# DOCUMENT RESUME

ED 184 840 SE 030 456

AUTHOR Wilson, James W., Ed.; And Others
TITLE Soviet Studies in the Psychology of Learning and Teaching Mathematics, Volume XII: Problems of Instruction.
INSTITUTION Chicago Univ., Ill.; Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 75
GRANT NSF-G-8847(MA)
NOTE 185p.; For related documents, see SE 030 451-458, ED 042 628, ED 042 632-633, and ED 080 382-384

EDRS PRICE MF01/PC08 Plus Postage.
DESCRIPTORS Elementary Secondary Education; Error Patterns; Geometric Concepts; Learning Theories; *Mathematics Education; *Mathematics Instruction; *Psychological Studies; Psychology
IDENTIFIERS School Mathematics Study Group; *Soviet Research; *USSR

ABSTRACT

This is one of a series that is a collection of translations from the extensive Soviet literature of the past 25 years on research in the psychology of mathematics instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English. The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession. The seven studies found in this volume are: An Experiment in the Psychological Analysis of Algebraic Errors; Pupils' Comprehension of Geometric Proofs; Elements of the Historical Approach in Teaching Mathematics; Overcoming Students' Errors in the Independent Solution of Arithmetic Problems; Stimulating Student Activity in the Study of Functional Relationships; Psychological Grounds for Extensive Use of Unsolvable Problems; and Psychological Characteristics of Pupils' Assimilation of the Concept of a Function. (Author/MK)

# SOVIET STUDIES IN THE PSYCHOLOGY OF LEARNING AND TEACHING MATHEMATICS

## VOLUME XII

Mary L. Charles
NSF

SCHOOL MATHEMATICS STUDY GROUP
STANFORD UNIVERSITY
AND
SURVEY OF RECENT EAST EUROPEAN
MATHEMATICAL LITERATURE
THE UNIVERSITY OF CHICAGO

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## PROBLEMS OF INSTRUCTION

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Financial support for the School Mathematics Study Group and for the Survey of Recent East European Mathematical Literature has been provided by the National Science Foundation.

# PREFACE

The series Soviet Studies in the Psychology of Learning and Teaching Mathematics is a collection of translations from the extensive Soviet literature of the past twenty-five years on research in the psychology of mathematical instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. The series is the result of a joint effort by the School Mathematics Study Group at Stanford University, the Department of Mathematics Education at the University of Georgia, and the Survey of Recent East European Mathematical Literature at the University of Chicago. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English.

Research achievements in psychology in the United States are outstanding, indeed. Educational psychology, however, occupies only a small fraction of the field, and until recently little attention has been given to research in the psychology of learning and teaching particular school subjects.

The situation has been quite different in the Soviet Union. In view of the reigning social and political doctrines, several branches of psychology that are highly developed in the U.S. have scarcely been investigated in the Soviet Union. On the other hand, because of the Soviet emphasis on education and its function in the state, research in educational psychology has been given considerable moral and financial support. Consequently, it has attracted many creative and talented scholars whose contributions have been remarkable.*

Even prior to World War II, the Russians had made great strides in educational psychology. The creation in 1943 of the Academy of Pedagogical Sciences helped to intensify the research efforts and programs in this field. Since then the Academy has become the chief educational research and development center for the Soviet Union. One of the main aims of the Academy is to conduct research and to train research scholars

*A study indicates that 37.5% of all materials in Soviet psychology published in one year was devoted to education and child psychology. See Contemporary Soviet Psychology by Josef Brozek (Chapter 7 of Present-Day Russian Psychology, Pergamon Press, 1966).

in general and specialized education, in educational psychology, and in methods of teaching various school subjects.

The Academy of Pedagogical Sciences of the USSR comprises ten research institutes in Moscow and Leningrad. Many of the studies reported in this series were conducted at the Academy's Institute of General and Polytechnical Education, Institute of Psychology, and Institute of Defectology, the last of which is concerned with the special psychology and educational techniques for handicapped children.

The Academy of Pedagogical Sciences has 31 members and 64 associate members, chosen from among distinguished Soviet scholars, scientists, and educators. Its permanent staff includes more than 650 research associates, who receive advice and cooperation from an additional 1,000 scholars and teachers. The research institutes of the Academy have available 100 "base" or laboratory schools and many other schools in which experiments are conducted. Developments in foreign countries are closely followed by the Bureau for the Study of Foreign Educational Experience and Information.

The Academy has its own publishing house, which issues hundreds of books each year and publishes the collections Izvestiya Akademii Pedagogicheskikh Nauk RSFSR [Proceedings of the Academy of Pedagogical Sciences of the RSFSR], the monthly Sovetskaya Pedagogika [Soviet Pedagogy]; and the bimonthly Voprosy Psikhologii [Questions of Psychology]. Since 1963, the Academy has been issuing collection entitled Novye Issledovaniya v Pedagogicheskikh Naukakh [New Research in the Pedagogical Sciences] in order to disseminate information on current research.

A major difference between the Soviet and American conception of educational research is that Russian psychologists often use qualitative rather than quantitative methods of research in instructional psychology in accordance with the prevailing European tradition. American readers may thus find that some of the earlier Russian papers do not comply exactly to U.S. standards of design, analysis, and reporting. By using qualitative methods and by working with small groups, however, the Soviets have been able to penetrate into the child's thoughts and to analyze his mental processes. To this end they have also designed classroom tasks and settings for research and have emphasized long-term, genetic studies of learning.

Russian psychologists have concerned themselves with the dynamics of mental activity and with the aim of arriving at the principles of the learning process itself. They have investigated such areas as: the development of mental operations; the nature and development of thought; the formation of mathematical concepts and the related questions of generalization, abstraction, and concretization; the mental operations of analysis and synthesis; the development of spatial perception; the relation between memory and thought; the development of logical reasoning; the nature of mathematical skills; and the structure and special features of mathematical abilities.

In new approaches to educational research, some Russian psychologists have developed cybernetic and statistical models and techniques, and have made use of algorithms, mathematical logic and information sciences. Much attention has also been given to programmed instruction and to an examination of its psychological problems and its application for greater individualization in learning.

The interrelationship between instruction and child development is a source of sharp disagreement between the Geneva School of psychologists, led by Piaget, and the Soviet psychologists.* The Swiss psychologists ascribe limited significance to the role of instruction in the development of a child. According to them, instruction is subordinate to the specific stages in the development of the child's thinking--stages manifested at certain age levels and relatively independent of the conditions of instruction.

As representatives of the materialistic-evolutionist theory of the mind, Soviet psychologists ascribe a leading role to instruction. They assert that instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages. The Russians study development in the changing conditions of instruction, and by varying these conditions, they demonstrate how the nature of the child's development changes in the process. As a result, they are also investigating tests of giftedness and are using elaborate dynamic, rather than static, indices.

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*See The Problem of Instruction and Development at the 18th International Congress of Psychology by N. A. Menchinskaya and G. G. Saburova, Sovetskaya Pedagogika, 1967, No. 1. (English translation in Soviet Education, July 1967, Vol. 9, No. 9.)

Psychological research has had a considerable effect on the recent Soviet literature on methods of teaching mathematics. Experiments have shown the student's mathematical potential to be greater than had been previously assumed. Consequently, Russian psychologists have advocated the necessity of various changes in the content and methods of mathematical instruction and have participated in designing the new Soviet mathematics curriculum which has been introduced during the 1967-68 academic year.

The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession, for it is generally recognized that experiment and research are indispensable for improving content and methods of school mathematics.

We hope that the volumes in this series will be used for study, discussion, and critical analysis in courses or seminars in teacher-training programs or in institutes for in-service teachers at various levels.

At present, materials have been prepared for fifteen volumes. Each book contains one or more articles under a general heading such as The Learning of Mathematical Concepts, The Structure of Mathematical Abilities and Problem Solving in Geometry. The introduction to each volume is intended to provide some background and guidance to its content.

Volumes I to VI were prepared jointly by the School Mathematics Study Group and the Survey of Recent East European Mathematical Literature, both conducted under grants from the National Science Foundation. When the activities of the School Mathematics Study Group ended in August, 1972, the Department of Mathematics Education at the University of Georgia undertook to assist in the editing of the remaining volumes. We express our appreciation to the Foundation and to the many people and organizations who contributed to the establishment and continuation of the series.

Jeremy Kilpatrick
Izaak Wirszup
Edward G. Begle
James W. Wilson

EDITORIAL NOTES

1. Bracketed numerals in the text refer to the numbered references at the end of each paper. Where there are two figures, e.g. [5:123], the second is a page reference. All references are to Russian editions, although titles have been translated and authors' names transliterated.

2. The transliteration scheme used is that of the Library of Congress, with diacritical marks omitted, except that Ю and Я are rendered as "yu" and "ya" instead of "iu" and "ia."

3. Numbered footnotes are those in the original paper, starred footnotes are used for editors' or translator's comments.

TABLE OF CONTENTS

## INTRODUCTION

James W. Wilson and Jeremy Kilpatrick

Educational research in the Soviet Union has produced a considerable literature devoted to problems of instruction in mathematics classes. The new school mathematics curriculum adopted by the Soviet Ministry of Education in 1967 drew upon and reflected this literature. The articles in this volume all predate the new curriculum and treat a variety of learning problems encountered in mathematics classrooms.

The lengthy paper by Shevarev reports a series of investigations on students' errors in learning algebra. The paper, published in 1946, follows from his earlier study of algebraic skills [2]. It uses a blend of psychological theory, observation of students in classes, and analysis of textbook material. The study was restricted to the errors occurring when students know a rule, are able to apply it, but nevertheless act contrary to it. The better understood kinds of errors--errors due to not knowing a rule or not knowing how to apply a known rule--were excluded.

Shevarev, like many Soviet psychologists, draws upon Pavlov for theoretical foundations. Algebraic expressions are operated on by rules learned and used by the student. As these rules become practiced, their conscious recall and application are diminished and instead a connection is actualized. To Shevarev, algebraic skills in students reflect connections, whereas the learning of skills begins with learning a rule. As the rule is applied and practiced, a connection evolves.

The distinction and interplay between rules and connections is demonstrated throughout Shevarev's paper. Errors are the result of incorrect connections, and he provides evidence on the source of these incorrect connections: For example, students often treated $(x^2)^3$ like $x^2 \cdot x^3$. They actualized a connection by perceiving general characteristics of the expression but ignoring some specific ones. The source of the incorrect connection was shown to be the sequence and imbalance of the two types of problems in the textbook and the algebra classes. The students seldom encountered both types of problems in the same lesson or same section, and considerably more practice was provided for problems like $x^2 \cdot x^3$ than for problems like $(x^2)^3$.

Incorrect connections leading to errors were also analyzed for algebraic expressions in radicals and for rational expressions. Shevarev confirmed that the source of errors was in the textbook presentation and classroom practices that followed from the textbook. One important point was that incorrect connections were generated while students were correctly solving algebra exercises.

Suggestions for improving classroom practices included the appropriate alternation of kinds of problems and the inclusion of some operative suggestions for students. The first deals with the structure and sequence of textbook materials. Shevarev argues that problems with similar general characteristics should be studied together so that distinguishing specific characteristics can be emphasized. This leads naturally to Shevarev's notion of teaching operative rules. It is natural that a complex mathematical operation becomes routine and algorithmic with practice. Shevarev argues that teachers must guide this process to avoid the formation of incorrect connections.

Gonobolin's study of pupils' comprehension of geometric proofs was aimed explicitly at improving the secondary school geometry textbook. The textbook in use at the time (1954) was noted for its "maximum brevity of presentation," and the pupils had difficulty understanding the proofs. Gonobolin conducted a series of investigations in several schools to discover the reasons for the lack of comprehension and some ways to overcome the problem. Items Gonobolin investigated that were shown to be needed in instruction included a sort of completeness of links in a chain of reasoning, a schematic presentation of material in a proof, and a generalized understanding of geometric material. Suggestions for improving the textbook were made.

At first glance, this study appears to be primarily a complicated analysis of textbook material. It is, however, considerably more. The repeated trials of rewritten material with students led to the identification of some general instructional concepts or principles. The completeness of links in the chain of reasoning is a principle or guideline by which a textbook can be revised by its authors or adapted by teachers who use it. The schematic presentation of material in a proof seems parallel to Polya's [1] advice on devising a plan. It is an instructional variable to be used by teachers, and of course the idea is not restricted to the particular material in the textbook Gonobolin was using.

The paper by Shevchenko is a treatise on the rationale for and methods of instruction in the history of mathematics in Soviet secondary schools. It is a substantial review of the Soviet literature on the use of the historical perspective as a pedagogical and motivational device. One section of the report is a series of lesson plans on historical approaches to certain mathematical topics. The extensive bibliography is an interesting feature of the article.

The paper by Bochkovskaya presents an analysis of three necessary conditions for preventing first-, second, and third grade students' errors in solving simple arithmetic problems. The paper is addressed to teachers and is a guide to instruction rather than a research report. Bochkovskaya proposes three conditions for an instructional system. First, the problems should be ordered according to their content and structure and the time between the introduction of certain kinds of simpler problems should be reduced. Thus the first grade student can study operations by being introduced to unknown addend and unknown subtrahend problems (rather than having them postponed until the second and third grades). Division of objects into equal groups should prepare for division into equal parts. Second, problem work should develop the habit of selecting arithmetical operations on the basis of the entire condition of the problem, not on the analysis or synthesis of separate parts. Third, the teacher should develop the students' self-reliance in solving problems.

A report of a teaching experiment on the interweaving of number concepts into the study of functional relationships is given by Goldberg, who used a series of questions to develop the properties of the function y = kx in eighth grade classes. The questions embodied certain number concepts such as positive versus negative, whole numbers versus fractions, absolute values, and decrease versus increase. Goldberg found that the use of number problems in the development of functional relationship enhanced both the knowledge of functions and the knowledge of number properties. Goldberg does not describe the traditional study of functions against which his approach was compared.

A short article by Grudenov argues for the "extensive" use of unsolvable problems in mathematics teaching. The psychological foundation of his argument is taken from work by Shevarev done some time after that reported in the first article of this volume, but done as a direct follow-up. An unsolvable problem will have a specific condition opposite to that of other problems having the identical general conditions. Thus the unsolvable problem will point up for the student any incorrect associations he may have formed. Grudenov has used unsolvable problems as a pedagogical device in a geometry textbook and workbook.

The final article, by Marnyanskii, concerns the learning of a function concept within the algebra course. It is the report of a series of exploratory studies in grades 8-10. The article is a fairly comprehensive analysis of one approach to the concept of function.

This collection of articles reveals the universality of certain problems of mathematics instruction. Although the Soviet curriculum was quite traditional before 1967, the art of analyzing pedagogical problems was well advanced. The reader may find helpful techniques of investigation and analysis, as well as practical advice on instruction, in these pages.

## REFERENCES

1. Shevarev, P. A. "On the Question of the Nature of Algebraic Skills," Academic Bulletin of the State Institute of Psychology, Vol. 2, 1941.

2. Polya, G. How To Solve It. Second Edition, New York, Doubleday, 1957.

# AN EXPERIMENT IN THE PSYCHOLOGICAL ANALYSIS OF ALGEBRAIC ERRORS*

P. A. Shevarev

Pupils' errors in solving algebraic "problems" can be divided into three groups: Sometimes a pupil does not know the rule that must be applied or does not know it precisely. Sometimes he knows the rule well but still does not know how to apply it. Finally, there are cases where a pupil knows the rule, is able to apply it, but nevertheless acts contrary to it. Of these three cases, the last is the most important and the most interesting. We shall deal with it in our work. In the first two cases, the basic causes of error are known; hence, the practical questions are easily resolved. The third instance is different; we have no satisfactory explanation of the causes of error. In this connection, practical questions of how to prevent such errors, and what to do if they have already occurred, also remain unanswered.

One might assume that the source of errors of this kind lies in the pupil's failure to recall the rule according to which he should operate. But it is not difficult to see that this supposition does not fully explain the causes of error. In the first place, why the pupil does not recall the corresponding rule remains incomprehensible. Second, the question of why the pupil makes precisely this mistake, and no other, remains unanswered. Third, recalling a rule, in general, is not an essential condition for correctly solving a problem using this rule.

In approaching the explanation of errors of the type indicated, one must consider several facts discussed in another work of ours [1]. Let us cite these facts. When a pupil is solving an algebraic "problem"

*Published in Proceedings [Izvestiya] of the Academy of Pedagogical Sciences of the RSFSR, 1946, Vol. 3, pp. 135-180. Translated by Albert Leong.

of a type well known to him, he operates in strict accordance with definite rules. From a logical point of view, the process of problem solving is a chain of deductions. In almost all cases, however, the pupil solving these problems is not aware of any rules and, in fact, makes no deductions. How is it possible for a pupil unaware of rules nevertheless to act in full accordance with them? Without making any deductions, how is it possible for him to behave as if he had performed them?

Analysis of data shows that special and distinctive combinations of intellectual processes play a substantial role here. In the simplest case, a combination of this type consists of two components. The first is the recognition of the features of the algebraic expression (sometimes only its defined part), or of the features of an operation just performed. The second component is an orientation toward performing a definite kind of intellectual operation.

Suppose we are solving a problem on multiplying the monomials, $3a^3b^2 \cdot 5a^2b$, in which we do not recall the corresponding rules. In this case, the following occurs:

> (1) Recognition that the algebraic expressions to be multiplied are monomials entails the rise of an orientation toward the mental selection and specification of coefficients.
>
> (2) Recognition that these coefficients are expressed (in the given case) by simple numbers entails the rise of an orientation toward recalling the products of these numbers.
>
> (3) Recognition that the number that arose in the mind is the product of coefficients stimulates an orientation toward recording it on the right-hand side of the equation.
>
> (4) Recognition that the product of coefficients is written on the right side of the equation prompts an orientation toward selecting and specifying identical letters in the monomials multiplied, and so forth.

Simple combinations, similar to those just indicated, are frequently united into a single whole to form complex combinations. In our example, the second and third combinations are almost always fused together; recognition that the coefficients are simple numbers prompts an orientation toward both recalling and recording the product of these numbers. Sometimes recognition that the expressions multiplied are monomials immediately produces an orientation toward specifying both coefficients and letters.

To designate combinations of the type indicated, simple as well as complex, we suggested the term connection [konneksiya]. One should note that both correct and incorrect connections are possible.

When we solve an algebraic problem by actualizing correct connections, our operations correspond exactly to those rules governing the given case. In elementary algebra, correct connections usually arise from operations based on the recognition of rules. When a pupil solves an algebraic problem of a specific type for the first time, he assimilates the general features of the expression given him, recalls the corresponding rule, and performs a series of deductions by which this rule is applied to concrete data. There arises in him an orientation toward executing a definite intellectual operation. Recalling rules and performing deductions, however, disappear during the repeated solving of problems of the same type. Only the peripheral links of the process remain; recognizing the general features of the given expression now immediately entails an orientation toward performing a definite intellectual operation.

Every incorrect connection also corresponds to a definite rule. But the rule to which an incorrect connection corresponds is a false rule. Analyzing how incorrect connections arise is one of the main tasks of our research, since these connections underlie errors of the kind indicated above.

Essential and sufficient conditions for actualizing a definite connection are:

(a) perceiving the expression or performing operations corresponding to the first component of this connection;

(b) recognizing the task that one can execute by actualizing this connection, although recognizing this task is sometimes quite indefinite;

(c) an orientation toward performing this task by actualizing connections, that is, without recalling rules.

In selecting work methods, we proceeded from the following considerations. We assumed that a broad application of experimental methods is not permissible in studying algebraic errors. In the first stage of study, isolated, small scale experiments cannot produce valuable results. Similarly, as the experiment revealed, observations of the processes of classwork and of a pupil's work outside class give very sparse and incidental material. Therefore, we decided first of all to perform a psychological analysis of those errors found in assignments.

I. K. Novikov, director of Moscow secondary school No. 110, kindly let us use the assignments of pupils from two parallel eighth grade classes. Each pupil did his assignments in the same notebook, which was kept in school between assignments--a circumstance that assured completeness of material. In school, the pupils kept special "collective notebooks" ["krugovye tetradi"], in which all problems solved in class and at home were recorded in chronological order. The facts that one can establish by studying these "collective notebooks" helped substantially in determining the causes of errors in several cases.

We were able to discover the causes of only a few mistakes. To understand the causes of errors made in assignments, several special conditions are essential, although not always present. Hence, there is no doubt that many other causes of algebraic errors still exist besides those we uncovered. Let us make one more preliminary remark. In analyzing errors, one may have different aims in mind. The task may be only to explain errors, that is, to determine which facts and regularities already known underlie a given error. In this approach, mistakes that cannot be reduced to facts and regularities already known must be set aside. But one can have another aim. In considering the facts already known that may be the cause of error, one can seek indications of facts and regularities still unknown. Not every mistake, of course, contains such indications. But, in this approach, precisely those mistakes in which these indications can be revealed offer greatest interest. The second of these two possible approaches seemed more fruitful to us. We considered in advance, of course, that the conclusions to which this approach leads will often have only hypothetical significance. In what follows, we shall not stress the hypothetical aspect each time, but one must always keep it in mind.

## Part I

In one assignment, eight pupils (from two parallel classes) made an error of the following type:

$$\left(\frac{a^m b^{2n}}{c^3}\right)^{2n-1} = \frac{a^{m+2n-1} b^{2n+2n-1}}{c^{3+2n-1}}.$$

Instead of multiplying the exponent of each letter by the exponent of the fraction, they added the fraction's exponent to the exponents of the letters. In other words, they made a mistake of the type

$$(a^M)^N \longrightarrow a^{M+N},$$

where N and M are any numbers, letters, or algebraic expressions.

One might suppose that the pupils who made this error had forgotten the corresponding rule and had proceeded from a false rule. This conjecture seems all the more probable, since the assignment in which this mistake was made had been given to the pupils on September 16. During the two preceding weeks, the pupils did not solve a single problem on raising a power to a new power, and not one complex problem that involved the elevation of one power to another. Therefore, the pupils' operations were determined by the same knowledge and habits that they retained from the preceding school year.

This assumption, however, is untenable. In the same assignment, the pupils had to formulate, in writing, the rule for elevating a power to a new one, and all the pupils in question formulated it correctly. One should note that the problem in which the error was made occurs first in the assignment, and the task of formulating the rule occurs third.

The notion may arise that the erroneous operation performed by the pupils was accidental, that is, was not connected with the features of the expression $(a^M)^N$. But this notion also does not accord with facts. If the connection were missing and the operation were accidental, then we would have a great variety of mistakes in solving the problem indicated. But, in fact, in solving this problem, none of the pupils made any other error besides the addition of exponents.

Where is one to seek the cause of error? We shall note that the pupils who made the mistake did not recognize, of course, the rule for raising a power to a new one. If they were aware of it, they would have solved the problem correctly. At the same time, the operation performed was no doubt connected with certain features of the expression $(a^M)^N$. Thus, it follows that the cause of error was the actualization of an incorrect connection. Its second component, apparently, was an orientation for adding exponents. It may be asked, how did this connection arise, and what was its first component? The two questions are intimately connected.

To resolve these questions, let us first note that a connection pertaining to problems on multiplying degrees of the type $a^M \cdot a^N$ no doubt existed in the minds of all eighth grade pupils. As our earlier research showed [1], this connection already existed in the minds of all seventh grade pupils. The first component of this connection is the recognition of the definite features of the expression mentioned; the second is an orientation for adding exponents. In this connection, one naturally supposes that the pupils in question somehow "confused" the expression $(a^M)^N$ with $a^M \cdot a^N$ and, therefore, perceiving the first expression, performed an operation pertaining to the second.

This conjecture correctly points the way to an explanation for the error. Indeed, some "mixing" of the two instances underlies the error; one cannot understand it otherwise. But this supposition still does not give a full explanation; it does not indicate those processes that occurred in the "mixing" of the cases mentioned above. In particular, it does not answer the questions outlined above. One must make this process concrete.

By way of concretization, one can suggest the following: in solving the corresponding problems, at first two completely correct connections arose in the pupils. In one connection, the first component was recognizing essential features of the expression $(a^M)^N$; the second component was an orientation toward multiplying exponents. In the second connection, the first component was recognizing essential features of the expression $a^M \cdot a^N$; the second component, an orientation toward adding exponents. But after the holidays, as a result of forgetting, the first components of these connections lost their specific features and merged with one another. A connection was formed, the first component of which was recognizing several features characteristic of both expressions mentioned above (the presence of two exponents, for example). As a result, an orientation originally connected with features of the second expression arose in perceiving the first one.

This assumption is supported by the well-known fact that specific features fade first in the process of forgetting. Hence, a fusion of what was separate earlier may be the result. One must admit that forgetting, of course, played a role in the rise of the incorrect connection examined. Nevertheless, this assumption should be discarded as not corresponding to reality.

In fact, if the first components of the two connections had lost their specific features and were combined, then what should be the second component of the single connection which arose this way? Apparently, both orientations (for adding and multiplying exponents) have equal chances of becoming the second component. Therefore, together with errors of the type we analyzed (adding exponents in raising a power to a new one), we should have had approximately the same number of mistakes of the opposite type (multiplying exponents in multiplying powers of one letter). However, in the assignments we studied, there was not one mistake of the second type. And, in general, we did not encounter errors of this type. Hence, the incorrect connection in question was formed precisely while the pupil was solving such problems, and not when he was dealing with problems of both types.

Therefore, the assumption naturally arises that the incorrect connection arose when the pupil was solving problems on elevating a power to a new one. But this assumption is indivisibly linked with three others:

(1) this connection arose when the pupil was solving the <u>first</u> problems of this type (since connections, correct or incorrect, generally arise in solving the initial problems of a specific type); therefore, in solving these first problems, perceiving the expression $(a^M)^N$ entailed an orientation toward <u>adding</u> exponents;

(2) this incorrect orientation arose because the pupil proceeded from the recognition of a distorted rule (for obviously, at the moment in question, no other reasons could induce this orientation); distortion of the rule occurred because the pupil "confused" the elevation of a power to a new one with the multiplication of powers of one letter;

(3) later, all those processes that would have inevitably destroyed the incorrect connection that had arisen and, at the same time, would have induced the rise of a correct connection were missing.

Let us check these last three assumptions with the facts. First, as our observations showed, while a pupil was solving his initial problems on applying a simple rule, cases of distorted recognition of this rule very rarely occurred. Hence, the assumption that all eight pupils under consideration had proceeded from a distorted rule in solving their initial problems has little likelihood of being true. Second, at the school in question, work on algebra was so organized that only in very rare cases could an error made by a pupil remain unnoticed, unrecognized, and uncorrected. At the same time, one must consider that each pupil solved dozens of problems on elevating a power to a new one. The supposition that a significant percentage of these problems was solved incorrectly under such conditions is quite improbable. Therefore, even if the initial problems were solved incorrectly, in almost all cases, a large number of problems followed that were solved correctly. Hence, the incorrect connection that arose in solving the initial problems could be preserved only as an exception. Even if the erroneous connection at first arose in all eight pupils, the probability that it would be preserved in all of them is totally negligible.

Thus, one must conclude that the last three assumptions, taken together, are not in accord with the facts. But, as we already said, these three assumptions stem necessarily from the first basic supposition. Therefore, one must admit that the first assumption also does not correspond to reality; that is, the incorrect connection in question did not arise in solving problems on elevating a power to a new one.

Only one possible conjecture remains: the connection we are examining arose while the pupils were solving problems on multiplying powers of one letter. If so, what was the first component of this connection?

To answer this question, let us note the similarities and differences between the expressions $a^M \cdot a^N$ and $(a^M)^N$. Let us call the features common to both expressions the general features of these expressions. Such common features are the absence of plus and minus signs on the base line and the presence of two exponents. The features peculiar to each expression we shall call the specific features of the expression. The specific feature of the first expression is the presence of two identical letters (bases) on the line; the specific feature of the second expression is the presence of only one such letter.

The connection we are analyzing arose, as we said, while the pupils were solving problems of the type $a^M \cdot a^N$. But in the eight pupils who made the mistake, it was actualized in solving a problem of the type $(a^M)^N$. Consequently, only the general features of these two expressions entered into the first component of the connection. If the specific features of the first expression had also entered into it, then this connection would not have been actualized in perceiving the second expression, which does not have these specific features.

Thus, the first component of the connection that induced the error we analyzed was a recognition of the general features common to the expressions $a^M \cdot a^N$ and $(a^M)^N$. Its second component was an orientation for adding exponents. When this connection was actualized in perceiving expressions of the first type, a correct result was obtained. But when it was actualized in perceiving expressions of the second type, a mistake arose.

However, one must modify what was just said. In the same assignment, immediately after the problem in question, came the problem:

$$\left(-\frac{4}{5}y^5x^2 + 4x^6 - \frac{1}{2}y^3x\right)^2.$$

All the pupils who made the mistake we analyzed solved this problem correctly; they correctly squared the powers of the given letters (five times in succession). Therefore, in these pupils existed a correct connection, of which the first component was a recognition of definite features of the expression $(a^M)^2$, and the second component, an orientation for multiplying exponents. To understand this fact, one must note that squaring powers occupies a special place in an elementary algebra course. Pupils are acquainted with this operation much earlier than elevating a power to a new one. During the course, they solve a large number of problems that entail the squaring of monomials. In particular, one must execute many such operations in solving problems on squaring binomials and polynomials.

In all probability, even in cases of the type $(a^M)^3$, a correct connection was also actualized, since such cases also occupy a special place (similar to cases of squaring) in an algebra course. But we can say nothing definite about those cases where the exponent equals 4, 5, 6, and so forth. Possibly, a correct connection was actualized even in these cases. But it is possible the pupils have already entered the sphere of an incorrect connection.

Concerning this, two additional assumptions pertaining to the first component of the incorrect connection examined are possible. Perhaps the recognition that N is neither 2 nor 3 also entered into the first component, together with the perception of general features characteristic of the expressions $a^M \cdot a^N$ and $(a^M)^N$. According to the second possible supposition, the recognition that N is a letter or an expression in letters (and not in numbers) also entered into the first component, along with the recognition of the general features indicated.

Soon we shall need these two assumptions, but temporarily, for simplicity of exposition, we shall consider only the general features of the expressions mentioned above. We have established that the incorrect connection arose while the pupils were dealing with problems on multiplying powers of one letter. But we still do not know how and why this erroneous connection arose. Apparently, it arose only when pupils, dealing with expressions of the type $a^M \cdot a^N$, recognized only general (in the sense indicated above) features. They nevertheless performed the correct operation, that is, added exponents. And this means that, at some stage of mastering multiplication of powers, conditions existed whereby: (a) situations requiring the pupil to recognize not only general, but also specific features of expressions of the type

$a^M \cdot a^N$, were missing; (b) whenever the pupil recognized only general features of this expression, he nevertheless solved the problem correctly, that is, added exponents.

To understand how these conditions arose, let us examine that period of time when the pupil was already acquainted with multiplying powers, but still knew nothing about dividing powers, or raising one power to another. During this period, in solving problems on multiplying powers, the question of what operation one must (or can) perform is very often decided even before the perception of data. The pupil has just familiarized himself with the rule for multiplying powers and has been solving the corresponding problems in class. He knows, therefore, that the homework given him will consist of problems on multiplying powers. In other words, he knows in advance, even before perceiving data, that he must add exponents. At the same time, the section of the textbook containing these problems is entitled "Multiplication of Monomials." But the pupil knows the rule for multiplying monomials; the rule states that one must add exponents of identical letters. Therefore, he again knows, before perceiving facts, what operation with exponents he should perform. Finally, a pupil must usually solve in succession several problems immediately following one another in the textbook. The pupil knows that problems in the textbook are arranged by section and by type. Therefore, after solving the first problem on multiplying powers, he knows he will have to add exponents in the next problem.

Thus, in all these situations, there is no need to recognize those features of an expression that indicate one must add, and not subtract or multiply, exponents. And these features, obviously, are the specific features of the expression--the presence of two bases and the presence of a dot between them (or the absence of any sign). In the process of solving the problem, of course, motives arise that prompt the pupil to recognize certain features of the data. But these features are either concrete features (for example, the fact that the base is expressed by a letter and the exponent by a number), which do not interest us now, or general features (the absence of plus or minus signs on the line or the presence of two exponents).

Thus, in fact, we see that a stage of mastering the multiplication of powers of one base exists in which: (a) certain impelling motives, prompting the pupil to recognize specific features of an expression, are missing, and (b) if the pupil recognizes only general features of this expression, he will nevertheless perform the correct operation. If so, during this period cases occur, no doubt, where this or that pupil solves a problem on multiplying exponents without recognizing the specific features of the expression indicated. At the same time, if he occasionally solves a problem without recognizing its specific features, the chance that he will solve it the same way another time is undoubtedly increased. As a result, the same incorrect connection discussed above will arise. Its first component will be recognition of only general features of the expression $a^M \cdot a^N$; and its second component, an orientation toward adding exponents.

While this explains how the incorrect connection arose during the solution of problems on multiplying powers, it still does not explain why the connection was preserved when the pupils, having mastered the multiplication of powers, after a while moved to the elevation of one power to another. One must think that the correct connection arose in the pupils while they were solving problems on raising one power to another, the first component being the recognition of all essential features of the expression $(a^M)^N$, and the second component, an orientation toward multiplying exponents. Why was this correct connection not actualized while the assignment was being performed? On the other hand, it would also seem that, while the pupils were assimilating the elevation of one power to another, the incorrect connection in question should have been reconstructed and converted into a correct one. For at this stage of instruction, the pupils were dealing not only with expressions of the type $a^M \cdot a^N$, but also with the type $(a^M)^N$. Therefore, to perform the correct operation in the first case, the pupil, it would seem, should have noticed not only the general, but also the specific features of the first expression. Why was the incorrect connection not reconstructed?

To find an answer to these questions, we counted the number of problems in the textbook (by Shaposhnikov and Val'tsev) on multiplying powers and on raising one power to another. We considered only <u>elementary</u> problems. For example, we counted three problems on multiplying powers in the case of $a^2b^3c^m \cdot a^nb^4c^2$. We tabulated not only problems found in the special sections "Multiplication of Monomials" and "Elevation to a Power," but also all the cases where the elementary problems on multiplying powers and on raising one power to another enter into more complex problems. Of course, the number of elementary problems entering into a complex one sometimes depends on the method of solving the complex problem and on what simplifications are performed after completing a given operation. We counted from that method of solution and those transformations containing the smallest number of elementary problems of interest to us. At the same time, we also recognized that cases where any power is squared or cubed no doubt form a special group. Therefore, in the first variant of our count, we excluded all cases in which a pupil was confronted with the task of squaring or cubing a power. As we indicated above, it is quite possible, however, that not only do these problems form a separate group, but in general all cases in which the exponent of the power to which one must raise a given power is expressed by a number form a separate group. Therefore, in the second variant of our count, we considered only those problems on elevating one power to another in which the exponent of the new power is expressed by a letter or by an algebraic expression. Of the number of problems on multiplying powers in this variant, we considered only those in which one or more exponents were expressed by a letter or an algebraic expression.

In the first variant, we obtained the following results: the first four chapters of the text contain no less than 620 problems on multiplying powers; in the fifth chapter, there are 58 problems on raising one power to another, and <u>not one</u> problem on multiplying powers; in subsequent chapters, there is <u>not a single</u> problem on elevating one power to another, but no less than 360 problems on multiplying powers. In the second variant of our tabulation, the results were: in the first four chapters, no fewer than 160 problems on multiplying powers; in the fifth chapter, 39 problems on raising one power to another; in subsequent chapters, not one problem on either multiplying powers or elevating one power to another.

The order in which the pupils solved the problems did not fully agree with the sequence in which these problems were arranged in the textbook. But, as we were able to establish, these deviations could be disregarded. Moreover, the pupils of course did not solve all the problems found in the textbook. At the same time, they solved several problems not in the book. These two circumstances could not, however, substantially change the relationship between the numbers of problems of various types that our calculations revealed. Therefore, the numbers mentioned above generally give a true picture of both the number of problems of different types solved by the pupils, and the order in which these problems were solved.

Consequently, the description of the incorrect connection we have given could remain totally unchanged by these data. It could have been modified and corrected if, at a definite stage of work, the pupils had alternately solved problems on multiplying powers and on raising one power to another. Under these conditions, the pupils would have been compelled to recognize not only general, but also specific features of expressions. But, as the data show, alteration of the two types of problems was lacking. At first the pupils solved problems on multiplying powers without solving problems on raising one power to another. The incorrect connection in question arose here. Later, without at the same time solving problems on multiplying powers, they solved problems on elevating one power to another. Possibly, at this stage of work, certain tendencies to reconstruct the incorrect connection in fact arose. But these tendencies could not appear or be reinforced, since the pupils were not dealing with problems on multiplying powers. Finally, after a long period of time, the pupils again solved problems on multiplying powers, without dealing with problems on elevating one power to another. At this stage, there again were no objective conditions that could have precluded the possibility of solving these problems by actualizing an incorrect connection. Therefore, the incorrect connection could remain unchanged even here.

One must note that, during this third period, the pupils dealt only with problems on multiplying powers whose exponents were numbers. But, as we already said, perhaps such cases form a special group and are not related to the incorrect connection in question. If so, the incorrect connection that arose earlier not only could, but also should remain unchanged in this third period.

Of course, after familiarization with problems on raising one power to another, it is quite possible that the incorrect connection formed during the multiplication of powers was reconstructed in several pupils. This could be caused by a conscious juxtaposition of $a^M \cdot a^N$ and $(a^M)^N$, and by the clarification of not only external, but also semantic distinctions between them. But the conditions we described for solving problems were such that they did not demand this reconstruction nor force its execution. Therefore, towards the end of the school year, the incorrect connection remained effective in several pupils. It apparently existed in those pupils who made the mistake we analyzed.

Our tabulation also answers another question: Why was the correct connection, which arose in solving problems on raising one power to another, not actualized in carrying out the assignment? As we have seen, the number of problems the pupils solved on elevating one power to another was several times less than the number they solved on multiplying powers. It is quite natural to suppose that the correct connection was significantly weaker than the incorrect one under these conditions. It is also possible that, during the holidays, the weaker correct connection weakened to a greater extent than the significantly stronger incorrect connection. Therefore, when the pupils in question, returning to school after the holidays, encountered an expression of the type $(a^M)^N$, the incorrect, and not the correct, connection was actualized in them.

To understand fully the causes of the error tolerated by the pupils, one should examine two essential questions: (1) Why didn't the pupils, knowing the rule governing cases of $(a^M)^N$ well, reproduce this rule in solving the problem? (2) Why, having recalled it immediately after solving the problem (when they had to formulate this rule in the same assignment), didn't they suddenly catch themselves, return to the problem, and correct the mistake?

We think that only one answer can be given to the first question. One must assume that the pupils' perception of the expression $(a^M)^N$ contained a feature which precluded the recall of the rule. The following considerations enable us to make this general supposition concrete. As we already clarified in our earlier work [1], a rule is recalled only when a problem is unfamiliar to us and, if only for an instant, the question arises: "What must be done?" In the absence of this question and the presence of a moment expressed by the words, "I know what must be done," the problem is solved by actualizing connections. In other words, recognizing that the task is familiar produces an orientation toward solving this problem by actualizing connections. Thus, we obtain an answer to the first question posed above. In perceiving the expression $(a^M)^N$, our pupils had an impression of the familiarity of this problem, an awareness that they knew how to solve it. These features of the perception prompted an orientation to perform the task by actualizing connections. Hence, the corresponding rule did not become conscious. The task in fact was executed by actualizing a connection.

Let us proceed to the second question: Why didn't the pupils, when they had to formulate the rule for raising one power to another in the assignment, later correct the mistake they had just made? One must seek the answer to this question in the content of the first component of the connection actualized in solving the problem. As we have established, its first component was the recognition of only general features of the expression $(a^M)^N$. The specific features of this expression, by which it becomes a power elevated to a new power, remained unrecognized. This means that, having written $(a^M)^N = a^{M+N}$, the pupil did not recognize that he was dealing with a problem on raising one power to another. Therefore, it is fully understandable why he did not recall the problem just solved after formulating the rule for elevating one power to another soon afterwards. He had no reasons for recalling it. In the pupil's consciousness, the problem in question was not related to the rule for raising one power to another, and vice versa.

To conclude our analysis of the error, we must investigate one more question that inevitably arises at almost every stage of analysis.

As we have seen, the causes for the rise and preservation of the incorrect connection were: (1) the conditions under which problems on multiplying powers of one letter were solved; (2) the small number of problems a pupil solved on elevating one power to another; and (3) the order in which problems of both types were solved. But all these conditions were the same for all the pupils. Why did the incorrect connection arise and persist in only eight pupils, and not in all pupils of both classes?

To answer this question, one must note one important circumstance that we have not considered yet. The recognition of specific features of an expression arises not only when the objective situation demands it, but also, sometimes, when we have a general orientation, habit, or custom of solving algebraic problems consciously. This orientation may lead us to carefully perceive all essential features of the "data," subsume these "data" under a corresponding concept, and only afterwards strive to perform specific operations. This orientation already begins to be formulated in the very first stages of studying algebra, but it is still quite unstable at first. It of course needs reinforcement. Moreover, the stability of this orientation is more significant in some pupils; in others, it has a tendency to fade away if it remains unsupported in practice. Naturally, pupils in whom it is more stable, in solving problems on multiplying powers, will clearly recognize all essential features of the data even when the situation does not demand this awareness of them. Hence, the correct connection arises in them. Other pupils, in whom the stability of an orientation toward conscious solving is insufficiently high, will perform the multiplication of powers without this awareness under conditions where compelling motives for recognizing all essential features of the expression $a^M \cdot a^N$ are missing. The incorrect connection in question will arise in them.

In the assignments we studied, one can find another series of errors whose causes are analogous to those of the mistake just analyzed. There is no need to examine all these mistakes. We shall describe only two of them.

In one assignment, the pupils had to reduce the root

$$\sqrt[3]{-x^7}$$

to its simplest form. They had rarely dealt with expressions of this type earlier. In the problems the pupils had solved prior to this assignment, there were 430 cases of extracting roots of monomials with a "plus" sign, and only 11 instances with a "minus" sign. In the month immediately preceding the assignment, they solved 295 problems on extracting roots of monomials with a "plus" sign, and not a single problem on extracting roots from monomials with a "minus" sign.

Nevertheless, almost all pupils solved the problem perfectly. Perhaps these pupils had recalled the rule of signs for extracting roots and acted according to this rule. It is possible, however, that the corresponding correct connection already existed in them. But two pupils made mistakes. They wrote

$$\sqrt[3]{-x^7} = \sqrt[3]{\frac{1}{x^7}} = \frac{1}{x^2}\sqrt[3]{\frac{1}{x}} .$$

In other words, they made an error of the type

$$-x^n = \frac{1}{x^n} .$$

Apparently, they did not have a correct connection corresponding to this type of problem. There is nothing surprising about this. We saw that the number of such problems the pupils solved was generally quite small, and that in the following month they did not solve such problems at all. There is also no doubt that they did not recall the rule that would have shown them how to operate.

In such situations, the question usually occurs to us: Is it possible, with the aid of a transformation we know, to reduce the given expression to a type familiar to us? This question, apparently, also occurred to the pupils. It was resolved as follows. The pupils were able to extract a root when a "plus" sign stood before a monomial, and they knew they could do this. But they didn't know how to operate when a "minus" sign stood before a monomial. Hence, the thought occurred to them: transform the expression under the radical sign so that a "plus" sign would appear before it. They decided to "remove" the minus.

This thought is not incorrect in itself. Proceeding from it, one could have performed the following transformations:

$$\sqrt[3]{-x^7} = \sqrt[3]{(-x^2)^3 \cdot x} = -x^2 \sqrt[3]{x} .$$

But this transformation did not, and could not occur to our pupils. They, of course, were seeking something familiar and well known; and this transformation, despite its simplicity, was not familiar to them (in the first part of the textbook by Shaposhnikov and Val'tsev, we did not find a single problem in which one had to perform this transformation to solve it). Hence the error was inevitable.

Why, however, did the pupils perform precisely the incorrect transformation indicated above and no other? The assumption that they had proceeded from the recognition of an incorrect "rule" no longer arises. If the pupils had recognized that astonishing "rule," according to which they in fact had operated, then they of course would have immediately noticed its absurdity. Therefore, the supposition remains that the error was prompted by the actualization of a connection.

Considerations analogous to those we mentioned in analyzing the first error show that this incorrect connection arose while performing transformations of the type

$$x^{-n} = \frac{1}{x^n} .$$

The first component of this connection was the recognition of the general features characteristic of the expressions $x^{-n}$ and $-x^n$; its second component is an orientation toward writing fractions in which the numerator is a unit and the denominator is a given expression without a minus.

It is not difficult to establish the causes for the rise of this incorrect connection. When the pupils were transforming the expression $x^{-n}$, they always knew in advance, even before perceiving data, what operations they had to perform. Therefore, compelling motives that would have demanded recognition of the expression's specific features were missing. At the same time, if a pupil recognized only general features, he nevertheless would solve the problem perfectly, since, he did not have to solve any problems having the same general features and different specific features, in which he had to perform other operations. Therefore, when problems on transforming powers with negative exponents (found in a special section of the text) were being solved, the incorrect connection was bound to arise in several pupils. Later the pupils did not deal with this kind of problem at all. Hence, those conditions which might have led to the correction of the incorrect connection were completely missing.

The cause of the error consists, however, not only in the fact that an incorrect connection existed in the pupils. If the pupils had understood the expression and the problem they were given (to remove the minus), they of course would not have erred. They would have either hit upon the correct transformations indicated above, or left the problem unsolved. The incorrect connection existing in them would have remained unactualized. Hence, the cause of error also lies in the fact that an orientation toward solving problems by actualizing a connection existed in these pupils. In the combined operation of both conditions, error was unavoidable.

Let us examine one more error of the same type, with several distinctive features. Five pupils, dealing with an algebraic fraction of the type

$$\frac{a^8 b^{12}}{a^6 b^{10}}$$

"reduced" it and obtained the fraction

$$\frac{a^4 b^6}{a^3 b^5}.$$

In other words, the exponents in the numerator and denominator were divided by their common factor, and not subtracted from one another. The pupils acted as if the numbers 8, 12, 6, and 10 were not exponents of powers, but factors of the fraction's numerator and denominator.

In the same assignment, they were to reduce the fraction

$$\frac{b^{y-1}}{b^{y-2c}},$$

or (in another variant of the assignment) the fraction

$$\frac{a^{y-a}}{a^{y-3a}}.$$

The pupils who made the mistake indicated above solved this problem correctly. Therefore, it is impossible to explain the error the five pupils made by claiming that they did not know the rule for dividing powers and had proceeded from a false rule. Thus, the immediate cause of error here was also the actualization of an incorrect connection.

Its first component was recognizing that two numbers with a common factor stand in the fraction's numerator and denominator; the position of these numbers did not enter into the content of the connection's first component. Therefore, this connection was actualized whenever the numbers were factors of the numerator and denominator. The second component of the connection was an orientation toward dividing these numbers by their common factor.

There is every basis to suppose that this connection arose while the pupils were still practicing the reduction of <u>arithmetic</u> fractions. In studying arithmetic, they of course did not deal with exponents and probably had never even seen expressions containing them. At that time, therefore, awareness that both members of the fraction were products and that one number stood in the numerator and the other in the denominator was sufficient basis for reducing the numbers entering into the fraction's structure. The fact that these numbers were <u>factors</u> of the fraction's members could remain unperceived. The situation for solving the problem contained no motives which would have demanded awareness of this fact. At the same time, if the pupil did not perceive this fact, he still could perform the reduction perfectly. Naturally, the incorrect connection indicated could be formed in him under these conditions.

But, in studying algebra, the pupils of course solved a great many problems on reducing, multiplying, and dividing algebraic fractions.

Reducing powers of identical letters in the fraction's numerator and denominator enters into almost every problem. In these problems, the pupils also had to reduce the coefficients of monomials forming the members of the fraction, that is, to execute an operation which they had performed in arithmetic. It would seem that, in the first place, the incorrect connection which existed in the pupils should have been reconstructed under these conditions; the specific features of the factors (coefficients) should have entered into its first component. But this connection could no longer be actualized in reducing powers of letters. Second, the correct connection, which pertains precisely to reducing powers of letters, no doubt should have arisen. Why was the incorrect connection not reconstructed, and why did it seem stronger than the correct one?

To answer this question, let us note that the incorrect connection under discussion can be reconstructed only in the presence of one essential condition. Such a condition is an alternation of cases of the type

$$\frac{6a}{3b} \quad \text{or} \quad \frac{a^6}{a^3}.$$

Of course, it is essential that each case be encountered often enough. Under this condition, the following processes will occur: (a) the pupil will notice that there are numbers with a common factor in the numerator and denominator; (b) he will note the position of these numbers: if they are factors of the fraction's member, the pupil will divide them by their common factor; if they are exponents, he will subtract the smaller from the larger. As a result, the incorrect connection will be reconstructed and will be actualized only in cases of the first type. At the same time, a correct connection pertaining to cases of the second type will arise.

But in all the problems contained in the first part of the text by Shaposhnikov and Val'tsev, there are only eight cases of the type

$$\frac{a^6}{a^3},$$

that is, cases in which exponents of reduced powers contain a common factor. At the same time, the number of cases of the first type is rather large. Obviously, the incorrect connection could not be reconstructed under these conditions. On the contrary, conditions favored its reinforcement.

But there is no doubt that the pupils very often correctly reduced exponents of a single letter contained in the fraction's numerator and denominator. In the first place, they were dealing with cases of the type

$$\frac{a^5}{a^3},$$

that is, with cases in which the exponents do not contain a common factor. Second, they were dealing with cases in which exponents were expressed by letters. As a result, the correct connection for reducing exponents arose in almost all pupils. Its first component was recognition of the following facts: (a) the fraction's numerator and denominator are monomials; (b) each monomial contains the same letter, which denotes the bases of powers which are factors of monomials; (c) the exponents are identical (the first variant) or dissimilar (the second variant). If the exponents of powers are identical, the second component of the connection is an orientation toward cancelling letters and their exponents (in the fraction's numerator and denominator). If the exponents are not identical, the second component of the connection is an orientation toward subtracting the smaller exponent from the larger.

This connection can be actualized in exponents of any magnitude, whenever exponents contain a common factor. But it cannot destroy the incorrect connection in question, since the first component of both connections differ sharply from one another. In particular, recognizing that the numbers standing in the numerator and denominator contain a common factor enters into the first component of the incorrect connection, but not into the first component of the correct connection. Therefore, both connections can coexist simultaneously.

This means that both true and false connections can be actualized in perceiving an expression of the type

$$\frac{a^8}{a^6}.$$

If, in perceiving this expression, the pupil notices that the numerals 8 and 6 are exponents or powers, then a correct connection will be actualized in him. If the fact that the numbers 8 and 6 contain a common factor strikes him first, then--in all probability--an incorrect connection will be actualized. Apparently, it was the second case which was realized in the consciousness of the pupils who made the mistake we analyzed.

In summary, we assume that the meaning of the results we obtained is by no means limited to the narrow circle of errors examined. Therefore, we shall try to list the general characteristics of what we discovered.

(1) Under certain conditions, cases can occur in which a pupil solving an algebraic problem does not notice several essential features of the data, but nevertheless arrives at perfectly correct results. In all probability, such cases are also possible in performing many other kinds of intellectual tasks (in arithmetic, spelling, and so forth).

(2) These cases are likely when the pupil knows in advance, even before an attentive examination of the data, what operation he must perform. Under the conditions of a typical school course in algebra (and arithmetic), pupils very often have such prior knowledge of forthcoming operations.

(3) When a pupil solves problems of a definite type in the manner just described, an incorrect connection arises in him. Its first component is recognizing only general features of the data with which he is dealing. The specific features of these facts do not enter into the content of the connection's first component. The second component of the incorrect connection is an orientation for performing an operation corresponding to a given type of problem. Thus, an incorrect connection of this type is distinguished from a correct one only by the fact that its first component is poorer in content than that of a correct connection. At the same time, the scope of the incorrect connection's first component is broader than it should be.

(4) As long as a pupil is dealing only with problems of the type in which the incorrect connection arose, he makes no mistakes. Therefore, the incorrect connection is strengthened more and more.

(5) Sooner or later the pupil encounters problems of another type. The data of these problems have the same general features as the first type, but different specific features. Therefore, to solve these problems, one must perform a different operation. However, as long as the data of problems of the second type have the general features indicated, the incorrect connection which arose earlier is actualized under certain conditions, even when problems of the second type are being solved. Error is the result of the actualization of the incorrect connection.

(6) Of course, even if a pupil has an incorrect connection, he does not always make a mistake under the conditions described above. Quite possibly, in dealing with problems of the second type, he will notice the specific features of the data, recall the appropriate rule, and operate according to it. In solving the initial problems of a new type, he usually proceeds in just this manner.

(7) The condition for actualizing any connection, including an incorrect one, is a special orientation toward performing tasks by actualizing certain connections or, in other words, solving problems without recalling rules. This orientation arises whenever the problem and its data are perceived as something familiar. Since data of the new (second) type of problem have the same general features as that of the first type, an impression of familiarity can arise in perceiving these facts. This will prompt an orientation toward actualizing connections.

(8) When a pupil solves problems of the new (second) type by recalling rules, a correct connection is produced in him; the incorrect one usually atrophies. However, a correct connection is strengthened once and for all only if the pupil has correctly solved a sufficiently large number of problems of this new type. If the number of problems of this type solved is significantly less than that of the first type the pupil solved earlier, an incorrect connection can be actualized in the orientation toward actualizing connections.

(9) Formation of a correct connection sometimes does not destroy the incorrect one. Such cases occur under the following condition: (1) data of both types of problems correspond to the first component

of a correct connection; but data of only one type (we shall call it critical), to the first component of an incorrect connection; (2) the pupil almost never solves problems of this critical type, but solves many problems of the other type; (3) in solving the second type of problem, the features distinguishing them from the critical type have no significance and therefore go unnoticed. As a result, a correct connection embracing both types of problems arises but, at the same time, the incorrect connection remains. Therefore, when a pupil encounters data of the critical type, both correct and incorrect connections can be actualized in him, depending on what features of the data "catch his attention."

## Part II

The characteristic feature of the incorrect connections examined above is that their first component is less full in content and broader in scope than it should be. Also possible, however, are contrary instances in which the connection's first component contains something which should not have entered into it and which narrows its scope. Perhaps it is impossible to call this connection incorrect. Whenever it is actualized, it leads to correct results. But it is not actualized in a series of instances in which it should have been actualized. In this case, problem solving can proceed incorrectly. Let us examine several facts which confirm these suppositions.

In one assignment, the pupils were to simplify the following fractions:

$$\frac{a^9y^5}{ay^{10}}\,;\quad \frac{x^{a+1}a}{xa^{1-y}}\,;\quad \frac{a^4(x-y)^5}{a(x-y)^{15}}$$

Four pupils correctly solved the first two problems, but not the third. Two of these pupils make no attempt to solve the problem; in their notebooks, a blank space follows the expression given them. Two other pupils tried to perform the task, but these efforts led to gross errors.

One of them wrote:

$$\frac{a^4(x-y)^5}{a(x-y)^{15}} = \frac{a^4x^5 - a^4y^5}{ax^{15} - ay^{15}}.$$

At first the other wrote:

$$\frac{a^4(x-y)^5}{a(x-y)^{15}} = \frac{a^4(x^5 - y^5)}{a(x^{15} - y^{15})}.$$

Then he erased what he wrote and went on to the next problem. Thus, none of these four pupils noticed that the third problem does not differ essentially from the two preceding ones, that here too one must reduce exponents of identical bases.

According to our observations, after studying the section of the course entitled "Reduction of Fractions," all pupils reduce fractions of the type

$$\frac{a^9y^5}{ay^{10}}$$

(we have in mind the first and second of the three problems mentioned above) by actualizing the corresponding connection. Therefore, there is every basis to suppose that these four pupils in question also performed this task by actualizing a connection. But apparently this connection was not actualized when they were dealing with the expression

$$\frac{a^4(x-y)^5}{a(x-y)^{15}}.$$

This means that these pupils subsumed under the connection's first component cases of the type

$$\frac{a^n}{a^m}, \text{ but not } \frac{(A)^n}{(A)^m},$$

where A is any polynomial. Consequently, the feature that the power's base is a <u>letter</u>, and not a polynomial enclosed in parentheses, entered into the first component of the connection. This attribute, of course, does <u>not</u> enter into a correct connection.

How did this narrow connection arise? To answer this question, we tabulated the number of elementary problems on reducing fractions contained in the fourth chapter of the text by Shaposhnikov and Val'tsev. We considered elementary problems on reduction; that is, we counted

$$\frac{a^6b^2c^3}{a^3b^2c}$$

as containing three such problems. Later, we eliminated from our count all instances in which both reduced exponents equalled one. We assumed that these cases were unique, and that their reduction proceeded in a special way. However, even if we had considered these cases, the relationship between the numbers found would have remained the same.

The tabulation showed that in the section of the text entitled "Reduction of Fractions," there were 28 elementary problems of the type

$$\frac{a^n}{a^m},$$

and only 11 of the type

$$\frac{(A)^n}{(A)^m}.$$

Subsequent sections of this chapter contain 81 elementary problems of the first type, and only 33 of the second type. Thus, the number of problems of the first type is approximately 2 1/2 times larger than that of the second type. There are almost no elementary problems on reduction in the remaining chapters of part one of the text. The relationship between the numbers of problems of both types the pupils actually solved, in all probability, did not differ essentially from that obtained by counting the problems in the text.

One must think that, while the pupils were solving problems from Chapter IV of the text, there arose in them correct connection encompassing all cases of reducing algebraic fractions. The total number of problems of the type

$$\frac{(A)^n}{(A)^m}$$

which the pupils solved was quite sufficient for these problems to be included in the connection. But, as we already said, in subsequent chapters of the first part of the text, the pupils rarely encountered elementary problems on reduction. In all these chapters, we could find only about 10 such elementary problems. Furthermore, in all these cases, one could solve a complex problem without reducing fractions. Thus, during a rather lengthy interval of time occupied with work on simple equations, the pupils did not deal at all with reducing algebraic fractions. Hence, the connection in question (still not very strong, of course) could suffer somewhat from forgetting. In the process of forgetting, the connection may undergo various changes. One of these changes is the shedding of several aspects of the data encompassed by the connection's first component. Moreover, it is natural that those aspects of the data with which the pupil dealt less should be shed first. In the case analyzed, the pupils dealt significantly less--as we saw--with data of the type

$$\frac{(A)^n}{(A)^m},$$

than with data of the type

$$\frac{a^n}{a^m}.$$

Before going further, let us examine one more similar case. In one assignment, the pupils were to solve the problem of simplifying

$$(x-y^2)^5\ (x-y^2)(x-y^2)^4.$$

In solving this problem, nine pupils made one mistake or another. In most cases, it took the following form:

$$(x-y^2)^5\ (x-y^2)(x-y^2)^4 = (x^5-y^{10})(x-y^2)(x^4-y^8).$$

Having performed this "transformation," the pupils then multiplied the binomials on the right side of the "equation."

There is no doubt that the pupils making this mistake totally failed to understand, even partially or in a distorted form, the rule for multiplying powers. At the same time, there is every reason to suppose that these pupils did not have a connection corresponding to this rule. For if a connection exists, it takes part in the operation whenever (a) a given expression possesses all the features which enter into its first component; (b) a task which can be performed by actualizing this connection is experienced one way or another; and (c) an orientation exists toward actualizing connections. All these conditions existed in our case. The algebraic expression given the pupils is totally subsumed under the first component of a connection adequate to the rule. The pupils were given the task of "multiplying," and the operations of most pupils show that they had this task in mind. Finally, from the character of the mistake the nine pupils made, it is clear that they had a general orientation toward actualizing certain connections. Nevertheless, despite the presence of all these conditions, the connection was not actualized. Apparently, the pupils did not have a connection which would have subsumed the expression given them under the first component.

One must compare this case with another. In the same assignment, immediately before the problem just discussed, the pupils were to solve the following problem on multiplying powers:

$$c^{x + 8} \cdot c^{6 - 2x} .$$

All the pupils solved this problem correctly. Moreover, there is no doubt they solved it by actualizing the corresponding connection, for all sixth-grade pupils already solve problems on multiplying powers of one letter without knowing the rule, that is, by actualizing a connection.[1]

Comparing these two cases, it is not difficult to see that we are dealing here with essentially the same case we encountered above. It is clear that the pupils possessed a connection which was actualized in the problem $a^n \cdot a^m$, but not in $(A)^n \cdot (A)^m$, the letter A being any polynomial. As in the preceding case, the feature that bases of powers are expressed by one letter evidently entered into the first component of the connection existing in the pupils. This feature, of course, does not enter into a connection adequate to the rule.

To clarify the reasons for the rise of this narrow connection, we again counted in the text the number of elementary problems of the types that interest us. We considered the problems contained in Chapters II,

[1]See pp. 28-29.

III, and IV of part one of the text. Whenever different methods of solving a problem were possible, we considered the most economical means. In most cases, it was the method the authors of the text no doubt had in mind.

First, we counted all problems of the types $a^n \cdot a^m$ and $(A)^n \cdot (A)^m$ in the first variant of the count. It turned out that the chapters indicated contain 604 elementary problems of the first type, and 16 elementary problems of the second type.

Quite possibly, however, cases in which the first powers of any quantity are multiplied form a special type in which special connections operate. In the problem we examined, the factors' exponents did not equal one. Therefore, in the second variant of the count, we considered only those cases in which the exponent of one or more factors does not equal one (that is, "there is" an exponent). The chapters indicated contain 332 problems of the first type, and 7 problems of the second type. Furthermore, all 7 problems occur only in sections 4 and 5 of Chapter II and are not encountered later.

Apparently, under these conditions there could have arisen a connection for multiplying powers in which expressions of the type

$$\frac{a^n}{a^m}\text{, but not }\frac{(A)^n}{(A)^m}\text{,}$$

are subsumed under its first member.

Let us now compare the narrow connections which we have exposed. It is not difficult to see that they are similar. Here and there, under its first component are subsumed cases in which bases of powers are expressed by a single letter, but not cases in which polynomials enclosed in parentheses appear as the base. The conditions for their rise are also similar: in both cases, the pupils were solving many problems of one type, and few of the other. Hence, the assumption naturally arises that a certain internal bond exists between these two connections, that a single condition underlies their rise.

To make this assumption concrete, to decide whether or not it is correct, one must examine a question with broader significance. Expressions of the types

$$\frac{a^n}{a^m}\text{ and }\frac{(A)^n}{(A)^m}$$

are both subsumed under the rule for reducing fractions. Similarly, expressions of the types $a^n \cdot a^m$ and $(A)^n \cdot (A)^m$ are subsumed under the rule for multiplying powers.

By the letter a, we mean any individual letter; by A, any polynomial. One can say that under each rule are subsumed two types of problems, which differ in that individual letters enter into one expression, whereas polynomials enclosed in parentheses enter into the other. This feature also distinguishes many other algebraic rules; problems of both types are subsumed under each rule. The question is: if a pupil solves only problems of the first type, can a correct connection embracing both types arise?

To answer this question, let us see how new connections are formed in a person who knows elementary algebra well, for example, a mathematics student. As an example, let us take a case in which a mathematics student is solving for the first time problems on computing determinants of the kind

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

where a, b, c, and d are any individual letters. In these cases, a connection arises in him, the first component of which is recognizing essential features on the left side of the equation, and the second, component an orientation toward performing operations whose result is recorded on the right side of the equation. Under the first component of the connection, however, are subsumed not only those cases in which separate letters stand in a definite order between the vertical lines. If we give the same student a determinant of the kind

$$\begin{vmatrix} (a + b) & (c + d) \\ (e + f) & (g + h) \end{vmatrix},$$

he does not recall the rule, and writes

$$(a + b)(g + h) - (c + d)(e + f).$$

Thus, although he solved only problems of the first type, he made a connection encompassing the second type and, in general, all problems on computing determinants of the second order.

The question is: why didn't a narrow connection rather than a connection adequate to the rule arise immediately? It is not difficult to answer this question. The mathematics student knows that each separate letter he deals with in solving the algebraic "problem" can designate any algebraic expression, and vice versa. And this knowledge does not seem abstract and verbal to him at all. He often had to substitute an algebraic expression for a letter in one or another formula and, conversely, replace an algebraic expression with a letter. Hence, in dealing with separate letters, he--so to speak--"sees" in them the designation of algebraic expressions, that is, he always understands them as such without any special considerations. And conversely, he can always "see" an algebraic expression as a special form of a written, constituent "letter." Therefore, in solving problems of the first type, he in fact has in mind problems of the second type. And, conversely, having encountered a problem of the second type, he immediately identifies it with the first type. Speaking more precisely, in solving problems of the first type, he essentially has in mind features common to all problems of computing determinants of the second order. Hence, there immediately arises in him a connection adequate to the rule pertinent here. When there is a generalizing conception of the problem's data, solving problems of one kind generally entails the rise of a connection corresponding to all kinds of problems solved according to the same rule. And at the same time, apparently, if we solve only one kind of problem, then a connection adequate to the rule can arise only in the presence of this generalizing conception of data.

Hence it follows that the nine pupils who did not solve the problem $(x - y^2)^5(x - y^2)^2(x - y^2)^4$ did not have a generalizing understanding of the expressions $a^n$ and $(A)^m$, both when the connection for multiplying powers was formed and when they had to solve the problem. The same can be said about those pupils who could not reduce the fraction

$$\frac{(x - y)^5}{(x - y)^{15}} .$$

Thus, both narrow connections are essentially linked. At the same time, it appears that the rise of these connections has two causes:

(a) the absence of a generalizing conception of the expressions $a^n$ and $(A)^n$ and (b) the more frequent solving of problems containing expressions of the first type. As we have shown, the second fact could generate narrow connections only in the presence of the first one, and vice versa.

Now we can answer one more question which naturally suggests itself in the initial analysis of the errors indicated above: in dealing with the problems

$$\frac{(x - y)^5}{(x - y)^{15}} \quad \text{and} \quad (x - y^2)^5(x - y^2)(x - y^2)^4,$$

why didn't the pupils in question recall the corresponding rules, since these rules were well known to them? When a pupil is dealing with the first problem, the essential condition for recalling the rule is understanding that the expressions $(x - y)^5$ and $(x - y)^{15}$ are powers of the base $(x - y)$. In other words, the essential condition for recalling the rule is the same generalized conception of the expressions $(x - y)^5$ and $(x - y)^{15}$, which was missing in these pupils. Apparently, the same can be said about the second problem.

Naturally, a generalized conception of the expressions $a^n$ and $(A)^n$ does not arise immediately. In those pupils who could not handle problems on reducing and multiplying powers of polynomials, this generalized conception was still missing. It was already developed, apparently, in the other pupils in the same grades. Therefore, although all pupils solved the same problems, narrow connections arose in only some of them. Only in them was this conception of data, the essential condition for recalling corresponding rules, impeded. In most pupils, on the basis of a generalized understanding of the expressions $a^n$ and $(A)^n$, either connections adequate to the rule had already arisen, or the recall of rules was accomplished while solving the problems.

Thus, our general conclusions can be formulated in the following propositions:

(1) While mastering algebra in school, a general ability to conceive each specific letter as designating any expression is cultivated. This does not mean that pupils having this ability necessarily think of a separate letter precisely as designating <u>any</u> expression each time they

see it. Usually this conception is missing, since there is no need for it. But the pupils of whom we speak are able to understand a letter in this way when necessary. The same can also be said of the ability to understand any expression as a letter, that is, to replace mentally this expression with a single letter.

(2) As the facts mentioned showed, eighth-grade pupils are far from possessing this general skill. In all probability, this is because pupils solved too few problems on substituting whole expressions for separate letters and vice versa, and too few problems in which they had to perform mentally, these operations in order to solve more complex problems.

(3) If pupils without this general skill solve problems of only one type out of those subsumed under a definite rule, a connection is formed which is too narrow. Into its first component enter not only features common to all kinds of problems subsumed under a definite rule, but also certain specific features found only in that type of problem with which the pupils were dealing. Therefore, this connection is not actualized in perceiving data of another type. At the same time, in perceiving such data, the pupils without a generalized conception of algebraic expressions do not recall the rule, since the essential condition for recalling it is precisely a generalized conception of facts.

## Part III

In the assignment given after studying the section "Irrational Expressions," nine pupils (from two parallel classes) made identical mistakes in removing factors from a radical. Let us cite a typical case. They were given the expression

$$3a\sqrt{\frac{b}{a}} - ab\sqrt{\frac{a}{9b}} + \frac{1}{3b}\sqrt{ab^3} + \frac{5}{a}\sqrt{\frac{a^5b}{16}}.$$

The pupil correctly transforms the first monomial and writes

$$3a\sqrt{\frac{b}{a}} = \frac{3a}{a}\sqrt{ab}.$$

In dealing with the second monomial, he makes an error; he places the factor removed from the radical not in the denominator, but in the numerator

$$2b\sqrt{\frac{a}{9b}} = 2b \cdot 3b\sqrt{ab} .$$

There is no error in transforming the third monomial; the pupil inserts the factor extracted from the radical in the numerator, where it should be

$$\frac{1}{3b}\sqrt{ab^3} = \frac{b}{3b}\sqrt{ab} .$$

But in removing factors from the fourth radical, he again errs; he places the factor removed not in the numerator, where it should stand, but in the denominator

$$\frac{5}{a}\sqrt{\frac{a^5b}{16}} = \frac{5}{4a \cdot a^2}\sqrt{ab} .$$

Thus, in performing the same operation four times in succession, the pupil executes it correctly twice, and twice incorrectly.

Is it possible to believe that the pupil making all these transformations had proceeded from the recognition of rules? Obviously not. For this would mean that he alternately proceeded now from a correct rule, now from an incorrect one. This supposition has no possibility whatsoever.

Perhaps he recognized the rule in two cases and then performed the transformation correctly, but operated without recognizing the rule and hence erred in the other two instances. This assumption, however, should also be discarded. Cases in which a pupil, solving similar problems, alternately recalls and forgets a rule are of course possible. But they occur only in special cases. For example, if a pupil solving a problem without recalling a rule makes a mistake and notices it, then he occasionally will remember the rule in solving the next problem of the same type. Similarly, if a pupil encounters a more complex problem of the same type, after solving a series of simpler problems, he may also recall the rule. In this case, such special conditions did not exist. The error made in transforming the second monomial went unnoticed;

transformation of the third monomial was simpler, not more complex, than the task performed just before. One cannot find any other basis for recalling a rule in executing the third transformation.

Hence, in all four cases, one must assume that the pupil operated by actualizing certain connections, not by recalling a rule. First we shall consider those cases (the second and fourth problems) where he erred. An incorrect connection was apparently actualized here. What are its components, and how did it arise?

If the pupil had operated in strict accordance with rules, he would have solved the second problem with the following chain of transformations:

$$2b\sqrt{\frac{a}{9b}} = 2b\sqrt{\frac{ab}{9b^2}} = 2b\,\frac{\sqrt{ab}}{3b} = \frac{2b}{3b}\sqrt{ab}.$$

The pupil did not perform all these transformations in writing. He no doubt executed the first one in his mind. The fact that the final result contains $\sqrt{ab}$ points to this. The pupil did not even perform the second and third transformations mentally. If he had, the problem would have been solved correctly. Hence, one must suppose that he "skipped" directly from the representation

$$ab\sqrt{\frac{ab}{9b^2}} \text{ to the "result" } 2b \cdot 3b\sqrt{ab}.$$

Thus, one must seek the incorrect connection in the transition from extracting a root from $9b^2$, to writing 3b before the root. Analysis of the second error yields similar results.

To establish the components of this incorrect connection, let us first note that both incorrectly transformed monomials share one feature--the same letter the pupil removed from the radical stands in front of the radical. Moreover, whenever this letter stands under the radical in the denominator, it stands before the radical in the numerator, and vice versa. These monomials are

$$ab\sqrt{\frac{a}{9b}} \text{ and } \frac{5}{a}\sqrt{\frac{a^5b}{16}}.$$

As we saw, in operating with them, the pupil obtained the following results

$$2b \cdot 3b\sqrt{ab} \text{ and } \frac{5}{4a \cdot a^2}\sqrt{ab}.$$

That is, he put the letter removed from the radical in the place where such a letter already stood before the radical. By so doing, the pupil in both cases placed the factor removed from the root in a position opposite to the correct one. Hence, two possible assumptions pertaining to the connection's components arise.

First, one can suppose that the second component of this incorrect connection was an orientation toward recording the letter removed from the root in the place where the same letter already stood before the root.

But could this connection arise? Let us note first that it decisively contradicts both the basic rules for extracting roots and generally the character of all algebraic rules. Therefore, it could have arisen as a result of transformations based on recognizing rules. It was conceivable that it arose because of an unsuccessful selection of problems the pupil solved earlier, that is, that problems predominated in which one had to write the letter removed from the root next to the same letter standing before the root. But a count of the problems the pupils solved[2] refutes this assumption too. These problems contained only 18 cases in which one had to write the letter removed from the root next to the same letter standing before the root, and 34 cases in which one had to record it in a position opposite to that of the same letter in front of the root. Thus, the first supposition no longer arises; the pupils had no orientation toward writing the letter extracted from the root in the place where the same letter stood before the root.

Another assumption is possible. It was conceivable that the connection's second component was an orientation toward recording the letter removed from a root in a position opposite to that which it occupied under the root. This supposition, however, should also be discarded. Such a connection could not arise as a result of operations according to rules, nor could it result from an unsuccessful choice of problems the pupil solved.

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[2] We used the "collective notebooks" for this count.

Hence it follows, first of all, that the position of the quantity from which the pupil extracted the root had no significance for him. He, of course, saw where this quantity stood: either in the numerator or the denominator. But its position did not determine the pupil's operations. And this means that the feature characterizing the position of the quantity from which the root was just extracted ($9b^2$ in this case) did not enter into the connection's first component. Its first component was recognizing only the fact that the quantity (3b in our example) found in the mind is the result of extracting a root.

Furthermore, it follows that recognizing the position in which one must write the factor extracted did not enter into the connection's second component. The second component of the connection was an orientation toward recording the factor somewhere before the root.

We shall try to clarify how a connection with insufficiently defined components could arise.

Before this assignment, the pupils had solved a sufficient number of problems on removing factors from a root for the rise of correct connections. About 120 such problems were solved in all. Of these, 74 were on removing from the root factors standing in the quantity's numerator (considering also those cases in which the quantity is not a fraction), and 46 problems on extracting from the root factors standing in the quantity's denominator.[3] The accuracy of the solutions was carefully checked. Therefore, each mistake a pupil made was recognized and corrected by him. Thus, there was no reason for the rise of incorrect connections; on the contrary, all conditions for the rise of correct connections were apparently present.

However, one should note one important circumstance. When the pupil solved his initial problems on removing factors from a radical, he of course performed all the intermediate transformations mentioned above. They were especially necessary whenever the quantity under the root was fractional. But the correct connection, which permits a "skip" across intermediate links, cannot arise in this method of solution. It begins

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[3]As before, we counted "elementary" problems and considered only those cases in which the factors removed were expressed by letters and not numbers.

to be formed only when an attempt is made for the first time to jump directly to the final result from the perception of data (or from the representation of their first transformation). For the correct connection to be formed, the pupil must solve a sufficient number of problems of this type the short way.

Hence, the conditions for the rise of a correct connection were less favorable than it seemed at first glance. Therefore, one must not consider all 120 problems the pupil solved, but only a certain part of them, that is, those problems the pupil solved by omitting intermediate links. We do not know precisely how many problems he solved the short way. But apparently they were insufficient for the correct connection to arise in him.

This does not mean that the other pupils of the same class did not have the correct connection. For the speed of formulating connections varies for different pupils. Besides, some pupils could pass from a detailed to a shorter solution sooner than others. Therefore, although the pupils all solved the same problems, it is quite possible that a correct connection arose in some, but not in others.

All this does not, however, resolve the question of the causes for the rise of an incorrect connection. Suppose our pupil still did not have a correct connection. Where did the incorrect connection come from? To answer this question, one must remember that an incorrect connection differs from a correct one only by its components' being less definite. The first component of the incorrect connection is recognizing a certain quantity as the result of extracting a root. The first component of a correct connection also contains the recognition of the position held by the radical quantity from which the root was extracted. The second component of the incorrect connection is an orientation toward recording the quantity obtained "somewhere in front of the root." The second component of the correct connection also contains an orientation toward writing this quantity in the position where it stood under the root. Now if we recall that the assimilation of anything, in many cases, begins precisely with the mastery of the general, less defined features of what is being assimilated, we are completely justified in supposing that the incorrect connection in question is only the first stage of formulating a correct one.

This supposition is confirmed by the following considerations. At a definite stage of assimilating the operation in question, the pupil attempts to perform the task the short way by omitting intermediate links. The very presence of these efforts shows that some kind of connection has already arisen in him. To develop and strengthen a correct connection, repeated correct execution of the task the short way is essential; but the pupil is only making his first attempt at performing it this way. Therefore, the connection now existing in him cannot correspond to rules. But at the same time, its components cannot contain anything missing from rules, or not resulting from rules, since the connection arose from detailed operations according to rules. Consequently, the lack of correspondence to rules can consist only in that several features corresponding to rules do not enter into the content of the connection's components.

Let us make these general assumptions concrete. We shall analyze a case in which a pupil in the first stages of mastering an operation solves a problem by the transformations indicated above:

$$2b\sqrt{\frac{a}{9b}} = 2b\sqrt{\frac{ab}{9b^2}} = 2b\frac{\sqrt{ab}}{\sqrt{9b^2}} = 2b\frac{\sqrt{ab}}{3b} = \frac{2b}{3b}\sqrt{ab}.$$

(1) (2) (3) (4)

How is the third transformation performed? At first the pupil writes 2b, then--convinced that $\sqrt{ab}$ defies transformation--he records this root in the same position as it was written in expression (2). Then he draws a line and turns to the denominator of the fraction in expression (2). He extracts the root from $9b^2$ and obtains 3b. In considering this expression, he of course recognized that it was obtained by extracting the root. An orientation arises toward recording it within the expression formulated (3). Must this orientation contain the feature characterized by the words "record in the denominator?" No, this feature may also be missing. The pupil has already finished writing the factor standing before the fraction and the fraction's numerator, and he is aware that he has completed both. Before him is the expression $2b\frac{\sqrt{ab}}{\phantom{3b}}$, with a noticeable empty slot. At the same time, he cannot place 3b before the fraction, nor in the fraction's numerator; there is no place for it.

Therefore, the operation will be performed correctly even if the orientation in question suggests placing the quantity existing in the mind somewhere in the expression formulated. Later, recognition that 3b was obtained from the quantity in the denominator apparently becomes superfluous in these cases. Even if it does arise, it does not determine the content of the orientation or of the subsequent operation, for the orientation does not contain the feature characterizing the place for writing 3b. Consequently, the rise of a connection is possible, under these conditions, with the following components: (a) recognition that the quantity existing in the mind is the result of extracting a root and (b) an orientation toward recording this quantity "somewhere" in an open slot in the expression which the pupil began to write.

Of course, this is still not the incorrect connection discussed earlier which is the cause of error. This is a connection which arises and is actualized whenever a problem is solved in detail. But the incorrect connection actualized in an abbreviated solution arises precisely from it. This probably occurs as follows. Having solved several problems on removing factors from a root, the pupil begins to understand that several quantities obtained in intermediate operations, especially in extracting roots, enter into the final result. In our example, for instance, 3b is such a quantity. Naturally, a tendency arises to record this quantity immediately in the place where it should stand in the final result. In our case, for example, after the expression

$$2b\,\frac{\sqrt{ab}}{\sqrt{9b^2}}$$

was written and the root from 9b was extracted mentally, in the pupil there arises a tendency to write the quantity 3b immediately before the root, to omit the subsequent intermediate link

$$2b\,\sqrt{\frac{ab}{3b}}$$

This tendency can be realized only by actualizing some connection. Recalling rules does not help here, since the rules indicated in the text pertain only to a detailed solution of a problem, not to an abbreviated one. At the same time, there already exists in the pupil a general orientation toward solving problems of a given type -- by actualizing connections which arose in him while solving problems in detail. However, what kind of connection can be actualized here? Apparently, only the one which arose in the detailed solution problems, in the transition from extracting a root to recording the result of this extraction. True, this connection should undergo some change; instead of an orientation toward writing the quantity "in some empty slot," an orientation toward recording it "somewhere before the root" should arise. But this reconstruction is implemented by itself, since the feature "before the root" enters into the problem's content. The remaining features of this connection are preserved; into its first component, as before, enters the recognition only that the quantity existing in the mind is the result of extracting a root, and into its second component, recognition that it should be written "somewhere."

These are the results of analyzing transformations in which errors were made. Let us turn to those which the pupil executed correctly. It will be recalled that he wrote these problems like this:

$$3a\sqrt{\frac{b}{a}} = \frac{3a}{b}\sqrt{ab} \quad \text{(first problem)}$$

$$\frac{1}{3b}\ ab^3 = \frac{b}{3b}\sqrt{ab} \quad \text{(third problem),}$$

Can one suppose that a correct connection, distinct from the incorrect one just analyzed, was actualized here?

Let us compare all four problems. It is not difficult to see that the first problem, performed correctly, does not differ in essence from the second, which the pupil executed incorrectly; nor the third, performed correctly, from the fourth, executed incorrectly. Indeed, in the first and second problems -- that is,

$$3a\sqrt{\frac{b}{a}} \quad \text{and} \quad 2b\sqrt{\frac{a}{9b}}$$

the root is extracted from the fraction's denominator, and the same letter removed from the radical stands before the root in the numerator.

In the third and fourth problems - - that is,

$$\frac{1}{3b}\sqrt{ab^2} \quad \text{and} \quad \frac{5}{a}\sqrt{\frac{a^5b}{16}}$$

the root is extracted from the numerator, and the same letter removed from the radical stands before the root in the denominator.

Let us further recall that correct and incorrect connections, as we have established, differ in that members of an incorrect connection do not contain several features entering into the elements of a correct connection. Therefore, supposing that the pupil possessed a correct connection which was actualized in both cases, we assume that both connections can exist simultaneously. Both the less and the more defined connections, however, are actualized under identical conditions: one time a correct connection is actualized in solving the first of two identical problems; another time, in solving the second problem, an incorrect connection is actualized. Such an assumption cannot correspond to reality. For in the first place, it follows that a correct connection is sometimes actualized and sometimes not under the same conditions. But this would mean that it simply did not exist. Apparently, the same can be said about an incorrect connection. Second, in considering the relationship between a connection's components (distinctions in definiteness), it is impossible to understand how an incorrect connection can exist and function separately from a correct one and not merge with it.

Thus, one must suppose that this pupil had no correct connection; there was only an incorrect one, whose structure and origin we revealed. It is not difficult to explain how the pupil solved two problems correctly, despite the presence of this incorrect connection. The incorrect connection's second member is an orientation of an indefinite character: "to write the quantity somewhere before the root." Apparently, when this orientation exists, the quantity in question can be written in either the numerator or the denominator of the fraction standing in front of the root. Moreover, there is an equal chance that it would be written in either the numerator or the denominator. This equality of chance is precisely reflected in our pupil: in two instances he wrote the quantity extracted from the radical in the fraction's numerator; in two cases, in the denominator.

Let us summarize the results of our analysis and the conclusions stemming from it.

(1) One must distinguish connections whose actualization forms a detailed solution from connections whose actualization constitutes an abbreviated solution of a problem. In the initial states of mastering a transformation, the task is executed in detail; later, it is carried out in a truncated form. Thus, the chain of connections that arose in the initial stages does not remain unchanged. During the exercise it is reconstructed, if not always, then in several instances.

(2) An abbreviated process of solution is characterized by the fact that several connections constituting a detailed process do not enter into a truncated one. However, the reconstruction in question does not consist only of discarding several connections. The rise of new connections, which "close" the gap formed as a result of discarding, should occur at the same time.

(3) These newly arising, "closing" connections differ in origin from the original ones.

The initial connections arise from performing a task by recalling and applying rules. Although recall and application of rules drop out, the components of a connection are nevertheless defined by the rule's content. The connection's first component is recognizing the expression's features indicated in the first half of the rule; the second component is an orientation toward performing operations indicated in the second half. For example, a short rule for multiplying powers states: to multiply powers of one letter, one must add the exponents of these powers. The first component of the connection arising on the basis of this rule is recognizing that the given expression consists of identical letters written side by side, without intermediate signs or divided by dots. The connection's second component is an orientation toward adding exponents. Thus, the original connection develops from the process of recognizing and applying a rule. Moreover, the rule's content enters wholly into the connection's structure, if special circumstances do not impede this. Therefore, when these circumstances are absent, the initial connection is always correct.

By contrast, the "closing" connection arises at the stage of assimilating a complex transformation when it is already executed without recognizing and applying a rule. Therefore, the closing connection cannot be the direct product of these processes, but can arise only by "short circuiting" the chain of initial connections: the first component of the original connection entering into this shedding chain is the first component of the closing connection, and the second component of the final connection entering into the same chain is the closing connection's second component.

(4) In the first stage of the rise of this closing connection, its components may seem insufficiently defined, since the components of the original connections which become components of the new connection cannot have all the features essential to the formation of a final, correct closing connection. The detailed execution of a task proceeds under conditions other than those for a truncated one. Therefore, several features essential to the formulation of a correct new connection cannot enter into closed links, but only into the discarded links of the process. In these cases, the actualization of a connection which is just arising (embryonic) can lead to errors in solving a problem. Moreover, as long as this embryonic connection differs from a correct one only by a lesser definiteness of components, one must expect that its actualization will lead to correct results in some cases, to incorrect ones in others. In the cases we analyzed, we found precisely this alternate execution of correct and incorrect operations.

(5) Since a correct closing connection replaces the whole chain of original connections, each of which corresponds to a rule, the closing connection itself also corresponds not to a single rule, but to a series of rules linked together so that each subsequent rule is related to the results of applying the preceding rule.

## Part IV

In twelve pupils, we found an error of the following type:

$$(a\sqrt{b} - b^2\sqrt{a})^2 = ab - 2ab^2\sqrt{ab} + b^2a.$$

In a correct squaring we would have

$$a^2b - 2ab^2\sqrt{ab} + b^4a.$$

Apparently, the pupils who made the mistake, in dealing with the monomial $a\sqrt{b}$, squared only $\sqrt{b}$, but not the letter, $a$. Similarly in dealing with the monomial $b^2\sqrt{a}$, they squared only $\sqrt{a}$, but not $b^2$. In one of the preceding assignments, the same pupils solved a problem of this type:

$$(-\tfrac{4}{5}y^2x^2 + 4x^6 - \tfrac{1}{2}y^3x)^2.$$

Of twelve pupils who made the mistake indicated, ten either solved this problem correctly or made an error unconnected with the one indicated above. In what follows, we shall consider only these ten pupils.

First let us note that, in solving the second problem, the pupils should have squared the degree of one letter or another five times. They did this correctly in all five instances. Thus, a problem of the type $(n^x)^2$, where $x$ is any number, was correctly executed five times in an earlier assignment, but twice incorrectly in a later one. How could this happen? Why did the pupils suddenly err after repeatedly performing the task correctly?

First of all, one must make this question more precise. In squaring trinomials, the pupils performed elementary tasks of the type $(n^x)^2$, of course, by actualizing connections. They had already solved a large number of problems on squaring binomials and on multiplying binomials according to the formula $(a + b)(a - b)$ in the seventh grade. Elementary tasks on squaring powers always enter into this type of problem. Consequently, there is every reason to suppose that the corresponding connection had arisen in them already in the seventh grade. After the holidays and during the first month of the new school year, it of course could weaken somewhat. But not long before the assignment in which they had to square trinomials, the pupils again solved a series of problems of the types indicated. This undoubtedly entailed the restoration of the connection.

The first component of this connection, apparently, would be recognizing that the given "letter" is the factor of a monomial which one must square; and the second component, an orientation toward copying letters with a doubled exponent. In the first assignment, in solving the problem

$$(-\frac{4}{5}y^5x^2 - 4x^6 + \frac{1}{2}y^3x)^2,$$

this connection was actualized five times; in the subsequent assignment, in solving the problem

$$(a\sqrt{b} - b^2\sqrt{a})^2,$$

twice it was <u>not</u> actualized. One cannot suppose that it somehow vanished during this time. Hence the question naturally arises: why was it not actualized in the second instance? Apparently, one must seek the answer to this question in the features of the second problem. But only one of these features is significant. In the first problem, one had to square monomials which did not contain roots; in the second, monomials containing roots. Therefore, when the pupil had to square the monomial $a\sqrt{b}$, the connection in question was not actualized in perceiving the letter $a$ precisely because $\sqrt{b}$ stood after this letter. If a rational factor, instead of a root, had stood in the monomial, the connection would have been actualized, and the pupil would have written $a^2$ as the answer. Similarly, in the monomial $b^2\sqrt{a}$, the presence of $\sqrt{a}$ caused the connection not to be actualized in perceiving the factor $b^2$. The <u>presence of a root hindered the actualization of the connection</u> in question.

But this explanation of the error is still by no means final. To obtain such an explanation, one must establish why and how the presence of root in a monomial could hinder the actualization of a connection.

As our survey of the "collective notebooks" showed, the pupils did not solve a single problem in which they had to square a square root before the assignment. Therefore, this operation was new and unfamiliar to them, and they had no connection according to which they could have performed it. Nevertheless, the pupils had at their disposal knowledge essential to the correct execution of this operation. They knew the definition according to which a number, raised to the power n to give a, is called the nth root of the number a. They also knew the identity $(\sqrt[n]{a})^n = a$. They had already memorized both at the very beginning of their work on the section entitled "Roots," and later often had to recall the definition itself, as well as the identity indicated. Squaring the roots $\sqrt{b}$ and $\sqrt{a}$ under these conditions apparently proceeded as follows: the pupils recalled either the verbal formulation of the definition of a root, or the written expression of this definition. They applied the definition to the given case and obtained the corresponding result. One should note that the definition played the role of a rule under these conditions; therefore, in what follows, we shall simply call it a "rule."

But why did the pupils recall this rule and draw the conclusion proceeding from it? Can one say that the cause of all this was just the aggregate of facts mentioned above: the absence of a corresponding correct connection, on the one hand, and knowledge of the rule, on the other? One must answer this question negatively. In analyzing other errors, we saw that the pupils who made these mistakes did not have correct connections, but knew the corresponding rules. Nevertheless, they did not recall these rules. Consequently, for the pupil to recall the rule, a special orientation toward solving a problem by recalling rules is essential. Our observations showed that this orientation arises whenever a pupil is dealing with a problem of a specific type for the first time, but thinks he can solve it with the rules he knows. The case in question was no doubt subsumed under these conditions. Hence, while perceiving $\sqrt{b}$ and $\sqrt{a}$, an orientation toward recalling a corresponding rule unquestionably arose in the pupils. But this orientation is directly opposite to the orientation toward solving a problem by actualizing connections and, without this second orientation, an actualization of connections is impossible. Hence it follows that the connection which should have been actualized in perceiving the letter a was not actualized precisely because an orientation toward recalling a rule arose in perceiving $a\sqrt{b}$.

It is not difficult to be convinced that no other reasons were significant here. Essential and, at the same time, adequate conditions for actualizing a definite connection are: (a) recognizing data subsumed under the first member of the connection; (b) recognizing the problem to be solved; and (c) a general orientation toward solving a problem by actualizing connections. The first two conditions no doubt existed in our example. The pupils of course recognized that the letter a entered into the structure of the monomial, and that this monomial had to be squared. Therefore, the connection was not actualized only because the third condition was missing, that is, an orientation toward actualizing connections. But why didn't this orientation arise? Squaring a monomial without roots was unquestionably a simple and familiar operation to the pupils; and, in these cases, an orientation toward actualizing connections always arises if there are no obstacles. In particular, it existed in squaring the rational trinomial indicated above. This time, therefore, certain processes linked with the perception of $\sqrt{b}$ impeded its rise. But only one of these processes could hinder it: an opposing orientation toward solving a problem by recalling rules and making deductions.

Here one objection, at first glance a substantial one, is possible. The pupil, of course, first squared the factor a and only afterwards moved to the factor $\sqrt{b}$. But if so, then the orientation toward recalling a rule linked with the perception of $\sqrt{b}$, apparently, could in no way affect operations with the letter a.

This objection prompts us to note several facts which we did not consider before. Various types of problems entered into the assignment, and the pupils of course did not know in advance what kinds of problems they would have to deal with and in what sequence. In cases of this kind, a tentative perception of the algebraic expression we are given always precedes the execution of any transformations. There is no doubt that this tentative perception also occurred in the case just analyzed. Moreover, since the task to square $\sqrt{b}$ was, as we know, new and unfamiliar to the pupil, he of course paid attention to this $\sqrt{b}$ Therefore, the perception of $\sqrt{b}$ preceded operations with the letter a, and hence the orientation toward recalling the rule connected with this perception could affect these operations.

However, our analysis of the error which interests us is still unfinished. The orientation toward recalling rules pertained to the factor $\sqrt{b}$. The question is: how could it impede the rise of an orientation toward actualizing connections in operations with other factors, with the letter a? We think there is only one answer to this question: when the pupil dealt with the letter a, this orientation did not pertain to the factor $\sqrt{b}$, but was related in general to all operations he was performing.

Quite possibly the content of this orientation was sufficiently undefined at the moment of its rise, that is, in a tentative perusal of the facts. In other words, it is possible that the pupil aimed at recalling any rules related to the given problem; but the clear character of the rules' features did not enter into the content of the orientation. The orientation of course pertained to the factor $\sqrt{b}$, but only because it was prompted by the perception of this factor. Therefore, when the pupil transferred his attention to the letter a, this relationship to $\sqrt{b}$ completely disappeared. Only an indefinite orientation toward recalling algebraic rules remained.

However, it is possible that, at the moment of its rise, the orientation toward recalling a rule was more or less defined by its content. Perhaps the pupil aimed at recalling the rule expressed by the identity $(\sqrt[n]{a})^n = a$, or at least some rule pertaining to roots. If so, when the pupil began operations on the letter a, the orientation had lost its definiteness and, at the same time, its relationship to $\sqrt{b}$. Only the same orientation toward recalling some algebraic rules remained.

But here an important new question arises. If the pupil had a general orientation toward recalling rules while performing operations on the letter a, why did he not recall those rules pertaining to the letter a in this case? Why did he not recall the rule for raising the degree of a monomial, or the rule for elevating a degree to a new degree? These questions, apparently, indicate one more essential feature of the orientation examined. This was an orientation that was not activated while operations on the letter a were being performed. It was a delayed [zaderzhannaya] orientation for recalling rules which was activated only when the pupil turned to $\sqrt{b}$ again.

To understand the source of this feature of the orientation examined, one should note one regularity which occurs in the solving of algebraic problems. Whenever possible, pupils perform operations on separate parts of the expression given them from left to right. This means that they have a general orientation toward performing operations in an effective order whenever mathematical rules do not dictate another sequence. This orientation unquestionably determined the pupils' operations in the case just analyzed. Therefore, the process of solution proceeded in the following manner. In perceiving $\sqrt{b}$ (and then a ), an orientation toward recalling a rule arose. But the actual recall of this rule, and deduction from it, were not realized, since these processes did not correspond to the orientation toward performing operations from left to right. And this means that the orientation toward recalling rules was delayed. The pupil moved to operations on the letter a with this delayed orientation toward recalling a rule. The orientation could not be realized, could not proceed to the actual recall of rules. But it nevertheless delayed the rise of an opposing orientation toward actualizing connections.

One should assume that the orientation's delay was not a simple interruption in realizing the process that had begun. Apparently the orientation undergoes certain qualitative changes at the moment of delay.

Let us summarize the results of our analysis and make several additional remarks:

(1) Under defined conditions, there is in us an orientation toward recalling rules and making deductions from them. Depending on circumstances, it evidently can have diverse characteristics.

At times it is a specific orientation: into its content enters a recognition of the typical features of an algebraic expression, and of the rule which should be reproduced. This specific orientation does not lose its defined objective reference even when perception ceases of the expression to which it pertains. The orientation's objective reference enters into the content itself.

In other cases, it is a general orientation which also has a defined objective reference. Recognizing the typical features of both the given expression and of the rule to be reproduced does not enter into the content of this orientation. However, despite the indefiniteness of its content, this orientation nevertheless has a defined objective reference to the expression immediately perceived.

Finally, a general orientation with an indefinite objective reference can occur. Recognition of the typical features of the given expression and the rule one must recall does not enter into the content of this orientation. Nor does it pertain to any definite expression. There is only a special intellectual condition conducive to recalling algebraic or other kinds of rules, and to making deductions.

(2) We have already seen that an orientation toward solving a problem by actualizing connections arises under defined conditions. By analogy to what was just said, one can imagine that this orientation might have a definiteness different in content and in objective reference.

(3) Characterized by a defined objective reference, an orientation toward recalling rules is incompatible with an orientation toward actualizing connections pertaining to the same object, and vice versa. However, the coexistence of orientations pertaining to different objects is apparently possible.

Characterized by an undefined objective reference, an orientation toward recalling rules is incompatible with an orientation toward actualizing connections. Similarly, an objectively indefinite orientation toward actualizing connections is incompatible with an orientation toward recalling rules. These orientations "impede" one another.

(4) Delayed orientations coexist with urgent ones. An urgent orientation is immediately activated; a delayed one is not activated until a definite moment. A delayed orientation, however, is not the

absence or disappearance of an orientation. Although delayed, the orientation continues to exist, that is, to affect the course of intellectual processes; but its influence is expressed only by its ability to impede the rise of certain processes.

(5) Orientations toward recalling rules and actualizing connections already arise in a tentative perception of an algebraic expression.

(6) Operations constituting the solution of a (number) problem are performed according to the order of elements entering into the expression (unless a rule of special circumstances demand another sequence). An orientation toward performing operations in a definite order may delay the orientation toward recalling rules that arose in a tentative scanning of the expression.

(7) A delayed orientation toward recalling rules can impede the rise of an orientation toward actualizing a connection. In this case, the connections existing in us are not actualized. At the same time, the corresponding rules are not recalled, since the orientation toward recalling rules is a delayed one. As a result, we do not perform the specific operation that should have been executed in the given situation.

## Part V

In our preceding work, we established that special combinations of mental processes arise in all pupils during the assimilation of elementary algebra. The first component of each combination is recognizing the general features of a specific part of the algebraic expression given us, or of an operation just performed. The second component is an orientation toward performing an operation defined by kind. We called these combinations connections.

The task of our work, the results of which we now summarize, was to analyze algebraic errors. The basic results of this work can be formulated as follows:

(1) In a series of cases, the error the pupil made resulted from the actualization of an incorrect connection which existed in him when solving a problem.

(2) Incorrect connections fall into several types according to their structure and to the conditions for their rise. We were able to discover three types. But it is possible that other types of incorrect connections also exist.

(3) The first type of incorrect connection is characterized by the fact that a specific feature that should have entered into its first component, does not. With such a connection, therefore, a particular operation which—according to rules—should be performed in relating facts of another kind. This connection can arise when a pupil, solving problems of a definite (first) type, knows what operation he should perform even before clearly perceiving the data. Under the usual conditions of school and homework in algebra, this prior knowledge of forthcoming operations arises rather often. Having this prior knowledge, the pupil often recognizes only certain features of the algebraic expression given him, and omits equally essential ones. He nevertheless performs the correct operation. If the pupil solves several (number) problems of a particular type in this way, then an incorrect connection arises in him. Into its first component enter only those features of algebraic expressions which he clearly recognized. If this method of solving problems is repeated often, the incorrect connection is reinforced more and more. One should note that it is impossible to discover the existence of the incorrect connection as long as the pupil deals only with problems of the first type. But later on, the pupil encounters problems whose data (a) have all the features he clearly recognized in solving problems of the first type; but (b) at the same time have several other features, by virtue of which one should perform an operation different from that performed in solving the first kind of problem. The incorrect connection which arose earlier may be actualized in solving problems of the second kind. The pupil sometimes performs the operation he should perform in solving problems of the first kind, and not the one he should perform for solving the second kind of problem. For example, in solving a problem on raising a power to a new one, the pupil does not multiply, but adds the exponents.

The existence of an incorrect connection which arose earlier in solving problems on multiplying powers of one letter is manifested in this error. Into the first component of this incorrect connection enter only these features: the presence of two exponents, the absence of two different bases, and so forth, that is, features common to problems of both kinds.

Of course, the rise of an incorrect connection in solving problems of one kind does not always entail errors in solving problems of the other kind. If the pupil recalls the relevant rule in solving the second kind of problem, and applies it to the data, he of course will not make a mistake. Then, if he has correctly solved a sufficient number of problems of the second type, the corresponding correct connection will arise in him. In many instances, the rise of this correct connection probably entails a reconstruction of the incorrect connection created earlier, its conversion to a correct one. When there is a sufficiently strong connection pertaining to problems of the second kind, especially after the reconstruction of the incorrect connection, the error that concerns us at this time becomes impossible.

However, the facts mentioned in our work show that an incorrect connection is sometimes nevertheless preserved and, moreover, seems stronger than a correct one. An incorrect connection is actualized more frequently whenever a pupil solves a problem by actualizing connections.

Hence, it is possible to successfully prevent errors of the type analyzed. First, while solving problems in class, one must try to have the pupil recognize clearly each time all essential features of the algebraic expression he is given. This self-evident methodological rule is not always observed. Second, one must further reduce the number of cases in which a pupil solves many problems of the same kind <u>in succession</u> without a teacher's supervision. Mathematics teachers often assume that independent, successive solution of many problems of the same kind reinforces the corresponding correct habit. But the opposite in fact occurs: conditions are created in which the rise of a false connection becomes more likely. Third, one must increase the number of cases in which a pupil solves in succession several problems of <u>different</u> kinds in which the operations to be performed are not indicated to him; he must either "simplify" the given expression or perform some transformation. Tasks of this kind will effect the rise of a general habit of beginning operations only after recognizing clearly the features of a given expression. Fourth, one must arrange assignments so that a correct connection pertaining to the second kind of problem will be stronger than an incorrect one that might have arisen while solving problems of the first kind. In other words, the number of problems of the second type the pupil solves should never be less than that of the first type. Cases we have seen testify that pedagogical practice occasionally does not satisfy this demand. Fifth, one must create conditions that assure the reconstruction of an incorrect connection, if it has arisen.

(4) Conversely, the peculiarity of the second type of incorrect connection consists in the fact that its first component contains a feature that it should not contain. Hence, the connection is actualized in perceiving one kind of data subsumed under a definite rule, but not actualized in perceiving another kind of data subsumed under the same rule. This connection arises under the following conditions: (a) during much of the course, the pupil deals only with data of one kind from those subsumed under a given rule; (b) moreover, the generalizing recognition of these facts, that is, the selection of their generic features, is missing.

Therefore, first, whenever problems of several different types are subsumed under a definite rule, it is essential for the pupil to solve a sufficient number of problems of each type. This demand is also self-evident, but--as we have seen--it is nevertheless not always carried out. Second, using special exercises, one should cultivate a general ability to conceive of each algebraic expression as a "letter" and, conversely, each letter as designating a complex algebraic expression.

(5) Several features that should have entered into both the first and second components of the third type of false connection do not. This connection may be formed in the first stage of

formulating a correct connection. It leads to the performance of both correct and incorrect operations under essentially identical conditions. It is probably impossible to prevent the rise of such connections. But one can and should hamper a pupil's premature attempts to solve problems of a definite type without recalling the corresponding rules.

(6) One important fact should especially be stressed: in the cases we analyzed, the incorrect connections arose precisely while the pupil was correctly (judging by objective results) solving problems of a definite type.

(7) Connections arising from the omission of several links of the already existing chain of original connections coexist with connections arising directly from operations based on recognizing rules. This "closing" [smykayushchya] connection does not correspond to a single algebraic rule, but to the entire chain. However, one can say that the closing connection corresponds to a single operative rule, that is, one according to which problems of a specific type, when the pupil's ability is high enough to solve these problems, are sometimes actually solved. Operative rules are not indicated in algebra courses, and the teacher at best gives only isolated, fragmentary indications corresponding to these rules. Hence, the ability to solve problems of a specific type is cancelled: that is, by omitting several links from the chain of deductions and transformations, the skill is spontaneously generated in the pupils without proper supervision. Therefore, cultivation of this skill leading to the rise of "closing" connections is often understandably accompanied by a large number of errors. These errors occur because the closing connections are insufficiently defined in the first stages of their formulation.

It seems to us, first, that one should introduce into the practice of teaching algebra several operative rules of complex transformations and, second, that one should consciously and systematically cultivate in pupils an ability to perform these transformations the "short" way.

(8) In independent work, a definite rule is recalled only when a special orientation for recalling rules exists. It ordinarily arises when a pupil is dealing with a problem new to him, or with a new type of data. Under various conditions, it possesses a definiteness different in content and in objective reference.

(9) One should distinguish actual and delayed orientations. An actual orientation is immediately converted into a corresponding operation. A delayed orientation, continuing to exist, is not activated until a definite moment. We discovered that delayed orientations for recalling rules arise under specific conditions. But other kinds of delayed orientations are probably also possible.

(10) The existence of a delayed orientation is manifested whenever processes contrary to this orientation do not arise, even though all conditions for their rise exist. In particular, the presence of a delayed orientation toward recalling rules under definite conditions impedes the rise of an orientation toward actualizing connections. A mistaken failure to perform an essential operation which, with no delayed orientation for recalling rules, should have been performed, may be the result of this inhibition.

One must keep these facts in mind during classwork with pupils, and especially when selecting problems for homework and assignments. In particular, one must carefully work through in class those types of problems in which the inhibiting influence of a delayed orientation for recalling rules might affect the solution.

(11) In revising the textbook, one should consider what we said above in (3) and (4). Several kinds and aspects of problems are presented inadequately in the existing text. There is no proper alternation of the kinds of problems that one must alternate. The book has too few sections containing various kinds of problems in a fortuitous order, and so forth. Apparently, one should also introduce into the text several indications of an "operative" character.

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# PUPILS' COMPREHENSION OF GEOMETRIC PROOFS*

F. N. Gonobolin

## The Problem and Methods of Research

In this article we investigate the psychological peculiarities of pupils' comprehension of study material in geometry. In particular their understanding proofs of geometric theorems is of interest.

As we know, the study of geometry in school begins in the sixth grade, with no preliminary introductory course on this subject in the lower grades. At the same time, the textbook presents proofs of geometric theorems very briefly and abstractly, a condition that often creates great difficulties for the younger school children when they try to understand the proofs. Moreover, the pupils of these classes do not always even recognize the necessity of the logical proof of geometric theorems, especially when these proofs are of a visually obvious character or can easily be established empirically.

Not understanding the value of the logical proof or theorems, the pupils feel no need to think about the structure of such proofs. "I don't understand," one sixth-grade girl told us, "why it's necessary to prove that in an isosceles triangle the angles at the base are equal; anybody can see that, especially if you use a protractor."

Besides that, another phenomenon is apparent. Comparing the pupils of the same class, one easily sees a considerable difference in the depth of their understanding of various study materials. While some of them are limited to merely reproducing the teachers' explanations or stating the material in the textbook, others sometimes find independent methods of proving theorems and handle solutions of geometry problems with comparative ease.

All this speaks for the necessity of a detailed study of this process from both the psychological and pedagogical points of view. We must ascertain the conditions for the correct and comprehensive

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*Published in Proceedings [Izvestiya] of the Academy of Pedagogical Sciences of the RSFSR, 1954, Vol. 54, pp. 175-192. Translated by David A. Henderson.

understanding of geometric material by pupils of the lower grades of secondary school and outline the most rational principles of presentation by teachers and authors of school texts.

In investigating the above problems, our initial theoretical premise was Pavlov's statement about thought and understanding being processes of education and the functioning of temporal connections:

> We must consider that the formation of temporal connections, i.e., of these "associations," as they are always called, is comprehension, it is knowledge, it is the acquisition of new knowledge. When a connection, that is, what is called an "association," is formed, this is surely knowledge of a matter, knowledge of definite relationships of the external world, and when you next use it, it is called "understanding," i.e., the use of knowledge acquired by connections is understanding [2:579-580].

To study the process of pupils' understanding geometric proofs, we turned to special observations and experiments, which we conducted in schools 187 and 240 in Moscow, in the Moscow Municipal Adult Secondary School, and the Moscow Regional Secondary School for Resident and Corresponding Adults. First the observations were conducted during the usual lessons of the teachers of these schools. Then, for the purpose of checking variations in teachers' explanations of the study material to the pupils, the observations were carried into the experimental lessons of one of the teachers of an adult school. During all the observations, we carefully registered the teachers' explanations and the following proofs of the theorems studied by the pupils.

Simultaneously with the observations, we conducted individual and group experiments. With this aim, the sixth- and seventh-grade pupils were given new material that was not yet used in school but for which they had already been prepared by the preceding course. Here the pupils were asked to independently read the text of a new theorem, together with its proof, as given in the standard geometry textbook. They were allowed to read it several times, after which they were to reproduce what they had read and answer the experimenter's questions. If it was observed that the pupils did not understand the proof of the theorem, the experimenter offered to read another variant of the explanation, which he himself had composed. If this yielded no positive results, the pupils were asked leading questions and given necessary explanations. Such experiments embraced six groups of sixth- and seventh-grade pupils, with four persons in each group (including two groups in the adult schools). To ensure a greater uniformity in the conclusions, average pupils were used in these experiments.

The pupils almost always had a hard time gaining an understanding of the proofs of theorems when studying them independently from the textbook. This may be considered normal, since the textbook was not designed for independent study of geometric proofs, but mainly to drill the study material after it has been presented by the teacher in a form intelligible to the pupil. Introduction of the pupils' independent study of the proofs of geometric theorems from the textbook, however, was only one of the investigational devices, and it permitted us to see more distinctly those difficulties that arise in pupils of the sixth and seventh grades when they study proofs of geometric theorems.

Besides the observations and experiments indicated above, in one of the schools we held systematic consultations with the pupils, who came to us for explanations of parts in the textbook they found difficult.

During the investigation it became necessary to define objective criteria for understanding, since it was not always possible to rely on the pupils' subjective statements in this respect. Pedagogical experiences indicate that pupils, stating their noncomprehension of some material, may understand all the basic details correctly, but may not understand a single, sometimes even unimportant, detail. And, on the other hand, there are times when pupils who say "I know" are very far from correctly understanding this material, as one learns after further checking.

Therefore, to determine the degree of the pupils' comprehension of the material given him in our experiment, we had to turn to more definite criteria. Such criteria were:

1) the pupils' ability to present the material they read, particularly the proofs of given theorems (in their own words);

2) the completeness of their answers to the experimenter's test questions; these questions related to the elucidation

of both the fundamental principle or logical scheme of the proof and its separate parts;

3) their use of acquired knowledge in new conditions, particularly in proving analogous theorems with modified data, or in proving new, more complex theorems, etc.

Further investigation confirmed the necessity of such a differentiated approach to evaluation of the pupils' comprehension of geometric proofs. On these criteria of understanding we based the analysis of our experimental materials.

The main cause of sixth graders' difficulty in understanding geometric material is the abstract nature of the proof of theorems. An analysis of the difficulties encountered here has permitted us to make this general premise concrete and to outline some important conditions that assure the pupils' best understanding of proofs of geometric theorems. Together with this, the observations and experiments made it possible to establish various levels of comprehension in the light of Pavlov's teaching.

The geometry textbook by Kiselev [1] that has been adopted in the schools undoubtedly has many merits. Among them are precise presentation of the material, accuracy of mathematical formulations, and maximum brevity of presentation. It should be remarked, however, that Kiselev's textbook takes little account of age peculiarities in the mental development of the pupils. It is uniformly designed for all secondary-school grades, although the level of the mathematical development of pupils who have just begun the geometry course and of those who have been studying it for several years is of course not identical. Therefore what is a merit for an upper-grade textbook becomes a defect for a textbook for beginners. In particular, the previously mentioned "maximum brevity" of presentation of the study material, while presenting no difficulties for pupils of the upper grades, makes the textbook difficult for sixth graders to understand.

The practical significance of our work is to direct the attention of teachers and authors of textbooks to the psychological difficulties that arise here and the possible ways to eliminate them. We do this by analyzing the particularities of pupils' comprehension of the proofs of geometric theorems.

## Two Conditions Necessary for Pupils' Comprehension of Geometric Proofs

From the formal logical point of view, comprehension of mathematical study material should begin when a new proposition comes forth from the propositions already known to the pupil. A conclusion is possible only when premises that are known and understood are given. In fact, however, this is not always so. The pupils can know some propositions separately and at the same time fail to see the connection between them.

We proposed to read Section 73, "Signs of Parallelness of Two Lines," from Kiselev's textbook [1] to four sixth-grade pupils (see Figure 1). They had not yet covered this material. At the preceding lesson they studied the theorem of two perpendiculars. Knowledge of this theorem was checked, as well as of the theorem stating that an exterior angle of a triangle is greater than either interior angle not adjacent to it. Thus the pupils had mastered all the material logically necessary for comprehension of Section 73.

After this, pupil K read the following from the textbook:

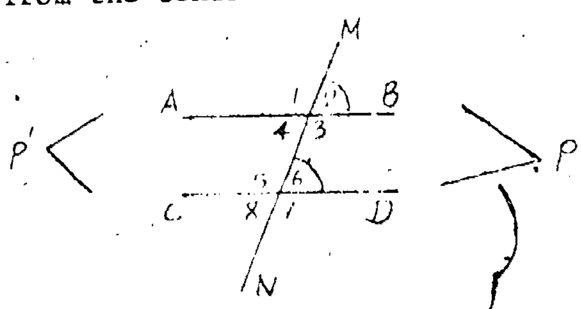


Figure 1

> Let there be given, for example, that the corresponding angles 2 and 6 are equal; prove that then AB || CD.
> Let us assume the contrary, that is, that lines AB and CD are not parallel; then these lines intersect at some point P to the right of MN, or P' to the left of MN. If the intersection is at P, a triangle is formed in which angle 2 is an exterior angle and angle 6 is the interior angle nonadjacent to angle 2; this means that angle 2 should be larger than angle 6 (Section 44), which contradicts the conditions; that is, there can be no intersection of lines AB and CD at any point P to the right of MN. If we assume the intersection to be at point P', then a triangle is formed in which angle 4, equal to angle 2, is an interior angle and angle 6 is the exterior angle nonadjacent to angle 4; then angle 6 should be greater than angle 4 and therefore greater than angle 2, which contradicts the conditions. This means that lines AB and CD cannot intersect at a point to the left of MN either. Therefore, these lines do not intersect at any point; that is, they are parallel [1:42-43].

Experimenter: Well, did you understand the proof?

Pupil K.: I didn't understand anything.

Pupil M: Me either.

Two other pupils said the same thing. The experimenter put all lines of the text that related to P in brackets and asked the pupils to read it again.

E: Did you understand something that time?

Pupil K: I don't understand why angle 2 should be greater than angle 6.

The experimenter made a drawing in which the extensions of AB and CD intersected at P. A triangle was formed.

E: What is angle 2 called with respect to the triangle?

The pupils were silent.

E: But look, it is <u>outside</u> the triangle.

Pupil K: Oh, yes, it's an exterior angle.

E: And what theorem do you know about the exterior angle of a triangle?

The pupils state this theorem from memory and say that now the proof is clear to them. Two of them repeat it correctly.

To another four pupils of a parallel class, who were approximately as good at geometry as the four mentioned above, we gave the same text, but after the words "a triangle is formed in which angle 2 is an exterior angle and angle 6 is the interior angle nonadjacent to angle 2; this means that angle 2 should be larger than angle 6" we inserted "according to the theorem we studied earlier which says that the exterior angle of a triangle is larger than the interior angle nonadjacent to it." All the pupils independently understood the theorem.

This shows that it is not enough to know the proposition required for a conclusion (the subjects all remembered the theorem about the exterior angle of a triangle well), but it is important to recall it in a <u>given</u> situation, under <u>given</u> conditions. The words recalling the theorem immediately made the situation clear where the theorem was to be used, but the simple reference in the text to Section 44, which contains this theorem, was hardly effective.

Pupil S, a sixth-grade girl, was first asked to read the proof of the theorem of the angles with parallel sides (Section 79 in Kiselev's book). Having drawn the two angles, she read in the text:

Having extended one of the sides of angle 2 until it intersected the side of angle 1 that was not parallel to it, we obtain angle 3, equal to angle 1 and to angle 2 (as corresponding angles formed by the transversal of parallel lines); therefore, angle 1 = angle 2 (see Figure 2).

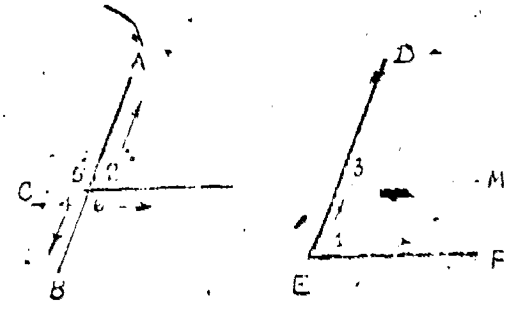
Figure 2

Pupil S: I don't understand why angle 1 = angle 2. How are they corresponding angles? Are 1 and 2 really corresponding angles?

E: Angle 2 is equal to angle 3 as corresponding angles formed by CM bisecting parallels AB and ED. Angle 3 = angle 1 as corresponding angles formed by ED bisecting parallels CM and EF. And if two angles are equal to a third, it means that they are also equal to each other.

The pupil said that now she understood the theorem, and she repeated its proof.

Pupil L, another sixth-grade girl, independently read the same theorem twice, then said that she did not understand the proof. Then the experimenter gave this explanation: "The proof is reduced to the fact that we first prove that angle 2 equals angle 3; then we prove angle 3 equal to angle 1." This remark was enough for the pupil to understand the proof.

When this theorem was given to several pupils of the Moscow Regional Secondary School for Resident and Corresponding Adults who had not yet covered this material, one of them, Mr. I, asserted categorically that he did not understand the proof. Here is his reasoning on this theorem:

Mr. I: What kind of "corresponding" angles are we talking about here? In parentheses it says "as corresponding angles formed by the transversal of parallel lines," "corresponding angles" [he emphasized the s of this word], and before this it says "angle 3, equal to angle 1 and to angle 2," so they should have written "As a corresponding angle" [about angle 3] or "to corresponding angles" [about angles 1 and 2] , but it just says "as corresponding angles."

E: But do you understand the course of the proof?

I: No, I don't. And it's hard to understand when they just casually mention something about corresponding angles.

The experimenter felt that Mr. I just did not want to understand the theorem, to penetrate into it, because of the feeling of dissatisfaction he got from such a presentation of this proof. When the experimenter stressed that the words "corresponding angles with parallels" related to the pairs of angles 2 and 3 and angles 3 and 1, Mr. I said: "But it doesn't say anything about that here. I can't guess what's not written. Some kind of hint is needed to explain the situation."

E: Angle 2 = angle 3. Understood?

I: Of course. As corresponding angles with parallel lines.

E: Angle 3 = angle 1. Yes?

I: Right.

E: So, what conclusion can you make?

I: If angle 2 is equal to angle 3 and angle 3 is equal to angle 1, that means angle 2 is equal to angle 1.

E: Well, great! You've understood it all. Isn't that so?

I: Now, yes, but it's very vague in this book.

In repeating the proof of this theorem, Mr. I first pointed out: "Three angles are equal: angle 2 to angle 3 and angle 3 to angle 1. Two angles are equal to a third, that is, they are equal to each other too." Thus he summed up the proof, stating its scheme. This helped him understand it and reproduce precisely the rest of the proof according to the textbook.

We let a group from School 187 examine for themselves the proof of the theorem that two lines are parallel: if the interior angles on the same side of the transversal add up to 2d (see Figure 3).

In Kiselev's textbook it is presented thus:

> Let it also be given that angle 4 + angle 5 = 2d. Then we should conclude that angle 4 = angle 6, since angle 6 added to angle 5 also equals 2d. But if angle 4 = angle 6, the lines cannot intersect, since otherwise angles 4 and 6 could not be equal (One would be an exterior angle and the other would be the interior angle nonadjacent to it) [1: 43].

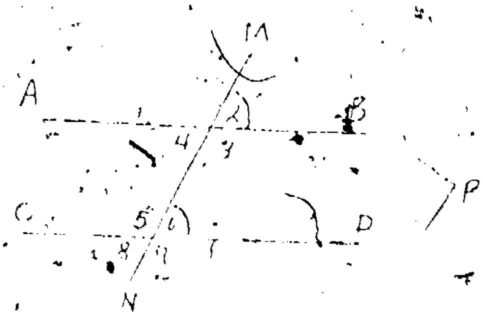


Figure 3

The pupils said they did not understand the proof. Then the experimenter read the text phrase by phrase with the pupils. It appeared that many elements on which the theorem was based were not understood by the pupils. Especially unclear for them was the statement that "the lines cannot intersect since otherwise angles 4 and 6 could not be equal." The pupils could not grasp the conglomeration of conclusions and inferences, and said so frankly. "What is this 'otherwise'? Why couldn't angles 4 and 6 be equal?" they asked.

Then the experimenter asked them to read the following variant of the exposition of the proof:

> Angle 4 + angle 5 = 2d. Let us look at angles 5 and 6. They also add up to 2d, since they are adjacent angles. And if angles 4 and 5 = 2d and angles 5 and 6 = 2d, what can be said of angles 4 and 6? They will be equal. Let us note this: angle 4 = angle 6. If the extensions of the lines were to intersect, a triangle would be formed and consequently angle 6 would be an exterior angle and angle 4 would be an interior angle. An exterior angle is always greater than the interior angle nonadjacent to it (you learned a theorem about this). But we have them equal, therefore our proposition is wrong: the lines cannot intersect, and this means that they are parallel.

At first glance this explanation is longer and more complex. The supposition is developed in detail here, and there is a conditional form of propositions. This text, nevertheless, was immediately understood by the pupils. Why is this?

In the textbook presentation, the material is given precisely, but too briefly--so briefly that each phrase was a problem for the pupils. They were immediately faced with several problems, and the course of the proof was incomprehensible.

The second variant of the proof, which we offered, was not mathematically different from the first. But in it there is, first, supplementary, guiding phrases of purely psycholgoical value: "let us look at angles 5 and 6," "what can be said about angles 4 and 6," "let us note this," etc. Second, the phrases in it are elaborated more fully, which results in the final conclusion being easily perceived by the pupils like an ordinary conversation. In addition, the fundamental idea of the proof--assuming the contrary--is lightly accented.

Thus in the second variant of the proof the pupils do not face two or three problems, as they do in the first variant, but only one--to understand the idea that the extensions of the lines cannot intersect. This idea is easily grasped by the pupils, and the whole proof is understandable to them.

We can state other similar facts. Pupils often announce their lack of understanding of an exposition in the textbook wherever the author makes a parenthetical reference only to the appropriate section number of the book, instead of repeating one of the previous statements. In the first part of that same theorem of the features of parallelism of lines the text says, ". . . angle 2 is an exterior angle and angle 6 is the interior angle nonadjacent to angle 2; this means that angle 2 should be larger than angle 6 (Section 44) [1: 40]." Here the reference is to the theorem of the exterior angle of a triangle.

The experimenter tested the knowledge of Pupil T, a sixth-grade girl, concerning previous material, particularly this theorem. The girl knew it well. Having read the proof of the theorem of the features of paralleiness of lines, the pupil said that she did not understand it. Then the experimenter requested her to read the proof again.

Pupil T: If the intersection is at P, a triangle is formed . . . .

E: Do you understand this?

Pupil T: Yes . . . [reads the book] ". ...in which angle 2 is an exterior angle and angle 6 is the interior angle nonadjacent to angle 1; this means that angle 2 should be larger than angle 6." There's something here I don't understand.

E: What don't you understand? You know the theorem of the exterior angle of a triangle, don't you?

Pupil T: Oh, yes. I know, I understand.

Later Pupil T read the proof to the end and was able to repeat it will full understanding of the matter.

Three sixth-grade girls were asked to examine independently the part in Kiselev's textbook concerning the theorem that each side of a triangle is smaller than the sum of the other two sides (See Figure 4). In the textbook its proof is presented thus:

> In triangle ABC let AC be the largest side. Extending side AB, we mark off BD = BC and draw DC. Since triangle BCD is isosceles, angle D = angle DCB; therefore angle D is less than angle DCA and, therefore, in triangle ADC side AC is less than AD (Section 47), i.e., AC < AB + BD. Replacing BD by BC, we get AC < AB + BC [1: Section 50].

The pupils understood the beginning of the proof (the first three sentences). It was clear to them that angle D = angle DCB. But they were confounded by the sentence: "Therefore angle D is less than angle DCA."

Pupil K: Why is angle D less than angle DCA?

Pupil S: But don't you see? Angle DCA is big and angle D is small.

Pupil P: Of course. Angle ACD is obtuse and angle D is acute.

It is clear that the link omitted by the author--the note that the greater side lies opposite the greater angle, mutely mentioned in the book by the reference to Section 47--did not come to the pupils' minds, and they left the logical path of the proof, crossing over to direct judgment from the drawing.

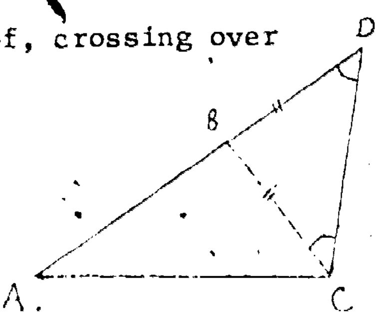

Figure 4

Also, the pupils were completely baffled by the words "and, therefore, in triangle ADC side AC is less than AD." Pupil K said this about these words: "Why 'therefore'? If it is simply obvious that AC is less than AD, then why prove it?" Evidently the pupil understood that a logical proof is required, but, in view of the break of the logical chain of deductions due to the omission of one element of the argument, the proof was unintelligible to her. Meanwhile, at the beginning of our conversation the pupils reviewed several of the preceding theorems, including the theorem of the greater side being opposite the greater angle in a triangle to which the textbook made reference. All the pupils knew this theorem and not only remembered its text, but also its proof. At the same time none of them thought to use it to prove the new theorem, so they did not understand the latter. They were perplexed by the formula AC < AB + BD. Also they did not understand why BD was replaced by BC. In a word, the theorem was not understood at all.

To escape from this situation the experimenter directed the pupils' attention to the parenthetical mention of Section 47. The girls read this section, said they knew it, and one of them, Pupil K, rereading the theorem, said that now she was beginning to understand the proof of the theorem.

It is interesting that the previously unintelligible phrase "angle D is less than angle DCA" was now correctly understood by the pupils. When the experimenter asked them why angle D was less than angle DCA, Pupil K answered: "Because it is equal to angle BCD, and angle BCD is less than angle ACD."

We observed the same thing in the experiments with adult subjects--the pupils of the resident and correspondence school. Thus, Pupil R, speaking of the proof of this theorem, said "Something here is so short that I can't grasp the main thing. Why should side AC be less than AD in triangle ADC? I don't understand anything." And she began repeatedly to reread the proof. Finally the pupil noticed the parenthetical reference to Section 47, looked at it, reread the theorem that was causing her trouble, said that now she understood the proof, and presented it independently.

E: Why did you look at Section 47? You already know this theorem, and it's clear that it's mentioned here.

Pupil R: I should have looked at Section 47 long ago. I would have proved it for you immediately.

E: But you remember this theorem, don't you?

Pupil R: Yes, but it's not directly mentioned here, and it's hard to remember when you are following an exposition.

It is quite obvious that in the cases mentioned above, the nonunderstanding arose as a result of the absence of a connection by the pupils between the separate mathematical statements referred to in the textbook. Usually the pupils remembered the theorem on which the proof was based, but because of excessive brevity of the exposition in the textbook, they did not find the link between the new and the old material, and consequently did not grasp the logical course of the proof of the new theorem.

It is fitting to recall the words of Pavlov about the fact that only thanks to a connection or association "a system, an organization is formed [3:47]." The excessive brevity of the exposition of proofs in Kiselev's geometry textbook hampered the utilization of necessary associations and added a new problem to the pupils' task, that of deciphering the book's references, which distracted them from the main problem--to follow the logical chain of references and conclusions in the proof of a theorem.

The following facts show just how difficult Kiselev's geometry textbook, because of its extreme laconicism, is for the independent comprehension of sixth grade pupils. Of the 12 pupils who independently studied the proof of the theorems given above according to this textbook (Sections 50, 73 and 79), only one pupil independently understood the proof of the first theorem; two, the proof of the second theorem; and four, the proof of the third theorem. For all the other pupils the proofs of these theorems were unintelligible without the experimenter's supplementary explanations; he had to reconstruct the omitted or insufficiently elucidated links of the proofs.

Omissions of intermediate links in reasoning have a strongly negative effect on the comprehension of proofs of geometric theorems, primarily on the comprehension of their logical structure. The obstacles that arise here in connection with restoration of the omitted arguments divert the pupils' thought from the main problem.

Pedagogical experience constantly shows that it is easier for the learner to solve one complex problem than several smaller ones piled atop each other. Because their attention is distracted by secondary problems, the pupils often do not grasp the main course of the idea and as a result cannot understand the material explained to them.

An interesting fact may be established here regarding the work of the memory. As already noted above, in all these cases the pupils had mastered the material needed for comprehension of a new idea. And still they did not understand, because they did not think to use earlier-learned theorems to prove new theorems. Psychologically speaking, the memory did not supply the needed material from past experience, mainly because the consciousness was entirely concentrated on the new material. This resulted in lack of understanding. There is a special difficulty here to understanding complex abstract material. There is reason for the pupils in this situation to say, "It's hard to remember when you are following an exposition." Only after clearly understanding the basic problem did the pupils begin to recall earlier material. Without this, they did not think of it.

From this a conclusion suggests itself: in pupil's thinking there is a characteristic tendency toward one-problemness, toward elucidating the basic problem with the greatest simplification of secondary problems whenever they are trying to understand new material. And this is completely natural, since if the system of basic connections is insufficiently elaborated, any additional irritations caused by mutual induction of the neural processes begins to slow down the main wakening centers. The <u>negative induction</u> of the neural processes in the cortex that then arise is the brake that hinders the pupils' comprehension of complex ("multiproblem") material.

Thus, one of the most important conditions for understanding study material is the precise isolation of its main idea, particularly the basic line of the proof, which should be examined in detail in all its intermediate elements, but without distracting particulars and secondary problems.[1]

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[1]The remarkable lectures by the greatest methodologists and mathematicians (Professors Bogomolov, Koyalovich, Kavun, whom the author was personally able to hear) were distinguished, among other reasons, by the fact that in them the listeners were constantly reminded of even the trifles that were necessary to understand the main point. Thanks to this, the listeners' minds, unburdened by numerous problems, went full speed along the road to elucidation of the main problem, as shown by the lector.

To explain abstract material such as the proof of geometric theorems understandably to sixth-grade pupils means to eliminate any vagueness in the details, leaving for solution only the one basic problem on which the listener concentrates. This is the first conclusion we make as a result of an analysis of our experimental material.

Further analysis of these materials showed that along with a detailed explanation of the basic line of proofs of geometric theorems (especially in examining the more complex ones) the pupils must show the scheme of the proof, turning their attention to the basic courses of thought, that is, to the connection between the main elements of the proof without analyzing them here in detail.

We have already cited the case of Pupil L, who for a long time did not understand the proof of the theorem of the angles with parallel sides. She understood it as soon as the experimenter gave her a scheme of the proof. We encountered other similar occurrences. We cite some of them below.

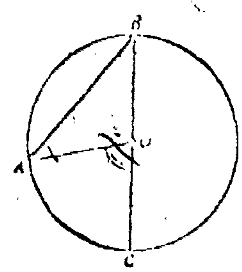

Figure 5

For a long time a group of pupils did not understand the theorem of the inscribed angle (Section 124 in Kiselev's book), mainly because of the great many data about various angles and figures given in the text. Then the experimenter decided to map out the main idea of the proof of the theorem like this: "Angle B is twice as small as angle AOC, the external angle, and angle AOC is measured by arc AC. Therefore angle B is measured by half of arc AC" (see Figure 5). After this the pupils easily understood the proof given in Kiselev's textbook.

Another group of pupils from School 187, during one of the consultations held by the experimenter, asked him to explain the theorem of the relationship between arcs, chords, and the distances of the chords from the center. The theorem reads:

> If two arcs, smaller than half a circle, are not equal, the larger of them is cut off by the larger chord, and of both chords the larger one is closer to the center [1: Section 109].

The proof was explained in class by the teacher, but still it remained unclear to these pupils. They understood the first part of the proof, but the second part seemed very complex and difficult to them. The experimenter's questions about each element of the proof disclosed that the pupils understood the constituent elements but could not unite them.

"I understand, but not quite," one girl defined her understanding. Then the experimenter gave this explanation:

> We must prove that OL is greater than OF. OL is greater than OM, and OM is greater than OF. Why? Look at right triangle OMF. In it OM is the hypotenuse and OF is a leg. The hypotenuse is always greater than a leg (Figure 6).

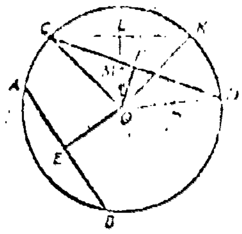

Figure 6

Essentially this was almost the same as what was said in the textbook, perhaps merely stated more schematically. The pupils announced that they now understood the proof given in the textbook and then presented it correctly.

This kind of schematic analysis is especially valuable for understanding the proofs of more complex theorems. Doing the experimenter's assignment, a group of eighth-grade girls first became acquainted with the proof of the Pythagorean Theorem in the textbook (Section 257). The pupils understood the separate elements of the reasoning given them, but the proof as a whole remained unclear for them--they could not summarize it. In this connection the experimenter decided to explain the scheme of the proof to them in this way:

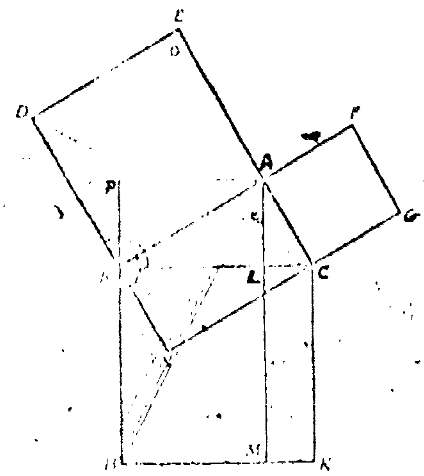

Figure 7

Experimenter: We must understand how the proof is constructed. I'll now summarize it very briefly for you, without even using letter designations. Look, this triangle [indicates triangle DBC in the textbook] (See Figure 7) is equal to half of this square [points out ABDE].

Pupil S: Why?

E: Don't ask now, I'll get to that .... And this triangle [indicates ABH] is equal to half of this rectangle [indicates BLMH]. But the triangles are equal; therefore this square [indicates ABDE] is equal to this rectangle [indicates BLMH]. We reason the same way with respect to the right square and rectangle. Now read the proof of the theorem from the textbook.

Having read it, the pupils said that they now understood the proof.

E: Do I have to explain why the area of triangle DCB is equal to half the area of square ABDE?

Pupil K: No, that is clear: triangle DCB has a base that is common to square ABDE and an altitude CN that is equal to the altitude AB of this square.

Thus the pupils completely comprehended the proof of the theorem immediately after the experimenter had given them a brief scheme of it, not even using literal designations of the figures and angles.[2]

The following data from our experiments speak for the value of this type of schematic exposition of geometric theorems. We asked 10 average pupils to analyze independently the proof of the theorem about the property of an angle bisector in a triangle [1: Section 109]. We asked 10 others to examine the theorem about the relationships between arcs, chords, and the distances of chords from the center [1: Section 109]. Of the 10 pupils, only 3 could prove the first theorem, and only 4 could prove the second. After a schematic exposition of the proof of these theorems by the experimenter, the pupils correctly analyzed their proofs in the textbook. Now eight pupils proved the first theorem and all proved the second.

At first glance, the data seem to be in sharp contradition to the earlier conclusion about the necessity of the detailed exposition of geometric proofs without any omissions of their intermediate elements. Really, however, there is no contradiction: detailed proof of geometric theorems which supposes the formation of connections

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[2]A similar method of "schematic" exposition of the proof of geometric theorems was applied by us only as one of the experimental devices of investigation and not as a general methodical device recommended to the teacher. From this point of view, the device of course needs a whole array of modifications.

between separate elements of a proof should be accompanied by generalization of the fundamental points of the proof, that is, by organization of a system of fundamental connections. This is immediately achieved in a schematic exposition of geometric proofs.

Sometimes such a schematic presentation of geometric proofs should be the concluding point, but more often it should be one of the initial ones. In particular, this would be true in the experimenter's explanation of the Pythagorean theorem cited above. In a schematic exposition of the proof of the Pythagorean theorem, Pupil S raised a question at the experimenter's first statement: "Why?" she asked. The experimenter answered, "Don't ask now, I'll get to that." Thus he simply made the pupil temporarily forget about substantiating the individual propositions in order to explain the structure of the proof as a whole. This helped the pupil understand the essence of the proof of the theorem.

Such an example is often used by the pupils themselves if they do not immediately succeed in understanding the proof of some theorem. The pupils, however, who do not think to use this device and at the same time who cannot go deeply enough into the material so as to analyze it independently, finally become confused and falsely convinced of their inability to solve a given problem. In such cases the teacher should come to the pupils' assistance and, in order to advance independent development of their thought, should advise them to make a scheme of the proof, omitting details.

Of course the nonunderstanding pupil may be forced to continue reading the material carefully, recalling what he should know, examining details, etc. Such demands may be useful sometimes, training the pupils' ability to overcome difficulties independently. Nevertheless, this route of intentional difficulties does not solve the question of rational devices for explaining incomprehensible material. The explanation, once it is given, should always be understandable to the pupils; otherwise it is useless. More than enough difficulties in mastering the study material always arise, and there is no need to create them artificially where they can be entirely eliminated.

Summarizing what has been said, it may be noted that among the factors promoting comprehension of the study material, that is, promoting establishment of the connection among its individual elements and isolation of the main group of them, these two should be distinguished above all:

1) a detailed presentation of the material with elimination of any secondary difficulties,

2) a brief schematic presentation of the material, permitting one to view the proof of the theorem as a whole, to understand the main idea in it, and to imagine its structure schematically.

It must be remembered here that the brevity and amount of detail of the presentation can never entirely be a matter of the purely quantitative side of the verbal formulation of the explanations given. The brevity we refer to here consists not only in the use of a small number of words, but is included in a special construction of the explanation of the material which reveals its scheme and transmits the essence of the matter in few words. Detail in the explanation does not signify its verbosity, but signifies a construction of the proof such that secondary difficulties hindering perception of the main idea are eliminated.

## Levels of Pupils' Comprehension of Geometric Proofs

In observing the indicated conditions for understanding--a detailed exposition of the study material and its subsequent schematic generalization--the pupils as a rule quickly grasp the proofs of theorems presented to them and correctly use them in further study of geometry. Here, however, such a level (degree or depth) of understanding is not always immediately achieved, and not by all pupils. One may often encounter more elementary levels of understanding. What are these levels, and what are their psychological characteristics?

To answer these questions, we return to an analysis of concrete materials.

Pupil V (a sixth-grade girl), proving the theorem of angles with correspondingly parallel sides, made a correct construction (see Figure 2, above), then made a mistake. Instead of saying, "We obtained angle 3 equal to angle 1 and angle 2 (as corresponding angles formed by the transversal of parallel lines); therefore angle 1 is equal to angle 2," she said, "Angle 2 is equal to angle 3 and angle 2 is equal to angle 1, therefore angle 1 is equal to angle 2." From what followed it was ascertained that this error was not a simple slip of the tongue, but the result of the pupil's not knowing what to equate; not understanding the logic of the proof, she could not summarize it properly.

However, individual links in the chain of reasoning were clear to her.

Experimenter: Why is angle 2 equal to angle 3?

Pupil V: As corresponding angles formed by parallel lines and a secant.

E: What can be said about angles 3 and 1?

P: They are equal.

E: Why?

P: Also as corresponding angles.

E: What do we have, then?

P: Angle 2 is equal to angle 3, and angle 3 is equal to angle 1. That means angle 2 is equal to angle 1.

Thus we see that the pupil's comprehension of the proof of the theorem existed, but it was still not profound enough for a correct presentation of it.

Further, the experimenter asked the pupil to prove that same theorem, herself, not for acute angles as had been done first, but for obtuse angles, with their sides directed to the left instead of to the right. Here the same method of proof should have been used. The pupil, however, extended the side of one obtuse angle until it intersected the side of the other angle, obtained an acute angle similar to the one in the preceding theorem, thought a moment, was about to construct a second acute angle, analogous to the one in the preceding drawing, but then, understanding the uselessness of such a construction, announced that she did not know how to prove it. It is obvious that here we have an incorrect transfer of the scheme of the drawing of the first proof to the second, without taking account of the conditions of the new problem. The pupil finally succeeded in proving this theorem only after the experimenter had erased the unnecessary extensions of the lines and directed her attention to the similarity of the proof of this theorem with both acute and obtuse angles.

Pupil K, a sixth-grade girl, was asked to read Section 38 in the textbook. Here the theorems that 1) in an isosceles triangle the bisector of the vertex is both the median and the altitude, and 2) in an isosceles triangle the base angles are equal, were proved (Figure 8). As is known, both theorems are proved by folding the drawing along the bisector BD.

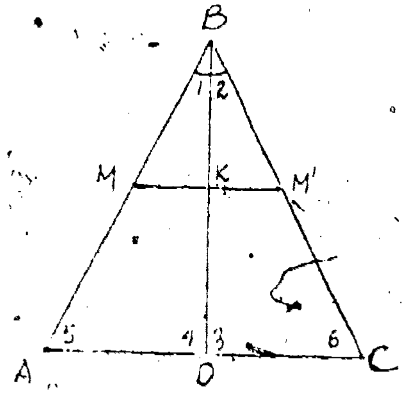


Figure 8

Having read the proof given in Kiselev's text twice, the pupil, when asked if she understood it all, answered that she understood the proof, but it was not clear to her why, as a result of the equality of angles 1 and 2 (at the vertex), side AB falls onto side BC (a lateral side).

The pupil further required an additional explanation of why, as a result of the equality of angles 3 and 4, BD is the perpendicular, that is, the altitude. After the experimenter's explanations the pupil announced that she understood everything in the theorem. But she still could not summarize its proof. The explanation had to be repeated before she could do this.

Another girl, Pupil P, also said that she understood all, but became confused in transmitting the proof. She said only that AB falls along BC and point A coincides with point C. And immediately, without proof, she concluded that angle 5 was equal to angle 6 and BD was the altitude and the median. However, to the question why AB falls along BC, she answered correctly: "As a result of the equality of angles 1 and 2 (at the vertex)." She also answered correctly the question why BD is the altitude ("As a result of the equality of angles 3 and 4."). It can be seen from this that the pupil, unable to transmit a connected proof of the theorem, correctly transmitted it by individual parts in the form of answers to questions.

Pupil Sh, reading the proof of this theorem, could not transmit it. When the experimenter asked her why AB falls along BC, why point A coincides with C, etc., she could not answer. Then the experimenter read her the proof by parts. Thus the proof was broken into a series of individual links: 1) Let us fold the drawing. 2) Because angles 1 and 2 are equal, AB will fall along BC. 3) Because AB and BC are equal, point A will fall on point C. 4) AD falls along BC and matches it. 5) Angle 4 matches angle 3. 6) Angle 5 matches angle 6.

After the proof was decomposed like this, the pupil announced that she understood it, although she could not summarize it independently. What she experienced subjectively as "understood," was really (objectively) related not to the whole proof, but only to individual parts of it, and as a result the pupil was unable to present the entire proof sequentially as a unit.

Comprehension of individual parts of the proof when it is impossible to present it as a whole is one of the initial stages of comprehending the study material. In this case the individual elements of the proof are perceived correctly but still without the obligatory interconnection, and as a result the comprehension has a scrappy, fragmentary character. This is the stage of formation of individual associations (or individual groups of associations), but it lacks their sequential combination to form one common chain and, moreover, a generalization of their system.

But many pupils with whom we conducted the work indicated another level of understanding.

Pupil S, having read the proof of the theorem of angles with parallel sides (Section 79) three times and correctly reproducing it under guidance of the experimenter, announced "I understand it all. The theorem is easy." The pupil, however, was hard pressed when the experimenter asked her to prove this theorem for obtuse angles facing in the other direction. Her attention had to be directed to the fact that here the same principle of proof as in the first case was to be followed, after which the pupil correctly proved this variant of the theorem.

Independent proof of the same theorem about angles facing in opposite directions from the vertex (Figure 9) was totally impossible for Pupil S. She began by extending AB to its intersection with DE and then began to prove the equality of the newly formed angle with the given one, but could go no further. She could not understand that the principle of proof here was the same as in the first two cases with the only difference being that here it was necessary to construct a supplementary vertical angle (Angle 3).

It was clear from further conversation that the pupil, although she repeated all parts of the proof in proper sequence, still had difficulty ascertaining its main idea: to compare two angles with a third and, if it turns out that the first two angles were each equal to the third, then therefore they are equal to each other.

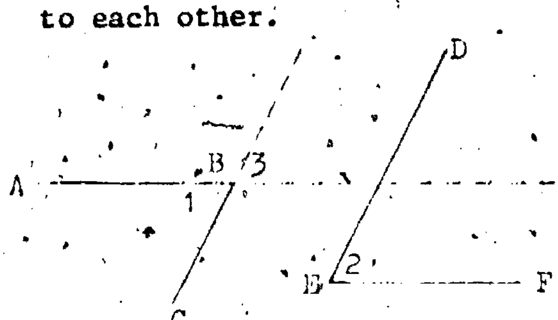

Figure 9

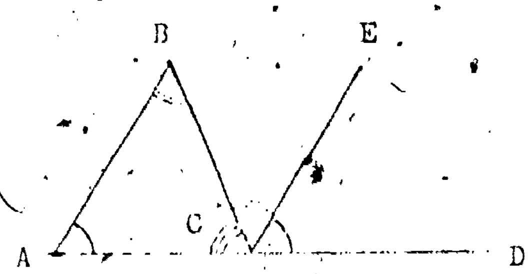

Figure 10

Pupil T read the theorem of the sum of the angles of a triangle and its proof (Section 81) twice. To the experimenter's question "Is the theorem clear?" she answered yes, correctly giving the proof of the theorem. All seemed to go well—complete comprehension. However, the experimenter's next question, "Why is angle A + angle B + angle C equal to 2d?" (Figure 10) confused the pupil; she was silent. Finally it was learned that the pupil did not understand that instead of angle A one could take angle ECD, and instead of angle B, angle BCE, since they were equal to each other. And here it appeared that the pupil, having correctly stated the proof according to the textbook, still did not isolate the basic principle, the main idea of the proof.

From this follows rapid forgetting of the proof and inability of transfer or generalization, the impossibility of constructing an analogous proof using the same idea on different material.

Pupil Kh, who correctly repeated the proof of the theorem about an isosceles triangle, was asked to repeat the proof a day later. She could not do it. She indicated only that the drawing had to be folded; she could not answer the experimenter's question, "But why does AB go along BC and angle A coincide with angle C?"

In all these examples we are dealing with a special level of understanding by which the pupils correctly state the proof of theorems but without isolating its main idea, or basic principle. Because of such understanding the pupils can correctly explain, for example, why one angle is equal or not equal to another, but they cannot reveal the logic of the proof or answer the question of why the proof is conducted in just this way and not some other way. Without isolating the scheme of a geometric proof, that is, the system of basic connections, the pupils naturally cannot use it (tranfer it) when proving analogous theorems.

Unlike the level of "fragmentary" understanding described above, by which the pupils grasped only individual elements of a proof without sequentially linking them with each other, this level of understanding may be called "<u>logically ungeneralized</u>." Its basic characteristic is the immediately sequential perception of the material given the pupils, without, however, isolation of the logical scheme. More precisely, it is without a generalization of the system of its main links.

In other words, at a given level of understanding we are dealing not with separate, uncoordinated associations and not with separate groups of them, but with a sequential "chain" or an elementary system of associations for which there has been no formation of their generalized systems.

In the study of geometry such logically ungeneralized understanding is quite widespread among the pupils, and they may sometimes be encountered even in the best pupils if an assignment surpasses their power. Thus, for instance, at the end of the school year the experimenter asked a pupil, an exceptional seventh-grade girl, to analyze independently one of the first theorems from the eighth-grade course (on the incommensurability of the diagonal with the side of a square, Section 149), a proof based on data known to seventh-grade graduates. Having read the proof in the textbook three times, the pupil correctly analyzed it, accurately stating the whole course of reasoning. She did not, however, immediately grasp the main idea of the proof. Supplementary leading questions and hints were needed for her to explain it.

It is an entirely different matter regarding the deeper understanding of the study material when the pupils not only grasp all elements of the proof in their sequential interconnection, but isolate the main principle of the proof, its logical scheme.

Two identically successful groups of pupils were asked to read different variants of the proof of the theorem that the exterior angle of a triangle is greater than each interior angle nonadjacent to the exterior one (as always, this was done before the theorem had been explained in class).

The first group was to read the proof given in Kiselev's text. There it begins thus:

> Let us prove that exterior angle BCD of triangle ABC (see Figure 11) is greater than each of the interior angles A and B which are nonadjacent to it. Through the midpoint of E of side BC we produce median AE and on its extension mark off segment EF = AE. Point F, obviously, will lie within angle BCD. We join F with C by a straight line . . . [1: Section 44].

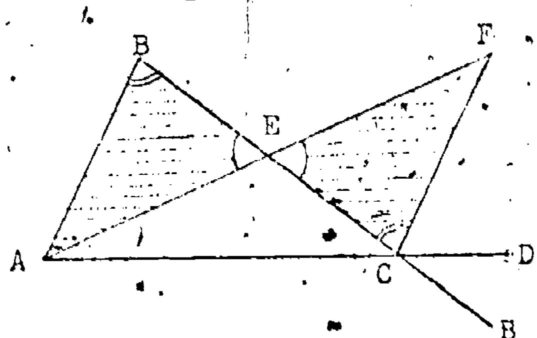


Figure 11

Thus, in the presentation a series of operations is given, some constructions are made, and the pupil is completely left in the dark as to why, for what purpose, all this is done. Following these unclear operations is the proof of the theorem.

The pupils read the proof of this theorem in the textbook three times and still were unable to analyze it independently. Of five pupils, only one could correctly reproduce this proof, but even she could not answer the experimenter's questions "Why did you draw the median? For what purpose was triangle EFC constructed?"

The experimenter gave the other group of pupils of the same grade the following variant of the exposition of the proof of this theorem:

> We must prove that the exterior angle BCD of triangle ABC is greater than interior angles A and B. We can prove that angle BCD is greater than angle B if we can construct angle C which would be equal to angle B and simultaneously be only a part of angle BCD. For this we construct an auxiliary triangle EFC whose angle C equals angle B. How can we do this? Through mid-point E of side BC we draw median AE and mark off EF = AE on its extension. We join F with C by a straight line . . .

The rest of the proof was presented as in Kiselev's book.

After reading the proof twice, the pupils correctly reproduced it. Then the experimenter made the problem more complex asking the pupils to prove that angle BCD is greater than angle A. At first, this presented difficulties for them. But scarcely had the experimenter extended side BC on the drawing when the pupils announced that they understood, and proved the required proposition correctly. Here it is quite clear that the pupils understood not only the sequential connection between individual elements of the proof but also its <u>logical structure</u>, the very <u>idea</u> or <u>principle</u> of the proof. This helped them use the scheme of reasoning in another, altered situation.

It is characteristic that when the experimenter asked Mr. G of one of the adult schools, to analyze the same theorem independently, only after lengthy work could he decompose the logical scheme of the proof from the textbook. In answer to the experimenter's question of how he analyzed it, he said:

> I concentrated on the fact that an angle equal to angle B had to be constructed at C. For this we construct triangle ECF, congruent to triangle ABE, and we prove that they are congruent and therefore that angle B is equal to angle ECF. Angle C is a part of angle BCD.

Then he independently proved that angle BCD is greater than angle A.

The pupil's arguments prove that he penetrated the essence of the theorem, discovered the scheme of the proof, and understood how it was constructed. For the aim and value of each operation to be clear to him, he had to do a great deal of work in deciphering the text.

Pupil K was asked to read this theorem in Kiselev's textbook:

> In a convex inscribed quadrangle the sum of the opposite angles is equal to two right angles [1: Section 139].

She also read the uncomplicated proof of this theorem. The pupil, however, spent a long time analyzing this problem, not at all independently. Judging from the questions she asked, her thought was extremely passive.

Pupil I found herself in a different situation when she was given the same theorem, at first in the form of a problem: "Can a circle be circumscribed about any quadrangle?" The pupil tried this, using a compass and ruler to draw, and announced: "No."

E: But are there any quadrangles that can be circumscribed by a circle?

I: Yes, probably.

Then the pupil drew a circle, inscribed a quadrangle in it. During this process she turned her attention to the fact that the angles were inscribed and, as such, they are measured by the arcs which they cut off. Then she considered that opposite sides of an inscribed quadrangle cut off arcs whose sum is $360^{\circ}$ and that therefore they measure $360^{\circ} \div 2 = 180^{\circ}$.

When the pupil was then asked to read the proof of this theorem from the textbook, she immediately analyzed it without difficulty.

Thus, in both pupils there was an understanding of the proof of the theorem, but their levels of understanding were quite different. This is shown by the fact that when both pupils were later asked to prove the converse proposition ("If in a convex quadrangle the sum of the opposite angles is equal to 2d, then a circle may be circumscribed about it"), the first pupil could not, whereas the second proved it correctly.

Pupil V, having twice read the theorem about angles with parallel sides, proved it and independently deduced a proof for obtuse angles and for angles with sides facing in opposite directions. Here completion of the different variants of the proof, the ability to use its device in different conditions, speaks for the pupil's profound understanding of the main idea of the proof.

In these last cases we are dealing with a new, higher level of the pupils' understanding of geometric proofs essentially different from the first two levels of understanding--fragmentary and logically ungeneralized--described above. If at the first level of understanding the pupils grasped the separate elements of the proof without their sequential interconnection and attained at the second level the sequential connection of the separate elements without decomposing the logical principle of the construction of the given proof, then at the new, third level of understanding the pupils have completely mastered the fundamental idea of the proof, they have understood the meaning of all the performed operations, and they have penetrated the essence of the material to such an extent that it is now possible for them to use the logic of a given argument for the proof of other similar theorems. On this basis the given level of understanding may be called logically generalized understanding, which reveals the idea of the proof and allows the use of this idea for the proof of other theorems.

From the physiological point of view this highest level of comprehension of the study material is characterized by the appearance not only of sequential "chains of associations," but also by the formation of a generalized and simultaneously sufficiently varied system of them which may be reorganized, depending on the concrete material. In other words, it is a matter of the formation of dynamic stereotypes of a higher level that allow the quick isolation of the basic elements of a proof and their correct use in solving various geometry problems.

The indicated levels of the pupils' comprehension of geometric proofs is substantiated, in the words of Pavlov, by "utilization of cognitions and acquired connections." The more knowledge the pupils have acquired and the more this knowledge has become systematized and generalized, the more profound is their understanding of geometric material. The achievement of this kind of logically generalized understanding of geometric proofs is one of the most important problems in teaching geometry in the school.

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# ELEMENTS OF THE HISTORICAL APPROACH IN TEACHING MATHEMATICS*

I. N. Shevchenko

## Introduction

A fruitful consideration of any question is possible only if the question is stated distinctly, its limits strictly defined, eliminating any arbitrary and vague commentaries. We therefore attempt, first of all, to indicate just what will be elucidated in this article. We shall not be speaking here of the value of the history of mathematics generally, of its significance for mathematics itself, of its usefulness for the scientific worker who is investigating a problem, or the like. Only the teaching of certain material from the history of mathematics in the general-education secondary school will be discussed here. This can be thought of either as a new subject included in the secondary-school curriculum--which is undoubtedly impossible, since the curriculum is already overburdened without it--or as separate material interspersed within the secondary-school mathematics course.

In the last few decades many authors of works on the questions involved in teaching mathematics have been saying that the study of mathematics in the schools should be accompanied by the communication of material on the history of this science. Those who discuss this include educators (in their methods manuals), mathematicians (in books on the history of mathematics), and sometimes authors of school texts. Opinions differ as to how this material should be communicated. A course in the history of mathematics might be taught parallel with the mathematics course; brief, episodic discussions might be conducted on individual curricular topics; lectures or reports on historical topics might be arranged within the cycle of out-of-class work, or finally, an attempt might be made to design the entire teaching program around a historical approach.

Although much has been said and written about this, still not enough attention is being given to the question of the historical

*Published in Proceedings [Izvestiya] of the Academy of Pedagogical Sciences of the RSFSR, 1958, Vol. 92, pp. 199-229. Translated by Joan W. Teller.

approach in mathematics teaching in our schools. In most schools some brief material on the history of mathematics is imparted only incidentally or haphazardly. In the main, historical questions are posed at sessions of the mathematics clubs and often with no relationship to the classroom sessions. Therefore we conclude that questions on the history of mathematics have not yet found a proper reflection in the school's teaching.

The reasons for this phenomenon consist in the following. First, the purpose of teaching the history of mathematics is unclear. This is a paramount reason. It is impossible to think that the history of mathematics can be taught in school (in whatever form) and not to know why it is being done. It is impossible to believe that goals for teaching the history of mathematics are clear to all, that it goes without saying. Depending on why we are doing this are the questions of what we shall impart from history and how it will be done, how much will be included, and when. We believe that if the mathematics teacher was convinced that elements of the history of mathematics are of some use, he would find both the time and the opportunity to do this.

The second reason the history of mathematics is not reflected in the school program is that teachers do not know what forms this work should take. The third reason is that the history of mathematics is not in the curriculum; it is something optional and this work is not controlled. The fourth reason is that the teacher does not know enough history of mathematics, he has not studied it, and he has no taste for this sort of question. Finally, the fifth reason is that there are few books on the history of mathematics.

Should questions in the history of mathematics be presented in school or not? The basic recommendations on this should be found in the courses in pedagogy. Let us turn to them. In a number of books on pedagogy, nothing at all is said on this topic, but in the course in pedagogy edited by Gruzdev [36] these ideas are presented:

> The next requirement for curricula is the <u>introduction of the historical approach</u>. A genuine scholarly knowledge of a subject is possible only with the application of a historical approach to the study of it.
>
> Not only does a historical approach to the study of phenomena deepen the understanding of them, not only does it form a dialectical-materialist world-view, but it heightens interest in science, it fosters respect for the great men of science, it helps one to perceive the reactionary role of religion in its struggle against the vanguard of science and

scientist, it helps one to understand the grandeur of the opportunities that are opening up for the application of science in the land of socialism. The historical approach plays an enormous role primarily with respect to the socio-political disciplines. A familiarity with social phenomena in their historical development, a study of the regularity in the development of social relations provides an accurate concept, with a scientific-Marxist foundation, of the future course of social development; and reinforces theoretically our movement forward, towards communism.*

Then the author devotes a number of lines to questions of the historical approach in teaching language, physics, chemistry, biology. And finally, after all this, he says: "In the algebra course for grade 10, before introducing complex numbers, it is desirable to draw the pupils' attention to the idea of the evolution of the concept of number and to impart brief historical material on this question."*

Here the fact that pedagogy regards the question that interests us as curricular attracts attention. But, unfortunately, it is not stated in the mathematics curricula which questions in the history of mathematics should be studied, where these questions should be placed, to what extent they should be examined, or approximately how much time the teacher should set aside for this. Recommendations on the historical approach in mathematics teaching can be found only in the course instructions for the secondary-school curricula. Here are these recommendations:

> One must draw the pupils' attention to the great cultural and historical value of mathematics, to its role in the study of other subjects (physics, chemistry, geography, astronomy, drafting, etc.), its applications in technology and in the practical work of building socialism. It is important that attention be given to imparting material on the history of mathematics and on the life and activity of prominent mathematicians (Euclid, Archimedes, Descartes, Euler, Gauss, Lobachevskii, and others) [67; 9].

Thus, the course instructions do not instruct the teacher to a sufficient degree in presenting historical material to pupils.

Nevertheless, we should take into consideration the recommendation of pedagogy, where it is stated directly that the historical approach is a curricular requirement. Therefore, if there are no questions of

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*Page not given in the original (Ed.).

history in the official curriculum, then apparently, they can be included by the teacher in his working curriculum.

The above quotation from a course in pedagogy does not discuss concretely enough the way in which questions of history are presented in mathematics lessons. The quotation states that the historical approach to the study of a subject provides a genuine scholarly knowledge of this subject. Perhaps this is true, but it is difficult to execute this sort of recommendation with respect to mathematics. To implement systematically a historical approach in the process of the study of mathematics, it is necessary to have a reliable textbook of this nature. At present there is no such textbook.

1. Questions of the Historical Approach in Methodological Literature

Now it interesting to observe how the authors of some of the works in mathematics methodology treat the historical approach.

In Methods of Teaching Mathematics by Professor V. M. Bradis, we read:

> Experience in teaching tells us with complete definiteness that the quality of mastery of mathematical material makes essential gains if each new concept, each new proposition is introduced so that a relationship of it with things that are already familiar to the pupils is apparent and so that the expediency of studying it is clear. What is the most convincing for pupils is the justification of each new concept and proposition through considerations concerning a practical activity that comes as close as possible to it . . . The genetic character of presentation yields especially good results in the lower grades, but its use should be recommended in work in the upper grades of school, in every way possible . . .
>
> The genetic character of presentation is contrasted with the axiomatic, in which science is presented in its more complete form. It is easiest to insure the genetic character of presentation on the basis of the history of a given branch of science, and therefore the historical elements in matters of teaching is of enormous value. It is said, not without reason, that a full understanding of any theoretical question is attained only when its history becomes clear. How the lesson will benefit if even brief directions on various circumstances related to the history of the question under scrutiny are given during the lesson--directions that can be developed in more detail during sessions of the mathematics clubs. The literature on the history of mathematics is great [17:44-45].

In Methods of Teaching Mathematics in the Seven-year School, edited by S. E. Lyapin, it is stated:

> Although the majority of the achievements by Russian mathematicians are inaccessible to the understanding of pupils in grades 5-7, the pupils should still be acquainted with the biographies of Russian scientists, with an indication of the value of their discoveries for world science. In the upper grades the pupils should be acquainted with the works of Russian mathematicians that are accessible to them.
>
> One can dwell on the following Russian scientists: L. Euler (1707-1783), N. I. Lobachevskii (1793-1856), B. Ya. Bunyakovskii (1804-1889), M. V. Ostrogradskii (1801-1861), P. L. Chebyshev (1821-1894), A. A. Markov (1856-1922), S. V. Kovalevskaya (1850-1891), A. N. Krylov (1863-1945). [53:61].

The book also recommends acquainting pupils with the biographies of contemporary Russian mathematicians: I. M. Vinogradov, S. L. Sobolev, V. I. Smirnov, A. N. Kolmogorov, P. S. Aleksandrov, and others. The author points out literature that gives much interesting information on the activity of the scientists listed above.

In Methods of Algebra by Professor I. I. Chistyakov it is said:

> Since, for the pupils, instruction in a subject is, as it were, an abbreviated reliving of the history of the development of the respective science, which occurs under a specialist's guidance, then it is essential that the leader use the lessons that the history of the subject's evolution provides, eliminating what retarded that evolution, and suggesting methods that assist its understanding and progress. There follows the necessity for a good familiarity with the history of mathematics and, in particular, of elementary algebra [22:6].

In N. M. Beskin's Methods of Geometry there is no special section on the use of the historical approach in teaching mathematics, but the entire first chapter sets forth "the evolution of views on the foundations of geometry." Here the possibility of tracing three periods in the evolution of these views is discussed: the pre-Greek (empirical), the Greek (intuitive-logical), and the contemporary, where geometry is presented as a logical system. In the last (fourth) section of this chapter, the author draws pedagogical conclusions consisting in the fact that:

> These three periods correspond to those three stages through which every pupil passes if he gets far enough in the study of geometry. The first--the pre-Greek period--corresponds to geometry instruction in the lower grades (. . .elements of geometry in the lower grades. . .enter into the arithmetic course). . . The systematic course in geometry, beginning in grade 6, corresponds to the second--the

> Greek—stage. The third—the contemporary—stage is not reflected in the secondary school. This is the step which is ascended only by those pupils who choose mathematics as their specialty. The appropriate questions in the foundations of geometry present good material for the work of the mathematics circle in the higher grades [12].

Thus, although the author does not discuss directly the historical approach in the teaching of geometry, he indicates that in the process of instruction the historical stages are repeated and that the educator can draw some lessons from the history of science.

In the journal, Mathematics in the School there is an article by A. M. Frenkel' (Arzamas), "Elements of the Historical Approach in Mathematics Teaching." In this article we read:

> It is scarcely necessary to prove again the immense educational value of elements of the historical approach in teaching any subject in secondary school. For the Soviet teacher this position has become an indisputable truth. Mathematics is a science in whose creation and development one of the most honorable places belongs to the Russian scientist; a science in which an enormous contribution was made by Soviet mathematicians presents broad opportunities in this respect. And it is entirely proper that the introduction of the historical element in the school mathematics course is envisaged by the course instructions for the working curriculum. The execution of this requirement is obligatory for the Soviet teacher.
>
> The following forms of introduction of the historical approach in mathematics teaching can be indicated:
>
> 1. A historical digression at the lesson, discussions from 2-3 to 8-10 minutes.
> 2. The communication of historical material organically connected with individual questions of theory or with problems.
> 3. Special lessons in the history of mathematics.
> 4. Mathematics circles (a special history-of-mathematics circle can be organized).
> 5. Historico-mathematical evenings.
> 6. The use of a wall newspaper or the organization of a special mathematics newspaper.
> 7. Reading in out-of-school hours.
> 8. Compositions and essays done at home by the students.
> 9. The making of albums and almanacs.
> 10. Work on collecting "folk mathematics."
> 11. Communication by the teacher, or by a pupil who is given advance preparation, at a class meeting.
> 12. Discussions, lectures, and talks by the teacher or by invited scientific workers.
> 13. The viewing of special scientific-historical films, slides, etc.

The first and basic form of introducing the historical approach to the teaching of mathematics is the episodic communication of historical information. The teaching of mathematics as of every other discipline, should be accompanied by historical digressions, comparisons, and historical problems. This information, as a rule, should take up little time and should not lead the pupils far afield from the immediate interests of the topic under study. Sometimes it is useful to give these historical digressions in beginning the coverage of subject matter, sometimes to relate to a definite question of a topic, lesson, or even problem, and sometimes to the conclusion of a lesson. This information can sometimes be limited to a few words, and sometimes the history of a question or biographical information on a mathematician can be elucidated in greater detail [27].

In the journal Mathematics and Physics in Secondary School, there is an article by F. Tsigler, "Elements of the History of Mathematics in Secondary School." In it the following ideas are expressed:

The history of mathematics has an enormous significance in teaching mathematics in secondary school primarily because of its educational value. The history of mathematics clarifies the mathematician's role and place in human practical activity, as a result of which our pupils are trained to approach properly the objects of our mathematical knowledge. But the history of mathematics has an even greater instructional significance. Historical material awakens a love for and an interest in the subject, a striving for scientific creativity, a critical attitude toward facts, and thoughtfulness. Finally, the history of a question is a key to the understanding of its logical essence. What sometimes seems to no purpose, a superfluous detail, acquires logical sense as soon as its historical development has been made clear. Mathematical concepts that seem fixed and paralyzed in the present, come to life in the past and, in their historical dynamics, become closer to the pupils.

But the question arises of how to construct a course in the history of mathematics. As an independent subject? As periodic reports on historical information? So that historical features penetrate the systematic presentation of the course, forming a single unit with it, as it were? I think that the last two of these three possibilities can take place in teaching mathematics in secondary school [75].

The transactions of the scientific-pedagogical conference of teachers of Leningrad, entitled The Ideological Education of Pupils in the Process of Instruction, carry a quite valuable work by Professor I. Ya. Depman, "The Historical Element in Teaching Mathematics in Secondary School [24]." It is true that this work is intended for the teacher and is mainly devoted to clarifying how much history of mathematics is essential for a teacher. Nevertheless, so many interesting and useful ideas are set forth in Depman's work that it is utterly impossible to pass it by.

The author says that the prerevolutionary mathematics teacher had to prove the significance for him of the history of mathematics, which the late Professor V. V. Bobynin did in his time, at the All-Russian conventions. The Soviet mathematics teacher does not have to prove the necessity for knowledge of the history of mathematics.

We shall make two remarks apropos of this.

First, Bobynin, in his talk at the First All-Russian Convention of Mathematics Teachers, proved not that the history of mathematics is necessary for the teacher, but that it should be taught in the secondary educational establishments.

Second, Depman says that the Soviet teacher does not need proof of the necessity of studying the history of mathematics. Alas! The teacher, the pedagogical institutions of higher learning, and the state institutions, on whom the supervision of the pedagogical institutions of higher learning depends, have a hundredfold need for such proof. If everyone understands the necessity of knowing the history of mathematics then what can explain the fact that the history of mathematics is in the background everywhere?

Professor A. I. Marushevich,* in a talk at the session of the Academy of Pedagogical Sciences of the RSFSR in June, 1949, stated the following:

> From the course in physics or chemistry, the pupil will find out about the most outstanding scientists of our and of other lands, and about precisely what they introduced into science. The material of the course itself is such that it is impossible not to talk about these discoveries and about their authors. It is impossible to glean similar information from the mathematics course. In an algebra textbook one might read one line in small type about the Arabian scientist al-Khowarizmi and not learn that this is the great Tajik mathematician, Mohammed, the son of Musa of Khwarezm. In the first part of a geometry text it is solemnly announced in large print that in 1873 the English mathematician Shanks found 707 decimal places for the number $\pi$. It might have been added that he did not succeed in carrying out even this vain enterprise properly: all the places beginning with the 528th are incorrect. On the other hand, one can read about Lobachevskii only in small print at the end of the second part, where the following is said, and I quote: 'In the first half of the 19th century, the

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*Source not given in the original (Ed.).

Russian mathematician and professor of the University of Kazan, Nikolai Ivanovich Lobachevksii, the Hungarian mathematician Janos Bolyai, and the German mathematician Karl-Fredrich Gauss expressed a bold idea . . . As a confirmation of their idea they constructed a new geometry . . . And so on, in the same vein.

Such is the level of 'information' on the history of mathematics that is communicated by our school textbooks. It would be much better if this 'information' were not there at all.

In the light of these requirements, the defects in the working curriculum are especially vividly revealed. No movement of the mathematical sciences is presented in it; it provides no background in which scientific discoveries might be shown.

We can draw the following conclusions from the works we have examined on mathematics methodology.

A complete understanding of any question is attained only when its history becomes clear (Bradis).

It is useful to acquaint the pupils with some biographies, of Russian and Soviet mathematicians in particular (Lyapin).

In teaching it is essential to use the lessons that are provided by the history of a subject's evolution (Chistyakov).

In the study of geometry the pupils should pass through those stages that mankind has passed through (Beskin).

Elements of the historical approach have an enormous educational and instructional value (Frenkel').

Historical material awakens a love for and an interest in the subject, a critical attitude towards facts, and thoughtfulness (Tsigler).

We can agree with these conclusions. But objection could be raised to some points made by these authors. For example, some authors name many different forms of introducing the historical approach into mathematics teaching, which would create an overload. We believe that the historical approach in mathematics teaching is a totally new matter which has arisen in unplanted ground. Here one cannot intimidate the teacher who has an abundance of tasks already facing him. It is necessary to involve the teacher in this work persistently but deliberately.

## 2. Questions of the Historical Approach in Works on the History of Mathematics

B. V. Gnedenko, in the pamphlet "Brief Discussions on the Origin and Development of Mathematics" says:

> The interest that a teacher succeeds in arousing in the pupil has a well-known value for the mastery of a subject. In the presentation of the mathematics course in secondary school there are many means that aid in maintaining a pupil's heightened interest in the subject, means that compel him to treat each stage of the course with unabated attention. These means might be a felicitous choice of problems, beautifully executed sketches, or a vividly distinguished applied and cognitive importance of the information that is imparted.
>
> We wish to dwell here on a method that attains its objective at all stages of instruction--brief digressions into the history of science. The history of mathematics is the richest source for these observations, digressions, comparisons, discussions, the selection of problems for exercises and for illustrating the exposition of theory. Digressions into the history of science are the more appropriate as they assist greatly in improving the general cultural level of the pupils and awaken them to a knowledge not only of what has become the object of historical treatises, but also of what is characteristic of modern science [33:3].

In the preface to G. P. Boev's book, Discussions on the History of Mathematics, Professor V. L. Goncharov wrote:

> At present it can be regarded as universally recognized that the teaching of mathematics is not at its proper level if it is unaccompanied by the communication of historical information that clarifies the general cultural significance of the question being considered. Teaching the history of mathematics, as such, in secondary school is, of course, out of the question, and therefore it remains for the teacher to use his opportunities during teaching itself, or apart from it, to make a historical digression. One cannot overestimate the value of these digressions, not only in a general-educational sense, but as an excellent means for enlivening the instruction and raising the pupils' interest as well. However, historical digressions present particular difficulty for the ordinary teacher, who does not have at his disposal an exceptional fund of knowledge of the history of science, culture, and technology; if he wishes to give the pupils proper and accurate information, he inevitably has to turn to rarely accessible and overly cumbersome sources just to read a necessary page or to find a few necessary lines. Moreover, teaching, as it is currently practised, does not give enough attention to elements of the historical approach in mathematics lessons and

has not established any tradition in this matter. The value of such a manual for the teacher is therefore clear--a manual that would, on the one hand, permit him to get the necessary references and would, on the other hand, suggest the idea to him of using certain features in teaching for historical digressions [35].

Professor G. P. Boev says, in his book, Discussions on the History of Mathematics:

> The introduction of elements of the history of mathematics into the school mathematics course can have two objectives: acquainting pupils with basic facts in the history of science, a notion of which makes up an essential part of a general education, and raising the pupils' interest in the curricular material. In the end, the secondary-school graduate should have a rough conception of how the sum of knowledge that is called mathematics was obtained by mankind, of what compels a mathematician to press forward, and of the kind of difficulties that have been overcome along the way.
>
> A basic difficulty in executing the historical approach in mathematics teaching is that mathematics did not develop historically in the same order as it is presented in the systematic course in secondary school. The method of imparting historical material to the pupils in connection with individual questions of the curriculum as these questions are covered should serve as a way out of this difficulty. Returns to the same historical epoch and some anticipation are thus inevitable. For example, in seventh grade, during the study of the unit on first-degree equations, it should be mentioned that simultaneously with first-degree equations there developed the theory of equations of the second, then of the third and fourth, degree. Not only is such anticipation not dangerous, it is useful, since it gives the pupils a perspective for their studies. The teacher will return to the history of equations for a second time apropos of quadratic equations in grade 8 and apropos of binomial equations in grade 10.
>
> Another difficulty in methods is the difficulty pupils have in understanding the dialectic aspect of the historical development of science. The pupil in grades 5-7 tends to conceive of history in an oversimplified way, demanding explicit answers to such questions as: Who was the first to solve quadratic equations? When was the number $\pi$ found? Taking into account the psychological features of each age group, the teacher should permit a certain sketchiness in history in the lower grades and then should gradually depart from that sketchiness and reveal before the pupils' gaze a picture of the contradictions, delusions, and interlacings in the paths along which the development of mathematical knowledge has proceeded. Of course, facts from the history of mathematics must be given, so far as possible, against a background of the general characteristics of the respective epoch.

Finally, there is one more serious difficulty in effecting the historical approach in the elementary mathematics course. This difficulty is the nonelementary nature of the problems of mathematics that have made up the front of its development throughout history. Notions of the profound connection between mathematics and astronomy and, consequently, of the development of spherical trigonometry, of the great algebraic discoveries during the Renaissance, of the 17th century discovery of the analysis of infinitely small quantities as a language of mechanics, of the origin of probability theory and its subsequent role in the natural sciences, of the great 19th century discovery--Lobachevskian geometry--all these should come within the scope of secondary education. The teacher is obliged to touch upon all these elements in some place in discussions of history, but the discussions on these topics should be of a descriptive nature and should given only elementary, popular notions of the appropriate mathematical concepts [15].

In the preface to G. Vileitner's book, <u>How Modern Mathematics Was Born</u>, the following is stated:

> Almost every instructor, as well as almost every pupil, now knows that one can fully understand science in its modern form only when one knows its history. This demand to understand how basic scientific ideas and methods arose is felt particularly keenly during the transition to new provinces of science devoted to distinctive methods that are new to the pupil and that sometimes require strained thinking of him (especially at first). How it was possible to reach such a depth of thought? How could such complex ideas arise 300 years ago? Why and how did these numerous curves, integrals, and differentials appear? These are questions that everyone who begins a study of higher mathematics inevitably asks himself.
>
> But it is not only the search for answers to these questions that induces the pupil to turn to a book on the history of mathematics. He also hopes to understand, with its aid, the true sense of abstract reasoning and of formal conclusions. However, disappointment usually awaits him here. It turns out that, to read books in the history of mathematics, a knowledge of mathematics itself in its contemporary form is essential, but these books are often so laden with petty factual material of a chronological, bibliographical, and even simply of an archival nature that it is almost impossible to use them for clarifying the most essential, typical features of the history of the development of basic mathematical ideas and methods [82].

I. Tropfke, in his <u>History of Elementary Mathematics in a Systematic Exposition</u> says:

> The high value of historical investigations in science in general, and in particular the importance of imparting historical information in teaching mathematics, are so universally recognized that it would be quite unnecessary to enlarge on this in detail ... In our elementary mathematics textbooks,

unfortunately, a place is seldom provided for historical material. Only a few of the newest manuals follow Bal'tser's example, which is worthy of imitation. The author would have been sufficiently rewarded for his work if the abundant material he presented had introduced certain changes in this respect; it is for this purpose that he has most willingly set aside the results of his labor. It could be considered a success and then some if the numerous inaccurate, but, unfortunately, too firmly rooted names finally disappeared from teaching--for example, diophantine equations, Cardan's formula, the golden section, the lunes of Hippocrates, Hudde's method, the Gauss plane, and many others--and if the latest accurate explanations...would supplant the inaccurate ones that are the favorites of everyone [72: Vol. I, Part I, Preface].

In G. N. Popov's book, History of Mathematics [62], there is a section called "The Necessity of Studying the History of Science and the Use of This Study." Here the author advances a number of arguments in favor of the necessity of studying the history of mathematics. He expresses many interesting ideas, but they are of value for teachers who are working creatively in the realm of mathematics, and not for the teacher in the schools.

V. P. Sheremetevskii uses, as an epigraph to his Essays on the History of Mathematics [68], the idea of the linguist A. Schleicher, "If we do not know about how something was generated, we do not understand it," and a quotation from the well-known mathematics historian, P. Tannery, "History by no means has as its only goal the satisfaction of vain curiosity: the study of the past should ultimately throw light on the future."

We can draw the following conclusions from our examination of works in the history of mathematics.

The interest that the teacher is able to arouse in his pupils has a great significance for the mastery of a subject. Brief digressions into the history of science belong among the methods that are capable of arousing interest. Historical "digressions" are all the more appropriate in that they assist greatly in raising the pupils' general cultural level (Gnedenko).

Mathematics teaching is not at its proper level if it is not accompanied by the communication of historical material that will clarify the general cultural value of the question under consideration (Goncharov).

Elements of history in the mathematics course can pursue two objectives: acquainting the pupils with basic facts of the history of science and raising the pupils' interest in the curricular material (Boev).

One can fully understand science in its contemporary form only when one knows its history (Vileitner).

One can become oriented among the numerous ramifications of science only when one knows how the main currents of scientific thought arose. In following the history of science, it is important to be persuaded of the indissoluble bond between its development and the progress in scientific achievements in other provinces of knowledge (Sheremetevskii).

### 3. Questions of the Historical Approach at the All-Russian Conventions of Mathematics Teachers

The question of elements of history in teaching mathematics, as we mentioned above, is not new. In this century the question was posed distinctly at the first All-Russian Convention of Mathematics Teachers (December 11, 1911-January 3, 1912). At the general meeting of the convention, an address was given by Bobynin, on the topic, "Objectives, Forms, and Means of Introducing Historical Elements into the Secondary-School Mathematics Course."

As is evident from the title, Bobynin clearly understood how this problem must be posed; he saw that one cannot speak of the historical approach in mathematics teaching without indicating "objectives," and that, after defining the objective, the forms of introducing historical elements in teaching should be discussed.

What objectives does Bobynin suggest in his address? First, he points out that in the general public, mathematics is often seen as a useless science, that not only persons remote from education but even educators of fame--Kapterev, for example--express the notion that "in the school general-education course there is not sufficient basis for making mathematics compulsory for everyone: it is too abstract and remote from life, too difficult for many." Finally, Bobynin points out that even such a person of world renown as L. N. Tolstoi "regards the sciences, including mathematics, as endless trifling, because there is a beginning and an end to work but there can be no end to trifling" (from Bobynin's address at the 12th Convention of Russian naturalists and physicians).

The speaker further said that pupils often have doubts about the usefulness of mathematics.

> Among pupils the question of the usefulness of mathematics arises when it arose in all mankind, that is, after the transition from lessons in the practical art of calculation and in measuring elementary geometric distance to the study of theoretical geometry and of the elements of theoretical arithmetic and algebra ... Up until this transition there was no room for doubt about the value or usefulness of mathematics, since everyday life experience, as well as the selection of problems, has shown the pupils its practicality. After this transition, the previous clarity of the value and usefulness of mathematics gave place to complete vagueness, not only for the secondary-school pupil but for such minds as that of Socrates and many other philosophers as well.

In their time, teachers and authors of textbooks have discussed the usefulness of mathematics, but without enough success. Bobynin pointed out that instead of all this, "to achieve an influence on pupils at least, concrete examples borrowed from the 'History of Mathematical Sciences' should be set in the direction under consideration." This, in Bobynin's opinion, is the first and the chief goal of teaching the history of mathematics in the school.

The second goal consists in the fact that "in the secondary-school mathematics course there are articles that, within the present organization of teaching, not only come to the pupils with difficulty in elementary study, but for most of them secondary schooling remains insufficiently and superficially mastered." As an example Bobynin cites articles (for arithmetic) devoted to systems of numeration (primarily the decimal), their laws and applications, and in geometry--the use of the method of exhausting the ancients' modifications of that method. Deepening the pupils' understanding of these subjects to a sufficient extent is possible only by a familiarization with the history of their development.

> Thus principal attention should be paid, in the first of the cited instances, to the history of the development of the systems of numeration and of their applications, the chief among which are verbal and written numerations, and in the second instance, to the presentation of the most characteristic and example-laden methods of exhausting the mathematical literature of ancient Greece...

> Also, as one of the forms of usefulness which pupils can elicit from the introduction of historical elements into mathematics teaching in secondary school, one should point out the bonds between the individual parts of elementary mathematics and real images produced by the personalities of scientists and by historical facts, as well as between the spiritual ideas from the province of logic and philosophy. This connection ... is a powerful means of fortifying in the pupils' memory the content of the elementary mathematics that has been taught to them...

Bobynin was of the opinion that historical elements can be introduced into mathematics teaching in secondary school in one of two ways: systematic form or else in an episodic study of the history of elementary mathematics.

In the discussions over Bobynin's address, various aspects of the issue he had posed were elucidated. Among the participants were fervent adherents to the historical approach, and spokesmen of a more moderate turn of mind. In particular, the following ideas were expressed: the history of mathematics cannot be regarded as a universal rememdy for all ills; at what age a pupil should approach elements of the history of mathematics must be indicated; it is useful to study the biographies of outstanding mathematicians; it is better to introduce the history of mathematics as a separate subject and not to include it in the mathematics course, in order not to divide the pupils' attention; the essay method of acquainting pupils with the history of mathematics should be used; it is useful to recommend works in the history of mathematics to the pupils for out-of-class reading; some elementary and more detailed books on the history of mathematics must be published.

More than forty years have passed since Bobynin's address at the first convention. During this time great social changes and very significant progress have taken place in people's consciousness. Now it is impossible to find persons who do not believe in mathematics or who do not know about it. Therefore, in the second half of the 20th century, Bobynin's first and strongest argument for using the historical approach in school mathematics teaching has fallen away. In our day, if an occasional pupil shows a skeptical attitude toward mathematics, the modern teacher does not follow the devious paths of the historical approach, but he cites the countless examples of the application of mathematics in physics, chemistry, astronomy, technology, and in all forms of human productive activity.

## Questions of the Historical Approach in Mathematics Teaching in the Modern Stage

One of the main reasons that elements of the history of mathematics find their way into the school with such difficulty is the circumstance that quite general, rather than concrete, arguments are often expressed on this topic.

It seems to us that there are three groups of historical questions that can find a place in the school instruction.

A. Historical information related to curricular questions of mathematics. The curriculum in mathematics for secondary school undergoes some changes almost annually, and it can undergo a great oscillation over a century. But these changes most often concern details and cannot be so radical as to change the aspect of school mathematics root and branch. The elements of arithmetic, elementary algebra, elementary geometry, and trigonometry are included in the school mathematics curriculum. To this can be added the modest material from analytic geometry and mathematical analysis, covering approximately 80-100 hours. Historical information that is related to these substantially settled questions of mathematics can be taught in the school.

B. Historical information related to the development of mathematics in our country. The mathematical facts that can be discussed here are related to the higher realms of mathematics and, as a rule, exceed the bounds of the school curriculum. Here mathematical facts related to the 19th and 20th centuries are chiefly taken into consideration.

C. Biographies of prominent mathematicians.

Let us examine each of these instances.

### Historical Information Related to Curricular Questions of Mathematics

In the sources we examined above, the aims of introducing the historical element into school mathematics instruction are not always distinctly indicated. Some authors do not indicate these aims at all. Tropfke [72] is among them. We cannot treat his statements with complete confidence. He says that there is no need to demonstrate the usefulness of historical information in mathematics teaching. On the other hand, Simon [69], an educator who is no less well-known in Germany, declares that teaching the history of mathematics in German schools is "in the background." It seems to us that the state of affairs cannot be such that everyone understands the necessity of the historical approach in mathematics teaching and at the same time this historical approach cannot carve its way into the schools.

We believe that the main task of every author who writes on this topic consists in formulating the aims of introducing the historical element into mathematics instruction. Under these conditions the teacher will find time both to acquaint himself with the history of mathematics and to devote attention to it in mathematics lessons.

These aims are presented to us in the following form.

1. The history of mathematics should indicate the reasons why the mathematical facts, concepts, and methods that are studied in school arose. Mathematics develops its theses by proceeding from objective reality, and then it cements the selected facts by means of logic, but by itself, it cannot indicate the reasons why certain constructs arose.

2. The history of mathematics should indicate the goals that the creators of the mathematical science were striving for when they introduced a certain fact or studied a certain mathematical phenomenon. This circumstance also escapes the field of vision of mathematics itself and can be reflected only in its history.

3. When it is a matter of some mathematical phenomenon which covering some time, changed, developed, or passed from one form to another, these facts can be examined only in a historical aspect.

4. The study of the history of mathematics has (if we may so express it) a methodological value. This idea must be understood in the following manner. Let us imagine that I want to present a mathematical question to pupils. How can this be done? One can even think of a nonelementary question, because elementary questions are presented in textbooks, and the way to present them is clear. Therefore let us take a question of a higher order. Suppose I wish to prove the transcendence of the number e. This proof is found in various books--in particular, in Klein's book [42] where it occupies seven pages and is conducted according to Hilbert. In an encyclopedia of elementary

mathematics by Weber and Wellstein [88] six pages were set aside for this proof. The proof occurs in a little booklet by Drinfield,* where it occupies four and a half pages. But the first proof of this truth was given in 1873 by Hermite and occupied significantly more space.

The question arises: What will methodology say apropos of using one proof or another? The first proof (Hermite's) is the "longest," but it is also the most natural. As you read it, it is as if you are present in a scientist's laboratory. But such a proof, precisely by virtue of its volume, does not get into textbooks, which give preference to short proofs. In textbooks, second, third--in short, later--proofs are cited, which are much easier to find, as soon as the first proof has been found.

Thus, from a pedagogical point of view the first proof is more valuable, but it does not get into an official textbook, which is always bound by the number of hours allocated for the topic in the curriculum. Thus, in answering the question posed above on how to present a new topic with methodological expediency, we should say: it ought to be learned from history. It is necessary to see how this topic was elaborated by pioneers in it, what way they used to reach definitive results. Of course, this path is thorny and not the shortest, but it is the most natural and the most instructive. The school, as a rule, is not in a condition to follow such a path, but it can venture, in certain especially important instances, to use this method.

We deliberately took as an example a question that goes beyond the limits of the school curriculum, and we did not use any theorem from the elementary course in geometry. It would have been hard for us to pursue these arguments with respect to the theorem on the sum of the squares of the diagonals of a parallelogram, for example, because the elementary geometry studied in secondary school was created all at once, so to speak, by the genius of Euclid (at least, that is the impression one gets).

Thus, the everyday presentation of minor mathematical theorems can be carried out in the textbook, using the second, third, perhaps even the tenth proofs, but in presenting major questions, large-scale problems,

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*Source not given in the original (Ed.).

it is useful to turn to the history of mathematics. There we sometimes meet unrigorous and insufficiently polished arguments, which are instructive in many respects, however. They show how searches for the truth proceed, not along an even road, but on the contrary, through tortuous paths, and that even a genius does not create instantaneously, and that the truths that are now common property evoked doubts, hestiation, and distrust in their day.

5. The history of mathematics is part of the history of culture, and in this sense the study of it has the same object as the study of the history of culture. That is, it acquaints a person with the facts of the cultural life of mankind and it shows the stages to which people have risen slowly, over a millennium, before they reached their present condition.

The elements of the historical approach in mathematics teaching are essential primarily for mathematics itself. But, in addition, they are essential for widening our pupil's horizon, for raising his cultural level. When examined from this aspect, the "elements of the historical approach" constitute part of the history of culture and therefore cannot be presented in isolation from that history. But, for the mathematics teacher to be able to present the history of culture easily and freely, he should be trained in this himself. He should be trained in this by the particular style of work of the pedagogical institutes. This style, by the way, should exclude easy, superficial reading of random, cheap, empty books. On the contrary, from the first year to the last it should be characterized by a study, against a background of the history of culture, of the monuments of world literature, architecture, sculpture, painting, music, technology, etc. An educator thus trained will be able not only to teach mathematics but to show the pupils its general cultural significance as well.

At the same time, we should make one stipulation. A mathematics teacher who is already established must not be treated too unmercifully. One must take into account the indisputable fact that the pursuit of the mathematical sciences over many years leaves a definite mark on a person--he has, we might say, worked out a particular style of behavior and thought. We might describe this style as deductive. In inviting our mathematician to switch over to the historical, we nevertheless compel him to renounce the style of work he is used to.

The fact is that the method of presenting facts, like the one used in history, often scares away our mathematics teacher, who is trained in Euclid. In mathematics he is accustomed to the logical method, and in history (even if it is history of mathematics) he should use the historical method.

The historical method is characterized by a study of events in a definite sequence, and therefore chronology acquires considerable significance here. The historical method links each event with a definite place (localization), and therefore it is impossible to study the history of mathematics without a geographical map and without imparting the geographical information to which a given mathematical phenomenon is related.

Certain persons inevitably participate in the historical process. Accordingly, the study of the history of mathematics presupposes a familiarity with the biographies of some historical personalities. The historical method can be characterized as inductive, since history has as its subject the study and description of facts that took place at a definite time and in a definite place. All this is, to a certain extent, alien to the mathematician, who resorts mainly to the method of deduction in his work. Nevertheless these quite necessary aims, which we have formulated above, should be fulfilled because only under these conditions do we extend the content of the information that constitutes the subject of mathematics.

The presentation of historical information cannot be diverted from the study of mathematics itself; it is set forth either before the study of a definite topic, or afterwards, but not separately from it. In the school there is no subject called history of mathematics, but there are individual historical discussions, lasting from 5 to 30 minutes. We think that a real history of mathematics might be taught in the school, but this is a luxury that no curriculum could bear.

Consequently, it is a question of organically blending mathematics and its history. Apparently, it should proceed in this way: the teacher presents some pieces of information, and then explains how they arose. It is clear that this cannot be done in every lesson. If the pupils are spending a number of lessons studying operations on algebraic fractions, or the solution of first-degree equations, or operations on radicals, the historical information is not imparted in every lesson, but once after a number of lessons, when generalizing or reviewing. We have already said that historical information can even be imparted before studying a definite topic.

Historical Information Related to Extracurricular Questions of Mathematics

It is no secret to anyone that the mathematics created in hoary antiquity is what is studied in school. This is clear. It is true with respect to arithmetic and geometry. As for algebra, we encounter facts of a later origin. However, these facts almost never go beyond the 17th century. We must remember that the limitation of mathematics to such remote information is utterly insufficient for a secondary-school mathematics education. At present the person who completes secondary school cannot manage without the concept of a function, without a knowledge of the elements of analytic geometry, or the ideas of the analysis of infinitely small quantities. We do not doubt that in the next few years the mathematics of the 17th century will penetrate the secondary school, but matters will not go farther than this date. The mathematics of the 19th centruy cannot be studied in school: there is not time for it, and now it is becoming a specialty, not a subject for general education.

However, teachers often tell the children in school about the achievements of mathematics in the 19th and even in the 20th centuries. How do they do it? What do they relate it to?

The historical information imparted by a teacher in connection with, say, the study of equations of higher degrees or of logarithms, should illuminate these questions from a historical, rather than a logical, point of view. In connection with this the names of Abel, Galois, Napier, Burgi, Stifel, and others will appear. But to introduce any 20th century mathematician in this way is rather difficult for the simple reason that he has not studied the questions of school mathematics.

Here we must find a different way. On finishing secondary school, the pupil should be familiar with the progress in science, technology, and the arts in the country of which he is a citizen. The mathematics teacher cannot accomplish this task alone; all the specialists, working in secondary school--physicists, chemists, historians, biologists, geographers, etc.--should solve it through combined efforts. They should allot among themselves the information on the history of culture in our land and should teach it to the children.

How and when can this be done?

We feel that it would be possible, for example, beginning with grade 7, to arrange annually, over four years, two discussions on questions in the history of culture and mathematics in our land. Material for these discussions can be found in Gnedenko's book, Essays on the History of Mathematics in Russia [34]. One can use the other books (they are rare, to be sure) that are cited in the accompanying list of literature.

But in conducting these lecture-discussions it is not enough to read one or two books on the history of mathematics. The lecturer should first describe the epoch, preparing the background out of which certain mathematical facts arose. He must thus become acquainted with the historical and socioeconomic literature.

But a basic difficulty is not how to find literature on a given question but how to present nonelementary things in a popular form to the pupils. Let us restrict ourselves to just one example. If a teacher wished to tell his pupils about M. V. Ostrogradskii, it would be senseless to say that he was concerned with the calculus of variations, but it could be said that he worked in the province of algebra and the theory of numbers. It should be pointed out that Ostrogradskii was not studying the algebra that is learned in school, nor the arithmetic that is set forth in a standard textbook, but was concerned with higher branches of both disciplines. It becomes clear that arithmetic as a science does not terminate in sixth grade, nor is algebra terminated in tenth grade of secondary school.

## Biographies of Prominent Mathematicians

In presenting individual facts from the history of mathematics, one must, of course, give various names (Euclid, Archimedes, Heron, Newton, Lobachevskii, Descartes, Napier, and others). In mentioning these names, the birth- and death-dates and nationality are cited. In a few cases two or three more sentences are added about the work of the given mathematician, about his place of residence, or about his mathematical interests. All of this is the sort of information that almost never sticks in pupils' memories. But in many schools (mainly during extracurricular time) more extensive biographical information is given to the pupils. An acquaintance with such information shows that it usually does not go beyond the bounds of biographical particulars. This means that not only are the dates of birth and death given, but the educational institutions from which a mathematician graduated, the institutions in which he served, and the years in which his most important works came out are indicated.

No special goal is set for imparting this information, and since there is no goal, there is no clarity with respect to what to consider important and what to consider not worth mentioning in a person's biography.

What considerations should guide the educator in the study of biography? The mentors of the rising generation have always used the biographies of great persons for educational purposes. For this, brilliant, prominent, extraordinary--rather than ordinary--personalities have been chosen. This is understandable. The school, in training the rising generation, should confront the pupil with great tasks and high goals, should form social ideals, for only under these conditions will it fulfill its purpose.

The school in our time places before its pupils the ideal of a thoroughly educated, trained person of high moral character, an active builder of communist society. On account of this, it is advisable to show that at different times and in different countries, personalities appeared who stood head and shoulders above their environment, who sometimes performed deeds that were incomprehensible to their contemporaries and who were evaluated only many years later, after their death, persons who lived in the name of science, sometimes renouncing private life, money, glory, every external advantage, and working all their lives in the name of the welfare of mankind.

Clearly, the biographies we are speaking of should be detailed. This means not one or two pages, but at least a booklet of 3-5 printer's sheets. How and when can they be studied and how much biography can be given to our pupil? We feel that it is not within his powers to study more than two biographies a year. When there are a few copies of a biography, they can be given to the pupils for preliminary reading at home, then a separate hour can be set aside for collective examination of what has been read. If there is a single copy of a biography, it can be narrated either by the teacher himself or by a pupil who is good at speaking (after suitable preparation). It would be good if all the pupils would start a notebook on each biography and would fill it up with their extracts.

One cannot read in earnest the biography of a great man at the so-called mathematical evenings, where theatrical productions are arranged, various tricks are shown, and even dances are organized. At these evenings there is no room for serious, sincere, and reflective work.

## 5. Historical Information of a Curricular Nature

In composing this section, we used G. P. Boev's book, Discussions on the History of Mathematics [15].

We have already said that at present it would be premature to require the teacher to construct mathematics teaching on historical bases. Such a requirement would be beyond his strength. But one cannot relinquish the elements of the historical approach in teaching. We should move forward with each year. What this means was discussed above. Let us briefly repeat our ideas: it means that a staff of teachers who know the history of mathematics well should be trained in the pedagogical institutes, more books on the history of mathematics should be issued for teachers and pupils, and historical information should be instilled in mathematics textbooks and in methodological literature. But at the same time, without waiting for the results of these measures, one should introduce elements of the historical approach into mathematics teaching in small doses, from time to time.

We indicate below just what materials should be imparted, and when it should be imparted, to pupils in the study of mathematics. We assume that some simple material from the history of culture, in a form accessible to children, is imparted in the first four grades. Not wishing to encumber our exposition, we shall not touch on this information, and we therefore begin with what it would be useful to present in secondary and upper grades.

Let us cite a model list of historical discussions.

### Arithmetic

1. The decimal system of enumeration.
2. Prime numbers.
3. Decimal fractions.

### Algebra

1. Historical introduction to algebra.
2. Negative numbers.
3. First-degree equations.
4. Equations of second and higher degrees.
5. Logarithms.
6. Functions and graphs.
7. Development of studies of number

### Geometry

1. Historical introduction to geometry.
2. Trisection of an angle.
3. Pythagoras and his time.
4. Squaring the circle.
5. Euclid's fifth postulate.
6. Duplication of a cube.
7. Conic sections.
8. Euclid's Elements. The foundations of geometry and other ways in which geometry developed after Euclid.

### Trigonometry

1. Historical introduction to trigonometry.
2. Development of modern trigonometry.

On the following pages we shall give explanations for each topic.

Topic: The decimal system of numeration.

Time of presentation: In grade 5, after a review of the decimal system of enumeration.

Literature: [9, 10, 83].

Content of topic: Roman numerals. Slavic numerals. Various systems of enumeration. Evolution of numerals. Auxiliary means of calculation (abacus). Large numbers with the Greek mathematician (Archimedes) and in Old Russia. The role of the Hindus in creating the decimal system. The penetration of Hindu numerals into Russia.

Topic: Prime numbers.

Time of presentation: In grade 5, during the study of the divisibility of numbers.

Literature: [23, 34].

Content of topic: Mathematicians were occupied with the study of primes in ancient times. First, it was interesting and of course important to solve the question of whether it is possible, by gradually going over the set of natural numbers, to reach the greatest prime number, after which only certain composites occur.

Euclid studied this question (300 B. C.) and established that the number of primes is infinite.

After Euclid, the mathematician and geographer Eratosthenes (275-194 B. C.) studied the question of the search for primes. He set himself the task of singling out the primes from the set of natural numbers. His method consists in first all numbers divisible by 2, except 2 - x, being cut off from the set of natural numbers from 2 to a, then all numbers divisibla by 3, except 3 - x, then all numbers divisible by 5, by 7, by 11, etc. (Eratosthenes' sieve).

The French mathematician, Pierre Fermat, proposed that the formula $a = 2^{2^n} + 1$ should yield the prime numbers. This hypothesis was refuted by L. Euler, who proved that when n = 5, the formula yields a number divisible by 641.

In the 19th century the question of the nature of the distribution of primes was studied by many mathematicians, but P. L. Chebyshev (1821-1894) achieved especially significant results in this area.

Topic: Decimal fractions.

Time of presentation: After covering ordinary fractions, and before the study of decimals.

Literature: [15, 21, 23, 72].

Content of topic: Basic attention in this discussion should be devoted to showing that decimal fractions were a true blessing for mankind, because they made calculations easier.

> In the study of a course in decimals, the teacher's chief goal should be for the pupils to arrive consciously at the inevitable conclusion that decimals are much easier than ordinary fractions in the method of their notation and for performing operations with them. In this respect decimal fractions have had no luck in school: pupils are firmly convinced that ordinary fractions are much 'better' and easier than decimals--this antipathy to decimals then remains the whole life long. For example, the English writer Jerome K. Jerome writes, in his Diary of a Pilgrimage: 'From Ghent we left for Bruges, where I found pleasure in throwing a stone at a statue of Simon Stevn, who tormented me a great deal in school, as he invented decimal fractions [28].

In presenting this question, the Uzbek mathematician and astronomer, al-Kashi (15th century), Christoff Rudolff von Jauer (16th century), and the Flemish (Belgian) engineer, Simon Stevin (1548-1620) are mentioned.

Topic: Historical introduction to algebra.

Time of presentation: In grade 6, before the study of algebra.

Literature: [21, 47, 68, 73, 82, 83].

Content of topic: For every beginner in the study of algebra, the question naturally arises of what algebra is and of what he will be learning in this science.

The school algebra course is primarily a continuation of arithmetic. This means the following: in arithmetic, integers and fractions and operations with them have already been studied. Still other kinds of numbers will be studied in algebra; after integers and fractions, first, negative numbers will be studied, then other kinds of numbers. As in arithmetic, we shall study operations on these new numbers: at first the same four operations examined in arithmetic--addition, subtraction, multiplication, and division--then still other operations.

Here is the sense in which school algebra is a continuation of arithmetic--it studies numbers and operations with them.

Equations are the next and at the same time the central question of algebra. Elementary notions of them can be given to the pupils by showing equations of the type $x + 10 = 15$, which are familiar to them from arithmetic. Historical material on equations will be given later, after familiarization with them.

Here one should touch on the matter of letter designation. the modern letter notation was the result of a long historical development. Three stages can be traced: a) rhetorical algebra, b) syncopated, and c) symbolic.

In presenting the topic, Diophantus, Mohammed ibn Musa al-Khowarizmi, Luca Pacioli, Cardan, Stifel, Vieta, Girard, Descartes, and Wallis can be named among the scientists.

Topic: Negative numbers.

Time of presentation: In grade 6, during the study of negative numbers.

Literature: [21, 48].

Content of topic: The gradual penetration of mathematics by negative numbers. The attitude of the Greek mathematicians toward them. The role of the Hindus. The views of Homammed ibn Musa al-Khowarizmi. Negative numbers among the algebraists of the 16th century: Luca Pacioli, Tartaglia. Stifel's establishment of the "rule of signs." Vieta's attitude toward negative numbers. The significance of the work of R. Descartes for consolidating negative numbers in mathematics. Girard's geometric interpretation of the negative roots of an equation.

Topic: First-degree equations.

Time of presentation: After composing first-degree equations, before passing on to systems of equations.

Literature: [15, 23, 47, 68].

Content of topic: Problems that lead to first-degree equations were known to the Egyptians. In the "Rhind Papyrus," written approximately 1700 years B. C. by Ahmes, there is a section devoted to the "calculation of piles." Methods of solving these problems. Problems on equations in a second-century collection, known under the title of the Palatine Apologia. Diophantus' "Arithmetica" as a significant step forward in the province of solution of equations. The appearance around 820 of a treatise on "algebra" by Muhammed ibn Musa al-Khowarizmi and the features of that treatise. Omar Khayyam and his algebraic works. Features of Leonardo Fibonacci's work, Liber Abaci (1202).

Topic: Equations of second and higher degrees.

Time of presentation: In grades 8-10 in connection with the study of the appropriate sections of the curriculum.

Literature: [15, 21, 68, 73, 74, 83, 88].

Content of topic: Information on equations above the first degree, up to the time of Diophantus. The quadratic equations of Diophantus. The quadratic equations of Muahmmed ibn Musa al-Khowarizmi. The role of Omar Khayyam. Subsequent development of the theory of solving quadratic equations (Stifel, Vieta). Solving equations of the third and fourth degree (Ferro, Tartaglia, Cardan, and Ferrari). Subsequent development of the theory of solving equations of higher degree (Gauss, Ruffini, Abel, Galois).

Topic: Logarithms.

Time of presentation: In grade 9 during the study of logarithms.

Literature: [1].

Content of topic: The first attempts at simplifying calculations with the aid of the composition of progressions in the 16th century (Stifel). The tables of Burgi and Kepler. The invention of logarithms by Napier. Napier's method of calculating tables. Briggs and his work in simplifying the system of logarithms. The latest tables of logarithms with a various number of decimal places. The calculation of logarithms with the aid of a logarithmic series.

Topic: Functions and graphs.

Time of presentation: The pupils should be familiarized with the proposed material during their entire stay in school.

Literature: [41, 55, 81].

Content of topic: In the exposition of this topic it is useful to dwell on the various forms of functional relationship. It is important to note the idea of the graphic representation of functions. Among the scientists, one can name: Galileo, Newton, Leibniz, Fermat, Descartes, Euler, Lobachevskii, and, depending on circumstances, some 19th century mathematicians.

Topic: Development of studies of numbers.

Time of presentation: In grade 10, in connection with the review of algebra.

Literature: [48, 81, 84].

Content of topic: During their stay in school the pupils have already become acquainted with natural, fractional, negative, and irrational numbers. Now, when the level of their mathematical development has attained its highest peak, a survey can be made of concepts familiar to them, but so to speak, from a higher point of view. This will be the last historical discussion in algebra, and in it the instructor can allow himself to make broader generalizations, without fearing that the pupils will not understand him. In presenting the question of complex numbers, the teacher can derive supplementary information from Simon's book [69], in Chapter V, "The Didactics of Arithmetic and Algebra," the section on "The History of Complex Numbers."

Topic: Historical introduction to geometry.

Time of presentation: Grade 6, before the study of the geometry course.

Literature: [15, 46].

Content of topic: Geometry in Ancient Egypt. The practical nature of Egyptian geometry. The role of geometry in land-surveying and in the construction of various buildings. The nature of Egyptian mathematics. The geometry of the Babylonians. Its origin in practical activity. Geometric design. Division of a circle into 6 equal parts and determining the side of a regular inscribed hexagon. The first Greek geometers--Thales and Pythagoras. The possible influence of Egyptian and Babylonian geometers on them.

Topic: Trisection of an angle.

Time of presentation: In grade 7 (at the end of the course).

Literature: [49, 68, 73].

Content of topic: The problem of dividing an angle into 3 equal parts with the aid of a compass and straightedge is related to a number of famous geometric problems of antiquity. Attempts at solving this problem date from the oldest times. The Pythagoreans knew a particular case of this problem--the division of a right angle into three equal parts. Hippias of Elis apparently was one of the first mathematicians to study the solution to this problem. According to the evidence of Proclus, he found a particular curve (the quadratrix), with the help of which the problem could not be solved. The same Proclus says that Nicomedes divided an angle into three equal parts with the aid of the conchoid. Pappus of Alexandria points out two more solutions to this problem. In modern times, Descartes studied the trisection of an angle. Besides Descartes, the solution to the problem was studied by Newton, Clairaut, Schall.

Topic: Pythagoras and his time.

Time of presentation: In grade 8, during the study of the metric correlations in a triangle and a circle.

Literature: [15, 21, 68, 73].

The characteristics of the epoch can be found in history courses. An exposition of Pythagorean teaching can be found in detailed courses in the history of philosophy.

Content of topic: Historico-biographical data on Pythagoras. Particulars of the Pythagorean brotherhood. The Pythagoreans' work in geometry. Their arithmetical works. Their discoveries concerning physics (laws of the vibration of strings) and astronomy (the daily revolution of the earth).

Topic: Squaring the circle.

Time of presentation: In grade 9, during the study of the area of a circle.

Literature: [15, 49, 64, 73].

Content of topic: This problem is related to the famous problems of antiquity. On Plutarch's evidence, even Anaxagoras studied the solution of this problem. The same problem interested Hippocrates of Chios. Antiphon's attempts. The use of the quadratices of Dinostratus and Nicomedes. Solution of the problem by Archimedes. The words of the Hindus. Later authors who studied the solution of the problem: Leonardo Fibonacci, Adriaen Metius, Ludolf van Ceulen, Huygens. The work of Lengendre and Lindemann.

Topic: Euclid's fifth postulate.

Time of presentation: In grade 9, before beginning solid geometry.

Literature: [88].

Content of topic: The Greeks' postulate of parallel lines. Proclus' information about the attempts at proof. The Arabs' postulate of parallel lines in the Renaissance and in the 17th century (Clavius, Borelli, Wallis). Forerunners of non-Euclidean geometry. Saccheri, Lambert, D'Alembert, De Morgan, Laplace, Legendre, Bolyai. The creation of non-Euclidean geometry. N. I. Lobachevskii.

Topic: Duplication of a cube.

Time of presentation: In grade 10, during the study of the volumes of polyhedra.

Literature: [49, 68, 73].

Content of topic: The problem of duplicating a cube (or the Delian Problem) belongs among the famous geometric problems of antiquity. Hypotheses concerning the origin of this problem. First attemps at solution by Hippocrates. Solution of the problem by Archytas of Tarentum. The method proposed by Menaechmus. Among the ancient mathematicians, Apollonius and Diocles studied the solution of this problem, and among later ones--Vieta, Descartes, Newton, and others.

Topic: Conic sections.

Time of presentation: In grade 10, during the study of solid geometry.

Literature: [68, 73, 81, 82].

Content of topic: The theory of conic sections prior to Apollonius. The conic sections of Apollonius. Properties of the conic sections discovered by Apollonius. Focal properties of conic sections. Polar properties. The subsequent application of conic sections: the trajectory of a projectile, orbits of planets, and others.

Topic: Euclid's "Elements." The foundations of geometry and other ways in which geometry developed after Euclid.

Time of presentation: The final discussion in grade 10.

Literature: [25, 88].

Content of topic: The discussion is the final one. Three questions can be touched on in it. First, the pupils can become familiar (quite briefly, of course) with the content of Euclid's "Elements." Second, if the pupils' mathematical preparation permits, they can become familiar (quite briefly) with modern constructions according to Hilbert. Third, to give the pupils an opportunity to feel the perspectives of the subsequent development of goemetry, they can be given a few words on the history of the development of projective or descriptive geometry.

Topic: Historical introduction to trigonometry.

Time of presentation: In grade 9, before the study of the systematic course in trigonometry.

Literature: [21, 73, 74].

Content of topic: Table of the chords of Hipparchus. The connection between astronomy and trigonometry. The tables of Ptolemy. The words of the Hindus in trigonometry. The works of the Central Asian scientists (Abu'l-Wefa).

Topic: Development of modern trigonometry.

Time of presentation: At the end of the first quarter of grade 10.

Literature: [15, 21, 73, 74, 81].

Content of topic: Some trigonometric facts were established in antiquity. As was indicated above, the first tables of chords were composed by Hipparchus in the second centruy B. C. Trigonometry takes on a more modern form almost 15 centuries later. First, the merits of Johann Muller (1436-1476), born in Konigsberg and known under the pseudonym of Regiomontanus, should be mentioned here. The Englishman, Bradwardine, can be named among his predecessors. Vieta (17th century) deduced a series of formulas in trigonometry. Besides the persons mentioned, many prominent mathematicians worked in the field of trigonometry: Napier, Pothenot, and Euler. The German mathematician Mollweide, whose name is connected with formulas that are familiar to the pupils, can also be cited. Furthermore, J. Bernoulli, De Moivre, and Lambert contributed to the progress of trigonometry. Euler can be regarded as the founder of the modern teachings on trigonometric formulas.

6. Historical Material of an Extracurricular Nature

We have observed that the historical information that belongs to the curricular material is presented together with this material. Historical information related to mathematics which does not enter into the school curriculum has other goals and cannot be presented in the scheme in which curricular questions are considered.

The information we are discussing in this section can be considered in connection with familiarizing pupils with the history of culture in our land. In order to impart to the pupils some information on the history of mathematics in our country, the teacher should not only read some books on the history of mathematics in Russia but should become acquainted, first and foremost, with civic history and the history of culture. This, of course, involves an expenditure of extra time, the selection of literature, and the mastering of a method of historical exposition that is unusual for a mathematician.

We have said that discussions on the history of culture in our land should be held in school not oftener than roughly twice a year, according to a definite, previously outlined plan. It is not hard to make such a plan, if one sticks to the division into periods which has taken shape in history. The circumstance that the first discussions on the history of culture will be held in fifth grade, and the last in tenth, is most important, pedagogically. Consequently, there should be an enormous difference among these discussions. In fifth grade the children have a quite limited stock of historical information, a poor vocabulary, and weak skills in thinking logically. Therefore the information cited above should be given to them in a maximally intelligible form. The teacher has to do a great deal of work in order to bring material from a book closer to the understanding of fifth graders. In the upper grades, the teacher should gradually make his discussions more complex, approaching a lecture style in grade 10, although in fifth grade an unbroken discussion covering 40-50 minutes would be tiresome.

## 7. The Study of Biographies

It was stated earlier that the study of biographies has an instructional value. The work of studying biographies, we feel, should be divided among the school teachers according to their specialties.

Which biographies do we recommend for study? It is difficult to answer this question for a number of reasons. In particular, our recommendation might be impracticable if the biography section of a school's library is very poorly represented. Then the teacher must use the biographies which he has near at hand.

In section 5 we recommended familiarizing pupils with the scientists of various epochs. This is useful in that pupils who become acquainted with a person's biography obtain information about the epoch in which that person lived and worked. It should be made a rule that a person's biography cannot be divorced from historical and geographical conditions.

Without trying to impose our views on the teacher, we shall cite a sample list of names whose biographies might be dwelt upon in school.

Euclid (300 B. C.) Biographical information about him is extremely scanty, but we give this name because an entire epoch is connected with it. Some information on Euclid can be found in courses in the history of mathematics and the history of culture. Vygodskii [85] provides some biographical information in his article on Euclid's Elements. In the presentation of this topic particular attention should be paid to the characteristics of the epoch--the flowering of the sciences in Alexandria in the epoch of the Ptolemies (the library, the museum, and academy of Alexandria).

Archimedes (287-212 B. C.) Euclid and Archimedes were close to each other in date. In discussing Archimedes, we wish to stress the difference between these great mathematicians of antiquity. Euclid was, so to speak, a theoretician in science. As for Archimedes, engineering or technological work (speaking in modern language) attracted him to a significant extent. There is literature about Archimedes. Besides biographical information in books on the history of mathematics, individual works can be cited [31, 38].

Ptolemy (87-168). Very little is known about his life. In presenting this topic, attention should be paid to the characteristics of the epoch and a survey of the works of Ptolemy. His main works:

the Almagest (the Arabic distortion of the author's original title), the Planetary Hypothesis, and the Risings of the Stars, Reference Tables. One can read about him in books on the history of mathematics, for example, Fatstsari [26], Tseiten [73], Sheremetevskii [68], and others. According to the evidence of Proclus, Ptolemy made an attempt to prove Euclid's fifth postulate. Ptolemy's geocentric system lasted up to Copernicus.

Brahmagupta (598-660). We are naming Brahmagupta because we feel that a discussion about the nature and acheivements of Indian mathematics can be developed around this name. The names Aryabhata and Bhaskara can be mentioned in this discussion. One can read about Indian mathematics in Tseiten's book [73]. Much material on the geometry of the Hindus can be found in Schall's history of geometry.*

Mohammed ibn Musa al-Khowarizmi (820). Biographical information about him is extremely scanty. But it is very desirable to dwell on his activity, since the very term "algebra" originates in his works. Literature can be cited: [23, 26, 47, 68, 73] and Essays in the History of Mathematics by Popov [61]. Mohammed's personality should be shown against a historical background. The role of the Arabic language should be pointed out.

Nasir ed-din (1200-1274). An astronomer and geometer, he directed the observatory in Maraga and wrote in Arabic. In presenting this topic, a number of mathematicians of different nationalities, who also wrote in Arabic, can be named. In the historical scheme, the Arabs' role in spreading and commenting upon Greek authors should be discussed. The works of Nasir ed-din embrace almost all branches of human knowledge. The translated Euclid, accompanying the translation with commentaries, and Archimedes into Arabic. There is an article by B. A. Rozenfel'd in Historico-Mathematical Research, issue IV.

Leonardo Fibonacci (13th century) and Luca Pacioli (1445-1514). We name together these mathematicians, who are separated by more than two centuries. They are little known to the general public, but they have a great significance in the history of mathematics. We dare say that, by not referring to them, we would permit a gap of almost

*No source given in the original (Ed.).

400 years in the history of mathematics: One can read about Leonardo Fibonacci (Pisano) in Tseiten's book [73]. There is much information on Luca Pacioli in volume I of History of Scientific Literature in Modern Languages [60].

Isaac Newton (1643-1727). The literature on Newton is substantial. Of the old editions we can cite the Pavlenkov biography and these books: Vavilov, Isaac Newton [80]; Marakuev, Newton, His Life and Works [54]; and Kudryavtsev, Issac Newton. For the 300th Anniversary of Newton's Birth. A Collection for the Tercentenary of his Birth [45]. An index to the literature can be found in A. P. Yushkevich, Soviet Anniversary Literature on Newton [90].

Gaspard Monge (1746-1818). In connection with the bicentennial of his birth, the USSR Academy of Sciences issued a collection of articles [70]. In addition, Starosel'skaya-Nikitina's book can be cited, Essays on the History of Science and Technology in the Period of the French Bourgeois Revolution, 1789-1794 [71].

Gaspard Monge was one of the men who introduced the metric system and was a member of the commission of five, which was formed by the National Convention in 1790.

Leontii Filippovich Magnitskii (1669-1739). Biographical information about Magnitskii is extremely scanty. On account of this it is useful to acquaint pupils with his "Arithmetic." Since this book is a bibliographical rarity, Baranov's book, The Arithmetic of Magnitskii [7], which is an exact reproduction of the original can be used. This book was published in 1914. Only a small part of Magnitskii's arithmetic, up to the division of whole numbers, appeared in Baranov's book, but an introduction in the book presents biographical information on Magnitskii. In addition, we can cite Galanin [30] and Section 4 of Gnedenko's book [34]. In the elaboration of this topic, the epoch of reform under Peter I should be pointed out.

Leonard Euler (1707-1783). Biographical information on Euler can be found in Gnedenko's book [34]. In the old edition of Pavlenkov there is a biography of Euler. The following sources can be cited: Krylov, Leonard Euler [44] and Luzin, "Euler" [52].

Nikolai Ivanovich Lobachevskii (1793-1856). From the literature on Lobachevskii we should first cite Materials for the Biography of

N. I. Lobachevskii. In general, there is rich literature both on Lobachevskii and on the non-Euclidean geometry he created [3, 37, 39, 40]. A series of books was issued for the centennial of the discovery of non-Euclidean geometry [65: Vol. 2, 3, 4].

Pafnutii L'vovich Chebyshev (1821-1894) is the most prominent mathematician of the 19th century, the closest to us in time. One can read about him in Prudnikov's book [63], in Molodshii's article [58], or Ginzburg's article [32]. There is a biography of Chebyshev in the first edition of his works.

## 8. The History of Mathematics in Extracurricular Lessons

In extracurricular lessons there are more opportunities to study the history of mathematics than in class. But extracurricular work, for all of its positive qualities, is fraught with one danger, which is that it is possible unwittingly to overburden the pupils with numerous variants of this work and of preparation for it. There is a danger that extracurricular work could turn into an end in itself and could eclipse (at least, in the student's budgeting of his time) the class work.

The class work should take first place both in the amount of time expended and in proportion. A secondary, and thus quite modest place should be given to extracurricular work, and this work should by no means burden the pupil, nor should it demand a considerable expenditure of time from him. These can be rather rare lessons, short in duration. Apparently, these assignments should be in the form of mathematics club classes.

In the programs for these classes, published in 1935 by the Administration of Elementary and Secondary Schools of the People's Commissariat of Education of the RSFSR, it is pointed out that "mathematics clubs can be of two kinds: 1) clubs of a mass nature, the tasks of which are to raise the mathematical culture and interest in mathematics among the pupils, to cause individual pupils' inclinations for the mathematical disciplines to be manifested, or to replenish and extend the pupils' knowledge in the elementary mathematics course; and 2) clubs for a limited number of pupils, the task of which is to aid in the acquisition of more extensive knowledge on individual topics by more gifted pupils or by those who have special interests in mathematics and technology.

In their lessons in the mathematics club, the pupils acquire:

a) knowledge in those branches of mathematics that do not enter into the secondary school curriculum and cannot be studied with the methods of elementary mathematics;

b) skill in reading independently a mathematics book of varying difficulty on the varying level of their mathematical development;

c) skill in choosing, abstracting, and systematizing mathematical material;

d) skill in presenting their ideas coherently on questions of mathematics, in lecture-form--written or oral (in circles for the upper grades);

e) skill in applying mathematical knowledge in practical life.

A place in the mathematics club can always be granted to the history of mathematics and the study of biographies. The history of mathematics is useful because it does not divert the pupils' thought from mathematics. In it the same equations, radicals, functions, logarithms, areas, constructions, and problems are discussed with which the pupil has become familiar during his class lessons. When he meets material that has been covered, this time in a historical scheme, the pupil reviews and reinforces that material.

Thus, lessons in the history of mathematics can take a high priority in the programs of the mathematics clubs, which cannot be said of other topics in the program for extracurricular lessons. For example, it is clear on many counts that dramatizations of sham-mathematical topics are arranged in the schools. In a great many schools, pupils perform on stage "the arithmetic lesson" from Fonvizin's comedy Nedorosl' (The Young Hopeful), and this dramatization is certain to be connected with the mathematical evenings. But what relationship to mathematics does this scene have? What is instructive about it, from a mathematical point of view? If the pupils wish to have amusement or to show off their acting skill, they can either play Nedorosl' entire on the school stage or select certain acts from it, but to put forward the aforementioned "lesson" as a mathematical number for a school evening is intolerable.

We wanted to show by this example that exceptional scrupulousness is needed in the choice of the questions that would be desirable and useful for presenting to children during extracurricular lessons.

Taking local conditions into account, the teacher should decide the question of the form of the historical lessons (a story by the teacher, lectures by the students, readings aloud, and so forth).

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# OVERCOMING STUDENTS' ERRORS IN THE INDEPENDENT SOLUTION OF ARITHMETIC PROBLEMS

O. T. Bochkovskaya*

Common mistakes made by students in solving certain categories of problems have been mentioned in print many times. These mistakes persist in the solving of simple problems of the second level of difficulty -- to which we shall for the time being relegate problems connected with the concepts of difference and ratio --, and in problems of the third level of difficulty -- by which we shall mean problems in finding unknown components of the arithmetical operations and problems whose content corresponds to the content of second-level problems, but whose relationships are not expressed in direct form. Students make a significant number of errors doing complex problems whose component elements are the simple problems mentioned above.

The causes of these mistakes have been established. To solve a problem, one must understand the diverse relationships and dependencies that comprise the substance of it. That is, analytic-synthetic mental activity takes place in problem solving. Underdevelopment of this activity, brought about by the system under which problems are worked out, causes students to make mistakes when they are solving problems by themselves. A fallacious system for working on problems is to use analysis and synthesis separately and to underestimate the importance of encouraging the students' intellectual activeness and independence.

Raising the level of students' skills and eliminating their mistakes in solving problems on their own is possible under a system that is free of the defects mentioned. This system must satisfy several conditions.

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*Of the Leningrad Municipal Institute for Teacher Improvement. Published in Reports [Doklady] of the Academy of Pedagogical Sciences of the RSFSR, 1960, Vol. I, pp. 23-26. Translated by Linda Norwood.

First Condition

An expedient order of introducing problems is needed. This order should take into consideration the difficulties determined by the mathematical content and structure of the problem, and use data from psychology to mark the way to overcome the difficulties arising in mastering problem solving. Furthermore, the broad gap between the introduction of certain kinds of problems needs to be eliminated.

From this standpoint, there is no justification for putting certain kinds of simple problems off to the second and even the third year of instruction. The calculation methods used in first grade in studying the operations give the pupils their first notion of finding both an unknown addend ("How much must be added to 7 to get 10?") and an unknown subtrahend.

Even dividing objects into equal groups (by content) is crucially important to perceiving correctly the first division into equal parts. Any pre-schooler can divide objects into equal groups without trouble, whereas dividing objects into equal parts requires mastery of the division technique. In order to divide a certain number of objects into, say, three equal parts, it is necessary to take a group of three objects each time and distribute them one at a time. If, however, each object is taken one at a time from the whole group of objects to be distributed, a mistake is inevitable because one does not always remember in which "small group" to put the object. And the second stage of division into equal parts under the heading of "verbal operations" is difficult without recourse to equal groups, of which the dividend is composed.

Our observations indicate that it is expedient to use division of objects into equal groups to prepare for division into equal parts, postponing the solution of division-by-content problems, insofar as it consists of two logical links. This sequence eliminates the gap between the two kinds of division and precludes the difficulties in mastering division by content that are continually observed in schools from the moment division by content is put off until second grade.

Eliminating the wide gap in familiarization with problems whose content is a single concept with different unknowns or differently expressed relationships helps children to form precise ideas of the meaning of a problem and of an arithmetical operation, and helps them to surmount the mistakes mentioned above.

Second Condition

The arrangement and order of problem work should utilize the opportunity of spurring the mind to action. This opportunity should be utilized to effect the second condition of a rational system of problem solving, which consists of systematically developing the students' mental analytic-synthetic activity and furthering their independent search for problem solutions.

In this plan, the first addition problems (finding the sum of two numbers), where the children's pre-school experience has paved the way for choosing operations, serve as a basis for understanding problems in which a number is increased by several units. Addition problems (finding a sum) and subtraction problems (finding a remainder) are the basis for understanding problems in which one must find the unknown components of these operations. Problems that include the concept of a difference and ratio (increasing and decreasing numbers by several units and increasing and decreasing them several times, where the connection is expressed directly) precede and are compared with the same problems, but with the relationship expressed indirectly. Simple problems of all categories form the basis for the corresponding complex ones and must precede them. Problems with fewer operations prepare the students for solving problems with more operations; problems with small numbers prepare them for problems using units of measure; problems with a directly worded relation prepare them for problems with a more difficult wording.

To develop the students' mental activity, the teacher's considered use of visual aids and questions is extremely important. Thus, in disclosing the concept of difference, one must not be limited to the visual perception of two groups of objects differing in number. Such a comparison, though ensuring a correct answer does not lead to an understanding of the particular features of this problem group and does not guarantee that the choice of the numerical data for doing the arithmetical operation will be correct. Comprehension can be ensured only by actual doing, when the pupil takes a portion of the objects away from the total number (for example, three saucers from five).

In developing proper conceptions of how quantities are related and how the elements of a problem are connected, it is no less important to prevent mistaken connections and generalizations, as well as to remove any possibility of extending relationships that are meaningful only for a limited number of problems to a larger group of problems. This could be said of the verbal stereotype that is used in analyzing problems: "To find out...we need to know...," which causes difficulty in applying the analysis method to solving problems in which a relationship is expressed in a complicated way.

Formation of this stereotype can be prevented by a series of exercises consisting of systematically varied questions about the same data, as well as different data with the same question. Such exercises promote flexible and mobile comprehension of relationships and lead to a formulation uniting the three factors used in solving problems: the question, the number data, and the arithmetical operation. Basically, this formulation could be expressed like this: "To answer the problem's question, knowing such-and-such, add or subtract, etc." or "Knowing such-and-such, we can obtain the answer to the problem's question by addition... ." Such a statement precludes separation of analysis from synthesis and still removes the obstacles that usually arise when one attempts to use analysis separately from synthesis to solve problems composed of simple problems of the third level of difficulty. Furthermore, when analysis and synthesis are no longer used separately, the habit of selecting arithmetical operations on the basis of the entire condition of the problem is developed.

<u>Third Condition</u>

Of no less importance in a system of teaching problem solving, is the development of self-reliance of the students in the problem-solving process. Observance of the first two conditions without the third is no guarantee that problem-solving skill will be developed. Spirited activity by every pupil, which is expressed initially by his industriously imitating the teacher and which gradually becomes creative self-reliance, is an indispensable condition for a system of problem work, the aim of which is to improve the quality of problem-solving skills and to root out the mistakes students make when they solve problems by themselves.

In the elementary grades, the pupils develop activeness and self-reliance as a result of the alternation of work with the class as a whole -- led by the teacher -- with individual work by every pupil. Knowing that they will have to do subsequent assignments by themselves mobilizes the children's attention when the material is first explained. Appropriate visual aids and the questions the teacher asks while supervising the children focus their attention on the essence and particular features of new problem forms, of which the children eventually become aware when they do the operations themselves. They will be encouraged to greater activeness and self-reliance if each age group is given forms of studying the text of a problem that it can understand. Initially, these forms are expressed in the selection of individual counting-kits and number-boxes corresponding to the condition of the problem, pictures or written numerical data; later, they consist in composing schematic notations of the condition, in drawings, in graphically isolating words that express the special features of the problem, and last, in independently selecting number data for problems and in making up problems.

The students can be made to become more active and self-reliant when solving a problem if they are taught forms of planning a solution and the habit of thinking through the entire solution of the problem beforehand. These forms will be varied gradually in proportion to the pupils' skill and knowledge. At first, thinking out the solution is accompanied by putting aside arithmetical operation signs or by writing them with an oral explanation following. Later it is expressed by using -- besides the operation signs -- other numerical data in the problem, then by ordinary questioning, and finally by devising a numerical formula for the solution.

Development of the students' self-reliance requires ruling out any chance to copy a problem's solution from the blackboard, even if it has already been solved collectively in verbal form. But an opportunity to check one's solution with the one done on the board is an essential aspect of learning how to solve problems. The solution of the problem can be written on the board and the possibility of its being copied can be prevented by using a screen or movable blackboard panels to cover up the solution -- which is done on the board while the class is writing -- until it is time to check. When comparing one's work with that on the blackboard, the ability to notice a discrepancy and to establish the nature of the mistake is of no small importance in eliminating errors.

Thus, errors made by students in solving arithmetic problems by themselves can be overcome by a system of problem work. Under this system, the purpose of the content of the problems, their order, and the method in which they are done is for the students to discover the manifold connections and relationships that make up the problem and to develop self-reliance and active minds. Thus, they are armed with techniques for tackling problems and with the ability to apply these techniques when actually solving and composing problems.

# STIMULATING STUDENT ACTIVITY IN THE STUDY OF FUNCTIONAL RELATIONSHIPS

Yu. I. Goldberg*

According to academic tradition, when functions and graphs are being studied, the development of numbers is not touched upon. That is, the study of functions is isolated from the study of numbers.

This investigation had the following purposes: to study the effect on the students' intellectual activity of including within the study of functional relationships matters related to the development of the number concept, and to study how interweaving number concepts into the study of functional relationships affects the mastery of knowledge of functions and numbers. To effect these purposes, a teaching experiment was organized in Moscow secondary schools No. 585 and No. 187. The investigation was carried out in the scientific-educational laboratory of the Lenin MSPI under Professor I. T. Ogorodnikov.

The students discovered the properties of the function $y = kx$ by giving the teacher answers to such questions as these:

1) Are the values of the function positive or negative when the values of the argument are positive? Why? When the values of the argument are negative? Why?

2) What numbers--whole numbers or fractions--express the values of function that correspond to whole number values of the argument? Why? To fractional values of the argument? Why?

3) How does the function vary when the argument is increased? Why? When the argument is decreased? Why?

4) How do the absolute values of the function vary when the absolute values of the argument are increased? Why? When the absolute values of the argument are decreased? Why?

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5) Which are greater, the absolute values of the argument or the absolute values of the corresponding functions? Why? Can they be equal? Why?

6) Are positive values of the argument greater or smaller than the corresponding values of the function? Why? Are negative values of the argument greater or smaller? Why? Is the zero value of the argument greater or smaller? Why?

7) How does the function vary when the absolute values of the argument are increased indefinitely? Decreased indefinitely?

Results of the experiment

After studying the function y = 3x, two eighth-grade classes studied the function y = -2.5x. In answering the above questions, many students incorrectly stated that when the argument is a fraction, the values for the function can only be fractions.

A large part of both classes had problems telling how the function y = -2.5x varies when the argument is increased (or decreased). Even after several particular values of the argument were taken and the corresponding function values obtained, many students answered uncertainly and clung to the positive argument. "If the argument is positive," declared Student V, "then it increases; but since the function here is negative, the function decreases." Here that long-standing misconception, so difficult to root out, was being expressed: No matter what and no matter where it occurs, students identify a plus sign with an increase and a minus sign with a decrease. This was exemplified in the answers Student L and Student T gave at the next lesson, when they said that when the absolute value of the argument is increased, the absolute value of the corresponding function does not change, since both are positive. Students N and G thought that the absolute values of a function are obtained by multiplying the absolute values of the corresponding argument values by -2.5. They were corrected by students L and N. When student N studied the same function, y - 2.5x, she completely disagreed that multiplying the absolute values of the argument by 2.5 increases them, but when Student G explained it again, N agreed.

Only individual cases of students' not knowing and not understanding elementary ideas concerning number and functional relationship have been cited here. There were many of these instances at the beginning of the investigation; they were encountered at every step. As the teaching experiment progressed, however, there were fewer and fewer instances; the students felt more sure of themselves in matters of functional relationship, after having spent time and energy (though not a great deal), than under the usual approach to these topics.

In the homework, they were to answer the first three questions in regard to the function $y = -4x$. A check of her homework revealed that Student P did not know that negative values of the function corresponded to positive values of the argument. Students A and T incorrectly declared that when the values of the argument are fractions, the function can have only fractional values. For the same function, Student L said, "When the argument increases, the function decreases, since the product of a negative number and a positive one is always negative, and a negative number is less than a positive." Here a good pupil failed to understand the idea of a function or the idea of a relation; she did not understand that when values of the argument vary, the corresponding values of the function change -- regardless of whether they are positive or negative.

Questions 1 - 7, which were examined in this investigation, force the students to actively comprehend the meaning of other mathematical ideas and facts and to penetrate their essence, as well as to study number evolution thoroughly. Student R brought up nothing about the argument's fractional values but talked only about the integral ones. Apparently, what was being expressed is the prevailing tendency of students to deal only with whole numbers as far as possible. Functional relationship is not fully understood. Without any explanation the student wrote "When the argument is negative, the function increases; and when it is positive, the function decreases." No reasons were given why the absolute values of the function would increase or decrease when the absolute values of the argument vary, and it is not values that were discussed, but a value. It is not clear how the function can increase or decrease when the argument has one and the same value. The pupil did not understand the notion of correspondence; he did not understand that the values of the function under consideration could vary only with different argument values, that only definite function values can correspond to values of the argument.

If matters of number development are raised, the students are able to consider matters of functional correspondence effectively and actively, and positive results may be arrived at with both topics.

Besides the functions y = 3x, y = -4x, and y = -2.5x, the functions y = x and y = -2/3x were considered in the experimental work with the students.

## Analysis of the results of the experiment

In the control classes, the majority of the students did not have a proper idea of the absolute value of a number; in the experimental classes, the students had mastered the concept of absolute value. An overwhelming majority in the control classes tended to allow the argument (the function) either positive or negative values only; in the experimental classes, the students considered all possible numerical values of the argument (function).

Almost all students in the control classes, even though they would sometimes discuss certain properties of certain numbers correctly, talked about them when there was no need, without any connection with those properties of the function being studied that they were trying to substantiate. Such instances occurred rarely in the experimental classes where they were observed only in a few students.

Most students in the control classes who managed to perceive certain property of a function correctly either completely failed to explain it by the number properties that conditioned this property, or explained it incorrectly. In the experimental classes, such instances were very rare toward the end of the work.

Many students in the control classes even compared the size of positive and negative numbers incorrectly. In the other two classes such cases were not observed.

Many students in the control groups failed to see the numerical values of a function, did not understand that both the function and the argument take numerical values. In the experimental classes such cases occurred only at the beginning of the year.

A decrease in anything at all was immediately associated with negative numbers and an increase with positive numbers only during the first half of the year in the experimental classes; in the control classes, on the other hand, such occurrences were observed quite frequently even toward the end of the year.

In the control classes, even at the end of the year, many students in their reasoning immediately connected multiplication by a fraction with a decrease in the multiplicand; this had stopped by the end of the year in the experimental classes.

Almost all students in the control classes actually did not understand the idea of a functional correspondence, discussing the value of a function or the value of the argument instead of the set of their values, and they did not understand -- even in the elementary cases discussed -- how the values of the function vary depending on the change in the values of the argument. The idea of even the simplest functional relationship -- direct proportionality -- remained something indefinite for them, although the classes did almost all the exercises in the standard problem-book and textbook that belonged to the section on functional relationship, with both teacher and pupils giving them serious attention. In the experimental classes, when functional relationship was being studied, attention was directed to the nature of the numerical values of the variables, to the fact that certain properties of the argument and function are conditioned by the properties of the numbers that are in the range of the function under discussion. Therefore the errors mentioned were not observed here in the students' comprehension of functional relationship.

Most students in the control classes did not know the limits to which the argument and function can vary; the students in the experimental classes knew them for the basic functions.

Most students in the control classes tended to ascribe only natural values to the argument (function), a smaller number mentioned negative values, and an even smaller number had in mind any values for the argument and function. These defects were observed in the experimental classes for a long time, but then the students began to discuss all possible values of the argument and function.

The control classes usually did not understand how to discover whether a function increased (or decreased) when the argument increased (or decreased). The students in the other two classes handled this with ease.

The entire course of the teaching experiment, the teacher's detailed record with notes on all the lessons conducted, the substance and results of the checked and analyzed homework, class work, and test papers all permit the following conclusions to be made:

1. The active study of functions, with analysis of the development of the number concept, is fully possible for all eighth-grade students.

2. The work is interesting for the students, provoking them to ask many serious questions, and the children work actively.

3. The work does not demand additional expenditure of school time compared to the traditional, isolated treatment of functions and numbers in secondary school; the interweaving of both subjects has a positive effect on the way students learn.

4. The control work for comparing the two experimental classes with ~~the two~~ control classes, proves that when the topics of numbers and functions are treated in isolation in the traditional manner, students demonstrate many grevious defects in their knowledge of both numbers and functions, while this was true only in isolated cases in the experimental classes.

Thus, under the method of studying functional relationship generally in use, the students learn the meaning and properties of functions passively and formally. Every step of the experiment was a confirmation of this. With every new step of the study, this idea was further substantiated: When number problems are woven into the study of functional relationship, the mental activity of the students is stimulated, and the nature of functional relationships, as well as the basic properties of numbers, is clarified.

# PSYCHOLOGICAL GROUNDS FOR EXTENSIVE USE OF UNSOLVABLE PROBLEMS

Ya. I. Grudenov*

The authors of several methodological works ([2: 72-3; 3: 14] and others) have come out in favor of using mathematical problems that have no solution. There is a very small number of such problems in Abugova's workbook [1] and in several other workbooks. Nevertheless, the fact that these problems are not contained in most workbooks evidently indicates that their authors either ascribe no special attention to them or view them negatively.

In this article we consider three problems: 1) proof of the expediency of using unsolvable problems, 2) the technical aspect of organizing the solution of such problems, and 3) the most expedient form in which to put these problems into workbooks.

The following statement of psychological Principle I appears in Shevarev [4]: If some repeated features of problems solved in the education process are such that cognition[1] of them is not, in a given situation, a necessary condition for the completion of correct operations, then the degree of the cognition of such features is more or less reduced.

Beginning with this principle, one may deductively prove the expediency of using unsolvable problems extensively in mathematical instruction. Say that the pupils solve n homogeneous problems while studying some mathematical topic. Let us denote the processes of the solutions of these problems schematically:

$$A_1B \rightarrow M_1;\ A_2B \rightarrow M_2;\ \cdots\ ;\ A_kB \rightarrow M_k;\ \cdots;\ A_nB \rightarrow M_n,$$

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*Of the Kursk Pedagogical Institute. Published in "New Research in the Pedagogical Sciences, II," Proceedings [Izvestiya] of the Academy of Pedagogical Sciences, 1964, Vol. 133, pp. 108-112. Translated by David A. Henderson.

[1]The word cognition is used here as a generic term embracing sensation, perception, conception, and thought.

where B is the common feature of these problems, whose cognition is not a necessary condition for completion of correct operations, $A_k$ is the set of all the other features of one of these problems, and $M_1$, $M_2$, ..., $M_n$ are various operations in the solutions. Under these conditions, according to Principle I, there can arise in many pupils an erroneous generalized association, whose first element contains cognition of features $A_1$, $A_2$, ..., $A_n$ but not of feature B. The root of the problem is that in the solution of homogeneous problems only of the type $A_kB$ the error of this association is not discovered, since regardless of whether the pupil recognizes feature B or remains completely unaware of it, he will obtain the same result--$A_kB \rightarrow M_k$. Therefore the pupil is only apparently successful; the teacher may not even know that many pupils who solve the problem of the type $A_kB$ correctly are not fully cognizant of the conditions. That is, they are unaware of feature B.

If pupils who make this erroneous association are given three different problems of the type $A_kB$, $A_k$, and $A_k\overline{B}$ , where $\overline{B}$ is a feature oppostite B but externally similar, it is very probable that they will solve these problems in the same way: $A_kB$ -- $M_k$, $A_k \rightarrow M_k$, and $A_k\overline{B} \rightarrow M_k$. In the last two instances the answer will be wrong. Thus the incorrectness of the association is revealed.

If, from the beginning of the study of a given topic, the pupils are given problems of the type $A_k$ (problems with insufficient data) and $A_k\overline{B}$ (problems with a contradictory condition or of opposite nature) together with problems of the type $A_kB$, then those pupils who make the incorrect association will make the mistakes $A_kB \rightarrow M_k$ and $A_k \rightarrow M_k$. Such mistakes will indicate to the teacher that the pupils are not aware of feature $\overline{B}$. Analysis of the mistakes permits an erroneous association to be corrected in the pupils before it can be reinforced.

It follows, therefore, that the negative influence described by psychological Principle I is most simply and expediently eliminated by introducing unsolvable problems and problems with superfluous, insufficient, and contradictory data into the system of exercises.

To confirm this statement, we cite some examples.

All problems that are usually solved in teaching the feature of the congruency of triangles having two corresponding sides and the angle between them equal have a constant factor: equal angles in the triangles are enclosed between equal corresponding sides. As long as only such problems are solved, a pupil can correctly solve any one of them—even if he is not cognizant of this feature. This means that, according to Principle I, the pupils' recognition of this feature is narrowed. This was confirmed by our observations in several sixth grades in which the pupils learned fairly well how to solve problems on proof according to prepared drawings of all the features of congruent triangles. In two of these classes, where unsolvable problems had not been used previously, not a single pupil was aware of the fact that in an ordinary problem on proof with a prepared drawing, the equal angles of two triangles were not contained between equal corresponding sides, although everyone formulated the appropriate theorem correctly. In the classes in which the pupils had previously encountered unsolvable problems, only a few of the better pupils noticed that the proposed problem had no solution. After several such problems had been solved in these classes, most pupils immediately noticed whether equal angles were contained between correspondingly equal sides of given triangles.

All problems usually solved in the study of the theorem about the property of the angle bisector at the vertex of an isoceles triangle have a constant factor: it is always a matter of only the bisector of the angle at the vertex of an isosceles triangle. As long as only such problems are used, cognition of this feature has no influence on the result of the solution; hence, according to Principle I, many pupils cease to recognize this feature when solving problems. Our investigations in several classes confirmed that if pupils are given a prepared drawing with the unsolvable problem in which one must calculate the segments into which the side of an isosceles triangle is cut by the bisector of a base angle, given that the sides are 6 cm each, almost all pupils will answer "into equal segments of 3 cm each." The solution of such problems has the result that most pupils begin to turn their attention to this feature.

What is technically involved in organizing the work with unsolvable problems?

The pupil himself should discover that a problem cannot be solved and why it cannot be solved. In such cases it is established how the condition must be changed so the problem will have a solution. The pupil corrects the condition of the problem and solves it.

<u>The entire essence of such problems lies in the fact that the pupils are not told beforehand about the contradictory or incomplete condition.</u> If a pupil does not turn his attention to the essential data, he "solves" the problem and falls into the trap.

When such problems are solved systematically, the pupils become more careful and try to read the conditon of a problem more attentively and thoughtfully. But if some pupil hurries and, without thinking out the condition of an unsolvable problem, "solves " it, this will evoke laughter from the class. Better than any comment by the teacher, this laughter will force the pupil to be more careful and attentive. Both he and the other pupils will now better remember the essential feature that made the problem unsolvable.

Unsolvable problems add interest and enliven the classwork, but only when at least some of the pupils have mastered this fact (the absence of a solution). The pupils who are first to discover the error in the condition of a problem usually wait with baited breath to see how their friends will react to the problem. And as soon as some "fool" begins to "solve" the problem, there is a burst of laughter in the class and a forest of hands goes up. Everyone hurries to tell what he has guessed and his method of changing the condition of the problem.

The effect of using an unsolvable problem is sharply curtailed when the teacher does not preserve an imperturbable, impassive appearance and smiles or asks a careless question hinting that the problem is unsolvable. Then the pupils have no need to be careful and attentive, for there is nothing for them to guess.

Such problems must, of course, be solved with a teacher's supervision so that any incorrect solution will be discovered and analyzed immediately and will not pass unnoticed by the pupils.

In the 1960-63 school years many teachers (about two hundred) of Kalinin Oblast used the System of Exercises in Geometry for the Fourth Grade, composed by the author of this article and printed by the local Teacher Improvement Institute. In this system of exercises there are a great many unsolvable problems. Almost all teachers who returned the questionnaires noted that the pupils showed increased interest in the unsolvable problems, as a result of which these pupils became more attentive and careful.

In the 1962-63 school year eight teachers from Kalinin (in 19 sixth and seventh grades) worked on the project of a geometry textbook and workbook (written by the author of the present article) containing unsolvable problems. Observations in these classes confirmed that unsolvable problems add interest to the classwork. All eight teachers participating in the experiment noted that extensive use of unsolvable problems resulted in the pupils' solving problems more thoughtfully, their critically approaching the conditions of problems, their learning to get out of difficult situations independently, etc.

Let us consider an illustration. In an eighth grade the pupils solve orally problems written on the blackboard:

1. Calculate the side of a square if its area is 578.2 sq. cm.
2. The area of a rectangle with equal adjacent sides is 28.34 sq. cm. Calculate its sides.
3. Calculate the side of a rectangle if its area is 435 sq. m.

In solving the second problem some of the pupils, of course, did not recognize the essential part "equal adjacent sides." But they obtained the correct result (which, unfortunately, occurs quite often). They solve the third problem. A minute's pause. Almost everyone who did not examine the condition of the problem was trying to find $\sqrt{435}$. One hand goes up, then another, then a third. Some pupils look at the condition again, glance at the imperturbable teacher, look at each other, and indecisively lower their hands. All the better pupils feel some misunderstanding intuitively. The poor and average pupils do not notice this, and the number of raised hands increases. One girl raises her hand, then lowers it. The pupil called to the blackboard says, "The side of the rectangle is 20.86." Immediate laughter. Everyone instantly grasps the difference between the second and third problems. A forest of hands goes up.

As mentioned at the beginning of this article, Abugova's Workbook 1 contains several problems (numbers 278, 317, 498, 503, and others) that the authors probably present as unsolvable problems. The texts of all these problems, however, are accompanied by questions such as: What condition do you have to supplement the problem with? Isn't there some contradiction in the problem's condition? What data in the condition of the problem is superfluous? In solving such problems the pupils no longer need be attentive and careful; there is nothing for them to guess at. On the other hand, if the unsolvable problems are not accompanied by any hints or answers, the erroneousness or incompleteness of the conditions of individual problems may be passed by unnoticed by the pupils. The experiment described above showed that it is profitable to follow up each unsolvable problem in a workbook (with one or two intervening problems) with problems or questions that return the pupils' attention to the unsolvable problem.

> No. 385. In triangle AEK, $\angle A = 62°$, $\angle E = 75°$, $\angle K = 53°$. Calculate the external angles of the triangle.
>
> No. 386 (or (387). Eliminate the error in problem 385 and solve it.. Can the amount of Data given in the problem be reduced?

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PSYCHOLOGICAL CHARACTERISTICS OF PUPILS' ASSIMILATION OF THE CONCEPT OF A FUNCTION

I. A. Marnyanskii*

Mathematics methodology has given much attention to problems in learning one of the central concepts of the school algebra course -- the concept of a function. Yet, this concept remains the Achilles' heel of mathematics instruction. One reason for the failure of the current methodology of the study of this concept is that the psychology of pupils' mastery of this complex concept has been poorly studied. True, in our educational psychology there are several investigations [3, 5] of the pupils' mastery of functional relationship, but these investigations deal only with the introduction to the concept of function.

Methodology of the experiment

To determine the psychological difficulties involved in secondary school pupils' mastery of the concept of function, the author (with the cooperation of teachers and students at the pedagogical institute) conducted experiments (1958-1962) in the schools of Rovno. The experiments were predominantly exploratory and were conducted in the form of written questions and discussions. The experiments encompassed 132 pupils of grades 8-10 in five schools.

Although the written questioning (in addition to the control work, the pupils were asked to answer one or two questions, explaining the motives for their answers in detail) was ineffective (usually the answers were too brief and the pupil's train of thought could not be established), it did show what points in the concept of function are incorrectly or imprecisely mastered. This made it possible for us to choose questions for the discussions.

The discussions were held with groups of from three to eight pupils, mostly above-average. On the day before the discussion the teacher told selected pupils of the topic to be discussed, so they could

*Of the Rovno Pedagogical Institute. Published in New Research in the Pedagogical Sciences, 1965, Vol. IV, pp. 79-86. Translated by David A. Henderson.

prepare for it. Before the discussion the pupils were told that, although their answers would be recorded, they could safely say anything they thought about the questions, since any errors in their answers would not affect the evaluation of their achievement in school. Experience showed that without this information an upper-grade pupil who was doubtful about the correctness of an idea would often answer very briefly or remain silent altogether.

Each pupil was given one or two questions, and, when he was ready to answer, the discussion was begun. The other pupils were also invited to take part, and thus we could determine all the pupils' opinions on questions given to each of them.

The discussion ended with an explanation of the correct answers and with control questions for establishing whether the pupils understood the nature of their mistakes.

Results of the experiment

One of the first tasks of the experiment was to check whether the pupils had a clear awareness [4:45] of the concepts on which the concept of function is based. Such concepts include the variable quantity, the set, and the functional relationship.

Almost all pupils understood the meaning of the concept of a variable quantity and gave examples of variables from geometry and physics. But, as we learned during the discussions, the pupils could not explain precisely what is meant by a quantity in general. This is not merely a matter of their inability to describe the concept of quantity correctly (the individual answers were close to the truth: "Quantity is what is measured," "Quantity is that which is large or small"), but of the tendency toward an improper expansion of the scope of this concept when concrete examples are considered: "Animosity can be a quantity since it can be great or small," "Studiousness is also a quantity -- it can be measured by desire," "Responsibility should be considered a quantity, since it is greater and lesser."

The subjects' understanding of the concept of a set was imprecise. Of the 46 pupils questioned, only three gave a positive answer to the question, "Can a set contain only one number?" Other pupils stated that "a set means very many," or "a set is an aggregate, and an aggregate means several."

Another error often repeated by the pupils during the discussions was a misconception of the properties of <u>infinite sets</u>. In particular, they did not imagine the existence of infinite number sets not containing the smallest number. Thus, many pupils stated unhesitatingly that in the set of all rational numbers there is the smallest number, since "the very first number should be there." They were often unable to distinguish the finite and the infinite set.

Thus, one of the ninth-grade pupils, answering the question of the finiteness or infiniteness of the set of all apples on the Earth, said: "The set of all apples on all the trees of the Earth is infinite, since while we are counting, new ones will grow." And one tenth-grader stated, "There is no infiniteness in itself; it is like an abstract number." Several tenth-graders gave the number $\pi$ as an example of an infinite number set.

In the pupils' understanding of the concept of <u>functional relationship</u> two tendencies were observed: unsubstantiated restriction of the scope of this concept (when relationship is taken to mean only several simple kinds of relationships, and only the causal connection in examples with concrete content from everyday life) or extreme expansion of its scope (when quantities are considered functionally dependent even when their connection is undefined, i.e., when there is no directed one-to-one relationship between their values).

These tendencies may be illustrated by three responses to the questions whether there is a functional relationship between the amount of rainfall and crop size: 1) there is no relationship, since with an increase in the amount of rainfall, the crop also increases at first, but when there is too much rain the crop begins to decrease; 2) there is a relationship, because in a drought the crop is bad, that is, low rainfall causes a poor crop; 3) there is a relationship because the crop is connected somehow with the amount of rainfall.

The subjects' vague awareness of the concepts examined above led, of course, to errors when they were solving the problem of the presence or absence of a function in a specific case. Thus, the subjects often regarded as functions things that were not quantities at all ("A pupil's success is a function of his attentiveness," "The quality of work is a function of the mood of the worker"), or did not consider quantities functions if the set of their values contained only one number ("The sum of the functions $x^2$ and $2 - x^2$ will not be a function, since it is equal to 2, and 2 is a constant"), or they disregarded the requirement that there be a one-to-one correspondence between the values of quantities ("The crop is a function of the area seeded, since the crop depends on the area of the field, on how the land is worked, and on the amount of rainfall").

The experiment showed that in most pupils' minds the "strongest" features of the function were not the one-to-one correspondence between the values of two quantities, but the changeability of the quantities and the presence of a general or a causal connection between them. For example, the subjects repeatedly made statements like: "The height of a man is a function of his age only up to age 25; after that his height stays the same, and there is no function because a function cannot stand still when its argument changes," "A function cannot assume identical values, since each value of the argument corresponds to a specific value of the function, i.e., another value," "There is functional relationship between the time in motion of an auto that makes a stop along the way and the route it travelled if the auto makes only short stops -- then the route is almost constantly changing." It is clear from the last argument that several pupils are beginning to understand intuitively the baselessness of their requirement of unconditional change. Here is one more interesting example, which shows that in the pupils' opinion there should be at least some visible evidence of the change: "A function taking only one value can exist, but then it must be written: $y = x^0$."

It is common knowledge that in solving problems with concrete content from everyday life, the subject may make secondary associations that hinder the actualization of his conceptions [6]. Hence, besides practical problems from everyday life, the pupils were given problems in which the functional relationship appeared in a pure form. But even here the number of incorrect answers was significant.

Thus, more than half the pupils questioned either did not answer or gave a negative answer to this question: If a quantity x takes only integral positive values, and quantity y is equal to one when the value of x is even, and equal to zero when the value of x is odd, is y a function of x? Pupils answering this question negatively (they constituted more than a third of those questioned) indicated that "here y can remain unchanged while x changes, but a function must change when its argument changes."

Finally, let us note the following circumstance, which is amazing at first glance. The experiment showed that the general conception of a function in tenth graders, who know a large number of functions is almost the same as that of eighth graders, who have just begun studying elementary functions.

Analysis of the results

Let us now try to give a general picture of the psychological difficulties in mastery of the concept of a function and ascertain the causes of individual misconceptions.

Teachers and methodologists often try to explain the pupils' poor mastery of the concept of a function by asserting that the concept is too abstract and hence difficult for the pupils to understand. But such an explanation is not convincing. First, the more abstract is not always more difficult psychologically. For example, the concepts "kinsman" and "polyhedron" are no less incomprehensible to the pupil than the concepts "grandnephew" and "truncated icosahedral pyramid." Second, in the school mathematics course there are concepts (e.g., the concept of the complex number) no less abstract than the concept of a function, but which the pupils master better than the concept of a function.

The main difficulty in mastering the concept of a function is that this concept exceeds the bounds of what is ordinary for the pupil, that it does not remind the pupil of any familiar concepts (unlike complex numbers, similar in many ways to real numbers, with which the pupil is quite familiar).

Up until the study of the concept of a function, the pupil thinks mainly in terms of individual images and is not used to thinking in terms of groups of objects. Practically, he remains trained in operations with constants. Actually, the introduction of functional relationship in school almost wholly consists of familiarizing pupils with facts such as the change in the sum when there is a change in an addend, the change in the size of a fraction when the numerator or denominator is changed, the dependence of the value of an algebraic expression on the number value of the letters entering into the expression, and so on. Here the main attributes of functional relationship -- the concepts of the set and the one-to-one relationship between elements of two sets -- remain hidden. Indeed, the pupil is in no position to reinterpret independently, for example, the fact of the relationship of the sum to the addend, when he sees an example of the one-to-one relationship of numbers of two sets (the set of values of the addend and the set of values of the sum). Pupils in general attribute no special importance to a statement of the type, "If we change the addend, then the sum is changed." Many of the pupils (as questioning of several fifth-graders showed) either do not know why they are told such facts, or perceive such a statement as one of the cautions against changing the addend arbitrarily, since this can lead to an incorrect result.

This is why the pupil, when becoming familiar with the variable quantity before he studies the concept of a function, perceives the variable as a unique object, is unaware of the set of numbers (as a rule, an infinite set) concealed behind it. Under these conditions the definition of the function given in contemporary school textbooks is entirely justified. This definition speaks of two variables, and not of two sets and their correlations. It is easier for the pupil to imagine the connection between two single objects (variable quantities) than between two sets. However, such a definition can serve only as a rough basis for forming the concept of a function per se in the pupil. The next step should be for the pupil to interpret the fact that the definition of function deals with the correlation of two sets consisting of values of quantities discussed earlier in this definition. But the pupil is not prepared for this by the introduction practiced at present.

For proper mastery of a concept (especially a complex one), being familiar with a large number of objects encompassed by this concept is not at all sufficient. In developing a precise notion of a new concept, it is important to compare it with other concepts somehow similar but not identical to the given one. Indeed, even concepts like "perpendicular," "polynomial," and "cone" are easily mastered by the pupil owing not only to their simplicity and the large number of examples considered, but also the the pupils' familiarity with objects like "incline," "monomial," "pyramid." Hence, for the best assimilation of the concept of a function, the pupil should have an opportunity to compare the function with other objects that are not concepts but whose peculiarities remind one of functions. But there are no such objects (they may be called "pseudofunctions") in the school mathematics course, and their construction and inclusion in the school textbooks is evidently considered useless by methodologists and teachers. Let us remark that it was pseudofunctions that enabled us not only to reveal a number of misconceptions in the pupils, but also to explain the meaning of the subjects' errors to the subjects themselves.

Sources of the pupils' errors

The pupils' vague awareness of the concept of magnitude, or quantity, is explained primarily by the textbook's failure to describe this concept precisely and by its free use of the term "magnitude" ("comparison of angles by magnitude," "the magnitude of fractions," "the absolute magnitude of a number"). Moreover, the examination of examples of pseudomagnitudes is not practiced in the school, although it essentially helps (as our experiment showed) delimitation of all essential peculiarities of this concept (the magnitude must somehow be measured and expressed by a number).

The pupils' refusal to regard a one-element set as a "legal" example of a set, as well as their identification of the concept of dependency with the concept of causal connection is explained mainly by a certain influence from the everyday meaning of the term ("A set is very many," "Salary depends on responsibility") on the mastery of the concept [1].

The pupils' confusion in questions dealing with infinite sets has several causes. Poorly informed in these questions, the pupils automatical ascribe to such sets properties, known from experience, of finite sets ("A set has to have the first number"). Such logical analogy is not difficult to explain psychologically: we observe here one of the phenomena of that common peculiarity of thought -- the inclination toward the stereotype. A striking phenomenon of this inclination in the study of the function is the pupils' frequently observed confusion when symbols with which they are accustomed to designating the argument and the function are replaced by new letters. In this connection let us note that a particular phenomenon of that same peculiarity of thought is that when solving geometry problems the pupils "follow the drawing." Therefore, we cannot agree with P. Ya. Gal'perin and N. F. Talyzina when they say that this "is explained not by the peculiarities of children's thinking, but by the peculiarities of the methodology of instruction [2:34]." It would be more correct to say -- without disputing the tendencies to follow the drawing -- that the effectiveness of opposing the negative influence of this peculiarity depends on the methodology of instruction.

It is also not hard to explain pupils' arguments of this type: "There is no infinity as such," or "The set of all apples on Earth is infinite." These arguments show that the pupils imagine infinity only as potential infinity. This is explained not only by the fact that potential infinity is in essence closer to the very familiar concept of the finite than to the concept of actual infinity; it also results from the pupils' conception of the variable as something gradually running through an unlimited number of values. For this reason, pupils' answers often show no actualization of knowledge of familiar examples of actual infinity (the set of all points of a segment, the set of all rational numbers, etc.). As for including $\pi$ in infinite sets, here the need to use an infinite number of decimal places for writing this number is associated with the process of changing a variable quantity, and the latter with the infinite set.

That many pupils consider unconditional variability of a function its most important property (as we established above) may be explained in two ways: First, in the textbook the concepts of variable and constant are presented only as opposites, and the variable is called a function. Second, the introduction to functions, focusing its attention on giving pupils an idea of what a variable is, creates a unique "variable hypnosis."

As we have observed, tenth-graders' general idea of the function is not much more precise than that of eighth-graders. This paradox occurs because, after having formally studied the definition of a function, the pupils do not use it in practice. They have no need to resort to this definition, either when seeking functions from among examples of all possible functional and nonfunctional relationships between objects (not a single subject could cite an example of a relationship that resembled a functional relationship but was not), or later, when trying to convince themselves that each of the concrete functions studied was indeed a function.

The present investigation allows several conclusions concerning methodological order to be made:

1. The introduction to functional relationship in the school should include familiarizing the pupils with the concepts of the number set and the correlation between the numbers of two sets.

2. There is a need to develop a system of exercises to help pupils recognize typical inessential features (e.g., the variability of the values of a function) and generalize the essential features of a function (single-valued part of a function or a one-to-one relationship). It is important that the pupil himself be able to cite an example of a function having some peculiarity, including functions having a "pathological" property, like remaining constant over some interval, or constancy in general, which would help to transfer the pupil's attention from variability to the single-valued, unchanging correlation.

3. To form a precise idea of the concept of function in the upper-grade pupils, it is necessary to explain to them the meaning of the term "variable" as the general designation for numbers of some set.

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