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ABSTRACT

This paper is divided into three parts. Part I connects the reality of the classroom with the idealism which arises from some of the problem solving literature. It is argued that a broader concept of problem solving is needed to provide a perspective for bridging the gap between the conceptions of problem solving in the literature and typical classroom practice. Part II examines what "problem solving" might mean in the context of the elementary school classroom. Part III considers how children can be helped to understand the non-arbitrary character of rules of arithmetic by examining the connectedness of mathematical ideas, rules, and procedures. Also included is a list of references and recommended readings, a list of specific pointers for teachers, and a conclusions section. (Author/MK)

UNDERSTANDING THE REALITIES OF PROBLEM SOLVING IN ELEMENTARY SCHOOL

With Practical Pointers for Teachers

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Editor's Foreword

"Problem solving" conjures up a variety of ideas, probably ranging from "word problems" to "heuristics", from "low scores on achievement tests" to "awfully difficult to know how to teach".

Linda Brandau and Jack Easley step aside from these conventional interpretations in this book to discuss the basic meaning for child and teacher: how to get an answer to that to which one does not know an answer. Using computation as the vehicle, they discuss two types of problem solving, providing real-life illustrations. Thus, it is apparent that their ideas and their practical suggestions for teachers are drawn from what goes on in classrooms -- and can, in turn, be applied in classrooms.

Real teachers should be able to help real children better as they consider the ideas in this book -- and put them to the test, one step at a time, with the children they are teaching. In the process, we may learn about how to help children solve word problems as well as other types of instructional problems.

We at ERIC/SMEAC are pleased to publish this book. Try it: we hope you'll learn from it!

Marilyn N. Suydam
Editor

To all the teachers and children who are graciously helping us understand their world.

L.B. & J.A.E.

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Introduction

"Problem solving is at the heart of doing mathematics."

How often have we read or heard statements like that? Almost every convention, professional journal, workshop, or textbook mentions the importance of problem solving as a goal in mathematics education.

Even with a long history of this kind of emphasis, recent reports from the National Assessment of Educational Progress (NAEP, 1979; Newsweek, Sept. 24, 1979) show that problem solving skills have declined from 1973 to 1978. The question seems to be: Why can children compute but not apply their knowledge to solving problems? (Bell, 1978). Elementary school teachers may realize that this means performance on word or story problems. However, some of them may not realize that many junior high school topics like approximations, proofs in high school geometry and trigonometry, and setting up functions in college calculus depend on a kind of mathematical problem solving whose occurrence may be quite rare in elementary schools. We will explore what it is and how it relates to the difficulties elementary students have with all kinds of problems.

We will also consider questions concerning the feasibility in most classrooms of problem solving activities suggested by authors in professional journals. (We are referring to the two features in each issue of the Arithmetic Teacher titled "Let's Do It" and "IDEAS." Similar game activities are suggested in each issue of Learning under the titles of "Idea Place" and "Swap Shop.") Teachers who try to teach this kind of problem solving may come away with a feeling of frustration. Part of that frustration comes from the realities of everyday classroom life that take precedence over many supplemental activities that may be planned.

In the early studies of Brownell, Buswell, and others, computation was studied as a problem solving phenomenon and meaningful instruction was found necessary for good performance. The influence of the psychological and philosophical tradition (Dewey, 1933;

Polya, 1957; Hadamard, 1945; Wallas, 1926; etc.) seems to have led research on arithmetical problem solving away from computation and treated verbal (story) problem solving instead. So computation was left to a few researchers (like VanEngen & Gibb, 1956) and information processing psychologists (like Suppes, 1967; Greeno, 1973; Resnick, 1976). Recent surveys of "problem solving research" (Lester, 1977; Kilpatrick, 1969; Hollander, 1978a,b) treat only verbal problem solving, and often (as in the collection of problems by Nelson and Kirkpatrick, 1975) refer only to non-traditional problem forms, similar in a way to the pennies problem that will be discussed in Part II. Because teachers in elementary schools tend to call computation "problem solving" and because computation problems are often quite difficult problems for children to solve (and especially to know whether they have really solved them or not), we have chosen to seek a larger meaning of the term.

It is because of our concern over the REAL day-to-day problems (of teachers and students) that this paper gives little attention to "word or story problems" and concentrates primarily on problems involved with computation. Too often the constraints of teaching are swept under the rug because they appear trivial or not relevant to theories of learning or theories of problem solving. But for the elementary school classroom teacher, who has to make assignments to a classroom full of active children, the constraints of teaching may be far more important than theories of problem solving. In a theoretical discussion, a problem solving activity may sound great, but in a class of 25 children there are so many events to attend to that nearly anyone would have trouble using the activity. It doesn't help to say, "You should have those kids under control." If you have those 25 kids working on problems they haven't previously learned how to work, there's likely to be some kind of trouble.

This paper is divided into three parts. Part I connects the reality of the classroom with the idealism which arises from some of the problem solving literature. In doing so, it is argued that a broader concept of problem solving is needed to provide a suitable perspective for bridging the gap between the conceptions of problem solving in the literature and typical classroom practice. Part II examines what different things "problem solving" might reasonably mean in the context of the elementary school classroom. Part III considers how children can be assisted in understanding the nonarbitrary character of rules of arithmetic by examining the connectedness of mathematical ideas, rules, and procedures. It is in this third part that various suggestions are made for what we hope are realistic, needed improvements.

PART I

CONNECTING THE IDEAL WITH THE REAL

* * * *

As one enters the doorway there are two tables of five children each on the right. Each child seems to be busily working on a page in a workbook. On the left, the teacher, Ms. James, is sitting at a table working with five other children. The children are about six years old. One girl (Jane) is working on filling in "boxes" for problems like

$$6 + 1 = \square \qquad 9 - 6 = \square$$

and

$$1 + \square = 7 \qquad 9 - \square = 6$$

The page has more of these, arranged in pairs like this. She has completed the page and has her hand raised. The teacher walks over to Jane's table, perhaps because several children seated there have their hands raised, and perhaps because they are getting noisy. This leaves the children at the table across the room with no supervision.

Teacher: Yes, Jane?

Jane: Do I have these problems right?

Teacher (silently looks over Jane's paper): You did very well, Jane. I'm going to put a little mark by the few that are wrong. Like this one here. (She points to $7 - \boxed{3} = 3$.) Can you tell me how you would do this problem?

(Jane erases the 3 in the box and gives the teacher a quizical look.)

Teacher: You have a seven here and a three here (points to the 7 and 3) and a minus sign. How do you find the number that goes in the box? How do you think about the problem?

Jane: You have 7, and after you take some away, you have 3. (Jane puts up 7 fingers using both hands. She then puts some down and leaves 3 still up. Then she counts the number of fingers she put down and writes a 4 in the box.)

A boy (Peter) is waving his hand wildly to get the teacher's attention. The teacher looks at him, perhaps wondering whether to leave Jane on her own to respond to Peter. What else can she do? But what a shame! The page Jane is working on has some perfect opportunities for talking about the relatedness of addition and subtraction--problems like 7 - 6 = 1 and 6 + 1 = 7. But that won't get done. Too many things are happening at once.

Noise is starting to float across the room from that table on the other side. The children seem to be finished with their activity. The teacher notices.

Teacher: I know you're done over there. Find a game to play with while I help the people here. We'll all get together in a few minutes.

Peter: Hey, Ms. James, I need your help.

Teacher (to Jane): You go on and fix those other ones. You're doing fine; I have to help Peter.

Peter: I don't know if I'm doing these right. (He's doing a page of vertical format problems like

$$\begin{array}{r} 3 \\ + \square \\ \hline 6 \end{array}$$

where the $\square$ is in different places and where some of the problems are subtraction. He has many mistakes. For this one he has a 9 in the box. For the ones with a minus sign, he has added. It's as if he has mechanically added the two numbers in the problem no matter where they appear.)

Teacher: Peter, look at this problem. (She points to

$$\begin{array}{r} 3 \\ + \boxed{9} \\ \hline 6 \end{array}$$

.) What does it ask you to do?

Peter: Add.

Teacher: What is the 6?

Peter: The answer.

Teacher: O.K., let's think about the problem. You have three of something and get some more and then have six in all. The mystery number is how many you got.

(Peter is silent and looks puzzled.)

Teacher: Let's use these dominoes. I'm going to take out 3 of them and put them here. Now I'm taking out 6 of them and putting them here (by her right hand), 'cause that's the answer.

Peter: You have 9 of them.

Teacher: Right, but that's not the problem we're trying to do. I want to know how many I can put with these 3 here to get 6 over here.

(Peter takes 3 from the pile of 6 by her right hand and moves them over to the other 3.)

Teacher: No, we can't do that. Those 6 have to stay over there. You have to find out how many new ones I need to take so that the pile on the left is the <u>same</u> as the pile on the right.

Peter: Oh, I see! (He takes 3 dominoes from the can and puts them on the left.)

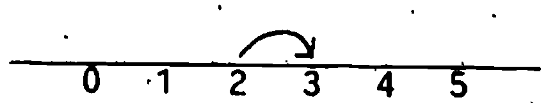
Teacher: So how do you fill in the box on the page?

Peter: I put a 3?

Ellie is wildly waving her hand and starts to jump out of her seat.

Teacher: Right. Now you do some more of those while I help Ellie. (She turns to Ellie.) Let's see how you're doing on those number line jumps. Nooo. 3 + 2 doesn't equal 3.

Ellie is working on a workbook page that has this sample at the top.

2 + 1 = 3 

The problems below the sample are addition problems with the blank at the right and a number line on which she has to draw the arrow. Ellie has done many of the problems wrong, sometimes jumping the incorrect number of times, sometimes starting at the wrong place.

Teacher: Remember, the first number on the left tells you where to start. Let's erase this. (She erases the arrow Ellie had drawn for 3 + 2.) Put your pencil on the 3.

(Ellie does so.)

Now the 2 tells how many to jump, so count, one, two. (She draws the arrow.) Where did I land?

Ellie: 5.

Teacher: Put the 5 as your answer over here. (She points to the space after 3 + 2 = .)

(Ellie writes the 5 where she's told to.)

Teacher: Now you try one yourself. I have to go to the table over there.

The teacher walks to the other side of the room. Ellie is trying to do some problems on her own, but is still having trouble. She mutters, "Boy, these are too hard for me."

Tom comes over to Peter and looks at his work. He is doing subtraction problems like

$$\begin{array}{r} 7 \\ - \square \\ \hline 2 \end{array} \qquad \begin{array}{r} 6 \\ - \square \\ \hline 1 \end{array} \qquad \begin{array}{r} \square \\ - 4 \\ \hline 3 \end{array}$$

Peter asks Tom for help. Tom says, "Those are easy. I can do them in my head. Here are the answers." Peter fills in Tom's empty boxes.

Teacher: O.K., let's have everyone put away your books. It's time to do something together.

There is a great commotion and bustle as children move around. They eventually go to the center of the room and all are seated on the rug in front of the teacher.

Teacher: Shh...quiet, so we can start. Tom... Peter... O.K. I'm going to put this container down here (places a plastic bowl on the rug), and I'm going to drop some chips, one by one, into it. You are to close your eyes and when you think you know how many chips I've dropped, raise your hand. O.K., now I need quiet. All eyes closed? Judy, close your eyes. O.K., I'm going to start. (She drops 6 chips, one at a time, then stops).

Terry: 6.

Teacher: Oh, I won't call on anyone who says it. Jane?

Jane: 6.

Teacher: Let's see if that's right. (She dumps out the chips on the rug.) Let's have Jane count them and we'll watch to see what she gets. Now count them by moving them one at a time.

Jane: One...two...three...four...five...six (moves them one by one as she counts).

Teacher: Is she right?

All: Yes.

Teacher: Let's do another one. Close your eyes now. (She drops 11 chips this time.) Count to yourself, in your head, Terry, not out loud. How many were there, Terry?

Terry: 11.

Teacher: Mike, you count them and see if Terry's right. Let's see if Mike counts them as I told you to.

(Mike counts them aloud and all watch as he moves the chips one by one.)

Teacher: Good, Mike. Let's do another one. Close your eyes. (She drops 8 chips in.)

Dan: 8.

Teacher: Now I'm going to have to do this all over and think of another number, 'cause someone said it. (She starts to drop chips.)

Josie: Look, Mike is watching you drop the chips.

Teacher: And why aren't <u>your</u> eyes closed, Josie? Let's start over again. (She drops 7 chips in, one by one.) I like the way Ellie is sitting, nice and quiet. How many, Ellie?

Ellie: 6.

Teacher: Jane?

Jane: 7.

Teacher: Ellie, let's count them. (She dumps out the chips.) One... two...three...four...five...six... (She looks at clock.) O.K. time to go soon. Let's line up by the door <u>quietly</u> now!

* * * *

This section of the paper sets the stage for what is to come. The above "slice of real classroom life" permits us to discuss the three major strands that run throughout the rest of the paper: the relationship of the ideal to the real, the negative effects of just teaching rules and procedures, and broadening the concept of problem solving.

Idealism vs. Realism

The teacher in the above scene, a friend of ours, read that transcript and commented, "At first, my heart just sank! What was going on was so much of what I wished wasn't going on." Understandably, other teachers have similar reactions to transcripts of their own teaching. These reactions reflect the dichotomy between our ideals and the reality of the situation. That is, we might have grandiose plans, but they go awry when we get into the classroom.

How can we reconcile this difference? One way is to realize that in reality we may never live up to our ideals, but at the same time we still need these ideals to strive for. Some teachers react ambivalently when reading suggestions given in the professional journals as to what they should be doing: "You read that stuff from the journals and you realize you're doing none of that; it makes you feel like a complete failure; and yet that's always what you're working towards." Although we may always be striving for the ideal, we need to see little pieces of that ideal in the reality of the daily school situation.

In Part III of this paper, we consider lots of little REAL alternatives that either do occur or can occur in the classroom to promote independence of thought and a sense of the connectedness of and the non-arbitrary character of arithmetic. There is no master list because they are techniques for improving an on-going practice. We will mention quite a few suggestions specific to situations and scenes recorded in actual classrooms. A list of these practical pointers is appended to the paper to assist the reader in locating them and as a reminder of their contextual character. They may serve to point the way for teachers (and those who would be helpful to teachers) to generate their own pointers. Perhaps using a tape recorder to record lessons and then selecting troublesome portions for written transcription is a way to promote the generation of appropriate, realistic techniques as a teacher discusses the lesson with another teacher, or a resource person of some kind. Mastery of any craft consists in internalizing a very large number of such pointers so they function almost automatically in daily interaction. We first need to recognize such alternatives to present practice. Then we can see how every one of these little alternatives brings us a little bit closer to the ideal.

Some might say that the ideal is to have a general problem solving attitude when teaching mathematics. But if only that suggestion is presented to teachers, they could understandably react, "Oh, that's an awfully big change that you're expecting of me." We need to consider carefully what kinds of changes are needed, for what reasons, and what kinds of changes are realistically possible without loss of other valuable conditions of schooling. Cognitively, we can understand why you can't work backwards from a "big change" proposal to determine the little things that are needed to accomplish that big change. It is much more reasonable to expect that, if lots of little changes are presented, teachers would choose some of them to try. They might respond with, "Oh, I can see myself doing some of those." And, when they do, they will be broadening their conception of problem solving.

Children's Views of Mathematics as a Set of Arbitrary Procedures

Later in this paper we will deal with some remedies to help prevent children from developing the view that mathematics is a set of arbitrary rules that you just memorize to survive. Sometimes this view isn't evident until the middle grades (Erlwanger, 1974, 1975) or even later, although it may emerge from dependence on teachers for checking answers. The following analysis of the problem, called "mathophobia" by Lazarus (1974), is especially clear.

> ...mathophobia can pass through a latency stage before becoming manifest...consider the plight of a high school student who has always relied on the memorize-what-to-do approach. Because his grades may have been satisfactory, this incipient problem may not be apparent to anyone, including himself; he may not even know that there is any other way to learn mathematics. But the time and effort that his approach demands will increase dramatically through high school, subjecting the student to a constantly increasing strain. Moreover, since it has now been some years since he last understood what he was doing, he is in no position to switch over to the more appropriate strategy of actually comprehending the material...he simply lacks the necessary knowledge.... When his grades finally drop, as they must, even his teachers are unlikely to realize that his problem is not something new, but has been in the making for years. And at this point, it is doubtful that casual, remedial work would accomplish much, because the student needs nothing less than a complete reeducation in mathematics. (p. 18)

Unfortunately, the phenomenon that Lazarus describes is often true. Some elementary school teachers here reported experiencing it themselves. But the argument of long-range cumulative effects of problem solving style is usually the one given to elementary teachers as the reason that they need to worry about "teaching for understanding" or "not emphasizing rules and computational skills." The implication is that, if elementary teachers do their job correctly, then students will not reach junior or high school with a case of "mathophobia." But elementary teachers are tired of hearing this argument, and, even when it is true, it is only part of the story. Certainly, the cumulative effects argument contributes little to help teachers in the classroom every day.

Teachers need to know some things they can do now and some more immediate reasons for worrying about "memorizing what to do." If student difficulties in the later grades are the result of cumulative effects (and there is no real evidence on this -- no longitudinal studies have been done relating to this idea), there must be some signs visible in elementary school pupils that teachers can look for.

Let's take another look at that classroom scene given on pages 3-7. This time, let's look for some possible immediate effects that can be related to the question: what happens if children develop the view that mathematics consists of a set of ARBITRARY rules?

We suspect that the following observations are related to that view:

1. The children develop a dependence on the teacher for correcting their work and for clues about when they've made errors. One of the clues children learn to listen for is when the teacher asks a question like, "Can you tell me how you did this problem?" (shown with Jane). In fact, when the teacher questions <u>any</u> exercise, children suspect an error because the ones not questioned usually are correct.

2. The children learn cues to get the teacher or someone to do (or at least show them how to do) the exercises for them. For example, the child learns that, if he or she is silent for more than a few seconds, the teacher will at least give some hints on how to do the problem if not actually do it for him or her (shown with Peter). The child also learns to give a brief answer to the teacher's questions, and not much more (shown with Peter and Ellie). In that way, the teacher will provide most of the information needed to do the problem.

If children are to have the courage to attack new problems by themselves, they need to see connections between ways of working mathematics problems.

<u>What is Problem Solving?</u>

The words "problem solving" conjure up many different meanings. Problem solving has been studied by philosophers (e.g., Dewey, 1933), psychologists (e.g., Duncker, 1945; Wertheimer, 1959; Newell and Simon, 1972; Krutetskii, 1976), social psychologists (e.g., Janis and Mann, 1977), mathematicians (e.g., Hadamard, 1945; Polya, 1957), and mathematics educators (at the elementary level, e.g., Brownell, 1942; Van Engen & Gibb, 1956; LeBlanc, 1968; Steffe, 1968, 1970), and each group tends to take a different perspective on the meaning of the term. It is not our purpose here to try to bring a literature that is so diverse in definitions and points of view into a coherent framework. Instead, we want to take a practical perspective on solving problems of all kinds in the context of elementary school mathematics classrooms.

In mathematics education, problem solving is usually discussed in the context of improving children's thinking ability or improving teaching to develop a variety of thinking strategies (Cambridge Conference, 1963; Trafton, 1975; NACOME, 1975). It is assumed that children will be provided with opportunities for problem solving that don't involve just following rules and procedures. Usually such drill-and-practice worksheets and story problems in textbooks are criticized in the problem solving literature as promoting an extremely limited view (Freudenthal, 1973; Lester, 1978; Bell, 1978). What is criticized is the fact that problems of that kind can usually be done by carefully following the right procedure for the problem type or the fact that solutions to such textbook problems almost always have nice whole numbers (or familiar fractions) as answers. It is also said that textbook problems promote "psyching out" the author rather than learning general problem solving skills. Furthermore, textbooks are criticized for not providing more problems or more guidance in choosing appropriate operations or techniques (Zweng, 1979). Because textbooks and worksheets are the mainstay of many teachers, the problem solving literature may appear to be quite idealistic. Our hope is to bridge this gap, to bring persons from both points of view into communication with each other.

In order to provide a perspective on the differences between the literature and much classroom practice, a broader concept of problem solving is needed. What is there about the scene on pages 3-7 that can broaden our perspective about problem solving? There are basically two different kinds of activity occurring within the scene. In one, the teacher is trying to work with a small group of children while the rest of the class is doing individual seat work. In the second, the teacher tries to do an activity with the entire class. Both the individualization and the group activity method have problems associated with them, from the teacher's and from the children's viewpoints.

That is, in individual seatwork, the teacher seems to be juggling time--the children at their seats need attention and so does the group at the table. The teacher tries to maximize the time spent with each individual child and with the group. This creates that immediacy problem so well described by Jackson (1968)--events are happening so fast for the teacher that decision-making has to be instantaneous and spontaneous. The children also seem to have problems when doing individual seatwork. They need to get the teacher's attention to correct their work. If mistakes were made, more problems occur--exercises have to be corrected. But to correct them they need more help from someone--usually the teacher. This means they either have to wait or get "help" from another student like Peter gets from Tom. This creates a dependency problem in which children lose their ability to work alone.

On the other hand, "entire class" activities are also fraught with problems. From the teacher's perspective, the problems are: "How do you get and keep all the children's attention? How do you control the situation so one child doesn't yell out the answer, thereby destroying the learning opportunities of the others?" So, although the "hurriedness" problems involved with the individual seatwork may have been solved or eliminated, new problems have been created.

Perhaps also involved with both individual and group work are some social problems. What about the child who isn't so bright and quick? Do the other children tease him or her? Even in an individual situation, children compare the work they are doing and create their own ladder of status so that some children end up at the bottom of the ladder and some at the top.

In our discussion of problem solving, we will not only refer to mathematical problems but also to other kinds of problems that are often encountered in the classroom, such as social problems, which involve decision making on the teacher's and/or students' parts. We don't want to make the strong claim that skills involved in solving social problems will transfer to solving mathematical problems. On the other hand, we are not willing to discount the idea that reflecting on the processes of solving social problems might help show new connections between the perspective in problem solving literature and the classroom teacher's perspective. We do want to claim that mathematical problem solving is not separable in the classroom from social problems. Let's consider, then, what there is about so-called general problem solving that is similar to mathematical problem solving.

PART II

TOWARD A BROADER CONCEPT OF PROBLEM SOLVING

General Problem Solving

In solving social problems, the problem itself isn't necessarily ever clearly defined. Frequently, the problem solving process starts when some situation or event occurs that perplexes us. We start to solve the "problem" by trying out familiar solutions. When one solution doesn't work, we may try another; when that doesn't work, we try another, and so on. Eventually, through reflections on our trials and errors, the problem becomes more defined; each effort at solving it improves our understanding of the situation.

For example, a teacher may have a student who interrupts with annoying frequency. This situation starts a process of first trying to reduce the annoyance and then to define the problem and solve it. The teacher may first try standard routines like telling the student to stop interrupting. This response may prove ineffective when the student keeps interrupting. Perhaps the teacher next will try ignoring the student, but finds this also doesn't work. The cause of the "problem" hasn't yet been located. Maybe the teacher then tries talking to the student alone. Now the teacher finds out that the student is currently living in the midst of problems at home. This suggests that she or he seeks extra attention from the teacher as compensation. Now the problem is more defined by this hypothesis, and a subordinate problem emerges of trying to give this student some extra attention, that is, more than others are getting. So the teacher who wants to help needs to solve this problem. But doing that may create additional problems.

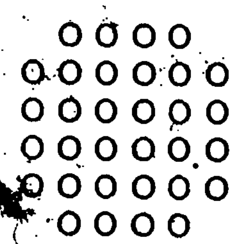
Mathematical Problem Solving

In the literature on mathematical problem solving, we learn that similar trial and error procedures may be used at the beginning stages. That is, if we haven't been taught procedures for doing a particular task, at first we'll try some procedures, hit and miss. As we go along, the problem becomes more and more defined. We read about examples like the following: Two children are playing with some cardboard pennies after finishing their assignment on "making change." They start to place them in triangular shapes such as:

O
OO
OOO

At the beginning they may make a triangle from 3 of them and then from 6 of them and then from 10 of them. One child may see immediately that the next larger triangle is created by adding a row on the bottom that is one penny longer than the previous bottom line. So this child's focus is on constructing the number pattern 3, 6, 10, 15, and so on. His or her problem is different from another child's, who breaks up one pattern before forming the next. For this other child, a problem starts to take shape when a triangular pattern of this sort cannot be formed using all of the 13 pennies available. This child may start to wonder which numbers of pennies can form "triangles" and which ones cannot. He or she finds some more pennies and begins to experiment in trial-and-error fashion, trying to create a figure like this from each number. The problem may be enlarged if the child tries to form square arrays with the pennies, or five-sided arrays, and so on.

Perhaps the second child will then have the idea of trying to find a rule to predict the next number of pennies that makes an arrangement of the type she or he is working on. The child eventually may want to make a table like the following. (By this time several students may be involved and different ideas emerge.)



Sides

3	4	5	6	7	8
3	4		7		12
6	9		12		16
10	16		19		21
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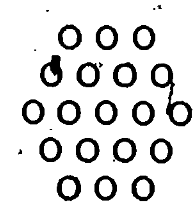


Someone may object that the six-sided figures with 12 pennies and the eight-sided figures with 16 and 21 pennies are not "regular"--that is, some sides are longer than others. Here, the problem definition could go two ways: irregular figures could be allowed in all columns or ruled out altogether. The children may eventually decide this definitional problem in terms of the rules that predict the next numbers in each column. At the beginning, though, they had little or no idea of what the problem was going to become and what solutions were going to be tried.

This mathematical example is similar to "social" problems in that informal exploration (trial and error) plays an important part in the process. Also, there are no fixed boundaries within which to work and no clearly specified procedures to use. Some followers of Dewey, Piaget, and other writers infer that, if someone had given the children boundaries and procedures to follow, whatever "problem" they might have been working on would have been removed. On the other hand, providing such guidance in a classroom would solve the "problem" that many teachers might perceive here as a lack of organization that is inappropriate for the classroom. This organizational problem could manifest itself by too many arguments between the children; too many questions to which the teacher's answer would be, "I don't know; why don't you find out?"; too many conflicts over what to do next and who should be working on which sub-problem; and the problem of how to decide when to terminate the "pennies" activity.

We could appreciate that removal of the "problem", by being given a procedure or even an answer, could be most welcome at times. Imagine that, in the earlier social problem (page 13), the school principal had come along and told the teacher that there had been a family disturbance and that the child, who was constantly interrupting, should be sent to the school counselor. That would have removed the teacher's problem -- at least, for the moment. There are times when that would be extremely welcome. However, there are other times when a teacher would rather solve that problem himself or herself. We think that problem removal (by being given clues, a procedure, or even the solution itself) needs to be recognized as "solving the problem."

There is another kind of problem solving in which specified procedures, rules, or standard routines that are somehow known to the problem solver are applied effectively to solve a problem. When they are well known and easily applied, we might say there is "no problem." However, while this skill is being acquired, remembering and figuring out how to use these processes may be quite a problem.

This kind of pre-structured problem solving is undoubtedly more typical of the kind that occurs in the mathematics classroom, both in the mathematical and social domains, perhaps because organization is needed in such a situation. There may be other good reasons for providing that kind of experience, as we shall argue. So, it seems unrealistic to suggest, as its neglect in some of the literature may imply, that the routine type of problem solving is "bad" and the more creative type is "good". There may be some very good reasons for increasing the time devoted to one kind of problem solving or the other, or for changing the types of occasions on which the two kinds tend to occur. However, we want to be very clear that both types of problem solving are still necessary in the complex social system of the typical mathematics classroom. If children are to make frequent use of the facts and principles of mathematics they learn in elementary school, they need opportunities to work without precise rules some of the time and to gain confidence that they can develop their own rules.

The more pre-structured type we will call problem solving 1 and the less pre-structured type we will call problem solving 2. We will first discuss problem solving 1 and next problem solving 2. Later, we will try to see how the first of these two types could be helped to evolve into the other in order to correct the common imbalance between the two. One of the reasons for correcting an excess of problem solving 1 is, as we have hinted earlier, to help pupils develop their full capacity for more advanced mathematical learning in secondary schools and college, and to avoid the common "math anxiety" that arises from dependence on someone to give directions before attempting an unfamiliar type of problem (Tobias, 1978). However, another more immediate reason is to develop a sense of the non-arbitrary character of arithmetical rules and procedures (Erlwanger, 1974, 1975).

Problem Solving 1

This is certainly the most common kind of problem solving occurring in mathematics classrooms, which, if we make a strict parallel, may also be true in "real life" situations. When an individual is faced with a bothersome or perplexing situation which needs a quick solution, the solution may be obtained by using one of a few standard well-known routines.

What we mean by "standard well-known routines" is that they come to mind immediately: Shall I add or subtract here? Shall I use this formula or that one? Shall I send this child out of the room or give him or her a new task in the room? We think we all understand standard routines because they are so frequently taught in mathematics classrooms and we use them to organize our lives. But, in reality, we may understand very little about them and their use. They may sometimes yield wrong or unfortunate results or be impossible to apply. We believe that problem solving 1 needs more attention than it has been given in the literature and than we are able to give it. It must be important, because so many of us (teachers or otherwise) use standard routines regularly.

The fact that a routine that worked so well up to now may not give the desired results in a certain situation leads one to wonder if the routine has been used appropriately. Often a slight modification in the routine or possibly a sudden recognition of the need for a different one is called for. For example, a child who has been using the standard division algorithm and tries to apply it to a problem like 3÷12 suddenly realizes, probably with some prodding from the teacher, that 3÷12 isn't the same problem as 12÷3. This leads to a different problem: "What do you do when 12 won't go into 3?"

Let's look at some more examples of problem solving 1.

1. Depending on clues. A child is using known algorithms to solve exercises in addition, subtraction, multiplication, or division. She or he needs to pay attention to the clues to decide which operation is wanted. The clues learned first (+, -, x, ÷) don't always work, as in the case of "missing addends." Then the child also has to be careful not to make any mistakes in carrying out the algorithms. Clearly, there is a lot to think about here. It's far from routine.

2. Solving one problem may create another. Bill is trying to work by himself and is being bothered and annoyed by another

As described by Lester (1978), "Any reference to a problem or problem solving refers to a situation in which previous experiences, knowledge, and intuition must be coordinated in an effort to determine an outcome of that situation for which a procedure for determining the outcome is not known" (p. 54). Note the emphasis on experience and intuition. Social intuition, as may be involved in regrouping children to minimize conflicts, may be an example with which many teachers are familiar. Such synthetic problems, in which new arrangements of things and/or people need to be created, involve different kinds of intuition than analytic problems in which a complex situation needs to be broken down into parts to discover what relates to what.

It is reported that problem solving 2 often involves some stage of incubation (Wallas, 1926; Hadamard, 1945; Poincaré, 1952), when conscious work on a problem comes to a halt, and the only progress that is made is made unconsciously. The idea of an unconscious incubation stage seems most important to recognize because of the various complexities of school situations that often require conscious and overt signs of pupil progress.

That is, a "solution" to a problem may suddenly seem to come without warning, as often occurs when one wakes in the middle of the night. Such sudden insights (whether they prove successful or not) must be the product of some unconscious process. Thus, problem solving 2 can be quite elusive in some respects. Often a clue to the solution may be recognized in some unrelated activity in which others, not working on the problem, would not see the connection between the solution and the problem. Since finding a solution by problem solving 2 takes an unpredictable amount of time, it may not occur until days, weeks, or months after the initial effort is made to solve the problem. Because of unconscious processes and unpredictable amounts of time, it is hard to evaluate how much progress someone is making toward getting a solution to a problem.

Problem solving 2 is difficult to observe under the pressures of normal classroom work for the above reasons. It may also have very disorganized beginnings, as in the cases previously described of children working with pennies. In a classroom, events seem to happen too fast to permit mulling over a problem, which may be necessary before any incubation can occur. In fact, frustration from trying many different approaches, without any success in grasping the source of the difficulty, may be a necessary precursor to incubation, and the creative process. However, this frustration can be much too difficult to contend with on a daily basis (from the viewpoint of both teacher and student) in the normal classroom of 20-30 students. Many teachers would certainly balk at promoting pupil frustration. Nevertheless, it does occur and can be observed in successfully solving mathematical and social problems. Another emotional aspect of problem solving 2 is the fact that one person's efforts to redefine a problem may "pull the rug out from under" another person, who depends on the very definition that is being questioned. Fear and anger can result, creating social problems for someone to solve. (In redefining "problem solving" in this paper, we may be creating fear and anger in some readers.)

Some examples of this kind of problem solving will perhaps clarify its properties and the conflicts that may arise with increased experience in mathematics classes of this kind of problem solving. These emotional aspects of problem solving 2 are familiar to those who enjoy the thrill of solving a really tough problem and seek out tough problems to solve. Whether these conflicts are themselves subject to resolution by some higher kind of problem solving is, of course, an interesting question.

1. Recognition of the triangular numbers. A child is counting cardboard pennies in class, finding different combinations of numbers that add to 13. While the child is involved in this task, the child starts to fiddle around with the pennies somewhat haphazardly and starts to make the same kinds of triangles we described earlier. As the pupil is doing so, the teacher tells the class that the time is up, and they must begin another task. The pennies are collected and the beginnings of the "triangular" number problem are momentarily forgotten.

Several weeks later the same student is again using pennies in learning about money relationships. He or she again starts to make shapes and thinks some more about "triangles." The child now formulates a problem: "What are the numbers of pennies I would need to make different sizes of triangles?". In spare moments throughout the succeeding days, the problem is worked on. But in these conditions, the solution may never occur until it is presented externally. For example, a year later when the next teacher, while talking about number patterns, gives the children this sequence, 0, 1, 3, 6, 10, 15, 21, ..., which is generated by successively adding 1, 2, 3, 4, ..., to the previous result, starting with zero. This particular child may immediately recognize it as containing the numbers of pennies that formed triangles, and recognize that the sequence 1, 2, 3, 4, ..., is the number in each row. It is easy to see how this child's problem solving would not be recognized as such because of the enormous amount of time between stages in the process, the change in teachers, and the triviality of the problem to anyone who understands it. In any case, it would take some explicit comment from the child to the teacher about these past experiences with this situation for the teacher to be aware of the connections and of this occurrence of problem solving.

2. A child's social decision making. In a previous example (in the Problem Solving 1 section), we discussed how a standard routine in a social problem situation was used. That is, Bill complained that Tony was "bugging him" and tried to persuade the

teacher to reprimand Tony. When that failed, Bill hit Tony. However, recognizing the failure of those techniques, Bill might begin to ask himself (and others) why Tony picks on him. After considering various possibilities, Bill finally might settle on the following unlikely sounding hypothesis -- Tony likes him and wants to interact with him. Approaching the problem head-on, he asks Tony, "What do you have against me?" "Nothing," comes the reply. "Then why don't we be friends?" He is surprised to hear, "Okay, see if you can come to my house after school."

3. Teaching about area. A teacher might use problem solving 2 in connection with a curriculum decision. In the situation described previously, when the teacher is trying to decide how to teach area, suppose he or she has found that teaching the use of formulas doesn't work very well with some children. In fact, the development of a concept of area, for most children, seems to require more than choosing and using appropriate formulas. The teacher thinks of counting one cm squares in various shapes that can be divided into such squares evenly. She or he worries about the children generalizing the process to shapes that don't divide evenly into squares.

One day, the teacher notices the children dividing up soda crackers, many of which are broken. The children decide to approximate whole crackers from the fragments. This gives the teacher an idea. She or he can use the cracker situation with the children to lead into the idea of finding the area of a rectangular trapezoid by counting squares and combining the partial squares to form whole ones. From there, they could work on the area of any shape. Perhaps this new sequence of activities will develop concepts of what areas are and might also help the children realize the usefulness of area formulas. This new arrangement of activities feels particularly promising to the teacher. There is a sense of excitement of a new discovery, an eagerness to try it out.

4. A teacher's social problem. Alma, a third-grade teacher, starts out the school year by splitting the class into groups. She works with one group at a time while the other groups are working on different assigned tasks. Increasingly, she feels more and more frustrated. She doesn't seem to be able to help one group because other children keep interrupting her. So the problem begins as frustration, then becomes defined as one of children's interruptions. She ponders the problem and further redefines it. The noise from the other children is the disturbing factor. How can those other children be motivated to keep quiet (or at least keep their noise down to an acceptable level)? She tries a point system for which students get a reward for doing their work quietly.

However, as the school year progresses, some children, Alma discovers, don't care whether or not they earn points and get the reward. They continue to be noisy; the teacher's problem still isn't solved. The problem needs redefining. Alma decides to enlist the children's help in reaching a more creative solution. Different possibilities are tried, but none seem to work. Finally, Alma and the teacher next door ponder the obvious fact that the children are bored and frustrated, and they decide to try group projects. But, they agree that, even though they may find their projects very interesting and thus avoid boredom, the children will still have to be trained to respect the rights of other persons and not interrupt them when they need help.

5. A project from home. Two 12-year-old children, Bill and Sue, are building a tree house in Bill's front yard. They have already attached to a tree one platform about six feet off the ground. They want to attach another one (3 feet wide) on the opposite side, to a board they have already nailed to the tree. This board is four feet up the tree trunk from the first platform (Figure 1). The problem is that, before they can raise it into position, they must attach braces that can be nailed to the edge of the lower platform to support the upper platform. They need to know how long the braces between the two platforms should be so they can be cut and nailed to the platform to help hold it up while they nail it to the board. They first try to think of a way of measuring directly, but the second platform must be too high to permit Bill to stand on a step-ladder and hold it in place while Sue measures. Also, it would be dangerous to try to hold the platform while standing on a long ladder placed against the tree.

They think of another idea: if they stand the 3-foot-wide platform on edge on the sidewalk, and mark the tree's position with chalk along the walk, then they could measure how long the braces should be (Figure 2). (The two braces should both be cut to the same length.) The problem is that the angle formed by the tree and the platform is unknown. So they don't know where to draw a line on the sidewalk to represent the tree. They decide to talk to their teacher. When their teacher heard the lengths, 3 feet and 4 feet, he immediately thought of a 3, 4, 5 triangle and began to talk to them about it, drawing a picture like that in Figure 3. "But our tree is leaning, so five feet will be too long," the two children said.

Bill and Sue talked to other people they knew and received the following suggestions: make a scale drawing; use two tape measures held up by strings and tied to the second story window of the house; build a large protractor to measure the angle between a spirit level and the tree. These ideas were not as easy as they sounded, however, and they felt that something more creative was called for -- something that would not take so much time and effort. One night, while resting on her bed before supper, Sue thought of a neat solution. A string with a rock tied to it could be fastened to one end of the board they had nailed to the tree for the second platform. It would hang down past the platform that was already there, and they could measure where the

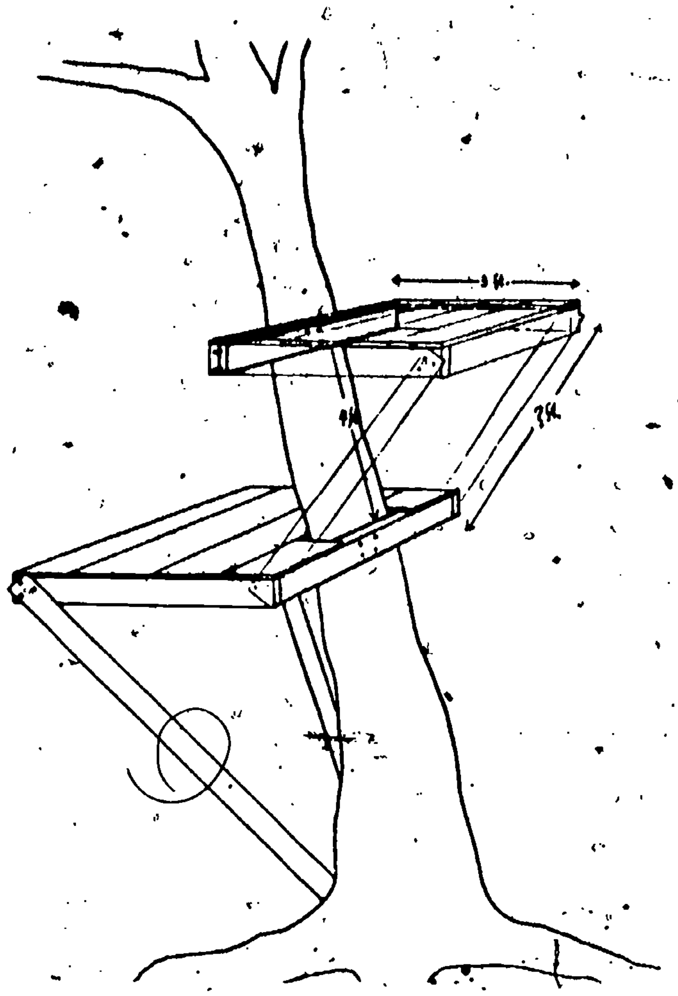


FIGURE 1

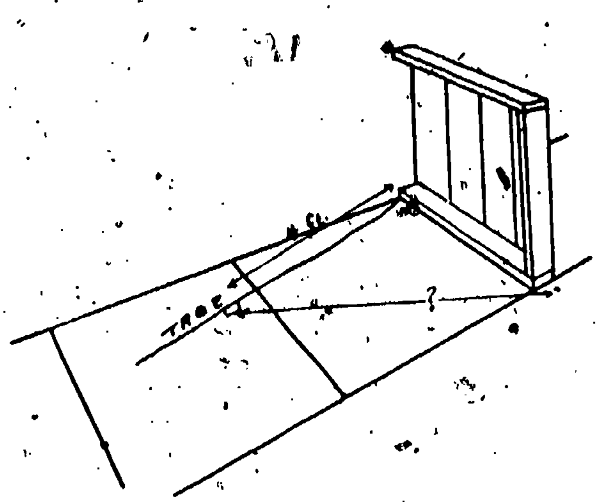


FIGURE 2

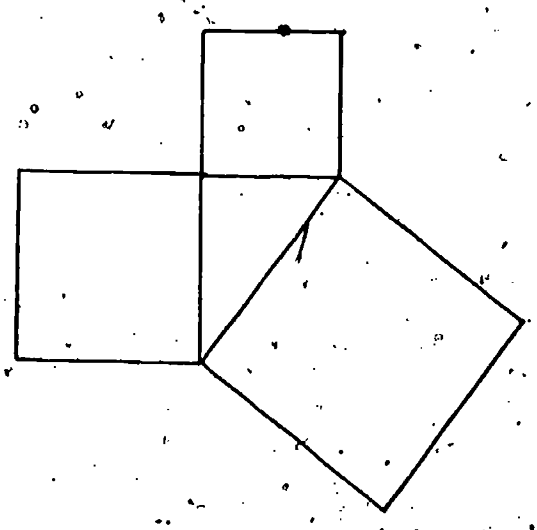


FIGURE 3

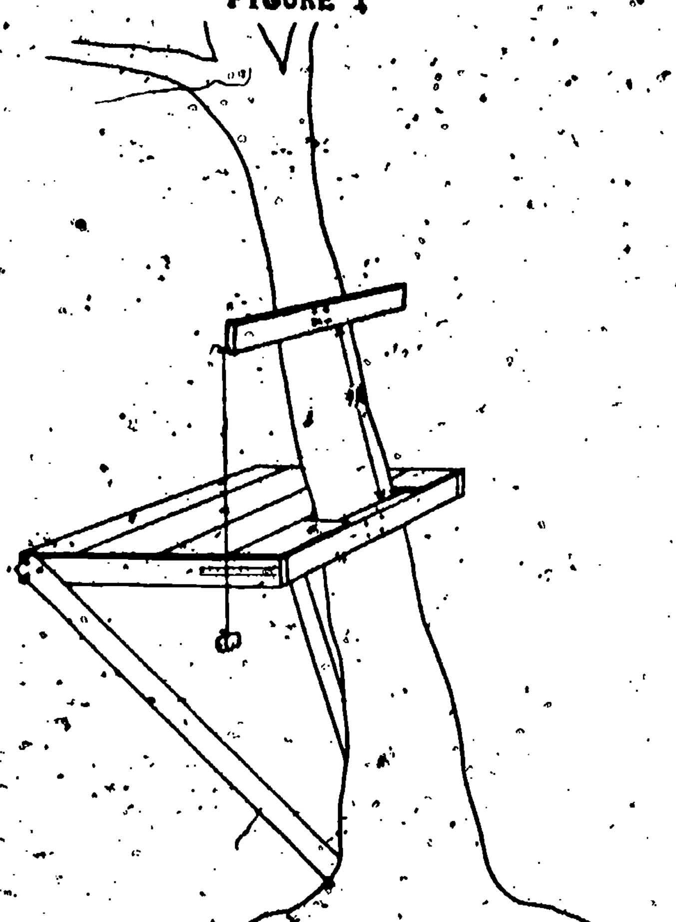


FIGURE 4

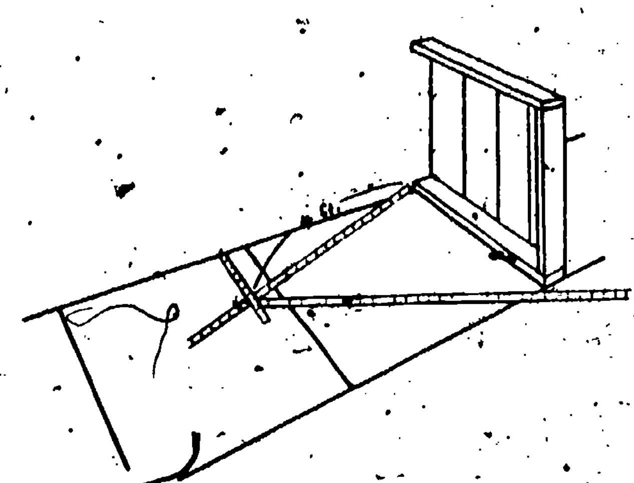


FIGURE 5

string with the rock on the end of it touched the platform (Figure 4). That way, they could copy the angle of the tree onto the sidewalk without measuring directly how many degrees it was.

After supper, Bill hung the string from the top board and Sue measured seven inches from the string to the edge of the platform. They then brought the platform out of the garage and stood it on edge on one of the cross-lines of the sidewalk. They measured where seven inches from the edge of the sidewalk crossed four feet from the platform. They next drew the four-foot line to represent the tree. They measured from the end of that line to the platform to find how long the braces had to be (Figure 5). They really felt good about solving their problem and told their teacher all about the solution the next day. That afternoon they sawed the braces to the right length, nailed them to the platform and, without too much trouble, nailed the second platform securely in place.

* * * *

Now that we've clarified some points about what we are calling problem solving 1 and problem solving 2, we are ready to consider some questions relating to these kinds of problem solving. What is there about problem solving 1 that's important? What happens if there's too much emphasis on problem solving 1 activities? What's important about problem solving 2? What happens if there is too much emphasis on problem solving 2 activities? What are some realistic ways of dealing with teachers' social and mathematical problem solving (both 1 and 2)? What about students' social and mathematical problems? And, perhaps of special concern to classroom teachers, are there practical ways of getting problem solving 2 to occur more often in the classroom?

In the next section, by examining everyday classroom situations, we hope to show why problem solving 1 is not sufficient for understanding arithmetic or even becoming skillful in it and what can realistically be done through problem solving 2 to remedy that situation.

PART III

UNDERSTANDING THAT ARITHMETIC PROCEDURES ARE NOT ARBITRARY

What does understanding arithmetic have to do with problem solving 1 and 2? When problem solving 1 methods are given too much attention, then children can survive by memorizing rules to get answers. Also, those methods are never connected to each other, as could be the case in problem solving 2. So children see mathematics as fragmented and as arbitrary rules (Erlwanger, 1975). No one is arguing for an exclusive diet of problem solving 2, which a few students approximate by their refusal to memorize anything they can figure out some other way. Probably that would produce an undisciplined mind, incapable of rapid learning. It is important to combine problem solving 1 with problem solving 2 methods, but not to entirely neglect either of them because each has its importance.

The Importance of Problem Solving 1

Although it may be assumed, from its neglect in the literature, that problem solving 1 is a simple process, there is growing evidence (Davis & McKnight, 1979) that the application of standard algorithms to numerical problems (or exercises) is not as simple as it is often assumed. It appears that fragments of algorithms are often learned independently and recombined incorrectly.

Much of what children are expected to learn in mathematics illustrates the importance of problem solving 1 (applying a few specified rules and procedures) for the classroom. [Further study might reveal its importance in other areas of life (see Stake and Easley, 1978, Chapter 16).]

Previously we gave, as an example, a child using known algorithms to solve exercises involving the four arithmetic operations. Obviously these algorithms are part of what is expected of mathematics teaching in elementary school. Even before we need to learn these procedures, though, there are other procedures which are prerequisites. For example, it is necessary to learn to recognize the numbers, to print them correctly, to say them correctly, to know the basic "facts of arithmetic," and so on. These necessities take much drill and practice of the problem solving 1 type.

Once children learn these "basics", there are other requirements: learning the procedures of adding, subtracting, multiplying, and dividing. Children need to learn how to write their computational exercises in a form that is communicable to others (and to themselves). For example, we all can think of the child who doesn't line up the numbers correctly when adding or multiplying and consequently gets wrong answers. We are reminded of the case of a very bright 13-year-old. His teachers couldn't understand why his standardized test scores in mathematical reasoning were so very high but in computation were so very low. Upon doing some individualized testing, it was discovered that he didn't line up his numbers correctly because of a perceptual problem.

The reader will doubtless be able to think of even better examples illustrating the importance of problem solving 1. We would appreciate hearing about them.

The Importance of Problem Solving 2

Perhaps it would seem strange to researchers to be discussing the importance of something whose significance is so much taken for granted in the literature. We said in Part I that problem solving 2 methods need more emphasis in elementary school so that students don't develop the idea that mathematics is just a set of arbitrary rules. It was also suggested that dependence on the teacher (as the authority) starts very early and perhaps more problem solving 2 could alleviate that. There is, however, reason to believe that complete dependence on authority is a major cause of inability to use mathematics or to develop confidence in one's mathematical ability (see Tobias, 1978, chapter 2).

The way students think about an over-emphasis on rules was evident in a conversation between some high school students. It went something like this.

Jane: I think the English parts of the SAT's are easier than the math parts.

Chris: Yeah, English is so much more open than math.

Joe: Yeah, math is just memorizing and remembering rules.

Teacher: Why do you think that?

Joe: I don't know...that's just the way it is.

Jane: I think I know. It's because whatever we've had to learn in mathematics class has been rules. First we learned the rules for adding, then subtracting, then multiplying and dividing. Then we got to high school and learned some more rules for algebra.

If ALL that students remember about mathematics class is having to memorize rules and procedures, then mathematics is of little use to them. What happens along the way to get this "rule applying" attitude? Perhaps this started in the primary grades where two different methods of counting the same collection (e.g., "skip counting" by fives and by tens) produce different answers. (See Stake, 1980, for an analysis of children's counting problems.) Because each method was memorized, and especially because no connection was made between methods, no discrepancy was observed by the child. The children conclude that getting different answers is O.K. So they (and teachers) can come away with the view that in mathematics 'when you apply a rule, there's nothing more to do.' (Learning to check one's own answer may be an essential step to the insights into the rational consistency that gives the emotional excitement and aesthetic beauty to mathematics and to making these connections.)

How Problem Solving 2 May Turn Into Problem Solving 1

We are only reflecting on a few years of observation, but we have the feeling that MOST early mathematical experiences are like that in the following scene from a second-grade classroom.

* * * *

Teacher: Let's get ourselves ready. Two feet on the floor. We're going to work for a while in your arithmetic books on putting some sets of different animals and objects together. Any time someone asks you, how many do you have in all, that's your clue that you need to add the two sets together. (Rule 1)* Any time you see these words, "how, many, in, all" (She writes each word on the board as she talks.) ...Let's do a problem. There's 7 red balloons and 2 blue ones. Someone says, 'How many do you have in all?' Can you do it in your head? Jimmy?

Jimmy: 8.

Teacher: Let's check it out. Come up here and draw a set of 7 balloons. Count 'em off as you go. (Rule 2)

(Jimmy comes to the board and draws 7 circles and counts along; as soon as he reaches 7, the teacher says...)

Teacher: There's your first set, right, John? Now how many more do you have to put with that?

---

* To help the reader identify rules in the scenes presented, we have inserted rule numbers in parentheses into the transcripts, underlined when they are introduced by the teacher and not underlined when they are followed by pupils.

Jimmy: 8.

Teacher: Look at the problem again. Over here you have how many in the first set? Look where I'm pointing. (She has drawn the numbers 7 and 2 on the board.) What numeral is that?

Jimmy: 7.

Teacher: And how many do you put with that?

Jimmy: 2.

Teacher: 2 more. How many do you need to draw then Jimmy?

Jimmy: 2. (Rule 3)

Teacher: How are you going to figure out how many you have in all? What are you going to do? (Rule 1.)

(Jimmy mumbles.)

Teacher: I can't hear you.

Jimmy (says louder): count...

Teacher: Do that please.

(Jimmy starts counting.)

Teacher: Louder so they can hear you in the back of the room.

Jimmy (starts over): One...two...three...four...five...six... seven...eight...nine...

Teacher: So you have how many in all?

Jimmy: 9.

Teacher: So we can say that 7 plus 2 equals 9. (She writes 7 + 2 = 9 as she talks.) Very nice, Jimmy. Do you want to sit down? Now let's try some in our math book.

Note that Jimmy isn't asked how he did the problem or whether he could do it another way to see if he got the same answer.

Teacher: We're going to make up a story for the first problem. We've got 3 blue ghosts for Halloween night. Do you see where I am? Put your finger on top of the page at the blue ghosts. Well, they were lonely. So friends joined them ... 5 golden ghosts: the question is, Adam, how many ghosts in all on Halloween night?

Adam: 3. (Rule 2)

Teacher: How many friends came along?

Adam: 5. (Rule 3)

Teacher: So your job is to add a set of 3 to a set of 5 (Rule 1) ... trace over the dotted 3. They already gave you the first part of the problem. Trace over the dotted 5 ... Supply how many they had in all. Adam, what number did you get for that?

Adam: 8. (Rule 1a?)

Teacher: Is he right, Arthur?

Adam: Yeah.

Teacher: I don't see your answer. There, Ruth. You need to show in your book that you know 8 is the answer. Let's go down the page.

In the spring of the second-grade year, adding one-digit numbers is not difficult for most children. What is difficult for Adam is knowing when to add. The question, "How many ghosts in all?" seems to come at the wrong time in his rule sequence. He was following Rule 2 at the time. However, the teacher gets into synchronization with him by asking the question for Rule 3 and then giving him Rule 1. (His sequence seems to be: Rule 2, Rule 3, Rule 1--possibly because Rule 1 is assumed for all these problems.).

Teacher: The next problem ... jack o'lanterns ... are we going to add or subtract, Ruth?

Ruth: Add. (Rule 1)

Teacher: Put the 2 sets together. (Rule 2 and 3) We're going to add. Go on and add it. Anyone want to use counting sticks?

(Several kids raise their hands. She passes some out to them.)

Teacher: Who can make up a problem for use with the lollipops?

Eve: 5 golden lollipops sitting in ... a man gave 2 (pause) root beer lollipops; how many in all?

Teacher: Eve, are we going to add or subtract? (Rule 1)

Eve: Add.

Teacher: Let's do it. Figure it out yourself, Jimmy, then we'll check it together. How many in the first set, Jimmy? (Rule 2)

(Jimmy mumbles.)

Here, we see that Rules 2 and 3 are subordinated to Rule 1, which explains the teacher's sequence. The teacher expects those (the majority) who can add to learn to use that skill in a new context, and Rules 2 and 3 are just reminders of what they already know. They do some more addition story problems, then go onto a page of subtraction ones.

Teacher: I'll make up the first story: 3 orange cats are living in a house and 2 of them decided to go out and sit on the fence and howl at the moon. If you had 3 cats and 2 of them got up and left, how many would still be inside, Sharon?

Sharon: 1.

Teacher: Read the sentence they gave us for the problem. Be quiet, Eve. Trace over the problem. 3 minus 2 equals 1. We're going to go across the page ... blue dogs ... got a story for us Karen?

Note that "3-2=1" is the "problem," not the "story" and not the task of making up the story. That is obviously because earlier the children had learned to call "3-2=?" a problem.

Karen: 2 dogs fighting over a bone and one of them ran away ... 2 minus 1...

Teacher: Think it out. You have 2 dogs; one of them gets the bone; how many don't get the bone?

Karen: 1.

Teacher: So problem will be what? ... read the whole thing...

Karen: 2 - 1 = 1.

Teacher: Very nice. Here's one about ducks.

* * * *

How does this scene illustrate our point? Well, what might have started out as a problem solving 2 experience for Jimmy and Adam turned into mostly routinized problem solving 1. How? Notice that the teacher wanted to give the children practice in composing story problems -- in this instance, problems related to pictorial situations already shown in their workbooks. What happens, though, is that ALL the problems shown pictorially on each page are the same. So, for the children, making up a story problem becomes making up a context for the situation "a less b". There is little room for mathematical creativity; the child only has to think of some reason for the objects being separated and then write the number sentence, called the "problem," out of deference to their habitual use of this familiar word.

We do not want to criticize this particular lesson. It meets needs that children will have in tests or textbooks so there are legitimate reasons for some practice. A legitimate concern, however, is that two of the children's mathematical problem solving opportunities were overlooked, and they may well never learn how checking their work helps them understand it. They had to synchronize their procedures with the teacher's and the class' procedure, and they lost a chance to learn to check their own work. They probably don't understand why their first answers were wrong.

Children (perhaps all of us adults, too) need a variety of experiences of both problem solving 1 and 2 to learn mathematics. Of course, teachers can't stop at every opportunity to pursue the creative type described as problem solving 2. But occasionally, doing so is necessary to avoid the conclusion that the right procedure is just a matter of following the teacher.

Turning Problem Solving 1 Into Problem Solving 2

Take our previous example of the teacher who was frustrated by being constantly interrupted by students. At first a problem solving 1 tactic is tried; the students are told to sit down and be quiet. When the first tactic doesn't work, the teacher's problem evolves into problem solving 2 and the problem becomes one of understanding the cause of the interruptions.

We have previously separated problem solving 1 and 2 in our discussion to show how different they can be, but we don't want to imply any sharp distinction between them. Sometimes when classroom events are closely observed, it is seen that situations exist that are ALMOST problem solving 2. If there were some small changes in the situation, then perhaps problem solving 1 could come closer to problem solving 2.

We want to point out that we realize that there are many difficulties with trying to find practical solutions to everyday classroom problems. We also need to point out that we can foresee some trouble with trying to change situations. One difficulty would be that a new situation is created and that new situation brings its own problems with it. So sometimes it's a trade-off of one set of problems for another. One then needs to decide which problems one would rather be faced with. For example, a teacher may read a suggestion that we have made and react: "If I do that with my class I know that I'll have trouble with John and Skip that I don't have now. On the other hand, it might solve the problems I now have with Randy and Sue."

We now want to consider some classroom events and discuss some of the most important problems that are involved with each of these events. In each case we will try to consider the event from both the teacher's and the student's perspective. (We may not consider the problem solving researcher's perspective, but we do want to bring our own perspective in.) We begin with examples from primary grade classrooms, because we are interested in determining how it happens that children so often fail to relate rules and procedures to each other. After a careful examination of typical primary classroom scenes, we will turn to special problems of the middle school years.

In these examples we want to emphasize one or more of the following points in relation to turning problem solving 1 into problem solving 2:

1. So procedures don't become arbitrary and disconnected in students' minds, some connections between them need to be shown. A way of doing this is to have children check answers to the same problem using different procedures. Sometimes answers will be in a different "form", but this needs to be talked about.

2. To increase problem solving 2 somewhat, we will make suggestions for using the student's present knowledge and extending it. Usually this can be done by a question like: If 6 + 2 is 8, what would 26 + 2 be? How does this relate to problem solving 2? Students will need to use their existing strategies and put them together in a new way to answer such questions. For 26 + 2, for example, students can think: if 6 + 2 = 8, then 26 has a 2 to the left of the 6; the answer will also have a 2 to the left of the 8; so 26 + 2 = 28. [Perhaps some would be disturbed by such a teaching strategy because mathematics is being taught "out of sequence." That is, the children are learning basic facts and haven't yet reached two-digit addition. But we would argue--along with Piaget (1968) and Lesh (1979b)--that learning doesn't actually occur in a nice orderly sequence. Sometimes we get intrigued by a complicated idea and set about "learning" things associated with that idea.]

3. Children are always looking for faster ways of doing things. Teachers could capitalize on these "shortcuts" by helping children find useful ones. This relates to problem solving 2, because old strategies are put together in new ways to find "shortcuts."

Too much specification. On one hand, we as teachers seem to feel that we must direct most classroom activity. Otherwise how would the children learn what they are supposed to learn? We all know that some children need more direction than others and some activities need to be more directed than others. Where is that subtle line then between too much direction and not enough? How do we decide where and when to draw it?

* * * *

Alice, the teacher, puts out an orange Cuisenaire rod on the table. She also places a unit rod and asks the 6 children how many of the small ones would make an orange one.

All: 10.

Alice: The orange one is called a ten stick. What would this one be called? (She puts down a blue stick.)

All: 9.

She passes out all the different colored rods, one of each color to each child.

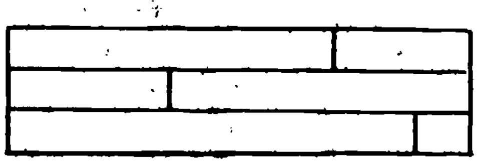
Mark: What are we going to do?

Alice: We're going to see what combinations will make 11. If you take the ten stick and the unit one, it makes?

All: 11.

Alice: I would like you to discover; here's the rules: using only two rods (Rule 1), how many ways can you make 11...using only two rods? You can use this (points to the length formed by the orange and unit rods) to measure against. (Rule 2).

Each of the children moved the sticks around to form patterns like the one shown below.



Alice writes on the board: 10 + 1 = 11 (Rule a)
9 + 2 = 11 (Rule b)
8 + 3 = 11 (Rule c)

Sally: Oh, I see...you can go 10... 9... 8... 7... 6... 5... 4... 3... 2... 1. (Rule 3)

Alice: That's what I'm going to do (She continues writing on the board.)...notice how these get smaller (points to the first addends) and these get bigger (points to the second addends). (Rule 4)

This activity continues and the teacher has the children complete some ditto sheets with combinations of numbers for 12 and 13.

* * * *

The problem, as stated by Alice (how many ways can you make 11?), can be interpreted as being overly directive because of (1) the limitation of the number of rods per child, and (2) the sequential procedure demonstrated. Getting the right answer is almost guaranteed for every child who is paying attention. (Perhaps some children aren't.) All the children have to do is to work until all the rods are gone or follow the procedure she has illustrated, combining the descending sequence starting from 10 with the ascending one starting from 1. So the event becomes a problem solving 1 task. However, the procedure for getting the pattern for the combinations of two numbers adding to 11 probably isn't as important as knowing that 9 + 2 = 11 (or any of the other number facts). However, the pattern in the resulting sequence of sums is even more important still.

What is unclear in this task is which facts are well-known to the children and which are being figured out somehow from previously known facts. For example, it might be that 5 + 6 = 11 is derived by means of 5 + 1 = 6 and 10 + 1 = 11. If things like this are happening, we might want to regard it as problem solving 2, since three rules (facts) are being combined in new ways.

To make the situation closer to problem solving 4, some writers might recommend that the teacher remove herself from the situation and let the children decide what problem was going to be solved and what questions were going to be answered. It seems doubtful that Alice would find this a useful suggestion.

For example, in this scene, Alice might know that these particular children need to succeed in following directions--something that is for them more important than insight into arithmetic procedures and relations. Whereas we can imagine that, with children whom she knows could handle a less directed (more ambiguous) situation, she would have organized the activity differently.

So at first glance, "stepping aside" may sound like an easy way to improve the situation. But it depends on the significance this situation has in the development of understandings and working relations in the class. And only the teacher involved can decide if it is appropriate to step aside.

Some other suggestions might be easier to follow. Alice could have withheld showing them the procedure, 10+1, 9+2, 8+3, etc. Following this suggestion might make Alice anxious that some children would get the wrong answer. But the answer to how many combinations there are is not as important as knowing, for example, that 10 + 1 = 11. She might say that the pattern of an ordered set of combinations was what she wanted them to see, not the answer to the question she has asked.

This seems to be a good place to extend the children's thinking. So if the pattern

10 + 1 = 11
9 + 2 = 11
8 + 3 = 11
7 + 4 = 11
etc.

is shown, then some numbers can be changed to show the pattern

20 + 1 = 21	or	10 + 21 = 31
19 + 2 = 21		9 + 22 = 31
18 + 3 = 21		8 + 23 = 31

or any other variation.

The next page was developed to give children (who presumably were well acquainted with facts like 3 + 2 = 5 and 5 - 3 = 2) an opportunity to extend these facts to new contexts for which procedures had never been given. It was tried with students in two first-grade classes who had mastered the eight cases in the boxes, but had only been introduced to missing addends in binary cases (on the left). The children were instructed to skip problems they couldn't do. After they reported they had done all they could, each child was given a colored pencil and was asked to go back and try the ones skipped.

The results indicated that most of the children (all but the slowest) were able to get some right that they had skipped. In other words, children were able to use the number facts 2 + 3 = 5, etc. The most difficult cases were the multiple addends in the upper right-hand part of the page, suggesting an inability to use a sum as an addend. Two-and three-digit sums and differences that were correct do not mean the <u>quantities</u> involved are always comprehended in a place value sense. However, some of the exercises might have been checked by other means -- using counters, for example.

Name ______________________

2 + 3 = □

3 + 2 = □

2 + □ = 5

3 + □ = 5

□ + 2 = 5

□ + 3 = 5

5 - 3 = □

5 - 2 = □

5 - □ = 2

5 - □ = 3

□ - 2 = 3

□ - 3 = 2

1 + 1 + 3 = □

1 + 2 + 2 = □

1 + 3 + □ = 5

1 + 2 + 3 = □

3 + 2 + 1 = □

1 + 1 + 1 + 1 + 1 = □

3 + 2 + □ = 5

3 + 2 + □ = 6

31	30	13
+20	+21	+ 2

12	51	51
+ 3	-20	-30

3	2	5	5
+2	+3	-2	-3

15	15	50
- 2	- 3	-50

30	20	50	50	32	22	222	233
-20	+30	-20	-30	+23	+33	+333	+322

232
+323

Another suggestion relates to "shortcuts". Using the pattern (Rule 4) to get the combinations for 11 is a shortcut. Another one is discussed next.

Discussing equivalent methods. In the following scene the teacher has a specific motive in mind, to try to show the children the "counting on" shortcut for doing addition problems.

* * * *

Bruce, the teacher, is working with 4 children.

Bruce: I'm going to show you a card and we will start with the number on top and use taps on the number line to get the answer. 1 + 1?

Marie: 2; that was easy. I didn't have to use the number line.

Bruce: Oh, they're going to get harder. (He takes out the "plus three" flash cards.) Marie? (He shows her 6 + 3 --- all problems on the cards are written vertically.)

(Marie is silent.)

Bruce: It's 3 taps after 6. (He points to 6 on the number line and with a pencil taps out 3 more, pointing to 7, 8, 9 as he goes.)

Marie: 9.

Bruce: Good, now Tommy? (Bruce shows him 1 + 3.) I'm going to tap out 3 times; 2... 3... 4.

Tommy: It ain't 3 taps...1... 2... 3... 4.

Bruce: You start with 1. Don't tap for 1, 'cause you started with 1, so you only tap 3 times.

Bruce goes around the table showing one flash card at a time to each child. When a child takes more than a second or two to answer, he shows them how to do the problem by tapping out the second addend.

* * * *

The method of "counting on" that Bruce is promoting is a very worthwhile procedure to use in adding. The problem in this lesson is that Bruce is not using the children's prior knowledge to develop new knowledge.

How could this lesson be changed to be closer to problem solving 2? One way would have been for Bruce to open the discussion by asking the children how they found the answer to each problem. If Tommy said that, to get 7 + 3, for example, he would count 1... 2... 3... 4... 5... 6... 7... 8... 9... 10, then Bruce could have presented the challenge: Find a quicker way to get the same answer other than counting both the 7 and the 3!

This kind of lesson certainly might take more time, but Bruce (if he needed to) could have placed a time limit on finding a quicker strategy. That could have solved his problem. Then the children might even have appreciated being given a solution because they would have worked on the problem long enough to want an answer.

As it is, Tommy seems to be doing the problems his own way. For Tommy, the problem is to persuade Bruce that he is wrong. Tommy, perhaps, is perfectly content to use his own strategy, as the other children are also, for the teacher hasn't said, "Let's find a shortcut."

Instead, the teacher feels compelled to teach a quicker method, because answers aren't given immediately. The intent is to help the child respond in a way that's closer to immediate recall than counting out the two addends and the sum as well. But it could be that there is a systematic way to build the "facts," and that teaching a new, faster algorithm is inefficient in terms of the time taken to teach it. That is, it would be systematic to use the facts the child <u>now</u> knows to build other ones. For example, if the child knows some doubles like 2 + 2 or 3 + 3, those facts can be used to do 2 + 3 = 5 (one more than 2 + 2 = 4) or 3 + 5 = 8 (two more than 3 + 3 = 6). An example of the teacher using this idea is on page 40. Perhaps what is more important is that <u>different methods</u> can be used to check answers for the same problem.

It seems that the emphasis in this scene is on getting an answer quickly, which IS important in learning basic facts. If the child is able to do so (by any method), then he or she is left alone--the objective has been met. But for children who do not produce the correct answer immediately, there is no attention to the rule that <u>they</u> are using. To give this attention, the teacher might have said, "No, even though both methods give the same answer, I don't want to tap how many there are altogether--just the ones to be added on." Or he might have combined the drill and practice with teaching some specific strategy by choosing a mixture of easy problems and hard ones for which a quick answer requires a shortcut. As we noted before, Bruce's strategy fails for one or more children. Those children may be forming some "interesting" hypotheses (as to what the new procedure is) and testing them: in short, they may be really doing problem solving 2.

The idea of discussing different strategies with pupils is related to the problem solving 2 process of testing different procedures. It is important to develop explicitly an understanding of which procedures are equivalent. Otherwise, if children depend on teacher feedback to know when they are right, they can very easily get the idea that their own methods for solving problems aren't important or trustworthy. They get the idea--implicitly but sometimes very explicitly--that only using the strategies and methods that the teacher <u>tells</u> you to is what is important. We don't want to imply that it <u>isn't</u> important. Rather, the important idea is to have <u>many</u> methods "up one's sleeve"--in a practical form, ready to use, ready to question if they don't fit, and ready to recognize ones that give equivalent answers.

In this next scene, some different methods for addition basic facts are used. Here the teacher tries some physical activities to get at the "counting on" strategy.

* * * *

Robert, the teacher, has the five children set out (on the rug) some numbered plastic cards in a row. There is an ordered sequence containing each number from 0 through 10. He then has the children sit on the floor. One by one he has them stand up and do a problem like 3 + 4 by stepping on the numbers. But there is a well-defined procedure for doing the problems.

Robert: OK..Randy, I want you to do 5 + 2. Get up and stand on the 5. (<u>Rule 1</u>) Then walk two steps (<u>Rule 2</u>) and tell us what number it is. (<u>Rule 3</u>)

(Randy starts at 5 and then goes to 8.)

Robert: You went three steps, not two, Randy.

Alan: He <u>kind</u> of went two, though. (He points to the two cards--6 and 7--that are in-between the 5 and 8.)

Robert: Oh, I see what you're saying. No, that isn't really going two steps. You count from where you started (<u>Rule 4</u>), like 5 then two more 6...7... So 7 is the answer. (<u>Rule 5</u>)

Note that here the teacher accepts Alan's interpretation of what Randy did and gives a fuller explanation of the procedure.

Robert: Robin, I want you to do 7 + 2. (She gets up and walks.)

Robert: OK, Robin, so what's 7 + 2?

Robin: 9.

Robert does some more of these with the five children. Then he says:

Robert: I'm going to give you your computer number line that you made last week and your "adding two" flash cards that you made. I want you to get in pairs and practice these problems and use your computer number line if you need to.

Ann and Robin work together. Ann uses her computer number line with a sliding window on it.

0	1	2	3	4	5	6	7	8	9	10

Robin, however, does her problems by adding on her fingers. Alan and Randy work together. Randy is still having trouble.

Randy: Is 6 + 2 equal to 9?

Robert: No...I see that you're doing it by using your fingers. Also don't count up to 6...start at 6 and then add on 2. Where's your calculator (meaning the computer number line)? Why aren't you using that?

* * * *

Everyone has to follow the "counting on" procedure in the physical activity. The first step is to place oneself physically on the first number in the addition problem. Even if one wanted to count from 1, it is prohibited by the situation. But why is no one using the old algorithm as a check on the new one? The use of the flash cards and the computer number line provides another time for the teacher to say explicitly that one kind of method (adding on your fingers) gives the same answer as another (the computer). The following scene again involves children having trouble with immediate recall of addition facts. The teacher now tries to broaden the student's repertoire of strategies.

* * * *

Robert, the teacher, is sitting at the table with four children. They are practicing the "plus 3's" with flash cards. Robert goes around the table and checks if a child can recite the answers quickly. If so, that child gets a check mark meaning that he or she can proceed to the "plus 4's." Mark has the wrong answer (11) for 9 + 3. Robert reminds him of the strategy they talked about before.

Robert: Remember the "secret of the 9's?" When you add to 9, you subtract 1 from the number you are adding and the answer is in the teens. So for 9 + 3, subtract 1 from 3 and get 2, so answer is?

Mark: 12?

Robert: Good.

Why would Robert suggest another rule without finding out which rule Mark was using? Supposedly this rule helps. But how is the old knowledge used in new situations?

Robert gives some of the other children the "plus 3's" and then comes back and gives them to Mark again. This time Mark does better. He then gives them to Tim, who has trouble, so now Robert tries another teaching strategy.

Robert: 4 + 3?

(Tim is silent.)

Robert: What was 3 + 3?

Tim: 6.

Robert: So what is 4 + 3?

Tim: 7.

Robert: Good--see how you can use 3 plus 3 and the idea of one more than?

* * * *

The teacher is trying to show how existing knowledge can be extended to get new knowledge. This certainly is a lot closer to problem solving 2 because the student's "bag of tricks" is increased to the point where there are too many overlapping rules to be able to make automatic decisions. So it becomes necessary for the child to focus on choosing the appropriate rule.

On the other hand, there still is some lost opportunity here for discussing what the children's strategies were and for making connections between strategies. Especially relevant is the opportunity, in all of the previous scenes, to discuss equivalent methods. That is, use of taps on the number line is similar to walking on the cards in that "counting on" is the shortcut. In fact, these brief scenes all show relatedness of several different procedures to get addition basic fact answers. We think it's important to point out explicitly this relatedness to the children and not to depend on their seeing the relatedness themselves. However, in the next section, the children do see a shortcut for themselves. But they also need to have the relationship clarified.

* * * *

A place value shortcut. The children are working on a workbook page that has exercises like the following:

☐ Tens (blue) + ☐ Ones (green)

Rose, the teacher, explains how to do the exercises.

Rose: Look across this way...Chris, [how many] bundles of 10 in the first picture?

Chris: 2.

Rose: Bundles!!

Chris: 1.

Rose: Sticks all by self, not in a bundle?

Chris: 2.

Rose: So put the number of bundles in the blue box (Rule 1) and the number of sticks by themselves in the green box. (Rule 2). Now you do the page by yourselves.

The children are working.

Chris (who has completed pictures of 11, 12, 13, and 14 sticks): Look; it goes 1...2...3...4...5...

Al: Yeah, all we have to do is fill in 1...2...3...4... 5... 6...7...8...9.

Chris and Al complete the page very quickly by filling in according to the pattern. Peter hasn't noticed the pattern yet, and is still working by counting the whole thing instead of just the ones. Chris and Al turn the page and do the next several ones like they are supposed to. Rose comes back and checks their answers. She tells them they did well and they can go play a game while the other children finish their work.

* * * *

The shortcut that Chris and Al found is quite important. Their strategy was based on the pattern in the format of the page. Whether or not this was what the authors of the workbook intended, the page could easily be completed as these boys did. Again, some open discussion of this strategy--a kind of sharing time with the class--would have called attention to the idea that thinking about strategies helps in being able to have a lot of them to choose from, or to be able to put new ones together in some way.

We realize, however, that opening up an explicit discussion of shortcuts may create many problems for the teacher. Other than the obvious problem of time constraints, there is another issue involved here. What if some children have strategies to discuss and others can't verbalize their own strategies? This can be both bad and good, of course. The ones who can't verbalize their strategies would at least learn something from the other children. And perhaps they would also learn that they were not alone in having few strategies or in not being able to state them.

Also, whenever you open up any topic for student discussion you are allowing opportunities for disagreement and challenge. This means that teachers have to decide for themselves whether to discuss something or not. One has to take into consideration what the topic is, who the students are in the class, and when the discussion is to be held.

Planning time for problem solving 2. We all treasure those moments when children get excited about the mathematics they're doing. If only they would occur more often! But when they do occur, time may have run out.

* * * *

Carol, the teacher, is using bundles of toothpicks to introduce place value. She puts out 10 bundles (of ten toothpicks each) and 10 single sticks.

Carol: How many do we have?

Mike: That's easy, 110.

Jeff: Right.

Carol: Can you put more than 10 bundles in the tens place? (Rule 1)

Al: We need a hundreds place.

Carol: Let's get a big rubber band. You said "one hundred", so we get one bundle. (She uses a thick rubber band to bundle the 10 bundles into a big one.) (Rule 2) So what's the number--just by looking? (She puts down the papers with the place value names as shown.)

hundreds	tens	ones

Mike: 100 and...10.

Carol: Dave? (Now, she puts out 1 bundle of 100 plus two 10 bundles and twenty single sticks in the ones place.)

(Dave counts the 140.)

Carol: Do you need to do anything?

Jeff: Yes.

Dave: You need to always start with the ones. (Rule 3)

Carol: Good, why?

Jeff: 'Cause you can change them to tens. (Carol continues to set out bundles and sticks for the children to figure out.)

(Dave is busy bundling sticks from the ones place into bundles of 10 each.)

Dave: 6 of these in the ones place.

Carol: In the tens?

Dave: 156.

Carol: Correct, wonderful.

She leaves the table to help a child in another part of the room. The boys are busily taking toothpicks and forming bundles of 10 and 100. She comes back.

Jeff and Dave: This is really fun. Let's do some more.

Jeff: I'm going to give you one.

Carol: Me!

Jeff: Let's take them all out! (He is pointing to the box containing about a thousand toothpicks.)

Carol (laughs): That would take all day! We do have time for one more.

Jeff and Dave: Ohhh... (sound disappointed).

She gives them one more problem.

* * * *

Jeff and Dave especially have gotten involved in this activity. Their excitement, however, gets squelched, at least for the moment. Time limits!

"We can't spend all day on this. Hurry up now, we have to move on to something else. We have to work in our books now." How often we all find ourselves saying things like this to our classes? We react like this because we as teachers are constantly aware (even if unconsciously), of the fact that there is only a certain amount of time in which to do so many topics. How does this relate to problem solving 1 and 2? Let's stop and think about it for a minute.

Remember that we discussed the idea that at times problem solving 2 involves extra time--time to define a problem, redefine it, think of strategies, and so on. Perhaps in this situation, if Jeff and Dave had some extra time, they would think of "skip counting" as a short cut to counting a thousand toothpicks one by one. (This could also be an opportunity to extend their knowledge of place value into the thousands.) Applying skip counting involves some new techniques and can easily lead to conflicting answers. But children can work out such conflicts themselves, inventing other ways to check which answer is right (Stake, 1980).

Place value concepts are notoriously difficult, and teachers could learn how to help by watching children struggle. So these are some reasons for giving them the extra time. But is this so easy? Perhaps the bell is going to ring and Jeff and Dave will need to leave class and go to gym. Or perhaps it is going to be lunch time in a minute or two. In such cases, giving them extra time at the moment would be impractical. Realistically, what could have been done?

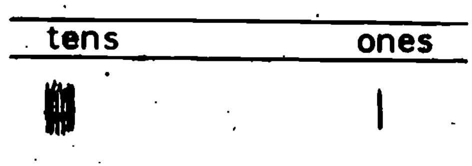
One strategy would have been the following. Carol could have said, "I like to see you so excited about what we're doing. I wish we had some extra time right now so that you could explore some of your ideas further. But we don't. Can you think of some time when you would be able to work on that? Perhaps we can arrange things so that you can have that time." In doing so, Carol would be acknowledging that there might be a serious reason for doing what they want, and is willing to accommodate them if possible. She also would not be making the decision for them, but rather placing it in their hands to think of ways to getting this extra time. This could be a solution to a social problem.

The next examples also involve place value and its relation to counting. Here, we will see how an extended definition of counting is needed.

Place value and counting again. What happens when children learn about place value and its application to money? A natural procedure for them to use is to count. However, as shown in the following scene, counting (as usually done by some children) can lead to trouble with place value.

* * * *

The teacher, Jim, has been using toothpicks to get at ideas of place value and why we write our numbers (past 9) as we do. For example, he has shown that 11 is represented as being 1 bundle (of ten) and 1 stick. He has the children practice by representing a number with the toothpicks on a card like



He then tells one of the children, Alan, to put down some toothpicks and have the others figure out the number. Alan puts down 1 bundle under the "tens" and a large bunch of single toothpicks under the "ones".

Jim: How many ones over here?

(All count aloud with Jim until they reach 25.)

Jim: Alan, put 25 sticks here. Look how we write this. What is the number?

All: 25.

Jim: The number all together?

Alan: 125.

Jim: Do we have 125 sticks here? No, we have 1 bundle and 25 here. We can only put 9 here. Every time we get 10 sticks what do we do?

Paul: Bundle it. [sic]

Alan: But it is 125. (He points to the 1 and says, "1 bundle" and then to the 25 and says, "25 sticks".)

Jim: I can see how you thought that; if we took the 1 away we would get 25 sticks.

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This is a good effort to interpret a child's error. However, here would be an ideal situation for asking Alan if he could find a way to check his own answer.

We can also see here how the child's natural counting has gotten him into trouble, momentarily at least. One problem is that Alan is correct in saying that there are 25 sticks and 1 bundle. (But he doesn't know yet that it's risky to mix his counting with place value until you have the idea of equivalence of ten ones and a bundle of tens.) He has the idea of bundling. But, his problem may be in thinking that it's like money, where dollars and cents can be seen as two integers rather than different place values. So, the definition of counting needs expansion. It's not that you don't count when doing place value, but that you count in a certain way and in combination with the decimal system for symbolizing large numbers. This is illustrated in the next scene.

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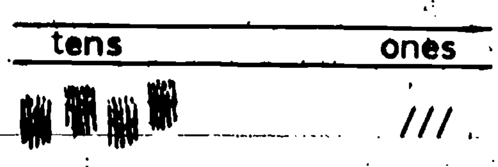
Jim: I want Chuck to make this number now. (He writes 43 on the board.)

Chuck puts 4 bundles under the "tens" card as:



Jim: Did he make 43?

All: 10... 20... 30... 40... 41... 42... 43.

Jim: O.K., Alan, make this number. (He writes 67 on the board.)

(Alan puts out the sticks.)

Jim: Let's see if he's right; are there 6 bundles? 10... 20... 30... 40... 50... 60...; are there 7 ones?

Chuck: Could go 67 or 6... 7.

Jim: That's the way we make 67; put a 6 here and a 7 here. (Jim puts down 13 extra toothpicks in the ones place, leaving the others that are already there.) Is that 67 now?

All: No.

Jim: So we need bundles. I'm going to give Ruth a big number. (He writes 92 on the board.)

(Ruth puts out the 9 bundles and 2 sticks.)

Jim: Let's see if she's right...10... 20... 30... 40... 50... 60... 70... 80... 90... or you can go...1... 2... 3... 4... 5... 6... 7... 8... 9.

* * * *

For 92 there are 90 sticks in the tens place, but the child needs to learn to write 9 not 90! The child also needs to extend the concept of counting to see that you can say 1, 2, 3, etc. (when counting tens), and think 10, 20, 30, etc. This procedure is not as obvious to children as it is to us adults! (Recall Alan's mistake of saying 125.) So these methods of counting must be explicitly made clear to show that they are equivalent and not arbitrary new rules for new "work sheets."

One problem is that, until they have more experiences with larger numbers, children are used to recording the numeral representing what they have counted. So if there were 16 objects, they wrote 16---no trouble. They knew how to "spell" that one without using place value. Now they have to learn the procedure of writing down only a certain part of the numeral <u>and</u> it has to be in a certain place in relation to other numerals written down. Mathematics is starting to get complex! What happens when they get to money? Read on.

* * * *

Jim: Instead of bundles we're now going to use...

Ruth: A dime.

Jim: How many pennies make a dime?

Ruth: 10.

Jim: So, it can be a bundle; a dime is the same as a bundle.

Ruth: So we can count money, too.

Jim: Don't count...tell me what to put in the one's place... How many dimes?

Alan: 10.

Jim: How many dimes? or bundles of pennies?

Alan: 10.

Jim: Do you see 10 dimes there?

Alan: 10 pennies.

Jim: How many dimes?

Alan: 1.

Jim: There's a way of doing money without counting... one dime (He writes a 1 on the board.); pennies in the one's place?

Chuck: 5.

Jim: Did you have to go 10... 11... 12... 13... 14... 15?

All: No.

Jim: Just put a 1 in the ten's place and a 5 in the one's place.

Alan: If a dime is there, put a 10?.

Jim: Just put a 1 'cause it's the same as 1 bundle.

* * * *

Jim does refer to the "counting on" algorithm when he asks, "Did you have to go 10, 11, 12, 13, 14, 15?" And then he says, "Just put a 1 in the ten's place and a 5 in the one's place." Presumably this is a third algorithm because the first two are counting the whole and place value composition. Notice how Jim tries to steer the children away from counting as they now know it. It might have been better to relate their counting method to this new way. Then the children could have seen the equivalence.

But how are they to know when to use the new method and when not to use it? Equivalent procedures can't be established if the children don't check them out in many cases (by seeing how you get the same answers).

It is important to note how the teacher is extending the children's knowledge and making corrections by relating place value with sticks to place value with money. In fact, it is explicitly pointed out to the children that bundles of ten are just like dimes in money.

Now that we've considered situations from the primary school, because they are mathematically simpler to discuss, we'd like to consider what happens in the middle grades (around 5th and 6th grades), when it seems that mathematics gets more and more difficult for many children.

What Happens When Mathematics Gets More Complex?

Is it the mathematics itself that has become more complex? Is it the way that it's taught that makes it seem complex? Is it the social structure of the classroom that is affecting the complexity?

It seems to us that many of the problems that teachers and students face in the primary classroom are similar to the ones faced in the middle grade classroom. Perhaps, however, some of these problems are intensified. That is, the range of abilities of the children may have gotten wider because some children have managed to "slide along" to 5th or 6th grade, dropping further and further behind each year. And at the other end, some children have progressed in their mathematical abilities more and more each year. Thus, for the teacher trying to plan activities, the problem of individual differences is intensified.

Let's first look at some children at this grade level and the types of problems they have in coping with the mathematical concepts being taught. The following examples, which explicitly show how children come to view mathematics as an arbitrary set of procedures, were taken from Erlwanger's study (1974) of children's mathematical thought. Then we will consider the issues involved with answering the questions: Has the mathematics become more complex? What is the middle-grade teacher to do with children who have these views?

Erlwanger studied six children whom he classified as ranging from dependent to independent. The figure on page 50 shows the children on this continuum.

Dependence (formal)					Independence (intuitive)
Mat(S)	Nel(S)	Rod(U)	Benny(S)	Lyn(S)	Tina(U)
121	99	113	115	123	128

In explaining his placement of these children, he says, "In the figure, S and U indicates successful and unsuccessful children in IPI (Individually Prescribed Instruction). The numbers indicate their IQ scores. The direction of the arrow indicates roughly a decrease in the children's dependence upon the program, and a corresponding increase in their attempts to think intuitively" (p. 31).

Benny's view of adding decimals. Let's look at a couple of Benny's ideas. (Underlining in the original is deleted here.)

* * * *

E: O.K. Now let's try addition. Suppose I had .3 + .4?

B: .07.

E: Now how do you decide that you should have .07?

B: Because you use two decimals and there is one number behind each decimal. So in your answer you have to have two numbers behind the decimal; and you just add them. (p. 57)

* * * *

The problem illustrated in that excerpt is probably familiar to all of us. Children have been taught that, for multiplication of decimals, the short cut in placing the deminal point in the answer is to count up the number of places in each number you're multiplying. Benny, like so many children, seems to have confused this rule with addition. But Benny's problem is not that simple, as shown in this excerpt.

* * * *

E: Your answer here (i.e., .3 + .4) is .07 and here (i.e., $\frac{3}{10} + \frac{4}{10}$) is .7.

B: Right,

E: You think that's right?

B: Because there ain't no decimals here (i.e., $\frac{3}{10} + \frac{4}{10}$).

E: You are not using decimals. But you are using decimals up here (i.e., .3 + .4); and that makes the difference. (p. 57)

* * * *

Benny knows the equivalences between decimals and common fractions. But to Benny, you can have different answers to mathematically equivalent problems.

Mat's ideas about adding fractions. Erlwanger first describes how Mat solves $\frac{3}{4} + \frac{1}{4}$ using two different methods. In one, Mat adds numerators and then denominators and gets $\frac{4}{8}$ or $\frac{1}{2}$. In the other, he draws some circles and shades them to arrive at $\frac{3}{4} + \frac{1}{4}$ equals 1. Erlwanger investigates how Mat can arrive at two different answers for the same problem (p. 78).

E. But when you did it the other way...you had $\frac{1}{2}$...how come?

M: I don't know! That's the way it is.

E: What method did you use when you got $\frac{4}{8}$?

M: I...uh...added the numerators and then the denominators... and...that's the one they taught me...I think it was E fractions.

E: How does this really work then ... that ... here you are doing it one way ...

M: [interrupting] WELL .... it's just like these [referring to examples on ordering fractions, in which he had obtained different answers] ... you get a different answer every method you use.

E: And then how do you decide which answer is right?

M: It depends on which method you are told to use ... [and] ... Well, see, sometimes they tell you what method to use. Like the ladder method in division, sometimes. And you use that method and you come out with the answer. And that's what answer is in the key, so that's what answer they use.

* * * *

So for Mat also, you can have different answers to the same mathematical problems -- depending on what method you use. Now the issue also becomes involved with the "ultimate resource of mathematical knowledge -- the answer key!" And since the answer key usually gives only one answer per problem, children never know if or when they have gotten an answer equivalent to the one given in the key. Because it is rarely explicitly discussed, they assume then that the answer in the key is the ONLY answer. Hence, since the key sometimes gives answers in different forms, children like Mat generalize that you can get different answers depending on different methods used. They may not think of the possibility that one method might yield different but equivalent answers.

Tina's views of multiplication and division. In this first excerpt, it seems that Tina has done many workbook pages where she has seen a " x " sign. Perhaps some of these pages were just multiplication exercises. But some of them had other purposes, such as having the children notice the distributive property. These other purposes, though, are not clear to children.

* * * *

E: What about multiplication?

T: It's O.K. But they teach too many different ways. They have this one way ... just the regular way ... like 8 x 8 = 64. But then ... they'll take it and say ... 64 equals 60 times blank ... and then 4 times blank. Then down here ... 60 times something ... goes in this blank .... goes in this blank, and 4 times something goes in this blank, and then you add up and get the answer. Why can't you just do it this way [8 x 8 = 64] ... the regular way. (p. 106)

* * * *

To Tina the purpose of filling in all those blanks is nonsense. She doesn't see how that relates to multiplication as she knows it.

She has interpreted it only as another way of doing multiplication!

Now consider how Tina thinks about dividing by 1/2.

* * * *

Tina was given 8 ÷ 1/2.

T: Divided by a half of 8 ... I think that would be 4.

E: What about 8 ÷ 2?

T: Divided by 2? ... is 4 ... Oh! A half of everything ... like ... you take a circle or square or something ... and you divide it in half ... there's two halves ... so that's 2 ... a half is always two of something ... so you multiply it by 2! (p. 120)

* * * *

What is the middle grade teacher to do?. We have given the above examples to show some middle grade children's conceptions of some of the usual mathematics done in classrooms at this age. What has happened to children as they've moved from primary to middle grades? Is it the nature of the mathematics that has overwhelmed them?

Reading these excerpts from an adult's point of view is difficult. We want to jump in immediately and try to "straighten out" these children. But it isn't that simple. These views of mathematics have grown and generalized since the primary grades. The child's view is that there are too many different and unrelated procedures to learn. When you try to learn more and more procedures, you eventually get some of them mixed up with each other. Since we don't yet have any longitudinal data, we can only speculate that children like those in Erlwanger's study have consistently done mathematical activities as separate and fragmented from each other. If connections between procedures were ever made, it was done by the child, and those connections were never made explicit. So the child could be inferring things similar to Benny, Mat, or Tina and no one finds out until it's too late. What's interesting is that Benny was doing very well in mathematics (in tests and grades) and his misconceptions would have gone undetected if not for Erlwanger's studying him!

Idealistically one would say that teachers should keep track of what each child knows and doesn't know. But realistically teachers know that this ideal is hard to meet. Small steps can be taken, perhaps. As we have noted, teaching children how to check their own answers by using different methods is one way.

Then, if answers don't check, the children can either receive help or can be encouraged to solve the problem of why they don't check.

Middle grade teachers, like primary grade ones, need to explicitly relate different methods and lessons to each other. (Note: Some could say that Erlwanger's studies were "unusual" in the sense that the classes were using IPI materials. But when we read these examples, we can think of similar children we've seen in regular classes who have these same conceptions of mathematics.)

Conclusions

Recent concerns over declining problem solving skills, while computation scores remain good, have led some to say that there's too much emphasis on "back to basics." But, from our examination of numerous elementary school mathematics classes, we feel that it's not that simple. We can't tell teachers to forget computational skill emphasis when they are being pressured by societal mandates to emphasize those skills. It is important to be able to do both what we've called problem solving 1 and problem solving 2. Our best students are the ones who know intuitively when it's appropriate to use which skills, and can integrate problem solving 1 techniques with problem solving 2 techniques. We need explicitly to help all students learn this appropriateness. And this may involve our explicitly giving instruction in what mathematically bright students learn on their own, for example, finding shortcuts, connecting procedures and methods, using patterns to extend knowledge, and seeing the relatedness that makes mathematics far from an arbitrary collection of rules.

Whether teachers use activities, individual instruction, entire class instruction, or whatever, is not that important. We have shown that increasing problem solving 2 doesn't necessarily depend on the method being used. What appears to some as inherently problem solving 1 may really be problem solving 2 already. So what seems to be an activity can actually be a rule-oriented lesson or what seems to be rote, algorithm learning can actually be, in some respect, creative problem solving.

We have also illustrated both social and mathematical examples of problem solving 1 and 2. Our classroom observations have convinced us that the connection between the two is strong, but we're not sure exactly what it is. Perhaps the following example will help us see this connection more clearly. We are thinking of a college student who was stunned to discover that to answer all the questions on a test, and thus have a chance at an A, he could not possibly follow the directions that said "compute" or follow other complex procedures specified. The "A" students followed another strategy: First read all of your possible choices and then work on trying to eliminate some of those choices. Then if--and only if--you have to, solve the original problem. Strategies like this are not usually discussed with students; they are left on their own to discover them.

With this student, the social problem solving (of always following directions) led to mathematical problem solving difficulties. He couldn't be successful in solving mathematical problems because he had never seen shortcuts.

We have emphasized the strategy of teaching children to check answers (using different procedures) as one way of combating the view that mathematics is a set of arbitrary rules. It seems to us that there are many good ways of combating this view, including using calculators to check answers (Morris, 1978). The problem is to cultivate the habit of looking for such ways.

* * * *

In closing, we would like to share some of the thoughts and problems we had to solve during the writing of this paper. We began with the idea of presenting some true-to-life classroom scenes -- ones in which we have actually participated. We thought we would then show how some improvements could be made that would bring more of the creative type of problem solving into everyday classroom activity. Sounds simple? At least in the beginning it did. Then we started talking with each other about these scenes and about what changes COULD realistically be made. Even the slightest change, from OUR perspective, became a major change when we thought about it from the teacher's perspective.

We like the following statement from Philip Jackson (1977), which is in reference to knowledge acquisition in the classroom being viewed as both disarmingly simple and frightfully complex at the same time. Even though it refers to classrooms in general, we feel it can just as well apply to mathematical problem solving.

> What is disarmingly simple, I suspect, is the advice we have to give teachers on the basis of what we have learned to date about the teaching and learning process. What is frightfully complex is our emerging understanding, in a theoretical or scientific sense, of how and why things work as they do. The simple and complex, therefore, are attached to quite different domains of intellectual concerns. As researchers, who are often tempted to slip into the role of advice givers, we would do well to keep the distinction in mind. (p. 402)

Perhaps in this paper we tried to give some advice and some discussion of understanding the situation. Whether or not doing both was appropriate, we leave to the reader to decide. We do hope the reader feels that we have been somewhat helpful in both aspects. We would appreciate hearing from teacher readers and receiving comments on this paper. We would also like to promote more discussion between practitoners and researchers. In doing so, perhaps we can come closer to helping teachers in solving the multitude of classroom problems.

The reader may finish and be disturbed about seeing in print the realistic scenes and the difficulties of making any substantial change within them. Why? Seeing classroom events written down makes it possible to think hard about them. And seeing the realities of classroom life may leave us feeling the futility of the classroom situation today. We try to help children, but see that, as much as we try, children still need more help! We see that our ideals get compromised when we are in the classroom. This is disturbing because we don't like to think that we aren't living up to our ideals.

But we can view these disturbing feelings as being positive recognitions of a problem. Once "reality" is out in the open, we can face it and try to do something about it.

A problem, however, is that when teachers want to improve things, they are usually handed seemingly "simple" advice. We hope that our advice isn't seen as simple, but as calling for the detailed reworking of instrumental practice in such a way that every little step makes sense and encourages another little step.

REFERENCES AND RECOMMENDED READINGS

(Those references marked by an asterisk are annotated as to their relevance to this paper. They are ones we recommend to elementary school teachers.)

*Arithmetic Teacher 25 (2, November), 1977.

This entire issue is devoted to the subject of problem solving. Articles range from the theoretical (a look at some of the psychological and brain-hemisphere research) to the practical (activity ideas). Of special interest is the Robinson article which deals with different types of problems (related to our problem solving 2) -- e.g., some which require reflective thinking or some which explore the context of the situation.

*Bell, M. Applied problem solving as a school emphasis: an assessment and some recommendations. Unpublished paper, January, 1978. (Available from Max Bell, Graduate School of Education, University of Chicago, Chicago, Illinois).

The author makes the argument that we cannot expect students to apply what they've learned if they haven't had any practice in doing so in the classroom.

Brownell, W. A. Problem solving. In The psychology of learning, Forty-first Yearbook of the National Society for the Study of Education, part II, 1942.

(Cambridge Conference). Goals for school mathematics, the report of the conference on school mathematics (Cambridge, Massachusetts, 1963). Watertown, Massachusetts: Educational Services, Inc., 1963.

Davis, R. & McKnight, C. Modeling the processes of mathematical thinking. Journal of Children's Mathematical Behavior 2 (2, Spring), 1979.

*Dewey, J. How we think. Boston: D.C. Heath & Co., 1933.

Dewey's theory that thinking begins when habitual behavior is interrupted by unanticipated results or other causes of confusion has been very influential in the whole field of problem solving research. A highly readable book!

Duncker, K. On problem-solving. Psychological Monographs 58 (5, Whole No. 270), 1945.

*Duckworth, E. Either we're too early and they can't learn it or we're too late and they know it already: the dilemma of "applying Piaget." Harvard Educational Review 49 (3, August): 297-312, 1979.

Besides discussing the dilemma mentioned in the title, the author tells about some current Genevan research. An interesting example is given about a child's activity that relates to the trial-and-error part of what we've called problem solving 2.

*Easley, J. A., Jr. & Zwoyer, R. E. Teaching by listening--Toward a new day in math classes. Contemporary Education 47: 19-25, 1975.

Two mathematics educators discuss the difficult art of really listening to the strange things pupils sometimes say in mathematics classes, and the consequences of doing so for understanding, poise, and confidence.

*Erlwanger, S. H. Case studies of children's conceptions of mathematics. Doctoral dissertation, University of Illinois, Urbana/Champaign, 1974.

Partially reprinted in the Journal of Children's Mathematical Behavior 1 (3):157-281, 1975.

These case studies reveal what many children think arithmetic is a set of arbitrary procedures they have to apply in arbitrary fashion to problems which are arbitrarily made up. These case studies are quite human expressions of the difficulties fifth and sixth grade children often have in making sense of arithmetic.

Freudenthal, H. Mathematics as an educational task. Dordrecht-Holland: D. Reidel Publishing Co., 1973.

*Gay, J. & Cole, M. The new mathematics and an old culture: a study of learning among the Kpelle of Liberia. New York: Holt, Rinehart, and Winston, 1967.

It is interesting in itself to read about how people learn mathematics in completely different cultures. This study is particularly relevant because not only mathematical learning but other traditions and customs of the Kpelle were studied. Of special interest is how the Kpelle solve social problems and the types of games they play. Many aspects of Kpelle culture are brought to bear on the question of how people in this society learn mathematics.

*Gladwin, T. East is a big bird. Cambridge, Massachussetts: Harvard University Press, 1970.

Pondering the question of teaching abstract mathematical skills to inner city children, the author goes off to the Caroline Islands to find out how high level skills are transmitted from generation to generation in a less developed society. What he finds is an intricate mixture of abstract and concrete information intricately intertwined in the crucial survival skill of oceanic navigation.

The reader might want to skim through some of the lengthy technical discussions and focus on how they solve navigational problems without having any so-called "formal" training as we would think of it.

*Gordon, T. Teacher effectiveness training. New York: Peter H. Wyden, 1974.

The strategies suggested for teachers to use with social problems stem from his Parent Effectiveness Training. We see some similarities between his problem solving approach and our problem solving 2 approach. Especially relevant is his idea of deciding who "owns" the problem and then letting that person (or persons) try to solve it. Also relevant is his idea of sharing the teacher's problem solving with the students.

Greeno, J. The structure of memory and the process of solving problems. In R. L. Solso (Ed.), Contemporary issues in cognitive psychology: The Loyola symposium. Washington, D.C.: V. H. Winston, pp. 103-133, 1973.

Hadamard, J. An essay on the psychology of invention in the mathematical field. Princeton, New Jersey: Princeton University Press, 1945.

*Hatfield, L. (Ed.). Mathematical problem solving: Papers from a research workshop. Columbus, Ohio: ERIC/SMEAC, 1978.

These papers are research focussed. We especially recommend teachers read the one on elementary school problem solving by Lester. He describes the work of the problem solving project based at Indiana University.

Hollander, S. A review of research related to the solution of verbal arithmetic problems. School Science and Mathematics 78 (January):59-70, 1978a.

Hollander, S. A literature review: Thought processes employed in the solution of verbal arithmetic. School Science and Mathematics 78 (April):327-334, 1978b.

Jackson, P. Life in classrooms. New York: Holt, Rinehart & Winston, 1968.

Jackson, P. Comments on chapter 11 by Berliner and Rosenshine. In R. Anderson, R. Spiro, and W. Montague (Eds.), Schooling and the acquisition of knowledge. Hillsdale, New Jersey: Lawrence Erlbaum Assoc., pp. 397-402, 1977.

Janis, I. & Mann, L. Decision making. New York: Free Press, 1977.

Kilpatrick, J. Problem-solving and creative behavior in mathematics. In J. Wilson and L. Carry (Eds.), Reviews of recent research in mathematics education, SMSG studies in mathematics, volume 19. Stanford, California: Stanford University, 1969.

*Krutetskii, V. A. [The psychology of mathematical abilities in school-children.] (J. Teller, trans., J. Kilpatrick and I. Wirszup, Ed.) Chicago: University of Chicago Press, 1976.

This is an especially important work for anyone interested in mathematics education. Through the longitudinal studies, the reader can get a glimpse at the development of mathematical giftedness. Individual case studies are presented in chapter 11.

Lazarus, M. Mathophobia: Some personal speculations. The Principal, Jan./Feb. 1974.

Le Blanc, J. The performance of first grade children in four levels of conservation of numerousness and three I.Q. groups when solving arithmetic subtraction problems. (Doctoral dissertation, University of Wisconsin, 1968). Ann Arbor, Michigan: University Microfilms (no. 67-68), 1968.

*Lesh, R. Mathematical learning disabilities: considerations for identification, diagnosis, remediation. Unpublished paper, April 1979a. (Available from Richard Lesh, School of Education, Northwestern University, Evanston, Illinois)

A great deal of the research on problem solving has dealt with identifying the abilities of bright children. In this paper the author discusses problem solving abilities of children with learning disabilities. Numerous examples are given from the author's research.

*Lesh, R. Applied problem solving in early mathematics learning. Unpublished paper, September 1979b. (Available from Richard Lesh, School of Education, Northwestern University, Evanston, Illinois)

This paper represents some current thinking and research from the author. The focus is on: what is "real" problem solving involving "real" situations?. He also addresses the importance of "knowing when to quit" and "knowing when an answer is good enough" in the context of applied problem solving. This extends some of our ideas about problem solving 2.

Lester, F. Ideas about problem solving: A look at some psychological research. Arithmetic Teacher 25 (2, November):12-14, 1977.

Lester, F. Mathematical problem solving in the elementary school: some educational and psychological considerations. In L. Hatfield (Ed.), Mathematical problem solving. Columbus, Ohio: ERIC/ SMEAC, 1978.

*Lundgren, U. P. Model analysis of pedagogical processes. Studies in Education and Psychology 2, Stockholm Institute of Education, Department of Educational Research. Lund: CWK, Gleerup, 1977.

We especially recommend reading about "piloting" children through new arithmetical processes. It is an example of our concept of "problem removal". He emphasizes that it is not a strategy consciously chosen by the teacher. It seems to grow out of problems demanding immediate attention in everyday classroom situations.

*Lundgren U. P. (Ed.). Code, context and curriculum processes. Studies in Education and Psychology 6, Stockholm Institute of Education, Department of Educational Research. Lund: CWK, Gleerup, 1979. (Especially chapters 1, 4, and 6).

Most striking is the similarity between art education and mathematics. In an area that we think of as being especially creative, it is reported that children also want to be told rules and be given explicit directions.

Maier, N.R.F. Problem solving and creativity in individuals and groups. Belmont, California: Brooks/Cole, 1970.

Morris, J. Problem solving with calculators. Arithmetic Teacher 26 (April): 24-26, 1978.

(NACOME). Overview and analysis of school mathematics, grades K-12. Washington: National Advisory Committee on Mathematical Education, Conference Board of the Mathematical Sciences, 1975.

NAEP (National Assessment of Educational Progress). Changes in mathematical achievement, 1973-78. Report No. 09-MA-01. Denver: Education Commission of the States, August 1979.

Nelson, L. & Kirkpatrick, J. Problem solving. In J. Payne (Ed.), Mathematics learning in early childhood, Thirty-seventh yearbook of the National Council of Teachers of Mathematics. Reston, Virginia: NCTM, pp. 69-93, 1975.

Newell, A. & Simon, H. Human problem solving. Englewood Cliffs, New Jersey: Prentice-Hall, 1972.

(Newsweek). Why Johnny can't add. p. 72, Sepember 24, 1979.

*Payne, J. (Ed.). Mathematics learning in early childhood, Thirty-seventh yearbook of the National Council of Teachers of Mathematics, Reston, Virginia: NCTM, 1975.

This is a collection of articles ranging from the history of early childhood mathematics curricula to papers, such as the one by Nelson and Kirkpatrick on problem solving, which presents a lot of practical ideas.

Piaget, J. Quantification, conservation and nativism. Science 162: 976-981, 1968.

*Piaget, J. The grasp of consciousness. Cambridge, Massachusetts: Harvard University Press, 1976.

A [illegible]y recent study by Piaget and some of his collaborators of the ways in which children are aware or are not aware of procedures they habitually carry out. One of the most interesting of the many books Piaget has written.

Poincaré, H. Mathematical creation. In B. Ghiselin (Ed.), The creative process. New York: The New American Library of World Literature, 1952.

*Polya, G. How to solve it. New York: Doubleday Anchor Books, 1957.

This classic study by a famous mathematician covers a lot of useful material, ranging from advice for people stuck on a problem to a dictionary of heuristic (the science of such helpful advice).

Resnick, L. Task analysis in instructional design: Some cases from mathematics. In D. Klahr (Ed.), Cognition and instruction. Hillsdale, New Jersey: Lawrence Erlbaum Assoc., 1976.

*School Science and Mathematics 78 (March), 1978.

This issue is devoted to the topic of problem solving. Of special interest to elementary teachers would be the article: "Informal Evaluation Strategies for Real Problem Solving."

Some creative techniques are given. One involves asking the students to write down what they've learned. The authors comment that a strength of this method is that there is no "right" answer involved.

Stake, B. Clinical studies of counting problems with primary school children. Doctoral dissertation, University of Illinois at Urbana/ Champaign, 1980.

*Stake, R. & Easley, J. Case studies in science education. Washington, D.C.: Government Printing Office, 1978.

Case studies are given in Volume I for eleven American school systems (grades K-12) which are representative of the diversity of schools in this country. Science, mathematics, and social studies are covered in considerable detail by the live-in observer-authors. Many quotations are given from teachers about their life in school and their views of these subjects. A second volume pulls the case studies together.

Steffe, L. The relationship of conservation of numerousness to problem-solving abilities to first grade children. Arithmetic Teacher 15 (January):47-52, 1968.

Steffe, L. Differential performance of first-grade children when solving arithmetic addition problems. Journal for Research in Mathematics Education 1 (May):144-61, 1970.

Steffe, L. & Johnson, D. Problem-solving performances of first-grade children. Journal for Research in Mathematics Education 2 (January):50-64, 1971.

*Sternberg, R. Stalking the I.Q. quark. Psychology Today, pp. 42-54, September, 1979.

This is a readable article in which the author relates I.Q and problem solving. He makes the point that good problem solvers know how to use their time wisely -- whether it's to encode data in certain problems or to do the computation in others. He concludes that it is important to teach people "higher order" strategies for constructing their own problem solving strategies.

Suppes, P. Some theoretical models for mathematical learning. Journal of Research and Development in Education 1 (Fall):5-22, 1967.

*Tobias, S. Overcoming math anxiety. New York: W. W. Norton & Co., 1978.

We find this a valuable book for several reasons. First of all, we all suffer from math anxiety in one form or another--

whether it hits us in first grade or in a graduate level math course. Secondly, the author discusses some issues that relate to some of the points we make in this paper. She considers the idea that math anxious people distrust their intuition and view mathematics as "getting the right answer." Although some might criticize the author as being too oriented to the general public, we think this is one of the book's strengths. It is highly readable and even the so-called mathematical sections will retain the reader's interest.

Trafton, P. The curriculum. In J. Payne (ed.), Mathematics learning in early childhood, Thirty-seventh yearbook of the National Council of Teachers of Mathematics. Reston, Virginia: NCTM, 1975.

Van Engen, H. & Gibb, E. G. General mental functions associated with division. Educational Services Studies, No. 2. Cedar Falls, Iowa: Iowa State Teachers College, 1956.

Wallas, G. The art of thought. New York: Harcourt, Brace, 1926.

*Wertheimer, M. Productive thinking. New York: Harper & Row, 1959.

Although this is not a recent work, it is invaluable reading. Especially interesting are the sections where the author presents specific examples of interviews with students or observations of classrooms. In one example, he shows how he created a pattern from which students would be led to a false generalization -- one made from too few cases. His creative approaches in interviewing children to see how they think contain good ideas.

Zweng, M. The problem of solving story problems. Arithmetic Teacher 27 (September):2-3, 1979.

PRACTICAL POINTERS WHICH CAN BE FOUND IN THE TEXT