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ABSTRACT

This study guide, designed for use at Oklahoma State University, contains lists of activities for students to perform based on the "mastery of learning" concept. The activities include readings, problems, self evaluations, and assessment tasks. The units included are: Lines in a Plane, Conics, Transformations, Polar Coordinates, 3-Dimensional Analytics, and Parameters. (MK)

6E MATH 1813 (PIPI)
ANALYTIC GEOMETRY

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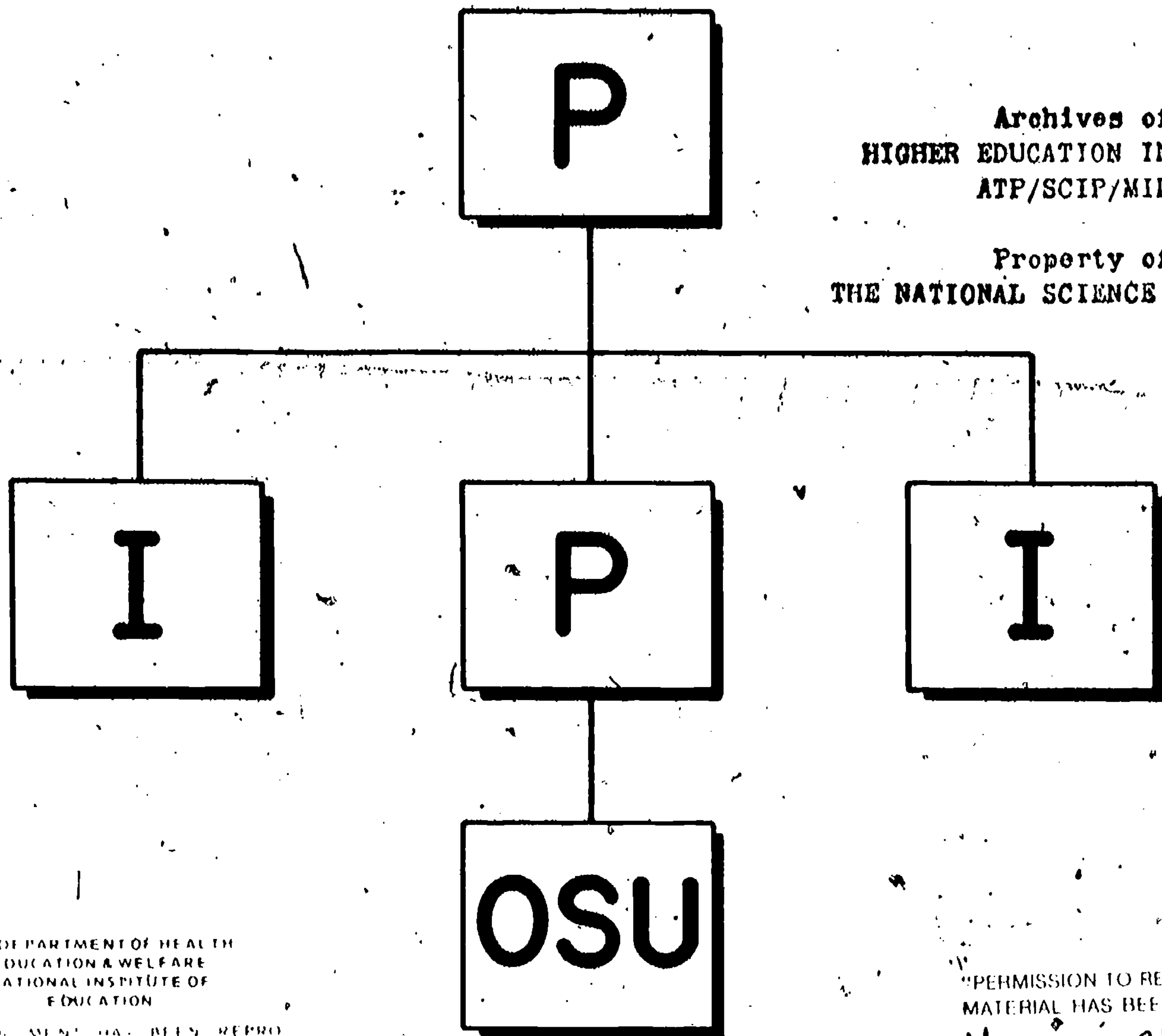
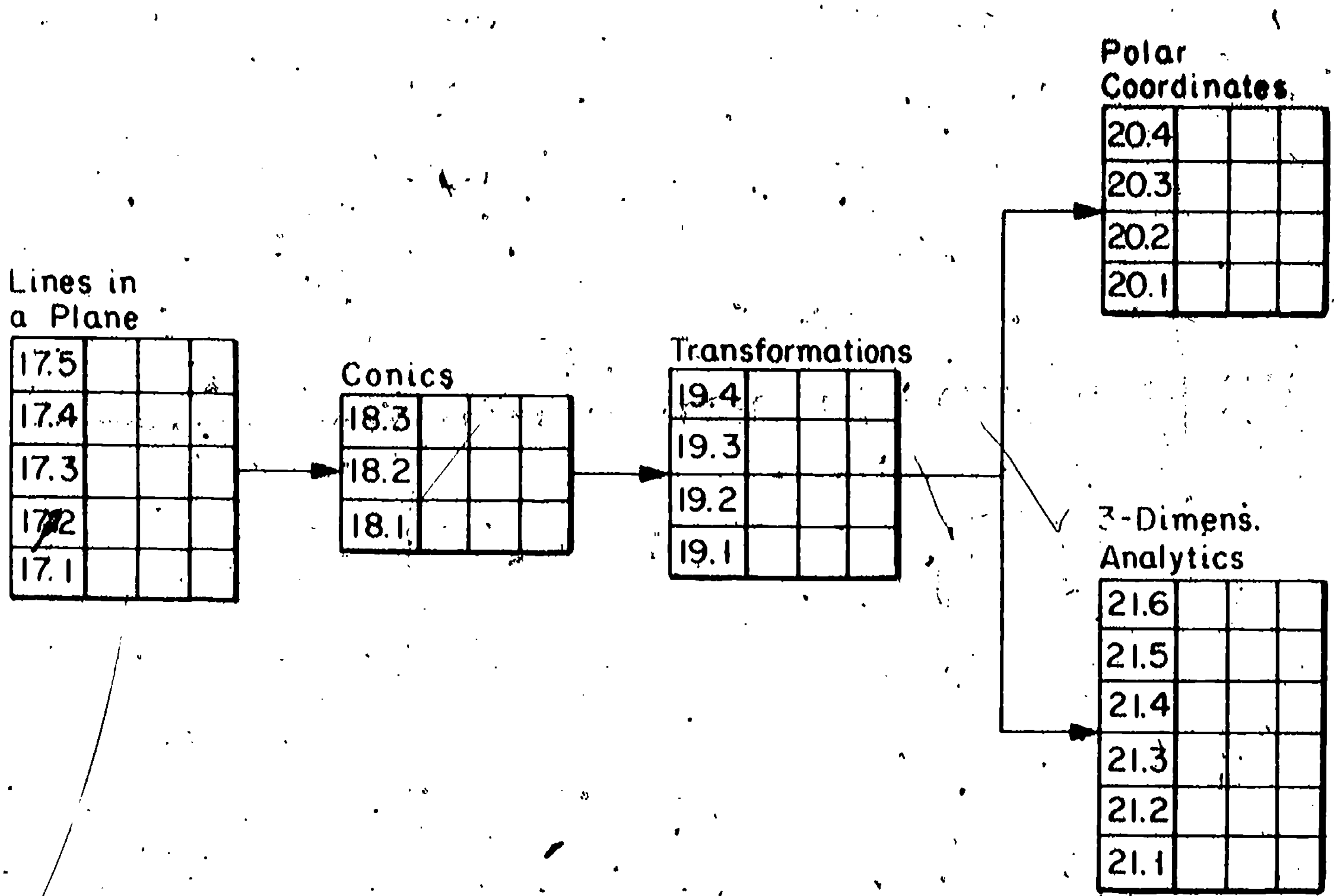


MATH 1813

SE 029 357

MATH 1813 (Analytic Geometry)



Study Guide

Unit 17

Line in a Plane

Unit 17 -- LINE IN A PLANE

(Contains Subunits 17.5 Through 17.1)

Rationale

By the end of this unit, you will be concerned with being able to do two things, (1) constructing the graph and interpreting it when you are given the equation of a line, and (2) when given some conditions or data, determining the equation of the line that satisfies those conditions.

An old Chinese proberb says "A picture is worth a thousand words". So the picture or graph of an equation is very valuable -- if you are able to use that picture to interpret information depicted by it. One of the pieces of information you should be able to determine from the graph of a line is the slope of the line; another would be the x and y intercepts of the line.

Since the line is a mathematical model for many physical phenomena, it is advantageous to be able to describe, in terms of an equation, that phenomena. For instance, in electronics, any electrical device whose I vs E curve exhibits the property that the ratio of the current through the device to the voltage across it is a constant (the graph is a line) is a linear resistance. In physics, Hooke's Law states, "The elastic deformation (of a spring) is directly proportional to the force if the elastic limit is not exceeded." Hence the graph of load vs elongation is linear.

This unit is broken into 5 subunits -- one of each dealing with points, line segments, and vectors and 2 dealing with lines.

Objectives

17.5 a) Write an equation in any of the requested standard forms (slope-intercept, point-slope, etc.) of the line which satisfies a given set of conditions and sketch the graph.

b) And given an equation of a line, determine the following

1) x and y intercepts
2) slope
3) direction numbers
4) 3 points on the line and the graph of the line

17.4 Determine the cosine and tangent of the angle(s) formed by the intersection of two lines.

17.3 Perform operations on plane vectors and analyze the results.

17.2 Determine the direction cosines of a directed line segment, and determine coordinates of a point that divides that segment into a given ratio.

17.1 Plot points in a rectangular coordinate system.

Prerequisite

Competency in Units 6 through 11 and Unit 14 (in particular objective 14.4 -- Solve a system of three linear equations using Cramer's Rule); or the equivalent of College Algebra and Trigonometry. Chapter one of Morrill, W. K., Analytic Geometry, is a good source for review material for trigonometry and linear algebra.

Unit Activities

Lectures 1 and 2

The lectures over this unit will have the following outlines:

1. The line segment and vectors
   A. Determining for the line segment
      1. scalar components
      2. distance between 2 points (length of segment)
      3. direction cosines
      4. coordinates of point that divides a segment into a given ratio
   B. Vectors -- operations and analysis
      1. addition
      2. scalar multiplication
      3. dot product
         a. parallel and perpendicular vectors
         b. cosine of angle between 2 vectors
      4. bar product and area of triangle

2. The Line
   A. Determining cosine and tangent of angle(s) formed by intersection of 2 lines
   B. Determining slope, intercepts and the graph from a linear equation
   C. Writing equation of a line

Lecture 1 covers objectives 17.2 and 17.3

Lecture 2 covers objectives 17.4 and 17.5

Procedural Options

This unit is important not only because it is a prerequisite for the later units, but also because of the work habits you develop for this unit. If you develop good work habits in this unit, you will probably continue with those habits throughout the course.

There are several options available to you in order to meet the objectives of this unit. Some of these are listed below.

Option 1.

Select a device or devices from the following chart which you think might help you acquire the skills and concepts for each objective beginning with 17.1 below. After you have completed an activity package for each objective, take the self-evaluation. If you score 100% and you are confident that you have mastered the objective, proceed to the next objective and continue until objective 17.5 is completed. You are now ready for the assessment task which measures all objectives 17.1 through 17.5.

Option 2.

Study the first objective carefully. If you feel confident take the self-evaluation for that objective. If you scored 100% go on to the next numbered objective and repeat this sequence until you are no longer familiar with the material or until you score less than 100% on the self-evaluation. Begin with the activity package in the objective in which you failed to meet the objective.

The average time for Unit 17 is about 3 weeks.

Objective 17.1

Plot points in a rectangular coordinate system.

Activities 17.1

(Most students should be able to take the self-evaluation immediately, if not, activity 1 should be sufficient.)

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 19-21.
   Exercises; pp. 11-12, problems 1 - 7, 12, 13; and 15.

2. Your Text and the Study Guide (Located at the end of the unit activities.)

3. Other Reading Sources

   1. Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, page 1.
   2. Eulenburg and Sunko: Introductory Algebra -- A College Approach, pages 173 - 178.

4. Individual Assistance

   Your instructors are available daily for individual help.

5. Informal Group Session

   The project room is a good place to study with other students in this program. Get acquainted with your fellow students.

Self-Evaluation 17.1

1. Draw and label the coordinate axes (right handed frame) and locate the following:

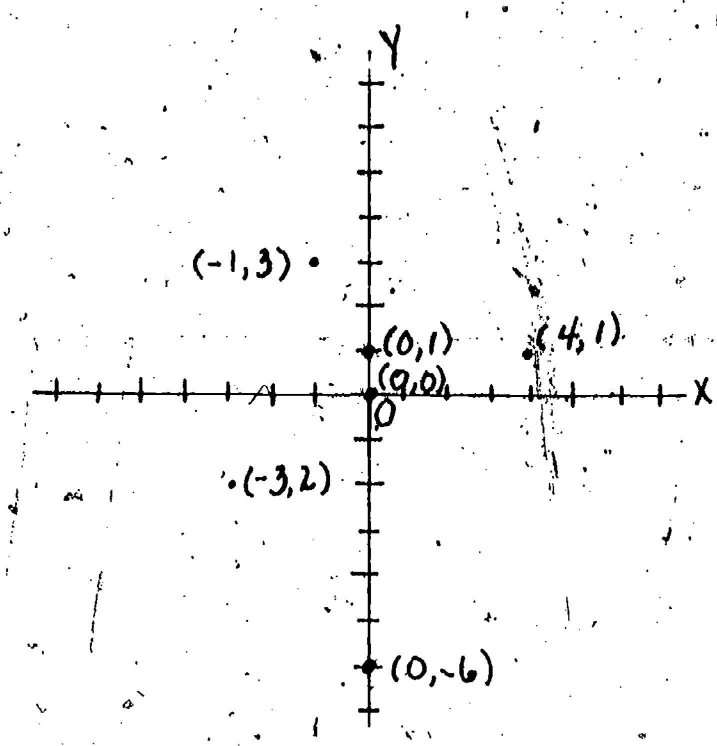
   (a) (0, -6) (0, 0) (0, 1) (4, 1) (-1, 3) (-3, -2)

   (b) abscissa is -3, ordinate is 4

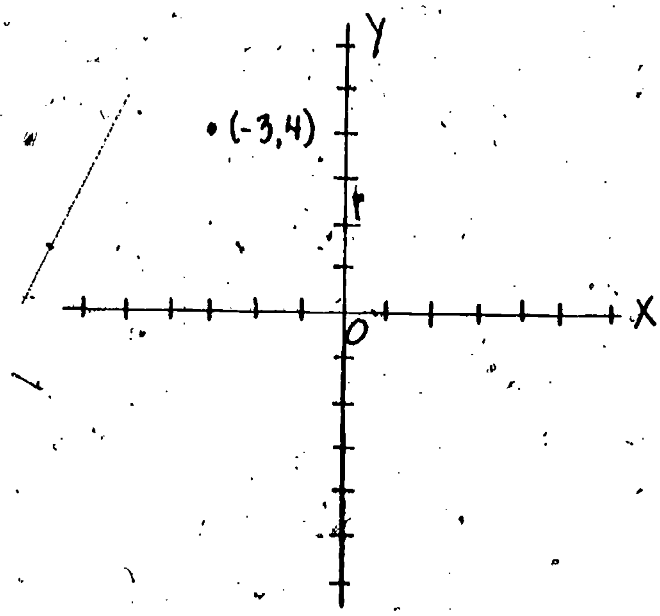
   (c) the set of points in quadrant III having ordinate -4

Answers:

1. (a)



(b)



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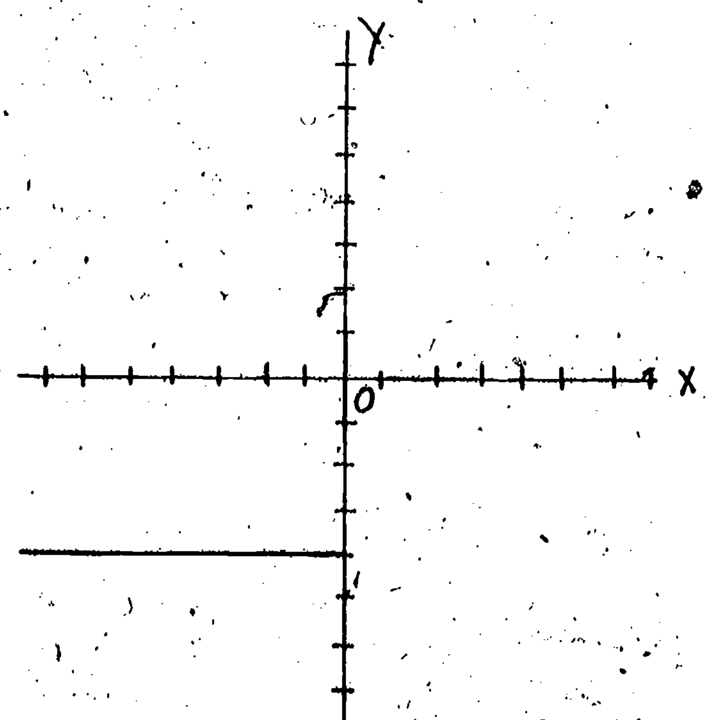

Objective 17.2

a. Determine the direction cosines of a directed line segment.

b. Determine coordinates of a point that divides a segment into a given ratio.

Activities 17.2 (2 and 7 are suggested)

1. Your Text (Part (a) of objective)

Morrill, W. K., Analytic Geometry, pp. 23-33
Exercises: pp. 25-26, problems 1, 3(a, b, c, d, f)
p. 27, problems 1-4(a, c, e of each)
pp. 29-30, problems 1(a, c, e, f), 2(a), 3, 4, 6(a, b, c), 10
pp. 32-33, problems 1, 3-6, 8(a, c, e)

If you follow the order of the text, do only part (a) of objective 17.2. Then proceed to Objective 17.3. After finishing Objective, 17.3, do part (b) of Objective 17.2.

2. Your Text and the Study Guide. (If you use the Study Guide, you cover Objective 17.2 (a) and (b).

3. Solved Problems

Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, pp. 3 and 4, problems 1 - 5 (distance between two points).

4. Other Reading Sources:

Protter and Morrey: Analytic Geometry, pp. 72-73 (Analytic Proofs).

5. Individual Assistance

6. Informal Group Sessions

7. Lecture 1.

Self-Evaluation 17.2 (a)

1. Find the projection of each of the following segments on the x and y axis, respectively.

a. From $P_1 = (5, 1)$ to $P_2 = (4, -3)$
b. From $P_1 = (0, -2)$ to $P_2 = (0, -10)$

2. Find the scalar components from $P_1 = (-4, 3)$ to $P_2 = (5, -2)$

3. Given the vertices of a triangle, $A = (6, -2)$, $B = (1, -2)$ and $C = (-2, 2)$, determine whether it is or is not

(a) right
(b) isosceles
(c) equilateral

4. Find the direction cosines for P_1P_2 where:

a. $P_1 = (2, 3)$ and $P_2 = (4, 1)$
b. $P_1 = (4, -6)$ and $P_2 = (-1, -3)$

Self-Evaluation 17.2 (b)

1. Find the point P that divides the segment P_1P_2 in a ratio of 2:1 where $P_1 = (5, 1)$ and $P_2 = (-2, 3)$.

Answers: (a)

1. (a) $\overline{AB} = 4 - 5 = -1$, $\overline{CD} = -3 - 1 = -4$

   (b) $\overline{AB} = 0 - 0 = 0$, $\overline{CD} = -10 - (-2) = -8$

2. $[9, -5]$

3. $AB = 5$, $AC = 4\sqrt{5}$, $BC = 5$; isosceles

   How do you know it is not a right triangle?

4. (a) $\ell = \frac{2}{\sqrt{8}}$, $m = \frac{-2}{\sqrt{8}}$ (b) $\ell = \frac{-5}{\sqrt{34}}$, $m = \frac{3}{\sqrt{34}}$

Answer: (b)

1. (1/3, 7/3)

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Objective 17.3

Perform operations on plane vectors and analyze results.

Activities 17.3 (2 is suggested)

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 33-42, 48-50
   Exercises: p. 36, problems 1, 2(a, c, & e), 9, 11(a, c, e, & g)
   pp. 41-42, problems 1(a & c), 2 - 5, 10, 11 15(a)
   pp. 50-51, problems 1, 2, 3(a), 6 - 9, 11(a)

   Part (b) of Objective 17.2
   Morrill, W. K., Analytic Geometry, pp. 42-46
   Exercises: pp. 46-47, problems 1 - 5, 9(a), 11, 13(a), 15

2. Your Text and the Study Guide

3. Other Reading Sources:

   Protter-Morrey, Analytic Geometry, pp. 80-93

   School Mathematics Study Group (SMSG), Analytic Geometry, (Unit No. 64, Revised Edition,) pp. 91-133. Also notice Proofs by Analytic Methods beginning on page 139.

   Fuller, Gordon, Analytic Geometry, pp. 183-188

4. Individual Assistance

   In a very short amount of time, your instructor can often help clarify points that are giving you difficulty. Do not overlook this very important device

5. Informal Group Sessions

6. Lecture

Self-Evaluation 17.3

Given the following points $P_1 = (4, 4)$, $P_2 = (1, 1)$; $P_3 = (5, 3)$

Determine:

(a) $\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$ and find cos θ where $\angle\theta = \angle P_1P_2P_3$

(b) $k\overrightarrow{P_1P_3}$ where $k = 3$

(c) are $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ perpendicular

(d) find a vector complimentary to $\overrightarrow{P_2P_3}$

(e) find the area of the triangle $P_1P_2P_3$

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On page 52-53 you will find a set of problems that will be a very good review for the first three objectives of this unit.

Answers:

(a) $u = P_1P_2 = [-3, -3]$

$v = P_1P_3 = [1, -1]$

$\cos\theta = \dfrac{u \cdot v}{|u|\ |v|}$

$\cos\theta = \dfrac{(-3)(1) + (-3)(-1)}{\sqrt{9+9}\ \sqrt{1+1}}$

$\cos\theta = \dfrac{-3+3}{\sqrt{36}} = \dfrac{0}{6} = 0$

(b) $k\,[1, -1] = 3\,[1, -1] = [3, -3]$

(c) yes

(d) $\overrightarrow{P_2P_3} = [4, 2]$ so vector complimentary to $\overrightarrow{P_2P_3}$ would be $[-2, 4]$

(e) $\dfrac{[3, 3] \mid [4, 2]}{2} = \dfrac{|6 - 12|}{2} = 3$ square units

Objective 17.4

Determine the cosine and tangent of the angle(s) formed by the intersection of two lines.

Activities 17:4 (2 and 7 are recommended)

1. Your Text:

   Morrill, W. K., Analytic Geometry, pp. 55-58 and pp. 70-72
   Exercises: p. 58, problems 1 - 4, 6(a, d, f, & h)
   p. 72, problem 10

2. Your Text and the Study Guide.

3. Solved Problems:

   Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, pp. 6 - 8

4. Other reading Sources:

   Fuller, Gordon, Analytic Geometry, pp. 9 - 14

   Protter-Morrey, Analytic Geometry, pp. 43 - 48

   Murdoch, David C., Analytic Geometry, pp. 41 - 44

5. Individual Assistance

6. Informal Group Sessions

7. Lecture 2

Self-Evaluation 17.4

1. Given the points $P_1 = (2, 1)$, $P_2 = (-2, -2)$ and $P_3 = (4, 1)$, determine:
   (a) a pair of direction numbers for the line containing the vector $\overrightarrow{P_1P_3}$.
   (b) the slope of the line containing the vector $\overrightarrow{P_2P_3}$.
   (c) the cosine of the acute angle θ between the lines containing the vectors $\overrightarrow{P_2P_1}$ and $\overrightarrow{P_2P_3}$.
   (d) the tangent of the acute angle θ between the lines containing the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.
   (e) are the lines containing the vectors $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_1P_2}$ perpendicular

Answers:

1. (a) $[2, -2]$ or $k\,[2, -2]$ for any $k \in R$

   (b) $s = 1/6$

   (c) $\cos\theta = \dfrac{u_1v_1 + u_2v_2}{|u|\,|v|}$ where $u = [4, 3]$, $v = [6, 1]$ $\quad \cos\theta = \dfrac{27}{5\sqrt{37}}$

   (d) $\tan\theta = \dfrac{s_2 - s_1}{1 + s_1s_2}$ where s_1 = slope of line containing the vector $\overrightarrow{P_2P_3}$

   s_2 = slope of line containing the vector $\overrightarrow{P_2P_1}$

   $s_1 = 1/6 \qquad s_2 = 3/4$

   $\tan\theta = \dfrac{3/4 - 1/6}{1+(1/6)(3/4)} = 14/27$

   (e) no: $s_1s_2 \neq 0$

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Objective 17.5

(a) Write an equation in any of the requested standard forms (slope-intercept, point-slope, etc.) of the line which satisfies a given set of conditions and sketch the graph.

(b) And given an equation of a line determine the following: a) x and y intercepts, b) slope, c) direction numbers, d) 3 points that lie on the line and the graph of the line.

Activities 17.5 (2 and 7 are suggested)

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 59-70, 73-94
   Exercises: pp. 61-62, problems 1, 2, 3(a, c, 3, of each), 5, 6, 9(a, c, e, g, and h)
   pp. 64-65, problems 2, 3(a, c, &e) 5, 6
   pp. 68-69, problems 1, 2, 3(a and b of each), 4, 6, 13, 14
   pp. 72-73, problems 3 & 6(a, c, & e of each)
   pp. 78-79, problems 1, 2, 10(a, b, c, d, & g), 11(a, c, &e), 20
   pp. 81-82, problems 1, 2, 3, 4(a & e of each)
   pp. 85-85, problems 1, 3(a & c), 4(a & c), 9, 11, 15
   p. 90 problems 1, 4

*2. Your Text and the Study Guide

3. Solved Problems

   Schaum's Outline Series: Theory & Problems of Plane and Solid Analytic Geometry, pp. 16 and 18, problems 9, 10, 12, and 13, and pp. 22-30.

4. Other Reading Sources

   Murdoch, David C., Analytic Geometry, pp. 44-46 and pp. 50-54

   Protter-Morrey, Analytic Geometry, pp. 49-56, pp. 62-64, and pp.68-71

   Fuller, Gordon, Analytic Geometry, pp. 28-38

5. Individual Assistance

6. Informal Group Activity

7. Lecture 2

Assessment Task

The assessment task for this unit will be in two parts. In part one you will be given an equation or equations of a line or lines and asked to determine certain information about the line(s). You will also be required to sketch the graph of the line or lines. In part two you will be given a set of conditions and asked to find the line(s) that meet those conditions.

The two problems below are examples of the type of problems you can expect. You cannot use your book or any notes on the assessment tasks. There will be twenty items on the assessment task. You must answer sixteen of them correctly.

1. Given: $\ell_1 = 3x + 2y = -4$ and $\ell_2 = x - 3y - 2 = 0$

   Determine (a) s_1 and s_2 (b) the tangent of the angle θ from ℓ_1 to ℓ_2 (c) the x and y intercepts (d) the normal form of ℓ_2 and (e) the graph of ℓ_1.

2. Given: $P_1 = (-2, 0)$, $P_2 = (-1, 3)$ and $P_3 = (2, -3)$

   (a) Write the equation of the line that contains $\overrightarrow{P_1P_2}$.

   (b) Write the equations of the lines that are parallel to and 3 units from the line containing $\overrightarrow{P_1P_2}$.

   (c) Write the equation of the line through the point P_2 and perpendicular to $\overrightarrow{P_1P_3}$.

Answers:

1. (a) $s_1 = 3/-2$; $y = -3/2\,x - 4/2$

$s_2 = 1/3$; $y = x/3 - 2/3$

(b) $\tan\theta = \dfrac{s_2 - s_1}{1 + s_1 s_2}$

$\tan\theta = \dfrac{11/6}{1/2} = 11/3$

(c) $k_1 = 2$

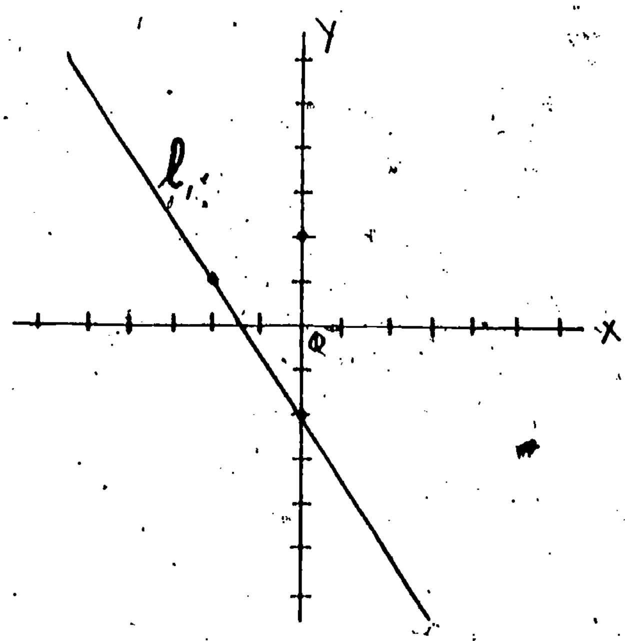
$k_2 = -2/3$

(d) $x - 3y = 2$

$\dfrac{x}{\sqrt{1+9}} - \dfrac{3y}{\sqrt{1+9}} = \dfrac{2}{\sqrt{1+9}}$

$\dfrac{x}{\sqrt{10}} - \dfrac{3y}{\sqrt{10}} = \dfrac{2}{\sqrt{10}}$

(e)



2. (a) $\overrightarrow{P_1P_2} = [1, 3]$, $[\Delta x, \Delta y] = [1, 3]$

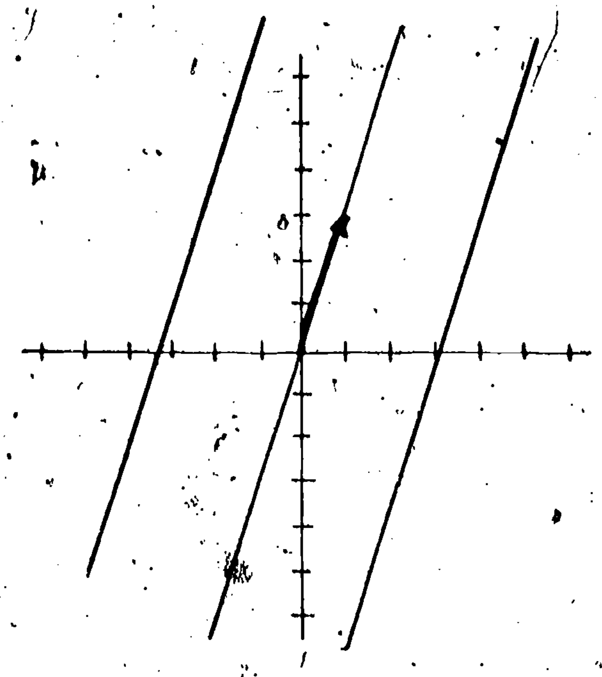
$x\Delta y - y\Delta x = x_1\Delta y - y_1\Delta x$

$3x - y = (-2)(3) - 0\,(1)$

$3x - y = -6$

(b) $\delta = \left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$ where $[a, b] = [3, -1]$

$\pm\,\delta = \dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$



$$\pm 3 = \frac{3(-2) + (-1)(0) + c}{\sqrt{10}}$$

$$\pm 3\sqrt{10} = -6 + c$$

$$\pm 3\sqrt{10} + 6 = c$$

$$\sqrt{10} = 3.16$$

$$+ 3(3.16) + 6 = 15.48$$

$$- 3(3.16) + 6 = -3.48$$

Hence the equations are:

$$3x - y + 15.48 = 0$$

$$3x - y - 3.48 = 0$$

(c)

$\overrightarrow{P_1P_3} = [4, -3]$ $P_2 = (-1, 3)$

Hence $ax + by = ax_1 + by_1$

$$4x - 36 = 4(-1) - 3(3)$$

$$4x - 3y = -4 - 9$$

$$4x - 3y + 13 = 0$$

Rationale

By the end of this unit, you will be concerned with being able to do two things, (1) constructing the graph and interpreting it when you are given the equation of a line, and (2) when given some conditions or data, determining the equation of the line that satisfies those conditions.

An old Chinese proberb says "A picture is worth a thousand words." So the picture or graph of an equation is very valuable--if you are able to use that picture to interpret information depicted by it. One of the pieces of information you should be able to determine from the graph of a line is the slope of the line; another would be the x and y intercepts of the line.

Since the line is a mathematical model for many physical phenomena, it is advantageous to be able to describe, in terms of an equation, that phenomena. For instance, in electronics, any electrical device whose I vs E curve exhibits the property that the ratio of the current through the device to the voltage across it is a constant (the graph is a line) is a linear resistance. In physics, Hooke's Law states, "The elastic deformation (of a spring) is directly proportional to the force if the elastic limit is not exceeded." Hence the graph of load vs elongation is linear.

This unit is broken into 5 subunits--one of each dealing with points, line segments, and vectors and 2 dealing with lines.

OBJECTIVES

Unit 17

Line in a Plane

17.5 a) Write an equation in any of the requested standard forms (slope-intercept, point-slope, etc) of the line which satisfies a given set of conditions and sketch the graph.

b) And given an equation of a line determine the following: a) x and y intercepts, b) slope, c) direction numbers, d) 3 points on the line and graph the line.

17.4 Determine the cosine and tangent of the angle(s) formed by the intersection of two lines.

17.3 Perform operations on plane vectors and analyze the results.

17.2 Determine the direction cosines of a directed line segment and determine coordinates of a point that divides a segment into a given ratio.

~~17.1 Plot points in a rectangular coordinate system.~~

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Subunit 17.1

OBJECTIVE: *Plot points in a rectangular coordinate system.*

Instructional Activities - Subunit 17.1

Task 1: Point Plotting

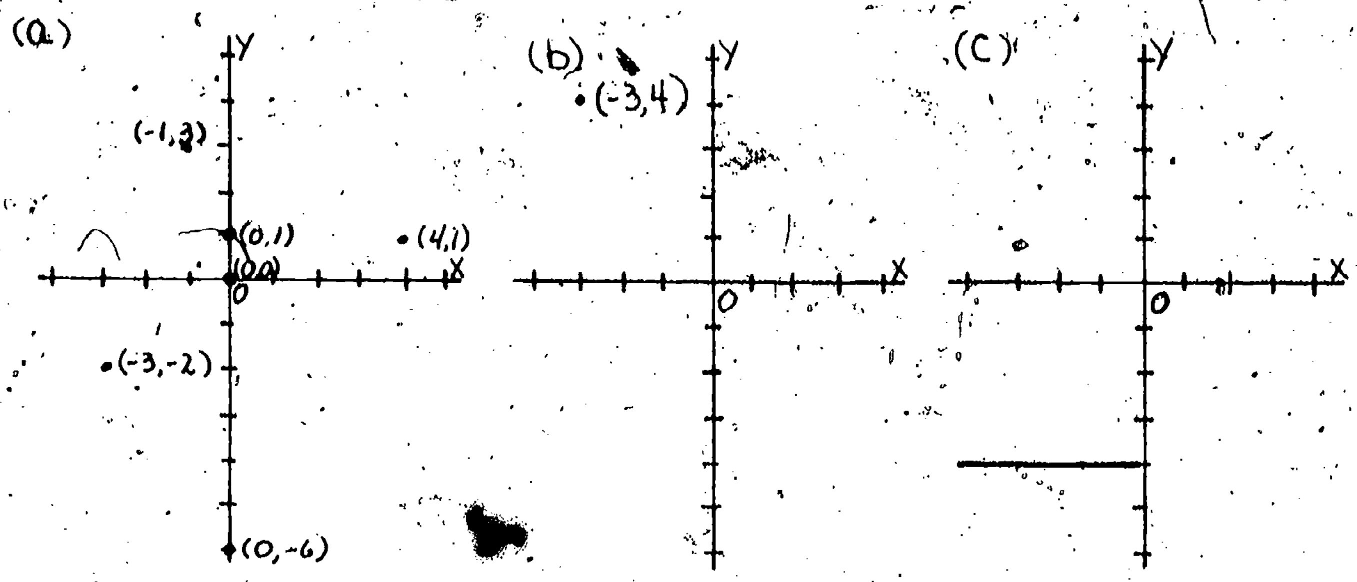
Read pages 19-21

Pay careful attention to the vocabulary of the section and then try the exercises on pages 21, 22. Some problems you should work are #'s 1, 2, 3, 5, 6, 10, 11, 12, 14, and 26. Also work any others that are of particular interest to you.

Self-evaluation - Objective 17.1

1. Draw and label the coordinate axes (right handed frame) and locate the following points:
   (a) (0, -6) (0, 0) (0, 1) (4, 1) (-1, 3) (-3, -2)
   (b) abscissa is -3, ordinate is 4
   (c) the set of points in quadrant III whose ordinate is -4

Answers: Self-evaluation 17.1



Subunit 17.2

OBJECTIVE: Determine the direction cosines of a directed line segment and determine coordinates of a point that divides that segment into a given ratio.

Task 2: Projections

Read pages 23-25 (projections) and on pages 25 and 26 work problems 1 and 3(a, b, c, d, and f).

In reading the material you might find the following additional information helpful.

At the top of page 24, in your text, you will find figure 2-8. The figure has been reproduced on the next page at the left with two lines (dotted) and one point B, inserted. In the figure 2-8 on the following page and to the left, the angle θ is obtuse. In figure 2-8' on the same page and to the right, the angle θ is acute.

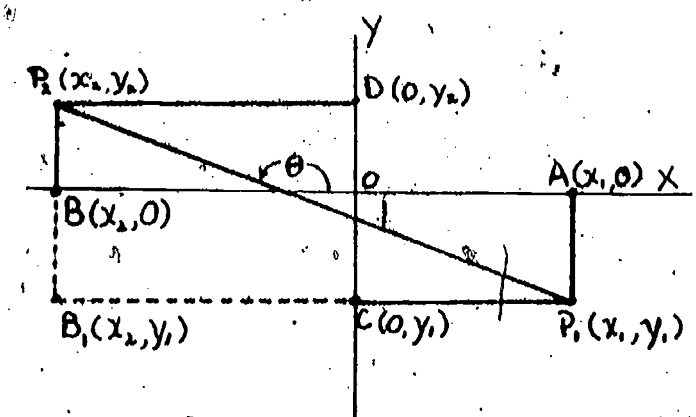


figure 2-8

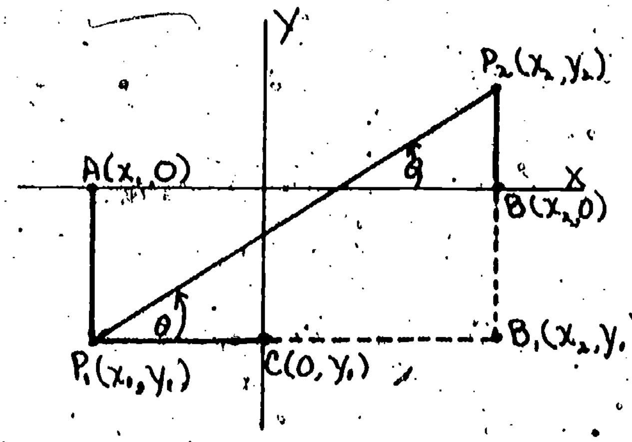


figure 2-8'

In figure 2-8, $\cos\theta = \frac{x}{r}$ where $x = \overline{AB}$ and where $r = |P_1P_2|$. Notice that since $\overline{AB}$ is measured to the left that $\overline{AB}$ will be ________________. (positive, negative, zero) Since $\overline{AB}$ is negative and θ is obtuse, the functional value of $\cos\theta$ will also be negative. Therefore $\cos\theta = \frac{\overline{AB}}{|P_1P_2|}$ and $\overline{AB} = |P_1P_2| \cos\theta$. In figure 2-8', $\cos\theta = \frac{x}{r}$ or $\cos\theta = \frac{\overline{AB}}{|P_1P_2|}$. Since $\overline{AB}$ is measured to the ____________ (right, left), $\overline{AB}$ is positive and since θ is acute, $\cos\theta$ will have a ________________ (positive, negative, zero) value. Hence $\overline{AB} = |P_1P_2| \cos\theta$.

Example 2-1

What are the projections of the segment P_1P_2 on the axis, where $P_1 = (0, 6)$ and $P_2 = (5, 3)$?

Solution: $\overline{AB} = x_2 - x_1 = (5-0) = 5$
$\overline{CD} = y_2 - y_1 = (3-6) =$ ____

Since $\overline{AB} = +5$, we know the direction from A to B is to the ____, and since $\overline{CD} = -3$, the direction from C to D is ____.

Answers: -3, right, downward, respectively

Self-evaluation - Task 2: Projections

1. $\overline{AB} = -5$ implies that the distance from A to B (where $\overline{AB}$ is the projection of P_1P_2 on the x-axis) is to the ____.

2. If the projection of the segment from P_1 to P_2 on the x-axis, AB is positive then the projection of the segment P_1P_2, $\overline{BA}$, on the x-axis is ____.

3. Find the projection of each of the following segments on the x and y axis respectively.
   (a) from $P_1 = (5, 1)$ to $P_2 = (4, -3)$
   (b) from $P_1 = (0, -2)$ to $P_2 = (0, -10)$

Answers: Task 2: Projections

1. left 2. negative 3. (a) $\overline{AB} = 4 - 5 = -1$ $\overline{CD} = -3 - 1 = -4$
(b) $\overline{AB} = 0 - 0 = 0$ $\overline{CD} = -10 - (-2) = -8$

Task 3: Scalar Components

Read pages 26-27 and on page 27 work problems 1, 2, 3, and 4 (do a, c and e of each problem).

Self-evaluation Task 3: Scalar Components

1. Given the following scalar components for the segments P_1P_2, what are the scalar componentents for the segment P_2P_1 ?

   (a) $P_1P_2 = [3, -4]$ (a) $P_2P_1 =$

   (b) $P_1P_2 = [-1, 2]$ (b) $P_2P_1 =$

   (c) $P_1P_2 = [-2, -5]$ (c) $P_2P_1 =$

2. Find the scalar components from $P_1 = (-4, 3)$ to $P_2 = (5, -2)$.

3. If the initial point of a segment is (-4, 6) and the scalar components are [0, -11], what is the terminal point of the segment?

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Answers: 1. (a) [-3, 4] (b) [1, -2] (c) [2, 5] 2. [9, -5]
3. [-4, -5]

Task 4: Distance Between Two Points

Read pages 27-29 and on pages 29-30 work problems 1(a, c, e, & f), 4, 5, 6 and 10.

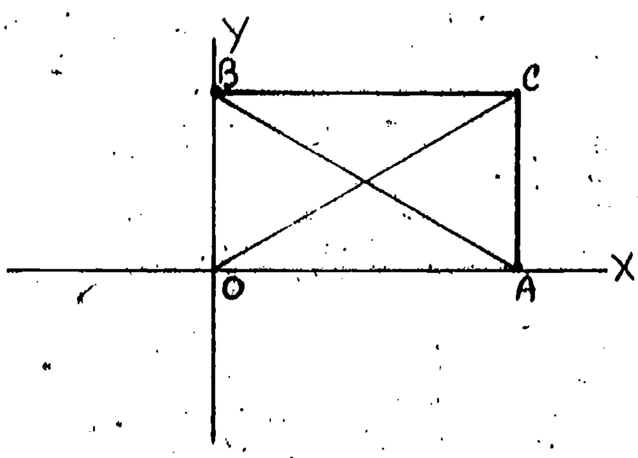
In reading the material, pay special attention to example 2-5 in the text and the example provided in this material. Both are analytic proofs of theorems that you proved previously in Plane Geometry. In many cases, an analytic proof is much shorter and easier than the same proof done geometrically in non-coordinatized system.

Problems 6, 9, 10, 11, 12, and 13 are more examples of geometric theorems that you have seen previously proved by another method.

Example 2-5 is an analytic proof of the theorem: The diagonals of any rectangle are equal in length.

Below is a proof of that theorem using congruencies, similar to one you would find in any plane geometry book.

Prove: The diagonals of any rectangle are equal in length.



Prove: $|OC| = |AB|$

Analysis: Need to prove $\Delta BOA \cong \Delta CAO$

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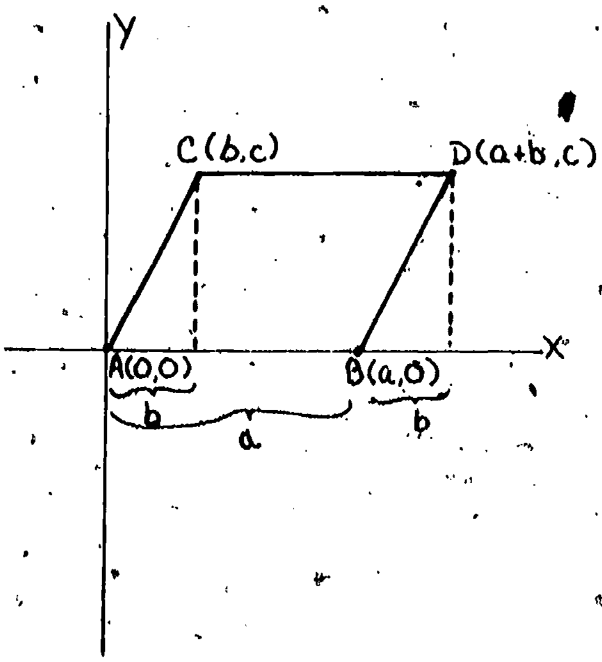
STATEMENTS	REASON
1. rectangle OACB	1. given
2. $\|OB\| = \|AC\|$ and $\|OA\| = \|BC\|$ $OB \parallel AC$ and $OA \parallel BC$	2. opposite sides of a rectangle are equal and parallel
3. $\angle BOA = \angle CAO$	3. All right angles are equal (in rectangles all angles are right angles)
4. $\|OA\| = \|OA\|$	4. same side
5. $\triangle BOA \cong \triangle CAO$	5. SAS
6. $\|OC\| = \|AB\|$	6. corresponding parts of congruent triangles

Hence the diagonals of any rectangle are equal in length.

Example: Prove the opposite sides of any parallelogram are equal.



Place the origin at A and the x-axis along AB. $A = (0, 0)$, $B = (a, 0)$, $C = (b, c)$, and then $D = (a + b, c)$

Prove $|AC| = |BD|$ and $|CD| = |AB|$

$AC = d_1 = \sqrt{(b - 0)^2 + (c - 0)^2} =$ ________

$BD = d_2 = \sqrt{} =$ ________

Since $|AC|$ and $|BD|$ both equal $\sqrt{b^2 + c^2}$ then $|AC| \cong |BD|$.

Using the same techniques on $|CD|$ and $|AB|$ then

$|CD| = d_3 = \sqrt{} =$ ________

$|AB| = d_4 = \sqrt{} =$ ________

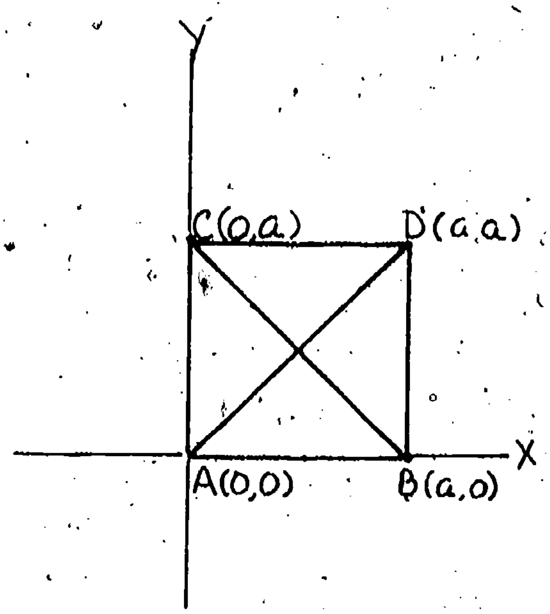
Since the length of both segments is a then $|CD| = |AB|$

Self-evaluation - Task 4: Distance Between Two Points

1. Find the distance from $P_1 = (2, 1)$ to $P_2 = (-4, 3)$.
2. What kind of triangle has vertices of $A = (6, -2)$, $B = (1, -2)$ and $C = (-2, 2)$? (a) isosceles (b) right (c) equilateral.
3. Prove analytically: The diagonals of a square are equal.

Answers: 1. $\sqrt{52}$ 2. AB = 5, AC = $4\sqrt{5}$, BC = 5, isosceles. How do you know it is not a right triangle (pyth. Thm.)

3.



Place the origin at A with AB on the x-axis, and AC on the y-axis. Then A= (0, 0), B = (a, 0), C = (0, a) and D = (a, a).

Prove: $|AD| = |CB|$

$|AD| = d_1 = \sqrt{(a-0)^2 + (a-0)^2} = \sqrt{2a^2}$

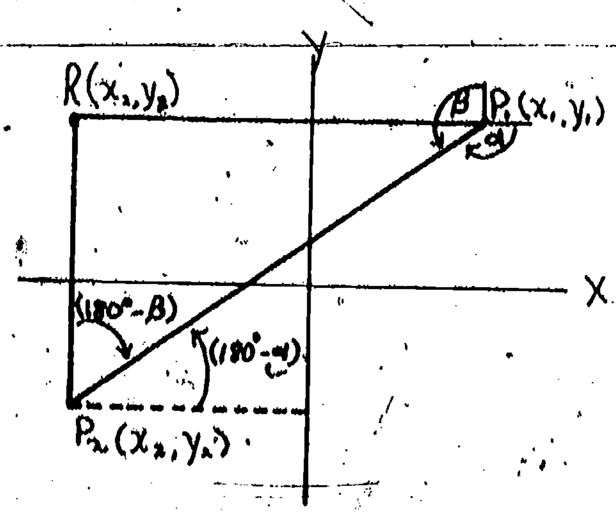
$|CB| = d_2 = \sqrt{(0-a)^2 + (a-0)^2} = \sqrt{2a^2}$

Since $|AD|$ and $|CB| = \sqrt{2a^2}$ then $|AD| = |CB|$

Task 5: Direction Cosines

Read pages 31-32 and on pages 32 and 33, work problems 1, 3, 4, 5, 6, and 8(a, c, and e). If necessary review pages 14-17 on Trigonometry. Also remember that a positive angle (α) is measured counter-clockwise while a negative angle ($-\alpha$) is measured in a clockwise direction.

You might find the following sketch and discussion helpful in understanding the discussion on direction cosines of P_2P_1, in the middle of page 31 in your text.



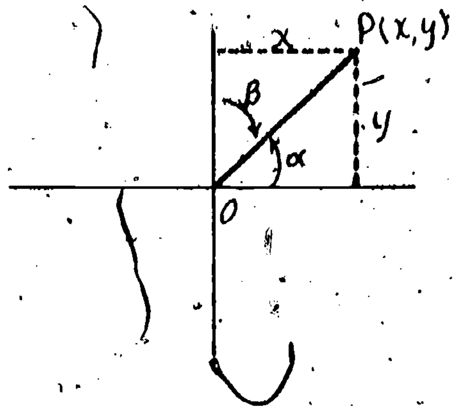
If $\ell = \cos\alpha$ and $m = \cos\beta$ for P_1P_2, the direction cosines for P_2P_1 would be

$\cos(180 - \alpha) = -\cos\alpha =$ ________

$\cos(180 - \beta) = -\cos\beta =$ ________

Hence the direction cosines of P_2P_1 are $-\ell$ and $-m$.

Continuing on to the bottom of page 31 in your text, where the origin is the initial point of the segment and P is any point,



ℓ = x/P since cos α = x/P and ℓ = ____

m = y/P since cos β = y/P and m = ____

p = |OP| = $\sqrt{\quad}$

Answers: $\ell = \cos\alpha$, $m = \cos\beta$, $p = |OP| = \sqrt{x^2 + y^2}$

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Self-evaluation - Objective 17. (a) Direction Cosines

1. Determine if the following could be direction cosines:

   (a) [1/3, 2/3]

   (b) [1/3, -3/2]

   (c) [0, -1]

2. Are the direction cosines of a segment $\overline{OP}$ the same as for the segment $\overline{PO}$?

3. Find the direction cosines for P_1P_2 where:

   (a) $P_1 = (2, 3)$ and $P_2 = (4, 1)$

   (b) $P_1 = (4, -6)$ and $P_2 = (-1, -3)$

Answers: 1. (a) no, (b) yes, (c) yes 2. no 3. (a) $\ell = 2/\sqrt{8}$ $m = -2/\sqrt{8}$ (b) $\ell = -5/\sqrt{34}$, $m = 3/\sqrt{34}$

Objective 17.2 (b)

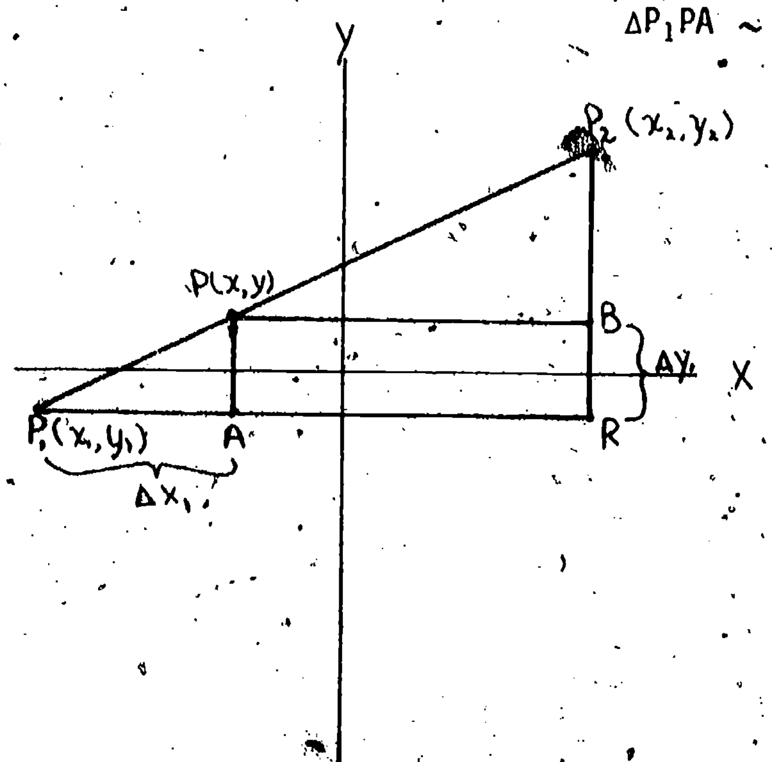
OBJECTIVE: Determining coordinates of a point P that divides a segment into a given ratio.

In this section you will learn how to find the coordinates of a point that divides a segment into a given ratio.

Let $P = (x, y)$ be any point on the line segment P_1P_2, where $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$

Let A be the projection of P on the line through P_1R and let B be the projection of P on the line through P_2R forming similar triangles

$$\Delta P_1PA \sim \Delta P_1P_2R$$



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Therefore $\frac{\overline{P_1P}}{\overline{P_1P_2}} = \frac{\Delta x_1}{\Delta x}$ and $\frac{\overline{P_1P}}{\overline{P_1P_2}} = \frac{\Delta y_1}{\Delta y}$

Let the ratios $\frac{\Delta x_1}{\Delta x}$ and $\frac{\Delta y_1}{\Delta y} = k$, then $\frac{\overline{P_1P}}{\overline{P_1P_2}} = k$ and

$$\overline{P_1P} = k\,\overline{P_1P_2}$$

The scalar components of $\overline{P_1P}$ are __________________.

and the scalar components of $\overline{P_1P_2}$ are ____________(see Task 3: Scalar Components)

So $\overline{P_1P} = k\,\overline{P_1P_2}$ becomes $[x - x_1, y - y_1] = k[\Delta x, \Delta y]$.

Solving for x and y yields $x - x_1 = k\,\Delta x$ $\quad y - y_1 = k\,\Delta y$

$x =$ __________ $\quad y =$ __________

Hence: $x = x_1 + k\,\Delta x$

$y = y_1 + k\,\Delta y$

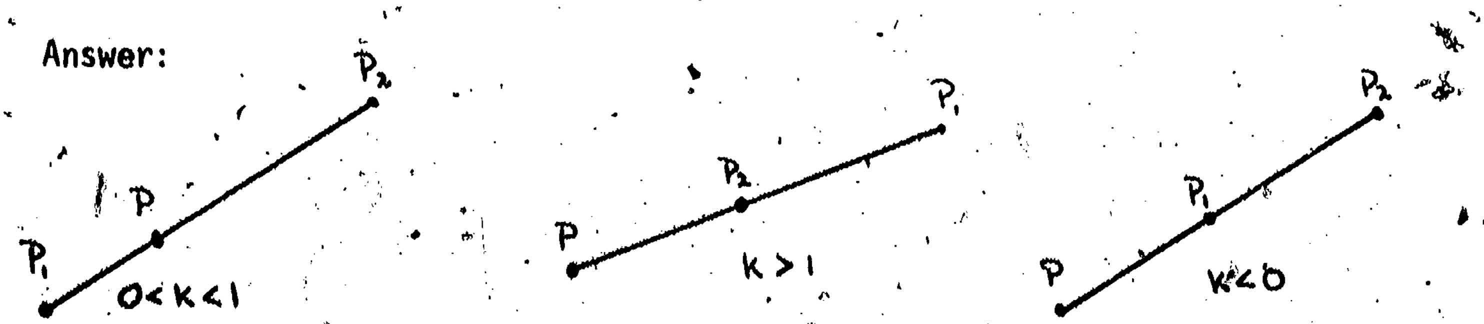
The point P can lie anywhere on the line through the segment. P_1P_2 and its location will depend on the value of k.

(a) If $0 < k < 1$ then P lies between ____ and ____.

(b) If $k > 1$ then P_2 lies between ____ and ____.

(c) If $k < 0$ then P_1 lies between ____ and ____.

Answer:



Midpoint of a segment

An important application occurs when P is the midpoint of the segment P_1P_2 or $k = \frac{1}{2}$. $k = \frac{1}{2}$ since P_1P is $\frac{1}{2}$ of P_1P_2.

When $k = \frac{1}{2}$, $P_1P = k\,P_1P_2$ becomes $P_1P = \frac{1}{2}\,P_1P_2$. or

$[x - x_1, y - y_1] =$ ________________________

solving for x and y

$x - x_1 = \frac{1}{2}\,\Delta x$ | $y - y_1 = \frac{1}{2}\,\Delta y$

$x =$ ____________ since $\Delta x = x_2 - x_1$ | $y =$ ____________________

and

$\Delta y = y_2 - y_1$ | $y =$ ____________________

Hence:

$$x = \frac{x_2 + x_1}{2}, \qquad y = \frac{y_2 + y_1}{2}$$

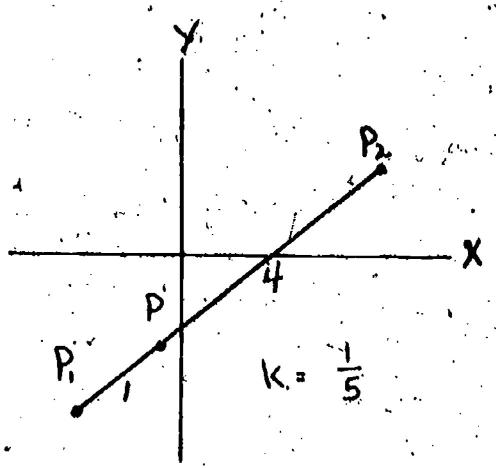
If $k = 0$, then

$P_1P = k\,P_1P_2$ becomes $P_1P = 0$ which implies that P = _____ (P coincides with P_1)

If $k = 1$, then

$P_1P = k\,P_1P_2$ becomes $P_1P_2 = P_1P$ which implies that P = _____ (P coincides with P_2)

Sometimes you are given the ratio r in which P divides the segment P_1P_2. (Remember k is the ratio of P_1P to P_1P_2). For example if P is the midpoint of the segment P_1P_2, then P divides the segment P_1P_2 in a 1:1 ratio r. (r is the ratio of P_1P to PP_2)



Any time you are given a ratio r it can be converted to the ratio k by observing the first element of the ratio r is the numerator of the ratio k and the sum of the elements of the ratio r is the denominator of ratio k.

For example:

(a) If the ratio is 1:1, then k = ____ $P_1P = 1$ $PP_2 = 1$ $P_1P_2 = 2$

(b) If the ratio is 2:3, then k = ____ $P_1P =$ ____ $PP_2 =$ ____ $P_1P_2 =$ ____

(c) If the ratio is 2:1, then k = ____ $P_1P =$ ____ $PP_2 =$ ____ $P_1P_2 =$ ____

Answers: (a) $k = \frac{1}{2}$ (b) $k = 2/5$, $P_1P = 2$, $PP_2 = 3$, $P_1P_2 = 5$, (c) $k = 2/3$, $P_1P = 2$, $PP_2 = 1$, $P_1P_2 = 3$.

Examples:

1. Find the midpoint of the segment P_1P_2 where $P_1 = (3, 5)$ and $P_2 = (-2, -6)$.

Solution:

$$x = \frac{x_2 + x_1}{2} \qquad y = \underline{\qquad}$$

$x =$ ____ $y =$ ____

$x = \frac{1}{2}$ $y = -\frac{1}{2}$

2. Find the coordinates of the point P which divides the segment from $P_1 = (1, -4)$ to $P_2 = (5, 6)$ in the ratio of 2:1.

Solution: Since $P_1P = 2$, and $PP_2 = 1$, (ratio 2:1)

$P_1P_2 = 3$, $k =$ ____, $\Delta x =$ ____, $\Delta y =$ ____

$x = x_1 + k \Delta x$ $\qquad$ $y = y_1 + k \Delta y$

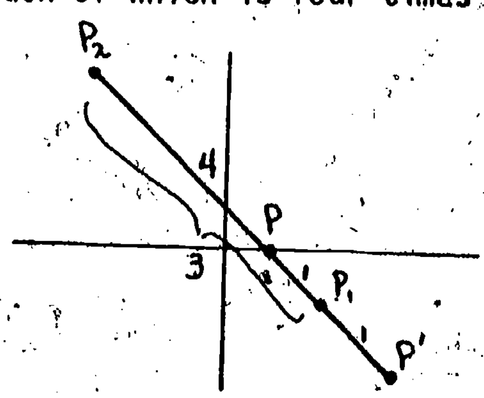
$x =$ ________ $\qquad$ $y =$ ________

$x = 11/3$ $\qquad$ $y = 8/3$

Answers: $k = 2/3$, $\Delta x = (5 - 1) = 4$, $\Delta y = (6 + 4) = 10$

$x = 1 + 2/3\ (4)$, $\quad y = -4 + 2/3\ (10)$

3. Find 2 points on the line containing $P_1 = (2, -1)$ and $P_2 = (-3, 4)$ each of which is four times as far from P_2 as it is from P_1.



Solution:

One position of P is between P_1 and P_2 where $k = 1/5$ (ratio 1:4). The other position is where P_1 is between P and P_2 or where $k = -1/3$ $\quad \Delta x =$ ____; $\Delta y =$ ____

For the first position, P internal ($k = 1/5$) $P_1P = k\ P_1P_2$

$x = x + k \Delta x$ $\qquad$ $y =$ ________

$x =$ ________

$x =$ ________ $\qquad$ $y =$ ________

Hence: $p = (1, 0)$

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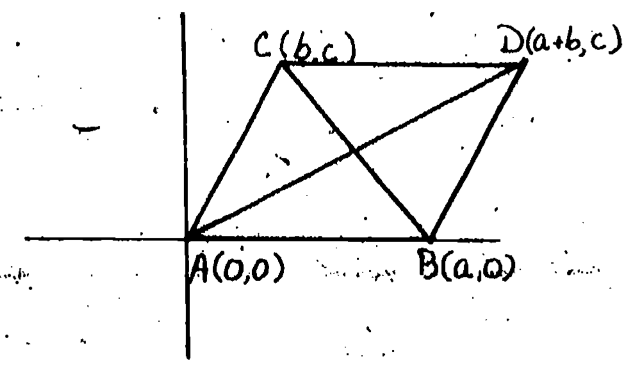
For the second position P' external (k = -1/3)

x' = ________ y' = ________

x' = ________ y' = ________

P' = (11/3, -8/3)

4. Prove that the diagonals of any parallelogram bisect each other.



Consider the parallelogram in the figure.

The midpoint of diagonal AD is ______.

The midpoint of diagonal CB is ______.

Since the two midpoints are both $\left(\frac{a+b}{2}, \frac{c}{2}\right)$, the diagonals bisect each other.

Do the following exercises on pages 46-47:

Problems 1, 3, 5, 7, 11, 12, 13(a & b), and 14.

Self-evaluation Objective 17.2 (b)

1. Find the midpoint of a segment P_1P_2 where $P_1 = (2, -7)$ and $P_2 = (-4, -9)$.
2. Find the point P that divides the segment P_1P_2 in a ratio of 2:1 where $P_1 = (5, 1)$ and $P_2 = (-2, 3)$.
3. Prove: The midpoint of the hypotenuse of a right triangle is equidistant from the 3 vertices.

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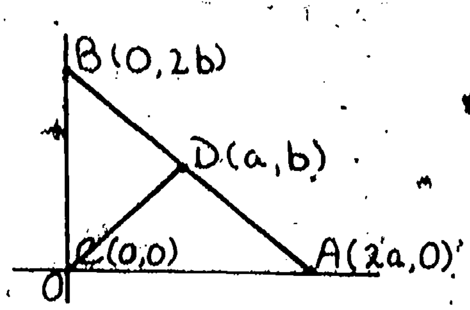
Answers: 1. midpoint (-1, -8) 2. (1/3, 7/3)

3.



Place the right angle C at the origin. Let A = (2a, 0) and B = (0, 2b). Then the midpoint D of the hypotenuse is (a, b).

Prove: $|AD| = |BD| = |DC|$

$|AD| = d_2 = \sqrt{(2a - a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

$|BD| = d_2 = \sqrt{(-a)^2 + (2b - b)^2} = \sqrt{a^2 + b^2}$

$|CD| = d_3 = \sqrt{a^2 + b^2}$

$|AD| = |BD| = |CD|$

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Summary - Subunit 17.2

This subunit is about line segments. How do you name a line segment; what is its length; what are its scalar components and direction cosines?

A segment is named in terms of its end points. A directed segment has a direction dependent on which end point is the initial point of the segment. The scalar components of the directed segment are defined to be the projections of the directed segment on the coordinate axes. And last, the direction cosines of a segment are defined in terms of the scalar components and the length of the segment.

The last part of the subunit is on finding the coordinates of a point that divides a given segment into a given ratio, and a formula for the midpoint is developed.

Subunit 17.3 -- Vectors

OBJECTIVE: Perform operations on plane vectors and analyze results.

Rationale

At this point we will discuss some of the operations performed on vectors and how those operations are applied to various geometric concepts.

Instructional Activities - Subunit 17.3

Read pages 33-36. Since the objective of this section deals with performing operations on plane vectors, you might find it helpful to list those operations defined in the reading material.

Operations:

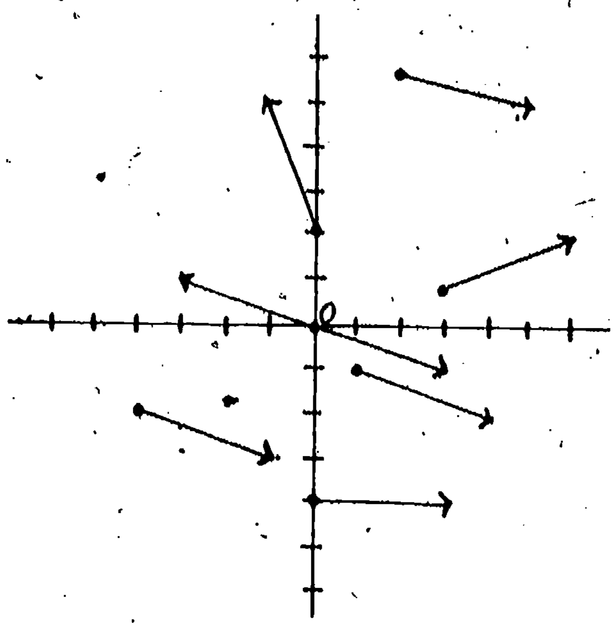
The operation at the bottom of page 34 defined as $k\ [u_1, u_2] = [ku_1, ku_2]$ is called scalar multiplication. The term scalar means that k is a number not a vector. So, scalar multiplication for vectors implies that you are multiplying a vector by a scalar. Notice that the product produced is a vector.

Now try the exercises on the next page.

Exercises:

Given: $u = [3, -1]$, $v = [-4, 2]$, $k_1 = 2$, $k_2 = 4$, and $k_3 = \frac{1}{2}$, answer the following questions.

1. Circle all of the geometric representations of the vector u in the drawing below:



2. Find:

   (a). $u + v$

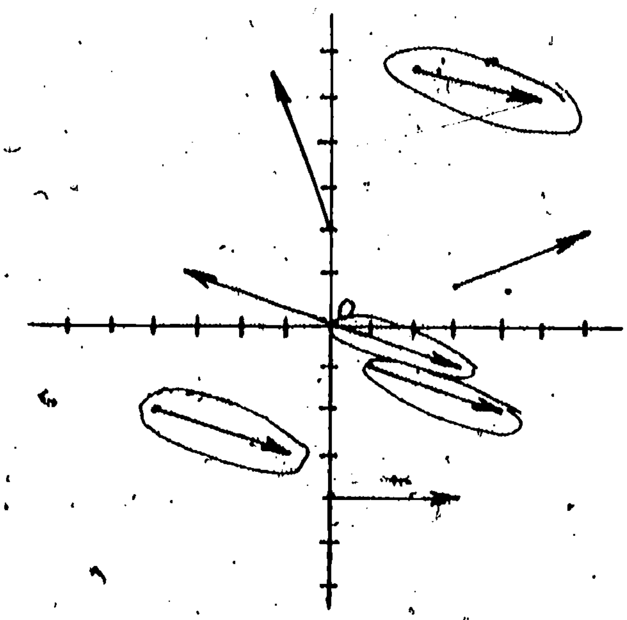
   (b) $u - v$

   (c) k_1u, k_2u, and k_3v and discuss the effect of scalar multiplication on the product vector. (When does it stretch or shrink a vector? Does the direction of the vector ever change?)

3. Represent u in terms of a unit vector.

Answers:

1.



2. (a) $u + v = [u_1 + v_1, u_2 + v_2]$

$= [3 + (-4), -1 + 2]$

$= [-1, 1]$

(b) $u - v = u + (-v) = [u_1 + (-v_1), u_2 + (-v_2)]$

$= [3 + 4, -1 - 2]$ $-v = [4, -2]$

$= [7, -3]$

(c) $k_1u = 2[3, -1] = [6, -2]$

$k_2u = -4[3, -1] = [-12, 4]$

$k_3v = \frac{1}{2}[-4, 2] = [-2, 1]$

Observe that multiplying by a scalar does several things

1. It either stretches or shrinks the vector

a { if $|k| > 1$, it stretches the vector
 if $|k| < 1$, it shrinks the vector

2. It either preserves the direction of the vector or it reverses the direction of the vector.

   b $\begin{cases} \text{if } k > 0 \text{ it preserves the direction of the vector} \\ \text{if } k < 0 \text{ it reverses the direction of the vector} \end{cases}$

3. $u = |u| \left[\frac{u_1}{|u|} , \frac{u_2}{|u|} \right] =$

$u = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

$u = |u| \left[\frac{u_1}{|u|} , \frac{u_2}{|u|} \right] = \sqrt{10} \left[\frac{3}{\sqrt{10}} , \frac{-1}{\sqrt{10}} \right]$

Notice that $\left[\frac{3}{\sqrt{10}} , \frac{-1}{\sqrt{10}} \right]$ is a unit vector and it is also direction cosines for the vector u, $\ell = \frac{3}{\sqrt{10}}$, $m = \frac{-1}{\sqrt{10}}$.

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On page 36, in your text, work problems 1, 2(a, c, e), 3(a), 4(a, c, e), 5(a, c, e), 9, and 11(a, c, e, g).

Read pages 37-41, in your text. In reading the first paragraph on page 37, do you know why the coorindates of P_1 and P_2 are the direction cosines of u and v?

Answer: Recall that any point P (x, y) on the unit circle (a circle whose radius is one unit) has the property that $r^2 = x^2 + y^2$. Since $r^2 = 1$, then $x^2 + y^2 = 1$, but this also is a property of the direction cosines of a vector.

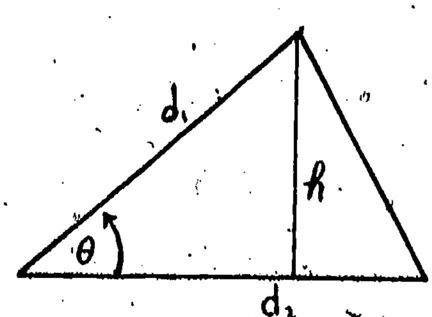
On page 41-42, in your text, work problems 1(a, c) 2, 3, 4, 5, 10, 11, and 15(a).

Read pages 48-50 in your text.

At the top of page 48 (2nd paragraph) you find the statement: let ℓ_1 and m_1 be the direction cosines of $u = [x_1, y_1]$. Then $\ell_1 = \cos \alpha$ and $m_1 = \sin \alpha$. Remember that $m_1 = \cos \beta$ where $\beta = (90^0 - \alpha)$ and $\cos (90^0 - \alpha) = \sin \alpha$.

AREA OF A TRIANGLE ----- page 49

$A = \frac{1}{2} bh$



$\sin \theta = h/d_1 \quad = \quad h = d_1 \sin \theta$

$A = \frac{1}{2} d_1 d_2 \sin \theta$

On pages 50-51, work problems 1, 2, 3, 4(a), 6, 7, 8, 9 (see 8(, 11(a).

Self-evaluation - Objective 17.3

Given the following points: $P_1 = (4, 4)$ $P_2 = (1, 1)$ $P_3 = (5, 3)$.

Determine:

(a) $\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$ and find cos θ where ∠ θ = ∠ $P_2P_1P_3$

(b) $k\overrightarrow{P_1P_3}$ where k = 3

(c) are $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ perpendicular

(d) find a vector complimentary to $\overrightarrow{P_2P_3}$

(e) find the area of the triangle $P_1P_2P_3$

On pages 52-53 you will find a set of problems that will be a very good review for the first three objectives for this unit.

Answers - Self-evaluation Objective 17.3

(a) $u = \overrightarrow{P_1P_2} = [-3, -3]$

$v = \overrightarrow{P_1P_3} = [1, -1]$

$$\cos\theta = \frac{u \cdot v}{|u|\ |v|}$$

$$\cos\theta = \frac{(-3)(1) + (-3)(-1)}{\sqrt{9+9} \cdot \sqrt{1+1}}$$

$$\cos\theta = \frac{-3+3}{\sqrt{36}} = \frac{0}{6} = 0$$

(b) $k[1, -1] = 3[1, -1] = [3, -3]$

(c) yes

(d) $\overrightarrow{P_2P_3} = [4, 2]$ so vector complimentary to $\overrightarrow{P_2P_3}$ would be $[-2, 4]$.

(e) $A = \frac{u|v}{2} = \frac{u_1v_2 - u_1v_2}{2}$

$$A = \frac{(-3)(-1) - (-3)(1)}{2}$$

$$A = \frac{3+3}{2}$$

A = 3 square units

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Subunit 17.4

OBJECTIVE: Determine the cosine and tangent of the angle(s) formed by the intersection of two lines.

Instructional Activity 17.4

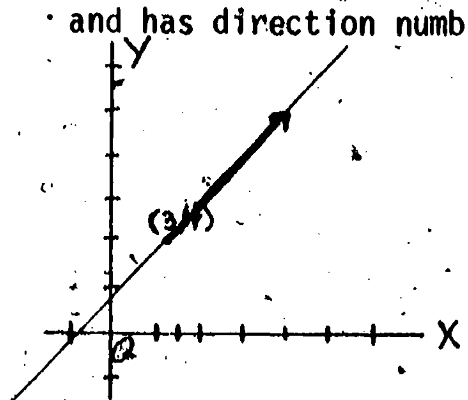
Read pages 55-58.

In example 3-1 on page 55, another pair of direction numbers can be found by considering the vector $-u = P_2P_1 = [-6, 8]$.

On page 58, work problems 1, 2, 3, (do a, c, and e in each problem), 5, 6(a, d, f, and h), 7(a & c), and 8(a & c). 6(b) is worked as an example below:

Example 6(b):

Construct the line that satisfies the following geometric conditions: It contains the point (3, 4) and has direction numbers [5, 6].



Locate the point (3, 4) and from that point draw the vector whose scalar components are [5, 6]. The line that contains that point (3, 4) and the vector [5, 6] is the required line.

Read pages 70-71.
Equation (2-7) is on page 38.

Notice that the intersection of two lines forms 2 pair of angles. One pair will be acute and the other pair will be obtuse (unless the lines are perpendicular.)

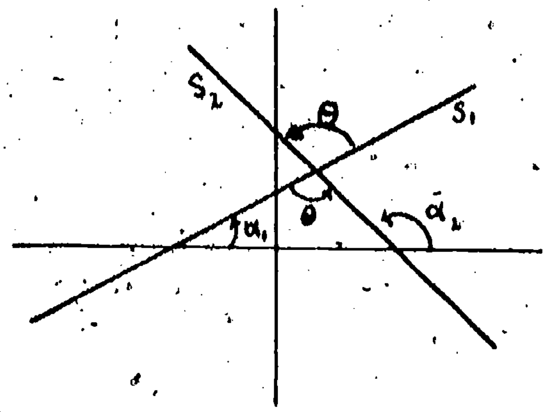


Figure 3-19

In figure 3-19

α_2 is an exterior angle whose measure is equal to the sum of the 2 opposite interior angles

$$\alpha_2 = \alpha_1 + \theta$$

Hence: $\theta = \alpha_2 - \alpha_1$

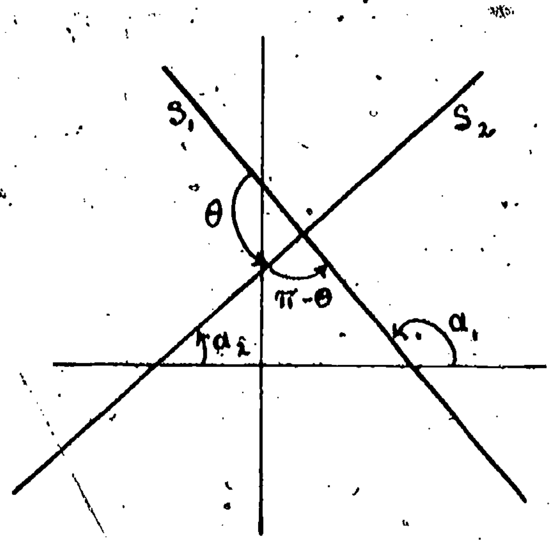


Figure 3-20

In figure 3-20

$$\alpha_1 = \alpha_2 + (\pi - \theta)$$

$(\pi - \theta) =$ ______________

$\tan(\pi - \theta) = -\tan\theta = \tan(\alpha_1 - \alpha_2)$

$\tan(\alpha_1 - \alpha_2) =$

____

$$-\tan\theta = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}$$

$$\tan\theta = \frac{-(s_1 - s_2)}{1 + s_1 s_2}$$

$\tan\theta =$

____

Answers: $\alpha_1 - \alpha_2$, $\dfrac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 - \tan \alpha_2}$, $\dfrac{s_2 - s_1}{1 + s_1 s_2}$ respectively.

On pages 72-73, work problems 1 and 2 (do a, c, e, & h of each problem) 5 and 10.

Self-evaluation - Objective 17.4

1. Given the points $P_1 = (2, 1)$, $P_2 = (-2, -2)$, and $P_3 = (4, 1)$ determine:
   (a) a pair of direction numbers for the line containing the vector $\overrightarrow{P_1P_3}$.
   (b) the slope of the line containing the vector $\overrightarrow{P_2P_3}$.
   (c) the cosine of the acute angle θ between the lines containing the vectors $\overrightarrow{P_2P_1}$ and $\overrightarrow{P_2P_3}$.
   (d) the tangent of the acute angle θ between the lines containing the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.

      (θ is the angle obtained by turning a counterclockwise direction from the line whose slope is s_1 to the line whose slope is s_2)
   (e) if the lines containing the vectors $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_1P_2}$ are perpendicular.

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Answers: 1. (a) [2, -2] or k [2, -2] for any $k \in R$

(b) $s = 1/6$

(c) $\cos \theta = \dfrac{u_1 v_1 + u_2 v_2}{|u|\ |v|}$ where $u = [4, 3]$, $v = [6, 1]$ $\cos \theta = \dfrac{27}{5\sqrt{37}}$

(d) $\tan \theta = \dfrac{s_2 \quad s_1}{1 + s_1 s_2}$ s_1 = slope of line containing vector $\overrightarrow{P_2P_3}$

s_2 = slope of line containing vector $\overrightarrow{P_2P_1}$

$s_1 = 1/6$ $s_2 = 3/4$

$$\tan \theta = \frac{3/4 - 1/6}{1 + (1/6)(3/4)} = 14/27$$

(e) no: $s_1 s_2 \neq 0$

Subunit 17

OBJECTIVE:

(a) *Write an equation in any of the requested standard forms (slope-intercept, point-slope, etc.) of the line which satisfies a given set of conditions and sketch the graph.*

(b) *And given an equation of a line, determine the following: (a) x and y intercepts, (b) slope, (c) direction numbers, (d) 3 points that lie on the line and (e) graph the line.*

Instructional Activities

There are 7 different forms of the equation of a line (a 1st degree equation) which will be introduced in this section. It is to your advantage to be able to develop any of the forms from any of the other forms. In some cases the information or conditions given will dictate the use of a particular form; you will need to recognize when one form is more advantageous than another.

Read pages 59-61. (Parametric Equations)

It is important that you realize that it takes both equations, $x = x_1 + t\,\Delta x$ and $y = y_1 + t\,\Delta y$ to define one line.

Read pages 62-64. (Direction Number Form)

To eliminate t from the parametric equations, solve the equations simultaneously.

$$\Delta y\,(x = x_1 + t\,\Delta x) \quad\rightarrow\quad x\,\Delta y = x_1\,\Delta y + t\,\Delta x\,\Delta y$$

$$-\Delta x\,(y = y_1 + t\,\Delta y) \qquad\quad -y\,\Delta x = -y_1\,\Delta x - t\,\Delta x\,\Delta y$$

this form is very useful

$$\boxed{x\,\Delta y - y\,\Delta x = x_1\,\Delta y - y_1\,\Delta x}$$

$$\frac{x\,\Delta y - y\,\Delta x}{\Delta x\ \ \Delta y} = \frac{x_1\,\Delta y - y_1\,\Delta x}{\Delta x\ \cdot\ \Delta y}$$

$$\frac{x}{\Delta x} - \frac{y}{\Delta y} = \frac{x_1}{\Delta x} - \frac{y_1}{\Delta y}$$

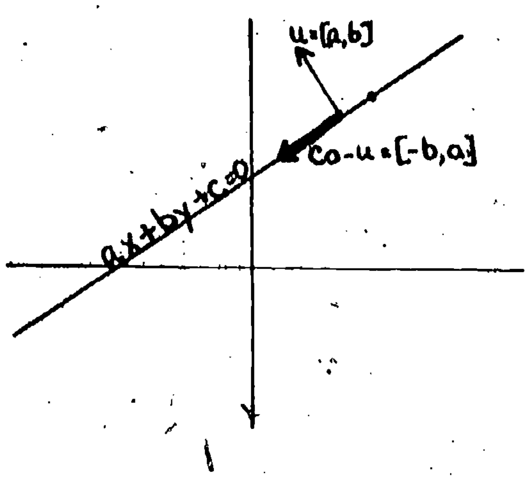
$$\frac{x - x_1}{\Delta x} = \frac{y - y_1}{\Delta y}$$

On pages 64-65 work problems 2 and 3 (a, c, & e), 5, and 6.

Read pages 65-68 (General Equation)

In example 3-12 you are asked to find a pair of direction numbers of the line whose equation is $2x - 3y + 4 = 0$. It is imperative that you know the difference between direction numbers for a line and the coefficient vector of the equation.

The direction numbers for a line are the scalar components of a vector ON the line. (In the figure, the vector co-u.)



The coefficient vector u = [a, b] is a vector perpendicular to the line.

At this point it would be a good exercise for you to compare the direction number form of the equation to the general form.

Comparison:

$x \Delta y - y \Delta x = x_1 \Delta y - y_1 \Delta x$ Direction Number Form

____________________________ General Form

The coefficient of x (a) in the general form corresponds to _____ in the direction number form. The coefficient of y (b) in the general form corresponds to _____ in the direction number form.

So the coefficient of x (a) and the coefficient of y (b) in the general form corresponds to the coefficient of x (Δy) and the coefficient of y ($-\Delta x$) in the direction number form.

Since u = [a, b] is a vector perpendicular to the line $ax + by + c = 0$, u = [a, b] is perpendicular to every vector contained on the line. One of the vectors on the line would be the vector co-u = ________ (see page 48). Since co-u = [-b, a] is a vector on the line, then the scalar components of co-u must also be a set of direction numbers for the line $ax + by + c = 0$.

$$\therefore [-b, a] = \_\_\_\_\_\_\_\_$$

$$\Delta x = -b$$

$$\Delta y = a$$

On pages 68-69, work problems 1, 2, and 3(do a, c, e in each problem), 4, 6, 11(a & c), 13, and 16.

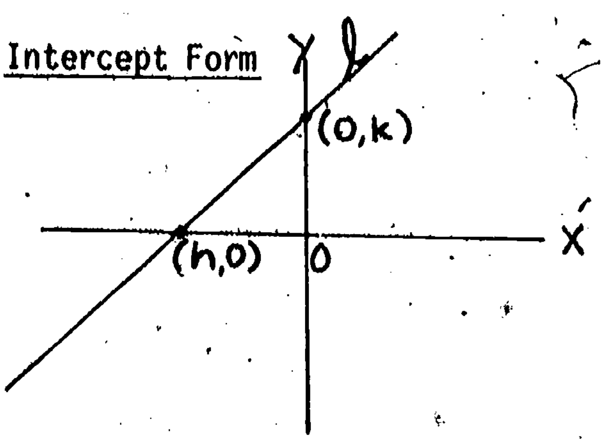
On pages 72-73, work problems 3 and 6 (do a, c, & e in each problem).

Read pages 73-75 (point-slope, intercept, slope-intercept -- don't read the section on the normal form.)

Point-Slope

In section 3-7, page 73, in the first paragraph, since the slope of a line, $s = \Delta y/\Delta x$, let $\Delta x = 1$, then $s = \Delta y$. Hence a pair of direction numbers $[\Delta x, \Delta y] = [1, s]$.

Intercept Form



In the first paragraph of section 3-8 on page 73 since h and k are the x and y intercepts then 2 points on the line would be $P_1 = (h, 0)$ and $P_2 = (0, k)$. and the slope s would be:

s = ________

Answer: $s = -k/h$

A pair of direction numbers $[\Delta x, \Delta y]$ would be ________.

The direction numbers are the scalar components of the vector co-u = ________.

A vector perpendicular to the line would be u = ________.

The equation of the line $ax + by + c = 0$ where $c = -ax_1 - by_1$ and

$u = [a, b] = [k, h]$ becomes ____________ or $kx + hy = kh$.

dividing both sides of the equation by hk yields:

$x/h + y/k = 1$ Intercept form

Answers: Direction numbers: $[h, -k]$, vector perpendicular: $u = [k, h]$,

equation: $kx + hy = -kh - h(0)$.

Slope-Intercept Form

As in the point-slope form, a pair of direction numbers is $[1, s]$ (where the slope of the line is s). The y-intercept is k, hence, a point on the line is $P = (0, k)$.

Then using the direction number form, $x \Delta y - y \Delta x = x_1 \Delta y - y_1 \Delta x$

the equation becomes ____________, or $sx - y = -k$

and solving for y, $y = sx + k$ Slope-intercept Form

On pages 78-79, work problems 1(a, b, c, & d), 2(a, c), 12, 14, 15, (HINT: Make use of the slopes of the lines) and 18.

Read pages 75-78 (Normal Form)

The figure 3-22 is provided for you on the following page so you can refer to the figure while reading the material on page 76. Also provided is further explanation of the derivation of the normal form of the equation of the line -- (corresponding to the top of page 76).

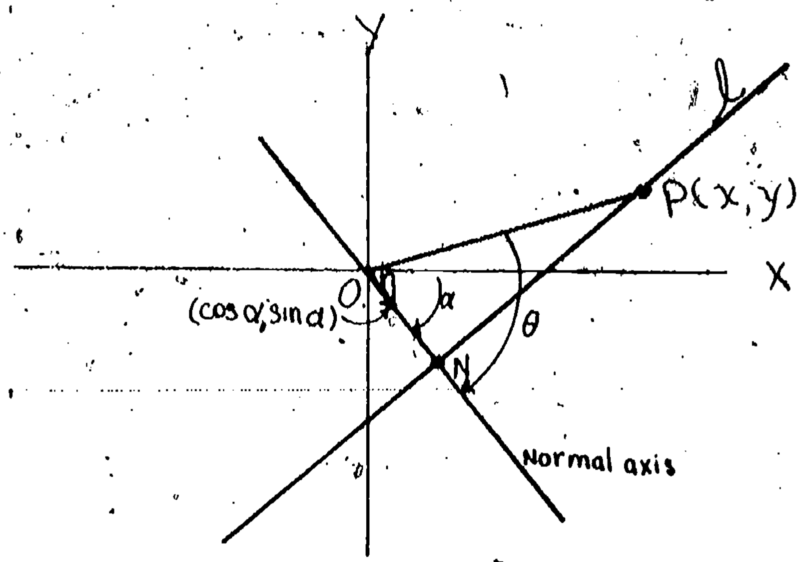


Figure 3-22

Since $\cos\theta = \frac{u \cdot v}{|u||v|}$ where $u = \overrightarrow{OP} = [x, y]$
$v = \overrightarrow{OQ} = [\cos\alpha, \sin\alpha]$

then $\cos\theta =$ ______________________

and since $\cos^2\alpha + \sin^2\alpha = 1$, $\cos\theta =$ ______________________

and $\sqrt{x^2 + y^2}\,\cos\theta = x\cos\alpha + y\sin\alpha$

BUT $\sqrt{x^2 + y^2} = \overrightarrow{OP}$

so, $x\cos\alpha + y\sin\alpha =$ ______________

and since $\cos\theta = \frac{|\overrightarrow{ON}|}{|\overrightarrow{OP}|}$, then $|\overrightarrow{OP}|\cos\theta = |\overrightarrow{ON}| = p$

HENCE

$x\cos\alpha + y\sin\alpha = p$ Normal Form

Answers: $\frac{[x, y] \cdot [\cos\alpha, \sin\alpha]}{\sqrt{x^2 + y^2}\sqrt{\cos^2\alpha + \sin^2\alpha}}$, $\frac{x\cos\alpha + y\sin\alpha}{\sqrt{x^2 + y^2}}$, $|\overrightarrow{OP}|\cos\theta$

Notice the two conditions (middle of page 76) that must be met before a linear equation is said to be in normal form. (Both must be met.)

They are: 1. ______________________________.

2. ______________________________.

Also notice that to change the general form $ax + by + c = 0$ to normal form you must change the coefficient vector $[a, b]$ to a unit vector (to meet condition 1).

Since $\left[\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right]$ is a unit vector, divide the equation $ax + by + c = 0$ by

____

Answer: $\sqrt{a^2 + b^2}$

Hence: $\frac{a}{\sqrt{a^2 + b^2}} x + \frac{b}{\sqrt{a^2 + b^2}} y = \frac{-c}{\sqrt{a^2 + b^2}}$.

To meet condition 2, the constant term was moved to the right hand side of the equation. Also the constant term must be positive. This is done by introducing a constant term $e = \pm 1$.

OR

$$\frac{a}{e\sqrt{a^2 + b^2}} x + \frac{b}{e\sqrt{a^2 + b^2}} y = \frac{-c}{e\sqrt{a^2 + b^2}}.$$

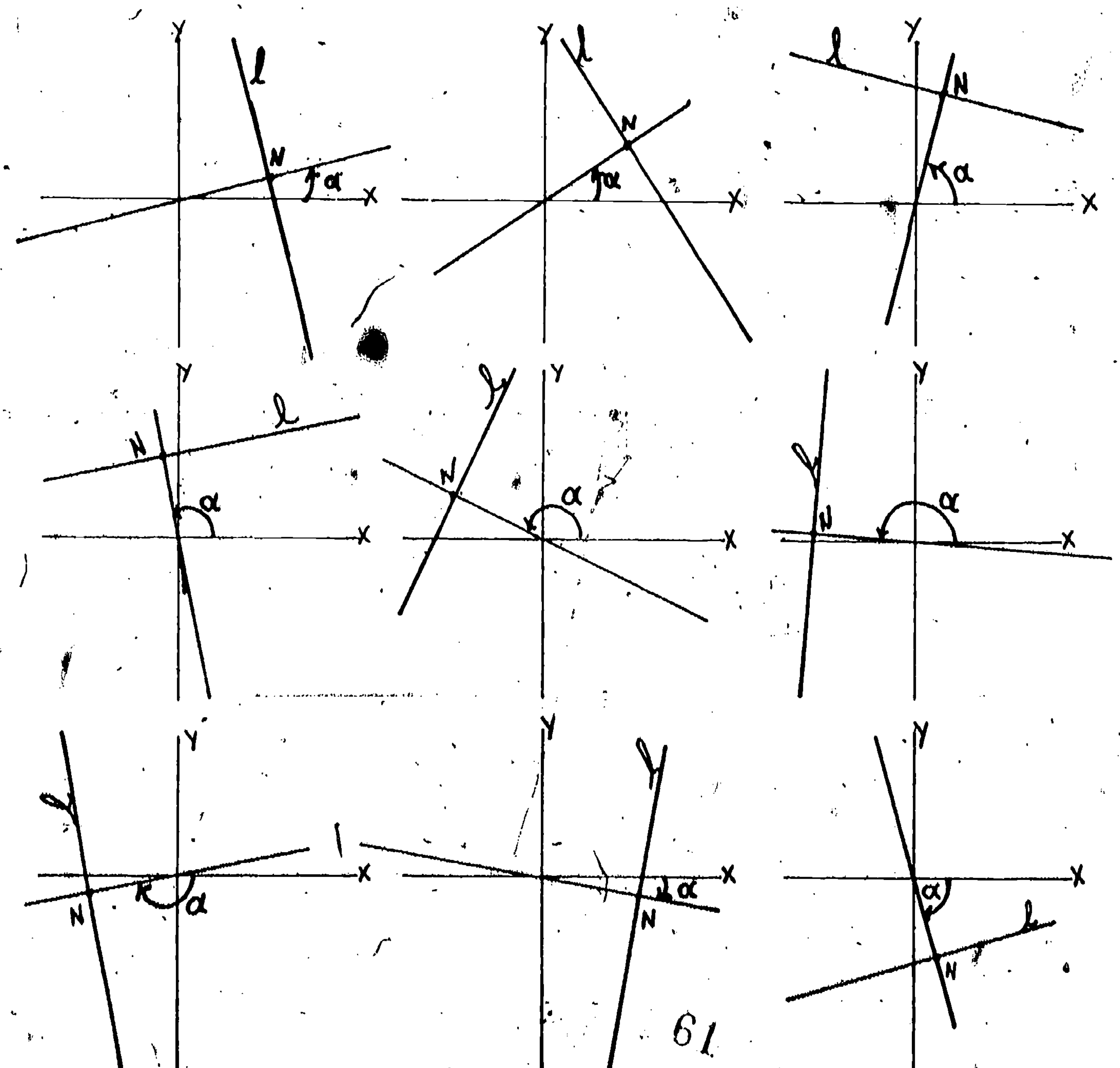
Notice $\left[\frac{a}{e\sqrt{a^2 + b^2}}, \frac{b}{e\sqrt{a^2 + b^2}}\right]$ is still a unit vector for either value of e.

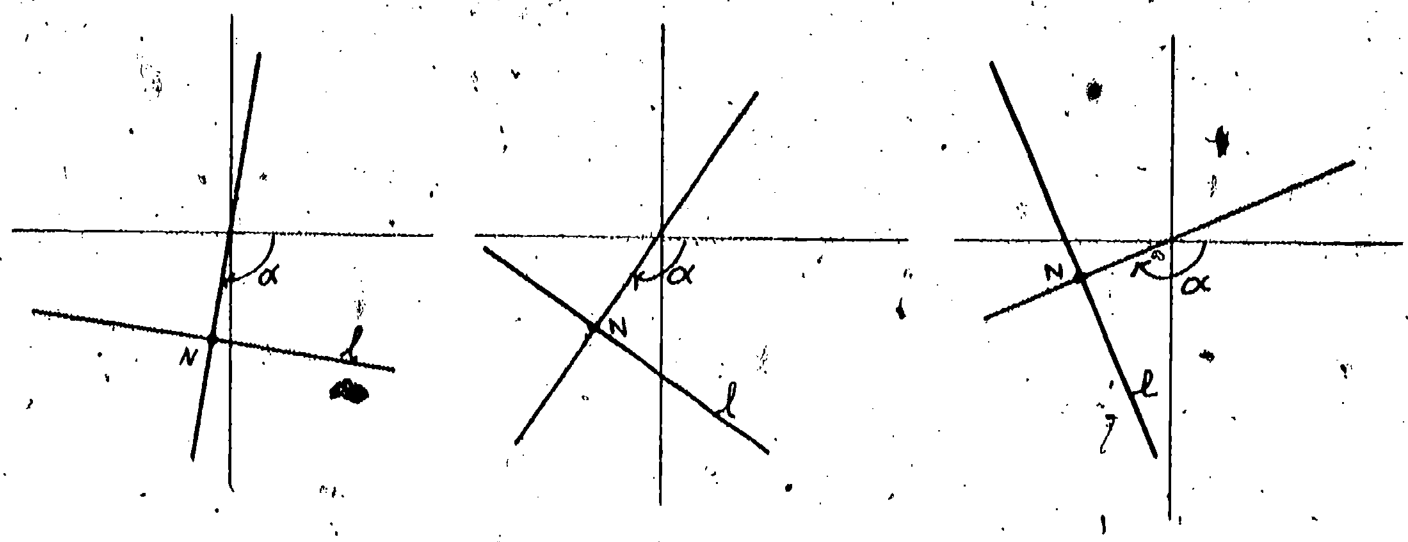
The value of e is determined by the sign of the constant term c. If the constant term c (in the general form) is positive, then e = _____. If the constant term c is negative, then c = _____.

Answers: -1, +1, respectively.

Read very carefully the bottom paragraph on page 76 and the top of page 77.

Since $-180^0 \le \alpha \le 180^0$, (you should convice yourself of this by observing the figures below) notice what happens as the normal axis is rotated through the angle $0 < \alpha \le 180^0$. What happens when $\alpha > 180^0$?





Sin α is positive when $0^0 < \alpha < 180^0$. (If $\sin \alpha = 0$, then $\alpha = 180^0$ or 0^0) α must be measured counterclockwise when $\sin \alpha$ is positive. Since $\sin \alpha$ is negative when $-180^0 < \alpha < 0^0$, the angle α is measured clockwise when $\sin \alpha$ is negative.

Cos α is positive when $|\alpha| < 90^0$; then the angle α must be acute when $\cos \alpha$ is positive. Since $\cos \alpha$ is negative when $90^0 < |\alpha| < 180^0$, the angle α is obtuse when $\cos \alpha$ is negative. ($\cos \alpha = 0$ when $\alpha = 90^0$ or -90^0).

In example 3-19 (page 77) notice that both conditions are met in the equation:

$$-\frac{3}{5}x + \frac{4}{5}y = 2$$

$\left[-\frac{3}{5}, \frac{4}{5}\right]$ is a unit vector, and the constant term is positive and is written by itself on one side of the equation.

Also notice in example 3-20 that the equation $-1/2 + \sqrt{3}/2\ y = 5$ is in normal form while the equation $x - \sqrt{3}\ y + 10 = 0$ is in the general form.

On pages 78-79, work problems 3(a, c, e), 10(a, b, c, d, g), 11(a, c, e) and 20.

Self-evaluation

1. Given the 2 points $P_1 = (-3, 2)$ and $P_2 = (3, 0)$ write the equation of the line through P_1P_2 in all of the 7 forms and draw the graph of the line.

2. Given the equation $3x - 2y + 5 = 0$, determine

   (a) the slope

   (b) the x & y intercepts

   (c) the normal intercept (p)

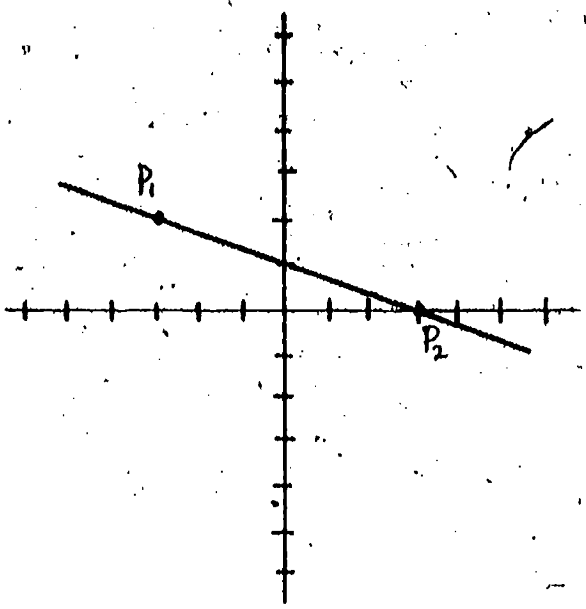
   (d) a vector perpendicular to the line

   (e) a set of direction numbers for the line

Answers:

1. $\overrightarrow{P_1P_2} = [6, -2]$ hence $\Delta x = 6$, $\Delta y = -2$

$s = \Delta y / \Delta x = -2/6 = -1/3$



(1) direction number form

$x \Delta y - y \Delta x = x_1 \Delta y - y_1 \Delta x$

$x(-2) - y(6) = (-3)(-2) - 2(6)$

$-2x - 6y = 6 - 12$

$2x + 6y - 6 = 0$

Notice the coordinates of P_2 and P_1 satisfy the equation, hence $2x + 6y - 6 = 0$ is the required equation.

(2) slope intercept form

$2x + 6y - 6 = 0$

OR $6y = -2x + 6$

$y = -1/3x + 1$

OR $3y = -x + 3$

(3) point slope form $y - y_1 = s(x - x_1)$

$P_1 = (-3, 2)$

$s = -1/3$

$y - 2 = -1/3 (x - (-3))$

OR $3y - 6 = -x - 3$

$x + 3y - 3 = 0$

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(4) intercept form $x/h + y/k = 1$

$k = 1$ (from equation above: y - 1[illegible]x + 1)

$h = 3$ ($P_2 = (3, 0)$)

$x/3 + y/1 = 1$

OR $x + 3y = 3$

$x + 3y - 3 = 0$

(5) parametric form $x - x_1 + t \cdot \Delta x$

$y = y_1 + t \, \Delta y$

$\Delta x = 6$

$\Delta y = -2$

$P_1 = (-3, 2)$

$x = -3 + 6t$

$y = 2 - 2t$

You should verify that the parametric form can be reduced to the equation $x + 3y - 3 = 0$ by eliminating the parameter t.

(6) normal form

$2x + 6y - 6 = 0$

$$\frac{2x}{\sqrt{2^2 + 6^2}} + \frac{6y}{\sqrt{2^2 + 6^2}} = \frac{6}{\sqrt{2^2 + 6^2}}$$

$$\frac{2}{\sqrt{40}} x + \frac{6}{\sqrt{40}} y = \frac{6}{\sqrt{40}} \qquad \sqrt{40} = 2\sqrt{10}$$

$$\text{SO } \frac{x}{\sqrt{10}} + \frac{3}{\sqrt{10}} y = \frac{3}{\sqrt{10}}$$

(7) general form $ax + by + c = 0$

$[\Delta x, \Delta y] = [6, -2]$ so $[a, b] = [2, 6]$ $P_I = (-3, 2)$

$c = -ax_1 - by_1$

$c = (+2)(-3) - (6)(2)$

$c = 6 - 12 = -6$

$ax + by + c = 0$ becomes

$2x + 6y - 6 = 0$

2. (a) $-2y - 3x - 5$ $x = 3/4$ $k = 5/2$

$y = 3/2x + 5/2$

(b) y intercept $(0, 5/2)$

x intercept $(-5/3, 0)$

(c) $3x - 2y + 5 = 0$

$$\frac{3x}{-\sqrt{9+4}} - \frac{2y}{-\sqrt{9+4}} = \frac{-5}{-\sqrt{9+4}} \qquad p = 5/\sqrt{13}$$

$$-3x/\sqrt{13} + 2y/\sqrt{13} = 5/\sqrt{13}$$

(d) [3, -2]

(e) [2, 3]

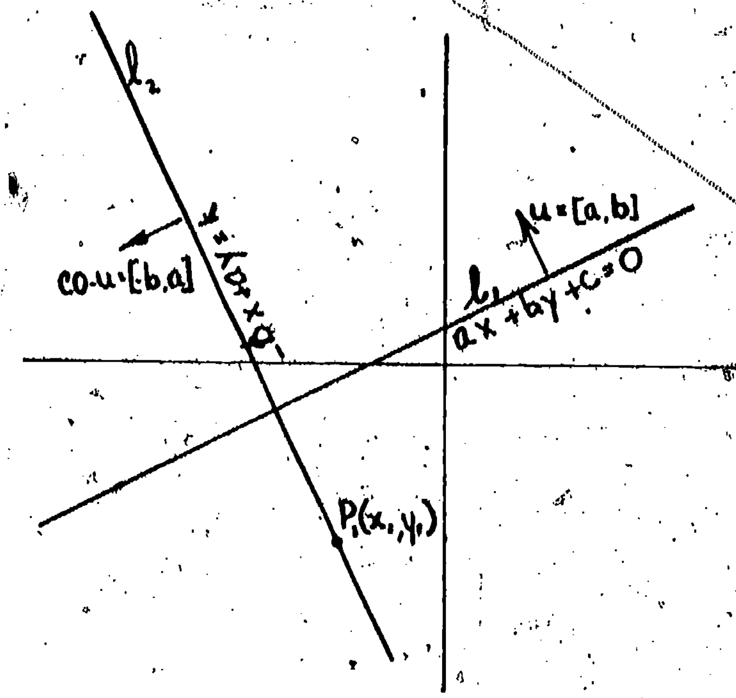
The objective (a) of this unit states, "*Write an equation in any of the requested standard forms of the line which satisfies a given set of conditions and sketch the graph.*"

In the previous material you have learned the forms and how to write a linear equation given some conditions. The conditions you have used so far are: 2 points; a point and the slope, the slope and y-intercept, direction numbers and a point, a coefficient vector and a point, or some combination of these conditions.

You might be asked to write the equation of a line parallel or perpendicular to a given line or to write the equation of a line parallel to a given line and at a given distance from a fixed point. Another problem you might encounter is that of writing the equation of the line that bisects a given angle (where you know the equations of the lines forming the angle.) So in this section you will investigate some of these conditions and learn how to use them to write the required equations.

Read pages 80-81 (Writing equations of lines parallel or perpendicular to given lines.)

In the discussion on writing the equation of a line perpendicular to the line $ax + by + c = 0$, the coefficient vector $u = [a, b]$ is perpendicular to the line $ax + by + c = 0$. Therefore, the vector $u = [a, b]$ is parallel to the required line and the vector $\text{co-}u = [-b, a]$ is the coefficient vector for the required line. (The one perpendicular to $ax + by + c = 0$).



In the figure, let $\ell_1 = ax + by + c = 0$ and ℓ_2 be the equation of the line perpendicular to ℓ_1. The equation of ℓ_2 is of the form $-bx + ay = k$, (co-u = $[-b, a]$ is the coefficient vector for ℓ_2). Since (x_1, y_1) is a point on ℓ_2, $k = -bx_1 + ay_1$.

Hence: $-bx + ay = -bx_1 + ay_1$, OR

$$bx - ay = bx_1 - ay_1$$

You may also use the direction number form since $u = [a, b]$ is parallel to ℓ_2. u is a pair of direction numbers for ℓ_2. and

$x \Delta y - y \Delta x = x_1 \Delta y - y_1 \Delta x$

Hence ______________________

Answer: $bx - ay = bx_1 - ay_1$

On page 81-82, work problems 1(a, c), 2(a, c), 3(a, c), 4(a, c), 5 and 9.

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Self-evaluation

1. Find an equation of the line that passes through the point (5, -1) and is parallel to the line $4x - 3y = 1$.

2. Find an equation of the line that passes through the point (2, -1) and is perpendicular to a line whose slope is 3/2.

Answers:

1. The equation is of the form $ax + by = k$, and since the line passes through the point $(5, -1)$, then $k = ax_1 + by_1$ so $k = 4(5) - 3(-1)$ or $k = 20 + 3 = 23$. Therefore, the required equation is $4x - 3y = 23$.

2. The slope of the given line $s = \frac{\Delta y}{\Delta x} = \frac{3}{2}$. Therefore the slope of the line perpendicular to the given line will be $-2/3$. A set of direction numbers for the required line is $[3, -2]$ and the equation

$$x \Delta y - y \Delta x = x_1 \Delta y - y_1 \Delta x$$

becomes

$$-2x - 3y = 2(-2) - (-1)(3)$$

or

$$2x + 3y = 1$$

Read pages 83-85 (Distance from a point to a line)

In the second paragraph, page 83, $ax_1 + by_1 + c \neq 0$ because only points on the line satisfy the equation of the line. Notice that the distance from a point to a line is perpendicular distance.

On pages 85-86, work problems 1, 3(a, c), 4(a, c) (Hint, since parallel lines are everywhere equidistant the distance from ANY POINT ON ONE LINE to the other line is the distance between 2 parallel lines), 9, 11, and 15.

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Self-evaluation

1. Find the distance from the point P - (3, 5) to the line
   $2x - 5y - 1 = 0$.

2. Find the height of a triangle whose base lies on the line
   $x + y = 2$ and the vertex of the angle opposite the base is at
   the point (5, 4).

3. Write the equation of the line parallel to the vector u = [4, 3]
   and at a distance of 4 units from the point (-1, -2).

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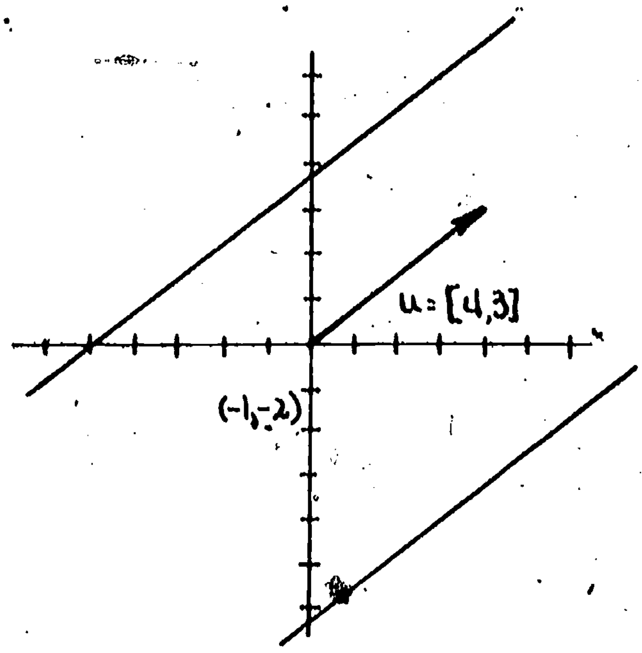
Answers:

1. $\delta = \dfrac{2(3) - 5(5) - 1}{\sqrt{9 + 25}} = 20/\sqrt{34}$

2. $\delta = \dfrac{5 + 4 - 2}{\sqrt{25 + 16}} = 7/\sqrt{41}$

3.



The equation of the line is $ax + by + c = 0$, where $[a, b] = [-3, 4]$ and

$$\delta = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right| \quad \text{where } \delta = 4$$

so, $\pm 4 = \dfrac{-3(-1) + 4(-2) + c}{\sqrt{9 + 16}} = \dfrac{3 - 8 + c}{5}$

$20 = 3 - 8 + c$

$-20 = 3 - 8 + c$

$c = 25, -15$

Hence, the equations of the lines are

$-3x + 4y - 15 = 0$ OR $3x - 4y + 15 = 0$

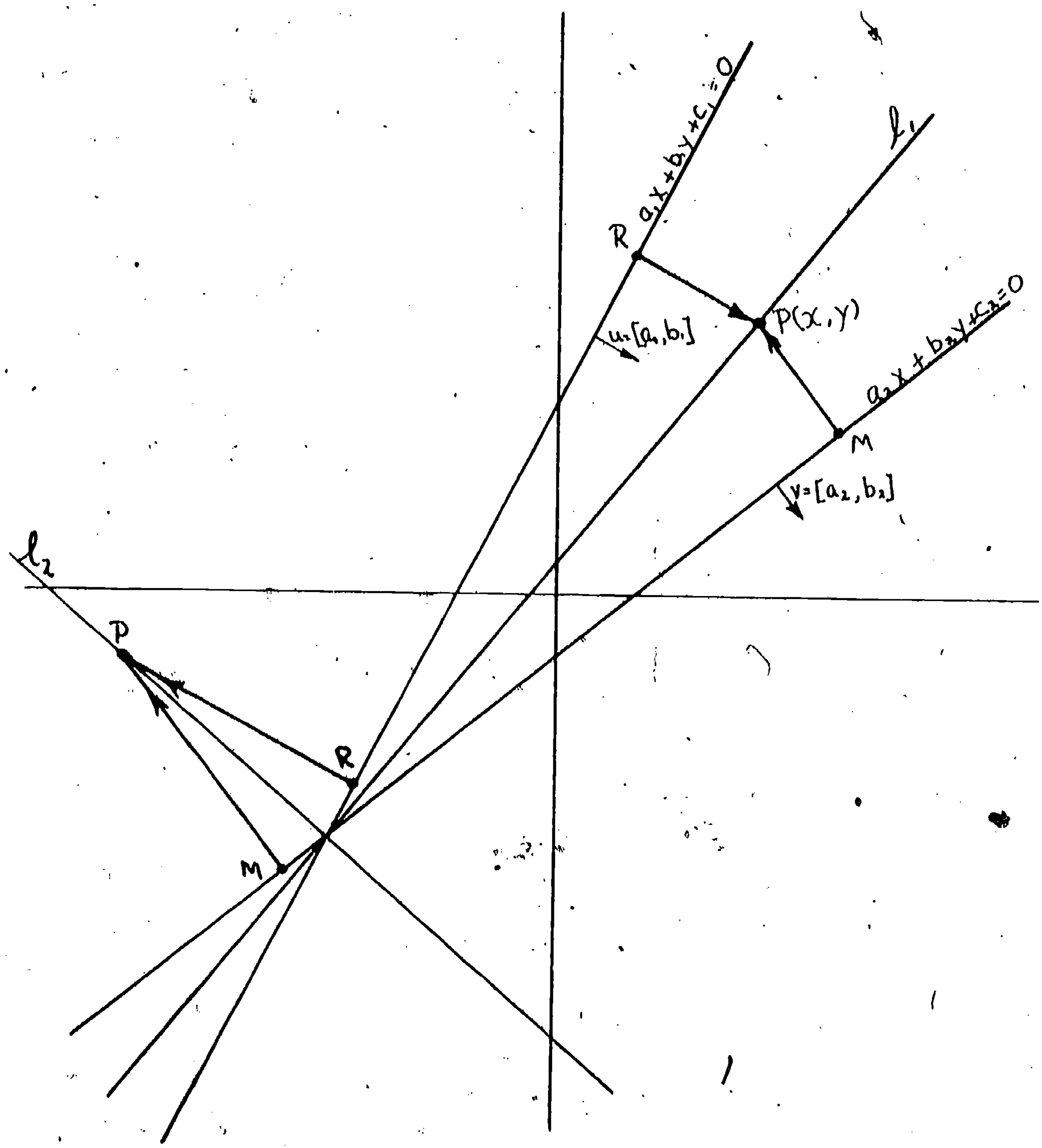
$-3x + 4y + 25 = 0$ OR $3x - 4y - 25 = 0$

Read pages 91-93 (Equations of the bisectors of angles).

Writing the equation of a line that is an angle bisector is a direct application of finding the distance from a point to a line.

In the first paragraph on page 71, if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then the lines intersect in a unique point.

The figure below is the same as figure 3-32 in your text with the other angle bisector included.



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For the point P on ℓ_2 (in the interior of the obtuse angle) $\overrightarrow{RP}$ and the coefficient vector u and $\overrightarrow{MP}$ and the coefficient vector v are both parallel in the opposite sense, therefore the equation of ℓ_1 is

$$\frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}$$

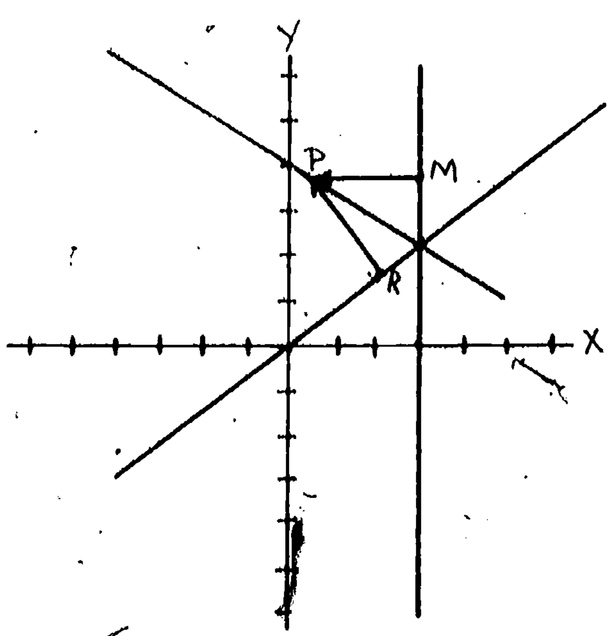
For the point P on ℓ_1 (in the interior of the acute angle) $\overrightarrow{RP}$ and the coefficient vector u are parallel in the same sense and $\overrightarrow{MP}$ and the coefficient vector v are parallel in the opposite sense.

Hence the equation of ℓ_1 is

$$\frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}$$

In example 3-26 the origin is contained in the interior of the angle but is not located on the bisector of the angle.

Problem 3, page 93, is begun below:



Find an equation of the bisector of the obtuse angle between the lines $x - 3 = 0$ and $3x - 4y = 0$. The coefficient vector u for $x - 3 = 0$ is ________. The coefficient vector v for $3x - 4y = 0$ is ________.

Answers: [1, 0], [3, -4] , respectively.

Since $\overrightarrow{MP}$ and $\overrightarrow{RP}$ are parallel in the opposite sense (see figure) to u and v, the equation is of the form

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a\ x + b\ y + c}{\sqrt{a_2^2 + b_2^2}}$$

or

$$\frac{x - 3}{\sqrt{1}} = \frac{3x - 4y}{\sqrt{25}}$$

Hence the required equation is

$$2x + 4y - 15 = 0$$

On pages 93-94, work problems 1, 2, 6, and 12.

Self-evaluation

1. Find the equation of the bisector of the acute angle formed by the intersection of the lines $4x - 3y + 3 = 0$ and $y = 0$.

2. Prove that the angle bisectors of the triangle formed by the coordinate axes and the line $3x - 4y + 12 = 0$ intersect in a point.

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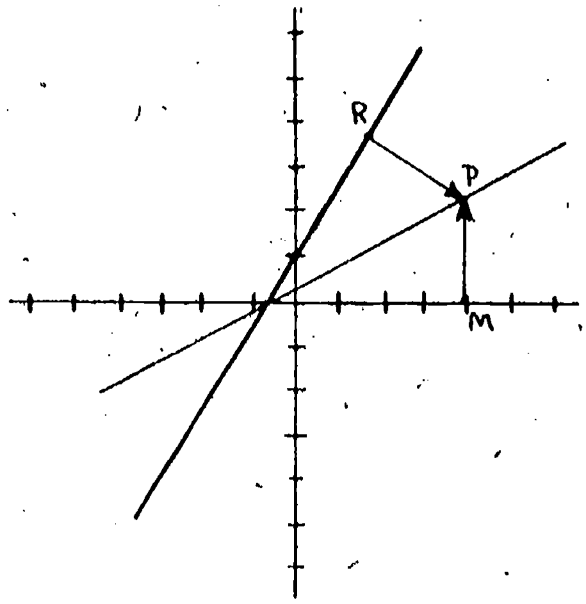
Answers:

1.



$u = [4, -3]$, $v = [0, 1]$. Hence, $\overrightarrow{RP}$ and u and $\overrightarrow{MP}$ and v are parallel in the same sense so the equation of the bisector is of the form

$$\frac{a_1x + b_1y + c_1 = 0}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

or

$$\frac{3x - 2y + 3}{\sqrt{16 + 9}} = \frac{x}{\sqrt{1}} ;$$

$$5x + 2y - 3 = 0$$

2. See your instructor.

Rationale:

You have seen that the graph of a linear function ax + by + c = 0 is a straight line. A natural step would be to consider the graph of second degree equations. When this is done, it is found that four new types of curves appear as possible graphs of such functions. These curves, the circle, ellipse, hyperbola, and parabola have been known and studied by mathematicians since ancient times. Collectively, along with certain combinations of two straight lines, they are called the conics or conic sections.

Conics also serve as mathematical models for many scientific applications. For example:

(a) The path of the projectile from a gun (or a suborbital rocket), when acted on by gravity alone, is a parabola.

(b) The orbits of the planets and their satellites are ellipses.

(c) The orbits of comets are either elliptical or hyperbolic.

(d) If the weight of a roadbed suspended from a cable and the weight of the cable are uniformly distributed horizontally, the cable assumes the shape of a parabola.

(e) In the construction of bridges, elliptic arches are used as well as parabolic arches.

So let us begin by defining a general conic.

Objectives:

18.5 Sort conics by name given the equation or enablers, or graph.

18.4 For a hyperbola, given a set of conditions, write the equation in simple form, determine the enablers pertaining to that conic (a, c, focal point, etc.) and graph the conic. (The condition can include the general equation.)

18.3 *For an ellipse, given a set of conditions, write the equation in simple form determine the enablers pertaining to that conic (a, c, focal point, etc.) and graph the conic. (The condition can include the general equation.)*

18.2 *For a parabola, given a set of conditions, write the equation in simple form, determine the enablers peratining to that conic (a, c, focal point, etc.) and graph the conic. (The condition can include the general equation.)*

18.1 *State the definition of a conic and apply the definition to determine the equation given the focal point(s), equation of the directrix and eccentricity.*

Prerequisites:

Unit 17, Unit 14 (Linear Algebra -- Objective 14.4 Solve a system of three linear equations using Cramer's Rule.)

Unit Activities:

Lectures 3, 4, and 5.

The lectures over this unit will have the following outline:

Lecture 1. The parabola

A. Definitions - parabola, and terms: focal point, focal radii, latus rectum, etc.

B. Determining enablers and graphing

C. Determining the equation of parabola

Lecture 2. The ellipse and circle

A. Definitions - ellipse, circle and terms: center, focal point, directrix, radius, etc.

B. Determining enablers and graphing

C. Determining equations

Lecture 3. The hyperbola

A. Definitions - hyperbola and terms: center, major axis, focal chord, etc.

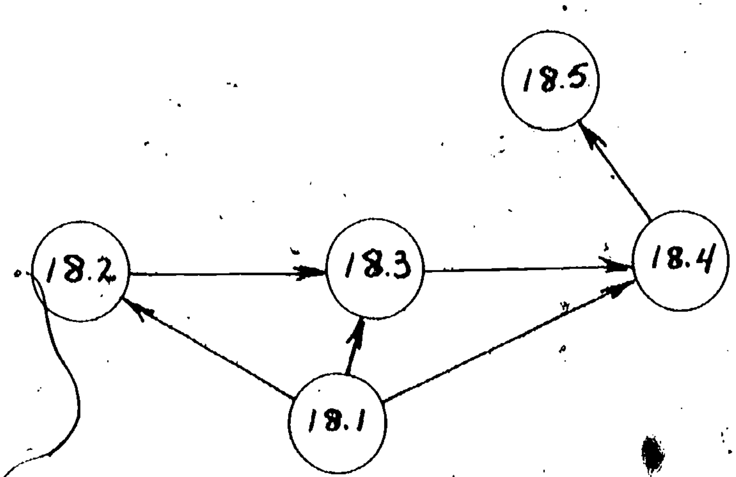
B. Determining enablers and graphing

C. Determining equations

D. Asymptotes, special hyperbolas (conjugate and equilateral.)

Procedural Operations:

In this unit you have the same options as in Unit 17. After objective 18.1 you can go to 18.2, 18.3 or 18.4, but you are advised to follow the sequence 18.2, 18.3, and 18.4 because this order follows the arrangement of the text. Below is a graphic representation of the order of the objectives:



Average time: 3 weeks.

Objective 18.1

State the definition of a conic and apply the definition to determine the equation given the focal point(s), equation of the directrix and eccentricity.

Activities 18.1 (1 is suggested)

1. Your Text and Study Guide

   Morrill, W. K., Analytic Geometry, pp. 138-139

2. Individual Assistance

3. Informal Group Session

Self-evaluation Objective 18.1

1. If the eccentricity of a conic is 5/4, the conic is a (an) ________.

2. Derive the equations of the following conics:

   (a) $e = 2/3$, $F = (6, 0)$, equation of directrix, $x = 27/2$

   (b) $e = \sqrt{2}$, $F = (0, 2)$, equation of directrix, $y = 1$

Answer:

1. hyperbola

2. (a) $\frac{x^2}{81} + \frac{y^2}{45} = 1$ (b) $\frac{y^2}{2} - \frac{x^2}{2} = 1$

Objective 18.2

(For a parabola) Given a set of conditions, determine the enablers pertaining to that conic (a, C, focal point, etc.) write the equation in simple form, and graph the conic. (The conditions can include the general equation.)

Activities - Objective 18.2 (2 and 7 are suggested)

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 22-23 (Symmetry, pp. 112-114 (Symmetry) pp. 142-147.
   Exercises: p. 23, problems 1 and 3
   pp. 145-146, problems 1, 3, 4, 7(a) and 8(a)
   pp. 147-148, problems 1, 3(a, b, c, f(and 7
   pp. 151-152, problems 4(a), and 9

2. Your Text and the Study Guide

3. Solved Problems

   Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, p. 47, problems 1 and 2, p. 48 problem 4.

4. Other Reading Sources

   Fuller, Gordon, Analytic Geometry, pp. 54-58 (Parabolas) pp. 61-62 (Symmetry).

   Murdoch, David C., Analytic Geometry, pp. 127-129.

   Protter, Morrey, Analytic Geometry, pp. 103-105

5. Individual Assistance

   Your instructor is available -- use this source of information when you need help. A short visit can probably help clear up many of your problems.

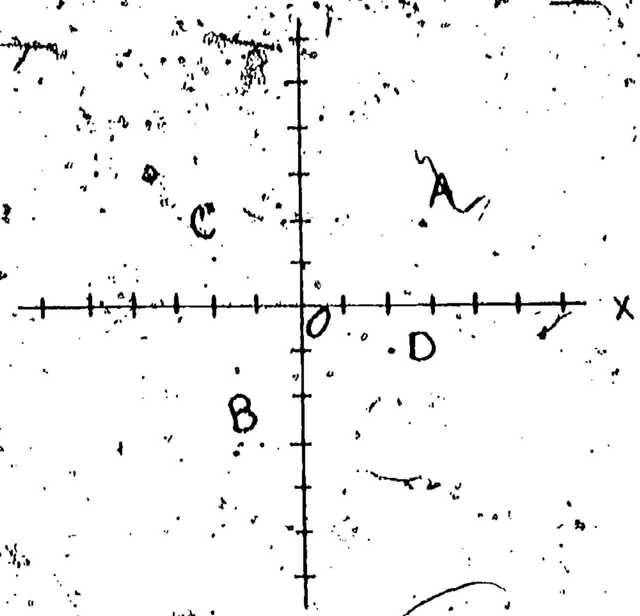
6. Informal Group Session

7. Lecture 3 (Parabola)

   (See the lecture schedule)

Self-evaluation - Objective 18.2

1. Sketch the image of the following set of points with respect to the y-axis. (Use space on the following page.)



2. Show that the curve whose equation is $y^2 = 8x$ is not symmetric with respect to the origin.

3. Match the equations of the directrices to the proper equations.

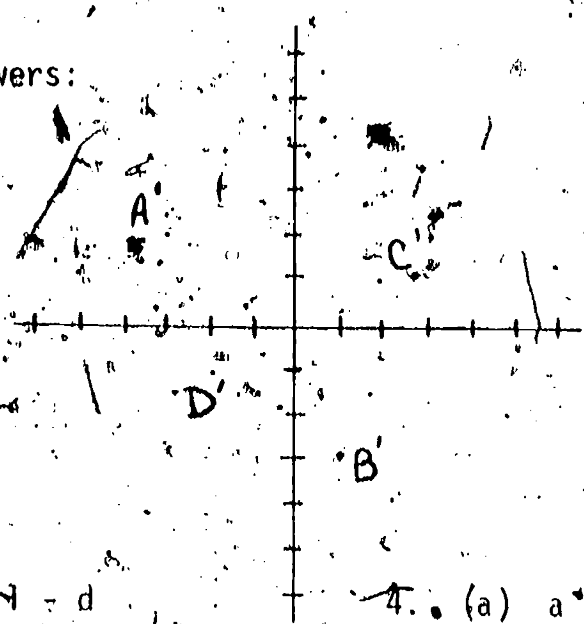
1. $y = 4$ _____ (a) $x^2 = 4ay$
2. $x = 3/2$ _____ (b) $y^2 = 4ax$
3. $y = -2/3$ _____ (c) $y^2 = -4ax$
4. $x = -4$ _____ (d) $x^2 = -4ay$

4. Given the following conditions, determine the value of a, determine the equation of the parabola that satisfies those conditions, and the graph of the parabola.
   (a) The vertex is a (0, 0) and the focus is at (-3, 0)
   (b) One of the coordinates of the intersection of the parabola and its latus rectum are (-4, -2) and the axis of symmetry is the y-axis, and the vertex is at the origin.

5. Discuss* and sketch the curve $2y^2 + 37x = 0$.

*"Discuss" means to give the coordinates of the vertex, focus, and end points of the latus rectum, and an equation of the directrix.

Answers:

1. 

2. Replace x with -x and y with -y.

$y^2 = 8x \quad ? \quad (-y)^2 = 8(-x)$

$y^2 = 8x \quad \neq \quad y^2 = -8x$

Hence, the curve is not symmetric with respect to the origin.

3. 1 - d
   2 - c
   3 - a
   4 - b

4. (a) $a = 3$

$y^2 = -4ax$

$y^2 = -12x$

(b) $a = 2$

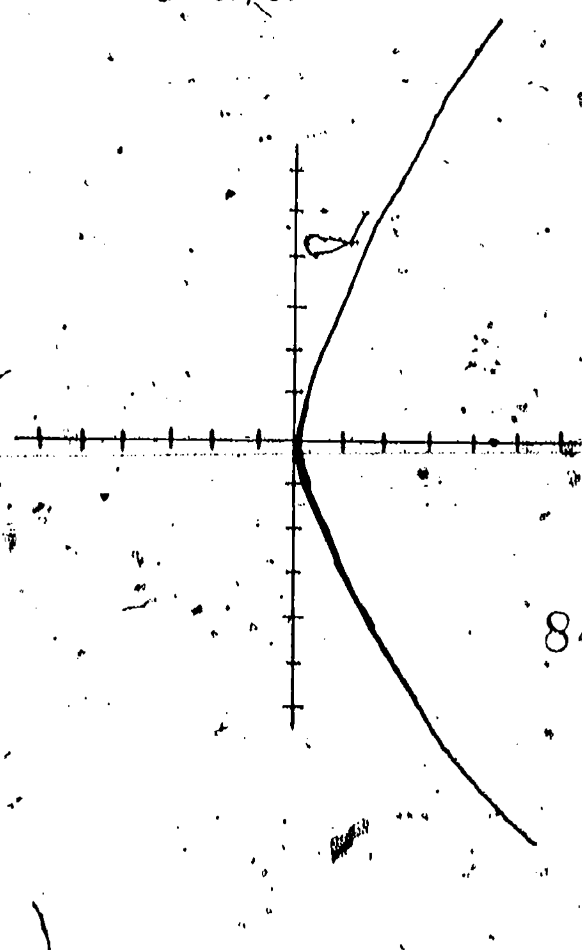
$x^2 = -4ay$

$x^2 = -8y$

5. $2y^2 - 37x = 0$

$y^2 = 37/2x$

Hence, $4a = 37/2$ and $a = 37/8$, the parabola opens to the right, coordinates of the focus are $(37/8, 0)$ coordinates of the end points of the latus rectum are $(37/8, 37/4)$ and $(37/8, -37/4)$, equation of the directrix is $x = -37/8$.



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Objective 18.3 Ellipses and Circles

(For an ellipse) Given a set of conditions, determine the enablers pertaining to the conic (a, c, focal point, etc.) Write the equation in simple form, and graph the conic. (The conditions can include the general statement.)

Activities - Objective 18.3 (2 and 7 are suggested)

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 153-163, pp. 165-166 (Method 1) and the Study Guide Task 6 (Equation of a Simple Circle).
   Exercises: p. 157, problems 2 and 5(a, b, c, d)
   pp. 158-159, problems 1, 4-8
   pp. 159-160, problems 1, 3, 5
   pp. 163, problems 1(a), 2(a, c, g), 3(a, d, f, g, h, 1), 4(a), 5 and 7

2. Your Text and the Study Guide

3. Solved Problems

   Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, pp. 52-57, problems 1, 2, 3, 4, 8, 9, 14, 15, 18.

4. Other reading Sources

   Protter, Morrey, Analytic Geometry, pp. 114-120.

   Murdoch, David C., Analytic Geometry, pp. 119-122.

   Fuller, Gordon, Analytic Geometry, pp. 66-69.

   NOTE: All three of these sources use a different definition for an ellipse than your text. Their definition is proved as a theorem (5-10) in your text.

5. Individual Assistance

6. Informal Group Session

7. Lecture 4 (Ellipse & Circle) (See the lecture schedule)

Self-evaluation - Objective 18.3

1. Use the string, tacks and cardboard (In the envelope on the back cover) to construct an ellipse whose equation is: $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

2. Discuss* the ellipse whose equation is $16x^2 + 9y^2 = 144$ and sketch the graph.

3. The earth's orbit is an ellipse with the sun at one of the foci. If the semi-major axis of the ellipse is 92.9 million miles and the eccentricity is 0.017, find the greatest and least distance of the earth from the sun.

*"Discuss" means to give the coordinates of the vertex, foci, the end points of the latus rectum, intersection of curve and minor axis, and equations of the directrices.

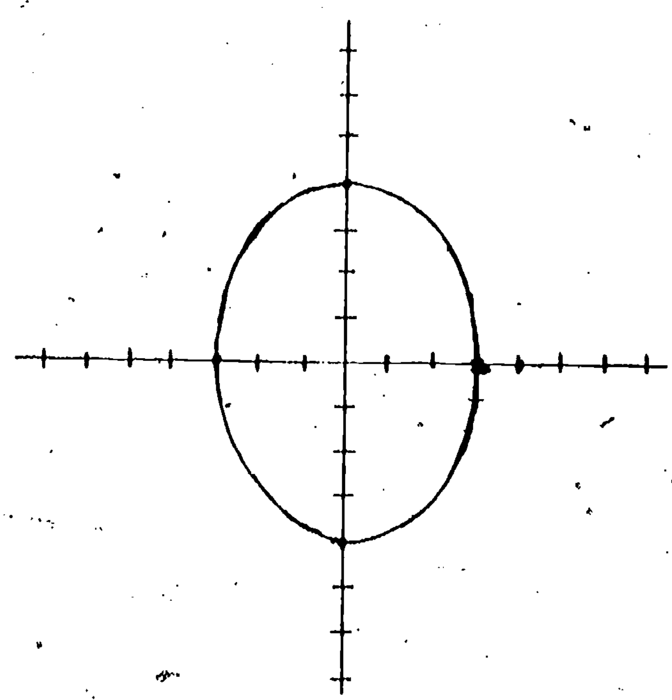
Answers:

1. See your instructor.

2. Equation is of the form: $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$, $a = 4$, $b = 3$, $c = \sqrt{7}/4$,

   coordinates of foci $(0, \sqrt{7})$, $(0, -\sqrt{7})$, coordinates of vertices, $(0, 4)$, $(0, -4)$, equations of directrices, $y = \pm \frac{16\sqrt{7}}{7}$, coordinates of intersection of curve and minor axis $(3, 0)$, $(-3, 0)$.

3. (94.5, 91.3) million miles



Objective 18.4 Hyperbolas

(For the hyperbola) Given a set of conditions, write the equation in simple form, determine the enablers pertaining to that conic (a, c, focal point, etc.,) and graph the conic. (The conditions can include the general equation.)

Activities 18.4 (2 and 7 are suggested)

1. Your Text:

   Morrill, W. K., Analytic Geometry, pp 168-182
   Exercises: p. 170, problems 1, 6(a, b, c)
   pp. 175-176, problems 1(a, b, c), 2(a, c, e, f, g, h, m, n), 7, 8, 11, 13
   pp. 178, problems 1, 4, 6
   p. 180, problems 1, 3
   pp. 181-182, problems 1(a, c, e), 2(a, g, i, o, r, k, u, w, and y)

2. Your Text and the Study Guide.

3. Solved problems:

   Schaum's Outline Series: Theory and Problems of Plane & Solid Analytic Geometry, pp. 60, problems 1, 2, 5, 4, 7, 8, 12, 14.

4. Other Reading Sources:

   Fuller, Gordon, Analytic Geometry, pp. 73-79
   Protter-Morrey, Analytic Geometry, pp. 124-131
   Murdoch, David, Analytic Geometry, pp. 123-126

Notice that the equations in the above material are developed from a different definition than your text uses. The definition used in the above sources is proved in your text as a theorem.

5. Individual Assistance

6. Informal Group Session

7. Lecture 5 Hyperbola

Self-evaluation - Objective 18.4

1. Discuss the following conic. $x^2 - y^2 = 25$

2. Write the equation of the hyperbola with its center at the origin, transverse axis on the y-axis, eccentricity $2\sqrt{3}$ length of the latus rectum 18 and sketch the curve.

Answers: 1. Vertices $(\pm 5, 0)$, foci $(\pm 5\sqrt{2}, 0)$, $e = \sqrt{2}$, latus rectum 10, equation of asymptotes $y = \pm x$. 2. $121y^2 - 11x^2 = 81$.

Objective 18.5

Sort conics by name given the equation, enablers or the graph.

Activities 18.5 (2 and 3 are suggested)

1. Your Text

   Morrill, W.K., Analytic Geometry, pp. 185,
   Exercises: p. 186, problems 1 and 2

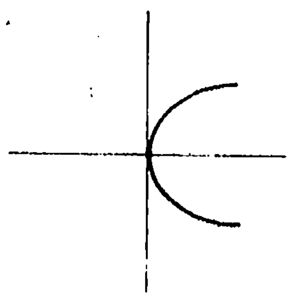
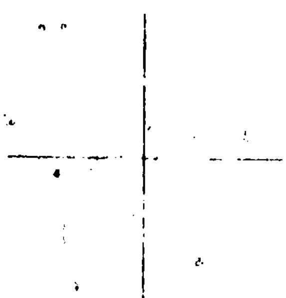
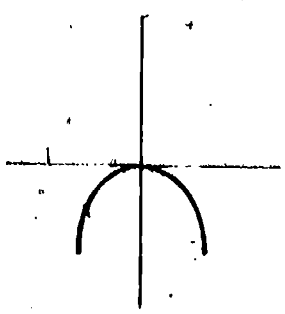
2. Your Text and Study Guide

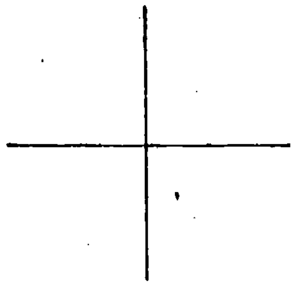
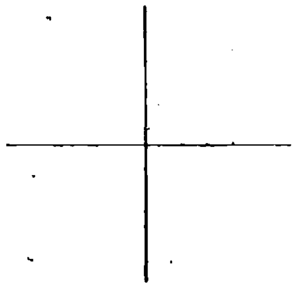
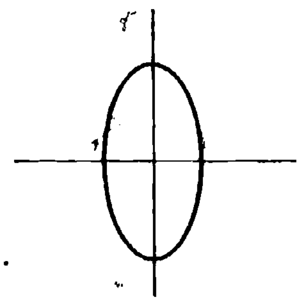
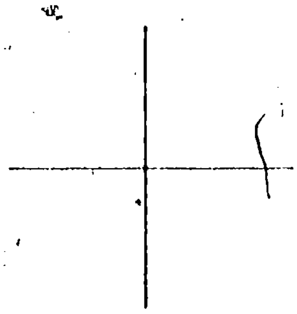
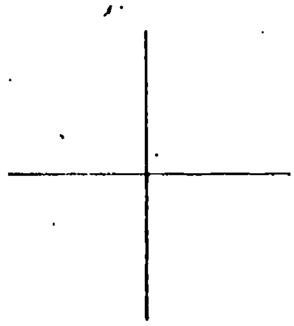
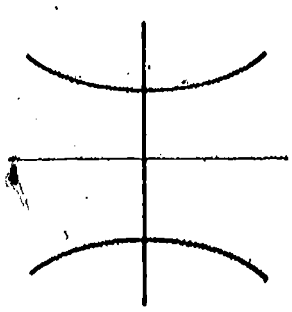
3. Individual Assistance

4. Informal Group Session

Self-evaluation - Objective 18.5

1. Fill in the blanks below. See (a) as an example.

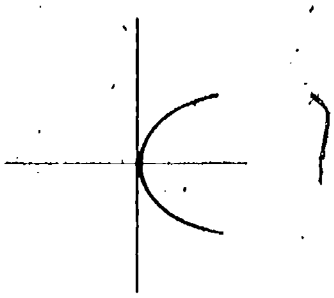
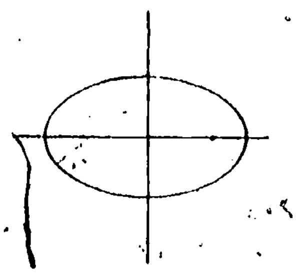
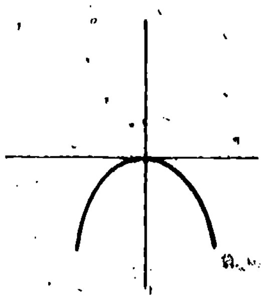
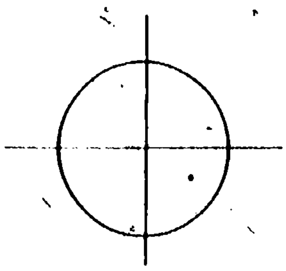
Name of Conic	Graph	Equation	Eccentricity
parabola		$y^2 = 4ax$	$e = 1$
ellipse		$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	________
parabola		________	________

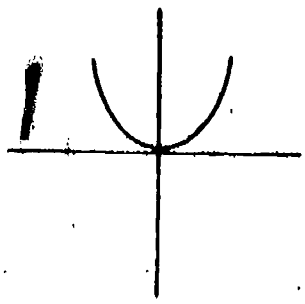
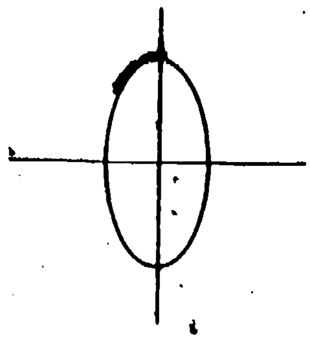
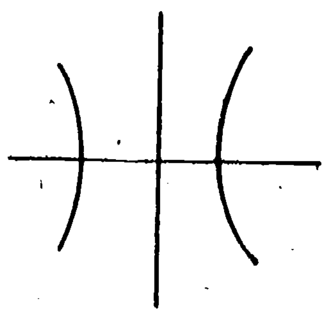
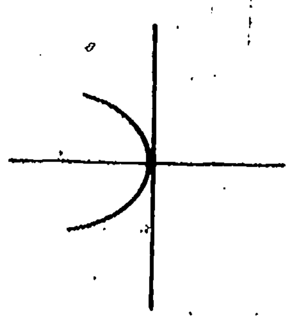
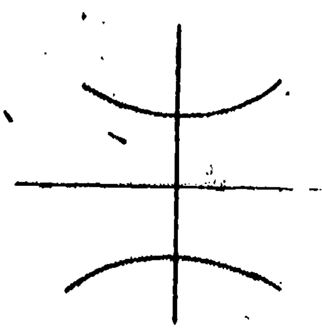
Name of Conic	Graph	Equation	Eccentricity
________		$x^2 + y^2 = r^2$	________
________		$x^2 = +4ay$	________
________		________	________
hyperbola		$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	________
________		$y^2 = -4ax$	________
________		________	________

Answers:

Self-evaluation - Objective 18.5

Fill in the blanks below. See (a) as an example.

Name of conic	Graph	Equation	Eccentricity
parabola		$y^2 = 4ax$	$e = 1$
ellipse		$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$e < 1$
parabola		$x^2 = -4ay$	$e = 1$
circle		$x^2 + y^2 = r^2$	$e = 0$

Name of conic	Graph	Equation	Eccentricity
parabola		$x^2 = +4ay$	$e = 1$
ellipse		$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$	$e < 1$
hyperbola		$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$e > 1$
parabola		$y^2 = -4ax$	$e = 1$
hyperbola		$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$e > 1$

Rationale:

You have seen that the graph of a linear function ax + by + c = 0 is a straight line. A natural step would be to consider the graph of second degree equations. When this is done, it is found that four new types of curves appear as possible graphs of such functions. These curves, the circle, ellipse, hyperbola, and parabola have been known and studied by mathematicians since ancient times. Collectively, along with certain combinations of two straight lines, they are called the conics or conic sections.

Conics also serve as mathematical models for many scientific applications. For example:

(a) The path of the projectile from a gun (or a suborbital rocket), when acted on by gravity alone, is a parabola.

(b) The orbits of the planets and their satellites are ellipses.

(c) The orbits of comets are either elliptical or hyperbolic.

(d) If the weight of a roadbed suspended from a cable and the weight of the cable are uniformly distributed horizontally, the cable assumes the shape of a parabola.

(e) In the construction of bridges, elliptic arches are used as well as parabolic arches.

So let us begin by defining a general conic.

Objectives:

18.5 Sort conics by name given the equation or enablers, or graph.

18.4 For a hyperbola, given a set of conditions, write the equation in simple form, determine the enablers pertaining to that conic (a, c, focal point, etc.) and graph the conic. (The condition can include the general equation.)

18.3 *For an ellipse, given a set of conditions, write the equation in simple form determine the enablers pertaining to that conic (a, c, focal point, etc.,) and graph the conic. (The condition can include the general equation.)*

18.2 *For a parabola, given a set of conditions, write the equation in simple form, determine the enablers pertaining to that conic (a, c, focal point, et.) and graph the conic. (The condition can include the general equation.)*

18.1 *State the definition of a conic and apply the definition to determine the equation given the focal point(s), equation of the directrix and eccentricity.*

Objective 18.1 General Conics

Objective:

State the definition of a conic and apply the definition to determine the equation given the focal point(s), equation of the directrix, and eccentricity.

Instructional Activity - Task 1

Read pages 138-139

Denote the focal point of any conic by F, any point on the conic by P, the projection of P on the directrix by D, and the eccentricity of the conic by e.

You should be able to translate the verbal definition into a mathematical statement. Try it below:

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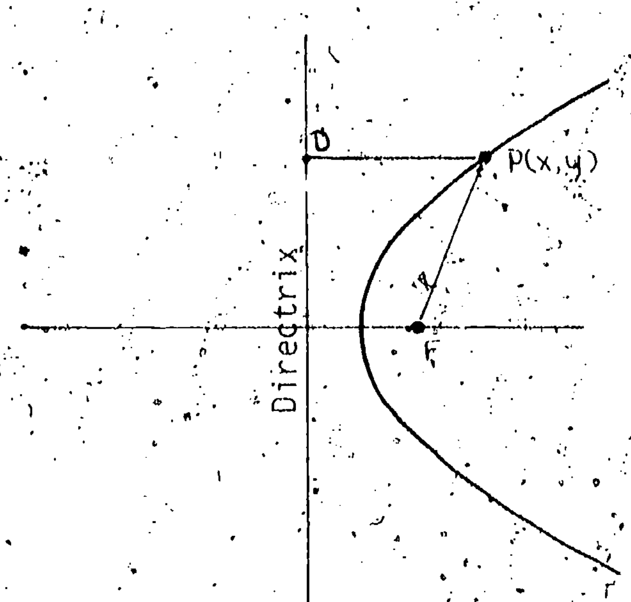
Answer: The distance from any point P(x, y) to a fixed point F can be denoted by |FP|.

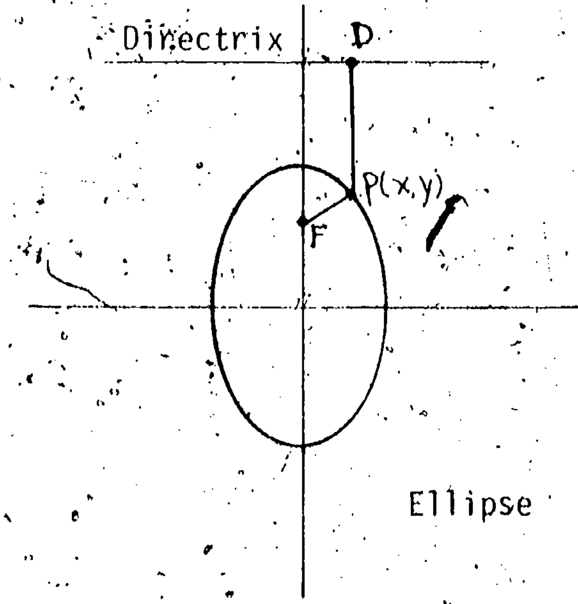
The distance from any point P(x, y) to a fixed line can be denoted by |DP|.

Hence: $\frac{|FP|}{|DP|} = e$ OR $|FP| = e\,|DP|$

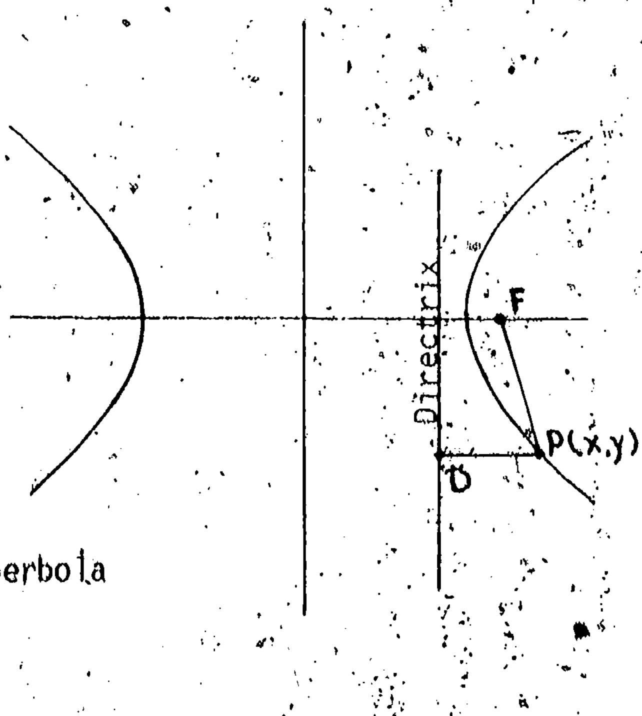
See the drawings below for a parabola, ellipse and a hyperbola.



Parabola



Ellipse



Hyperbola

So to determine the equation by applying the definition of a conic you must set $|FP|$ (the distance from the focal point to any point $P(x, y)$ on the conic) equal to $e|DP|$ (the eccentricity times the distance from any point $P(x, y)$ to the projection of $P(x, y)$ on the directrix.) This is an application of 2 tasks in Unit 17 (the distance between 2 points, Objective 17.4, and the distance from a point to a line, Objective 17.1).

Examples:

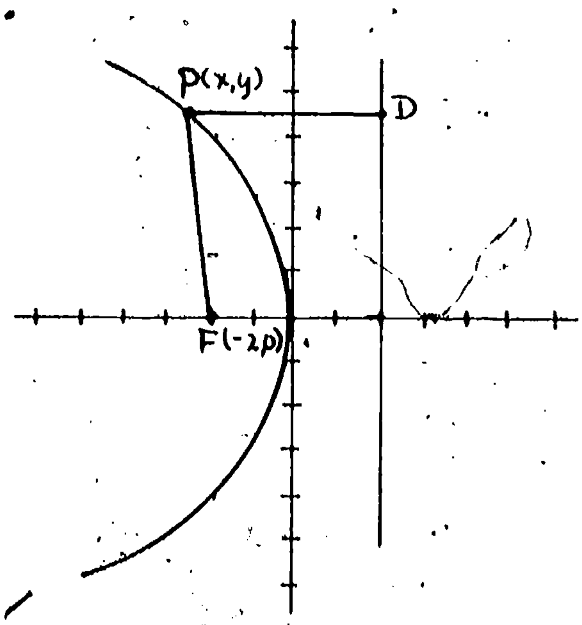
Determine the equation of the following conics:

(a) $e = 1$, $F = (-2, 0)$, equation of directrix: $x = 2$

(b) $e = 5/3$, $F = (0, 5)$, equation of directrix: $y = 9/5$

(c) $e = 1/3$, $F = (3, 0)$, equation of directrix: $x = 27$

(a)



Since $e = 1$ the conic is a ____________.

$|FP| = e|DP|$ OR $|FP| = 1|DP|$

$|FP| = \sqrt{(x-(-2))^2 + y - 0)^2} =$ ____________

$|DP| = \dfrac{|x - 2|}{\sqrt{1}} =$ ____________

$|FP| = |DP|$

$\sqrt{(x + 2)^2 + y^2} = |x - 2|$

$(\sqrt{(x + 2)^2 + y^2})^2 = (|x - 2|)^2$

(squaring both sides)

Hence the equation of the parabola is:

____

Answers: parabola, $\sqrt{(x + 2)^2}$; $|x - 2|$, $y^2 = -8x$, respectively

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(b) The drawing isn't necessary, so it will be omitted in this example.

Since $e = 5/3$ the conic is a ____________________.

$|FP| = e\,|DP|$ OR $|FP| = 5/3\,|DP|$

$|FP| = \sqrt{(x - 0)^2 + (y - 5)^2} =$ ____________________ $F = (0, 5)$

$|DP| = \dfrac{|5y - 9|}{25} = \dfrac{|5y - 9|}{5}$ $y = 9/5$, equation of directrix OR $5y - 9 = 0$

$|FP| = 5/3\,|DP|$

$\sqrt{x^2 + (y - 5)^2} = 5/3\ \dfrac{|5y - 9|}{5}$

$\left(\sqrt{x^2 + y^2 - 10y + 25}\right)^2 = \left(\dfrac{|5y - 9|}{3}\right)^2$ squaring both sides

Hence, the equation of the hyperbola is ____________________.

Answers: hyperbola, $\sqrt{x^2 + y^2 - 10y + 25}$, $\dfrac{x^2}{9} - \dfrac{x^2}{16} = 1$

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(c) Since $e = 1/3$ the conic is a (an) ________________.

$|FP| = e|DP|$ OR $|FP| = 1/3\ |DP|$

$|FP| =$ ______________________________

$|DP| =$ ______________________________

$F = (3, 0)$

$y = 27$ equation of directrix

OR $y - 27 = 0$

$|FP| = 1/3\ |DP|$

Hence:

Hence the eqution of the ellipse is ________________.

Answers: ellipse, $\sqrt{(x - 3)^2 + y^2}$, $|x - 27|$, $\frac{x^2}{81} + \frac{y^2}{72} = 1$

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Self-evaluation

1. State the definition of a conic (general conic).

2. If the eccentricity of a conic is 3/4 the conic is a (an) ______________.

3. If the eccentricity of a conic is 3 the conic is a(an) ______________.

4. If the eccentricity of a conic is 1 the conic is a(an) ______________.

5. Derive the equations of the following conics:

   (a) $e = \sqrt{2}$, $F = (0, 2)$, equation of directrix is $y = 1$.

   (b) $e = 2/3$, $F = (6, 0)$, equation of directrix is 27/2.

   (c) $e = 1$, $F = (0, -3)$, equation of directrix is $y = 3$.

Answers:

1. See bottom page 139 -- General Definition of a Conic.
2. ellipse
3. hyperbola
4. parabola
5. (a) $|FP| = \sqrt{(x - 0)^2 + (y - 2)^2}$ $\quad |DP| = \frac{|y - 1|}{\sqrt{1}} = |y - 1|$.

$$|FP| = \sqrt{2}\,|DP|$$

$$\sqrt{x^2 + y^2 - 4y + 4} = \sqrt{2}\,(|y - 1|)$$

$$(\sqrt{x^2 + y^2 - 4y + 4})^2 = (\sqrt{2}\,|y - 1|)^2$$

$$x^2 + y^2 - 4y + 4 = 2(y^2 - 2y + 1)$$

$$x^2 + y^2 - 4y + 4 = 2y^2 - 4y + 2$$

$$x^2 - y^2 = -2$$

$$\frac{x^2}{-2} - \frac{y^2}{-2} = \frac{-2}{-2}$$

$$\frac{y^2}{2} - \frac{x^2}{2} = 1 \quad \text{(hyperbola)}$$

(b) $|FP| = \sqrt{(x - 6)^2 + (y - 0)^2}$ $\quad |DP| = \frac{|2x - 27|}{\sqrt{4}} = \frac{|2x - 27|}{2}$

$$|FP| = 2/3\,|DP|$$

$$(\sqrt{x^2 - 12x + 36 + y^2})^2 = \left(2/3\,\left|\frac{2x - 27}{2}\right|\right)^2$$

$$x^2 - 12x + 36 + y^2 = \frac{4x^2 - 108x + 729}{9}$$

$$9x^2 - 108 + 324 + 9y^2 = 4x^2 - 108x + 729$$

$$5x^2 + 9y^2 = 405$$

$$5/405x^2 + 9/405y^2 = 405/405$$

$$x^2/81 + y^2/45 = 1 \quad \text{(ellipse)}$$

(c) $|FP| = \sqrt{(x - 0)^2 + (y - (-3))^2} = \sqrt{x^2 + (y + 3)^2}$

$|DP| = \frac{|y - 3|}{\sqrt{1}} = |y - 3|$

$|FP| = 1|DP|$

$\sqrt{x^2 + (y + 3)^2} = |y - 3|$

$(\sqrt{x^2 + y^2 + 6y + 9})^2 = (|y - 3|)^2$

$x^2 + y^2 + 6y + 9 = y^2 - 6y + 9$

$x^2 = -12y$ (parabola)

Objective 18.2 (Parabola)

Objective 18.2

(For a parabola). Given a set of conditions, write the equation in simple form, determine the enablers pertaining to that conic (a, c, focal point, etc.) and graph the conic. (The condition can include the general equation.)

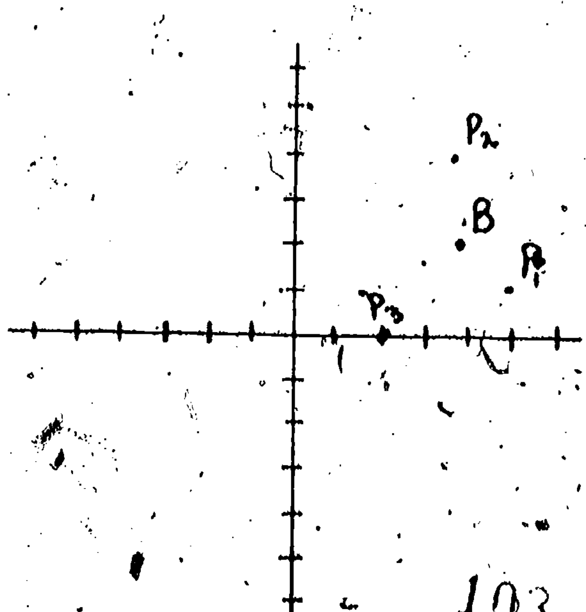
Instructional Activity Task 2 (Symmetry)

Read pages 22, 23 (on Symmetry) and work the Problems 1 and 3 on page 23.

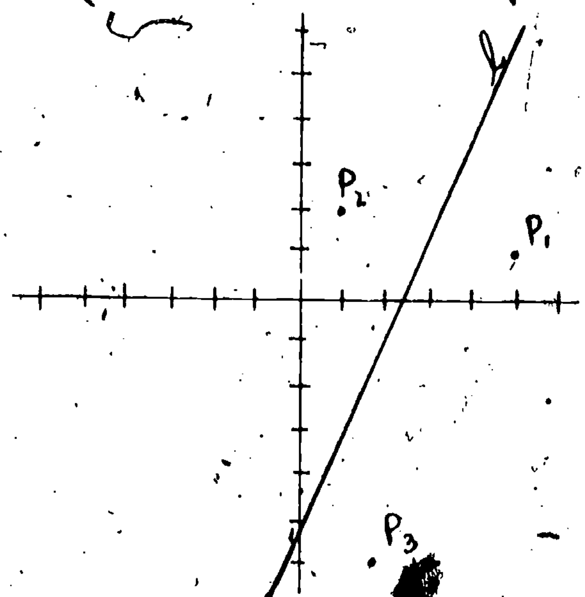
Self-evaluation

1. B is the image of A provided C is the ______________ of the segment joining A and B.

2. Sketch the image of the following points P_1 P_2 and P_3 with respect to the given point B.

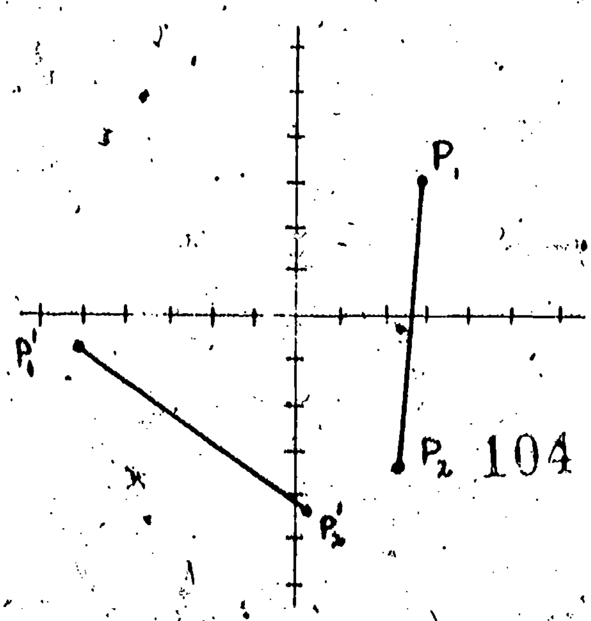


3. Sketch the image of the following points P_1, P_2 and P_3 with respect to the line 1.



4. If the following line segments $P_1 P_2$ and $P'_1 P'_2$ are images of each other sketch the line l that is the axis of symmetry.

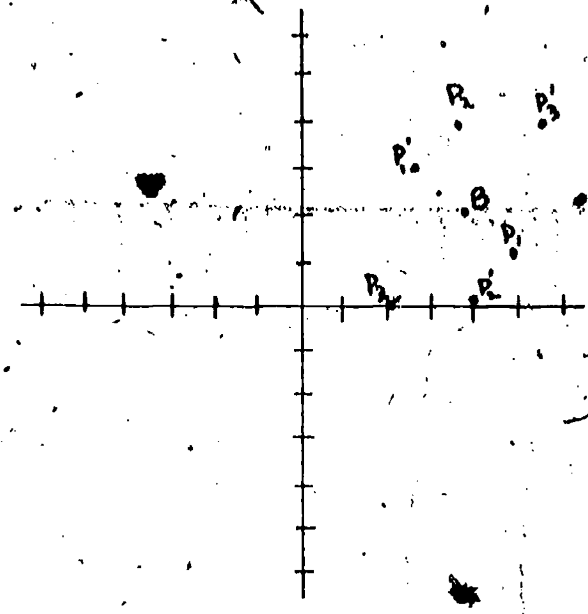
NOTE: Remember $P_1 P_2$ and $P'_1 P'_2$ are sets of points hence P'_1 is the image of P_1 and P'_2 is the image of P_2. Likewise any point P_i on P_1P_2 contains an image P'_i on $P_1' P_2'$.



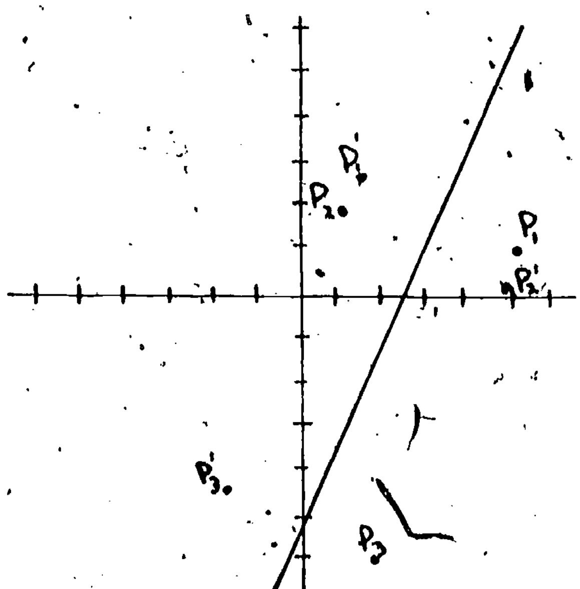
Answers:

1. midpoint

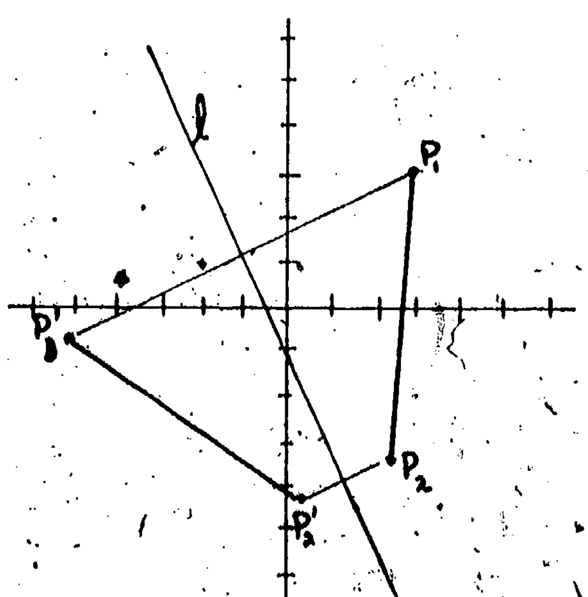
2.



3.



4.



Read pages 112-114 (Symmetry)

Example 1.

Show that the parabola $y^2 = 8x$ is symmetric with respect to the x-axis but not with respect to the origin.

Solution: (a) a curve is symmetric with respect to the x-axis provided $f(x, -y) = 0$ is equivalent to $f(x, y) = 0$.

If you replace y with -y in the equation $y^2 = 8x$ the result is $(-y)^2 = 8x$ or $y^2 = 8x$: hence the equations are equivalent and the parabola is symmetric with respect to the x-axis.

(b) If the curve is symmetric with respect to the origin provided $f(x, y) = 0$ and ________________ are equivalent: Replacing x with -x and y with -y in the equation $y^2 = 8x$ yields ________________. Hence the curve is not symmetric with respect to the origin.

Answers: $f(-x, -y) = 0$, $y^2 = -8x$.

Self-evaluation

1. Show that the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is symmetric with respect to the (a) origin, (b) the x-axis, (c) and the y-axis.

Answer:

1. (a) Replace x with -x and y with -y.

$(-x)^2/16 + (-y)^2/9 = 1$ OR $x^2/16 + y^2/9 = 1$

(b) Replace y with -y.

$x^2/16 + (-y)^2/9 = 1$ OR $x^2/16 + y^2/9 = 1$

(c) Replace x with -x.

$(-x)^2/16 + y^2/9 = 1$ OR $x^2/16 + y^2/9 = 1$

Instructional Activity Task 3 (Deriving equations)

Read pages 142-145

You should notice that you only need to know two pieces of information in order to write the equation of a simple parabola.

1. you must know the value of a

2. you must know how the curve is located with respect to the axes, (it opens up, down, right or left) and the corresponding equation.

If you know the focal point this gives you both pieces of information. (The coordinates of the focal point are F = ________ or F = ________ hence the location of F tells you how the curve is located.

Answers: (0, ± a) or (± a, 0)

If you know the equation of the directrix you also have the needed information since the equation of the directrix is either ________ or ________.

Answers: $y = \pm a$ or $x = \pm y$

Do the following exercises on page 145-146 do problems 1, 3, 4, 7, (a) and 8(a).

Exercises:

1. Match the following focal points to the proper equations:

1. (3, 0) ____	(a) $x^2 + 4ay$
2. (0, -2) ____	(b) $y^2 = 4ax$
3. (0, 4) ____	(c) $x^2 = -4ay$
4. (4, 0) ____	(d) $y^2 = -4ax$
5. (-2, 0) ____	

2. Match the equations of the directrices to the proper equations.

1. $y = 4$ ____	(a) $x^2 = 4ay$
2. $x = 3/2$ ____	(b) $y^2 = 4ax$
3. $y = -2/3$ ____	(c) $y^2 = -4ax$
4. $x = -4$ ____	(d) $x^2 = -4ay$

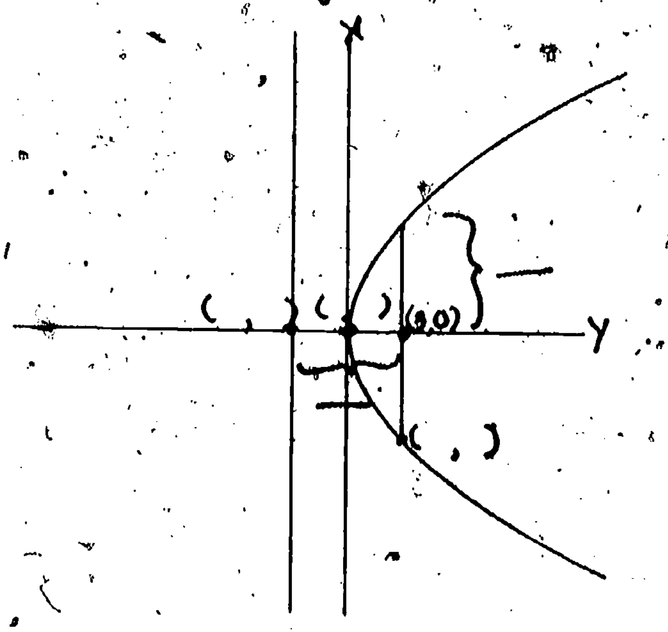
Answers: 1. (1. (b), 2. (c) 3. (a) 4. (b) 5. (d))
2. (1. (a), 2. (c) 3. (a) 4. (b))

Instructional Activity Task 4 (Determining enablers and graphing)

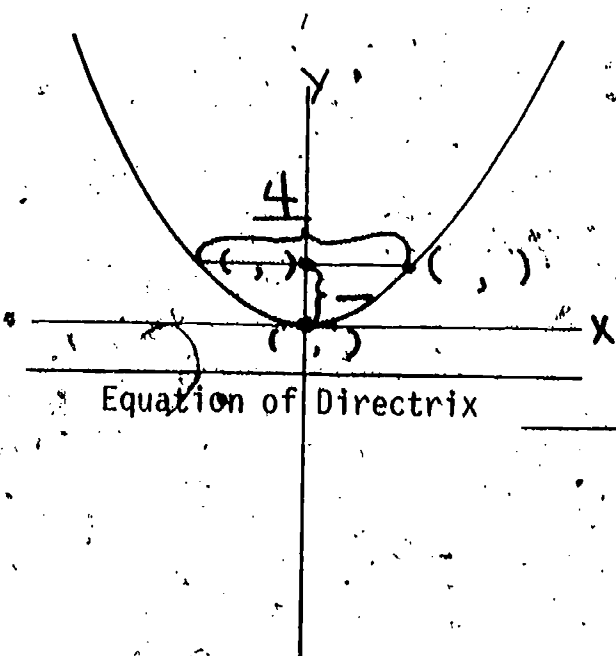
Read pages 146-147. Do the exercises on the following page and on pages 147-148, work problems 1, 3(a, b, c, f) and 7. A labeled drawing for problem 7 is at the end of the Self-evaluation for this objective. (Don't use it unless absolutely necessary.) On pages 151-152 work problems 4(a) and 9.

Exercises:

1. Fill in the blanks with the coordinates of the points, the distances, or the required equations.

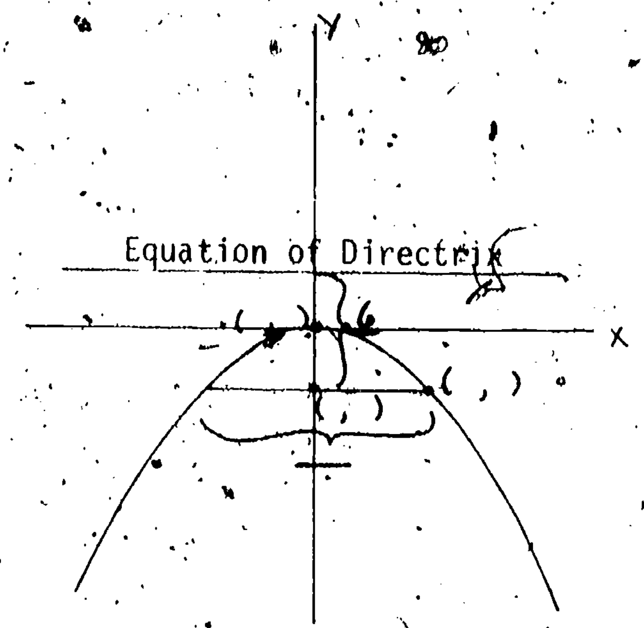


equation of parabola ______________

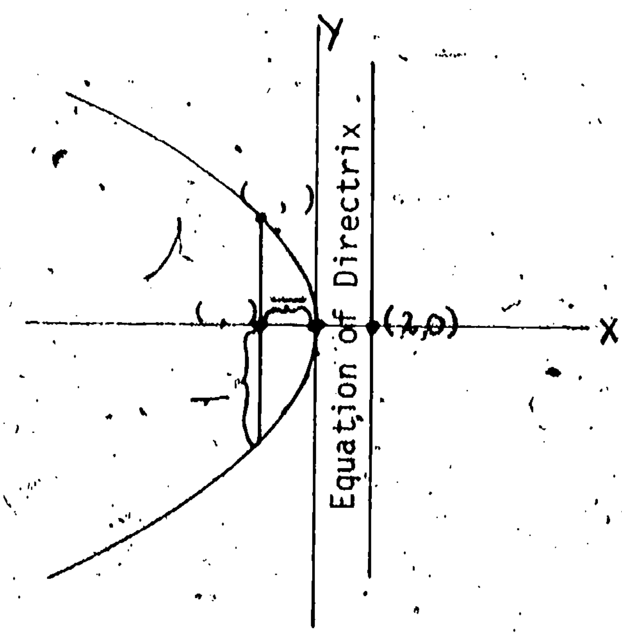


Equation of Directrix ______________

equation of parabola ______________



equation of parabola ______________



equation of parabola ______________

Answers: See your instructor.

Self-evaluation

1. Given the following conditions, determine the equation of the parabola that satisfies those conditions. Determine the value of a.

   (a) the vertex is at (0, 0), and the focal point is at (0, -4)

   (b) the coordinates of the intersection of the parabola and its latus rectum are (-4, -2) and the axis of symmetry is the y-axis.

2. Discuss* and sketch the curve $2y^2 = 37x$.

*"Discuss" means to give the coordinates of the vertex, focus, and end points of the latus rectum, and an equation of the directrix.

Answers:

1. (a) $a = 4$

   $x^2 = -4ay$

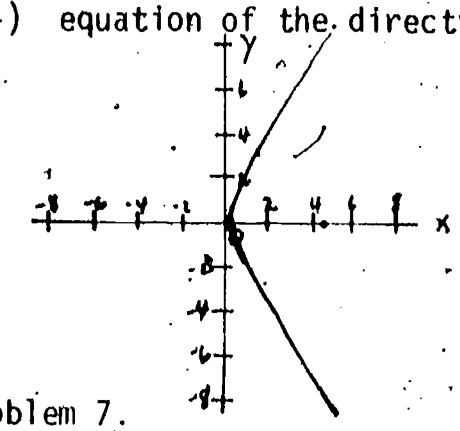
   $x^2 = -16y$

   (b) $a = 2$

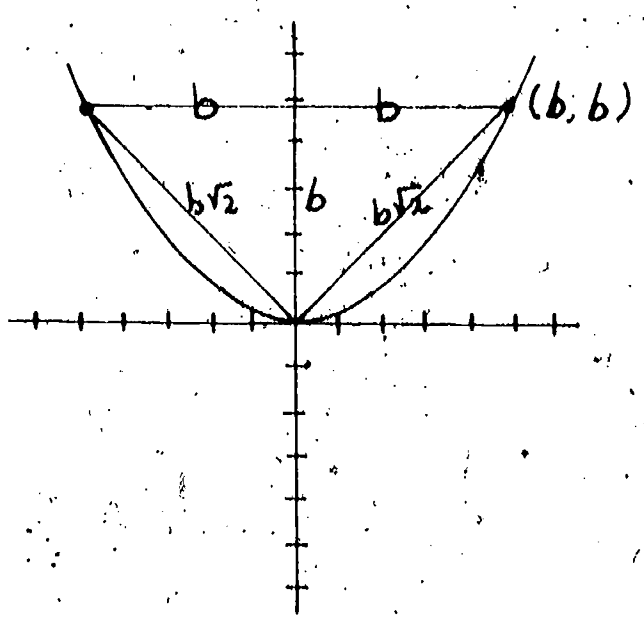
   $x^2 = -4ay$

   $x^2 = -8y$

2. $2y^2 = 37x$ or $y^2 = 37/2x$ $4a = 37/2$ hence, $a = 37/8$. The parabola opens to the right, coordinates of focal point are $(37/8, 0)$. End points of latus rectum $(37/8, 37/4)$ and $(37/8, -37/4)$ equation of the directrix is $x = -37/8$.



Drawing for problem 7.



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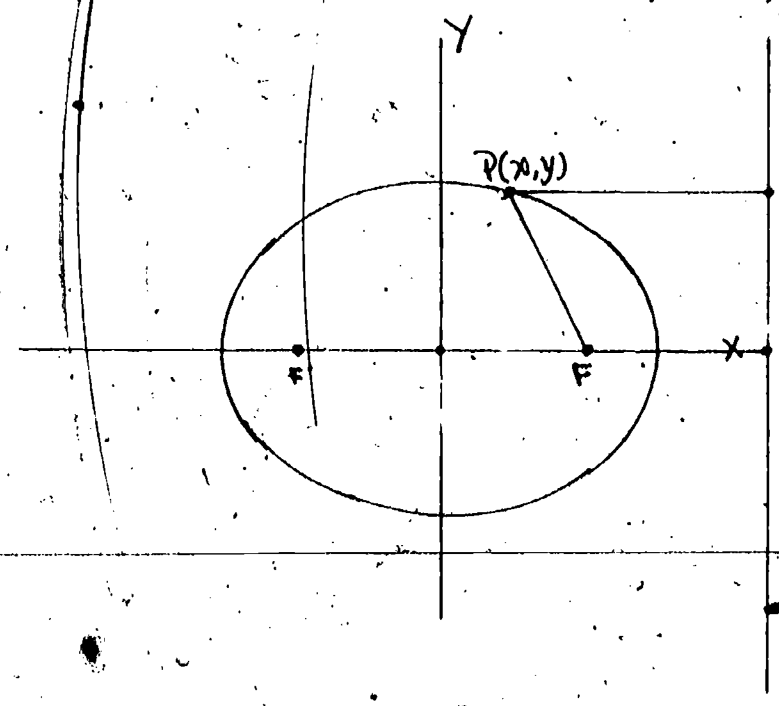
Objective 18.3 (Ellipses and circles)

Objective:

(For an ellipse) Given a set of conditions, write the equation in simple form, determine the enablers pertaining to that conic (a, c, focal point, etc.) and graph the conic. (The condition can include the general equation).

Instructional activity Task 5 (Equation of an ellipse)

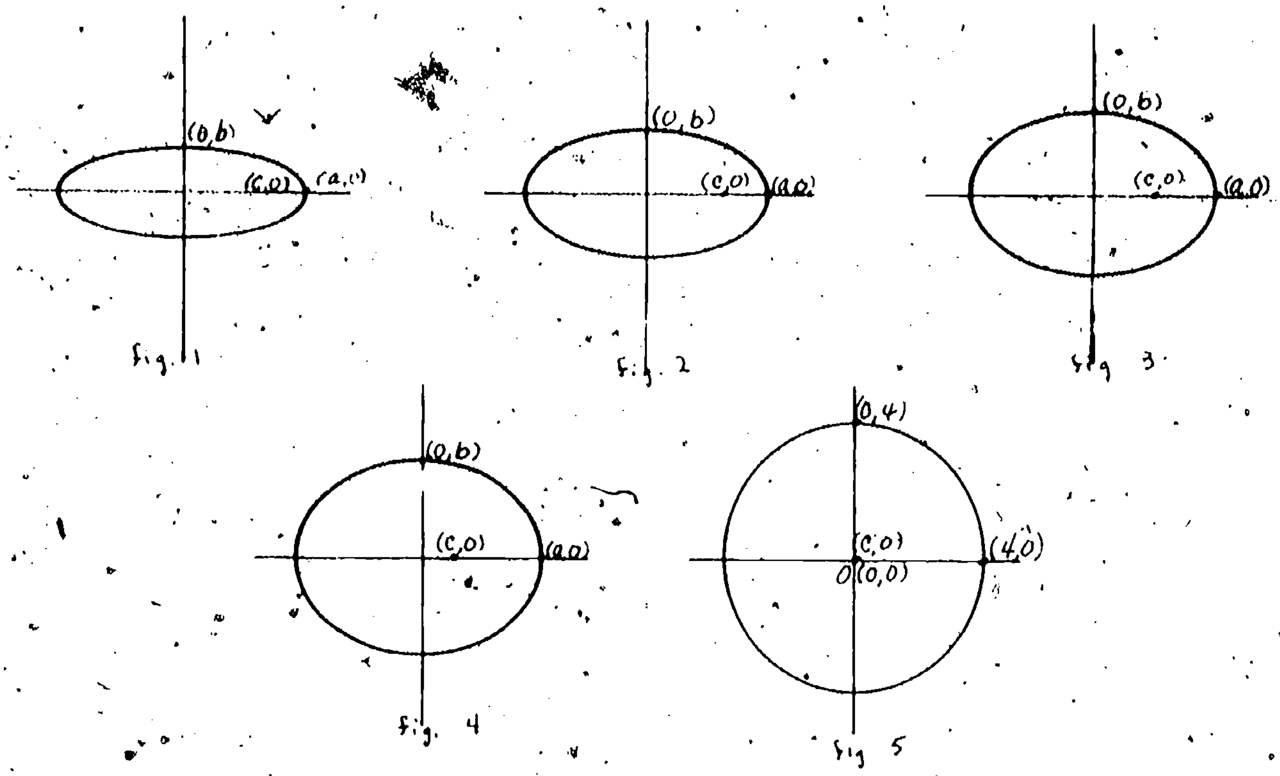
Read pages 153-156. Below is a more detailed drawing of figure 5-19.



On pages 157 work problems 2, and 5(a), (b), (c), and (d).

Instructional Activity Task 6 (Equation of simple circle)

Study the set of figures below.



Notice that as the value of b approaches the value of a the graph of the ellipse becomes more circular and when b = a the graph is in fact a circle. b = a when c = 0 because $b^2 = a^2 - c^2$.

Since c is the distance from the center to the foci, when the foci coincides with the center, the curve is a ______.

Answer: Circle

114

Exercise:

Develop the equation of the circle in figure 5 by using the equation for an ellipse ($\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$).

Answer: $\frac{x^2}{4} + \frac{y^2}{4} = 1$ OR $x^2 + y^2 = 4$

Definition A circle is the locus of all points and only those points which are equidistant from a fixed point. The constant distance is called the radius and the fixed point is called the center of the circle.

When the origin is the center of the circle as in figure 5 above, the equation of the circle is

$$x^2 + y^2 = r^2$$

where r is the radius of the circle. This equation is called the simple equation of the circle.

Exercises:

1. Find the equation of the circle whose center is at the origin and whose radius is 4.
2. Find the equation of the circle whose center is at the origin and whose diameter is 6.
3. The end points of the diameter of a circle are (3,4) and (-3, -4). What is the simple equation of the circle?

Answers:

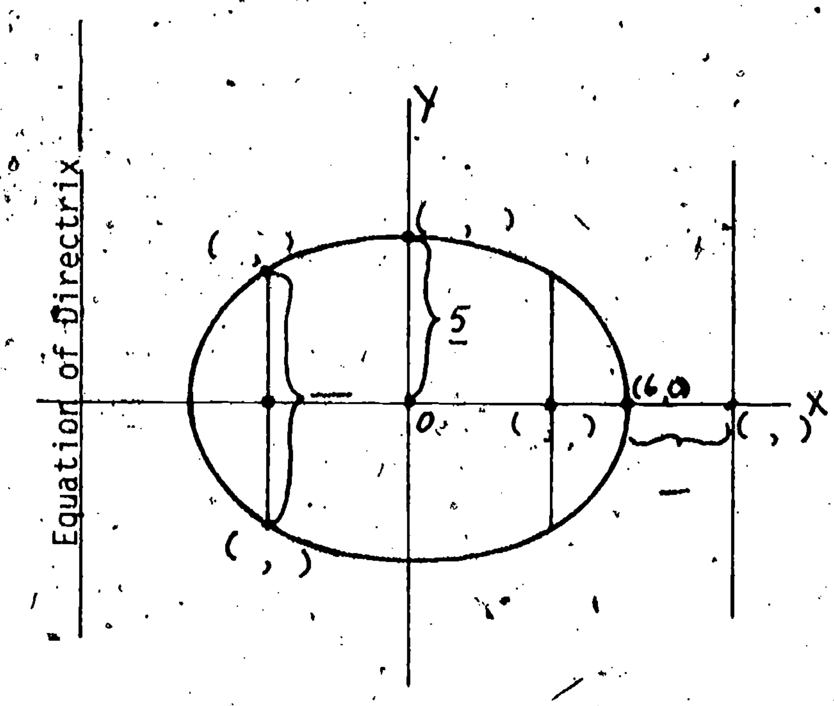
1. $x^2 + y^2 = 16$
2. $x^2 + y^2 = 9$
3. $x^2 + y^2 = 25$

Instructional Activity Task 7 (Terms for ellipse)

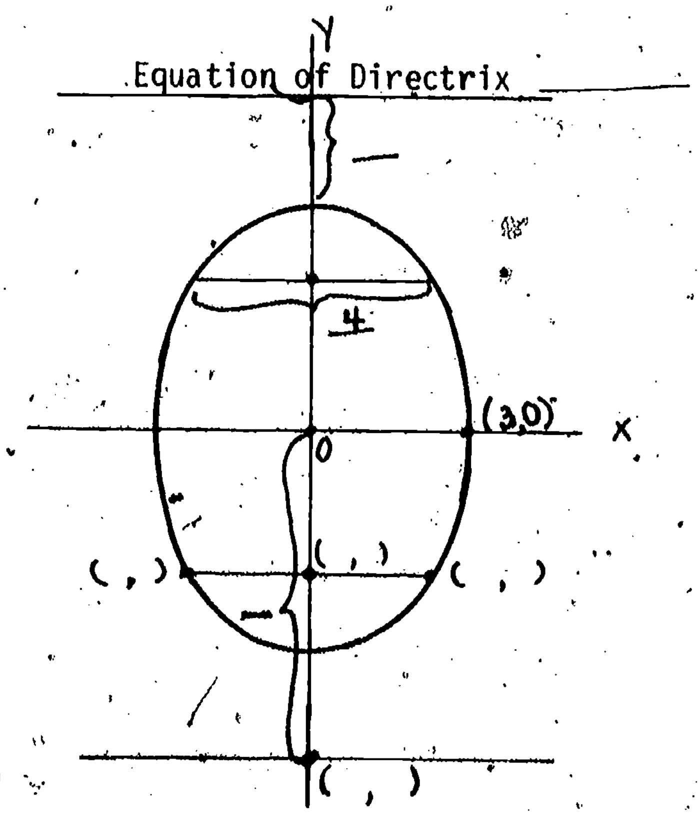
Read pages 157-159.

Exercise:

Label the drawings on the following page by filling in the blanks (required equations and distances) and by filling in the coordinates.



equation of ellipse ______________



Answers: See your instructor.

equation of ellipse ______________

To determine the equation of an ellipse you must know or be able to find three items. (the value of a, the value of b, and which axis is the major axis). If you look at the exercises on page 158-159, you will see the various conditions you might be given. For example problem 1 gives you the values of a & b and tells you which is the major axis. (The foci & verticies are on the major axis.) Problem 3 gives you the value of a and the value of e, hence you can find c. When you have the value of c you can find the value of b.

You might find it helpful to make a table similar to the one below before you attempt to find the equation of an ellipse.

a =	coordinates of foci	( ), ( )
b =	coordinates of vertices	( ), ( )
ae = c =	coordinates of intersection of curve and minor axis	( ), ( )
ae =		
e =		

Example: Problem 3 page 158.

$e = 2/3$, focus is at (6,0)

a = *9*	coordinates of foci	(6, 0), *(-6, 0)*
b = *45*	coordinates of vertices	*(9, 0), (-9, 0)*
ae = c = 6	coordinate of intersection of curve and minor axis	*(0, 45), (0, 45)*
a/e = *27/2*	equation of directrices	$x = 27/2$, $x = -27/2$

$ae = 6$ $\qquad b^2 = a^2 - c^2$ $\qquad a/e = \dfrac{9}{2/3} = 27/2$

$a(2/3) = 6$ $\qquad b^2 = 81 - 36$

$a =$ [illegible] $\qquad b^2 = 45$

Hence the equation of the ellipse that satisfies the given condition is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{OR} \qquad \frac{x^2}{81} + \frac{y^2}{45} = 1$$

On pages 158 & 159 work problems 1, 4, 5, 6, 7, and 8.

Read pages 159-163.

On pages 159-160 work problems 1, 4, and 7.

On page 162 work problems 1, 3, and 5.

On page 163:

1. Find the intersection of the ellipse whose equation is $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$, and its latus rectum.

2. Work problems 1 (a), 2 (a) (c) and (g), 3 (a) (d) (f) (g) (k) and (L), 4 (a), 5 and 7.

Answers: 1. $(\pm \frac{b^2}{a}, c)$ $(\pm \frac{b^2}{a}, -c)$

Instructional activity Task 8 (Constructing an ellipse)

Read pages 165-166 (method one)

Exercise:

Use the string, tacks and cardboard that are enclosed to construct an ellipse whose equation is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

Self-evaluation - Objective 18.3

1. Discuss* the ellipse whose equation is $16x^2 + 9y^2 = 144$ and sketch the graph.

2. Given the following conditions determine an equation that satisfies those conditions: focus (0, 3), directrix $y = 25/3$.

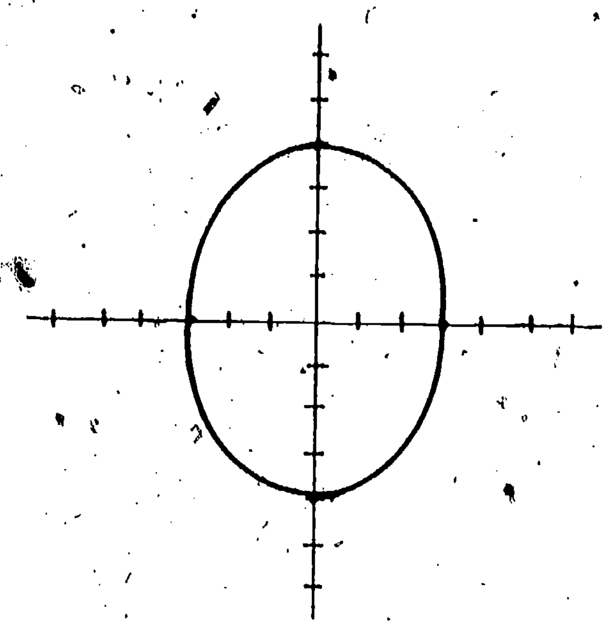
*"Discuss" means to give the coordinates of the vertex, foci, the end points of the latus rectum, intersection of curve and minor axis, and equations of the directrices.

Answers:

1. $\frac{x^2}{9} + \frac{y^2}{16} = 1$ (Hence, $b^2 = a^2 - c^2$ $\quad$ $a = 4,\ b = 3,\ c = \sqrt{7}$

$9 = 16 - 7$

coordinates of the foci $(0, \pm\sqrt{7})$, coordinates of vertices $(0, \pm 4)$, coordinates of intersection of curves minor axis $(\pm 3, 0)$



2. $\frac{x^2}{16} + \frac{y^2}{25} = 1$

Objective 18.4 - Hyperbola

Objective:

(For the hyperbola) Given a set of conditions, determine the enablers pertaining to the conic (a, c, focal point, etc.) Write the equation in simple form, and graph the conic. (The conditions can include the general statement.

Instructional Activity - Task 9 (Deriving Equations)

Read pages 168-170

Exercises:

1. Verify that $|CL| = a/e$ and $c = ae$
2. Work problems 1, 6(a, b, c) on page 170

Instructional Activity - Task 10 (Determining enablers and graphing)

Read pages 171-175

Notice as in the discussion on the ellipse, to determine the equation of a hyperbola you must know three things:

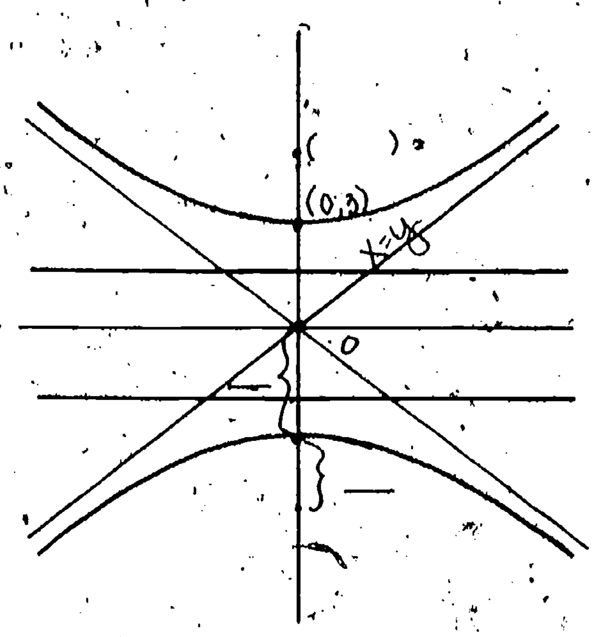
1. the value of a
2. the value of b
3. how the curve is located with respect to the axis, (which form of the equation to use)

The relationship between a, b, and c is $c^2 = a^2 + b^2$. Notice that a can equal b. If $a = b$, then the hyperbola is called an equilateral hyperbola.

The table used for the ellipse will also be helpful for the hyperbola, but you should include the equation of the asymptotes in the list.

Exercises:

1. Label the drawing below. (Fill in the blanks with the equations of the hyperbola and its asymptotes, the distances, and the coordinates of the points.



equation of hyperbola ______

2. Since $y = \pm b/ax$ are the equations of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, what are the equations of the asymptotes of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$?

3. Work problems 1(a, b, c), 2(a, c, e, f, g, h, m, n), 7, 8, 11, 13
4. Fill in the blank with the equation of the proper form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{OR} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(a) the equation of a directrix is $y = 4/5$. ______

(b) a vertex is at (0, 5). ______

(c) the foci are (6, 0) and (-6, 0). ______

(d) the transverse axis is the y-axis. ______

(e) the conjugate axis is the y-axis. ______

(f) the equations of the asymptotes are

$3x - 2y = 0$ AND $3x + 2y = 0$. ______

Answers: See your instructor.

Read pages 176-182

Exercises:

1. Work problems 1, 4, and 6 on page 178
2. Work problems 1 and 3 on page 180
3. Work problems 1(a, c, e), 2(a, g, i, k, o, r, u, w, and y)

Self-evaluation:

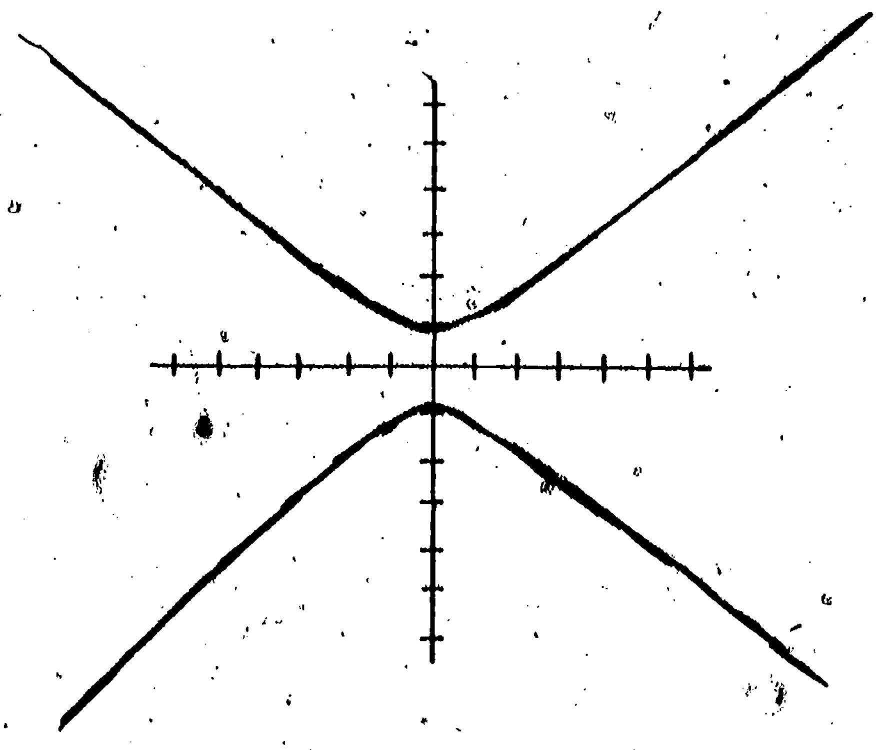
1. Discuss* the conic $x^2 - y^2 = 25$.

*"Discuss" means to give the coordinates of the vertex, focus, and end points of the latus rectum, an equation of the directrix, and the equations of the asymptotes.

2. Write the equation of the hyperbola with its center at the origin, transverse axis on the y-axis, eccentricity $2\sqrt{3}$, length of the latus rectum 18, and sketch the curve.

Answers:

1. vertices $(\pm 5, 0)$, foci $(\pm 5\sqrt{2}, 0)$, $e = \sqrt{2}$, latus rectum 10, equation of asymptotes $y = \pm x$ 2. $121y^2 - 11x^2 = 81$



Objective 18.5

Objective:

Can sort conics by name given the equation, enablers, or graph.

Instructional Activity - Task 11 (Recognizing conics)

Read pages 185-186 on the comparison of conics. The circle could be included as in number 4.

4. Circle

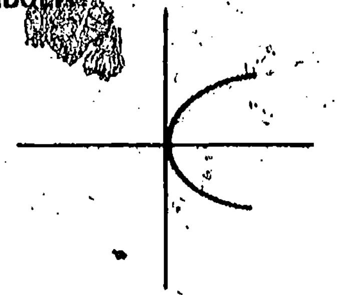
$Ax^2 + Cy^2 + F = 0$, where $A = C$ and A and C are not = to 0.

(a) If A, C, and F all have the same sign, the circle is imaginary.

(b) If F = 0, the locus is a point circle.

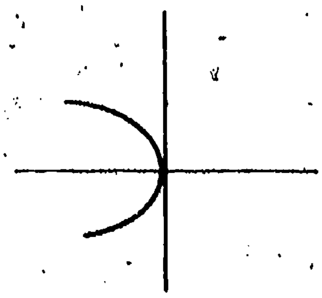
(c) If F is different in sign from A and C, the locus is a real circle with the center at the origin.

The following three pages contain a comparison of the parabola, ellipse, and hyperbola.



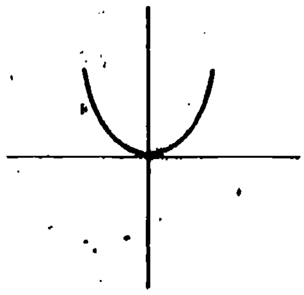
$y^2 = 4ax$

coordinates of focus (a, 0)



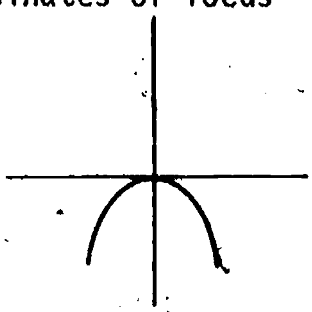
$y^2 = -4ax$

coordinates of focus (-a, 0)



$x^2 = 4ay$

coordinates of focus (0, a)



$x^2 = -4ay$

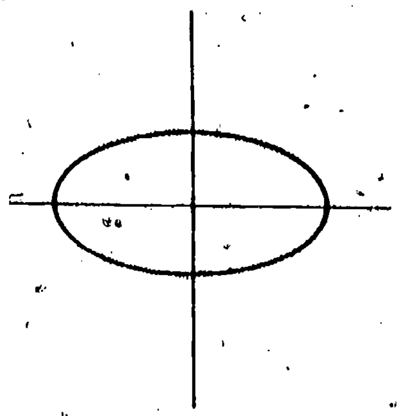
coordinates of focus (0, -a)

a is the distance from the vertex to the focus -- also the distance from the vertex to the directrix.

the vertex is at the origin.

the length of the latus rectum is 4a.

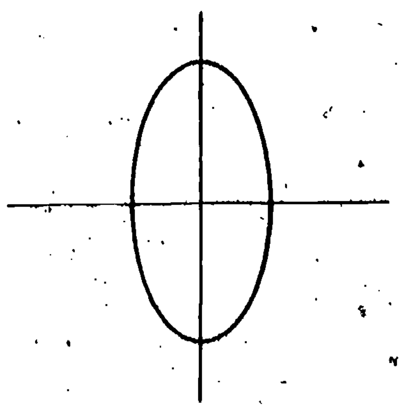
ELLIPSE



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

coordinates of the vertices (a, 0), (-a, 0)

coordinates of the foci (ae, 0), (-ae, 0)



$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

coordinates of the vertices (0, a), (0, -a)

coordinates of the foci (0, ae), (0, -ae)

a is the distance from the center to the vertex

c = ae is the distance from the center to the focus.

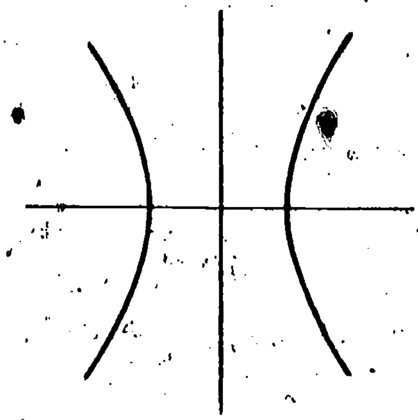
the center is the focus.

the center is at the origin.

a/e is the distance from the center to the directrix

$b^2 = a^2 - c^2$ $a > b$

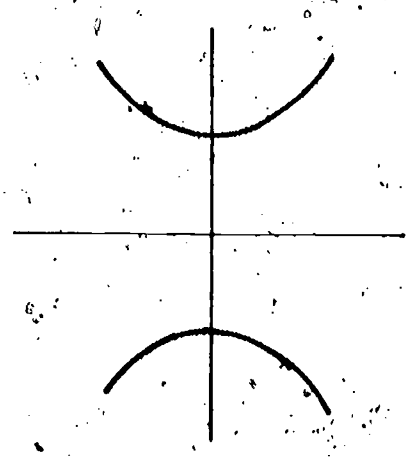
HYPERBOLA



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

coordinates of vertices (a, 0), (-a, 0)

coordinates of foci (ae, 0), (-ae, 0)



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

coordinates of vertices (0, a), (0a, -a)

coordinates of foci (0, ae), (0, -ae)

a is the distance from the center to the vertex.

c is the distance from the center to the focus.

a/e is the distance from the center to the directrix.

the center is the origin

$b^2 = c^2 - a^2$

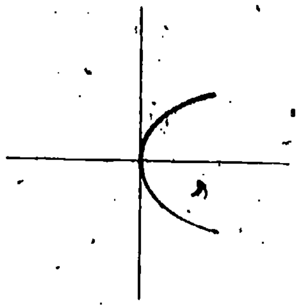
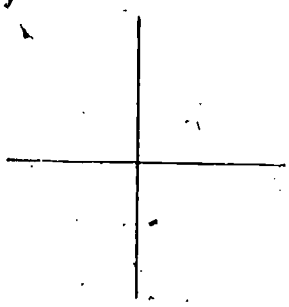
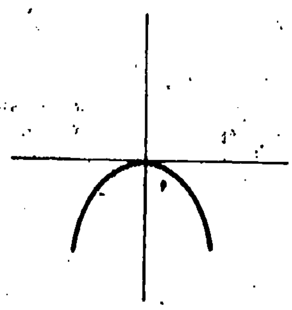
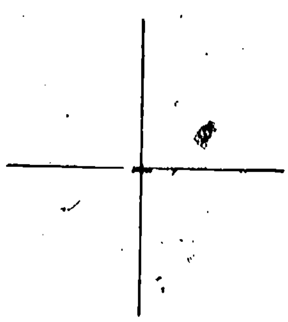
a and b are not ordered. In fact a = b if the hyperbola is an equilateral hyperbola.

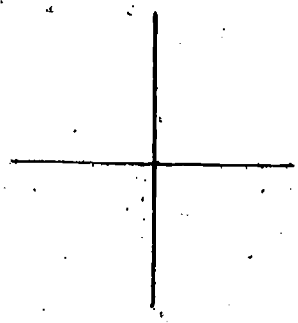
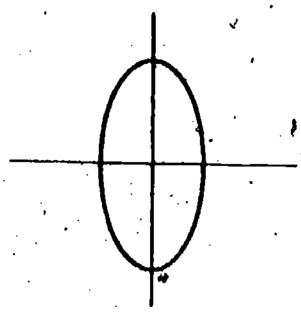
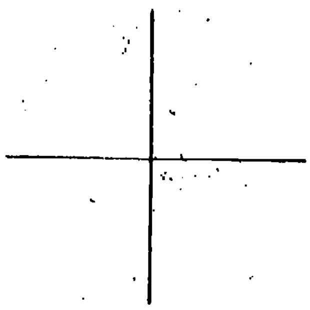
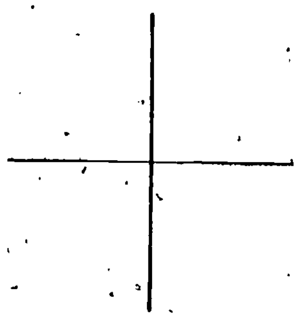
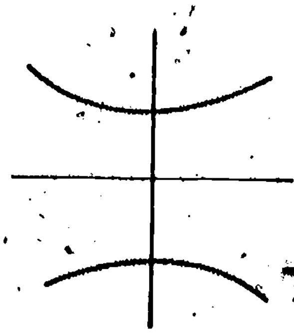
Exercises:

1. Work problems 1 and 2 on page 186.

Self-evaluation - Objective 18.5

Fill in the blanks below. See (a) as an example.

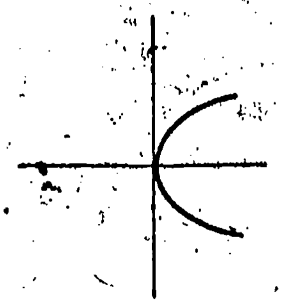
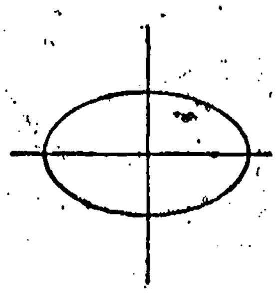
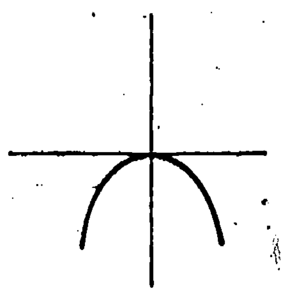
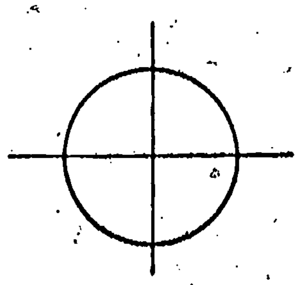
Name of conic	Graph	Equation	Eccentricity
parabola		$y^2 = 4ax$	$e = 1$
ellipse		$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	________
parabola		________	________
________		________	________

Name of conic	Graph	Equation	Eccentricity
______		$x^2 = +4ay$	______
______		______	______
hyperbola		$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	______
______		$y^2 = -4ax$	______
______		______	______

Answers:

Self-evaluation 4 Objective 18.5

Fill in the blanks below. See (a) as an example.

Name of conic	Graph	Equation	Eccentricity
parabola		$y^2 = 4ax$	$e = 1$
ellipse		$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$e < 1$
parabola		$x^2 = -4ay$	$e = 1$
circle		$x^2 + y^2 = r^2$	$e = 0$

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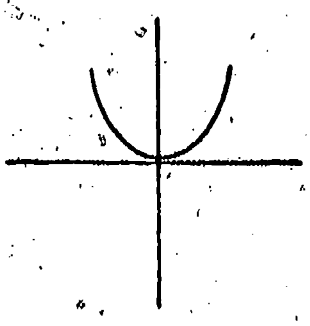
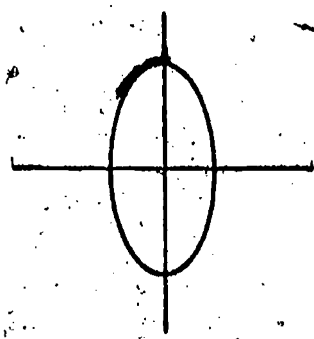
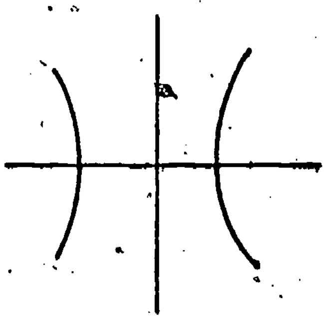
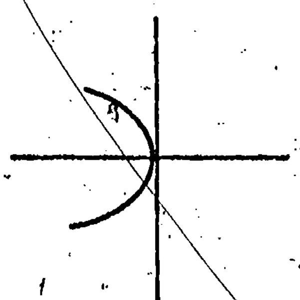
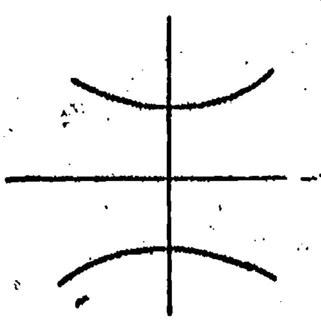
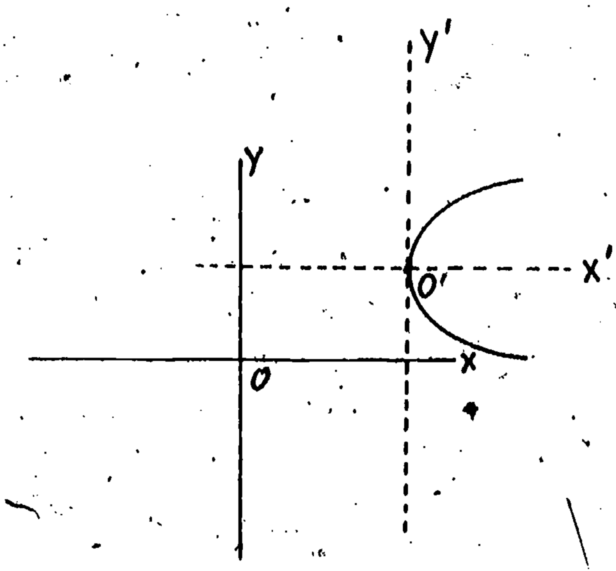
Name of conic	Graph	Equation	Eccentricity
parabola		$x^2 = +4ay$	$e = 1$
ellipse		$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$	$e < 1$
hyperbola		$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$e > 1$
parabola		$y^2 = -4ax$	$e = 1$
hyperbola		$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$e > 1$

Unit 19 -- TRANSFORMATIONS

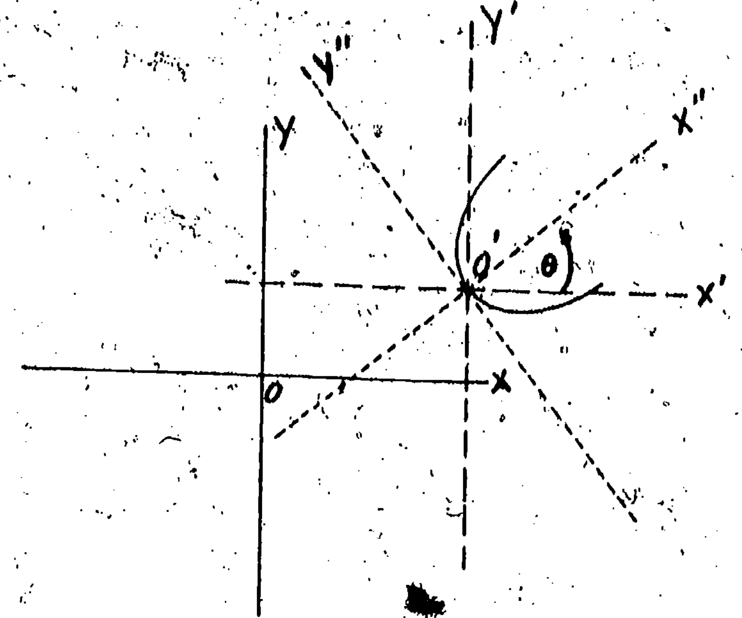
Rationale:

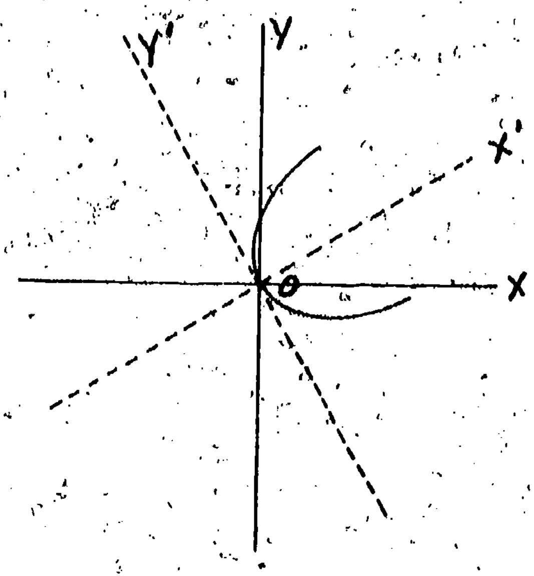
The conics you considered in Unit 18 were very special--they were all simple conics. (Their centers, in the hyperbola, ellipse, and circle, and the vertex in the parabola, were all at the origin.) When the equation of the conic is a simple equation, it is very easy to discuss the curve and sketch the graph, but unfortunately not all conics are simple conics. The methods of handling non-simple conics are introduced in this unit.

The two methods introduced are called translation and rotation. A translation is performed when the axis of symmetry of the non-simple conic is parallel to the coordinate axes. (See drawing below.)



A rotation is performed when the axis of symmetry of the non-simple conic has been rotated through some angle with respect to the coordinate axes, but the center or vertex remains at the origin. (See the drawing to the left on the following page.)





Sometimes you must perform both a translation and a rotation (see the drawing to the left above). This occurs when the axis of symmetry of the non simple conic has been rotated and the center is not at the origin. In all the cases the translation, the rotation, or the translation and rotation converts the equation of the non-simple conic to a simple equation. Then you can use the methods and skills of Unit 18 to discuss the curve and sketch the graph.

Objectives:

19.5 *Given the image of a conic, describe the transformation and sketch the image*

(a) *with respect to a rotation and translation*

(b) *with respect to a rotation*

(c) *with respect to a translation*

19.4 *Determine the equations of the images of conics relative to a given transformation and sketch the curve*

(a) *with respect to a rotation*

(b) *with respect to a translation*

19.3 *Determine the simplified equations of images of polynomial functions relative to a translation*

19.2 *Determine the coordinates of the images of points relative to a given transformation and sketch the points with respect to the new axes*

(a) *with respect to a rotation*

(b) *with respect to a translation*

19.2 *Determine the coordinates of the images of points relative to a given transformation and sketch the points with respect to the new axes*

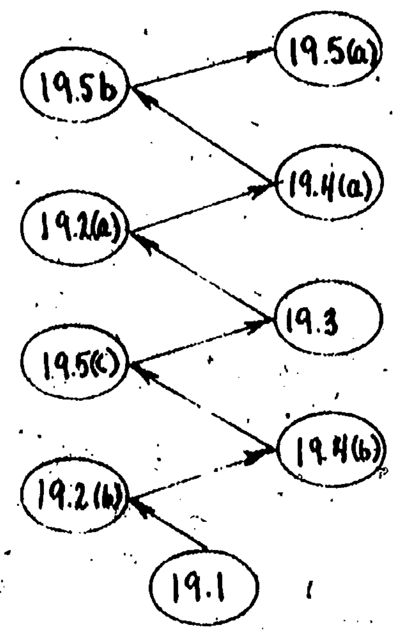
(a) *with respect to a rotation*

(b) *with respect to a translation*

19.1 *Determine the equation of any conic by applying the definition of that particular conic*

Hierarchy:

In order to follow the order of the text, you should use the hierarchy below:



Unit Activities:

Lectures 6 and 7

The lectures over this unit will have the following outline:

Lecture 6. Translations

1. When do you use a translation to simplify an equation
2. How do you perform a translation with respect to a new set of axes.
   A. Translation of a point
   B. Translation of a conic
3. How to choose the proper translation for a given equation

Lecture 7. Rotations, Rotations and Translations

1. When do you use a rotation to simplify an equation
2. How do you perform a rotation through a given angle
3. How to choose the proper rotation for a given conic
4. How to perform both a translation and a rotation in order to simplify a given conic

Procedural Options:

The procedural options for this unit are the same as for Unit 17. If you are having trouble meeting the assessment task requirement on your first attempt, you should talk to your instructor and discuss some alternate approaches.

Average time: 2 weeks

Objective 19.1

Determine the equation of any conic by applying the definition of that particular conic.

Activities 19.1

1. Your Text

Morrill, W. K., Analytic Geometry, pp. 139-142, pp. 103-112
Exercises: p. 141 problems 1 - 8
pp. 105-106, problems 1 - 3, 4(a), 5(a & c), 6(a, c, & e), 7(a), 9
p. 108 problems 1, 3, 9, 11, 15, 17
pp. 111-112, problems 1(a & e), 2, 3, 4,(a, d, f, g), 5, 7, 9, 13

2. Your Text and the Study Guide

3. Solved Problems

Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, pp. 35-42, pp. (circles) p 18, problem 17; p 19 problem 20; p. 48; problem 5; p. 53, problem 4; p. 53, problem 6; p. 55, problem 12; p. 62, problem 7.

4. Other Reading Sources

Fuller, Gordon, Analytic Geometry, pp. 43-49. (circles)

Protter-Morrey, Analytic Geometry, pp. 94-96. (circles) pp. 106-107, (example 2)

5. Individual Assistance

Go visit your instructor, and let him know how well you are doing.

6. Informal Group Sessions

Take it upon yourself to help your fellow students with the previous unit. You will learn as much as they will.

Self-Evaluation Objective 19.1

1. Determine the equation of the ellipse with a focus at (-1, -1), directrix x = 0, and $e = \frac{\sqrt{2}}{2}$.

2. Derive the equation of the parabola with its focus at (4, 3) and its vertex at (4, 6).

   (Hint: you need some of the skills you developed in Unit 18 to determine the equation of the directrix)

3. Derive the equation of the locus of a point P(x, y) which moves so that the distance from c = (3, 2) is always 4.

Answers:

1. $x^2 + 2y^2 + 4x + 4y + 4 = 0$
2. Equation of the directrix is $y = 9$

   Equation of the parabola is $x^2 - 8x + 12y - 56 = 0$
3. $(x - 3)^2 + (y - 2)^2 = 16$

Objective 19.2 - B

Determine the coordinates of the images of points relative to a given transformation and sketch the new coordinate axis and points with respect to the old coordinate axis under a translation.

Activities 19.2 - B (2 and 7 are suggested)

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 192-195
   Exercises: pp. 194-195, problems 1, 2

2. Your Text and Study Guide

3. Solved Problems

   Schaums Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, p. 67, problem 1

4. Other Reading Sources

   Fuller, Gordon, Analytic Geometry, pp. 49-52

5. Individual Assistance

6. Informal Group Sessions

7. Lecture 6

Self-Evaluation Objective 19.2 C

1. Determine the coordinates of each of the points P when the axis is translated so the new origin is 0'.

   (a) P = (3, 2) 0' = (4, 1)

   (b) P = (-2, -1) 0' = (-3, 0)

   (c) P = (4, -3) 0' = (5, 3)

Answers:

1. (a) $P' = (-1, 1)$ (b) $P' = (1, -1)$ (c) $P' = (-1, -6)$

Objective 19.4 B

Determine the equations of the images of conics relative to a given transformation and sketch the curve with respect to a translation.

Activities 19.4 B (2, 3, and 6 are suggested)

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 193-194 (Example 6-2)
   Exercises: p. 195, problem 3

2. Your Text and the Study Guide

3. Solved Problems:

   Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, p. 67, problem 1.

4. Individual Assistance

5. Informal Group Activity

6. Lecture 6

Self-evaluation 19.4 B

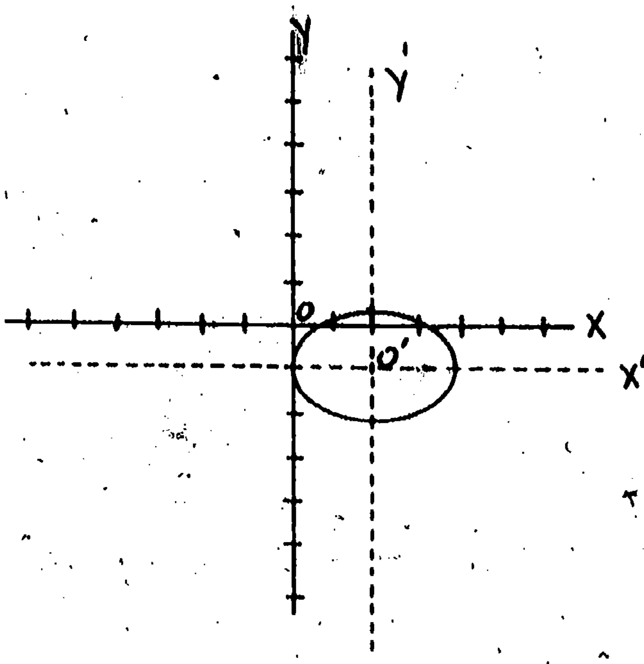
1. Transform each of the following equations by using the given point as a new origin, and sketch the graph.

   (a) $3x^2 - 12x + 4y^2 + 8y = -4 \qquad O' = (2, -1)$

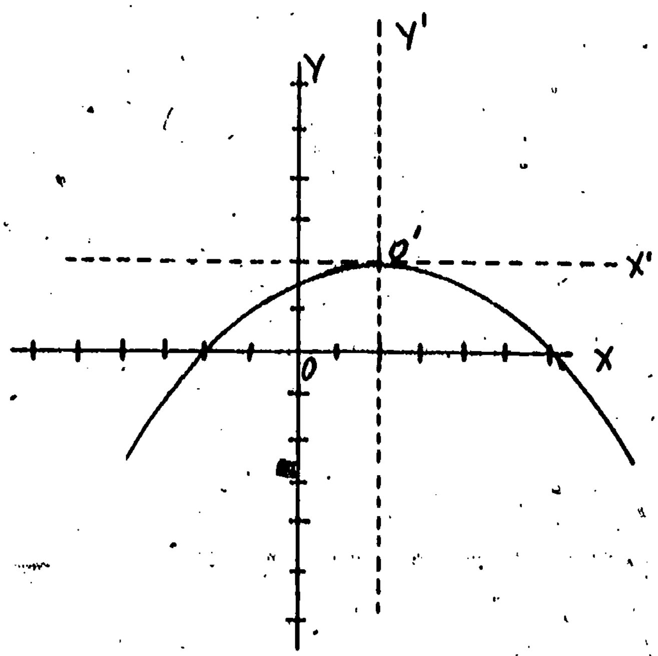
   (b) $x^2 - 4x + 8y = 12 \qquad O' = (2, 2)$

Answers:

(a) $\frac{x'^2}{4} + \frac{y'^2}{3} = 1$



(b) $x'^2 = -8y'$



Objective 19.5 C

Given the image of a conic, describe the transformation and sketch the image with respect to a translation.

Activities 19.5 (c)   (2 and 7 are suggested)

1. Your text:

   Morrill, W.K., Analytic Geometry, pp. 195-199 (omit example 6-3, page 197).

   Exercises: p. 199, problems, 3, 5, 7, 8, 12, 14, and 15.

2. Your text and study guide.

3. Solved problems

   Schaums Outline Series: Theory and Problems of Plane & Solid Analytic Geometry, p. 67, problem 2; p. 69, problem 8.

4. Other Reading Sources:

   Protter-Morrey, Analytic Geometry, pp. 135 (middle of page, paragraph beginning, "We now illustrate...) - 137.

   Fuller, Gordon, Analytic Geometry, pp. 82-84.

5. Individual Assistance.

6. Informal Group Sessions

7. Lecture 6

Self-evaluation 19.5 C

Simplify and name the following equations and sketch the graph of problem 3.

1. $5x^2 - 40x - 3y^2 + 65 = 0$

2. $x^2 + 4x - 16y + 52 = 0$

3. $9x^2 - 18x + 4y^2 + 32y + 37 = 0$

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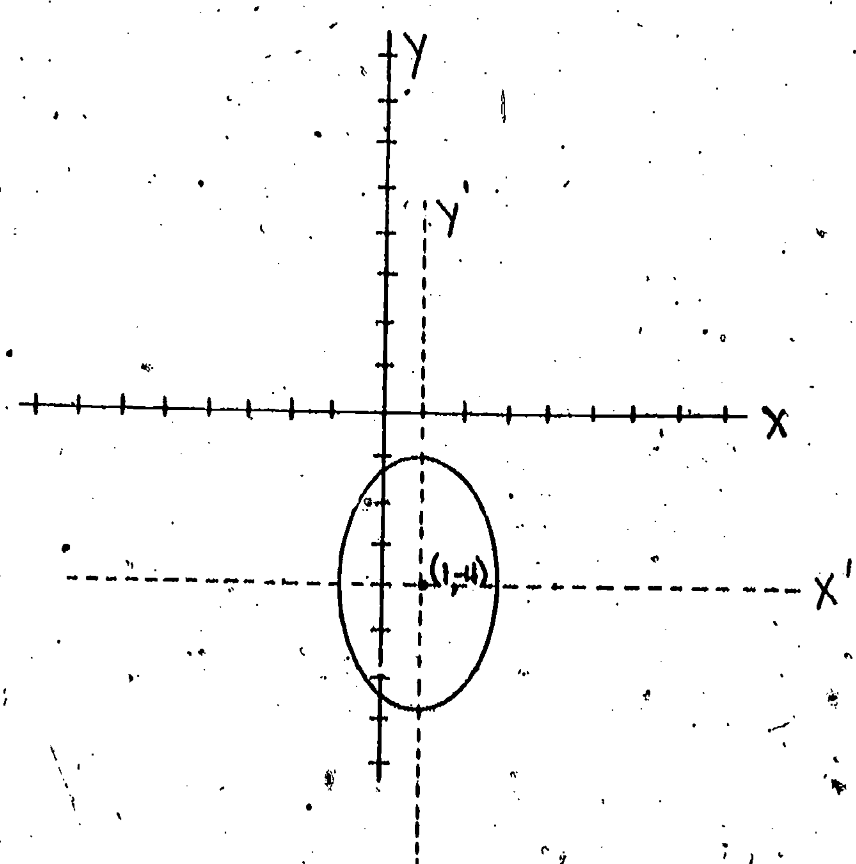
Answers:

1. $\frac{(x-4)^2}{3} - \frac{y^2}{5} = 1$ (hyperbola)

2. $(x+2)^2 = 16(y-3)$ (parabola)

3. $\frac{(x-1)^2}{4} + \frac{(y+4)^2}{9} = 1,$ (ellipse)



OBJECTIVE 19.3

Determine the simplified equation of images of polynomial functions relative to a translation.

Activities 19.3 (2 is suggested)

1. Your Text

   Morrill, W. K., Analytic Geometry, p. 197, example 6-3
2. Your Text and Study Guide
3. Individual Assistance
4. Informal group sessions

Self-evaluation Objective 19.3

Simplify the following polynomial function:

$$y = x^3 - 9x^2 + 2x + 30$$

<u>Answer:</u>

$$y' = x'^3 - 25x'$$

OBJECTIVES 19.2 A and 19.4 A

19.2 A Determine the coordinates of the images of points relative to a given transformation and sketch the points with respect to a new axis and with respect to a rotation.

19.4 A Determine the equation of the images of conics relative to a given transformation and sketch the curve -- with respect to a rotation.

Activities 19.2 A and 19.4 A (2 and 7 are suggested)

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 205-209
   Exercises: pp. 209-210, problems 1(a, c, e), 2(a, c, e, g)

2. Your Text and Study Guide

3. Solved problems

   Schaums Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, p. 67, problem 3.

4. Other Reading Sources:

   Fuller, Gordon, Analytic Geometry, pp. 85-87 (down to example 2).

5. Individual Assistance

6. Informal group sessions

7. Lecture 7

Self-evaluation Objectives 19.2 A and 19.4 A

(You may use a trig table)

1. Determine the coordinates of the following points when the axes have been rotated through 60^0.

   (a) (1, 4)

   (b) (2, 6)

2. Determine which of the following equations must be rotated in order to simplify the equation.
Simplify that equation by rotating the axes through the angle $\theta = 135^0$.

(a) $3x^2 + 3y^2 - 12x + 12y - 1 = 0$

(b) $x^2 - 6y - 4x + 5 = 0$

(c) $x^2 - 4xy + y^2 = 5$

Answers:

(a) $\left(\frac{1 + 4\sqrt{3}}{2}, \; 2\sqrt{3} + 2 \right)$

(b) $\left(2 + \sqrt{3}, \; -1 + 2\sqrt{3} \right)$

(c) $3x'^2 - y'^2 = 5$

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Objective 19.5 (b)

Given the image of a conic, describe the transformation and sketch the image with respect to a rotation.

Activities 19.5 (b) (2 and 7 are suggested)

1. Your test:

   Morrill, W. K., Analytic Geometry, pp. 210-212 (omit last paragraph page 212.)

   Exercises: p. 216, problem 11 (a, c, e, j and n)

2. Your text and study guide.

3. Solved problems:

   Schaum's Outline Series. Theory & Problems of Plane & Solid Analytic Geometry, p. 68, problem 4; page 70, problem 10.

4. Other Reading Sources:

   Protter-Morrey, Analytic Geometry, pp. 140-142 (from middle of page 140 to middle of page 142.)

5. Individual Assistance.

6. Informal Group Sessions.

7. Lecture 7.

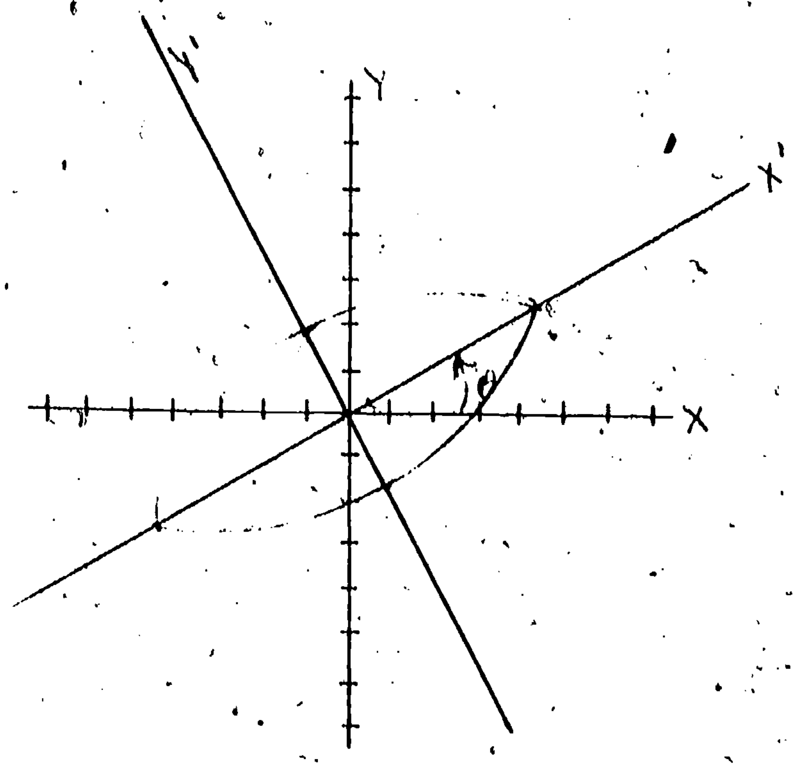
Self-Evaluation: 19.5 (b)

1. Remove the xy term by rotation of axes, identify, and sketch the curve.

   $41x^2 - 84xy + 104y^2 = 500$

Answer Objective 19.5 (b)

1. $\frac{x'^2}{25} + \frac{y'^2}{4} = 1$, ellipse



Objective 19.5 (a)

Given the image of a conic, describe the transformation and sketch the image with respect to a translation and a rotation.

Activities 19.5 (a):

1. Your text:

   Morrill, W. K. Analytic Geometry, pp. 212-215.

   Exercises: p. 215, problems 5, 7 and 9.

2. Your text and study guide.

3. Solved Problems;

   Schaum's Outline Series, Theory & Problems of Plane & Solid Analytic Geometry, p. 68, problem 5; p. 70, problem 9.

4. Other Reading Sources:

   Fuller, Gordon, Analytic Geometry, pp. 88-91.

   Protter-Morrey, Analytic Geometry, pp. 142-145.

5. Individual Assistance.
6. Informal Group Sessions
7. Lecture 7

Self-Evaluation 19.5 (a)

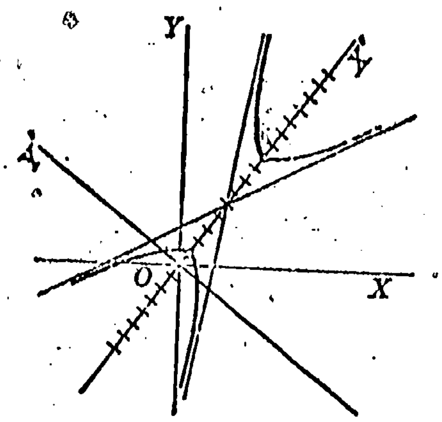
1. Simplify the following equation; identify, discuss and sketch the conic.

   a. $11x^2 - 24xy + 4y^2 + 30x + 40y - 45 = 0$

Answer Objective 19.5 (a)

$\frac{x''^2}{16} - \frac{y''^2}{4} = 1$. Hyperbola

a=4, b=2, c=$2\sqrt{5}$, with respect to the new axis the vertices are (4,0) and (-4,0) the foci are ($2\sqrt{5}$, 0) and ($-2\sqrt{5}$,0). The equation of the asymptotes are 4y''= 2x'' and 4y'' = -2x''.



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STUDY GUIDE

UNIT 19

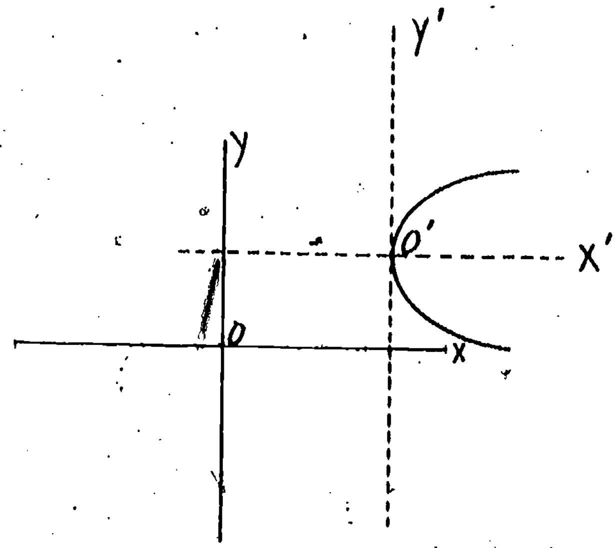
UNIT 19

Title: TRANSFORMATIONS

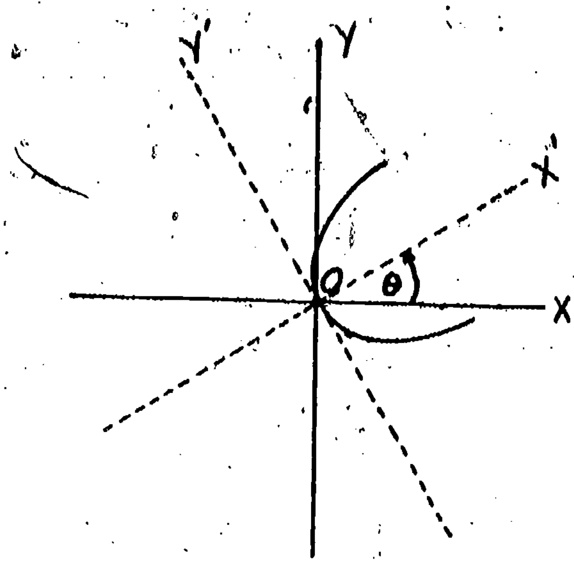
Rationale:

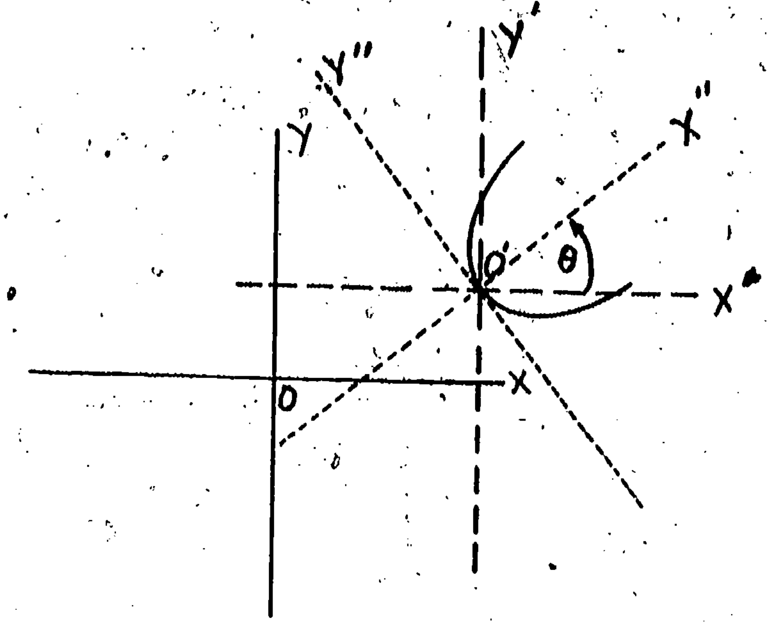
The conics you considered in Unit 18 were very special--they were all simple conics. (Their centers, in the hyperbola, ellipse, and circle, and the vertex in the parabola, were all at the origin). When the equation of the conic is a simple equation, it is very easy to discuss the curve and sketch the graph of the conic, but unfortunately not all conics are simple conics. The methods of handling non-simple conics are introduced in this unit.

The two methods introduced are called translation and rotation. A translation is performed when the axis of symmetry of the non-simple conic is parallel to the coordinate axes. (See the drawing below).



A rotation is performed when the axis of symmetry of the non-simple conic has been rotated through some angle and with respect to the coordinate axes, but the center or vertex remains at the origin. (See the drawing to the right on the following page).





Sometimes you must perform both a translation and a rotation. (See the drawing to the right above.) This occurs when the axis of symmetry of the non simple conic has been rotated and the center is not at the origin. In all the cases the translation, the rotation, or the translation and rotation converts the equation of the non-simple conic to a simple equation. Then you can use the methods and skills of Unit 18 to discuss the curve and sketch the graph.

Prerequisites:

Units 17 and 18 and objective 10.2 in Algebra and Trigonometry. (Specifically that section that deals with "completing the square")

Objectives:

19.5 Given the image of a conic, describe the transformation and sketch the image

(a) with respect to a rotation and translation

(b) with respect to a rotation

(c) with respect to a translation

19.4 Determine the equations of the images of conics relative to a given transformation and sketch the curve

(a) with respect to a rotation

(b) with respect to a translation

19.3 Determine the simplified equations of images of polynomial functions relative to a translation

19.1 Determine the equation of any conic by applying the definition of that particular conic.

Instructional Activity:

Task 1 (equations of ellipses, hyperbolas, and parabolas)

Objective 19.1

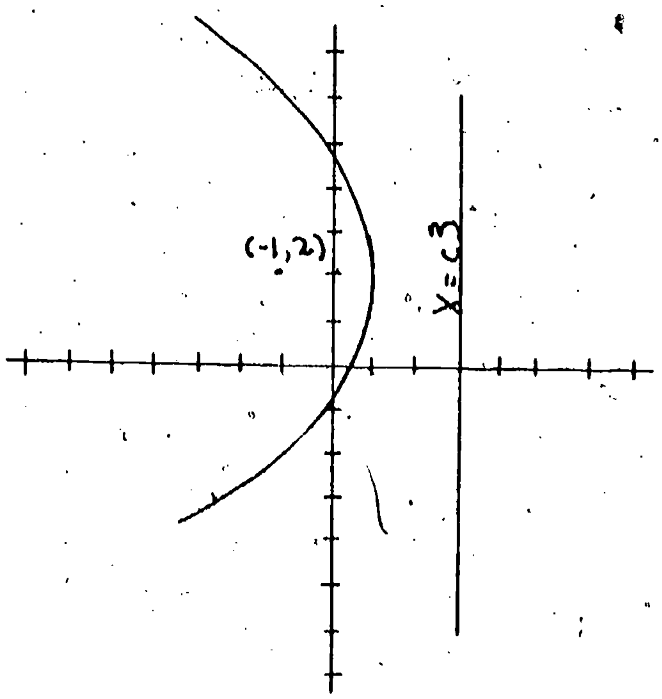
Determine the equation of any conic by applying the definition of that particular conic.

Read the General Definition of a conic on page 139 in your text again then examine the example 5-1 on page 140. Work problems 1 - 8 on page 141, problem 1 is begun for you on the following page.

Problem 1, page 141:

Derive an equation of each of the following conics:

1. Focus is at (-1, 2); directrix is the line x = 3, e = 1



Since e = 1, the conic is a ____________

and |FP| = |PD|

$|FP| = \sqrt{(x - (-1))^2 + (y - 2)^2} =$ ____________

$|DP| = \frac{|x - 3|}{1}$

Since |FP| =   and |DP| =

|FP| = |DP| is

and squaring both sides ____________

collecting terms ____________

hence the equation is ____________

Did you get $y^2 - 4y + 8x - 4 = 0$. Good! If not, check you calculations over again. If you are still having problems see your instructor. Now try problems 2 - 8 on page 141.

Task 2 (determining equation of a circle)

Read pages 103-105 and then write the definition of a circle and the mathematical statement (equation) that is described by the definition. (Use space provided at the top of the next page).

The mathematical statement $(x - h)^2 + (y - k)^2 = r^2$ is called the ______________ equation of the circle. How does the standard equation (your answer above) differ from the simple equation?

Again as with the simple conics in Unit 18, the circle whose equation is simple has its center at the origin, while the circle whose equation is in standard form can have its center anywhere on the coordinate plane.

Notice in example 4-4, page 104 in your text, you must apply the concept of "determining the distance from a point to a line" to find the radius of the circle. (The point is the center of the circle; the line is the tangent line.)

On page 105, work problems 1 through 3, 4(a), 5(a, c), 6(a, c, e), 7(a), 9. (Also work any others in which you are particularly interested.)

Problem 3 is started for you to finish.

3. What is the equation of a circle if the end points of the diameter are (2, 3) and (-1, 5). Draw the circle.

the center is the midpoint of diameter (see Objective 17.2 b)

$$x = \frac{x_1 + x_2}{2} = \frac{2 - 1}{2} = \frac{1}{2}$$

$$y = \qquad = \qquad =$$

radius - distance between 2 points -- part of Objective 17.2

radius = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(2 - \frac{1}{2})^2 + (3 - \quad)^2} =$

Hence the equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$.

or ______________________.

Answers:

center of the circle or midpoint of diameter $(\frac{1}{2}, 4)$, $r = \frac{13}{2}$

equation of circle $(x - \frac{1}{2})^2 + (y - 4)^2 = \frac{13}{4}$

When you have finished working the problems, read the next section on the general equation of the circle, pp. 106-108. In order to understand the reading material and work the problems in this section, you should be able to work the problem below. Try it. If you have problems, see a college algebra text (look under "completing the square"). Robison, J. Vincent, Modern Algebra & Trigonometry contains this information on pages 220 and 221.

Your instructor can help you find more information on this subject.

Group the x-terms together and the y-terms together and complete their square:

$$x^2 + 5x + y^2 - 2y + 3 = 0$$

Answer: $(x + 5/2)^2 + (y - 1)^2 = 17/4$

If you are unable to work this problem, get help before you continue.

If you are able to work the problem on the previous page, (by completing the squares), continue with the material below:

After reading this section you should be able to list the conditions necessary in order for an equation to be that of a circle.

They are:

1. ____________________________________
2. ____________________________________
3. ____________________________________

You should have listed:

1. The equation is of 2nd degree
2. The coefficients of $x^2 + y^2$ are equal
3. There is no xy term

Now, using your criteria, pick out the circles from the list below:

(a) $x^2 + 3y^2 + 7x + 2y + 3 = 0$

(b) $x^2 + y^2 + 2x + 5 = 0$

(c) $x^2 + y^2 + xy + 3x = 0$

(d) $2x^2 + 2y^2 + 4x + 7y + 3 = 0$

(e) $5x + 10y^2 + 3x + 7y - 9 = 0$

You should have picked out (b) and (d). Did you?

You should also be able to determine when the graph of the circle exists.

(a) If $D^2 + E^2 - 4F > 0$, the graph of the circle exists.

(b) If $D^2 + E^2 - 4F = 0$, the graph of the circle is a ________.

(c) If $D^2 + E^2 - 4F < 0$, the graph of the circle is said to be ________.

Answers: (b) point circle (c) imaginary

On page 108, problems 1 - 14, determine when the graph exists, when the circle is a point circle, or imaginary circle by evaluating $D^2 + E^2 - 4F$.

Also work problems 1, 3, 5, 7, 9, 11, and 17 on page 108.

After working the problems successfully, continue to section 4-3 (Circle determined by 3 conditions). Before beginning, look at the top of page 109. Can you perform these operations? (Determine D, E, and F) If not, review the Linear Algebra Unit Objective 14.1 (Solving equations using Cramer's Rule) or see the Introduction in this text (Morrill) section 1-5, 1-6, and 1-7.

When you are able to solve 3 equations in 3 unknowns using Cramer's Rule, read pages 108-111. The examples 4-7 and 4-8 in your text should be very helpful.

On page 111-112, work problems 1(a, c), 3, 4(a, c, e, f, g). Problems 1(a) and 4(c) are begun for you.

1, (a) Find an equation of the circle which passes through each of the following points. (2, 0) (1, -3) and (3, 1)

All three points must satisfy the equation of the circle;

$$x^2 + y^2 + Dx + Ey + F = 0$$

Hence for the point:

(2, 0)	4 + 0 + 2D + 0 + F = 0
(1, -3)	1 + 9 + ____ D + ____ E + F = 0
(3, 1)	____ + ____ + ____ D + ____ E + F = 0

And solving for the D, E and F:

$$D = \frac{\begin{vmatrix} -4 & 0 & 1 \\ -10 & -3 & 1 \\ -10 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 1 \\ 1 & -3 & 1 \\ 3 & 1 & 1 \end{vmatrix}} = \underline{\qquad\qquad} =$$

$$E = \frac{\begin{vmatrix} \quad \end{vmatrix}}{\begin{vmatrix} \quad \end{vmatrix}} = \underline{\qquad\qquad}$$

$$F = \frac{\begin{vmatrix} \quad \end{vmatrix}}{\begin{vmatrix} \quad \end{vmatrix}} = \underline{\qquad\qquad}$$

Hence the equation of the circle is ______________________.

4. (c) Find an equation of the circle satisfying the given conditions

Its center is on the line $x - y + 4 = 0$ and it touches both axes.

Since the circle touches both axes, the lines whose equations are $x = 0$ ________ and ________ are tangents to the circle, and the coordinates of the center of the circle must be either (h, h) or (,) ________.

Since the center is on the line $x - y + 4 = 0$, then either (h, h) or (h, -h) must satisfy the equation.

If (h, h) is the center, then

$h - h + 4 = 0$ or $4 = 0$

(Hence (h, h) is not the center of the circle)

If (h,-h) is the center, then

$h - (-h) + 4 = 0$ or $h = -2$

(Hence the center of the circle is (h, -h) = (,).

And the radius must be the distance from the center (-2, 2) to either of the coordinate axes (x = 0 or y = 0) .

r = ________ =

Hence the equation of the circle is ( ) + ( ) =

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Self-Evaluation Objective 19.1

1. Determine the equation of the ellipse with a focus at (-1, -1), directrix x = 0, and $e = \frac{\sqrt{2}}{2}$.

2. Derive the equation of the parabola with its focus at (4, 3) and its vertex at (4, 6).

   (Hint: you need some of the skills you developed in Unit 18 to determine the equation of the directrix)

3. Derive the equation of the locus of a point P(x, y) which moves so that the distance from c = (3, 2) is always 4.

4. Derive the equation of the locus of a point P(x, y) the sum of whose distance from (3, 4) and (3, -4) is always 10. (You must use skills you developed in Units 17 and 18 to work this problem.)

Answers:

1. $x^2 + 2y^2 + 4x + 4y + 4 = 0$
2. Equation of the directrix is $y = 9$

   Equation of the parabola is $x^2 - 8x + 12y - 56 = 0$
3. $(x - 3)^2 + (y - 2)^2 = 16$
4. $\frac{x^2}{25} + \frac{(y - 4)^2}{16} = 1$

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Instructional Activity

Task 3 (Translation of points and curves)

Objective 19.2 (b) and 19.4 (b)

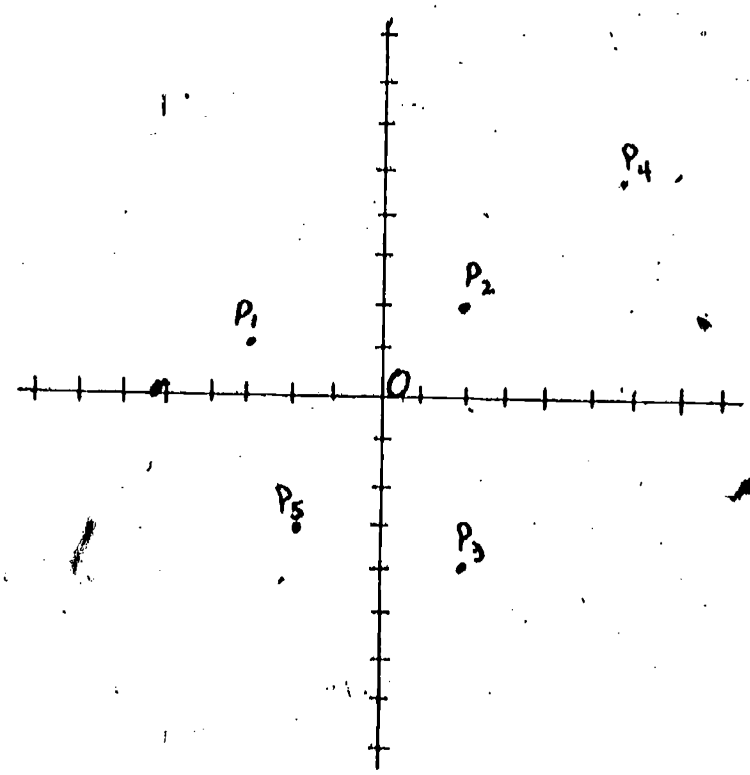
19.2 (b) Determine the coordinates of the images of points relative to a given transformation and sketch the points with respect to the new axis under a translation.

19.4 (b) Determine the equations of the images of conics relative to a given transformation and sketch the curve.

Read pages 192 - 194 in your text.

Do the exercies below and then do the problems 1-3 on pages 194 and 195 of your text.

1. Below is a coordinate system. Sketch a new set of coordinate axes with the point P = (3, + 4) as the new origin. Now using your drawing determine the coordinates of the points P_1, P_2, P_3, P_4, and P_5 with respect to the new coordinate axes.



Answer:

1. $P_1 = (-6, -3)$

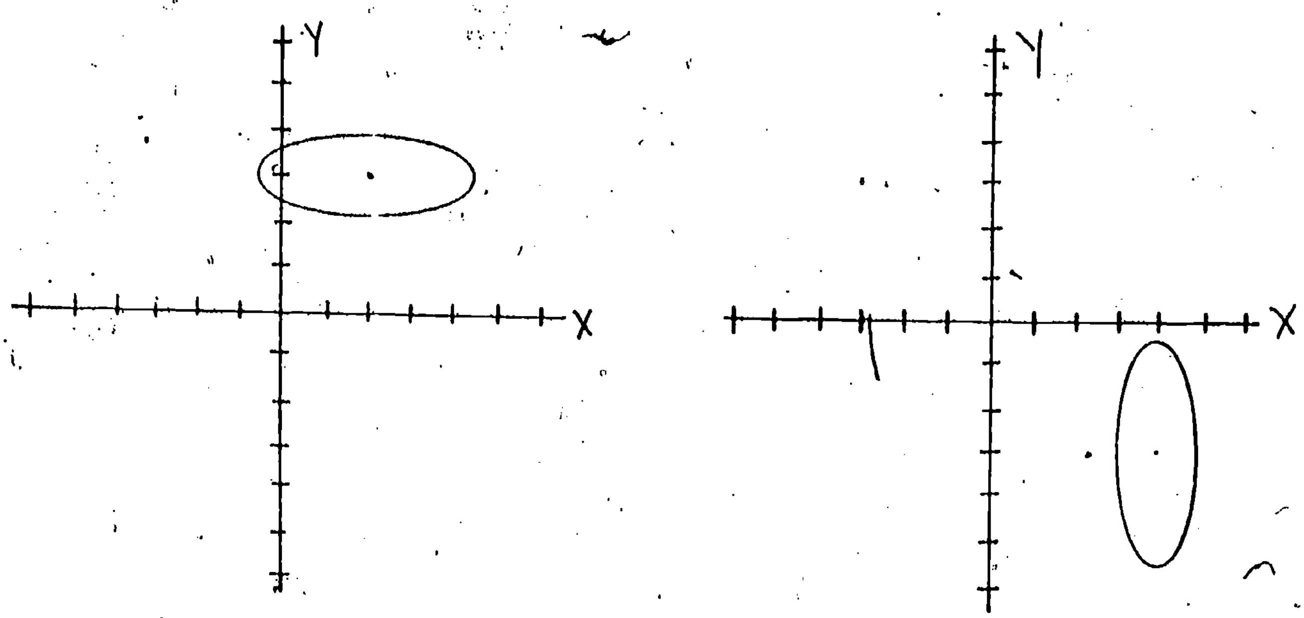
   $P_2 = (-1, -2)$

   $P_3 = (-1, -8)$

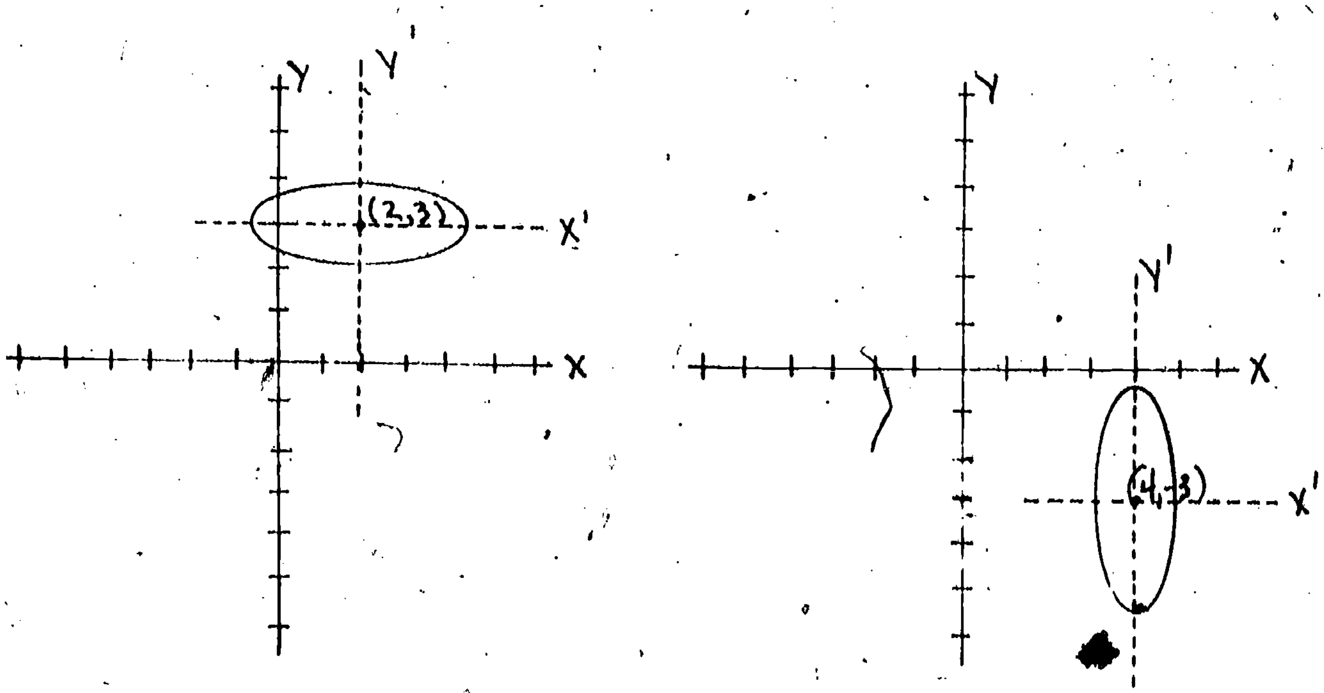
   $P_4 = (3, 1)$

   $P_5 = (-5, -7)$

In the drawings below, draw new sets of coordinate axes so that the centers of the conics are at the new origin. What are the coordinates of the new origin with respect to the original set of axes.



Answers:



Self-Evaluation Objectives 19.2 (b) and 19.4 (b)

1. Determine the coordinates of each of the points P when the axis is translated so the new origin is 0', and sketch the old and new coordinate axes.

(a) P = (3,2) 0' = (-3,0)

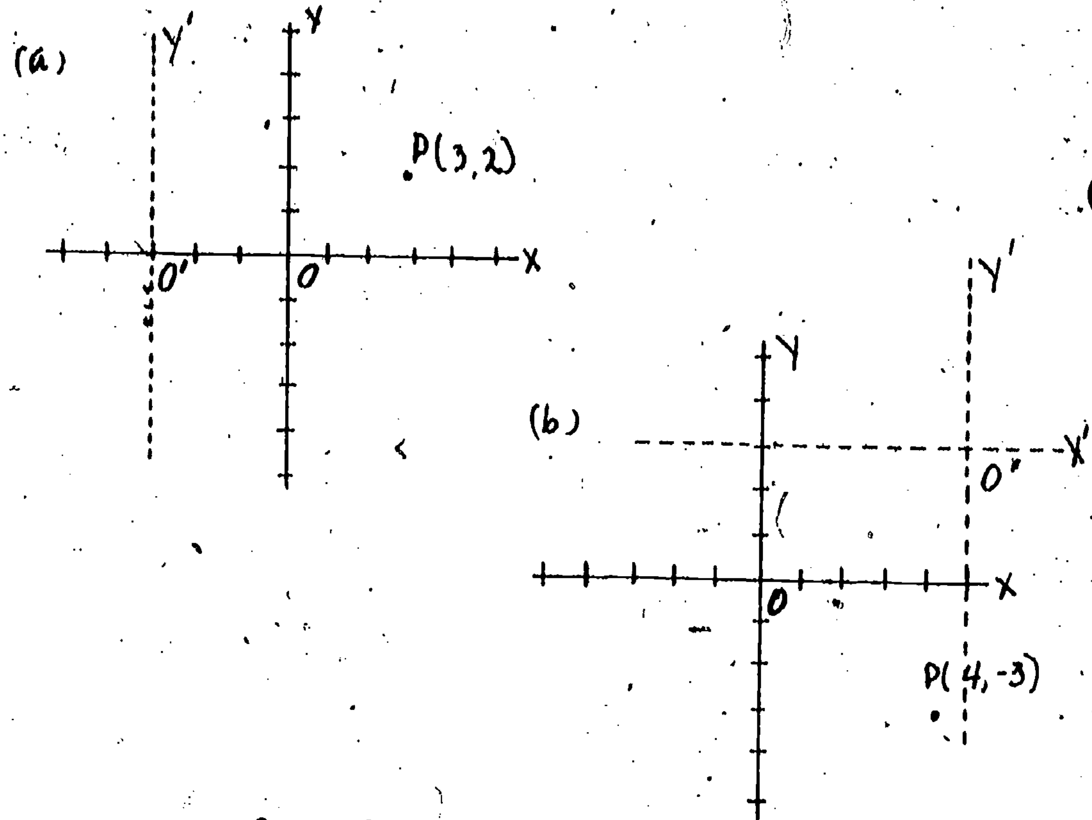
(b) P = (4,-3) 0' = (5,3)

2. Given the circle whose equation is $(x + 2)^2 + (y - 3)^2 = 4$, determine the center of the circle, and transform the equation of the circle to a simple equation by using the center as the new origin.

Answers:

1. (a) (6, 2)

   (b) (-1, -6)



2. Center (-2,3). New equation $x'^2 + y'^2 = 4$

Objective 19.5 (c)

Given the image of a conic, describe the transformation and sketch the image with respect to a translation.

Instructional Activity - Task 4 (Determining the translation for a given conic.)

Read pages 195-199 in your text. (Omit example 6-3, page 197.) On page 199 work problems 3, 5, 7, 8, 12, 14, and 15. Problems 3 and 5 are started for you.

Simplify and discuss each of the following equations and draw its graph when it exists.

3. $9x^2 + 4y^2 + 72x = 0$

$9x^2 + 72x \qquad + 4y^2 = 0$ (Collect x and y terms)

$9(x^2 + 8x \qquad) + 4(y^2 \qquad) = 0$ (remove common numerical factors)

$9(x^2 + 8x + 16) + 4(y^2 + 0) = 0 + 9(16)$ (complete the square)

or $9(\qquad)^2 + 4(y + 0)^2 =$ ________

Let $x' =$ ________ and $y' =$ ________

Then the equation becomes:

$9x'^2 + 4y'^2 = 144$ or

$\frac{x'^2}{16} + \frac{y'^2}{36} = 1$ which is the equation of a(an) ________
(what type of conic)

Now using the skills and methods you developed in Unit 18, discuss and sketch the graph of the ellipse.

5. $4x = 8y - y^2$

$y^2 - 8y \qquad = -4x$ (collect x and y terms)

$(y^2 - 8y + \qquad) = -4x +$ ______ (complete the square)

$(y -$ ____ $)^2 = -4(x$ ______ $)$

Let $y' =$ ______ and $x' =$ ______

The equation then becomes ________________________,

which is the equation of a ______________ (what type of conic).

Use the skills and methods you learned in Unit 18 to discuss and sketch the graph of the parabola $y'^2 = -4x'$.

After finishing the problems on page 199, read pages 200-204 in your text.

On page 205, work problems 2 (a, b, d, g, j, l, p, s, t, and w). Problems 2(a) and 2(b) are started for you.

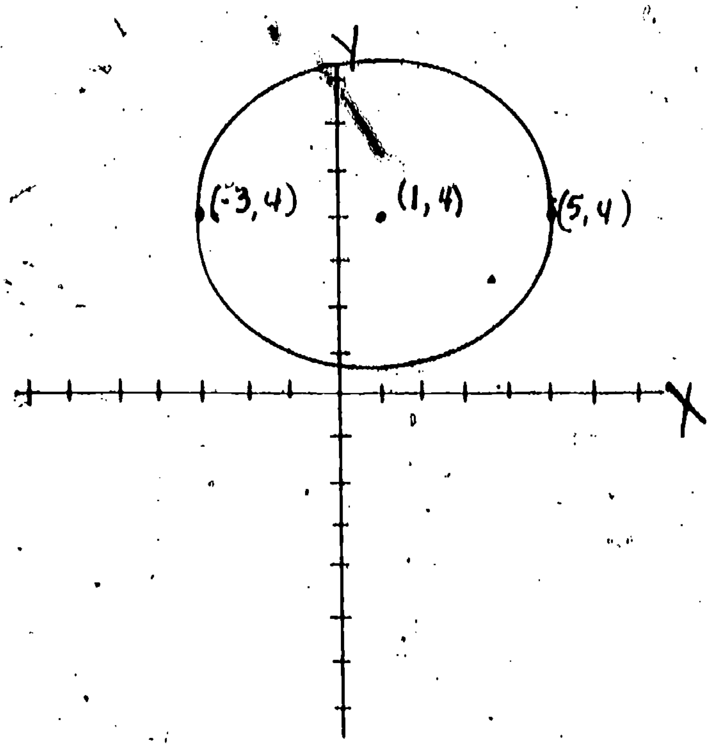
Problem 2. Find a standard equation of each of the following conics and sketch each.

(a) e = 1/2, and the vertices are at (-3,4) and (5,4).

Since $e = \frac{1}{2}$ the conic is a (an) ____________________.

Since the x' axis is the major axis, the equation of the ellipse is of the form ____________________.

$$\left(\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1 \right)$$



The length of the major axis (2a) is ______________.

Since 2a = 8, then a = ____

and ae = ______________.

Since c = ae = 2, then

$b^2 = a^2 - c^2$ implies that

b^2 = ______________.

The center (h, k) = (____, ____)

Hence the equation of the ellipse:

$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ is ______________________.

Answer: $\frac{(x - 1)^2}{16} + \frac{(y - 4)^2}{12} = 1$

(b) Center is at (-3, -1), a = 4, b = 5, and foci are on y = -1.

Since the conic has a center it is either a __________, a __________, or an __________. It is not a circle since $a \neq b$. Since $b > a$ the conic cannot be an ____________. Hence the conic is a ______________ and the equation is of the form ______________________, since the x axis is the transverse axis.

Answers: Circle, ellipse, hyperbola, ellipse, hyperbola.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Hence the equation:

$$\frac{(x+h)^2}{a^2} - \frac{(y+k)^2}{b^2} = 1$$

becomes

$$(x + \_\_)^2 - (y + \_\_)^2 = 1$$

Answers: $\frac{(x+3)^2}{16} - \frac{(y+1)^2}{25} = 1$

Self-Evaluation Objective 19.5 (c)

Simplify & name the following equations and sketch the graph of problem 3:

1. $5x^2 - 40x - 3y^2 + 65 = 0$
2. $x^2 + 4x - 16y + 52 = 0$
3. $9x^2 - 18x + 4y^2 + 32y + 37 = 0$

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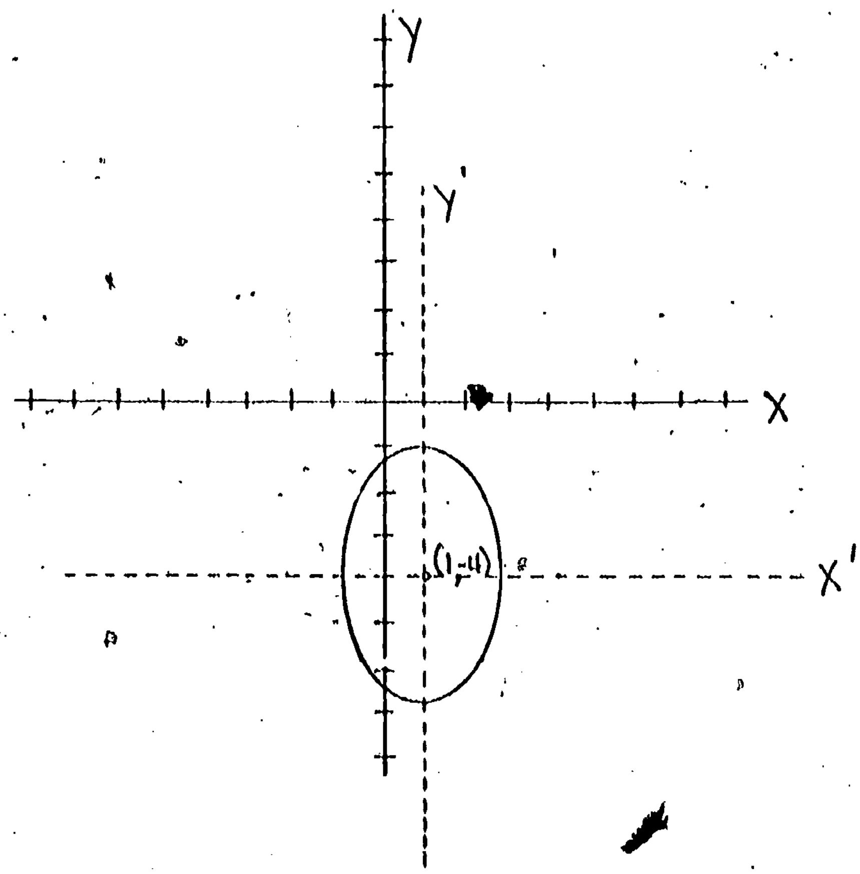
<u>Answers</u>:

1. $\frac{(x - 4)^2}{3} - \frac{y^2}{5} = 1$ (hyperbola)

2. $(x + 2)^2 = 16(y - 3)$ (parabola)

3. $\frac{(x - 1)^2}{4} + \frac{(y + 4)^2}{9} = 1$ (ellipse)



Objective 19.3

Determine the simplified equations of images of polynomial functions relative to a translation.

Instructional Activity - Task 5 (translation of polynomial functions).

On page 197 in your text, study example 6-3. Notice that the equation is a cubic and is the polynomial function $f(x) = x^3 - 3x^2 + 2x + 2$.

It is possible to find the zeros of the function and to sketch the graph of the function by using standard methods and skills acquired in college algebra (unit 9) polynomial functions), but the polynomial function can be simplified by a transformation.

After studying the example you will notice that there are four steps in transforming a polynomial function.

They are:

1. Make a substitution: $x = x' + h$ and $y = y' + k$.
2. Simplify the equation by expanding and collecting terms.
3. Determine h and k so that one of the coefficients and the constant term is zero.
4. Write the transformed equation.

Now, you try the problems below.

1. $y = x^3 + 6x^2 - 3x - 12$

2. $y = x^3 - 6x^2 + 11x - 10$

Self-Evaluation:

Simplify the following polynomial function.

$y = x^3 - 9x^2 + 2x + 30$

Answer: $y' = x'^3 - 25x'$

INSTRUCTIONAL ACTIVITY

Objectives 19.2 (a) and 19.4 (a)

Determine the coordinates of the images of points relative to a given transformation and sketch the points with respect to a new axis, (b) with respect to a rotation.

Determine the equation of the images of conics relative to a given transformation and sketch the curve, (b) with respect to a rotation.

Task 6 - Rotation of points and conics.

Read pages 205 - 209 in your text. In the reading material at the top of page 206 you might need to refer to the first chapter of your text for a review on trigonometry. Also you might wish to refer to page 31 in your text (the discussion on direction cosines.) Equation 2-6 is found on page 38.

Examples 6 - 9 and 6 - 10 in your text should be very helpful.

After reading the material on pages 209 - 210, work problems 1 (a,c, & e) and 2 (a,c,e, &g). Problems 1(a) and 2(c) are started for you.

1. Find the coordinates of each of the following points after the axes have been rotated through the given angle.

   (a) (3,-1) and $\Theta = 30^0$

Solution:

	x'	y'			x'	y'
x	$\cos 30^0$	$-\sin 30^0$	or	x	$\frac{\sqrt{3}}{2}$	____
y	$\sin 30^0$	$\cos 30^0$		y	$\frac{1}{2}$	____

Hence:

$x' = [x,y] \cdot [\cos 30^0, \sin 30^0]$

$y' = [x,y] \cdot [-\sin 30^0, \cos 30^0]$

$x' = [3,-1] \cdot \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\right] =$ ______________________

$y' =$ ______________________ $=$ ______________________

$(x', y') =$ ______________________

Answers: $\frac{-1}{2}, \frac{\sqrt{3}}{2}$, $x' = \frac{3\sqrt{1} -1}{2}$, $y' = \frac{-3 -\sqrt{3}}{2}$.

2. Rotate the axes throught the given angle and determine what the following equation becomes.

   $5x^2 + 4xy + 8y^2 = 36$ and $\theta = \cos^{-1}\frac{3}{\sqrt{10}}$

* If you are confused about the notation $\theta = \cos^{-1}\frac{3}{\sqrt{10}}$.

refer to a textbook on trigonometry. Look under the topic "Inverse Trig. Functions"

Since $\cos\theta = \frac{3}{\sqrt{10}}$ and $\sin\theta = \frac{1}{\sqrt{10}}$

and

	x'	y'
x	$\frac{3}{\sqrt{10}}$	______
y	______	______

$x = [x', y'] \cdot \left[\frac{3}{10} + \_\_\_\_\_\_\right]$

$y = [x', y'] \cdot \left[\_\_\_\_ + \_\_\_\_\right]$

Hence the equation becomes:

$5(\qquad)^2 + 4(\qquad)(\qquad) + 8(\qquad)^2 = 36$

expanding and collecting terms yields:

The simplified equation is:

______________________________________ which is the equation of a ________________________.

Before you continue, take a minute to look at some of the equations you have either translated or rotated. Why is it necessary to perform a translation on one equation while on another you perform a rotation? Go back to page 199 and look at the equations there. (These you translated.) Now look at the problems you just finished (on rotations).

Below are some characteristics of the two types of equations. Read them and pick out the ones that are characteristic of the problems on page 199. 1. ___and____. Pick out the ones that are characteristic of the problems on rotation. 2. ___and ____. Pick out the two that an equation you translate and an equation you rotate do not have in common. 3. _____ and _____.

(a) Equation is second degree.

(b) Equation contains linear terms in x and y.

(c) Equation contains a term in xy.

Answers: 1. a & b 2. a & c 3. b & c

Since you performed a translation on the problems on page 199 and a rotation on the set of problems in the study guide, decide what you should do to simplify the problems below. (Answer with either translate or rotate.)

a. $x^2 + 3x + y^2 - 2y + 12 = 0$ __________

b. $3x^2 - 2xy + 5y^2 - 9 = 0$ __________

c. $x^2 + y^2 - 2x - 4 = 0$ __________

d. $x^2 - 4y + 2 = 0$ __________

e. $3y^2 + xy - x^2 + 4 = 0$ __________

Answers: a. translate, b. rotate, c. translate, d. translate, e. rotate.

SELF-EVALUATION. Objectives 19.2 (a) and 19.4 (a). (You may use a trig. table.)

1. Determine the coordinates of the following points when the axes have been rotated through 60°.

   a. (1,4) b. (2,6)

2. Determine which of the following equations must be rotated in order to simplify the equation. Simplify that equation by rotating the axes through the angle $\theta = 135^\circ$.

   a. $3x^2 + 3y^2 - 12x + 12y - 1 = 0$

   b. $x^2 - 6y - 4x + 5 = 0$

   c. $x^2 - 4xy + y^2 = 5$

Answers:

1. a. $\left(\frac{1 + 4\sqrt{3}}{2}, 2\sqrt{3} + 2\right)$ b. $(2 + \sqrt{3}, -1 + 2\sqrt{3})$

2. c. $3x'^2 - y'^2 = 5$

Instructional Activity

Task 7 (Determining a rotation for a given conic.)

Objective 19.5 (b)

Given the image of a conic, describe the transformation and sketch the image with respect to a rotation.

Read pages 210-212 (stop at last paragraph on 212). You might wish to refer to the first chapter in your text, page 16, or see a trigonometry text in trig. identities.

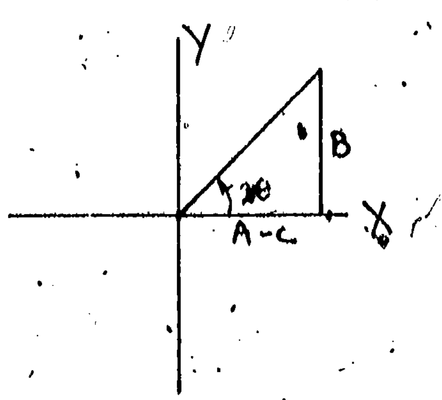
At the top of page 211 in your text, the author states, "If $\sin 2\theta = 0$, we must have $B = 0$. Why is this so?

____

____

To obtain equation 6-26 see the drawing below.

$$\text{Cot } 2\theta = \frac{A - C}{B}$$



What is the length of the hypotenuse of the triangle pictured at the left?

____

$$\text{Hence Cos } 2\theta = \frac{A - C}{\pm\sqrt{\quad\quad}}$$

Answers: $\sqrt{B^2 + (A-C)^2}$, $\quad \text{Cos } 2\theta = \dfrac{A - C}{\pm\sqrt{B^2 + (A-C)^2}}$

Why is θ a positive acute angle if $\cot 2\theta$ and $\cos 2\theta$ have the same sign? ______________________________

____

Answer: The only quadrants where $\cos 2\theta$ and $\cot 2\theta$ have the same sign is in quadrants 1 and 2. Hence $0^0 \leq 2\theta \leq 180^0$ and $0^0 \leq \theta \leq 90^0$

After reading the material and studying example 6-11, try working problem 11(a, c, e, j, and n) on page 216. Problem 11(a) is started for you:

11. Remove the x y term, identify the conic, and draw the graph of each of the following equations.

(a) $xy = 2$

$$\cot 2\theta = \frac{A - C}{B} \qquad \cos 2\theta = \frac{A - C}{\pm\sqrt{B^2 + (A - c)^2}}$$

$$\cot 2\theta = \frac{0}{1} = 0 \qquad \cos 2\theta = \frac{0}{\pm\sqrt{\quad}} = \underline{\qquad}$$

$$\sin\theta = \sqrt{\frac{1 - 0}{2}} = \sqrt{\frac{1}{2}} \qquad \cos\theta = \sqrt{\frac{1 + 0}{2}} = \sqrt{\frac{1}{2}}$$

Hence θ = ________ o.

	x'	y'
x	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
y	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

Hence $x =$ ____________ and $y =$ ____________

Substituting for x and y in the equation $xy = 2$ the equation becomes:

(____________)(____________) = 2

or

____

which is the equation of a(an) ____________

Answers: $\theta = 45^{\circ}$, $x = \frac{x' - y'}{\sqrt{2}}$, $y = \frac{x' + y'}{\sqrt{2}}$, $xy=2$ becomes

$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) = 2$ or $x'^2 - y'^2 = 4$ which is

the equation of a hyperbola.

Objective 19.5 (a)

Given the image of a conic, describe the transformation and sketch the image with respect to a translation and a rotation.

Instructional Activity

Task 8 (Describing a Transformation)

Read pages 212 (last paragraph) through 215 in your text.

After reading pages 212-215, you should be able to answer the following questions:

1. It is necessary to translate and to rotate the axes in order to obtain the equation in its simplest form when both ____________ terms and the ________ and/or ________ terms appear in the equation.

2. In order to determine the order of transformation (translate and then rotate, or rotate and then translate) you should evaluate ______________________________.

3. If the value of the answer to problem 2 above is zero, you ______________ the axes and eliminate the xy term before translating.

4. If A = C in the equation, the angle θ through which the axes should be rotated to eliminate the xy term is ________ °.

Answers:

1. xy, x and/or y

2. $4AC - B^2$

3. rotate

4. 45^{o}

Now work problems 5, 7, & 9 on page 215.
Problem 5 is begun for you.

Simplify and discuss each of the following equations and draw the graph.

5. $24xy - 7x^2 - 120x - 444 = 0$ OR $-7x^2 + 24xy - 120x - 144$

Since $4AC - B^2 \neq 0$, you should ______________ the axes first.

$$h = \frac{\begin{vmatrix} 120 & 24 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} -14 & 24 \\ 24 & 0 \end{vmatrix}} = \frac{0}{(24)^2} = \underline{\qquad\qquad}$$

$$k = \frac{\begin{vmatrix} -14 & 120 \\ 24 & 0 \end{vmatrix}}{\begin{vmatrix} -14 & 120 \\ 24 & 48 \end{vmatrix}} = \frac{24\ (120)}{(24)^2} = \underline{\qquad\qquad}$$

After translation of the axes to the point (0, -5) as a new origin, the given equation becomes $-7(x' + 0)^2 + 24(x' + 0)(y" - 5) -120(x' + 0) - 144 = 0$ which reduces to ____________________

Next rotate the axes to eliminate the x'y' term.

$\text{Cot } 2\theta = \dfrac{A - C}{B} =$ __________

$\text{Cos } 2\theta = \dfrac{A - C}{\pm\sqrt{B^2 + (A-C)^2}} =$ __________

$\sin\theta = \sqrt{\dfrac{1 - \cos 2}{2}}$ $\cos\theta = \sqrt{\dfrac{1 + \cos 2}{2}}$

$\sin\theta =$ __________ $\cos\theta =$ __________

Check at this point to see if your answers for sin θ and cos are feasible by using the identity $\sin^2\theta + \cos^2\theta = 1$.

	x"	y"
x'		
y'		

and x" =

y" =

The new equation becomes:

Rewriting the equation in simple form you have ____________

The curve is a(an) ____________. Now use the skills and methods you acquired in unit 18 to discuss and sketch the curve.

Answers: translate, h=0, k=-5, after translation the equation reduces to $-7x'^2 + 24x'y' - 144 = 0$, $\cot 2\ = -\frac{7}{25}$, $\cos\ = \frac{3}{5}$ $\sin\theta = \frac{4}{5}$, $x' = \frac{3x'' - 4y''}{5}$, $y' = \frac{4x'' + 3y''}{5}$, the new equation becomes $-7\left(\frac{3x'' - 4y''}{5}\right)^2 + 24\left(\frac{3x'' - 4y''}{5}\right)\left(\frac{4x'' + 3y''}{5}\right) - 144 = 0$

Writing the equation in simple form you have $\frac{x''^2}{16} - \frac{y''^2}{9} = 1$ which is the equation of a hyperbola.

Self-Evaluation Objective 19.5 (a)

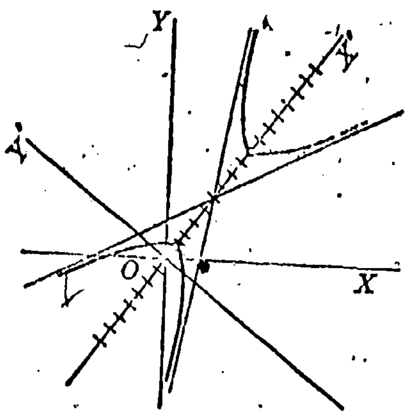
1. Simplify the following equation and identify and discuss and sketch the conic.

   a. $11x^2 - 24xy + 4y^2 + 30x + 40y - 45 = 0$

Answer Objective 19.5 (a)

$$\frac{x''^2}{16} - \frac{y''^2}{4} = 1$$. Hyperbola

a=4, b=2, c=$2\sqrt{5}$, with respect to the new axis the vertices are (4,0) and (-4,0) the foci are ($2\sqrt{5}$, 0) and ($-2\sqrt{5}$,0). The equation of the asymptotes are $4y'' = 2x''$ and $4y'' = -2x''$.



Unit 20---Polar Coordinates

Rationale:

The rectangular coordinate system, based on a grid composed of two mutually perpendicular lines in a plane, is the most common coordinate system. It is not however, the only coordinate system, nor is it the best system for every problem.

In this unit we will take a look at another system, the polar coordinate system. The polar coordinate system is based on a grid composed of a system of concentric circles and a system of rays radiating from the common center of the concentric circles.

The common center, called the pole, and a fixed ray, called the polar axis, is the frame of reference for this system. The polar coordinates of a point P are written as an ordered pair (p, θ), where p is the polar distance (The distance from the pole to the point P) and θ is a measure of the polar angle (located by rotating a ray about the pole, from the polar axis in either direction and terminating the rotation in a position such that the ray contains the point P.

Objectives:

20.4 *Find the coordinates of the intersection of two polar equations.*

20.3 *Graph a polar equation and name the curve among (spiral, rose, limacon, conic, lemniscate).*

20.2 *Transform rectangular equations to polar equations and transform polar equations to rectangular form.*

20.1 *Transform the coordinates of a point in polar form to rectangular form and from rectangular form to polar form.*

Prerequisites

Units 17, 18, & 19

Procedural Options

The procedural options for Unit 20 are the same as for Unit 17. You should begin with objective 20.1 and then go to 20.2, 20.3, and 20.4 in that order.

Unit Activities

Lectures 8 and 9

The lectures over this unit will have the following outline:

Lecture 8 Transforming equations

1. Transforming coordinates
   (a) from polar to rectangular form
   (b) from rectangular to polar form
2. Transforming equations
   (a) from polar to rectangular form
   (b) from rectangular to polar form

Lecture 9 Graphing polar equations

1. How to graph polar equations

Notice objectives 20.1 and 20.2 are covered in lecture 8, and objective 20.3 is covered in lecture 9.

Objective 20.1

Transform the coordinates of a point in polar form to rectangular form and from rectangular form to polar form.

Instructional Activities 20.1

1. Your text

Merrill, W. K. Analytic Geometry, pp. 228-232.

Exercises: p. 230, problems 1, and 2 (a,c,e,g,i,l, & n)

2. Your text and Study Guide
3. Other Reading Sources

   Fuller, Gordon, Analytic Geometry, pp. 118-122 (down to example 3)

   Protter-Morrey, Analytic Geometry, pp. 180-181.

4. Individual Assistance
5. Informal Group Sessions
6. Lecture 8

Self-Evaluation:

1. Convert the following points from cartesian to polar coordinates:

   (a) $(4\sqrt{2}, -4\sqrt{2})$

   (b) $(0, 3)$

2. Convert the following points from polar coordinates to cartesian coordinates:

   (a) $(-2, 60^\circ)$

   (b) $(3, 45^\circ)$

Answers

1. (a) $(8, -45^{\circ})^{*}$ (b) $(3, 90^{\circ})^{*}$ * not unique

2. (a) $(-1, \sqrt{3})$ (b) $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$

Objective 20.2

Transform rectangular equations to polar equations, and transform polar equations to rectnagular form.

Instructional Activities

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 232-234
   Exercises, pp. 234, problems 1 and 2

2. Your Text and Study Guide

3. Solved Problems

   Schaum's Outline Series, Theory and Problems of Plane and Solid Analytic Geometry, pp. 76- problem 13, 14, 15, and 16.

4. Other Reading Sources

   Protter-Morrey, Analytic Geometry, pp. 187-188.

5. Individual Assistance

6. Informal Group Sessions

7. Lecture 8

Self-Evaluation Objective 20.2

1. Transform the following equations to rectangular form

   (a) $p^2 \sin 20 = 16$

   (b) $p^2 = 4 \tan 20$

2. Transform the following equations to polar form

   (a) $2xy = 8$

   (b) $x^2 + y^2 - 2x = 0$

Answers Objective 20.2

1. (a) $xy = 16$ (b) $x^4 - y^4 = 8xy$

2. (a) $\rho^2 \sin 2\theta = 8$ (b) $\rho = 2 \cos \theta$

Objective 20.3

Graph a polar equation and name the curve among (spiral, rose, limacon, conic lemniscate)

Instructional Activities

1. Your Text

   Morrill, W. K., Analytic Geometry, pp. 234-253
   Exercises, pp. 236, problems: 1, 4, 7, 10, 13, 16, 19, 22, &25
   P. 238, problems: 1(a, b), 2(a , c), 3, 4(a, c, e), 5(a, c, e), 6(a, c, e,)
   pp. 247-248, problems: 1(a, c, e, g, j, p), 2(a, c,e, g, i, k, m, o, q)
   p. 251, problems: 1(a, c), 2(a, c,e)

2. Your Text and Study Guide

3. Solved Problems

   Schaums Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, pp. 74-80, problems, 3, 4, 7, 8, 9, 11, 12, 21, 22, 23, 24, 25, and 26.

4. Other Reading Sources

   Protter-Morrey, Analytic Geometry, pp. 182-186
   Fuller, Gordon, Analytic Geometry, pp. 126-133

5. Individual Assistance

6. Informal Group Sessions

7. Lecture 9

Self-evlauation Objective 20.3

1. Graph and name the curve $p^2 = 4 \sin 2\theta$.

Answers: Objective 20.3

Intercepts

θ	0^0	90^0	180^0	270^0
p	0	0	0	0

Symmetry:

Since the exponent of p is even, the curve is symmetric with respect to the pole.

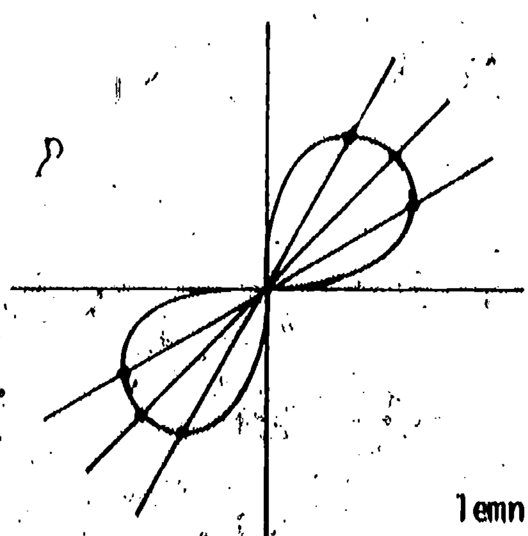
Extent: $\sin 2\theta \leq 1$ and $\sin 2\theta = 1$ when $\theta = 45^0$

p is imaginary when $\sin 2\theta$ is negative

Hence, the curve lies in quadrants I and III

Plotting:

θ	0^0	30^0	45^0	60^0	90^0
p	0	1.9	2	1.9	0



lemniscate

Objective 20.4

Find the coordinates of the intersection of two polar equations.

Instructional Activities

1. Your Text

   Morrill, W. K., Analytic Geometry , pp 253-258
   Exercises, p. 258, problems 1, 3, 5, 7, 9, 11

2. Your Text and the Study Guide

3. Individual Assistance

4. Informal Group Sessions

Self-Evaluation Objective 20.4

1. Find the points of intersection of

   $p = \sin\theta$ and $p = \cos\theta$

Answers:

1. $(\frac{1}{\sqrt{2}}, 45^{o})$ * and the pole. * not unique

STUDY GUIDE

Unit 20---Polar Coordinates

Rationale:

The rectangular coordinate system, based on a grid composed of two mutually perpendicular lines in a plane, is the most common coordinate system. It is not however, the only coordinate system, nor is it the best system for every problem.

In this unit we will take a look at another system, the polar coordinate system. The polar coordinate system is based on a grid composed of a system of concentric circles and a system of rays radiating from the common center of the concentric circles.

The common center, called the pole, and a fixed ray, called the polar axis, is the frame of reference for this system. The polar coordinates of a point P are written as an ordered pair (p, θ), where p is the polar distance (The distance from the pole to the point P) and θ is a measure of the polar angle (located by rotating a ray about the pole, from the polar axis in either direction and terminating the rotation in a position such that the ray contains the point P.

Objectives:

~~*20.4 Find the coordinates of the intersection of two polar equations.*~~

20.3 Graph a polar equation and name the curve among (spiral, rose, limacon, conic, lemniscate).

20.2 Transform rectangular equations to polar equations and transform polar equations to rectangular form.

20.1 Transform the coordinates of a point in polar form to rectangular form and from rectangular form to polar form.

Instructional Activities:

Objective 20.1

Transform the coordinates of a point in polar form to rectuangular form and from rectangular form to polar form.

Task 1. (Plotting Polar Coordinates)

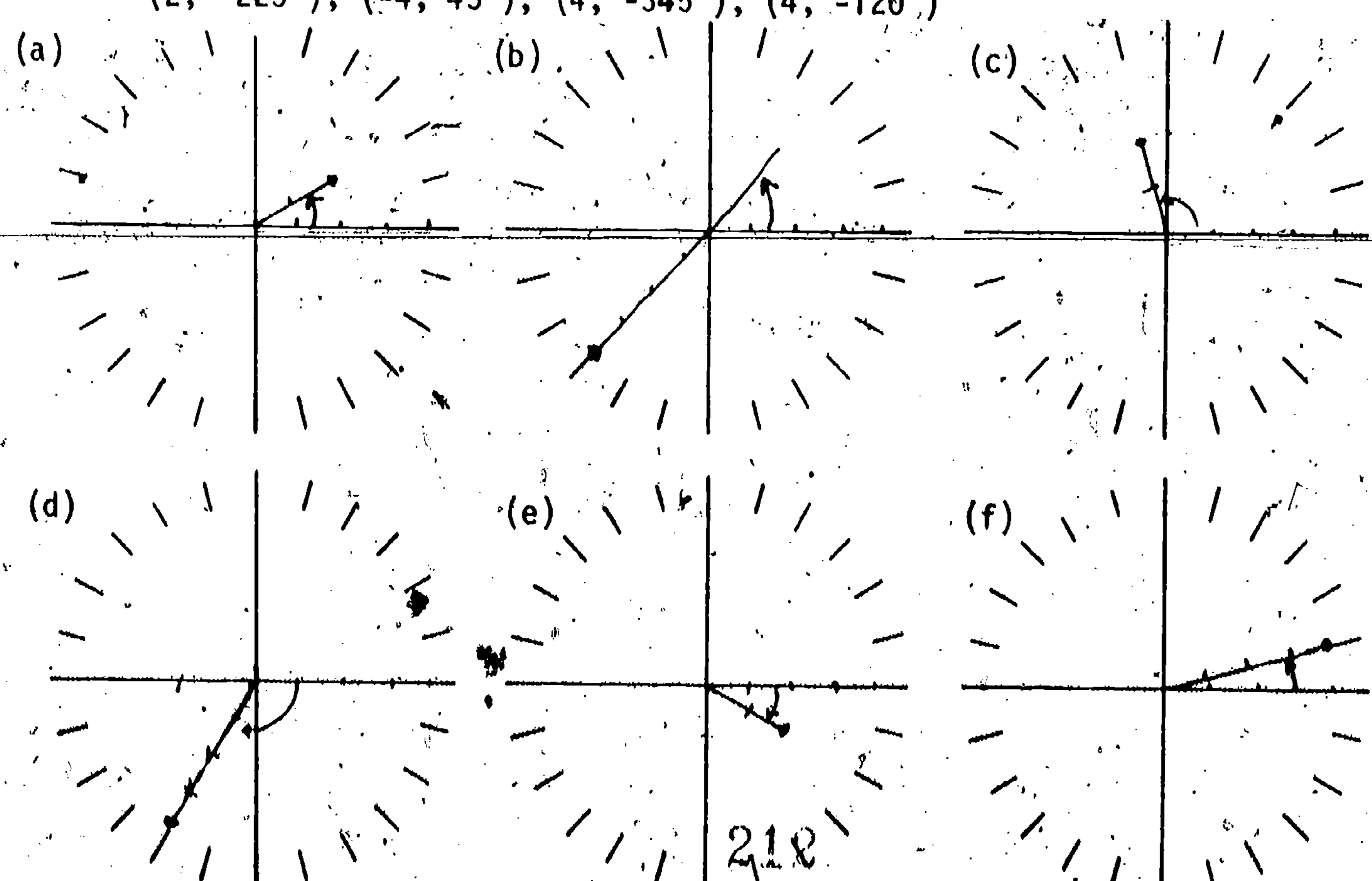
Read pages 228-230 in your text.

After reading the pages (228-230), work exercises 1 and 2 below and work problems 1 and 2(a, c, e, g, i, l, and n) on page 230.

Exercise 1.

Below is a set of figures (a-f) and a set of coordinates (set A). Pick out <u>all</u> the coordinates (from set A) that name the given point in each figure (a-f). NOTE: Each mark = 15^0.

$A = \{(4, 15^0), (0, \theta), (2, -30^0), (4, 30^0), (-4, 60^0), (2, 30^0), (4, -135^0), (-2, -150^0), (4, -150^0), (2, 105^0), (-2, 150^0), (2, -225^0), (-4, 45^0), (4, -345^0), (4, -120^0)\}$

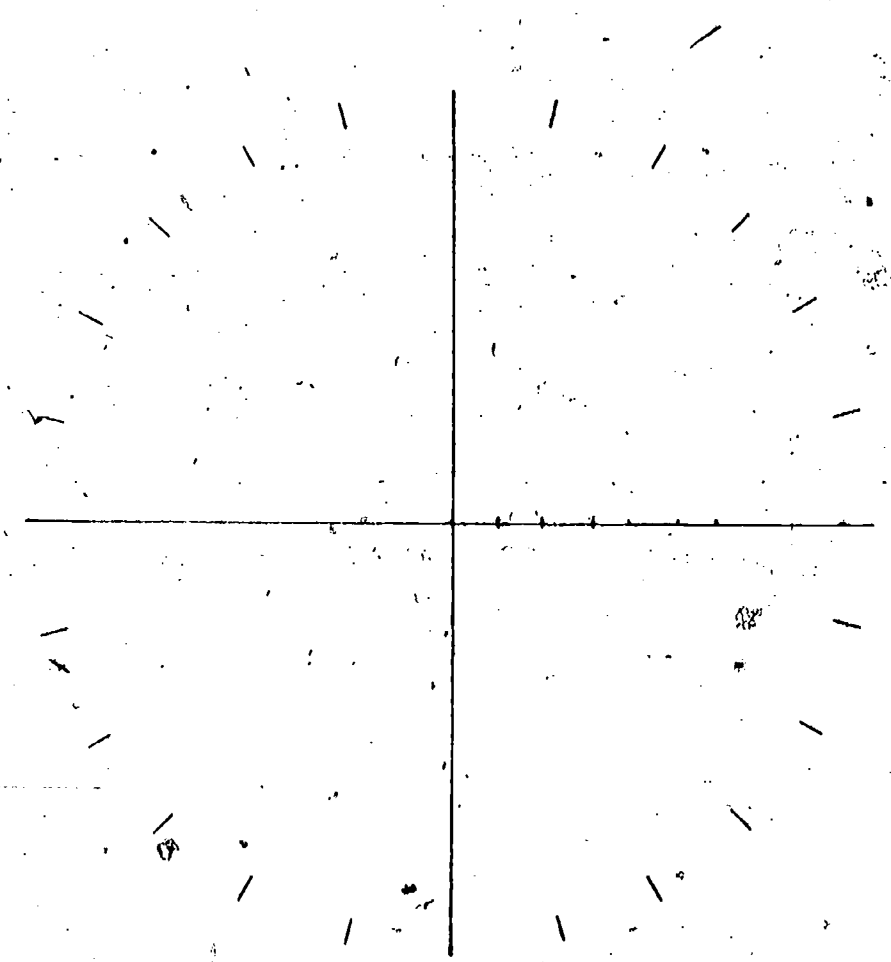


Exercise 2.

Plot the following polar coordinates on the polar coordinate system furnished below:

(a) $(3, -60^0)$ (b) $(2, \pi)$ (c) $(5, 0^0)$ (d) $(-6, -30^0)$

(e) $(4, -\frac{3\pi}{2})$ (f) $(-3, -300^0)$ (g) $(0, 75^0)$ (h) $3, -240^0)$



Answers

Exercise 1.

(a) $(2, 30^0)$, $(-2, -150^0)$

(b) $(-4, 45^0)$, $(4, -135^0)$

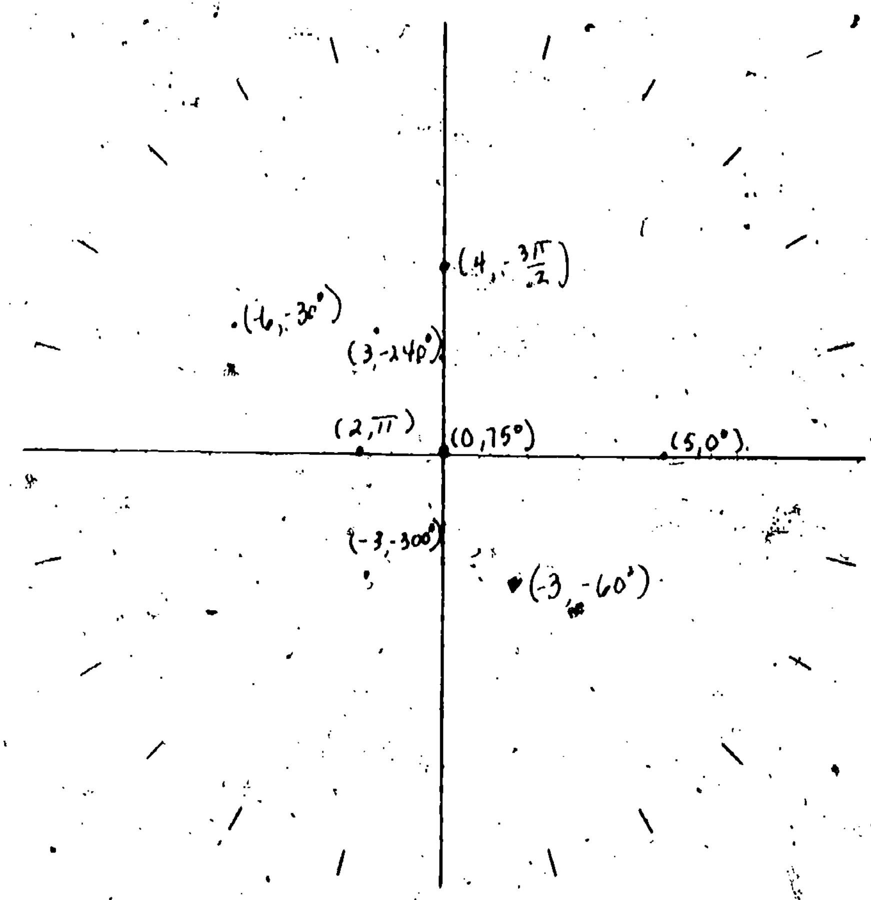
(c) $(2, 105)$, $(2, -225^0)$

(d) $(-4, 60^0)$, $(4, -120^0)$

(e) $(2, -30^0)$, $(-2, 150^0)$

(f) $(4, 15^0)$, $(-4, -175^0)$, $(4, -345^0)$

Exercise 2.



Task 2. (Converting Polar Coordinates to Cartesian)

Read pages 230-232 (down to Ex. 7-3)

(You will need a trig table for the exercise in this task.)

Work Exercises 1 and 2 below.

Exercise 1.

Convert the following cartesian coordinates to polar coordinates.

(a) $(3, 3)$ (b) $(-2, -2)$ (c) $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ (d) $(-\frac{3\sqrt{3}}{2}, \frac{3}{2})$

(e) $(-\frac{5}{2}, -\frac{5\sqrt{3}}{2})$

Exercise 2.

Convert the following polar coordinates to cartesian coordinates.

(a) $(3, 180^0)$ (b) $(2, 60^0)$ (c) $(4, 90^0)$ (d) $(-2, 210^0)$

(e) $(-3, -30^0)$

Answers

Exercise 1.

(a) $(3\sqrt{2}, 45^0)$ (b) $(2\sqrt{2}, 225^0)$ (c) $(1, -30^0)$ (d) $(3, 120^0)$

(e) $(5, 240^0)$ (answers not unique)

Exercise 2.

(a) $(-3, 0)$ (b) $(1, \sqrt{3})$ (c) $(0, 4)$ (d) $(\sqrt{3}, 1)$ (e) $(-\frac{3\sqrt{3}}{2}, \frac{3}{2})$

Self-Evaluation Objective 20.1

1. Convert the following points from cartesian coordinates to polar coordinates:

   (a) $(4\sqrt{2}, -4\sqrt{2})$ (b) $(0, 3)$

2. Convert the following point from polar coordinates to cartesian coordinates:

   (a) $(-2, -60^{0})$ (b) $(3, 45^{0})$

Answers:

1. (a) $(8, -45^0)$* (b) $(3, 90^0)$* *not unique

2. (a) $(-1, \sqrt{3})$ (b) $(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$

Objective 20.2

Transform rectangular equations to polar equations and transform polar equations to rectangular form.

Instructional Activities.

Task 3 (Transforming rectangular equations to polar equations.)

Read page 232 (Example 7-3) in your text. Also at the top of page 233 is another example (part 2 of example 7-4).

Notice that to transform a rectangular equation to a polar equation, you have several approaches.

First you know that:

$x = p \cos \theta$ and $y =$ ______________________

Also you know that $p = \pm\sqrt{x^2 + y^2}$

Answer: $y = p \sin \theta$

You also have all of the skills you acquired in college algebra and trigonometry available to you in order to help you simplify the equation.

In this set of problems practice is the best teacher.

On page 234 work all of the exercises in problem 1, problems (a) and (c) are begun for you.

1. Transform each of the following equations to polar coordinates.

   a. $x^2 + y^2 = 16$

      Since $p = \pm\sqrt{x^2 + y^2}$

      $p^2 = \pm (x^2 + y^2)$

      Hence $x^2 + y^2 = 16$ becomes ______________.

   c. $y^2 - x^2 = 4$

      Since $x = p \cos \theta$ and $y =$ ______________

      $y^2 - x^2 = 4$ becomes $(\_\_\_\_\_\_\_\_)^2 - p^2 \cos^2\theta = 4$

      and simplifying yields ______________________.

Answers: (a) $p = 4$ or $p = -4$, (b) $p \sin \theta = 6$, (c) $p^2 \cos 2\theta = -4$

(d) $p^2 - 2p \cos \theta + 3 \sin \theta = 6$ (e) $p^2 (\cos^2\theta + 1) = 16$

(f) $p^2 \sin^2\theta = 4 ap \cos \theta$ (g) $p^2 \cos^2\theta - 32p \sin \theta - 32p \sin \theta - 256 = 0$

Task 2 (Transforming from polar equations to rectangular equations.

Read pages 232-234 (begin with Example 7-4). On page 234, the last paragraph in section 7-3 is very important - read it carefully.

Now work all of problem 2. (a) and (g) are begun for you.

2. Transform each of the following equations to cartesian coordinates.

(a) $p \sin \theta = 4$

Since $p \sin \theta = y$

$p \sin \theta = 4$ becomes ________________.

(g) $p = 3 \cos 2\theta$

$p^2 (p = 3 \cos 2\theta)$

$p^3 = 3p^2 \cos 2\theta$

Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$ the equation becomes

$p^3 = 3p^2$ (________________).

or $p^3 =$ ____________________

substituting $x^2 =$ __________ and $y^2 =$ ______________

yields $p^3 =$ ____________________.

Now squaring both sides

yields $(p^3)^2 =$ (____________________)^2

or $p^6 =$ ____________________

Substituting for $p^2 =$ ________________

yields ______________________________.

Now you must verify that your answer $(x^2 + y^2)^3 = 9(x^2 - y^2)^2$ can be converted from rectangular form to polar form.

Answers: 2 (a) $y = 4$ (b) $x^2 + y^2 = 3x$ (c) $x^2 + y^2 = -15$ (d) $2y = x$
(e) $x^2 + y^2 = x$ (f) $(x^2 + y^2)^3 = 9(x^2 - y^2)^2$
(g) $3x^2 - y^2 + 12x + 9 = 0$ (h) $4x^2 + 4y^2 = (2y + x^2 + y^2)^2$

Self-Evaluation: Objective 20.2

1. Transform the following equations to rectangular form.

   (a) $p^2 \sin 2\theta = 16$

   (b) $p^2 = 4 \tan 2\theta$

2. Transform the following equations to polar form.

   (a) $2xy = 8$

   (b) $x^2 + y^2 - 2x = 0$

Answers:

1. (a) $xy = 16$ (b) $x^4 - y^4 = 8xy$

2. (a) $p^2 \sin 2\theta = 8$ (b) $p = 2 \cos \theta$

INSTRUCTIONAL ACTIVITIES:

Objective 20.3.

Graph a polar equation and name the curve among (spiral, rose, limacon, conic, lemniscate).

Task 4: Polar (Equations of the line and circle).

Read pages 234-236 and work problems 1, 4, 7, 10, 13, 16, 19, 22 and 25 on page 236.

Problems 1 and 4 are begun for you.

Identify the locus of each of the following equations, draw its graph, and transform it to cartesian coordinates.

1. $p \sin \theta = -4$ is a horizontal line through $(-4, 90^{0})$
   Sketch the graph in the space provided below:

$p \sin \theta = -4$

Since $p \sin\theta$ = ____________

$p \sin \theta = -4$ becomes ____________________

4. $p \cos (\theta + 45)^{0} = -2$

Since $\alpha = 45^{0}$ the normal axis makes an angle of 45^{0} with the positive x-axis and the normal intercept $p = -2$. Sketch the normal axis and the line in the space provided at the left.

$p \cos (\theta + 45)^0 = -2$

expanding yields

$p \cos\theta \;\cos 45^0 + p \sin \theta \sin 45^0 = -2$

Since $p \cos\theta$ = __________ and $p \sin \theta$ = __________

the equation becomes ______________________.

After working the problems on page 236, read pages 236-238 and work problems 1 (a and b), 2(a and c), 3, 4(a, c, & e), 5(a,c, & e), and 6(a,c, & e). Problem 6(a) is begun for you.

6. Find the center and radius of each of the following circles and draw its graph.

(a)

$p = 3 \cos \theta - 2 \sin \theta$

$p^2 = 3p \cos \theta - 2p \sin \theta$ (multiplying both sides of the equation by p).

Since $p \cos\theta$ = __________, $p \sin \theta$ = __________ and p^2 = __________

the equation becomes ______________________.

Answers: x, y, $x^2 + y^2$, $x^2 + y^2 = 3x - 2y$.

Hence the center can be located by completing the square.

$(x^2 - 3x \quad\quad) + (y^2 + 2y \quad\quad) =$ ______________________.

The center is __________ and the radius is __________.

Answer: Center (3/2,-1), radius (13/4).

After finishing the problems on page 238, read pages 239-247. On page 247 and 248, you will find a set of problems. Pick out and work several problems from problem 1 and several from problem 2. In problem 2, you should pick out one each of the circle, line, cardioid, limacon, rose, a conic, and spiral.

Examples 7-11, 7-12, and 7-13 in your text should be very helpful to you. Notice that problem (g) $p = a \pm b \cos \theta$ is example 7-11 when a = 1 and b = 2. Problem (j) $p^2 = a^2 \sin 2\theta$ is example 7-12 when $a^2 = 4$, and problem (m) $p = \frac{k}{a \pm a \cos \theta}$ is example 7-13 when k = 2 and a = 1.

After finishing the problems on pages 247 and 248 read pages 249-251. On page 251 work problems 1 (a & c), 2 (a, c, e, & g), 3(a, c, e, & g) and 4(a, c, & e). Example 7-14 should be helpful to you.

Self-Evaluation Objective 20.3.

1. Graph and name the curve $p^2 = 4 \sin 2\theta$

Answers: Objective 20.3

Intercepts

θ	0^0	90^0	180^0	270^0
p	0	0	0	0

Symmetry:

Since the exponent of p is even, the curve is symmetric with respect to the pole.

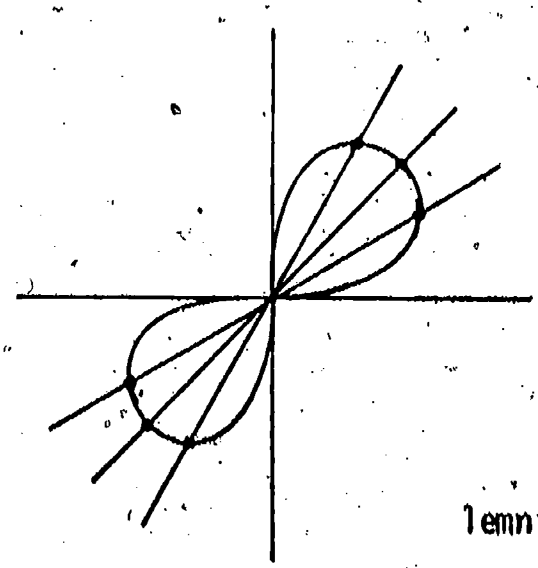
Extent: $\sin 2\theta \leq 1$ and $\sin 2\theta = 1$ when $\theta = 45^0$

p is imaginary when $\sin 2\theta$ is negative

Hence, the curve lies in quadrants I and III

Plotting:

θ	0^0	30^0	45^0	60^0	90^0
p	0	1.9	2	1.9	0



lemniscate

Objective 20.4

Find the coordinates of the intersection of two polar equations.

Instructional Activities

Task 5 (Determining intersection of polar equations)

Read pages 253-255 (Stop at Example 7-17). After reading this material, you should be able to answer the following questions:

1. To find all the points of intersection of 2 curves there are three things to do. They are:

   (a) Determine if the pole lies on both curves by solving the equations $f(0, \theta) = 0$ ~~and~~ g(_______) = 0 separately. If one or the other of the equations has no real solution, the pole doesn't lie on the curve.

   (b) Solve the equations $f(p, \theta) = 0$ and g(__________) = 0

   (c) Solve the equations $f(p, \theta) = 0$ and ______________.

2. In Example 7-16, why is $a \sin(\theta + 2k\pi) = a \sin \theta$?

3. In Example 7-16 (middle of page 255), why is $-a \sin(\theta + \{2k + 1\}\pi) = a \sin \theta$?

Answers:

1. (a) $(0, \theta)$ (b) $(p, \theta + 2k\pi)$ (c) $g(-p, \theta + \{2k + 1\}\pi) = 0$
2. Since k is an integer, the angle $(\theta + 2k\pi)$ becomes $(\theta, 0)$, $(\theta + 2\pi)$, $(\theta + 4\pi)$, and $(\theta + 6\pi)$ for the values of k = 0, 1, 2, and 3 respectively. Hence: $\sin(\theta + 2k\pi) = \sin \theta$
3. Likewise, the angle $(\theta + \{2k + 1\}\pi)$ becomes $(\theta + \pi)$, $(\theta + 3\pi)$, $(\theta + 5\pi)$, $(\theta + 7\pi)$ and $(\theta + 9\pi)$ for the values k = 0, 1, 2, 3,

and 4, respectively. Hence: $a \sin(\theta + (2k + 1)\pi) = -a \sin\theta$ and $-a \sin(\theta + (2k + 1)\pi) = -(-a \sin\theta) = a \sin\theta$.

Now read pages 255-258.

In studying the table at the top of page 257, you should notice that the 5th column, $p = \pi - \theta$ is found by using the value obtained for $\theta + 2k\pi$ in the fourth column. Hence for $k = 1$, $p = \pi - \theta$ becomes $p = \pi - \frac{3\pi}{2} = -\frac{\pi}{2}$.

After reading these pages, work problems 1, 3, 5, 7, 9, and 11 on page 258. Problem 1 is begun for you.

Find the coordinates of all points common to each of the following pairs of curves.

1. $p = 3 \cos\theta$ 2. $p = 3 \sin\theta$

Checking to see if the pole is common to both curves:

$0 = 3 \cos\theta$ $0 = 3 \sin\theta$

$\cos\theta = 0$

$\theta =$ ________ $\theta =$ ________

Hence the pole ________ (is, is not) a point of intersection.

Now let $f(p, \theta) = p = 3 \cos\theta$ and $g(p, \theta) =$ __________

Using the first pair of simultaneous equations

____________ $= p = 3 \sin(\theta + 2k\pi) =$ __________

Simplifying yields: __________

Hence $\theta =$ ________ and ________.

Substituting these values of θ in the equation $p = 3 \cos \theta$ yields

$p =$ ______________ and $p =$ ______________

Hence another point of intersection is __________ or ____________

NOTE: Answers above should be the same point.

Using the second pair of simultaneous equations:

______________________ = ____________________

yields ____________________

and again $\tan \theta = 1$, hence there are no new points of intersection.

Answers:

$\theta = \frac{\pi}{2}$, $\theta = 0$, is, $g(p, \theta) = p - 3\sin\theta$ $p - 3 \cos \theta$, $p - 3 \sin \theta$,

$\tan \theta = 1$, $\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$, $p = \frac{3\sqrt{2}}{2}$ and $p = -\frac{3\sqrt{2}}{2}$,

$(\frac{\pi}{4}, \frac{3\sqrt{2}}{2})$ or $(\frac{5\pi}{4}, \frac{-3\sqrt{2}}{2})$, $p - 3 \cos \theta = p + 3 \sin (\theta + (2k + 1)\pi) =$

$p - 3 \sin \theta$ or $-3 \cos \theta = -3 \sin \theta$, yields $\tan \theta = 1$

Self-Evaluation Objective 20.4

1. Find the points of intersection of

$p = \sin \theta$
and
$p = \cos \theta$

Answers:

1. $(\frac{1}{\sqrt{2}}, 45^{o})$ * and the pole. * not unique

Unit 21---3-D Analytics

Rationale:

Assume you are the pilot of an airplane, and you wish to locate your position with respect to a certain city. How would you describe that position? You could say you were so many miles north, east, south, or west (or any compass direction) from that city. That still would not adequately describe your position since you might be at a 100 Ft. altitute or on the ground. Therefore you must also add an altitude reading to your position.

As you see there are some cases where a "plane" coordinate system (a system of 2 mutually perpendicular lines) is not sufficient. Therefore, this unit will deal with problems in 3-space.

Objectives:

21.6 *Given sufficient conditions to describe three planes, determine:*
1. *the equation of the planes*
2. *the solution (and describe if unique, a line, or coplanar)*
3. *the distance from a point to any of the planes*
4. *the cosine of the angle formed by two planes*

21.5 *Given sufficient conditions to describe a line in three space, determine the equation of the line in any requested form.*

21.4 *Determine the equation of a plane and sketch the plane*
1. *given three conditions*
2. *given the intercept form*
3. *given the general form* $ax + by + cz + d = 0$

21.3 *Perform the following operations on space vectors.*
1. *alternating or triple scalar product*
2. *vector or cross product*
3. *determine the cosine of the angle between 2 vectors*
4. *dot product*
5. *addition*
6. *scalar multiplication*

21.2 *Determine the direction cosines of a directed line segment in three space and the direction cosines of a space vector.*

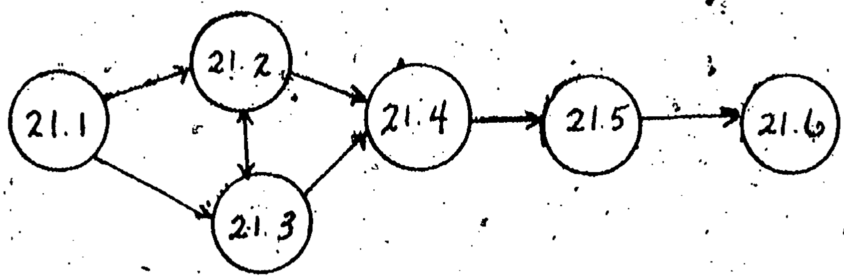
21.1 *Match ordered triples with its graphic representation in three space.*

Prerequisites

units 17, 18, & 19

Procedural Options

The procedural option for unit 21 is the same as for the previous units. A hierarchy for the objectives in unit 21 is provided below.



Unit Activities

Lectures 10, 11, & 12
The lectures over this unit will have the following outline.

Lecture 10. The Space Vector
1. Definition

2. Determining magnitude and direction cosines and cosine of angle between 2 vecotrs.

3. Operations
   a. addition
   b. scalar multiplication
   c. dot product
   d. vector product
   e. triple scalar product

Lecture 11 The Plane
1. Equation of a plane
   a. general form
   b. intercept form

2. Determining equations
   a. given 3 conditions

3. Distance from a point to a plane

Lecture 12 The Straight Line in Space
1. Equations of a line in space

   1. parametric
   2. symmetric
   3. general equations

Objective 21.1

Match ordered triples with its graphic representation in three space

Instructional Activities

1. Your text
   Morrill, W. K. Analytic Geometry, pp. 297-298
2. Your text and Study guide
3. Other reading sources

   Protter-Morrey, Analytic Geometry, pp. 194-195

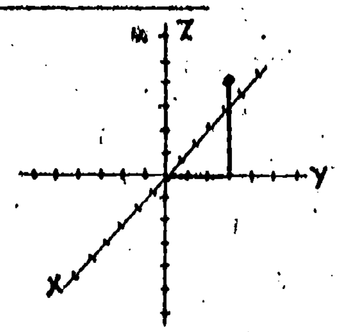
   Murdoch, David C., Analytic Geometry, pp. 19-20

   Fuller, Gordon, Analytic Geometry, pp. 166-167
4. Individual Assistance
5. Informal Group Sessions
6. Lecture 10

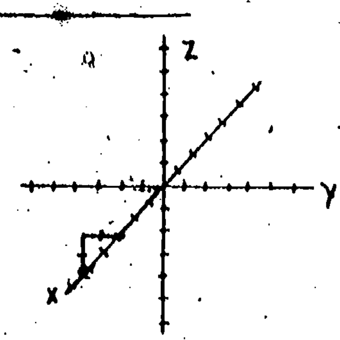
Self-Evaluation Objective 21.1

1. Match each graphic representation below with its ordered triple. Choose your answers from the set of ordered triples provided.

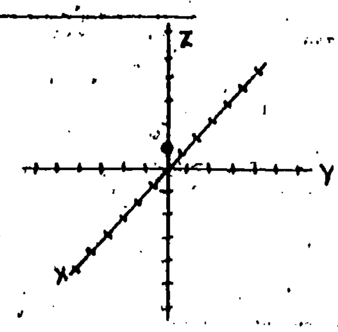
a. ________



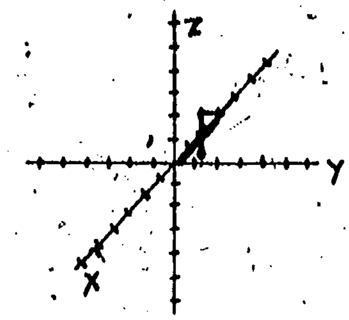
b. ________



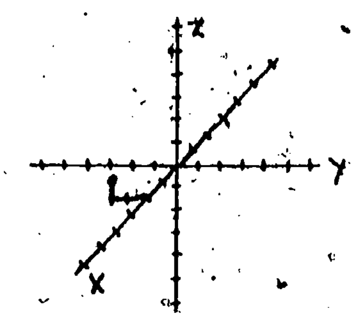
c. ________



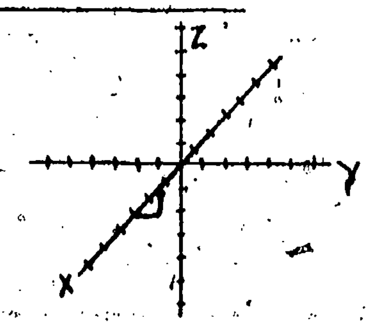
d. ________



e. ________



f. ________



Choose your answers from this set. Notice not every ordered triple has a graphic representation shown above.

(2,-3,1), (0,3,4,), (1,1,-2), (3,-2,-2) (-1,0,2) (0,0,1)

(4,2,-3) (-3,-1,-2) (2,-2,1) (3,1,1)

Answers Objective 21.1

(a) (0, 3, 4) (b) (3, -2, -2) (c) (0, 0, 1) (d) (-3, -1, -2)
(e) (2, -2, 1) (f) (3, 1, 1)

Objective 21.2

Determine the direction cosines of a directed line segment in three space and the direction cosines of a space vector.

Instructional Activities

1. Your text
   Morrill, W.K., Analytic Geometry, pp. 298-304, pp. 305, 306 (1st paragraph), 307 (last paragraph) and 308.

   Exercises: p. 300 problems 9, 10, and 11a
   pp. 304-305 problems 1 (a and c), 3,5 (a and c), 6, 9, 12, and 13.
   p. 308 problem 10

2. Your text and Study Guide

3. Solved Problems
   Schaum's Outline Series, Theory and Problems of Plane and Solid Analytic Geometry, pp. 106-107 problems 1, 2, 3, and 5.

4. Other Reading Sources

   Protter-Morrey, Analytic Geometry, pp. 198-200

5. Individual Assistance

6. Informal Group Sessions

7. Lecture 10

Self Evaluation

1. Determine the direction cosines of a directed line segment from P_1 to P_2 where:

   (a) $P_1 = (1,3,4)$ and $P_2 = (-3, -1, 0)$

   (b) $P_1 = (0,4,-2)$ and $P_2 = (3, -2, -1)$

2. Determine the length and the direction cosines of the following vectors.

   (a) $u = [-1, 0, 3]$

   (b) $v = [2, 3, -1]$

   (c) $w = [0, 0, 3]$

Answers: Objective 21.2

1. (a) $\ell = \frac{-1}{\sqrt{3}}$ $m = \frac{-1}{\sqrt{3}}$ $n = \frac{-1}{\sqrt{3}}$

   (b) $\ell = \frac{3}{\sqrt{46}}$ $m = \frac{-6}{\sqrt{46}}$ $n = \frac{1}{\sqrt{46}}$

2. (a) $|u| = \sqrt{10}$ $\ell = \frac{1}{\sqrt{10}}$ $m = 0$ $n = \frac{3}{\sqrt{10}}$

   (b) $|v| = \sqrt{14}$ $\ell = \frac{2}{\sqrt{14}}$ $m = \frac{3}{\sqrt{14}}$ $n = \frac{-1}{\sqrt{14}}$

   (c) $|w| = 3$ $\ell = 0$ $m = 0$ $n = 1$

Objective 21.3

Perform the following operations on space vectors.

1. Alternating or triple scalar product.
2. Vector or cross product
3. Determine the cosine of the angle between 2 vectors.
4. Dot product
5. Addition
6. Scalar multiplication

Instructional Activities

1. Your text
   Morrill, W. K. , Analytic Geometry pp. 306-312, 323-326

   Exercises, p. 308, problems 3 (a,c, and e), 4 (a and c), 5, 6, 7, 8, and 9
   p, 312 problems 1, 2a, 3 (a and d), 4a, 6, 9, and 10
   pp. 326-327, problems 1, 3, 5 (a), 6 and 7.

2. Your text and Study Guide

3. Solved Problems
   Schaum's Outline Series, Theory and Problems of Plane and Solid Analytic Geometry, pp. 107 problems 6, 7, 8, and 9

4. Other Reading Sources
   Murdoch, David C. Analytic Geometry, pp. 74-79.

5. Individual Assistance

6. Informal Group Sessions

7. Lecture 10

Self Evaluation

1. Given the following vectors, u = [1,0,-4], v = [2,5,-3], and w = [-1, 3,-5] find:

   (a) the cosine of the angle θ between u and v.
   (b) v· w
   (c) 2v + 3 u
   (d) v x w
   (e) u· (v x w)
   (f) a vector perpendicular to v x w.

Answers: Objective 21.3

1. (a) $\cos\theta = \dfrac{14}{\sqrt{17}\ \sqrt{38}}$

(b) $v \cdot w = 19$

(c) $2v + 3u = [12,20,-18]$

(d) $v \times w = [-16, 13, 11]$

(e) $u \cdot (v \times w) = -60$

(f) w or v

Objective 21.4

Determine the equation of a plane and sketch the plane
1. given three conditions
2. given the intercept form
3. given the general form ax + by + cz + d = 0

Instructional Activities

1. Your text

   Morrill, W. K., Analytic Geometry, pp. 318-323.

   Exercises, p. 320, problems 2,4,6(a & c), 7(a,c, & e) 8(ac, & e), 9,11,13,15, and 16(a)
   P. 322, problems 1,2, and 3(a & c of each)
   p. 323, problems 1(a & c) & 2(a & c)

2. Your text and Study Guide
3. Solved Problems

   Schaum's Outline Series, Theory and Problems of Plane and Solid Analytic Geometry; pp. 116-119.
   problems 1,2,3,4,5,12 & 15.
4. Other Reading Sources

   Protter - Morrey, Analytic Geometry, pp. 208-210.
   Murdoch, David C., Analytic Geometry, pp. 87-90.
5. Individual Assistance
6. Informal Group Session
7. Lecture 11

Self-Evaluation Objective 21.4

1. Given the equation 3x - 2y + z - 7 = 0 write the equation in the intercept form and find the intercepts. How sketch the plane using the intercepts as an aid.
2. Determine the equation of the plane that satisfies the following conditions.

(a) Find the equation of the plane through the point (-3, 5, 1) perpendicular to a line with direction numbers 4, 1, -3.

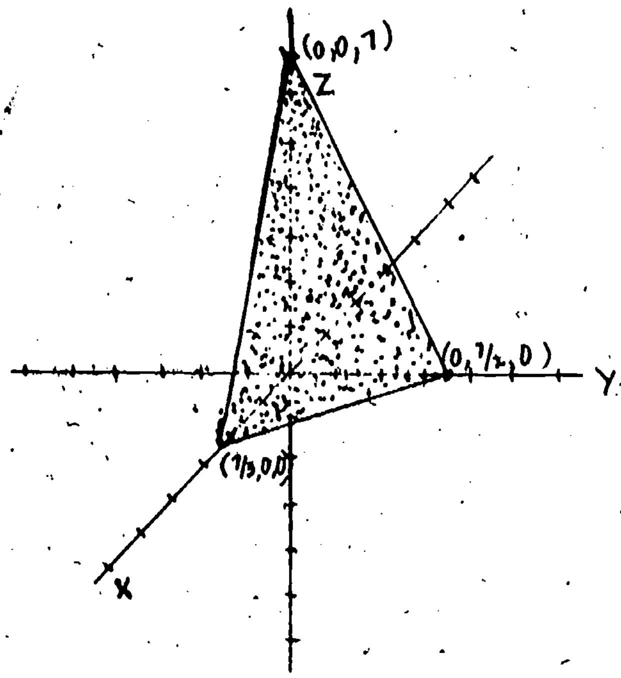
(b) Find the equation of the plane through the points (0, 0, 0), (1, 1, -1), (0, 2, 1)

Answers:

1. $\frac{x}{7/3} - -\frac{y}{7/2} + \frac{z}{7} = 1$ x intercept = 7/3

y intercept = 7/2

z intercept = 7



2. (a) $4x + y - 3z + 10 = 0$

(b) $3x - y + 2z = 0$

Objective 21.5

Given sufficient conditions to describe a line in three space, determine the equation of the line in any required form.

Instructional Activities

1. Your text
   Morrill, W. K., Analytic Geometry, pp 338-344.
   Exercises, p. 339, problems 1(a & c), 2(a & c), 3, & 6
   p. 343-344 problems 1(a & c), 2, 3(a & c), 4(a & c), 7(a & b) 8,9, & 10.
2. Your text and Study Guide
3. Solved Problems
   Schaums Outline Series, Theory and Problems of Plane & Solid Analytic Geometry, pp. 124-127 problems 1,5,9,10,12,13, & 14
4. Other Reading Sources
   Protter- Morrey, Analytic Geometry, pp. 204-206
   Murdoch, David C., Analytic Geometry, pp. 91-93
5. Individual Assistance
6. Informal Group Sessions
7. Lecture 12

Self-Evaluation Objective 21.5

1. Determine the equations of the line containing P_1 (1,1,2) and P_2 (-1,2,3,) in symmetric form.
2. Write the parametric equations for the line determined by the following conditions:
   (a) Through P = (1,2,3) perpendicular to the plane 3x + y -z -6 = 0
   (b) Through the origin, parallel to $\overrightarrow{QR}$, Q = (-3,2,1) and R = (2,-3,1)
3. Find the direction numbers of the line represented by the following pair of equations and write the equation of the line in parametric form.
   2x + y - 6 = 0 and 3x + 2z + 12 = 0

Answers:

1. $\frac{x - 1}{-2} = \frac{y - 1}{1} = \frac{z + 2}{5}$

2. (a) $x = 1 + 3t,\quad y = 2 + t,\quad z = 3 - t$

   (b) $x = 5t,\quad y = -5t,\quad z = 0$

3. direction numbers = [24, -42, 1] a point on the line is (-4, 14, 0) and a set of equations is:

$$x = -4 + 24t$$

$$y - 14 - 42t$$

$$z = t$$

ANSWERS TO THIS SET ARE NOT UNIQUE

Objecitve 21.6

Given sufficient conditions to describe three planes, determine:

1. the equations of the planes
2. the solution (and describe if unique, a line, or caplanar)
3. the distance from a point to any of the planes
4. the cosine of the angle formed by two planes

Instructional Activities

1. Your text

   Morrill, W. K., Analytic Geometry, pp. 331-336, pp. 345-349.

   Exercises, p. 333 problems 1(a & c), 3,7, and 11.
   p. 336, problem 1
   p. 347, problems 1,3, & 5
   p. 349, prolbems 1,3, & 5
2. Your text and Study Guide
3. Solved Prolbems

   Schaum's Outline Series, Theory and Problems of Plane and Solid Analytic Geometry, pp. 117-118. Problems 7,8,9,10,11, & 14
4. Other Reading Sources

   Protter - Morrey, Analytic Geometry, pp. 212-215.
5. Individual Assistance
6. Informal Group Sessions

Self-Evaluation Objective 21.6

1. Given the planes $2x + y - z - 1 = 0$, $3x - y - z + 2 = 0$, and $4x - 2y + z - 3 = 0$, determine the solution.
2. What is the distance from the point (3,1,0) to the plane $2x + y - z - 1 = 0$
3. Find the equation of the plane that passes throught the point (3,-2,4) and is perpendicular to the planes $7x - 3y + z - 5 = 0$, and $4x - y - z + 9 = 0$.
4. Find the cosines of the angles between the planes $2x - y + z = 7$ and $x + y + 2z - 11 = 0$

Answers: Objective 21.6

1. (1, 2, 3)

2. $\frac{6}{\sqrt{7}}$

3. 4x + 11 + 5z - 10 = 0

4. $\cos\theta = \pm\frac{1}{2}$

STUDY GUIDE

Unit 21--3-D Analytics

Rationale:

Assume you are the pilot of an airplane, and you wish to locate your position with respect to a certain city. How would you describe that position? You could say you were so many miles north, east, south, or west (or any compass direction) from that city. That still would not adequately describe your position since you might be at a 100 Ft. altitute or on the ground. Therefore you must also add an altitude reading to your position.

As you see there are some cases where a "plane" coordinate system (a system of 2 mutually perpendicular lines) is not sufficient. Therefore, this unit will deal with problems in 3-space.

Objectives:

21.6 *Given sufficient conditions to describe three planes, determine:*
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3. *the distance from a point to any of the planes*
4. *the cosine of the angle formed by two planes*

21.5 *Given sufficient conditions to describe a line in three space, determine the equation of the line in any requested form.*

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4. *dot product*
5. *addition*
6. *scalar multiplication*

21.2 *Determine the direction cosines of a directed line segment in three space and the direction cosines of a space vector.*

21.1 *Match ordered triples with its graphic representation in three space.*

Instructional Activities

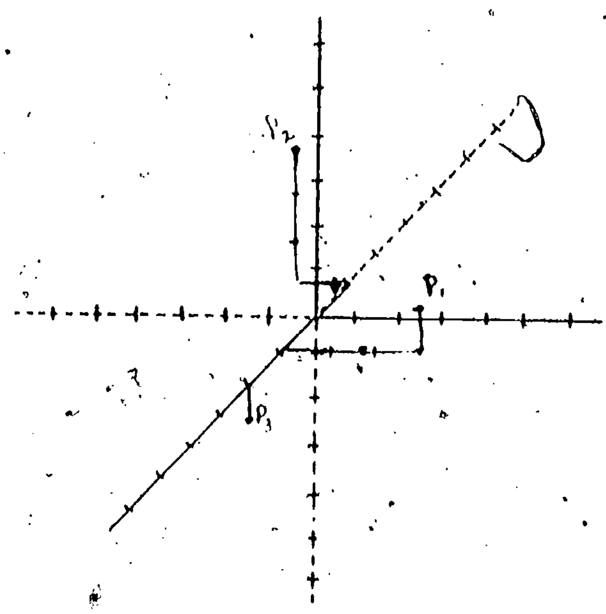
Objective 21.1

Match ordered triples with its gaphic representation in three space.

Task 1. (Points in three space)

Read pages 297-298 in your text.

After reading the pages, label the coordinate system below such that it is a right handed system and the positive X axis is "toward you." (out or the page)



After labeling the system name the coordinates of the points P_1, P_2, P_3.

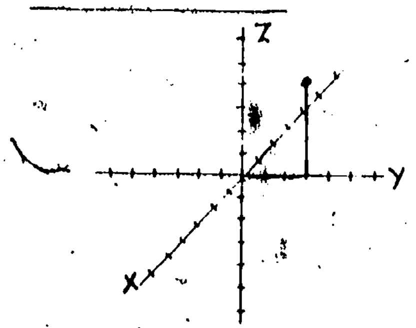
<u>Answers</u>: $P_1 = (1,3,1)$ $P_2 = (-1,-1,3)$ $P_3 = (2,0,-1)$

You work problems 1,2,3,4,5, and 7 on page 300.

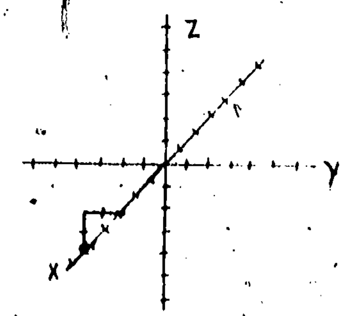
<u>Self-Evaluation Objective 21.1</u>

1. Match each graphic representation below with its ordered triple. Choose your answers from the set of ordered triples provided.

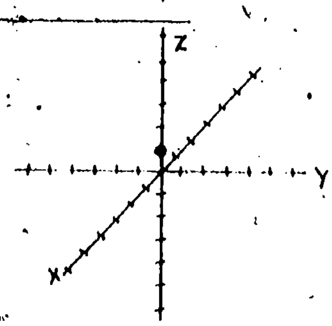
a. ____________



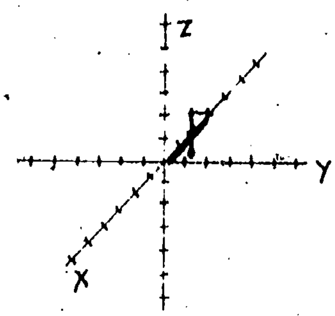
b. ____________



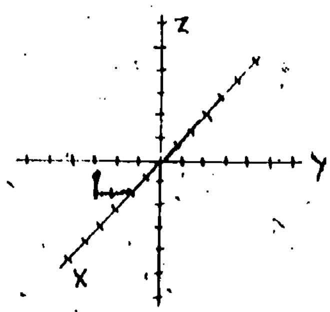
c. ____________



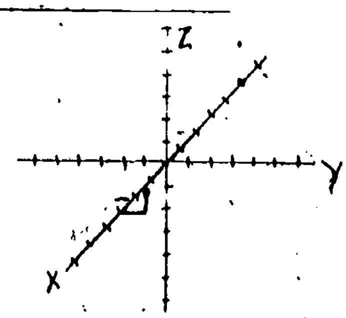
d. ____________



e. ____________



f. ____________



Choose your answers from this set. Notice not every ordered triple has a graphic representation shown above.

(2,-3,1), (0,3,4), (1,1,-2), (3,-2,-2) (-1,0,2) (0,0,1)

(4,2,-3) (-3,-1,-2) (2,-2,1) (3,1,1)

Answers Objective 21.1

(a) (0, 3, 4) (b) (3, -2, -2) (c) (0, 0, 1) (d) (-3, -1, -2)
(e) (2, -2, 1) (f) (3, 1, 1)

Instructional Activities:

Objective 21.2

Determine the direction cosines of a directed line segment in three space and the direction cosines of a space vector.

Task 2. (Projections in 3 space)

Read pages 298-299.

After reading the material work problems 9,10, and 11a on page 300. The first part of problem 9 and part a of problem 10 is begun for you.

9. Determine the projections of the following points on the xy-plane the xz-plane and the yz-plane, respectively.

(1,-1,3).

The projection on the xy-plane is (1,-1,0)

The projection on the xz-plane is (1,0,3)

The projection on the yz-plane is (_________)

10. (a) Determine the projections of the following segments on the x-axis, y-axis, and the z-axis respectively.

P_1 (3,-2,4) to P_2 (-1,1,-1)

The projection on the x-axis (Δx) = $x_2 - x_1$

The projection on the y-axis (Δy) = _____

The projection on the z-axis (Δz) = _____

Hence $\Delta x = x_2 - x_1 = -1-3 = -4$

$\Delta y = y_2 - y_1 =$ _______ = _______

$\Delta z =$ _______ = _______ = _______

Answers; $\Delta y = y_2 - y_1$, $\Delta z = z_2 - z_1$, $\Delta y = 1 - (-2) = 3$

$\Delta z = z_2 - z_1 = -1 - 4 = -5$

Task 3 (Scalar components and magnitude)

Read pages 301-303. After reading the material work problems 1(a & c) and 3 on page 304.

Self-Evaluation Tasks 2 and 3

1. Find the projections of the segment P_1P_2 on the x-axis the y-axis and the z-axis, where $P_1 = (0,1,4)$ and $P_2 = (-3,2,1)$.

2. Find the scalar components of the segment P_1P_2 where $P_1 = (-3,4,1)$ and $P_2 = (-2,1,2)$.

3. Find the length of the segment P_1P_2 where $P_1 = (3,1, 0)$ and $P_2 = (-4,3,-1)$.

Answers: Tasks 2 and 3

1. $\Delta x = -3 \quad \Delta y = -1 \quad \Delta z = -3$
2. $\Delta x = 1, \quad \Delta y = -3, \quad \Delta z = 1$
3. $\sqrt{54}$

Task 4 (Direction Cosines)

Read pages 303-304.

On Page 305 work problem 5(a&c), 6,9,12, and 13. Problem 5a is begun for you.

5. Given the initial points and scalar components construct each segment, and find the terminal point, the magnitude and the direction cosines of the segment.

(a) $P_1 = (1,-3,2)$; scalar components [2,1,-1]

$\Delta x = x_2 - x_1 \qquad \Delta y = y_2 - y_1 \qquad \Delta z = z_2 - z_1$

$2 = x_2 - 1 \qquad 1 = y_2 - \_\_\_\_ \qquad z_2 = \_\_\_\_\_\_$

$x_2 = 3 \qquad y_2 = \_\_\_\_\_\_$

Hence $P_2 = (3,-2,1)$

$|P_1P_2| = \sqrt{x^2 + y^2 + x^2}$

$|P_1P_2| = \_\_\_\_\_\_\_\_\_\_$

and $\ell = \dfrac{\Delta x}{|P_1P_2|} \qquad m = \dfrac{\Delta y}{|P_1P_2|} \qquad n = \dfrac{\Delta z}{|P_1P_2|}$

Hence:

$\ell = \frac{2}{\sqrt{6}}$ $m =$ ____________ and $n =$ __________

Answers:

$y_2 = -2$, $z_2 = 1$, $|P_1P_2| = \sqrt{6}$, $\ell = \frac{2}{\sqrt{6}}$ $m = \frac{1}{\sqrt{6}}$ $n = \frac{-1}{\sqrt{6}}$

You construct the segment in the space provided below.

After you finish the problems, read page 305 (section 9-6) and read the first paragraph on page 306. Also read the last paragraph on page 307 beginning with "The direction cosines of a non-zero vector ..." and 308. On page 308 work problem 10.

Self-Evaluation

Objective 21.2

1. Determine the direction cosines of a directed line segment from P_1 to P_2 where:

   (a) $P_1 = (1,3,4)$ and $P_2 = (-3,-1,0)$

   (b) $P_1 = (0,4,-2)$ and $P_2 = (3,-2,-1)$

2. Determine the length and the direction cosines of the following vectors.

(a) u = [-1,0,3]

(b) v = [2,3,-1]

(c) w = [0,0,3]

Answers: Objective 21.2

1. (a) $\ell = \frac{-1}{\sqrt{3}}$ $m = \frac{-1}{\sqrt{3}}$ $n = \frac{-1}{\sqrt{3}}$

   (b) $\ell = \frac{3}{\sqrt{46}}$ $m = \frac{-6}{\sqrt{46}}$ $n = \frac{1}{\sqrt{46}}$

2. (a) $|u| = \sqrt{10}$ $\ell = \frac{-1}{\sqrt{10}}$ $m = 0$ $n = \frac{0}{\sqrt{10}}$

   (b) $|v| = \sqrt{14}$ $\ell = \frac{2}{\sqrt{14}}$ $m = \frac{3}{\sqrt{14}}$ $n = \frac{-1}{\sqrt{14}}$

   (c) $|w| = 3$, $\ell = 0$, $m = 0$, $n = 1$

Instructional Activity

Objective 21.3

Perform the following operations on space vectors.

1. Alternating or triple scalar product
2. Vector or cross product
3. Determine the cosine of the angle between 2 vectors
4. Dot product
5. Addition
6. Scalar multiplication

Task 5 (vector operations - addition & scalar multiplication)

Read pages 306-307

Notice that subtraction for vectors is defined in terms of addition.

(to perform the operation u - v you add the inverse of v to u.
$u - v = u + (-v)$

Also notice that the operation defined by

$$ku = k \left[u_1, u_2, u_3\right] = \left[ku_1, ku_2, ku_3\right]$$ is scalar multiplication.

Scalar multiplication in 2 dimensions was defined as

$$k \left[u_1, u_2\right] = \left[ku_1, ku_2\right]$$

On page 308, work problems 3(a,c & e) 4(a & c) 5,6,7,8, and 9

Self - Evaluation Task 5

1. If $u = \left[3,1,2\right]$ and $v = \left[-1,2,-1\right]$ determine the following.

   (a) $u + v$

   (b) $u - v$

   (c) $3u$

   (d) $\frac{1}{2}v$

   (e) express u as a unit vector times a constant

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Answers:

(a) $u + v = [2,3,1,]$ (b) $u - v = [4,-1,3]$

(c) $3u = [9,3,6,]$ (d) $\frac{1}{2}v = [-\frac{1}{2},1,-\frac{1}{2}]$

(e) $u = \sqrt{14}\left[\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right]$

Task 6 (Cosine of angle between 2 vectors and dot product)

Read pages 309-312.

Now work problems 1, 2a, 3 a& d, 4a, 6, 9, and 10. Problem 3a is begun for you.

Find the cosines of the angles of the following triangles whose vertices are: $P_1(2,-1,-1)$, $P_2(-3,4,2,)$, and $P_3(1,-1,2)$.

$u = \overrightarrow{P_1P_2} = [-5,5,3]$

$v = \overrightarrow{P_1P_3} = [-1,0,3]$

$$\cos\angle P_2P_1P_3 = \frac{u \cdot v}{|u||v|} = \frac{5 + 0 + 9}{\sqrt{59} \cdot \sqrt{10}} = \frac{14}{\sqrt{590}}$$

$u = \overrightarrow{P_2P_1} = [5,-5,-3]$

$v = \overrightarrow{P_2P_3} =$ __________

$$\cos\angle P_1P_2P_3 = \frac{u \cdot v}{|u||v|} = \_\_\_\_\_\_\_\_\_\_$$

$u = \overrightarrow{P_3P_1} =$ __________

$v = \overrightarrow{P_3P_2} =$ __________

$\cos\angle P_1P_3P_2 =$ __________

Answers:

$\cos \angle P_1P_2P_3 = \frac{45}{\sqrt{59}\sqrt{41}}$, $\cos \angle P_1P_3P_2 = \frac{-4}{\sqrt{410}}$

Self-Evaluation Task 6

1. Find the cosine of the angle between $\vec{P_1P_2}$ and $\vec{P_1P_3}$ where $P_1 = (2,1,-3)$, $P_2 = (3,-1,0)$, and $P_3 = (4,1,1)$.

Answer: $\cos\angle P_2P_1P_3 = \frac{7}{7,0}$

Task 7 (Vector product & triple scalar product)

Read pages 323-326.

So far every operation in 3 space has had a corresponding operation in the plane. There has been no operation defined in the plane which corresponds to the vector product. You should also notice that the results of the operation is a vector.

The vector product is defined as follows:

$$u \times v = \left[u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\right]$$

OR

$$u \times v = \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix},\ \begin{vmatrix} & \end{vmatrix},\ \begin{vmatrix} & \end{vmatrix}\right]$$

Find the vector product of the vector $u = [1,-3,2]$ and the vector $v = [3,4,-2]$.

Answer: $u \times v = \left[-2,8,13\right]$

In reading the material you should have found the statement on page 326 last paragraph, "The triple scalar product is a number." This is true since the triple scalar product $u \cdot (v \times w)$ is the dot product of two vectors, since $(v \times w)$ the vector product is a _____________________.

Answer: vector

Now work problems 1, 3, 5(a), 6, 7, on pages 326-327.

Self-Evaluation -- Objective 21.3

1. Given the following vectors u = [1, 0, -4], v = [2, 5, -3] , and w = [-1, 3, -5], find:

   (a) the cosine of the angle θ between u and v.

   (b) v · w

   (c) 2v + 3u

   (d) v x w

   (e) u · (v x w)

   (f) a vector perpendiuclar to v x w

Answers: Objective 21.3

1. (a) $\cos\theta = \dfrac{14}{\sqrt{17}\sqrt{38}}$

   (b) $v \cdot w = 19$

   (c) $2v + 3u = [12, 20, -18]$

   (d) $v \times w = [-16, 13, 11]$

   (e) $u \cdot (v \times w) = -60$

   (f) w or v

Instructional Activities Objective 21.4

Determine the equation of a plane and sketch the plane

1. given three conditions
2. given the intercept form
3. given the general form $ax + by + cz + d = 0$

Task 8 (General Form of a Plane)

Read pages 318-319. You should note that the general form ($ax + by + cz + d = 0$) of the plane is developed like the general form of a line by the use of a vector perpendicular to the plane.

On page 320, work problems 2, 4, 6(a & c), 7(a, c, & e), 8(a, c, & e), 9, 11, 13, 15, and 16(a).

Example 16(a):

Find the equation of the plane parallel to the yz plane and containing the point (1, -1, 2).

Solution Example 16(a)

Since the required plane is parallel to the yz plane, a normal vector is $u = [1, 0, 0]$. Hence the equation of the plane is of the form $ax + d = 0$ where $d = -ax_1 - by_1 - cz_1$. Let $u = [1, 0, 0]$ then $d = -1(1) - 0 - 0 = -1$. Hence the required equation is $x - 2 = 0$.

Problem 11 is begun for you:

Find an equation of the plane that is parallel to the x-axis and contains the points $P_1(2,-1,3)$ and $P_2(-1,0,5)$. Since the required plane is parallel to the x-axis, a normal vector is $u = [0, b, c]$.

Hence the required equation will be of the form $by + cz + d = 0$

also $u \cdot \overrightarrow{P_1P_2} = 0$

$\overrightarrow{P_1P_2} =$ ______________ and

and $u \cdot \overrightarrow{P_1P_2} =$ ________________.

Hence $b + 2c = 0$ and $b = -2c$.

Let $c = 1$, hence $b =$ _______.

and the equation is of the form $-2y + z + d = 0$.

Since $d = -ax_1 - by_1 - cz_1$, $d =$ _________.

Hence the equation is $2y - z + 5 = 0$.

After finishing the problems on page 320-321, read page 321 and work problems 1, 2, and 3 (do a & c in each) on page 322.

Task 9

Read pages 322-323 and work problems 1(a, c) and 2(a, c) on page 323.

Task 10

Read pages 328-330. The two examples 10-10 and 10-12 should be very helpful. Now work problems 1(a, c), 2, 3, 4, 8, 9, and 12a.

Self-Evaluation Objective 21.4

1. Given the equation $3x - 2y + z - 7 = 0$. Write the equation in the intercept form and find the intercepts. Now sketch the plane using the intercepts as an aid.

2. Determine the equation of the plane that satisfies the following conditions:

   (a) Find the equation of the plane through the point (-3, 5, 1) perpendicular to a line with direction numbers 4, 1, -3.

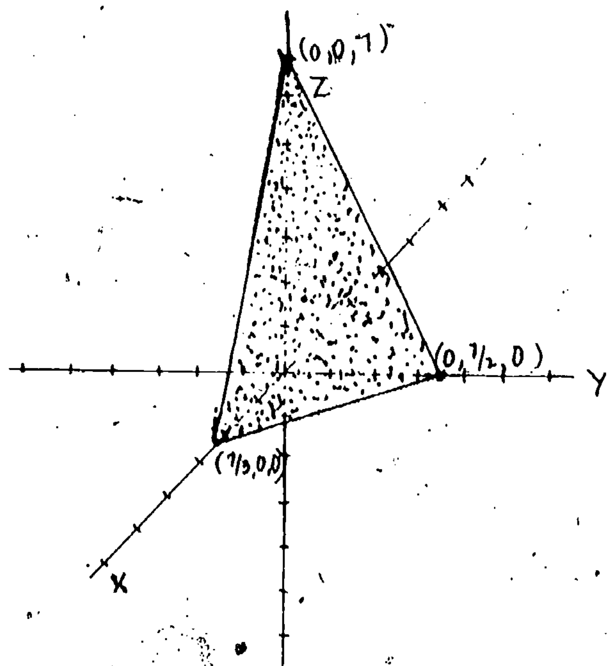
   (b) Find the equation of the plane through the points (0, 0, 0), (1, 1, -1), (0, 2, 1)

Answers:

1. $\frac{x}{7/3} - -\frac{y}{7/2} + \frac{z}{7} = 1$ x intercept = 7/3

y intercept = 7/2

z intercept = 7



2. (a) $4x + y - 3z + 10 = 0$

(b) $3x - y + 2z = 0$

Objective 21.5

Given sufficient conditions to describe a line in three space, determine the equation of the line in any required form.

Instructional Activities

Task 11 (Direction numbers & Direction cosines of a line)

Read pages 338-339 and work problems 1(a, c), 2(a, c), 3, and 6 and page 339.

Task 12 (Parametric Equations of a Line)

Read pages 340 and work problems 1(a & c) and 2 on page 343.

Task 13 (Symmetric Equations of a Line)

Read page 341 and work problems 3(a, c) and 4(a, c) on page 343.

Task 14 (General Equations of a line)

Read page 341-342. You should be aware of the fact that in each form, parametric, symmetric, and general, it takes two equations to determine a line. Now work problems 7(a, b), 8, 9, 10, on pages 343-344. Problems 7(a) and 8 are begun for you.

Problem 7. Find direction number of the line represented by the following pair of equations and write each line in parametric form.

(a) $2x - 3y + z - 6 = 0$ and $x - y + 2z + 4 = 0$

Solution:

Let $u = [2, -3, 1]$ and $v = [1, -1, 2]$. Then

$$u \times v = \left[\begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} & \\ & \end{vmatrix}, \begin{vmatrix} & \\ & \end{vmatrix} \right] = \underline{\qquad\qquad}$$

Hence the direction numbers are ______________.

Let $z = 0$, equations of the line become $2x - 3y - 6 = 0$ and ______________

Solving for x and y yields.

$x =$ ______________ $=$ ______ and $y =$ ______________ $=$ ______

The point $P = (\_\_\_, \_\_\_, 0)$ lies on both planes. So parametric equations of the line are ______________, ______________ and ______________.

Answers:

$$u \times v = \left[\begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}\right] = [-5, -3, 1]$$

direction numbers: [-5, -3, 1]

equations of the line $2x - 3y - 6 = 0$ and $x - y + 4 = 0$

$$x = \frac{\begin{vmatrix} 6 & -3 \\ -4 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix}} = -18 \qquad y = \frac{\begin{vmatrix} 2 & 6 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix}} = -14$$

$P = (-18, -14, 0)$

parametric equations are $x = -18 - 5t$ $\quad y = -14 - 3t$ $\quad z = t$

NOTICE: To obtain the answers in the back of the text let $z = 1$ instead of $z = 0$.

8. What are the equations of the line that passes through $P(2, 3, -1)$ and is parallel to the line represented by the equations $3x + 4y - 2z + 6 = 0$ and $x - 2y - z - 2 = 0$?

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Solution:

Since the required line is parallel to the line determined by the planes $3x + 4y - 2z + 6 = 0$ and $x - 2y - z - 2 = 0$, the required line is represented by the equations $3x + 4y - 2z + d_1 = 0$ and $x - 2y - z - d_2 = 0$.

Solving for d_1 and d_2 yields:

$d_1 = -a_1x_1 - b_1y_1 - c_1z_1$ $\quad\quad$ $d_2 = -a_2x_1 - b_2y_1 - c_2z_1$

$d_1 =$ ________ $\quad\quad$ $d_2 =$ ________

Hence the equations are;

$3x + 4y - 2z - 20 = 0$ and $x - 2y - z + 3 = 0$.

Self-Evaluation Objective 21.5

1. Determine the equations of the line containing $P_1(1, 1, -2)$ and $P_2(-1, 2, 3)$ in symmetric form.

2. Write the parametric equations for the line determined by the following conditions:

   (a) Through $P = (1, 2, 3)$ perpendicular the the plane $3x + y - z - 6 = 0$

   (b) Through the origin, parallel to $\overrightarrow{QU}$, $Q = (-3, 2, 1)$ and $U = (2, -3, 1)$

3. Find the direction numbers of the line represented by the following pair of equations and write the equation of the line in parametric form.

$$2x + y - 6 = 0 \quad \text{and} \quad 3x + 2z + 12 = 0$$

Answers:

1. $\frac{x-1}{-2} = \frac{y-1}{1} = \frac{z+2}{5}$

2. (a) $x = 1 + 3t, \quad y = 2 + t, \quad z = 3 - t$

   (b) $x = 5t, \quad y = -5t, \quad z = 0$

3. direction numbers = [24, -42, 1] a point on the line is (-4, 14, 0) and a set of equations is:

$$x = -4 + 24t$$

$$y = 14 - 42t$$

$$z = t$$

ANSWERS TO THIS SET ARE NOT UNIQUE

Objective 21.6

Given sufficient conditions to describe three planes, determine:

1. the equation of the planes
2. the solution (and describe if unique, a line, or coplaner)
3. the distance from a point to any of the planes
4. the cosine of the angle formed by two planes

Instructional Activities

Task 15 (Distance from a plane to a point)

Read pages 331-332 and work problems 1(a, c), 3, 7, and 11 on page 333.

Task 16 (Angle formed by two planes)

Read pages 334 and work problem 1 on page 336.

Task 17 (Determining solution-given three planes)

Read pages 345-347 and work problems 1, 3, and 5 on pages 347.

Now read pages 347-349 and work problems 1, 3, and 5 on page 349.

Self-Evaluation Objective 21.6

1. Given the planes $2x + y - z - 1 = 0$, $3x - y - z + 2 = 0$, and $4x - 2y + z - 3 = 0$, determine the solution.

2. What is the distance from the point $(3, 1, 0)$ to the plane $2x + y + z - 1 = 0$.

3. Find the equation of the plane that passes through the point $(3, 2, 4)$ and is perpendicular to the planes $7x - 3y + z - 5 = 0$ and $4x - y - z + 9 = 0$.

4. Find the cosines of angles between the planes $2x - y + z = 7$ and $x + y + 2z - 11 = 0$.

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Answers: Objective 21.6

1. $(1, 2, 3)$

2. $\frac{6}{\sqrt{7}}$

3. $4x + 11 + 5z + 10 = 0$

4. $\cos \theta = \pm \frac{1}{2}$

Unit 22 -- PARAMETERS

Rationale:

Quite often it is helpful in the solution of problems to express two or more variables in terms of a single variable. This procedure is especially useful when time is involved. Consider the following problem and solution.

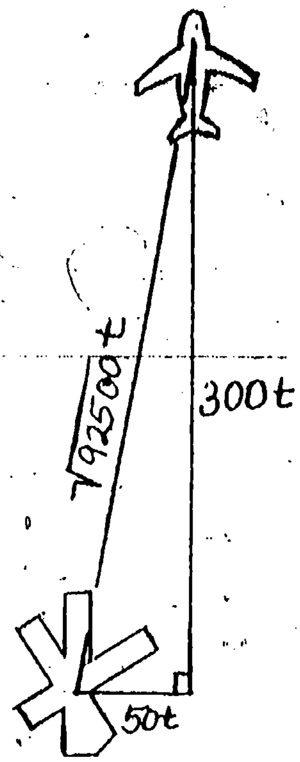
Problem: A passenger train traveling 60 miles per hour covers a certain distance in 2 hours less time than a freight train traveling 50 miles per hour. Find the distance traveled by each train.

Solution: Let $d_1 = 60(t - 2)$ represent the distance traveled by the passenger train and $d_2 = 50t$ the distance traveled by the freight train for time t. Since $d_1 = d_2$ then $60t - 120 = 50t$ or $10t = 120$ and $t = 12$ hours. Thus, $d_2 = d_1 = 600$ miles.

In the solution to the above problem, d_1 and d_2 were both expressed parametrically, i.e. both were expressed in terms of time.

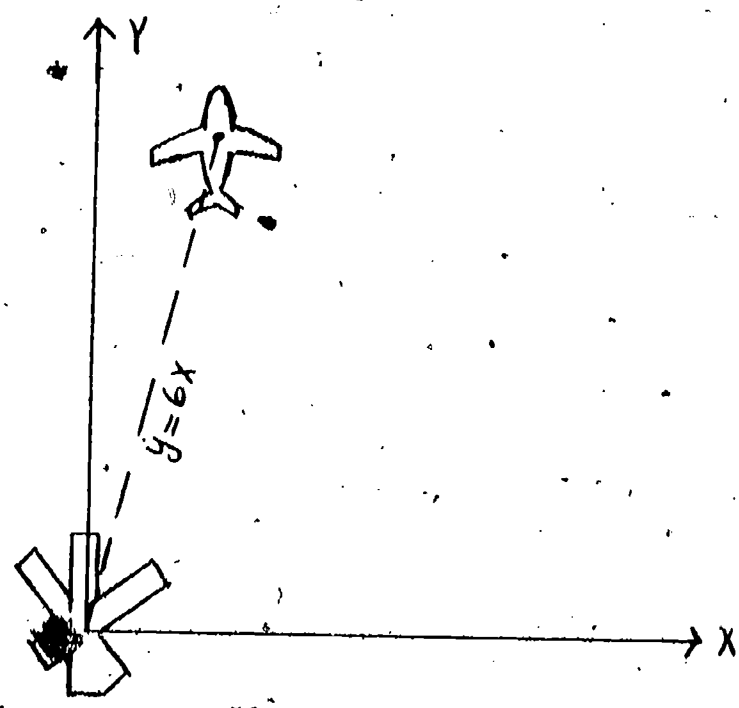
As another example of how parameters can be used, consider the following situation.

An airplane heading due north flies directly over an airport, there is a crosswind blowing due east. If the plane is headed north at 300 mph and the wind is blowing east at 50 mph, determine the function which determines the distance from the airport at any time. Let $d_1 = 300t$ be the plane's direction vector at time t. $d_2 = 50t$ be the wind vector at time t. Then $d(t) = \sqrt{(300t)^2 + (50t)^2}$ is the plane's path vector. Thus, $d(t) = \sqrt{92500t^2}$ represents the distance from the airport at time t.



The above problem illustrates another use of a parameter. Two variables were expressed in terms of a parameter, allowing one to establish still another variable in terms of the parameter.

Suppose, in the above problem, one was asked to establish the equation of motion of the airplane considering the airport to be the origin, due east as the positive x-axis and due north as the positive y-axis.



Then for a fixed time t, $x = 50t$ and $y = 300t$. Thus, $\frac{x}{50} = t$ and $\frac{y}{300} = t$. Hence, $\frac{x}{50} = \frac{y}{300}$ or $6x = y$ or the equation of motion is $6x - y = 0$ and the motion is along the line described by this equation.

Thus, it is again useful to use a parameter. In this case x and y were expressed in terms of t and then the parameter was eliminated to obtain the result in x and y.

Objective 22.1

Can write parametric equations of lines satisfying given conditions and given parametric equations of lines can graph the lines and determine characteristics of the lines.

Activities 22.1

1. Your Text

   Morrill, Analytic Geometry, pp. 59-61
   Exercises: pp. 61-62, problems 1, 2, 3, 6, 9

2. Other Reading Sources

   Love and Rainville, Analytic Geometry, p. 180.

   Shuster, Elementary Vector Geometry, p. 77.

   Hart, Preparation for Calculus, p. 262.

   Wade and Taylor, Contemporary Analytic Geometry, pp. 51, 274, 109.

   Protter-Morrey, Analytic Geometry, p. 56.

3. Individual Assistance

   Instructor available in PIPI Center.

Self-Evaluation Objective 22.1

1. Find the parametric equations of the line through each of the following pairs of points.

   (a) (2,-1), (4,0)

   (b) (3,1), (3,7)

   (c) (-5,1), (2,2)

2. Find the direction numbers, a point on the line, graph, and eliminate the parameter of the lines below.

   (a) $x = -3 + 2t$, $y = 4t$

   (b) $x = 5t - 3$, $y = t + 5$

3. (a) What is the slope of a line parallel to $x = 3t - 5$, $y = 2t + 8$?

   (b) What is the slope of a line perpendicular to $x = -t + 1$, $y = 3t - 6$?

   (c) Find the x intercept in 3b.

4. Write the parametric equation of a line containing the point (1,-3) and parallel to [-1,-1].

Answers:

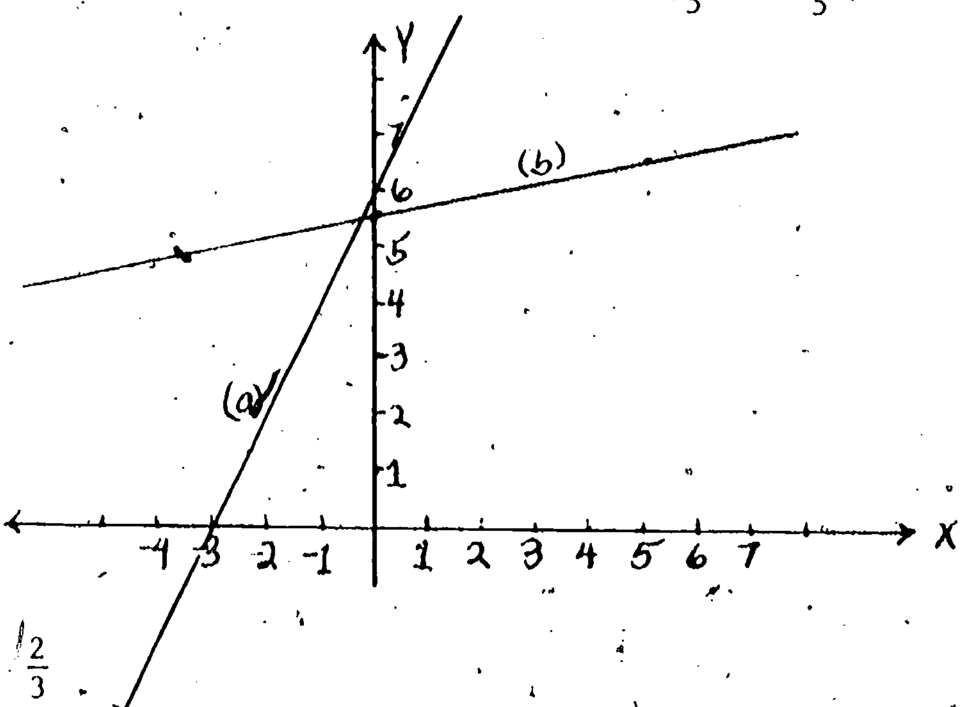
1. (a) $x = 2 + 2t, \quad y = -1 + t$

   (b) $x = 3, \quad y = 1 + t$

   (c) $x = -5 + 7t, \quad y = 1 + t$

2. (a) [2,4], (-3,0), $x = -3 + \frac{y}{2}$ $(y = 2x + 6)$

   (b) [5,1], (-3,5), $x - 5y = -28$ $(y = \frac{1}{5}x + \frac{28}{5})$



3. (a) $\frac{2}{3}$

   (b) $\frac{1}{3}$

   (c) -1

4. $x = 1 - t, \quad y = -3 - t$

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Objective 22.2

Can write parametric equations for conics and given the parametric equations for a conic can identify and sketch the graph of the conic.

Activities 22.2

1. Your Text

   Morrill, Analytic Geometry
   Exercises: pp. 149-150, problems 1, 2
   pp. 167-168, problems 1, 2, 3, 4
   pp. 184-185, problems 1, 2, 3

2. Other Reading Sources

   Love and Rainville, Analytic Geometry, p. 181.

   Wade and Taylor, Contemporary Analytic Geometry, pp. 51, 157, 159, 160.

   Protter-Morrey, Analytic Geometry, p. 175.

3. Solved Problems

   Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, pp. 98-99, problems 15, 17, 18; pp. 102-103, problems 51, 56, 57, 58, 63, 66.

4. Individual Assistance

   Instructor available in PIPI Center.

Self Evaluation Objective 22.2

1. Determine the following conic sections:

   (a) $x = 3t, \quad y = 4t^2$

   (b) $x = 3 \sin \theta, \quad y = 3 \cos \theta$

   (c) $y = -t + 2, \quad x = t^2 - 4$

   (d) $x = 3 \cos \theta, \quad y = 4 \sin \theta$

   (e) $x = 2 \sec \theta, \quad y = 5 \tan \theta$

   (Write the equations, find e, and the coordinates of the focus.)

2. Write the following equations parametrically:

(a) $y^2 = 3x$

(b) $x^2 + y^2 = 10$

(c) $\frac{x^2}{16} + \frac{y^2}{25} = 1$

Answers:

1. (a) $y = \frac{4}{9}x^2$, $e = 1$, $F = (0, \frac{1}{9})$

(b) $x^2 + y^2 = 9$ circle

(c) $(y - 2)^2 = (x + 4)$, $e = 1$, vertex $(-4,2)$, $F = (-3\frac{3}{4}, 2)$

(d) $\frac{x^2}{9} + \frac{y^2}{16} = 1$, $e = \frac{\sqrt{7}}{4}$, $F = (0, \pm\sqrt{7})$

(e) $\frac{x^2}{4} - \frac{y^2}{25} = 1$, $e = \frac{\sqrt{29}}{2}$, $F = (\pm\sqrt{29}, 0)$

2. (a) $y = t$, $x = \frac{t^2}{3}$

(b) $x = \sqrt{10}\cos\theta$, $y = \sqrt{10}\sin\theta$

(c) $x = 4\cos\theta$, $y = 5\sin\theta$

Objective 22.3

Can sketch the graph and find the Cartesian equation for a curve given the parametric equations of the curve.

Activities 22.3

1. Your Text

Morrill, Analytic Geometry, pp. 283-291.
Exercises: p. 291, problems 1-14.

2. Solved Problems

Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, pp. 99-101, problems 16, 19, 20, 21, 22, 23; pp. 102-103, problems 50, 52, 53, 54, 55, 59, 60, 61, 62, 64, 65, 67, 68.

3. Other Reading Sources

Hart, Preparation For Calculus, p. 270.

Protter-Morrey, Analytic Geometry, p. 157.

4. Individual Assistance

Instructor available in PIPI Center.

Self Evaluation Objective 22.3

1. Eliminate the parameter in the following pairs of equations.

(a) $x = 2t + 1$
$y = 3t^3 + t - 7$

(b) $\left.\begin{aligned} x &= \frac{3}{t} + 1 \\ y &= t^2 + 2t \end{aligned}\right\} t \neq 0$

(c) $\left.\begin{aligned} x &= 3 \sin \theta \\ y &= \frac{\theta}{1 - \cos^2 \theta} \end{aligned}\right\} \theta \neq k\pi, \quad k \in z$

(d) $\left.\begin{aligned} x^2 &= \frac{t}{t+1} \\ y &= t^2 + 2t + 7 \end{aligned}\right\} 0 \leq t$

Answers:

(a) $y = 3\left(\frac{x - 1}{2}\right)^3 + \left(\frac{x - 1}{2}\right) - 7$

(b) $y = \left(\frac{3}{x - 1}\right)^2 + 2\left(\frac{3}{x - 1}\right) \quad x \neq 1$

(c) $y = \dfrac{\sin^{-1}\left(\frac{x}{3}\right)}{\frac{x^2}{9}} \quad x = 0$

(d) $y = \dfrac{x^4}{(x^2 - 1)^2} + \dfrac{-2x^2}{(x^2 - 1)} + 7 \quad x \neq \pm 1$

Objectives 22.4

Can write the parametric equations for lines and planes in space.

Activities 22.4

1. Your Text

   Morrill, Analytic Geometry, p. 340.
   Exercises: p. 243, problems 1, 2, 7
   pp. 349-351, problems 1, 2

2. Solved Problems

   Schaum's Outline Series: Theory and Problems of Plane and Solid Analytic Geometry, p. 123; p. 126, problem 10; p. 129, problem 16e.

3. Other Reading Sources

   Protter-Morrey, Analytic Geometry, pp. 204-205.

   Hart, Preparation for Calculus, p. 314.

   Brady-Mansfield, Analytic Geometry, p. 220.

   Wade-Taylor, Contemporary Analytic Geometry, p. 274.

4. Individual Assistance

   Instructor available in PIPI Center.

Self Evaluation Objective 22.4

1. Find the parametric equations of the plane through the points:
   $P_1(-1,1,1)$, $P_2(2,-1,-1)$, $P_3(0,3,2)$

2. Find the parametric equation of the plane $3x - 2y + 7z = 7$.

3. What is the parametric equation of the line which is the intersection of the planes:
   $3x - 2y + 7z = 5$ and $2x + 2y - 2z = 0$

4. Use the solution of problem 3 to find the point of intersection of the planes in problem 3 with the plane $x - 5y + 3z = 6$.

Answers:

1. $x = -1 + 3t + t'$
   $y = 1 - 2t + 2t'$
   $z = 1 - 2t + t'$

2. $x = t + 3t'$
   $y = -2t + t'$
   $z = 1 - t - t'$

3. $x = 1 - t$
   $y = 2t - 1$
   $z = t$

4. $(1 - t) - 5(2t - 1) + 3t = 6$
   $t = 0$
   $\therefore x = 1, \quad y = -1, \quad z = 0$ is solution

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