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ABSTRACT

The purpose of this paper is to identify how the theory and techniques of research in cognitive development can be applied to the study of learning and instruction in mathematics. Five basic research paradigms are characterized; major directions of research on number, measurement, geometry, and adolescent reasoning are identified. Most of the research was built upon the work of Piaget and his associates. However, recently translated Soviet research and information processing techniques are also discussed. Potential education applications discussed include: content and sequencing of mathematical topics, matching instruction to appropriate levels of development, and choosing instructional strategies. (Author/MK)

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Theoretical Paper No. 73

COGNITIVE DEVELOPMENT RESEARCH AND MATHEMATICS EDUCATION

by

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Report from the Project on
Studies in Mathematics

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ABSTRACT

The purpose of this paper is to identify how the theory and techniques of research in cognitive development can be applied to the study of learning and instruction in mathematics. Five basic research paradigms are characterized; and major directions of research on number, measurement, geometry, and adolescent reasoning are identified. Most of this research was built upon the work of Piaget and his associates. However, recently translated Soviet research and information processing techniques offer promising alternatives.

If cognitive development research is going to have a significant impact on education, its theories have to be recast into an educational context and principles of cognitive development have to be applied directly to educationally significant questions. The role of mathematics educators should not be to validate or develop aspects of different theories of cognitive development but to determine how useful these theories are in explaining childrens' learning of mathematics concepts. Specifically it is important to focus on content that is central to the mathematics curriculum. Furthermore, it is necessary to empirically establish how the descriptive information from research on childrens' thinking can be applied to prescribe instruction. One of the most promising directions for such research is to attempt to determine how content can be designed and sequenced to reflect or build upon childrens' informal mathematical concepts and strategies. A second potentially productive line of research is attempting to identify how instruction may be individualized to match childrens' level of development.

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Cognitive Development Research and Mathematics Education

The basic concern of research and theory in cognitive development is to describe the growth of basic concepts of children over time and explain the processes by which these concepts are acquired and applied. Cognitive development can be characterized in a number of different ways. A useful distinction between two different conceptions of cognitive development has been proposed by Reese and Overton (1970). One is based on an organismic model and is represented by the works of Piaget and his followers. This model takes as its analog the biological organism and is concerned with the development of complex cognitive systems. The other conception of cognitive development is based on a mechanistic model and is essentially an extension of behavioristic theories to explain development. The theories of Gagne (1968, 1977) regarding development are representative of this orientation. The mechanistic model is based on the machine and is concerned with the development of discrete chainlike associations. From the mechanistic perspective the only distinction between learning and development is the duration of time involved. Development deals with change in behavior over weeks, months, or years whereas learning theory deals with changes in behavior over much shorter periods of time.

Even within the organismic framework cognition is an elusive concept. However, although any attempt to characterize cognition in detail remains open to argument, certain fundamental premises of the organismic model can be identified. The basic premise is that any intelligent behavior must be explained by reference to internal psychological mechanisms of some kind (e.g., groupings, transitive inference, logical grammars, etc.). There is far from universal agreement as to what internal mechanisms are most appropriate for explaining intelligent behavior, but the general organismic view is that these internal mechanisms are organized into well integrated structural systems rather than consisting of a series of independent associations. Furthermore, it is generally recognized that these systems are not restricted to higher order conscious behavior but operate on a wide range of mental functions, including such functions as perception and memory.

From the organismic perspective, the study of cognitive development is the study of the development of these cognitive systems. Whereas learning involves the application of intellectual structures to new events, development entails transformations in the cognitive structures themselves. The primary focus of developmental research is not to identify what specific knowledge a child possesses at a given point in time but to study how the child processes or operates on information. There is relatively little concern with finding out which addition facts are known by most second graders or identifying the age at which most children master addition of two-digit addends; the focus is on concepts like conservation, transitivity, and seriation that involve the application of logical inference and seem to be closely linked to underlying cognitive structures. The interest in a concept like conservation is not simply that it is an important bit of factual knowledge, that is, that quantities remain invariant under certain transformations. Rather performance on conservation tasks is viewed as a measure of underlying cognitive structures that the child can apply to a wide variety of problems. In other words a child's performance on a conservation task does not simply demonstrate a knowledge of an isolated fact about the physical world. It is indicative of the way that child processes information in a variety of problem situations. This is a central issue in cognitive development. Those who attach relatively little significance to concepts like conservation generally regard them as no more than bits of factual knowledge, whereas those who attribute a central role to such concepts regard them as measures of basic cognitive processes.

Mechanistic models do not recognize the integrated cognitive systems that are the essence of cognition within organismic models. Although internal mediating responses are acceptable, these are organized into chainlike associations rather than integrated into complex systems. Mechanistic models also are more concerned with product than with process.

As with cognition, the two types of models provide radically different conceptions of the nature of development. In organismic models the individual actively participates in the construction of knowledge. New information is not received passively. The subject actively assimilates it into and interprets it in light of the existing cognitive structures. The mechanistic position is that knowledge is a copy of reality and that people are essentially reactive rather than active in acquiring knowledge.

Experience plays a key role in development in both types of models, but experience is characterized in different terms in each type of model. Within organismic models experience traditionally has been regarded as a function of the sum of all an individual's experiences. The environment is considered to be something of a black box within which specific cause and effect factors are undifferentiable. Research based on organismic models traditionally has been observational and correlational rather than experimental, and until recently there has been little attempt to manipulate experimental variables. Note that this holistic view of experience is an integral part of a theory that hypothesizes the existence of an integrated cognitive system in which individual elements cannot be significantly altered without changing the structure of the entire system.

Mechanistic models, on the other hand, focus more on the specific effects of training. In the last 10 to 15 years there have been an increasing number of training studies that have been based upon traditional organismic variables. Indeed, even from an organismic perspective it is becoming accepted to attempt to identify specific mechanisms of development. Even Piaget and his associates have been conducting training studies in the past few years (Inhelder, Sinclair & Bovet, 1974). Thus, organismic models do not preclude the analyzing development or studying part processes. However, the parts must ultimately be interpreted in context of the whole of which they are a part.

The ultimate source of cognitive mechanisms is another point of disparity. Most mechanistic models hypothesize that behavior can be explained strictly on the basis of environmental determinants, and it is assumed that all internal mechanisms originate solely from experience. However, organismic models allow that some structure exists at birth and that others develop through maturation and the interaction of present structures and the environment.

Organismic models generally view development as proceeding through an irreversible, fixed sequence of qualitatively different stages; and the mechanistic model views development as essentially continuous, reducible to quantitative change. Thus, for organismic models development results from change in the organism itself; and for mechanistic models development reduces to quantitative increments.

Organismic models tend to be teleological in that they are goal oriented in their characterization of cognitive development. For example, a child is inexorably developing toward a stage of formal operations. Mechanistic models, on the other hand, do not rely on teleological causes to explain development. Organismic models also tend to be more species specific. Their proponents maintain that to understand human behavior it is necessary to study cognitive development of people, whereas proponents of mechanistic models are more likely to generalize from research with simpler organisms.

Many of the significant issues in cognitive development reduce to differences over which type of model of cognitive development is more appropriate. Issues involving the stage concept, the effect of training, and the significance of conservation all revolve around the question of which type of model of cognitive development has been adopted. Experimental paradigms have been proposed (Watson, 1968) and a wide range of studies (Beilin, 1971) have been conducted that attempt to resolve the conflicting theories that result from adopting one or the other of these two models. In general, these studies have done little to resolve the basic issues or establish the validity of either model.

Reese and Overton (1970) propose that these models represent two independent world views that are based on different sets of assumptions and are essentially irreconcilable. In essence, this means that it is futile to attempt to synthesize an organismic approach like Piaget's with a behavioristic, mechanistic approach. In fact, the central question is not which model is valid or accurate. It is not being asserted that the model describes the way that cognition actually develops, only that the model accurately describes behavior. To quote Reese and Overton (1970), "It is not being asserted [in a model] that the real world is thus and so, only that the real world behaves as if it were thus and so" (p. 120). Thus, the relevant question is pragmatic. Which model is more fruitful for adequately explaining and predicting behavior? Since any model limits the domain of problems that are susceptible to investigation, the choice of a model essentially involves a value judgment as to which problems are most significant to solve.

Since neither model appears to be sufficient to account for the whole range of human behavior, several eclectic theories have emerged. For example, White (1965) maintains that at about the age of 5 to 7 years qualitative change occurs in children's behavior. A mechanistic, associative model best accounts for early behavior while a cognitive model is most appropriate after this transition. Kohlberg (1968) and Uznadze (1966) also attempt to integrate the two models to explain simple and more complex behavior.

Many of the lower level skills in mathematics, like learning addition facts, are readily reduced to associations and lend themselves to analysis in terms of mechanistic models. However, most of the more interesting complex cognitive processes are more adaptive to cognitive, organismic models. Although it is not always clear what sort of model a particular researcher has adopted, most of the cognitive development research that is of particular interest for the learning and teaching of mathematics is based on organismic models, and they will be dealt with most completely in this chapter. For a more complete discussion of the mechanistic position with respect to cognitive development, see Baer (1970) or Bijou and Baer (1961).

Cognitive development is an extensive field that is impossible to characterize adequately here. Consequently it is necessary to assume that the reader has some familiarity with the work in this area. Most of the research that has the greatest potential significance for the teaching and learning of mathematics has been based on the work of Piaget. Several excellent summaries of his voluminous works exist elsewhere, and no attempt has been made to duplicate or summarize these efforts. Piaget's basic positions have not changed significantly in recent years and early studies on number, space, and geometry still provide the basis for much of the research on the teaching and learning of mathematics. Consequently, Flavell's (1963) summary is still one of the best statements of Piaget's basic theories and research available. For more recent and somewhat more general discussions of theories and research in cognitive development, see Flavell (1970, 1977), Gelman (1978), and Ginsburg and Koslowski (1976).

Major Paradigms of Cognitive Development Research

Wohlwill (1973) has outlined a hierarchical model for the study of development problems which with certain modifications provides a useful framework for characterizing cognitive development research in mathematics education. Wohlwill places certain restrictions on the criteria for suitable behavioral dimensions that would disallow many of the problems of central interest in mathematics education. In research that is primarily concerned with explaining the general cause of cognitive development, variables that develop independently of specific experiences or specific school curriculum are generally most appropriate. Although the foundations of many mathematical concepts may fit these criteria, most mathematical concepts are acquired under the influence of instruction. It can be argued that concepts that are influenced by instruction are not truly developmental, but this distinction does not seem very productive. If certain mathematical concepts show the same stagelike characteristics as pure developmental concepts, the paradigms and techniques of research in cognitive development can prove useful in their study and should be applied. Experience plays a role in the development of most concepts. When instruction is a factor, there is simply a greater potential for variability in the development of the given concepts. The important point is that at each level of Wohlwill's model, it is necessary to identify the specific effects of instruction when appropriate. If this attention to the effects of instruction is built into the model, Wohlwill's general model provides a useful framework to describe the major paradigms of basic research in the development of mathematical concepts.

Wohlwill identifies five basic phases in research on development: (a) the discovery and synthesis of developmental dimensions, (b) the descriptive study of change, (c) the correlational study of age change, (d) the study of the determinants of developmental change, and (e) the study of individual differences in development. Although there is an implied hierarchy, research has generally been conducted at each of the levels simultaneously. Ideally, the results at lower levels provide a foundation for research at higher levels, and conclusions reached at lower levels are consolidated and possibly revised on the basis of work at advanced levels.

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The Discovery and Synthesis of Developmental Dimensions

The first task in investigating the development of mathematical concepts is to identify the dimensions to be used to describe development. In the study of the development of mathematical concepts, the specific variables to be investigated have been derived from two primary sources: (a) the mathematical axioms and theorems underlying the concepts under investigation and (b) the general study of cognitive development.

The study by Wagman (1975) is an example of an investigation employing variables derived from the mathematical structure of area measure. Although the study is similar to those based on psychological considerations, Wagman maintains that most studies in the general cognitive development tradition have not investigated some of the significant aspects of area measure. The implication is that by beginning with the mathematical foundations of a subject one is more likely to provide a complete picture of the development of a mathematical concept. The studies reported by Lovell (1971a, 1971c) on the growth of the concept of a function and the development of the concept of mathematical proof provide additional examples of mathematically derived dimensions.

Most of the research on the development of mathematical concepts that has evolved from the general study of cognitive development has been based on the work of Piaget and his associates. For Piaget, certain logico-mathematical structures (e.g., groupings) provide excellent models of actual cognitive processes used by older children and adults. (For a more complete discussion of groupings, group-lattice structures, etc., see Flavell, 1963.) According to this theory the major elements of cognitive development can ultimately be described in terms of the development of these logico-mathematical structures. Thus, for Piaget these structures provide the major dimensions for the study of cognitive development. However, although such structures are appealing for their mathematical elegance, to hypothesize the existence of these structures requires a high degree of inference, and their usefulness for explaining the development of basic mathematical concepts is open to question (Steffe, 1973).

Most research in the development of mathematical concepts has focused directly on principles like conservation, transitivity, seriation, and class inclusion, that are more readily observable than grouping structures. However, although these principles are less obscure than grouping structures, researchers are still plagued by the problem of constructing criteria that are necessary and sufficient to establish whether or not a child has attained an operational level in applying them.

Developmental dimensions should be sufficiently situation independent to generate valid, reliable measures of development. Researchers are faced, however, with the well documented problem of horizontal décalages. Although it seems that operations with the same logical structure would readily transfer from one problem situation to another, this is not the case. For example, conservation of mass is attained as young as 7 years, while conservation of weight is not attained until at least 9 years, and volume is not conserved until 11 or 12 (Piaget & Inhelder, 1941; Elkind, 1961). Thus, although it is desirable to define developmental dimensions in terms as general as possible, some specification of the domain of application seems unavoidable. It is not sufficient to identify children as conservers; it is necessary to specify whether they conserve one-to-one correspondence, continuous quantity, weight, or volume.

Unfortunately, the problem does not disappear with the specification of the domain in which the operation is applied. Methodological variations account for significant differences in children's performance on tasks testing logical operations. Differences in the criteria for success, the use of verbal or nonverbal procedures, the presence or absence of conflict, and variations in materials or protocols all significantly affect children's level of performance (cf. George, 1970; King, 1971; Sawada & Nelson, 1967; Shantz & Smock, 1966; Stone, 1972; Uzgiris, 1964). Furthermore, these differences are not trivial. Methodological variations account for a four-year age differential in the acquisition of transitivity (Bailey, 1971; Smedslund, 1963). This has led to sustained debate over appropriate research methodology (cf. Braine, 1959, 1964; Smedslund, 1963, 1965). Methodological issues frequently involve basic philosophic differences that seem to go back to fundamental mechanistic-organismic distinctions. This makes any empirical resolution virtually impossible.

One proposal for dealing with experimental variability and the décalage issue has been put forth by Flavell and Wohlwill (1969). The whole problem centers on what performance is necessary to demonstrate competence for a given logical operation. Flavell and Wohlwill propose that an analysis of cognitive development should incorporate a competence-performance distinction similar to Chomsky's model for language acquisition. The competence component of the model is the logico-mathematical structure of the domain, and the performance component represents the psychological processes by which the structures in the competence component are accessed and applied to specific tasks. The competence component is an idealized abstract representation of what is known or understood, whereas the performance component must account for the reality of stimulus variations, conflicting information, and memory limitations.

In Flavell and Wohlwill's model, a child's performance for a given operation should be specified in terms of three parameters: P_a, the probability that the operation will be functional in a given child; P_b, the probability of the operation being applied to a given task; and k, the weight to be attached to P_b in a given child at a given age. The equation for the probability of a given child solving some particular task is:

$$P(+) = P_a \times P_b^{1-k}$$

Any description of development must account for P_b as well as P_a. In other words, developmental dimensions cannot be based completely on logical operations. They must also be defined in terms of attributes of potential problem situations that affect performance.

The Descriptive Study of Change

Once developmental dimensions have been selected, the task is to describe the course of development along these dimensions. This is the descriptive phase of the research program that characterizes the initial efforts in almost any field of scientific endeavor. Educational research in general has been marked by a disdain for this phase of the scientific process. Major curriculum projects and elaborate theories of instruction have been grounded on extremely limited empirical foundations. Rather than beginning with a careful observation of children learning mathematics, research has too often been initiated with narrowly defined hypotheses tested in carefully controlled settings using standardized, objective instrumentation. Standardization and experimental control certainly have their place in research. But if controlled experimentation is preceded by careful observation, the experimenter has a much better basis for explaining specific results. In addition, the experimenter should gain a clearer idea of what elements are not being tapped with the standardized instruments and should be able to design items that get at the most significant variables.

In many scientific fields carefully controlled research is not initiated until the experimenters have a sufficient empirical basis to be virtually certain of their results. That this paradigm has seldom been applied in educational research may in part account for the syndrome of no significant difference and the general lack of real progress in identifying significant educational variables.

Research in cognitive development has been somewhat less guilty in this regard than educational research in general. In fact, demonstrating the usefulness of clinical interview techniques and the wealth of information that is contained in incorrect responses may be one of the most significant contributions of Piaget to research in the learning of mathematics.

Ginsburg (1976) has made one of the strongest cases for the use of clinical-observational techniques in studying the learning of mathematics. He maintains that standard tests often misrepresent children's competence. Consequently a greater emphasis on the flexible observation of children's mathematical thinking is required. This point is aptly illustrated by Erlwanger's (1973, 1975) evaluation of IPI using flexible interview techniques. Although standardized tests generally indicated that certain children were successfully progressing through the IPI program, clinical interviews uncovered a number of serious misconceptions that Erlwanger attributed to the program's specific nature.

Although the potential richness of clinical observation and interactive interview techniques is generally acknowledged, serious questions concerning their validity and reliability have been raised. To overcome these objections, most of the replication studies based on Piaget's original research have attempted to standardize protocols and procedures.

Frequently the standardized procedures impose less stringent conditions for assuming a child has attained a given operation which results in identifying earlier ages of emergence of the operation. Piaget and his associates require that in order for a child to be judged operational for a given concept the following criteria must be met: (a) they must make the correct judgment with respect to the given operation, (b) they must justify their response, (c) they must resist verbal counter suggestion, and (d) their performance must transfer to related tasks (Inhelder & Sinclair, 1969).

Brainerd (1973a, 1977) contends that these criteria are too restrictive and result in too many false negatives. He proposes that children be required only to give correct judgments and not to justify their answers. Others have proposed using nonverbal techniques that minimize verbalization of both the experimenter and the child (Braine, 1959; King, 1971; Miller, 1976; Sawada & Nelson, 1967).

Although standardized techniques are definitely needed at some point in the research process, much of the richness uncovered using more subjective techniques is lost. Standardized protocols seldom uncover the transitional stages of performance on a given task that are identified by Piaget and others using interactive methods. The proponents of standardization could respond that these transitional stages are simply illusory anyway,

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whereas those favoring interactive procedures would maintain that the standardized techniques do not tap genuine operational competence.

This division tends to split along philosophical lines. Those favoring a mechanistic model believe that important cognitive outcomes should be specifiable in terms of overt behavior. Therefore they take a hard line on standardized techniques; those favoring interactive methods tend to fall in the organismic camp. To some degree these methodological issues present a false dichotomy. It is not a question of either-or. Both paradigms have their strengths and weaknesses. Some standardization is necessary in studies comparing instructional treatments and in studies comparing the relative difficulty of two or more tasks. In these kinds of studies, objectivity is of central concern. Standardization is also ultimately needed to test hypotheses and determine the prevalence of specific responses. On the other hand interactive methods also have their place, and clinical case study research should be recognized as legitimate research endeavor. Something is lost and something is gained with each paradigm, and both clinical and controlled studies are needed.

Up to now, researchers favoring one type of study or the other have tended to rely exclusively on the design of their choice. Because of the variability introduced by differences in materials and procedures, it has been difficult to equate the two bodies of research. Researchers would be well advised to incorporate both types of design in their programs.

In either case the goal of this phase in the research program is to describe the course of development along the dimensions that were laid out in the first phase discussed above. For the most part, the developmental dimensions that have been of greatest interest in describing learning in mathematics have been qualitative rather than quantitative. As a consequence, development cannot be described in terms of a mathematical function like one might generate from a test of word recognition. Instead, the problem becomes one of describing invariant sequential patterns of qualitatively different responses. The main task reduces to specifying the sequence in which behaviors appear during the course of development along with determining how invariant this sequence is for a given sample of children.

Some attempt has been made to establish age norms for the emergence of specific responses. However, in addition to the variance introduced through experimental variation, cultural and socioeconomic factors create an almost overwhelming obstacle in this regard. Although estimates of such age norms are useful as benchmarks and they do provide some measure of the duration of different stages of development, caution should be exercised in their application.

The Correlational Study of Development

The development of most mathematical concepts of real interest, like number or measurement, are not readily described along a single dimension. These concepts involve the synthesis of a number of logical operations, and therefore multiple measures are required. Furthermore, it is impossible to understand the development of a concept by considering it in terms of isolated, independent dimensions.

> One of the major aims of cognitive-developmental study is to identify and interpret the temporal relations that may hold among conceptual acquisitions. For any pair of acquisitions A and B, the most interesting of such relations

are invariant concurrence (A and B develop synchronously in all children) and invariant sequence (e.g., A develops earlier than B in all children). (Flavell, 1970, p. 1034)

Developmental sequences. By analyzing the structure of various problems, one can hypothesize that certain concepts must be learned before others because the development of the first mediates or, in some way contributes to the development of, the second. The difficulty is that by analyzing tasks in different ways different sequences can be identified. For example, one can construct a reasonable argument for conservation preceding transitivity or for transitivity preceding conservation (cf. Brainerd, 1973d). For this reason logically derived sequences must be compared with the actual sequences of development, so that hypotheses can be tested regarding factors that contribute to the development of a concept and the processes that a child is using to solve a given problem.

Although developmental sequences are an integral part of Piaget's theory, his method of comparing mean ages of development for different samples of children is inadequate for verifying the existence of such sequences. Repeated measures on the same subjects are required, the most effective of which would look for sequence reversals. If the development of B depends on the development of A, there should be an invariant A - B sequence; and B should precede A only in cases of measurement error. Where more than two tasks have been involved, scalogram analysis techniques have frequently been applied (Kofsky, 1966; Wohlwill, 1960, 1973). Longitudinal design provides certain information that is inaccessible using cross-sectional methods; and invariant sequences identified in cross-sectional studies should be confirmed with longitudinal study, where the sequence of development can be observed directly within individual subjects. However, because of the practical problems involved, relatively few longitudinal studies relating to the development of mathematical concepts have been conducted (cf. Almy, Chittenden & Miller, 1966; Almy, Dimitrovsky, Hardeman, Gordis, Chittenden & Elliot, 1970; Carr, 1971; Dudek & Dyer, 1972; Little, 1972; Niemark & Lewis, 1968; Hooper & Klausmeier, Note 1.)

The objective of the study of developmental sequences is to establish some functional relationship between tasks that accounts for observed invariant sequences. A key problem in this endeavor is the sensitivity of the tasks used to measure the individual concepts. An observed A - B sequence may simply result from the fact that the task measuring B is less sensitive than the task measuring A and consequently yields a greater number of false negatives. If there is sufficient time lag between the development of different concepts as with conservation of mass, weight, and volume, there may be no serious problem. But most sequences of greatest interest occur over shorter periods of time.

For example, the sequence of development of conservation and transitivity is of some potential significance for understanding the development of number concepts because it may reflect the ordinal-cardinal controversy (see Brainerd, 1976). Piaget and Inhelder (1941) initially proposed that the two concepts develop synchronously, but with the exception of a study by Lovell and Ogilvie (1961), most of the initial replications found that conservation develops before transitivity (Kooistra, 1964; McManis, 1969; Smedslund, 1961, 1963, 1964; Steffe & Carey, 1972). These studies, however, have been criticized by Brainerd (1973a) for failing to equate the relative sensitivities of the assessment tasks. Each of the studies employed perceptual illusion in the transitivity tasks. Using tasks that did not involve perceptual illusion, Brainerd (1973d) found that the development of transitivity precedes the development of conservation. It is not clear that Brainerd's procedures are any more equitable than the others, since the conservation tasks involved perceptual illusion and the transitivity tasks did not. At this point, the most reasonable conclusion seems to be that the sequence of development appears to depend on what evidence one requires for the respective operations. If one compares the standard conservation tasks to the weaker definition of transitivity, then it appears that transitivity develops earlier. If one insists on stronger criteria for transitivity, then it appears that conservation develops earlier. Unfortunately there are no valid empirical procedures to resolve this issue. No task has any special claim to be the measure for a given operation. The competence-performance distinction is involved again, and it appears necessary to account for the performance dimension in the characterization and explanation of developmental sequences.

Developmental concurrences. Piaget's theory hypothesizes that new cognitive structures that can be applied to a wide range of problems emerge within a given stage of development. Furthermore, these operations are integrated into unified structural systems. This would seem to imply that development would be marked by the synchronous development of a variety of abilities, which should be mainfested by consistent failure or success across a number of different tasks. Not only should tasks with the same inherent structure be mastered concurrently, but because of the hypothesized interconnectedness of logical operations, similar concurrences should be found for related operations.

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The stage concept is potentially useful because it proposes to predict behavior over a wide range of tasks. Thus, performance on a small set of tasks should be sufficient to predict performance on a large domain of related tasks. Unless this sort of generalization is possible, the stage concept has little practical value for education. Unfortunately, very few consistent concurrences have been found. Although certain logical operations may ultimately be integrated into a <u>structure d'ensemble</u>, they appear to emerge asynchronously and initially generalize to a restricted number of problem situations.

In spite of the almost insurmountable obstacles in terms of horizontal décalages and methodological variability, the correlational study of development is central to the applying research in cognitive development to education. Decisions involving the sequencing of content and matching instruction to appropriate levels of children's development both rest on such study. For a more complete discussion of this topic the reader is referred to the articles by Flavell (1970, 1971, 1972), Pinard and Laurendeau (1969), and Wohlwill (1973).

<u>The Study of Developmental Change</u>

The most widely used approach to investigate the factors affecting developmental change is the training study. Extensive reviews of Piagetian training studies can be found in articles by Beilin (1971), Brainerd (1973), Brainerd and Allen (1971), Hatano (1971), Glaser and Resnick (1972), Strauss (1972), and Wohlwill (1970). The typical training study has employed a relatively short period of training. Most treatments have consisted of a single short training session, and few have involved more than 10 half-hour sessions. Typically, the treatments and pretest and posttests have been administered individually or in small groups; and training has involved a single logical operation like conservation, seriation, or transitivity.

Beilin (1971) identifies three generations of training research. In the first generation, studies were designed to substantiate basic elements of Piaget's theory of cognitive development. One group of studies has attempted to induce logical operations by creating a state of disequilibrium with respect to the given operation. Other studies of this genre have focused on mental operations such as addition-subtraction or reversibility, that are presumed to be involved in the natural development of the concept to be trained.

The second generation of training studies were based on the hypothesis that Piaget's stage theory is overly rigid in the limitations it places on cognitive development. A number of these investigators believe that the acquisition of logical structures can be accelerated and reject the equilibration model as the sole explanation for their acquisition. They do not accept, for example, that reversibility and compensation are the essential mechanisms leading to conservation. Some studies have trained children to attend to relevant attributes and disregard or ignore misleading perceptual cues. Another group of studies has relied on verbal rules or feedback in training. Others have employed techniques of conformity training, pairing nonconservers with conservers or exposing nonconservers to expert models.

Studies of this second type continue to be a major force in Piagetian research. However, there is a third generation of studies whose objectives are different from the other two. The aim of these studies, which are conducted by the Genevans themselves, is to investigate the psychological mechanisms that underlie the transitions between stages. These studies attend more closely to the stage of development of subjects before training and specify in much greater detail than the earlier studies the specific effects of training. The perspective of these studies is that training only extends the domain of application of operational structures. It does not initiate the development of new operations. According to this view, "The development of operativity is malleable only within the limits imposed by the nature of development" (Beilin, 1971, p. 101). Evidence of the development of an early emerging operation like the conservation of number is prerequisite for successful training of more advanced concepts. In terms of the Flavell and Wohlwill (1969) model, training operates on the performance component rather than the competence component of the model.

Although many individual studies failed to demonstrate significant training effects, almost every type of training procedure has been able to accelerate the acquisition of logical operations. However, they have failed to identify the specific mechanisms that lead to the development of the operation One difficulty is that researchers often fail to agree on the specific mechanism that is operating in a given training procedure. One researcher may attribute the effect of training to learning to attend to relevant dimensions, and another may identify latent reversibility training as the significant variable (cf. Brainerd & Allen, 1971).

There has also been a failure to distinguish between necessary and sufficient conditions for the development of a given operation. A basic assumption underlying many training studies is that if training accelerates the acquisition of a given operation then the conditions of the training must be crucial for the natural development of the operation. Just because a training condition has been sufficient to accelerate the acquisition of an operation does not mean that condition is necessary for the development of the operation. With regard to this point Wohlwill (1973) concludes:

> Thus, we have had a parade of training conditions which to varying extents and degrees of consistency have shown themselves to be sufficient to induce conservation, at least given a child within a particular age range. But the relationship between these conditions and the process of conservation as it takes place naturally--that is the question of the plausibility that these conditions could in fact have been operating in the child's extra laboratory experience--has rarely been examined. If it had been, it would quickly have become apparent that most of them, from rule learning to reversibility training, from cognitive conflict to reinforced practice, are of dubious relevance to that experience. (p. 323)

Virtually all training studies have found that training transfers to novel materials not used in the training procedure. This specific transfer applies to situations in which the tasks are similar to those used in the training and only the specific materials are changed. For example, toy cars may be used in conservation training and poker chips in the specific transfer task. Nonspecific transfer applies to situations in which the traired logical operation extends to a new domain of application (e.g., transfer of training on length to area). This type of transfer has been more difficult to achieve, although several studies have reported considerable success (e.g., Bearison, 1969; Gelman, 1969). The difficulty in finding nonspecific transfer is not especially surprising given the prominence of observed décalages in the natural development of operations. Transfer between logical operations (e.g., conservation to transitivity) has been even more difficult to achieve.

Wohlwill (1970) proposes that cognitive development can be thought of as a combination of horizontal and vertical transfer. The larger the number of vertical steps the learner must climb to reach his goal, the narrower the span of generalization or horizontal transfer. Wohlwill also observes that the amount of transfer appears to be a function of the breadth and intensity of training.

Several studies have tested for retention over periods ranging from one to seven months. Almost universally they have found that the trained concepts have been retained. The picture is somewhat different when a specific effort is made to extinguish a given concept. According to the stage theory of development, once an operation is fully attained it should be extremely resistant to extinction. An early study by Smedslund (1961) found that trained conservers readily abandoned conservation judgments when they were deceived with an example in which it appeared that weight was not conserved. Natural conservers were much more resistant to such extinction. This seemed to imply that the trained conservers were giving only superficial responses and had not attained a genuine operational level. Recent studies have failed to confirm these results. They have found no appreciable differences between trained and natural conservers in their resistance to extinction or countersuggestion. In general, earlier developing concepts like number have proved to be quite resistant to extinction for both groups, whereas later developing concepts like weight are quite easy to extinguish for both natural and trained conservers (Brainerd, 1973c).

Although the goal of much of the training research has been to understand the specific mechanisms of development, not to accelerate development per se, a number of studies have been conducted whose only apparent goal is to demonstrate that training of a specific operation is possible. Many studies conducted by researchers interested in problems dealing with the learning of mathematics have been of this type. The assumption underlying these studies seems to be that a specific operation like conservation or seriation is apparently important for learning basic mathematical concepts. Therefore, if these operations can be successfully trained, the subsequent learning of basic mathematics will be facilitated. Although these studies have frequently been successful in training the specific operations, none have demonstrated that any significant savings transfer occurs in the learning of subsequent mathematical topics. In fact, in a follow up to one of the more successful conservation training studies, Bearison (1975) concluded that the training had no effect on the subsequent learning of number skills.

Acceleration development has been a major issue in cognitive development. Piaget has questioned why Americans are so interested in accelerating development when the basic operations develop naturally anyway. This concern has been echoed by Glaser and Resnick (1972), who have questioned whether early stimulation will lead to richer growth or just faster growth.

Elkind (1971) and Wohlwill (1970) have hypothesized that the longer formal instruction is delayed, up to reasonable limits, the longer the period of plasticity resulting in a richer ultimate level of achievement with greater flexibility and creativity. Elkind (1976) has also proposed that development is a whole-organism phenomenon and that accelerating any single part of it may encourage maladaptation. Both hypotheses remain to be proven, but at this point there is scant empirical evidence that any attempts to accelerate development result in any desirable educational outcomes.

It has been amply demonstrated that training using a variety of different training procedures is possible. Future training studies need to be designed so that they provide a greater understanding of the specific mechanisms of development. Such studies should provide answers to the following questions: (a) What are the prerequisites for attaining a given level of cognitive development? (b) What are the specific experiences that contribute to the development of a given concept? (c) Once a concept has been learned, to what extent does it generalize? This involves (a) measuring subjects' entering knowledge and level of development; (b) carefully designing training that is based upon a reasoned theoretical rationale; and (c) measuring specific outcomes, including transfer and retention. Results should not be reported using global measures of group success or failure. Instead, some attempt should be made to account for the differential effects of instruction on individual subjects. Future training studies will contribute to our knowledge only in so far as they can help us to understand how development proceeds in individual children.

An example of a study that incorporates many of the recommendations

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listed above is the teaching "experiment" of Steffe, Spikes, and Hirstein (Note 2). The purpose of their study was to investigate whether two clusters of Piagetian variables, class inclusion and conservation, are prerequisites for first-grade children's learning certain number concepts. Twenty-nine individual measures were clustered into the two readiness variables and seven achievement variables. Subjects were divided into two groups. For one group, instruction was carefully designed and monitored. The other group received regular classroom instruction. The treatment consisted of approximately 40 hours of instruction over a three-month period. The results are complex and difficult to summarize, but evidence indicated that conservation was not a prerequisite for learning some number skills, the learning of conservers was qualitatively different from the learning of nonconservers. Specifically, the conservers could transfer their learning to an unfamiliar task, whereas the non-conservers generally could not. This conclusion was possible only because of the completeness of the dependent and independent variables and the duration of the instructional treatment.

Individual Differences in Development

For the most part, individual differences have been virtually ignored in the study of cognitive development. Wohlwill (1973) observes that "the real problem appears to be the failure of psychologists at either end to come to grips with the question, how developmental and differential foci may effectively be integrated into a coherent whole" (p. 333). Such an integration may take several forms. One involves the study of individual differences in development. A second involves the study of the development of individual differences. In the first case the variables of interest are those that have traditionally been of interest in the study of cognitive development. There is overwhelming evidence that individuals differ significantly in their cognitive development. This is attested to by the difficulty researchers have encountered in attempting to identify reliable stages of cognitive development. Any complete theory of cognitive development must include ways to account for and describe these individual differences. In the second case the emphasis is on individual differences. Here the question is what is the origin of individual differences? How do they develop, and how consistent are they over the course of development? For a more complete discussion of individual differences in cognitive development see Kagan and Kogan (1970) and Wohlwill (1973).

The Development of Mathematical Concepts

One of the unique features of research in cognitive development that has made it especially relevant for mathematics education is the fact that much of the research deals with the development of specific concepts, many of them mathematical in nature. Although developmental psychologists are concerned with the development of cognitive structures that transcend the formation of any specific concept, these experiements are designed at least in part to describe the development of specific concepts. The development of number, measurement, space and geometry, and adolescent reasoning are areas that have received particular attention. What follows by way of summary is highly selective. For more complete accounts, see Brainerd (1973b, 1976), Bryant (1974), Churchill (1961), Flavell (1963, 1970), and Ginsburg (1975, 1977a) on number; Carpenter (1976) and

Carpenter and Osborne (1976) on measurement; Lesh and Mierkiewicz (1977), and Martin (1976) on space and geometry; Flavell (1963, 1977) and Neimark (1975) on adolescent logical reasoning; and the general discussions by Beilin (1969) and Wallach (1969) on conservation.

Number

Much of the early research on number assessed children's ability to perform conventional arithmetical operations (counting, adding, subtracting, etc., cf. Brownell, 1941). Current research is no longer concerned simply with identifying which problems are most difficult or how many children at a given age can solve a certain type of problem. The focus has shifted to an attempt to explain the development of basic number concepts and to characterize how children solve problems, not simply whether they can solve them.

Current research on the development of early number concepts can be categorized into two major lines of investigation. The first attempts to explain the development of primary number concepts in terms of the development of underlying logical operations. The second is based on the hypothesis that the development of number results from the integration and/or increasingly efficient application of certain number skills like counting, estimating, subitizing, comparing, and matching. Although there are prominent exceptions and some of the research is difficult to categorize, much of this research tends to be mechanistic in character, whereas number research based on primitive logic is almost exclusively organismic.

Logical foundations of number. Although McLellen and Dewey (1896) called attention to underlying mathematical assumptions over 80 years ago, it is the work of Piaget (1952) that provides the focus for current attempts to explain children's concept of number in terms of the development of logical reasoning abilities. Piaget's influence has been so great that it has led Flavell (1970) to observe, "Virtually everything of interest that we know about the early growth of number concepts grows out of Piaget's pioneer work in the area" (p. 1001).

For Piaget number is a synthesis of class and asymmetrical relation. In assigning a cardinal number to a set, one disregards the differences between elements and treats all the elements of the set as though they were members of a common class, ergo the class or cardinal component of number. However, in counting the set to arrive at its cardinal value, it is necessary to order the set--count one element first, another second, and so on. This ordering represents an asymmetrical relation. As a consequence of this analysis, a principle focus of Piaget's research on the development of number has been the study of seriation and class inclusion and the coordination of cardinal and ordinal concepts. The segment of Piaget's investigation of the development of number concepts that has had the greatest impact on subsequent research involves the principle of conservation. Piaget contends that some form of conservation is necessary for any mathematical understanding, and almost a third of his book, The Child's Conception of Number, is devoted to studies of conservation of one kind or another.

Piaget describes a stagewise development of number concepts in which conservation, seriation, and class concepts develop in close synchrony. In the first stage children are dominated by immediate perceptual qualities of an event and give little evidence of logical reasoning. Consequently they do not conserve, are incapable of seriation, and do not understand simple class-inclusion relationships. At this stage only gross quantitative judgments, based on dominant perceptual attributes, are possible. The second stage is a transitional stage. Some progress is made on all fronts, so that children can construct series and correspondences. But they still have difficulty when either is transformed. Cardinal and ordinal concepts have developed to a great extent, but since they have not been integrated, children cannot relate them to each other. Finally, in the third stage, the development of conservation, class inclusion, and seriation is complete; and the child achieves an operational concept of number.

Most of the replications of Piaget's research on number have concentrated on a single task, most frequently conservation. On the whole, these studies have confirmed Piaget's account of the progression of behaviors exhibited for each of the individual tasks. Furthermore, these replications have demonstrated that the errors exhibited by young children are not experimental artifacts and do not result simply from children's failure to understand the questions asked. On the other hand, studies that have included a variety of Piaget's number tasks have found a great deal less synchrony than described by Piaget (cf. Dodwell, 1960, 1962; Wohlwill, 1960).

A different organization of the logical operations that underlie number is proposed by Brainerd (1973b, 1973e, 1976) who takes issue with Piaget's contention that an operational understanding of natural number results from the concurrent development of cardinal and ordinal concepts. He contends that such a theory is inadequate from a logical perspective and is contrary to the results of a number of empirical studies that he has conducted. Brainerd proposes that the concept of ordinal number is psychologically more basic than that of cardinal number, and that the former plays a more important role in the early growth of arithmetic concepts and skills than the latter. In fact, Brainerd proposes that much basic arithmetic is learned before cardinal concepts are acquired; and he concludes that the developmental sequence is ordinal number, natural number, and finally cardinal number.

These conclusions are based on a series of different studies. In one set of studies Brainerd found that children perceive ordinal sequences by 3 years of age, but cardinal number does not begin to emerge until about 5. In another set of studies, first-graders were significantly more successful on ordinal tasks than on cardinal tasks, and ordinal number concepts (but not cardinal concepts) were almost uniformly mastered by students who were proficient with basic addition and subtraction facts. Finally, in another set of studies it was found that training was significantly more successful for ordinal number than for cardinal number concepts and that there was significantly greater transfer to basic arithmetic achievement.

Brainerd's results have uniformly supported his position. However, in spite of the range of experimental paradigms he has employed, he has tended to use the same basic items to characterize cardination and ordination. The ordination problems have generally involved some form of transitivity task; the cardination problems are a sort of pseudo-conservation task in which subjects are asked to compare the number of elements in two sets arranged in what can best be characterized the final state of a typical conservation task. It is questionable whether these tasks validly represent either cardinal or ordinal numbers.

Another basic question is whether the observed ordinal-cardinal sequence is a function of basic competence or simply reflects differences in difficulty of the selected tasks. Brainerd (1976) cites the results of a study by Gonchar (Note 3) as essentially supporting his position. However, although Gonchar found the same developmental sequence for Brainerd's tasks, the sequence was reversed when more difficult ordinal tasks were used. This led Gonchar to conclude that Brainerd's ordinal-cardinal sequence is primarily a performance distinction between the tasks used to measure each concept.

Research based on number skills. In counterpoint to the logically based theories of Piaget and Brainerd, there is a growing body of research based on the assumption that the development of number concepts can best be explained in terms of the development of specific number skills. This may involve the hierarchical integration of a number of different skills as illustrated by the work of Klahr and Wallace (1976) and Schaeffer, Eggleston, and Scott (1974); or it may involve the increasingly efficient application of a single skill or a small number of skills. This approach, which usually focuses on counting strategies, is illustrated by the work of Davydov (1975), Gelman (1972a, 1972b, 1977), and Ginsburg (1977a, 1977b).

The sequence of development of different number skills has not been clearly established. For example, Klahr and Wallace (1976) cite evidence to suggest that children subitize (directly perceive) the number of elements in small sets before they count. Gelman (1972a, 1972b, 1977), however, asserts that counting precedes subitizing.

Although there is no consensus on which skills are most productive

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to study or how different skills are hierarchically integrated, there is some agreement that the growth of the ability to count is a central factor in the acquisition of number concepts. Children first learn to count by memorizing a rote sequence of numerals (D'Mello & Willemsen, 1969; Wang, Resnick, & Boozer, 1971). They initially have a great deal of difficulty counting the number of elements in a set and make a variety of errors like counting an element more than once, skipping an element, or counting on after all the elements in the set have been exhausted (Potter & Levy, 1968). Younger children also do not recognize that the number of elements in a set is unaffected by the order in which the set is counted (Ginsburg, 1975). Children first learn to assign numbers to small sets and gradually extend their range (Gelman, 1972a, 1972b, 1977; D'Mello & Willemsen, 1969; Wang et al., 1971). Once they can accurately assign numbers to sets of a given size, number becomes a salient feature of those sets; and they have some understanding of the effect of different transformations on those sets but not on larger ones. However, younger children still have some difficulty attending to relevant attributes in more complex situations, and counting does not insure correct responses in typical conservation problems (Carpenter, Note 4).

Although this line of research does not accept that the development of basic number concepts depends on underlying logical operations, the existence of such constructs as conservation is generally acknowledged. In fact these theories often try to explain the development of concepts like conservation in terms of the application of number skills. For example, Gelman (1969, 1972a, 1972b, 1977) hypothesizes that conservation failures do not reflect an immature conception of number, but that they occur because children center on different attributes of the array and do not attend to numerousness. Conservation emerges as the child learns to attend to the appropriate attribute. Gelman denies that conservation is a prerequisite for understanding basic number concepts. Instead she proposes that conservation develops through a growing sophistication to apply counting and estimating strategies. Unlike the very ephemeral conception of number that Piaget attributes to young children, Gelman contends that number is a stable and salient property of a set, provided that the number of elements is within a range that a child can reliably count.

Gelman contends that children first learn to deal effectively with small numbers. Provided that they apply counting or estimating strategies, they will conserve and recognize the effect of adding or subtracting an element for sets with a small number of elements. They will fail, however, to generalize these operations to larger sets. In other words, these responses are restricted to a domain that the child can count. As the ability to count is extended to larger numbers, there is a commensurate increase in the domain of understanding the effect of different transformations. When children finally realize that numbers are infinitely constructible by the continued addition of units, they can generalize the basic operations and conserve number in all situations.

Gelman (1972b) makes a critical distinction between "estimators" and "operators".

> The cognitive processes by which people determine some quality, such as the numerosity of a set of objects, are termed <u>estimators</u>. The cognitive processes by which people determine

the consequences of transforming a quantity in various ways are termed <u>operators</u>. (p. 116)

A similar distinction is made by Ginsburg (1975), who identifies three cognitive systems that children possess. System 1 includes conservation and other processes that are used to make quantitative judgments without counting. System 2 involves the various counting strategies that children develop independently of formal instruction. System 3 involves the formal knowledge transmitted through instruction. Ginsburg proposes that in individual children the three systems may be relatively independent of one another or may show some degree of integration. He suggests, however, that even though the study of System 2 and System 3 concepts will help explain children's learning of mathematics, the study of System 1 concepts is not productive in explaining children's learning of mathematics concepts.

<u>The Development of Arithmetic Operations</u>. It is not immediately clear how the research of Piaget on the development of early number concepts might be extended to study children's acquisition of arithmetical operations. One attempt has involved the correlation of performance on a test of Piagetian tasks with some measure of mathematics achievement (cf. Cathcart, 1971; Dimitrovsky & Almy, 1975; Kaminsky, 1971; Kaufman & Kaufman, 1972; Nelson, 1970; Rohr, 1973; Smith, 1974; Steffe, 1970; LeBlanc, Note 5). These studies have uniformly found high positive correlations, even when IQ is held constant (Kaminsky, 1971; Steffe, 1970; LeBlanc, Note 5). Furthermore performance on Piagetian batteries administered in kindergarten appear to be excellent predictors of mathematics achievement as much as two years later (Bearison, 1975;

Dimitrovsky & Almy, 1975).

One limitation of correlational studies that is often overlooked in the rush to identify educational implications of research is that they do not specify cause and effect relationships. High positive correlations between performance on Piagetian tasks and arithmetic achievement does not imply that mastery of these tasks is a prerequisite for learning arithmetic skills. In fact Mpiangu and Gentile (1975) found that training on arithmetic skills did not have a differential effect on the learning of conservers and nonconservers. In other words, although conservation was correlated with overall arithmetic achievement, nonconservers benefited as much from Mpiangu and Gentile's instruction as conservers. Thus, conservation was not necessary to benefit from instruction. However, to conclude on the basis of this study that the lack of conservation does not limit children's ability to learn computational concepts would be inappropriate. As did most of the correlational studies, Mpiangu and Gentile's studies relied on superficial measures of arithmetic achievement. Even Piaget would not deny that nonconservers can be taught a variety of arithmetical calculations. From a Piagetian perspective, the important question is, What meaning do the operations have for children? This requires that the concepts have a certain degree of generalizability, transfer, and resistance to extinction.

The significance of the kind of learning measured is demonstrated by the teaching experiment by Steffe et al. (Note 2). Their results indicate that although nonconservers learned many of the same counting strategies as conservers, they learned them in a much narrower sense and could not transfer them to related problems. It is not clear what implications these results hold for instruction. There was no evidence that the nonconservers were harmed by instruction or would have benefited from having instruction deferred. A great deal more research is needed before we understand how the development of conservation affects the learning of other mathematical concepts and operations.

The link between the development of counting strategies and the learning of arithmetic operations is easier to establish. Children's earliest notions of addition and subtraction are built on counting; and even before they receive formal instruction in addition and subtraction, they can solve simple problems using a variety of counting strategies.

Even after several years of instruction on addition and subtraction algorithms, children continue to employ a variety of counting and heuristic strategies. Different strategies involve varying degrees of sophistication and efficiency. For example, younger or less capable children tend to count all the elements in sets representing addition or subtraction problems, whereas older or more capable children may use appropriate counting on or counting back strategies.

Several techniques have been used to study the processes that children use to solve problems involving the application of arithmetic operations. Perhaps the most productive involves the use of clinical interview techniques (Davydov, 1975; Ginsburg, 1976, 1977a). A second approach has been to use response latencies to infer what sort of strategies children apply to the solution of different problems (cf. Groen & Parkman, 1972; Groen & Poll, 1973; Rosenthal & Resnick, 1974; Suppes, 1967; Suppes & Morningstar, 1972; Woods, Resnick & Groen, 1975). This approach involves breaking operations down into a series of discrete steps (e.g., counting by ones). It is assumed that the time required to solve a given problem using a particular strategy is a linear function of the number of steps needed to reach the solution. By finding the best fit between response latencies for subjects solving a variety of problems and the regression equations of possible solution strategies, the most appropriate model can be inferred. For example, to solve 9 - 6, children might count down 6 units from 9, or they might count up from 6 until they reach 9 and keep track of the number of units. For this particular problem this latter strategy would require fewer steps; the counting down strategy would be more efficient for 8 - 2. The evidence to date suggests that there is a developmental trend for children to move from using a single model exclusively to a more heuristic strategy by which they attempt to choose the most efficient strategy.

Most of the research on number has concentrated on the early development of number concepts. There have been only a few clinical studies of the processes that children use to solve more advanced problems, (cf. Erlwanger, 1975; Lankford, 1974, Note 6). Developmental psychologists tend to be primarily interested in concepts that develop somewhat independently of the school curriculum. Presumably this accounts for their singular lack of interest in all but the primary number concepts. This is one of the ways that the focus of cognitive development research in mathematics education should be different from that in psychology. We are primarily interested in school learning; and the limits that children's levels of cognitive development place on their ability to apply and understand algorithms for whole numbers, fractions, and decimals should be a central focus for such research.

Measurement

Although the development of measurement is frequently subsumed under the development of geometry, in many ways recent measurement research seems more closely aligned with research on basic number concepts than with research on space and geometry. The work of Piaget (Piaget, Inhelder, & Szeminska, 1960) provides the focus for much of the recent research on the development of measurement concepts. He and his colleagues found that the general stages of development of number concepts also characterize major phases in the development of measurement. However, for measurement the second and third stages are each divided into two substages and a fourth stage is added.

As with number, conservation is the central idea underlying all measurement. The attainment of conservation and the corresponding notion of transitivity is the hallmark of the first level of the achievement of measurement concepts (Stage IIIA). Measurement further depends on the synthesis of change of position and subdivision so that unit iteration is possible (Stage IIIB). Finally, the development of formal measurement operations is complete with the onset of the ability to coordinate the measures of several linear dimensions so that areas and volumes can be calculated directly from their respective linear dimensions (Stage IV).

Piaget et al. (1960) assessed the development of measurement concepts with a great variety of measurement and premeasurement tasks. In the earliest stages children do not conserve and are unable to apply any sort of measurement operations correctly. In later stages they begin to apply some rudimentary forms of measurement, but they will use unreliable measures like the span of their arms and still rely extensively on visual comparisons. By trial and error they gradually discover that if it takes more units to cover A than B, then A is greater than B. But initially they fail to understand the importance of the size of the units and often count a fraction of a unit as a whole or equate two quantities that measure the same number of units with different sized units of measure.

Conservation and transitivity are attained at about 7 to 8 years. Although this marks a significant stage in the development of measurement concepts, operational measurement is still not achieved. Children in this stage can use a moving middle term transitively but only if it is as long as or longer than the original. Children at this stage can conserve and therefore can compare units. Similarly, they recognize that a quantity is the sum of its unit covering. However, these ideas have not been fused. Children in this stage continue to ignore the size and completeness of units of measure, and consequently unit iteration is not possible. It is also interesting that although children at this stage conserve included area and volume they fail to conserve complementary area or occupied volume. In other words, they recognize the equality of areas and volumes contained within certain boundaries but do not recognize that the amount of space occupied by the object in relation to other objects around it must also be equal. The eventual coordination of change of position and subdivision makes unit iteration possible, but it is not until the onset of formal operations at the age of about 11 to

12 that development is complete and the calculation of areas and volumes is possible. As with the number research Piaget and his colleague's (1960) description of the range of children's responses to individual tasks involving conservation and transitivity has been confirmed, but the relationship between tasks and their place in the development of measurement has been questioned.

Some of the most interesting research on measurement has revolved around the use of different units of measure. A unique feature of the measurement process that distinguishes it from simply counting is the unit of measure. In assigning a number to a set, the units are the individual elements of the set. However, in the measurement process the individual units that are counted may not be distinguishable, and different units may be used to measure the same quantity. This second feature of units of measure has been the subject of a variety of studies.

One study employed a series of conservation and measurement tasks in which children were provided both measurement and visual cues regarding the relationship between two liquid quantities (Carpenter, 1975). In some tasks children had to focus on the visual cues; the liquid was in identical containers and was measured with different units. In others the same unit was used; so children had to focus on the numerical cues since the visual cues were misleading. This study found that, contrary to earlier hypotheses, virtually all first- and second-grade children respond to numerical measurement cues at least as readily as to perceptual cues. However, the majority still center on a single dominant dimension, numerical or perceptual, depending on the problem situation. This leads to both correct and incorrect judgments. But the errors appear to result from an inability to attend to the relevant cues, not misconceptions regarding the relevance of measurement operations. Almost all errors resulted from children responding to the most recent cues, whether they were numerical or perceptual. In fact, there was a greater tendency to focus on numerical cues than visual cues, and virtually all children correctly responded on the basis of number in the simplest measurement situations. These results are consistent with Gelman's (1972a) hypothesis that number is a salient property in arriving at quantitative judgments.

Furthermore, just as counting and estimating operations formed a basis for the development of conservation with discrete sets in Gelman's studies, there is evidence that measurement operations may extend this domain to include continuous quantity. Three of the most successful conservation training studies have used measurement activities to train conservation (Bearison, 1969; Fusaro, 1969; Inhelder et al., 1974). A longitudinal study by Wohlwill, Devoe, and Fusaro (Note 7) found a significant correlation between performance on a set of measurement tasks and performance on a conservation test administered approximately nine months later. Although the data are somewhat tenuous, they support the hypothesis that measuring activities actually contribute to the natural development of conservation and are not limited to laboratory training sessions.

All in all, a fairly consistent, if somewhat illogical, sequence emerges in the development of number and measurement concepts. It is clear that children's logic is not congruent with adult logic. Children who do not conserve length are also incapable of reasoning that this conservation failure should have any consequences for their measurement activities. If children are not asked specific conservation questions, the questions do not occur to them and they blissfully count the units just as any adult would do. As a consequence conservation is not a prerequisite for successfully performing certain measurement tasks.

Children come to school with a well established notion of counting. Number is a salient cue that children readily attend to. However, they still have difficulty controlling their attention and tend to center on a single dominant dimension, sometimes numerical and sometimes perceptual. This leads to a number of correct and a number of incorrect judgments.

Although measurement with a single unit is possible for quite young children, difficulties are encountered relating measures using different units. Here one of the incongruities in the development of measurement concepts is found. It seems logical that children would learn to identify the effect of measuring with different units by observing the results of actual measurement with different units. However, children know that an inverse relationship exists between the size of the unit and the number of units measured long before they are able to apply this knowledge to measurement problems involving several different units (Carpenter & Lewis, 1976). This may account in part for the equally incongruous finding of Inhelder et al. (1974) and Montgomery (1973) that measurement training involving comparisons of measures made with different units of measure are successful with relatively young children.

Space and Geometry

The work of Piaget and his colleagues (Piaget & Inhelder, 1956) also provides a central focus for much of the recent research on young children's spatial and geometric concepts. A central feature of Piaget's characterization of the development of spatial concepts is his distinction between perceptual and conceptual space. "Spatial concepts are internalized

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actions and not merely mental images of external things or events" (Piaget & Inhelder, 1956, p. 454). A young child might be able to perceive the differences between a circle and a triangle but be unable to deal with these differences conceptually. For example, the child may be unable to represent these differences in a drawing or to distinguish between the figures tactically.

Piaget and Inhelder describe three main series of spatial studies--one dealing with topological concepts, one dealing with projective concepts, and one dealing with Euclidean concepts. They propose that certain topological properties like proximity, separation, order, enclosure, and continuity are primitive spatial concepts from which projective and Euclidean concepts emerge. These properties are unaffected by a variety of transformations and, hence, do not require conservation. In projective space, objects are no longer considered in isolation but rather from particular points of view. Thus, the studies in this series characterize children's growing ability to describe objects viewed from a perspective other than their own. Since straight lines are preserved in a projective space, children's ability to construct straight lines is considered to be another measure of their knowledge of projective space.

From a Euclidean perspective, space is viewed as a common medium containing objects with well-defined spatial relationships between them. At an operational level, distance, area, and volume are conserved and measurement is possible. In addition to concepts of distance, relations between objects depend upon a reference system of horizontal and vertical lines. Thus, for Piaget, the ability to conserve and measure and an understanding of the properties of horizontal and vertical lines are the hallmarks of the emergence of an operational view of Euclidean space.

Smock (1976b) characterizes the differences between the three spatial domains as follows: "In short, topological space deals with the internal relations of the isolated object. Projective space deals with the relations of objects to subjects. Euclidean space deals with the relations of objects to objects" (p. 48).

Because of the great variety of tasks Piaget used to assess children's concept of space and the concurrent development across three related spatial domains, it is even more difficult to characterize briefly Piaget's work in this area than in the areas of number and measurement. For a more complete account see Smock (1976b).

Certain parallels exist between Piaget's research on number and measurement and his studies of spatial concepts. Whereas number concepts were grounded in basic logical class and relational concepts, Euclidean space was built on the logically more basic concepts of topology. The course of development also follows parallel paths starting with a stage of gross global judgments and proceeding through an intuitive trial and error stage to a final operational stage. In fact, some of the same underlying factors seem to account for errors in all three realms.

A primary feature of development that seems to affect children's concepts in all areas is the growing ability to control attention, to attend to relevant attributes. Whereas difficulty in controlling attention leads to conservation errors in number and measurement problems, it also appears to contribute to children's difficulty constructing straight lines and their failures on tasks testing their ability to construct horizontal and vertical lines. Failures in both areas tended to result from children's inability to ignore the irrelevant characteristics of the surrounding medium. Children tended to construct straight lines following the edge of the table on which they were constructing them, even when the table was round; and they represented the level of water in a jar as being parallel to the bottom of the jar, even when the jar was tilted at an angle.

As with number and measurement, replications of the Piagetian tasks have found the same range of responses identified by Piaget. The tendency for replications to find a great deal less order and symmetry than described by Piaget holds true for spatial investigation (cf. Dodwell, 1963).

Several comprehensive attempts to expand Piaget's investigations of space have been reported. Laurendeau and Pinard (1970) describe a detailed experimental analysis of five tasks derived from Piaget's work in an attempt to construct a scale of spatial development. A second line of research has been conducted by the Genevans themselves. Laurendeau and Pinard's research has been directed at critically analyzing and validating the earlier work of Piaget and Inhelder (1956), whereas the recent work of the Genevans has attempted to expand the domain of the research to new tasks that deal with new concepts. These include the study of children's ability to deal with reflection and rotation transformations, several studies of children's understanding of the relationship between changes in area and perimeter, and a study dealing with children's ability to describe the characteristics of a Moebius ring. Although the complete report of these studies is not available in English translation, a summary has been reported by Montangero (1976).

Another characterization of the development of geometry concepts has been proposed by the van Hieles (Freudenthal, 1973; Wirszup, 1976). They pick up where Piaget leaves off and describe a development sequence culminating in abstract geometric systems. They propose that the development of geometry proceeds through five levels. In Level I children perceive geometric figures in global terms. Although they recognize and can reproduce squares, rectangles, and parallelograms, they cannot isolate specific attributes of the figures. They also are unable to identify relationships between different figures and do not recognize that all squares are rectangles, all rectangles are parallelograms, and so on. This is similar to Piaget's observation that young children have difficulty constructing class hierarchies in general.

At Level II children can isolate individual attributes of figures. But these are established empirically, and the child does not see that certain properties imply that other properties must also be present. In other words children at Level II may recognize that the opposite sides of a parallelogram are both parallel and congruent but these properties are simply considered to occur concurrently. The child does not recognize that any quadrilateral with opposite sides congruent must be a parallelogram. Children at this level can identify the common attributes of different figures but still do not discern the class hierarchy between figures like squares, rectangles, and parallelograms.

Level III is a transitional level between the essentially empirical geometry of the first two levels and the formal systems of the next two. Deduction must be supplemented with empirical demonstration. Students at this stage see that certain properties must follow from others and understand the multiple classification of geometric figures. But the student's ability to use deduction is still limited and requires the support from the teacher or textbook.

At Level IV a deductive system at the level of Euclid's Elements is complete. But it is not until Level V that an understanding of abstract systems divorced from concrete representations is acquired.

The van Hieles propose that there are distinct discontinuities between levels and that the levels cannot be skipped. Unlike Piaget they propose that the levels develop primarily under the influence of school instruction. Therefore, instruction should be geared to lead students deliberately from one level to the next. Wirszup (1976) reports on the efforts of two Soviet researchers who have based a program of geometry instruction on the work of the van Hieles with striking success.

If the van Hieles' analysis is correct, it would have serious implications for instruction in geometry. Formal instruction in 10th grade geometry begins at Level IV and is preceded in earlier grades by relatively feeble efforts that certainly would be insufficient to lead students through Levels II and III. However, although there is an almost a priori logic to the sequence of development described by the van Hieles, it is not yet clear that the course of development is as rigid as they propose. At this point, relatively little research has been conducted to validate their conclusions. Although Wirszup (1976) reports that the Soviets have conducted extensive pedagogical investigations based on the van Hieles' work, it has yet to attract the attention of American researchers; and the implications for American curriculum are still unclear. It is clear, however, that many, if not most, students fail to master even the basic elements of formal geometry; and the van Hieles' work provides a beginning framework for research in this area.

Adolescent Reasoning

Although the study of cognitive development in children is currently a major focus of research in both psychology and education, the parallel study of adolescence has not fared so well. In general the study of adolescent reasoning is characterized by "the paucity of systematic evidence, by the limited generality of what evidence there is, and by the almost complete failure to relate intellectual development to other concomitant developmental changes which mark this period" (Neimark, 1975, p. 541).

The scarcity of available research makes it impossible to specify with any confidence the precise nature of adolescent thinking. However, on the basis of a comprehensive review of current research, Neimark (1975) concluded that there is a stage of cognitive development beyond, and different from, the concrete operational stage of middle and late childhood. Although Piaget initially proposed that this stage emerges between the ages of 12 and 15, it appears to develop later in many children. In fact, it is not attained at all by some individuals. Furthermore, there is a great deal more variability in the application of the formal reasoning structures of this period than is the case for the concrete operations of earlier stages. Even adults operate at a formal operational level on some tasks but fail to do so on others. Piaget (1972) himself concedes that at this stage individual aptitude, interest, and experience appear to play a significant role in determining which tasks an individual can complete successfully. Although it is conceded that training should be a significant factor at this stage, the specific effects of training are largely unexplored.

The most fundamental property of formal thought is the ability to consider the possible rather than being restricted to concrete reality (Inhelder & Piaget, 1958). At this stage adolescents can identify all possible relations that can exist within a given situation and systematically generate and test hypotheses about these relations. They are also capable of evaluating the logical structure of propositions independent of any concrete referents, and they are able to reflect upon their own thought processes. Formal operations are also characterized by an ability to use more complex classification strategies and to shift the basis of classification more readily. In this stage adolescents are increasingly aware of the demands that tasks place on memory and use more efficient strategies for dealing with them. They also have much greater comprehension of key logical connectives and quantifiers.

In general, the capabilities of formal operational thought appear to be necessary for success in most mathematics beyond basic arithmetic. The construction of formal proofs and the learning of general heuristic strategies certainly appear to depend on formal reasoning processes. These are both areas in which many high school students experience little success. To what degree this failure results from these students' inability to operate at a formal operational level is largely a matter of conjecture, since there is little empirical evidence one way or another.

In many ways, the potential significance of cognitive development research for education may be greater at adolescence than at the earlier stages, where it has been concentrated. Although concepts like conservation and transitivity are logical prerequisites for most of number and measurement, they are not an integral part of what children actually do when generating addition facts, learning algorithms, or making simple measurements. As a result, in spite of an overwhelming number of studies involving these concepts, their consequence for the learning of elementary mathematical concepts remains ambiguous. However, the abstract reasoning skills of the formal operational stage are precisely those that are needed for any real success in high school mathematics. Furthermore, the development of these skills in any given area appears to be much more a function of specific experience than in earlier stages. Consequently it is likely that specific, relatively short term training should have a more profound effect than has been demonstrated by the myriad of conservation training studies. Since many adults fail to attain formal reasoning levels in many areas, it would also be easier to argue that such instruction has some educational value in its own right. While it is generally conceded that experience and training should be a significant factor in the development of formal reasoning, the specific effects of training are largely unexplored at this level.

The study of the development of formal reasoning is a potentially rich area that has not seen the concentration of studies that there have been on conservation and early number concepts. A number of individual studies have dealt with the development of various mathematical concepts at a formal operational level. For example, see the studies on proportionality by Ginsburg and Rapaport (1967), Lovell and Butterworth (1966), Lunzer and Pumfrey (1966), and Pumfrey (1968); the studies on probability by Lovell (1971d); the study of limits by Taback (1975); and the studies on the concept of a function by Lovell (1971c) and Thomas (1975). These studies have just begun to unravel the basic question of how the development of formal reasoning skills affect the learning of mathematics, and the study of formal operations should be a prime area for research in mathematics education in the future. One of the major problems in this regard is the construction of good measures of formal operational thought.

New Directions

Recent research in cognitive development has been dominated by the research and theories of Piaget. In the areas of number, measurement, geometry, and formal reasoning almost all the research of major interest has been conducted either on the basis of or in reaction to his theories (Flavell, 1970). His influence has been so extensive that it has lead Neimark (1975) to observe: "There is only one comprehensive theory of cognitive development, Piaget's. All other contenders are so deeply influenced by and derived from the work of Piaget as to be better classified as shifts in focus or extensions" (p. 575).

Recently, however, several alternative approaches to the study of cognitive development have emerged. Klausmeier, Ghatala, and Frayer (1974) suggest that the general principles of concept learning outlined in their Concept Learning and Development model might be useful in studying the development of basic concepts; and Scandura (1977) has proposed that structural learning theory may provide a productive framework for the analysis of developmental phenomena. Another interesting line of research is proposed by Wheatley, Mitchell, Frankland, and Kraft (1978) who have been examining the implications of hemispheric specialization for cognitive development. Recently translated Soviet research provides an especially rich source of ideas for cognitive development research in mathematics education. Of special note are the recently available works of Vygotsky (1978) and Krutetski (1976) and the fourteen volumes of the Soviet Studies in the Psychology of Learning and Teaching Mathematics. Another potentially productive approach involves the application of information-processing theories to the study of cognitive development.

Soviet Studies

Like Piaget's research much of the Soviet research has relied on qualitative methods and has focused on mental operations and other processes that children use to solve problems. However, whereas Piaget and most Western psychologists have focused on concepts that presumably develop independently of the school curriculum, the Soviets maintain that cognitive development and school learning are inexorably linked. "In the final analysis, a pupil's mental development is determined by the content of what he is learning. Existing intellectual capabilities must therefore be studied primarily by making certain changes in what children learn at school" (El'Konin & Davydov, 1975, p. 2). Thus, stages of development are not viewed as absolute; and it is believed that changes in the curriculum can result in significant changes in the nature of the developmental stages through which a child passes. The types of misconceptions that Piaget identifies in early stages of development are attributed to shortcomings in the curriculum, and much of the Soviet research is directed at identifying such misconceptions and reconstructing the curriculum so that they do not develop. The view is also held that the various logical reasoning processes used in mathematics are not strictly a function of maturation and general real world experiences but can be learned through appropriate instruction. The instructional treatments that are used in Soviet research are not the short clinical studies typical of most Western research. Much of the instruction occurs in school settings over extended periods of time, sometimes as long as an entire academic year.

The Soviet studies do not provide the unified theory found in the work of Piaget. Although 6 of the 14 volumes deal with issues involving cognitive development, the studies reported represent the work of many different authors attacking a variety of different problems. Only the works of Krutetskii (Krutetskii, 1976; Kilpatrick & Wirszup, 1969b), Vygotsky (1962, 1978) and possibly El'Konin and Davydov (Steffe, 1975) are presented in sufficient detail to provide anything approaching a unified theory.

Several examples that illustrate the general orientation and techniques of Soviet research follow. The first example reports the results of a study by Gal'perin and Georgiev (1969) dealing with the learning of measurement concepts by young children. The second involves a discussion of several theoretical constructs of Vygotsky's that have potential implications for research in mathematics education. A brief summary of Soviet research in instructional psychology can be found in Volume I of the Soviet Studies series (Menchinskaya, 1969).

The study reported by Gal'perin and Georgiev clearly illustrates the difference between the Soviet and Piagetian points of view. Gal'perin and Georgiev identified many of the same types of conservation and measurement errors found by Piaget. But rather than accepting these errors as developmental phenomena, they attributed them to the traditional emphasis in school mathematics programs on number concepts, which incorrectly characterized units as discrete entities.

To test their hypothesis, they administered a series of measurement problems to the "upper group" of a Soviet kindergarten. They concluded that young children who are taught by traditional methods lack a basic understanding of a unit of measure. They do not recognize that each unit may not be directly identifiable as an entity and that the unit itself may consist of parts. They are indifferent to the size and fullness of a unit of measure and have more faith in direct visual comparison of quantities than in measurement by a given unit.

On the basis of this study, Gal'perin and Georgiev devised a program of 68 lessons that focused on measurement concepts and systematically differentiated between units of measure and separate entities. The lessons were divided into three parts. The first part dealt with forming a mathematical approach to the study of quantities. This section focused on replacing the habit of direct visual comparison with systematic application of measuring units. Appropriate units for measuring different quantities were identified, and measuring skills were studied directly with special attention being directed to the deficiencies identified in the pretest. A variety of units was used, including units consisting of several parts (two or three matches, spoons, etc.) or some fractions of a larger object (half a mug or stick). All these concepts were presented without assigning numbers to the quantities.

Not until the second part was the concept of number introduced. Thus, Gal'perin and Georgiev introduced most of the basic measuring skills and spatial concepts before they introduced numbers. In the third part, the inverse relationship between the size of the unit and the number of units was introduced.

Although the investigation was not conducted with strict experimental controls, the students who participated in this program showed striking gains over the performance of the previous year's students. Whereas fewer than half the students in the previous year could answer most of the items on the measurement test, performance was close to 100% for the experimental group.

Another example of Soviet research that provides a counterpoint to Piaget is found in the work of Vygotsky (1962, 1978). In a recent paper Fuson (Note 8) has discussed at some length how Vygotskian theory might be applied to the study of number concepts. Several of Vygotsky's constructs may also provide a useful framework for cognitive development research in other areas of mathematics education.

One potentially useful construct involves the distinction between spontaneous and scientific concepts. Spontaneous concepts are generated by each child on the basis of concrete experience and the child's own mental effort. Scientific concepts, on the other hand, are the product of direct instruction or interaction with adults. Vygotsky proposes that it is the interplay between spontaneous and scientific concepts that leads to development. The formal structure of the scientific concepts helps to organize the child's spontaneous concepts into a coherent system, and the experiential basis of the spontaneous concepts provides meaning to the scientific concepts at a more elementary concrete level. The significant role attributed to instruction in this theoretical development is characteristic of Soviet psychology and offers a distinct alternative to Piagetian theory.

Another Vygotskian construct that has potential significance for cognitive development research in mathematics education is the zone of proximal development. This is defined by Vygotsky (1962) as "the discrepancy between a child's actual mental age and the level he reaches in solving problems with assistance" (p. 103). This measure, which Vygotsky suggests is an excellent predictor of children's ability to learn from instruction, provides an alternative method of measuring and characterizing development that may be especially appropriate for educational applications.

A variety of other interesting studies deserve the attention of Western researchers. The work of El'Konin and Davydov (Davydov, 1975; Steffe, 1975) on children's early number concepts is especially noteworthy. Although the focus of Krutetskii's work is on individual differences, many of his techniques and results are of interest from the perspective of cognitive development (Krutetskii, 1976; Kilpatrick & Wirszup, 1969b). Volume III of the Soviet Studies Series (Kilpatrick & Wirszup, 1969a) contains four papers discussing the thinking processes children use in arithmetic and algebra, and Volume V (Kilpatrick & Wirszup, 1971) is devoted to the development of spatial abilities.

Information Processing

Whereas Soviet cognitive development research has operated from an entirely different perspective than Piaget, information-processing approaches have generally attempted to build upon Piagetian research. The nature of this contribution is best understood in terms of Flavell and Wohlwill's (1969) performance-competence model. Whereas Piaget has been primarily concerned with questions of competence, information-processing approaches have attempted to incorporate the performance component of the model into their accounts of cognitive development. Instead of analyzing behavior in terms of the logical and algebraic properties of the problem, tasks are analyzed in terms of their information-processing requirements.

> Tasks must be analyzed in much more detail than is provided by a description of their conventional logical structure. The general problem is to determine exactly how the input is encoded by the subject and what transformations occur between encoding and decoding. The objective task structure alone does not yield a valid description of the solution performance, and it is necessary to diagnose the actual psychological processes in great detail to obtain minute descriptions or well supported inferences about the actual sequences and content of the thinking process. (Klahr & Wallace, 1976, pp. 3-4)

A wide range of information-processing theories exist. Although they are all based on an analogy with the computer and are therefore essentially mechanistic, some carry this analogy farther than others. At the most task specific level, the goal is to construct a running computer program that models some segment of behavior. At the other end of the continuum, the computer acts as a sort of metaphor to describe general processing mechanisms. The most extensive attempt to generate computer simulations of developmental phenomena is provided by the work of Klahr and Wallace (1970, 1972, 1973, 1976). Their general modus operandi can be described as follows:

> Faced with a segment of behavior of a child performing a task, we pose the question: "What would an information-processing system require in order to exhibit the same behavior as the child?" The answer takes the form of a set of rules for processing information: a computer program. The program constitutes a model of the child performing the task. It contains explicit statements about the capacity of the system, the complexity of the processes, and the representation of information--the data structure--with which the child must deal. (Klahr & Wallace, 1976, p. 5)

The prominent features of the general architecture of such a system include a short-term memory, which is extremely limited in capacity, and a long-term memory, which is potentially unlimited in capacity. The information-processing system also has access to the external environment and some sort of mechanism for controlling attention that determines which sensory information is selected for processing. The long-term memory contains conditions or rules for processing information. All processing occurs in the short-term memory, and information from the external environment or long-term memory must enter the short-term memory before it can be acted upon.

The strategy is to produce programs that fit the general architecture of the processing system described above and accurately model the general patterns of success and failure at different stages in the development of a given task. Then the question becomes what sort of transition mechanisms are necessary to transform one model into the next. Klahr and Wallace have been relatively successful in modeling different performance levels, but a number of questions regarding transition proceses remain.

At the metaphorical level, one of the most viable information-processing models has been proposed by Pascual-Leone (1970, 1976). The principle focus of this theory regards the capacity of the central processor. Pascual-Leone (1970) hypothesizes that the basic intellectual limitation of children is the number of schemes, rules, or ideas they can handle simultaneously--a capacity that increases regularly with age. The maximum number of discrete chunks of information that a child can integrate is assumed to grow linearly in an all-or-none manner as a function of age. From the early preoperational stage (3 to 4 years), a child's information-processing capacity, or M-power, grows at the rate of one chunk every two years until the late formal operational stage (about 15 to 16 years).

Children frequently do not operate at full capacity and it is proposed that some children have a tendency to operate well below capacity. The ability to operate near capacity is hypothesized to be linked to individual differences in field dependence-independence. Studies by Case (1972a, 1972b, 1974) and Scardamalia (1977) have provided substantial support for the predictive value of Pascual-Leone's model.

Information processing provides a fresh approach to the study of cognitive development that may help resolve some of the paradoxes that have plagued Piagetian theory. For example, by holding constant the information-processing requirements of the tasks, Baylor, Gascon, Lemoyne, and Pothier (1973) were able to eliminate the well documented décalage between seriation of length and seriation of weight. On the other hand Scardamalia (1977) was able to produce décalages in logically isomorphic tasks by varying the information-processing demands of the tasks.

Information-processing approaches may also help account for the rather illogical sequence of development of certain number and measurement concepts and children's ability to complete successfully certain instructional sequences for which they lack the logical prerequisites.

It might be hypothesized that the effectiveness of instruction is more a function of the information-processing demands of the specific tasks than of the development of logical prerequisite operations. In other words, children may benefit from instruction as long as the information-processing demands of the tasks do not exceed their limits, in spite of the fact that they do not possess the prerequisite logical operations. Children's logic is not the same as adult logic. Given appropriate instruction, they may be able to attend to certain relevant dimensions of a stimulus situation and ignore the fact that their judgments depend on certain prerequisite knowledge that they lack. For a further discussion of how the demands of instruction might be geared to the information-processing capacities of the learner, see Case (1975).

Finally, from an information processing perspective of cognitive development, training studies potentially take on a different interpretation.

Although a variety of training procedures have been successful in accelerating the acquisition of various Piagetian operations, there is no evidence that similar training can increase information-processing capacity. One might speculate that the traditional training studies have simply shifted the domain in which established information-processing levels can be applied. For example, one might hypothesize that a lack of sufficient information-processing capacity is the primary cause of conservation failures. Children fail to conserve because they are unable to focus on several dimensions simultaneously. As a consequence they center on a single dominant dimension and fail to conserve. From this perspective it might be hypothesized that the successful training studies have simply taught children to focus on the appropriate attribute but have not accelerated cognitive development in the sense of actually changing basic cognitive structure.

An analysis of cognitive development in terms of information-processing variables seems especially well suited for dealing with educational problems. The emphasis on the existence of internal logical structures and the debate over what evidence is necessary to demonstrate the existence of these structures has never seemed especially germane to the problems in education. We are primarily interested in performance and can leave the question of underlying competence to the psychologists. Our primary concern is whether a child can attend to and learn from a particular instructional sequence. An analysis of both the mathematical skills and the instructional sequence in terms of their information-processing demands provides a potentially productive method for relating the mathematics curriculum to one measure of cognitive development.

Two information-processing variables that show clear developmental trends are children's memory and their ability to control attention (Flavell, 1977; Hagen, Jongeward & Kail, 1975; Peck, Frankel, & Hess, 1975). Many errors in traditional concept development tasks result from children attending to inappropriate dimensions of the problem. Individuals are faced with an overwhelming quantity of information from the environment that must be routed through the central processor in order to be acted upon. This can create a tremendous bottleneck, and the mechanisms of attention which determine which information will be selected for processing are exceedingly important in characterizing information-processing capacity. To plan instruction, it is essential to know what stimuli children can, and naturally do, attend to and how capable they are of shifting their attention from one dimension to another.

Memory is also an important information-processing variable. As children mature, they use increasingly efficient coding, storage, and retrieval strategies and are increasingly aware of the demands that specific tasks place on memory and their own abilities to handle these demands. Mathematical problems place significant demands on memory, and an inefficient use of memory may clog the central processor when its full capacity may be needed to solve the given problem. For example, it is quite difficult for most adults to multiply in base 8, even when they are given preliminary instruction in different number bases and are provided with a multiplication table. To some degree this simulates an inefficient memory strategy.

Potential Educational Applications

Basic research and theory on cognitive development is not focused primarily on educational practice or the teaching and learning of mathematics. However, since the study of cognitive development involves the study of basic intellectual functioning in children and since the specific content under investigation frequently has involved fundamental mathematical concepts, potential applications for the teaching and learning of mathematics naturally come to mind. There are numerous general discussions of the relevance of this body of research for educational practice (e.g., Athey & Rubadean, 1970; Beard, 1969; Brearly & Hutchfield, 1969; Bruner, 1960; Furth, 1970; Ginsburg & Opper, 1969; Hooper, 1968; Kohlberg, 1968; Schwebel & Raph, 1973; Sigel, 1969; Stendler, 1965; Sullivan, 1967; Hooper & DeFrain, Note 9; Klausmeier & Hooper, 1974). Others have specifically addressed the relevance of cognitive development research for the teaching and learning of mathematics (e.g., Copeland, 1974; Huntington, 1970; Inskeep, 1972; Lovell, 1971b, 1972; and Steffe, 1971).

Some authors have attempted to draw specific inferences for educational practice directly from the general research (e.g., Copeland, 1974; Huntington, 1970). Sullivan (1967) and Weaver (1972) have made a strong case that such extrapolation from pure research based exclusively on psychological considerations is inappropriate. What is needed is what Glaser (1976a, 1976b) calls a "linking science" to establish the relationship between the descriptive science of cognitive development and the prescriptive science of instructional design. In other words, fundamental instructional issues cannot be resolved directly on the basis of pure research. Research on cognitive development is descriptive, not prescriptive. It does, however, provide a basis for initiating certain lines of instructional research that could address three basic curricular issues: (a) What content should be taught, (b) when particular content should be taught, and (c) how should it best be presented to the student?

The Content and Sequencing of Mathematical Topics

The first question involves two distinct issues. One involves the question of what content is most important to teach. Kohlberg and Mayer (1972), for example, argue that the aim of education should be to foster development and, in so far as possible, insure that students progress through the basic stages of development identified by Piaget, Kohlberg, and others. A similar argument is found in the van Hieles' proposals regarding the learning of geometric reasoning skills. Although certain elements of the assumptions upon which these proposals are based are potentially subject to empirical validation, the basic issue of what content is most important to teach seems to be based primarily on value considerations. Consequently the implications of this issue for research are minimal.

The second issue involving what to teach is more pedagogical and is potentially of greater consequence for research in mathematics education. Once a specific objective has been identified, there is still the problem of choosing the most effective way to develop the topic mathematically. This involves choosing the mathematical approach, definitions, or models to be used and deciding how to sequence topics.

During the late 1950's and early 1960's this choice was based almost exclusively on the logical structure of the subject. There is a growing awareness, however, that one must also account for the psychology of the child learning the subject.

The basic paradigm involves attempting to trace the natural development of a concept in children and to reflect this natural development in constructing the curriculum. The assumption is that the foundations of many basic mathematical concepts develop naturally, independent of any specific instruction. Through careful investigation, one can identify this sequence and design a curriculum that builds this basic foundation and takes advantage of what a child already knows. In other words it is proposed that certain approaches for developing mathematical topics will be more congruent with children's cognitive development. The task for research is to identify the approaches that are potentially the most productive.

Caution must be exercised in applying this paradigm. There is some question of whether one can identify natural foundation concepts that are independent of current school practice. If children's conceptions diverge significantly from the development in the school curriculum, a reasonable case can be made that this pattern of development is generally independent of the specific curriculum. But if the development of children's understanding of a mathematical concept parallels its development in the school curriculum, it is difficult to separate out the effects of the current curriculum. This might not be as great a problem as it appears. If children's development follows the curriculum, it may not be possible to isolate the specific contribution of the curriculum; but one can have some confidence that the curriculum is not in opposition to the natural sequence of development. Thus, curriculum changes would be needed only if children's pattern of learning differed significantly from the curriculum. However, any attempt to identify the "natural" development of a given concept should include a careful analysis of the potential contribution of the current school curriculum.

A line of investigation that illustrates the application of this paradigm is the work of Brainerd (1973b, 1973e, 1976, 1979) discussed above. He proposes that basic natural number concepts can be developed logically either from an ordinal perspective as evidenced by the work of Dedekind and Peano or from a cardinal perspective in the tradition of Russell and Whitehead. On the basis of his research with young children, Brainerd contends that ordinal number concepts develop before cardinal number concepts, and ordinal number concepts are more closely connected with the initial emergence of arithmetic. He recommends, therefore, that serious consideration be given to abandoning the traditional cardinal development of natural number in favor of ordinal definitions.

Even if Brainerd's conclusions regarding the sequence of emergence of ordinal and cardinal concepts were valid, his recommendations would represent unwarranted extrapolation. No attempt was made in his studies to design and test instruction based on the ordinal definition of number. Furthermore, the examples of ordinal and cardinal concepts included in his studies represent only a very narrow sampling of the concepts involved in the development of either ordinal or cardinal numbers.

It might be more productive to design curriculum to take into account the explicit concepts, processes, and skills that children exhibit throughout their acquisition of a topic rather than attempt to completely redesign it to be consistent with the development of certain underlying logical concepts as proposed by Brainerd. One set of explicit strategies that might be incorporated in the mathematics curriculum involves children's use of various counting strategies to solve simple addition and subtraction problems. Traditionally instruction has failed to take into account the richness and growing sophistication of these strategies. As illustrated by the teaching experiment of Steffe et al. (Note 2), curricula could be developed to build on these strategies rather than portray operations exclusively in terms of set operations.

An alternative to focusing on children's naturally developed concepts and successful strategies is to analyze their errors. By identifying serious misconceptions or significant prerequisite concepts or skills that children are failing to master, instruction can be designed to compensate for these deficiencies. The series of studies by Gal'perin and Georgiev (1969) discussed above is an excellent example of this type of research. Another example is provided by a study of Zykova (1969), in which children were found to lack certain geometric processing skills and instruction was designed specifically to teach these skills.

Matching Instruction to Appropriate Levels of Development

A second potentially significant contribution of cognitive development research for the teaching and learning of mathematics deals with the issue of readiness. The basic problem is to provide instruction that is appropriate for an individual student's level of cognitive development. The question is not a matter of constructing the sequence of instruction, but rather of identifying the specific place in the instructional sequence that is appropriate for an individual student at a given stage of development.

From a mechanistic perspective, this becomes a problem of identifying a hierarchy of prerequisite skills, whether they involve a knowledge of addition facts or an understanding of the principle of conservation, and insuring that students have mastered all essential prerequisites. From the organismic point of view, stages of development are a function of integrated cognitive structures that are not readily altered by instruction. Therefore, it is not sufficient simply to identify a sequence of prerequisite skills or knowledge and insure that a child has mastered it. One must also account for the child's ability to process information. The problem is to match instruction to a child's level of cognitive development rather than simply fit the child into the appropriate step in a sequence of instruction.

A critical difference between the two approaches is that mechanists believe that mental processes operate essentially unchanged throughout development, whereas the organismic view is that there are qualitative differences in the processes available to children at different stages of development. In the mechanistic approach all learning and development is reducible to its component parts and is susceptible to instruction. In the organismic approach certain fundamental processes like conservation or transitivity are representative of basic levels of cognitive functioning that are not reducible to isolated pieces or susceptible to instruction. The level of development puts certain limits on a child's ability to learn from particular instructional situations. These basic limitations cannot readily be removed by specific instruction. Therefore, the problem for cognitive development research is to identify the specific limits for each stage of development and to describe how instruction that is consistent with these limits can be designed.

There are three central problems involved in this endeavor. First, it is necessary to specify the basic dimensions of the individual stages of development. Second, the cognitive developmental requirements of each mathematical topic must be identified so that individual topics can be matched with appropriate levels of cognitive development. Third, it is necessary to devise some means to insure that individual students are provided with instruction appropriate for their level of cognitive development.

It is possible to insure that instruction is appropriate for an individual student's level of cognitive development in several ways. One is to identify age norms for the attainment of given levels of development and then sequence the curriculum so that a topic is taught at the appropriate grade. This approach is illustrated in the article by Huntington (1970) criticizing the grade placement of geometry topics in the SMSG curriculum on the basis of Piaget, Inhelder, and Szeminska's (1960) description of children's development of measurement concepts.

A second approach deals with levels of development on an individual basis. Since it is generally recognized that there are wide individual differences in the rate of cognitive development, this approach should provide a much better match between an individual child's level of development and the child's mathematics instruction. The critical problem for this approach is to develop a valid, reliable measure of cognitive development.

A classic series of studies that addresses the problem of readiness is reported by Washburne (1939). Although these studies are dated and the methods and variables under investigation would not be considered the most appropriate today, this series of studies is noteworthy for its comprehensive attack on the problem. Over 30,000 students participated in the studies over a period of more than 10 years. The purpose was to identify the appropriate placement of arithmetic topics in terms of individual children's levels of development. Development was defined in terms of mental age, and recommendations were based on the initial mental age at which 75 to 80% of the students successfully learned a controlled teaching unit on a specific topic.

The measure of mental age used by Washburne (1939) was, to some degree, a primitive measure of cognitive development. However, tasks used in instruments measuring mental age are chosen for their psychometric properties and may be based on a hodgepodge of different reasoning processes; so it is not possible to characterize different levels of development in terms of specific cognitive skills. Therefore, Washburne was able to establish only an empirical relationship, not a logical one, between mental age and the ability to perform different mathematical tasks.

Current attempts to construct scales of development are based on theories of cognitive development (usually Piaget's) rather than on normative procedures. Piaget hypothesizes that intellectual development proceeds through an invariant sequence of stages. The stages are characterized by the emergence of integrated systems of new cognitive structures that can be applied to a wide range of problem situations.

The hypothesized invariant sequence of development should allow the construction of a series of tasks that characterize sequential levels of development and form a good Guttman scale.

This means that the tasks can be sequenced in an ordinal scale so that it is possible to identify specific tasks an individual can and cannot do successfully by locating the student at a point on this scale. The student should be able to do all of the tasks scaled below that point and none above it. Because individual tasks are representative of levels of development, these results should generalize to other problems that are characteristic of a given level of development. By analyzing school mathematics topics in terms of their cognitive development requirements, it should be possible to specify which ones are appropriate for an individual student's level of cognitive development.

Washburne's argument was somewhat circular and devoid of any cause and effect justification. Chronological age or height or weight could just as well have served as measures of development. The argument was simply th. because a give topic was not mastered by most students until a given age using standard teaching practices, this was the appropriate age to teach the topic. On the other hand ordinal scales could potentially identify the presence or absence of specific logical reasoning processes that are necessary for learning a given topic.

That's the theory. In practice it has proved a great deal more difficult to construct an ordinal scale of development than was originally supposed. A number of researchers have been working on the problem for the last 10 to 15 years with only mixed success (Green, Ford, & Flamer, 1971; Pinard & Laurendeau, 1964; Pinard & Sharp, 1972).

Standardized tests using Piagetian tasks have been constructed (e.g., Goldschmid & Bentler, 1968). But these only substitute Piagetian tasks for traditional psychometric items. They are not true Guttman scales, since the number correct is not reproducible from the ordinal position of the most difficult item passed.

There are two major problems. First, factor analytic research indicates that logical reasoning is not a one-dimensional domain (Kaufman, 1971; Stephens, McLaughlin, Miller & Glass, 1972; Wohlwill, 1973). Therefore it is unlikely that the major dimensions of cognitive development could be incorporated into a single scale. However, this problem could be resolved by profiling development in terms of several scales measuring different factors of cognitive development. Pinard and Sharp (1972) report an effort to coordinate five ordinal scales--space; causality; classification, seriation, and number; conservation; and time, movement, and speed--into an overall test of cognitive development.

The second problem is more severe. It has simply proved extremely difficult to scale any set of relevant tasks into an invariant sequence. The problems of horizontal décalage and the variability introduced by methodological variations have created almost overwhelming difficulties.

> The evidence thus far obtained has about extinguished whatever hope we might once have held that we could place each child on a single developmental continuum equivalent to mental age, and from his score predict his performance on content of whatever kind. (Tuddenham, 1971, p. 75)

This may be overly pessimistic. To date, all efforts to construct ordinal scales have been based on a purely Piagetian rationale. By including information processing variables or some measure of task difficulty in the equation, some of these problems may be at least partially resolved.

Furthermore, it is not clear that task variability poses the same stumbling block for education that it does for psychology. The reason rests in a construct validity-predictive validity distinction. Psychologists have been intent on constructing a scale that locates a child at a given point in Piaget's sequence of developmental stages. They have felt constrained to develop an instrument that clearly identifies a given operation and have become enmeshed in competence-performance issues. In order to identify what mathematics a child is capable of attending to, competence measures are unnecessary. All that is required is to identify a level of performance that generalizes to a range of mathematical tasks. Performance distinctions should be included in such a measure because they are also a part of the mathematical problems and the manner in which they are presented.

In addition, the variability that has been introduced through differences in experience and familiarity with different stimuli may pose less of a problem in constructing an ordinal scale for curriculum purposes. Psychologists have attempted to construct a scale that is reasonably pure in that it is minimally affected by variations in school curriculum. As a consequence they have used stimulus situations that are maximally independent of the content of the school curriculum. Children have a wide range of experiences outside of school, and this creates a great deal of variability in stimulus familiarity. By sticking closer to the curriculum and using terminology and stimulus materials that are part of it, one gains at least some control over one segment of experience, and perhaps can eliminate some of the extraneous error.

If ordinal scales could be constructed, they might provide a relatively efficient measure of cognitive development, which would explicitly characterize individual children's ability to operate with a wide range of intellectual tasks. This characteristic makes them extremely attractive for application to school practice. But although the problems involved in constructing ordinal scales may not be insurmountable, they certainly are substantial; and little progress has been made in constructing ordinal scales that have potential for classroom use.

Although ordinal scales provide an appealing elegance and ease of interpretation, their construction is not the central problem. What is essential is the construction of good measures of children's thinking and the identification of specific relationships between performance on those measures and the learning of particular mathematical concepts. Whether these measures fall into an ordinal scale is not critical. It is important, however, that the measures of children's thinking predict with some accuracy children's ability to learn specific mathematical concepts and skills.

Several alternative directions for developing such measures are possible. They might be based on fundamental developmental variables like conservation, class inclusion, and transitivity that are presumed to develop outside of formal instruction. As noted above, several of these measures have been shown to correlate highly with mathematical achievement. However, with the exception of the study by Steffe et al. (Note 2), little progress has been made in relating these measures to children's ability to learn specific mathematical concepts and skills. In other words, it is not sufficient to demonstrate that there is a difference in overall achievement between conservers and nonconservers. It is necessary to document exactly how they differ and what instruction is appropriate for each group.

A second alternative would be to focus more explicitly on the concepts and processes that children apply directly to the mathematics they are learning. This analysis should go beyond standardized aptitude or achievement tests. Even tests specifically constructed to measure whether children have mastered specific prerequisite skills are inadequate. What is needed are measures of the specific concepts and processes that children apply to the content of instruction and the specific errors they may make. It is very difficult to get this type of information from paper-and-pencil tests. An application of clinical interview techniques discussed by Ginsburg (1976) seems to be the most promising approach. However, in order for this approach to have any impact on educational practice, efficient procedures for applying it in educational settings need to be developed.

A third potential measure is Vygotsky's (1962, 1978) zone of proximal development. Since this measure actually involves adult interaction which represents a form of instruction, it should provide an excellent measure of children's ability to benefit from instruction.

A closely related technique is the application of teach-test procedures to ascertain children's ability to deal with certain types of instruction. Teach-test procedures have frequently been used with mentally retarded children to measure their susceptability to traditional forms of instruction (cf. Budoff, 1967) but have seldom been used with normal children. The basic format involves a short, controlled training session over certain novel and presumably unfamiliar tasks followed by a test on the instructed material. Unlike other measures the initial knowledge or ability to do the task is not the primary concern. What is of interest is the degree to which subjects are able to profit from the instructional sequence. By manipulating the form of the short training session, one potentially can generate a measure of children's ability to attend to, and learn from, different instructional sequences.

A study that illustrates the application of this technique is reported by Montgomery (1973). This study was an aptitude-treatment interaction study that examined the interaction of second- and third-graders' ability to learn unit of length concepts with two treatments based on area and unit of area concepts. Aptitude was measured using a teach-test procedure that partitioned subjects on their ability to learn to compare two lengths measured with different units. Subjects were randomly assigned to one of two nine-day instructional treatments on measuring and comparing areas. The difference between the treatments was the emphasis placed on the unit of measure. In one treatment, subjects always measured with congruent units and compared regions covered with congruent units. In the other treatment, subjects measured with noncongruent units and compared regions covered with different units. On both a posttest and a retention test, the treatment that used different units was significantly more successful in teaching children to assign a number to a region (measure) and to compare two regions using their measures. However, there was no significant difference between the two treatments on a transfer test that included problems involving measurement with different units, and no significant interactions were found between aptitude levels and treatments. The failure to find significant results may, in part, reflect certain anomalies in the development of measurement concepts that were not taken into account (see the discussion on measurement above). But it does illustrate the difficulties and pitfalls in attempting to construct good measures in order to match children to appropriate instruction.

Given the difficulty in characterizing stages of development and constructing good measures of development, it is not surprising that little has been accomplished in analyzing specific mathematical topics in terms of their cognitive development requirements. A very rough first approximation of this task is provided by the Nuffield Checking Up booklets (Nuffield Mathematics Project, 1970, 1972).

Choosing Instructional Strategies

The third potential application of cognitive development theory to problems of education involves the choice of instructional strategies. Cognitive development theory, that of Piaget in particular, has been used to justify a wide range of instructional programs that are based on open classroom, discovery, or activity learning approaches. Two of the most reasoned attempts to formulate general principles for instruction on the basis of cognitive development can be found in Elkind (1976) and Smock (1976a). Hooper and DeFrain (Note 9) report on a number of attempts to apply Piagetian theory to the design of preschool programs.

In general relatively little is known about the specific mechanisms that contribute to cognitive development or how they operate; and in spite of the fact that there exists an abundance of training studies even less is known about how instruction can be designed to take advantage of basic developmental mechanisms. Much more basic research is needed before specific application of the theory to identify optimal instructional strategies is appropriate. On the whole, this does not appear to be one of the more productive avenues for cognitive development research. Although instruction should be consistent with established theories, basic research in cognitive development cannot specify exactly what types of instructional strategies are most appropriate.

Conclusions

It was observed earlier that most cognitive development research is only incidentally concerned with the learning of mathematics. Variables have been selected for their potential value in explaining the general course of cognitive development; and although mathematical topics have frequently been studied, their inclusion has not been motivated by a desire to improve instruction in mathematics. In fact, specific content has been chosen for investigation because it is presumed to develop very much independently of the school curriculum; and much of the content of school mathematics has been virtually ignored.

Rohwer (1970) has argued that if cognitive development research is going to have a significant impact on education, its theories will have to be recast in an educational context and principles of cognitive development will have to be applied directly to educationally significant questions. Thus, the objective for mathematics educators should not be to verify some aspect of a general theory of cognitive development. Rather we should attempt to identify how the theories and techniques of cognitive development can be applied to deal with issues that are significant for the teaching and learning of mathematics. Instead of selecting variables for investigation because of their independence of school experience, we should be primarily concerned with problems that are significant from the perspective of the school mathematics curriculum.

This refocusing of cognitive development research in mathematics should be aimed at the construction of what Shulman (1974) has called middle-range theories. Such theories fall between the task-specific working hypotheses that are generated to explain individual behaviors, errors, and the like and comprehensive theories, such as those of Piaget, that attempt to encompass all of cognitive development. It is not clear that general cognitive structures like Piaget's groupings are especially useful in understanding children's learning of mathematics, and it is a profligate expenditure of limited resources for those of us in mathematics education to expend our energies identifying or validating the existence of such all-encompassing structures. If we can generate middle-range theories that can adequately explain aspects of children's mathematical behavior over limited periods of time we shall have accomplished a great deal indeed.

For example, it should not be the role of mathematics educators to resolve the conflict between the theories of Brainerd and Piaget regarding the development of the logical foundations of early number concepts. A more important question for education is how useful are the theories in explaining children's learning of concepts that are part of the mathematics curriculum. Thus, the question is not whether conservation is a valid construct but whether it tells us anything about children's ability to learn and apply number and measurement skills.

Two general applications of cognitive development research have been identified that seem to hold the greatest promise for influencing educational practice. The first involves selecting and sequencing of content. The second concerns individualizing instruction on the basis of each student's level of development of appropriate concepts and processes. Both applications require a good cognitive map of the development of key mathematical concepts and processes. This map must take into account both individual differences and the effects of instruction. Thus, a major objective for research in mathematics education should be to characterize the processes and concepts that children acquire at significant points in the learning of important mathematical topics. Furthermore, it should describe how these concepts and processes evolve over the course of instruction. This involves describing the different processes and errors that individual children exhibit on key tasks at each stage of instruction. It also should include an analysis of performance on related tasks. Although significant individual differences should be anticipated, it should be possible to identify clusters of children who exhibit similar profiles of performance over a range of tasks. If so, then key problems can be used to identify how individual children will perform over the complete range of tasks.

Finally it is necessary to describe the change in concepts, processes, and errors over the course of instruction. Piaget assumes that all children go through essentially the same stages of development. Therefore, it is necessary only to characterize each stage to describe development. The evidence suggests, however, that there is a great deal of variation in the pattern of acquisition of most mathematical concepts. Consequently to characterize development of these concepts it is necessary to describe how change takes place within individual children, or at least groups of children, over the course of instruction. This means analyzing how certain processes, concepts, or errors at a given stage have evolved from the processes, concepts, or misconceptions of earlier stages. For example, if a child makes certain errors at a given stage, will they be resolved as the child acquires more mature concepts and skills, or will these errors be magnified as new concepts are built on these earlier misconceptions? To assess change effectively within individual children, it is necessary to follow them over the relevant instructional periods. This does not mean that the only appropriate studies are longitudinal ones that continue over the entire course of the development of a given concept. But any study that purports to measure intraindividual change must at least have repeated measures on the same subjects over the time that change is being measured.

Individual children master concepts at different points in an instructional sequence. An important question is whether all children go through essentially the same basic sequence of development in learning certain concepts even though they may pass through a given stage at different points in the instructional sequence. In other words are there certain key prerequisite concepts or processes that all children achieve before they master a given concept? Research should be especially sensitive to identifying such key prerequisites.

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Teaching experiments seem especially appropriate for the research program outlined above. They may involve a few children or many. But they should systematically monitor children's progress through a carefully designed instructional sequence so that the children's specific experiences can be identified. In addition the concepts that children have learned, the processes they are applying, and the systematic errors they are making should be regularly assessed. Clinical interview techniques seem most appropriate for this purpose. Finally, this research should not only describe children's knowledge or performance at a given point in time but should also attempt to characterize their ability to attend to, and benefit from, instruction.

Assuming that it is possible to characterize the development of a given mathematical topic, it is not at all obvious how this information should be appl the design of instruction. Consider the problem of selecting and ,uencing appropriate content. Several alternatives are possible. One is to identify a minimum set of concepts and skills that all children exhibit at one point or another in the acquisition of a given topic and to build instruction around this basic set. This approach is not especially elegant and seems to reduce instruction to the least common denominator. However, one might assume that if one teaches the minimal set of skills that is logically complete and that can be understood by all students, the better students will continue to generate their own more complex strategies. A study by Groen and Resnick (1977) offers some support for this hypothesis.

An alternative approach would be to identify the most efficient processes that children use and/or the processes that are used by the most capable students and teach those specific processes. Although this approach has the appeal of attempting to make the most efficient strategies available to all the students, there are potential drawbacks. The slower students may not have the cognitive capacity to understand or apply the complex processes of the better students, and the complex processes may be very difficult to teach explicitly.

Clearly these extremes do not represent the only choices, and there is a great deal of middle ground. Furthermore, as Resnick (1976) proposes, appropriate instruction should not necessarily copy the natural development of the concepts in children. Instead, it should put learners in the best position to invent or discover appropriate strategies themselves. There is no simple answer to the question of how to select and sequence content, and it is unlikely that a single approach will be effective with all content or for all learners.

Similar problems exist with respect to problems of individualization. Should instruction be congruent with a child's level of development, so that the instructor can be sure the child can attend to the appropriate aspects of instruction? Or should instruction lead development, as suggested by Vygotsky (1962) and others?

Research in mathematics education cannot stop with the description of the development of mathematical concepts. We must initiate Glaser's (1976a, 1976b) linking research to establish how the descriptive information from research into children's thinking can be applied to prescribe instruction. Furthermore, this program of linking research cannot wait until a complete description of the development of a given concept is available. If viable programs of basic and applied research existed, they could interact to the mutual benefit of both. Research into children's thinking could provide the framework for initiating instructional research, and instructional research could identify the types of information about children's thinking that are most useful for making educational decisions.

Like most of the significant problems in education, the problems of characterizing children's thinking and applying this information to the design of instruction are not simple problems that can be answered by a collection of isolated studies. If any real progress is to be made toward resolving these problems, there is a critical need for coordinating the efforts of researchers sharing ideas, identifying and attacking critical problems, and standardizing research techniques.

Four working groups that are dealing with problems relevant to the application of cognitive development to the learning of mathematics are currently operating under the general direction of the Georgia Center for the Study of Learning and Teaching Mathematics at the University of Georgia. They include a working group on number and measurement, one on rational numbers, one on space and geometry, and one on models for learning mathematics. These working groups, which constitute a somewhat loose consortium of individuals at different institutions, offer one of the best mechanisms currently available for unifying our attack on educational problems.

Papers from a series of research workshops at which these working groups were established have been published (Lesh, 1976; Martin, 1976; Osborne, 1976) and several monographs reporting the efforts of different working groups are in preparation.

Cognitive development research in mathematics education must not only unify its efforts to attack significant educational problems; an effort must be made to insure that the results of this research have some impact on school practice. Rohwer (1970) has observed: "The relevance of cognitive development for education is easier to establish than the assertion that a substantial contribution to education will be made from its study" (p. 1380). Although we must avoid premature conclusions and clearly establish the links between cognitive development research and classroom practice, we must not bury our resutls in research journals. Part of the consortium orientation should be directed at including curriculum developers and developing curriculum materials. Unless we can convince teachers and curriculum developers to begin to see some of the problems of education in cognitive development terms, research in cognitive development will have little practical value for the teaching of mathematics.

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