had worked on other Project topics with the class 20 weeks previous to the filming. [Film running time: 44 min.]

Discussion after the film:

I. When the class was doing the problems

$$(4 * 5) * 100 =$$
and $$4 * (5 * 100) =$$

they were asked to predict the difference between the answers. The next day the teacher again brought up this question, and Robert S. gave this answer:

> Both the 100's only got added once, the two 5's got added twice, and the only difference can be in the 4's — one of them is added four times and the other just twice, which is why the difference is 8.

What does he mean? Was his thinking correct?

II. The last problem in the film was to use the numbers 100, 0, and 51, and one √, one *, and one ( ) to get the biggest number. Did anybody figure it out? Can you get the biggest number with more than one arrangement?

III. Q: While the class was working on the problem

$$(\square * \square) * \square = 21$$

and the teacher was getting whispered answers, she said that some people were doing this problem instead:

$$\square * (\square * \square) = 21$$

What answer were they getting and how were they getting it?

A.: It is surprising that they did work the problem

$$\square \ast (\square \ast \square) = 21$$

because in some ways this is a more difficult problem than the one they were given. To find the answer, we can analyze the problem the way Bruce did for $(\square \ast \square) \ast \square = 21$.

$$\square \ast (\square \ast \square) = 21$$

$$\square \ast (\square + \square + \square) = 21$$

$$(\square + \square) + (\square + \square + \square) = 21$$

so, $$\square = 4\frac{1}{5}$$

Note: Some members of your group might like to try an extension of this problem. Here is one:

$$((\square \ast \square) \ast \square) \ast \square = 60$$

IV. The class found that the difference between the answers for

$$(10 \ast 3) \ast \frac{1}{2} =$$

and $$10 \ast (3 \ast \frac{1}{2}) =$$

is 20 and the difference for the pair

$$(50 \ast \frac{1}{3}) \ast 4 =$$

and $$50 \ast (\frac{1}{3} \ast 4) =$$

is 100. Then the teacher asked the class to predict the difference for

$$(4 \ast 5) \ast 100 =$$

and $$4 \ast (5 \ast 100) =$$

Did you think this question was premature? Were you able to predict the difference at that point? Did the class predict correctly?

V. The following points are not questions that are likely to come up, but they may be of interest to your group.

(a) After giving the problem $-17 \checkmark -18 = ?$, the teacher gave the problem $-18 \checkmark -17 = ?$. Is it evident that the operation $\checkmark$ is commutative?

(b) The operation $\checkmark$ is a frequently used binary operation, commonly known as "max" (maximum).

☆(c) Can anyone find general methods for solving problems like these? (That is, find all numbers that work in $\square$ and $\triangle$.)

$$((((((\square * \square) * \square) * \square) * \square) * \square) * \square) * \square = \triangle$$

$$\square * (\square * (\square * (\square * (\square * (\square * (\square * \square)))))) = \triangle$$

What if there were more boxes?

Twelfth Session
Written Lesson Discussion Notes

Page 2, problems 6 - 8.

6. $(3 \ \delta \ 4) \ \delta \ 5 = \square$

7. $3 \ \delta \ (4 \ \delta \ 5) = \square$

8. Find a number for $\square$ so that $(\square \ \delta \ 4) \ \delta \ 5$ does not equal $\square \ \delta \ (4 \ \delta \ 5)$.

Problems 6, 7 and 8 deal with the question of whether the operation delta is associative. Since you get the same answer for problems 6 and 7, it looks as if the location of the parentheses does not change the answer. However, finding only one number so that $(\square \ \delta \ 4) \ \delta \ 5$ does not equal $\square \ \delta \ (4 \ \delta \ 5)$ proves that the operation is not associative. Ask participants for fast methods of solving problem 8.

Page 5, problems 18 - 22

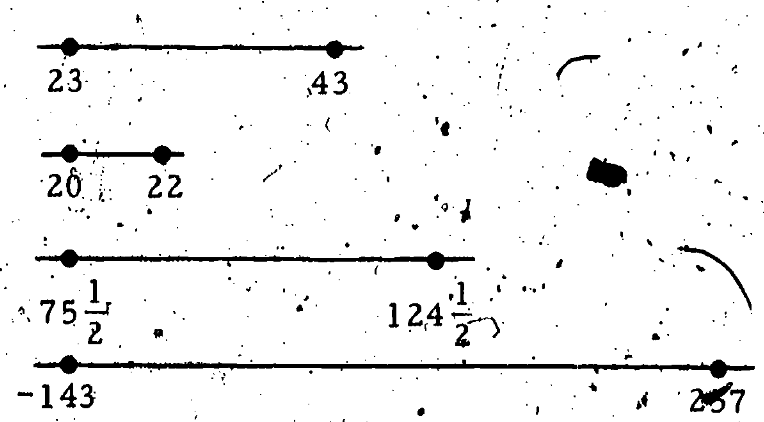
It may be helpful for your group to graph on a line the numbers that work in each box:

$33 \checkmark \square = 5$

$21 \checkmark \square = \frac{1}{2}$

$100 \checkmark \square = 12\frac{1}{4}$

$\square \checkmark 57 = 100$

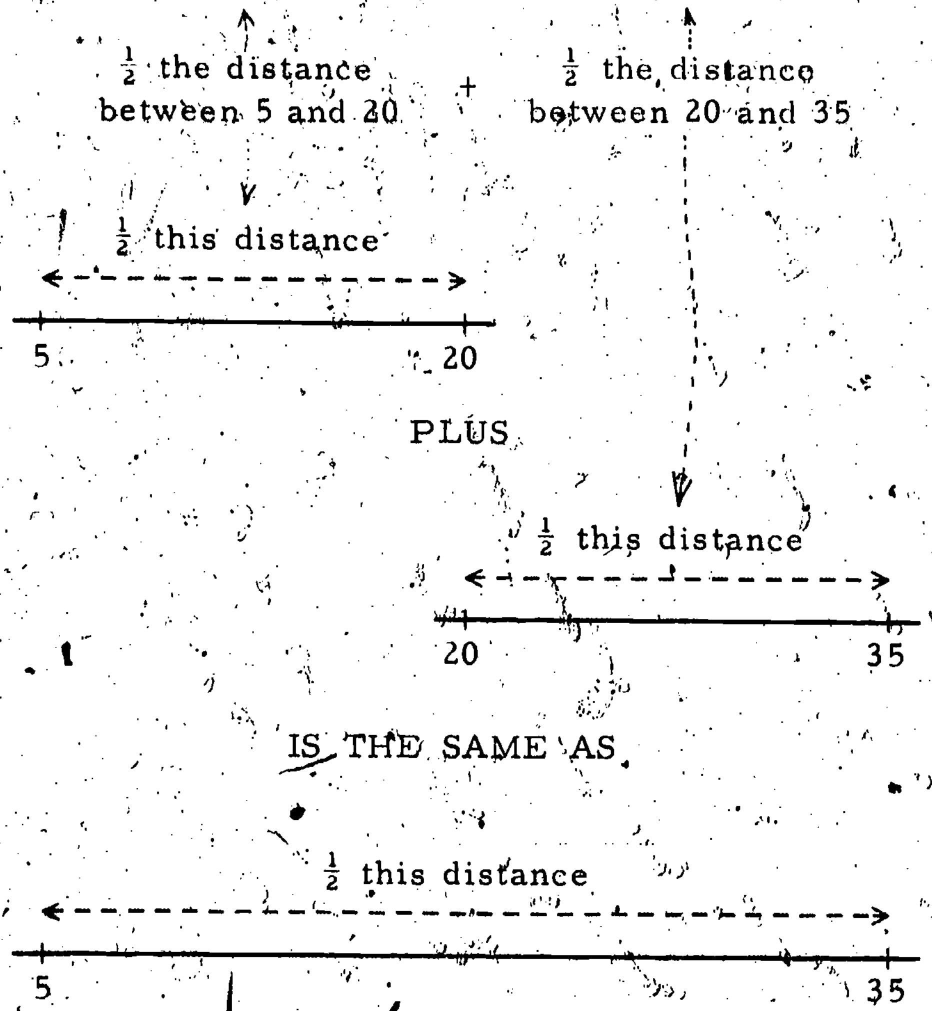


What is the distance between the two numbers that work in each of the problems? Are there any patterns worth discussing? Can you make up a problem with the operation "check" similar to those above so that the numbers that work are 56 and 96? Can you make up a problem so that only one number works?

Page 6, problems 28 - 32

One way to solve these problems is by computing the answers, using a trial and error approach for problems 31 and 32. An alternative approach follows:

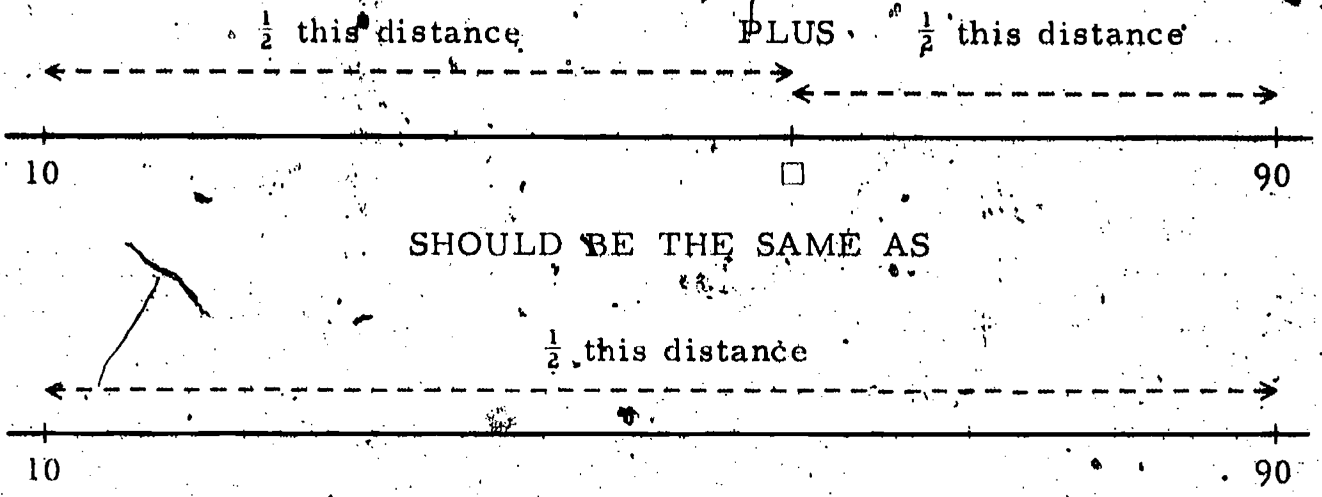
28. $(5 \checkmark 20) + (20 \checkmark 35) =$ ______



Problems 29 and 30 can be illustrated in a similar manner. Notice that in all three problems the numbers 20, $21\frac{1}{2}$, and $31\frac{5}{8}$ occur inside the region between 5 and 35.

31. (10 √ □) + (□ √ 90) = 10 √ 90

For this problem, suppose □ is a number between 10 and 90. The picture would look like this:



Now suppose that □ is a number outside the region between 10 and 90. If you draw a similar picture, you will see why no number outside the region between 10 and 90 will give the same results.

Page 10, problems ☆10 and ☆11

☆10. □ $ △ $\stackrel{df}{=}$ | □ ⊙ △ |

☆11. □ ⊙ △ $\stackrel{df}{=}$ | □ | ⊙ △

The operation in problem ☆10 is commutative. The operation in problem ☆11 is not commutative. Ask participants who attempted these problems to describe why this is so.

Page 11, problems ☆12 and ☆13

☆12. □ δ △ = □ ⊙ △

☆13. □ √ △ = □ ⊙ △

For teachers who are puzzled by these problems, it might be helpful to translate the mathematical symbols into words.

> □ δ △ = □ ⊙ △ may be translated as: Look for numbers whose average is either zero if they're different, or one if they're the same.

> □ √ △ = □ ⊙ △ may be worded as: Look for numbers so that half the distance between them equals their average.

Film Discussion Notes
"Frames and Number Line Jumping Rules"

Preliminary Information:

In this film you will see a heterogeneously grouped class of fifth graders. The teacher is Lee Osburn. He had met with the class for about an hour prior to this film. [Film running time: 38 min.]

Discussion after the film:

Question: Early in the film the teacher writes the problems

$$\square + \square + \square = 21$$

and $$\square + \square + \square = 24$$

with a gap between them, and later writes

$$\square + \square + \square = 22$$

directly under the first problem. What advantages, if any, does this have over writing each problem directly under the preceding one, like this:

$$\square + \square + \square = 21$$
$$\square + \square + \square = 24$$
$$\square + \square + \square = 22$$

Probably the chief advantage of writing the problems as the teacher did is to give an additional clue to the answer. Writing the third problem ($\square + \square + \square = 22$) physically between the first two emphasizes the fact that numerically 22 lies between 21 and 24, and hence that the answer to the third problem may lie between the answers to the preceding problems.

Later on, of course, the child would be expected to find the answers to problems like □ + □ + □ = 22 without hints like these.

> Question: When the class is trying to find an answer to □ + □ + □ = 22, the teacher asks if the answer is going to be larger or smaller than 7. This interchange ensues:
>
> Andy: Smaller.
>
> Teacher: Give me a number smaller than 7.
>
> Andy: 6.
>
> Teacher: (Writes) 6 + 6 + 6 = 22.
>
> Andy: No, larger.
>
> Was Andy right in concluding from this evidence that the answer had to be larger than 7?

The answer to this depends on what Andy was thinking when he saw 6 + 6 + 6 = 22—which is something we'll never know. If he realized that 6 + 6 + 6 is smaller than 22 then he is justified in changing his answer. If, on the other hand, he saw only that 6 + 6 + 6 is not equal to 22, he could deduce only that 6 in the boxes is not right. There is also the possibility that Andy changed his answer on the basis of totally non-mathematical cues. He might have been thinking, "Only two answers make sense—larger or smaller. Evidently the teacher didn't like the one I gave, so I'll change it."

> Question: When Richard suggests $7\frac{1}{3}$ for □ + □ + □ = 22 the exchange is:
>
> Teacher: $7\frac{1}{3} + 7\frac{1}{3}$?
>
> Richard: 15.
>
> Teacher: That would be $7\frac{1}{2}$.

Richard: $14\frac{2}{3}$.

Teacher: And another $7\frac{1}{3}$?

Richard: 22.

Why does the teacher have Richard add $7\frac{1}{3}$ and $7\frac{1}{3}$, and then another $7\frac{1}{3}$ when adding the 7's and then the $\frac{1}{3}$'s is so much easier?

There is no doubt that adding the 7's and then the $\frac{1}{3}$'s is far easier. On the other hand, the problem is $7\frac{1}{3} + 7\frac{1}{3} + 7\frac{1}{3}$, not $7 + 7 + 7 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$. Perhaps the teacher suggested that Richard add the first two $7\frac{1}{3}$'s, hoping that Richard would say he'd rather add the 7's and then the $\frac{1}{3}$'s, instead. If this was the teacher's intention, it didn't work. But it is interesting to note that Richard said $7\frac{1}{3} + 7\frac{1}{3}$ is 15; we can conclude from this that if Richard checked his answer at all, he didn't check it by adding the $7\frac{1}{3}$'s, but rather by adding the 7's and then the $\frac{1}{3}$'s.

> Question: When Andy explains how he did 96 + 96 + 96 = □, he insists twice that he subtracted 3 fours from 96. He got the right answer (288) so clearly he did not subtract 3 fours from 96, but from 300 instead. The teacher lets this explanation go by. Should he have?

Very frequently children get right answers but have trouble in explaining their methods. This is a classic example. Two courses were open to the teacher. First, he could have pursued Andy's explanation, making him see that he wasn't saying what he meant. Possibly some children would have benefited from this, but it would have slowed down the class. The teacher's on-the-spot decision was to follow a second course, namely, to ignore a faulty explanation of a correct answer. It is important to remember that no particular filmed class represents the last bit of mathematics instruction these children will ever have—there is always another day when verbalizations can be refined.

<u>Question:</u> When the teacher writes the jumping rule

$$\square \longrightarrow 3 \times \square - 10,$$

the arrow seems extraordinarily long. Why does he do this?

If the arrow in a jumping rule is drawn too short it is frequently confused with a minus sign. Particularly at the beginning, it is a good idea to make the arrow quite distinct from other symbols.

Here are some questions you may want to raise:

When the problem is $\square + \square + \square = \square + 12$, Paul suggests 3. Why is this a likely wrong answer?

When the arithmetic is done, with 3 in the boxes, the teacher says, "9 = 15. True or false?" Paul says, "True." The answer 3 is understandable, but what could have made Paul think that nine equals fifteen?

For the problem $\square + \square + \square = \square + 50$, Julio suggests $20\frac{1}{2}$. Where do you think this answer comes from?

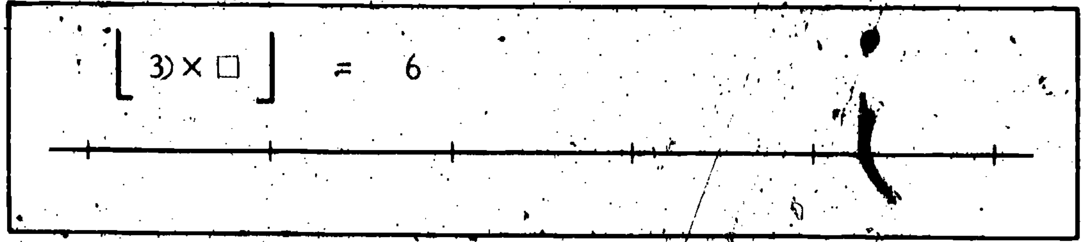
Andy suggests 3 as the place to start in order to get a jump of one space when using the rule $\square \longrightarrow 3 \times \square - 10$. Why is this a likely wrong answer?

Thirteenth Session

Written Lesson Discussion Notes

Since the first twelve problems involve similar ideas, here are some examples of likely errors.

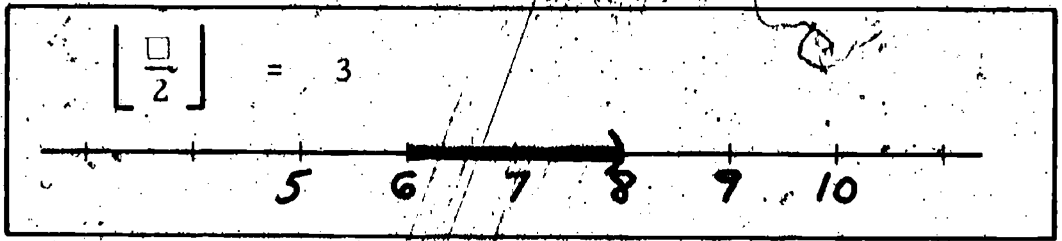
Page 2, problem 5



A common wrong answer is to graph the numbers from 2 to 3, not including 3. The right answer is that all the numbers from 2 to $2\frac{1}{3}$ work, not including $2\frac{1}{3}$.

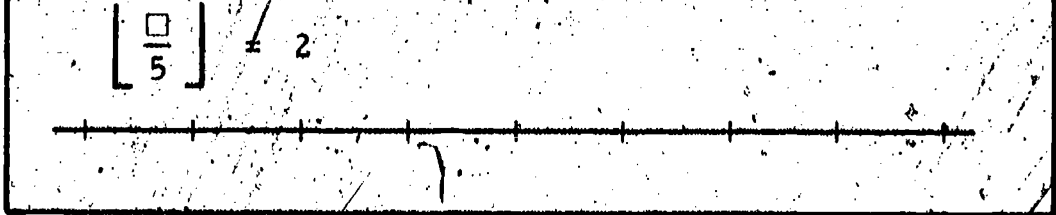
Perhaps the best way to answer questions, if there are any, is to try the participants' suggestions for numbers that work.

Page 3, problem 10



A common wrong answer here is to graph the numbers from 6 to 7, not including 7. The right answer is pictured above. Again trying numbers would probably be the best approach. If no one suggests it, the moderator may need to suggest a number like $7\frac{3}{4}$.

Page 3, problem 11



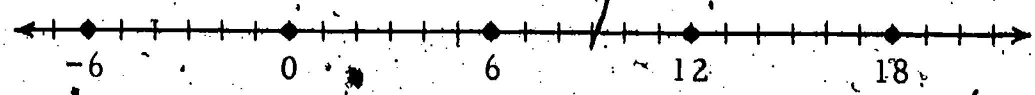
It may be worthwhile to draw some comparisons between problems 10 and 11. What is the same in the two pictures? What is different?

Most of the other problems in this written lesson are starred, and it is often the starred problems that participants ask about. Following are some examples and general comments about them.

Pages 4 and 5, problems 1, 2 and 3

Many participants, in graphing these problems, omit zero. The reason is probably due to the division sign. Teachers may be thinking, "Whenever there is a division sign, do not use zero." Zero will work in these three problems because the division is not by zero. Whether the moderator wants to comment on this is left to his discretion.

Page 5, problem ☆5

> The pattern of dots on the number line below continues in both directions. Make up an equation so that these dots (and only these dots) will work.
>
> 
>

As with any problems where an equation is asked for, one must be sure that the equation does not give more numbers than the pattern of points indicates. As an example, $\lfloor \square \div 3 \rfloor = \square \div 3$ will give all the points indicated above. However, this equation will give more than the pattern of points indicated. (For example, multiples of 3 work also.) This type of error appears to be very common among participants. It is also possible that the equation will not include all the indicated points, but this type of error is less likely to occur. A correct answer to problem ☆5 is $\lfloor \square \div 6 \rfloor = \square \div 6$.

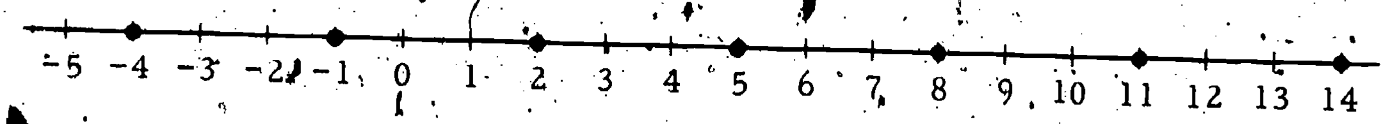
Page 6, problems ☆6 and ☆7

If there are questions on these two problems, it is probably best to have participants suggest possible equations and try them. Some participant may have a method for finding an equation that will work.

For each of these problems there are an infinite number of equations that will work. Here is a brief explanation of how to find the equations.

Each side of the equality will be the same except one side will have lower brackets. The numerator will be □ plus the distance necessary to move any of the points up to zero, or □ minus the distance necessary to move any of the indicated points down to zero. The number you divide by (denominator) will be the distance between any two consecutive points that are indicated by the graph. (Note to moderator: Perhaps your group would like to consider why this method works.)

In problem ☆7 we know that dots will be at 11 and at every third space away from 11.



To move any indicated point up to zero we need to add 1, or 4, etc., so the numerator could be □ + 1, or □ + 4, etc. To move any indicated point down to zero we need to subtract 2, or 5, etc. Therefore, the numerator could also be □ − 2, or □ − 5, etc. The distance between any two consecutive points is 3; therefore, the denominator will be 3.

Any of the following equations, as well as many others, will work for problem ☆7:

$$\left\lfloor \frac{\square + 1}{3} \right\rfloor = \frac{\square + 1}{3}$$

$$\left\lfloor \frac{\square + 4}{3} \right\rfloor = \frac{\square + 4}{3}$$

$$\left\lfloor \frac{\square - 2}{3} \right\rfloor = \frac{\square - 2}{3}$$

$$\left\lfloor \frac{\square - 5}{3} \right\rfloor = \frac{\square - 5}{3}$$

Film Discussion Notes
"Graphing With Square Brackets"

Preliminary information:

These discussion notes are somewhat different from previous film discussion notes. There are two reasons for this:

1. Many teachers viewing films from the Project have asked, "What else can we do with the ideas presented in the film?"
2. It has been our experience that participants have not asked many questions directly about this film.

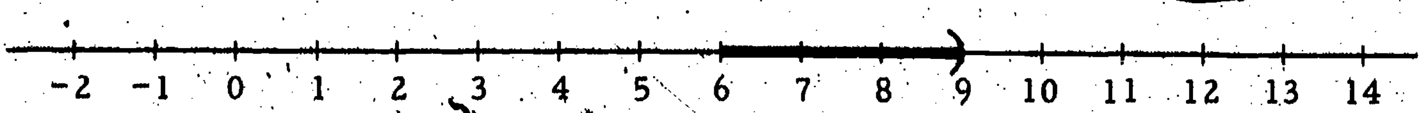
Therefore, these discussion notes are various extensions of the ideas in the film. We suggest that if any of these ideas are used, the moderator present them in much the way a teacher would use them in a classroom. As with all films in the course, however, participants' questions have priority.

This is a fifth grade class from the James Russell Lowell School in Watertown, Massachusetts. The class had met with the teacher, Professor David A. Page, for approximately 50 hours prior to this film. They had worked with square brackets previously, but had not done much graphing. The square brackets notation was used by the Project before lower and upper brackets were adopted. Square brackets means exactly the same as lower brackets. (For further information, see page 100 of these notes.)
[Film running time: 28 min.]

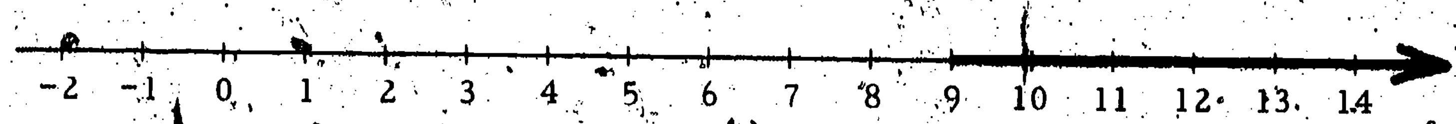
Extensions

Following are some ideas that a moderator could use after the film. It is left to the discretion of the moderator which, if any, of these ideas are used.

I. Here is the graph of the numbers that work in $\lfloor \square \div 3 \rfloor = 2$

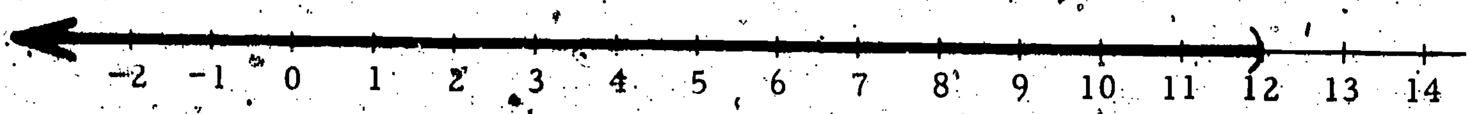


How does the graph change if we use $\lfloor \square \div 3 \rfloor > 2$?

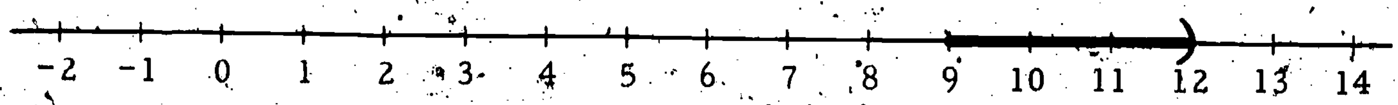


(The arrow here indicates that all numbers after 14 work also.)

How about $4 > \lfloor \square \div 3 \rfloor$?



Now put the previous two problems together. $4 > \lfloor \square \div 3 \rfloor > 2$



Here are other problems the moderator could use.

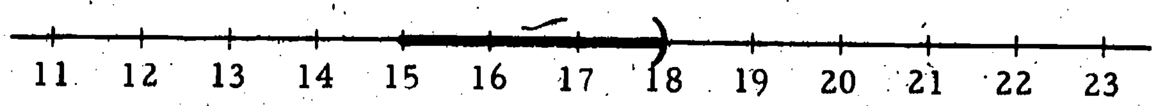
$$7 > \lfloor \square \div 2 \rfloor > 3$$

$$7\frac{1}{3} > \lfloor \square \div 5 \rfloor > 2$$

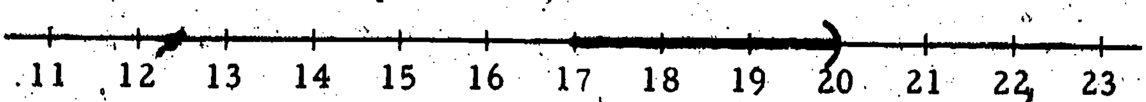
$$7 > \lfloor \square \div 5 \rfloor > 2\frac{1}{100}$$

$$7 > \lfloor \square \div 5 \rfloor > 6$$

II. Another idea that can be pursued is how to slide the graph up or down the number line. Here is the graph of $\lfloor \square \div 3 \rfloor = 5$.



(a) How can you change the equation so that the graph would be:



An answer is to subtract 2 from the box: $\left\lfloor \frac{\square - 2}{3} \right\rfloor = 5$. A likely wrong answer is to add 2 to the box, since the graph was moved up the line 2 units.

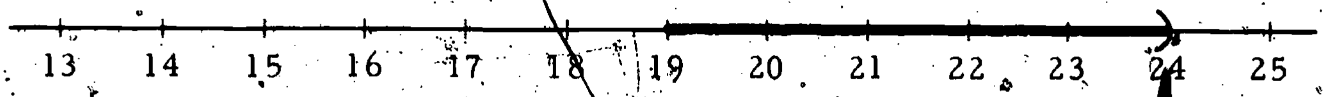
(b) How could you move the original graph down the number line one unit?

(c) How could you move it up the line 3 units? (For this question either $\left\lfloor \frac{\square - 3}{3} \right\rfloor = 5$ or $\left\lfloor \frac{\square}{3} \right\rfloor = 6$ would work.)

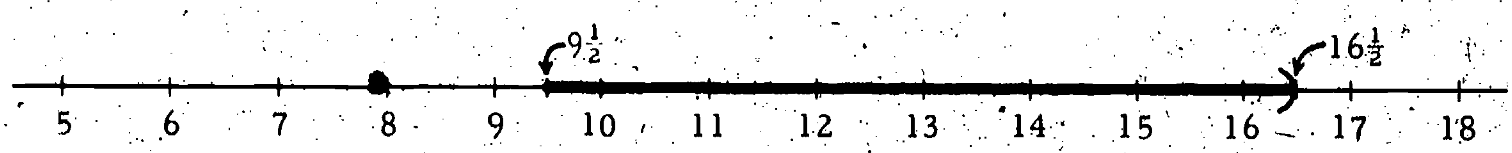
(d) How would you move the original graph down the line 7 units? Up the line 100 units?

(e) Predict what the graph of $\left\lfloor \frac{\square - 3}{7} \right\rfloor = 6$ would look like.

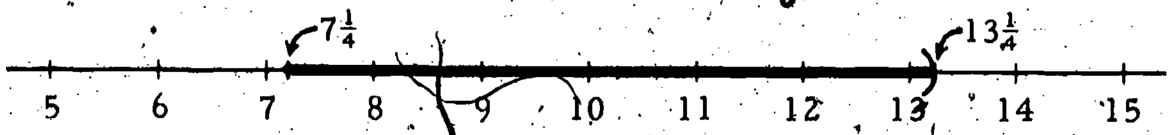
III. Here are some graphs. What are some equations that would give these graphs? In each case one possible answer is given.



$$\left\lfloor \frac{\square - 4}{5} \right\rfloor = 3$$



$$\left\lfloor \frac{\square - 2\frac{1}{2}}{7} \right\rfloor = 1$$



$$\left\lfloor \frac{\square + 4\frac{3}{4}}{6} \right\rfloor = 2$$

(a) What happens if you multiply the box by some number rather than add or subtract some number?

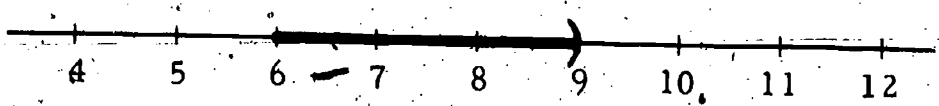
(b) Compare the graphs of $\left\lfloor \frac{\square}{3} \right\rfloor = 5$ and $\left\lfloor \frac{2 \times \square}{3} \right\rfloor = 5$.

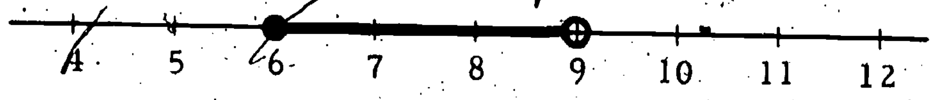
(c) How about $\left\lfloor \frac{\square}{7} \right\rfloor = 2$ and $\left\lfloor \frac{4 \times \square}{7} \right\rfloor = 2$?

(d) What happens if we multiply the box by some number and then add or subtract some number before we divide?

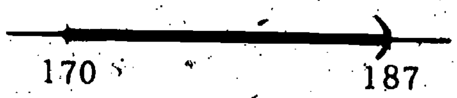
It probably is worth pointing out that there are two fairly standard ways of showing the graph of an equation on a number line. One way was used in the film; the other is to use a shaded loop or open loop. Both graphs below show the same thing; namely, the numbers that work are 6 up to but not including 9.

$$\left\lfloor \frac{\square}{3} \right\rfloor = 2$$





<u>Some questions that might be raised:</u>

1. The last problem in the film was $[\square \div 17] = 10$ and the answer was  How did the students figure that so fast?

   Could you solve $[\square \div 200] = 57$ the same way?

   What is a general method for solving this type of problem?

2. In the film, the students try to find the biggest number that works for $[\square \div 3] = 4$. Why can't they find the biggest number? What is the smallest number bigger than 12 that does <u>not</u> work?

Information about square brackets

Square brackets was not made up by the Project; it is a standard mathematical function. If students pursue mathematics they will perhaps see square brackets again, although they will probably find it presented as "the greatest integer not greater than" function. Because these words can get in the way of the idea, "square brackets" was used instead.

Since this film was made, the Project has modified the idea of square brackets to upper and lower brackets. The participants have worked with upper and lower brackets in the written lessons. Lower brackets are defined in the same way as square brackets; upper brackets leave integers alone and take any non-integer up to the next higher integer.

If some participant is interested in and insistent about where the idea of brackets can be used, here are two examples the moderator could give.

1. In a book store some books cost \$2.89 . If I have \$10.00 , how many books can I buy? One way to find the solution is to solve the equation

$$\left\lfloor \frac{10}{2.89} \right\rfloor = \square$$

2. The post office uses upper brackets for postage. First-class mail costs 8¢ an ounce (at this writing at least), but a $2\frac{1}{4}$ oz. letter will cost 24¢ to mail.

$$8 \times \left\lceil 2\tfrac{1}{4} \right\rceil = \square$$

These reasons alone are not sufficient for teaching brackets. For a discussion of why brackets are taught, participants are referred to the Epilogue of the written lesson in Book 8.

Fourteenth Session
Written Lesson Discussion Notes

This lesson deals with finding rules that will send two given points to two given points. When you ask for rules that do certain things, you need to be careful because there may be many rules that will do the same thing. Not only may there be different rules that do the same thing but the same rule may be written in many forms.

For example,

$$\square \longrightarrow \square + 0$$
$$\square \longrightarrow 1 \times \square$$
$$\square \longrightarrow \square$$
$$\text{and} \quad \square \longrightarrow \square + \square - \square$$

are different ways to write the same rule.

If a rule is proposed for a particular problem, it should be tried with the given numbers to ascertain whether it will work.

Following are the problems that appear to be the most difficult for people.

Page 3.

8. Write a rule:

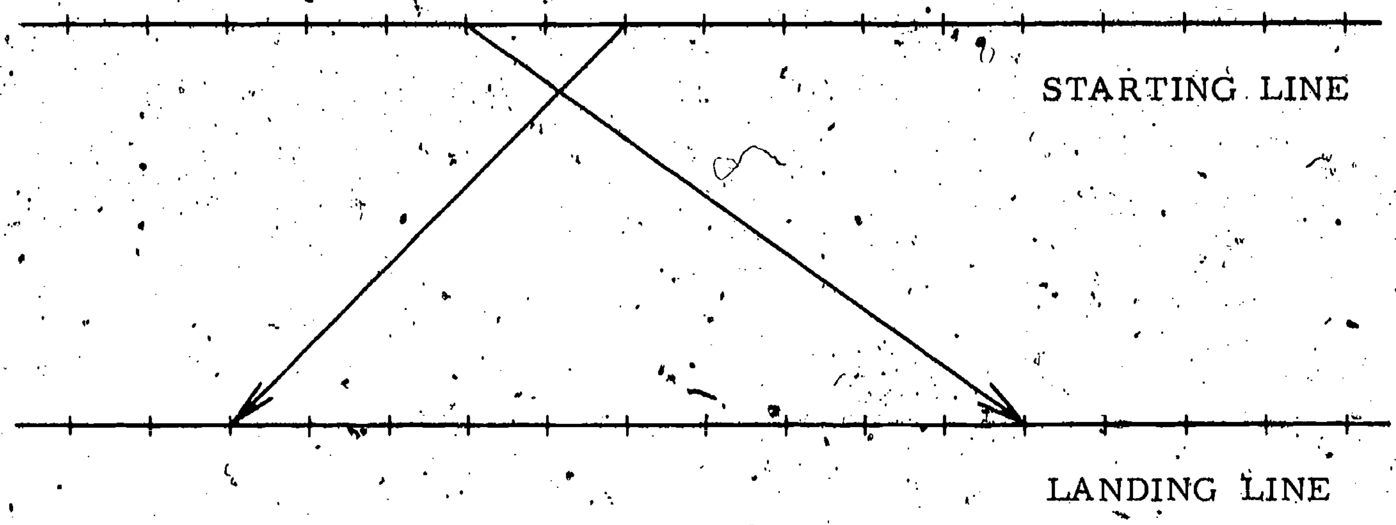
START		LAND
$\frac{3}{20}$	$\longrightarrow$	$1\frac{1}{60}$
$\frac{7}{15}$	$\longrightarrow$	$1\frac{1}{3}$

$\square \longrightarrow$

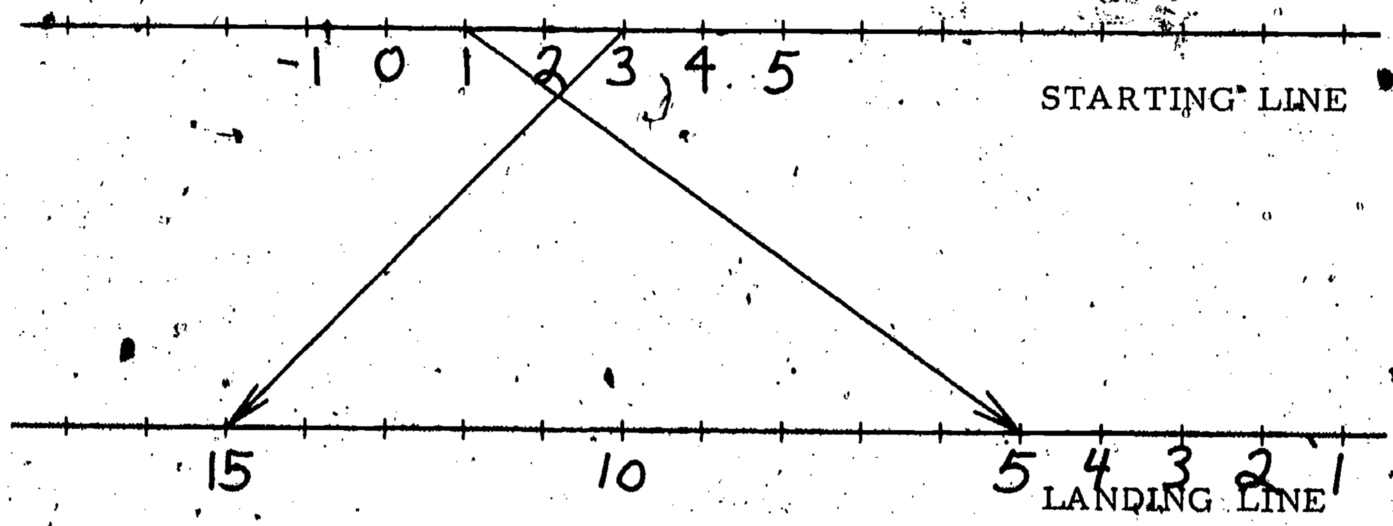
It is mostly the numbers in this problem that cause the difficulty. Since the differences between the starting points and landing points are the same, you only need to find how much is added to the starting point to reach the landing point. If a question is asked about this problem, then it may be best to work it out as a group.

Page

☆20. Find a rule such that two of its jumps might be illustrated by the following diagram. You should number your starting and landing points, and you may calibrate your number lines any way you wish.



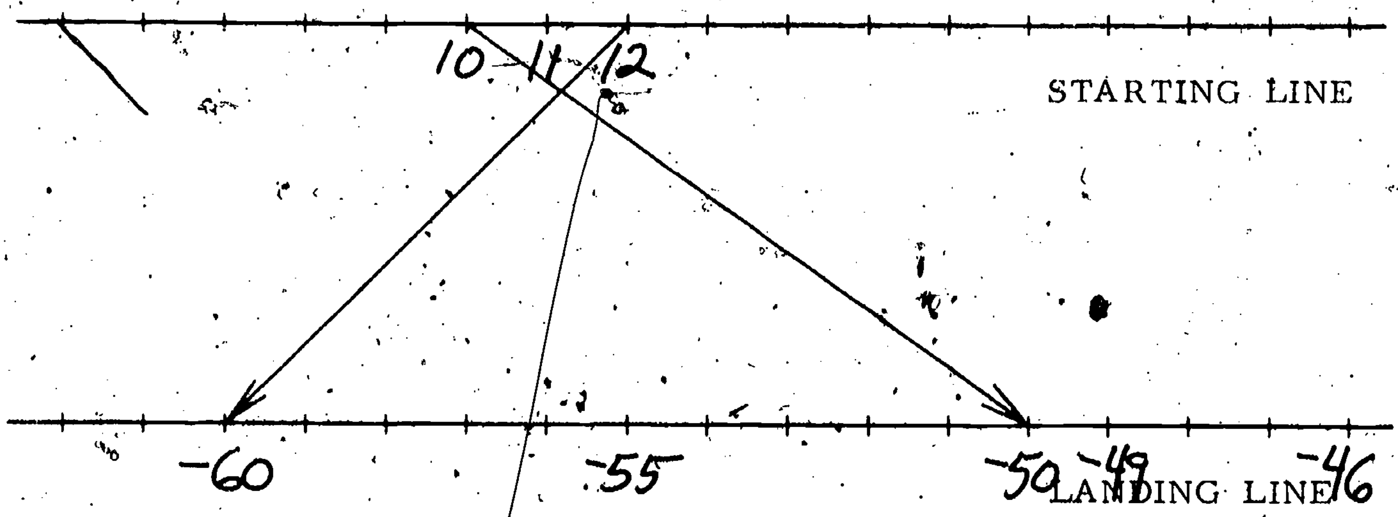
Participant: I attempted this problem and I don't know what I did wrong. The difference between the starting points is 2 and the difference between the landing points is 10 so the rule should be □ ⟶ 5 × □. I picked 1 and 3 as my starting points. With my rule 1 ⟶ 5 and 3 ⟶ 15. The only way my numbers will work in the diagram is if I calibrate the landing line from right to left like this:



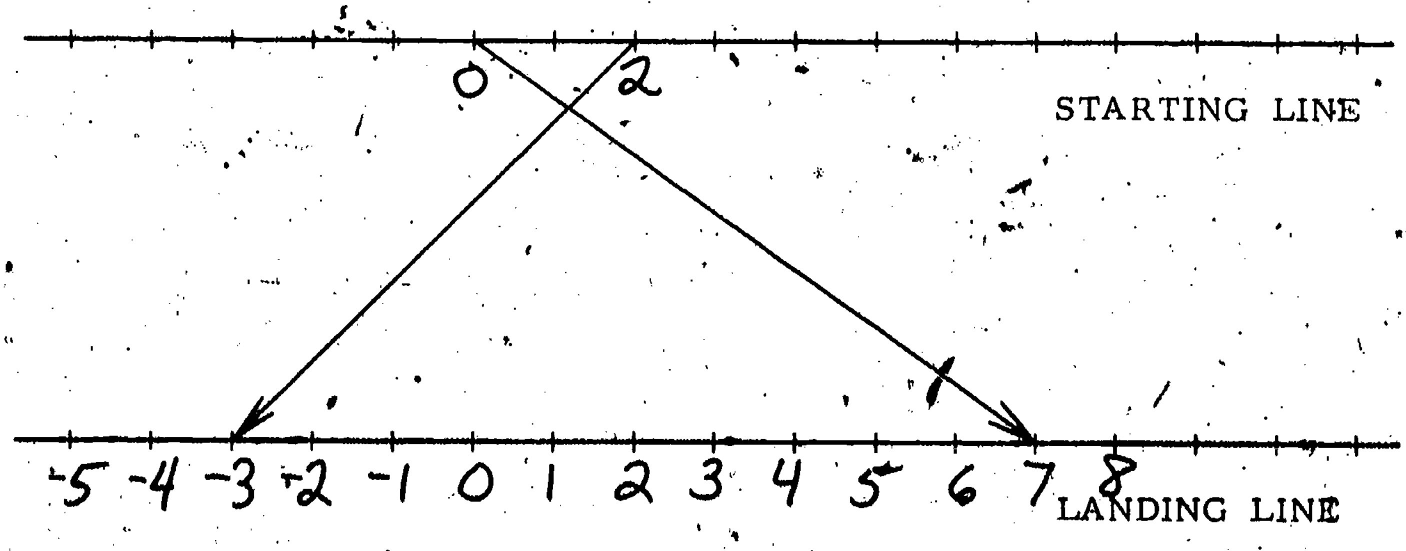
Is that correct? Is there a way to do the problem calibrating both lines from left to right?

Moderator: The problem states that you can calibrate the number line any way you wish, so your solution is acceptable. Did anyone do it numbering both lines from left to right?

Participant: Well, I assumed that both lines were numbered in the usual way. I figured that the 1st starting point minus the 2nd starting point is 2 while 1st landing point minus 2nd landing point is −10. I knew I wanted the rule □ ⟶ −5 × □. Next I picked 10 and 12 as my starting numbers. My diagram looked like this:



Participant: I thought the number lines should be numbered so that 1 on the landing line was directly under 1 on the starting line. Picking 0 and 2 as my starting numbers, my diagram looked like this:



I knew that I needed a rule which took you from 0 to 7 and from 2 to -3. By what came before in the lesson I thought that the rule should be $\square \longrightarrow -5 \times \square$.

With this rule $\begin{matrix} 0 \longrightarrow 0 \\ 2 \longrightarrow -10 \end{matrix}$ and I wanted $\begin{matrix} 0 \longrightarrow 7 \\ 2 \longrightarrow -3 \end{matrix}$

Comparing the landing points 0 and -10 with the landing points that I wanted, 7 and -3, I figured out that the rule should be $\square \longrightarrow -5 \times \square + 7$.

<u>Note to moderator:</u> If this last approach to the problem is used, the rule that works will be of the form $\square \longrightarrow -5 \times \square + k$ where k stands for a number. The value of k will depend on the starting numbers that are picked.

Film Discussion Notes
"Competing Number Line Rules"

Preliminary information:

The class you are about to see is a heterogeneously grouped fifth grade from the James Russell Lowell School in Watertown, Massachusetts. The film was made in the early spring. The class had worked with the teacher, David A. Page, for about fifty previous hours on various topics of the Project. [Film running time: 33 min.]

Here are some possible questions which might arise after the teachers view this film. Note that the responses given are merely sample answers. Answers are not provided to some questions. A variety of opinions and approaches ventured by teachers in the group can be sought.

Question: The class seemed to be quite at ease using jumping rules. Had they ever done this kind of topic before?

Response: Yes, the class had certainly worked with jumping rules before. Some of the topics they had explored were finding standstill points, making consecutive jumps, using rules in combination, and finding inverses for rules. However, they had not pursued this particular topic—that is, combining rules to obtain the largest number possible.

Question: When the rules were:

$$\square \xrightarrow{a} \square + 7$$
$$\square \xrightarrow{b} 6 \times \square$$
$$\square \xrightarrow{c} \square \times \square$$

the teacher asked where the next interesting number would be. What is an "interesting number"?

Response: In this case, the interesting starting number was 6, because you get the same landing number (36) whether you use rule b or rule c. At another time, some other number might be considered interesting.

<u>Question:</u> Aside from the computation involved, why is this particular topic important for children?

<u>Response:</u> In pursuing which combination of rules to choose to get the largest number, the class began to get an idea of what each of these rules accomplishes.

Consider the following rules:

$$\square \xrightarrow{a} \square + 8$$
$$\square \xrightarrow{b} 4 \times \square$$
$$\square \xrightarrow{c} \square \times \square$$

When Lorraine said, "From 5 on, it's cc," she indicated that squaring any number over five will yield a larger result than either adding 8 to it or multiplying it by 4. And squaring the result (in effect, finding the fourth power of the original number) will certainly yield the maximum. Lorraine and many of the other students eliminated much unnecessary testing of numbers as soon as they realized how strong the squaring rule was.

<u>Question:</u> I heard Janet say that in order to find the largest landing point, you should get the biggest number you can before you hit the squaring rule. Is her method reliable at all times?

<u>Response:</u> Given the rules:

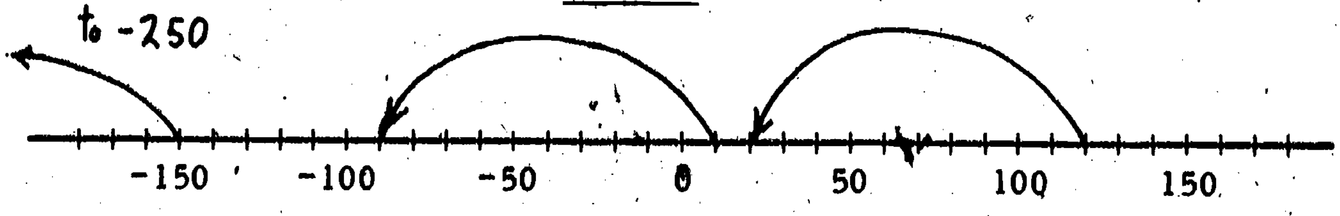
$$\square \xrightarrow{a} \square + 7$$
$$\square \xrightarrow{b} 6 \times \square$$
$$\square \xrightarrow{c} \square \times \square$$

Janet's method does work most of the time. But before adopting this strategy for every problem, let us change rule a to:

$$\square \xrightarrow{a} \square - 100$$

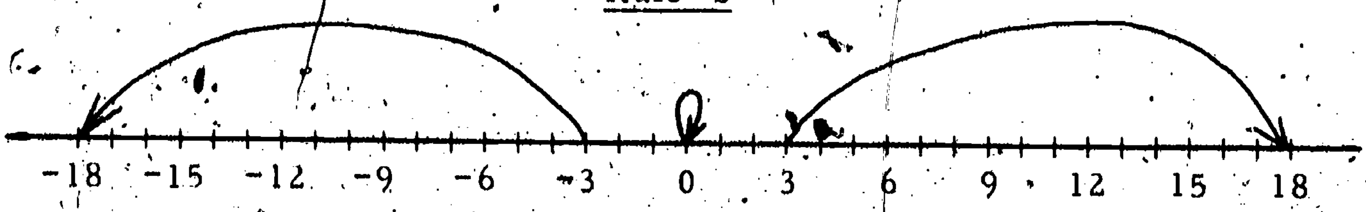
Now rule a always moves us down the line 100 units, no matter where we start.

<u>Rule a</u>



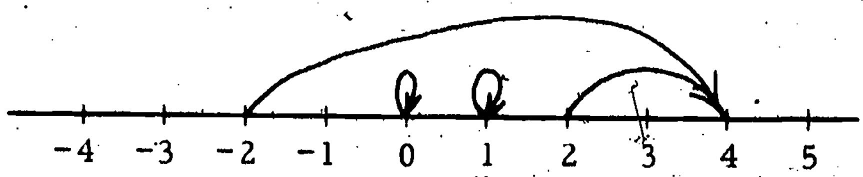
Rule b multiplies numbers by 6. Positive numbers are taken further up the line, but negative numbers are taken further down the line:

Rule b



Rule c takes positive numbers greater than 1 up the line; it takes negative numbers up to the positive portion of the line also.

Rule c



Which two rules would you use starting at 0? Here, the best strategy would be to use rule a first. That gets you 100 spaces to the left of zero. Then you would use rule c, taking you to $(-100)^2$, or positive 10,000. Hence, we first want to get as small as possible before using the squaring rule. You may wish to try some other starting numbers to see where things change.

An interesting topic to pursue may be to find out for which starting numbers Janet's method fails with rule a changed back to: $\square \xrightarrow{a} \square + 7$.

Another variation worthy of group discussion may be: what general strategies can be developed if rule a is again changed, this time to: $\square \xrightarrow{a} 51 - \square$. Rules b and c remain the same.

Question: Near the end of the film, when the rules were:

$$\square \xrightarrow{a} \square + 1$$
$$\square \xrightarrow{b} 2 \times \square$$
$$\square \xrightarrow{c} \square \times \square$$

Alfred said that he'd choose cc starting at $2\frac{1}{5}$ because 2 times $2\frac{1}{5}$ is not as much as $2\frac{1}{5} \times 2\frac{1}{5}$. He seems sure of this, even though he may not know the precise answer for $2\frac{1}{5} \times 2\frac{1}{5}$. Is he correct?

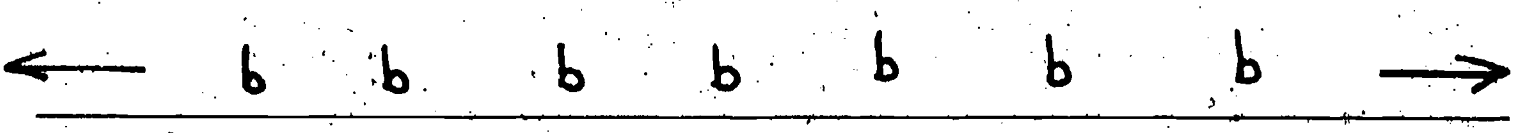
Question: Suppose a child does not see Alfred's shortcut. Are there other ways of finding out which rules to use at $2\frac{1}{5}$ and at $1\frac{4}{5}$ without doing a lot of computation?

Question: I would like to try this topic, but my students do not know very much about negative numbers. They also would not be able to handle the squaring rule. How can I adapt this material to my class?

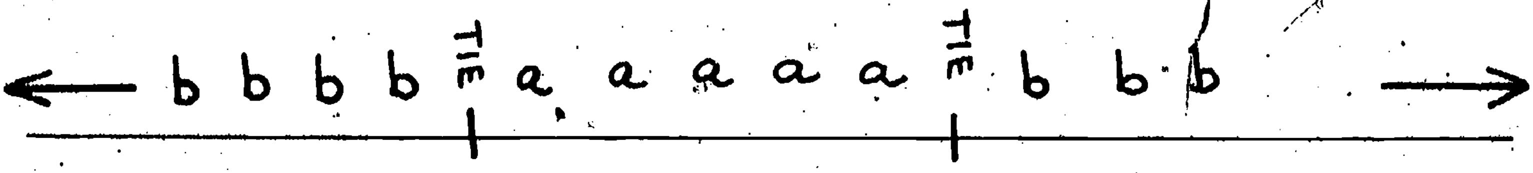
* * *

In the discussion notes for the film "Which Rule Wins?" (Book 2), three situations were described. For the sake of completeness, we repeat them here and append a few others.

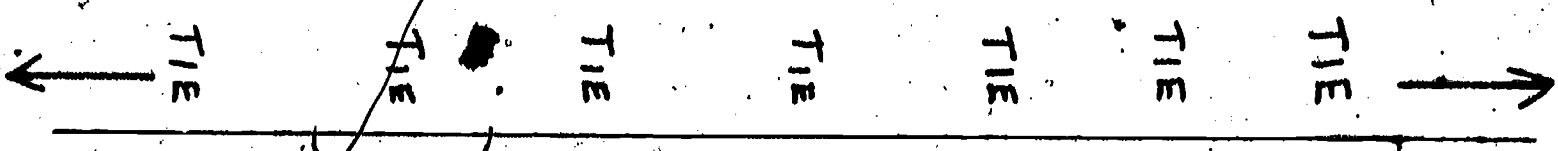
(a) Are there two rules which never tie? That is, could we have a picture like this?



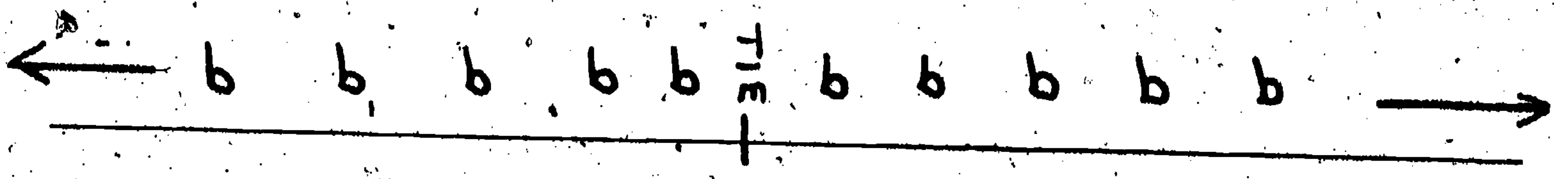
(b) Are there two rules which tie exactly twice, giving a picture like this?



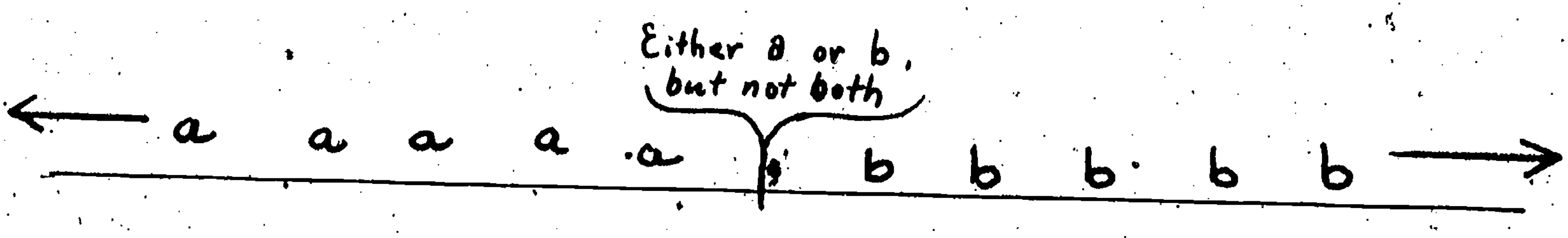
(c) Are there two different rules which always tie?



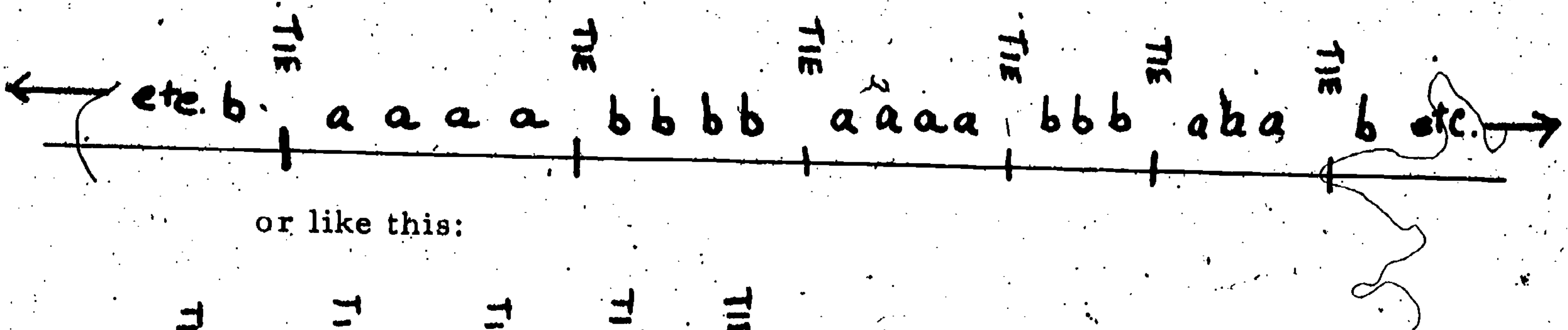
(d). Are there two rules (a and b) where b wins everywhere except for one tie point? Picture:



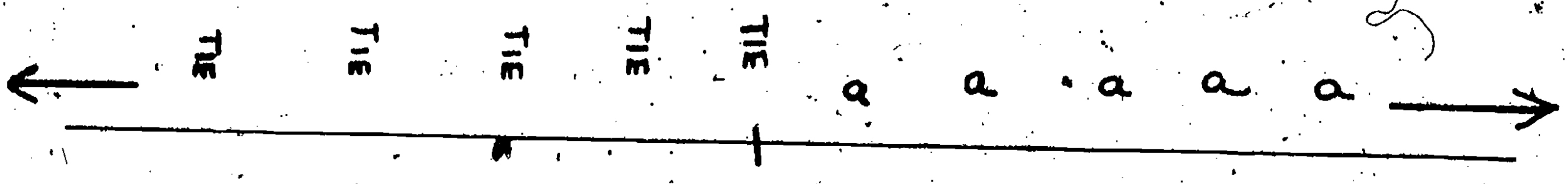
(e) Are there two rules (call them a and b as usual) where sometimes a wins and sometimes b wins but where they never tie? The picture might look like this:



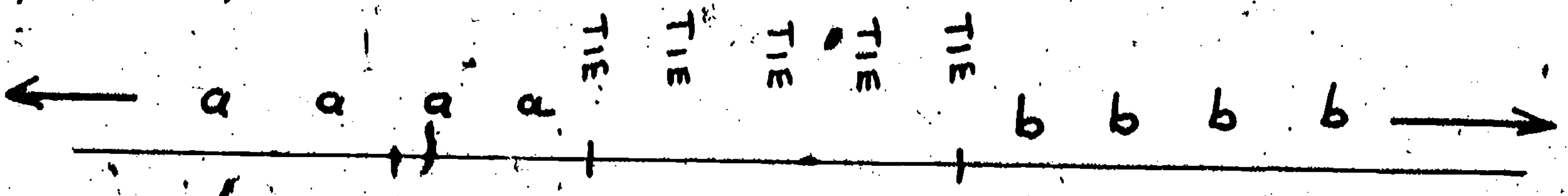
(f) Are there two rules which tie infinitely often but not always? The picture might look like this:



or like this:



or like this:



Finding a pair of rules which works for situation (d) is far easier than finding pairs for (e) or (f), but the latter two problems can be done. By the end of the course the participants will have the tools to find rules which work, so it might be a good idea to bring up these points again later on.

* * *

The summary for this film includes a note about irrational tie points. In connection with this, you should refer to the Written Lesson Discussion Notes for Book 11. For those interested, further notes about quadratic vs. linear functions may be found in the Epilogue to the written lesson in Book 15, and in the supplement "Graphing Number Line Jumping Rules" (Books 13 and 14).

Fifteenth Session

Written Lesson Discussion Notes

For the most part, these notes give a straightforward exposition of the more difficult problems in the lesson. We hope that the moderator will use them not as they are presented, but rather in a manner which will elicit the appropriate responses from the members of his group.

Page 2

	Starting Points	Landing Points	1st Starting pt. minus 2nd Starting pt.	1st Landing pt. minus 2nd Landing pt.	Rule
9.	1 ⟶ 8 2 ⟶ 13				

This problem may be troublesome for some people because the differences are −1 and −5 and the rule is □ ⟶ 5 × □ + 3 . If the two negatives led people to think the rule would involve a negative, it might help to work through the following problem:

START		LAND
2	⟶	13
1	⟶	8

Page 3

What do problems 5 through 9 have in common?

	Starting Points	Landing Points	1st Starting pt. minus 2nd Starting pt.	1st Landing pt. minus 2nd Landing pt.	Rule
5.	______ → 50 ______ → 25				$\square \longrightarrow \frac{1}{6} \times \square$
6.	300 → 57 150 → 32				
7.	300 → $62\frac{3}{4}$ 150 → $37\frac{3}{4}$				
8.	300 → ______ 150 → −29				$\square \longrightarrow \frac{1}{6} \times \square - 54$
9.	$29\frac{1}{2}$ → $17\frac{1}{2}$ →	Leave Blank		2	

Probably the participants will recognize that although the starting points in number 9 differ from those in the other problems, the equation

$$\frac{\text{1st Landing pt. minus 2nd Landing pt.}}{\text{1st Starting pt. minus 2nd Starting pt.}} = \frac{1}{6}$$

is true for each problem.

The moderator might want to ask what rule works in problem 9. Probably there will be at least two different rules given. If not, a rule such as $\square \longrightarrow \frac{1}{6} \times \square + 2$ or $\square \longrightarrow \frac{1}{6} \times \square - 1$ might be tested to show that any rule of the form $\square \longrightarrow \frac{1}{6} \times \square + n$, where n is a number, will satisfy the conditions of this problem.

☆2. Write a rule:

START		LAND
11	$\longrightarrow$	10
17	$\longrightarrow$	30

$\square \longrightarrow$

If someone asks about this problem, the group might benefit from working it out. Proceeding as in the table on page 111, you would have:

1st Starting pt. minus 2nd Starting pt.	1st Landing pt. minus 2nd Landing pt.
−6	−20

This suggests the rule $\square \longrightarrow \frac{20}{6} \times \square$ or $\square \longrightarrow 3\frac{1}{3} \times \square$.

Starting at 11 with this rule you land at $36\frac{2}{3}$.

How should the rule be adjusted so that you land at 10 instead of $36\frac{2}{3}$?

The rule

$$\square \longrightarrow 3\frac{1}{3} \times \square - 26\frac{2}{3}$$

gives a jump from 11 to 10. With this rule, does the other starting point, 17, land at 30 ?

Page 10

7. Rules:

$\square \xrightarrow{c} \square + 10$

$\square \xrightarrow{d} \frac{1}{2} \times \square$

Write rules cd and dc :

$\square \xrightarrow{cd}$ ______________

$\square \xrightarrow{dc}$ ______________

(Note that rule dc is very different from rule cd . If in doubt check both rules.)

The fact that rules cd and dc are different illustrates that composition is not always commutative. In other words, the order in which you do the composition makes a difference. It would be profitable for the group to investigate whether composition is ever commutative, and if so, when. Examining the rules used in this lesson, one finds examples of both commutative and non-commutative composition. The following information may be helpful.

Page 5 Rules ab and ba :

$\square \xrightarrow{ab} 3 \times \square - 4$

$\square \xrightarrow{ba} 3 \times \square - 12$

Page 6

$\square \xrightarrow{cd} 7 \times \square + 8\frac{1}{2}$

$\square \xrightarrow{dc} 7 \times \square + 59\frac{1}{2}$

Page 7

$\square \xrightarrow{ef} \square + 30$

$\square \xrightarrow{fe} \square + 30$

Page 8 Rules gh and hg are the same.

Page 9 # 4

$\square \xrightarrow{ab} 6 \times \square + 5$

$\square \xrightarrow{ba} 6 \times \square + 15$

5

$\square \xrightarrow{cd} 2 \times \square + 6$

$\square \xrightarrow{dc} 2 \times \square + 3$

Page 10 # 6

$\square \xrightarrow{ab} 5 \times \square + 15$

$\square \xrightarrow{ba} 5 \times \square + 3$

After consideration of a few of these problems, see if your group can predict when the composition of two rules is commutative and when it is not.

Film Discussion Notes
"Rules Moving Two Points"

Preliminary Information:

The film shows a heterogeneously grouped fifth grade from the James Russell Lowell School in Watertown, Massachusetts. The teacher is David A. Page. Prior to this film he had met this class for about 50 hours. About five minutes of the previous day's lesson is included so that the viewers may know what led up to this film. [Film running time: 50 min.]

The notes for this film are written in three parts. Part one consists of questions that have been asked by teachers about this film and some answers to these questions. The second part consists of questions you may want to raise. The third part is a transcript of the beginning of an actual discussion after the film was viewed in one of the Project's in-service institutes.

I.

> Question: The teacher had been asking for rules that made the distance between the landing points either longer or shorter than the distance between the starting points. Why did he suddenly switch to finding a rule that kept the distance the same?

Finding rules that keep the distance between landing points the same is rather easy and not very interesting. The teacher tried to make it more interesting by making both landing points negative. It appears in the film that nearly everyone could stretch or shrink the distance and that it was time to move on to more complex things. In order to do that, the class needed the idea that adding or subtracting something does not affect the distance. The teacher thought that one example, at this point in the lesson, would get across this idea.

> Question: Right after David said the landing numbers would be two units apart, Lisa said David was wrong because there was a standstill point. Does the fact that a rule has a standstill point influence how far apart the landing points will be?

What, exactly, Lisa had in mind is debatable. It is conceivable that she thought David said that all jumps would be two units long. In other words, she was probably computing the distance from a starting point to its landing point, rather than the distance between two landing points.

The fact that a rule has a standstill point does not have anything to do with the distance between the landing points. However, knowing that a rule of the form

$$\square \longrightarrow a \times \square - b$$

(where a and b are numbers with a not equal to 1) has a standstill point, and that as you move farther away from that standstill point the jumps become longer, implies that the landing points of two particular jumps will not always be 2 units apart. Such an intuition comes after working with jumping rules for a long time (which this class had done).

> Question: Are there always only two rules that take two points to two points?

[Although this question may be raised by someone in your group after seeing the film, most people will not be able to discuss the question adequately until they have finished the fifteenth lesson. In that lesson starting points are sent to the landing points in a specific order, while in the film the order is not specified. The question as treated here applies to the type of problem given in the film, while in the Epilogue to the fifteenth lesson the question is answered for the kind of problem given there. The inclusion of the following material at this point is not to suggest that this is the proper time to discuss it, but only to give you background for handling the question when you wish to do so.]

There will be only two simple (linear*) rules that will take two given points to two given points but there are many other kinds of rules that could be used. There are various requirements that one can make in asking the questions. The following is an attempt to show what restrictions can be placed on the question and some possible solutions.

---

* A linear rule is one that can be written in the form $\square \longrightarrow a \times \square + b$ where a and b are numbers.

(a) Start at 3 and at 5. Find a rule so that the landing points are 18 apart.

$\square \longrightarrow 9 \times \square$

$\square \longrightarrow -9 \times \square$

$\square \longrightarrow 9 \times \square + 11\frac{1}{3}$

$\square \longrightarrow 9 \times \square - 102$

$\square \longrightarrow -9 \times \square + 75$

$\square \longrightarrow -9 \times \square - \frac{71}{97}$

etc.

There are infinitely many rules of the form

$\square \longrightarrow a \times \square + b$

that will do what the problem asks; a must be 9 or −9 and b can be any number.

Another rule that will work:

$\square \longrightarrow \square \times \square + \square - 3$

(This is not a linear rule because of the $\square \times \square$.)

(b) Start at 3 and at 5. Find a rule so that the landing points are 18 apart, and one of them is 37.

$\square \longrightarrow 9 \times \square + 10$

$\square \longrightarrow 9 \times \square - 8$

$\square \longrightarrow -9 \times \square + 64$

$\square \longrightarrow -9 \times \square + 82$

Here there are four linear rules that will do the job. The first and third rules above send 3 to 37, while the second and fourth rules send 5 to 37. One could eliminate two of the rules above by specifying which of the starting numbers you wanted to go to 37. Another rule, not of the same form, that works:

$\square \longrightarrow \square \times \square + \square + 7.$

(c) Start at 3 and at 5. Find a rule so that the landing points are 23 and 41.

$$\square \longrightarrow 9 \times \square - 4$$

$$\square \longrightarrow -9 \times \square + 68$$

Here there are only two rules of this form that will work. Again notice that the first rule will send 3 to 23 while the second will send 5 to 23. You could eliminate one of the rules by specifying which starting point you wanted to send to 23.

Another rule that works:

$$\square \longrightarrow \square \times \square + \square + 11$$

In most classes the students would not propose any of these other rules; however, if a student has a rule that appears to be different, probably it is worth checking.

II.

Some questions you may want to raise:

1. The previous day Ricky had predicted correctly what the distance between the landing points would be, yet the next day it appeared no one knew what was going on.

Wouldn't it have been better if the teacher had spent more time leading up to this topic?

2. At one place two rules were given: $\square \longrightarrow \frac{1}{2} \times \square$ and $\square \longrightarrow \square \div 2$. I know that they both do the same thing, but why did the teacher choose $\square \longrightarrow \frac{1}{2} \times \square$ rather than $\square \longrightarrow \square \div 2$?

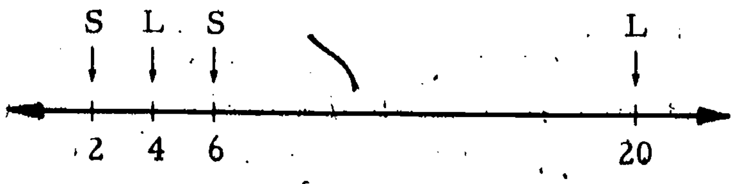
3. At one point the rule $\square \longrightarrow 3 \times \square - 10$ was put on the board and the students were asked to predict what would happen to the distance between any two starting points. David said that every pair of landing numbers would be two units apart. Where did he get that?

4. Late in the film the starting points 2 and 6 were given. The problem was to find a rule that would take these starting points to the landing points 4 and 20. What question could a teacher ask if his students did not find a rule after a reasonable length of time?

III.

Following is the beginning of a discussion by 25 teachers led by a Project staff member after the teachers had seen the film.

Moderator	Participants
Did anybody find the other rule for the last problem in the film? Starting points: 2, 6. Landing points: 4, 20.	
	Is there another rule? The rule they found made 2 go to 4. Now you want to get the 6 to go to 4?



Alfred's rule was □ ⟶ 4 × □ − 4

2 ⟶ 4
6 ⟶ 20

Moderator	Participants
What would the other rule do?	
	Now you want 2 to go to 20, and 6 to go to 4.
	It has to be a multiple of 2.

Moderator	Participants
I didn't understand what you said.	
	It has to start off 4 times something to keep the distance.
	How about $4 \times \square - 20$?
$\square \longrightarrow 4 \times \square - 20$ Let's try 6.	
	It would bring you to 4.
$6 \longrightarrow 4$ Now try starting at 2.	
	You land at -12, but we wanted 2 to go to 20.
So this one doesn't work. You want $6 \longrightarrow 4$ and $2 \longrightarrow 20$. Is it impossible?	
	Don't tell us the rule. Tell us how to start the rule.
	It is $4 \times -4 + 28$.
Tell me the whole rule.	
	$\square \longrightarrow -4 \times \square + 28$

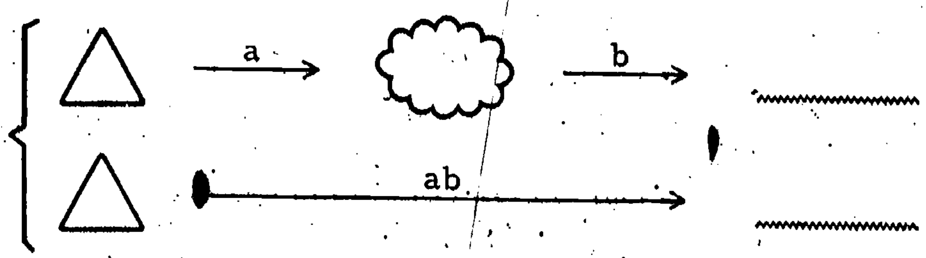
Moderator	Participants
Let's check it. With 2, what happens?	
	You land on 20.
What about 6?	
	You land on 4. I figured it had to be the opposite.

Sixteenth Session

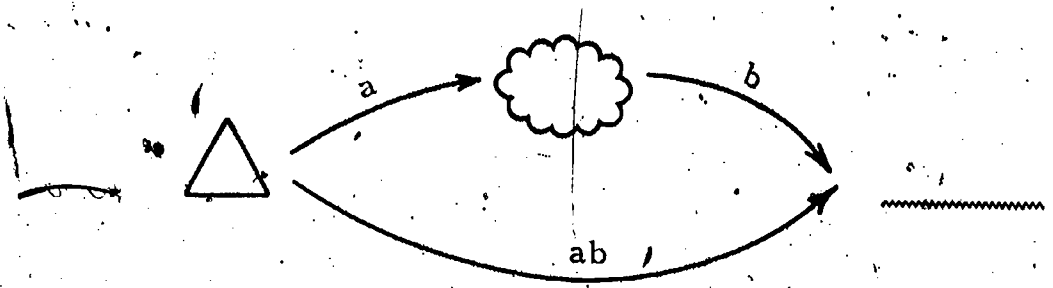
Written Lesson Discussion Notes

I.

Throughout this written lesson participants are directed to check their answers. The check involves a setup like the following:



or it could look like this:



Since the answers should agree for both parts of the check, there is a strong tendency for participants to do just part of the check and copy that answer in the other part. This defeats the purpose of checking. Participants are urged to be thorough in making checks.

II.

If some participants have difficulty finding the composite of two or more rules, other participants should be urged to explain how they went about finding it. One of the nice things about this topic is the variety of ways to approach the problems. To illustrate, each of the following problems is solved in a different way.

Page 6, problem 8

Write rule b :	$\square \xrightarrow{a} \square + 5$
	$\square \xrightarrow{b}$ ______
	$\square \xrightarrow{ab} 2 \times \square + 10$

Solution: Compare $\square + 5$ with $2 \times \square + 10$. The result obtained from using rule ab is double the result obtained from using rule a. Therefore, rule b must be a rule that doubles: $\square \xrightarrow{b} 2 \times \square$

Page 6, problem 9

Write rule d :	$\square \xrightarrow{c} \square + 5$
	$\square \xrightarrow{d}$ ______
	$\square \xrightarrow{cd} 2 \times \square + 6$

Solution:

(1) Change rule c to $\square \longrightarrow \square$ by subtracting the 5.

$$\square \longrightarrow (\square + 5) - 5$$

or $\square \longrightarrow \square$

(2) Now change $\square \longrightarrow \square$ to $\square \longrightarrow 2 \times \square$ by multiplying by 2.

$$\square \longrightarrow 2 \times \square$$

(3) Finally change $\square \longrightarrow 2 \times \square$ to $\square \longrightarrow (2 \times \square) + 6$ by adding 6. This final change produces rule cd.

(4) Rule d is the rule which brings about the indicated changes. (It subtracts 5, multiplies by 2, and adds 6.)

$$\square \xrightarrow{d} 2 \times (\square - 5) + 6$$

or $\square \xrightarrow{d} 2 \times \square - 4$

Page 8, problem 3

□ $\xrightarrow{a}$ □ − 5
□ $\xrightarrow{b}$ 3 × □
□ $\xrightarrow{c}$ □ + 5
□ $\xrightarrow{abc}$ ______

Solution: Since rule abc is formed by using rule a, then rule b, and then rule c, we can write rule abc directly from these:

subtract 5
then multiply by 3
then add 5

□ $\xrightarrow{abc}$ ((□ − 5) × 3) + 5

Simplifying, we get:

□ $\xrightarrow{abc}$ 3 × □ − 10

III.

Suggested discussion points:

Pages 6 and 7, problems 8 - 11

In each of these problems the first rule given is □ ⟶ □ + 5, and the four problems form a sequence. How can the work done in solving problem 8 be used to help solve problems 9, 10, and 11?

Page 6, problem 8

In problem 6 on page 5 rules gh and hg were the same. If you were to find rule fe using rules e and f of problem 5 on page 4, you would see that it is the same as rule ef. In this problem rule ab is □ ⟶ 2 × □ + 10. Is rule ba the same as rule ab? (No.) Why not?

Page 9, problem 8

□ $\xrightarrow{bcabb}$ ______
(Hint: The answer is not □ ⟶ 9 × □.)

What problem might have been done to get an answer of □ ⟶ 9 × □?

IV.

The following problems give practice in working with composition. Use these rules:

$\square \xrightarrow{a} \frac{1}{2} \times \square$ $\square \xrightarrow{d} 2 \times \square$

$\square \xrightarrow{b} \square + 2$ $\square \xrightarrow{e} 2 \times \square + 4$

$\square \xrightarrow{c} \square - 3$ $\square \xrightarrow{f} \square$

Make the equations below true for all numbers in the boxes by writing one or more letters. The first one has been done as a sample. There are many correct answers for each problem.

1. $\square \xrightarrow{dac} \; = \; \square \xrightarrow{c}$

2. $\square \xrightarrow{dbb} \; = \; \square \xrightarrow{\quad}$

3. $\square \xrightarrow{ebcc} \; = \; \square \xrightarrow{\quad}$

4. $\square \xrightarrow{\quad} \; = \; \square \xrightarrow{f}$

5. $\square \xrightarrow{bbbcc} \; = \; \square \xrightarrow{\quad}$

6. $\square \xrightarrow{badb} \; = \; \square \xrightarrow{\quad}$

Film Discussion Notes
"Introduction to Composition"

Preliminary information:

The class you will see is a fifth grade from the Phillips School in Watertown, Massachusetts. Although the students had worked only one hour with Mrs. Hermann prior to this class, they had worked during much of the year on Project topics, including jumping rules. Four students had had a Project teacher the preceding year and can be seen in the film "Some Artificial Operations". The class had not previously worked on composition. [Film running time: 35 min.]

Discussion after the film:

1. Early in the film the class was given the problem of finding rule b.

$$\square \xrightarrow{a} \square + 5$$
$$\square \xrightarrow{b} \underline{\hspace{5em}}$$
$$\square \xrightarrow{c} 2 \times \square + 10$$

David B. suggested the rule $\square \xrightarrow{b} 2 \times \square + 5$. How do you suppose he got that answer? Besides testing David's rule with some starting number, is there any other way to determine whether it is correct?

2. In the film many numbers were tried to check the rules. (Did the teacher try too many?) In time the children began to sense that they could continue selecting numbers indefinitely, but that this was not necessary, and that, in fact, trying only a few possibilities should "prove" that the rules would work for all numbers. At one point Michele said:

> You only have to do three things to find out if the rule is correct. I mean you don't have to take all numbers——well, maybe four. Take a plain number, a number with a fraction, just a plain fraction, and a negative, and if you try those you can probably tell by then if all of the numbers work.

Later on in the film, Danny said:

If we tried 1,804, that would prove it.

Each of these statements was an attempt to categorize the numbers in some way. Danny thought that large numbers might act differently from small numbers. Michele divided the numbers into rough categories of positive whole numbers ("plain numbers"), fractions, and negative numbers. The moderator may wish to discuss these statements and how participants might have responded if they had been teaching the class.

Since all the rules used in the film are linear rules, you need only two numbers that work to show that a certain rule is the composite of two other rules. These two numbers can be any kind of number you wish to choose: negative, positive, fractional, whole, large, or small. A brief proof of this is given in the Epilogue to the written lesson in Book 16.

Eventually students should be led to see that only two numbers are needed to test the rules. How would teachers plan to accomplish this? (Discussion of this question might be started now and continued next week.)

3. Several answers were given for the rule fge at the end of the film:

$$\square \xrightarrow{fge} (\square - 1) \times 5$$
$$\square \xrightarrow{fge} 5 \times \square - 7 + 2$$
$$\square \xrightarrow{fge} 5 \times \square - 5$$

How might a child in your class decide that these are different ways of writing the same rule?

4. After working with the rules a and b seen in the film, the teacher gave this problem that was omitted from the film:*

Given the rules $\left\{ \begin{array}{l} \square \xrightarrow{cd} 3 \times \square + 21 \\ \square \xrightarrow{dc} 3 \times \square + 7 \end{array} \right.$, find rules c and d:

$$\square \xrightarrow{c} \underline{\qquad\qquad}$$
$$\square \xrightarrow{d} \underline{\qquad\qquad}$$

*See the Summary of Problems for the Film "Introduction to Composition".

What are rules c and d ?* Would it have helped to write the problem like this:

$\square \xrightarrow{c}$ ______________

$\square \xrightarrow{d}$ ______________

$\square \xrightarrow{cd} 3 \times \square + 21$

$\square \xrightarrow{dc} 3 \times \square + 7$

Why or why not?

5. In doing the problem above, the rules $\square \longrightarrow 1\frac{1}{2} \times \square + 3\frac{1}{2}$ and $\square \longrightarrow 1\frac{1}{2} \times \square + 10\frac{1}{2}$ were suggested for rules c and d, respectively. Why might a student suggest such rules?

---

*There is more than one possibility for rules c and d. What would rule c be if rule d were $\square \longrightarrow \square - 7$?

Seventeenth Session
Written Lesson Discussion Notes

Possible discussion topics:

Page 5, problem 9

$$\begin{cases} \square + \triangle = 118 \\ \square - \triangle = 0 \end{cases}$$

If $\square - \triangle = 0$, what conclusion can you draw about $\square$ and $\triangle$? ($\square$ and $\triangle$ must be equal.) If $\square$ and $\triangle$ are the same, then the first equation can be rewritten as either $\square + \square = 118$ or $\triangle + \triangle = 118$. Now it is easy to find the number for box or wedge, depending on which equation you use.

Page 5, problem ☆12

$$\begin{cases} \square + \triangle = 36 \\ \square + \square + \triangle + \diamond = 96 \\ \square + \diamond + \diamond + \square + \triangle = 139 \end{cases}$$

If this is discussed the participants should be urged to give their methods. One approach follows.

Look at the second and third equations. If the third equation is rewritten as $\square + \square + \triangle + \diamond + \diamond = 139$, then the first four frames exactly match the frames of the second equation:

$$\square + \square + \triangle + \diamond = 96$$
$$\square + \square + \triangle + \diamond + \diamond = 139$$

Thus, the first four frames of the third equation can be replaced by 96. We have $96 + \diamond = 139$, or $\diamond = 43$. Can you use similar reasoning to determine numbers for box and wedge? (Hint: Use the first and second equation and the fact that $\diamond = 43$.)

For the problems on pages 14 through 18 it sometimes helps to look at the pattern of the numbers. For example, in the problem on page 15, the points given are (0,0) and (5,2). If we look at the first number in each pair we see a difference of 5; the second numbers have a difference of 2. Since the line is a straight line (all lines referred to in this lesson are straight lines), the difference must remain in the same ratio. (Why?) This can be summed up by using a table.

first number	second number
0	0
5	2
10	4
15	6

What happens if we want to know the coordinates of a point on the line when the first number is not a multiple of 5? If the first number increases (or decreases) by 5, then the second number increases (or decreases) by 2. How much will the second number increase if the first number increases by 1? By 2?

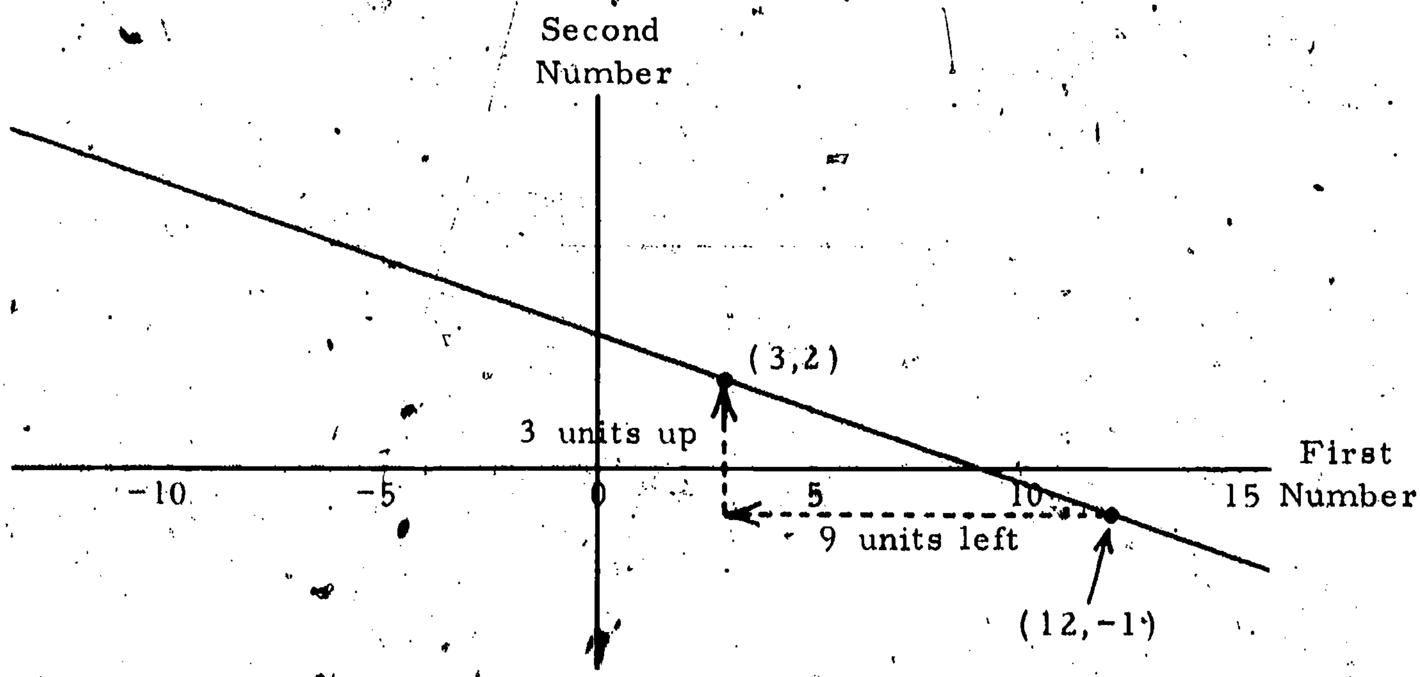
In problem ☆12 on page 19 the number lines are not numbered. Most participants will probably count each mark as one space. Some participants may use a different scale and therefore might have a different number pair. It may be worth discussing what happens if the lines are numbered other than by ones.

Note to moderator:

The problems on pages 14 through 18 are an attempt to give the participants an intuitive idea of slope. The slope of a line is defined as the ratio of how far up (or down) you move to how far right (or left) you move when you go from one point to another point on the same line. It turns out that the points you choose to find the slope of a line will not make any difference: If you pick two points and someone else picks two different points on the same line, then both of you will arrive at the same slope.

If you move down, that is considered a negative direction, and if you go left, that is also a negative direction. This means that some lines will have a positive slope and others will have a negative slope. Lines parallel to the first number line have a slope of zero. The slope of a line parallel to the second number line is undefined because division by zero is involved, and division by zero is undefined.

Here is an example of how one can find the slope of a line. We want to find the slope of the line that passes through the points (3 , 2) and (12 , −1). To get from (12 , −1) to (3 , 2) we must go 3 units up and 9 units left. Thus the slope is $\frac{3}{-9}$ or $-\frac{1}{3}$.



What is the slope of the line that passes through (4 , 3) and (12 , 5)?

You may want to draw some lines on a graph board and have the participants find the slope of the lines you have drawn.

Many geometrical theorems can be proven by using the idea of slope and some algebra. The branch of mathematics in which theorems are proven by this method is called analytic geometry.

(The slope of the line that passes through (4 , 3) and (12 , 5) is $\frac{1}{4}$.)

Film Discussion Notes
"Graphing Absolute Value Equations"

Preliminary information:

This is a second grade class from the James Russell Lowell School in Watertown, Massachusetts. The teacher is Mrs. Marie Hermann. The class had worked with Project teachers since the beginning of first grade. The students had done some graphing of equations and had worked with absolute value. [Film running time: 33 min.]

Note: You should have a graph board available for the discussion. You may want also to have graph paper for the participants. Participants should refer to Book 6 to refresh their memories concerning absolute value.

Discussion after the film:

In the film the students graphed $|\square| + \triangle = 5$ and found that it formed an upside-down V. The students then moved the upside-down V vertically, and they also flipped it over. There are two other ways to move the figure that would be worth discussing.

Can you rotate the upside-down V so that it looks like this (<) or like this (>)? Participants will probably not have much trouble predicting the answer to this question. Placing the absolute value bars around the wedge will rotate the picture. The graph of $\square + |\triangle| = 5$ will look like >. The graph of $\square - |\triangle| = 5$ will look like <.

Can you move the upside down V left or right? This is a harder question and various suggestions should be tried. The equation $|\square - 3| + \triangle = 5$ will move the graph 3 units right. To move left 3 units, add 3 rather than subtract 3.

Here are other equations that your group may profit from graphing.

$|2 \times \square| + \triangle = 6$

$|\square| + 2 \times \triangle = 6$

$|\square + \triangle| = 5$

$|\square| + |\triangle| = 5$

$|2 \times \square + \triangle| = 5$

$||\square - 3| - 5| = \triangle$

$||\square - 5| - 3| = \triangle$

Eighteenth Session

Written Lesson Discussion Notes

Some questions for discussion:

Page 3, problem ☆6(c)

> No computing. Starting at (6 , −3), do you think you could land at (3,089 , 187)? ______ Comment:

It's impossible to get from (6 , −3) to any point whose coordinates are both even or both odd because rules a , b , c and d change the parity (evenness or oddness) of either both coordinates or else neither coordinate. But is it possible to get from (6 , −3) to every other point with whole-number coordinates? Try to get from (6 , −3) to (8 , −3). What is a convincing argument that you can (or cannot) get from (6 , −3) to (3,429,817 , −4,216), for example? (Hint: In answering these questions there is no need to consider rule d at all.)

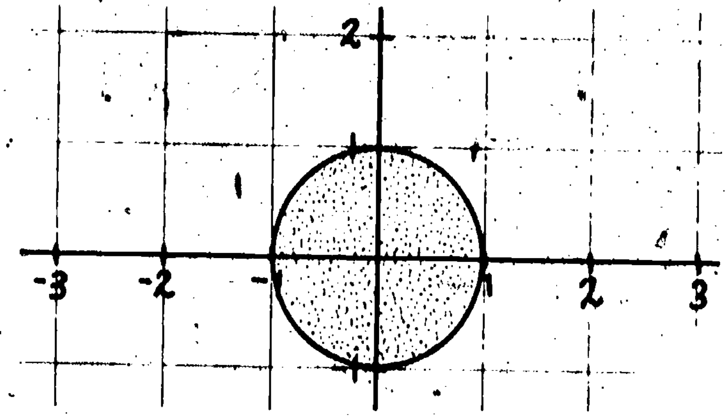
Page 5, problem 1

> Start at (10 , 4) and make a jump with rule A ; another jump with rule A , a jump with rule B , and a jump with rule C . Plot the landing points below, connect successive points with straight lines, and find the area of the resulting figure.

Is making a jump with rule A , followed by another jump with rule A , followed by a jump with rule B , and then followed by a jump with rule C the same as making a single jump with the composite rule AABC ? (No. The intermediate landing points, not just the final landing point, are crucial to this problem; the composite rule is $(\square,\triangle) \xrightarrow{AABC} (\square,\triangle)$, which doesn't go anywhere.)

Page 7, problem 2

Give a starting point within or on the edge of the shaded disc shown below so that eventually successive jumps will take you to (-8 , -8).



(a) Could you get to (-8 , -8) if you were not allowed to start on the boundary of a region? (Yes. Start on ($-\frac{1}{2}$, $-\frac{1}{2}$), for example.)

(b) Could you get to (-8 , -8) if you had to start above the first number line and inside the circle? (Yes. Start on ($-\frac{5}{16}$, $\frac{3}{16}$), for example.)

(c) Could you get to (-8 , -8) if you had to start above the first number line and also to the right of the second number line? (No. Both numbers would be positive and the rule would keep taking you to points where both coordinates are positive.)

(d) What are all the starting points that satisfy the conditions of the problem? (All points inside or on the circle and which have coordinates whose sum is -1 or the opposite of some power of $\frac{1}{2}$, such as $-\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{1}{8}$, etc.; stated efficiently, all points (□ , △) such that

$$\square + \triangle = -\left(\frac{1}{2}\right)^n \quad \text{for some integer } n$$

and where $$\square^2 + \triangle^2 \leq 1 \text{)}$$

Page 8, problem 3

> Continue with the rule: (□ , △) → (□ + △ , □ + △)
>
> Give all standstill points:
>
> ______________________________
>
> Prove that you have them all:

If you play around with the rule, it is pretty easy to find that (0 , 0) is a standstill point. If you experiment further you may feel sure that (0 , 0) is the only standstill point. Now give a proof! (How rigorous and detailed should such a proof be?)

SAMPLE PROOF

First note that a point (□ , △) and a point (◇ , ✿) turn out to be exactly the same point when the two first numbers are the same, □ = ◇ , and the two second numbers are the same, △ = ✿ . Furthermore, the points are the same only when □ = ◇ and △ = ✿ .

Let's try starting points other than (0 , 0) as possible standstill points.

I. One way a point (□ , △) can be different from (0 , 0) is for the first number □ to be different from the first number 0 . If □ is not 0 , then we can show that the landing point (□ + △ , □ + △), as given by the rule, is different from the starting point (□ , △). We can show that (□ + △ , □ + △) is different from (□ , △) by comparing the two second numbers, □ + △ and △ . Since we are assuming that □ does not equal 0 , we know that adding □ to △ will change the △ to □ + △ , which is different from just △ .

So we have shown that if the first number □ of a starting point (□ , △) is different from 0 , the landing point (□ + △ , □ + △) is different from the starting point; for this case, where $□ \neq 0$, we cannot have a standstill point.

II. Another way a point (□ , △) can be different from (0 , 0) is for the second number, △ , to be different from 0 . If △ is not 0 , then we can show that the landing point (□ + △ , □ + △), as given by the rule, is different from the starting point (□ , △). We show this by comparing the two first numbers, □ + △ and □ . Adding a number △ which is not 0 to a number □ will make □ + △ come out different from just plain □ .

So now we have shown that if the second number $\triangle$ of a starting point $(\square, \triangle)$ is different from 0, the landing point $(\square + \triangle, \square + \triangle)$ is different from the starting point; for this case, where $\triangle \neq 0$, we cannot have a standstill point.

III. Combining the results of I and II, we have that if either $\square$ or $\triangle$ is not 0, then the starting point $(\square, \triangle)$ is not a standstill point. So the only way to have $(\square, \triangle)$ be a standstill point is for $\square$ to be 0 and also for $\triangle$ to be 0.

We have proved that (0, 0) is the only standstill point for the rule $(\square, \triangle) \longrightarrow (\square + \triangle, \square + \triangle)$. There is certainly no point in going into such details with an elementary school child. But you might find an occasional child who would say something like, "Well, unless both starting numbers are zero, then when you add them the way the rule says you'll get a different two numbers from the ones you started with, because adding them changes the numbers." Such a child understands the essence of the argument.

* * *

A moderator may want to bring up some or all of the points in either or both of the next two paragraphs.

A jumping rule is called <u>one-to-one</u> if no landing point comes from more than one starting point. The jumping rule on page 6,

$$(\square, \triangle) \longrightarrow (\square - \triangle, \square - \triangle),$$

is not a one-to-one jumping rule since both $(8, 6) \longrightarrow (2, 2)$ and $(100, 98) \longrightarrow (2, 2)$. Jumping rule A on page 5,

$$(\square, \triangle) \longrightarrow (\square + 3, \triangle - 4),$$

is a one-to-one jumping rule: each landing point comes from the starting point 3 spaces to the left and 4 spaces higher, and from no other starting point. Which of the other rules in the lesson are one-to-one? (All the rules on pages 1 through 5; the rule $(\square, \triangle) \longrightarrow (\square + 2 \times \triangle, \triangle)$ on page 10; the rules which participants will probably write on pages 11 and 12.)

Some of the rules in the lesson have inverses and others do not. Recall that the inverse of a rule is a rule that sends each landing point back to its starting point. For example, the inverse of $(\square, \triangle) \longrightarrow (\square + 3, \triangle - 4)$ is the rule $(\square, \triangle) \longrightarrow (\square - 3, \triangle + 4)$. The composite of these two rules is $(\square, \triangle) \longrightarrow (\square, \triangle)$. The rule $(\square, \triangle) \longrightarrow (\square - \triangle, \square - \triangle)$ does not have an inverse, for the inverse would have to send $(2, 2)$ both to $(8, 6)$ and also to $(100, 98)$, and to infinitely many other points as well. A jumping rule cannot do this! Which rules in the lesson have inverses and which do not? (The rules which have inverses are exactly the ones which are one-to-one. The inverse of $(\square, \triangle) \longrightarrow (\square + 2 \times \triangle, \triangle)$ is the rule $(\square, \triangle) \longrightarrow (\square - 2 \times \triangle, \triangle)$, which may come as a surprise.)

Book 18

Film Discussion Notes
"Jumping Rules in the Plane, Part I"

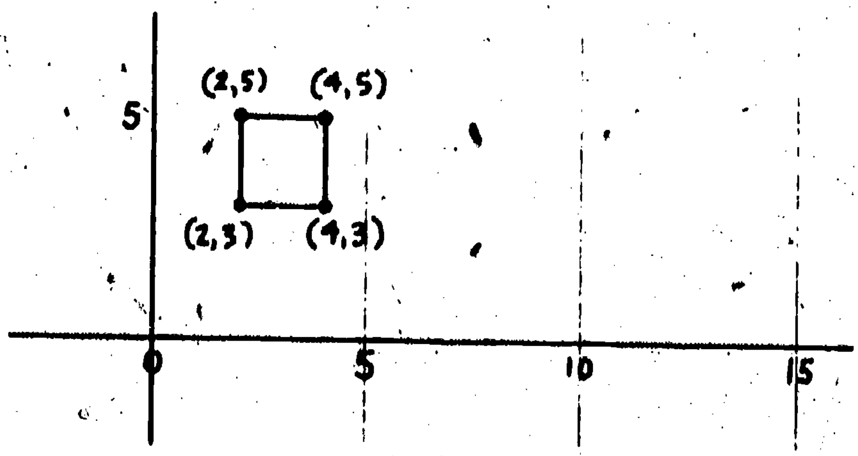
A large graph board should be on hand for discussion of this and the two subsequent films.

Preliminary information:

The class you are about to see is a sixth grade group from the James Russell Lowell School in Watertown, Massachusetts. The teacher is Lee Osburn. Some students had worked with Project materials in fourth or fifth grade. This class had worked with various Project topics for about eight weeks prior to filming. The pupils were already familiar with jumping rules on a line, plotting points in the plane, and graphing equations in the plane. They had not worked with jumping rules in the plane. The film was made in the spring of the year. [Film running time: 25 min.]

1. Near the end of this film, the students were predicting that the area of the given square would not change with the rule $(\square, \triangle) \rightarrow (\square \times 3, \triangle)$ since multiplying by 3 would just move the square three times as much. Their arguments are convincing, but incorrect. Why?

2. Institute participants have come up with various questions which can be asked about how to move figures in the plane and how their areas change. Do encourage your participants to formulate problems for themselves and their classes, and perhaps to begin to solve them during discussions. Sample questions from participants follow. All questions refer to the original square as given in the film.



2. (cont.)

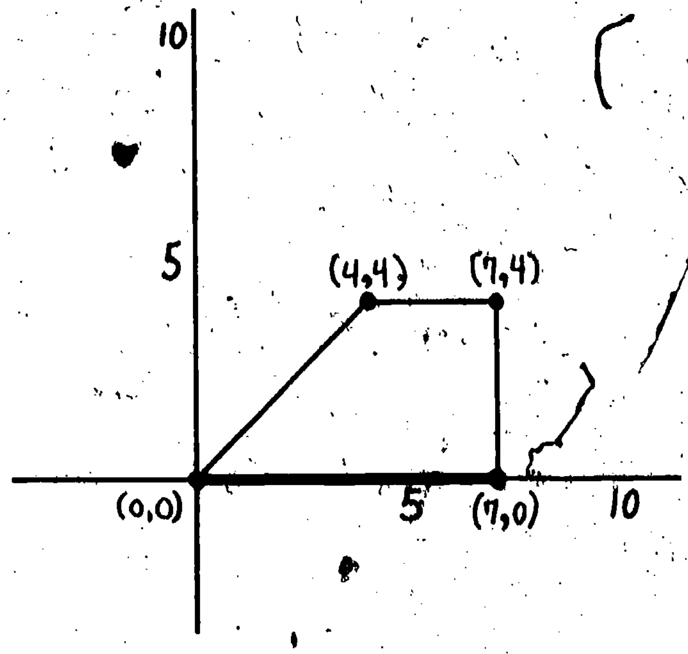
a. Write a rule which makes the area 9 times bigger, and keeps the figure a square.

b. Write a rule which makes the area 25 times bigger, and still yields a square.

c. Write a rule so that the resulting figure is a square whose area is 9 square units.

d. Write a rule which makes the resulting figure a rectangle which is $\frac{1}{4}$ as big (one square unit instead of 4 square units), and is situated inside the original square.

3. Consider using a number plane jumping rule on a figure other than a square: for example, the following trapezoid.



a. Using the rule (□ , △) ⟶ (□ ÷ 2 , △ ÷ 2) predict what the resulting figure will look like. Put some of the points through the rule and check your predictions.

3. (cont.)

b. Predict the shape and area when you put the points of the given trapezoid through this rule: $(\square , \triangle) \longrightarrow (\square \times 3 , \frac{1}{2} \times \triangle)$ Check your predictions.

c. Write a rule that will change the original trapezoid to a figure that is taller and narrower than it is now.

See the following page for possible answers to some of these questions. If there is any doubt about a rule which someone in your group proposes, you should, of course, check out some of the points to convince yourselves of what is happening.

Sample Answers and Hints

2a. The rule (□ , △) ⟶ (3 × □ , 3 × △) will fit the requirements. So will this one, which places the resulting square in the third quadrant: (□ , △) ⟶ (−3 × □ , −3 × △)

2c. Since you want the area to increase from 4 square units to 9 square units, you need to make the area of the figure $2\frac{1}{4}$ times bigger. But you also need to stretch the sides uniformly to keep the figure a square. The rules (□ , △) ⟶ ($1\frac{1}{2}$ × □ , $1\frac{1}{2}$ × △) and (□ , △) ⟶ ($-1\frac{1}{2}$ × □ , $-1\frac{1}{2}$ × △) will both work.

2d. One method of attack is to shrink the area by dividing the box or the wedge part of the rule by 4. Then add or subtract to move the rectangle inside the original square.

3c. To make the trapezoid narrower, one needs to shrink it horizontally by dividing the box component by some number bigger than 1 (which, of course, is the same as multiplying by a number between 0 and 1). To make the trapezoid taller, one must multiply the wedge component by some number bigger than 1. One such rule is (□ , △) ⟶ (□ ÷ 7 , △ × 2).

Of course, many other rules do what the problem requires.

Nineteenth Session
Written Lesson Discussion Notes

The moderator should read these notes before the discussion begins, if he plans to use them. A large graph board (or graph paper and an overhead projector) should be available. Do not attempt to "cover" all the suggestions. [Some hints and answers appear in brackets.]

Section I of the written lesson shows just a few of the many variants of the first rule, $(\square, \triangle) \longrightarrow (\square + 1, 10 - \triangle)$. Participants should be encouraged to find others.

(a) How can you change the rule so that the farther you <u>are</u> to the right, the farther the jump <u>goes</u> to the right?

[You don't want to add a constant amount to the first component. Try $(\square, \triangle) \longrightarrow (\square \times 2, 10 - \triangle)$.]

Start at (2 , 9) and draw several consecutive jumps. What happens if you start at (−2 , 9)? Where are the standstill points? [There is only one.] Is this rule an isometry? [No.]

(b) How can you change the rule so that successive jumps get closer and closer to the second (vertical) number line?

[Try $(\square, \triangle) \longrightarrow (\square \times \frac{1}{2}, 10 - \triangle)$.]

Again make several consecutive jumps from (2 , 9) and from (−2 , 9). Standstill points? [There is only one.] Is this rule an isometry?

(c) How can you further modify the rule $(\square, \triangle) \longrightarrow (\frac{1}{2} \times \square, 10 - \triangle)$ so that the vertical distance of each successive jump gets shorter and shorter and so that the landing points get closer and closer to (0 , 5)? A good guess would be $(\square, \triangle) \longrightarrow (\frac{1}{2} \times \square, \frac{1}{2} \times (10 - \triangle))$. Start at (20 , 10) and make four or five consecutive jumps. Surprisingly enough, successive landing points are <u>not</u> approaching (0 , 5). What is the standstill point of $(\square, \triangle) \longrightarrow (\frac{1}{2} \times \square, \frac{1}{2} \times (10 - \triangle))$? Replace the 10 by some other number so that the standstill point <u>is</u> (0 , 5). Now make four or five consecutive jumps starting at (20 , 10). What point do you seem to be approaching? What happens if you start at (−20 , 10)? What happens if you start at (0 , 10)?

(d) Another variation on $(\square, \triangle) \longrightarrow (\frac{1}{2}\times\square, 10 - \triangle)$ is the rule $(\square, \triangle) \longrightarrow (-\frac{1}{2}\times\square, 10 - \triangle)$. Try to predict what will happen if you start at (20, 10) and make several consecutive jumps. If you draw the arrows as straight as you can, an interesting pattern of intersecting arrows emerges, which will enable you to draw as many jumps as you want without doing any computation.

(e) Now use the rule $(\square, \triangle) \longrightarrow (-\frac{1}{2}\times\square, \frac{1}{2}\times(10 - \triangle))$. Again try to predict what will happen if you start at (20, 10) and make several consecutive jumps. Draw the arrows as straight as you can. What is happening?

(f) If we change our original rule, $(\square, \triangle) \longrightarrow (\square + 1, 10 - \triangle)$, by replacing the first component by its opposite, we get the rule $(\square, \triangle) \longrightarrow (-(\square + 1), 10 - \triangle)$. Start at (2, 9) and make several consecutive jumps. Start at (2, 8), or at (3, 9), or anywhere else and make consecutive jumps. How could you tell that the rule $(\square, \triangle) \longrightarrow (-(\square + 1), 10 - \triangle)$ is an isometry, before making any jumps?

[The rule is the composite of $(\square, \triangle) \longrightarrow (\square + 1, \triangle)$ followed by $(\square, \triangle) \longrightarrow (\square, 10 - \triangle)$ followed by $(\square, \triangle) \longrightarrow (-\square, \triangle)$. Each of these rules is an isometry; so the composite is also.]

What kind of isometry is it? What are the standstill points?

(g) How can the original rule be changed so that the zig-zag pattern of consecutive jumps will look like this:

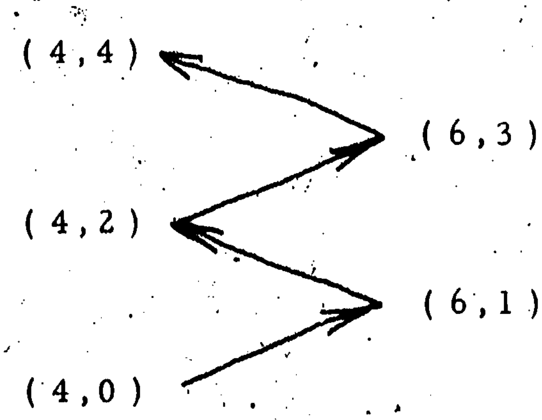


Fig. 1

or like this:

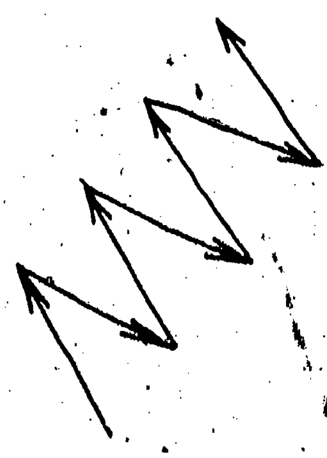


Fig. 2

[Follow a translation with a reflection whose axis of symmetry points in the same direction as the direction of the translation. To get the jumps of Figure 1, first use the translation (□ , △) ⟶ (□ , △ + 1), which moves points straight up. The reflection (□ , △) ⟶ (10 − □ , △) flips the plane around the vertical line five spaces to the right of (0 , 0). The composite of these rules, (□ , △) ⟶ (10 − □ , △ + 1), gives the jumps of Figure 1. Participants may suggest a rule like (□ , △) ⟶ (10 − □ , △ + 1) because of its similarity to the rule (□ , △) ⟶ (□ + 1 , 10 − △). To get the jumps shown in Figure 2, we need a translation which moves points diagonally, together with a reflection about a diagonal line. The composite of (□ , △) ⟶ (□ + 1 , △ + 1) and (□ , △) ⟶ (△ , □) will do it.]

* * *

It would be worthwhile for the moderator to compare isometries in the plane with isometries on the line, where the situation is similar but simpler. Just as in the plane, an isometry on the line is a jumping rule which preserves distances, although now, of course, we are talking about number line jumping rules.

Rules like □ ⟶ □ + 3 and □ ⟶ □ − 4 are translations. No translation except □ ⟶ □ has any fixed points. (All points on the line are fixed points for □ ⟶ □.) Just as in the plane, the composite of two translations is also a translation.

The only other isometries of the line are the reflections around a point, rules like □ ⟶ 10 − □ and □ ⟶ −8 − □ and □ ⟶ −□. (Sometimes these rules are called "flip-flop" rules.) Each reflection has a single standstill point; for the three examples above they are 5, −4, and 0, respectively. A reflection can just as well be thought of as a half turn.

The composite of two reflections is not a reflection. Consider the two reflections

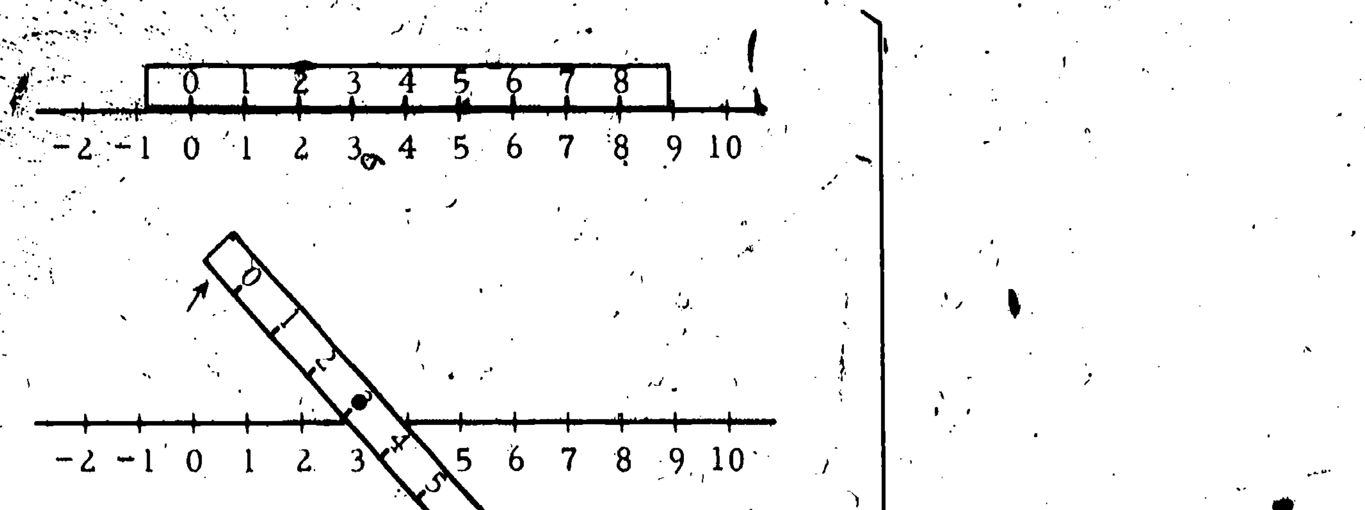
$$\square \xrightarrow{a} 12 - \square$$

and

$$\square \xrightarrow{b} 4 - \square$$

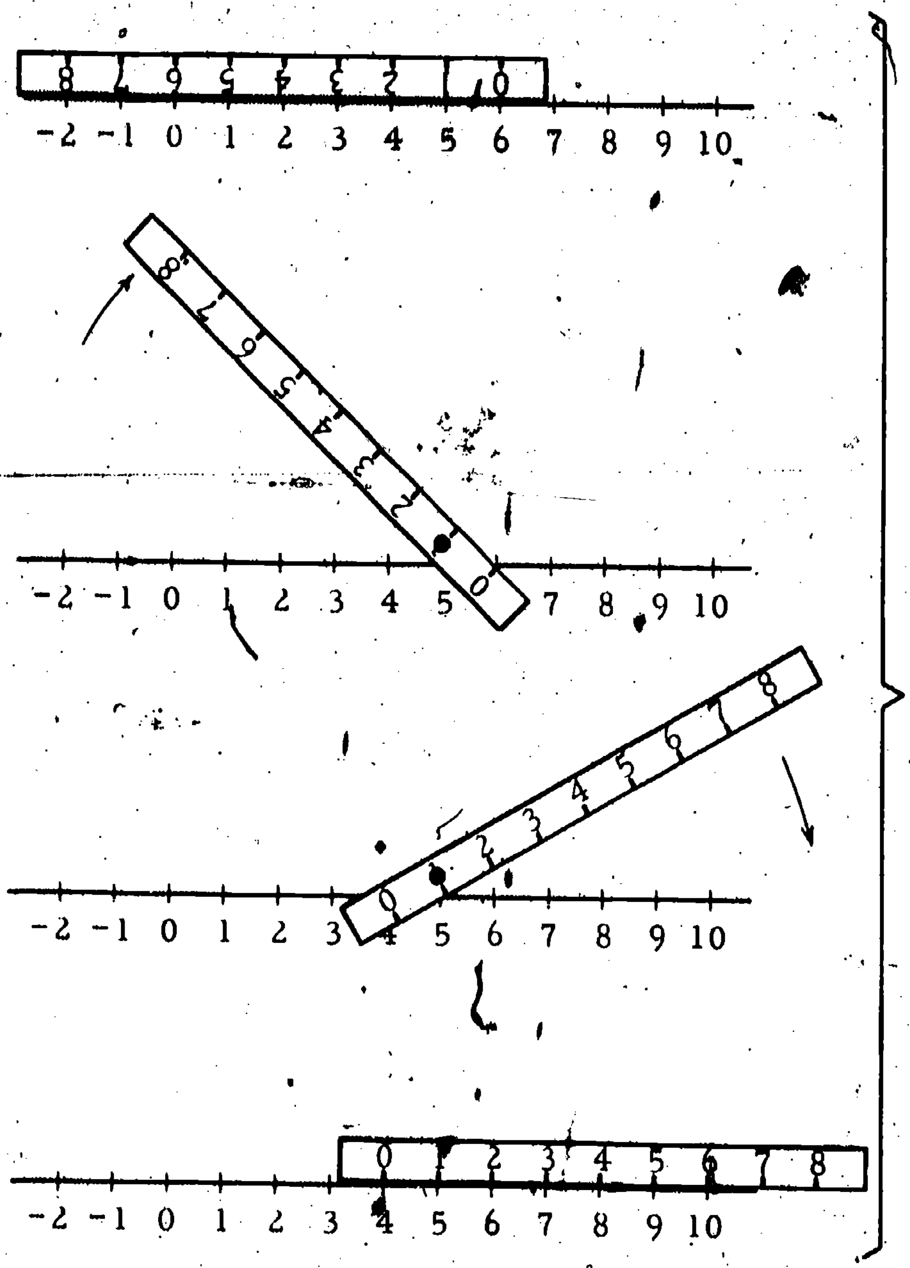
The composite ab is $\square \xrightarrow{ab} 4 - (12 - \square)$, or $\square \xrightarrow{ab} \square - 8$, which is a translation and not a reflection. The composite ba is $\square \xrightarrow{ba} \square + 8$, another translation, but different from ab.

Another way of showing what isometries of the line do is by using a long, rigid stick, like a yardstick or a blackboard pointer. A number line should be marked on the rigid stick to match another number line drawn on the blackboard. Translations are illustrated simply by sliding the stick to the right or the left; reflections are shown by giving the stick a half turn while keeping the standstill point fixed. Starting numbers appear on the stick and landing numbers on the blackboard. The drawings below and on the next page indicate how the reflection □ ⟶ 6 − □ followed by the reflection □ ⟶ 10 − □ would be done. Since the starting points for the second rule are the landing points for the first rule, we pivot around 5 on the number line when using the second rule.



(Diagram continued on next page.)

Diagram continued from page 146:



□ ⟶ 10 − □

Standstill point is 5

Composite rule: □ ⟶ □ + 4

Advantages of the stick method:

1. It shows immediately why the composite of an odd number of reflections is a reflection and why the composite of an even number of reflections is a translation: just count the number of times the stick gets flipped.

2. It encourages participants to consider what isometries do to the <u>entire line</u>, not just to individual points on the line.

It's somewhat harder mechanically to do the same sort of thing with the entire plane, but certainly small figures cut out of cardboard can be translated, rotated or reflected against a graph board background.

Film Discussion Notes

"Jumping Rules in the Plane, Part II"

A large graph board should be on hand for discussion of this film.

Preliminary information:

This is the second part of the same class which was seen last session. The students are sixth graders from the James Russell Lowell School in Watertown, Massachusetts. The teacher is Lee Osburn.

[Film running time: 25 min.]

Discussion after the film:

Participant A: I don't understand how to get the answer to the last question in the film.

Participant B: I think I can help. You want to use the rule on the board:

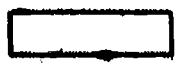
$$(\square, \triangle) \longrightarrow (2 \times \square, \frac{1}{2} \times \triangle)$$

Draw a rectangle someplace on the graph and put its points through the rule. You want the resulting figure to be a square.

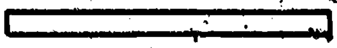
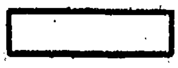
Participant A: Will a rectangle that is 1 by 4 do it?

Participant C: I figure that the 2 times box in the rule will stretch it out to be 2 units wide, and the $\frac{1}{2}$ times wedge would shrink the 4 units to 2 units. The square you get will be 2 units by 2 units.

Moderator: Perhaps a 4 by 1 rectangle would do it too.



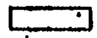
Does it?

Participant D: No, that rectangle won't. The resulting figure will just be a narrower rectangle, probably shaped like this: 
If you want to use this figure  and get a square, you'd have to change the rule to $(\square, \triangle) \longrightarrow (\frac{1}{2} \times \square, 2 \times \triangle)$.

Participant F: Does it matter where on the graph you put that rectangle? Can you put it any place you want, as long as it is 1 unit by 4 units?

Moderator: Yes, you can.

Participant C: I think that a rectangle that is 2 units by 8 units will also become a square.

Participant E: Probably any rectangle whose sides have a ratio of 1 to 4 and which is placed this way  rather than this way  will be all right. Even $\frac{1}{2}$ by 2 will work.

Moderator: Yes, you are correct.

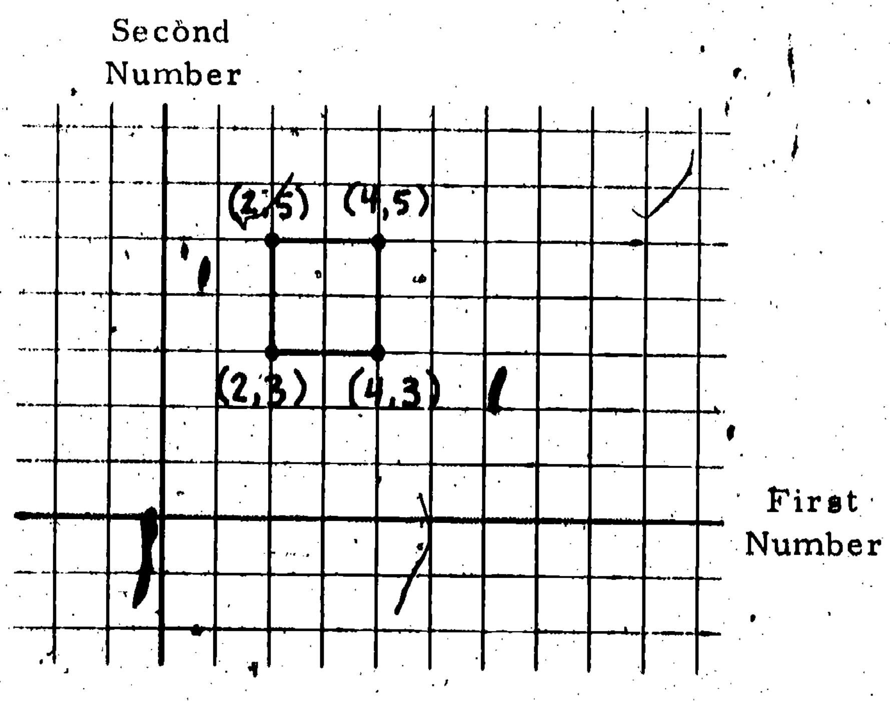
(Note to moderator: At this point, other questions can be asked. Draw a rectangle so that after you use the given rule twice you get a square. Suppose you had a rectangle whose sides were in the ratio of 1 to 64. How many times would you have to apply the given rule $(\square, \triangle) \longrightarrow (2 \times \square, \frac{1}{2} \times \triangle)$ so that you would get a square? Same problem with a rectangle whose sides are in the ratio of 1 to 32.)

Participant G: By the way, were those rules we saw as the teacher walked around the class really written by the students?

Moderator: Yes, they were copied precisely from the students' notebooks.

* * *

Moderator: Do you recall the problem of writing a rule that made the area of the given square 15 times bigger and put the figure in the third section of the plane?



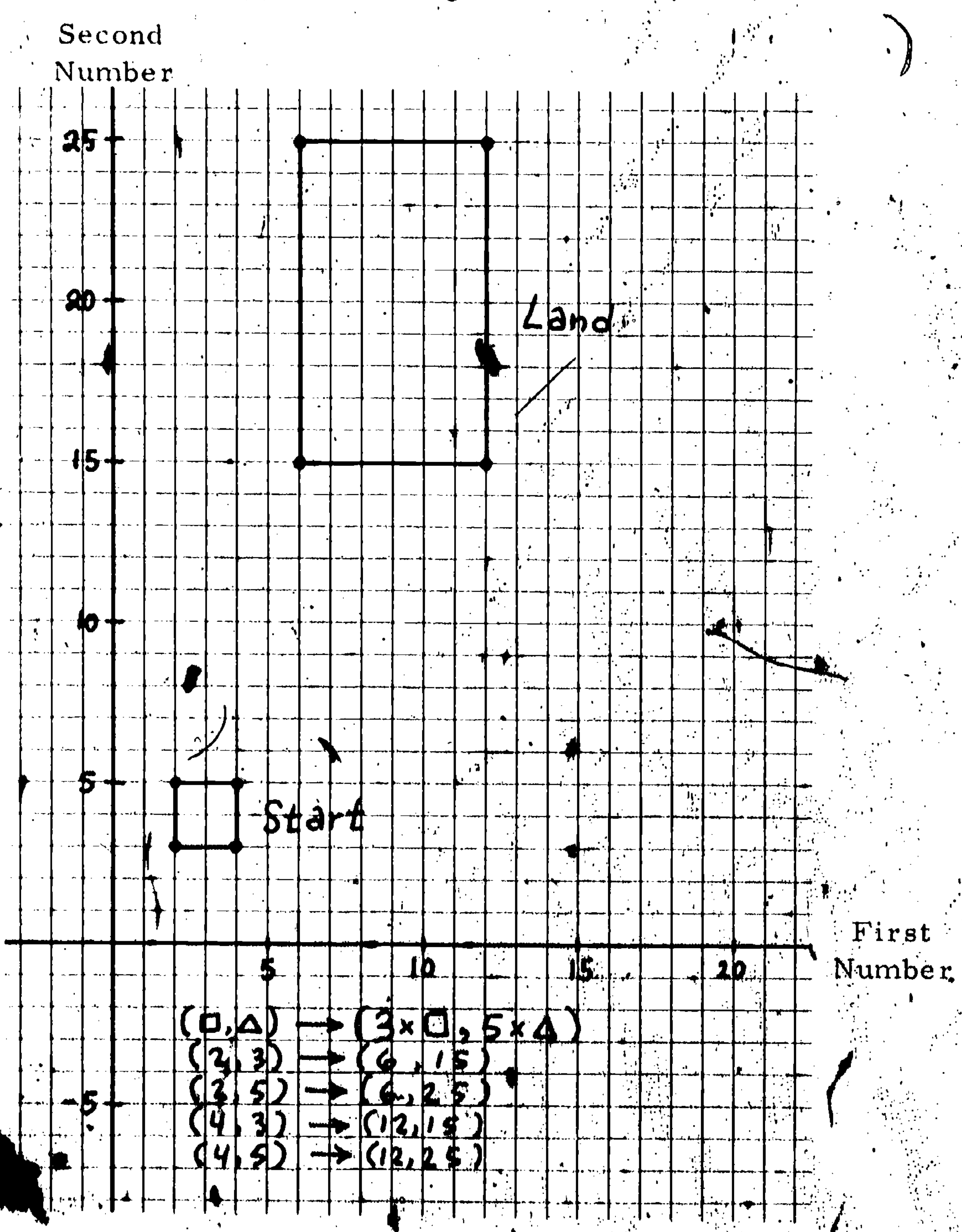
Moderator: The class wrote one rule which satisfied the conditions. It was: $(\square, \triangle) \longrightarrow (15 \times -\square, -\triangle)$.

But there are other rules which will do it, too. Does anyone have an idea of what such a rule might be?

Participant H: I think that you have to make the figure 15 times bigger by multiplying, and then move it down and to the left by subtracting.

Moderator: What would your rule look like?

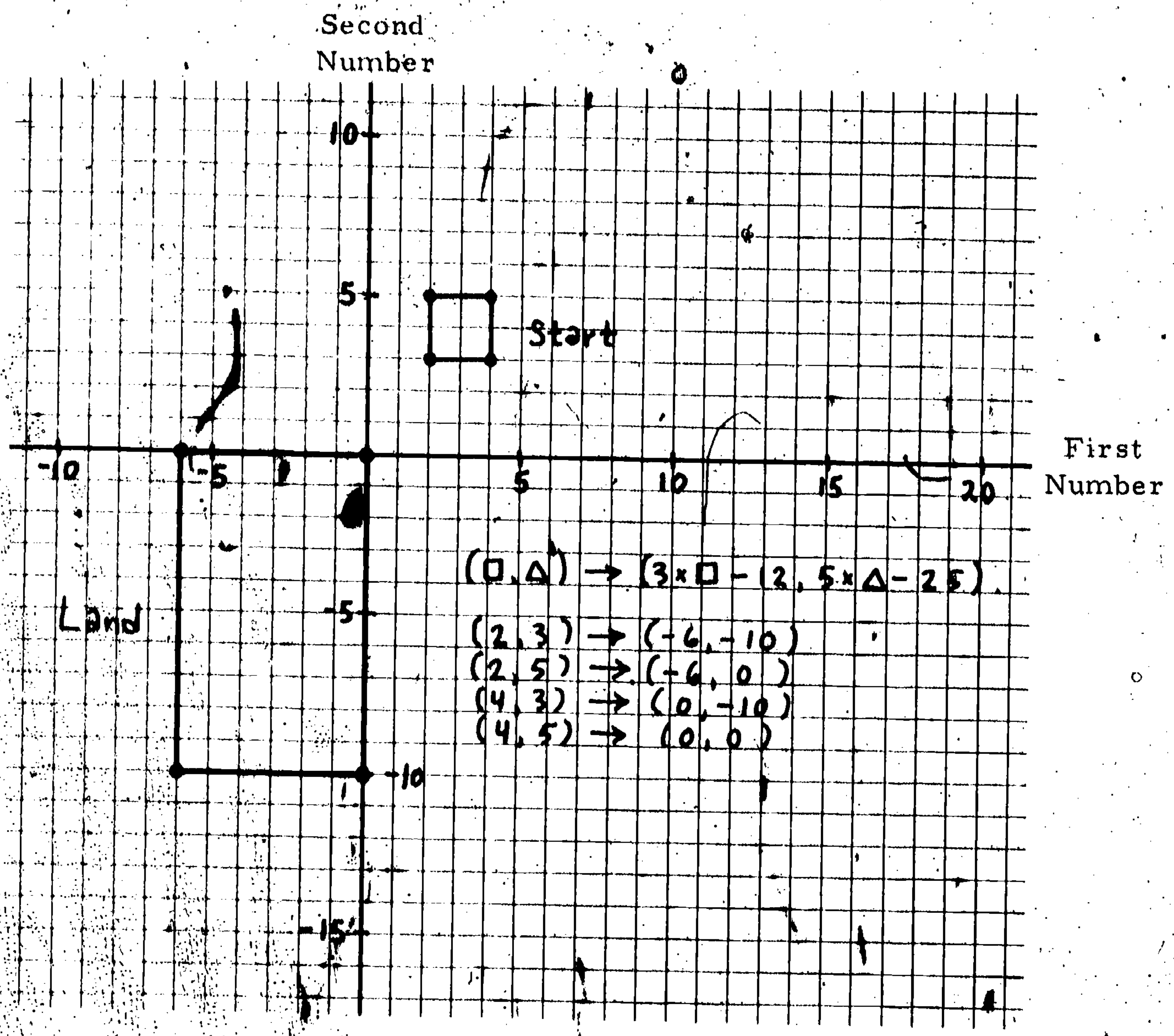
Participant H: $(\square, \triangle) \longrightarrow (3 \times \square, 5 \times \triangle)$ would make the area 15 times bigger. The figure would look like this:



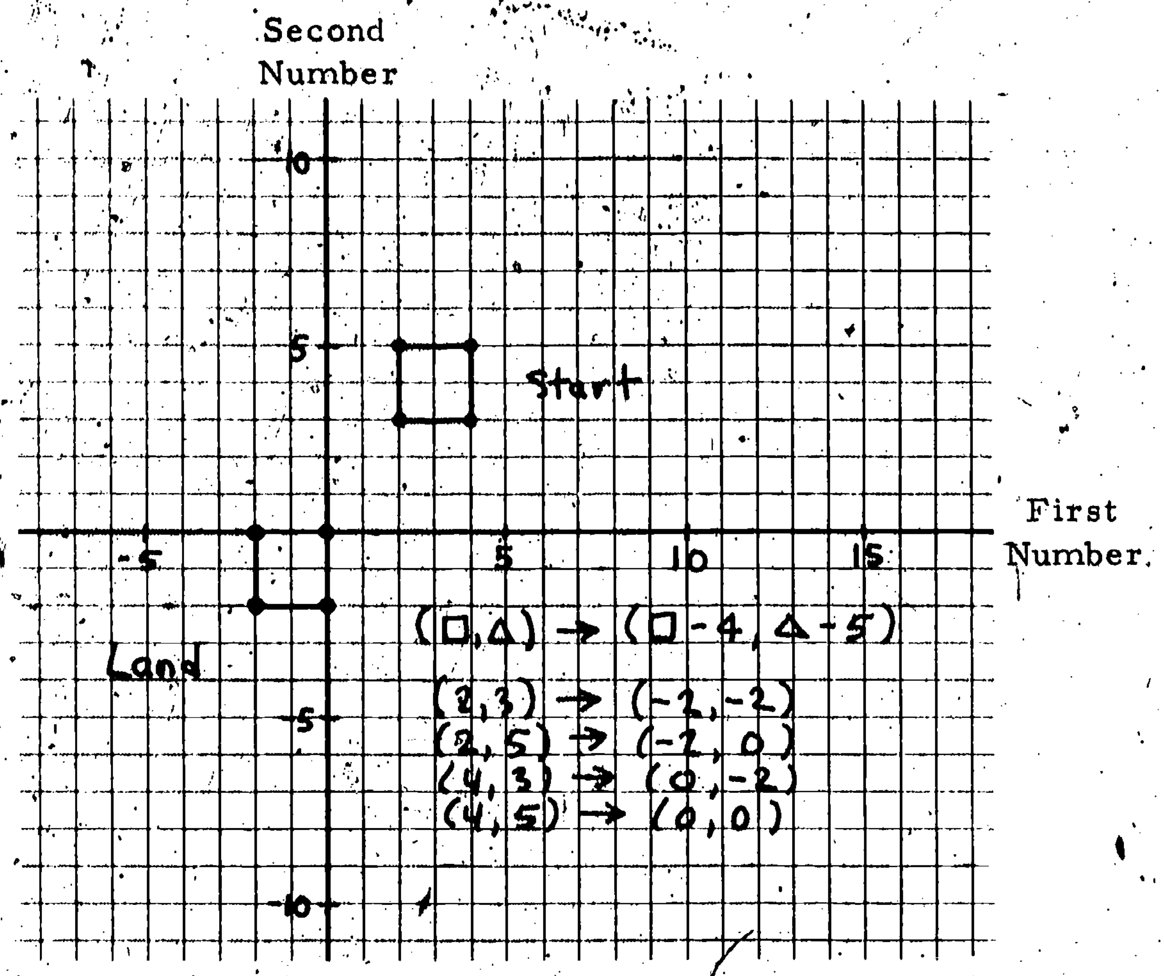
Participant H: Then I would move it left by subtracting at least 12 from the first component, and I would need to move it down by subtracting at least 25 from the second component. So if I made the rule

$$(\square, \triangle) \rightarrow (3 \times \square - 12, 5 \times \triangle - 25)$$

the new figure would look like this:



Participant J: You can move the figure down first by subtracting, and then you could multiply to get it 15 times bigger.
(□ , △) ⟶ (□ - 4 , △ - 5) will move it down like this:



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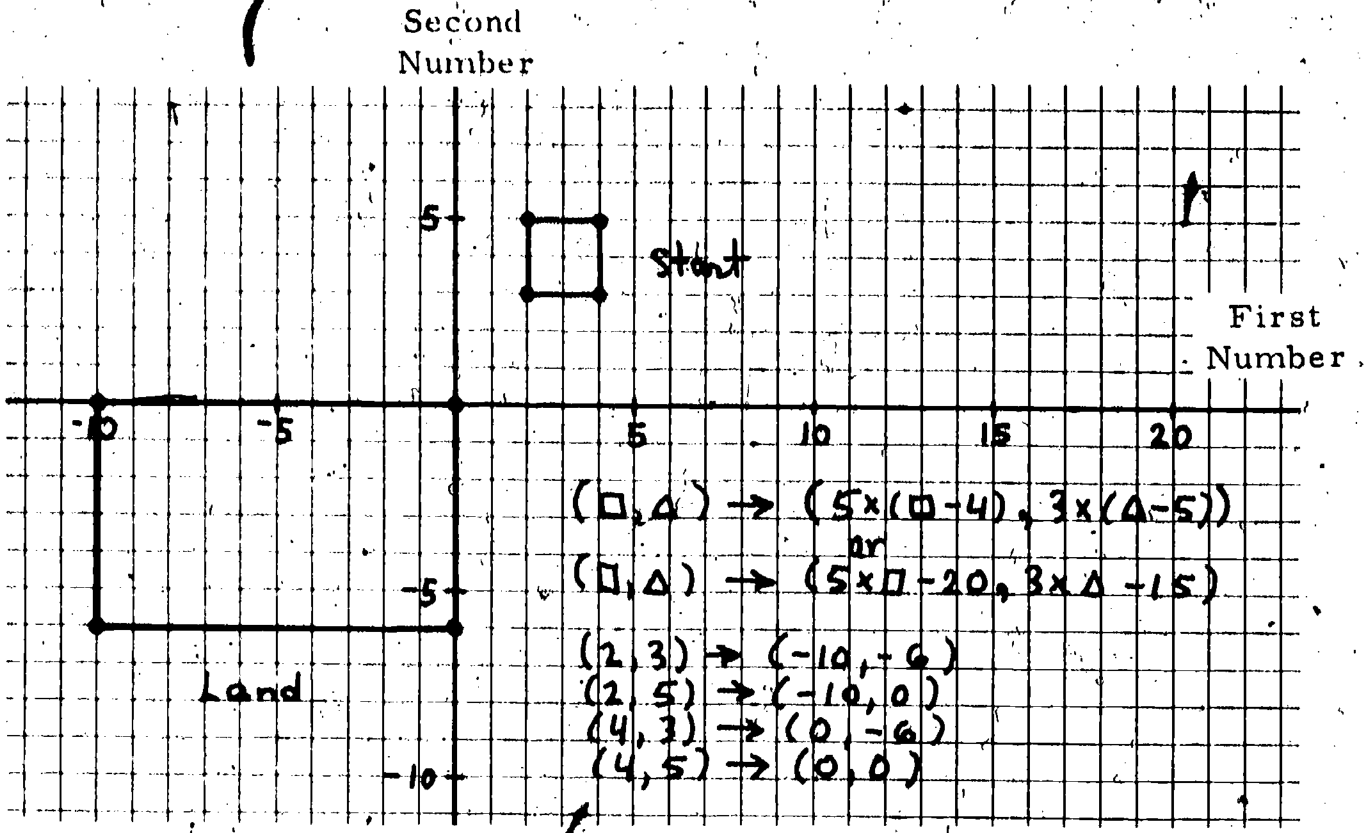
Participant A: And multiplying will make the area 15 times bigger:



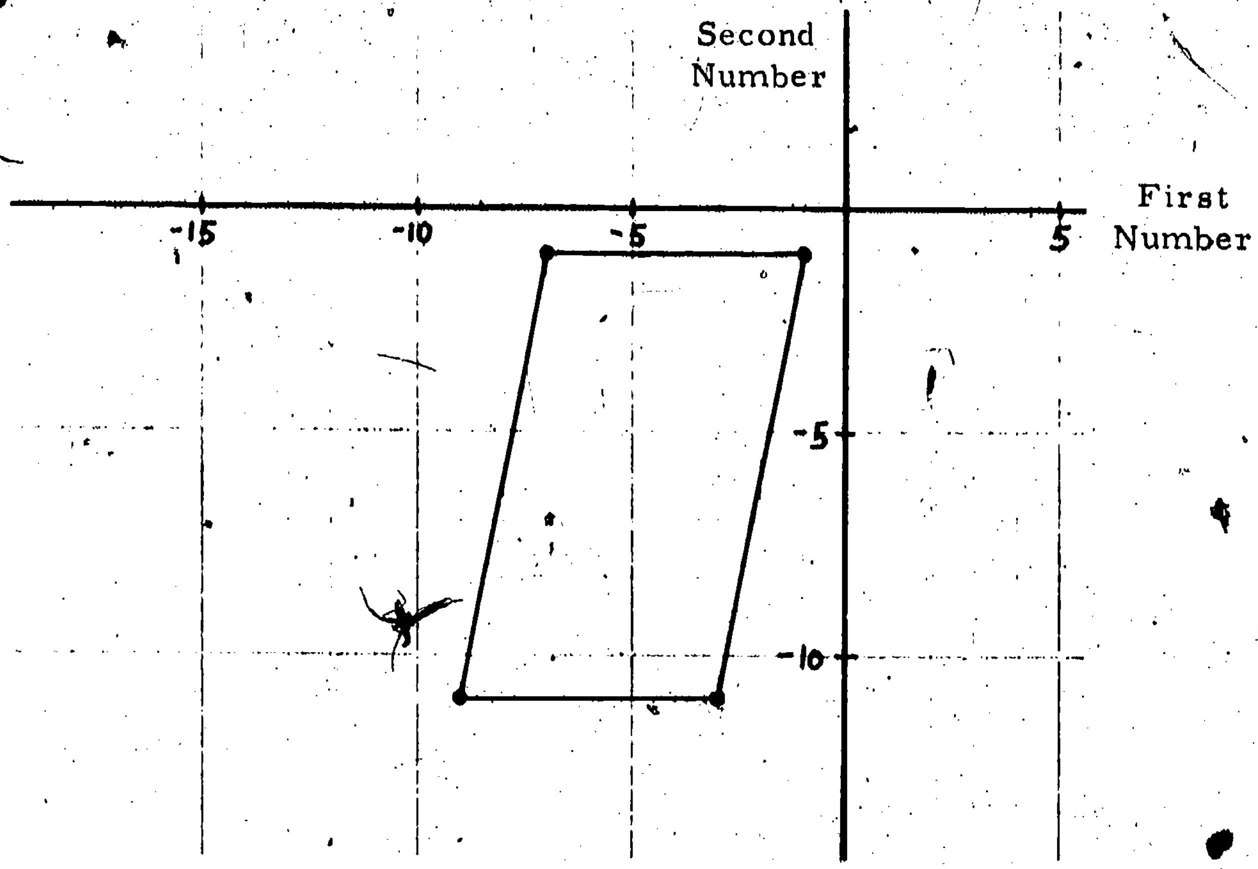
Note to moderator: Since there are many rules which will fulfill the conditions of this problem, here are some general guidelines for what to look for in the rules proposed in the discussion.

If the rule is of the form $(\square, \triangle) \longrightarrow (a \times \square + b, c \times \square + d)$, where a, b, c and d are numbers, $|a \times c|$ must be 15, and b and d must be chosen so as to slide the rectangle into the third quadrant.

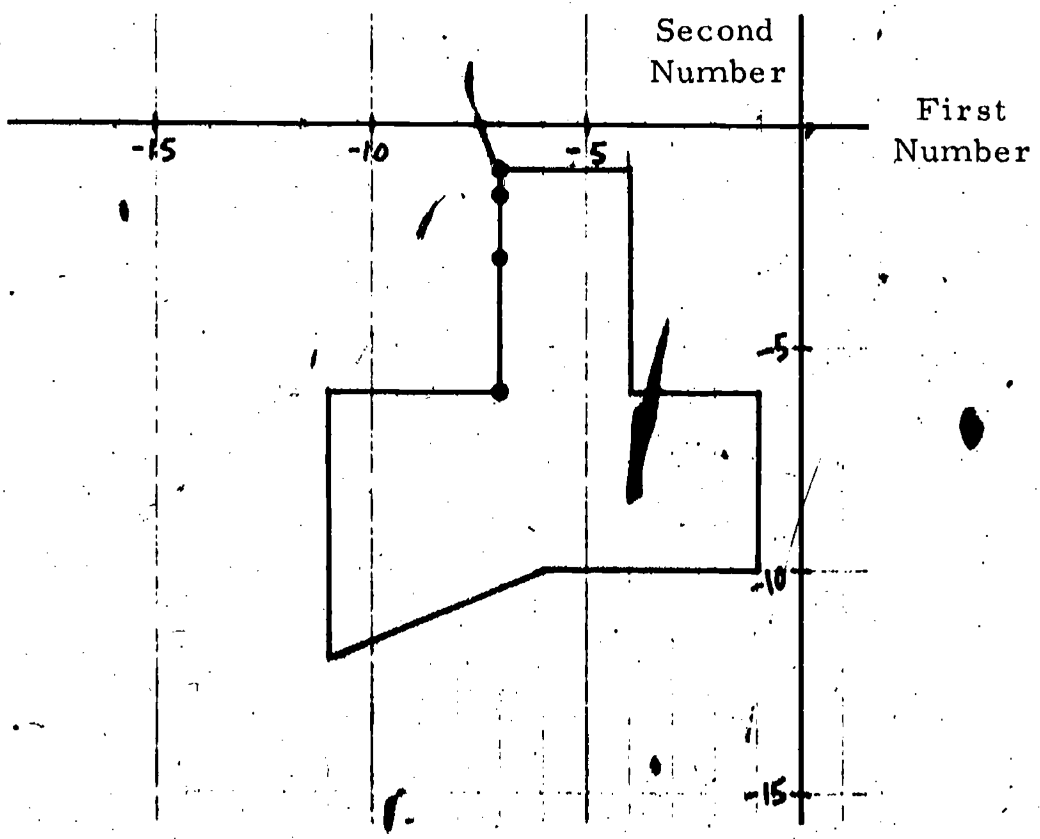
But other kinds of rules will work too. For example:

$$(\square, \triangle) \longrightarrow (3 \times \square + \triangle - 18,\ 5 \times \triangle - 26)$$

makes the square into the parallelogram shown on the next page. The area of this figure is 60 square units.



There is even a rule which will take the square into this figure.



If someone proposes a rule whose effect is not immediately evident, it is worthwhile to check at least the corner points of the original square to see how the proposed rule affects its area and location.

Film Discussion Notes
"Rotations in the Plane"

Preliminary information:

You will see a fifth grade class from the Seldon L. Brown School in Wellesley, Massachusetts. The students were familiar with some materials from this Project. Their regular classroom teacher was attending a Project institute, and she had had the class do many of the written lessons (not including those on number plane rules). The teacher is David A. Page.

[Film running time: 37 min.]

Possible discussion topics

The rule $(\square, \triangle) \longrightarrow \left(\frac{2 + \square - \triangle}{2}, \frac{\square + \triangle}{2} \right)$ has only one standstill point: (1 , 1). What is a convincing argument that there are no other standstill points for this rule? (Hint: It might help to think about finding standstill points for number line jumping rules. The graphs on the second page of Book 20 may provide a geometric argument.)

When the class was asked, "Do you think we could ever hit the standstill point?" they did not agree on an answer. Jorie said, "I thought we would land about the standstill point." What might she have had in mind? If participants seem in doubt about the answer to this question, here is a notion which may be worth considering: if one could hit the standstill point in some number of jumps, where would one have landed just before that?

Before attempting to answer the questions near the end of the film, it may be worthwhile for the group to play around with other number plane jumping rules. Some rules are listed on the next page. For any particular rule, choose a starting point and make at least ten consecutive jumps.

Be on the lookout for standstill points and the relative lengths of jumps.
For which rules do you hit a point that you've hit before?
How many jumps does it take to get back to your starting point?
Which rules never take you to a point you've already touched?

> For some rules, it's easy to tell exactly where you would be if you took one hundred jumps from some point. For other rules, try to predict generally where on the plane you would land after one hundred jumps.
>
> For any given rule, are there points from which it is illegal to start?
>
> Are there some points one can never reach?

These questions cannot all be answered rapidly; a considerable amount of time and thought will probably be necessary in order to find out what a given rule does. But participants should now have sufficient skills to begin to think about these questions, to make jumps, and to venture predictions.

Here are some number plane jumping rules:*

$$(\square,\triangle) \xrightarrow{a} (\square - \triangle \,,\, \square + \triangle)$$

$$(\square,\triangle) \xrightarrow{b} \left(\frac{6 + \square - \triangle}{2} \,,\, \frac{\square + \triangle}{2}\right)$$

$$(\square,\triangle) \xrightarrow{c} (\triangle - \square \,,\, -\square)$$

$$(\square,\triangle) \xrightarrow{d} \left(\frac{\square}{\triangle} \,,\, 2 \times \square\right)$$

$$(\square,\triangle) \xrightarrow{e} \left(\frac{\square - \triangle}{\frac{1}{3}} \,,\, \frac{\square + \triangle}{\frac{1}{3}}\right)$$

$$(\square,\triangle) \xrightarrow{f} \left(\frac{2 + \square - \triangle}{\sqrt{2}} \,,\, \frac{\square + \triangle}{\sqrt{2}}\right)$$

It is this last rule, rule f, which meets the requirements posed at the end of the film. That is, this rule balances evenly between spiraling in and spiraling out. From wherever you start, you will come back to the same starting point after eight jumps.

If we make the denominators of rule f some number larger than the square root of 2 (for example, a denominator of 2, as used in the film), we get a spiraling in toward the standstill point. If we make the denominators positive but less than the square root of 2 (such as in rules a or e above), we get a spiraling away from the standstill point.

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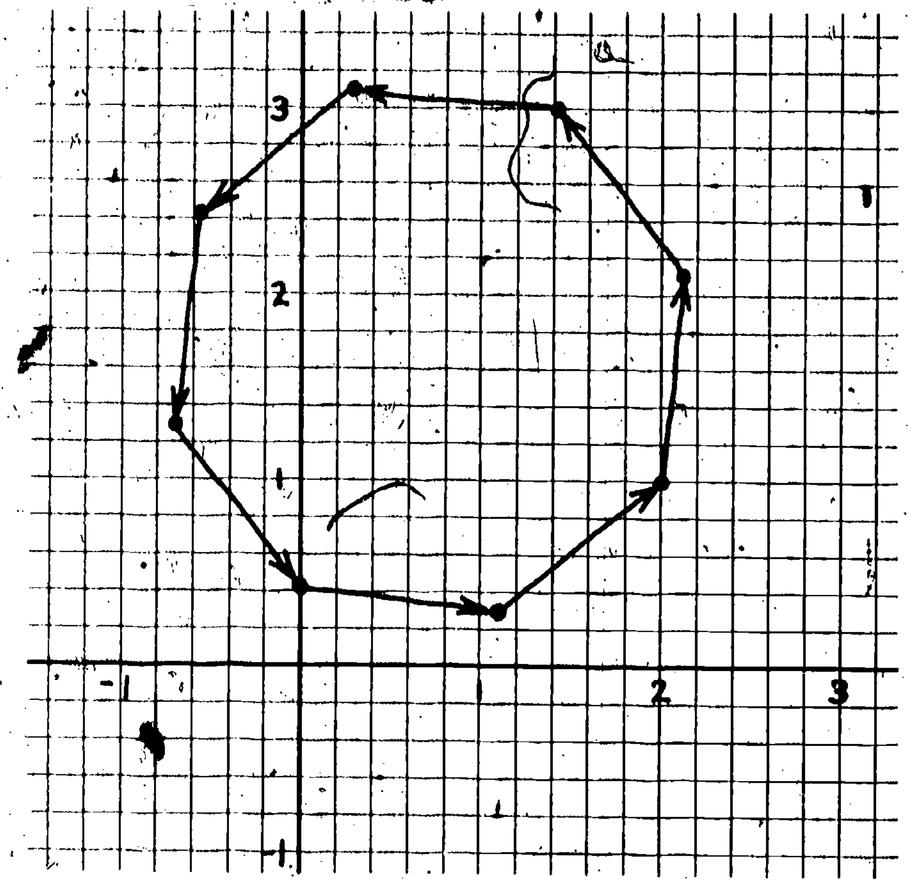
*For an analysis of rules simpler than those considered here and in the film, see the supplement, "More Work With Number Plane Rules" (Book 19).

Here is what happens when you use rule f

$$(\square, \triangle) \xrightarrow{f} \left(\frac{2 + \square - \triangle}{\sqrt{2}} , \frac{\square + \triangle}{\sqrt{2}} \right)$$

and start at the point (2, 1).



Start: $(2, 1) \longrightarrow \left(\frac{3\sqrt{2}}{2} , \frac{3\sqrt{2}}{2} \right) \longrightarrow (\sqrt{2}, 3) \longrightarrow$

$\longrightarrow \left(-\frac{\sqrt{2}}{2} + 1, \frac{3\sqrt{2}}{2} + 1 \right) \longrightarrow (\sqrt{2} - 2, 1 + \sqrt{2}) \longrightarrow$

$\longrightarrow \left(-\frac{\sqrt{2}}{2} , 2 - \frac{\sqrt{2}}{2} \right) \longrightarrow (0, \sqrt{2} - 1) \longrightarrow$

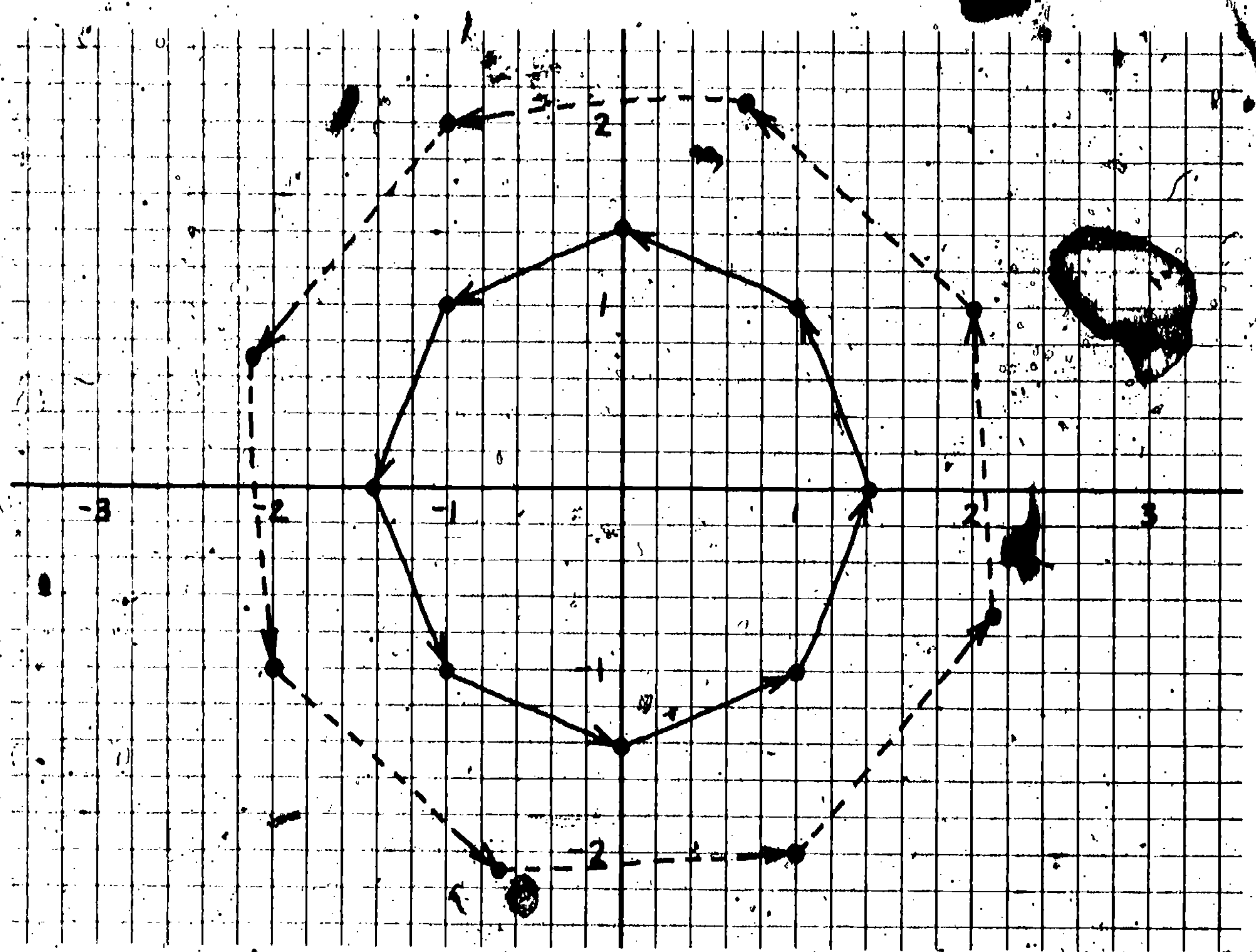
$\longrightarrow \left(\frac{3\sqrt{2}}{2} - 1, 1 - \frac{\sqrt{2}}{2} \right) \longrightarrow (2, 1)$

Standstill point: $\left(\frac{\sqrt{2}}{2} , 1 + \frac{\sqrt{2}}{2} \right)$

The solid line shows what happens when you use the rule

$$(\square, \triangle) \xrightarrow{g} \left(\frac{\square - \triangle}{\sqrt{2}}, \frac{\square + \triangle}{\sqrt{2}} \right)$$

and start at $(1, 1)$. The dotted line shows what happens when you start at $(2, 1)$.



Start: $(1, 1) \rightarrow (0, \sqrt{2}) \rightarrow (-1, 1) \rightarrow (-\sqrt{2}, 0) \rightarrow (-1, -1) \rightarrow$

$\rightarrow (0, -\sqrt{2}) \rightarrow (1, -1) \rightarrow (\sqrt{2}, 0) \rightarrow (1, 1)$

Start: $(2, 1) \rightarrow \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) \rightarrow (-1, 2) \rightarrow \left(-\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \rightarrow (-2, -1) \rightarrow$

$\rightarrow \left(-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) \rightarrow (1, -2) \rightarrow \left(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \rightarrow (2, 1)$

Standstill point? (,)