DOCUMENT RESUME

ED 184 839 SE 030 455

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TITLE Soviet Studies in the Psychology of Learning and Teaching Mathematics, Volume XI: Analysis and Synthesis as Problem-Solving Methods.
INSTITUTION Chicago Univ., Ill.; Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 75
GRANT NSF-G-8847(MA)
NOTE 186p.; For related documents, see SE 030 451-458, ED 042 628, ED 042 632-633, and ED 080 382-384.

EDRS PRICE MF01/PC08 Plus Postage.
DESCRIPTORS Elementary Education; *Elementary School Mathematics; Learning Theories; *Mathematics Education; *Mathematics Instruction; *Problem Solving; Psychological Studies
IDENTIFIERS School Mathematics Study Group; *Soviet Research; USSR

ABSTRACT

This is one of a series that is a collection of translations from the extensive Soviet literature of the past 25 years on research in the psychology of mathematics instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English. The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This volume differs from the others in the series in that the entire volume records the search for a method of problem-solving instruction based on the analytic-synthetic nature of the problem-solving process. In this work, Kalmykova traces the history of the use of the analytic and synthetic methods in her country, explores elementary classroom situations involving teachers who had various degrees of success in problem-solving instruction, makes hypotheses regarding the use of certain techniques, and concludes with suggestions for "productive" methods to be used in the classroom. (Author/MK)

G-8847 (MA)

SOVIET STUDIES IN THE PSYCHOLOGY OF LEARNING AND TEACHING MATHEMATICS

VOLUME XI

SCHOOL MATHEMATICS STUDY GROUP
STANFORD UNIVERSITY
AND
SURVEY OF RECENT EAST EUROPEAN
MATHEMATICAL LITERATURE
THE UNIVERSITY OF CHICAGO

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SERIES
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VOLUME XI

ANALYSIS AND SYNTHESIS
AS PROBLEM-SOLVING METHODS

VOLUME EDITOR
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Financial support for the School Mathematics Study Group and for the Survey of Recent East European Mathematical Literature has been provided by the National Science Foundation.

PREFACE

The series Soviet Studies in the Psychology of Learning and Teaching Mathematics is a collection of translations from the extensive Soviet literature of the past twenty-five years on research in the psychology of mathematical instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. The series is the result of a joint effort by the School Mathematics Study Group at Stanford University, the Department of Mathematics Education at the University of Georgia, and the Survey of Recent East European Mathematical Literature at the University of Chicago. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English.

Research achievements in psychology in the United States are outstanding indeed. Educational psychology, however, occupies only a small fraction of the field, and until recently little attention has been given to research in the psychology of learning and teaching particular school subjects.

The situation has been quite different in the Soviet Union. In view of the reigning social and political doctrines, several branches of psychology that are highly developed in the U.S. have scarcely been investigated in the Soviet Union. On the other hand, because of the Soviet emphasis on education and its function in the state, research in educational psychology has been given considerable moral and financial support. Consequently, it has attracted many creative and talented scholars whose contributions have been remarkable.*

Even prior to World War II, the Russians had made great strides in educational psychology. The creation in 1943 of the Academy of Pedagogical Sciences helped to intensify the research efforts and programs in this field. Since then the Academy has become the chief educational research and development center for the Soviet Union. One of the main aims of the Academy is to conduct research and to train research scholars

*A study indicates that 37.5% of all materials in Soviet psychology published in one year was devoted to education and child psychology. See Contemporary Soviet Psychology by Josef Brozek (Chapter 7 of Present-Day Russian Psychology, Pergamon Press, 1966).

in general and specialized education, in educational psychology, and in methods of teaching various school subjects.

The Academy of Pedagogical Sciences of the USSR comprises ten research institutes in Moscow and Leningrad. Many of the studies reported in this series were conducted at the Academy's Institute of General and Polytechnical Education, Institute of Psychology, and Institute of Defectology, the last of which is concerned with the special psychology and educational techniques for handicapped children.

The Academy of Pedagogical Sciences has 31 members and 64 associate members, chosen from among distinguished Soviet scholars, scientists, and educators. Its permanent staff includes more than 650 research associates, who receive advice and cooperation from an additional 1,000 scholars and teachers. The research institutes of the Academy have available 100 "base" or laboratory schools and many other schools in which experiments are conducted. Developments in foreign countries are closely followed by the Bureau for the Study of Foreign Educational Experience and Information.

The Academy has its own publishing house, which issues hundreds of books each year and publishes the collections Izvestiya Akademii Pedagogicheskikh Nauk RSFSR [Proceedings of the Academy of Pedagogical Sciences of the RSFSR], the monthly Sovetskaya Pedagogika [Soviet Pedagogy], and the bimonthly Voprosy Psikhologii [Questions of Psychology]. Since 1963, the Academy has been issuing collection entitled Novye Issledovaniya v Pedagogicheskikh Naukakh [New Research in the Pedagogical Sciences] in order to disseminate information on current research.

A major difference between the Soviet and American conception of educational research is that Russian psychologists often use qualitative rather than quantitative methods of research in instructional psychology in accordance with the prevailing European tradition. American readers may thus find that some of the earlier Russian papers do not comply exactly to U.S. standards of design, analysis, and reporting. By using qualitative methods and by working with small groups, however, the Soviets have been able to penetrate into the child's thoughts and to analyze his mental processes. To this end they have also designed classroom tasks and settings for research and have emphasized long-term, genetic studies of learning.

Russian psychologists have concerned themselves with the dynamics of mental activity and with the aim of arriving at the principles of the learning process itself. They have investigated such areas as: the development of mental operations; the nature and development of thought; the formation of mathematical concepts and the related questions of generalization, abstraction, and concretization; the mental operations of analysis and synthesis; the development of spatial perception; the relation between memory and thought; the development of logical reasoning; the nature of mathematical skills; and the structure and special features of mathematical abilities.

In new approaches to educational research, some Russian psychologists have developed cybernetic and statistical models and techniques, and have made use of algorithms, mathematical logic and information sciences. Much attention has also been given to programmed instruction and to an examination of its psychological problems and its application for greater individualization in learning.

The interrelationship between instruction and child development is a source of sharp disagreement between the Geneva School of psychologists, led by Piaget, and the Soviet psychologists.* The Swiss psychologists ascribe limited significance to the role of instruction in the development of a child. According to them, instruction is subordinate to the specific stages in the development of the child's thinking—stages manifested at certain age levels and relatively independent of the conditions of instruction.

As representatives of the materialistic-evolutionist theory of the mind, Soviet psychologists ascribe a leading role to instruction. They assert that instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages. The Russians study development in the changing conditions of instruction, and by varying these conditions, they demonstrate how the nature of the child's development changes in the process. As a result, they are also investigating tests of giftedness and are using elaborate dynamic, rather than static, indices.

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*See The Problem of Instruction and Development at the 18th International Congress of Psychology by N. A. Menchinskaya and G. G. Saburova, Sovetskaya Pedagogika, 1967, No. 1. (English translation in Soviet Education, July 1967, Vol. 9, No. 9.)

Psychological research has had a considerable effect on the recent Soviet literature on methods of teaching mathematics. Experiments have shown the student's mathematical potential to be greater than had been previously assumed. Consequently, Russian psychologists have advocated the necessity of various changes in the content and methods of mathematical instruction and have participated in designing the new Soviet mathematics curriculum which has been introduced during the 1967-68 academic year.

The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession, for it is generally recognized that experiment and research are indispensable for improving content and methods of school mathematics.

We hope that the volumes in this series will be used for study, discussion, and critical analysis in courses or seminars in teacher-training programs or in institutes for in-service teachers at various levels.

At present, materials have been prepared for fifteen volumes. Each book contains one or more articles under a general heading such as The Learning of Mathematical Concepts, The Structure of Mathematical Abilities and Problem Solving in Geometry. The introduction to each volume is intended to provide some background and guidance to its content.

Volumes I to VI were prepared jointly by the School Mathematics Study Group and the Survey of Recent East European Mathematical Literature, both conducted under grants from the National Science Foundation. When the activities of the School Mathematics Study Group ended in August, 1972, the Department of Mathematics Education at the University of Georgia undertook to assist in the editing of the remaining volumes. We express our appreciation to the Foundation and to the many people and organizations who contributed to the establishment and continuation of the series.

Jeremy Kilpatrick
Izaak Wirszup
Edward G. Begle
James W. Wilson

EDITORIAL NOTES

1. Bracketed numerals in the text refer to the numbered references at the end of each paper. Where there are two figures, e.g. [5:123], the second is a page reference. All references are to Russian editions, although titles have been translated and authors' names transliterated.

2. The transliteration scheme used is that of the Library of Congress, with diacritical marks omitted, except that Ю and Я are rendered as "yu" and "ya" instead of "iu" and "ia."

3. Numbered footnotes are those in the original paper, starred footnotes are used for editors' or translator's comments.

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INTRODUCTION

Mary G. Kantowski

There has been a recent surge of interest in the process of problem solving among mathematics educators in this country. Such a trend is consistent with the emphasis on education as a process that Bruner [3] saw emerging more than a decade ago, and with Brownell's thesis [2] that educators can best guide learning if they are familiar with how students think in the face of new learning tasks.

The problem-solving process has been recognized as analytic-synthetic since the time of the early Greeks. Pappus's suggestion of making an analytic plan to be carried out by a deductive synthesis served as the inspiration for Polya's well-known four phases in the solution of a problem. Polya [6:75-85] speaks of decomposing a problem into its parts and recombining these parts into a different whole, the solution of the problem. To this end he suggests techniques that may aid the problem solver in finding relationships among the data, the unknown, and the condition of a problem, thus implying a link between the use of these techniques and the processes involved in solving the problem.

The present volume differs from the others in the series in that the entire volume records the search for a method of problem-solving instruction based on the analytic-synthetic nature of the problem-solving process. In this work, Kalmykova traces the history of the use of the analytic and synthetic methods in her country, explores elementary classroom situations involving teachers who had various degrees of success in problem-solving instruction, makes hypotheses regarding the use of certain techniques, and concludes with suggestions for "productive" methods to be used in the classroom.

In Chapter I, the Overview, Kalmykova verbalizes the universal complaint of mathematics teachers: the inability of students, even many of high ability, to solve complex, nonroutine problems independently. She proposes the need for a "rational" method of problem-solving instruction and views this study as an initial attempt to attain that goal.

The second chapter begins with the theoretical framework for the study: the Pavlovian concept that knowledge is acquired by continual abstraction and generalization resulting from the analytic-synthetic activity that is the essence of thought. The introduction of protocols of the solutions of verbal problems by successful and by unsuccessful students sets the stage for the typically Soviet clinical studies described in the succeeding chapters. A protocol from a strong student illustrates high analytic-synthetic activity, which is characterized by analyzing each new datum in relation to the problem's solution, operating with entire complexes rather than with individual pieces of data, and carrying out syntheses only when these syntheses bring the problem solver closer to the unknown. A protocol from a weak student, on the other hand, exhibits random manipulation of data and many "superfluous syntheses," characteristics of low analytic-synthetic activity.

Kalmykova distinguishes between the analytic-synthetic process that is involved in all problem solving and the analytic and synthetic methods used in problem breakdown. An extensive review of the literature of prerevolutionary as well as Soviet methodologists reveals no agreement as to the best "rational" approach to problem solving in the classroom. Notable among the prerevolutionary writers are Latyshev and Egorov, who argue against the confining "classical analysis" routine breakdown of problems in favor of techniques similar to those suggested by Polya. Among the Soviet writers, Skatkin agrees with this criticism and, moreover, views the analytic method as possible only after the solution is already clear and not as a means of obtaining a solution.

Chapter III includes studies involving the use of the purely mechanical analytic method of problem breakdown in actual classroom situations. The emphasis is on the breakdown of the problem rather than on finding the solution. Three levels of mastery are considered, the first two of which are artificial and do not imply any understanding of the relationship between the data and the unknown. In general, Kalmykova found that the ability to master the analytic method of problem breakdown is a function of the type of instruction, the amount of practice, and the experience of the teacher. Pupils who were not given regular instruction could not master the method, even for simple problems. The use of diagrams and earlier introduction of the method (in the second rather than in the third grade) were conducive to mastery. Longitudinal clinical studies that followed experiments involving entire classes supported the conclusions, but the claim that most pupils would eventually attain the level of mastery reached by the better students seems unwarranted in light of the evidence presented.

In Chapter IV, Kalmykova examines student performance on more complex problems. Her results in this aspect of the study support those prerevolutionary and Soviet writers who condemned the use of "classical analysis" as an unproductive method of instruction for developing the ability to solve nonroutine problems independently. In fact, in some instances, the confining classical methods had negative effects on independent problem-solving ability. A balance between the use of analytic methods and the use of synthesis as soon as data are isolated is seen as necessary in problem-solving instruction.

These results lead quite naturally to the heart of the study, the climactic fifth chapter that contains Kalmykova's recommendations for a productive method of instruction. Her suggestions are based on observations of an elementary teacher, V. D. Petrova, and of problem-solving behavior demonstrated by a group of older students and adults. Petrova's classes were chosen because of her history of success in training independent problem solvers. Her deceptively simple techniques emphasize making sure students thoroughly understand the problem, seeing that they carefully think through the solution, and allowing them time to arrive at the solution independently. In the analysis of the protocols of the older subjects, five "auxiliary methods" were noted and categorized as: (1) concretization, (2) abstraction, (3) modification, (4) graphical analysis, and (5) analogy. Kalmykova proposes that instruction emphasize these auxiliary techniques as well as the modeling techniques used so extensively and effectively by Petrova. Applying her suggested method, Kalmykova worked with four very weak students over a four-month period and found significant improvements in their problem-solving ability.

Although Kalmykova's suggestions for instruction closely parallel the heuristic methods being investigated in this country [4,5], there are aspects of the study that should be of interest to researchers. In particular, observations of successful teaching and problem-solving behavior could prove to be a fruitful source of hypotheses for studies relative to instructional techniques in the classroom.

If, for example, teachers who are successful in training efficient problem solvers are observed to have recourse to common techniques, to assign parallel types of problems, to pace instruction or to order problems in similar ways, these common denominators might suggest dependent variables to be investigated experimentally. Likewise, patterns of processes employed in search of solutions by students with some expertise in problem solving could provide clues for modeling methods to be tested and behaviors to be developed. Close scrutiny of such behaviors could also furnish important and needed data relating aptitudes to instruction.

This volume continues to emphasize the message of the Soviet researchers: that instruction is the key to proficiency in problem solving, and that investigating process is imperative for the improvement of instruction.

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PROCESSES OF ANALYSIS AND SYNTHESIS

IN THE SOLUTION OF

ARITHMETIC PROBLEMS

by

Z. I. Kalmykova

From: Proceedings [Izvestiya] of the Academy of Pedagogical Sciences of the RSFSR, 1955, Vol. 71. Pp. 3-112.

Translated by Terry Merz.

Chapter I
OVERVIEW

We are facing enormous tasks in making a gradual transition from socialism to communism. Stalin has pointed out the necessity "of achieving such a cultural growth that every member of the society would be assured of thorough development of his physical and mental capabilities [39: 68]" as one of the three basic preconditions for this transition. The 19th Congress of the Communist Party submitted a decision on measures for guaranteeing a transition to compulsory ten-year education and for introducing polytechnical schools. These decisions of the party congress involve Soviet teachers in the prodigious task of educating the generation of young persons who will build the communist society. Therefore each teacher should evaluate the effectiveness of the methods he uses and should introduce the most productive means to guarantee a significant improvement in his work.

Instructors of mathematics, one of the leading school disciplines, are no exception. In a mathematics course, problems are particularly important. In problem solving, mathematical concepts are formulated and different arithmetical operations are interpreted. Problems teach the pupils to disclose the mathematical content of concrete data. But problems are especially important as a means of developing logical thinking and the ability to determine proportional relationships between quantities and to draw the proper conclusions. As the students solve problems, they will be developing ingenuity, as well as the ability to work independently, without a model, but with creative initiative.

Analyzing students' test papers and oral answers over the last few years has shown that our schools are improving from year to year. The students are learning to solve the problems demanded by the curriculum and to give rather detailed explanations. However, when a problem's solution deviates slightly from the ordinary, when more independent work or creative thought is required, many students, even good ones, are incapable of finding the way to solve it and easily slip into an unproductive manipulation of the numerical data, into the method of "blind trial and error." They act according to the prescription of one of the pupils: "When I cannot arrive at the answer to the problem," he said, "I begin to add, subtract, multiply, or divide the numbers until I obtain the right answer." (The answers to the problems are ordinarily given in the back of the book.)

This easy lapsing into a seldom-productive method of solution as soon as the difficulty of the problem increases shows a weakness in the students' analytic-synthetic activity and an inability to break down a problem thoroughly.

In order to teach pupils to solve rather complicated problems independently, one must lead them to rational methods of analysis and synthesis. "Children are not taught the methods and techniques of thinking creatively in school. The methods of analysis and synthesis used in school miss the mark [16: 132]," Menchinskaya wrote several years ago. Are these criticisms still valid? To what extent do the methods of teaching problem solving that are used in school provide the pupils with the proper means of analysis and synthesis? Where should we look for more effective methods of teaching analysis and synthesis in problem solving?

This article is meant to help teachers to resolve these questions. For many years the author has taught the process of independent solution of rather complicated problems to students in different grades and to adults, and, in particular, taught the use of the so-called analytic method of problem breakdown. The experience of one of the foremost teachers in Moscow, V. D. Petrova, was studied in detail from the standpoint of the methods of analysis and synthesis that were used. Several methods of analysis were also taught experimentally.

Using Pavlov's reflex theory, the author attempted to find the essence of analysis and synthesis in problem solving. Several productive types and methods of analysis are described, on the basis of the investigations that were conducted. A number of rational means of teaching analysis and synthesis in problem solving to students are also described.

Much of this work is devoted to verifying experimentally the productivity of the so-called analytic method of problem breakdown. It should be noted that the basic methodological literature approves of this method and recommends using it widely (cf. the methods of Chichigin [6], Lyapin [14], and others). However, although this method has been known for a long time, it has not seen widespread use in schools. Is practice lagging behind the progressive theory? Then the method of analysis must be publicized and forced into practice in the schools. Should the proponents of the theory themselves (for methodological literature should carry scientific theory to the masses) perhaps reconsider their position in approving this method?

To answer this question, the author first turned to the methodological heritage of the past. There was no general agreement on this problem in the methods literature and experts have been voicing doubts and protests about this method both in pre-revolutionary and in Soviet Russia. Then the author turned to the experimental study of the use of this method in the schools, thus acquiring a sufficiently detailed notion of the psychology of teaching the analytic method of problem solving. This work will show the position this method should occupy and where to look for new, productive means of problem breakdown. Undoubtedly the study of productive means of analysis and synthesis should be continued. The present work is only a beginning.

The work was done in the education laboratory of the Psychology Institute of the Academy of Pedagogical Sciences of the RSFSR, under the supervision of N. A. Menchinskaya.

Chapter II

SUBSTANTIATION OF THE PROBLEM OF ANALYSIS AND SYNTHESIS

1. A Physiological Basis of Analysis and Synthesis Processes

We can begin with the familiar tenet in dialectical materialism that thought is both analytic and synthetic and that the analytic and synthetic processes are bound together inseparably.

Analysis and synthesis compose the physiological basis of the analytic-synthetic mental activities carried out by the cerebral cortex. Pavlov [21] showed repeatedly that the cortex simultaneously and continuously carries on both analytic and synthetic activity. The analytic process decomposes the "complexities of the world" into "separate parts," to use Pavlov's words; it isolates individual facets of the environment. In synthesis, connections between the separate parts that were isolated in the analysis and in the appropriate activity of the organism are "tied together."

Pavlov points out the differences in the degrees of complexity of the analytic and synthetic processes. Only certain elements of the environment can be isolated and tied together with a definite activity of an organism. When an entire complex of stimuli influences the elements composing it during a rather long period of time, a connection is established and, later on, this complex of stimuli will be isolated by the analytic process (i.e., the elements synthesized earlier will be isolated). As the investigations of Pavlov and his associates have shown, a relationship between stimuli can be distinguished, and this relationship can be identical even though the composition of the stimuli might be qualitatively different (e.g., the rhythm of sounds and the rhythm of a light bulb's flashing might be identical). To isolate this identical relationship and to abstract oneself from the qualitative difference in its elements undoubtedly demands a high level of analytic-synthetic activity.

Temporary reflex-conditioned connections are established during the simultaneous or sequential influence of objects of the environment on an organism. However, many of these connections can prove to be coincidental, unrelated to reality, by an objective correlation of the objects and phenomena of the environment. As Pavlov has shown, we must form not only temporary connections to obtain a proper relationship to the external world; but we must also be continually and rapidly correcting these connections, when they are not justified by reality, i.e., we must be ready to revoke them. This revocation of temporary connections is carried out with the aid of inhibition that detaches whatever does not correspond to reality and is one basis for the highest forms of analysis.

As a result of this activity, the true relationship of the animal to its environment becomes more and more precise. We see that the analytic-synthetic activity of the cortex in animals becomes highly complex. It reaches an immeasurably higher level in man with his "extraordinary addition" -- the second signal system, speech.

"Speech is for man just as real a conditioned stimulus as all the other common ones are for animals, but at the same time so all-embracing that no others in animals approach it either quantitatively or qualitatively. Speech, thanks to the entire previous life of an adult, is connected to all internal and external stimuli entering into the large cerebral hemispheres; speech signals and replaces all of them, and therefore is able to evoke all of those actions, the reactions of an organism, which cause those stimuli [21:429]."

Due to the signalling of the first signal system in speech, as Pavlov has shown, a new principle of neural activity is introduced -- abstraction together with generalization of the innumerable signals of the preceding system, in its turn again analyzing and synthesizing these new generalized signals -- a principle conditioning the unlimited orientation in the environment and creating a higher adaptation for mankind -- knowledge.

The generalized and indirect knowledge of reality realized in analytic-synthetic activity is the essence of thought.

2. The Interrelationship of Analysis and Synthesis in Problem Solving

Solving arithmetic problems, like any other thought process, is an analytic-synthetic process.[1] The concrete subject matter of problems

[1] It must be emphasized that when we consider thought as a complicated analytic-synthetic process, we are using the terms "analysis" and "synthesis" in their broadest sense. They include such thought forms as judgment and deduction, and such processes as comparison, generalization and abstraction. We consider the processes of analysis and synthesis in exactly the same broad sense in which Pavlov used them when speaking of the analytic-synthetic principle of brain activity.

is extremely varied, but definite mathematical relationships underlie them, as well as known mathematical principles that relate the data to one another and to the unknown quantity.

The purpose of analytic-synthetic activity in solving a problem should be to expose its mathematical content as described by the concrete situation in the hypothesis as well as to ascertain the relationships of the data to one another and to the unknown quantity, and, having isolated the appropriate principles, to determine the value of the unknown from the known data.

The analysis begins with the decomposition of the text of the problem into "separate parts." Thus, individual words, numbers, or elements of the problem might be isolated (Menchinskaya called such an analysis elemental).[2] Thus, in the problem:

> Twenty-five birch trees were felled in the forest, but only 1/5 as many linden trees. How many trees were felled altogether?

The student might isolate the numbers (25 and 1/5) and the words "as many" and then start to carry out the operations on the basis of these elements taken from the text. A synthesis which was carried out on the level of such an "elemental" analysis might frequently be mistaken or superfluous, or might not lead to determining the value of the unknown quantity.

So long as the unknown and the data of the problem and their interrelationships are defined not by isolated words but by combinations of them (forming definite complexes), a productive solution of the problem demands a synthesis on the level of "complex analysis." Thus, the following complexes should be isolated in the problem stated above: "25 birches," "one-fifth as many lindens (as birches)," "how many trees in all (birches and lindens) were felled," and then the relationships between these complexes should be found.

If the problem's structure is familiar to the student (that is, if he has solved several similar problems in the past), then he will abstract himself from the details of the concrete situation and isolate the appropriate relationship easily and quickly, and solving the problem will present no particular difficulty.

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[2]The descriptions of analysis as "elemental," "complex," and "anticipatory" were given by Menchinskaya [17].

Thus, Galya K., a fifth-grade honors pupil, reads Problem No. 4:

> Fourteen m of wide lace and 9 m of narrow lace were purchased. The wide lace cost six rubles 30 kopeks more than the narrow lace. How much is 1 m of wide lace and 1 m of narrow lace, if it is known that 1 m of wide lace costs 20 kopeks more than narrow lace?

"Aha," she says, "They paid 6.30 rubles more for the wide lace. ...That's for 5 m extra... and for 20 kopeks more per meter. So for 14 m they would have paid 2.80 rubles additional for wide lace, and hence 5 m of narrow lace would be 6.30 - 2.80 ...."

Although Galya had never solved this problem about lace, she had frequently met problems that were structurally similar. This familiarity served as a basis for decomposing the relationships between the data and the unknown that are characteristic of problems with this sort of structure, and assured a higher level of analytic-synthetic activity in solving this problem.

We can see that in perceiving the problem's conditions, Galya separated the relationship between the data and the unknown and determined a course of solution, and then a way to find the unknown quantity was immediately clear to her.

The relationships between the complexes (of the isolated data to one another and to the unknown) stand out immediately in the analysis. This particular aspect of analysis can be attributed to <u>anticipatory analysis</u>, provided that it is directed at the succeeding operations.

The physiological basis of the given process is the formation of definite systems of temporary associations that become increasingly fixed by uniform repetition of the conditions, by virtue of which the appropriate process is brought about more easily and automatically. Thus, to solve problems with a familiar structure, one reproduces associations that have been formed and consolidated earlier.

The analysis becomes significantly more detailed in the solution of more or less complicated problems that are new for the pupils (<u>task-problems</u>). Task-problems are what we call problems, both model and non-model, whose course of solution is unknown to the pupils, and therefore they must find it, i.e., find the relationship between the unknown and the data.

"People reduce every drill, every training, habit formation, orientation in the environment, among natural events, either to the formation of new connections or to the finest analysis," Pavlov points out [21: 333]. The solution of problems whose structure is familiar depends mainly on the reproduction of old, well-consolidated associations. The solution of task-problems presupposes the formation of associations based on the finest analysis.

However, Sechenov says, "Not a single thought passes through the mind of a man during his entire life which does not arise from elements registered in his memory [33: 441-442]." New associations can be preserved only on the basis of old ones already amassed by past experience. Analysis in the solution of task-problems is directed at isolating these old associations, these "elements registered in the memory," through which the new ones can be established.

In solving problems with a familiar structure, the pupils sometimes isolate the relationships between the unknown and the data while reading the problem, but in task-problems the data and the unknown appear apart and, to find the relationship between them, an entire series of intermediate elements must be isolated in the process of the finest analysis, which will connect the data to one another and to the unknown. A special analysis of the unknown and the data, and the functional relationships between them, is then necessary.

In the analysis of the unknown and the data their content is made more precise, the composition of the complicated data is revealed, their basic properties and features are isolated, i.e., the pupils answer the questions: "What is this?", "What is it composed of?", "What features does it have?"

The analysis of functional connections is directed at isolating the principles on the basis of which the interrelationships between the data and the unknown might be established, as well as at isolating the very relationship on whose basis the unknown quantity might be found.

For example, consider the following problem:

> Vitya bought a notebook for 20 kopeks, 3 pencils for 30 kopeks each, and he had four pyatachki left. How much money did Vitya have at the start?

First, the first datum should be subjected to a special analysis. The pupils should explain that a pyatachok is a coin whose value is 5 kopeks, and consequently Vitya had 4 coins of 5 kopeks each. In disclosing the composition of this datum, the pupils easily determine its magnitude also.[3]

A special analysis of the unknown is also necessary here to reveal the composition of the unknown. The problem asks how much money Vitya had at the start.

"When was it?"

"Before the purchases."

"And what did he buy?"

"Notebooks and pencils."

"Did he spend all his money?"

"No, he still had 4 pyatachki, that is, 20 kopeks."

Analyzing the isolated data and the functional relationships between them makes it possible to establish definite relationships between the data and to synthesize them.

The content of the data can be extremely diverse, and the interrelationships that they can have are also diverse. In the problem given above, one can compare the costs of the notebooks and of all the pencils, and determine their difference; one can find the cost of one pencil and find out how much more expensive the notebook is; one can compare the money spent and the money remaining. Which of these possible operations must be carried out?

The only operation, the only synthesis, that will be productive is one that will bring the pupil closer to finding the unknown quantity. The pupil should choose from all possible relationships those that will "correspond to reality" ---to the situation of the problem.

A problem is a question whose answer should be found by determining definite <u>relationships</u> between the unknown and the data. The appropriate relations should be chosen with a view toward determining these relationships (i.e., the pupil should rise to the level of anticipatory analysis). To determine how much money Vitya had at the start, one must find out how much money he spent and how much money he had left; it would be superfluous to determine the difference in the costs of pencils and notebooks.

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[3]The author observed that without a special analysis of this datum, the students operated with it as a known quantity, regarding 4 pyatachki as 4 kopeks.

A proper choice of productive relationships is possible only on the basis of the analysis of all the data and the unknown. The appearance in the solution of "superfluous syntheses" (a term used by the methodologist Bellyustin [1]) — operations which do not bring one closer to the unknown--involves passing to isolated analysis of the separate elements of the problem. The pupil isolates individual words or data and begins to combine them on the basis of his past experience without correlating them with the other data or with the unknown (a synthesis on the level of an elemental analysis).

It must be remembered that words are "all-embracing stimuli." The same word in one problem might be connected with one operation, and in another problem with its opposite. Thus, in the problem: "Twenty fir trees grew in a forest plot; eight were cut down. How many remained?" the words "cut down" and "remained" are connected with subtraction. But in another problem, stating that "eight fir trees were cut down, and 12 remained on the plot. How many fir trees were there?" -- both of the verbs are connected with addition.

If the pupil is in the habit of choosing arithmetical operations by depending on isolated elements in the problem's text--isolated words taken from the text--then, in the second problem, if he isolates the words "cut down" or "remained," he will try to subtract, since "remained" has been associated most frequently with subtraction in his experience.

A productive synthesis can be performed only on the basis of a comprehensive analysis of the data, the unknown, and the functional relationships between them.

Problem solving is an analytic-synthetic process. Analysis and synthesis are interdependent. Investigation of independent problem solving showed a gradual substitution of these processes for each other both in pupils and in adults. In analysis the pupils isolate sufficient bases for the synthesis, and the synthesis is carried out immediately afterwards. The new datum obtained as a result of the synthesis is again subjected to analysis, and the connection between them and the known data is re-established. Thus, in this activity, the well-known proposition of Engels on the indissoluble ties between analysis and synthesis is substantiated: "Thought consists as much in decomposing the objects of consciousness into their elements as in unifying the elements which are related to one another. There is no analysis without synthesis [8: 40]."

Thus, analysis in solving arithmetic problems includes, first, decomposition of the problem into individual data and the unknown, discovery of the content of the unknown and the data, isolation of their individual features and facets, isolation of principles connecting them, and isolation of their functional relationships. Secondly, analysis is included in the selection of the "appropriate realities" of productive associations that lead to the determination of the unknown quantity. Such a selection is possible only with a thorough analysis of all the data and the unknown of the problem.

3. A Contrast of Successful and Weak Students in Problem Solving

That pupils differ in their success in arithmetic is clearly shown by the different levels of their analytic-synthetic activity in solving task-problems independently. We shall compare the methods of solving task-problems by students who were successful in arithmetic with the methods used by weak students.

An outstanding fourth-grade student in the 172nd School, Valya K., was told to solve the following problem independently:

> Two workers received the same sum of money for work that they had done. One was paid 20 rubles per day, and the other was paid 12. Determine how many days each laborer worked if it is known that the second worker put in 6 more days than the first.

The record of her solution follows:

> Valya read the problem slowly, distinguishing clearly by her intonation one datum from another. (She was conducting the primary analysis.) She repeated the question of the problem twice and singled it out particularly. The problem was difficult, and Valya did not hurry. She turned to the text and re-read it more thoughtfully.
>
> "The second laborer was paid 8 rubles less. But they both received the same amount of money -- so he had to work 6 days more." She isolated the essence of the problem, thus determining the main functional relationship: "He was paid less but worked more days. The second laborer worked an <u>extra</u> 6 days." She continued, "How much money did he earn in these 6 days? 72 rubles... 72. The second laborer earned in 6 days...." She re-analyzed the data she had obtained.

"And how did I find that out?" Valya asked herself, and made it more precise: "The second worker received 8 rubles less (20 - 12) and therefore worked 6 days more... In 6 days he received 72 rubles, and in one day 8 rubles less...72 ÷ 8 = 9 ... days... What kind of days are these? Who worked during them? ... The first laborer worked 9 days. And the second laborer? He was paid less; in 9 days he wouldn't make 72 rubles, and he worked 6 days more." Having isolated this relationship, Valya easily found the second unknown: "The second laborer worked 15 days."

As this record shows, an extremely high level of analytic-synthetic activity is characteristic of Valya K. She operates not with elements merely taken from the text of the problem, but with entire complexes. She carries out the synthesis only when she has a sufficient basis for it. Each new datum she obtains is analyzed, and its significance for approaching the basic goal—the unknown— is evaluated. This procedure for solving task-problems is typical of pupils who have high grades in arithmetic. Successful pupils show a high level of analytic-synthetic activity in problem solving.

Pupils with poor problem-solving skills solve rather difficult problems, task-problems, in a different way. We shall consider a solution by a weak fourth-grade student in the 69th School, Oleg A. A problem was given to him (Problem No. 14):

> Four pieces of material, each 50 m in length, were brought into a shop. Twenty suits and several overcoats were made from the material. How many overcoats were made if 4 m were used for one overcoat and 3 m for a suit?

The problem was not easy for Oleg (although this type of problem is solved at the beginning of the third grade).

> Oleg read the text of the problem through superficially, without expression, and without precise decomposition of the separate data. Then, putting aside the paper on which the problem was written, he took up his pen.
>
> "Four m of material went into the overcoat, and 3 m into a suit." He remembered a pair of the known data and asked, "How many meters were used for one suit and one overcoat? Seven," he calculated and turned to something else: "They bought 20 suits; how many meters went into the suits? Sixty." Then, still without having posed the question, he divided 60 by 4. After getting 15, he stopped.

"What did you find out?" the experimenter asked.

"This is how many meters go into one overcoat," Oleg answered, without thinking (obviously, he had forgotten that this was given in the problem) and wrote down "m" (meters) by the 15.

"How many meters went into all the suits?" Oleg continued.

$$15 \times 3 = 45$$

"Explain what you're doing," said the experimenter.

"Three m went into one suit, and we found out how many went into them all ...."

There is no point in quoting the record further. The nature of the solution is clear. At the beginning Oleg isolated individual data and carried out operations with them, although they did not correspond to the other data or to the unknown (and then he performed a superfluous synthesis). Furthermore, he began to operate with only the numerical quantities, assigning a significance to them quite arbitrarily, and he did not compare these quantities with each other in actuality. Oleg manipulated the numerical data, in fact, and used blind trial and error.

This sort of solution is very clearly depicted by Nosov [19] in his work <u>Vitya Maleev in School and at Home</u>:

> There were 8 saws and 3 times as many axes in a store. Half the axes and 3 saws were sold to a crew of carpenters for 84 rubles. The remaining axes and saws were sold to another crew of carpenters for 100 rubles. How much does one saw and one axe cost?

Vitya shortened the problem, simplifying it:

> 12 axes and 3 saws cost 84 rubles.
> 12 axes and 5 saws cost 100 rubles.
> How much does one saw and one axe cost?

Vitya explained, "I could not shorten it any more, and I started to think about how to solve the problem. At first I thought that if 12 axes and 3 saws cost 84 rubles, then the saws and axes must be added together and 84 should be divided by whatever was obtained. I added 12 axes and 3 saws, and got 15. Then I started to divide 84 by 15, but I got a remainder. I understood that some sort of mistake had occurred, and I began to look for another way."

"I found one: I added 12 axes and 5 saws, got 17, and began to divide 100 by 17, but again there was a remainder. Then I added all 24 axes together and added 8 saws to them, and also added the rubles together and began to divide the rubles by the saws and axes, but the division still didn't come out right. Then I began to subtract the saws from the axes and divide the money by what I obtained, but all the same I got nothing. Then I tried again to add up the saws and the axes separately, and then subtract the axes from the money and divide what was left by the saws, but whatever I tried, no sense came of it ... [19: 145]."

Every instructor knows how typical Vitya Maleev's method of solving problems is for pupils who are not making normal progress. The level of analytic-synthetic activity is too low. They add up the axes and the rubles easily, without thinking of the meaning of what they obtain. Their solution-process frequently leads to a simple manipulation of the numerical data taken from the text of the problem.

To instill an ability to solve problems, one must teach the pupils to carry out the synthesis only on the basis of a thorough analysis of the problem; one must provide them with productive means of analysis, which they can use while solving rather complicated problems, task-problems, independently.

4. A Comparison of Analytic and Synthetic Methods

How can we teach pupils to analyze a problem? To resolve this question, mathematics instructors first turn to the methodological literature.

Ordinarily, in considering the question, the authors of methods handbooks describe and compare the two methods of reviewing a problem—the analytic and the synthetic.

Let us recall the way to break down a problem by each method. Suppose that the following problem [25: 71] is given:

> An airplane travelled 1940 km the first day, and travelled 340 km further on the second than on the first; on the third day it travelled 894 km less than on the first two days together. How far did the plane travel in the 3 days?

Reviewing this problem by the analytic method, we start with the question: "The question asks how many kilometers the plane flew in 3 days." To answer it, we must know how many kilometers the plane travelled each of the three days. We know that it travelled 1940 km on the first day, but how far it travelled on the second and third days is unknown.

To find how many kilometers it travelled on the second day, we must know how many kilometers it travelled on the first day and how many more it travelled on the second. Both data are known.

To determine how many kilometers the airplane travelled on the third day, we must know how many kilometers it travelled on the first and second days together, and how many kilometers less on the third day. We know the latter, but not the former.

To find the distance it travelled on the first and second days together, we must know how many kilometers it travelled on each day separately. All the necessary data are at hand, and the breakdown is completed.

With the synthetic method of breakdown, we begin with the known data:

1. We know that on the first day the plane travelled 1940 km, and 340 km further on the second. Consequently, we can find out how far the plane went on the second day. We must add 1940 km to 340 km, to obtain 2280 km.

2. Now we know that on the first day the plane travelled 1940 km, and 2280 km on the second, and therefore we can find out how far it travelled on both days...etc., all the way up to finding the unknown.

Thus, in the synthetic method, the concrete data of the problem are the starting point, and we ask a question about them, the answer to which should bring us closer to finding the unknown.

In the analytic method, the unknown is the initial step in the discussion. In complex problems it might not be defined directly by combining the known data (intermediate steps would be necessary). The solver selects data for the unknown that are not contained directly in the text but which he deems necessary for finding the unknown quantity. These data are expressed in an abstract, logical form, since their numerical value is unknown.

In the synthetic method, the breakdown of each link ends in resolving the question and in finding a new link, and the solver relies on the concrete values of the intermediate data. In the analytic breakdown, the entire chain of the discussion should be built up from abstract, logical categories; only in the last steps do the abstract data receive numerical values.

The breakdown of a problem by the method of analysis is a logically valid argument whose initial step is the final question of the problem, the unknown, and each link follows regularly from the preceding one. This logical order, the strict validity of the links of analysis, attract methodologists to this method.

Are pupils capable of constructing such a rigorous, logically valid argument as the method of analysis proposes? In what grade can this method be introduced? To what extent does practice in the construction of this type of logical argument enable the pupils to solve problems and keep them from the superfluous, invalid operations to which they are sometimes so susceptible? To which method -- the analytic or the synthetic -- and when, must priority be given?

It should be noted there is broad disagreement on these questions. I do not propose to give a systematic exposition of the views of all authors; but I shall try to outline the basic solutions to the questions in the works of pre-revolutionary and Soviet methodologists.

5. Views of Methodologists on Analysis and Synthesis in Problem Solving

The Pre-Revolutionary Methodologists

A description of two methods of solving problems--from the data to the unknown and from the unknown to the data, without introducing the terms "analytic" and "synthetic"--can be found in the 1896 methods textbook of Evtushevskii [10]. He writes:

> At first, in an elementary arithmetic course, it is both more natural and easier to carry out the solution of a problem from the given numbers to the unknown, which is clearer and more understandable for the pupil; subsequently it is useful to pass to the reverse solution gradually, that is, starting from the unknown, determining its connections with the numbers given in the problem...The change from one method to the other is one of the most powerful tools in the development of the pupil's thinking and simultaneously prepares a method, which the pupil will need later, in composing formulas and equations from problems...[10: 88-89].

Evtushevskii regards as a necessary condition for transition to the method of analysis (to use the modern term) the student's acquisition of the skill in remembering the entire content of the problem and the presence of a precise conception of the data and their connections in conformity with the text of the problem.

He demonstrates a review of the problem "from the unknown" (i.e., by the method of analysis) in three operations with an extremely clear relationship between the data. He gives the review of a more complex problem (involving six operations) differently. He suggests isolating the question of the problem and the known data, then listing all the intermediate data that should be found before the unknown quantity can be determined. Listing these data will proceed in the same order as the performance of the arithmetical operations. That is, the review proceeds by the synthetic method, but at first the entire plan of solution is composed, and then the appropriate arithmetical operations are carried out.

Thus, just as the method of synthesis is easier and more natural for the student, so the breakdown of more complicated problems is carried out in the same way. But this method should be combined with the method of analysis, which influences the development of logical thought in the pupil in a positive way. This is Evtushevskii's opinion of both methods.

The greatest methodologist of the 19th century, Latyshev [13], emphasized that a school should not only inculcate definite habits, but should "teach one how to think" and develop ingenuity in its pupils. The author held a low opinion of the method of analysis. He considered it possible to use analysis only after a series of exercises, especially exercises on the selection of data for the unknown. Latyshev said:

> In teaching students to determine by which numbers unknowns of a known type can be found one can compose a plan, after the problem's solution, beginning from the unknown and describing by what type of data it was found... The composition of similar plans after the solution of problems is easily accessible to students. And only when the pupils can compose such plans very well and have become accustomed to determining by what data the unknown can be found should problems even be solved, beginning with the unknown and discussing by what data it can be determined... (italics mine -- Z.K.) ...one cannot give materials for practicing the analytic solution of questions in the study of arithmetic (with children); geometry supplies an abundance of similar material, and therefore the study of geometry is very opportune for learning the analysis of solving questions, starting from the unknown.

> In arithmetic problems only in isolated instances can questions which are posed from the unknown actually help the pupil in solving the problem, and then they ordinarily can only direct him toward explaining the content, but cannot guide him through the entire solution [13: 132-135].
>
> Exercises in this method of solving problems *should not take up much time*..., since the *time* which the students spend mastering it *is not justified by its usefulness to them*; it is more advantageous to spend time on other exercises which are more closely related to the nature of the required work, to the general direction of the course [13: 101] (italics mine — Z. K.).

As one of these more useful methods, Latyshev suggests introducing an "explanatory discussion" with a "Why?" question as a basis. "Superficially," he explains, "a feature of the method is expressed in the establishment of the question - the students intend to treat 'Why?' in some way or other [13: 101]." In concrete examples, the author shows his recommended method of breakdown. "Why did the second purchase cost more, why did the second train catch up with the first, why did he receive less than he assumed?" All of these "whys" are directed at uncovering a basic link in the problem, at uncovering the basic principles linking the unknown to the data.

It is not the analytic method of breakdown in its "classical" form, but a versatile breakdown of the problem, a dismemberment of its main link, exposing the functional relationships of the data to one another and to the unknown, with the aid of a "why?" question—that is the way which, in Latyshev's opinion, largely permits the goal of "teaching thinking," teaching independent solution of a problem, to be attained.

"One must teach all students to use mainly the synthetic method," asserts the distinguished methodologist of the late 19th and early 20th centuries, Shokhor-Trotskii [34]. "The synthetic method," he explains, "is in general somewhat easier than the analytic, since pupils applying the synthetic method *use their common sense and their ingenuity more freely*, starting with the given numbers, and not with the unknown numbers. [34: 287]" (italics mine -- Z. K.). However, in Shokhor-Trotskii's opinion, "the synthetic means of solution can distract the pupils from the necessary operations and *lead them to operations* which are *unnecessary* for solving the problem. It meets the mark when the problem is not very complicated [34: 285]."

The analytic method "demands considerable mental acuity and considerable power over the reasoning proceeding from general considerations." This method "leads slowly to the goal, but also never denies its help ... (it) leads more reliably to the goal [34: 285]" (italics mine -- Z. K.).

"It is important," Shokhor-Trotskii remarks, "that pupils learn to solve problems independently of the rules of solution by the tiresome analytic or by the not-always-reliable synthetic methods [34: 287]." The author recommends training the pupils at first in the solution of a sufficient number of simplified problems, i.e., problems in which the conditions are laid out in the order which corresponds most closely with the sense of the problem and with the order of the required operations. Then he suggests having the students change these simplified problems into unsimplified ones, in which the order of the conditions does not correspond to the order of the operations.

We see that neither the analytic nor the synthetic method of problem-breakdown satisfies Shokhor-Trotskii.

In Methods of Arithmetic by Egorov [7] we find an extensive comparison of the methods of analysis and synthesis. Egorov, foremost among methodologists, clearly stated the notion that solving problems is an analytic-synthetic process. He regarded the methods of analysis and synthesis in their pure form as methods of exposition of the way to solve problems.

In his search for rational means for teaching problem solving, Egorov followed in Latyshev's footsteps. He recommended simplifying the data of a problem and investigating how one of the data might be formed with the help of the other data and the unknown. He considers it helpful sometimes to assume that one of the intermediate unknown quantities is known, for then it would be easier to determine the link between the unknown and the data; in problems on merging, it must be determined why a profit or a loss was sustained, and in what way, and so on. Egorov's instructions for rational means of finding solutions to problems are also of interest today.

Arzhenikov resembled Egorov in his opinions. He also believed that guess-work and mistakes are possible in the analytic method of problem-breakdown as well as in the synthetic. At the same time, according to him, the method of analysis is so difficult that it can have only partial use in elementary school, where the method of synthesis should predominate.

The notion of the analytic-synthetic nature of thought in problem solving was developed further by Bellyustin [1]. He said, "Neither isolated synthesis nor isolated analysis can be considered means of solving problems. Problems should be solved by the combined use of analysis and synthesis" (italics mine Z. K.).

In order to solve a problem, one must combine its data. In a more complicated problem "superfluous syntheses" (Bellyustin's term) are possible, that is, combinations of the data which do not lead to finding the unknown. Bellyustin says:

> In such a case, in order to arrive at the necessary synthesis more quickly and accurately, and, consequently to arrive at the problem's solution, one can utilize the reverse method of breakdown--analysis. Most children do an analysis of problems, but in an abridged way. They ordinarily carry it out silently, to themselves, often vaguely, with jumps in their logic, digressions and backward steps. During this time, the mind crosses from one combination of data to another, sometimes not pursuing it to its end, because it carries out superfluous synthesis and does, finally, an entire series of decompositions of the question. All of this work is useful to a great extent. ...In order to impart to them the ability to do this, one must repeat problems that have already been solved or perform an analytic breakdown of them [1: 54-55].

According to Bellyustin, a complete analysis is rarely encountered. He writes:

> The reason consists in the difficulty and the complication of such a breakdown. It is useful only as a new form of logical thought and as a refresher for the synthetic method. Its best place is in problems which have already been solved synthetically. The analysis of a problem after it has already been solved is not difficult and is accessible to children; it clarifies and supplements the synthesis [1: 54].

The author considers "abridged analysis" more natural and easier when a difficult problem is separated, not into simple ones as in complete analysis, but into two less complex ones, which are sometimes already familiar to the solvers. Asking analytic questions that follow from the main question of the problem facilities finding the solution.

("Why was a profit obtained?", "Why did one traveller overtake the other?", etc.) If a problem is difficult, he recommends inventing similar problems with small numbers ("the method of induction").

Thus, although he emphasizes the analytic-synthetic character of thought in solving problems, Bellyustin regards the method of analysis as a method of breaking down problems that have been solved, useful "only as a new form of logical thought and as a refresher for the synthetic method."

Shpital'skii [35] criticized the views of Egorov [7] (and at the same time of Bellyustin [1]). Without denying that problem solving is an analytic-synthetic process, he still found that the "main direction of thought unconditionally exists"and is for the most part neither synthetic nor analytic. "The strict use of a particular method of thought," the author emphasized, "for the very solution of the problem is necessary for purely pedgogical goals [35: 11]."

An educational method that largely corresponds to a child's natural course of thought would be more expedient — and the method of analysis is such, according to Shpital'skii. Analysis should not take place after the solution, as Egorov asserted, for then, Shpital'skii believed, its value would be decreased, and the children would lose interest in the problem. The analytic method should be used during solution. According to Shpital'skii, finding the missing link in the chain of reasoning is especially difficult while solving the problem; analysis helps to find this necessary link.

Analysis "eliminates the possibility of any guess-work [35: 13]," Shpital'skii wrote. The analytic method makes the solution of problems easier, but "this method is by no means meant to be an automatic way to solve problems [35: 32]." It "provides a plan, but a plan of the process of thought is, of course, just as consistent, not only for all problems, but in general for all instances of thought [35]." Without analysis, without being separated into simpler problems in a definite order, a problem cannot be solved. Shpital'skii emphasized that the teacher's task is to turn his attention to this natural process and to order and reinforce.

In contrast to other methodologists, Shpital'skii saw the advantage of the method of analysis in its difficulty, since thought is developed by overcoming difficulties. At the same time, he strived, as far as possible, to lessen the difficulty of the analytic breakdown of problems by introducing a graphic scheme, to make the analysis more visual, to show it in its entirety, to emphasize the logical sequence of each link, to discipline thinking, and to provide the pupil with a method of thought that will help him to solve other vital problems.

Shpital'skii supported his discussion of the value of the analytic method by describing a small experiment in training twenty students in different grades (from preparatory students to older ones, exactly which is not stated) during two summer months he prepared them to pass the examinations (all of them had to be re-examined after failing). He asserted that the use of the "analytic method in connection with the graphic method of solving problems" aroused the pupils' interest, forced them to regard a problem's solution conscientiously and to follow the logical train of thought. The pupils acquired the ability to solve problems of any difficulty.

Ern [9] gave considerable attention to the methods of analysis and synthesis. Comparing the methods, Ern emphasized that each places the solver in a position where he must choose from among several different combinations. To make the selection with synthesis, we should "anticipate" and think about the possibility of composing successive simple problems. And, in the analytic method of breakdown, we must anticipate, in selecting the data for the unknown, in order to choose the necessary ones.

Choosing data presents greater difficulties, however, for the analytic method than for the synthetic, according to Ern.

> Here one must frequently select data that are not yet in the problem, i.e., unknowns in their own right; therefore it is much more difficult here than in synthesis to foresee which of the possible combinations will be the most advantageous in the given instance [9: 97].

Although he observed that solving problems analytically "demands high mental development in the pupils, as well as skill in abstract thinking, and therefore should be carried out at a higher level of instruction [9: 102]," Ern still objected to using the synthetic method alone in elementary school.

> Analysis in the solution of arithmetic problem is the first and simplest use of those means of thought which have such great significance in algebra (in composing equations) and in geometry (in solving problems and in the proofs of theorems). Therefore acquaintance with analysis in its elementary form is desirable in elementary school, if this school is to be a preparation for secondary school [9:102].

Ern recommended starting with the breakdown of problems that have already been solved, in teaching the analytic method. A problem that is <u>algebraic in nature</u>, he pointed out, "cannot be solved by the ordinary means of synthesis and analysis [9: 117]"; it demands special and quite artificial means (the method of assumption is basic).

<u>Summary - Pre-Revolutionary Methods Literature</u>

Thus we see that the greatest methodologists in Russia in the late 19th and the early 20th centuries agree that the analytic method develops logical thought in pupils, but that it is difficult. Various conclusions can be drawn from this. Latyshev and Shpital'skii state the most extreme views.

Latyshev [13] asserts that because the analytic method is difficult, it is of little use in elementary school. It would not be worth the time spent in teaching it. To teach the reasoning involved in starting with the unknown is much more natural and more productive to use geometric material.

Shpital'skii [35] we recall, argues that the method of analysis is useful because it is difficult. This method teaches one how to think, "eliminates all guess-work," simplifies the solution of problems. This is the method of searching for solutions to problems.

Shokhor-Trotskii [34] introduces the notion that the analytic method "leads more truly to the goal," although the pupils "use their common sense more freely" with the synthetic method.

Egorov [7], Bellyustin [1], and other methodologists show convincingly the possibilities of error in both the synthetic and the analytic breakdown of problems. Indeed, both methods place the solver in a position where he must select one of several different combinations, and this selection is much more difficult in a breakdown by the analytic method (and, consequently, errors occur more frequently).

Solving problems is, according to them, a complex analytic-synthetic process. It is impossible to solve a problem by the method of analysis, which, according to them, is a way to break down problems that have already been solved. With this kind of breakdown, the pupils are trained in logically rigorous argument, which facilitates the later mastery of algebra and geometry. Ern [9] points out the inapplicability of the methods of analysis and synthesis in solving algebraic problems.

Many authors who are critical of the analytic and synthetic methods introduce other, more productive (in their view), means for promoting problem-solving skills. Evtushevskii [10] recommends a very precise decomposition of the problem into the known data and the unknown, and the composition of a complete plan of solution, up to the execution of the arithmetical operations themselves. Latyshev [13], along with Egorov [7], Bellyustin [1], and others, suggests introducing an "explanatory discussion," using a "Why?" question to expose the basic kernel of the problem and the basic relations between the data and the unknown. Egorov recommends simplifying difficult data in the problem, exposing their content, altering them, observing how the other data are then changed, using a "Why?" question broadly, and so on.

These proposals for more productive means of working on problems merit attention at the present time. We see that the pre-revolutionary methods literature contains no unanimous opinion of the relative value and place of the application of the analytic and synthetic methods of breaking down problems. Let us turn to the work of Soviet methodologists and see (again only in its basic features) how they resolve this important issue of analysis and synthesis in solving problems.

The Soviet Methods Literature

As was noted above, a Soviet school strives not only to give its pupils a fund of knowledge and skill; it strives primarily to develop well-rounded persons who will build a communist society. "Arithmetical information should be imparted to pupils by methods which promote their thorough mental and moral development," writes Pchelko [[illegible]: 4].

It is quite understandable that, with such a task, the analytic method with its rigorous system of a logical chain of conclusions is particularly attractive to Soviet methodologists. In arithmetic teaching manuals we find many lines devoted to the question of analysis and synthesis. We shall investigate the positive and negative features that various school systems have noted in the analytic and synthetic methods of problem breakdown, which are of interest to us.

Kavun and Popova [11] assert:

> Whether certain data should be combined is decided with synthesis as well as with analysis. However, the students determine this subconsciously, without realizing it. With the analytic method the process tends to be <u>more conscious</u>. ...the "analytic method" which takes the pupils in the experimental, inventive directions in problem solving (exactly how is not said -- Z. K.), develops their thinking--trains them in strictly sequential thought--but due to its abstraction and verbosity, it is very tiring and complicated and therefore <u>frequently cannot be used</u> in school and demands special preparatory exercises (solving problems with missing data, selecting the data for the question, etc.).

It is noted in <u>Methods of Arithmetic</u> [41], that the analytic method is more helpful in developing the method of synthesis:

> The students, after asking themselves blind questions, which either encourage them slightly or do not help them at all, can begin to carry out the most unlikely manipulations with numbers, adding people to rubles, etc. With analysis the pupil approaches the goal by a strict chain of logical conclusions, but the need for this chain is incomprehensible to him...Composing such a chain is difficult for him. If the pupil is aware of the basic sense of a problem, he will solve it rapidly synthetically [41: 50].

Because the method of analysis is difficult, the editors believe that it should not be introduced until third or fourth grade, and a teacher should not intend to develop problem-solving skills by this method in the primary school.

The educational significance of the method of analysis is emphasized by Snigirev and Chekmarev [38]. "It trains the pupil in strict sequential thinking [38: 49]." Acknowledging the difficulty of this method, the authors recommend introducing a series of preparatory exercises in the second grade so that the pupils will be trained for full analytic breakdown of problems in the fourth grade.

The value of the method of analysis is obvious to Berezanskaya [2]:

> There are doubtless advantages for using the analytic method of solving problems, in which the ability of the students to reason and to attain considerable rigor is developed, as well as deliberation and consistency in drawing conclusions, in contradistinction to the synthetic method, where through a series of questions one arrives at the goal coincidentally, as it were [2: 245] (italics mine -- Z. K.).

At the same time she demonstrates that mistakes are also possible with the analytic method, and that this method is difficult even for fifth-graders, particularly if a problem can be separated into more than three simple problems. With sufficient skill, the synthetic method often leads to the goal more faithfully and easily. According to Berezanskaya, various methods must be used in school: the analytic, the synthetic, and the analytic-synthetic.

We find a very detailed exposition of the peculiarities of both methods and a detailed comparison of them by Chichigin [6]. He also shows a preference for the analytic method, where the thought process is stricter and more consequential and "guarantees the correct method of solving a complex problem [6: 292]." But this method is difficult, especially at the start. It is difficult to understand that one unknown can be determined with the help of other unknowns (in the analysis of a complex problem). One must teach only partial breakdown of problems by the method of analysis at first. Chichigin advises using the complete analytic breakdown of quite complex problems (with 5-9 questions) in secondary school.

The author mentions the value of an analytic-synthetic method of problem breakdown. But this is actually a mechanical combination of the ordinary methods of analysis and synthesis. At first the full analysis of the problem is carried out, and then it is synthetically broken down and solved.

Lyapin [14] gives a similar evaluation of the methods of analysis and synthesis. Although he recognizes the analytic-synthetic nature of thought, he emphasizes that

> The analytic method of reasoning trains pupils in strict consistency of thought and promotes the development of logical thinking more than does the synthetic; this method should always be used whenever the instructor is not sure that the plan of solution is clear to the pupils [14: 208].

Polyak considers the analytic and synthetic methods of breakdown of problems in a number of works, although his views have changed radically since his Basic Questions in Methods of Arithmetic [28] appeared. In this work of 1930, he asserted that the method of analysis teaches the technique of solving problems and helps the work of low-achievers. At the same time he wrote:

> The method of analysis possesses an essential inadequacy in spite of all its externally positive qualities, which is that the pupils are deprived of the possibility of thinking through the problem, since the teacher's questions often prevent the pupils from concentrating on the problem's content.

Polyak also stressed that the analytic breakdown is time consuming.

In a 1940 work [31], Polyak emphasizes the positive aspects of the method of analysis. This method guarantees realization of the solution and prepares the pupil for practical life, since in life it is more frequently necessary to solve a problem by selecting the data necessary for solving it than to explain what problems can be solved on the basis of the existing data. Therefore, Polyak believes that this method ought to be an introductory method in all grades in school. Without denying the difficulty of analysis, Polyak demonstrates a method of simplifying it in elementary school (using partial analysis, analysis of previously solved problems or of analogous problems, etc.).

In his work of 1948 [29] Polyak speaks of the analytic-synthetic method, and gives preference to it. This method guarantees a conscious solution, as long as the composition of a plan of solution precedes the performance of the arithmetical operations. As a matter of fact, just as with Chichigin [6], this method (in Polyak's description) is a combination of the analytic breakdown of a problem in its "classical form" and the succeeding solution by the ordinary synthetic method. Polyak considers it possible to introduce such a breakdown even in first grade.

In his 1950 work [30], Polyak does not introduce anything new in resolving the question at hand.

Thus, Polyak, who began with a low opinion of the method of analysis, later speaks of its great value and the necessity of making this method introductory; in his later works (1948 and 1950) he argues that the analytic-synthetic method should be introductory, but retains the form of the analytic breakdown of problems with no change.

"The analytic-synthetic process in breaking down problems is a method of finding a way to solve them [23: 61]" — this idea is very precisely formulated in the works of Pchelko. In his Methods of Teaching Arithmetic in Elementary School, which went through several editions, he writes:

> Analysis is impossible without synthesis. In fact, when we start with the question of the problem and select the data for it (analysis), we note these data not abstractly, but, starting from the text of the problem, from numerical data and from the concept of the problem as a whole (synthesis). On the other hand, when we start the breakdown with the numerical data and select the question for it (synthesis), we constantly verify the advisability of combining certain data through the main question of the problem (analysis).

Realizing that the breakdown of a problem is always an analytic-synthetic process, Pchelko believes that, depending on the starting-points (the data or the unknown), either analytic or synthetic features can come to the fore.

> When a pupil begins the breakdown with the question analytic features come to the fore...When a pupil, having read the problem and having singled out the question, begins the breakdown by combining the numerical data, constantly checking against the problem's question, synthetic features are prevalent [23: 61-62].

It should be noted that in the first and second editions of Methods, Pchelko included a number of critical remarks on the analytic method:

> The synthetic method of breakdown corresponds more closely to the natural mode of thought in pupils who always begin with the data in solving problems. The analytic method is more abstract, more verbose, and more artificial... The analysis of complex problems is difficult for elementary school pupils in its detailed aspect [24: 90; 25: 84].

In the third and fourth editions [26, 27] these remarks are removed, and the positive aspects of the analytic method are emphasized largely. "The analytic method in its conventional interpretation (the method of analysis -- Z. K.) is always useful for developing abstract thought and speech in pupils [23: 62]" -- this is how Pchelko answers teachers' questions about the value of the method.

In the 1953 edition [27] he moves away from his positive opinion of the method of analysis and to a still greater extent emphasizes the significance of the analytic-synthetic method of problem-breakdown.

Skatkin's polemic article [36] is somewhat related to the statements introduced above. The author, referring to statements by Egorov, Bellyustin, and others, emphasizes that the methods of analysis and synthesis "are a model of a logically rigorous exposition of a problem's solution, and not a reflection of a method of finding a solution [36]." He does not deny the use of such a coherent exposition of a solution, but believes that it

> does have a direct relationship to the question of the methods pupils can use to solve a problem. In order to show a student how to solve a composite (complex) problem, one can introduce an example of the solution of a concrete problem, explaining how to disclose and dismember the content of the question of the problem and how to compare the data which are necessary for answering the problem's question with the data already at hand, as well as how to compare the possible combination of the numerical data with the requirement expressed in the problem's question [36: 17].

In order to teach pupils to solve arithmetic problems by themselves, one must, according to Skatkin, "reveal those modes of thought which will most hopefully lead the pupils to find the ties between the unknown and the numerical data of the problem [36: 17]."

The editorial staff of the journal, without denying that problem solving is an analytic-synthetic process, nonetheless consider that the denial of the methods of analysis and synthesis as a means of preliminary breakdown of problems, which point the way to solve them, "weakens our position in the fight for the correct solution of one of the main issues in the methodology of solving problems [36]." Unfortunately, other comments on Skatkin's article did not follow.

In 1949 a collection of articles entitled Solution of Problems in Elementary School appeared, under the editorship of Pchelko. In it a whole series of authors dwelt on the question of analysis and synthesis in problem solving, and several points-of-view were expressed as a solution to this question. Popova, in her article [32] asserts that "reasoning based on synthesis is constructed without ties with the main question, without the necessary purposefulness, and nonetheless gives way logically to reasoning based on analysis [32: 81]." She is aware of the difficulty of the analytic method. Students frequently skip isolated intermediate logical links in the breakdown of problems by this method. In order to simplify the breakdown for students, Popova recommends composing a graphic scheme of analysis.

She considers the second year of instruction the best time for starting the method of analysis. In the third grade one should introduce complete analysis along with partial analysis and in the fourth grade "quite frequent use of complete analysis, during which time it should become habitual [32: 8]," is necessary (my italics -- Z. K.).

Noting that many teachers take a negative view of complete analysis on the grounds that it is inapplicable to difficult problems, and easier problems can be solved without a breakdown by the method of analysis, Popova says: "One should use complete analysis both for problems of medium difficulty and especially for easy problems, since such problems in and of themselves do little to develop thought in the pupils [32: 82]". (my italics -- Z. K.). For example, the following problem is given:

> An airplane flew 260 km in the first hour, 30 km more in the second. In the third hour it flew 250 km less than in the first two hours together. How many kilometers did the airplane fly altogether?

For fourth-grade pupils this is not a difficult problem, and the way to solve it is clear. Popova believes that for the solution of this problem to be more useful in developing the thought of the students, one must give a full breakdown of it by the method of analysis, composing a graphic scheme, where quantities necessary for solution but unknown are mentioned, and the values of the known data are inserted. Popova introduces an analytic breakdown of the given problem and a scheme for it (Fig. 1).

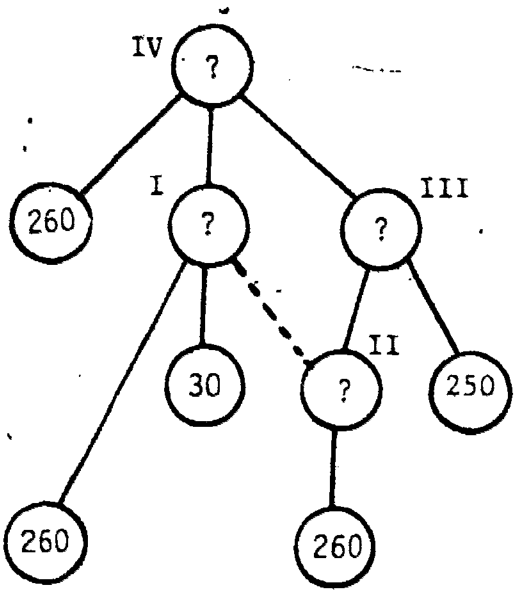

Fig. 1

The analytic breakdown of the problem and the schematic diagram are much more difficult to construct than the solution of the problem, but it is precisely in this that Popova sees the value of the method of analysis [32].

A few model problems, according to the author, yield themselves entirely to complete analysis (problems on motion in opposite directions, the rule of three, etc.). "However, there are problems," Popova [32: 107] notes, "which at first glance will not yield to complete analysis" (my italics -- Z.K.), and "analysis of a special type" is needed. What is this analysis?

In problems on proportional division the analysis of the question is prefaced by the ordinary breakdown of the problems by the method of analysis. It is explained that the question assumes two answers, and an ordinary analysis of the problem follows.

In problems on finding the unknown by the difference of two quantities, and on exclusion of one of the quantities by means of subtraction, the "Why?" question must be asked at first (like Latyshev's recommendation). This question helps to disclose the causal-consequential ties between the data and the unknown, and the way to solve the problem then becomes clear. Then an ordinary breakdown of the problem by the method of analysis is given.

In problems on finding terms by a sum and a difference, a short notation of the problem is made, giving the relationship of the parts, decomposing the question of the problem, and, when the way to solve the problem is understood, an ordinary, "classical" analysis of it is made.

Thus, Popova acknowledges that the method of analysis in its "classical form can be used only when the course of solution of the problem is clear. It is inapplicable to difficult problems [32: 107]." "Analysis of a special type" is directed at explaining the course of solution of a number of model problems, and when the course of solution has become clear, the problem is investigated by the ordinary method of analysis. The classical method of analysis is used as a special logical exercise that complicates the process of the problem's breakdown and thereby increases the value of the problem for developing logical thought.

Bochkovskaya also uses the idea that the method of analysis can be introduced only when the interrelationship between the unknown and the data is clear to the solver. She writes:

> The decomposition of a complex problem into simple ones during the problem's breakdown, is an analytic- synthetic process. Whether analysis or synthesis will dominate in this process is determined by the structure of the problem and by the extent to which the pupils have mastered the relationships between the quantities entering into the problem [3: 74].

Until the pupils have mastered these relationships, the problem must be investigated by the method of synthesis.

> The use of analysis in solving problems gives the best results when the pupils become familiar with it through problems with few operations and a clear structure and when they clearly understand the mutual ties between the unknown quantity and the numerical data [3: 65].

According to Bochkovskaya, both methods can be used at different periods for problems with identical structures: synthesis when the pupils are learning how to solve problems of a given structure, and analysis when the way to solve these problems is already familiar to the pupils.

Bochkovskaya allots less space to the method of analysis than does Popova. For developing logical thought, she recommends, along with the method of analysis, a number of different exercises — comparing problems with different structures, describing the solution of problems by a formula, and thinking up problems as examples.

Novoselov [20] comes out against the monopoly of the analytic and synthetic methods.

> The breakdown of the process of problem solving shows that it is not a strict logical process, as the methodologists depict it when speaking of the analytic and synthetic methods of solving problems. This breakdown shows the great significance of memory and of knowledge of simple problems; more precisely, acquaintance with a given aspect of the relationship of quantities. It shows that *both the question and other problems resemble missing data* [20: 197] (italics mine — Z. K.).

According to Novoselov there is no "general" method for solving different problems. Problems vary in their conditions and difficulty and demand the use of different methods. The breakdown of problems by the method of analysis "proceeds smoothly only with a person who already knows how to solve the problem...[20: 196]" (here he agrees with Popova and Bochkovskaya). New problems should be presented by different methods, to reveal the concrete situation described in the problem (an explanation of words and expressions, a visual demonstration, a graphic scheme, a sketch, etc.). The presentation, according to Novoselov, should be short and should give only the necessary support for independent solution of the problem.

Shor's and Bogolyubov's works read at the Pedagogical Lectures are of considerable interest. The work of Shor, relied upon by instructors in pedagogical schools, gives no evaluation of the methods of analysis and synthesis. Analysis in its traditional form is included as one of the factors of the work on a problem's text, while it precedes the more serious work on it (a short notation of the problem, a graphic breakdown of it, etc.). At the same time, Shor proposes other quite rational means of working on problems (simplifying the text of a problem, solving it by different methods, generalizing model problems, etc.).

Bogolyubov, in summarizing many years of work in schools and pedagogical institutes, describes both the analytic and synthetic methods of solving problems in some detail. He evaluates both methods negatively, since, in his opinion, they both operate in isolation, thus interfering with the natural train of thought, where these processes are inseparably connected. The method of analysis is especially complex in this regard, since it assumes operations first with a series of unknown data, whereas in using the method of synthesis, the pupils are relying on known data. Bogolyubov disapproves of analytic schemes, which, in his opinion, complicate rather than simplify the breakdown of problems and, since they are so artificial, can hardly serve to develop logical thought. The author contrasts these methods with the analytic-synthetic method of problem-breakdown, where the processes of analysis and synthesis interact.

Summary - Soviet Literature

Thus, in resolving the question of analysis and synthesis, Soviet methodologists acknowledge quite properly the analytic-synthetic nature of thought in problem solving. At the same time, Pchelko and other methodologists indicate that, depending on the starting-points for the solution (from the unknown or from the data), either analysis (breakdown by the method of analysis) or synthesis (breakdown by the method of synthesis) comes foremost, and accordingly they select the comparative value of one method or the other. There is still no unanimity in evaluating these methods, especially analysis. Most methodologists mention the value of the method of analysis. It develops logical thought, is distinguished by its rigor, sequentialness, movement toward a goal (Berezanskaya [2] and others), it leads in inventive, investigative directions (Kavun and Popova [11], it is better than the synthetic, it prepares one for practical life (Pchelko [22, 23, 24, 25, 26, 27], Polyak [28, 29, 30, 31] and so forth.

At the same time, shortcomings of this method are noted in the methodological literature. It is difficult, appears to be too abstract, wearisome, artificial (Kavun and Popova [11] and Pchelko [24, 25] up to 1947), incomprehensible to the pupils, and hence the long chain of conclusions is necessary [41]; the construction of this chain hinders the students in thinking independently (Polyak [28, 31] in articles before 1941), and so on.

Just as there is disagreement about the difficulty of this method, there is debate about when to introduce it into schoolwork. Polyak [29] believes that it can be introduced ~~in~~ first grade, and that later this method should become the leading one. Popova [32] believes that familiarization with this method should begin in second grade so that complete analysis of problems can become habitual by fourth grade. Snigirev and Chekmarev [38] indicate that the method of analysis, without a preliminary breakdown of the problem by the synthetic method, can be used only in the fourth grade. Berezanskaya [2] considers it difficult even for fifth-graders.

The question about when and for what type of problems analysis should be used in its "classical form" is also solved in various ways. Pchelko, Polyak, Chichigin, and others, recommend this method for preliminary breakdown of problems, as a method of searching for the solution. According to Chichigin [6], it guarantees the proper course for solving a complex problem. Pchelko [23] believes that this method should be used with uncomplicated problems (with 3 or 4 questions), unreduced, non-algebraic ones.[4] Popova [32] believes that it is impossible to investigate difficult problems by the method of analysis. Because it makes the process of breaking down the easier problems more complex, the method of analysis increases the value of these problems for developing logical thought. According to Popova, a number of model problems demand "analysis of a special type": the causal-consequential ties are revealed through questions (of the "Why?" type), the relationships of the data to one another and to the unknown are disclosed, and then, when the way to solve the problem has become clear, thanks to this breakdown, the problem yields to the ordinary "classical" analysis.

Skatkin [36] in general disclaims the method of analysis as a method of preliminary breakdown of problems; it could be a useful logical exercise after solving the problem. Novoselov [20] likewise believes that one can give a complete breakdown of a problem by the method of analysis only if he knows its solution and not if he is seeking it. In searching for a solution, one does not think in the logical sequence that the methods of analysis and synthesis assume. Different aspects of problems require widely varied means of breakdown, which are determined primarily by the conditions of the specific problem.

These, then, are the basic differences in evaluating the method of analysis in teaching problem solving. How can we explain the existence of these disagreements? First, there is no scientific, experimental verification of the psychological and educational value of the methods of analysis and synthesis. Most methodologists assert that the method of analysis is useful: It develops logical thought. What is the basis for this assertion? It is actually made by pure logic: Since the breakdown of problems by the method of analysis is a logically rigorous chain of deductions, any exercise in constructing such a chain should act to develop logical thinking in students. Is this conclusion correct? To what extent does constructing such a chain of deductions help one to reveal the meaning of a specific problem or to find a way to solve it? Is a full breakdown by the method of analysis of an unfamiliar problem possible? Is exercise in analytic breakdown useful for problems that

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[4] By algebraic problems Pchelko means problems with artificial means of solution (equation, substitution, etc.).

that have already been solved? And, if so, to what extent? How does it influence the ability to solve problems independently? These and a number of similar questions can be answered only by special investigations.

Investigations of this kind have been made by the author of this work over several years. The materials obtained provide a basis for drawing conclusions to these questions. A description of these investigations and their results will be given in the next section.

Chapter III

EXPERIMENTAL INVESTIGATIONS OF THE USE OF THE METHOD OF ANALYSIS IN SCHOOL

1. The Need for Specific Instruction in Analysis

Most methodologists have no doubt about the possibility of a "classical" analysis of a non-model problem with 2-4 questions. The great bulk of these problems are solved in the second and third grades. Therefore, we planned to carry out the investigations in these grades.

We also planned to observe, as the first step in the investigation, how pupils grasp the method of analysis in the most typical conditions in the mass schools. Ordinarily, the schools acquaint the pupils with the method of analysis, and, more or less frequently, they turn to the breakdown of problems by this method, but without working systematically with the pupils on mastering it.

To observe how the analytic method in breakdown of problems was mastered under these conditions, we conducted individual experiments in teacher V.'s class, grade 3C in the 47th School for Girls. The pupils had become familiar with the method of analysis in second grade, and the breakdown of problems by this method was practiced from time to time without any system. When the teacher corrected the mistakes the pupils made in breaking down problems, she did not work systematically to eliminate the errors and did not exact precision in the formulations of the breakdowns.

The absence of sequential work on the analytic method of breaking down problems was naturally reflected in the level of the pupils' mastery of this method. As our experiments showed, only the very best pupils in the class began isolating the unknown and indicated the data that could help to find the unknown. For example, Problem No. 14[1] indicates that the number of overcoats which were made is to be determined, and that one know how many meters went into one overcoat and how many into all the overcoats. Moreover, if the method of analysis is followed,

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[1] See p. 13.

one should indicate which data are necessary for determining the number of meters used for overcoats and that one needs to know this in order to determine how many meters went into one overcoat. However, these pupils, actually interrupting the analytic breakdown, simply indicated what must be determined in the first question of the problem, in the second, etc. That is, they passed to the synthetic method, which was more habitual with them.

The greater portion of the pupils in this class did not even attempt to give an analysis of the unknown. Average and even good pupils simply indicated the unknown and passed on to composing a plan of solution. Here is the record of the "analysis" of the above problem by Valya Z.:

> We must find out how many overcoats were made, and therefore we must know how many meters there were altogether in the shop and how many meters were used for the suits, and we have to know how many meters of material were left for the overcoats.

As we can see, only the indication of the unknown remained here from the method of analysis, and there follows a listing of those questions whose solution leads to finding the unknown. This is also analysis, of course, but it does not answer the demands of the analytic method, the mastery of which we were verifying in the experiment.

Weak students, when asked to give a complete analysis of the problem (they had been acquainted with these terms), repeated the question and enumerated all the known data in the text. This is how Lena M. "analyzed" the same problem:

> "We must find out how many overcoats were made," she said. "For this we must know how many pieces of material there were, how many suits were made, and how much material went into one suit and into one overcoat." She overlooked one of the known data here and simply enumerated in the form of questions those data which were given in the conditions.

Thus we see that pupils in a class where special systematic work on the method of analysis is not done do not master the method. Only external elements, and few of them, are taken from the new method — isolated formulations in speech ("We must find out," "This we cannot determine immediately"). Not even the best students in the class could give a full breakdown of this little problem by the method of analysis.

2. Mastery of An Analysis Method - School No. 64.

As long as we were intending to observe the mastery of the method of analysis by pupils in elementary school and to determine the influence of this method on the problem-solving ability of the pupils, we had to turn to classes where systematic work was done for mastering it, where the method was considered extremely valuable. Following the instructions of experts, we turned to School No. 64, where the school administration insisted that the teachers use the analytic method in breaking down problems, regarding it as particularly influential in developing the pupils' mathematical thought. Therefore most instructors at the school taught the pupils to break down problems by analysis. According to the administration, this work was particularly productive in teacher G.'s class, and so we chose it for the experiments. G. acquainted the pupils with this method in first grade and began to teach it to her students regularly, especially in second grade. In the third and fourth grades, when we conducted our investigations, analyzing a problem was a habit for them.

How did the students in this class master this method of analysis? As the experiments showed, the following rather distinctive form of breakdown was quite typical. We shall introduce the most typical record of the breakdown by the method of analysis of Problem No. 14 by the best pupil in the class, Sasha B. Isolating the question of the problem, Sasha begins his "analysis":

1. "We cannot answer the question of the problem immediately, since we don't know how many meters of material remained after the suits had been made.

2. This we cannot find out because we don't know how many meters went into all the suits.

3. We cannot find out how many meters went into all the suits because we do not know how much material there was altogether. But we can find this out."

We see that Sasha indicates only one datum as necessary for determining the unknown - the one whose value is determined in the preceding operation. For determining the latter, Sasha indicates a datum as necessary whose value he determined in the solution of the second question of the problem. In the first and second instances these assertions correspond to reality. Both data needed to determine the amount of material used for the suits are contained in the problem. However, here also Sasha asserts: "We cannot find out how many meters were used for all the suits because we do not know how much material was brought in," i.e., he indicates as necessary that datum whose quantity is determined by solving the preceding (here the first) question of the problem. Other good and average students analyzed the problem the same way (weak students did the analysis as an ordinary breakdown after the first question). Thus, a very distinctive way of breaking down was composed by good and average pupils in this class. At the start the "main" unknown is indicated in the analysis, whose value is determined by solving the last question of the problem (in the given case, the fourth). Then that datum is indicated as necessary whose value is determined by solving the next-to-last question (here the third), and then by solving the one preceding it (here the second) and hence right up to the datum whose value is determined by solving the first question.

The assertion that it was impossible to determine the value of the necessary datum was made without referring to the text of the problem. The pupils repeated this assertion, but when both necessary data are present, determining the value of the intermediate datum was entirely possible.

The pupils simply remembered the course of solution, disposed of the simple problems solved in this way in the reverse order, and, combining them mechanically in this order, asserted, repeating the standard formulation, that determining the unknown of one simple problem is impossible without determining the unknown in the preceding one.

When the experimenter indicated the real possibility of determining the intermediate data, the pupils either repeated their original assertion or, convinced that both required data were present, considered the analysis complete (i.e., interrupted it at this intermediate link).

The presence in the problem of both data necessary for determining the needed unknown was connected in the pupils' minds with the completion of the "analysis."

Under what educational conditions would it have been possible to explain why this quite distinctive form of "analysis," whose sole occurrence was noted with the pupils in the 47th and 19th Schools, predominated here?

In her work the teacher G. often found "superfluous syntheses" being made by students, syntheses not required by the course of the problem's solution. In the analytic method of breaking down problems, the necessity of synthesizing precisely these determined data is based on logic.

According to G.'s assumption, the analytic method of breakdown, unlike the synthetic, safeguards the students from these groundless "superfluous syntheses." Because she valued this aspect of the analytic method highly, G. emphasized over and over during the analytic breakdown the necessity of determining some particular data in order to determine the succeeding data.

In second and third grades "chain" problems are frequently solved in which the datum obtained by solving the first question is immediately used again to solve the following one, and so on (work-books for these grades abound in such problems).

Partial analysis is practiced most often in class — with an indication of only one unknown datum in the conditions (this form of analysis is recommended by the experts for the primary grades). In partial analysis of this type of problem, the datum obtained for the solution of a succeeding question cannot be determined without solution of the preceding one. Superficially, partial analysis of such problems takes on the form of problem-breakdown "from the end," the reverse of the habitual method.

Because superficial view of analysis of a definite type of problem was repeated so often, the pupils singled it out as an essential feature of an analytic breakdown.

A graphic scheme of analysis was rarely used. Here, when the teacher broke down a problem by the method of analysis, she ordinarily drew that type of scheme on the board and did not ask the students to construct it. In oral breakdown, the specific nature of the breakdown of problems by the method of analysis (with an indication of both data necessary for determining the unknown) escaped the pupils' attention. The corrections introduced by the teacher did not attain their object either -- there were not too many of them because, in the chain-problems, the partial analysis coincided superficially with the breakdown "from the end" and did not lead to a mistake. Therefore this distinctive form of "analysis" was reinforced in the students in this grade, and a firm corresponding stereotype was worked out.

In the fourth grade the teacher did more work on complete analysis and introduced graphic schemes more frequently, but the established stereotype was reconstructed only with difficulty, and when analyzing the problems by themselves, even strong pupils frequently reverted to the form of breakdown which had become habitual.

3. Mastery of An Analysis Method - School No. 653

The method of analysis in second and third grades in the 653rd School was learned under similar conditions. The teacher D.,[2] trying to teach pupils to make a well-founded review of the data for a synthesis, turned to partial analysis (without a graphic scheme), just as did the teacher G. (in the 64th School).

If the values of both of the data necessary for determining the intermediate unknown were contained in the text of the problem, teacher D. explained: "This can be determined, but it must be determined in order" -- and an indication of the next link in the analysis followed. By this the teacher wanted to emphasize the necessity of a definite order, a certain sequence of arithmetical operations, wanted to work out a definite, rigorous, logically valid system of problem breakdown. The teacher understood the role of training in mastery of the new system, and from the very beginning of second grade, she did this type of analysis of problems day in and day out.

As a result, the students in this class worked out a very precise and monotonous form of "analysis" of problems. We shall introduce one of the records of the analytic breakdown of Problem No. 14 by a good student in the class, Marusya K.

Having read the text of the problem and thought about how to solve it, Marusya began her "analysis":

> "This problem is complex, and it is impossible to answer the question of the problem immediately, because we do not know how many meters of material were left for the overcoats.
>
> "We cannot know how many meters were left for the overcoats because we do not know how many meters were used for 20 suits. This we cannot determine either, since we do not know how many meters there were in the 48 pieces of material.

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[2]Her class was recommended for study by the workers at the Institute of Methods of Education of the Academy of Pedagogical Sciences, since the method of analysis was widely used in this class.

"We can find how many meters there were in 48 pieces. We must multiply 25 x 48..." The repetition of the course of the problem's solution followed.

The experimenter indicated that it is possible to determine the number of meters of material used for the suits, since it is known that 3 meters went into one suit and 20 suits were made and he asked the pupil to repeat the "analysis."

Marusya again began the breakdown with the same sentences. Arriving at the number of meters used for all the suits, she said "This we can find out, but then it would not be in the right order, because we must first find out how many meters are in 48 pieces..."

We see that this breakdown is very similar to the one conducted by Sasha B., from the 64th School. Here, too, analysis is understood as a breakdown of the problem "from the end," as an enumeration of the data necessary but not given in the text, in the reverse order compared with the usual one, and included in the appropriate standard speech form.

As our observations and individual experiments have shown, this form of "analysis" was very typical of the overwhelming majority of pupils in this grade.

Thus we see that under similar educational conditions the same form of "analysis" of problems is worked out by the students.

In spite of lengthy practice in the breakdown of problems by the method of analysis, the pupils in the classes of both teachers, G. and D., did not become proficient in these methods. They mastered mainly its superficial side: breakdown of a problem "from the end," in reverse order from the habitual. As we saw, a too abstract breakdown, without relying directly on known numerical data (students arrive at them only at the end in an analytic breakdown), a purely verbal breakdown of problems by this method is extremely difficult for pupils, and they do not master the method, even after lengthy special training.

4. Teaching the Method of Analysis

The difficulty of the method of analysis is acknowledged by all experts. However, as was shown above, many believe that instructors should search for ways of eliminating these difficulties, since, although the method is difficult, learning it exerts a positive influence on the development of logical thinking in the students, as well as on their ability to solve problems.

This assumption had to be verified. In order to avoid, as far as possible, any negative influence on the pupils' mastery of the method of analysis due to poor teaching, the second grade of the 47th School was chosen as a basis, since it was taught by S., a very experienced teacher. She was recommended by the Institute for Teacher Improvement as an outstanding expert in the analytic method, which she was using widely in her work. An Honored Teacher of the Republic, a recipient of the Order of Lenin, a lively, creative worker with 25 years of experience in education, S. approached the experimentation topic with great interest. She wanted to elucidate the value of this method for the psychology of education, to which she had devoted so much attention in her work.

The investigation in S.'s class was carried out over two years. Observations of the arithmetic lessons were made regularly in the class throughout this period. In addition, a group of seven students was selected (3 good, 2 average, and 2 poor ones), with whom individual experiments were carried out at regular intervals, in order to follow their mastery of the analytic method. A series of group experiments was carried out in the class with the same goal (experiment in analyzing problems).

At first we shall observe how the method of analysis was introduced, and then we shall explain, on the basis of the experimental material, how the good, average, and poor students mastered it, and how the class as a whole mastered it.

Mastery of the method of analysis demands that the pupils have some skill in operating with abstract concepts. S. had begun to develop this skill in them in first grade. She taught them to use a type of abstract concept, i.e., price, quantity, cost, etc. She taught them to define the type of simple problems, a task which demands quite a high level of abstraction. At the beginning of the second year of study her pupils could easily identify all eleven types of simple problems, and were able to think up their own problem of any type, select a problem as an example, and write down as an example (formula) the solution of a non-complex problem; they could easily think up possible questions for given data.

Thus, when the analytic method was introduced, the pupils in this class had already developed the ability, quite well for their age, of operating with abstract concepts. The class was distinguished by its wonderful discipline and its high capacity for work.

It was decided to introduce the new method to them at the end of the first quarter of the academic year. In preparing for this, the teacher faced considerable difficulties. She had to show the pupils the specific character of this method (the breakdown, starting from the unknown), contrast it to and differentiate it from the synthetic method of breakdown already familiar to the pupils. But how should this be done?

It would be impossibly complex to give an analytic breakdown of a more complicated problem whose solution would have hampered the pupils. The method of analysis had to be shown in the breakdown of an easier problem. But the pupils could solve such a problem easily in their heads, and they they would not understand why such complicated reasoning was required of them when the problem was actually already solved.

P. Glagolev very successfully depicted the course of such a lesson in his article, "In the World of Numbers":

> In class 'a', the teacher gave the following problem: 'From one potato-bed 11 baskets of potatoes were dug up, and 3 fewer baskets from another. How many baskets of potatoes were dug up in all?' The imaginative boys and girls guessed immediately: 11 and 8 is 19. They put up their hands to answer. But the teacher said: "Put down your hands! Let's review the problem. Tell me, children, can we find out immediately how many baskets of potatoes were dug up?" And the children, although in actuality they had solved the problem immediately, said, simulating the teacher's tone:
>
> "No, we cannot find out immediately." (They knew how to answer in such instances. They were trained regularly and systematically in the notion that they could not solve any problem immediately.)
>
> "Why then, children, can we not find out immediately how many baskets of potatoes were dug up?" the teacher continued, pedantically following the analytic method.
>
> Then many children were nonplussed. Even though they had already solved the problem, how were they to answer as though they had in fact not done so, and think up a reason why they had not? Only the most alert children, who understood the method of study they were using intuitively, guessed how to answer correctly.
>
> "We cannot answer immediately because we do not know..." etc.

The difficulty so well illustrated here, an exception to the "classical"[3] analysis of the method of breakdown learned earlier, was very successfully overcome by S. at the first lessons when the pupils were becoming acquainted with the new method. She asked the children to solve a "guess-problem":

> A girl bought a pen and a pencil. How much money did she spend?

The pupil called on by the teacher lost her head. After a pause, she answered: "How can you find out? It's impossible to solve!"

This question followed naturally: "Why is it impossible to solve?" One of the pupils answered that you have to know how much money the girl had. The teacher suggested that they think up an amount by themselves. One girl restated the problem:

> A girl had 50 kopeks. With this money she bought a pen and a pencil. How much money did the girl spend?

Having obtained such a problem, the girl was convinced that the data she had thought up were not appropriate to the question. "She had 50 kopeks and spent it all. What's there to find out?"

They were convinced that not an arbitrary, but a <u>rigorously defined</u> datum was <u>necessary</u>.

The teacher made the question of the problem more exact: How much money did the girl spend for the pen and the pencil?

This simplified the choice of the necessary data, and the girls easily determined that to answer this question one must know how much the pen and the pencil cost separately. After adding the necessary data, they solved the problem and then repeated in general form the data necessary for finding the unknown: "We must know the price of a pen and the price of a pencil."

The teacher immediately contrasted this new form of breakdown with the familiar establishment of a question for the data:

> A boy bought a notebook and a book for 70 kopeks. The book cost 50 kopeks. What can be found out?

---

[3] The breakdown of problems by the method of analysis in its complete form we shall call "classical" analysis, to distinguish it from others.

The girls supplied the question (how much did the notebook cost?) and solved the problem. This question was repeated in its general form as well:

> The cost of the entire purchase and the price of a book are known. We must find out the price of a notebook.

The following problem, given to the pupils, has one datum: "A girl bought 3 notebooks. How much money did she spend?" In trying to solve the problem, the pupils were convinced that not one, but two data were required to determine the unknown.

Then they continued to solve a series of problems with one or both data missing. The solution of each problem was concluded by its breakdown in a more general, abstract form (What must be found out? What must be known for this?). The solution of such "guess-problems" was of great interest to the pupils. It demanded a greater degree of active thought from them. Noting that it was impossible to answer the question of the problem did not contradict the facts (as generally happens with the analytic breakdown of easy problems -- see above, under Glagolev). The pupils actually could not answer the question of the problem without the definite data. In solving different variants of the problems, they became convinced that definite rather than arbitrary data were required, and that there should be at least two for determining the unknown. It was also natural that at the start the necessary data were named in their general form and only then were concrete values ascertained (thought up) for them.

The "classical" analysis of problems was contrasted with the ordinary breakdown, the starting points of which were the concrete data of the problem (corresponding to the method of synthesis).

Furthermore, it was necessary to fortify the pupils' ability to decompose problems by the method of "classical" analysis and to acquaint them with that type of breakdown of complex problems.

"Classical" analysis of complex problems presupposes the construction of a rather long chain of deductions that connect the unknown with the data contained in the problem. The construction of such a chain of conclusions is doubtless difficult for pupils, as we had previously been persuaded, especially in elementary school. In order to help the pupils maintain the order in their judgments and to give visual support to them, many experts (Pchelko [27], Popova [32], and others) recommend constructing a graphic scheme of analysis when using the analytic method. Both the known and the unknown data are introduced visually in such a scheme, as well as their interrelationships. According to a number of experts, this type of graphic scheme can serve as a visual support in the construction of a long chain of deductions and can prevent missing links in the analysis.

That is why we decided to introduce this scheme as a support in our experimental studies, trying to convince the students to use it in their analytic breakdowns as well. (Use of the scheme by the teacher alone, as the experiment with the teacher G. showed, was less effective.)

The teacher S., guided by her previous experience, only slightly changed the scheme given in the methodological handbooks. She introduced, superficial differences in representing the known and the unknown data. For the unknown data she sketched small squares ("huts," as the children called them), and the data whose values were known from the problem she wrote down outside the squares. Such a superficial change was to promote the differentiation of these data to a still greater extent.

At first the children became acquainted with the construction of this type of scheme[4] for simple problems, but their attention was turned primarily to the correctness of reasoning in the analytic breakdown of problems; the scheme served as a superficial aid for such reasoning.

For three weeks the teacher practiced "classical" analysis of simple problems with the pupils and made efforts so that the large majority of the pupils would master it. They could make a correct analysis of a problem (corresponding to the demands of the method) and construct a graphic scheme for it. A somewhat unusual method of breakdown, demanding reverse reasoning, awakened their interest. The teacher always emphasized the importance of such a breakdown.

Finally, the teacher acquainted them with the breakdown of complex problems by the method of analysis.

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[4]Examples of the schemes will be given below.

At the start a simple problem was analyzed:

A pen costs 3 kopeks. How much did Natasha pay for all the pens?

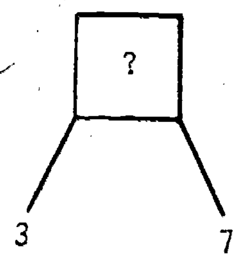

Fig. 2

The pupils ascertained that to answer the question one must know the price of one pen and the number of pens that were bought, then they added the missing data (7 pens) and, repeating the reasoning, composed the scheme of the analysis (Fig. 2).

Then the teacher said: "Lyalya bought erasers. How much money did she pay for them?" The children again selected the necessary data (2 erasers at 4 kopeks each) and drew the scheme (Fig. 3) on the basis of the appropriate reasoning.

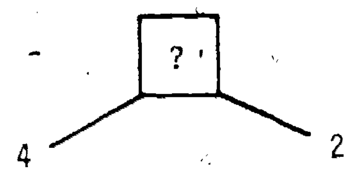

Fig. 3

After repeating the problems about Natasha and Lyalya, the teacher proposed composing one problem from both of them, thinking up one more question.

"How much money did both girls spend?" asked one of the pupils. It was explained that this question needed those data which entered into the other two problems as unknowns and which were already represented in the scheme as "huts" with question marks in them, then both "huts" in the scheme were combined into a new "hut," drawn for the unknown of the complex problem; then they introduced the appropriate operation sign.

Thus a graphic scheme of the analysis of a complex problem was obtained (Fig. 4). Using the scheme, the pupils repeated the entire reasoning demanded by the method of analysis. Having thus solved the problem, they put the data in the "huts," repeated the full analytic breakdown of the problem, and verified their scheme.

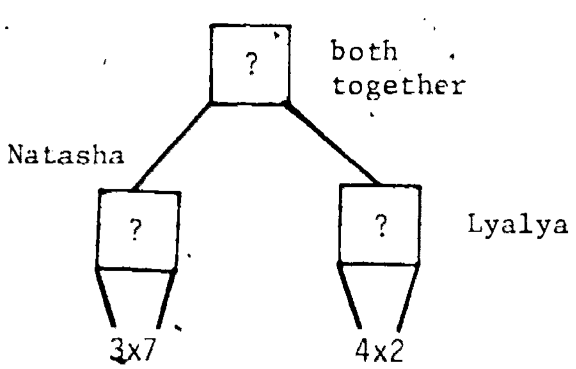

Fig. 4

Thus, S. introduced the analysis of complex problems very naturally by showing that the unknowns of the simple problems making up the complex one serve as necessary data for determining the unknown of this complex problem.

Under the teacher's supervision, the conclusion was drawn that the number of "huts" introduced for the unknown data corresponds to the number of questions in the solution of the problem.

In order to clarify the influence of the analytic method on the pupils' ability to solve problems, the pupils should have mastered this method. That is why S. continued systematic work on it. Of course, as an experienced teacher, she used other, varied means of working on problems, using visual aids widely, and so on. In conformity with the goal, however, she gave much attention to the method of analysis in its classical form throughout the second and third years of study. As the experiments introduced earlier showed, the pupils cannot master this method, and thus we cannot verify its influence on them.

The problems that most frequently undergo breakdown by the analytic method are those which are solved either at home or in class, or problems whose structure is familiar to the students; sometimes "classical" analysis is carried out with the teacher's help, when problems are encountered that are new for the pupils. Problems with 3 or 4 arithmetical operations are analyzed in the third grade by the analytic method. A graphic scheme is ordinarily constructed in the breakdown of more complex problems; easier problems are analyzed orally by the students, without a scheme, while the teacher follows attentively, insuring the completeness and correctness of the breakdown.

It must always be emphasized to the pupils that one must "think a lot" in the breakdown of problems by the analytic method, and that the breakdown of problems by this method is valued more highly, being most frequently recommended for analyzing problems independently and for composing appropriate graphic schemes for them (as a check). All of this increases the pupils' interest in the new method, and evokes their desire to master it.

The analytic breakdown of problems whose structure is familiar rings all the more true at lessons where at the slightest difficulty the teacher and pupils always come to the aid of the problem solver, showing him what direction to take in the breakdown. The class, with the teacher's skillful and systematic guidance, gradually master the new and, for them, quite complex method of breakdown.

5. Experiments with Individual Students

Let us turn to the material of the experiments with individuals and observe the way in which both good and poor students learn this difficult new method of breakdown.

The individual experiments were conducted in the following manner. After the arithmetic lesson two problems were given to the pupils individually -- one problem with the same structure as was analyzed that day in class, and the other similar to ones analyzed in previous classes. Thus, the basic mass of experimental problems did not present any difficulty. These were problems whose structure was familiar. After the pupils had become acquainted with the text of the problem (if necessary, essential clarifications of its content were given), they were told to decompose it by the method of analysis for solution. Then they were asked to solve the problem and to analyze it again. At the experimenter's request, the pupils accompanied the analytic breakdown of the problem with the construction of a graphic scheme. The scheme was meant, first, to make the analytic breakdown easier for the pupils, and second, to give the experimenter visual evidence of the children's notions of the connections between the data. If constructing the scheme became difficult, the problem was analyzed without it. Assistance was given only in case of difficulty. Six of these experiments were carried out, with two problems in each one.

Before proceeding to the analysis of the results, we shall look into the task facing our pupils in mastering the method of analysis. Along with reinforcing the systems of problem breakdown which they had learned earlier (the method of synthesis), the pupils also had to master the new system of breakdown (corresponding to the method of analysis), which was often in contrast to the one they already knew.[5]

The starting point for both the method of synthesis and the method of analysis is the text of the problem -- the data and the unknown. In breakdown by the method of synthesis the pupils choose several data, ask a question about them, and then, having chosen the appropriate arithmetical operation, they calculate and obtain the value of a new datum. They

[5]Again I emphasize that here we are speaking not of the processes of analysis and synthesis, without which no problem can be solved, but of mastery of the method of analysis and synthesis as particular systems of problem-breakdown.

operate with concrete data, known quantities, and go from them to the unknown, and reasoning alternates with computation. Only in the final step do they arrive at the unknown. After some practice with such a system of breakdown, a definite system of temporary connections and a definite dynamic stereotype is formed, and the process of breakdown becomes much easier.

The initial link in breaking down a problem by the method of analysis is the unknown, for which data are selected whose values are not given in the problem, and hence can be looked at only as abstract concepts (path, speed, etc.).

The reasoning is constructed by a definite scheme, and it is very detailed; all its steps are interconnected. Only at the end of the reasoning does one arrive at the data contained in the conditions and the corresponding arithmetical operations.

Mastery of this method of breakdown, which is new for the pupils, assumes formation of a new system of connections and of a new dynamic stereotype; although this new system has links in common with the old one (in the breakdown of specific problems, the data in the problem and the unknown and their interrelationships are common to both), they will be differentiated by the order in which the data are combined and by the scheme of reasoning. The construction of a graphic scheme of analysis was connected with the breakdown of problems by the method of analysis with pupils in this class.

The new system of ties, corresponding to the classical method of analysis, is formed by strengthening the already complicated system of connections, having several links in common with it (as was shown above), and the latter exerts a certain influence on the process of developing the new system of connections. Certain efforts are required for the new system of connections, with the mastery of the method of analysis as a basis, to be formulated and differentiated from the previously formulated system which was based on the breakdown of problems by the method of synthesis.

As the investigation showed, although this process has a common line of development in both analysis and synthesis, the mastery of the method of analysis by poor and good pupils does have some peculiarities. First, we shall observe how poor and good students master the new method, and then, on the basis of all the resulting experimental material, we shall trace a general course in the mastery of this method and in the formation of the system of connections underlying it.

The Records of Weak Students

The poorest pupil in the class (of the teacher S.) where the experiments were conducted, Mila F., was a calm, studious, slightly sluggish, and passive girl. Under the guidance of her teacher, she did a great amount of extra work and managed with difficulty to get passing marks in the basic subjects. Arithmetic problems gave her particular difficulty. She went readily to the experimenter for "supplementary study."

The first series of experiments was conducted during the first week after introducing the method of analysis. Mila was given this problem:

> A girl bought 3 candies at 4 kopeks each and 2 cakes at 10 kopeks each. How much did she spend?

When the experimenter asked if she could decompose the problem again as they had done that day in class, Mila answered affirmatively and silently began to draw a diagram (Fig. 5).

"I'll put a minus sign here, multiply here, and divide here," she explained.

She constructed this scheme for the problem's breakdown. It corresponds exactly to the one drawn during the lesson, except that the last operation sign was changed (obviously forgotten).

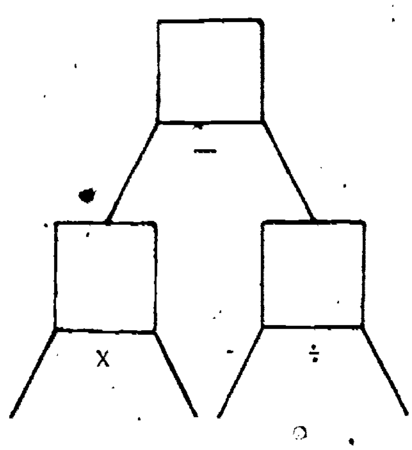

Fig. 5

Having constructed the scheme, Mila turned to the problem. "We do not know," she said, "how much money the girl spent. For this (for the answer to the question of the problem) we must multiply 3 X 4 = 12, and then 10 X 2 = 20. Here we must also multiply" and she corrected the operation sign, then completed the problem, and, correcting the minus to a plus to correspond to the last operation, put the data into the scheme. Since the problem given her was identical in structure to the one done in class, all the data were easily placed in the previous scheme which she recalled, and she was completely satisfied with her "analysis."

The second problem given to her had a slightly different structure. The breakdown of such problems had been conducted earlier in class, but not on the day of the experiment. The problem was:

> A girl bought 4 notebooks at 20 kopeks each and gave 3 rubles to the cashier. How much change did she receive?

One datum that is needed to determine the unknown is not given here, and therefore the asymmetric scheme in Fig. 6 would have to be constructed in analyzing the problem.

Again, Mila began, not with the breakdown of the problem, but with the scheme.

> She silently drew a scheme analogous to the preceding one, but simplified (without the the operation signs or the lines for several data) (Fig. 7). The problem was harder for her, and she concentrated on solving it. She did so with some help and then returned to the breakdown. "We do not know," she said, "how much change the girl received. For this we find out how much she paid for all the notebooks (she wrote down all the data). Then we find out how much change she received." She put the sign of operation by the third "hut" and wrote the answer in the first rectangle (Fig. 8).

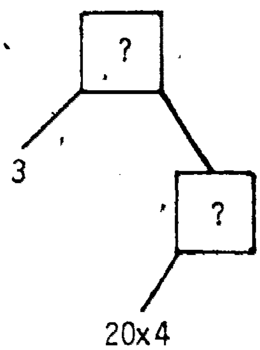

Fig. 6

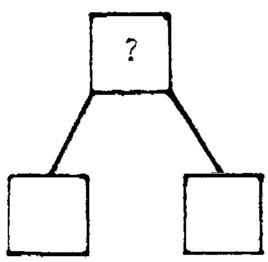

Fig. 7

Thus, there was no reconstruction of the ordinary, synthetic analysis of the problem. The graphic scheme of breakdown somehow seemed ~~purely~~ external, having no influence on the nature of the argument.

When asked to correct the scheme, since it was not exact, Mila repeated her performance and refused to change anything. "I can't do it," she kept repeating.

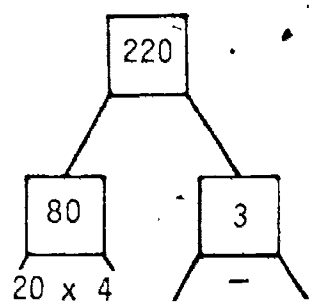

Fig. 8

Thus, in the initial period of mastery of the method of analysis, Mila F. had learned that the breakdown begins by repeating the question and that this is accompanied by the construction of a graphic scheme. She still did not isolate the relationship between each element in this scheme and the corresponding datum of the problem. She reproduced the scheme before the breakdown of the problem, which had been drawn in class (at first even with the signs of the operations). The scheme was filled in after solving the problem. Only the first element of this scheme, the "main hut," was filled in with an unknown according to the new method. Furthermore, the problem was broken down and the different elements of the scheme were filled in their ordinary order -- from the first question to the last. The elements of the new breakdown were joined to the system that had been learned earlier, without any change in the latter.

After six weeks Mila began to show some reconstruction of her process of reasoning reflected in the graphic scheme of analysis she constructed. For example, this is how she "analyzed" Problem No. 3:

> Mother spent 50 rubles while shopping. She bought a scarf for 20 rubles, and she bought 3 pairs of mittens with the remaining money. How much does 1 pair of mittens cost?

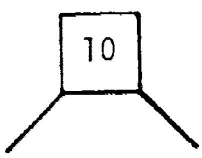

Fig. 9

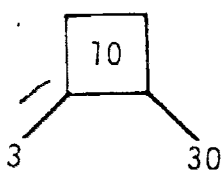

Fig. 10

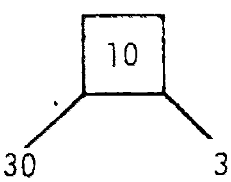

Fig. 11

Mila solved the problem first, then, having repeated the question, drew a square and wrote the unknown which she had just found in it. (Fig. 9). She said, "For this we have to know how many scarves she bought: Three scarves (which she writes down in the diagram). And how much she spent for them. 30 rubles" (Fig. 10).

"I did it wrong," she noted, replacing the data (Fig. 11). "Now we must know," she continued, "how much she paid for the scarf. 50 - 20."

She turned again to the diagram. A pause ensued--there was no "hut" in the diagram; both data were known. Mila put one of the data in the square and drew a new line to the other with a new square, where she wrote in the known datum (Fig. 12).

"And where is the 20?" asked the experimenter.

Mila gave up this diagram and made a new one (Fig. 13).

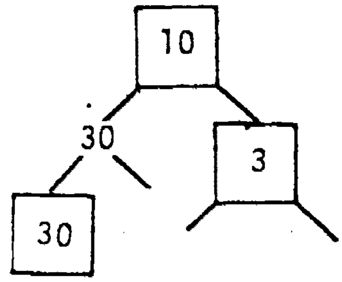

Fig. 12

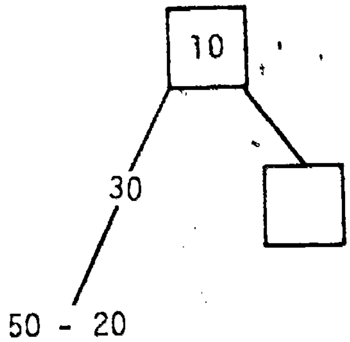

Fig. 13

Here we see a substantial change in the reasoning and the construction of the diagram: The diagram does not come first here; the breakdown does, and the diagram is subordinate to the nature of the breakdown and is reconstructed in conformity with it.

The breakdown itself is reconstructed slightly. Mila mentions not only the unknown, but at the same time selects also those data on the basis of which the unknown can be determined.

Thus the entire first link of an analytic breakdown has been mastered. Furthermore, this link is joined automatically to the ordinary system of breakdown which was mastered earlier. Mila repeated the first variant and the corresponding data (in their numerical form) and tried to join them to the diagram.

How did subsequent mastery of the method of analysis proceed? Did Mila succeed in mastering its remaining links and finally in differentiating it from the system of breakdown she had learned previously (the method of synthesis)? As the experiments showed, this task proved to be beyond the capacity of poor pupils under this system of working on the new method. Mila mastered the analysis of isolated simple problems that made up a complex one (combining the unknown with the appropriate data), but was unable to select a basis for joining these isolated links of the analysis into a single chain, and the links remained disconnected.

For example, this is how she "analyzed," towards the end of the fourth quarter (of the second grade), Problem No. 9.

> Young apple trees and young pear trees were planted in a garden. The apple trees were planted in 10 rows of 5 trees each, and the pear trees were planted in 15 rows of 3 trees each. Which type of tree were there more of, and how many more?

Mila solved the problem without difficulty and started an analytic breakdown at the teacher's suggestion.

> "We must find out how many more trees were planted of which kind," she started correctly, isolating the unknown. "We must multiply 5 trees by 10, which would be 50. That's how many apple trees there are. Then we must multiply 15 by 3. This is the number of pear trees. Then we must subtract 45 trees from 50 trees... and hence there are 5 more apple trees."

As we see, Mila simply recalled the course of the problem's solution. Then the experimenter suggested that she break down the problem and again try at the same time to construct a scheme for it.

Mila repeated the problem's question and drew a "hut" for the unknown. She continued, "We need to multiply 5 trees by 10... to obtain 50." She then wrote down these data in the diagram (Fig. 14).

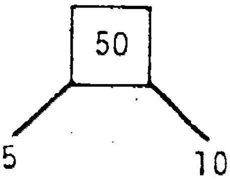

Fig. 14

Mila stopped in indecision. How should the diagram be continued? The "main hut" was already filled in, and the unknown was yet to be found. "We don't have to write 50 anywhere." She had found a way out and drew a new diagram (Fig. 15).

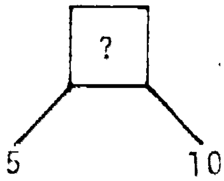

Fig. 15

"Now, I should find out," she continued, "how many pear trees were planted. We must multiply: 3 x 15 = 45 pear trees," and she wrote these data into the diagram (Fig. 16).

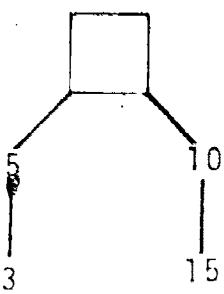

Fig. 16

"Then I must find out how many more apple trees were planted. For this I must subtract 45 trees from 50 trees = 5 trees," and she entered the unknown in the top square.

"And where do you have 50 and 45 in your diagram?" asked the experimenter.

"But we calculated them," Mila answered. "5 X 10 = 50."

The experimenter reminded her that there should be a place for every datum in the diagram. Then Mila repeated the course of the problem's solution and drew a diagram for each simple problem. (Fig. 17). When the experimenter asked if it was possible to join these into a single diagram, she said it was not.

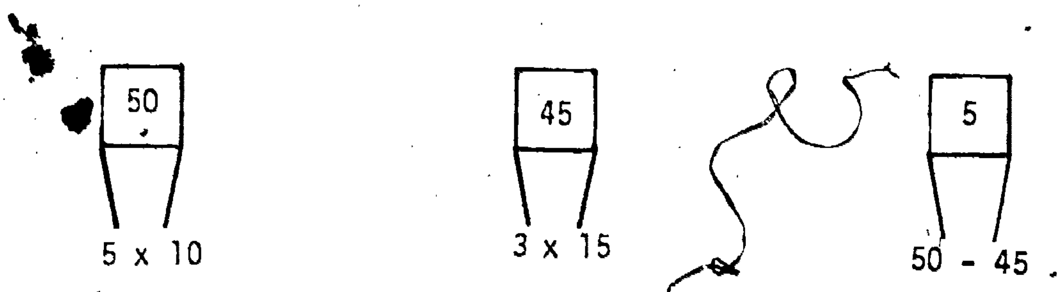

Fig. 17

Without having isolated the principle of combining the separate links into a single chain, Mila gave up, as we saw, and drew a diagram of simple problems that was completely comprehensible to her, disposing of them in the ordinary order to correspond to the course of solution.

At the end of the third grade, Mila F. remained at this level of mastery of the method of analysis.

We see that the "rivalry" between the old and the new systems in Mila ended up with a triumph over the old system; only the slightly important links from the old system were retained (the breakdown began with a repetition of the question of the problem, and the drawing of the "huts" for the unknown data). The method of analysis for this weak pupil proved so complicated that by the end of the third grade she had not surpassed the level of analysis of individual simple problems composing a complex one, as described above, and she could not combine the separate links of the reasoning into a single chain.

The principle of combining the links of analysis into a chain was inaccessible for another poor pupil in this class, Alla G. At the end of the experimental period (the last quarter of the third grade), Alla "analyzed" Problem No. 14 in the following manner. The problem says:

> Eight pieces of material, each 25 meters long, were brought into a store. From all the material 20 suits and several overcoats were made. How many overcoats were made if 3 meters of material were used for a suit and 4 meters for an overcoat?

Alla tried at first to give an analysis of the problem and then, having gone astray, gave it up and solved the problem. Afterwards she returned to the breakdown:

> "We must find out how many overcoats they made. We must know how many meters of material were used for one overcoat and how many meters were used for all the overcoats." She gave a precise analysis of the first link. "For one overcoat," she continued, "four meters of material were used, and how much

was used for all the overcoats is unknown. This cannot be found immediately...We must also know how much was used for the suits and how much material there was in all, and how many pieces...."

Although she gave a correct analysis of the first link, Alla was misled later, as we see; she could not isolate the next link in the analysis and she confused its elements.

"There were 8 pieces," she continued, "No, how many meters were used for the suits?...That's not known... Three meters were used for one suit, and how many for all the suits is unknown... For this we must know how much material there was in all...."

An attempt to rely on the graphic diagram, made at the experimenter's suggestion, did not bring success. Alla repeated the breakdown cited earlier and put together the elements of the diagram unsystematically.

She had mastered somewhat more of the new method of breakdown than had Mila F. Her analysis of the unknown was much more detailed than was Mila's. But, since she was unable to isolate the principle of combining the isolated links of the analysis, she also slipped back into the ordinary synthetic breakdown.

Thus, the investigation showed that poor pupils were unable to master the method of analysis, even over a long period (more than a year and a half), with the given method of study, and hence were unable to work out and reinforce the corresponding system of ties, or to work out the necessary dynamic stereotype. However, the work on this method was not without its effects. Individual links of the new method of breakdown were isolated and were rewoven into the ordinary order, although without actually reconstructing it.

At first the new method of breakdown was connected with the presence of the diagram (i.e., with the visual elements), although the diagram itself was brought in from elsewhere, from other problems, as something previously given. Moreover, the specifically verbal beginning of the breakdown was mastered--from the unknown: "It is asked in the problem. This we cannot find out immediately...For this we must know...." At first, the naming of the unknown was directly combined with the indication of the first question of the problem and the corresponding data. That is, the ordinary order of solution was used as soon as the unknown was indicated.

Later on, the weak pupils selected the specific character of the analysis of the unknown--they indicated (remembering the solution) precisely those data on whose basis the value of the unknown was determined. They had learned the first link in the analysis; most often these data were indicated more concretely, in their numerical expression. The pupils could also give a proper analysis of isolated intermediate links (isolated simple problems making up the complex one). However, they did not rise to the principle of combining these links.

Obviously another method of study and more exercises are needed if the method of analysis is to be fully mastered by the weak students.

The Records of Good Students

Good pupils mastered the "classical" method of analysis differently. Let us observe how one of the best pupils in the class, Lyusya G., learned the method. Lyusya was a quiet, attentive, and studious girl. She loved arithmetic, and ordinarily received excellent marks in it.

At the start of the experiments Lyusya had already developed a verbal beginning for her analysis of problems. In solving Problem No. 2, she started confidently:

> "The problem asks how much change the girl received... This we cannot find out immediately, because we do not know how much money she paid for all the notebooks"--she indicated one of the data, and not the second (the amount of money given to the cashier). "We must find out," she continued, how much money the girl paid for all the notebooks. For this we must multiply: 20 kopeks X 5 = 100 kopeks, 1 ruble." She repeated the solution of the first question and introduced all of these data into the diagram (Fig. 18).
>
> Then she posed a second question, recalled the solution and wrote it down correctly, as in the preceding diagram, in the proper "hut" for this link; the answer was again introduced into the uppermost box (Fig. 19).

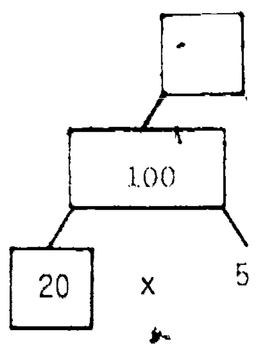

Fig. 18

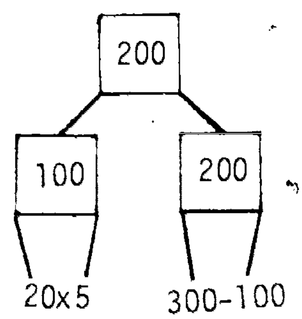

Fig. 19

Thus, Lyusya combined the partial analysis of the unknown (indicating only one of the two necessary data) with the ordinary breakdown of problems, which dictates the order for combining the elements in the diagram. (This is particularly clear on the second diagram Lyusya drew.)

The analytic breakdown is more complicated when problems demand reconsideration of some data such as renaming of the data or requiring a transfer from one system of associations to another.

Lyusya met this difficulty in analyzing Problem No. 3. Solving it gave her no trouble.

> Having isolated the unknown, she indicated: "In order to answer the question, we must know how much all the gloves cost," and she made a partial analysis of the first link and drew a new "hut." "We know that she had 50 rubles and that she spent 20 rubles. How many rubles did she have left?" She introduced both data and stopped in indecision. She even "built a hut" in order to introduce "how much all the mittens cost," and, having subtracted 20 rubles from 50, she found out "How many rubles did Mother have left?"
>
> "How did you find what one pair of mittens costs?" (The experimenter came to her assistance.)
>
> "She bought 3 pairs of mittens," Lyusya answered, "and I want to know how much one pair of mittens costs."
>
> "What operation do we introduce? What data do you take?" (Again the experimenter helped.)
>
> "We must divide 30 by 3," Lyusya remembered, but still could not continue the analytic breakdown.
>
> "What is the '30'?" asked the experimenter.
>
> "Thirty rubles Mother had left...She bought gloves with them... 3 pairs of gloves." Lyusya has found the necessary link. "30 rubles is what all the mittens cost. We must divide 30 by 3, that is 10 rubles," and she introduced all of these data into the diagram.

We see that Lyusya obtained the numerical datum, 30 rubles, when she realized how many rubles Mother had left, i.e., it was advanced as a remainder of money. A number of helpful questions were required before Lyusya could come to regard these 30 rubles as the cost of three pairs of gloves. This rethinking was carried out as she refreshed her memory about the appropriate arithmetical operations (50 - 20 = 30; 30 : 3 = 10).

Because Lyusya gave a correct partial analysis of the unknown at the first stage, one would expect that after more exercises she would master analysis of the first link rather quickly--indicating both data necessary for finding the unknown. According to the requirements of the method of analysis, in doing so she would make note of both necessary data in abstract rather than numerical form, on the basis of a knowledge of the functional relationship betwen the quantities. She could give a correct analysis of simple problems included in a complex one. However, the principle of joining the separate links of the analysis into a single, logical chain was still unclear to her, and she joined the succeeding links by the customary method (from the first question). For example, consider Problem No. 10:

> There were 450 eggs in each of two boxes. A dining hall used up these eggs in 5 days. How many eggs did the dining hall use daily, if they used the same number of eggs every day?

She analyzed the problem in this way:

> "The problem asks how many eggs they used daily. In order to find this out, we must find how many eggs there are in two boxes and how many days they were used."

Thus she gave a correct analysis of the unknown. Furthermore, instead of indicating the data known from the problem's text and the unknown data, then selecting the corresponding data for the unknown, Lyusya enumerated all the data contained in the problem:

> "It is known that there were 450 eggs in each box, that there were two boxes and that the eggs were used up in 5 days. We must find out how many eggs there were in both boxes," and she passed to the ordinary course of breakdown.

When asked to repeat the breakdown, accompanying it with a diagram, Lyusya reproduced what she had said earlier and drew a diagram (Fig. 20).

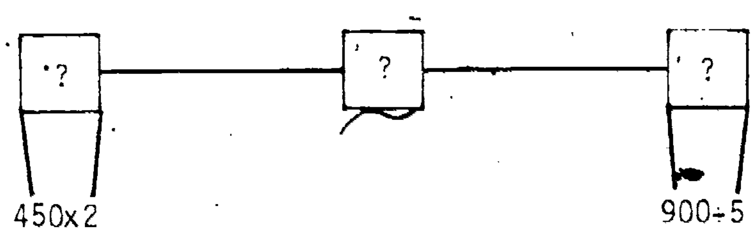

Fig. 20

This diagram, as we can see, corresponds completely to the course of her reasoning: The composition of the individual simple problems that make up the complex one is given, and the dismemberment of them corresponds to the ordinary course of solution (from the first question).

In the future Lyusya also isolated the principle of joining separate links of the analysis (the data of one link whose value is not known appears as the unknown in the succeeding link, and, entering simultaneously into both links, joins them to each other). At the end of the experimental period, Lyusya could do a correct analytic breakdown of a problem with four questions if the way to solve the problem was familiar to her. This is how she did the breakdown of Problem No. 14 by the method of analysis: She immediately made an analysis of the first link (the unknown). Then she interrupted the breakdown, solved the problem, and then correctly completed its breakdown by the method of analysis.

Thus, when good pupils contrast a new system of breakdown with one composed earlier, the new system triumphs.

At first, as with the poor pupils, the more visual elements of the new system are mastered (the standard phrases "We must find out," "This we cannot find out immediately," etc., and the presence of a diagram), then the analysis of the first link (the unknown) while it is mechanically combined with succeeding links in the ordinary order of breakdown (from the first question). Finally, the principle of joining the links in the analysis is singled out, and the entire breakdown is accordingly reconstructed. The new system of breakdown is finally differentiated from the one previously learned, and the pupil can give a complete analytic breakdown of a problem in four arithmetical operations.

Other good students mastered the method of analysis analogously, with greater or lesser difficulties.

As the experiments showed, there is some difference in the way that poor and good pupils learn the method of analysis. Still, we can note <u>common features</u> in the mastery of the method -- which is new and quite difficult for the student.

At first only a few superficial factors are isolated in the new method -- individual standard phrases, the beginning of the breakdown by repeating the problem's question, the construction of a diagram with the breakdown--grasped quite superficially, without the basic connections between its elements. These elements of the new breakdown are mechanically combined with the ordinary synthetic breakdown, without re-structuring it.

Next the pupils master the analysis of the unknown (i.e., the first link of the breakdown). At first they simply recall the last arithmetical operation by which the value of the unknown was found, and indicate it as necessary data corresponding to the numerical quantities. Later they begin to find the data necessary for determining the unknown, as the method of analysis proposes, without reproducing the appropriate arithmetical operations, on the basis of a knowledge of the functional relationships between the data. Then this first link of the analysis is mechanically joined with the ordinary method of breakdown (first the first question is indicated, then the data necessary for solving the problem, and so on).

The principle of joining the separate links of the analysis (the unknown and the appropriate data) into a single chain turns out to be the most complicated. Poor pupils do not master this principle and remain on the level of a disconnected analysis of isolated simple problems. Good pupils master it, and then the new system of breakdown is finally separated from the old and begins to co-exist with it independently.

Thus three levels of mastery of the method of analysis are noted:

I. isolation of some superficial elements of the new method, connecting them mechanically to the old system of breakdown;

II. mastery of the analysis of the first link (the unknown), mechanically connecting it with the usual breakdown of a problem (from the first question);

III. the singling out of the principle of combining the links of the analysis and, on the basis of this, the mastery of the method of analysis.[6]

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[6] It should be emphasized that this is only the basic line of development in the mastery of the analytic method. Not all pupils, as we saw, attained the third level of mastery during the experimental period. The weaker pupils stopped at the second level of mastery.

These levels of mastery of the method of analysis are also clearly connected to the way the diagram is constructed. At the first level the pupils make diagrams whose elements are joined by random signs, and perhaps a number of data will not be reflected in the diagram. At the second level the first link is clearly singled out--the unknown is joined with those data needed for determining it, and the remaining links are available and are joined together in the order of the solution (from the first question to the last). The disconnected diagrams of individual simple problems that enter into a complex one also belong to this level of mastery. The third level is characterized by a correct analytic diagram.

This description of the process of learning the analytic method is given on the basis of individual experiments with a relatively small amount of experimental material. Naturally the question arises of the extent to which these features of mastery of the analytic method are characteristic for the rest of the pupils in the class. To answer this question, we shall turn to the results of the group.

6. Group Experiments

The group experiments were conducted in the beginning period of mastery of the method of analysis (in second grade) at the end of the first and second school years (i.e., in second and third grades). Since we could not require second-and third-grade pupils to write a detailed breakdown of problems by the method of analysis, we decided to limit them to the composition of diagrams of the analysis of the problems.

Even in the individual experiments it was established that the nature of the problems broken down by the method of analysis told particularly on the features of the construction of a diagram. Therefore we thought it possible to judge the level of mastery of this method according to the nature of the diagrams.

Problems whose course of solution was clear to the pupils were given to them (i.e., problems whose structure was familiar). After solving these problems, the pupils were to compose a diagram of the analysis (they had composed these schemes both in class and in doing homework). But, unlike the schemes they were used to composing, the pupils were to write down those data which they considered necessary, near the squares allotted in the diagram to unknown data. Such inscriptions clarified the content of their diagrams.

Doubtless, a group experiment can provide only orientational material for understanding the learning process. However, the general direction of this process and its most characteristic features can be isolated.

The following problem was given for the first time to the pupils, who were to compose a diagram:

> In one class the pupils sat in two rows, with 12 pupils in each row, and in another they sat in three rows, with 13 pupils in each. In which class were there more pupils, and how many more were there?

At the start of the group experiment the pupils had analyzed a significant number of similar problems, and the problem should not have presented any difficulties.

The results showed that not all the pupils had mastered the method of breakdown to the same extent. In five papers the weakest pupils introduced a false solution into a superficially correct diagram of the analysis. See the diagram turned in by the pupil M. (Fig. 21). The other pupils performed different arithmetical operations, but in all five the diagram of the analysis was analogous to the one that had been most frequently constructed in class during the breaking down of problems. One can assume that these pupils were on a very elementary level of mastering the method of analysis, having mastered individual elements of it (in particular, the presence of the diagram) without reconstructing the process of the breakdown. They produced a directly visual form of the diagram that had been impressed upon them on the basis of the classical exercises, with no connection to the process of reasoning.

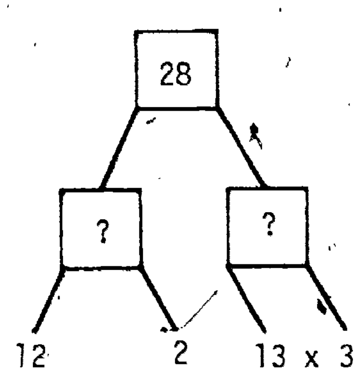

Fig. 21

In six papers disconnected diagrams of the simple problems that made up the complex one were handed in, which indicates mastery of the analysis of isolated links of the problem. Five papers showed complex diagrams whose links were mechanically connected in the order of the problem's solution (from the first question). It is interesting to note the ingenuity with which the pupils reconstructed the diagram according to the ordinary course of solution. Thus, Valya S. made the following diagram (Fig. 22), where alongside the unknown (5) she placed the final question (how many more?) and then, corresponding to the first operation, how many students were in the first class? and, corresponding to the second operation, how many students were in the second class? Valya S. did not find any place for the third operation, and she wrote only its result, the value of the unknown, in the "main hut."

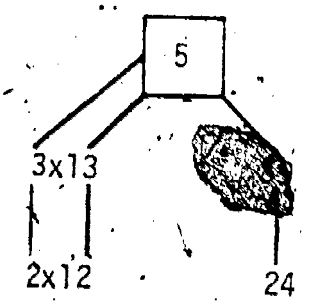

Fig. 22

Nadya T. managed to include all three arithmetical operations in her diagram (Fig. 23):

1) How many pupils in one class?
2) How many pupils in the other class?
3) How many more pupils?

All 11 papers were characteristic of the second level of mastery of the method of analysis.

The remaining 20 papers (56 percent of the class) gave a correct diagram of analysis. The group experiment occurred at the end of the academic year in second grade.

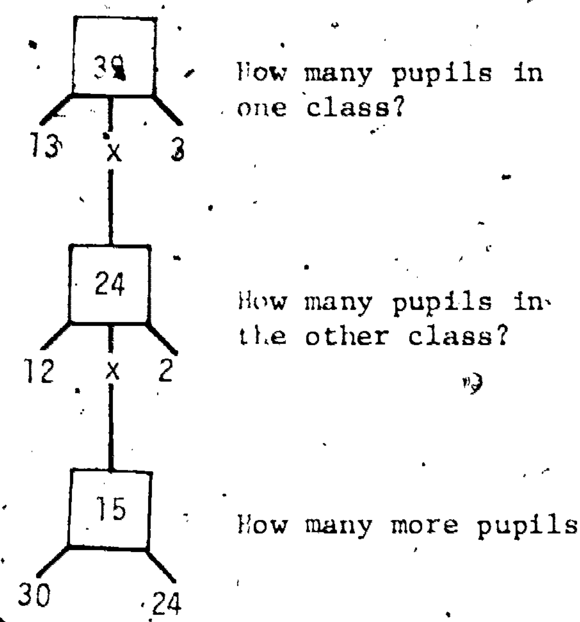

Fig. 23

A more complex problem was next given to the pupils, who were to compose a diagram[7] of the analysis:

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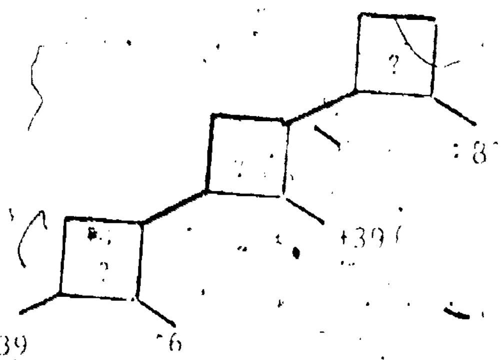
[7] The diagram of the analysis of the problem is not symmetric:



Before lunch some woodcutters cut down 39 trees, and after lunch they cut down 6 fewer trees. All the trees were carried away to a farm in 8 wagons. How many trees did each wagon hold?

An analysis of the papers showed some progress in mastering the method of analysis. There were no diagrams characterizing the lowest level of mastery, no superficially correct diagrams with incorrect solutions, and no disconnected diagrams of simple problems.

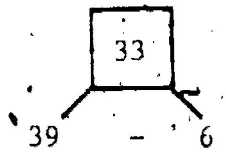


Fig. 25

In three papers the construction of the diagram was subordinate to the accustomed order of breakdown (the synthetic), but in two of them variants of the diagram were given, which reflected somewhat of a reconstruction of this process. Thus, Nadya T. at first made a diagram of the analysis of the first simple problem (Fig. 25). This diagram was crossed out and another was given, where the second simple problem was constructed on top of the first (Fig. 26). The third simple problem (for the unknown) was not given in the paper, but it was included in another paper, that of Zina S., where the last simple problem was constructed on top of the first two.

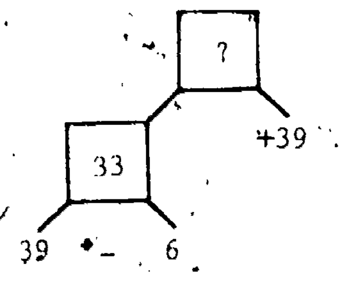


Fig. 26

The presence of crossed out diagrams allows us to deduce the way these pupils constructed the diagrams of analysis. The diagram itself was done in the ordinary order--first the layout was constructed, then the first simple problem was included, followed by the second and third (i.e., the synthetic method). However, these schemes were arranged one above the other, and thus the scheme for the third simple problem (where the unknown is determined) appeared on top, under the second and the first, i.e., the reverse of the order of solution.

Obviously, this arrangement, the reverse of the ordinary order of arranging the links "from the end," was selected as a sign of the analytic breakdown by these students, as the pupils in the 64th and 653rd Schools characteristically did (this was not noticed in the individual experiments). This is apparently an intermediate stage of mastery, between the first and second stages.

In five papers a correct analysis of the first link (the unknown) was given, but the second link[8] of the analysis was missing, and a new diagram was given in four of them, where all the links of the analysis were established. These papers were characteristic of the second level of mastery of the method of analysis.

In 24 papers out of 34[9] (i.e., 65.8 percent) the correct graphic diagram of the analysis was given. Thus, the number of pupils who diagrammed the analysis correctly (although the problem was more complicated) actually increased. One may assume that these pupils had mastered the analytic breakdown of problems whose structure was familiar.

The third and last group experiment was conducted at the very end of the academic year (in the third grade, in May). The problem, which was to be diagrammed using analysis, was analogous to the one given in the individual experiments (Problem No. 14).

Many of the pupils managed the composition of this diagram of analysis, which was much more complicated than the preceding one had been. Of 33 pupils,[10] 25 immediately diagrammed the analysis correctly, at the first level of mastery. Two papers showed diagrams that had at first omitted individual links of the analysis, but that then gave a correct diagram of the analysis alongside. Three pupils had the correct analysis of the first link (the unknown), combined with the arrangement of the links in the order of an ordinary breakdown. All five papers indicate the second level of mastery.

---

[8] Let us remember that, according to many experts, the method of analysis is not admitted in the breakdown of individual simple problems composing a complex one.

[9] Two girls were sick.

[10] A number of pupils were absent in the third grade, and a new pupil was not asked to compose a diagram.

In two papers a false solution was given: a link in the analysis was thoughtlessly connected. One paper showed a diagram of only the first simple problem, although the problem was solved correctly. These three papers belong to the third level of mastery. Table 1 shows the mastery of the analytic method we have just described:

TABLE 1

Mastery of the Method of Analysis by Pupils in Teacher S.'s Class[11]

Group Experiment:	Group Experiment Number 1		2		3	
Level of Mastery	No. of Pupils	%	No. of Pupils	%	No. of Pupils	%
I	20	55	24	77	25	76
II	11	31	5	15	5	15
III	5	14	3	8	3	9
No. of Pupils	36		32		33	

Thus, the group experiments showed that those features of mastery of the method of analysis that were noted on the basis of our individual experiments were also present in other students in the class. Here the same levels of mastery of the analytic method were noted, the same general direction for mastery of the given method. The group experiments showed that at the end of the experimental period the great mass of pupils had grasped the analytic breakdown of more or less complex problems whose structure was familiar to them (after solving them).

7. Experiments in Third Grade - School No. 19

The series of experiments described above (both the individual and the group ones) confirms the opinion of most authorities that the method of analysis is difficult for primary school pupils, but that these

[11]The table shows the number of pupils appearing at the given level of mastery at the time of the first, second, and third group experiments.

difficulties can be overcome with appropriate educational guidance. The bulk of the pupils in the teacher S.'s class overcame the difficulties arising during mastery of the method of analysis towards the end of the experimental period (i.e., after the second and third years of study).

But perhaps some of these difficulties arose because the method of analysis was introduced too early. Perhaps those methodologists are right who consider it impossible to introduce the breakdown of problems by the analytic method before the third grade.

In addition, no matter how experienced the teacher is, the peculiarities of the method he uses and the peculiarities of the given group of children must exert an influence on the mastery of the method of analysis. We felt the need to see how the analytic method is learned under the guidance of other teachers, when it is introduced in third grade.

The experimental instruction of the method of analysis, as was indicated above, was conducted in two third grades of the 19th School for Girls. Both teachers in these classes were vivacious and creative, with a great interest in anything that might help them in their work. The teacher of grade 3A, Miss R., was very experienced. The teacher of grade 3B, Miss K., did not have a long service record, but was well prepared and worked successfully and in close contact with Miss R. These were good teachers, of which there are many in our schools.

The analytic method of problem-breakdown, widely recommended in the methodological handbooks, had attracted their attention long before, and both teachers were willing to study the method of analysis with the children regularly, assuming that this study would reflect auspiciously on the pupils' general development and on their ability to solve problems.

Before the experimental training the pupils had been taught to ask a question about the data and to select the appropriate data for the question, but special work on the method of analysis had not been done.

The lessons on acquainting the pupils with the "classical" analysis of simple problems were modelled after the first lessons in the class of the teacher S. The pupils were also given a diagram of the analysis, the kind recommended in the methodological handbooks (both the known and the unknown data are written in circles or squares).

It was not possible to dwell on the analysis of simple problems, since this might reflect negatively on the mastery of the curriculum material.

At the same time, it was necessary to liquidate as quickly as possible the interval between the problems that were analyzed and those that were solved in conformity with the curriculum for the third grade. Therefore the breakdown of complex problems in two and then in three operations was demonstrated rather rapidly. Work on classical analysis of problems continued through the entire year.

In both grades along with observations at lessons, both group and individual experiments were done analogous to those in S.'s class.

The relatively rapid transfer to the analytic breakdown of complex problems doubtless created certain difficulties for the pupils. In addition, it must be mentioned that these classes did not receive a large amount of special work on developing an ability to operate with abstract concepts, as did S.'s classes, and this, too, was reflected in the mastery of the new method.

Therefore, the beginning period for mastery of the method of analysis was probably particularly difficult for the pupils in these classes (as the experiments showed). The first group experiment was done four months after the introduction of the "classical" method of analysis; the following problem with three operations, easy for third-grade pupils, was given:

> Nine bicycles and 8 sewing machines were delivered to a country store for a sum total of 8425 rubles. How much did one sewing machine cost if it is known that one bicycle cost 625 rubles?

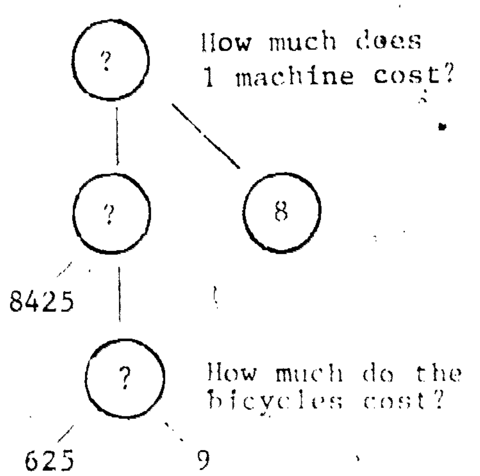


Fig. 27

The students were supposed to diagram the analysis to this problem, just as in the class of the teacher S. The nature of this diagram allowed us to judge the level of mastery of analysis.

An analysis of the pupils' work showed that only six pupils in grade A (out of 40) and two in grade B (out of 39) were able to diagram the analysis properly after solving the problem. The correct diagram of the analysis is given in Figure 27.

A significant number of the class A pupils (20) and quite a few from class B (10) diagrammed the analysis to correspond to the course of solution, only introducing the unknown into the foreground (in the uppermost circle).

In the papers (of the 30 indicated above) an attempt was made to arrange the known data in the reverse order of the way they were obtained (the opposite of the way they were solved). As an example we introduce the work of Zina Z. (Fig. 28).

1) "For how much did they sell one sewing machine?" She explained her diagram.
2) "How much did the sewing machine cost?"
3) "How much did the bicycles cost?"

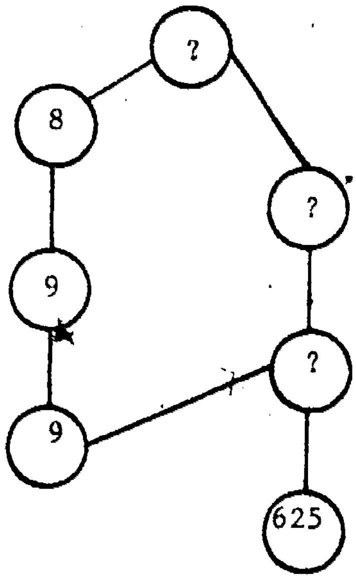


Fig. 28

All of these pupils subordinated the combination of the elements of the diagram to the ordinary order of the (synthetic) breakdown, and we can note a totally arbitrary combination of the elements of the diagram with the other pupils (11 from class A and 22 from class B).

Thus, Tanya S. introduced only the data known from the conditions into the circles (it is precisely these data that pupils select first in a breakdown). She drew this type of diagram (Fig. 29):

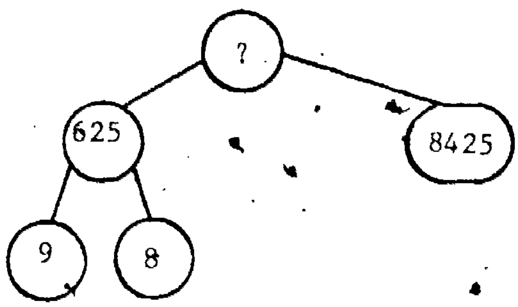


Fig. 29

Valya T. also introduced the known data into the circles, but supplemented them with circles containing question marks (Fig. 30).

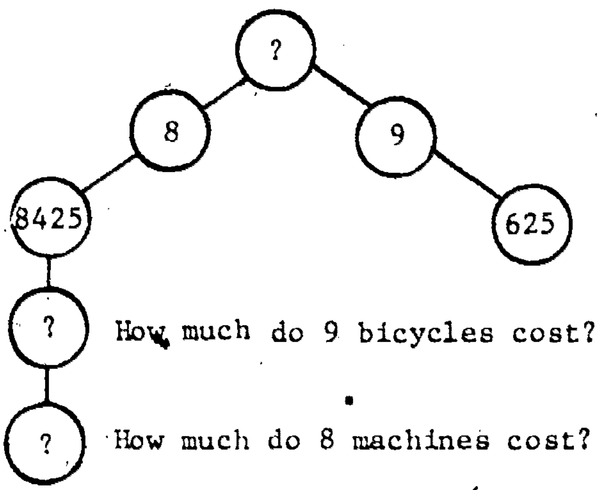


Fig. 30

Asya M., apparently trying to make the diagram symmetrical, disposed of the known data on one side, and the unknown on the other (Fig. 31).

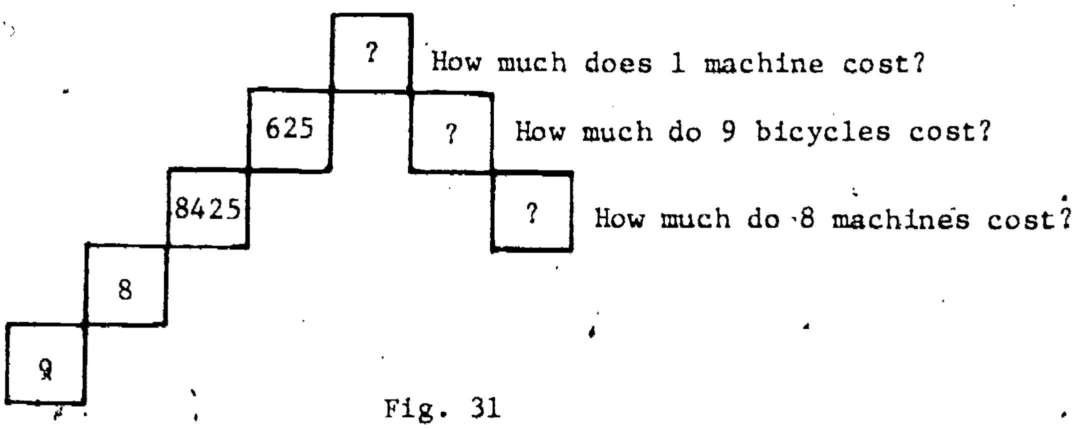


Fig. 31

In all the diagrams of this type, the fortuitousness of combining the data in the diagram is clearly manifest. All of these pupils singled out very little from the new method--the presence of the circles with question marks and data was subordinate to the determined order, apparently because they did not understand that the arrangement in the diagram reflects a definite relationship between them.

On three papers from class A and on five papers from class B only the rudiments of the diagrams were given--one or two circles. These pupils, obviously, refused to complete the problem, considering it too difficult.

Thus, almost all the pupils in these classes turned out to be on the first level of mastery of the method of analysis, having learned very little of the essential elements of the new method.

At the end of the experimental period (the end of the fourth quarter of the third academic year), the picture was changed slightly.[12] The number of pages showing the correct diagram of the analysis increased significantly (indicating the third level of mastery): 17 papers from class A and 16 from class B. In five papers from class A and in two from class B the correct analysis of the first link (the unknown) was combined with the arrangement of the remaining data corresponding to the course of solution, i.e. it was still difficult to isolate the principle of combining the links of the analysis with one another, and therefore these pupils had attained the second level of mastery of the method of analysis. In the remaining papers the combination of the data was either subordinate to the ordinary method of solution, or else the elements of the diagram were combined in an extremely random manner. In six papers, only the rudiments of the diagrams of the analysis were given.

For comparison, the appropriate figures are entered in Table 2.

Table 2

Mastery of the Method of Analysis by Pupils in Third Grade

	Start of Experimental Period		End of Experimental Period		Class of Teacher S.	
Level of Mastery	No. of Pupils Class A	No. of Pupils Class B	No. of Pupils Class A	No. of Pupils Class B	No. of Pupils	%
I a	20	10	11	6	2	6
b	14	27	8	16	1	3
II			5	2	5	15
III	6	2	17	16	25	76
No. of Pupils	40	39	41	40	33	

Note: Mastery level I includes a) combining the elements of the diagram corresponding to the ordinary course of solution, and b) combining totally at random with only a rudimentary diagram.

The individual experiments confirmed this general direction for the mastery of the method of analysis, and nothing new in principle was introduced.

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[12] A problem in three operations was given, which was easier than in the class of the teacher S.,

It is necessary to note some difference in the mastery of the method by the pupils in class A, where the teacher was more experienced, and those in class B. Although the methods of work and the selection of problems in both classes were generally the same, in the class taught by the less experienced teacher, more of the papers (particularly at first) fell into a lower level of mastery of the new method. The influence of the teacher's skill, her ability to make more complex questions accessible to the pupils, both are involved here.

The data from the last group experiment in the class of the teacher S. (in the last quarter of the third grade) are given in Table 2, for comparison. Even though a more complicated problem was given for analysis in S.'s class, the greater number of correct papers indicates a higher level of mastery of the new method by the pupils in this class. Only in two papers was the joining of the elements of the diagram subordinate to the mode of solving, and in only one paper was there just a rudimentary diagram. In S.'s class, even in the initial learning period, there were almost no papers in which the elements of the diagram were joined altogether arbitrarily, by a random method (symmetry, etc.).

Thus, gradually introducing the "classical" method of analysis, beginning in second grade (under a sufficiently experienced teacher), proves to be much more auspicious than introducing it in the third grade.

The highest level of mastery was attained by the bulk of the pupils in the class of the teacher S. and by the better pupils in classes A and B of the 19th School. There is every reason to assume that with further work on the analytic method of breakdown of problems, most of the pupils in the 19th School could attain the same level as the better ones.

8. Summary

Thus, we investigated the peculiarities of mastery of the method of analysis under different educational conditions:

1. In the absence of regular work on mastering it (the class of the teacher P., third year of study);

2. With systematic work on this method but without regular use of diagrams (the classes of the teacher G. -- 3rd and 4th grades, and of the teacher D. -- 3rd grade);

3. With systematic work on this method of analysis and with the use of diagrams:

a) introduced in second grade;

b) introduced in third grade (the classes of the teacher S., and of teachers R. and K. of the 19th School).

The investigations showed that, depending on the peculiarities of the educational process, the pupils attained different levels of mastery of the method of problem-breakdown, which was new and quite complicated for them. In classes where analysis is studied from time to time but without regular special work on this material, the pupils retain completely the old, habitual system of breakdown; only some isolated, superficial elements of the new one become intertwined with it (a few standard phrases). Even the best pupils do not attain mastery of the analysis of separate links (except the first) of the problem (grade 3 B, 47th School).

Where special exercises in the analysis of problems are done regularly, where partial analysis is practiced frequently, emphasizing the necessity of determining one datum for funding the value of the following one, where complete analysis is rarely practiced and diagramming is not used, where students single out a superficial facet of the analytic method of some type of problem ("chain"-problem) as essential, the method of analysis is approached as a "breakdown of the problem from the end." The corresponding stereotype, worked out on the basis of lengthy training, proves extremely stable and yields to reorientiation with difficulty (the 64th School and especially the 653rd).

The introduction of the complete analytic breakdown of problems with diagramming in third grade, when the curriculum material is quite complex and does not allow one to dwell on the analysis of simple problems, involves considerable difficulties. Since the initial period--work on the analysis of simple problems--is by necessity too short, many pupils single out only a few purely superficial elements from the new method--the standard verbal beginning, the availability of superficial graphic elements. Not having isolated the principle of combining these elements, they join them together in a very happenstance manner (in the order of their arrangement in the problem, and the like). With persistent continuing work on analysis, the best pupils are able to analyze uncomplicated problems by the end of the year. The other pupils (good and average), although they had mastered the analysis of separate links, could not grasp the principle of combining these links and most frequently joined them on the basis of the habitual order of breakdown (i.e., following the analysis of the unknown are the data which correspond to the first, second, and succeeding arithmetical operations).

Poor pupils turn away from analysis or reduce it to the ordinary solution (19th School, grades 3A and 3B).

Finally, when the method of analysis is introduced skillfully in the second grade, with lengthy practice in the analysis of less complex problems, the pupils master the analytic method with difficulty, but more completely, with a gradual increase in its complexity. At the end of the third grade not only the good pupils but also the average ones can give a correct analysis of problems with three or four arithmetical operations (grade 3B, 47th School)[13] More than 75% of the pupils gave this correct analysis.

At the same time, one can note a general trend in the mastery of this new method of breakdown in all those classes where work was conducted on the analytic method of breakdown of problems. In general, all the pupils pass through the same levels of mastery of the method of analysis. Under inauspicious educational conditions most pupils are detained on a lower level of mastery, and under auspicious conditions, they attain the highest level.

The introduction of the method in second grade, with diagrams and with regular length training, are auspicious conditions. This is the conclusion of this part of the investigation.

[13] The class of the teacher S.

Chapter IV

EXPERIMENTAL INVESTIGATIONS OF ANALYSIS AS A METHOD OF SEARCHING FOR A SOLUTION

1. Difficulty as a Function of Problem Structure

It was shown above that the method of analysis is difficult for pupils. How difficult it is depends largely on the mathematical structure of the problem.[1]

Even the analysis of simple problems, as the investigations of Menchinskaya [15: 90] have shown, turn out differently according to the degree of difficulty, depending on the structures of the problems. In this investigation first-grade pupils were supposed to reinstate data which had been left out of the problem. Reinstating the missing datum did not cause any difficulty if the question of the problem contained an indication of the missing datum ("How much did the top and the drum cost?") and of the datum contained in the problem which was identical to the missing one (the top costs, the drum costs).

The analysis is more complicated where the question does not contain an indication of the missing datum and the latter is not identical with that contained in the condition ("There were 17 desks in a classroom. How many desks remained if they removed 5?" There were 17 desks, they removed 5).

Problems that demand the recreation of a new and still abstract datum in order to make the solution of the problem possible present still greater complications for analysis. ("A boy bought a top and a drum. The top cost 2 rubles. How much did the drum cost?") Here, in order to recreate the missing datum, the cost of the drum must be related somehow to the cost of the top.

Thus, the analysis of simple problems is the more complicated as the indications contained in the question of the problem for the necessity

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[1] One must include the features of interrelationships between quantities (the interrelationship between numbers and the relationship of these numbers to the unknown number) under mathematical structure of a problem.

of finding the unknown data are less concrete, and as the interrelationships of these quantities are more abstract.

If in the analysis of simple problems their structure influences the difficulty of the analysis, then it will undoubtedly exert a still greater influence in analyzing complex problems. As was shown by the individual experiments in which analysis was applied to problems of different structures, those problems were easiest whose question contained a direct indication of both data necessary for determining the unknown; both of the last data are of the same kind and can be determined on the basis of the knowledge of more or less habitual functional connections between the data. (We shall call them problems of the first structure.)

Problem No. 1 is an example of a problem that is easy to break down analytically.

> A girl bought 3 candies at 4 kopeks each and 2 cakes at 10 kopeks each. How much money did she spend?

Its question requires determining how much money a girl spent on candy and cakes. On the basis of this question the pupils determine easily that they need to know how much she spent for the candy and how much for the cakes (i.e., the data required for determining the unknown can be indicated on the basis of the analysis of the question itself).

The second and third links of the analysis are of the same type--one must determine the <u>cost</u> of the candy and the <u>cost</u> of the cakes. These unknown data necessary for finding the unknown can be determined on the basis of the ordinary interrelationships of cost, price, and quantity. The values of the latter two are known from the problem.

Good pupils are able to give a correct analytic breakdown of problems with such a structure during the initial period of mastery of the method of analysis.

Those problems in which the question does not indicate the two data necessary for determining the unknown, but in which one must re-establish the course of solution of the given problem, as long as it cannot be determined on the basis of the most ordinary functional interconnections, are more complicated to break down analytically (these are problems of the second structure).

Thus, in Problem No. 3 one must find the cost of one pair of mittens.

> Mama spent 50 rubles shopping. She bought a scarf for 20 rubles and 3 pairs of mittens with the remaining money. How much does one pair of mittens cost?

In order to select the data necessary for the first link (to the unknown), we must recall those abstract interrelationships which involve the unknown and from which it can be determined.

The price is determined on the basis of the ordinary connections between price, quantity, and cost, and it is precisely these functional ties which must be recalled.

In classes where sufficient work is done on reinforcing knowledge of the functional connections between the data, the pupils quickly overcome this difficulty and then give a correct analysis of the first link without any particular difficulty.

The analysis of the second link presents much greater difficulty in the problems, in which the unknown cannot be determined on the basis of these ordinary combinations. For selecting the necessary data, one must turn to the specific text of the problem.

In two-step problems (composed of two simple problems) both data necessary for determining the unknown are at hand, but they are connected with the unknown in a less common relationship.

Thus, in Problem No. 3, the unknown of the second link is the cost of three pairs of mittens. To find it, one must subtract the cost of the scarf (20 rubles) from the total sum for the entire purchase (50 rubles). Thus, the cost of three pairs of mittens enters in as the difference of the two (50 - 20 = 30 rubles; 3 pairs of mittens cost 30 rubles). The pupils should recognize that this remainder is the cost of three pairs of mittens.

Both my in-class observation and the experiments showed that this re-thinking, this switching over from one system of ties to another, presents great difficulties for pupils, even good ones.

If, in a problem with two arithmetical operations, the pupils can rely on the concrete data of the problem in the analysis of the second link (as long as both necessary data are known), in problems with three or four arithmetical operations both required data are, in their turn, unknown. In order to select the data necessary for determining the unknown of the second link, one must recall the course of solution of the given problem, and thus re-establish the required data.

Thus, the analysis of the first link of Problem No. 14[2] does not cause any difficulties. The pupils easily determine that in order to determine the number of overcoats made, one must know the amount of material used in making them and the number of meters used in each overcoat--the given functional relationship is familiar to them. The value of the first datum is not contained in the problem; it must be determined.

In the second link of the analysis the students again have to deal with a quantity of material, but here this quantity is not determined on the basis of the familiar functional relationship. In order to show the data necessary for determining it, one must recall the course of the problem's solution, those arithmetical operations by which it was found, (200 - 60 = 140 m), and one must change their abstract significance into numerical quantities, i.e., show that to determine the number of meters of material used for the overcoats, one must know the total amount of material (200) and the number of meters used to make the suits (60).

As the experiments showed, the pupils who had already mastered analysis of the unknown on the basis of the ordinary combination of data had great difficulty with the analysis of that type of unknown and, when they came across it, slipped easily back to the ordinary course of breakdown (in the order of solution).

Only towards the end of the third grade could good and average pupils (of the teacher S. from the 47th School) and the best pupils in the 19th School give a correct analysis of that type of problem (after solving it).

Thus, the less the pupil can lean on the direct, concrete data of the problem on the one hand, and on familiar common functional relationships between quantities on the other, the more complicated is the breakdown of problems by the method of analysis.

2. Analysis as a Method of Searching for a Solution

Up to this point we have considered the learning process and the degree of complexity of analytic breakdown of problems <u>after</u> they have been solved--after the question of courses of solution has been disposed of and the solver can concentrate on the construction of the breakdown itself. Thus the analytic breakdown is the <u>final stage</u> of work on a problem. But is it possible to break down still unsolved problems by

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[2] See p. 13.

the method of analysis? Can the method of analysis be a method of searching for the solutions?

Let us recall that opinions also differ on this question, in both pre-revolutionary and Soviet literature on methods. Some authorities regard the method of analysis as a way of looking for a solution, and others regard it as a method of breaking down problems that one already knows how to solve.

The experimental data confirm the latter view. The investigations showed that good and average pupils who had mastered the method of analysis of problems they had already solved were quite capable of coping with the preliminary analysis of problems of the first structure. These were the problems whose question contained a direct indication of the necessary data, or else the unknowns (both the main one and the intermediate ones) could be determined on the basis of the usual combination of data by familiar functional ties. In short, the pupils could readily use analysis for problems whose solutions were clear to them.

A preliminary analysis of problems of the second structure, whose course of solution was not confined to the ordinary combination of data, proved impossible. Moreover, the attempt to rely on the method of analysis while looking for the solution ended in failure and forced the pupils into mechanical trial and error.

Thus, an average pupil in S.'s class, Zoya K., was given the breakdown of Problem No. 14 by the method of analysis and correctly indicated both data necessary for determining the unknown. Here she was guided by the common interrelationships of the data.

> "In order to determine how many overcoats were made," she said, "we must know how many meters went into one overcoat and how many meters went into all the overcoats. Four meters went into one, but the number of meters used for all of them is unknown."
>
> She continued, "We must find how many meters were used for all the overcoats. For this we must know how many overcoats were made and how many meters went into one overcoat." Zoya paused. The effort to rely on the ordinary combination of the data had led her in a logical circle. She repeated the first link of the analysis, and then contended that to determine the amount of material used for the coats, it was necessary to know how many meters were in each piece of material and how many pieces there were.

Here, as we see, she turned to the solution, and indicated as necessary those data which were involved in the solution of the first question of the problem (how many meters were brought into the shop).

"We must multiply 25 X 8," she said, and drew a diagram to correspond to her reasoning (Fig. 32).

Introducing both data, she continued: "And we still must know how many meters were used for the suits. For this we must multiply 3 meters by 20..." -- Zoya has indicated the data necessary for solving the second question of the problem. Again she turned to the diagram and hesitated -- Where should these new data be put?

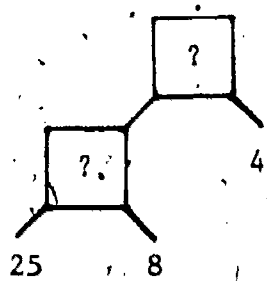

Fig. 32

Not finding a place for them in her diagram, Zoya sketched a new "hut" at one side (Fig. 33). Then, looking at the diagram, she carried out the operation indicated in it: 25 X 8 = 200 m. She wrote "200 m" in the appropriate box. Now, according to the scheme, she should have divided 200 by 4. She did the division and wrote the result in the place for the unknown (50 pieces). She wrote "60 m," obtained from multiplying 3 X 20, in the "hut" at the side.

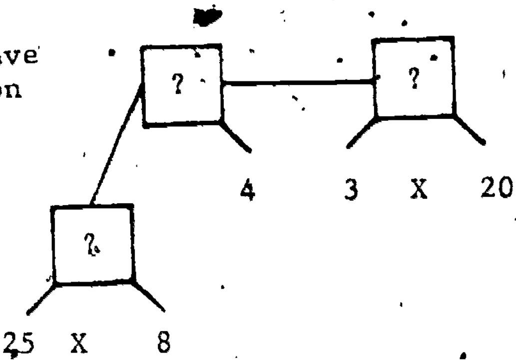

Fig. 33

The experimenter suggested that she read through the problem again, attentively. She did so and again drew a "hut" and suggested finding how many meters were used for 1 suit and 1 overcoat -- 3 m + 4 m = 7 m, i.e., it was evident that she was using blind trial and error. The experimenter's question -- What happened to the 200 meters? -- helped Zoya to gain an understanding of the situation. She solved the problem and then gave a correct breakdown by the method of analysis.

The efforts to give an analytic breakdown of this problem before solving it proceeded analogously with a number of other pupils. Two of the best pupils, after having given a correct analysis of the first link, hesitated, re-read the text, solved the problem in a whisper, and then completed its breakdown with assurance. In cases where the first link of the analysis was not constructed on the basis of the ordinary combination of data, these pupils gave an analysis only after solving the problem.

But perhaps these pupils still had not mastered the method of analysis adequately, and therefore this type of breakdown, which was difficult for them, did not simplify their search for a solution of the problem? What if the analytic breakdown of the task-problems were to be carried out in class, under the teacher's guidance? To what extent would a preliminary breakdown of a problem by the method of analysis, done in class under the teacher's guidance, clarify the essence of the problem for the pupils and help them to find the way to solve it? We conducted a supplementary series of experiments to answer this question.

A year after the method of analysis had been introduced (at the end of the first quarter) in the third grade in the 47th School, the experiment was done in the class of the teacher S.

The following problem was given to the pupils in class:

> Forty-five workbooks were delivered to a bookstore. On the first day they sold 30 rubles' worth, and on the second 45 rubles' worth, after which 20 workbooks were left. How much did all the books cost?

This was the first time that the pupils had seen a problem of this sort --it was a task-problem for them. The way to solve it was not contained in the framework of the ordinary interrelationships. It was a problem of the second structure.

After the ordinary work on the text of the problem--repeating it and isolating the known data and the question--the pupils set about breaking it down by the method of analysis.

The first link of the analysis was constructed on the basis of familiar functional relationships between the data, i.e., it was recalled easily. The price of one workbook appeared in the second link as an unknown. This datum was determined on the basis of the same functional relationships between quantity, price, and cost. However, neither datum necessary for determining the price of one workbook was indicated in the text. The first datum appears as a sum of money received during the sale of the books on the first and second days (30 rubles + 45 rubles) and the second, the number of workbooks sold, as the difference between the total number of workbooks and the number of workbooks remaining (45 - 20 = 25).

Both data should be reconsidered on the basis of the analysis of the text of the specific problem, and this caused great difficulty for the pupils.[3]

> "We do not know how much one workbook costs," said one girl who was called to the board. "For this we must know how many rubles were spent on the second day." (Pause.)
>
> "How many were sold on the second day?" was heard.
>
> "How many remained?"
>
> "One must know how many workbooks there were in all and how many rubles they were sold for on the two days."

We saw that the girls suggested completely different data as necessary: Only one of them is actually necessary—that datum which is determined in the first operation while solving the problem. The analytic method of breakdown, obviously, does not eliminate mistakes or guessing.[4]

> The teacher reminded them:
>
> "We must know the <u>price</u> of the workbooks; what must be known for this?"
>
> "The number of workbooks must be known."
>
> "For how much money were they sold on the second day?"
>
> "How much on both days?"
>
> "How many workbooks were left?"
>
> "How much money was received on both days and how many workbooks were sold?"
>
> The girls had finally found the necessary data and filled in the diagram. The last links in the diagram were filled in easily.

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[3]On the difficulties of reconsideration, see above, p. 29.

[4]Recall the opposite assertion of a number of authorities -- E. Shpital'skii and others. The experiment indicates otherwise.

Thus, even here, in a classroom situation, the preliminary breakdown of task-problems by the method of analysis is too difficult just at that link which requires not so much recollection of the familiar relationship of the quantities as an awareness of the text of the given problem. Only after a series of data given at random did the students finally find the necessary ones.

During the analysis the pupils drew the appropriate diagram on the board. A repetitive analysis of this diagram caused no difficulties (with good pupils). The analysis was repeated twice, and then the work in class on the problem was discontinued.

Then seven good pupils were chosen to solve this problem individually. Of all the subjects, only one, Lyusya G., solved it easily. The solution caused some difficulty for all the others. The greatest difficulty was caused by the realization that the 75 rubles received on both days was the cost of 25 workbooks. That is, the essence of the problem was incomprehensible to the pupils.

Thus, Galya Z. found both data (75 rubles and 25 workbooks) and got lost:

> "How much...did one workbook cost? All the books cost 75 rubles. How much did...we must find out how much one workbook cost? For this we must know how much...,"--she attempted to remember the course of the analytic breakdown, but then returned to the text. After re-reading the problem once or twice, she repeated it, again solved the first and second questions and, finally, joined the required data correctly.

Galya Z. barely coped with the difficulties that arose, but others, even superior students, would require helpful questions or a direct hint from the experimenter in order to find the course of solution. It should be noted that when difficulty arises, the pupils quite easily cross over to looking for the right course by the method of trial synthesis: They find the difference between 45 rubles and 30 rubles or divide 45 by 15, 75 by 25, etc.

If the analytic breakdown of a problem introduced into the class did not clarify the course of its solution even for good pupils, then a fortiori it would remain unclear to the weaker pupils.

Similar results were obtained when this type of experiment was done with fourth-grade pupils in the 64th School (pupils of teacher G.).

A problem was taken for analysis in class whose full analytic breakdown is presented in the book edited by Znamenskii, et. al. [41]:

> A collective farm assumed that some hay stockpiled for cattle would last for 198 days, but the hay lasted for 217 days since it was of the highest quality and they used 171 kg less per day than they thought they would. How much hay had been prepared on the farm?

This problem was doubtless very difficult for this class (the way to solve it was not familiar to them).

The teacher constructed an analytic breakdown of it in complete accordance with the form in the methods handbook. Difficulties were encountered in the breakdown, but they were overcome by common effort. The analysis was repeated twice according to the diagram sketched on the board and, just as in the 47th School, no more work was done on the problem.

Individual experiments with the ten best students in the class showed that the analytic breakdown done in class did not clarify the way to solve the problem. Not one of the ten could solve the problem unassisted. Only the beginning of the solution was mastered thoroughly. (The analysis in class was concluded with these data.) Furthermore, the course of solution turned out to be different. Thus, Zhenya G. tried to determine how much hay was given out in one day by dividing 171 by 19 (he took the daily saving of hay to be the entire saving). His result did not worry him: "On the collective farm they gave 9 kg of hay daily to all the cattle"! He tried to continue the solution: "9 X 217. . ."

The experimenter indicated the absurdity of the answer: 9 kg daily for all the cattle! Then Zhenya began to carry out new arithmetical operations, without even asking questions:

171 X 217 = 37007

171 X 198 = 33758

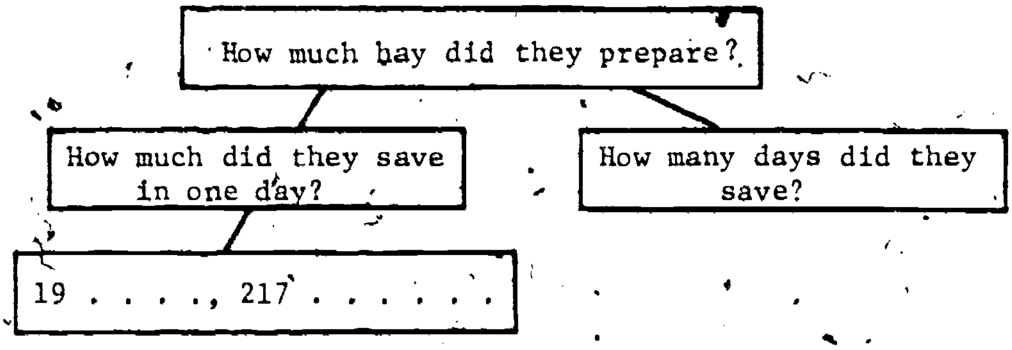
37007 - 33758 ...

that is, he began using blind trials.

Analogous attempts could be noted with the other pupils.

When difficulties were encountered, the pupils usually did not try to apply the analytic breakdown of the problems. Only Vanya C. made an attempt:

"May I start with the analysis?" he asked, and, with permission, he wrote down the question of the problem. As data needed to determine the unknown, he indicated those which were necessary for the solution and drew this diagram:



Then he stopped, tried to remember the breakdown done in class, and refused to continue the analysis.

Thus, the one attempt to be guided by the method of analysis when difficulties arose did not meet with success.

Only after the experimenter asked: Why did they feed the cattle extra days with the old stock? How long did it take them to use up the saving? (questions leading the solver to isolate the basic connections between the data)--did the course of solution become clear for them.[5]

Thus, the experiments showed that a task-problem cannot be broken down independently by the method of analysis until the way to solve it is clear. The preliminary breakdown by the method of analysis of a problem which is new for the pupils, done in class with the teacher's help, clears up its essence only a little for the pupils, and does not reveal sufficiently the course of solution.

Consequently, <u>the method of analysis cannot be a method for preliminary breakdown of rather complex problems, task-problems</u>.

3. The Effect of Practice in Using Analysis

The investigation showed that the method of analysis is undoubtedly difficult for pupils. For this method to be mastered by the bulk of the pupils requires extensive, persistent, systematic work on it under the

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[5]As a control, the preliminary breakdown of these problems was carried out in the third grade of the 183rd School and in the fourth grade of the 69th School, but by different methods. The pupils had a good understanding of the essence of the problems, and even the weakest pupils solved them easily.

guidance of an experienced teacher. The breakdown of problems by the method of analysis demands much time, and therefore the number of problems solved in class will be decreased. At the same time, the method of analysis cannot be a method of preliminary breakdown of rather complex problems. Consequently, the method of analysis should be the final stage in working on a problem, as a method of breaking down problems that have been solved or that are solved easily.

Let us remember that a number of experts (Skatkin [36], Popova [32], and others) who recognize the impossibility of preliminary breakdown by the method of analysis of rather difficult problems still consider the analytic breakdown of solved problems or of easy problems extremely useful. A thorough analysis of the text of a problem should prevent the pupils from superfluous syntheses and from slipping into mechanical manipulation of the data while searching for a solution, as students are inclined to do.

A special series of experiments was conducted to determine the influence of lengthy practice in "classical" analysis on the ability of pupils to solve problems.

Rather complicated problems were given to pupils in classes where quite a long time was spent on practice in the "classical" analysis of problems; these problems were to be solved without help. Help was given only in extreme circumstances, when the pupil himself could no longer continue the solution. For comparison, pupils were taken from classes where little attention was paid to the "classical" method of analysis.

Only the best pupils were chosen as subjects (the experiments were individual), since they would have mastered the method of analysis better than the others, and it was the influence of this mastery that was to be clarified. In the "control" classes, also, only the best pupils were chosen (from seven to ten from each class). The experiments were conducted during the last quarter of the school year.

In the third grades the following problem (No. 16) was assigned for the experiments:

> Twenty-five members of an artel were supposed to make 1,950 vases per month according to the plan. In the first 10 days each made 3 vases daily, and in the remaining days each made one vase more. By how many vases did the artel overfulfill the plan, if it is known that in the first 10 days they all made 75 vases daily, and 100 vases daily in the remaining days?

The problem is a little unusual--intermediate data are included in it (75 vases and 100 vases), which could be obtained by combining other data contained in the problem (25 X 3; 25 X 4). Although the pupils were only slightly familiar with problems with superfluous data, we thought it possible to introduce such data as an experimental task. By including them, we hoped to clarify the pupils' ability to analyze a problem, to preserve the logic of their reasoning without straying into superfluous, unproductive arithmetical operations, with superfluous syntheses.

If a pupil is capable of analyzing the problem, he should first explain the content of the data contained in it, and then, still before solving it, should understand that 75 and 100 are derived, intermediate data. In addition, the problem can be solved in two ways—by using as an unknown either the data about the productivity of one worker (3 and 4 vases), or by using the data about the productivity of the entire artel (75 and 100 vases); but the second way will be shorter and more rational. (The ability to analyze the text of a problem should also influence the choice of a more rational means of solution.)

If we discard these intermediate data (75 and 100), we obtain an ordinary problem, which should not give any difficulty to third-grade pupils, especially those in the last quarter of the school year.

How did the independent solution of this problem proceed in the third grade of the teacher S. in the 47th School, where "classical" analysis had been studied for about two years?

As an example, I shall introduce the record of the way the problem was solved by an excellent student, Dora K.; the record was quite characteristic of the pupils in this class.

Dora: 1. How many did they make in 10 days?

3 X 10 = 30 vases.

In 10 days they made 30 vases.

2. How many did each man make?

They made 1950....

Experimenter: But they were supposed to make 1950 according to the plan; that was the plan given to them.

Dora: Ten days.... In the second question we can find out how many still had to be made before they fulfilled the plan.

$$1950 - 30 = 1920 \text{ vases}$$

3. How many did they turn out in 10 days, if they overfulfilled the plan?

$$75 \times 10 = 750 \text{ vases.}$$

4. How many did they make altogether? By how many more, when they overfulfilled the plan?

Experimenter: Who made 3 vases?

Dora: This was one man who made 3 vases....75 is also one... No, in 10 days....

The experimenter explained that all 25 men made 75 vases in one day, and suggested she read the problem more attentively. Dora re-read the text and began the solution with the first question:

Dora: 1. How many did they make on each of the remaining days?

$3 + 1 = 4$ vases, 4 vases every day.

Exp.: This is all of them?

Dora: One ... no, each of them.

2. How many did they make in 10 days?

$$4 \times 10 = 40 \text{ vases.}$$

Exp.: For 10 days each of them made 3. (Dora corrected:

$$3 \times 10 = 30 \text{ vases.)}$$

Dora: 3. How many did they make in the remaining days at 4 per day? But over how many days is not known...

Exp.: What is known about the time?

Dora: It's not stated here.

Exp.: In how much time should they have fulfilled the plan?

Dora: It's not stated here. (Began repeating the problem.) In a month.

Exp.: Which is how many days?

Dora: Thirty.

The experimenter explained that there are 26 working days in a month.

Dora wrote 26 - 10 = 16 days.

4. How many did they make in the remaining days?
16 days...

4 X 16 .... no, 16 ÷ 4.

There is no sense in citing the entire record—Dora's course of solution is clear: It is the mechanical manipulation of data, in which the weakness of the analysis undoubtedly shows.

This primarily indicates only slight attention to the text of the problem. After having re-read the problem, Dora continued to operate with the data she had selected on a first reading, without attempting to verify the correctness of her operations (by re-reading the problem), and she returned to the text only after being directly instructed to by the experimenter.

She showed a weakness in the analysis of data, in the process of which the content of the data should have been ascertained. The essential features should have been isolated to serve as a basis for indicating connections between them for the synthesis. Dora did not always isolate precisely the data contained in the problem. She sometimes missed words related to one datum, and then the significance of these data were distorted. Thus, it is stated in the problem that "each made 3 vases daily." Dora missed this "each" and considered 3 vases as the output of the entire artel (perhaps because the entire artel had been referred to earlier in the problem). "In 10 days they made 30 vases," she said.

Later on, she missed the indication that "they all made 75 vases daily" and considered the 75 vases as the output of one worker ("75 is also one").

It is interesting to note that the datum about the working period ("per month") was not expressed numerically, and Dora did not isolate it at the start; even after a re-reading she continued to assert that nothing was known about the time.

No data were considered from the standpoint of their real significance and the specific quantities of the arithmetical data. In asserting that the entire artel (25 men) made 3 vases in a day, Dora was not bothered by this low productivity, even when she knew that the monthly plan was for 1,950 vases (she subtracted the 30 "which they did in 10 days" from 1,950 in order to find out "how many still had to be made before they fulfilled the plan").

The analysis of the data assumed that these significant features will be isolated, to serve as a basis for determining relationships between the data, and thus it assumes a comparison of the data. It was this comparison which Dora lacked—she regarded the data as more or less isolated from one another. Therefore she did not notice the derivative character of the data about the daily productivity of the entire artel (75 and 100 vases) and she operated with these data alongside the data on the productivity of one worker, without setting up connections between them. She found the productivity of the entire artel at the start by this means: 4 X 16 = 64; 64 X 25 = 1600 vases; but later, multiplying 100 by 16, she failed to see the significance of these two data.

The analysis of functional relationships was also weak here. In such analysis, the relationship between the data should be revealed and those principles on the basis of which the productive synthesis[6] can be realized should be isolated. Frequently, Dora, having determined the connection between certain data and having carried out an arithmetical operation, did not isolate the principle underlying the operation. Therefore she sometimes did arithmetical operations that did not correspond to the question she had posed, slipping easily into using another. Having asked the question -- "How many vases did they make on the remaining days at 100 vases per day?" -- she divided 750 by 100, and then, obviously understanding that this operation was impossible, she tried to divide by 10, and only the experimenter's question guided her to the proper operation. Later she multiplied 4 by 16, then immediately changed and wanted to divide 16 by 4. The direction of the analysis towards finding the unknown was not displayed here either. Dora had forgotten what the problem asked and, having determined the output of the artel for 16 days, considered the solution completed. "That's all," she announced to the experimenter.

---

[6]By productive synthesis we mean the determination of those relationships which bring us nearer to finding the unknown.

Dora, who was one of the best pupils in the class, had mastered completely the method of "classical" analysis (we had confirmed this in the individual experiments). She solved and analyzed problems easily, as they had been analyzed in class. However, in solving problems which were somewhat different from those selected in class, as we saw, she revealed a weakness of genuine analysis of the text of a problem and easily slipped into the more or less random combination of data.

The peculiarities noted in Dora K.'s independent solution of task-problems were also present to some extent in the other pupils in this class.

Here is the solution of the same problem by the best pupil, Galya Z. She also believed the 30 (i.e., 3 X 10) and 64 (i.e., 4 X 16) vases were made by "them," i.e., the entire artel of 25 men. She thought that 750 vases (75 X 10) was the output of the entire artel in a month, and it did not trouble her that the same artel made 1600 vases in 16 days — she did not correlate or compare the concrete values of these data.

Later Galya found how many vases the artel made in all (750 + 1600 = 2350 vases) and did not understand the period in which this was done. Later she decided that 2350 vases was the plan which was given to the artel and attempted to answer the question of the problem—by how many vases did the artel overfulfill the plan?— by the following operation: 2350 - 94 (i.e., 30 + 64).

When the experimenter asked, "What was the number required by the plan?" she quickly answered, "1950 vases." Consequently, she remembered this quantity, but in the solution she had changed it into another, the one which she obtained when the question about overfulfilling the plan was posed (this was a datum obtained in solving the preceding question, and Galya included it in the solution of the next question). Only with questions from the experimenter, directed at clarifying the significance of individual data, their content, and the principles connecting them, did Galya solve the problem.

Another pupil who was making normal progress, Tanya G., after having determined how many vases one man made in 10 days (30), still stubbornly tried to add 30 to 75 (75 vases was the output of the entire artel in one day), and then she thought she would be able to find out "how many they made in 10 days."

Many similar examples are found in the records of the solution of this problem by the other pupils in this class.

But perhaps the somewhat unusual text of the problem is the reason for the vaguely realized course of solution that many pupils in this class followed. Perhaps the problem would have been too difficult for third-grade pupils anyway. How would this problem be solved with pupils in a different class?

For a control and for comparison, analogous experiments were carried out with third-grade pupils in the 172nd School (taught by Miss P.),[7] where "classical analysis" had occupied a very modest place.

As the experiments showed, the poor pupils as well as the good ones in this class coped with the problem. Out of 7 good pupils (4 with "5" and 3 with "4" as grades in arithmetic), only one attempted to multiply 75 by 25 immediately, and a question from the experimenter (What does 75 vases indicate?) was required for her to give up the false operation, and then her solution was perfectly correct.

Several incorrect operations were observed in three good pupils, but they immediately corrected themselves, indicating precisely where they had gone astray.

Two weaker pupils (with "3" as grades) were enlisted for the experiments in this class. They turned out to be more predisposed to a successful operation. One of them attempted to multiply 100 by 25, and believed that she had to find the productivity for 36 (26 + 10) days, but the experimenter's question (What do these data indicate?) quickly led her to the correct way to solve the problem. The second made similar mistakes.

The initial period of these pupils' work on the problem merits attention. All of them re-read the problem more than once, then repeated it under their breath, re-read the phrases about the output of one worker in one day (3 vases) and of all the workers in one day (75 and 100 vases). During the solution, the experimenter asked the following questions: "What is bothering you in the problem?" "What prevented you from starting the solution?" and always received a clear answer.

"Here they speak of 3 vases, and at the end, of 75...This is 75 vases made by all of them together, right?" said Valya L.

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[7] More details about the operation of this class will be given below.

"I wanted to multiply 3 by 25, but then I saw that this was already known," answered Zhenya K.

The other pupils gave similar answers.

It is completely characteristic that only one of the pupils started the solution with an explanation of how many vases were made by one worker in 10 days (3 X 10) - taking the less rational way - but then reconsidered and solved the problem by the more economical means. However, after the solution all the pupils (except one) indicated this less rational but entirely possible method of solution (3 X 10 = 30; 30 X 25 = 750 vases, in place of 75 X 10). One good pupil took a more original course: She found how many vases the artel had left to make to fulfil the plan after the first 10 days, how many days the artel took to fulfil the whole plan, and then how many vases they made in the remaining days of the month.

The datum about time ("per month"), not expressed numerically, was isolated by all the pupils, and before the solution they ordinarily asked: "Is a month 30 days?" and some of them said themselves: "They worked 26 days."

The entire course of solution of this problem showed that the pupils had been taught to analyze the text of the problem carefully and to be guided by this analysis in their solution.

In order to demonstrate more clearly the difference in the process of independent solution of the task-problems by the pupils in both classes, we shall introduce some quantitative data.

It is very difficult to carry out the process of solution in the language of numbers. What should be chosen as indices? Two indices were chosen: the number of "superfluous syntheses" and the number of "mistakes in analysis." With a thorough analysis of the data and of their functional interrelationships, the solver will not carry out "superfluous syntheses"-- operations that are not required for determining the value of the unknown. A great many "superfluous syntheses" indicate a weakness in the analysis of the problem.

As a second, supplementary, index, the number of "mistakes in analysis" was chosen. These mistakes occur as a result of a partial composite analysis. The solver isolates individual data from the problem, rejecting a number of descriptive words, and thus the significance of a datum is distorted, which entails an incorrect solution. Let us remember

Dora K.'s method of solution. She isolated these data from the problem: 3 vases and 75 vases, and operated with them as if they signified the same thing (this was "what they made"), although 3 vases is the output of one worker and 75 is the output of all 25 workers.

This type of mistake we called "mistake in analysis." The greater their number, the weaker the analysis of the problem's text.

The number of "superfluous syntheses" by pupils who had studied "classical" analysis for a long time (grade 3B of the 47th School) and the pupils in the control class, where little attention was paid to this method (grade 3B of the 172nd School), is shown in Table 3.

TABLE 3

Distribution of the Number of "Superfluous Syntheses" in Independent Solution of the Experimental Problem

Number of "superfluous syntheses"	0	1-2	3-4	5-6	7-8	9-10	11-15	16-26	Total
Grades:									
3B of the 47th School, No. of Pupils:	-	-	-	-	1	1	2	3	111[a]
3B of the 172nd School, No. of Pupils:	3	4	-	-	-	-	-	-	6

[a] The number 111 shows how many "superfluous syntheses" were made by 7 pupils in the given class. That is, the seven students made 8, 10, 12, 15, 17, 23, and 26 "superfluous syntheses," respectively, making 111 in all.

The indices for 7 good pupils in each class are included in the table. We see that of 7 pupils in the teacher S.'s class, where the "classical" method of analysis of problems was studied regularly, not one solved the problem without unproductive operations. Two pupils showed a large number of this type of synthesis (from 7 to 10), and three pupils showed even more than 17. Altogether, the seven pupils in this class made 111 superfluous syntheses.

The large number of superfluous syntheses is evidence that these pupils looked for the solution mainly by the method of mechanical manipulation of the numerical data, and this indicates weakness in analyzing the concrete text of the problem.

We see something else in the control class. Three pupils immediately solved the problem correctly, and four carried out one or two superfluous syntheses (six for all four pupils). These data indicate the ability of the teacher P.'s pupils to construct their solution on the basis of a thorough analysis of the text of the problem.

The number of "mistakes in analysis" is shown in Table 4, which substantiates the same conclusion: In the class where much attention was given to the method of analysis, the pupils showed little ability to analyze the concrete data of the problem. They appeared much less well prepared for independent analysis (and consequently for solving) of problems than the pupils in the class where a very limited place was devoted to this method.

TABLE 4

Distribution of the Number of "Mistaken Analyses" in the Solution of the Experimental Problem

"Mistaken analyses"	0	1-2	3-4	5-7	7-	Total
Grades:						
3B, 47th School No. of pupils:	-	1	1	-	5	44[a]
3B, 172nd School No. of pupils:	6	2	-	-	-	2

[a]The number of mistaken analyses made by each of the seven students was 2, 4, 7, 7, 8, 8, and 8, respectively.

The question arises quite naturally: Will the same features of independent solution of task-problems show up with pupils in other classes where much time is spent studying the "classical" method of analyzing problems? To clarify this, analogous experiments were carried out in two more schools--the 64th and the 69th. Let us recall that the pupils in grade 4B of the 64th School (teacher G.) studied "classical" analysis systematically, beginning with the second year of study. Grade 4B of the 69th School (teacher E.) was taken as a control; no particular attention was paid to the "classical" method of analysis in it.[8]

The following problem was assigned for independent solution:

> One worker saved 696 rubles every year. His older brother started work at the same factory after 14 months, and after 28 months he had accumulated as much as his brother had done from the beginning. How many rubles did his brother save yearly? [5]

This problem was a little new for them, but its only difficulty was to determine the length of work of the first worker: he worked 14 months before his brother came and 28 months together with him before their respective savings were equal. An expression of the time (yearly) is indicated in the problem, but it is not expressed numerically. The following question was given to the pupils to avoid misunderstanding of the term: "Yearly—how many months is that?" (The question was asked before the solution.)

Only the best pupils were enlisted for the experiment, as was the case with the preceding (again 10 pupils).

Let us select one of the records of solution of this problem by a pupil in grade 4B of the 64th School, Petya Z.

Having read the problem, Petya began alertly:

> "We cannot find how much money the older brother laid aside yearly: For this we must know...how many more months the older brother worked than the young..." (the aim at an analytic breakdown did not come easily).
>
> "May I...simply solve the problem first?" Petya asked, then, with permission, he began the solution:
>
> 1. 28 − 14 = 14 months -- this is the length of time the laborer worked.
>
> 2. How many rubles did the laborer receive in 1 month?
>
> 696 : 14 = ...(he tried to divide)
>
> It doesn't come out! And if I multiply 696 X 14? (And he began to multiply).

5

[8] Teacher E is considered a good teacher, of which there are many in any of our schools.

Exp: After what period of time did the laborer set aside 696 rubles?

Petya: After 14 months!

Exp.: Read more carefully.

Petya: (Re-read the problem) That is yearly.

Exp.: That is, after what period?

Petya: After 12 months. 696 ÷ 12 = 58 rubles. How much in each month-- 58 ÷ 14 = ...

Exp.: Read the problem.

Petya: (Read and wrote in silence) 28 + 14 = 42...

58 - 42 = 16; 696 ÷ 16...

Exp.: And then?

Petya: What one obtains by dividing by 28.

Exp.: No, not like that.

Petya: (again in silence)

16 - 14 = 2; 696 ÷ 2 = 348... 348 ÷ 14 = ...

(and he attempted to divide). It doesn't come out! Dividing by 28 will work?

There is no sense in citing the course of the solution any longer; it proceeds in the same way, with questions from the experimenter (How many months did the first laborer work? How much money did he save in this period?) re-directing Petya's thought into the proper channel.

We had before us a clear model of searching for the solution by the blind method of trial and error, manipulation of the numerical data, which doubtless shows weakness in the analysis of the data and of their functional interrelationships.

Patya first picked the numerical expressions of the data (696 rubles, 14 months, 28 months) out of the problem, ignoring the phrases describing these data, and thus the data turned out to be inadequte for the situation (this is a partial complex analysis). He did not single out the datum "monthly," not expressed numerically, but 696 was, to him, a sum which the worker "received" and did not save, in 14 months.

Having isolated the numerical data, Petya began to manipulate them, not isolating accurately the principles that connect them with each other.

If it was impossible to carry out an operation (14 does not divide 696 evenly), Petya easily slipped into another one ("And if we multiply 696 by 14?"). He subtracted 42 months from 58 rubles and divided 696 by the result. He even stopped asking questions about the operations he carried out.

The entire process of solution shows an extremely low level of analysis of the concrete text of the problem.

Was this the only such method of solution? Alas, no! True, it was the clearest example of looking for the solution by the method of blind trial and error, but 6 pupils (out of 7 who did not solve the problem alone) showed analogous transitions to this method when difficulties arose in the solution.

There were also mistakes in the solutions of two other pupils (696 ÷ 14; 58 X 14, and not X 42, among others), but when they obtained a result, they made precise exactly what they had obtained, compared it with the other data and they themselves detected the mistake in their operations.

Their approach to the solution was close to the one characteristic of the pupils in the control class (4B, 69th School). For them, more lengthy and more careful work on the text of the problem was characteristic: They read it several times, and in difficulty returned by themselves to a repeated reading. When they received a new datum as a result of an arithmetical operation, they made its significance precise, and compared it with what had already been obtained, which allowed them to detect more easily the mistakes they had sometimes made. Even in error, they did not transfer to the level of simple manipulation of the numerical data.

I introduce a segment from the record of the solution of the problem by Vanya D., for whom the problem was more difficult than for the rest. He did not isolate the word "yearly" as a datum and decided that 696 rubles was accumulated in 14 months. He explained how many rubles the laborer saved in one month and, after making an error, divided 696 by 14 and then multiplied the result by 28 (i.e., he did not understand the period of time the younger brother worked).

Exp.: After how many months did the worker save 696 rubles?

Vanya: After 14 months.

Exp.: Read it!

Vanya (reads): Ah! Yearly! 12 months, and he worked 14 months.

Exp.: How long did this laborer work?

Vanya: ...14, ...no, 12 and 14 months.

Exp.: And how long did his older brother work?

Vanya: (reading the entire problem) 28 months, and then 14 and 28 months — 42 months.

42 ÷ 12 and that many times at 696 rubles,...No, better to do it 696 ÷ 12 = 58, and then 58 X 42 = 2436....

And he continued to solve it correctly to the end.

At first we see gross mistakes with this pupil, showing an insufficient analysis of the problem, as a result of which "superfluous syntheses" appeared in his solution. But here there is no mechanical manipulation of numbers: The arithmetical operations he performed proceed from his understanding of the data. If 696 rubles was regarded as a sum accumulated over 14 months, he correctly turned to division to find out the monthly saving. He wanted to divide 42 by 12, in order then to determine the sum accumulated by the laborer after 24 months. This is a mistaken operation, but only because 42 is not divisible by 12, and Vanya himself easily found a different course of solution.

In order to compare more easily the results of independent solution of the problem by pupils in the class with the "classical" analysis and in the control class, where not much attention was given to the "classical" analysis, as we did last time, we turn to quantitative data—to the number of "superfluous syntheses" and "mistaken analyses" made by the subjects in both classes (see Table 5). We see that although the difference between the classes is not as marked as it was with the 47th and 127th Schools, the nature of the corresponding data is exactly the same.

In the class with the "classical" analysis, half the examinees (5 out of 10) made more than 8 superfluous syntheses while searching for a solution. In all, in this class, there were almost twice as many unproductive superfluous syntheses and "mistaken analyses" as in the control class. Thus, the experiments did not confirm the opinion of many experts that lengthy practice on "classical" analysis will have a good influence on problem solving.

TABLE 5

Number of "Superfluous Syntheses" and "Mistaken Analyses" in Independent Solution of Problems (10 pupils tested in each class)

	No. of Superfluous syntheses								No. of Mistaken analyses					
Grades	0	1-2	3-4	5-6	7-8	9-10	11-16	Total	0	1-2	3-4	5-6	7-8	Total
4B 64th School	1	1	1	1	1	2	3	76[a]	1	1	2	2	4	45
4B 69th School	1	3	1	3	2	-	-	40[a]	1	4	4	1	-	23

[a]The number of "superfluous syntheses" for each student in the respective classes were:

4B 64th School: 0, 1, 4, 5, 8, 10, 10, 11, 16
4B 69th School: 0, 1, 2, 2, 4, 5, 6, 6, 7, 7

Thus giving totals of 76 "superfluous syntheses" in the first and 40 in the second.

Contrary to expectations, the students in those classes where much time was spent on classical analysis showed less ability to analyze the text of a problem and in general to be guided by analysis while looking for a solution of a somewhat complicated (for them) problem, during an independent solution. A slight trend to analyze the concrete condition was very characteristic for them. Having isolated the numerical data from the problem, they rushed to begin operating with these data, carrying out many unproductive, superfluous operations, which shows that they are poorly oriented toward solving the basic question of a problem -- finding the unknown.

The "classical" method of analysis does not teach the productive analysis of the text of the problem--such is the slightly paradoxical conclusion to be drawn from the data.

Long practice in "classical" analysis turns out to have a negative influence on the pupils' ability to solve problems.

4. Long-term Effects of Practice in Using Analysis

How is lengthy work on the "classical" method of analysis of problems reflected in the pupils' later work? When difficulties are met, is there the same ease in transition to manipulation of the numerical data as appeared in elementary-school pupils who studied the method of analysis for a long time?

In order to clarify this, experiments were done with pupils in the sixth grade of the 47th School, former pupils of the teacher S. (grade 4B). For three years (grades 2-4, inclusive), S. had taught these pupils to break down problems by the "classical" method of analysis. During final exams in fourth grade, according to reports by the principal of the lower forms of the school and by the methodologist from the Institute for Teacher Improvement, her pupils pleased the onlookers by their ability to carry out "classical" analysis of the problems assigned them.

Parallel with this class, grade 4C for four years (in primary school) had been in the hands of a quite experienced teacher, who did not, however, devote much attention to the method of analysis.

Thus, at the promotion into the fifth grade we had two classes (B and C); in one an experienced teacher had spent much time teaching the "classical" method of analysis of problems, and in the other the pupils had little practice in the breakdown of problems by this method. Teacher T. taught both of these classes in middle school; she did not usually use the method of "classical" analysis. Well organized, industrious, having mastered the primary school curriculum thoroughly, class B (S.'s pupils) was considered the best of the fifth, then of the sixth grades. However, in solving more complex problems, where data were to be regarded in a slightly new way, where "gumption" was needed, as the teacher said, the advantage was with the other class (C). The teacher noted with surprise that working on mathematics with class B was more difficult for her than with the parallel class, although its pupils knew the rules more soundly, calculated more quickly, and solved model problems with considerable ease.[9]

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[9]The instructors in other disciplines considered class B to be better.

As indicated, the experiments were carried out when the pupils were in the sixth grade. Ten of the best pupils in mathematics were taken from each class. They were asked to solve a series of arithmetic problems without assistance (on the order of an individual experiment). The problems were taken from a fourth-grade workbook,[10] but ones were chosen that could present a certain difficulty for these pupils, i.e., they were task-problems for them.

Since all the experimental problems (there were four) were solved in basically the same way, we can use as an example the solution of Problem No. 12:

> Two laborers earned the same amount of money. One received 20 rubles per day, and the other 12 rubles per day. How many days did each laborer work if it is known that the second laborer worked 6 days longer than the first?

The basic difficulty in solving this problem is in determining that the daily difference in wage (8 rubles) will be covered by the money earned by the second laborer in 6 work-days (i.e., 12 X 6 = 72 rubles; 72 : 8 = 9 days). For the majority of pupils in both classes, determining this relationship was not very easy--many had to search actively for a way to solve this problem.

As an example we shall take the record of solution by Galya L. (a pupil in grade 6B, which had extensive practice in "classical" analysis):

> Having read the text, Galya immediately started solving the problem:
>
> 1. How much more did the second laborer earn in 1 day?
>
>   20 - 12 = 8 rubles.
>
> 2. How much money did <u>each</u> laborer <u>earn</u> in all?
>
>   8 X 6 = 48 rubles.
>
> 3. How many days did the first laborer work?
>
>   48 : 12 = 4 days.
>
> 4. How many days did the second laborer work?
>
>   4 . . . 48 : 20 . . .

[10] The numerical values of the data were somewhat simplified.

The second laborer, 8 rubles more and 6 days more .... 48. ...We found out how much each laborer earned, the same amount of money, 48 rubles...I can't say, but I understand...after 6 days...--48, after one day--8...

Exp.: Why should the second worker have worked 6 extra days?

Galya: He earned less...6 days extra. The other, 6 days earlier than he did...48 rubles....

Exp.: How many extra days did he work?

Galya: Six.

Exp.: And how much did he earn per day?

Galya: Twelve rubles...The first received more, but worked less... They started out the same...

Exp.: The first finished his work, and the second worked 6 days more, in order to receive the same amount...

Galya: 20 X 6 = 120 rubles.

12 X 6 = 72 rubles.

If the first worker worked 6 days more, he would have received 120 rubles, and the second -- 72...

Exp.: The second worked 6 extra days and earned 12 rubles each day, and earned 72 rubles in all on these 6 days. Why did he have to earn these extra 72 rubles?

Galya: And how much would the first have had to work for this money?

72 : 20.

Exp.: Why? He didn't have to work these days... The second wanted to get the same amount as the first. He worked together with the first for one day and received 8 rubles less, another day--again he received 8 rubles less.

Galya: And he was supposed to work 6 days in all. It comes out that he worked 6 days.

Exp.: No, he worked 6 days more than the first. How much money did he earn on these 6 days?

Galya: 48 rubles.

Exp.: 12 rubles per day; in 6 days 72 rubles.

Galya: Aha!...72 rubles...72 : 8 = 9 days. I understood that he earned 8 rubles per day.

Let us review the record of the foregoing solution. First it must be noted that there was little attention to the text of the problem itself. Having read through the problem, Galya rushed to carry out the arithmetical operations, without turning to the text of the problem unless prompted by the experimenter, and without attempting to verify whether her reasoning corresponded with the data obtained.

The weakness of the analysis of the data and the functional connections between them shows up clearly in the method of looking for a solution. The first question she asked was solved correctly. But then the difference of 8 rubles which she had just found entered in as the complete wage of each worker. This re-thinking occurred, probably, under the influence of the problem's indication that the laborers received the same sum of money. Only this "same-ness" was isolated, since the indication relating to these data was ignored, and it was joined arbitrarily to other data ("equally--8 rubles each; equally--48 rubles each..."). The difference in the length of time worked (one worked 6 days more than the other) she regarded as the time the work took. Thus, particular complex analysis was characteristic of Galya.

Even after answering correctly the question of why the second worker had to work an extra 6 days, Galya could not use the experimenter's direct hint--"6 extra days at 12 rubles per day"; obviously, it did not evoke productive connections in her mind. The pupil finds the sum received in 6 days by each laborer, then tries to divide 72 (12 X 6) by 20, and so forth, i.e., she carries out a series of arithmetical operations, not realizing and not isolating those principles underlying the operations. She looks for support in the numerical value of the data themselves, and not in the isolation of the essential meaning of these data or in comparing them to determine the relationships between them.

If we compare Galya L.'s solution of the problem with the solution described above by the primary school pupils in those classes where much time was spent on "classical" analysis, we can notice much in common. We see the same inattention to the breakdown of the concrete problem, the same weakness of genuine analysis. As a result-- a partial isolation of the data not adequate for the conditions (a partial complex analysis), a tendency to combine them on the basis of superficial, sometimes arbitrary, significations, an attempt to cross over to mechanical manipulation of the numerical values of the data when difficulty arises (i.e., a synthesis on the level of an elemental analysis).

Galya L.'s solution of the problem was very characteristic of other pupils in this class as well. For all of them a rapid synthesis on the level of a lower form of analysis was characteristic: Partial complex analysis, and under difficulties, elemental analysis as well.

In the parallel class the pupils showed more ability to analyze the concrete text of the problem, and thus far fewer "mistaken analyses" were found, as well as fewer unproductive operations.

In order to represent more clearly the difference between the grade where a good portion of the time was spent on the analytic method (grade 6B) and the grade where a modest place was devoted to it (grade 6C), we turn to Table 6. In it the number of pupils committing a certain number of superfluous operations and "mistaken analyses" in their solution of Problem No. 12 is shown.

TABLE 6

Number of "Superfluous Syntheses" and "Mistaken Analyses" In the Solution of Problem No. 12 by Pupils In Middle School

	"Superfluous syntheses"						"Mistaken analyses"				
Grades	0	1-2	3-4	5-6	7-8	Total	0	1-2	3-4	5-6	Total
6B ...	2	3	1	2	2	34[a]	4	3	2	1	18
6C ...	1	7	2	–	–	15[a]	5	5	–	–	6

[a]The number of "superfluous syntheses" by each respective pupil was:

Grade 6B: 0, 0, 1, 1, 1, 4, 5, 6, 8, 8
Grade 6C: 0, 1, 1, 1, 1, 1, 1, 2, 3, 4

Thus giving totals of 34 and 15 "superfluous syntheses"

We see that in the class which paid much attention to "classical" analysis (6B), there were three times as many "mistaken analyses" as in the control class (6C), and more than twice as many superfluous syntheses, while 4 of the 10 pupils did 5-8 of these superfluous operations (which show efforts to find a solution by mechanical manipulation of the numerical data).

Thus, the negative influence of the "classical method" of analysis on the pupils' ability to solve task-problems, where a less ordinary combination of the data is required, turns out to be very persistent and appears even a year and a half after they have ceased studying it.

5. The Effects of Analysis in Solving Geometry Problems

Does this negative influence spread to the mastery of other mathematical disciplines--in particular, to the solution of geometric problems? Solving geometric problems demands great ability to analyze the text of the problem. Several authors of methods handbooks, as was noted above, assume that practice in "classical" analysis of arithmetic problems should have a good effect on the mastery of geometry.

In order to clarify the effect of "classical" analysis on the solution of geometry problems, a new series of individual experiments was carried out. The pupils of grades 6B and 6C (the 10 best mathematics pupils in each grade) were given four geometry problems for independent solution, corresponding to the level of their knowledge.[11]

Let us take as an example the record of the solution of the following problem by one of the pupils in grade 6B, Valya C.:

An isosceles triangle ABC is given. The side AB is extended upwards, and an arbitrary point D on it is joined with the point C. The perimeter of the triangle ADC is 55 cm, and the perimeter of the triangle DBC is 45 cm. Find AC.

Figure 34 was attached to the text. Valya C., having read the problem, wrote down the basic data: P (perimeter) ADC = 55 cm; P DBC = 45 cm; AC = ? and started the solution immediately: 55 - 45 = 10....

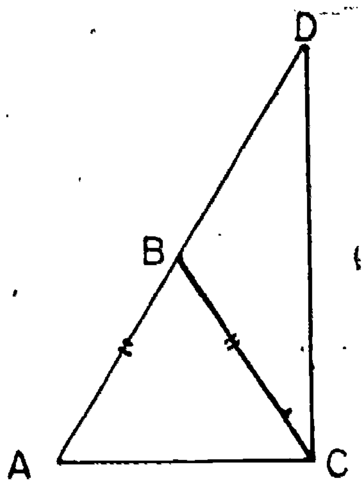

Fig. 34

There was a long pause. The experimenter asked what this 10 cm signifies. Another pause...Finally, Valya answered: "This is the perimeter of ADC...No, it's AB + AC..., and BD + DC = 45 cm, since 55 - 10 = 45 cm."

[11] The texts of the problems were approved by the teacher of these grades.

She looked at the sketch for a long time and covered the triangle BCD with her hand. There was another pause....

The experimenter suggested reading through the problem again. Valya read it and exclaimed: "It turns out that we have to find AC, and I thought it was AB!...55... 45... BD + DC + BC = 45 ... AB + AC = 10..." There was another pause.

Exp.: Are you sure of that?

Valya: Yes...

Exp.: Write down what the perimeters consist of.

Valya wrote:

AD + DC + AC = 55 cm.

BD + BC + DC = 45 cm.

Exp.: Now compare them.

Valya: DC is the same in both of them.

Exp.: What other conclusion can you draw?

Valya: BD + BC... no, no. BC = 10.

Exp.: Why?

Valya: This I ... AB + AC = 10.

Exp.: No. Compare the perimeters. DC is the same in both triangles, and now compare the remaining elements.

Valya: AD = BD and BC.

Exp.: Why?

Valya: Because AD is the side.

Exp.: Think carefully!

Valya: Because AD is the base.

Exp.: Compare AB and BC, and look at the problem.

Valya: Aha! AD = BD + BC, i.e. BD + AB...So, AC = 10 cm.

The problem was finally solved.

What is characteristic of this course of solution? First, the hesitation to study the text of the problem merits attention. Having read the problem, Valya immediately isolated the numerical data and begins to operate with them. Actually only one operation is possible here: 55 - 45 = 10, since the length of a side is required, and the sum of the perimeters does not yield anything here. Valya carried out the subtraction, and obtains the difference of 10 cm. However, what do the data obtained signify? Valya tried to see the answer to this question directly in the figure. She carried over the arithmetical operation done above to the figure: Mentally taking away DBC, she got the answer 10 cm -- this is the perimeter of the triangle ADC.

Later she corrected herself -- 10 cm is AB + AC, now taking away the line BC with the triangle BDC. Then Valya covered the triangle BDC with her hand, since the line BD appears covered, and again asserted that AB + AC = 10 cm. Having included the side BD in one triangle, she could not include it in the other, and could not switch over from one system of connections to another.[12]

Valya was completely sure of her conclusions, drawn from direct visual observation. Thus, along with the weakness of the verbal-logical analysis, an overestimation of the visual image also occurred here.

Valya sometimes did not compare the data she had obtained with each other or with the problem and therefore she asserted that the perimeter of the triangle ADC was equal to 10 cm, although in the text it is stated that it equals 55 cm. It did not bother her that the sum of the sides BD + DC (45 cm) is 4 1/2 times greater than the sum of the sides AB + AC (10 cm).

Valya was not always aware of the law on whose basis she drew a conclusion. For example, she asserted that BC equals 10 cm, and when asked why she thought so, she said, "This I-" and withdrew the conclusion. In drawing the correct conclusion that AD = BD + BC, Valya based it on the fact that AD is the side or the base. Only with the help of the experimenter did she substantiate her conclusion correctly.

Thus, in solving geometry problems Valya C. showed the same weakness in analysis of the data and their functional interconnections as was characteristic of her solution of arithmetic problems.

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[12]The difficulty of this transition was also noted during the solution of arithmetic problems.

We find analogous peculiarities, sometimes even more clearly expressed, with the other pupils in this class. Galya L. (whose solution of an arithmetic problem was introduced above), in finding the difference between the perimeters of the triangles (10 cm), began to fit the quantity obtained to different combinations of the data: "AB + BC = 10 cm... No, BD + DC = 10 cm"...Further on, she turned the figure so that the line DC turned out on the bottom and asserted: "The triangle BDC is isosceles, BD and BC are equal, then AB is equal to BD, each 10 cm -- 20 cm...55 - 20 = 35 cm," and again she changed the position of the sketch, trying to find the method of solution in it.

Vera P. found that ΔADC is isosceles, because "It's drawn like that." Zina S. also drew the same conclusion while looking at the sketch. When the experimenter demanded another basis, Zina answered: "AD = DC, since we extended AB with a straight line from the vertex B to the point D, and only one straight line may be drawn between two points," which in no way substantiates her conclusion.

This overevaluation of the visual image, in direct conjunction with weak verbal-logical analysis, is very characteristic of pupils in this class, even good ones.

In grade 6C the pupils also made mistaken reasonings, relying on an insufficient analysis, and sometimes looked for the answer in the direct visual image, in the figure. However, there were considerably fewer of these attempts than in class B.

To show more graphically the difference between these classes, I introduce Table 7, which gives the number of mistaken deductions as a result of weakness of the analytic process.[13]

[13] Models of this type of reasoning were introduced above, in the breakdown of the record of solving the geometry problem by pupils in class B.

TABLE 7

The Number of False Conclusions Made by Pupils in Solving the Geometry Problem (Ten students were taken from each class)

Grade: \ Number of False Conclusions	0	1-2	3-4	5-6	7-8	Total number
6B	-	4	4	-	2	36[a]
6C	5	4	1	-	-	10[a]

[a]The number of false conclusions made by the respective pupils in each class was

Class 6B: 1, 1, 2, 2, 3, 3, 4, 4, 8, 8
Class 6C: 0, 0, 0, 0, 0, 1, 2, 2, 2, 3

We see that in class C 4 pupils solved the problem completely correctly, and the remaining 6 pupils made only 10 false conclusions in all. In class B (where the pupils had studied "classical" analysis in the past), all 10 of the best pupils, in solving the experimental problem, made quite a few false conclusions while looking for a solution, 3 1/2 times more so than in class C, and not a single one of the pupils was able to solve the problem correctly at once.

The significantly greater number of false conclusions in the pupils in class B shows a lower level of analytic-synthetic activity than in the pupils in the parallel class. Since the same teacher had taught them their middle school mathematics, this difference can be explained only by their habits of analysis of the concrete text of a problem which they had acquired in the primary grades.

As we see, this series of experiments also supports the earlier conclusion that lengthy practice in the breakdown of problems by the method of analysis exerts a negative influence on the ability to solve more or less complex problems unassisted.

Thus, the investigation showed that:

1) The "classical" method of analysis cannot be a method of looking for a way to solve a task-problem.

2) Lengthy practice in the breakdown of problems by the method of "classical" analysis of problems that have already been solved or of easy problems, while it uses up much time and energy, exerts a negative influence on the elaboration of an ability to analyze the concrete text of a problem thoroughly, and holds the pupils back on a lower level of analysis (partial complex analysis and elemental analysis); the influence is also adverse for the solution of geometry problems.

6. The Need for a Balance of Analysis and Synthesis

What can account for the fact that lengthy practice in the breakdown of problems actually holds back the pupils on lower levels of analysis, that it actually influences very negatively their ability to solve more or less complex problems independently? The basic reason is that this method is very artificial, and that it contradicts the natural mode of thought in solving problems (as many methodologists have so rightly pointed out). The thought process in solving problems is analytic-synthetic. In it, analysis is closely intertwined with, and inseparable from, synthesis. Synthesis is carried out as soon as the bases for it are isolated in the process of analysis. The problem thus is simplified (as long as the number of simple problems entering into the complex one is decreased), and this further simplifies the subsequent analysis of the problem; analysis and synthesis always support one another.

The "classical" method of analysis assumes that the processes of analysis are isolated from the processes of synthesis. The solver is supposed to carry out a complete analysis at the start guided by the unknown, find all the necessary data for determining the unknown, and only then turn to synthesis. Such an artificial isolation of the processes of analysis from those of synthesis cannot be fruitful.

Various concrete situations are described in problems, but underlying them are certain relationships, known mathematical laws. Finding the way to solve a problem means discovering these laws and determining the relationships between the unknown and the data.

Both the unknown and the data can be related to other data in various ways, and the unknown might be determined through a combination of different data. From all of these possible combinations of different functional relationships, the pupils should choose precisely those which "will correspond to reality" (Pavloy) —the text of the given problem--and those which will serve as a basis for determining the value of the unknown.

The solver can make this selection correctly only if he is guided not only by the analysis of the unknown, but also by the analysis of each of the data contained in the condition. The analysis of the unknown cannot be divorced from the analysis of the concrete data and the functional interconnections.

The traditional method of analysis, in transferring the center of gravity to analysis of the unknown, separates the analysis of the unknown from the analysis of the data and distracts the attention from the analysis of the concrete data of the problem. The entire process of reasoning proposed by the method of analysis is constructed not on operating with the concrete data of the problem, but on data abstracted by choice and still unknown in the problem, precisely those whose combination might give, in the opinion of the solver, the value of the unknown. Only in the final steps of the reasoning does the solver arrive at the known data. Naturally, the lengthy practice in the traditional analytic breakdown of problems develops skill in constructing reasoning in isolation from the specific problem, which leads to an underevaluation of analysis of the text of the problem. Hence arise the numerous mistakes caused by a weakness of analysis of the data and of their functional interrelationships.

On the basis of the analysis of the unknown and of the concrete data of the problem, the solver should set up the possible relationships between them and choose the productive ones from among them, the ones which will lead to finding the value of the unknown. While solving problems that are new for him, task-problems, a person is not in a position to see immediately the entire course of solution. As numerous psychological investigations have shown, a method of solving a problem is sought while constructing and verifying (ordinarily by mental experiments) different hypotheses (propositions). The solver, guided by the analysis of the unknown and of the data, plans the method of solving the problem in his head and begins to carry it out. If he fails, he analyzes the mistakes, clarifies why the chosen method did not lead to the goal, and attempts to correct it, or else takes a different way.

Sometimes he temporarily reconstructs the problem, discards some datum, simplifying the determination of the necessary relationships between the data and the unknown.[14] The creative work of thought appears in this construction and choice of possible courses of solution. Only while solving problems whose structure is familiar does the solver not make a choice, determining in his memory the necessary relationships between the unknown and the data. Creative thought is minimized here.

The analytic method of breaking down problems excludes the factor of choice, presuming that the solver will immediately determine the necessary relationships and choose the data necessary for determining the unknown. Such a method of breakdown is possible only with problems whose structure is familiar and whose solution is made according to a pattern. Problems which require that the data correspond in a slightly different way cannot be dismembered by the method of analysis before being solved (as confirmed by the experiments).

The method of analysis does not teach the pupils to evaluate their propositions critically, and this undoubtedly has a negative effect on the nature of independent problem solving (they do not analyze their reasoning or their mistakes).

The method of analysis does not teach means which should simplify the analysis of the problem when difficulty arises. The pupils who have difficulty in their solution and do not know methods for overcoming it, as we saw, often turn to unproductive mechanical manipulation of the numerical data of the problem.

Lengthy practice in "classical" analysis, wasting time and effort, (and thus decreasing the number of problems solved), has a poor effect on the pupils' ability to solve problems and therefore should not have any place in school.

This does not mean that the "classical" method of analysis in general should be removed from school practice. Both the observations in the school and the data of the experiments showed that the method of analysis can be productive where and when the knowledge of the functional relationships between the data needs reinforcement. For example, the pupils should learn that to determine speed one must know distance and time, and that if we know the speed and the distance covered, we can

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[14] On helpful methods of analysis, see the next chapter.

determine over what period of time a trip took place. Without a sound knowledge of such functional ties between the data, the solution of more complex problems becomes impossible.

In solving more complex problems it is sometimes useful to turn to the partial analytic breakdown and remember what kind of combinations of the data might be used to determine the unknown, and further, to return to searching for the method of solution by different methods. This is precisely how many adults behave while solving problems; this is also the way teachers act when they are dismembering the course of solution of a new problem with the children.

With the older groups (grades 4 - 6) it is worthwhile for the teacher to acquaint the pupils with the breakdown of one or two problems by the method of analysis in order to show more clearly the logical necessity of the operations they have performed in the solution, without requiring this type of breakdown from the pupils themselves.

However, the teacher should give primary attention to developing in the pupils the ability to analyze the concrete text of problems and to arming them with helpful methods which might simplify independent solution of rather difficult problems.

Chapter V

PRODUCTIVE METHODS OF ANALYSIS AND SYNTHESIS

1. Introduction

The terms "analysis" (as a thought process) and "method of analysis," "synthesis" and "method of synthesis" have converged in the methodological literature; they are frequently used as synonyms; a discussion of the processes of analysis and synthesis frequently changes into a discussion of the methods of analysis and synthesis. Speaking of the difficulties, and sometimes of the inaccessibility of analysis for pupils, as opposed to the accessibility of synthesis (meaning the methods), the methodologists to some extent force teachers to underestimate the processes of analysis. On the other hand, in quite justifiably emphasizing the value of analysis for solving problems, individual methodologists, as was indicated above, have exaggerated the role of the method of analysis in teaching problem solving. Our investigations have shown that this method cannot and should not occupy an important place in school.

Does an underestimation of the processes of analysis then arise? Of course not. We have frequently emphasized that there is no synthesis without analysis. The method of synthesis is not overestimated either. Although we have not done any special investigation of the value of the method of synthesis for the psychology of education, we are inclined to believe that this method should not be the leading one in school either, since it assumes some isolation of the processes of analysis and synthesis. Any isolation of the processes of aralysis from those of synthesis is doomed to failure. Nowadays, we speak more and more of the analytic-synthetic method.

The solution of even the most elementary problem assumes separation of its text into individual complexes and isolation of the relationships connecting the unknown with the data, i.e., an analysis. The results of the investigation of the peculiarities of problem solving by low achievers, introduced at the beginning of this volume, show that a basic cause of their mistakes was their low level of analysis. A partial complex analysis or even an elemental analysis was typical for them. Consequently, to teach pupils to solve problems, one must teach them proper means of analysis and the correct combination of analysis with synthesis. One can find a number of productive methods of teaching problem solving described in the methodological literature.[1]

The best teachers in our schools use the methodological heritage, reworking it creatively and developing it further. Study of the experience of the foremost teachers should, in turn, significantly enrich the psychology of education and methods. I also turned to this fruitful source, in studying the most productive methods of teaching analysis and synthesis in solving problems. I have made only a few steps in this direction.

2. Observations of V. D. Petrova's Class

The pupils of V. D. Petrova (172nd School, fourth grade) attracted my attention. They showed (in individual experiments) an excellent ability to solve problems without assistance. The model for independent solution of problems that were quite complex for fourth grade was a pupil in this class, Valya K., who was mentioned in the first section of this work. Her solution was detailed and well substantiated. Valya's style of problem breakdown was characteristic of both good pupils and average pupils in this class. When difficulty arose, V. D. Petrova's pupils returned to the text of the problem, reread it, and looked through the solution they had done. They corrected most of the errors they made by themselves. They were able, in case of failure, to change the method of solution they were using or to replace it by a new one. They could outline a different plan of solution for a single problem. All of this shows a high level of development of the analytic-synthetic activity for the given grade. It should be noted that V. D. Petrova's pupils did not lower their level of achievement, as so often happens. Of the 36 pupils (all were passed) in the first quarter of the fifth grade, 11 received excellent marks in all their subjects, 18 pupils had 4's and 5's; and 7 had 4's and 3's. The mathematics instructor in the middle school mentioned the ability of V. D. Petrova's pupils to analyze the text of problems that were new for them.

Undoubtedly, the ability of these pupils to solve problems is determined by their entire system of working. However, it is also certain that their success in solving problems largely depends on the correct

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[1] A description and an evaluation of them can be found in the work of Menchinskaya [18].

means of analysis that they used. These proper methods of working on problems and analyses of them is frequently instilled from the very beginning of instruction.

I observed V. D. Petrova systematically while she was teaching problem solving during 1952-1953 in the first grade. In this section (to conform with the principal topic) I shall dwell on the method she used in teaching first-grade pupils the methods of analysis and synthesis in solving problems.

Emphasis on Reading the Problem

The work on the analysis of a problem begins with reading it properly, with intonational expression, and in this process the first primary separation of the text occurs, as well as the isolation of the unknown and the individual data. V. D. Petrova devotes much attention to this — she teaches her pupils how to read problems.

From the beginning the teacher continually emphasizes that each word in the problem, regardless of how small it is, has its importance. If one changes "tiny little words" — "in" and "on" — the entire sense of the problem is changed. These small but very important words must be emphasized by intonation. V. D. Petrova makes the pupils vary their intonation when they see punctuation marks (pauses), to aid in breaking down the problem into its component parts. She requires special expressive emphasis of the problem's question. All of these demands are mastered by the pupils, and they begin to demand it of themselves and of their classmates. A pupil read a problem:

> "Ten aspen...logs were put into a stove and-"
> She put in a pause which destroyed the logic, and the teacher immediately called her attention to this: "Where is the comma?" she asked.
>
> "After the word 'stove,'" answered the girl.
>
> "Then a voice pause should be there also," the teacher reminded her and demanded another reading with the correct intonation.
>
> The girl read: "Ten aspen logs were put into a stove, and six fewer birch logs were put in than aspen. How many logs were put into the stove in all?"
>
> The teacher asked the class what other mistakes this girl had made in reading.

The pupils noticed:

"She read the word 'than' poorly."

"She did not emphasize the word 'fewer.'"

"She did not emphasize the number..."

The problem was read again, in an attempt to meet all of the teacher's demands. The monotony of this repeated reading did not deaden the girls' attention, because they were on the alert to notice if the reader made any mistakes and were to correct any inaccuracies.

The teacher emphasized that the solution itself largely depends on a correct reading of the problem: "Valya, here, read the problem poorly and cannot explain its solution; and Katya was mistaken because she missed this important little word 'than' when she read the problem at home," she explained. The pupils developed a genuine respect for this stage of the work on a problem. Gradually they formed a sound habit of reading the text of a problem with the correct intonation, and the teacher devoted less and less attention to this stage.

<u>Emphasis on a Breakdown of the Text of a Problem</u>

Although the individual data and the unknown are isolated while reading the problem, the teacher did not limit her class to this. In the initial period of teaching the separation of the text of the problem into individual data and the unknown, she singled this out as one stage in the work on problems. Having read the problem, the pupils were to enumerate each of the data and isolate the unknown in particular. Here is how this breakdown of the above problem was done by one of the pupils, Katya S.:

> "It is known," she says, "that they put 10 aspen logs into the stove. It is also known that six fewer birch logs were put in than the number of aspen logs, but it is unknown how many logs were put into the stove in all."

Here the text of the problem has been repeated in a slightly different form, and one datum was distinguished from another very precisely—this is one thing that must be taught. The investigation of the peculiarities of problem solving by young pupils has shown that, if they are only able to reproduce the text of a problem verbatim, they sometimes separate it into its components incorrectly, isolating partial complexes whose operations lead to mistakes (this type of mistake was described above).

At the end of the second quarter all the pupils in this class, even the weak ones, could break down the problem in this manner, and in the future the teacher became more and more inclined to skip this stage, returning to it only in more difficult cases.

Having separated the text of the problem, the pupils started a more detailed analysis of each datum and the unknown.

"What kind of logs were put in the stove?" the teacher asked.

"Birch and aspen."

"How many aspen logs were put in?"

"Ten."

"Read again what it says about the birch logs."

"Six fewer birch logs were put in than aspen logs."

"Fewer than what?"

"Than aspen logs."

"And how many aspen logs?"

"Ten."

"What is asked in the problem?"

"How many logs in all were put in."

"In all--this is, consequently, what kind of logs?"

"Aspen and birch."

After this type of breakdown the way to solve the problem will become clear.

If unfamiliar words are found in a problem, the teacher reveals their meaning in detail, so that the pupils can imagine very clearly the articles referred to in the problem.

Once, the number of pine trees that were sawed up into boards was mentioned in a problem. In repeating the problem, one girl used the word "logs" in place of "boards." The teacher then explained the difference between the concepts. She cut up a stick into a "board" and a "log" and discussed the uses of boards and logs.

With this type of work, the problem evoked a more vivid, clear concept in the students and became part of life for them. Thus the solution was simplified for them, and they could verify their results more realistically. This is very important now, as our schools are resolving the issues of polytechnization.

Emphasis on Differentiation of Concepts

The teacher always dwelt particularly on similar concepts which should be differentiated from one another. The confusion of concepts sometimes hinders the conscious solution of the problem (for example, a problem about 5-kopek pieces and 3-kopek pieces in the workbook).

The analysis of concepts that express a quantitative relationship between objects (fewer than..., so many times bigger than, etc.) was guided by a great deal of systematic work by the teacher on these concepts. (I do not have the opportunity to explain this in more detail here.)

The teacher elaborates proper concepts about quantitative relationships between objects by visual material. In the analysis of problems she observes whether the pupils understand the meanings of all these "small, but important, words," so that they can imagine clearly the relationships described by the problem, and she turns to visual aids when difficulties arise.

Emphasis on Substantiation

In solving problems, V. D. Petrova demands a substantiation of the method of solving of a problem from the text. She demands that the pupils point out the part of the text that determined the operation performed by the pupil. Thus, in solving the above problem about the aspen and the birch logs, the pupils asked, "How many birch logs were brought?" To answer it they proposed subtracting six logs from 10 logs. The teacher requested rereading the part of the problem that stated that subtraction should be carried out ("six logs fewer"). Through this type of work the pupils became accustomed to conducting the solution on the basis of an analysis of the text, thus controlling their choice of operation.

The analysis is subordinate not only to the data contained in the problem but also to the intermediate data obtained during the solution.

In doing the appropriate arithmetical operation, the pupils indicate precisely what kind of data they obtained and connect these data with those contained in the problem.

"Now we know," a pupil said, in solving the problem about the logs, "that they put 4 birch logs into the stove. We also know, that they put in 10 aspen logs...Now we can find out...." He continued with the statement and resolution of the next question.

Emphasis on the Question of a Problem

V. D. Petrova devoted much attention to "work on the question of a problem," on the unknown. The pupils isolated and separated the unknown data. The teacher emphasized that determining the value of the unknown and answering the question of the problem are the goals of the solution and that there should be no superfluous operations—all operations should serve the one basic goal of determining the unknown.

When they found the value of the unknown, the girls explained: "We have solved the problem because we have answered its question. In the question the following is asked: How many logs in all were put into the stove? We found out that 14 logs were put into the stove."

"Katya here," said the teacher, "solved this problem in one operation: She subtracted 6 logs from 10 logs. Did she complete the problem?"

"No, Katya did not read the question of the problem. She didn't answer it," the children explained.

This kind of work established the purpose of looking for the unknown in the solution and prevents superfluous syntheses.

V. D. Petrova teaches the pupils to ask themselves questions for different known data ("What can we find out if we know...") to select data for the question ("What must we know in order to determine...").
Sometimes she asks if they can immediately, with one operation, find the answer to the problem's question, and has them explain why this is impossible. Thus, she includes elements of the method of analysis, but still does not teach the pupils to conduct the "classical" analysis of problems on their own.

She also teaches them different formulations of questions referring to a single operation. If one reads: "A boy had 20 notebooks and gave half of them to his sister," the following questions could be posed:

1) How many books did he have left? or: 2) To what is half the books equal? 3) How many notebooks did he give his sister? The answer to these questions, as the pupils explain, is found in one operation: 20 ÷ 2. This type of work makes the pupils' thought more flexible; the transition from one system of connections to another will not be so difficult for them (recall the difficulty Lyusya G. had in solving Problem No. 3).

Emphasis on Analysis of Errors

V. D. Petrova also trains the pupils to analyze their mistakes, showing that the basic source of the mistake is a superficial analysis of the text. For example, they had the problem:

> Twenty birch trees were planted in a park, then 10 more poplars than birches, and as many linden trees as poplars and birches put together were planted. How many linden trees were planted?

Valya S. found how many lindens were planted by adding 20 and 10.

The teacher, in reviewing the solution, asked the girls to indicate where Valya had gone wrong, and what part of the problem she had ignored ("10 more poplars than birches").

Nina K. solved a problem about nickels. To answer a question about the number of pennies in a nickel, she wrote the operation thus:

3 N. X 5 p.

The teacher again demanded that the pupils show why Nina was wrong, what rule she had forgotten. The teacher requires independence in her pupils' problem solving and does not consider homework done unless the girl shows the sheet of scratch paper on which she solved the problem. The teacher told the girls how this pupil solved the problem, and then they all singled out the errors she made and told why the problem was not solved.

The pupils were gradually trained in controlling the operations they used and in correcting their mistakes; this training, as we saw, is reflected in their ability to solve problems.

Emphasis on Developing Language Powers

A person thinks with words. "Naked thoughts, free from linguistic material, free from linguistic 'natural material,' do not exist [40: 39]," says J. V. Stalin. Developing the pupils' speech also involves developing logical thought and increasing the level of their analytic-synthetic activity.

V. D. Petrova worked very hard to develop the speech powers of her pupils. She broadened their active vocabulary, including words to signify abstract concepts (weight, quantity, etc.). She constantly required detailed, precise answers, without extra words, to her questions. By the first half of the year her first-graders could, without the teacher's helping questions, transmit the content of a problem and explain how to solve it.

An Individual Experiment

At the end of the year an individual experiment with 15 pupils was conducted (5 having 5's, 5 having 4's, and 5 having 3's in arithmetic). The following problem was given for independent solution:

> Forty meters of satin were brought into a store. Half the material was sold on the first day, and 7 meters less was sold on the second day than on the first. How many meters of satin were sold on the second day?

Of the 15 pupils, only one required some help from the experimenter; the rest solved the problem all by themselves, giving a detailed explanation of the course of solution. As an example of such a solution and the explanation of this problem, I introduce the record of Zhenya S. (she had a "4" in arithmetic). After repeating the text of the problem, Zhenya began its breakdown:

> "We know that 40 meters of material were brought into the store, and we know that half of the material was sold on the first day, but it is unknown how many meters were sold on the second day. This we shall have to find out. In order to find out how many meters were sold on the first day, we must divide: 40 m : 2 = 20 m. Twenty m of material were sold on the first day.
>
> "Now we know that 20 meters were sold in one day, and that 7 m less were sold on the second day, but it is unknown how many meters were sold on the second day.
>
> "In order to find out how many meters of material were sold on the second day, we must subtract: 20 m - 7 m = 13 m. Thirteen meters of material were sold on the second day."

Such a detailed, well-founded solution was very typical of the pupils in V. D. Petrova's class.

The initial knowledge of functional relationships between data was given by V. D. Petrova in first grade during the solution of simple problems. Later, in grades 3 and 4, she broadened the scope of the connections to be learned and achieved a sound mastery of them. She worked hard to decrease the time spent in translating the concrete data of the problem into more abstract mathematical concepts.

Thus, in solving quite a complex problem involving finding the difference in gasoline consumption of two different automobiles, the fourth grade pupils indicated that they must explain which of the automobiles consumed more gasoline and by what amount.

"What shall we do?" asked the teacher.

"We should compare the amount of gasoline used by the automobiles. In order to do this, we must subtract the amount of gasoline consumed by one automobile from the amount of gasoline consumed by the other, and thus find the difference between them."

"The difference of what?" The teacher demanded precision.

"We shall find the difference in gasoline consumption," a girl answered.

Thus, the entire course of the solution was traced in an abstract formulation.

Emphasis on Alternative Solution

V. D. Petrova ordinarily considers and evaluates, from the point of view of their productiveness, various possible solutions of a single problem and the solution of problems without using numbers or numerical formulas.

These means of training instill in pupils the ability to plan different courses of solution of a problem and the ability to choose the most rational of them. This was very clearly shown in the solution of Problem 16 by fourth-grade pupils in the 172nd School. The girls planned two or three possible ways to solve it and indicated the best one. All of this shows a high level of analytic-synthetic activity.

Her success in her work was determined, as I have already indicated, by her entire approach, not just in arithmetic lessons. The ability to think logically, to reason, to express thoughts precisely and accurately, all of this she taught the pupils in all the lessons. The methods used by this teacher are described in the methodological literature. The definite, strictly-thought-out system of using them was valuable in her work, and as a result, this method was mastered not only by the best pupils, but by the weaker ones as well, and these became means of creative thinking for them. As was indicated above, the fourth-grade pupils of V. D. Petrova, made very few errors in analysis when they solved a difficult problem on their own, and when difficulties arose they turned to a repeated analysis of the text, analyzed their reasoning and mistakes. All of this shows the productiveness of the system described here.

3. An Experiment with Auxiliary Methods

Above, as one of the examples, an experiment was described in which the teacher was training her pupils in elementary means of analysis and synthesis in problem solving. However, these elementary means could not always guarantee finding a way to solve more complicated task-problems, the solution of which is a process of creative thinking. When difficulties arise, the solver will search creatively for some auxiliary means that will simplify finding the course of solution. The knowledge of this type of method can doubtless be very useful.

For the purpose of finding the auxiliary methods that simplify analytic-synthetic activity in solving problems, we investigated the process of independent solution of arithmetic problems by adults. Ninth- and 10th-grade pupils with excellent grades, college students, mathematics instructors, and a number of scientists were used as subjects (30 in all); each of them solved 10 problems, and 300 solutions were collected.[2] Quite complex problems were given to them, many of which caused difficulties in solution. To overcome these difficulties, the solvers ordinarily introduced a number of devices which made it easier for them to find the relationships between the unknown and the data. The following aids were the most widely used: concretization, abstraction, modification, graphic analysis, analogy, and analytic questions.[3]

Concretization

The method of concretization is used when the solver introduces

[2]For a more detailed description of the results of this investigation, see Proceedings [Izvestiya] of the Academy of Pedagogical Sciences of the RSFSR, Issue 61.

[3]These means presuppose a very complex analytic-synthetic activity. However, they are primarily directed at exposing the content of the complex data, at isolating the relationships between them, i.e., at an analysis. Therefore we conventionally call these means "methods of analysis."

into the problem a series of concrete details, which make the situation outlined in the problem more comprehensible and therefore simplify the determination of the relationships between the unknown and the data. Many turned to this method, for example, while solving Problem No. 1 (the statement of which is quite abstract):

> 1260 rubles profit was received for goods sold. If 4800 rubles more had been paid for the goods than was paid the first time, and if they had been sold for twice the price, then the profit would have been the same. What were the buying and selling prices of the goods?

Many subjects made the problem concrete in this way: "This store sells something from its stock to a buyer. There is a resale. The store itself paid 4800 rubles more the second time."

The abstract formulation of the problem has become more concrete (here: the store, its stock, and the buyer), helping many to separate the processes of buying and selling, and to realize the profits as the differences between the buying price and the selling price.

However, the investigation showed that not every concretization was useful. One person who solved this problem clearly imagined the merchant measuring off material for a buyer, and these life-like pictures, by his own admission, led him astray from the solution.

"In order to make the situation of the problem more life-like, one must almost feel it," said one of the solvers, quite accurately conveying the attempt to rely on a visual aid (on the first signals of reality). But this visual aid should not be too burdened with details detracting from the basic goal--the relationship between quantities. As was shown, the most generalized schematic forms are the best support.

Abstraction

In conjunction with making the problem concrete, or frequently after it, the solvers turn to the method of abstraction: They discard all the details and strive to express the situation in the most abstract concepts, thanks to which the functional ties between the data stand out and their mathematical relationship is exposed.

"More was paid, they were sold for more, and the profit was the same"--this is how one of the solvers conveyed the content of the problem indicated above. Consequently, its formulation reflects the basic mathematical relationship on which the problem is constructed. Such an abstract formulation simplified disclosing the appropriate mathematical laws on the basis of which the value of the unknown can be found. In the given problem, on the basis of such a formulation, it was easier for the solver to draw the conclusion that the profit therefore remained unchanged, that the higher the price they asked, the more they had paid for the goods (i.e., 4800 rubles), and by that very means he was able to find the key to the solution. The method of abstraction assumes, apparently, guidance by the most general associations of the second signal system.

Schematic Notation

An intermediate position between these two methods is occupied by the method of a schematic notation of the problem. The adult solver ordinarily turns to notation when the problem has many data and it is difficult to understand immediately the connection between them.

Thus, for the following problem:

> 425 rubles were paid for 40 m of fabric of the best variety, and for 30 m of the second best. If 30 m of the first had been bought, along with 40 m of the second, then the cost of all the fabric would have been 415 rubles. How much does 1 m of the best fabric cost, and 1 m of the second best?"

Most solvers made a notation, grouping the data in the following way:

40 m.	30 m.	--	425 rubles
30 m.	40 m.	--	415 rubles
10 "	10 "		10 "

Or:

40		30	
	(425 R.)		(415 R.)
30		40	

Here all details are discarded, and only the numerical data are singled out. This makes this method similar to the method of abstraction. However, the grouping of these data is presented visually. The solver looks for guidance in this visual grouping to determine the interrelationships between the data; in this way the schematic notation of the problem approaches the method of concretization.

Graphic Analysis

A similar intermediate position is also occupied by the method of "graphic analysis of the problem." Using conventional shapes having no objective similarity to the specific content of the situation, the solver strives to represent the relationship between the unknown and the data, and thus to simplify finding the necessary relationships.

Thus, G. could not find the way to solve the problem indicated above (No. 1) about profits. Then he represented by one rectangle (1) the old cost of the goods, by a second (2) the sale, where the 1260 rubles of profit are noted; then he wrote down a rectangle for "the new purchase" (3), and the new sale was given as the sum of the first two (4). The following diagram was obtained (Fig. 35):

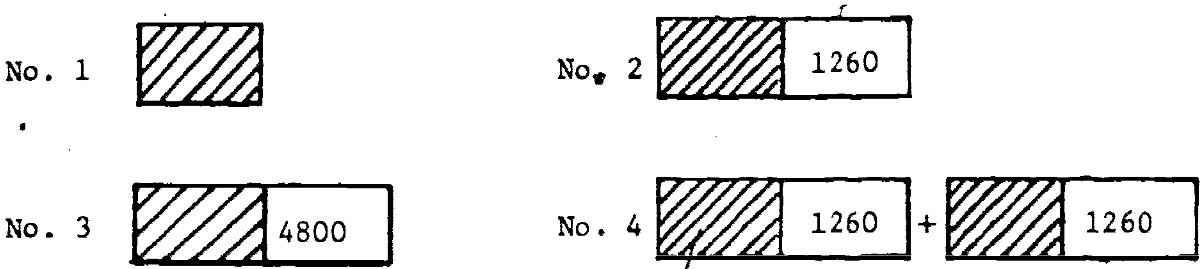

Fig. 35

Looking at his diagram, G. made the necessary conclusion: "Aha! If we add 1260, then they are the same. The old price plus 1260 is equal to 4800. 4800 minus 1260, this is the old price, and they made the sale for 4800 rubles."

Such a visual separation of the data simplifies comparison of them and of the isolated parts, and therefore simplifies finding the laws connecting the data.

N. very effectively changed the method of graphic analysis in relation to the following problem:

> A boy had a few kopeks. When someone gave him 14 more kopeks, he took all of the money to buy 4 pencils, paying for each of them twice what he had at the start. How much did the boy have before he received the 14 kopeks?

He drew a circle—this represented the sum of money spent for the pencils. "The boy bought 4 pencils—1/4 of the circle for each pencil," he noted. "His own money sufficed for 1/2 a pencil, which is 1/8 of the total sum. For the remaining 7/8 he was given 14 kopeks." Guided by his diagram, N. easily found the required relationship. (Fourteen kopeks is 7/8 of the cost of the pencils.)

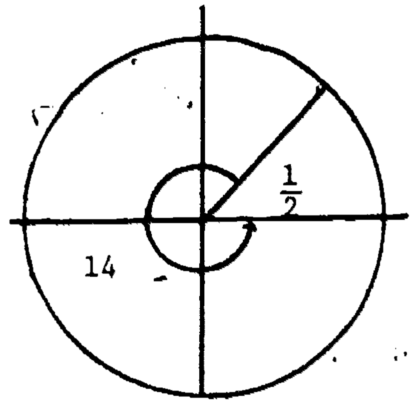

Fig. 36

The relationship between the data and the graphic diagram is represented visually; at the same time the diagram itself, along with its parts, has an abstract, generalized significance. Thus, in the diagram the use of visual aids interacts with the abstract components of our thought (the interaction of the signals of the first and the second signal systems).

In the method described above, the solver actively intervened in the situation, introducing some details or else discarding something contained in the problem. The problem is altered even more when the method of variation is used.

Variation

The method of variation actually represents a mental experiment, in which the solver rejects a datum temporarily or arbitrarily changes its numerical value so that later, on the basis of logic, he might explain the consequences of this transformation, what happened when this datum was isolated from the rest. By this change it is easier for him to see the role of this datum and of the rules which connect this datum with the others and with the unknown, i.e., it is easier to determine the course of the problem's solution.

It was precisely this method which was used by the subject P. in solving Problem No. 1. P. determined that the price doubled and was more than 4800 rubles. However, he was not able to cross over from this position to the concrete arithmetical operation for determining the unknown. Then he introduced the following assumption: "Suppose they had not increased the price by twice the amount. Then," he concluded, "they would have suffered a loss. What would this loss have been? It would be 4800 - 1260 = 3540. However, they did not suffer a loss," and

P. again compared his conclusion with the situation in the problem. "They made a profit. Consequently, doubling the price covered the loss by 3540." Then the answer and the choice of operation was clear to him: 3540 + 1260 this is the sum for which the goods were bought.

Here the solver has temporarily discarded one of the data. In solving another problem, a conditional change of value of one of the data proved effective. For example, this method was used successfully by the student Yu. in solving the following problem:

> A collective farmer's wife had money for buying ribbon. It turned out that if she bought ribbon for 3 rubles 42 kopeks per meter, she would not be able to pay for 2 meters, but if she bought ribbon at 2 rubles 85 kopeks, she would have enough money left to buy another 2 meters. How much ribbon did she buy and at what price did she propose to buy it?

Yu. determined the cost of 2 m of the cheaper (5 rubles 70 kopeks) and the more expensive ribbon (6 rubles 84 kopeks) and the difference in price between one meter of the more expensive and one meter of the cheaper ribbon. Having done this, Yu. was not able to understand the ties between the data. "Here are three data. Is it possible to join them? Of what are they composed?" he asked himself and turned to the method of variation. He imagined that the price of 1 m of the cheaper ribbon was increased.

"With a gradual increase in the price," he reasoned, "the farmer's wife would spend all her money and 6 rubles 84 kopeks beyond that. But if the cheaper one is bought, then 5 rubles 70 kopeks would remain. The price of 1 m went up and up, and there was not enough money. If it went down, money would be left over. We must add them: 6 rubles 84 kopeks and 5 rubles 70 kopeks, and they are composed of these 57 kopeks," and Yu. drew the correct conclusion about the connections between the data.[4]

It should be noted that the methods of variation taken from the solution of the model problems are very widely used by adults in solving the most varied problems.

Analogy

When serious difficulties arise, the solvers sometimes use the method of analogy. They compose a problem analogous to the one they are solving,

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[4] The subject expressed himself in very compressed, abbreviated formulations: The difference in cost was formed thanks to the difference in prices - 1 m - 57 kopeks ("They are composed of these 57 kopeks.").

but with small data and with an unknown whose value they can calculate in advance. The small size of the data and the known value of the unknown make it considerably easier to compare them and disclose those relationships that connect them to each other. The relationship determined in the new problem is then transferred to the initial problem.

Thus, G. forgot the usual means of solving Problem No. 6:

> A train includes two-, three-, and four-axled cars, and the number of three-and four-axled cars is the same. There are 36 cars in all, and 111 axles. Determine the number of each type.

Then he took these arbitrary data:

> "Suppose there were 3 wagons with 4 axles each: that's 12 axles. And 5 cars with 2 axles each — 10 axles; these 8 cars would have 22 axles."

> G. had not only the initial data, but the intermediate data and the unknown as well. It was necessary only to find the relationships of these data. And G. looked for the necessary relationship by comparing the numerical values of the data: "22 ÷ 3; 22 ÷ 4; 22 ÷ 7..."

> He rejected all of these variants quickly; the presence of the value of the unknown simplified the checking significantly. "It's not divisible. We have to multiply," G. decided, and continued his trials.

> "If they were all four-axled, there would be 32 axles (4 X 8); if they were all two-axled, there would be 16 axles: 32 - 16 = 16. Here we have to determine the difference and divide." G. has found the proper course of solution and he transferred it to the original problem.

As the investigations showed, the method of analogy can be productive only when the solver has sufficiently analyzed the data contained in the problem. Otherwise the problem he constructs will not be analogous to the original one, therefore will not lead to the necessary course of solution.

In the solution process adults ordinarily ask themselves analytic questions which direct their thought to the analysis of the content of the data and the functional ties between them. Why did the profit remain the same although the price of the goods changed? Why was the second purchase more expensive than the first? What is the content of these data? Are they the same? These and similar questions help the solver to isolate the basic relationship in the problem, its kernel, and help him to concentrate on precisely this basic thing.

These are the most widely used methods which make it easier for adults to find the way to solve quite complex task-problems.

4. Developing Auxiliary Methods

The question justifiably arises of the source of these methods. They are undoubtedly worked out gradually, in solving problems and especially in academic study.

The auxiliary methods described here are not, essentially, new ones to teachers of mathematics. Teachers are acquainted with these methods and use them to some extent in their work. The schematic notation of the problem is quite widely known in school, but usually the teacher himself does it.

In trying to make a difficult problem more comprehensible, the teacher makes its situation concrete, sometimes proposes an analogous problem, but with small numbers and with a more transparent relationship between the data (the method of analogy). Many teachers ask analytic questions during the breakdown, to guide the pupils' thinking in the necessary direction. Often, especially for certain groups of problems (e.g., those on motion), they draw a diagram (the method of graphic analysis). An entire group of problems is actually solved by the method of variation (problems on substitution, equalization, supposition).

However, the number of problems for which these methods is used is too small, and the main thing is that teachers who use the methods themselves ordinarily do not intend to teach the pupils to use them independently and to introduce them in problem solving when difficulties arise. Only rarely does the literature on methods indicate the necessity of teaching the pupils some method or other (if so, it is often the method of schematic diagrams of the problem and graphic analysis).

Our investigation of independent solution of task-problems by third- and fourth-grade pupils has shown that, although the teachers do not ordinarily teach these helpful methods of analysis, the pupils can master them on their own and use them when difficulty arises; however, only the best pupils attain this level.

The weaker pupils, as the experiments have shown, are not in a position to single out and master these methods. As was shown above, they ordinarily revert to simple manipulation of the numerical data when difficulties arise.

Since the weaker pupils do not master the helpful methods of analysis, we shall observe how the better pupils master them. We made an experimental study of the independent solution of rather difficult problems by excellent third-, fourth-, and sixth-grade pupils. Twenty pupils were taken from each grade (from different teachers, three or four of the best pupils from each class).

As the investigations showed, even good pupils do not isolate the auxiliary methods of analysis at the first stage of learning, when they are still inexperienced in solving problems. Third-graders show no effort to change the method of solution, even when the chosen course turns out to be inexpedient. If they fail, they either reject completely any further search for a course of solution, or else they proceed to manipulation of the problem's numerical data without any correlation with their objective significance. The solution itself proceeds by separating individual parts of the problem, and these parts, on account of the weak analysis, are not always adequate to the situation. For example, Kolya S. solved Problem No. 12:

> One worker saved 696 rubles every year. His older brother started work at the same factory after 14 months, and after 28 months he had accumulated as much as his brother had done from the beginning. How many rubles did his brother save yearly?

> After asking himself how many rubles the younger brother saved monthly, Kolya tried to divide 696 by 14 (and not by 12 months), i.e., he regarded the 696 rubles as the amount saved by the laborer in 14 months.

> When the teacher indicated the mistake, Kolya corrected it. He found the sum saved by the younger brother in 14 months (812 rubles), and wanted to determine the older brother's monthly saving by dividing this sum by 28 months. Again the experimenter had to point out the mistake.

> Then Kolya announced: "I can't do it," and waited for help from the experimenter.

It should be noted that for most third-graders, suggestions made by a direct hint from the experimenter to use one of the auxiliary methods that their teachers had used in class to explain difficult problems were unproductive.

For example, the method of schematic notation of the problem would be most effective in helping to solve Problem No. 16:

A store was supposed to receive 115 kg of boiled butter at 18 rubles per kg, and 135 kg of cream butter, at a total cost of 4770 rubles. But only 75 kg of boiled butter were delivered, and all the remaining money was used to buy cream butter. How much cream butter did they buy?

Noticing that Vitya B. was making the solution of the problem more difficult than necessary, the experimenter suggested that he write down the problem in abbreviated form, as his teacher had done. Vitya did so but left out isolated words and introduced abbreviations, without trying to systematize the data somehow. Volodya T., under similar conditions, did it in a more orderly fashion:

$$4770 \text{ R} \begin{cases} \text{115 kg bld. but., 75 kg bld. but., at 18 R.} \\ \text{135 kg of cream but.} \end{cases}$$

However, these diagrams helped neither Vitya B. nor Volodya T. Then the experimenter suggested the following diagram:

115 kg bld. but. at 18 R. & 135 crm. but. -- 4770 R.

75 kg bld. but. at 18 R. & ? -- 4770 R.

This diagram simplified the course of solution considerably for each of them.

Analogous facts are also noted with respect to the use of visual diagrams. A visual diagram can be very productive for Problem No. 20:

1560 rubles was deposited in each of two departments of a savings bank. A certain sum of money was taken from one of the departments, and from the second as much was taken as remained in the first. How much money remained in both departments of the bank?

A basic difficulty of this problem for young pupils is that there is only one numerical datum present. "This problem is impossible to solve. There are no numbers here," third-graders often announce after they have read the problem.

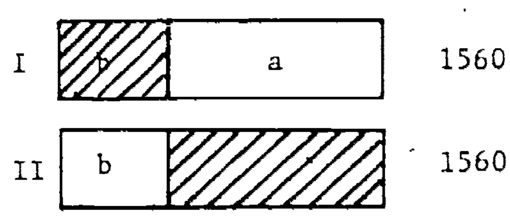

Fig. 37

Translating the problem into a visual scheme ordinarily serves as a good support for obtaining the required conclusion from the problem. Here is the form for such a diagram (Fig. 37). The sums of money removed (or, more precisely, the relationship between them) are made concrete in the diagram, and one can easily see that the sum of the remainders is equal to 1560 rubles.

When a group of third-graders had difficulty solving this problem, the experimenter suggested sketching a diagram for it. He gave a series of specific hints:

1) First sketch two separate boxes where the money is.

2) Mark off the amount (arbitrary) taken from the first box.

3) Read what is said about the second box in the problem and mark off the part taken from the second box on the diagram.

The pupils carried out the first two instructions without any difficulty but stopped in indecision at the third.

"What is this? I don't see how to show it. The same amount as remained in the first," Volodya N. kept repeating and, finally, marked off the same portion of the second "box" as of the first! Only the teacher's direct hint led him out of the difficulty. The pupils were not able to translate the abstract "the same amount as remained in the first department of the bank" into visual form. Obviously the mastery of these quite complex methods of visual aids (graphic analysis, diagrams) demands a higher level of analytic-synthetic activity than the pupils had.

However, the method of concretization turns out to be more accessible and effective for third-graders. Kolya S. arrived at this method by himself by accident.

> When the experimenter suggested sketching two boxes for the money, he drew rectangles and wrote the appropriate sum in each. Later it was suggested that he mark off how much money was taken from them. He crossed out the last three digits from the sum 1,560, i.e., 560 rubles, and subtracted this sum from the second, noting that there was a remainder of 560 rubles. Then he added the remainders and thus found the unknown.
>
> "But if you take a different sum?" the experimenter asked.
>
> Kolya took 300 rubles and after he had carried out the appropriate operations, obtained the same answer.
>
> "And if you take a different sum again?" the experimenter continued. Kolya again carried out the subtraction and, finally, arrived at the generalized conclusion: "It doesn't matter what you subtract; there will always be 1,560 rubles."

The method of concretization was productive here. Obviously, numbers are of more help to third-graders than a very conventional visual diagram.

A refusal to search actively for a method of solution, the transition when difficulty arises to manipulation of the numerical data without reference to their real, objective significance, the inability to use the indicated auxiliary methods--all of these were characteristic for 16 of the 20 excellent third-graders we investigated.

Four of the good pupils showed a slightly different approach to searching for a method of solving the problems. When one method failed for them, they were induced to try other methods of solution known to them, and this effort proceeded on the basis of features isolated in the process of analysis, and came from problems known to them. For example, Lenya F. solved Problem No. 13:

> 1280 rubles were paid for a child's overcoat and suit. An overcoat costs 60 rubles, and a suit is 20 rubles cheaper. How many overcoats and suits were bought if there were 7 more suits than overcoats?

Lenya correctly determined the cost of one overcoat and one suit (100 rubles) and then tried to divide the cost of the entire purchase by it (1280 ÷ 100). The impossibility of this operation forced him to turn to the problem's text.

"I made a mistake...Here it says 7 more"--he indicated, quite correctly.

"Seven more..." he repeated, and asked: "How many pieces were there?"

Thus, the isolation of the datum "7 more" gives a false basis for the use of the method of solution by sum and multiple relationship.

He carried out the problem breakdown (7 + 1 = 8 parts: 1280 ÷ 8 = 160 coats) and transformed the problem accordingly-- "160 coats and 7 times as many suits." After the experimenter's indication ("7 more"), Lenya added 7 to 160, but, realizing that the answer was wrong, turned again to the text. He correctly determined the cost of the 7 "extra" suits. Finding how many rubles were paid for the entire purchase if the same number of suits were bought as overcoats (1000 rubles), he then tried to solve this problem as problems in proportional division are ordinarily solved (they were solving this type of problem at that time in class). He divided 100 in half and tried to divide 500 by 40 and 60. With some help from the experimenter, Lenya changed the method of solution somewhat and determined the value of the unknown.

Three other third-graders searched for the solution of new problems in an analogous manner. The question naturally arises how the method of solution described above should be assessed. Even Lenya used a model method without sufficient basis for it, without a thorough analysis of the data, sometimes with only an arbitrary similarity between some isolated elements of the problem. In older pupils we undoubtedly could not evaluate positively such a method of solution. With third-graders it must be regarded in a different manner. For the bulk of the pupils in this grade, a rejection of active searching for the solution was characteristic when the first method they had chosen did not lead to the goal. When difficulties arose, they sought for support in the numerical values of the problem, not in the problem's content.

All of these pupils (and only the best were chosen) were able to solve model problems corresponding to the curriculum requirements for the third grade. However, their use of one method precluded the possibility of other methods. A definite stagnation of possible systems of associations corresponding to each type of problem was characteristic of them--if one system of associations had just been introduced into the operation, this precluded the possibility of actualizing another system, when the stimulus (the situation of the problem) remained the same.

A small group of four third-graders showed greater flexibility. Under similar conditions, they were able to cross over from one system of associations to another, although it was perhaps invalid, with only one element in common.

This freer use of model methods assumes a higher degree of abstraction, greater generality, and isolation of them as <u>methods</u> of solution, which creates the prerequisites for modifying the model means of solution to correspond with the situation, and this is one of the means of solving task-problems.

In fourth grade the number of problems solved is enlarged, the number of model methods known to the pupils is broadened, and the students' experience is enriched. Likewise, there are fewer good students who, like the bulk of the third-graders, will turn away from introducing new methods when difficulty arises and seek support for the solution in the absolute value of the numbers (large or small numbers, whether they divide each other evenly, etc.). There turned out to be only 9 of these, out of 20 examinees, i.e., 45%.

The percent of the pupils who began trying habitual methods when difficulty arose increased somewhat (25%—5 pupils). It should be noted that some of these pupils accompanied the use of familiar model methods by a more detailed discussion, on the basis of which they sometimes modified the method themselves and found the way to solve the problem. Thus, Valya K. tried to solve Problem No. 13 (see above) in the same way as problems on proportional division were ordinarily solved (1000 : 2; 500 ÷ 40; 500 : 60). The impossibility of dividing told her of the incorrectness of the course she had chosen. Then she looked over her solution again.

> "We know," she said, "that 7 overcoats cost 280 rubles. All the overcoats cost 500 rubles...No...1000 rubles is the cost of all the overcoats and the suits. They are now in equal number, but they cost different amounts." [Valya found the mistake in her previous reasoning.] "So, fewer overcoats should be obtained, since they are more expensive than the suits. They all together cost 1000 rubles, and their quantity is the same. How much does one overcoat and one suit cost together?—100 rubles. How many pairs are there?"

From this account we see that Valya found the mistake in her previous reasoning and changed to another method of solution appropriate to the problem.[5]

It should be noted that problems similar to this one had not been solved in class before the experiment. Valya herself introduced a method similar to the model one and modified it to correspond with the given problem; thus she showed a high level of analytic-synthetic activity.

In comparison to these third graders, a group of fourth grade pupils isolated some new individual auxiliary methods of analysis and turned to them when difficulties arose in the solution.

> Sasha D. chose a visual diagram as an auxiliary method. He constructed such a diagram for any kind of difficulties arising in his problems. Here he had read Problem No. 20, and, having isolated the one datum contained in it, drew a sketch (Fig. 38).

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[5] Valya K., along with two other fourth-graders who were equipped with the ability to change the chosen method of solution on the basis of reasoning, were pupils of V. D. Petrova, 172nd School.

Fig. 38

Sasha connected the 1560 with both "departments." Having missed the little word "each," Sasha considered 1560 to be the total sum of money put in both departmental boxes. Re-reading the problem, Sasha now isolated those parts which refer to the sums of money removed. He was acquainted with the use of the letter "x" from the solution of examples; "x" signifies, as he knew, <u>any</u> unknown quantity, and he noted it in his sketch:

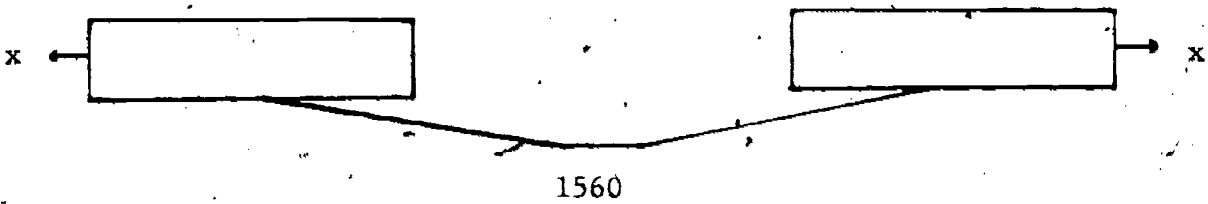

Fig. 39

Here both <u>x's</u> refer to a sum that had been removed (he did not differentiate them). "1560 rubles <u>each</u>" Sasha singled this out while continuing the analysis. "I thought it was in both!" He discovered his mistake and introduced the appropriate change in his sketch:

Fig. 40

He reread the problem in silence. There was a long pause. "Note the precise sum of money removed," suggested the experimenter. Sasha immediately took this hint, noted the correct sum removed from the first and second boxes:

Fig. 41

He then considered the sketch for a long time.

"What is being asked in the problem?" the experimenter reminded him. Sasha read and again considered the sketch and, finally, gave the correct answer: "They took out an entire department...1560 rubles was taken out of both boxes, and 1560 rubles remained."

We see that Sasha returned to the sketch himself, he made rapid use of a little help from the teacher, and, guided by the sketch, found the correct method of solution. We remember that a number of good third-grade pupils could not use very detailed help from the experimenter and could not make productive use of this method.

Although the use of the diagram was productive for Sasha in that problem, in both of the other problems that caused him difficulty (he solved two of the five assigned problems very easily), the diagram did not propel him onto the correct course of solution. He modified his diagram, trying to translate the problem's situation more precisely into it, but did not try turning to another method; more detailed help from the experimenter proved necessary.

When asked who had taught him to make the diagrams, Sasha answered: "Our teacher sketches like that in class." Indeed, the teacher of this class used this method extensively, and of the four subjects from this class, two singled out this method, and it became a method of independent thinking for them.

A fourth-grade pupil of the 172nd School (in V. D. Petrova's class) also singled out her teacher's favorite method as an auxiliary method--this was asking <u>analytic questions</u>. Thus, for Problem No. 12, Galya K. asked the following question at the start of the search for the course of solution: "Why did the laborers receive the same sum of money?" Later she tried to explain why a difference of 48 rubles was obtained and, finally, she asked the germane question, concentrating on the basic relationship of the data: "Why did the second laborer have to earn another 72 rubles?" Asking these questions made it easier to determine the unknown, making the search for a course of solution more precise. Wherever analytic questions did not simplify the search for finding a method of solution, Galya tried out known methods.

A fourth pupil (Kolya S.) used the diagram as his chief auxiliary aid (in solving five problems, he turned to this method four times).

Finally, we noticed one or two other methods alongside a chief auxiliary method in two pupils in the fourth grade (also V. D. Petrova's). The primary method used by both girls turned out to be one their teacher often used, the posing of analytic questions. But along with this method, both girls turned to the method of concretization, and even to the more complex method of variation (more complex since it presupposes a thorough understanding of functional relationship). This can be illustrated by the approach of Galya K. to solving Problem No. 14.

> A first woman put 8 logs into the common fireplace; a second put in 13, and a third, not having any logs, gave the first two women a ruble. How should the first two women divide the ruble?

In addition to asking analytic questions, she turned to the method of concretization. Finding the difference between the number of logs put on the fire by the first and second women, she asked this question: "Why do we need this datum? Why did I find it? One woman put in 5 logs more than the other. And if she had given these five logs to the other woman? Then 5 logs would have cost 1 ruble 50 kopeks." When she subtracted, she became convinced of the unproductivity of the chosen datum (she did not discover the way to determine the necessary connections) and sought another course of solution. This method turned out to be particularly productive in solving Problem No. 20. By subtracting arbitrarily chosen data, Galya found the unknown easily ("It will always be 1560 rubles," she said).

Sonya K., in addition to asking analytic questions, turned to diagrams of the problem (e.g., Problem No. 14); we also noted an attempt at using the method of variation. In solving Problem No. 12, she arbitrarily changed the number of days after which the laborers fulfilled the given amount of work. "If both laborers worked the same number of days, then the first would have earned 72 rubles more in the six days... The other earned 72 rubles in 6 days and 8 rubles more in one more day." From here she proceeded to determine the desired connections [72 : 8 = 9 (days)]. She also turned to variation in solving Problem No. 14, but she did not lead her supposition to its conclusion and turned to a different method instead.

Singling out the method of variation assumes that one is abstracting not only from the features of solving isolated concrete problems, but also from the features of solving concrete model problems--the most general feature of the entire group of model problems is dismembered (equalizing, exclusion, change, etc.)--the proposed change of the value of one of the given conditions exposes the functional ties between the data and the unknown. If this method is singled out, it turns out to be productive for many problems. Using this method shows a high degree of abstract thought. In the fourth grade we see this process only in embryo. For older pupils and adults the method of variation is one of the foremost auxiliary methods of analysis.

Thus, in the fourth grade, the number of good pupils who introduce different model methods when difficulty arises increases; pupils appear who have selected one or two auxiliary methods of analysis and who use them widely.

The experience of the pupils is increased still more during the 5th and 6th years of study. According to the curriculum, the pupils should master the solution of problems of all types and be able to solve quite complex arithmetic problems. Precisely those more complex problems which are solved by introducing not one but a whole series of model methods compose the great bulk of problems solved in the fifth, and especially in the sixth, grade. Corresponding to this, the teacher should also introduce an entire series of model methods while explaining these more complex problems. All of this serves as a basis for the pupils to select from a greater variety of both model methods and auxiliary methods of analysis.

Among the good pupils in sixth grade, we can notice a very small group (4 in 20 -- 20%) who, in searching for a solution to task-problems, depend on model methods, hardly using auxiliary methods of analysis at all. Here also, however, one can note an essential difference in their solutions, as compared with fourth-grade pupils. Due to the increased experience of these pupils, the number of model methods introduced for solving difficult problems increased. Thus, fourth-grade pupils used one or two model methods for solving Problem No. 12 and sixth-grade pupils used three or four. Their reasoning becomes more detailed.

However, most characteristic of good sixth-grade pupils is the singling out and the wide use, along with trials of model methods, of an entire series of auxiliary methods of analysis. This can be observed in

16 of the 20 pupils who took part in the experiment. Six of them showed a tendency to prefer using chiefly one or two methods, but this did not preclude the possibility of their introducing other methods.

Thus, when Victor I. came across a difficulty, he turned first to a diagram or a drawing; the method of concretization was particularly characteristic of Edika G., who also used a detailed discussion, ordinarily leading to finding the course of solution; and so on. The remainder (10 pupils--50%) used different model methods widely as a trial, as well as a number of auxiliary methods, crossing over from one to the other easily, until they found what was most productive in solving the problem for determining the connections between the unknown and the data.

Here is the record of solution of Problem No. 4 by Valerii A.:

> Fourteen m of wide braid and 9 m of narrow braid were bought. Six rubles 30 kopeks more was paid for the wide braid than for the narrow. What is the price of 1 m of wide braid and of 1 m of narrow braid, if it is known that 1 m of wide braid is 20 kopeks more expensive than 1 m of narrow?

> This problem turned out to be quite difficult for him, and therefore he did not find a way to solve it quickly. While reading through the problem, he immediately determined the basic method of solving it: "It will be necessary to make the quantities equal," he said, but used this method incorrectly at first. Having singled out the difference in price of one meter, he then figured out the difference of 14 meters and subtracted it from 6 rubles 30 kopeks. Having obtained 3 rubles 50 kopeks, Valerii realized that this datum is the cost of equal quantities of braid, i.e., corresponding to the method chosen by him at the start -- "3 rubles 50 kopeks would be the cost if the quantities were equal," he said (although actually making the prices equal, and not the number of meters). Then he made the concept of "equal quantity" more concrete: "Nine m of wide braid and 9 m of narrow," he continued, "and the wide braid is 20 kopeks more expensive than 9 m of the narrow." This repetitious isolation of 20 kopeks in the neighborhood of 9 m evoked a new synthesis: 9 m-- 1 ruble 80 kopeks. "For 9 meters there is 1 ruble and 80 kopeks difference, and I have 3 rubles 50 kopeks in all," and thus he compared both data. "What's the matter?" (He was bothered by the small sum -- 3 rubles 50 kopeks, which even he realized as the cost of 9 m of both kinds of braid). "I made the quantities equal," he continued, "and now I must make the costs equal. Then one can equate the prices and divide by the meters." He noted again in generalized form the future development of the course he had chosen, but he did not discover his mistake: 3.50 - 1.80 = 1 ruble 70 kopeks. "I equated the cost...9 m of one and 9 m of the other-- 1 ruble 70 kopeks." Now it was obvious that the

data obtained did not correspond (18 m of braid costs 1 ruble 70 kopeks), and Valerii turned to the problem's text. Rereading it, he exclaimed: "Aha! The thing is that the problem says it costs this much more and not just this much, as I had thought!" The mistake was found, but the course of solution was still not clear. Valerii silently began to draw, representing visually the differences in the number of meters (Fig. 42).

He explained: "There were 14 m and 9 m, and 20 kopeks extra." He repeated, "And there are still 6 rubles 30 kopeks. No, 3 rubles 50 kopeks." He has conveyed the problem in a more abstract form, isolating its basis and introducing the datum obtained earlier. "There are always 20 kopeks extra, and this increases to 3 rubles 50 kopeks"--he has determined the functional relationship, although he recognized the 3 rubles 50 kopeks incorrectly here (and therefore the connection itself is false). He represents visually the situation which he extracted and introduces precision into his drawing (Fig. 43).

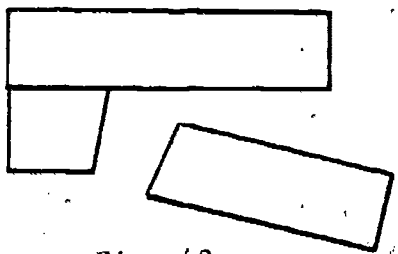

Fig. 42

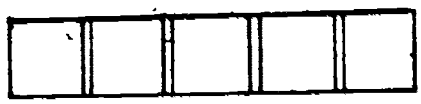

Fig. 43

The narrow rectangles here signify the repeated "extras" of 20 kopeks each.

"I don't understand," said Valerii, "why it increased by 3 rubles 50 kopeks for 9 meters? In 9 m they (i.e., the 'extras' of 20 kopeks) should only be 1 ruble 80 kopeks." By asking an analytic question precisely, Valerii determined the erroneousness of his reasoning.

"If at the start..." (He seeks a new course). "If these 20 kopeks are equated, only 5 m remain." And he finally isolates a very important datum, but is still unable to use it.

"Six rubles and 30 kopeks more, and not the cost of all the braid, 3 rubles 50 kopeks does not correspond to these 20 kopeks...." Not seeing the procedure, Valerii turns to the method of analogy: "I have to take an easier example of a similar problem," he said. "Ten apples of the best and 10 apples of the second quality. The former are 20 kopeks more expensive. If we took 9 portions and another 9 portions, increased by 20 - 9 + 3.50 - 9 x 20"--he crosses over from the analogy to the original problem, while he tries to introduce the method of portions (according to the type of problem on sum and multiple relationship), but disregards it, not seeing a way out, and rereads the text. "What did I equate?" he asks himself an analytic question. "I equated the qualities. There were 14 m of the wide, and now we can consider everything to be narrow, and hence 23 m of the narrow." He has produced an auxiliary synthesis and, finally, found the right course: "Five m difference. Nine m cost more than the 14 by 3 rubles 50 kopeks. And that's all," he announced happily, without even carrying out the subsequent calculations -- the rest of the procedure is clear.

The record introduced here (and it is characteristic of the other subjects in this category) testifies to a high level of analytic-synthetic activity in solving arithmetic problems. The method of equating chosen by Valerii on the basis of the solution is very generalized (equating in general, arbitrary data). This high level of generality allows it to be used more widely (he attempted to equate at first the quantity, then the cost, etc.).

He had selected a whole series of auxiliary methods along with the trials of the model methods. He turned to the auxiliary methods when difficulties arose. In solving the given problem, Valerii introduced the methods of concretization, abstraction, analogy, diagrams, and others, as we saw. The use of a method helped Valerii to make his analysis more thorough and precise, to discover mistakes and thereby to come closer to finding the unknown. Both the model and the auxiliary methods became methods of creative thought for him. Active enlistment of trials of model methods as well as of auxiliary methods of analysis in seeking a solution, and ease in transferring from one method to another can be observed in all of the pupils in this category.

To see more clearly the basic trend in isolating the methods of analysis by the pupils, see Table 8. In it we have shown the distribution of good third-, fourth-, and sixth-grade pupils[6] (the ones used in the experiments) according to groups, depending on the methods of analysis they chose.

We see that a very small number of third-grade pupils turned to the trial of model methods when difficulty arise. In fourth grade the best pupils used isolated auxiliary methods along with the model methods. The

[6]The fifth-grade pupils occupied an intermediate position between the fourth- and sixth-grades. I noticed nothing basically new in the records of their solutions. Therefore I do not introduce their data here.

great bulk of the good sixth-grade pupils actively used an entire series of auxiliary methods when difficulty arose.

TABLE 8

Use of Methods of Analysis by Pupils of Grades 3-6

Grade	3		4		6	
Level	No. of pupils	%	No. of pupils	%	No. of pupils	%
I. Do not use either trial of model methods or auxiliary methods	6	80	9	45	—	—
II. Search for the solution by trial of model means	4	20	5	25	4	20
III. Use chiefly auxiliary methods	–	–	4	20	6	30
IV. Use a number of auxiliary methods along with trials of model methods	–	–	2	10	10	50

The comparison of the records of solution of arithmetic sum-problems by excellent students from six grades and by adults has shown that there are no essential differences. This might be because the direction of further development does not proceed towards improving arithmetic methods of solving problems, but towards the mastery of new algebraic methods of solving them. These methods require a higher level of analytic-synthetic activity and are much more rational. It is natural that once having mastered them, older pupils and especially adults will use these methods in solving problems. Requiring that a problem be solved by arithmetic methods causes difficulty for many of them. Thus, in sixth grade (at the moment of completion of the arithmetic curriculum), the mastery of arithmetic means of solving problems (and, consequently, of analyzing them) attains its highest level.

Thus, the investigation showed that the isolation of auxiliary means of analysis and the transformation of these into means of creative thought is a complicated analytic-synthetic process.

At the first stages of learning, when the pupils are still inexperienced in solving problems, even good pupils will not single out auxiliary methods of analysis: They cannot use them even with a direct hint from the teacher (or the experimenter). With few exceptions, they do not turn to the trial of model methods when seeking a solution. The use of one method for a given problem precludes the possibility of using another method, which is evidence of a significant stagnation in forming systems of connections.

With the broadening of the pupils' experience, they begin to introduce various model methods of solving problems at the start on the basis of coincidental, sometimes inessential, elements of similarity between the problem being solved and the corresponding model.

In the future this trial of model methods in conjunction with a higher level of analysis leads to the modification of the most suitable model method corresponding to the situation of a specific problem. At this level the model method itself is included in a more abstract form, which is a result of the high degree of abstraction from the specific features of the original problem. The pupils develop a quite mobile, dynamic system of connections (corresponding to the isolated methods of solution), and inter-system connections arise between them, guaranteeing the possibility of transferring from one system to another.

Auxiliary methods of analysis are enumerated and mastered, along with the isolation of the model methods. At first these are isolated methods; later they are an entire series of methods used widely in seeking a course of solution.

In solving a group of model problems (equations, transformations, etc.) good pupils single out their common elements—arbitrary (in the order of a proposition) changes of the data contained in the problem. Abstracting themselves from other model features of the solution of these problems, they use this general sign extracted by them as the basis of an auxiliary method of variation. Other methods (graphic analysis, etc.) are chosen by the pupils from the methods the teacher has used in the breakdown of difficult problems.

5. Developing Auxiliary Methods with Weaker Students

Independent isolation and mastery of auxiliary means of analysis in classes where it is not especially taught turns out to be beyond the powers of the best pupils. The weaker pupils even in the higher grades (6-7), as the investigation showed, were not in a position to isolate and master these methods by themselves; in the best conditions they used the trial of different model methods. (without modifying them).

Can weaker pupils master the methods of analysis if the teacher especially trains them? What difficulties arise then? How is the mastery of these methods reflected in the independent solution of problems?

I conducted a training experiment with a small group of pupils, to investigate this area somewhat. The three poorest pupils (2's in mathematics) in grade 4B of the 64th School in Moscow were chosen for the experimental study. Sixty lessons were conducted with them, from October through March. In the second quarter they received passing marks, and they passed their examinations well.

One of them, Zhora, was a quite diligent, quiet, but slow boy. Although he did not show any great interest in mathematics, he could study it for a long period. To understand a new problem, he needed an extremely detailed explanation, along with an orderly transition from the visual situation (sometimes even with the elements of dramatization) to the generality. He frequently tried simply to remember the course of solution of problems, and he often recalled the solution of similar problems.

The other two pupils, Felix and Tolya, had much in common: They were quite quick-witted, especially Tolya; they were proud, lively, easily excitable, with a very unstable attention span. When they had a personal interest, they learned the material comparatively easily, and searched for the course of solution themselves, but they were very much given to rapid, superficial conclusions. As soon as the personal interest declined, serious difficulties arose in the solution, and they put aside the work easily, or else they began to solve by manipulating the numerical data, adjusting the solution to fit the answer (if it was known). The basic reason for their poor progress, obviously, was their very unstable attention and intellectual passivity.[7]

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[7] On this type of poor progress in pupils, see the article by Slavina [37].

To avoid this passivity and to force them to work harder, we had to introduce some elements of play in the first lessons. Each pupil acquired a form (a "table," as they called it). An answer to a question or the solution of an explanation of the method of solving a problem was evaluated with marks; each of them entered these marks in the "table" and added them at the end of the lessons; good answers increased the number of marks, and bad ones were subtracted. By the sum of the marks it was possible to judge who had done the best during the lesson.

This system of marking, by introducing competition, forced the pupils to work more actively and at the same time not to answer hastily with ill-considered answers; the boys began to work more attentively. Gradually an interest in the subject itself was awakened in them, as well as an interest in solving rather complex problems, and the necessity for the "tables" disappeared.

The first lessons showed that all of these pupils were on the level of a seldom-productive "elemental" analysis. They isolated first the numerical quantities from the text of a problem and combined them, guided primarily by the comparison of their absolute values (their multiples, etc.) and secondarily by the presence in the text of isolated words which, in their experience, were closely connected with definite arithmetical operations. For example, the selection of the word "more" always evoked an attempt to carry out addition or multiplication; if both numbers were large, the pupils ordinarily chose addition; if one was small, multiplication was usually chosen. The selection of the word "less," by the same token, led to the choice of either subtraction or division. It is interesting to note that wherever there was some conflict between the numerical data and the verbal components of the problem, the first always won out. Thus, the numerical data 25 and 5 were isolated, as well as the words "greater than." The number 25 was easily and usually divisible by 5, and the words "greater than" required multiplication--so all three boys chose division. Here, obviously, the habitual, well-established combination of numerical data was active, and the more secure stereotype went into operation.

In solving problems these pupils tried not so much to understand the essence of the operations as to memorize their sequence. "I understood," Tolya consoled himself somehow at one of the first lessons. "First you subtract, then divide and multiply."

Arithmetic problems were examples to them, where one had to carry out certain arithmetical operations with numbers. They did not try to imagine the real, vital significance of the data in the problem. In considering a problem, they could easily assert an airplane's speed as 8 km/hr., and a train's speed as 9 km/hr., and the like. Nevertheless, in a conversation not connected with solving problems, the boys could name the characteristic speeds of these vehicles. In choosing data for problems they composed, they simply did not think about their real significance; they chose numbers with which it would be easy to operate when carrying out the arithmetical operations (division, multiplication, etc.)

Mastering the purely superficial aspect of the solution fairly easily, the boys, especially Tolya and Felix, learned how to solve one type of problem rather quickly. But what vague conceptions were sometimes covered by these superficially correct solutions!

For example, at one of the first lessons the following problem was assigned:

> From two communities situated 84 kilometers apart, a horseman and a pedestrian were sent out in the same direction. How much later did the horseman overtake the pedestrian, if the pedestrian travels at 6 km/hr. and the horseman at 13 km/hr.?

This type of problem was being solved at that time in class, and the boys quite adeptly carried out the appropriate operations:

1) $13 - 6 = 7$ km.

2) $84 \div 7 = 12$ hours.

To clarify to what extent they had realized the course of the problem's solution, they were required to draw the distance between the cities and note the paths of the horseman and the pedestrian (diagrams had already been used in connection with solving problems of motion).

Tolya did the problem in the following manner (Fig. 44). In spite of the teacher's direct instructions, both the pedestrian and the horseman (judging from the drawing) left from the same point, and the sense of the problem was distorted.

Felix omitted the distance between the pedestrian and the horseman, and it was too small and did not correspond to the distance of 84 km which he indicated (Fig. 45).

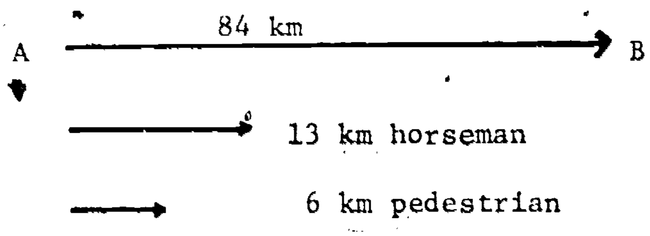

Fig. 44

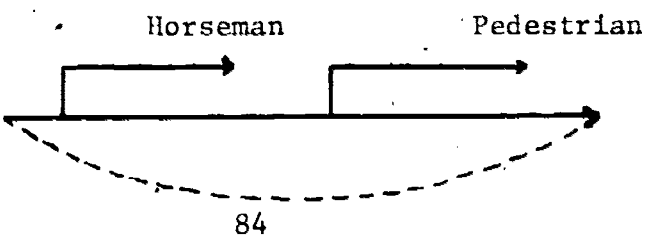

Fig. 45

Zhora, introducing a line, marked off 13 km on one end and 6 km on the other (hardly any shorter) (Fig. 46).

Fig. 46

"And where is the 84 km? Mark it off," the experimenter requested, and Zhora added it (Fig. 47).

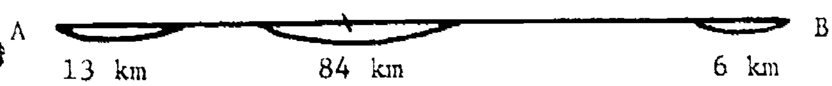

Fig. 47

All three drawings show very clearly that in solving the problem the pupils did not imagine the real situation described in it, and they carried out only the familiar order of the arithmetical operations.

The choice of the arithmetical operations, as was noted above, is sometimes determined by the nature of the numerical data. Thus, after solving a series of analogous problems, the following was assigned:

> One meter of satin costs 5 rubles, and 5 meters of satin cost as much as 4 meters of linen. How much does 1 m of linen cost?

Logically, the solution should be carried out by the following operations:

1) 5 X 5 = 25 rubles
2) 25 rub. : 4

But 25 "is not divisible" easily by 4 (one ruble must be expressed in terms of kopeks). From the standpoint of ease of performing the arithmetical operations, these data are more easily combined in this way:

5 X 4 : 5;

and the latter prevailed; all three, in spite of the logic of the solution, combined the data in just this way. The "hypnosis" of the numerical data proved stronger than common sense.

All three pupils did not realize the real significance of the question of the problem, did not consider the problem itself as a question whose answer might be obtained on the basis of the data contained in it. The course of solution of the problem was largely dependent on the nature of the preceding problems. Varying the question of the problem did not cause a corresponding change in the arithmetical operation in them. If they chose addition in the preceding problem to answer the question, "How many were there in all...," in the next problem they would also add, although the problem might require finding how much greater a certain datum was than another (i.e., a certain stagnation occurred, a striving for a pattern).

For the question "How far did the train travel?" Tolya thought up the following "problem": A train covered 240 km in 6 hours," including in the "problem" precisely what it should have asked.

For the question "What is the airplane's speed?" Zhora thought up this "problem": "The airplane covered 32 km in 4 hours (!). Its speed is 8 km." Here too, as we see, the unknown is included in the text of the problem; it is known.[8]

This sluggishness and striving for a pattern are easily combined with an ease in transferring the method of solution from one type of problem to another on the basis of random, sometimes solitary signs of similarity in the situations of the problems. The presence in a problem of a multiple comparison (so many times bigger than...) for example, is a basis for transferring the solved problem to problems on "portions."

The nature of the preceding problem has an especially great influence here. The pupils solved this problem on sum and difference:

> There were 300 books in 3 boxes. In the second box there were 40, and 20 books more in the third than in the first. How many books are there in each box?"

Then the following problem was assigned:

> There were 420 books on 3 shelves. On the first there were 100 books, and 50 more on the second than on the first. How many books were there on the third shelf?"

All three of them solved it in this way:

1) 100 + 50 = 150 books.
2) 420 - 150 = 270 books.
3) 270 : 3 ... etc.,

i.e., as the preceding problem on sum and difference had been solved.

A low level of analytic-synthetic activity, in combination with a certain sluggishness and striving for a pattern (which is determined by the presence of inflexible systems of connections), seen in the pupils chosen for the experiment, is very typical for pupils who are progressing slowly in arithmetic, and therefore it was described here in some detail.

Much painstaking work was required to teach the pupils to make an attentive breakdown of problems and to analyze them productively.

First it was necessary to show them the function of the question of a problem, to show the relationship of the nature of combining the data from the final question, and to dissuade the pupils from "superfluous synthesis," unwarranted by the situation of the problem. A very positive role was played by trick problems in this. The following problem, for

[8]Menchinskaya [15] noted similar examples with first-graders.

example, was assigned:

> Three bicyclists covered 30 km in 3 hours. How long would it take one bicyclist to cover the same distance?

Zhora immediately divided 30 by 3, and Tolya started to multiply 30 by 3. Only Felix at first was silent, then smiled and did nothing, explaining to his classmates what their mistake was.

Such trick problems, offered along with regular ones, forced the pupils to remain on guard (they did not want to be caught in a ridiculous situation), and showed them that there is a definite relationship between the data and the question of a problem. Problems with insufficient and superfluous data forced them to turn their attention to the analysis of the data, their evaluation, and the choice of the necessary data to correspond with the final question.

In order to emphasize the importance of the question still more (it is the question, the unknown, which determines the direction of the analysis), much attention was given to the choice of different questions for the same problem (the method of varying the question). The pupils were convinced that a change in the final question also involves a change in the nature of the arithmetical operation; with a change of the unknown some datum might be superfluous, and some necessary data might not be sufficient. The pupils themselves composed problems to go with a question, or they chose a question to go with data, and then, changing the question, introduced the appropriate change in the problem.

A problem now appeared to the pupils to be a single whole, all parts of which have a special significance and are tied to each other by a principle. They became convinced that manipulation of isolated data, taken at random from the problem, does not lead to the desired goal. They began to be trained to evaluate the data and their possible combinations from the standpoint of their significance for finding the unknown.

A method of schematic notation of the problem, to simplify its analysis, was demonstrated to the pupils. This method is useful where the text of the problem is cumbersome, where it is difficult to arrive at the relationships between the data. A standard form for writing down the problem was not given. The pupils were asked to find the most reasonable one themselves for the given problem. Thus, the following problem[9] was given:

> 12,000 cubic meters of firewood were brought in: 4000 cubic meters of it were birch, and 2000 cubic meters more were aspen, and the rest were spruce. After some time, 3000 cubic meters of the aspen logs had been sold, and 2500 cubic meters of the spruce. How many cubic meters of each type of firewood remained?

With the teacher's help, the following chart was made:

Firewood	Brought in	Sold	Remained
Birch	4000 cubic m.	?	?
Aspen	birch + 2000 cubic m.	3,000 cubic m.	?
Spruce	?	2500 cubic m.	?
Total . .	12,000 cubic m.	?	?

This chart helped them to compare the known and the unknown data more easily and simplified finding the course of the solution.

Sometimes this type of diagram was supplemented (at the teacher's suggestion) with a sketch. For example, in the problem:

> It is known that the area of Lake Baikal is 34,000 square kilometers, and the area of the Aral Sea is 30,000 square km bigger than Lake Baikal, and that the area of the Caspian Sea is 335,000 square km bigger than the areas of Lake Baikal and the Aral Sea together. What is the area of the Caspian Sea?

The pupils made the following diagram:

I	Area of Lake Baikal	-- 34,000 square km
II	" of the Aral Sea	-- area of Lake Baikal + 30,000 sq. km.
III	" of the Caspian Sea	— area of Lake Baikal + area of the Aral Sea + 335,000 square km.

What is the area of the Caspian Sea?

[9]The problem was taken from the standard textbooks and problem books of Ya. F. Chekmarev and others.

However, this is cumbersome and does not give a clear representation of the relationship of the quantities. Another sketch was made (Fig. 48).

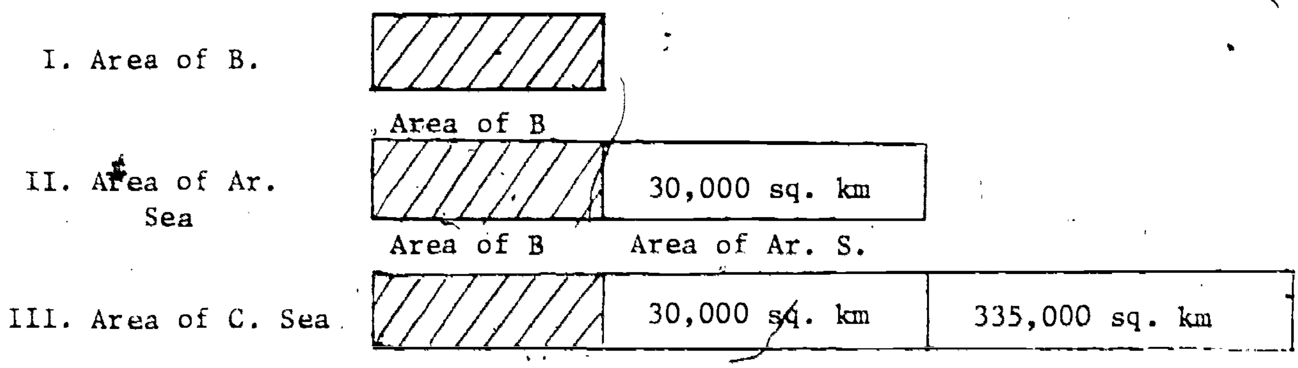

Fig. 48

In constructing this type of scheme the pupils carried out the necessary analysis, and the method of solution became clear to them.

The pupils very quickly recognized the value of the sketch, and they began to turn to it themselves when difficulties arose. "If you write it down better, then it's easier" is the conclusion which Felix drew, and he himself sought a more expressive diagram.

At first the following problem was difficult for him:

> A store was supposed to receive 115 kg of boiled butter at 18 rubles per kg and 135 kg of cream butter, the total coming to 4770 rubles. But only 75 kg of boiled butter was delivered, and all the remaining money was spent to buy the cream butter. How much cream butter did they buy?

At first he wrote down the conditions thus:

115 kg of boiled butter at 18 rubles per kg
135 kg of cream butter — 4770 rubles
75 kg of boiled butter

This notation did not satisfy him, and he himself attempted to improve it, without any coaxing from the side:

1 kg at 18 rub--115 kg of bld but. 75 kg
135 kg of cream butter — 4770 rubles

Then, with the teacher's help, the diagram was made precise:

115 kg of bld butter at 18 rubles — 75 kg
135 kg of cream butter at ? — ? — 4770 rubles.

For finding a method of solving less complex problems, this type of visual analysis is sometimes sufficient (with the help of diagrams and sketches).

In more complex problems, when it was difficult for the pupils to separate the relationship basic to the solution between the unknown and the data, they had to turn to a much more detailed visual-schematic analysis of the problem.

I introduce an example of the analysis of this type in the solution of the following problem:

> Two bags of potatoes were bought. In the first there were 30 kg more than in the second. Forty-two kg were taken from the first, and 10 kg from the second. In which bag were there more potatoes, and how much more?

At first the conditions were written like this:

bag 1-- 30 kg more	42 kg
bag 2-- "	10 kg

In which bag were there more potatoes and how much more?

Then it was depicted schematically:

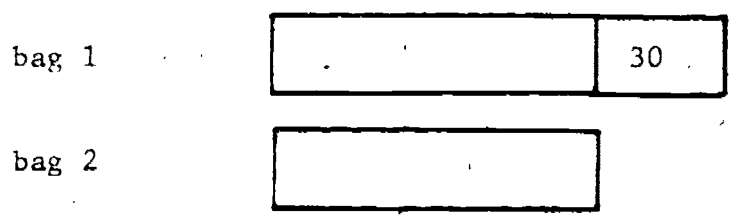

Fig. 49

Then they introduced the second part of the problem into the scheme:

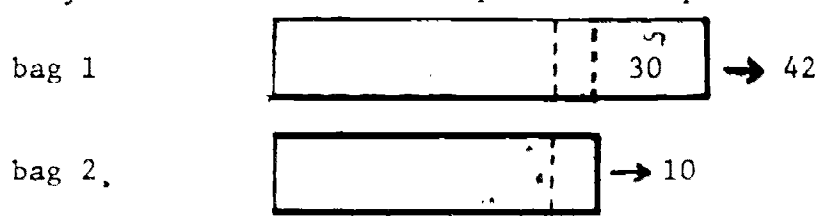

Fig. 50

However, the pupils could not relate the unknown to the data on the basis of these diagrams. It was necessary to turn to a more detailed articulation. At the teacher's instructions, the pupils returned to the original data, and, noticing the 10 kg removed from the second bag, carried out an auxiliary synthesis, elucidating what difference there was then between the bags:

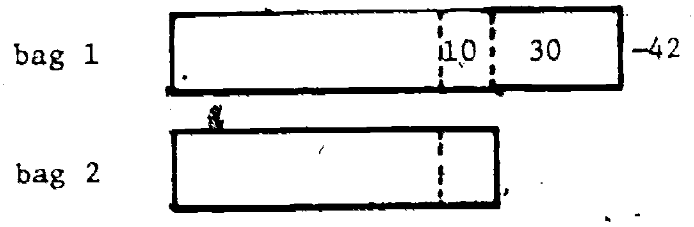

Fig. 51

Then they turned to the 42 kg removed from the first bag. What was it composed of? They considered this sketch:

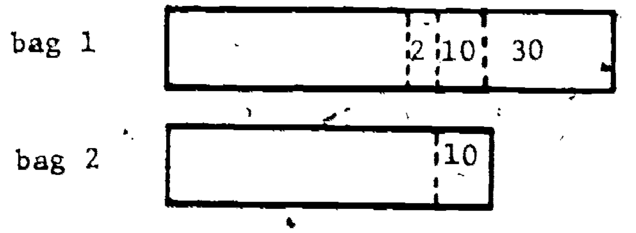

Fig. 52

In considering the sketch, the pupils determined that the numerical datum 42 kg includes these superfluous 30 kg, and 10 kg more, and 2 kg more. Only after all the indicated portions had been noted on the sketch did the pupils who were considering it find the necessary relationship between the unknown and the data and determine the value of the unknown.

Work was begun to make the analysis of the data and the functional ties between them more thorough.

The significance of each concept in the text of a problem was explained in detail, and visual material was introduced wherever necessary. Much attention was given to words expressing the relationship between the data (so much bigger than, so many times smaller than..., many similar ones).[10] After solving a problem containing, say, the words "so much bigger than...," the pupils substituted the words "so many times smaller than" and explained how this substitution was reflected in the method of solution. This type of practice taught them to treat each word in a problem carefully.

Much attention was devoted to the elaboration of the proper conceptions and the reinforcement of the knowledge of the functional relationships between the data. We shall show how this work was conducted by

[10] More details on work with words in solving problems can be seen in the article by Bogolyubov [4].

an example of solving a problem on motion.

A simple problem on motion was chosen:

> A train traveled 100 km at a speed of 25 km/hr. How long did it take to cover this distance?

The pupils wrote down the data:

Distance -- 100 km

Speed -- 25 km/hr

Time -- ?

After explaining what the distance, speed, and time were (some explanation of these concepts had been given earlier), we began to compose a table where the known quantities were marked by a plus sign, and the unknown ones by a question mark. In this problem the distance and speed were known, and this was noted in the table:

Distance	+
Speed	+
Time	?

We then explained possible variants of this problem, and we introduced the appropriate signs into the table:

	I	II	III
Distance	+	+	?
Speed	+	?	+
Time	?	+	+

The other types of problems were thought up by the pupils themselves, and then they were asked to read through a number of problems on motion in the workbook and to group them according to the table. This type of exercise is useful not only for reinforcing the knowledge of the given functional relationship but also for broadening the content and scope of the pupils' mathematical concepts. In making this grouping of the problems, the pupils should subordinate a narrower visual concept to a wider, generic one. Thus, in the various problems, seconds, minutes, hours, and sometimes days are referred to. The pupils, in noting the unknown and the known data in the table, should place these different concepts under the generic concept of "time."

After a sufficient number of exercises, a generalized conclusion can be drawn about how to find the time when one knows the distance and the speed, about determining the distance by the time and the speed, about what data are necessary for determining the speed, etc. That is, the knowledge of functional ties is reinforced.

How one datum changes with respect to changes in another is also clarified. To do this we leave one of the data unchanged and, giving an arbitrary value to another datum, determine how this increase or decrease is reflected in a third datum. For example, we give a train a constant speed of 20 km/hr and explain what distance it covers at this speed in 6, 7, 8, etc., hours. We introduce the appropriate data into the table:

Speed	Time:	5 hours	6 hours	7 hours	10 hours
20 km/hr.	Distance:	100 km.	120 km.	140 km.	200 km.

We conclude that with an increase of the time in motion, the distance covered also increases, and that this is a direct relationship.[11]

"And what will happen with the time if we change the speed but the distance stays unaltered?" We pose the question and begin to seek an answer to it. This is an inverse relationship, and it is much more difficult for the pupils; therefore we are guided by an auxiliary sketch while composing the table:

Distance --300 km

	Train				Airplane		Bicycle
Speed (km/hr)	20	30	50	100	150	300	10
Time (hrs.)	15	10	6	3	2	1	30

Speed 150 km/hr.

x ———— 1 hr. ———— x ———— 1 hr. ———— x 300 km

Speed 100 km/hr.

x ——— 1 hr. ——— x ——— 1 hr. ——— x ——— 1 hr. ——— x 300 km

Speed 300 km/hr.

x ———————— 1 hr. ———————— x 300 km

etc.

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[11] The concepts of direct and inverse relationships I have already introduced and consolidated.

Again the generalized conclusion of the inverse relationship is drawn here.

Passing on to more complex problems (on meeting), we constructed the work according to the same plan: We explained that here, in contrast to the other previous problems, two speeds are given, corresponding to the two moving objects; the distance traveled by each object separately can also be determined. Again we explain the possible variants of the problem; we may think them up or select them from the workbook.

The same type of work was done on another type of quantities that were functionally interrelated. This work created a basis for the use of the methods of variation, so widely used by adults.

Analytic questions were used extensively in the analysis of problems, calling the pupils' attention to explaining the basic interrelationships between the data, to exposing the basic link of the problem. Thus, the question, "Why can the express train catch up with the passenger train which left earlier?" directed the pupils' attention to the difference in the speeds of both trains and simplified finding a way to solve the problem. With this type of question the pupils were taught to isolate the main part of the problem, its "kernel."

I have made only the first steps in training pupils to use auxiliary methods which simplify seeking for a method of solution of more complex task-problems. The results show the possibility, in principle, of reaching a level where not only the best but even the weaker pupils can master these methods with special training. However, we attained this only under the conditions of the experiment, with individual training.

Can this type of method be taught in the ordinary conditions in class? To verify this possibility, experimental instruction in one of the auxiliary methods under classroom conditions was done by the teacher A. E. Kozlova[12] (132nd School in Moscow).

As an experimental method, we chose the method of asking the analytic question "Why?" recommended by the pre-revolutionary methodologist Latyshev. A. E. Kozlova (teaching second and third grades) asked this question every time a problem's structure permitted it, and turned the children's attention to the fact that this question made it easier to understand the problem.

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[12] Her work has been published in Roads to Improved Progress in Mathematics (see [4]).

Later, in third grade, the teacher asked the pupils to ask a "Why?" question themselves when they had isolated appropriate data. The answer to this question helped the pupils to disclose the relationship between the unknown and the data. Such work was done regularly.

To what extent did the pupils master this method? Were they able to ask this question independently? Does this simplify the search for a method of solution? A controlled experiment was conducted at the end of the year to answer these questions. The pupils were given a problem on exclusion of one of the quantities, whose method of solution was not familiar to them, as a trial experiment showed. This problem was asked individually of seven pupils in the class. Five of them, having asked the question, "Why was more paid for the second purchase?", could then find the correct method of solution, using this question. The other two, having asked the question, could not. For them this method had still not become a method of creative thought. We can still maintain that Kozlova's experiment showed the possibility in principle of teaching, under classroom conditions, methods which simplify finding a method of solution of more complex task-problems.

In the investigation, only some of the possible ways of teaching methods of working on problems have been noted. Undoubtedly, there are other, perhaps more productive, methods being used in the work of the best teachers. We must disclose these methods and teach how to use and popularize them.

With respect to each method investigated, it is necessary to determine at what age level (in what grade) and with respect to which group of problems the method can be used, and how to teach the method so that it will become an independent method of thought for the pupils.

It would be particularly desirable for the teachers themselves to study the investigation of the problems indicated here. Combining the researchers and the teachers into one body makes the work more productive, answering the inquiries of practice indirectly.

The author hopes that her research will help teachers to regard their methods of teaching problem solving critically and to address themselves to seeking the most productive methods of breaking down problems.

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Published by the School Mathematics Study Group, Stanford University, and available from A. C. Vroman, Inc., 2085 E. Foothill Blvd., Pasadena, California 91109.