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ABSTRACT

The investigation reported is primarily a description of how college students solve or fail to solve algebra equations. Protocols were collected from two groups of college students; one group was expected to be good solvers, the other group contained many poor solvers. The investigators sought to identify and classify the difficulties students had, and they tried to guess the mechanism which produced those difficulties. They compared the work of successful and unsuccessful solvers, looking for ideas that might help make more solvers be successful, and they tried to identify what must be learned by the student of equation solving. The discussion of findings includes: identification of three types of errors, and sections devoted to kinds of knowledge in algebra, meaning in algebra, errors and the psychology of skill, and characteristics of good solvers. (MK)

Psychology of Equation Solving

An Information Processing Study

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Final Technical Report

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PREFACE

It is our shared opinion that what is learned in school mathematics differs sharply from the outcomes desired by mathematics educators. We feel that curriculum and instruction in school mathematics is heavily influenced by the reflections of individuals who, having successfully completed their own study, possess well organized and coherent conceptualizations of a very large body of knowledge. Within that body of knowledge is a tightly conceived logical network which guides the organization of curriculum materials and instructional procedures for school mathematics. Such a logical organization, the reasoning goes, should produce the desired outcomes.

Unfortunately many students study mathematics for years and in the end, exhibit jarring voids in their own ability to use mathematical tools in very simple and presumably logical ways. Here we find a dilemma.

The investigation we have reported is primarily a description of how college students actually solve or fail to solve algebraic equations. We have focused on describing the behaviors exhibited together with the spoken descriptions of process. Where possible we have provided a theoretical framework for these data. The reader will find evidence of the thought processes which operate. They appear to us to be very different, even among good solvers, from those thoughts and actions which seem to be logical expectations. Algebra may be generalized arithmetic from the mathematician's point of view but our data suggest that few college students have that perspective.

Perhaps as researchers accumulate more information of the sort reported here, the basis of curricula and instructional decision will shift in direction away from "what logically ought to be the outcome" toward "what is the observed outcome". This is our hope.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

Introduction

Many high school graduates today are restricted in career choices by the lack of adequate and proficient knowledge in mathematics. One key component needed for entry especially into scientific careers is fluency in algebraic equation solving. This study was intended to develop a coherent description of effective and efficient algebraic equation solving and of those factors that interfere with such performance. This is an early stage in a line of investigation which should ultimately yield implications for the learning and teaching of algebra.

This report presents detailed information about the way in which certain university students solve and fail to solve equations in elementary algebra. Understanding the solution process as it is actually carried out by students will hopefully aid teachers to transmit successful methods and prevent the development of unsuccessful ones. Equation solving involves a complex interplay of many forms of knowledge. We tried to use our study of this task to shed new light on such perennial issues in the psychology of problem-solving as the role of understanding, as well as more recently raised questions in the psychology of skill.

In interpreting the data, we looked for three kinds of conclusions relevant to education. First, we tried to identify and classify the difficulties students had, and to guess the mechanism that produced those difficulties, whether the mechanism was lack of a specific piece of knowledge or the failure to carry out a process. Such ideas about mechanisms should be useful in suggesting countermeasures for these errors: an error that results from a specific wrong idea cannot be overcome by emphasis on careful or neat work, while an error in execution might be. Second, we compared the work of successful and unsuccessful solvers, looking for ideas that might help make more solvers successful. We wanted to find differences in the way solvers worked on the problems that go beyond the presence or absence of errors.

Third and more generally we tried to identify what must be learned by the student of equation solving, following this question wherever it led. We started from the legal moves of the algebra game, were led immediately to the knowledge that underlied an appropriate choice of move, and eventually to the knowledge that permits the legal moves and illegal moves to be distinguished. We have also suggested that there is information of a higher order relating these three kinds of knowledge that may be very important to learners and teachers.

Psychologists of problem-solving have long distinguished insightful behavior from uninsightful, without being able to precisely trace the boundary between them. Equation solving is an interesting task partly because it is possible to envision both uninsightful and insightful ways of performing it. We have found human performance mixing these two approaches, and have seen in some specific cases what understanding does and does not do.

Recent work in the psychology of skill by John Seely Brown and others, some of it directed specifically to algebra, has suggested how the ability to perform a complex task is developed and how failures occur and are handled. We have examined segments of behavior which can be used as examples in evaluating and perhaps extending these ideas.

Related work

There has been little empirical study of ordinary algebra, despite its importance. Brown and his associates (Brown, Burton, et al 1975) collected and interpreted errors from college students in remedial courses. The present work can be seen as an attempt to carry forward the analysis of solving behavior begun by them. Matz (1979a, 1979b) has also been pursuing and extending this work, concentrating on mechanism underlying errors. Davis and Cooney (1977) have collected errors from high school students. We have been able to supplement their findings with data from a wider class of problems. The theoretical work of Bundy (1975) in developing an equation solving computer program has been discussed at some length below.

The enterprise of developing detailed accounts of behavior has been carried out more widely in other areas of mathematics, especially arithmetic. Most closely related is the work of Suppes and Morningstar (1972), that has been followed by Brown and Burton (1978) and Larkin (1978). The techniques of representing procedural knowledge developed in this later work, and in artificial intelligence by Sacerdoti (1977), are now being extended to simpler skills such as counting by several investigators. Van Lehn and Brown (1978) discussed these techniques. Neves (1979) is developing a model of the process of learning to solve equations from examples.

Overview of the study

We collected protocols from two groups of university students solving elementary algebra equations. One group of students was selected for proficiency from a population of engineering and mathematics education students, who were expected to be good solvers. The other group was an unselected sample of volunteers from an introductory psychology course. Many students in this group were poor solvers, so we were able to compare good and poor performance. The written work of each student and any spoken comments were retained for analysis, and the comments were keyed to the written work using video recording made of the solving session.

To organize this large, rather unstructured body of data, we used an artificial model of the solution process, based on the work of Bundy (1975). This artificial model, described in detail below, is perhaps the simplest system realistically capable of equation solving. The complex data actually obtained from human solvers can be compared with the predictions of this simple model to bring into relief those aspects of human solving that deviate from it. This served to separate characteristics of the human performance which simply reflect the demands of the task, and hence agree with the simple model, from those that indicated the human contribution.

The body of the report is organized as follows. Following a description of the materials and data collection procedure, including the recruitment of subjects, summary of performance on the task is given. The Bundy model is then described, so that the details of performance can be related to it. Strategy, errors, and special features of skilled performance are considered, followed by discussion of aspects of the data that require departures from the simple model.

CHAPTER 2

Development of the Research Instruments Procedures and Performance Summary

The equation solving behavior under investigation centers around the type of equations (linear, quadratic, and formulas) found in elementary and intermediate level high school algebra courses. Since the contrast between good and poor performance was to be made with college students, there were questions concerning the appropriateness of the equations to be used. A series of twelve (12) equations containing such features as parentheses, fractions, signs, and literals was pilot tested (see Table 2.1) with a small group (about 20) of subjects representing the population of less proficient solvers.

Table 2.1

Equations Used in Pilot Testing

1. $4x - 2 = 18$
2. $11 - 2x = 3x + 17$
3. $x - 2(x + 1) = 14$
4. $A = p + prt$; solve for p
5. $6(y-2) - 3(4-2y) = y-12$
6. $3 - \frac{3x}{2} = \frac{8-4x}{7}$
7. $A = \left(\frac{b+c}{2}\right)h$; solve for b
8. $x + 2\left[x+2(x+2)\right] = x+2$
9. $\frac{5}{x+4} = \frac{4}{x-4}$
10. $x^2 + 3 = (x+5)(x+9)$
11. $\frac{1}{x} + \frac{1}{x+3} = \frac{2}{x-3}$
12. $\frac{1}{R} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$; solve for x

Pilot data were examined for equations which were challenging, but not cumbersome. On this basis, certain of the pilot equations were omitted and new equations added. The final instrument is shown in Table 2.2.

Table 2.2

Equations used in First Session

1A. $A=p+prt$; solve for p

1B. $2x = x^2$

2A. $\frac{1}{3} = \frac{1}{x} + \frac{1}{7}$

2B. $\frac{1}{R} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$; solve for x

3A. $9(x+40) = 5(x+40)$

3B. $7(4x-1) = 3(4x-1) + 4$

4A. $xy+yz = 2y$; solve for x

4B. $\frac{x+3+y}{x^2} = 1$

5A. $\frac{5}{10} = \frac{x-10}{x+5}$

5B. $\frac{1 - x^2}{1 - x} = 2$

6A. $x+2(x+1) = 4$

6B. $x+2(x+2(x+2)) = x+2$

7A. $x-2(x+1) = 14$

7B. $6(x-2) -3(4-2x) = x-12$

This instrument consists of seven (7) pairs of equations. Each pair was designed to probe for specific errors observed in the pilot data. These 14 problems were given to all subjects in the first of two sessions. In some cases the equations share specific structural features, other times they vary in some systematic way.

Corresponding to the seven pairs of equations used in Session 1, the seven triples of similar equations shown in Table 2.3 were constructed for Session 2. The latter were intended to permit an error seen in Session 1 to be elicited for discussion in Session 2.

Four pairs of more complex equations of the same general type as those for Session 1 were also prepared for use in Session 2, to allow the performance of skilled solvers to be assessed on more difficult material. These equations are shown in Table 2.4

Table 2.3

Session Two Equations
for Unselected Group

1C	$A = 2k$[illegible] $+2kh + 2wh$ solve for h
1C	$x-x^2 = x^2-3$
1E	$ax = b-x$
2C	$\frac{22}{7} - \frac{7}{3-x} = \frac{1}{7}$
2D	$\frac{a}{x+1} = b + 1/c$
2E	$\frac{1/x + 1}{2} + 3 = 0$
[illegible]C	$9(7x-15) = 8(7x+15) + 7$
3D	$2(4x-2) +3(2x-1) = 4$
3E	$2(3+x) = 4(x+3) - 1$
4C	$xy + yz + xz = 1$
4D	$bcx + acx + abx + abc = 0$
4E	$ax + bx + ab = 2x + 2a$

Table 2.3 continued

5C $\frac{3+x^2}{1+x} = 2$

5D $(4+2/x)\left(\frac{x}{2+x}\right) = 7$

5E $\frac{x^2 + (x+3)}{3(x+3)} = x/3$

6C $x+1\left(\frac{x+1}{2}\right) = 1$

6D $\frac{x+2(x+2)}{x+2} = 2$

6E $(3+x)\ 3+x = 1$

7C $x-2(x-2(x-2)) = x-2$

7D $x^2 - (x+1)(x-2) = 9$

7E $7 - \frac{x+3}{x} = 5$

Table 2.4

Session Two Equations
for Selected Group

A1	$\frac{1/x + 1/x^2}{1/x + 2x^2} = 3$	A2	$3x+5(x-3) = 5x+3)-3(x-2)$
B1	$1/x = \frac{2}{4-x}$	B2	$\frac{a}{x-b} = c + d$
C1	$2(4x+2) - 3(1+2x) = 0$	C2	$3(x+(a+b))+2(b+(x+a)) = 1$
D1	$\frac{(r+y+z)x}{1/p+1/q} = d$	D2	$y+2\left(\frac{3x}{x+1}\right) - (4+y)\left(\frac{3x}{x+1}\right) = 1$

An instrument for screening the more proficient, fluent solvers on the basis of speed and accuracy was also developed. Equations like those in Table 2.2 or with added complexity, were selected. The equations for the screening instrument are shown in Table 2.5.

Table 2.5

Equations Used to Screen
Selected Group

1. $3x - 2(x-5) = 2$
2. $(2x-3) -3(x-6) = 4$
3. $\frac{1}{x+2} - \frac{1}{2-x} = \frac{3x+8}{x^2-4}$
4. $5-3\left[2x -2 \ (5-3x)\right] = 4(2-3x)$
5. $\frac{x-2}{4} - \frac{2x+3}{5} + \frac{5x}{2} = 13$
6. $\frac{2}{9} = \frac{15}{x + 3} - \frac{1}{6}$
7. $(2x+1) \ (8x-3) = (4x-1)^2$
8. $\left(\frac{4}{x} \quad -x\right) \ \left(\frac{x}{x-2}\right) = 3$
9. $A = \pi r^2 + 2\pi rh$; solve for h
10. $\frac{1}{a} - \frac{1}{b} + \frac{1}{c} = \frac{1}{d}$; solve for b
11. $\left(\frac{x}{6} + x - \frac{x+1}{4}\right) -3 = \left(2x - \frac{1}{12}\right) -1$
12. $3x-x^2 = 0$
13. $x-3(x+3) = 0$
14. $\frac{1}{4} \ (x+2) -\frac{5}{2} \ (x-8) = \frac{1}{3}(x+12)$
15. $\frac{x^2-4}{x-2} = 4$
16. $\frac{2}{x^2-1} = \frac{1}{x-1}$
17. $x = 10 + (10\%)x$
18. $\frac{x^2 + 7y^2}{y^2} = \frac{3y-4x}{y}$; solve for $\left(\frac{x}{y}\right)$
19. $5x - \frac{x+1}{x+1} + 6 = 0$

Subjects. Two groups of university students served as subjects. The unselected group consisted of nineteen volunteers from an Introductory Psychology course, who received course credit for their participation. These students were not selected by the experimenters; preliminary work had shown that this pool of subjects included many poor equation solvers. Participation was voluntary, and three other students declined to participate after the study was described to them. Data from three more students were lost to technical failure.

The unselected group varied in mathematical background. All had had two years of algebra in secondary school, but some had continued to take mathematics in college and others had not. The time elapsed between initial learning or last use and the time of this study must be kept in mind in considering their performance in the study. Specifically, it is likely that memory retrieval difficulties played a greater role of this group than it might in a corresponding study with high school students.

The fifteen students in the selected group were recruited as follows. A screening test (Table 2.5) was administered to 91 students in a junior-level electrical engineering laboratory course and 20 student mathematics teachers. The test was given in a classroom setting, and the students were asked to record the time they started and finished.

All tests were scored for number of equations correctly solved.

Performance of the group ranged from 4 to 19 correct solutions and from 8 to 30 minutes to complete the test.

The 24 students who had 16 or more correct solutions were selected for telephone contact. Of these 4 could not be contacted, 4 declined to participate and 16 served as subjects. One student's data was lost due to technical failure.

These students were paid $5.33 for a 1 1/2 hour session.

Procedure

All students served individually in the experiment. On arrival, each student heard a general description of the experiment and its goals, and decided whether or not to participate. General instructions were then given, in which the student was asked to solve a series of algebra equations, by whatever method the student chose while being video-taped. No simplification of answers was required, and the student was free to write down as many or as few steps of the solution as desired. The student was asked to resist the temptation to be unusually careful or clear in working on the problems and not to worry about making mistakes. It was explained that no record of the student's identity would be kept, and that the video tapes would not show the face. The student was free to skip or abandon an equation at any time. A pen was provided to prevent erasing. Questions were encouraged at any time during the procedure.

The instructions also called for spoken comments during the solution process. For the first seven equations, the student was told "As you work, try to describe your problem-related thoughts. Don't worry about feeling foolish, but just try to say whatever comes to mind as you work." For the second seven equations the student was asked to explain the solution process as if to a person studying algebra who asked for help on homework. Comments on how to decide what to do with an equation were specifically requested.

The first two sets of seven equations were drawn from the set shown in Table 2.2. These were presented in a random order, with the priviso that one equation in each of the pairs shown in the table fall in the first seven and the other in the second seven. Working time for these 14 equations ranged from 10 to 30 minutes.

After completing the work on the first two sets, the students in the Selected group were given two more sets of four equations each. These sets were drawn from Table 2.4, with the requirement that one equation from each pair fall in each set. The first set was solved under the "describe your thoughts" instructions, and the second under the "explain" instructions just as for the initial sets of seven. Working time for these 8 equations ranged from 15 to 30 minutes.

The Unselected group was treated differently. The first sets of seven equations completed the first session, and they returned a week later for the second session. In the interval their initial work was examined, and sets of equations related to those on which they had made errors were selected from Table 2.3. Three sets of three equations were prepared for each student, with equations that had been little used to that point, being used to fill out the count for students who had not made errors on three different types. These three sets were presented in random order, with the equations within each set also randomized. The students were asked to comment on their solutions under the "explain" instructions as used in the first session. In addition, the experimenter asked for comments on particular features of their solutions or comments. This second session lasted about 25 minutes.

All solutions and comments were recorded in the following manner. Each equation was written at the top of a blank 8½ x 11 sheet of paper. The student was seated at a table with this sheet placed in a convenient position for writing. The experimenter was seated on the opposite side of the table, controlling a video camera with a zoom lens. The camera was aimed at the area of the sheet on which the student was writing. A small TV monitor was placed on the table in such a way that the experimenter could follow the written work and guide the camera by viewing the monitor. A desk lamp was used to fill in shadows cast by the student's writing hand; otherwise no special lighting was used. The student was told not to worry about blocking the camera or moving the worksheet as the experimenter could move the camera. A microphone placed on the table near the student recorded the comments of both student and experimenter on the video tape. Additionally, an audio cassette recorder with a lapel microphone worn by the student was used to capture the comments in a form more convenient for transcription than the video tape.

Spoken comments of subject and experimenter were transcribed, with pauses indicated by slashes timed by a metronome running at 40 beats per minute. Numbered flags were placed on the written work for each solution indicating the order in which lines or parts of lines were written, by reference to the video recording. Subscripts corresponding to these flags were placed in the transcript of the comments to indicate the relative timing of comments and written steps. Flags also were used to show where the subject pointed when no mark was made.

Overall description of performance

Table 2.6 shows the performance of the subjects on the Session 1 equations. The symbol "e" in the table indicates that the subject completed the problem but made one or more errors in the solution. The symbol "n" indicates that the subject did not complete the problem. A number indicates that the correct solution was attained, and gives the number of written steps in the solution, excluding false starts and backtracking. The steps counted were the lines showing successive transformations of the equation, with marks added to such a line not counted.

Subjects are designated in the table and throughout the report by the codes assigned them when they arrived to participate in the study so there are gaps in the numbers for subjects whose data were lost or who declined to participate after being assigned a code. Codes starting with S were assigned to the unselected solvers, and those starting with E to the selected solvers: the letters may be taken to stand for Solver and Expert. These subject codes will be joined with equation numbers to identify examples discussed in the report. The designation S1 4B would identify the work of subject S1 on equation 4B.

Table 2.6 displays the subjects grouped according to the number of Session 1 equations they were able to solve. In Table 2.6 and in the grouping one error discussed below is not counted: Equation 5B has one as an extraneous root, but in Table 2.6, one was counted correct. This was done to avoid placing undue weight on correct handling of the rare extraneous root problem in ranking the solvers. As long as any partial credit is given for one as a solution to 5B, the grouping shown is unchanged.

Interestingly, there was little difference overall in the lengths of solutions generated under the "describe your thoughts" and "explain" instructions. The average across equations of the average solution length for "describe your thoughts" solutions to the Session 1 equations was 3.4 steps, and for "explain", 3.7. The "explain" solutions were longer for 9 of the 14 equations and shorter for the other 5. Though this indicates that the difference was not consistent for equation 3B, the increase in length under "explain" instructions was significant (rank sum test, T' = 154.5 $p<.001$).

Tables 2.7 and 2.8 show the correctness of solutions for the Session two equations. Here "c" means correct, "e" error as in Table 2.6, and "n" not completed. The symbol "t" indicates a problem not completed because of insufficient time.

Table 2.6

Performance on Session 1 Equations
Top Ten Solvers

Equation

	1A	1B	2A	2B	3A	3B	4A	4B	5A	5B	6A	6B	7A	7B	number correct
E3	2	3	2	2	3	7	2	6	4	3	4	3	5	5	14
E5	2	3	3	3	3	4	2	4	4	5	4	5	3	4	14
E8	1	4	3	2	3	3	2	4	3	6	3	4	3	3	14
E9	2	3	4	3	3	e	2	7	3	10	3	5	3	3	13
E13	3	e	2	2	3	3	2	4	3	4	3	3	3	3	13
E14	2	1	4	3	3	5	2	6	e	e	3	4	3	4	12
E16	2	4	4	4	3	4	2	6	4	6	3	5	3	4	14
S8	2	3	e	2	3	3	3	5	6	5	4	e	3	4	12
S17	2	4	5	e	3	5	2	6	4	5	6	4	6	4	13
S19	2	e	3	3	e	3	1	4	3	3	3	4	3	4	12
No. Correct	10	8	9	9	9	9	10	10	9	9	10	9	10	10	
Average No. Steps	2.0	3.1	3.3	2.7	3.0	4.1	2.0	5.2	3.8	5.2	3.6	4.1	3.5	3.8	

e - denotes error

Table 2,6 continued

Middle 14 Solvers

	Equation														
	1A	1B	2A	2B	3A	3B	4A	4B	5A	5B	6A	6B	7A	7B	number correct
E1	2	3	4	4	3	5	2	e	4	5	e	e	3	4	11
E2	2	3	3	4	2	3	2	e	4	e	3	5	3	e	11
E4	2	e	4	2	3	3	1	7	4	e	3	e	3	4	11
E6	2	e	4	2	3	3	2	3	3	4	3	e	e	3	11
E7	2	e	3	3	3	3	2	n	4	2	3	6	e	e	10
E10	2	e	3	3	e	3	2	4	3	e	2	4	2	4	11
E11	2	e	2	2	e	3	2	4	4	3	2	e	3	4	11
E12	2	4	3	3	2	e	2	e	3	4	3	e	4	6	11
S7	2	e	5	5	4	5	n	e	4	e	5	5	5	6	10
S10	e	e	4	e	3	5	2	5	4	6	3	5	3	5	11
S11	5	3	5	e	3	4	2	6	3	3	e	e	3	3	11
S12	4	e	e	n	4	3	e	7	6	7	3	5	4	4	10
S18	2	e	5	n	e	3	2	3	3	4	3	e	4	4	10
S23	n	4	5	n	3	4	2	e	6	6	3	5	4	e	10
No. Correct	12	5	13	9	11	13	12	8	14	10	12	7	12	11	
Average No. Steps	2.4	3.4	3.8	3.1	3.0	3.6	1.9	4.9	3.9	4.4	3.0	5.0	3.4	4.3	

e - denotes error

n - denotes not complete

Table 2.6 continued

Bottom Ten Solvers

	Equation														
	1A*	1B	2A	2B	3A	3B	4A	4B	5A	5B	6A	6B	7A	7B	number correct
S1	3	e	e	e	3	5	e	n	e	e	e	e	e	4	4
S3	e	3	n	e	3	4	n	e	n	n	4	5	3	4	7
S5	n	e	6	e	4	4	2	e	e	e	e	e	e	5	5
S6	2	n	5	n	3	4	2	e	4	3	e	e	3	4	9
S13	e	e	5	n	e	e	n	4	4	5	3	e	3	5	7
S15	e	e	6	n	3	e	2	5	5	6	4	4	e	4	9
S16	e	e	e	e	e	e	e	e	e	e	4	e	e	e	1
S20	e	e	e	n	4	e	n	e	e	e	e	e	e	e	1
S21	e	n	e	n	3	e	2	n	e	n	3	4	3	4	6
S22	e	e	e	e	3	e	e	e	e	e	e	e	e	e	1
No. Correct	2	1	4	0	8	4	4	2	3	3	5	3	4	7	
Average No. Steps	2.5	3.0	5.5	-	3.2	4.2	2.0	4.5	4.3	4.7	3.6	4.3	3.0	4.3	

e - denotes error

n - denotes not complete

Table 2.7

Performance on follow-up equations
for unselected group

Equation

	1C	1D	1E	2C	2D	2E	3C	3D	3E	4C	4D	4E	5C	5D	5E	6C	6D	6E	7C	7D	7E
S1	e	n	e				c	e	c							e	c	c			
S3	n	c	n	c	n	c	e	c	c												
S5				c	e	e							n	e	e	n	e	t			
S6													n	c	c	c	c	c	c	c	e
S7							c	c	e	c	c	c							c	c	c
S8				c	c	c	c	c	c							c	c	c			
S10				c	c	c	c	c	c				c	c	c						
S11	c	c	c							c	c	c				e	e	c			
S12	c	n	c	t	t	t				e	e	c									
S13	c	e	e	c	e	c													c	c	c
S15	t	c	e										c	n	c				c	e	c
S16										t	t	e	e	e	e				t	t	e
S17							c	c	c				c	c	c				c	c	c
S18	c	e	e				c	c	c							c	c	c			
S19							c	c	c				c	c	c				c	c	c
S20										n	n	n	e	n	n	n	e	e			
S21				e	t	t				n	n	n	e	n	e						
S22				e	e	e				e	e	e	e	n	n						
S23	c	c	c	c	c	c				c	c	e									

c - denotes correct
e - denotes error
n - denotes not completed
t - denotes insufficient time

Table 2.8

Performance on Session Two Equations for Selected Group

	A1	A2	B1	B2	C1	C2	D1	D2	Total Correct
E1	c	c	c	c	c	c	c	e	7
E2	c	c	c	c	c	c	c	c	8
E3	c	c	c	c	c	c	c	c	8
E4	e	c	e	e	c	c	e	n	3
E5	e	c	c	c	c	c	c	c	7
E6	e	c	c	c	c	c	c	e	6
E7	c	c	c	c	c	c	c	e	7
E8	c	c	c	c	c	c	c	e	7
E9	c	c	c	c	c	c	c	e	7
E10	c	c	e	c	e	c	c	e	5
E11	e	c	c	c	e	c	c	e	5
E12	e	c	c	c	c	c	c	c	7
E13	c	e	c	c	c	c	c	c	7
E14	e	e	c	c	c	c	c	c	6
E16	c	c	c	e	c	c	c	n	6
Total Correct	9	13	13	13	13	15	14	6	

c - denotes correct

e - denotes error

n - denotes not completed

CHAPTER 3

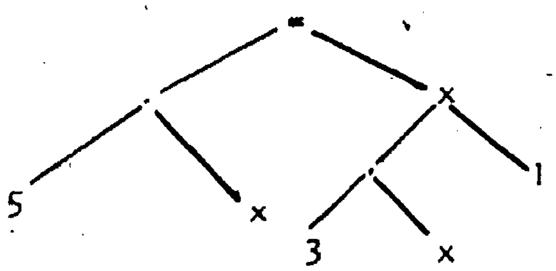
The Bundy Model

Bundy (1976) presented an outline of the organization of a program to solve equations. Though his model is not intended as a psychological model, it nevertheless makes a convenient starting point for considering human performance.

The program operates on tree representations of equations, as in Fig 3.1. This representation avoids the need for parentheses and makes operator, operand relationships easily apparent.

Figure 3.1

Tree representation of the equation 5x - 3x + 1



The trees are transformed by the application of operators (called axioms by Bundy) selected from a store. The operators are straightforward uses of arithmetic principles. Examples appear in Fig. 3.2.

Figure 3.2
Example Operators

$-u = V \qquad u = -V$

$(u+V)-V \longrightarrow u$

$w \cdot (u+V) \qquad w \cdot u + V \cdot w$

$w \cdot u + V \cdot w \qquad w \cdot (u + V)$

Each operator has an old pattern, which must be matched in applying the operator, and a new pattern, showing how the operator changes the tree to which it is applied. The program has a pattern matcher which is able to tell when an old pattern can be matched to a tree and establish what parts of the tree correspond to the variables in the old pattern. The pattern matcher embodies principles like commutativity and associativity of addition and multiplication. This means that operators need not be stated in different forms to apply to trees which differ simply in order of addends, for example. The pattern matcher is also able to call the equation solver recursively to determine complex matches. An example is presented in Figure 3.3 below.

Several operators, typically, are applicable to a given equation tree. The selection of which operator should be used is governed by simple heuristics that partition the operators into groups appropriate to reaching specified subgoals. These subgoals break up the solution process into phases, which can be described as follows for an equation that initially has two occurrences of the unknown.

The first phase is attraction, which has as its goal the rearrangement of the occurrences of the unknown in such a way that one can be eliminated by a further operation. Attraction is guided by heuristics that identify the smallest subtree of the equation containing the occurrences and that try to reduce the number of links of the tree connecting the occurrences. Operators that have this potential are designated "helpful to attraction" in the operator store.

Collection is the phase in which occurrences of the unknown are actually eliminated. To limit search, only single operators that could eliminate an occurrence are considered. As for attraction, attention is focused on the smallest subtree containing the occurrences of the unknown when matches are sought, and operators useful to collection are marked in the operator store.

If more than two occurrences of the unknown are present, attraction and collection must be repeated. If only one is present, they can be skipped, and the final phase, <u>isolation</u>, entered at once. The goal of isolation is to remove any structure in which the unknown is embedded, leaving the equation tree in solved form.

At the end of each phase a rudimentary simplification is performed. This is limited to the removal of zero from sums and other such routine simplifications. No multiplication, factoring or other rearrangements are performed. As with the phase described above, operators appropriate for simplification are designated in the operation store.

Figure 3.3 shows an example of a solution illustrating the operation of these phases.

Figure 3.3

The Bundy Program - Linear Equation

<u>EQUATION</u>	<u>OPERATION</u>	<u>PHASE</u>
$5x - 3x + 1$	$u = w-v$ $\downarrow$ $u + v = w$	attraction
$5x +(-3x) = 1$	$w \cdot u + v \cdot w$ $\downarrow$ $w \cdot (u+v)$	collection
$x\ (5 + -3) = 1$	arithmetic done in simplification	
$x\ (2) = 1$	$u \cdot v = w$ $\downarrow$ $u = w/v$ for $v/0$	isolation
$x = 1/2$		

Figure 3.4 shows a more complicated example taken from Bundy (1975) in which the pattern-matcher calls the equation solver recursively to solve a quadratic equation.

Figure 3.4

The Bundy Program - Quadratic Equation

EQUATION	DISCUSSION
$ax^2 + bx + c = 0$	Collection tries the operator $u^2 + 2\cdot u\cdot v + v^2 \longrightarrow (u+v)^2$ to collect the x's in $ax^2 + bx$. Choosing u as the variable to be matched with x, the structure containing the u's, $u^2 + 2\cdot u\cdot v$ is identified, and the operation changed to $u^2 + 2\cdot u\cdot v \longrightarrow (u+v)^2 - v^2$. In matching $x^2 + 2\cdot x\cdot v$ to $ax^2 + bx$ the program tried to find a w that can be multiplied times $x^2 + 2xv$ so that $w\cdot 1 = a$ and $w\cdot 2\cdot v = b$. These simultaneous equations are solved, giving $w = a$, $v = \frac{b}{2a}$. So the operation performed is $ax^2 + bx \longrightarrow w\left[(u+v)^2 - v^2\right]$ which is $a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right]$
$a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{2a}\right)^2\right] + c = 0$	Isolation now proceeds, since there is just one x.

Figure 3.4 continued

$$a \left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 \right] = -c$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

$$x = \frac{-b}{a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

This is the equivalent to the familiar

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The objective of Bundy's program is the solution of the search problem presented by the large number of operators applicable to a given equation. Other systems for algebraic manipulation (Moses, 1971) solve the problem by reducing expressions to canonical forms that can be manipulated in standard ways. Bundy argues that his heuristic approach, which is extended in his paper to cover elementary function symbols as well as the four arithmetic operations, may solve a wider range of problems. In any case, since human solvers do not appear to use canonical forms, it seems that Bundy's use of heuristics for operator selection offers a more plausible point of reference for considering human solvers.

In considering Bundy's model as an account of human solving, some changes seem called for immediately. First it seems reasonable to replace the equation trees by the ordinary written syntax used by the solvers, so that operations are formulated as operations on strings. This makes it more difficult to determine the distance between occurrences of a symbol, as used in the attraction phase, but this information is nevertheless still available. As will be seen, parentheses as used in the string notation seem to have some effects on solvers' behavior, so their inclusion is desireable.

Second, Bundy's model is intended to produce only correct solutions, so that all operations are mathematically valid, pattern-matching always respects the syntax of expressions and the actual execution of operations is always accurate. Of course, these requirements must be relaxed in adapting the framework for a psychological model.

In relating human performance to this simple model we will first discuss two areas in which useful agreement is found: strategy, that is, the selection of operators to reach the goal of solving the equation, and errors, the ways in which mathematically invalid results are obtained. We then discuss some difference between skilled and less skilled performance that relate to the Bundy model. Finally, we turn to a discussion of several aspects of human performance which seem to be outside the scope of the simple model. We wish to repeat here that the Bundy model is not intended to be a psychological model, and our discussion is not a criticism of it but an attempt to use Bundy's insights to clarify the patterns in our data.

CHAPTER 4

Strategy Analysis

The strategies used by solvers provides one important way of describing observable differences in solving behavior. In this chapter we consider the strategies used by the top ten solvers, the middle 14 solvers and the bottom ten solvers and discuss the relationship of those strategies to those suggested by the Bundy model.

The Bundy model establishes priorities to guide the choice of operations to apply to an equation. Recall that the model first attempts to attract instances of the unknown, moving them together, then to collect instances so that only one is left. If there is only one instance, the model isolates the unknown, stripping off any operations in which it is embedded.

These priorities seem to agree reasonably well with those of human solvers, though there are some differences. Table 4.1 displays the choices made by the solvers for each of the 14 equations in Session 1. The choices mentioned are illustrated in Table 4.2. The choices given are the initial choices presented by the equation in its original form, and a box in Table 4.1 indicates the choice favored by the basic strategy of the Bundy model. No box appears for equations 4B, 5A, and 5B because these equations have x in the numerator and denominator of a fraction, and it is not clear how the Bundy model would proceed in these situations. The needed operation is to clear the fraction, but no operator to do this is listed as useful for attraction in Bundy's tentative list, perhaps because it does not reduce the distance between the occurrences of x in Bundy's expression tree notation. It seems likely that such an operator

Table 4.1

Strategic choices

Where the Bundy Model makes a clear choice,
that is indicated by a box.

Equation and choice	Top 10 Solvers	Middle 14 Solvers	Bottom 10 Solvers	All
1A				
$\boxed{\text{Collect p's}}$	10	11	0	21
Other	0	3	10	13
1B				
$\boxed{\text{Attract}}$	6	4	3	13
Cancel or divide	3	5	2	10
Other	1	5	5	11
2A				
$\boxed{\text{Isolate}}$	7	8	4	19
Clear or combine fractions	3	5	2	10
Other	0	1	4	5
2B				
$\boxed{\text{Isolate}}$	7	10	3	20
Clear or combine fractions	3	3	1	7
Other	0	1	6	7
3A				
$\boxed{\text{Attract}}$	0	3	0	3
Multiply out	9	7	9	25
Cancel or divide	1	4	1	6
3B				
$\boxed{\text{Attract}}$	1	3	0	4
Multiply out	7	10	9	26
Cancel or divide	0	1	1	2
Introduce variable	2	0	0	2

Table 4.1 continued

Equation and choice	Top 10 Solvers	Middle 14 Solvers	Bottom 10 Solvers	All
4A				
Isolate	8	8	3	19
Factor out y	2	6	3	11
Other	0	0	4	4
4B				
Collect x's	2	3	4	9
Clear fractions	7	11	4	22
Other	1	0	2	3
5A				
Clear fractions	7	11	5	23
Simplify 5/10	3	3	2	8
Other	0	0	3	3
5B				
Collect by cancelling	3	6	4	13
Clear fractions	7	8	3	18
Other	0	0	3	3
6A				
Distribute and collect	10	13	6	29
Other	0	1	4	5
6B				
Distribute and collect	3	3	0	6
Distribute twice	7	6	5	18
Cancel or divide	0	5	5	10
7A				
Distribute and collect	10	14	6	30
Isolate x	0	0	1	1
Other	0	0	3	3

Table 4.1 continued

Equation and choice	Top 10 Solvers	Middle 14 Solvers	Bottom 10 Solvers	All
7B				
Distribute, distribute, collect	8	12	9	29
Distribute, distribute, other, collect	2	1	0	3
Other	0	1	1	2

Table 4.2

Illustration of strategic choices listed in Table 4.1

1A	collect p's	$A=p+prt \longrightarrow A = p\ (1+rt)$
1B	attract	$2x=x^2 \longrightarrow 2x-x^2 = 0$
	cancel or divide	$2x=x^2 \longrightarrow 2=x$
2A	isolate	$\frac{1}{3} = \frac{1}{x} + \frac{1}{7} \longrightarrow \frac{1}{x} = \frac{1}{3} - \frac{1}{7}$
	clear fractions	$\frac{1}{3} = \frac{1}{x} + \frac{1}{7} \longrightarrow 7x = 21 + 3x$
	combine fractions	$\frac{1}{3} = \frac{1}{x} + \frac{1}{7} \longrightarrow \frac{1}{3} + \frac{7+x}{7x}$
2B	As for 2A	
3A	attract	$9(x+40)=5(x+40) \longrightarrow 9(x=40)-5(x+40)=0$
	multiply out	$9(x+40)=5(x+40) \longrightarrow 9x+360=5x+200$
	Cancel or divide	$9(x+40)=5(x+40) \longrightarrow 9=5$

Table 4.2 continued

3B	attract	$7(4x-1)=3(4x-1)+4\longrightarrow 7(4x-1)-3(4x-1)=4$
	multiply out	$7(4x-1)=3(4x-1)+4\longrightarrow 28x-7=12x-3+4$
	cancel or divide	$7(4x-1)=3(4x-1)+4\longrightarrow 7=3+\frac{4}{(4x-1)}$
	introduce variable	$7(4x-1)=3(4x-1)+4\longrightarrow 7y=3y+4$
4A	isolate	$xy+yz=2y\longrightarrow xy=2y-yz$
	factor out y	$xy+yz=2y\longrightarrow y(x+z) = 2y$
4B	Collect x's	$\frac{x+3+x}{x^2}=1\longrightarrow \frac{2x+3}{x^2} = 1$
	Clear fractions	$\frac{x+3+x}{x^2} = 1\longrightarrow x+3+x = x^2$
5A	clear fractions	$\frac{5}{10}=\frac{x-10}{x+5}\longrightarrow 5(x+5) = 10(x-10)$
	simplify 5/10	$\frac{5}{10} = \frac{x-10}{x+5}\longrightarrow\frac{1}{2} = \frac{x-10}{x+5}$
5B	collect by cancelling	$\frac{1-x^2}{1-x} = 2\longrightarrow 1+x = 2$
	clear fractions	$\frac{1-x^2}{1-x} = 2\longrightarrow 1-x^2 = 2(1-x)$
6A	distribute and collect	$x+2(x+1) = 4\longrightarrow 3x+2=4$
6B	distribute and collect	$x+2(x+2(x+2))=x+2\longrightarrow x+2(3x+4)=x+2$
	distribute twice	$x+2(x+2(x+2))=x+2\longrightarrow x+2x+4x+8=x+2$
	cancel or divide	$x+2(x+2(x+2))=x+2\longrightarrow x+2(x+2) = 1$
7A	distribute and collect	$x-2(x+1) = 14\longrightarrow -x-2=14$
	isolate x	$x-2(x+1)=14\longrightarrow x = 14+2(x+1)$
7B	distribute, distribute, collect	$6(x-2)-3(4-2x)=x-12\longrightarrow 12x-12-12=x-12$
	distribute, distribute, other, collect	$6(x-2)-3(4-2x)=x-12\longrightarrow 6x+6x-x=12+12-12$

should nonetheless be included for use in attraction, since it is sometimes necessary to set up collection. In equation 5B, collection is possible by factoring $1-x^2$ and dividing but it is uncertain that the pattern-matcher would discover this, so no favored choice is listed.

Of the 11 equations for which a choice is favored by the Bundy strategy, the most common choice made was the favored one for eight equations. Equations 3A and 3B are two of the exceptions, possibly because the favored choice requires dealing with repeated subexpressions as a unit. Equation 3A also offers a popular cancellation choice, which is not available as such in the Bundy model.

In the other exception, equation 6B, human solvers prefer to complete the clearing of parentheses before proceeding to combine terms, whereas the Bundy model would collect x's as it went along. In fact the multiplying out of the parenthesized quantities would be just a part of a collection operation and would not be done separately. It seems from solvers' comments that clearing of parentheses or multiplying out is itself a subgoal for many human solvers.

Although the Bundy model often identifies the most popular choice, there are a number of minority choices that deviate from it. Cancellation, already mentioned ~~for~~ equation 3A, is popular on equations 1B and 6B. The favored collection in equation 1A is apparently difficult, and even though there is no really viable alternative many solvers did not use it. Clearing and combining fractions are important for equations 2A and 2B, even though they increase the number of occurrences of x. Clearing fractions is also the most popular choice for 4B, 5A, 5B, where agreement with the Bundy scheme cannot be assessed.

There is some indication that better solvers may tend to agree more with the Bundy choices than do poorer solvers. For the eleven equations where a choice can be favored, there is only one case, 7B, where the bottom 10 solvers agreed with Bundy more often than the top ten, Equation 3A is a tie. Some of the lack of agreement of the poorer solvers comes from their having available spurious operations which lend themselves to deviant strategic choices.

<u>Consistency of strategic choices</u> Do solvers' strategic choices reflect a stable hierarchy of preference, like that in the model, so that similar choices would be made consistently for similar equations? This question deserves more attention, but can readily be examined only for equations 2A and 2B among the Session 1 problems, since only these two offer closely matched choices. As shown in Figure 4.1, there is a strong tendency for solvers to make similar choices on the two problems.

Figure 4.1

Consistency of Strategic choice for equation 2A & 2B

		Equation 2B		
		Isolate	Decollect	Other
Equation 2A	Isolate	16	1	2
	Decollect	2	6	2
	Other	2	0	3

Quadratics As shown in Fig. 3.4, the Bundy system is able to solve quadratic equations without any adjustment of strategy or operators, essentially by inventing the method of completing the square for each such equation. No solver used this method in the study. Instead, either of two methods which do require a strategic adjustment and special operators were used. The commoner method is factorization, in which the equation is put in the form $(ax + b)(cx + d) = 0$, and then split into two equations, $ax + b = 0$ and $cx + d = 0$, which are then solved separately. The other method is use of the quadratic formula, in which the equation is placed in the form $ax^2 + bx + c = 0$ and the roots are obtained by use of a formula in a, b and c. Table 4.3 shows the frequency of use of these two methods for the Session 1 equations. Cases are included whether or not the method was successfully completed and whether or not its use depended on an error. For example, equations 6A and 6B are not quadratic equations, but by error they were sometimes turned into quadratics. As can be seen, only five students used the quadratic formula. Of the five, three were among the top ten solvers, and two among the middle 14. Four of the five were members of the more experienced screened group.

Table 4.3

Frequency of use of methods for solving quadratic equations in Session 1

	Cases	Students
Factoring	48	26
Quadratic formula	7	5

Both of these methods require special handling. In both, the equation must be put in a form with zero on one side, while normal strategy for linear equations would often result in a non-zero constant on the opposite side from the terms containing x. Further, both the splitting operation and the use of the formula would have to be specified by special operators.

Strategy difficulties Some students appeared to have trouble selecting an appropriate operator to apply. In the Bundy scheme, such difficulties would reflect flaws in the control structure that sequences the phases of the solution process.

Table 4.4 collects cases in which students isolated one occurrence of the unknown when there was another occurrence that had not been collected. the many cases in which they occurred for equation 1A, the difficulty was probably brought on by inability to recall the correct collection operation.

Table 4.4

Premature Isolation

1A	S5,S6,S12 S13,S15,S20 S21,S22,S23	$A=p+prt \longrightarrow p = expr$ or $-p = expr$ where expr includes p
1B	S1,~~S5~~,S18	$2x=x^2 \longrightarrow x = expr$, where expr includes x
7A	S15	$x-2(x+1)=14 \longrightarrow x=14-2(x+1)$

Table 4.5 shows difficulties associated with the transition between the strategy of separating unknown terms from constants, which is appropriate for linear equations, and the strategy of gathering all terms on one side with zero on the other, which is appropriate for quadratic equations. The moment of decision is shown in the first case, S6 6B. The remaining cases under the "quadratic" heading are failures to set equal to zero at the appropriate moment. In the first two, E4 4B and S12 4B, the solvers did go on to do this. In the remaining cases, S7 4B and S16 4B, the solvers did not, but instead used invalid operations to "solve" the problems. So these solvers may not have made any strategic distinction here.

The cases under the heading "linear" are ones in which the "set equal to zero" strategy was applied to linear equations. This is not mathematically invalid, assuming it is done accurately, but it does involve unnecessary steps. In the first example shown, found in the work of two solvers, setting equal to zero is followed by factoring, as it might be for a quadratic. Again, this is mathematically valid, but of questionable value. The move might be worthwhile if it allowed the correct conclusion to be drawn about the role of y: that if y is zero than any value of x satisfies the equation. Neither student drew that inference, though S13 did get as far as "I'm not sure about this right here but I think you can get y equals zero and then x plus z plus two equals zero" before abandoning the problem.

Table 4.6 shows some more dramatic failures of strategic control. In each case, an earlier error resulted in the elimination of all occurrences of the unknown from the equation. A strategy that aimed at

Table 4.5

Strategy shift between linear and quadratic equations

Quadratic			
S6	6B	$x^2+4x+4=1$	"I'm trying to figure out if you can carry it further. I don't think so. I bring the four to the other side. I get x squared plus four oh ah, you could bring the one to this side so it'll be equal to zero."
E4	4B	$2x+3=x^2 \longrightarrow 3=x^2-2x$	"get the x's together"
S12	4B	$2x+3=x^2 \longrightarrow x^2-2x=3$	
S7	4B	$x^2 = x+3+x \longrightarrow x^2-x-x=3$	
S16	4B	$2x+3=x^2 \longrightarrow 2x+x^2=3$	
Linear			
S3, S13	4A	$xy+yz=2y \longrightarrow xy+yz-2y=0 \longrightarrow$ $y(x+z-2)=0$	
S13	2B	$\frac{1}{R} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \longrightarrow 0= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{R}$	
S13	5A	$5x+25=10x-100 \longrightarrow 0=5x-125$	
S13	7B	$6(x-2)-3(4-2x) = x-12 \longrightarrow$ $6x-12-12+6x-x+12 = 0$	

collecting and isolating the occurrences of the unknown should have signaled trouble immediately, but in all but one of these cases the students continued to work for some further steps. In the first case, S21 2C, it appears that the solver lost track of the left-hand side of the equation, zero, and interpreted the purely numerical expression obtained when the x's were cancelled as an expression for the answer. In the remaining cases the solver abandoned the equations when they were simplified. In the last case no simplification was needed after the elimination of p, and the solver simply proceeded to the next problem.

Table 4.6

Working with no unknown

S21 2C	$0 = \frac{-2-3x}{3-x} \longrightarrow \frac{-2}{3} - \frac{3}{1}$ $\longrightarrow \frac{-2}{3} - \frac{9}{3}$ $\longrightarrow - \frac{11}{3}$	"x goes into negative x goes into, x goes into that 3 times minus negative two over three. I'm destroying this problem I think. And just set that two over three make it two thirds, and that's negative eleven over three. I don't know how. That's what I got." stops
S20 3B	$7(4x-1) = 3(4x-1)+4 \longrightarrow$ $7=3(1)+4 \longrightarrow$ $7=3+4$ $7=7$	No comment
S20 5A	$\frac{5}{10} = \frac{\cancel{x}-10}{\cancel{x}+5} \longrightarrow$ $\frac{5}{10} = \frac{-10}{5} \longrightarrow$ $25 = -100$	Comments unintelligible
S20 1A	$\frac{-p}{p} = \frac{prt}{p} \longrightarrow -1=rt$	"That's about as far as it goes."

Just as having too few unknowns leads to trouble, so does having too many. Table 4.7 lists cases in which students proposed solutions expressed in terms of the unknown. As indicated in the table, in some cases there is indication that the solver is dissatisfied with the solution, but the table does not include cases in which the solver stated clearly that the last equation they reached did not give a solution. It seems likely that at least some of the solvers represented in the table do not clearly grasp the fundamental unacceptability of solution in terms of the unknown; they may be influenced by the similarity in form between these solutions and correct ones.

Proper handling of equations containing fractional expression requires an addition to the normal strategy that is not present in the Bundy model. It may happen that a number or expression obtained by normal manipulation as a root of an equation renders the equation undefined when substituted for the unknown, because of division by zero. Such a number or expression is not a solution of the equation, since it does not make the sides of the equation equal when substituted and is called an extraneous root. So in solving equations containing division by expressions containing the unknown, it is necessary to verify that any putative roots do not create this problem. Table 5.14 shows the equations used in the study for which this issue arises, and the subjects whose solutions included extraneous roots. As can be seen, many solvers do not perform the needed check. They may identify the goal of obtaining solutions to an equation with the slightly different goal of obtaining the results of normal manipulations.

Table 4.7

Solution containing unknown

S1 1B	$x = \frac{x^2}{2}$	"I guess that's simple enough. It's not really but that's the best I can do."
S3 1A	$\frac{A}{prt} = p$	"Am I supposed to get that p out of there? I don't know."
S5 1B	$x = \frac{1}{2} x^2$	Comments not recorded
S13 1A	$\frac{A}{rt} -p = p$	"That's as far as I can go since there's no number values for any of these variables."
S13 2D	$ac-cbx-cb-1 = x$	No comment
S13 1E	$x = \frac{b-x}{a}$	E: "Is there a rule that you can give me for knowing when I'm finished with the problem, or what?" S: "Ok, well there's I can't think of any way to get rid of this x over here so it would just have to be..." tries other manipulations which do not get rid of the other x "So as soon as you get x on one side, that's that's about it." E: "If I could get rid of that should I try to, or what would you say about that?" S: "Yuh, that's what I try to do at at first. To get rid of the x on the one side. But I couldn't, couldn't see."
S16 5E	$\frac{x^2}{3} = \frac{x}{3} \longrightarrow$ $x^2 = \frac{1}{2}x$	"x squared would be one half x. I'm going to leave that as my answer". Later: "I had x^2, but the answer denotes x, so it would have to be half of x^2."
S18 1B	$x = \sqrt{2x}$	No comment
S22 1A	$A-prt=p$	No comment
S23 2B	$\frac{x^2}{r} - \frac{x^2}{y} - \frac{x^2}{z} = x$	"real dumb answer."

The strategy pursued by the Bundy model is largely a direct translation of the demands of the solving task. It is necessary to eliminate all but one occurrence of the unknown, and it is necessary to isolate the single remaining occurrence. The model's strategy uses these necessities to organize the phases of the method. In light of this, it is not surprising to find some agreement between human performance and the behavior of the model. It is more surprising to find greater agreement in the performance of better solvers, though as noted above this is probably due only partly to differences in strategy itself between better and poorer solvers, since some contribution is made by spurious operators available to poorer solvers.

The common departures from the Bundy strategy are interesting. It appears that clearing fractions and removal of parenthesesare important operations. This suggests that human solvers may to some extent use a canonical form in the solving process, a form free of fractions and parentheses. This could also explain why repeated subexpressions were rarely used: they disappear when the problem is canonicalized.

Another departure is in the handling of quadratics. Unlike the Bundy model, it appears that human solvers need at least two strategies, one for linear and one for quadratic equations, with a means of selecting the appropriate one.

The difficulties students have in strategy seem broadly to be what one would expect when one allows flaws in the Bundy scheme, except that some of the flaws seem very fundamental. It is remarkable that some students continue to do algebra as usual after the unknown has vanished, and that others propose solutions in terms of the unknown. A lack of knowledge of the goal of equation solving is implicit in such performance, and it would be interesting to know if explicit instruction would help.

CHAPTER 5

Analysis of Errors

The Bundy model is a model of correct performance, but it nevertheless provides the basis for a theory of errors. Errors would be produced if incorrect operators were placed in the operator store, if correct operators were applied where their conditions of applicability are not met, or if the actions called for by an operator were not fully or accurately carried out. We present here a tabulation of the errors exhibited in the study, separated into three categories: operator errors, applicability errors, and execution errors. Within each category an effort has been made to collect errors that share various more particular characteristics.

Many errors could be assigned more than one origin. There is no obvious way to distinguish the application of an incorrect operator from the imperfect execution of a correct one, for example, on the basis of an isolated occurrence. The reader should therefore attend to the errors themselves as much as to the classification, treating the classification as a suggestion concerning mechanism behind the error that might be explored more fully. Also, the grouping in many cases reveals a pattern of similarities among errors that may be important in identifying mechanisms.

Operator Errors Table 5.1 shows a collection of errors which involve the deletion of elements from expressions. The mathematically valid operations which these errors apparently approximate are subtraction from both sides of an equation, division of both sides, simplification of quo-

Table 5.1

Operator Errors

Group 1	Simplification of quotients
S1 1C	$\frac{2kw + 2h\ (k+w)}{h} \longrightarrow 2kw + 2(k+w)$
S1 1E	$\frac{x + ax}{a} \longrightarrow x+x$
S1 1D	$\frac{2x^2 - x}{2} \longrightarrow x^2-x$
S5 5B	$\frac{1-x^2}{1-x^1} \longrightarrow x^1$
S5 5A	$\frac{x-10}{x+5} \longrightarrow -2$
S5 5D	$\frac{x}{2+x} \longrightarrow \frac{1}{2}$
S16 5E S20 5E	$\frac{x^2 + (x+3)}{3(x+3)} \longrightarrow \frac{x^2}{3}$
S16 5D	$\frac{8x}{2+x^2} \longrightarrow \frac{8}{2+x} \longrightarrow \frac{4}{x}$
S20 5A	$\frac{x-10}{x+5} \longrightarrow \frac{-10}{5}$
S20 4B	$\frac{x+3+x}{x^2} \longrightarrow \frac{x+3}{x}$
S20 5B	$\frac{1-x^2}{1-x} \longrightarrow \frac{1-x}{1}$
S20 4C	$\frac{xy+yz+xz}{x} \longrightarrow y + yz + z$
S20 4E	$\frac{ax + bx + ab}{a} \longrightarrow x + bx + b$
S20 4E	$\frac{2x + 2a}{a} \longrightarrow 2x + 2$
E4 5B	$\frac{(x+1)\ \cancel{(x+1)}}{(1+x) - \cancel{(x+1)}} = -2 \longrightarrow x+1 = -2$

Table 5.1 continued

S20 6C	$\frac{x^2+2x+1}{2} \longrightarrow x^2+x+1$
S22 5E	$\frac{x^2+x+3}{x+3} \longrightarrow x^2$
S16 5B	$\frac{1-x^2}{1-x} \longrightarrow x^2$
S20 6D S11 6D	$\frac{x+2(x\ \ 2)}{x+2} \longrightarrow x+2$
S20 5B	$\frac{1}{1}-x \longrightarrow -x$
S5 6B	$\frac{x+2}{x+2} \longrightarrow 0$
S5 1A	$\frac{p}{p} \longrightarrow 0$
Group 2	Simplification of quotients with variants of deletion operator replace null numerator by zero instead of 1
S20 5D	$\frac{x}{2+x} \longrightarrow \frac{0}{2}$
	if numerator is null, result is denominator
S12 4D	$\frac{bc}{-abc} \longrightarrow -a$
S1. 1C	$\frac{A}{h} \Big/ A \longrightarrow h$
	replace s by 1 when it appears as a term in a sum
S20 4B	$\frac{x+3}{x} \longrightarrow 1+3$
S12 1A	$p-A = -prt \rightarrow \left(\frac{1}{p}\right)$ $p-A = -prt \left(\frac{1}{p}\right) \longrightarrow$ $1-A = -rt$
	replace s by 0 when it appears as a factor of a product

Table 5.1 continued

S15	1E	$\frac{ax}{a} \longrightarrow 0$
		treat $\frac{x}{x}$ as x
S5	5D	$\frac{x}{x} \longrightarrow x$
	5A	$\frac{x-10}{x+5} \longrightarrow x-2$
S16	5B	$\frac{2-2x^2}{2-2x} \longrightarrow x^2$
S21	5A	$\frac{x-10}{x+5} \longrightarrow x-2$
		subtract s from terms containing it
S20	7B	$\frac{12x-24}{12} \longrightarrow x-12$
S1	4A	$\frac{2y}{y} \longrightarrow y$
S22	5C	$\frac{(3+x)x}{1+x} \longrightarrow 2+x$
		perform deletion only on numerator
S5	1A	$\frac{prt}{p} \longrightarrow \frac{rt}{p}$
		when deleting a power from a term with a lower power as a factor, subtract exponents
S5	4B	$\frac{2x^1 +3}{x^2} \longrightarrow 2x^{-1} +3$
Group 3		deletions from both sides of an equation
S17	3E	$2(3+x) = 4(x+3)-1 \longrightarrow x+3=2(x+3)-1$
S20	3B	$7(4x-1)=3(4x-1)+4 \longrightarrow 7=3(1)+4$
S16	4E	$\begin{array}{ccccccccc} a^2 & + & b^2 & + & x^2 & = & 2x & + & 2a \\ -a & & & - & x & & -x & & -a \\ a & + & b^2 & + & x & = & 2 & + & 2 \end{array}$

Table 5.1 continued

S20	7A	$x^2-x\frac{-2}{-2} = \frac{14}{-2} \longrightarrow x^2 -x = -7$	
	6A	$x^2+3x+\frac{2}{2}=\frac{4}{2}\rightarrow x^2+3x=2$	
S22	1B	$x^2= 2x\longrightarrow x=2$	Comment:"could I subtract an x from these ones"
	6B	$x+2(x+2(x+2))=x+2\longrightarrow x+2(x+2)=0$	Comment:"subtract"
	4B	$x+3+x = x^2\longrightarrow 3+x = x$	
	7B	$\frac{12X}{12} = \frac{x+12}{12} \longrightarrow x = x$	
S22 E4 E12	6B	$x+2(x+2(x+2))=x+2\longrightarrow x+2(x+2)=0$	
S5	6B	$(x+2)(x^2+4x+4)=x+2\longrightarrow 0=x^2+4x+4$	
Group 4		Deletion from one side of equation	
S3	1A	$A=p +prt\rightarrow \frac{A}{prt} = p$	
S18	1E	$x(a+1)=b\longrightarrow x=b-a-1$	
S16	1A	$A=p+prt\longrightarrow (rt)A = p+p$	
S22	6A	$x+2(x+1)=4 \longrightarrow x+1 = 4(x+2)$	
S22	5D	$(4+2/x)\left(\frac{x}{2+x}\right) = 7\longrightarrow 2/x) \left(\frac{x}{2+x}\right. = 3$	
Group 5		Subtraction	
S22	4B	$x^2 -x\longrightarrow x$	
S16	4A	$2yz\longrightarrow z$ $-2y$	
E2	7B	$6x-\cancel{12} -\cancel{12} +6x = x-12\longrightarrow 11x=-12$	

tients, and subtraction. As figure 5.1 shows, all of these correct operations do produce deletions in common situations. Inspection of the errors suggests that students identify these correct operations with a single generic deletion operation which often produces incorrect results. The following is a tentative description of this deletion operator and its use.

Figure 5.1

Some legal operations producing deletions

$$\underset{-a}{x+a} = \underset{-a}{b+a} \longrightarrow x = b$$

$$ax = ab \longrightarrow x = b$$

$$\frac{ax}{a} \longrightarrow x \qquad x + a - a \longrightarrow x$$

The deletion operator transforms an expression e by deleting a specified subexpression s from it. If the resulting expression has operation signs which are missing an operand because of the deletion, these signs are deleted. Similarly, unnecessary parentheses are deleted. If s is a constant and a constant k=c.s appears in e, the effect of the deletion of s is to replace k by c in e. Figure 5.2 shows examples of the deletion operator in action.

The deletion operation is used to simplify quotients when a subexpression s appears in the numerator and denominator, either explicitly or as a factor of a constant. The simplification proceeds by deleting s from the numerator and from the denominator. If the resulting numerator and denominator are both nonnull, the result is their quotient.

Figure 5.2

Examples of deletions

delete h from $2kw + 2h(k+w)$: $2kw + 2(k+w)$

delete x from $x-10$: -10

delete 5 from -10 : -2

delete x from $2+x^2$: $2+x$

delete 2 from $2+x$: x

If the denominator but not the numerator is null, the result is simply the numerator. If the numerator is null, but not the denominator, the numerator is replaced by 1. If both numerator and denominator are null, the result is zero. Figure 5.3 shows examples of this process, which can produce the first group of errors in Table 5.1. As shown in Figure 5.4, the procedure also correctly handles some simple examples. Some errors call for variations of the scheme; these are indicated in the second part of Table 5.1.

Figure 5.3

Use of deletion to simplify quotient

$\frac{x-10}{x+5}$ delete x from numerator: -10

delete x from denominator: 5

delete 5 from numerator: -2

delete 5 from denominator : 0

denominator is null, so result is numerator: -2

Figure 5.3 continued

$\frac{8x}{2+x^2}$

delete x from numerator: 8

delete x from denominator: 2 + x

delete 2 from numerator: 4

delete 2 from denominator: x

result: 4/x

$\frac{x}{2+x}$

delete x from numerator = 0

delete x from denominator: 2

numerator is null, so replace by 1

result: 1/2

Figure 5.4

Examples for which deletion operation correctly simplifies a quotient

$\frac{ax}{a} \longrightarrow x$ (shows deletion of a and dropping of null denominator)

$\frac{x}{ax} \longrightarrow \frac{1}{a}$ (shows need to replace null numerator by 1)

The same deletion operation may be used to transform an equation, if a subexpression s appears on both sides. The procedure is to delete s from each side, replacing the side by 0 if it is null. The third group of errors in Table 5.1 is produced by this procedure, while the fourth group calls for the indicated variations. As in the case of simplification of quotients, this solver does transform some equations correctly, as in Figure 5.5.

Figure 5.5

Examples for which deletion operation correctly simplifies an equation when applied to both sides

$$ax = ay \longrightarrow x = y$$

$$x + a = y + a \longrightarrow x + y$$

Another case in which deletion may be used arises when a subexpression s is to be removed from one side of an equation, but does not appear on the other side. Here s can be deleted from one side, but must be subtracted from or divided out of the other side. It is not clear how the choice of subtraction or division is made in such a case. Examples of errors which may arise in this way are shown in the fourth group of Table 5.1. Note that the last errors in this group involve a multiplication done on the other side rather than a subtraction or division.

The final group of errors in Table 5.1 may result from the use of deletion to carry out an indicated subtraction within one side of an equation.

Table 5.2 illustrates a constraint on the deletion operator stated by one subject. The subject indicated that cancellation could not be used to simplify either quotient because the entire denominator did not appear in the numerator.

Table 5.2

Examples with cancellation blocked

S5	5E	$\frac{x^2+ (x+3)}{3(x+3)}$
S5	5E	$\frac{x^3 + 3}{3x + 9}$

Another constraint is shown in Figure 5.6. Two excerpts from the protocols of the same subject are shown, in which the subject states that items cannot be cancelled unless they are multiplied.

Figure 5.6

Examples of Erroneous Cancellation

	Cancels h's
S1 1C $\frac{A}{h} = \frac{2kw + 2h(k+w)}{h}$	E: "How do you know when something cancels and when it doesn't like that?" S: "Ok, if it's in the, if it's in the denominator, if it's on the top and bottom. Then ah, and if it's multiplied, if it's added you can't do it. Like if I had a w there (points to second w) and a w down here (points to denominator) I couldn't cancel that w out (points to second w) but I could cancel that one (points to just w)" E: "OK" S: "Because that's multiplied and not added."
S1 6D $\frac{x+2\ (x+2)}{x+2} = 2$	S: "Well, if I treat this as one unit right here (points to second x+2 and puts parentheses around denominator) I could cancel this out (points to second x+2 and denominator) because that's one unit (points to second x+2) that's being multiplied by another unit (points to first x+2) right there. (Puts parentheses around first x+2.) Since they're all the same you can do that, I guess, and then that would still apply to my rule, you know, you've got to be multiplying."

This restriction avoids many simple errors, like transforming (x+a)/a to x, but this may actually be unfortunate, since it makes the invalidity of the operator less likely to be spotted. It is interesting that this constrained operator is quite a reasonable guess at an operator that transforms ax/a to x but does not transform (x+a)/a to x, and illustrates what might happen in learning from a few examples.

Transposition errors Table 5.3 shows some errors, all from one subject, that involve an erroneous operation in moving elements from one side of an equation to another but seem not to involve the deletion operator. In the first three examples a term is subtracted from the side of the equation where it is found, thus deleting it, and is added to the other side. The last example, Problem 7B, follows this pattern in the handling of the term x, but not in the handling of -24.

Table 5.3

Transposition errors

S16 3B	$28x - 11 = 12x - 3 \longrightarrow 40x = -14$
S16 6B	$\begin{array}{rcl} 7x + 8 & = & x + 2 \\ \underline{+x} & & -x \\ 8x+8 & = & 2 \end{array}$
S16 3A	$9x+360 = 5x+200 \longrightarrow 9x+5x = 360+200$
S16 7B	$12x-24 = x-12 \longrightarrow 12x + x = 24 + -12$

Recombination operations Table 5.4 shows errors involving the recombination of elements of two or more terms. These errors may arise from an interpretation of addition and multiplication in which both represent a generic combining operation. For example, $x+x$, $2x$, $x \cdot x$, and x^2 may all be thought of as combination of two x's. Since many different expressions are given the same interpretation in this way, it

Table 5.4

Recombination errors

S5	5E	$x^2 + x + 3 \longrightarrow x^3 + 3$
S5	5E	$-3x^3 + 3x^2 \longrightarrow -3x$
S16	4A	$y + yz \longrightarrow 2yz$
S16	1A	$p+p \longrightarrow p^2$
S16	4B	$2x + x^2 \longrightarrow 3x^2$
S16	5E	$x^2 + (x+3) \longrightarrow 2x^2 + 3$
S16	5E	"x is one half of x^2"
S16	4E	$ax + bx + ab \longrightarrow a^2 + b^2 + x^2$
S22	4A	$xy + yz \longrightarrow x+2yz \longrightarrow xz + 2y$
S22	4C	$xy + yz + xz \longrightarrow x^2yz \longrightarrow x^2 \quad 2yz$
S22	4D	$bcx + acx + abx + abc \longrightarrow 3\ abc + 3x$
S22	4E	$ax + bx + ab \longrightarrow 2a + 2b + 2x$

is possible to rearrange the elements of an expression quite freely without changing its interpretation. Figure 5.7 shows how some of the errors in Table 5.4 can be produced.

Figure 5.7

Recombination of elements of expressions

$x^2 + x + 3 \longrightarrow$ "two x's" and "one x" and "three"

$\longrightarrow$ "three x's" and "three"

$\longrightarrow x^3 + 3$

$y + yz \longrightarrow$ "one y" and "one y" and "one z"

$\longrightarrow$ "two y's" and "one z"

$\longrightarrow 2yz$

Two subjects mentioned restrictions on the possible recombinations. In rearranging ax + bx + ab as $a^2 + b^2 + x^2$, the subject noted, "Ok what you're doing here is you're multiplying all three of these things together; the a to the x, the b to the x, the a to the b. So it's a case of multiplication not a case of addition. An x plus an x would be an x squared instead of x + x, and a times an a would be an a squared, and a b times a b would be a b squared. But if this was a plus in here (writes plus between b and x in original expression) instead of multiplication then you couldn't do this." The suggestion may be that elements which are recombined into products should have appeared in products in the original expression. Another subject, asked to explain a similar recombination, said "I don't really know the rules. It makes it easier if you combine everything that's the same. You have, like, the same thing if you're working with numbers and you have 1,1,1 it's just easier to say three." The same subject had rearranged ax + bx + ab as 2a + 2b + 2x. The response was, "No, cause here you're multiplying. 2ab is the same as 2 times a times b and here it's all addition." If the subject means by "here" the rearranged expression 2a + 2b + 2x rather than the original expression, this may reflect the same constraint on products described by the subject above. The subject went on, "No, it's not anyway. It just makes it easier to keep them separate." In the context of the problem, the term 2a could be cancelled at the next step, so the subject's preference for one rearrangement over another may be based on that.

<u>Combining Fractions</u> Table 5.5 shows errors arising in adding or multiplying fractional expressions. Most of the operators apparently employed

Table 5.5

Errors in combining fractions

S1 6C	$\frac{x}{1} + \frac{x+1}{2} \longrightarrow \frac{x+x+1}{2}$
S15 5D	$(4+2/x) \left(\frac{x}{2+x}\right) \longrightarrow \frac{4(x(2+x)+2(2+x)(x^2)}{x(2+x)}$
S16 7E	$\frac{7}{1} - \frac{x+3}{x} \longrightarrow \frac{7-x+3}{1-x}$
S16 5D	$\frac{4}{1} + \frac{2}{x} \longrightarrow \frac{8}{x}$
S22 2D	$\frac{b}{1} + \frac{1}{c} \longrightarrow \frac{b}{c}$
S22 2C	$\frac{1}{7} + \frac{7}{3-x} \longrightarrow \frac{1}{3-x}$
S16 2B	$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \longrightarrow \frac{3}{x+y+z}$
S22 2B	$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \longrightarrow \frac{1}{xyz}$
S20 2A S21	$\frac{1}{3} - \frac{1}{7} \longrightarrow \frac{10}{21}$

involve suboperations and patterns of operations that are seen in correct operations on fractions. These elements are shown in Figure 5.8, which also shows how they can be incorrectly assembled to form operators that produce some of the errors in the Table. In one case, a student who combined 4/1 + 2/x to obtain 8/x said that the operation involved was "multiplication of fractions", and said that it was only appropriate when fractions were to be added. Here the entire correct operation for multiplication is carried over for use in addition.

Figure 5.8

Elements of correct operations on fractions

$$\frac{a}{b}+\frac{c}{d} \longrightarrow \begin{array}{l}\text{form common}\\ \text{denominator}\end{array} \quad \frac{ad}{bd}+\frac{bc}{bd} \longrightarrow \frac{\text{sum of numerators}}{\text{denominator}} = \frac{ad+bc}{bd}$$

$$\frac{a}{b}\cdot\frac{c}{d} \longrightarrow \frac{\text{product of numerators}}{\text{product of denominators}}$$

Note that this is a component-wise operations: the indicated operation is performed separately in numerator and denominator.

How these elements may be combined to give errors.

$$\frac{x}{1}+\frac{x+1}{2} \longrightarrow \frac{\text{sum of numerators}}{\text{product of denominators}} = \frac{x+x+1}{2}$$

$$(4+2/x)\left(\frac{x}{2+x}\right) \longrightarrow \begin{array}{l}\text{form common}\\ \text{denominator}\end{array} \quad \left(\frac{4(x(2+x))}{x(2+x)}+\frac{2(2+x)}{x(2+x)}\right)\left(\frac{x^2}{x(2+x)}\right)$$

$$\longrightarrow \frac{\text{combination of numerator under indicated operations}}{\text{denominator}}$$

$$= \frac{(4(x(2+x))+2(2+x))(x^2)}{x(2+x)}$$

$$\frac{7}{1}-\frac{x+3}{x} \longrightarrow \text{component-wise subtraction} = \frac{7-x+3}{1-x}$$

$$\frac{4}{1}+\frac{2}{x} \longrightarrow \frac{\text{product of numerator}}{\text{product of denominator}} = \frac{8}{x}$$

A possible related family of errors is shown in Table 5.6. These may be seen as the result of inverting the component-wise addition of fractions seen in some examples in Table 5.5.

Table 5.7 shows two errors from one subject which may be related to the errors in combining fractions just discussed. In each the subject inverts one fraction of a sum. This operation may be based on the inversion used in the common method of dividing fractions. This inter-

Table 5.6

Errors in decomposition of quotients

S16 5C S20 5C S21 5C	$\frac{3+x^2}{1+x} \longrightarrow \left|\frac{3}{1}\middle|\frac{+}{+}\middle|\frac{x^2}{x}\right| \longrightarrow 3+x$	
7E	$\frac{7-(x+3)}{1-x} \longrightarrow 7+x+3$	Comment "negative x into negative x gives positive x"
S21 2C	$\frac{-2-3x}{3-x} \longrightarrow \frac{-2}{3} - \frac{3}{1}$	
S22 5B	$\frac{1-x^2}{1-x} \longrightarrow 1-x$	

pretation is supported by the fact that the second rearrangement was in fact transformed to x/7, which is the product of the two fractions after inversion. This follows the division procedure, except that the second fraction would be the one inverted if the notation a/b ÷ c/d were used.

Table 5.7

Inversion errors

S22 2A	$\frac{1}{x} + \frac{1}{7} \longrightarrow \frac{1}{x} + \frac{7}{1}$
	$\frac{1}{x} + \frac{1}{7} \longrightarrow \frac{x}{1} + \frac{1}{7}$

<u>Cross-multiplication</u> The correct cross-multiplication operation transforms an equation of the form a/b = c/d into ad = bc. Errors involving this operation are shown in Table 5.8. It appears that the operation is sometimes used on an equation but altered to produce a fraction as a re-

Table 5.8

Errors involving cross multiplication

S1	5A	$\frac{1}{2} = \frac{x-10}{x+5} \longrightarrow \frac{x+5}{2(x-10)}$	
S1	5B	$\frac{1-x^2}{1-x} \div \frac{2}{1} \longrightarrow \frac{1-x^{2}}{2-2x}$	
S5	2C	$\frac{22}{7} - \frac{7}{3-x} \longrightarrow 66 - 22x - 49$	
S5	2D	$\left(\frac{1}{b+\frac{1}{c}}\right)\left(\frac{a}{1}\right) \longrightarrow \frac{1}{a\left(b+\frac{1}{c}\right)}$	comments: says is cross multiplication
S5	6C	$x+1 \quad \left(\frac{x+1}{2}\right) = 1$	comments: begins to cross multiply and writes 2x+2 = before remarking "can't have two equal signs"
S1	2A	$\frac{1}{x} + \frac{1}{7} \longrightarrow 7+x$	
S1	2A	$\frac{1}{x} + \frac{1}{7} \longrightarrow \frac{x}{7}$	
S20	6E	$\frac{x}{1} = \frac{-8}{x} - 6 \longrightarrow x^2 = -8 - 6$	

sult, perhaps by analogy to other operations on pairs of fractions, and may also be applied to a sum, difference, or product of fractions rather than to an equation, again by analogy to other operations on pairs of fractions. In both situations, the cross-multiplication operation yields two terms, ad and bc, which must be used to form the result, so one might be used as numerator and the other as denominator as in S1 2A and S5 2D, or they might be combined using the indicated operation on the original pair of fractions, as in S5 2C and S1 2A. Even when the operation is used correctly to transform an equation to an equation trouble is possible if the conditions of application are too loose, as shown in the last example in the table.

Errors S5 2C and S1 2A are examples of what Matz (1979b) has called the "Lost Common Denominator Bug." We include them here because, as indicated above, they could result from the adaptation of cross multiplication as an operation that combines pairs of fractions. Case S5 6C seems to indicate this possibility quite clearly, since the solver was using the cross multiplication operator on two fractions that were multiplied, and stopped when the operator generated an extra equals sign. In the earlier case S5 2D the cross multiplication was completed.

Matz attributes this error to partial execution of the addition of fractions operation. Our analysis of course does not mean that Matz is wrong, since the same surface form can have many origins. Our analysis does, however, complicate efforts to pin down the origins. Specifically, Matz suggests that a reduced incidence of the error when adding numerical fractions would favor the partial execution explanation because the processing load would be reduced for this simpler problem. But subjects have probably never applied cross-multiplication to purely numeric examples, so the frequency of erroneous use of cross-multiplication would also presumably be less.

Splitting equations with fractions Table 5.9 shows the use of an operator which splits an equation with fractions into an equation in which the numerators are set equal and one in which the denominators are set equal. This splitting is occasionally valid, as in the case of the equation $x/(x+1) = 2/3$, but in general it is not, since the two equations in general do not have the same solution. This use of this operator is discussed below, where it is argued that it reflects an eccentric view of how equation solving works, in which it assumed that a solution can be obtained by trying to make the sides of the equation match, rather than by the usual manipulations.

Table 5.9

Splitting equations with fractions

S16	5A	$\frac{5}{10} = \frac{x-10}{x+5} \longrightarrow \begin{array}{l}5=x-10\\10=x+5\end{array}$
	5E	$\frac{2x^2+3}{3x+9} = \frac{x}{3} \longrightarrow \frac{2x^2+3=x}{3x+9=3}$
	4B	$\frac{2x+3}{x^2} = 1 \longrightarrow 2x+3=1$
S10	5A	$\frac{5}{10} = \frac{x-10}{x+5} \longrightarrow \begin{array}{l}5=x-10\\10=x+5\end{array}$
S16	5B	$\frac{1-x^2}{1-x} = \frac{2}{1} \longrightarrow x^2 = 2$

The reciprocal operation Table 5.10 shows errors resulting when the reciprocal of a sum is taken improperly. In each case, the error reflects an assumption that the function $f(x) = 1/x$ is linear, so that $1/(a+b) = 1/a + 1/b$. This sort of assumption has been noted by Matz (1979b) in connection with this and other functions, including square-root.

Division by zero If an equation is transformed by dividing both sides by an expression whose value is zero an inequivalent equation may result. Neglect of this condition on division may result in losing solutions,

Table 5.10

Errors in forming reciprocals

S1 2B	$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \longrightarrow \frac{1}{x+y+z}$
S5 2D	has $\frac{1}{x+1}$ = expr but drops the 1/ , getting x = expr -1. Corrects this to $x = \frac{1}{\text{expr} -1}$
S10 2B S11 2B	$\frac{1}{R} - \frac{1}{y} - \frac{1}{z} = \frac{1}{x} \longrightarrow R-y-z=x$
S12 4D	$\frac{1}{x} = -a -b -c \longrightarrow x = -\frac{1}{a} - \frac{1}{b} - \frac{1}{c}$ $\frac{1}{x} = -\frac{1}{a} - \frac{1}{b} - \frac{1}{c} \longrightarrow x = -a-b-c$
S22 2A	notes $\frac{1}{3} = \frac{1}{-4} + \frac{1}{7}$

as illustrated in the first group of errors in Table 5.11. Thus the division operator should carry with it actions besides the division itself to provide for detecting any lost roots. One way to do this would be to create two new equations rather than one: a = b produces a/s = b/s or s = 0. The final error shown in the first group is a subtle one perhaps stemming from such an elaborated division operation. The subject may have derived a/s = b or s = 0 from a/s = b, whence it does not follow. This is because if division by s is indicated in an expression, s cannot be zero if the expression is to be defined.

Group 2 of Table 5.11 shows subjects who lost roots on the equation $2x = x^2$, but not by division. These subjects simply noted that 2 was a root, but failed to notice the other root.

Table 5.11

Loss of roots

Group 1	Division by zero
1B E4,E6,E10 E11,E13 S7,S10,S13 S19,	$2x=x^2 \longrightarrow 2=x$ (zero is also a root)
3A, E3,E6,E9, E10,E11,S19	$9(x+40) = 5\ (x+40) \longrightarrow 9=5$ (-40 is a root)
S13 3A	$9(x+40) = 5\ (x+40) \longrightarrow \frac{9(x+40)}{5(x+40)} = 1$ $\longrightarrow \frac{9}{5} = 1$
E14 5B	$\frac{\cancel{(1-x)}(1-x)}{\cancel{(1-x)}} \longrightarrow \begin{cases} 1-x=0 \\ 1+x=2 \end{cases}$ Comment: "cancel these, 1-x by 1-x. If 1-x is equal to zero, and x=1. And the other answer would be..."
Group 2	**Solution by Inspection**
1B E7,S12,S20	$2x=x^2 \longrightarrow 2=x$

Splitting factored quadratic equations Table 5.12 shows two errors, from one subject, in which the operations for solving a quadratic equation by factoring are closely followed. In the key step shown, however, the student splits the equation into two without having first placed it in the form expr = 0. The splitting operation, as used by this subject, lacks an essential check on its application.

Square root Table 5.13 shows some errors associated solving equations of the form $x^2 =$ expr. The last two examples may simply reflect a choice of only one root, and so may belong with the loss of root errors just discussed, but they may also be related to the puzzling error of the same subject on Problem 5B.

Table 5.12

Splitting quadratic not equal to zero

S7	5B	$x(x-2)=-2 \longrightarrow x = -2$ or $x-2 = -2$
S7	4B	$x(x-2)=3 \longrightarrow x = 3$ or $x-2 = 3$

Table 5.13

Errors associated with square root

S19	1B	$2x = x^2 \longrightarrow 2=x \longrightarrow \pm 2=x$	comments: "but this is squared so it could be plus or minus two"
S16	5B	$x^2 = 2 \longrightarrow$ answer is -1	
S16	4B	$x^2 = 1 \longrightarrow x = -1$	
S16	4B	$3x^2 = 3 \longrightarrow x = -1$	

Extraneous roots It may happen that a number or expression obtained by normal manipulation as a root of an equation renders the equation undefined when substituted for the unknown, because of division by zero. Such a number or expression is not a solution of the equation, since it does not make the sides of the equation equal when substituted and so is called an extraneous root. So in solving equations involving division by expressions containing the unknown, it is necessary to verify that any putative roots do not create this problem. Table 5.14 shows the problems used in the study for which this issue arises, and the subjects whose solutions included extraneous roots. As can be seen, the false assumption that the results of algebraic manipulations are always solutions is widely held. Subjects may identify the goal of obtaining solutions to an equation with the slightly different goal of obtaining the results of normal manipulations.

Table 5.14

Extraneous roots

5B E1,E5,E6 E12,E13,E14 S6,S10,S11 S12,S15,S19	$\frac{1-x^2}{1-x} = 2$	extraneous root = 1
S10 5D	$(4+2/x)\left(\frac{x}{2+x}\right) = 7$	extraneous root = 0

<u>Arithmetic errors</u> Table 5.15 shows errors which seem to involve simply incorrect arithmetic on unsigned numbers.

Table 5.15

Arithmetic errors

S15 3B	"eleven minus three nine"	
S21 3B	$7(4x-1) \longrightarrow 21x-7$	
S5 2E	$(2) - 3\frac{1}{2} \longrightarrow 6.5$	multiplying

Operator gaps Just as invalid operators might be included in the operator store in the Bundy model, so might valid operators be missing. As would be expected from this, some solvers seem to lack certain operators, and so have trouble when they reach a point in a solution where such an operator is needed. Table 5.16 collects cases in which solvers abandoned work on a problem or backtracked to an earlier point in their solution, when a legal operator exists that would have permitted progress towards solution. In these cases it is plausible that the solver lacks the needed operator, though it is also quite possible that the solver knows the operation and actually considered using it but decided not to. Where this latter possibility is suggested by the solver's comments the case is not included in the Table.

The cases in the table are grouped according to the operator that is not used. The first group involve the inversion operator: the solver transforms the equation to the form 1/x = expr, where expr is free of x, but cannot finish the job by putting x = 1/expr. Actually, the lack of this operator probably is more common than these examples show, because only cases in which the solver found no operator to use are included. Some subjects transformed 1/x = expr to 1 = expr to x = 1/expr, and it is likely that some of these did not have the inversion operator available.

The second group of cases involve other expressions with x in the denominator. Here is seems unlikely that the solvers do not possess the needed operator, multiplying both sides by the denominator, but rather that they do not use it. Such difficulties could be described in the Bundy framework as improper labeling of operators: the multiplication operation needed might not be marked as useful for isolation or attrac-

Table 5.16

Operator gaps

Group 1	Inversion	
S6 2B	$\frac{-1}{x} = \frac{-yz+Rz+Ry}{Ryz}$	
S12 2B	$\frac{-1}{x} = \frac{1}{y} + \frac{1}{z} - \frac{1}{R}$	
S21 2A	$\frac{10}{21} = \frac{1}{x}$	
S23 2B	$\frac{1}{x} = \frac{1}{R} - \frac{1}{y} - \frac{1}{z}$	
Group 2	Other cases with x in denominator	
S1 4B	$\frac{2x+3}{x^2} = 1$	
S1 2B	$\frac{1}{R} = \frac{1}{x+y+z}$	
S16 2B	$\frac{x}{R} = \frac{3}{x+y+z}$	
S21 5B	$\frac{1-x^2}{1-x} = 2$	not attempted
4B	$\frac{x+3+x}{x^2} = 1$	not attempted
2B	$\frac{1}{R} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$	not attempted
Group 3	Dead ends solving for p in A=p+prt	
S3 1A	$\frac{A}{prt} = p$	
S5 1A	$p = \frac{prt}{A}$	
S5 1A	$-p = -A+prt$	
S6 1A	$-p = -A+prt$	

Table 5.16 continued

S6	1A	$A=p(rt)$	
S10	1A	$A -prt = p$	
S13	1A	$\frac{A-p}{rt} = p$	
S16	1A	$rt(A) = p^2$	
S21	1A	$-p=prt-A$	
Group 4		Collecting terms	
S3	2B	$\frac{xyz}{Rxyz} + \frac{Ryz}{Rxyz} + \frac{Rxz}{Rxyz} + \frac{Rxy}{Rxyz}$	Comment: "That's all you can do to that. You can't add them. None of them are alike."
S15	2B	$xyz - xzp - xyp = pyz$	Comment: "I don't really see how you can reduce that anymore."
Group 5		Quadratics	
S3	5A	$50x\ (x-20) = 6250$	
S3	4B	$x^2\ (2x+2) = x^2$	
S5	7A	$x^2 -x -16 = 0$	
S15	1B	$0 = x^2-2x$	
S20	7A	$x^2-x = 7$	
S20	6A	$x^2+x = \frac{2}{3}$	
S22	7A	$x^2-x-2 = 14$	
S22	6B	$x+2(x+2(x+2))=x+2 \longrightarrow x+2(x+2)=0$	(step spoken, not written)
S20	6B	$x+2(x+2(x+2)) = x+2 \longrightarrow \ldots \longrightarrow$ $x^2+4x+4 = 1$	Comment: "I can factor this back down but that wouldn't do me much good"

Table 5.16 continued

Group 6		Other gaps	
S5	4B	$2\ x^{1-2} + 3 = 1$	Comments: "I'm not sure how to get rid of a negative exponent so I won't try."
S6	1B	$x^2 - 2x = 0$	Unable to factor to get constant term zero

tion. In fact, the multiplication needed in the first case in the group is not marked useful for attraction in Bundy's tentative list of operations: perhaps human solvers have the same lack.

Group 3 collects dead ends reached by solvers on a single problem in which the key step is transforming p + prt to p(1 + rt). As can be seen, this step was difficult for many solvers. Some never succeeded, and others succeeded only after considerable exploration. Two interesting solutions are shown in Figure 5.9. As with the multiplication operation discussed above it is doubtful how many solvers really lack this factoring operation. The difficulty may be in perceiving that it is applicable to a sum of terms in which one has only one explicit factor: 1 must be seen as an implicit factor for the ordinary inverse of distribution to be applied.

Group 4 shows a few cases in which solvers were unable to collect terms. The comments suggest that the solver may have been looking for a common factor of all the terms, and failed to consider combining only a subset of the terms, as is actually needed.

Group 5 shows quadratic equations abandoned by solvers. It does not include cases in which factorization or use of the quadratic formula

Figure 5.9

Two solutions for p in A=p+prt

S6

$A= p+prt$

$-p = -A+prt$

$A-p=prt$

$A=p+prt$

$A=p(rt)$

$A=p+prt$

$A=p(1+rt)$

S:"You want to get this p (points to second p in second equation) over there (points to -p) somehow but this A doesn't have p in it. what you might do is go ahead and leave this A over here"

writes 3rd line

"and you'll have prt so that way you can divide. No, that still won't work."

rewrites original problem

"Oh how dumb. A = ok go ahead and factor out this p here while it's on this side since they both have p's in them and you have p times it, oh wait a minute that's wrong"

rewrites original problem

"OK you want to cancel out this p so when you do you'll have a one left here" solves

S11

$A=p+prt$

$\frac{A}{p} = \frac{p}{p} + \frac{prt}{p}$

$\left(\frac{1}{A}\right) \frac{A}{p} = 1+rt \left(\frac{1}{A}\right)$

$\frac{1}{p} = \frac{1+rt}{A}$

$p+prt=A$

$p = \frac{A}{1+rt}$

S:"If you divide both sides by p you'd get p over p leaving A over p equal to 1+rt. Then dividing both sides by A".

add 1/A to both sides

"would give you one over p would be equal to one plus rt over A. Um, distressing, I'm going to cross multiply, would give you p times 1 plus prt, what I originally had, make that equal to A. But this is factored now into these two components and if I just want the p I divide by one plus rt, divide both sides by that and that would give p equal to A over one plus rt.

was unsuccessfully attempted, but only those in which there was no indication of a next step. A partial exception is the last one. The solver does apparently consider factoring, but decides against it because it would simply undo the multiplication just performed. This suggests that at the very least the solver is missing the "set equal to zero" part of the procedure of solution by factorization and that if solution by factorization were available at all it would have been used before multiplying out, when the equation was in factored form, except for the zero. Also, factorization was not attempted by this solver in two other cases included in the group.

Applicability errors   Errors can result if a correct operation is applied to an expression or equation that does not satisfy its conditions of application. This could result from faulty checking of these conditions, or from improperly interpreting the syntax of an expression or equation. Table 5.17 shows some errors that may arise in this way. The first group involves treating a quantity as if it were parenthesized, either in a division operation or a multiplication. Note that often a subject misanalyzed only one part of an expression when two analogous parts were present. In the expressions shown with 2 parts, one misanalysis allows a cancellation and another forces distribution of a binomial. Of these, the first is much more common, suggesting that the error may not arise from misinterpretation of syntax but from use of an incorrect deletion operation, in these cases.

The second group of errors in Table 5.17, all from a single subject, involve the interpretation of fractions in which x appears as a factor of the denominator. Asked to explain operations which transformed 2(1/2x)

Table 5.17

Applicability errors

Group 1	Misinterpretation of grouping
E1,E11 S6,S18 6B	$x+2(x+2(x+2))=x+2 \longrightarrow x+2(x+2)=1 \longrightarrow x+2x+4=1$
S5 6B	$x+2(x+2(x+2)) \longrightarrow (x+2)\ (x^2+4x+4)$
S8 6B	$x+2(x+6) \longrightarrow (x+2)(x+6)$
S12 S20 6B	$x+2(x+2(x+2))=x+2 \longrightarrow x+2(x^2+4x+4) = x+2$ $\longrightarrow x^2+4x+4=1$
S11 6B	$x+2(3x+4) = x+2 \longrightarrow 3x+4=1$
S5, S6 S11,S20 6A	$x+2\ (x+1) \longrightarrow x^2+3x+2$
S22 6A	[illegible] $(x+1) \longrightarrow x^2+3x + 3$
S11 6C S20	$x+1\ \left(\frac{x+1}{2}\right) \longrightarrow \frac{x^2+2x+1}{2}$
S20 6E	$(3+x)3+x \longrightarrow 9+6x+x^2$
S5 7A S22	$x-2(x+1) \longrightarrow x^2-x-2$
S20 7A	$x-2(x+1) \longrightarrow x^2+x-2x-2$
S6 7C S13	$x-2(x-2(x-2)) = x-2 \longrightarrow x-2(x-2)=1 \longrightarrow x-2x+4=1$
Group 2	
S5 2A	$\frac{21}{21x} \longrightarrow \frac{1x}{1}$
S5 2B	$\frac{1}{x} \longrightarrow \frac{1}{1x} \longrightarrow x$

Table 5.17 continued

S5	2E	$2\left(\frac{1}{2x}\right)\longrightarrow x$	S: "not sure whether this x stands for one half x"
S5	5D	$\frac{2}{x}\longrightarrow 2x$	
Group 3			
S3	3C	$-120+7\longrightarrow -127$	
Group 4			
E4	6B	$x+2x+4 = 0\longrightarrow x= \frac{-2\pm\sqrt{4-(4)(4)}}{2}$	

to x, the subject said, "I'm not sure whether this x stands for 1/2x or whether it's one over 2x but I think that does not make a difference. I think x applies to the whole term before it." The next group of errors in the table may arise from the application of a subtraction operation to a part of an expression that is not a subexpression of the expression.

The final error in the table may result either from a misinterpretation of the syntax or from incomplete checking of conditions for the operator that applies the quadratic formula.

Table 5.18 shows two cases in which misinterpretation of parentheses is not implicit in a multiplication or division operation but is made explicit by insertion of parentheses.

Table 5.18

Errors in grouping

S5	6B	$x+2(x+2(x+2))\longrightarrow(x+2)(x+2(x+2))$
S1	6D	$\frac{x+2(x+2)}{x+2}\longrightarrow\frac{(x+2)(x+2)}{x+2}$

Execution errors The errors in Table 5.19 may result from incomplete execution of correct operation. In each case, the possibility exists that it results instead from the complete execution of an incorrect operation.

Table 5.19

Partial Execution

Group 1	Distribution
S1 6A	$2(x+1) \longrightarrow 2x+1$
S1 6B	$2(x+2) \longrightarrow 2x+2$
S1 7A	$-2(x+1) \longrightarrow -2x-1$
S16 5D	$\left(\frac{8}{x}\right)\left(\frac{x}{2+x}\right) \longrightarrow \frac{8x}{2+x^2}$
S16 7D	$-(x+1)(x-2) \longrightarrow -x^2-2$
S21 5D	$\frac{2}{x}\left(\frac{x}{2+x}\right) \longrightarrow \frac{2x}{2+x\ 2}$
E6 6B	$x+2(x+2(x+2)) \longrightarrow x+2x+4x+4$
E14 5A	$\frac{1}{2} = \frac{x-10}{x+5} \longrightarrow x+5 = 2x-10$
Group 2	Errors in setting sign in transposing
E1 E2 4B	$2x+3=x^2 \longrightarrow 0= x^2-2x+3$
E12 4B	$2x+3=x^2 \longrightarrow x^2-2x+3$
S12 2A	$\frac{1}{3} = \frac{1}{x} + \frac{1}{7} \longrightarrow \frac{1}{x} = \frac{-1}{3} + \frac{1}{7}$
S15 4A	$xy+yz = 2y \longrightarrow xy = 2y+yz$
S22 2C	$\frac{22}{7} - \frac{7}{3-x} = \frac{1}{7} \longrightarrow \frac{22}{7} = \frac{1}{7} - \frac{7}{3-x}$

Table 5.19 continued

S22 2E	$6\frac{1}{x}+7=0 \longrightarrow 6\frac{1}{x}=7$	
S22 5A	$5x+25=10x-100 \longrightarrow 5x=10x-75$	
Group 2	In arithmetic operations or signed quantities	
E2 5B	$2-2x-1+x^2 \longrightarrow x^2-2x-1$	
S7 3E	$+12-1 \longrightarrow -11$	
S15 5C	$x^2-2x+3=2 \longrightarrow x^2-2x-1=0$	
S18 3A	$4x=-160 \longrightarrow x=40$	
S13 3B	$28x-7=12x-3+4 \longrightarrow 16x+4-4=0$	
S15 7A S16	$-2(x+1) \longrightarrow -2x+2$	
	In other operations	
E4 5B	$1-x^2 \longrightarrow (x+1)(x-1)$	
E6 7A E7	$-x=16 \longrightarrow x=16$	
E10 5B	$\frac{-(x^2-1)}{1-x} \longrightarrow \frac{-(x+1)(x-1)}{x-1}$	
Group 3	Other operations	
S16 2B	$\frac{1}{r}=\frac{3}{x+y+z} \longrightarrow (x)\quad \frac{1}{r}=\frac{3}{x+y+z} \longrightarrow \frac{x}{r}=\frac{3}{x+y+z}$	
S23 4E	$3(4)+2(2) \longrightarrow 14$	S: "So that would be 12 + 2 which is 14"
S17 2B	$\frac{1}{R}-\frac{1}{x}=\frac{1}{y}+\frac{1}{z} \longrightarrow \frac{1}{x}=\frac{1}{y}+\frac{1}{z}-\frac{1}{R}$	part of a series of steps to isolate x

The first group of errors in the table involve distribution, in which a needed product is not formed. The second group consists of errors in which only the sign attached to a result is incorrect. These may result from incomplete execution of operations in which the determination of sign is a final step, as in multiplication of signed quantities.

Factoring Table 5.20 shows incorrect factorizations. In each case the factorization is such that at least the x^2 term of the product is correctly generated. In the first and last examples the x term is also correctly formed. In the fourth example, both terms of the desired product is generated~~, but~~ there are unwanted x terms. In view of these facts, it is possible that these errors occur through incomplete checking of the product. If checking is regarded as part of a complex factorization operator, these errors would arise from incomplete execution.

Table 5.20

Errors in factoring

S13	1D	$2x^2-x-3 \longrightarrow 2(x+1)(x-2)$
S22	5C	$3+x^2 \longrightarrow (3+x)x$
S1	1D	$x^2-x \longrightarrow x(x-x)$
S22	5C	$3+x^2 \longrightarrow (3+x)(1+x)$
E1	4B	$x^2-2x+3 \longrightarrow (x-3)(x+1)$

Part of the process of applying most operators is replacing part of an equation by a transformed, usually equivalent, part. In the

Bundy system, using a tree representation for expression, subexpressions can be replaced directly by equivalent ones. The replacement problem is more complicated in the linear notation in common use. The error in Table 5.21 shows what happens when the subexpression ((x+3)/x)x, resulting from multiplying both sides of the equation by x, is replaced by the equivalent subexpression x + 3. The rules of syntax require that the replacement be enclosed in parentheses in this context, though not in other contexts.

Table 5.21

Replacement

S6 7E $-\frac{x+3}{x} = -2 \longrightarrow -x+3 = -2x$

Some other errors which seem to reflect difficulties in actually carrying out what might be correct operations applied appropriately, are shown in Table 5.22. These are quite diverse, and so the table contains comments on each. The grouping in the table separates errors associated with the writing or reading process, those related to mixing of operation, and others.

The mixing of operations group includes two errors arising in clearing the denominator of one or more fractions. Note that in each case the fraction was multiplied by the denominator, the denominator is eliminated, but the numerator is also multiplied. It is plausible that this represents a mixture of the operation of clearing the denominator and the operation of multiplying numerator and denominator by the same number (Richard Young, personal communication). It is not evident

Table 5.22

Failures of Control

Group 1	Errors related to writing or reading	
S15 7D	$-(x+1)(x-2)\longrightarrow(-x\mp1)(x-2)$ $\longrightarrow -x^2-3x-2$	
E9 3B	$7(4x-1)=3(4x-1)+4\longrightarrow 7y=3y+1$	Introducing variable for 4x-1
E11 6B	$x\cancel{+}2(x+2(x+2))=\underset{1}{x\cancel{+}2}\longrightarrow(x+2x+4)=0$	
S8 6B	$x+2(x+2)\longrightarrow x+2+4$	says "two x"
S22 3B	$3(4x-1)+4\longrightarrow 12-3+4$	says "twelve x"
S23 7B	$6(x-2)-3(4-2x)=x-12\longrightarrow$ $6x-12+(-12+6x) = x-2$	
S3 4B	$x^2\ \dfrac{2x+3}{x^2}\longrightarrow 2x^3+2x^2$	
Group 2	Mixing of operations	
S3 4B	$x^2\ \dfrac{2x+3}{x^2}\longrightarrow 2x^3+2x^2$	Clearing denominator and multiplying numerator and denominator by same quantity
S8 2A	$\dfrac{1}{3}-\dfrac{1}{7}=\dfrac{1}{x}\longrightarrow\dfrac{7}{21}-\dfrac{3}{21}=\dfrac{21}{x}$	forming common denominator and multiplying through
S20 6C	$\dfrac{x^2+2x+1}{2}=1\longrightarrow 2x^2+4x+2=2$	clearing denominator and multiplying numerator and denominator by same quantity.
Group 3	Other	
S3 2B	$\dfrac{1}{R}=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\longrightarrow$ $\dfrac{xyz}{Rxyz}+\dfrac{Ryz}{Rxyz}+\dfrac{Rxz}{Rxyz}+\dfrac{Rxy}{Rxyz}$	

Table 5.22 continued

E1	6A	$3x=2 \longrightarrow x=3/2$	
E7	7B	$11x=12 \longrightarrow x= 11/12$	
S22	2E	$\frac{1/x+1}{2} + 3 \longrightarrow 6\frac{1}{x} + 6$ corrected to 7	Comment: "multiply that (points to 2) times that (points to 3) and add that (points to numerator) to it, so it would be 2 times 3 times that (points to numerator) is 6x + 6" "2 times 3 plus 7 (changes 6) 2 times 3 plus 1"
S22	6A	$x+2(x+1) \longrightarrow x^2 +3x+3$	
S8	2C	$\frac{22}{7} - \frac{7}{3-x} = \frac{1}{7} \longrightarrow \frac{-7}{3-x} = \frac{-23}{7}$	
S7	5B	$1-2 \longrightarrow -2$	

whether this mixture should be thought of as an incorrect operator, possessed by a subject as a stable entity, or whether it results from a control failure during execution of what might be separate, more-or-less correct operators. One subject, when asked to explain the operation did not do so but proposed an alternate method.

Error Summary Table 5.23 shows the overall frequency of occurrence of all the types of errors found, excluding the follow-up problems for the selected group. Frequencies for the session 1 problems alone, which were presented to all students, are shown in Table 5.24, for the top, middle, and bottom groups of solvers. There is a suggestion of a shift in the prevalence of the error types as accuracy increases, with execution errors relatively more frequent and operator errors less

Table 5.23

Error frequencies, excluding follow up problems for selected groups

Category	Frequency
OPERATOR ERRORS	
Deletion	60
Transposition	4
Recombination	12
Combining fractions	10
Decomposition of quotients	6
Inversion	2
Cross multiplication	8
Splitting equations with fractions	5
Reciprocals	7
Loss of roots	20
Splitting quadratic not equal to zero	2
Square root	4
Arithmetic	3
APPLICABILITY ERRORS	
Applicability	29
Grouping	2
EXECUTION ERRORS	
Partial execution	30
Factoring	5
Replacement	1
Failures of control	17
TOTAL	227

Table 5.24

Number of errors and percentage of errors by accuracy of solvers for Session 1

Category	Top 10 Solvers		Middle 14 Solvers		Bottom 10 Solvers	
OPERATIONAL ERRORS						
Deletion			5	11%	31	35%
Transposition					4	4%
Recombination					4	4%
Combining fractions					3	3%
Decomposition of quotients					1	1%
Inversion					2	2%
Cross multiplication					4	4%
Splitting equations with fractions			1	2%	3	3%
Reciprocals			2	4%	2	2%
Loss of roots	6	50%	11	23%	3	3%
Splitting quadratic not equal to zero			2	4%		
Square root	1	8%			3	3%
Arithmetic					2	2%
TOTAL OPERATOR ERRORS	7	58%	21	45%	62	70%
APPLICABILITY ERRORS						
Applicability	1	8%	8	17%	12	13%
Grouping					1	1%
TOTAL APPLICABILITY ERRORS	1	8%	8	17%	13	15%
EXECUTION ERRORS						
Partial execution	1	8%	12	26%	9	10%
Factoring			1	2%		
Replacement						
Failures of control	3	25%	5	11%	5	6%
TOTAL EXECUTION ERRORS	4	33%	18	38%	14	16%
GRAND TOTALS	12		47		89	

frequent. This is loosely consistent with the finding of Davis and Cooney (1977) that more accurate solvers made relatively more errors they called "computational". They intended by "computational" errors of arithmetic, and as it can be seen, arithmetic errors are very rare among the subjects, if sign errors are otherwise classified. Most of their "computer" errors would fall in a "execution" category. It is important to note that this observation may be due in part to artifact, however, since any subject who was prone to an error that would be repeatedly elicited on these particular problems would automatically fall among the poor solvers.

Discussion We have seen that a large number of errors can be roughly located in the framework of the Bundy model. In attempting to account for errors in the framework of the Bundy model, or any other, one is challenging a common intuition that errors are the result of perturbations of correct performance due to inattention, random forgetting, or other unsystematic cause. There are reasons to persevere in analysing errors.

First, some errors tend to occur consistently for given students: a student who makes one error of a given type is likely to make another. Three of the errors were made five or more times by the same subject. Of deletion errors, S1 had 5, S5 10, S16 7, S20 16 and S22 10. S16 made 6 recombination errors. S5 made 7 applicability errors and S5 5. Figure 5.10 compares the obtained distribution of deletion errors in session 1 only, where all solvers saw the same problems, with the Poisson distribution. As can be seen, the errors are clumped together.

This suggests that a student has not an unsystematic tendency to make errors, but a tendency to make errors of a particular type.

Figure 5.10

Poisson fit to deletion error frequencies
for session 1

Errors	No of Solvers	Poisson Probability	Expected No. of solvers
0	23	.33	11.22
1	6	.36	12.24
2	1	.20	6.80
3	0	.074	2.52
4	0	.02	.68
5	1	.004	.14
6	0	.0008	.03
7	1	.0001	.00
8	2	.0000	.00

mean errors per solver: 1.06

Second, some errors seem to be consistent with the solver's statements of what should be done, indicating that it is not just the solver's execution but also his knowledge that is faulty. While such cases suggest that errors can originate from faulty knowledge, two cautions are called for in interpreting them. First, they do not help to establish the proportion of errors that have such a systematic origin, above a minimum. Second, it is possible that some or even all of these statements are explanations after the fact of what was done, rather than true accounts of knowledge underlying the errors. (See Nisbett and Wilson, 1977, for a discussion of this problem in interpreting self-reports.) So this evidence is not by itself very strong.

Third, as a perusual of the tables of errors presented above reveals, errors are quite systematic in form. Very few errors seem at all like random distortions of correct performance. For one thing, the same or very similar errors appear in the work of different subjects. For another, the errors tend to reflect features of actions that would be appropriate in other situations. Even if errors do stem from unsystematic perturbations, therefore, a study of them must reveal something about the system being perturbed.

Although the Bundy model provides a rough classification of errors, it does not allow us to interpret the detailed structure of the errors in the way this last observation suggests we must. Accordingly, we take up this analysis in a later section, along with a consideration of other aspects of behavior that call for modification and extension of the simple model.

CHAPTER 6

Features of Skilled Performance

Within the Bundy framework there appear to be two ways in which performance could be improved beyond the elimination of errors. First, skilled solvers may develop more powerful operators, that can produce solutions in fewer steps. Second, the pattern-matching ability of skilled solvers may be better developed, so that the skilled solver is able to see the usefulness of operators that would be passed over by a less skilled solver. We can examine these possibilities by comparing the solutions of more and less accurate solvers.

If skilled solvers are using more powerful operators, their solutions should have fewer written steps. The number of written steps is also influenced by other factors, such as the number of steps performed mentally, but such action would also be expected to shorten the solutions of skilled solvers.

Average length of solutions As seen in Table 6.1, there is a tendency for more accurate solvers to find shorter solutions, but this is not a large difference and it is not consistent across problems. Figures 6.1 and 6.2 show the solutions found by students among the most and least accurate solvers for two problems. In figures 6.1 and 6.2, each phrase separated by commas indicates a single written line. When more than one operation was performed in the transition between lines, these are separated by colons. "Isolate" is used to refer to a step that results in a single term containing the unknown occupying one side of the equation by itself. If the term has a negative sign this is indicated, because it generally requires an additional step to change the sign. "Dist"

Table 6.1

Average Lengths of Solutions

Equation	Top 10 Solvers	Middle 14 Solvers	Bottom 10 Solvers
1A	2.0	2.4	2.5
1B	3.1	3.4	3.0
2A	3.3	3.8	5.5
2B	2.7	3.1	-
3A	3.0	3.0	3.2
3B	4.1	3.6	4.2
4A	2.0	1.9	2.0
4B	5.2	4.9	4.5
5A	3.8	3.9	4.3
5B	5.2	4.4	4.7
6A	3.6	3.0	3.6
6B	4.1	5.0	4.3
7A	3.5	3.4	3.0
7B	3.8	4.3	4.3

Figure 6.1

Solutions to Problem 2A

$$\frac{1}{3} = \frac{1}{x} + \frac{1}{7}$$

Top Ten Solvers

E3	isolate, invert
E5	isolate, multiply by x, divide by coeff of x
E8	multiply by x, collect terms, divide by coeff of x
E9	multiply by 21x, clear fractions, combine terms, divide by coeff of x
E13	isolate, invert
E16	multiply by 21x, clear fractions, combine terms, divide by coeff of x
S17	isolate: multiply by x, distribute: multiply by 21, clear fractions, collect terms, divide by coeff of x
S19	isolate, add fractions, invert

Bottom Ten Solvers

S5	isolate -1/x, form common denominators, add, change sign, multiply by x, divide by coeff of x
S6	isolate -1/x, add, simplify, change sign, invert
S13	multiply by 3, multiply by x, multiply by 7, collect terms, divide by coeff of x
S15	isolate, add, simplify: multiply by x, simplify, multiply by 21, divide by coeff of x

Figure 6.2

Solutions to Problem 7B

$6(x-2)-3(4-2x) = x-12$

Top Ten Solvers	
E3	dist 6: dist 3, dist -, simplify, transpose: combine terms, divide by 11
E5	dist 6: dist -3, set equal to zero: combine terms, transpose 12, divide by 11
E3	dist 6: dist -3, combine terms: transpose: transpose: combine terms, divide by 11
E9	identical to E8
E13	dist 6: dist 3: transpose: transpose: transpose, combine terms: combine terms, divide by 11
E14	dist 6: dist -3, transpose, combine terms, divide by 11
E16	dist 6: dist -3, simplify, transpose: combine terms, divide by 11
S8	dist 6: dist -3, simplify, transpose: combine terms, divide by 11
S17	dist 6: dist -3, simplify, transpose: combine terms: multiply by 1/11, simplify
S19	dist 6: dist -3, simplify: transpose.24, transpose: combine terms, divide by -1

Figure 6.2 continued

Solutions to Problem 7B

Bottom Ten Solvers

S1	dist 6: dist -3, simplify, transpose: combine terms: divide by 11, simplify
S3	dist 6: dist -3, simplify, transpose: combine terms, divide by 11
S5	dist 6: dist -3, simplify, transpose, combine terms, divide by 11
S6	identical to S3
S13	set equal to zero, dist 6: dist -3, simplify, transpose, divide by 11
S15	identical to S3
S21	dist 6: dist -3, simplify: transpose 24, transpose x: combine terms, divide by 11

abbreviates "distribute". Figure 6.1 shows equation 2A, the only one for which the difference in length of solution was significant. As can be seen, the difference in length seems to be due in part to poor choice of initial step by some of the poor solvers, partly to more frequent use of the efficient inversion operation by the better solvers, and partly to some combining of steps by the better solvers that were written out by poorer solvers. The solution of S13 is a clear example of the expansion of a single operation, clearing fractions, into three separate multiplications. So there is some indication of the expected differences between better and poorer solver here.

On the other hand, inspection of the solutions to equation 7B in Figure 6.2 shows that the difference is not dramatic. This problem requires a number of steps, offering scope for combining of operations, but as can be seen the solutions of the better solvers are for the most part very similar to those for the poorer solvers. Subject E8, E9 and E13 do show more combining of operations than the others, but it is clear that this is not typical for the group of better solvers. It may be that these three subjects give some indication of what solving is like for solvers who are more proficient than those who participated in the study.

Use of subexpressions In many operator applications the pattern-matcher only needs to handle simple terms: the unknown, constants, or products of the unknown and constants. Occasionally it is useful to deal with larger subexpressions as units, and it seems possible that more skilled solvers might be better able to do this, rather than breaking down the problem into smaller units.

Three problems in the initial set were designed to permit the solver to exploit the presence of repeated subexpressions and treat them as units, as shown in Figure 6.3. In equation 7B the repeated expression is not apparent in the surface form of the problem, and identifying it requires sophisticated matching. As shown in Table 6.2, there is a tendency for the more accurate solvers to use the subexpressions more often. The table includes all cases in which the subexpression was used, correctly or not. The commonest use in equation 3A, $9(x+40) = 5 (x+40)$ was to cancel $x +40$ from both sides, which is incorrect. The hidden feature of equation 7B was used by only one subject, E12. As with the solution lengths, we can speculate that there may be more proficient solvers than our top ten group that would use such features, but it is also possible that in working with simple equations such as these the extra analysis is just not worthwhile.

These two areas of investigation, solution length and use of repeated subexpressions, have not turned up dramatic differences between the most and least accurate solvers in the study. It seems fair to say that within this group of solvers the more accurate solvers differ mainly in possessing correct operators, rather than in having more powerful ones or applying them more effectively.

Figure 6.3

Problems with repeated subexpressions, with solutions

E7 3A $9(x+40)=5(x+40)$

$(9-5)(x+40)=0$

$x = -40$

E14 3B $7(4x-1)=3(4x-1)+4$

$7(4x-1)=3(4x-1)=4$

$(4x-1)(7-3)=4$

$(4x-1)\cancel{4}=\cancel{4}$

$4x-1 = 1$

$4x= 2$

$x = \frac{1}{2}$

E12 7B $6(x-2)-3(4-2x) = x-12$

$6(x-2)+6(-2+x)=x-12$

$12(x-2) =x-12$

$12x-24=x-12$

$11x=24-12=12$

$x = \frac{12}{11}$

Table 6.2

Use of repeated subexpressions in equation 3A and 3B

	Top Ten solvers	Middle 14 solvers	Bottom ten solvers
number of students using feature on either equation	4	6	2
number not using feature	6	8	8

CHAPTER 7

Model Discrepant Solving Behavior

In this section we consider some aspects of subjects' performance that do not fit well into the framework of the Bundy model. Part of the evidence to be discussed is drawn from the structure of the errors presented earlier: these suggest something about the organization of the knowledge subjects are using. Additional evidence comes from the comments and activities that surround solving itself: subjects do not simply solve, they also explore, evaluate and check.

Evidence of hierarchial organization In the Bundy scheme there is a single pool of operators, no operator being part of another. Students' remarks seem to indicate that their operators are hierarchically organized, so that some operators must be expanded into a sequence of other operations when executed. For example, the operation "factor" must be expanded into a complicated and unstandardized sequence of trial-and-error attempts. The operator "multiply out", as applied to binomials, expands into a series of four simple multiplication operations, which may be written separately and may be carried out in a standard order. Table 7.1 collects examples of operators appearing in subjects' comments that seem to have expansions.

The expansions of operators into parts seems indicated also by some of the errors discussed above. In combining fractions, for example, it appears that erroneous operators arise from the combination of pieces of correct operations. The assimilation of addition and multiplication to a generic combination scheme, and of division and subtraction to a deletion scheme, also indicate that these operations are not unanalyzed in the system. However, the analysis suggested by this evidence is not an expansion into other operations, but a more general decomposition of the knowledge of the operators' separable parts, so that the basic combination aspect of addition may be retained while more specific knowledge about its action may be lost or not learned.

Table 7.1

Examples of complex operations mentioned in protocols

find the common denominator

get the x's together

put all the variables on one side and the numbers on another

cross multiply

multiply both sides by _____

subtract _____ from both sides

multiply two binomials

multiply what's inside the parenthesis

multiply everything out

distribute

factor

invert everything

Matz (1979b) and Brown (1979) have discussed the way in which partial information about operations, embodied as "critics", might be used by students attempting to learn or recall a given operation. The "critics" might block an attempted addition operation if it lacked some key feature associated with addition. Robert Neches (personal communication) has suggested that the use of partial information to constrain an operation might be a basic part of the memory retrieval process, in which the remembered procedure is built by the memory system to embody as many remembered features as possible. Whatever mechanism is responsible for the effect, it does seem clear that both abstract features of an operation (deletion for subtraction or division) and pieces of related procedures (combination of fractions) can influence the procedures students use.

It would be interesting to know whether erroneous operators are built up at the time they are used, or during a learning process that might precede their application. Only a few passages in the protocols suggest strongly an operation being devised on the spot. These are shown in Table 7.2. Internal evidence from the erroneous operators suggests, in some other cases, that the operator was created just before application. These are cases in which the operator seems to have been shaped by the particular goal being sought at the moment of application. Table 7.3 shows examples.

In the first two cases in the table, operators are devised that have the desirable property of moving x from the denominator, where it is hard to deal with, to the numerator. Having made the move shown here, S5 went on to use variants of it on three subsequent problems. The third case,S5 5A, shows adjustment of the cancellation operator to avoid the problem of the vanishing unknown. The last two examples, involving recombination, may not reflect construction or adjustment of

Table 7.2

Constructing Operations

S1 1D

$\frac{3}{2} = x^2 - x$

$= x \cdot (x-x)$

S: "I'm not sure about that one. Plus the properties are very confusing. So if you do x-x thats obviously zero. Oh wait a minute.

That's x times x minus x so if x minus x is zero it'd be x times zero. It'd be three halves equals zero. Can't do that. (pause) Well, if this won't work right here then I'm trying to figure out some alternative way of doing it."

E: "What would the rule be - suppose you could do what you just did, you know, putting the parentheses in like that. Is there a general rule that you're using there that would allow you to do that?"

S: "Um, I believe it's the associative. I'm not real positive but I think it is."

E: "Could you give like another example of it besides this one that might indicate what the rule would be like?"

$3 \cdot (2+4) = 18$

$(3 \cdot 2)+4 = 10$

$3+(3-2)$

S: "It would be like ah um, three times two. I'm just trying to figure out if this one will work. Three times two plus four that'd be three times six that equals eighteen. Three times two plus four no that doesn't work. Ok, the property I'm trying to think of I guess it just does for, if the mode of, whatever, if it's addition or whatever is the same, like three plus three minus two, that would work I think when you take the reciprocal of a number you have to change the signs but I'm not quite sure about that."

S5 2D

$\frac{1}{3} = \frac{4}{2}$

$\frac{-3}{6} = \frac{-2}{4}$

"I was trying to give numerical values to the variables but I didn't think about the equal sign when I made 6/3 equal to 4/2 but 2 is equal to 2. But that's not exactly what I want.

$\frac{1}{x^{-2}} \rightarrow x^2$

"I was trying to see if takir if taking the reciprocal meant that I had to change the sign to the opposite. What I'm thinking of is when you have an exponent on the bottom that's negative, when you take the reciprocal it becomes positive. But, I really am not thinkin too much.

Table 7.2 continued

S1	2A	$\frac{1}{3}=\frac{1}{x}+\frac{1}{7} \longrightarrow \frac{1}{3}=7+x$ $\frac{1}{3}=\frac{x}{7}$	S: "This one I can do. Cross multiply". It doesn't seem to me like that'd be right. Ok all right, it's not right" changes to $\frac{1}{3}=\frac{x}{7}$
		$\frac{5}{10}=\frac{x-10}{x+5} \longrightarrow$ 5=x-10 10=x+5	
		$\longrightarrow$ 15=x 5=x	S: "You get 5 equals x. But x should be the same. HaHa". "So I'm thinking it's messed up. Maybe you could set the 2 equations, but they're not equal to each other. Reduce 5/10 to one half and set x minus 10 equal to one and x plus five equal to five" (gets x=11, x=-3) "You could solve it as a system and say x minus 10 equals 5, x plus 5 equals 10, and then subtract this whole equation from this one. Maybe we should add the two equations." (gets 2x-5=15, x=10, checks) "So that wouldn't work either. I don't know. I don't understand why it does not work out."

the recombination operator, but may simply illustrate its flexibility. All three errors in the two examples seem to be well calculated to attain a useful end. The first error separates x. This is however next undone to permit cancellation of the 2y term. In the last example the recombination chosen allows x to be easily isolated.

Table 7.3

Goal directed errors

S22 2A	$\frac{1}{x}+\frac{1}{7}\longrightarrow\frac{1}{x}+\frac{7}{1}$ first, then changed to $\frac{x}{1}+\frac{1}{7}$
S5 2A	$\frac{4}{21}=\frac{21}{21x}\longrightarrow\frac{4}{21}=\frac{1x}{1}$
S5 5A	$\frac{5}{10}=\frac{x-10}{x+5}\longrightarrow\frac{1}{2}=-2$, then later $\frac{x-10}{x+5}=\frac{1}{2}\longrightarrow x-2=\frac{1}{2}$
S22 4A	$xy+yz=2y\longrightarrow$ $x+2yz=2y\longrightarrow$ $xz+2y=2y\longrightarrow$ $xz=0$
S22 4D	$bcx+acx+abx+abc=0\longrightarrow$ $3abc+3x=0\longrightarrow$ $3x=-3abc\longrightarrow$ $x=-abc$

<u>Some characteristics of Erroneous Operators</u> As just discussed, erroneous operators might be put together using existing knowledge, and then applied. Another way some erroneous operations could arise is by switching between correct operations in mid-execution. Thus the student might not possess the incorrect operator as a stable entity. If complex operations were implemented by sets of productions in a production system one could readily see how invalid hybrid operations might be carried out by the system in the event of loss of control information

from memory. A stack discipline, in which a stack is used to hold pending parts of expanded operators, could also fail in such a way as to produce mixing. Some of the errors in combining fractions in Table 7.4 might lend themselves to this kind of account (Richard Young, personal communication). Such mixing of operation, if it occurs, would necessitate an analysis of operations into parts.

Table 7.4

Errors in combining fractions

S1 6C	$\frac{x}{1} + \frac{x+1}{2} \longrightarrow \frac{x+x+1}{2}$	
S15 5D	$(4+2/x) \quad \frac{x}{2+x} \longrightarrow \frac{4(x(2+x)+2(2+x)(x^2)}{x(2+x)}$	
S16 7E	$\frac{7}{1} - \frac{x+3}{x} \longrightarrow \frac{7-x+3}{1-x}$	
S16 5D	$\frac{4}{1} + \frac{2}{x} \longrightarrow \frac{8}{x}$	
S22 2D	$\frac{b}{1} + \frac{1}{c} \longrightarrow \frac{b}{c}$	
S22 2C	$\frac{1}{7} + \frac{7}{3-x} \longrightarrow \frac{1}{3-x}$	
S16 2B	$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \longrightarrow \frac{3}{x+y+z}$	
S22 2B	$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \longrightarrow \frac{1}{xyz}$	
S20-2A S21	$\frac{1}{3} - \frac{1}{y} \longrightarrow \frac{10}{21}$	
S22 2E	$\frac{1/x+1}{2} + 3=0 \longrightarrow 6\frac{1}{x} + 7$	Comment: "multiply that maybe (points to 2) times that (points to 3) and add that (points to numerator) to it". Maybe based on $\frac{1}{a} + b \longrightarrow \frac{ab+1}{a}$

One case of mixing of operations in correct performance is shown in Figure 7.1. This large step may arise as a combination of n distributions, n transpositions, and n combinations of terms. The result was written strictly left-to-right. Although the operation accomplishes all the effects of the mentioned operations, it may or may not be a mixture of them in the sense discussed above. As pointed out by Robert Davis, personal communication, it is possible to define a single procedure which gives the desired effect, and the student might simply be using such a procedure without reference to the smaller operations. So this example, like the cases of combination of fractions, does not establish that pieces of operations may be mixed during execution.

Figure 7.1

A complex operation

E13 7B

$$6(x-12)-3(4-2x) = x-12$$

$$6x+6x-x = 12+12-12$$

$$11x = 12$$

$$x = \frac{12}{11}$$

The kinds of control mechanisms that might produce mixing of operations might account for other kinds of errors. The idea of partial execution presupposes a control system in which it can happen that only part of an operation is carried out. Both the production system and stack discipline have this as a natural mode of failure. In a production system, if not all the tests that make up a condition were accurately checked, a production that should have awaited the completion of an operation being carried out by a set of other productions might apply prematurely. It is not necessary to assume that the action of any single production could be only partly performed.

If a stack discipline were used, it would be necessary to assume that items are sometimes simply lost from the stack. The prevalence of sign errors might suggest that some pieces of operations, such as sign setting, are more likely to be lost than others. However, only some losses will show up as partial executions, since loss of really vital early parts of an operation would result in inability to perform the later parts at all, and perhaps lead to reloading of the stack.

For human solvers Operators are apparently not units of knowledge that either are known or unknown, as they are in the Bundy model. Students frequently expressed doubt about the correctness of operators they used. These doubts seem to lead to behaviors that would be unnecessary in a solving system that had only definite and certain knowledge of its operators. First, students monitor the progress of their solution, making evaluations of the states they reach. They may backtrack if it appears that what they have done was not leading in the right direction.

Table 7.5 collects examples showing evaluation of the situations reached in the solution process, grouped according to the aspect of the situation that is attended to. During the course of solution, the complexity of the equations formed is noted, though it is unclear what determines the judged complexity. Also, the legality of moves is assessed: solvers may plan or even carry out moves which are then retracted because they are judged not valid. Solvers also try to avoid returning to earlier states. Group 3 of the table shows examples. When the solution process is complete, the solution or solutions obtained are evaluated. It may be that the conclusion is that there is no solution; since this is rare, solvers may use an alternate method to be safe. If a solution is obtained it may be checked, as discussed below, and if it is found wanting, a new method may be used. Some solvers seem to be aware that a quadratic equation has two roots, though they may not be distinct. The Table shows two cases in which failure to obtain two distinct roots may have led to new attempts.

Not all backtracking is controlled by this sort of evaluation, of course. Solvers may reach a dead end, a state to which no appropriate operation can be applied. This forces backtracking. The final group in the Table shows a few examples.

A second behavior made necessary by imperfect knowledge is checking. Students may check their answers or the solution process, and may check the process as a whole or in part, by a variety of methods.

Methods of checking may be divided into local and global methods. Local methods indicate whether some particular step in the solution process is correct or not. Global methods indicate whether the solution process as a whole is correct. Within these categories, different methods with different characteristics exist. Table 7.6 lists the methods used by students on the fourteen first-session problems.

Considering first the local methods, by far the most common is retracing the solution process. In this method the solver examines each step of the solution to see whether the intended operation was accurately carried out. The problem is not solved again, because the checking is done by referring to the written trace of the solution. This means that

Table 7.5

Evaluations

Group 1	Complexity	
E10 3B	$7(4x-1)=3(4x-1)+4$	
	$7\cancel{(4x-1)}=3\cancel{(4x-1)}+\frac{4}{4x-1}$	S:"Well actually I don't believe that's going to help me any. It'd complicate things more in the long run."
S5 2A	$\frac{1}{3}=\frac{1}{x}+\frac{1}{7}\rightarrow\cdots\rightarrow x=\frac{4}{21}$	S:"That's not, that doesn't look ok. I'm going to do it another way."
S12 2B	$\frac{1}{R}-\frac{1}{y}-\frac{1}{z}=\frac{1}{x}\longrightarrow$	S:"I'm trying to see if I can take x squared and multiply this (points to 1/x) by x squared and get x" does so "I'm not approaching it. I'm not even coming to it."
	$\frac{x^2}{R}-\frac{x^2}{y}-\frac{x^2}{z}=x$	
	$\frac{1}{R}-\frac{1}{x}=\frac{1}{y}+\frac{1}{z}\rightarrow\cdots\longrightarrow$	"I'm going to try a different way. I don't like that. It's too complicated."
	$yzx-yzR=Rxz+Rxy$	
S15 2B	$\frac{1}{R}-\frac{1}{y}-\frac{1}{z}=\frac{1}{x}\longrightarrow$	
	$\frac{1x}{p}-\frac{1x}{y}-\frac{1x}{z}=1\longrightarrow$	S:"It just doesn't seem like it simplifies anything."
	$\frac{xyz-xzp-xyp}{pyz}=1$	S:"I think I made it more complicated."
Group 2	Legality	
E4 2A	$7=\frac{21}{x}+3\longrightarrow x=2$ (incomplete)	S:"Take the x over to this side and bring this seven through. Can't do that."
S1 2A	$\frac{1}{3}=\frac{1}{x}+\frac{1}{7}\longrightarrow\frac{1}{3}=7+x$	S:"It doesn't seem to me like that'd be right. Ok all right, it's not right" changes to $\frac{1}{3}=\frac{x}{7}$

Table 7.5 continued

Group 3		Return to earlier state	
S5	6A	$x+2(x+1)=4 \longrightarrow \cdots \longrightarrow$ $x^2 + 3x-2=0$	S:"I can't go back because I come up with this (points to original left hand side) if I try to factor down so I'm just going to leave it like that."
S13	2B	$yz(x-R)=Rx(z+y) \longrightarrow$ $yzx-yzR = Rxz+Rxy$	S:We're going to factor out, naw, that will give me the thing I started with."
S20	6B	$x+2(x+2(x+2))=x+2 \longrightarrow \cdots \longrightarrow$ $x^2+4x+4=1$	S:"I can factor this back down but that won't do me much good."
Group 4		No solution	
E3 Similar: E6,E9,S11 S10,3A	3A	$9(x+40) = 5(x+40)$	S: cancels, then remarks "9=5 doesn't exist, so I'm going to work out the problem to see what I really got."
E9	5B	$\frac{1-x^2}{1-x} = 2$	S: solves using quadratic formula, gets x=1. Checks. "And the first thing you notice is that when you plug in one for x up here you're dividing by zero. So there is no solution." Then solves by cancelling. "You notice that there's again no solution. All right."

Table 7.5 continued

Group 5	Number of roots	
E14 5B	$\frac{1+x}{1-x}=2$ $\frac{\cancel{(1-x)}(1+x)}{\cancel{(1-x)}}=2$ $\longrightarrow$ $1-x=0$ or $1+x=2$	S:"Cancel these 1+x by 1-x if 1-x is equal to zero and x equal to one. And the other answer would be 1+x is equal to 2. x equal to 2 minus one would be also one. Therefore no we can't say that that's equal to zero because that's (points to 2 on right hand side of equation with cancellation) not zero. Ok that's a good question." Solves by multiplying by 1-x, obtaining $(x-1)^2=0$. "Ok that's the reason we got 2 answers x=1 up here because we have x=1 have as an answer for both of them.
S11 1B	$2x=x^2 \longrightarrow 2=x$	S:"I believe I can divide by x. Let's try that" checks both negative and positive 2 as solution. "So it's positive 2 only I guess. Another way, can I solve this another way?" forms $x^2-x=0$, solves, finds both roots. "That's correct." Crosses out 2=x.
Group 6	Dead ends	
E7 4B	$x^2-2x+3=0$	S:Equation obtained by an error. "This one doesn't seem to factor very well. That's why, I made a mistake." corrects
S6 6A	$x^2+3x-2=0$	similar

Table 7.5 continued

S1	5A	$\frac{1}{2} = \frac{x-10}{x+5} \longrightarrow \frac{x+5}{2(x-10)}$	S:"I know what I can do I think. Multiply, cross multiply."
S1	5B	$\frac{1-x^2}{1-x} = \frac{2}{1} \longrightarrow \frac{1-x^2}{2-2x}$	
S20	4B	$\frac{x+3+x}{x^2} = 1 \longrightarrow 1+3 = 1$	S:"Broke down the equation but ah its not going to equal. I think I'm going to have to skip that one."
S20	1A	$A=p+prt \longrightarrow \ldots \longrightarrow -1=rt$	S: Solve for p "That's about as far as it goes."

there is danger that an error made in the solution may be repeated during checking, either because of a stable conceptual error or because of some temporary lapse. Excerpt E8 1A in Table 7.7 shows one student's handling of this second difficulty.

There are a few cases, collected in the next category in Table 7.6, in which only a single step of the solution process is checked by retracing. This was done three times to check the product of binomials, once to check clearing of denominator, and once to check conversion of a fraction to a new denominator.

A local method which has the advantage that it could detect even stable conceptual errors is checking a step by carrying out the inverse operation of the step. This was seen seven times, each time to check factorization by multiplying the factors. The method could be applied to other operations, such as simplification of quotients by cancelling, using multiplication of numerator and denominator by the same quantity

Table 7.6

Method Checking

Local Methods	Cases			Number of cases	Number of students
Retrace solution	E8 1A E8 7A E8 6B E8 2A E8 5A E8 3A E8 4A E9 6A	E9 7B E9 1A E9 4B E9 5A E9 4A E9 7A E9 2B E9 3B	E9 6B E14 2B E16 7B S3 6A S3 3B S5 7B S17 7B	23	7
Retrace single step	E16 2A S3 2A	S3 7A S3 7B	S5 6B	5	3
Perform inverse operation	E5 5B E16 5B S1 1A	S11 5B S11 4B S13 6B	S13 2B	7	5
Numerical Substitution	S5 5A			1	1
Invalid Variant	S5 6B			1	1
Analogous Problem	S5 6B			1	1
Global methods					
Substitute Answer	E3 5B E3 3A E3 3B E3 6B E3 4B E3 5A E5 4B E8 4B E8 4B E11 5B E12 5A E16 5B	E16 4B E16 2A E16 1B S3 1B S5 7A S8 7A S10 5A S11 1B S17 5B S17 4B S17 2A	S17 6A S17 5A S18 5A S23 1B S23 7B S23 5B S23 3B S23 2A S23 7A S23 6A S23 5A	34	14
Consistency	S5 2A S10 5A	S12 1A	S13 2B	4	4

Table 7.7

Protocol excerpts on checking

E8	1A	Student checks at end of series of problems. E: "Do you have a particular reason for checking at the end of the whole series rather than checking each one as you go along?" S: "Yes, I'm not as likely to make the exact same mistake twice: After you've done something one time it tends to fix in my mind and if I try to check it immediately I'm liable to just do it over again."	
S5	7A	$x^2 -x-16=0 \longrightarrow x-16=0$ $\longrightarrow x = 16$	S:"I don't know whether I should do this. Let me try this." substitutes 16 in original equation. "Definitely not."
S23	2A	$\frac{1}{3} = \frac{1}{x} + \frac{1}{7} \longrightarrow \cdots \longrightarrow$ $\frac{4}{21} = \frac{1}{x}$	S:"I'm trying to think if I can cross multiply to find the answer or invert the fraction. See if that would be the answer. I don't remember if that works or not. It's been a long time since I've had algebra, so God, I don't know what to do. Um, if I cross multiply I'd have 21/4 for x and that let's see, 4 21st and 3 7 21sts and that's the answer so that's what I'd do."
S5	2A	$\frac{1}{3} = \frac{1}{x} + \frac{1}{7} \longrightarrow \frac{7}{21} = \frac{1}{x} + \frac{3}{21}$ $\longrightarrow \frac{7}{21} - \frac{3}{21} = \frac{1}{1x}$ $\longrightarrow \frac{4}{21} = \frac{21}{21x}$ $\longrightarrow \frac{4}{21} = \frac{1x}{1}$	"Four 21 is equal to x. That's not, that doesn't look, Ok I'm going to do it another way" does so

Table 7.7 continued

	$\frac{1}{3} = \frac{1}{x} + \frac{1}{7} - \frac{1}{x} = \frac{1}{7} - \frac{1}{3}$	cross multiplies
	$x = \frac{21}{4}$	S: "But it's exactly the opposite of what I did over here. I must have done something goofy." rechecks cross multiplication "This is strange. If I worked it two ways I should come out with the same answer considering all I did was switch to the other side of the equal sign. I must have made. Oh I see what I did. (Points to $\frac{1x}{1}$ I made it x over one. No, that is correct."
S13 2B	S: "I'm trying to get the same thing up there to see if I did it right or not." gets agreement "Ok, I suppose that could be the answer. There's too many variables. You can't work it out."	
S5 3A	Checks answer by substitution, get 0 = 0. S: "Um that still looks a little awkward. Normally you don't come out with zero is equal to zero. So I'm going to select a number, let's say 20, and put it in and see if any other number will come out other than -40." substitutes "And I have 60 times 9 is 540 and this is 60 times 5 is 300 which is not the same, so ah basically looking at it I would say that -40 is the one and only solution."	
S17 7B	S: "Let's see I'm just checking over this. It doesn't I'm looking for a quick solution like could I divide 12 straight through. Well I couldn't because there is an x there. And what I'm looking for is sometimes these problems are made where you're looking for a simple way to get some factor out so I'm just seeing if maybe I made a mistake. That's a quick way for me to catch an error."	

Table 7.7 continued

S17 6B	S: "There's no fraction so I don't see why to check it, and I'm beginning to feel a little bit more confident."
S17 4B	S: "No we need to go back and check on cause we have a fraction here."
S23 7B	S: "And I probably wouldn't even go back and put it in because it came out even and it's too much trouble with all those numbers."

as the inverse. The requirements are that an inverse exist, that it be known by the student to be the inverse, and that the student be able to perform the inverse operation accurately.

A more generally useful method, that could be used to check all steps that replace an expression by an equivalent expression, is trial evaluation. If expr is to be replaced by expr', then all unknowns and literal constants in expr and expr' are assigned numerical values. The expressions are then evaluated, and if expr and expr' are equivalent the values obtained must be the same. The method is only of heuristic value, since it may happen that the values obtained are the same even when expr and expr' are not equivalent. One student verified that x/x could be simplified to x by finding that 1/1 is 1. But prudent choice of numerical values, and the use of more than one assignment in especially doubtful cases, can make the method quite reliable. As shown in the table, this method was used only once in session 1.

The same solver used an invalid variant of this method in one other case. To check whether a factor could be cancelled from the two sides of an equation the student set up the "equation" $1- (1) (4)$, and then cancelled the 1's. Since the original "equation" is not valid it is hard to see how the cancelltion can be checked.

The same solver used one more local method to investigate the same cancellation operation. This time a simpler problem of the same form as the problem being attempted was created, and the doubtful step attempted on that problem: the original problem was $x+2=x+2(x^2+4x+4)$, the doubtful step being cancellation of $x + 2$, and the analogous problem was $a = (a) (b+c+d)$. In this case the analog was not a useful check for two reasons: first, the student was no more confident of the correctness of the step as applied to the simpler problem and second, the analogous problem is not really analogous, since it embodies a false interpretation of the grouping of terms in the original.

Turning to global methods, the commonest method of all is checking by substituting the answer into the original equation and seeing whether it reduces to an identity. This and other global methods have the drawback that they provide no information about what step in the solution process is faulty, if the answer does not satisfy the equation. Commonly, when substitution indicates the presence of an error, the student retraces the solution process hoping to spot a mistake. As noted above, retracing can fail because of a stable error, so it may be necessary to attempt an alternate solution method rather than patching up the original solution.

In two cases, S5 7A and S23 2A, students appeared to use substitution checking as a local check: There was just one step they were doubtful of, which they checked by obtaining an answer and substituting. Excerpts from these protocols are included in Table 7.7.

The only other global method used is consistency checking. Here the student compares the results of two different solution methods which should agree. The protocol S5 2A, excerpted in Table 7.7, illustrates the difficulty of using a global method to find an error. The student locates the incorrect step in one of two conflicting solutions, but is confident it is correct.

The other three cases of the use of this method also deserve special comment. In one, S13 2B, the solver had started and eventually abandoned one line of work. A second line was also bogging down. Before abandoning the problem the student checked for errors in either line by working from the end of the second line of work to obtain the equation reached at the end of the first line. This protocol is excerpted in Table 7.7. In S10 5A the two alternate solution methods begin from the two equations obtained by splitting an equation of the form a/b = c/d into a=c and b=d. Having obtaining a solution to the numerator equation the student solved the denominator equation, and, since the solution was different, realized that an error had been made. That the error arose in the splitting process was not realized. In case S12 1A, discussed more fully below, the student solved the equation, which contained literal constants, by solving an analogous equation with only numeric constants and then decomposing the numeric quantity obtained as a solution into an expression in the original literal terms. Rather than check this solution by substitution, which would have entailed the manipulation of literals that were avoided in the solution, a check was attempted by repeating the solution process using a different assignment of numerical constants. Unfortunately, an error in the second solution led to rejecting of the answer obtained earlier, which was in fact correct.

As mentioned above, methods differ in the type of error they can detect. Since the type of error different solvers are likely to make differ, different methods will be appropriate for different solvers.

Specifically, one would expect better solvers to have less need to check for stable conceptual errors, while this would be crucial for solvers with many conceptual confusions. Accordingly, a method like retracing would be more useful for good solvers than for poor ones, though a method like substitution of answers which catches both stable and unstable errors would be better for both groups.

Table 7.8 shows the frequency of use of the various checking methods broken down according to the accuracy of the solvers. As can be seen, checking is much more common among better solvers, especially when number of cases of checking rather than number of students checking is considered.

Table 7.8

Checking and Accuracy

	Top 10 Solvers		Middle 14 Solvers		Bottom 10 Solvers	
Method	cases	solvers	cases	solvers	cases	solvers
Local						
retrace	20	5	0	0	3	2
retrace part	1	1	0	0	4	2
inverse	2	2	2	1	3	2
trial evaluation	0	0	0	0	1	1
invalid variant	0	0	0	0	1	1
analog	0	0	0	0	1	1
Global						
substitution	19	6	13	6	2	2
consistency	0	0	2	2	2	2
Any	42	8/10	17	7/14	17	4/10

An argument could be made that trial evaluation would be the method of choice for poor solvers. Like substitution, it can detect stable conceptual errors of the sort poor solvers confront, but unlike substitution it can be used to test a single suspect step and so avoids the problem of determining where in an entire solution things went wrong. As can be seen, this desirable method is virtually unused.

The initiation of checking can apparently be controlled in various ways. Some students appear to check as a general policy, while others will check only when they have some reason to be doubtful about some step they have taken or some unexpected feature of the answer. Some excerpts from protocols that bear on this question appear in Table 7.7. Case E3 3A shows a check within a check: checking by substitution doesn't usually come out with 0 = 0, so the student considers the possibility that the equation might be satisfied by any value, which would render the check by substitution trivial. In case S17 7B the solver expected that the solution would involve some trick and since none was found an error is suspected. The two cases S17 6B and S17 4B indicate that the solver regards problems with fractions as especially needing checking. It is not clear whether this is because of their difficulty or whether the specific problem of extraneous roots is behind this idea. This solver did detect an extraneous root, and so was aware of the problem. Finally, case S23 7B indicates that checking will not always be done even when it might be desirable.

Another difference one might expect to find between human solvers and the Bundy model arises from the fact that equations have no meaning for the model. The model incorporates no knowledge about algebra that connects algebra to any other domain of knowledge. Greeno (1976) has argued that such connections are an ingredient of what we call understanding, and play a role in learning, recall, and transfer.

As shown in Figure 7.2, there are two domains in which the equations manipulated in algebra can be assigned meaning. Expressions and equations represent numbers, functions, and relationships among these. These entities, in turn, are often used to represent physical quantities and their relationships, or other relationships between quantities in the world, such as the relationship between discounts and prices. Having an interpretation for the objects being manipulated can be useful in checking results obtained. No physical or other practical interpretation was supplied in the present study, and there were no references by solvers to any interpretation of that kind.

Figure 7.2

Domains for interpreting equations

Symbolic domain	equations, expressions
Mathematical domain	numbers, functions relationships among these
The real world	physical processes and other quantitative relationships

The mathematical domain also provides potentially useful interpretations for equations and their parts. Expressions represent functions, and equivalent expressions are just those that represent equivalent functions (Note that "a+b-a" and "b" represent equivalent functions in the sense needed here, but not identical functions, since the first function has an extra variable.) Since equivalent functions have the same value for corresponding arguments, it is possible to test the equivalence of expressions by evaluation: this is just the trial evaluation method described above, which was used by one student. This check is semantic in that it implicitly used the correspondence of expressions and functions. It is questionable that there was any explicit knowledge of this semantic relation, however, since this student tried to use the method inappropriately at least one other time, in a way that violated the necessary correspondence between expression and interpretation.

There is not much indication, then, that meaning plays a large role in equation solving. That does not suggest that it could not do so, however. One might hope that the prevalence of ridiculously invalid operations could be reduced if students knew enough about the meaning of equations and expressions to assess the correctness of their actions. We return to this point in the discussion.

As a final point of departure from the Bundy model, we present some non-standard ways to solve equations. These serve as reminders that people, unlike the Bundy model, are not specially adapted to algebra, so one can expect invented methods to appear here as they do elsewhere in mathematics (see Resnick, 1979).

The ordinary way to solve an equation is to transform it into an equivalent equation or set of equations which is in the form x = expr, with expr free of x. This is the method used by the Bundy system. Two other methods appeared in the study, however. These methods avoided transforming the equation.

One of these methods occurred only in the single example shown in Figure 7.3. The student had attempted to solve the equations in the normal way by transforming it, but had trouble with the formal manipulations required. So he transformed the problem to an analogous one with numeric coefficients. This could be transformed using arithmetic instead of some of the formal operations, and solved. Then the solution to the original equation could be formed by tracing the solution of the numeric version.

Figure 7.3

Solving a numerical version of a problem

S12 1A

$A = p + prt$

restarting after
abandoning $1-A = -rt$

$A= 3$

$r= 2$

$t= 3$

$3-x+x(2)(5)$

$3=x+10x$

$11X$

$x = \frac{3}{11}$

$p = \frac{A}{rt+1}$

The other method was used more often, but its use is obvious only on problems for which it leads to errors, since on many problems it looks like solution by inspection. The method requires the assumption that a solution of an equation will, when substituted into the equation, make the two sides of the equation look the same. Thus 2 is a solution of $2x = x^2$, because 2·2 looks the same as 2·2. Zero would not be a solution, or perhaps not as good a solution, because 2·0 and 0·0 do not look the same. The method becomes distinctive when applied to equations involving fractions. Examples are shown in Table 7.9. As can be seen, the equation is split into two equations, one of which will when solved make the numerators the same, and the other of which would make the denominators the same. Unfortunately, it is not clear how users of the method deal with the usual situation in which these two equations have different solutions. In one case, S10 4B, it appears that it was assumed that either equation would yield a solution. In other cases the investigation of the two equations became confused and inconclusive, because of errors.

Table 7.9

Splitting equations with fractions

S16 5A	$\frac{5}{10} = \frac{x-10}{x+5} \longrightarrow \begin{array}{l}5=x-10\\10=x+5\end{array}$
5E	$\frac{2x^2+3}{3x+9} = \frac{x}{3} \longrightarrow \frac{2x^2+3=x}{3x+9=3}$
4B	$\frac{2x+3}{x^2} = 1 \longrightarrow 2x+3=1$
S10 5A	$\frac{5}{10} = \frac{x-10}{x+5} \longrightarrow \begin{array}{l}5=x-10\\10=x+5\end{array}$
S16 5B	$\frac{1-x^2}{1-x} = \frac{2}{1} \longrightarrow x^2 = 2$

CHAPTER 8

Discussion

We now return to the five general issues raised in the Introduction, and consider what light has been shed on each by the results of the study.

Errors and mechanisms We have argued that the errors we observe can be divided into three types: operator, applicability, and execution. The three types have different preventive or remedial measures.

Operator errors seem to reflect incorrect knowledge, or incomplete knowledge that is overextended under the pressure of solving. The erroneous steps seem to be the distorted and fragmented, but seldom completely unrecognizable, images of correct operations. The task of preventing such errors seems to divide into three parts: making initial learning more successful, discouraging the later construction or reconstruction of incorrect operators, and correcting existing incorrect operators. We have little to suggest about this last part of the task, beyond remarking that the kind of detailed diagnosis we have attempted may be essential in dealing with students' difficulties. This study included no intervention, and so we do not know what would have happened if S1, who gave such an articulate account of the cancellation errors in Figure 5.6, had been given at that moment an accurate analysis to consider.

The same detailed analysis of errors may be important in improving initial learning. With respect to deletion and recombination errors in particular it seems students are inclined to take a rather generalized view of the operations they are learning, and with the nature of the likely over generalization in mind it might be possible to choose examples in teaching that would bring out the needed discriminations.

A second attack on the problem of initial learning is through meta-knowledge of right and wrong. If the student can distinguish a correct operator from an incorrect one, then he or she can edit out the wrong guesses and generalizations that must inevitably flow from limited examples and ill-understood explanations. We will return to this in discussing the kinds of knowledge students should have.

In discouraging the construction or reconstruction of erroneous operations knowledge of right and wrong is still clearly important, since the problem of filtering correct operations from wrong guesses (Matz, 1979a) is fundamentally the same in construction as it is in initial learning. There is a second attack on this problem, however, that may be worth considering, though it is disturbing: Perhaps students should be discouraged from trying to figure out algebraic operations they do not understand. It may be that trying to stretch inadequate knowledge results in creating, patching up, and so preserving false notions, as well as preventing requests for help. Such an approach raises the problem of helping the student see the boundary of what it is sensible to try to work out and what it is not, but that problem is a real one and cannot just be ignored.

The bulk of applicability errors involve mishandling of parentheses, with the terms in an expression being assigned a false grouping. These errors are quite common even among accurate solvers, so it is unlikely that they reflect a real misunderstanding of the syntax of expressions. Rather, it seems that the patterns of parentheses is just not as salient as it should be, and when solvers are blocking out the structure of expressions it may be overridden by such other grouping factors as the repeated form in equation 6B. Unfortunately, it seems that it is the absence of parentheses, rather than presence, that is most frequently overlooked, so devices to draw attention to the parentheses might be ineffective. Perhaps students could rewrite expressions with extra spacing between terms that are not within parentheses.

Some students, such as the one who confused "one over two x" and "one-half x", do have trouble with the syntax of expressions. Probably more confusions about syntax are masked by the generic deletion and recombination operators, which blur many distinctions. Since expressions are representations of calculations, and the syntax is intended to capture the necessary information of order of operations and assignment of arguments to arithmetic operations, students might benefit from exercises in which complex calculations were to be written as expressions. Programs for a computer or calculator might be good representations of calculations from which to translate to expression form.

Like applicability errors, execution errors are relatively common even among accurate solvers. Also, there is not the clumping of these errors that is seen with some of the operator errors: no solver had more than 3 partial execution or control failure errors in Session 1. It is plausible that there is a tradeoff of speed and accuracy in equation solving as with other tasks, and that the price in efficiency of carrying out all operations flawlessly would be very large. More data would be needed, however, before it could be concluded that students' execution accuracy could not be improved without loss of efficiency.

<u>Good solvers</u> Not much was learned about the differences between more and less accurate solvers. There were a few glimpses of the kind of per-

formance one might have expected of expert solvers, but no sharp contrasts on a group basis between the most and least accurate solvers in the study beyond the difference in accuracy itself. There were suggestions of tighter strategic control, indexed by agreement with the basic Bundy strategy, more economical solutions, and greater use of repeated subexpressions as units of analysis. Further work with more challenging problems and more experienced solvers might sharpen and add to these indications.

Kinds of knowledge in algebra Equation solvers have to know a set of correct operators. They also have to know what to do to an equation, to move it closer to solution, and what operators will help. To decide what to do to an equation, the solver must know what features of an equation, such as number of instances of the unknown in a denominator, are important in deciding what to do. We have presented difficulties that reflect voids or distortions in these bodies of knowledge.

We argued that operators are not simple units of knowledge. It appears that there are operators like "factor" that have other operators as parts so that the student can use the notion "factor" in planning, and then expand it into its constituents when it must be carried out. Even operators that do not seem to have other operators as parts seem to have structure, in that a student may have only partial knowledge of them. Part of the knowledge of adding and multiplying seems to be that they accumulate things, while subtraction and division both take away things. We have argued in the case of recombination and deletion errors that these pieces of partial knowledge can determine the form and occurrence of errors.

Strategic knowledge is also not simple, or at least not as simple as in the Bundy model. It appears that a solver needs to distinguish linear from quadratic equations, and have a different plan for each. When the unknown is present in a denominator, a check for extraneous roots must be appended to the normal plan. The solver must know that if there are no occurrences of the unknown something has gone wrong, and that multiple occurrences of the unknown must be reduced to one before the equation is solved. At a more tactical level, the solver has to know what operators to use to collect occurrences, including the situation in which one occurrence has an implicit "1" as its coefficient. Another important area of tactics is the handling of x in the denominator especially the often baffling form "1/x = expr". Many solvers have explicit knowledge of all of these things, many have implicit knowledge, but many have no knowledge of some of them.

So much for basic equipment. A student who had the above knowledge would be a good solver. But it appears that there are other things that a solver should know, that are not parts of the skill itself, but are important in creating and maintaining the skill.

We have already touched on the importance of knowing how to tell a correct operator from an incorrect one. It appears that every student must encounter wrong ideas of operators. These may be formed by the student while learning in class or from a text, or while attempting to apply already learned material. They might originate in a misanalyzed example, in a false inference, or simply through memory failure. The student must be able to detect and reject these false ideas.

Despite the importance of this ability, some students appear to have no idea why some operators that might occur to them are valid and others are not, beyond appeal to authority. Asked about an error in combining fractions, one student said, "Well, I can't tell you what the rule is but I've seen it done before." The more extensive remarks of another student are excerpted in Figures 8.1.

Figure 8.1

Comment on knowing the rules

S21 4E

E: asks about what rule is used transforming 'ax+bx-2x = 2a-ab' to 'a+b-2 = 2-b'

S: "Mostly what I've been doing is just mainly from memory or its been trying to be from memory. I know there's a lot of rules in algebra that you have to learn first before you can go on to you know, that's the most important thing is to learn those rules...

If I had a book maybe...

I personally can't remember anything about those.. You have to go ask the teacher."

S21 5E

E: "Ah, so again I'm asking you for help. Is there a rule you could tell me that would tell me when I could cancel something of that sort...?"

S: "That I could tell you, that I could tell you, no. The book could tell you, yes"

E: "I mean apart from asking someone if there is any way that I could figure out whether a particular thing that I'm doing would be correct or whether it wouldn't be? Can you give me any advice along those lines?"

S: "It's always safe to ask the teacher if you don't know something. Maybe consult with a friend who is doing well in the subject or knows what he's doing."

E: "If I'm all by myself in a locked room or something is there some way I could figure it out, like is there some way I could relate it to other things that I might know about algebra or do I have to, you know, are the rules just things that you have to know or..."

S: "You're going to have to know the rules. Whether or not you can know if its the end of the problem pretty much depends on how much you have studied and how much you know. You know again it relates back to the rules."

There seem to be two ways students might test the validity of their ideas, use of principles and checking. "Principles" refers to the basic mathematical properties of the arithmetic operations that underlie the manipulations of algebra: that multiplication distributes over addition, that multiplication and division are inverses, and so on. A student who knew these principles well could reject operations that could not be derived from them. Unfortunately, the basic principles are almost as numerous and complex as the algebraic operators based on them, so students will be doubtful about them. Further, the testing of an operation by use of the principles is an exercise in proof, and may require some creativity. It is not transparent how cross multiplication is related to the basic principles, for example, because the operation as it is performed suppresses the underlying multiplications.

In our earlier discussion of checking we distinguished global and local methods, where local methods are those that can provide information about the correctness of a single step. Of these, trial evaluation seemed to be the method to prefer, for poor solvers, because of its generality and ability to detect both stable and transient errors.

While the basic trial execution method is applicable directly only to steps that replace an expression by an equivalent one (<u>reductions</u> in the terminology of Matz, 1979a), it can be extended to most <u>deductions</u>, steps that transform the entire equation, as follows. Most deductior can be analyzed as performance of the same operation on both sides of an equation, and simplification. Under this decomposition, it is the <u>simplifications</u> where many errors occur, and the simplifications are <u>reductions</u>, that can be checked by trial evaluation. Figure 8.2 shows how this method could be used to detect an invalid operation on reciprocals.

Two deductions to which the method cannot be applied are splitting of a quadratic equation by factoring, and use of the quadratic formula. These cannot be analyzed as performing the same operation to both sides of the equation. Cross multiplication is a borderline case. It can be analyzed as multiplying both sides of the equation by the product of the denominators, and then simplifying, but as was said above this analysis is not obvious.

Figure 8.2

Use of trial evaluation to check
an operation on reciprocals

Problem: Can $\frac{1}{\frac{1}{a}+\frac{1}{b}}$ be replaced by $a + b$?

Check:

Substitute $a = 2$, $b = 3$

$$\frac{1}{\frac{1}{a}+\frac{1}{b}} \longrightarrow \frac{1}{\frac{1}{2}+\frac{1}{3}} \longrightarrow \frac{1}{\frac{5}{6}} \longrightarrow \frac{6}{5}$$

$$a + b \longrightarrow 2 + 3 \longrightarrow 5$$

$5 \neq \frac{6}{5}$, so replacement is not valid.

Despite these limitations, it appears that "checking by trial" education might help students filter good ideas from bad. But would they use it? One promising indication is that students do seem often to be aware that they are performing doubtful operations, as some of the comments collected in this report show. Thus they recognize some occasions when local checking would help them, if they knew about it.

Lack of knowledge of how to check is probably not the only reason students do not check, however. Checking is taught as a means of increasing accuracy. As long as checking is just a way to be more sure of the answer to a given problem, students are free to use it or not according to the accuracy and time pressures they feel. For good solvers, it is often rational not to check for this reason. Poor solvers, however, need to see checking as a way of evaluating their knowledge, not their answers.

This leads to the last kind of knowledge of algebra that we will discuss. Students need to know how to distinguish incorrect operators from correct ones but they also need to see how this ability fits into the learning task. They need to know that they will form misconceptions and make errors, despite their best efforts, and that there are specific actions they can take to deal with these difficulties. Testing and correction of knowledge should be seen as a normal and important activity, not as an option, or worse, as something good students could avoid. Discussion of common errors, with emphasis on the often sensible analysis that lies behind them, might help develop the necessary perspective on learning.

Meaning in algebra As the Bundy program illustrates, one can do algebra without understanding it, in the sense of being able to assign any meaning to the entities being manipulated. We have seen that there is very little evidence that the human solvers in the study made use of meanings in their work. We have suggested nevertheless, that meaning is important when knowledge of algebra is learned or recalled. This seems true not just because of the helpful redundancy that any interconnections in a body of knowledge seem to provide, but because of the specific role meaning can play in allowing the validity of operators to be tested, as in the trial evaluation method. There the relation between expressions and calculations is central. Another area in which we have suggested this relation could be useful is in learning the syntax of expressions.

The expression-calculation system is an interesting domain of application of Greeno's (1978) ideas on understanding. Greeno suggests an analogy with language comprehension, in which to understand an entity, say a sentence, is to have an internal representation of it that is coherent, connected to other relevant knowledge, and accurately captures the essential features of the entity. In the expression-calculation domain, understanding an expression means being able to construct an internal representation of a calculation? The calculation that the expression represents.

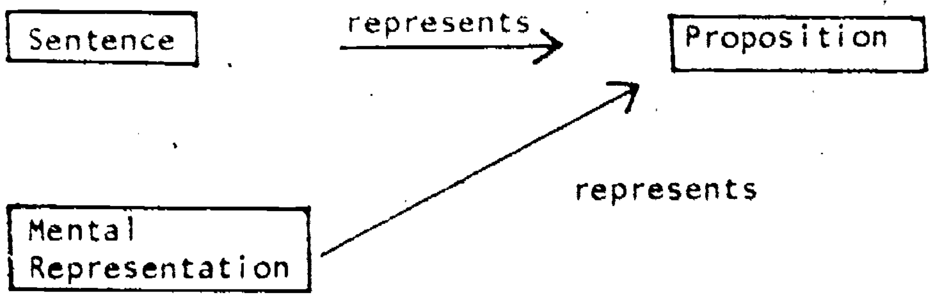
To straighten out these many uses of "represent", let us try to trace the parallel with sentence comprehension more closely. When a sentence is understood, the internal representation of it has to be more than a coding of the sounds, say, that make up the sentence. It is common to claim that the representation should be a proposition (or set of propositions). But propositions are not things that can exist in someone's head, so the representation of the sentence is not really a proposition but a representation of a proposition. In the same way, the internal representation of an expression would not be a calculation, but a representation of a calculation. Finally, what proposition is it whose representation is the appropriate representation of the sentence? Just the proposition that the sentence itself represents. Figure 8.3 shows the analogy between sentence and expression and calculation graphically.

This linguistic analogy may be helpful in clarifying how establishing calculations as meanings for expressions could help students. There is research indicating that having meanings assigned helps enormously in the learning of the syntax of artificial languages (Moeser & Bregman, 1972, 1973). The problem of learning operators that preserve the meaning of expressions might seem to be parallel to the problem of learning meaning-preserving transformations, but it is not. The operations of algebra do not alter the rules of syntax for expressions: an expression with zero added to it has the same syntatic rules of formation of any other expression. A passivized sentence may likewise have the same surface grammar as sentences generally. But a sentence like "John gave Mary the book," does not follow the ordinary rules of syntax. So learning meaning-preserving transformations cannot be separated from learning the syntax of surface strings in the same way that learning algebraic manipulations can be separated from learning the syntax of expressions.

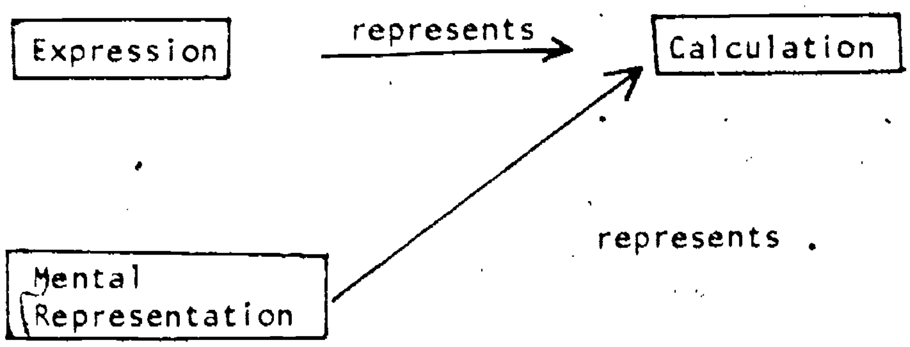
<u>Errors and the psychology of skill</u> Matz (1979a) and Brown (1979) have outlined related theories of the origin of errors in skilled performance. In both theories, errors arise when incomplete knowledge is extended to cover a new problem, and the nature of the error that is made in the extension process is influenced by the partial knowledge

Figure 8.3

Parallel between understanding a sentence
and understanding an expression



To understand a sentence, one needs a mental representation of the proposition.



To understand and expression, one needs a mental representation of the calculation.

that is available, and by ideas the subject has about the character of the operations being learned or applied.

In Matz's theory, operators are constructed from the partial knowledge available, and then filtered by a family of "critics" which accept or reject the construction according to whether it has certain desirable general properties. For example, Matz suggests that an operator in algebra should "touch every part": it should not ignore any piece of the expression or expression being transformed. Most actual operators have this property, though there are exceptions. For example, replacement of 0 expr by 0 does not require any testing, analysis, or copying of expr. Matz argues that students are likely to make generalizations ~~like~~ "touch every part" and that their constructed operators will be made to obey the generalizations.

In Brown's theory, developed most fully for algorithms in arithmetic, students try to apply their existing partial algorithms and run into missing or impossible steps. They then use their accumulated general knowledge to "patch" their algorithm so that it can be executed to completion. In the patching process they use knowledge about the character of familiar and likely steps, information about the desired outcome of the algorithm, and also about features of the execution process itself: the algorithm should not loop, for example.

While these theories specify mechanisms for the production of errors, the mechanisms are quite flexible, and depend heavily on the specific knowledge possessed by a solver to shape errors. Consequently, it is difficult to support or falsify these ideas by an examination of errors such as those we have selected. Considering students' comments, we did not find students articulating principles like "touch every part", and these would have to be implicit knowledge for our solvers.

Students' comments do show, however, that the sort of stretching of knowledge called for in both theories does occur for many solvers, and a few of the protocol excerpts presented indicate some of the knowledge solvers bring to bear on the problem. For example, in case S5 6C in Table 5.8 shows the erroneous application of cross multiplication halted by the appearance of a second equal sign, which violated the students' knowledge of the form of an equation. Above the level of individual operators, we have examples showing a number of indicators of progress or trouble that students use, collected in Table 7.5.

We have obtained some general support, then, for the ideas of Matz and Brown, and some specifics. It might be possible to fill in more of the details in these schemes by asking students to evaluate operators proposed by the experimenters. This might elicit more comments from the students we obtained, as well as getting a more complete picture of the criteria students may be using by going beyond just self-generated errors.

Two particular categories of errors which might repay further analysis are deletion and recombination errors. While these are perhaps consistent with the Matz or Brown mechanisms, it seems that these errors originate from general ideas of the task, rather than just being passed by critics that embody this knowledge, as in Matz's view, or resulting from a patch on a mutilated algorithm. Since algebra can be thought of as an exercise in pure symbol manipulation, it is tempting to imagine that students might organize their knowledge of operators around the key notions of deletion and rearrangement, and so develop operators to perform these functions. The existence of several different deletion operations, and of restrictions on rearrangement might be seen as detail that could be suppressed. If this analysis is correct, it places the origin of some errors closer to the center of the learning process than the critics or patching mechanisms of Matz and Brown.

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