DOCUMENT RESUME

ED 184 883 SE 030 558

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TITLE Kinds of Knowledge in Algebra.
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SPONS AGENCY National Inst. of Education (DHEW), Washington, D.C.; National Science Foundation, Washington, D.C.
PUB DATE Apr 80
NOTE 19p.; Paper presented at the Annual Meeting of the American Educational Research Association (Boston, MA, April 7-11, 1980). Not available in hard copy due to marginal legibility of original document.

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *Algebra; *Cognitive Processes; *College Mathematics; *Educational Research; Error Patterns; Higher Education; Interviews; Mathematics Education; *Mathematics Instruction; Problem Solving

ABSTRACT

Solving equations in elementary algebra requires knowledge of the permitted operations, and knowledge of what operation to use at a given point in the solution process. While just these kinds of knowledge would be adequate for an ideal solver, human solvers appear to need and use other kinds of knowledge. First, many errors seem to indicate that single operations of algebra are often represented as collections of parts which may have independent status. Second, operations seem to be connected, often inappropriately, with a general scheme for symbol manipulation. Third, various kinds of knowledge are brought into play to detect errors. Evidence about the role of these kinds of knowledge, found in the solutions and comments of college students, is discussed, and some implications for instruction are considered. (Author/MK)

Kinds of knowledge in algebra

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AERA Annual Meeting, April, 1980

Abstract. Solving equations in elementary algebra requires knowledge of the permitted operations, and knowledge of what operation to use at a given point in the solution process. While just these kinds of knowledge would be adequate for an ideal solver, human solvers appear to need and use other kinds of knowledge. First, many errors seem to indicate that single operations of algebra are often represented as collections of parts which may have independent status. Second, operations seem to be connected, often inappropriately, with a general scheme for symbol manipulation. Third, various kinds of knowledge are brought into play to detect errors. Evidence about the role of these kinds of knowledge , found in the solutions and comments of college students, is discussed, and some implications for instruction are considered.

I'd like to talk about some of the things students know about solving equations in elementary algebra. Obviously, successful solvers have to know the legal moves of the algebra game, and when to make them. But there's apparently more to solving than that, at least as people play the game: other kinds of knowledge are involved. I want to discuss some of this extra knowledge. For evidence and examples I'll be drawing on a set of protocols that John Bernard, Ray Carry, and I collected from college students at the University of Texas at Austin.

This work was supported by a grant from the joint NIE-NSF program for Research on Cognitive Processes in Science and Mathematics, and by IBM.

Pieces of procedures

Let's look first at the legal moves solvers have to know. It appears that some students don't know these as neat little packets, with each packet holding a procedure. They seem to know about a collection of pieces of procedures that can be put together to make legal moves. (At least, the hope is that they can be put together this way.) Here is an example of this kind of knowledge.

Look at the errors shown in Figure 1. In each case, two fractional expressions are combined incorrectly. But the incorrect combinations are not arbitrary. In fact, as shown in the Figure, the procedures used in each case can be dissected into pieces, each of which is found in some correct procedure for combining fractional expressions. This suggests that what these students know is not a way to add these expressions, say, but rather a repertoire of operations that may be drawn on in adding them.

It's interesting that Karen Larkin found the same thing happening when she studied adults' procedures for doing arithmetic on fractions (Larkin, 1978). It is plausible that students carry over to the algebraic combination of fractional expressions the operations they learned in doing arithmetic on fractions.

A question is raised by the suggestion that students just know pieces of procedures. How do we know when the pieces are put together? It could be that at some point in the

$$\frac{x}{1} + \frac{x + 1}{2} \rightarrow \frac{x + x + 1}{2}$$

Procedure: Add numerators, as in addition of fractions with common denominator. Then place over common denominator.

$$(4 + 2/x)\left(\frac{x}{2 + x}\right) \rightarrow \frac{4(x(2 + x) + 2(2 + x)(x^{2})}{x + 2x}$$

Procedure: Convert all terms to common denominator of $x(2 + x)$. Then combine numerators according to the indicated operations and place result over common denominator.

$$\frac{7}{1} - \frac{x + 3}{x} \rightarrow \frac{7 - x + 3}{1 - x}$$

Procedure: Combine numerators and denominators according to the indicated operation, as in multiplication.

$$\frac{4}{1} + \frac{2}{x} \rightarrow \frac{8}{x}$$

Procedure: Apply method for multiplication.

Figure 1. Some errors in combining fractions

past the student (incorrectly) built up a procedure for combining fractions from disparate pieces, but at the moment we are watching, this incorrect procedure is neatly packaged, with the knowledge about the pieces long forgotten. It would be nice to have more direct evidence than we do on this point. Larkin found changes in the procedures used by a given person that suggest there was no established procedure available. We collected one protocol

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$$\frac{1}{3} = \frac{1}{x} + \frac{1}{7}$$

This one I can do. Cross multiply.

$$\frac{1}{3} = 7 + x$$

It doesn't seem to me like that'd be right.

OK all right, it's not right.

$$\frac{1}{3} = \frac{x}{7}$$

Figure 2. Protocol showing attempts to combine fractions

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fragment, shown in Figure 2, that may show an operation on fractional expressions being built. As shown in Figure 3,

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$$\frac{1}{x} + \frac{1}{7} \dashrightarrow 7 + x$$

"Cross multiplication" may occur as part of the addition procedure, a/b + c/d --> (ad + bc)/bd, in which it is used to form the numerator, or as a misapplication of the operation a/b = c/d --> ad = bc, with "=" replaced by "+".

$$\frac{1}{x} + \frac{1}{7} \dashrightarrow \frac{x}{7}$$

Invert and multiply, as in division of fractions.

Figure 3. Analysis of the operations attempted in Figure 2.

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the procedures the student considered do draw on the repertoire of pieces of legal operations on fractions.

So here's one kind of extra knowledge students seem to have. They sometimes know pieces of legal moves, not the legal moves themselves. Of course, this knowledge is "extra" only in the sense that if one did know the legal moves one wouldn't need this other stuff.

How to manipulate symbols

Another kind of "extra" knowledge is knowledge about symbol manipulation. We are accustomed, I think, to deplore "blind symbol manipulation" as an unproductive approach to mathematics for students. We may perhaps also note (as I will below) that viewing mathematical operations as just operations on symbols throws away redundancy in the system that can be used to detect errors. But I want to suggest that seeing algebra as symbol manipulation is even more pernicious, in that it actually promotes wrong ideas.

The key notions in symbol manipulation are deletion, rearrangement, replacement: those are the things you do with symbols. Viewing algebra this way, you try to get rid of things, rearrange, and replace, until you have "x = something" with no x in the something. What's dangerous, as opposed to just suboptimal, in this view? It makes salient those aspects of the operations of algebra that have to do with deleting, rearranging, and replacing at the expense of other critical aspects.

Consider deletion. The common operations that have the effect of deleting things are subtraction and division, as

$x + a \longrightarrow a$ by subtracting x

$ax \longrightarrow x$ by dividing by a

Figure 4. Subtraction and division as deletions.

illustrated in Figure 4. In the symbol manipulation view of algebra, these become close companion operations-- they do "the same thing"-- and we might expect their differences to be ignored. But that's just what happens-- look at the

$x^2 = 2x \longrightarrow x = 2$ "Could I subtract an x from these ones?"

$x + 2(x + 2(x + 2)) = x + 2 \longrightarrow$

$x + 2(x + 2) = 0$ "Subtract"

$x + 3 + x = x^2 \longrightarrow 3 + x = x$

$\frac{12x}{12} = \frac{x + 12}{12} \longrightarrow x = x$

Figure 5. One student's confusion of subtraction and division, with his comments.

examples in Figure 5.

The confusion may give rise to a generic deletion operation, that combines the features of correct subtraction and division. In the last three examples in the figure, it may appear that the student is breaking a basic rule of algebra, in that he is doing different things to the two sides of the equation. But given the generic deletion operation, his misconception is seen to be quite different, and perhaps less basic. He <u>is</u> doing "the same thing" to both sides. He's deleting.

The generic deletion operation can also account for the

Correct example:

$$\frac{ax}{ay} \longrightarrow \frac{x}{y}$$

Incorrect examples:

$$\frac{x^2 + (x + 3)}{3(x + 3)} \longrightarrow \frac{x^2}{3}$$

$$\frac{x + ax}{a} \longrightarrow x + x$$

$$\frac{x^2 + x + 3}{x + 3} \longrightarrow x^2$$

Figure 6. Simplifying fractional expressions by deleting elements common to numerator and denominator.

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very common errors shown in Figure 6. What the students are doing is deleting the same thing in numerator and denominator, not dividing. Division is just a special case.

Figure 7 shows some examples of correct rearrangement and replacement of symbols. Let's call this "recombination". In each case, symbols are counted and replaced by an expression that embodies the identity of the symbol, and the count. Well, this useful pattern of recombination seems to stand out above the details, for some students, like the

$x + x + x \rightarrow 3x$

$x\ x\ x \rightarrow x^3$

Figure 7. Recombination operations.

affinity of subtraction and division. The distinction between addition and multiplication gets lost. Look at the

$x^2 + x + 3 \rightarrow x^3 + 3$

$y + yz \rightarrow 2yz$

$p + p \rightarrow p^2$

"x is one-half of x squared"

$ax + bx + ab \rightarrow 2a + 2b + 2x$

Figure 8. Recombination confusions.

examples in Figure 8.

In these examples, and the deletion cases discussed before, it appears that what students know about algebra is shaped, and in fact distorted, by what they know about symbol manipulation. This view is consistent with Matz's proposals (1979) about the origin of errors in algebra learning. Matz emphasizes the role of internal "critics" that try to make sure that the operations performed are consistent with what the student knows about the legal moves. It seems likely that the symbol manipulation model forms an important part of the knowledge these critics embody, for some students.

<u>How to find mistakes</u>

The last kind of knowledge that I want to discuss is knowledge that's used to detect errors. We can distinguish two kinds. First, Figure 9 shows some remarks that indicate that students monitor their progress as they work on problems, being sensitive to some indications that something may have gone wrong. Second, there are a number of ways students have of checking their work. Figure 10 describes

Method	Number of students (n=34)
Substitute answer in equation	14
Retrace solution or part of solution	7
Use an inverse operation (e.g., multiply to check factoring)	5
Try a second solution method	4
Other	2

Figure 10. Methods of checking observed among 34 college students.

the methods we observed students using.

Although it's clear that many students do have and use knowledge about monitoring their progress and checking, Complexity:

$$\frac{1}{R} - \frac{1}{x} = \frac{1}{y} + \frac{1}{z} \longrightarrow \ldots \longrightarrow$$

$$yzx - yzR = Rxz + Rxy$$

"I'm going to try a a different way. I don't like that. It's too complicated."

Legality:

$$7 = \frac{21}{x} + 3 \longrightarrow x = 2 \ldots$$

"Take the x over to this side and bring this seven through. Can't do that."

Backtracking:

$$yz(x - R) = Rx(z + y) \longrightarrow$$

$$yzx - yzR = Rxz + Rxy$$

"We're going to factor out, naw, that will give me the thing I started with"

Lack of solution:

$$9(x + 40) = 5(x + 40) \longrightarrow 9 = 5$$

"Nine equals five doesn't exist, so I'm going to work out the problem."

Dead end:

$$x^2 - 2x + 3 = 0$$

"This one doesn't seem to factor very well. That's why, I made a mistake."

Figure 9. Monitoring the progress of the solution process.

nevertheless there are gaps in this knowledge that makes it less effective than perhaps it might be. What students need to be able to do, especially poor students, is not so much detect an error on a given problem, but rather detect an error in their procedure for solution. Unfortunately, the checking methods students use are not well adapted to use for this kind of knowledge refinement. The retracing method won't detect stable, conceptual errors. The substitution method can't be used until an answer is obtained, and won't pinpoint which step in a solution is faulty. What's needed is a method that can detect stable errors, but unlike the substitution method can be used in the middle of the solution process to give feedback on a single doubtful step.

So students do have knowledge they can use to detect errors. But their knowledge is not well tailored to their needs.

Implications

We've now talked about three kinds of knowledge students seem to have, above and beyond the obviously necessary:

.) They know about the legal moves, but not always the moves themselves

.) They know things about symbol manipulation that may mislead them about algebra

.) They know some ways to detect their own errors

These points are perhaps interesting from a natural history standpoint but they have implications for pedagogy as well.

Let's go back to knowing pieces of legal moves. Given that students look on operations on fractions as collections of more or less familiar pieces whose specific combinations are poorly understood, maybe we can help them with the specific problem of keeping track of what pieces go with which operation. One way to do this is to learn a couple of simple examples as patterns, like one-half plus one-quarter. Figure 11 shows how this pattern can be used

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Pattern: $\frac{1}{2} + \frac{1}{4} \dashrightarrow \frac{3}{4}$

Procedure: Add numerators, place over common denominators, as in

$$\frac{x}{1} + \frac{x+1}{2} \dashrightarrow \frac{x+x+1}{2}$$

Test: $\frac{1}{2} + \frac{1}{4} \dashrightarrow \frac{1+1}{4} \dashrightarrow \frac{2}{4} \text{ not } = \frac{3}{4}$

Figure 11. Testing a procedure for combining fractions.

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to detect one of the errors in Figure 1.

Let's look more closely at how this use of patterns works. As shown in Figure 12, the student sets up the

```
Domain of symbols and      I  Domain of numbers and
operations on them.        I  and operations on them.
                           I
                           I
      1/2..........represents..........one-half
                           I
      1/4..........represents..........one-quarter
                           I
    procedure              I
    under test.....represents..........addition
                           I
    result of              I
    procedure,             I
       2/4....should represent.........three quarters
                           I
                           I
```

Figure 12. Logic of test shown in Figure 11.

operands, 1/2 and 1/4, and applies the procedure that is to be tested. The result is 2/4. Now this system, of operands, procedure, and result, is irredundant: there's no way to tell what the result should be, so there's no way to tell whether the procedure is correct. What has to be added is relation between this system and actual numbers, as opposed to written symbols. The objects in the symbol domain <u>represent</u> numbers, and the symbolic procedure is supposed to <u>represent</u> addition. Now the system is redundant. The result of applying the procedure has to represent the result of adding in the number domain. But it doesn't, so the procedure is wrong.

If we now turn to the errors spawned by the symbol-manipulation view of algebra, we find that the same general idea of <u>representation</u> can be used to expose these.

```
Domain of symbol             I  Domain of
strings                      I  calculations
                             I
                             I
      y + yz.......represents..........Multiply y and z
                             I          and add y.
                             I
      2yz..........represents..........Multiply 2, y, z.
                             I
                             I
                             I                For y=2 and z=3,
                             I                the first calculation
                             I                gives 8, while the
                             I                second gives 12.
                             I                They are therefore
                             I                not equivalent.
                             I
```

Figure 13. Detecting a recombination error.

The analysis is shown in Figure 13. In this case, the problem is to tell whether two expressions, which through a recombination confusion might be thought equivalent, really are equivalent. As symbol strings, there's no way to know whether y + yz and 2yz can be regarded as equivalent or not. But the expressions <u>represent</u> other things, just as 1/2 represents a number. An expression represents a calculation, a recipe for combining numerical ingredients arithmetically. For example, y + yz represents a calculation which might otherwise be expressed as "Multiply y and z, and add y." Two calculations are equivalent just when they always give the same result for the same ingredients. But this gives a simple way to tell whether two calculations are equivalent: just carry them out on the same ingredients and compare the results. Sometimes different calculations will look the same for a few cases, but if two calculations agree in two cases without a lot of zeroes and ones in the ingredients they probably are equivalent.

This comparison method for calculations now allows us to tell whether two expressions are equivalent, as shown in the figure. Putting the representation relations in this system makes it redundant in just the same way as in the case of operations on fractions.

Now we've made some suggestions about the first two kinds of knowledge we're considering: knowing pieces of procedures, and knowing how to manipulate symbols. But in fact this has led us to talk about the third kind of knowledge, knowing how to detect errors. In fact the checking procedure we've just been discussing is just the sort of procedure we argued students don't know and should. This procedure -- let's call it "trial evaluation"-- can detect both stable and unstable errors, and can be used to evaluate a single step in the middle of a solution. It can test the validity of any step that can involves the replacement of an expression by an equivalent one, and most steps have that character.

If students knew about trial evaluation, would they use it? That remains to be seen. Students often express doubt about the correctness of steps. but they don't know what to do about the doubt. One student's views are given in Figure

Experimenter: Is there a rule you could tell me that would tell me, that would tell me when I could cancel something of that sort...?

Student: That I could tell you, that I could tell you, no. The book could tell you, yes.

Experimenter: I mean apart from asking someone if there is any way that I could figure out whether a particular thing that I'm doing would be correct or whether it wouldn't be? Can you give me any advice along those lines?

Student: It's always safe to ask the teacher if you don't know something. Maybe consult a friend who is doing well in the subject or knows what he's doing.

Experimenter: If I'm all by myself in a locked room or something is there some way I could figure it out, like is there some way I could relate it to other things that I might know about algebra, or do I have to, you know, are the rules just things that you have to know or...

Student: You're going to have to know the rules.

Figure 14. A conversation about checking knowledge.

14. Trial evaluation may help.

Meaning

The argument for trial evaluation, or for using patterns to check operations on fractions, happens to fit a familiar form: if students knew the meaning of the things they manipulate they could avoid mistakes. But I think it is important not to put the case this way. It suggests that "meaning" is some generally good thing that we should exploit, and perhaps there are many ways to do that. One might try (as we have) to convey "meaning" by teaching the mathematical abstractions, like "distributivity", that describe arithmetic. But in fact this knowledge cannot be used in a practical procedure to compare two expressions, and what is important is not "meaning" but <u>what it lets you do</u>. "Meaning" doesn't make trial evaluation good; rather, trial evaluation is one thing that makes meaning good.

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