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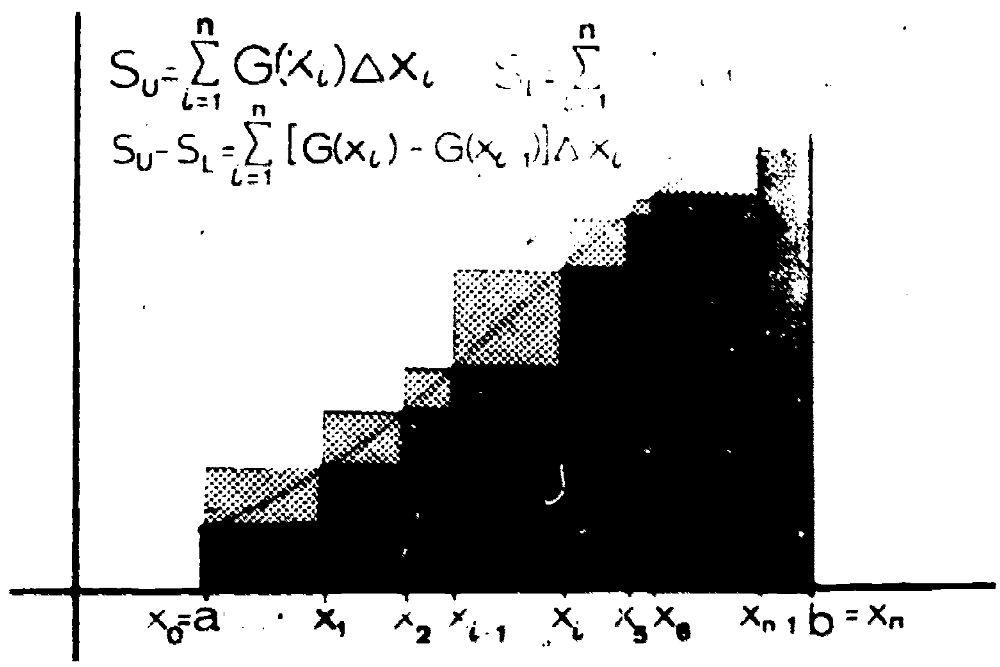
ABSTRACT

This textbook is designed to be used in an independent study calculus course at South Dakota State University. A unique feature of the course is a heavy reliance on color slide transparencies that are used to develop basic concepts graphically. These slides and accompanying commentaries are available for students at an educational media center. Besides doing programed work in the text, students are required to attend a one-hour meeting each week. (MK)

SINGLE VARIABLE
CALCULUS



An independent study course using audio-visual-seminar instruction

--GORMAN R. NELSON

PREFACE

The original motivation for developing this course was to increase the efficiency of classroom presentation of mathematical concepts through the use of graphic aids and photographic projection. The basic concepts occurring in single variable calculus were developed graphically in sequential drawings, then photographed in color slide transparencies. In classroom lecture presentations these slides were used either totally or supplemented with chalk board as detail expansion required. Results of this innovation have been published.

Commentaries on all (about 400) slides were compiled and this, with audio tapes, became the basis for another innovation: this independent study course for college calculus. Results of this are ready for publication.

All teaching innovations for these courses have been conducted at South Dakota State University, which has very excellent audio-visual facilities, including special classrooms and dial access equipment. This has been especially helpful in making this innovation possible.

A grant from the National Science Foundation has provided funding for the time and materials necessary to complete this work over the past two years.

Gorman R. Nelson
South Dakota State University
Brookings, South Dakota
August, 1971

RATIONALE FOR COLLEGE CALCULUS COURSE

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A beginning course in College Calculus should reveal a structure in which the student sees the specific categories of subject matter as parts of a larger system. Such a structure is inherent in the course but generally it is not obvious, and unless some overt motivation exists to synthesize the Table of Contents into a unified structure, the student ends up with a fragmented proficiency in the course. He may well be able to perform specific manipulative skills without ever understanding what larger significance it has.

Felix Klein, the German mathematician, mentions three objectives for the math student:

1. A scientific survey of the structure.
2. Skill in handling problems.
3. An appreciation of the significance of mathematical thought for a knowledge of nature and modern culture.

These objectives are still generally acceptable in math analysis courses, but their relative importance will vary according to the students area of specialization.

Course design factors must be concerned with the environment of the learning process. I wish to consider two situations: 1. The classroom situation with students, teacher and teaching aids. 2. A teaching media laboratory and small unit seminars. The course design is more critical in the second situation since it possesses less of the classroom versatility for supplementation. Course material designed for the second situation will be fully adequate for the first but not necessarily vice versa.

Course content and organization is essentially determined by the nature of the course but its pedagogical appeal is limited only by the skill and imagination of the author and teacher. The content can be judged immediately and the pedagogical appeal can be estimated but generally this requires testing in use.

A Table of Contents for this course requires the insertion of some means for identifying the structure of the system without diminishing the objective of the immediate task.

The appeal here is neither to the historical nor chronological development of the ideas but rather to the idea of mathematics as a tree of knowledge, which one sees as a related whole.

Since the idea of the function is basic to all concepts introduced in this course, such as: limit, derivative, continuity, extreme, integral and many others, it is natural to use this concept as the unifying structure and wellspring of other concepts.

There is a growing opposition today to scientific education of the individual as just a reliable component in a technological society. In particular, eduation in mathematics must extend beyond the boundary of proficiency in technology as a means to furnishing industry with a component in its stratagem.

Scientific education should involve behavioral change. How we look at the universe says something about us. How we see the universe and ourselves in it is enlarged by understanding the mathematical abstractions taken from the universe. A preoccupation with technical proficiency diminishes this understanding and awareness of self in this structure. I do not imply by this that technical proficiency is without value---only that understanding the abstractions must precede it.

The mathematical concepts found in Calculus are abstractions evolving from centuries of man's observation of the universe. These abstractions put into mathematical thought are not easily comprehended from a definition. It is questionable that an abstraction can ever be taught explicitly but by presenting the concept in several forms the student grasps the abstraction as the common property of several phenomena. Then a definition has meaning.

Motivation for understanding math concepts should not be left to chance but should be inherent in the mathematical environment to which the student is exposed.

THE FOLLOWING INFORMATION IS GIVEN TO AID YOU IN DIRECTING YOUR SCHEDULE TO FINISH AT THE END OF THE TERM:

Tests are scheduled as shown for each unit. These tests are to serve you as a measure of your proficiency over the unit. If your tested proficiency is not a "C" level or better you may take an additional test after further study.

You must attend the scheduled meeting for one hour each week. Time and place will be determined to fit your schedule. This period will be used for personal aid as needed and discussion of material from the Educational Media Center. You may spend as much time in the Center as is needed to master the material presented. It is important that you respond to all programmed material pertinent to our weekly meetings.

If necessary, additional personal help will be provided to meet your special needs.

All programs in the Educational Media Center are listed in the E.M.C. Directory.

Don't make the mistake of doing your programmed work "tomorrow". This will inevitably lead to poor comprehension and retention of knowledge.

If you don't understand -- ASK !!

	UNIT	No. of Programs	No. of Prog. in E. M. C.	No. of Sets of Problems	Tests Dates
1.	Functions	12	4	4	
2.	Limits	20	7	6	
3.	Derivatives	17	6	5	
4.	Applications of Derivatives	18	8	5	
5.	Integral	18	8	5	
6.	Applications of Integral	5	3	2	

Fifteen school days are allowed for each unit. You may go faster if you wish, but a slower speed will make it difficult to finish by the end of the semester.

MATH ANALYSIS I: Calculus and Analytic Geometry has more to do with concepts as a way of thinking than a supply of recipes to be used as the occasion demands.

<u>These are the resources available to you in this course</u>:

1. Textbook: Refer to the Reading Program at the end of FUNCTION PROGRAM.
2. The Educational Media Center. Most of the ideas or concepts will be presented here in audio and visual means. Sample problems will be worked also. You may use this facility any time it is open.
3. This course guide will direct your work and coordinate it with the E.M.C.
4. Seminars will be held once a week (1 hour) to discuss any question relative to this work.
5. Private tutorial help will be available for your special needs.
6. Work which you hand in will be corrected and returned for your benefit.
7. A basis for grading will be discussed at a seminar meeting.
8. You can proceed at your own pace, but you <u>must</u> finish by the end of the semester. You may attain any degree of proficiency you choose.

FUNCTION PROGRAM

The function concept is basic in all fundamental ideas of calculus. In fact, calculus may be considered as a special analysis of functions. The program which follows will guide you in becoming proficient in the use and understanding of the function concept. Follow it carefully.

You must take whatever time is necessary to meet the implied objectives. Programs are numbered at the left margin. Refer to the Reading Program for reading assignments.

PROGRAM

1.Lecture: Educational Media Center

Subject: Basic Function Concept. (Lecture)

1. Develops meaning of a function
2. Defines function.
3. Shows how a function is developed.
4. Algebraic and Transcendental Functions.
5. Classes of Function.
6. Range and Domain.

Objectives for each concept are implied in the text and lecture content and reinforced in work sheets and problems.

2. Read: Refer to Reading Program #2 at the end of the function Program.

3. Problem Study: **Educational Media Center**

You may, if you prefer, work through the following examples before going to the center for confirmation of your work. Or, you may work in the center and confirm your work after each example.

These examples and the problems which follow are intended to help you understand and apply the function concept.

Example 1.

Given: $f(x) = x^2 - 3x + 1$

Find: $\dfrac{f(x + h) - f(x)}{h} \qquad h \neq 0$

Function notation directs you to find $f(x + h)$ when the function given is

$$f(x) = x^2 - 3x + 1$$

Hence,

$$f(x + h) = (x + h)^2 - 3(x + h) + 1$$

Simplify and subtract $f(x)$. You may divide by h since $h = 0$ is not included in the domain.

Example 2.

Find the equation of a line passing through the points $(2, f(2))$ and $(5/2, f(5/2))$ where f is defined by:

$$f(x) = 2x^2 - 5x$$

Use the two point form for the equation of a line. Use function notation to find the two points. For instance, (2,f(2)) becomes (2,-2). Put your answer in the form:

$$ax + by + c = 0$$

Example 3.

Write the equation defining a function G such that G(A) is the surface area of a sphere, and A is the great circle area of the sphere.

The surface area of a sphere is

$$A = 4\pi r^2$$

Example 4.

If: $A_{n+1} = \frac{1 - A_n}{1 + A_n}$ (n + 0,1,2.....and A = x)

Express:

A_1, A_2, A_{10}, A_{11} as functions of x.

4. Problems: Refer to Problem Assignment Program -- Functions.

1. If $f(x) = x^2 - x + 1$, find:

   a. f(0) b. f(-2) c. f(2½)

   d. f(a) e. f(a + h)

2. If $f(x) = x^2 - 5x + 6$, find:

   a. $\frac{f(x) - f(a)}{x - a}$, $x \neq a$

   b. $\frac{f(x + h) - f(x)}{h}$, $h \neq 0$

3. If $F(x) = \frac{x^2 - x - 2}{x - 2}$, $x \neq 2$

$G(x) = x + 1$

$$H(x) = \begin{cases} x + 1, & x \neq 2 \\ \frac{1}{2}, & x = 2 \end{cases}$$

Explain carefully how the functions defined by these equations differ and how they are similar.

4. Find the zeros of the functions defined by:

a. $F(x) = x^2 + 2x + 1$

b. $f(x) = x^2 + x - 1$

c. $h(x) = \frac{x^2 - 1}{x-1}$

d. $g(x) = x^3$

3. $H(x) = 4 - x^2$

5. Find the slope of the line connecting the two points (1,f(1)) and (-1,f+1)) for the function defined by:

$f(x) = x^2 - 3 + 1$

6. An open top box is constructed from a flat sheet 8" x 4" by cutting out corners a" square. Develop the equation defining a function of all such boxes. What is the domain and range for this function?

7. If $H(x) = \frac{x^2 - 1}{x-1}$ $x \neq 1$

Find H(1), H(.1), H(.01), H(.001)

As x approaches 1, what does H(x) appear to approach?

Refer to Program 4. for problem assignment at the end of this Function Program.

Seminars: Every week (1 hour)

5.Read: Refer to Reading Program # 5 --Functions

6.Problem Study: Educational Media Center: Refer to Center Directory.

Since functions are so important in this course, anything that can be done to increase the comprehension of the nature of functions is also important. The graphing of function is a visual means of grasping some of the basic qualities of functions and is the sole reason for graphing. At this level we must resort mostly to positioning points (ordered pairs) on the cartesian plane and connecting these to form the graph.

You will probably be unfamiliar with most functions defined in the set of problems. This is part of the exercise, to develop in yourself the ability to analyze different functions graphically. Later in the course we will supplement point plotting with additional techniques in graphing.

Consider now:

<u>Example 5.</u>

The defining equation is

$$g(t) = |t + 1|.$$

The independent variable is t, the dependent variable is g(t). The most difficult part is how to handle absolute value notation. We should naturally be motivated to recall how this is defined, i.e.

$$|t + 1| = t + 1 \text{ if } t + 1 > 0, (t > -1)$$

$$|t + 1| = -(t + 1) \text{ if } (t + 1) < 0 , (t < -1)$$

Now try to graph this in two parts to accomodate the two cases, i.e., $t > -1$ and $t < -1$.

Example 6.

Graph the function defined by:

$$C(x) = 2^x + 2^{-x}$$

This problem like many others in this section is important not only as an exercise in graphing but as a means of becoming familiar with a function which will be used later. So, while doing the work, you should also observe the nature of the function revealed by your graph. Don't get so absorbed in the technique of graphing that you fail to observe the nature of what you have done.

7.Problems: Graphs of Functions. Refer to assignment Program #7 -- Functions.

8. Read: Refer to Reading Program # 8 -- Functions

9.Problem Study: Educational Media Center

Combination of Functions. This is a difficult section and requires more than the usual amount of effort.

Example 7.

Probably the most difficult part of this section is that dealing with the composite function. In this problem we are to find the composite function f o g where,

$$f(x) = x^2 - 1$$

$$g(x) = 3x + 1$$

f o g means to find f(g(x)). If the math notation for functions is observed carefully, then when

$$f(x) = x^2 - 1$$

$$f(g(x)) = (g(x))^2 - 1$$

Complete the problem.

Example 8.

The purpose of this problem is essentially to enable you to become familiar with the manipulation involved in composite functions.

If $f(x) = \sqrt[3]{x}$,

then $f(g(x)) = \sqrt[3]{g(x)}$

and we must find g(x) such that

$$\sqrt[3]{g(x)} = x$$

Example 9.

The objective again is to provide you with means to master the manipulative technique for the composite function.

If $F(x) = x + 1/x$

then $F \circ F = (x + 1/x) + \dfrac{1}{x + 1/x}$

Simplify this and determine the domain.

10.Problems: Combination of Functions. Refer to Assignment Program -- Functions. Hand in.

11.Review: Repeat PROGRAM 1. Refer to Assignment Program -- Functions.

12.Problems: Refer to Assignment Program -- Functions. Hand in.

READING PROGRAM -- Functions

All reading assignments refer to your textbook, Calculus with Analytic Geometry, Johnson and Kiokemeister, 4th Edition.

Program # 2. Definition and Types of Functions. Pages 58 - 60

5. Graphs of Functions. Pages 63 - 66.

8. Combination of Functions. Pages 67,68.

PROBLEM ASSIGNMENT PROGRAM -- Functions

All references are to a. Textbook - Johnson and Kiokemeister, and b. your Program Manual.

Program # 4. Work and hand in Problems 1 - 7, Program Manual.

7. Work and hand in Problems I. 1 - 6, II. 1,3,5,7, 9.11. Textbook, Page 66.

#10. ork and hand in Problems I. 1 - 8. Text Page 68.

#12. Work and hand in Problems 1, 2. Textbook, Page 70.

The following commentary on Functions may be used with PROGRAM 1.

LECTURE --- FUNCTIONS

Slide 1. We are defined by how we look at the universe.

2. All of calculus is based on an understanding of the function concept. Without this, very little of what follows will be understood.

3. The mathematical idea of a function is one of the most fundamental concepts in theory and application of mathematics.

Since all the new mathematical concepts which occur in calculus have their genesis in the function, it is natural to evolve these from the nature of the function.

But first, what is the mathematical idea of a function?

4. We begin with a set of real numbers which is shown as set A.

5. From the elements of set A we will generate another set of elements (y) according to some rule. Call the new set B. Note that if we wish to speak of one element of set A without identifying it we call it x. That is, we can say the element x which is contained in set A is either 1, 2, 3 or 4. The same notation is used for set B. As yet we don't know what the elements y of set B are.

6. A simple rule, which is represented by the lower case f, is chosen. The rule f is: Multiply each x in A by 2 and add 1 to form a corresponding y in B. We now have a specific means for generating set B. From this procedure some mathematical notation is also generated to help us understand what is going on.

7. The flow of action is from set A to set B. Each element in A is operated on according to rule f. The corresponding element in set B is identifiable by the notation used here. For instance, in set B, the letter f followed by 1 in parentheses, which is read f of 1 means that rule f is to operate on the element 1 of set A to produce the corresponding element y in set B. f(2) means rule f operates on the number 2 of set A to form its corresponding element y in set B. In like manner f(3) and f(4) are elements in B corresponding to 3 and 4 in set A.

By use of this functional notation we are able to observe immediately the correspondence of elements between sets A and B. That is, we know 1 goes with f(1), 2 with f(2) and so forth.

Some new descriptive words are often used here to express this idea of corresponding elements. For instance, we can say that f(1) in set B is the image of 1 in set A. This is a pictorial way of saying if element 1 is exposed to rule f it reflects the image of 1 as f(1). In like manner f(2) is the image of 2 and so on.

Still another descriptive picture evolves from this process of forming set B. We can say that 1 of set A maps into f(1) of set B, 2 maps into f(2) and so on.

The distinction should be noted here that the notation for the rule is f while the notation for each element y of B is f(1), f(2) or in general f(x).

These various mathematical notations are important to a clear understanding of the literature of mathematics.

8. We simplify again by using the mathematical notation to express the rule f. The expression 2x + 1 says very succinctly what rule f is.

9. Henceforth we will generally use only the mathematical form to express the rule.

Performing the operation indicated by f(1) will produce 3. f(2) will produce 5, f(3) = 7 and f(4) = 9.

10. The elements y of B derived from the elements x and rule f are shown here in correspondence with the elements of set A. The rule f could be written y = 2x + 1 which as an equation then defines the y value for a given x value. If we wish to represent an equation in a general form, that is, without stating the rule precisely, we say y = f(x). The variable nature of x and y in the form of an equation is indicated but we are stating precisely what values x may have here by listing them.

11. It is possible now to form ordered pairs of numbers from the corresponding elements of each set. In doing this there is the natural tendency to put the x element first and the y element second as: (1,3), (2,5), (3,7) and (4,9). The notion of ordered pairs evolves from this.

The set f of ordered pairs is called a function. The elements of set f are the ordered pairs. Each ordered pair has a first and second component. The set of first components of each ordered pair are given the name of domain. These are 1, 2, 3, and 4. The x notation which represents each member of this set is often given the descriptive name of "independent variable" for the reason that the y value is determined by the x value. The set of second components 3, 5, 7 and 9 taken from the set f is called the range. The y notation which represents these numbers is called the <u>dependent</u> variable since its value <u>depends</u> on the value of x.

No mention has yet been made of restrictions in either set A or B. The idea is now presented that whatever rule or device is used to form the set it must contain the restriction that the final set does not have duplication of first components. That is, no first components can be repeated. For instance, if our first two ordered pairs had been (1,3) and (1,5) then this set would not be called a function. The reason for this certainly does not evelve from anything we have done thus far. The explanation lies in reasons not yet apparent. One immediately acceptable reason is that we want no ambiguity as to what value of y we have corresponding to any value of x in the function. This does not preclude having any duplication of x components in the ordered pairs. Additional reasons for this restriction will become apparent as our investigation continues.

Notice that the same notation f is used to identify the set of ordered pairs as was used to identify the rule. This will cause no ambiguity in understanding and is a generally accepted notation. These two meanings are certainly not one and the same. One cannot always tell from the rule what the set of ordered pairs will be. But for convenience we will accept the duplication of letter identity for both meanings. In context this causes little or no concern.

12. For convenience we define a function as: "A set of distinct ordered pairs having no two first components the same." Identical ordered pairs are accepted as one ordered pair.

You will notice that the elements of this set are the ordered pairs and that the parts of the ordered pairs are called components. The set of first components of each ordered pair constitutes the domain and the set of all second components constitutes the range.

13. Here are shown set f, g and h. Are they functions? Consider set f which contains three elements of ordered pairs. No two first components are the same, hence it is a function.

Consider set g. Here again we have the elements of ordered pairs, but since the first two are not distinct they count as only one ordered pair. The two distinct ordered pairs do not have repeated first components, hence this is also a function.

Consider set h. Again we find three elements of ordered pairs but now we also find the first component of each ordered pair are the same; hence this set is not a function.

14. Consider now a new rule which we identify by the lower case letter g. The rule is in mathematical shorthand and explicitly states that for every x we consider we will find the corresponding g(x) by squaring each x and subtracting 1. However, nothing is said about what values of x should be chosen. When a function is defined without a specified domain we will assume it defines the set for all real values of domain and range so defined.

15. We define the domain here as the numbers 2, 1, 0, -1 and -2. What are the corresponding elements g(2), g(1), g(0), g(-1) and g(-2)?

For g(2) we square 2 and subtract 1 to get 3.

16. In like manner we get g(1) = 0, g(0) = -1, g(-1) = 0 and g(-2) = 3 to form the ordered pairs as shown.

This is the function g.

17. The zeros of a function are those values of the domain

for which the corresponding value of the range is zero. We observe here that the zeros are at 1 and -1. Algebraically these may be found by setting $g(x) = 0$ and solving $x^2 - 1 = 0$ for the corresponding values of x.

18. The domain is here defined explicitly as being those values of x in the closed interval [-2,2]. Since all possible numbers included between and including -2 and 2 are an infinite set they cannot be enumerated. Hence the missing ordered pairs are indicated by three dots as shown.

19. Another means for defining the set of ordered pairs is shown here. This is read "The set of ordered pairs $(x,g(x))$ such that $g(x) = x^2 - 1$ and x has all values in the closed interval [-2,2]".

20. The number of different functions is unlimited. However, most of the functions we work with can be classified by obvious characteristics which they possess. Most of these you are familiar with in varying degrees. There is no intention here to imply these are the most important classes, but only to observe that many functions have qualities which permit them to be classified by name. For instance, 1. is representative of linear functions.

All of these classes of functions will be examined later.

21. A function defined by an equation for which no specific

domain and range is given has a domain implied as being those real numbers x which define a real number y.

For instance, consider the equation $y = x + 1$. It is apparent that each real number x defines a real number y. In this case the implied domain is all real numbers. Such a domain yields all real numbers as value of y, hence the range also contains all real numbers.

22. The implied domain of the function defined by the equation $y = \sqrt{x^2}$ is not quite so obvious. First, it is necessary to know that $\sqrt{x^2}$ is, by definition, always positive. That is; $\sqrt{x^2}$ is x if x is positive, $\sqrt{x^2}$ is -x if x is negative, and zero if x is zero. Hence the domain may be any real number x. The range implied by this domain will be all real numbers greater than or equal to zero.

23. The implied domain for the function defined by

$$y = \sqrt{x^2 - 1}$$

differs from the previous function in that the radicand $(x^2 - 1)$ must be greater than or equal to zero if it is to have a real number as its square root. For instance, if $x = 0$ the radical -1 yields no real number. The condition to be satisfied by the radicand must be

$$x^2 - 1 \geq 0$$

or

$$x^2 \geq 1.$$

This implies $x \geq 1$ or $x \leq -1$ as the largest domain. For these values of x the corresponding y values include all positive numbers and zero.

24. The three equations:

1. $H(x) = \dfrac{x^2 - 1}{x - 1} \qquad x \neq 1$

2. $F(x) = x + 1$

3. $G(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & x \neq 1 \\ 3/2 & x = 1 \end{cases}$

all define functions. Careful consideration of these equations will reveal that F(x), G(x) and H(x) have the same implied domain excepting at x = 1. G(x) and F(x) both include x = 1 in the domain but have different corresponding values of G(1) and F(1). H(x) does not include x = 1 in its domain. You should observe carefully that the functions, as sets of ordered pairs, defined by these three equations are identical except for the ordered pairs at x = 1. H has no ordered pair at x = 1. For G the ordered pair is (1,3/2) and for F the ordered pair is (1,2).

F(x) and G(x) have the same implied domain, that is, all real numbers. Their ranges differ only by one value.

Record and study these equations. They will be used later in consideration of limits.

25. Some of the greatest scientific discoveries are couched in the language of the function. Equation 1. defines a

function. t is the independent variable producing values of x. Science and engineering are predominantly engrossed in the study and investigation of functions as mathematical models.

Equation 3. is Einstein's discovery relating energy and mass. What a profound discovery to be expressed in such simple mathematical language.

These are but a few of the many equations shown here defining functions with which you are probably already familiar.

Consider equation 5.

26. This is Newton's discovery in which he expresses force of gravitation as a function. Assume m_1 to be the mass of the earth, m_2 the mass of the moon. Then if G is constant, and the independent variable d is given, the set of ordered pairs (d,F) expresses the phenomena of gravitational force. The means of communicating this law of gravitation is mathematical. Probably one of the strangest phenomena in all of science is the capacity of mathematical thought to portray aspects of nature. This is somewhat less surprising if one considers that these abstractions were derived from observation of nature.

27. Title -- Elemental Creation of a Function

Most functions which are developed in science and engineering are probably not the major scientific discoveries but involve more of detailed analysis. To show what sense a function is used as a mathematical model we first take a simple condition and show how a function can be generated to portray one aspect of this condition and from the defining equation we will observe in detail how this function is used to further comprehend the nature of the condition it describes.

28. First consider a sheet of material having the dimensions shown.

29. We form a box by cutting squares from each corner as shown here. We can intuitively discern that in some way the box volume depends on the size of the cut-outs. It is this condition we concern ourselves with.

30. Call the side of the cut-out squares "a". You can observe at this stage what the dimensions of the box will be. In terms of the given dimensions and the dimension "a", what will be the length, width and the height of this box?

31. We are to find L, W and H in terms of the given dimensions. L is equal to 8 - 2a, W equals 6 - 2a and H is just a.

32. The volume of the box is length times width times height.

33. By taking this product we have

$$\text{Volume} = (8 - 2a)(6 - 2a)a$$

or if the product operation is performed then

$$V(a) = 48a - 28a^2 + 4a^3.$$

We now have an equation which defines a set of ordered pairs, having an implied domain and range of all values of a and V. However, the nature of the problem reveals that the ordered pairs are without meaning if dimension "a" is more than 3 and certainly it cannot be less than zero. So, for the condition of our concern we take the domain as all values in the closed interval zero, three. Intuitively we observe that the volume of the box will be zero at both of these end points, but what happens to the volume for the remaining values of "a" is defined by the ordered pairs.

In developing this function we gave meaning to the mathematical abstraction of numbers. That is, we gave a number of length, of width and of height and another number the meaning of volume.

34. We observe here several sizes of a box corresponding to different values of "a".

35. Title: A Graphic Portrayal of Functions.

Probably everyone has at some time seen and performed the game of connecting numbers on a paper surface with lines and observed the emergence of some recognizable object. The numbers alone gave little clue to their pattern but in connected form meaning emerged.

In a similar manner it is possible to take a set of ordered pairs, which by themselves give little evidence of a pattern, but reveal a meaningful pattern when connected graphically. The sole purpose of this game is to reveal the structure of the function. Again, it is assumed everyone listening to this lecture knows, to some degree, the technique of graphing, but its purpose and what can be read from it might not be so obvious.

36. The Cantor-Dedekind axiom makes the deceptively simple statement that all real numbers can be put into one-to-one correspondence with the points on an infinite straight line. Numbers are here given the meaning of length or distance which they do not inherently have. This must not be interpreted as meaning that the sum of an infinite number of points equals a length but rather that segments or intervals of a line can be represented by numbers. Segments refer to lengths or parts of a line, while an interval is considered as the set of numbers included between two numbers.

All integers, rational and irrational numbers and zero are assumed to have an ordered position on this line.

37. The mathematical notation of two numbers enclosed in brackets as shown here is called a closed interval and indicates the set of numbers on the line shown in red between 0 and 2 and including 0 and 2.

38. The open interval (2,5) does not include the end points

but does include all points (numbers) between these two numbers.

39. The number (1,3] is called a half open or half closed interval. The parantheses indicates 1 does not belong to this set of points (numbers) and the bracket indicates 3 does belong to the set.

40. Shown here is a segment of the infinite line. The segment is the portion from a to b. We are directing attention here to the portion of the line indicated instead of a set of numbers indicated by an interval. Direction of the line segment can be implied by labelling the segment from a to b as against the opposite direction from b to a. The magnitude of the segment is given as b - a, and is simply the magnitude from zero to "b" minus the magnitude from zero to "a".

41. The magnitude of the segment indicated by red dimension lines would be b - a. Notice that even though a is, in this case, a negative number the magnitude of the segment is still b - a.

42. The segment can also be written using the notation of absolute values to assure the positive sense of magnitude.

43. Irrational numbers may be positioned as shown here using the hypotenuse of a right angled triangle.

44. The Cantor-Dedekind axiom permits locating all real numbers in order on the horizontal line called the horizontal coordinate axis. Only the integers are indicated, the position of the remaining numbers are assumed.

45. Suppose now another line is drawn, perpendicular, to the horizontal line such that their zero positions are coincident. This arrangement is called a cartesian or rectangular coordinate system and permits a graphic portrayal of ordered pairs of numbers.

Assume the infinite vertical coordinate axis also containes all real numbers.

46. Ordered pairs of numbers are positioned on this coordinate system plane in this manner: assign the value of the first number to its appropriate position on the horizontal coordinate axis. For instance, if the ordered pair (1,-3) is to be positioned, the point on the horizontal axis corresponding to 1 is found, and assumed carried all along the perpendicular line shown. Then the second number (-3) is positioned on the vertical coordinate axis and carried along its vertical line. The point of intersection of the two lines is given the position (1,-3).

It is important to note that as a consequence of the Cantor-Dedekind axiom this system of positioning does not have different ordered pairs occupying the same position on the plane. That is, there is a one-to-one correspondence between all ordered pairs and the points on the infinite plane described by this coordinate system.

Ordered pairs defined by an equation such as:

$$f(x) = x^2 - 4$$

may be positioned on the coordinate plane . Obviously, all the ordered pairs can't be placed since f is an infinite set, but, by placing a few points and connecting these points, a very close estimate of the set can be observed. For instance, the set

$$\{(-5/2,2\tfrac{1}{4})(-2,0)(-1,-3)(0,-4)(1,-3)(2,0)(2\tfrac{1}{2},2\tfrac{1}{4})\}$$

is a subset of f.

47. If these are positioned on the coordinate plane and connected as shown, a fairly good picture of the entire set is obtained.

The second component of each ordered pair is often called "the value of the function", or just "the function". By looking at the graph of f we may observe:

1. Where "the value of the function" is negative or positive.
2. The intercepts or zeros of the function.
3. Where the "function" is increasing as x is increasing.
4. How fast it is increasing for various domain values.
5. Where in the domain the "function" changes from decreasing to increasing.

It is probably not obvious why these observations are important, but it should become so as we advance in this study.

Several functions are next exhibited which are useful in later discussions.

48. Constant function

$K(x) = a$

49. $F(x) = |x|$

50. $G(x) = |x - 1|$

51. $H(x) = \frac{x}{|x|}$

52. $f(x) = [x]$

53. $$F(x) = \begin{cases} 1, & x \text{ is irrational} \\ 0, & x \text{ is rational} \end{cases}$$

54. All equations so far considered have a common special quality: from the defining equation the ordered pairs are generated by the algebraic operations of addition, multiplication, extraction of roots and raising to powers. For instance, the equation

1. $g(x) = x^2 - 1$

is an algebraic equation because the ordered pairs are generated by the operation of multiplication $(x \cdot x)$ and addition

(-1). Other equations which do not have this quality are those trigonometric, logarithmic or exponential functions such as:

2. $h(\theta) = \sin \theta$

3. $y = \log x$

4. $y = c^x$

It is not possible to generate ordered pairs by algebraic operations from such equations. The reason is, of course, that these functions are not algebraic in origin. For instance, the trigonometric functions are geometric in origin while others require operations that are neither algebraic or nor geometric. Functions which are not algebraic are called transcendental functions.

55. Before proceeding with generating the trig functions we need one more number, i.e. the constant π. Again we consider the circle with a radius of 1 unit.

56. Divided into tenths as show.

57. Take this length. Lay it along the circumference as shown. Mark the initial point (1,0) and the terminal point.

58. Extend the radius measure again along the circumference and again mark the terminal point.

59. Continue in the same way. We now have three units of measure on the circumference. Continue once more.

60. The diameter mark is crossed at .14. Hence, the total length of the half circle measured in units of the radius is about 3.14. This number is called π and is the ratio of circumference of a circle to its diameter. Or, as shown here, is the ratio of half the circumference to half the diameter (=r).

Since r may be any number the number π is valid for any size circle. That is, π is dimensionless.

61. The length of arc along the circumference for this unit circle is indicated at positions π/6, π/4, π/2, 3π/4, π, 7π/6, 3π/2 and 2π, all measured from the start position. That is, from the start position to π/6 or about 1/2 the length of the radius, π/4 is about 3/4 of the radius along the arc, etc. These values are commonly used since they represent fractional parts of the circle, such as π/6 is 1/6 of a semicircle or 30°.

We could use any real number to represent a position along the circumference. Each number would represent the measure of radii along the circumference from the start position. A positive number is measured c.c.w. and a negative number is measured c.w.

62. Ordered pairs may now be generated from this unit circle.

A length along the circumference is represented in units of radius and called θ (theta). For instance, from A to B is a length π/4 times the length of the radius. This length is called $\pi/4$ radians. At position B on the unit circle the value of x on the coordinate system may be determined from the broken line drawn perpendicular to the x axis. This value appears to be about .7. The ordered pair $(\pi/4, .7)$ is called the cosine function. In equation form this is represented by $x = \cos\theta$. All the ordered pairs found for θ and its corresponding x value constitute the cosine function. The elements shown here are a subset.

A(0,1), B(π/4,.7), C(π/2,0), D(5π/6,-.86), E(π,-1)
F(8π/6,-.5), G(3π/2,0), H(-π/4,.7)

The sine function is generated in a similar manner using the y value with each θ.

The equation representing this function is

$y = \sin\theta$.

This is also an infinite set. The set

{A(0,0), B(π/4,.7), C(π/2,1), D(5π/6,.5), E(π,0),
F(8π/6,-.86), G(3π/2,-1), H(-π/4,-.7)}

is a subset of the sine function.

A fairly extensive set for the sine and cosine function are found in the set of "trig tables".

From these two functions, the remaining trig functions can also be generated.

LIMIT PROGRAM

The mathematical concept of the limit of a function provides a valid mathematical basis for the concepts on derivative, integral and others. Without a firm understanding of limits, it is not possible to understand basically what follows.

PROGRAM

1.Lecture: Educational Media Center. See the E.M.C. Directory for Dial access and Slide location.

A slide commentary follows the Limit Program.

2.Read: Introduction to Limits. Refer to Reading Program # 2 at the end of the Limit Program.

3.Problem Study: Educational Media Center

Examples 1., 2., 3. See E.M.C. Directory for Dial Access and Slide location.

Example 1.

Find:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Before you observe the solution to this problem try writing out each step giving detailed reasons explaining what you are doing and why; then observe the solution.

Example 2.

If: $f(x) = x^2 - 2x + 3$

Find:

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

Perform the operations indicated in the numerator. Try dividing the denominator into the numerator and then find the limit.

Example 3.

Given: The parabola $y = x^2$

Find: The limit as x approaches 1, of the secant line passing through the two points (1,1) and (x,x^2).

4. Problems: Refer to Problem Assignment at end of Limit Program.

5. Read: Definition of Limit of a Function. See Reading Program #5

These Theorems are frequently used in theory and problem solving. They should be remembered:

Theorem 1. $\lim_{x \to a} mx + b = ma + b$

Theorem 2. $\lim_{x \to a} b + b$

Theorem 3. $\lim_{x \to a} x = a$

Theorem 4.

If: $f(x) = g(x)$ for every x in domain K except at $x + a$ within K

Then: $\lim_{x \to a} f(x) = \lim_{x \to g} g(x)$

Record here the definition of limit given in the text.

6. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access.

Subject: Verifying limits from definition.

Examples 4,5.

Example 4.

Show by use of the limit definition that

$$\lim_{x \to -1} 4x - 1 = -5$$

We must show that:

for $f(x) = 4x - 1$, $L = -5$, $a = -1$.

Definition:

1. For <u>every</u> $\varepsilon > 0$
2. There is a $\delta > 0$ $(x \neq -1)$

Such that:

3. $f(x)$ is in $(1 - \varepsilon, L + \varepsilon)$
4. when x is in $(a - \delta, a + \delta)$

Proof:

From definition 3. $(f(x) = 4x - 1, L = -5, a = -1)$

1. $4x - 1$ is in $- 5 - \varepsilon, - 5 + \varepsilon$

or 2. $- 5 - \varepsilon < 4x - 1 < - 5 + \varepsilon$, is equivalent.

Simplify 2., first add 1 to both inequalities.

3. $- 4 - \varepsilon < 4x < - 4 + \varepsilon$

Then divide by 4.

4. $- 1 - \frac{\varepsilon}{4} < x \quad - 1 + \frac{\varepsilon}{4}$

It is important here to note from step 4. that

$f(x)$ is in $(-5-\varepsilon, -5+\varepsilon)$

when x is $(-1-\frac{\varepsilon}{4}, -1+\frac{\varepsilon}{4})$.

Compare this with the Limit definition steps 3. and 4. Since $a = -1$, we need only choose

$$\delta < \frac{\varepsilon}{4}$$

and then we have shown

1. For every $\varepsilon > 0$
2. There is a $\delta \leq \frac{\varepsilon}{4}$

such that

3. $f(x)$ is in $(-5-\varepsilon, -5+\varepsilon)$
4. When x is in $(-1-\delta, -1+\delta)$ $\quad \delta \leq \frac{\varepsilon}{4}$, $x \neq -1$

This procedure is not a method for finding the limit, only for verifying if the limit exists. The value of the problem is in its use of the limit definition

Example 5.

Prove the limit L does not exist.

$$\lim_{x \to -2} \frac{|x+2|}{x+2} = L\ , \quad f(x) = \frac{|x+2|}{x+2}$$

$$L = L\ , \quad a = -2$$

$$f(x) = \begin{cases} \frac{x+2}{x+2} & x > -2 \\ -\frac{(x+2)}{x+2} & x < -2 \\ \text{not defined} & x = -2 \end{cases}$$

Hence, for $x > -2$

$$\lim_{x \to -2^+} \frac{x+2}{x+2} = \lim_{x \to -2^+} 1 = 1 \qquad \text{Why?}$$

for $x < -2$

$$\lim_{x \to -2^-} \frac{-(x + 2)}{(x + 2)} = \lim_{x \to -2^-} -1 = -1 \qquad \text{Why?}$$

Since $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

$\lim_{x \to a} f(x)$ does not exist.

7.Problems: Refer to Problem Assignment Program # 7.

8.Read: Refer to Reading Program # 8

Continuity of a Function

Give particular attention to definitions and theorems. The idea of continuity is important as a quality of functions, since some concepts of calculus apply only to functions which have this quality.

1. The following limit theorem should be remembered:

If: $\lim_{x \to a} f(x) = b \qquad \lim_{x \to a} g(x) = c$

Then:

1. $\lim_{x \to a} (f + g)(x) = b + c$
2. $\lim_{x \to a} (fg)(x) = bc$
3. $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{b}{c} \qquad c \neq 0$
4. A polynomial function is continuous at every number.
5. A rational function is continuous in its domain.

2. This is a good time to review that part of the limit lecture (Program 1) in which continuity is discussed.

9. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide Location.

Subject: Limit Problems.

Examples 6, 7.

Example 6.

Find:

$$\lim_{x \to a} (x^5 + 4x^2 - 2\sqrt{x}) \qquad (a \geq 0)$$

Application of the limit theorem which states that the limit of a sum of functions is the sum of the limit of the functions permits writing the problem

$$\lim_{x \to a} (x^5 + 4x^2) - \lim_{x \to a} 2\sqrt{x}$$

$$(\lim_{x \to a} \sqrt{x} = \sqrt{a})$$

Example 7.

Discuss the continuity of the function defined by:

$$f(x) = \frac{|x|}{x}$$ Sketch the graph.

Do this in two parts:

1. For $x > 0$ $|x| = x$

   And $f(x) = 1$

2. For $x < 0$ $|x| = -x$

   And $f(x) = -1$

10.Problems. Refer to Problem Assignment # 10

11. Read: One-Sided Limits. Refer to Reading Program # 11.

12.Problem Study: Educational Media Center. See E.M.C. Directory.

Subject: One-sided limits.

Examples 8, 9.

Example 8. (See Example 5.)

Find the limit if it exists:

$$\lim_{x \to -2^-} \frac{|x + 2|}{x + 2}$$

Note for $x < -2$ the problem is simply:

$$\lim_{x \to -2^-} (-1) = -1$$

Give reasons for each step.

Example 9.

$$\lim_{x \to 1^+} \frac{2x\,|x - 1|}{x - 1}$$

For $x > 1$ the problem is written

$$\lim_{x \to 1^+} \frac{2x\,(x - 1)}{(x - 1)}$$

13.Problems: Refer to Problem Assignment Program # 13.

14.Read: Infinite Limits and Limits at Infinity.

Refer to Reading Program # 14.

15. Problem Study: Educational Media Center. Refer to E.M.C. Directory for Dial Access and Slide Location.

Examples 10, 11, 12.

Example 10.

Find:

$$\lim_{x \to \infty} \frac{x^2}{1 - x}$$

Write this in the form

$$\lim_{x \to \infty} \left(\frac{1}{\frac{1}{x^2} - \frac{1}{x}} \right).$$

and apply appropriate limit theorems to get:

$$\frac{\lim_{x \to \infty} 1}{\lim_{x \to \infty} \frac{1}{x^2} - \lim_{x \to \infty} \frac{1}{x}}$$

Example 11.

Find:

$$\lim_{x \to 1^-} \frac{[x^2] - 1}{x^2 - 1}$$

Recall that,

$$x^2 = n, \quad \text{if } n \leq x^2 < n + 1$$

Hence,

$$\lim_{x \to 1^-} x^2 = 0$$

and,

$$\frac{\lim_{x \to 1^-} [x^2] - \lim_{x \to 1^-} 1}{\lim_{x \to 1^-} x^2 - \lim_{x \to 1^-} 1} = -\infty$$

Example 12.

Determine the vertical and horizontal asymptotes, and sketch,

$$F(x) = \frac{2x}{(x + 2)^2}$$

A vertical asymptote occurs at x = a where

$$\lim_{x \to a^+} F(x) \quad \text{or} \quad \lim_{x \to a^-} F(x) \text{ equals } \pm \infty.$$

Observe:

$$\lim_{x \to 2^+} \frac{2x}{(x + 2)^2} = \lim_{x \to 2^-} \frac{2x}{(x + 2)^2} = -\infty$$

The horizontal asymptote will exist at y = b where:

$$\lim_{x \to \infty} \frac{2x}{x^2 + 4x + 4} = 0$$

16.Problems: Refer to Problem Assignment # 16.

17.Read: Limit Theorems. See Reading Program # 17.

1. In addition to the limit theorem previously given, you must know, understand and be able to apply the following theorem on limit of composite functions:

Theorem:

Given: The composite function

$$f(g(x))$$

If: (a). $\lim_{x \to a} g(x) = b$

(b). f is continuous at b

Then: $\lim_{x \to a} f(g(x)) = f(b)$

2. Theorem:

If: $\lim_{x \to a} f(x) = b$,

then:

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{b}$$

if $\sqrt[n]{b}$

18. Problem Study: Educational Media Center. See E.M.C. Directory.

Examples 13, 14.

Example 13.

$$\lim_{y \to -3} \sqrt[3]{(y + 2)^5}$$

Write this in the form:

$$\sqrt[3]{\left(\lim_{y \to -3} (y + 2)\right)^5}$$

Example 14.

Find: $\lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x}$

Simplify by rationalizing to:

$$\lim_{x \to 0} \frac{(\sqrt{4 + x} - 2)\ (\sqrt{4 + x} + 2)}{x(\sqrt{4 + x} + 2}$$

19. Problems: Refer to Assignment Program # 19.

20. Review:
1. All Lab Lectures. Use the script where it is helpful in going over slides where more time is required.
2. All Problem Study examples.
3. Sample tests.
4. Problems at chapter end.

READING PROGRAM --LIMITS

PROGRAM

2. Introduction to Limits.

Johnson & Kiokemeister, Pages 73 - 77.

5. Definition -- Limit of a Function

Johnson & Kiokemeister, Pages 78 - 83

8. Continuity of a Function

Johnson & Kiokemeister, Pages 85 - 87

11. One-sided Limits

Johnson & Kiokemeister, Pages 89 -91

Review the Limit Lecture, Program 1, discussing left and right hand limits.

14. Infinite Limits and Limits at Infinity.

Johnson & Kiokemeister, Pages 92 - 97.

Review Limit Lecture Program # 1.

17. Limit Theorems.

Johnson & Kiokemeister, Pages 98 - 103.

PROBLEM ASSIGNMENT PROGRAM -- LIMITS

All references are to Textbook (Johnson & Kiokemeister unless otherwise indicated. Hand in all programmed problems when completed.

Program # 4. Page 77. Problems I,1, 3, 5, 7, 9, 11, 13.

7. Page 84. Problems I, 1 - 12.

10. Page 88. Problems I, 1 - 10.
Give detailed reasons where required.

13. Page 91 ff. Problems I, 1 - 6.

16. Pages 97, 98. Problems I, 1 - 6, 7, 9, 14.

#19. Page 103. Problems I, 1 - 11.

LECTURE -- LIMITS

Slide 1. In the following discussion on limits an intuitive development is considered first and then related to the definition of the limit of a function. The idea of the limit of a function has immediate importance in understanding the basic concepts of Calculus which follow. You will be confronted with some rather illusive logic. If it isn't clearly understood the first time through, go over it again. The final definition of a limit must carry with it an understanding of how this applies to functions.

We begin first with an intuitive idea of a limit in which only an implied function is involved. That is, the function is not at first defined by an equation and, in fact, is not interpreted as a function. The equation is then introduced and the limit concept is related to this.

2. We begin with the intuitive idea of a limit.

The rod at position B is three units high made as shown by stacking three one-unit rods, two red and one blue. The other blue rod at position A is one unit high.

Suppose, at position B, the top one-unit rod is cut in half, the top half is removed and placed on top of the one-unit rod at position A.

This operation makes the B rod 2 1/2 units in height and the A rod 1 1/2 units in height as shown in the next drawing.

3. The operation is repeated by cutting in two the remaining half-unit blue rod at position B. The top half of this is removed and again placed on the rod at position A.

4. This makes the rod at position B 2 1/4 units high and at position A the rod is 1 3/4 units high.

5. If the operation is repeated again and again then a process is described whereby the remaining portion of blue rod at position B, as shown, is cut in half, and the top half placed on the accumulated sections at position A. If this process is continued ad infinitum, what is the smallest rod at position B found in the process?

The situation is real, the question is valid, but no precise number answer can be given. The best answer seems to be: there is no smallest rod at B found in this process.

However, a precise answer can be given if the question is changed to this:

What number L has these two qualities:

1. Every rod found in the process at position B is larger than L, and
2. No rod is smaller than L?

The number L = 2 has these two qualities. Every rod is larger than L = 2, and no rod is smaller than L = 2.

A similar situation exists at position A, where no <u>largest</u> rod is found in the process but where the number L = 2 is such that:

1. Every rod found in the process at A is

smaller than L, and

2. No rod is larger than L = 2.

The limit of the process at both positions A and B is said to be L = 2 even though 2 is never found in the process.

This idea of limit can be put in a more useful form if it is couched in a mathematical language. To do this, some mathematical notations are needed to encompass the ideas.

6. First, suppose the green band has the width shown. Epsilon (ε) is a number greater than zero representing the width above L = 2, and below L = 2. Hence, on a vertical scale the green band is from $L - \varepsilon$ to $L + \varepsilon$.

For the value of ε shown every height of rod at A and B in the continuing process will lie with $L - \varepsilon$ to $L + \varepsilon$. And in fact there seems to be, intuitively, some point in the process where this is true for every $\varepsilon > 0$ no matter how small ε becomes.

If, for every $\varepsilon > 0$, there is some point in the process such that, for every continuing step, rod heights A and B will lie within $L - \varepsilon$ to $L + \varepsilon$, then the limit of the process is said to be L. Observe that, since $\varepsilon > 0$ the length L is never required in the process, but the limit is said to be L if the described condition holds.

The same process is presented next in a slight variation, accomodating still more mathematical notations.

7. A line F is drawn from the top of the original unit rod at position A to the top of the original three-unit rod

at position B. The positions x^- are all less than 1 and positions x^+ are all greater than 1 and represent lengths along the base line from the zero position. The corresponding rod heights A_1 and B_1 are where the rods intersect the line F. In this case the height of A_1 at $x^- = 1/2$ is 1 1/2 and height of B_1 at $x^+ = 3/2$ is 2 1/2, and corresponds to the first operation of removing the top half of the top unit rod at B and placing it on top of A. The idea of a function is now more clearly defined by the ordered pairs, x and corresponding rod height.

8. The heights of A_2 and B_2 correspond to the $x^- = 3/4$ and $x^+ = 5/4$ positions of the base line. The ordered pairs $(3/4, A_2)$ and $(5/4, B_2)$ are indicated in this step.

9. The height of A_3 at x^- and B_3 at x^+ are shown. As the process continues by permitting x^- and x^+ to approach 1 by an amount half of δ (delta), then of course the height of A and B approaches 2. Observe that δ represents the distance from x^- to 1 or 1 to x^+.

Again, however, there is no largest A or smallest B.

10. The green band shown here has a width $L - \varepsilon$ to $L + \varepsilon$. For the value of δ shown every x^- and x^+ in the continuing process will have the corresponding heights of A and B within this green band.

In fact, <u>if</u> for <u>every</u> $\varepsilon > 0$, however small, there is a corresponding value of $\delta > 0$ such that heights A and B are within $L - \varepsilon$ to $L + \varepsilon$ for all x^-, x^+ values between $1 - \delta$ and $1 + \delta$, then the limit of the process is $L = 2$. Note that the limit $L = 2$ does not require that $L = 2$ be found in the process.

11. Suppose in the beginning the top half of the remaining blue section at B is discarded instead of placing it on top of A. Then for all values of x^-, A remains 1 unit and as x^+ approaches 1, B decreases to approach 2.

In this process it is no longer possible to say that for every $\varepsilon > 0$ there is some point (δ) in the process such that all values of x^- and x^+ have corresponding values of A and B in the green band. Hence, this process has no limit.

12. The intuitive limit $L = 2$ found by the process of removing the top half of the top unit from B and placing it on A is a suitable basis for the important concept of the limit of a <u>function</u> as defined by an equation. The idea is basically the same but the process is now described mathematically and is much more flexible and rigorous.

First, the equation

$$H(x) = \frac{x^2 - 1}{x - 1}$$

defines a set of ordered pairs for all values of x, except $x = 1$. If $x - 1$ is divided into $x^2 - 1$, then

$$H(x) = x + 1$$

and if $x = 1$ is removed from the domain of this equation then the <u>same</u> set of ordered pairs is defined by both equations. For instance, if $x = 2$, $H(2) = 3$ for both equations. Or, if $x = 0$, $H(0) = 1$ for both equations. These ordered pairs, (2,3) and (0,1) describe the height of the original rods B and A at the beginning of the process. However, instead of using rod heights to describe the process the value $H(x)$ of the ordered pairs defined by the equation

$$H(x) = \frac{x^2 - 1}{x - 1}$$

will convey the same idea if x is permitted to take values of half the remaining interval to 1 from both x^- and x^+ positions. That is, x^- takes on values of 1/2, 3/4, 7/8, 15/16 etc. and x^+, 3/2, 5/4, 9/8, 17/16, etc. The corresponding values $H(x)$ will then describe the process.

13. For instance, for $x^- = 1/2$, the broken red line represents as length the value $H(x^-)$ or in this case $H(1/2)$ and from the defining equation $H(1/2) = 3/2$ units. Likewise, the value if $x^+ = 3/2$ has the corresponding $H(x^+)$ or $H(3/2) = 5/2$ from the defining equation.

To convey the same idea of diminishing B and increasing A, the values of x^- and x^+ and corresponding $H(x^-)$ and $H(x^+)$ must be chosen so x^- and x^+ approach 1 by 1/2 the remaining interval. If this is done then the original process is described exactly but by using a mathematical notation for the idea. The limit of the process of increasing $H(x^-)$ and decreasing $H(x^+)$ is still $L = 2$.

This idea is next put into better mathematical form.

First, the symbol δ is used to provide an interval about $x = 1$ but not including $x = 1$. This interval is from $1 - \delta$ to $1 + \delta$ as shown here, $x \neq 1$.

14. The symbol ε provides a band width about $L = 2$ from $L - \varepsilon$ to $L + \varepsilon$. The mathematical idea of the limit of the <u>function</u> is now expressed in this form:

For the function $H(x) = \frac{x^2 - 1}{x - 1}$, the limit L of this function as x^- and x^+ approach 1 is $L = 2$ if:

1. For <u>every</u> $\varepsilon > 0$, however small, forming a band width $L - \varepsilon$ to $L + \varepsilon$

15. 2. there is a corresponding $\delta > 0$ about 1, (not including 1.) forming an interval of $1 - \delta$ to $1 + \delta$, such that;

16. whenever x^-, x^+ is in the interval, $1 - \delta$, $1 + \delta$, the height $H(x)$ is in the interval $1 - \varepsilon$ to $L + \varepsilon$.

Of course, as $\varepsilon > 0$ is chosen smaller, so must the value of δ be chosen smaller. For this function, that always appears possible by simply choosing $\delta < \varepsilon$.

In simplified form then: by calling x^- and x^+ just x,

$$\lim_{x \to 1} H(x) = L$$

if: for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that when x is in $1 - \delta$ to $1 + \delta$, $H(x)$ value is between $L - \varepsilon$ to $L + \varepsilon$.

It isn't enough to show a corresponding δ for the one

$\varepsilon > 0$ shown here. It must be possible to show that for every $\varepsilon > 0$, however small, there is a corresponding $\delta > 0$ such that when x is $1 - \delta, 1 + \delta$, H(x) is in $L - \varepsilon, L + \varepsilon$.

As shown here, whenever x (either x^- or x^+) is in the $1 - \delta, 1 + \delta$ interval the corresponding H(x) (vertical red lines) terminates in the $L - \varepsilon, L + \varepsilon$ interval.

17. The condition where rod A remains a constant height equal to 1, and B decreases as before is described mathematically by the equation;

$$H(x) = \begin{cases} 1, (0 \leq x < 1) & \text{Domain} \\ 1 + x, (1 < x \leq 2) & \text{Domain} \end{cases}$$

The graph of this equation looks like this. x values are measured on the base line and values of H(x) are vertical distances from this. For values of x between 0 and 1 H(x) is a constant value equal to 1.

18. $$\lim_{x \to 1^-} H(x) = 1$$

The notation means the limit as x approaches 1 from values of x less than 1 (corresponding to x^- values) and hence is called a left hand limit.

For every epsilon (ε) about H(x) = 1, every x^- in the left hand interval $1 - \delta$ to 1 has its corresponding $H(x^-)$ in the interval $1 - \varepsilon$, to $1 + \varepsilon$.

Also

$$\lim_{x \to 1^+} H(x) = 2$$

implies that only values of x greater than 1 are considered, or only x^+ values, and the limit is a right hand limit, since:

For every $\varepsilon > 0$, there is a $\delta > 0$ such that every x^+ in the interval 1 to $1 + \delta$ has its corresponding $H(x^+)$ in the interal $(2 - \varepsilon, 2 + \varepsilon)$.

Clearly the limits are not the same, since

$$\lim_{x \to 1^-} H(x) = 1$$

and

$$\lim_{x \to 1^+} H(x) = 2$$

although the left and right hand limits exist, since

$$\lim_{x \to 1^-} H(x) \neq \lim_{\to 1^+} H(x)$$

The $\lim_{x \to 1} H(x)$ does not exist.

19. For a function $H(x)$ to have a limit at $x = a$

$$\lim_{x \to a} H(x) = \lim_{x \to a^-} H(x) = \lim_{x \to a^+} H(x)$$

20. In finding the limit of the function

$$H(x) = \frac{x^2 - 1}{x - 1}$$

as x approaches 1, observe that $x = 1$ is never used and yet the

$$\lim_{x \to 1} H(x) = 2$$

Since $x = 1$ is not used in describing the limit process, the function can have <u>any</u> value or no value at $x = 1$.

For instance, the function shown here

$$G(x) = \begin{cases} x + 1, & x \neq 1 \\ 3/2 \;\;, & x = 1 \end{cases}$$

defines exactly all ordered pairs defined by H(x) <u>and</u> one more, i.e., $x = 1$, $G(1) = 3/2$ or $(1,3/2)$.

21. Returning to the original process of decreasing one rod and increasing the other, the question is asked: What number L has these qualities:

1. Every rod length $G(x^-)$ is smaller than L,
2. No rod length $G(x^-)$ is larger than L?

Again, $L = 2$ has this quality even though $G(1) = 3/2$ which does not equal L.

Also,

1. Every rod $G(x^+)$ is larger than $L = 2$,
2. No rod $G(x^+)$ is smaller than $L = 2$.

Since the value of the function G at $x = 1$ is not essential to describing $L = 2$, G(1) can have any value or no value at $x = 1$, and

$$\lim_{x \to 1} G(x) = 2$$

22. The mathematical description of the <u>limit</u> remains unaltered. That is:

1. For <u>every</u> $\varepsilon > 0$, however small,
2. There is a $\delta > 0$.

such that:

3. When x is in $(1 - \delta, 1 + \delta)$, $x \neq 1$
4. G(x) is in $(L - \varepsilon, L + \varepsilon)$.

Hence,

$$\lim_{x \to 1} G(x) = 2$$

23. Finally, suppose the function defined by

$$H(x) = \frac{x^2 - 1}{x - 1}$$

is changed by adding the ordered pair (1,2). Call this function

$$F(x) = x + 1.$$

It differs from H(x) by only the one ordered pair (1,2).

24. Again L = 2 satisfies the two conditions;

1. Every $F(x^-)$ is smaller than L = 2.
2. No $F(x^-)$ is larger than L = 2.

Also,

1. Every $F(x^+)$ is larger than L = 2.
2. No $F(x^+)$ is smaller than L = 2.

However, there is one unique quality about the function F(x) and that is:

$$F(1) = L = 2.$$

When this quality exists for any function, then that function is said to be continuous at that value of x..

25. The mathematical description of the limit again remains unaltered. That is;

1. For every $\varepsilon > 0$
2. There is a $\delta > 0$

such that;

3. When x is in $(1 - \delta, 1 + \delta)$ $x \neq 1$

4. $F(x)$ is in $(L - \varepsilon, L + \varepsilon)$.

26. In this case:

$$\lim_{x \to 1} F(x) = F(1)$$

This statement implies:

1. That the limit exists.

2. That $x = 1$ is in the domain of $F(x)$.

3. That the limit L of $F(x)$ is $F(1)$.

When these three conditions hold, the function is continuous.

27. Note that for

$$H(x) = \frac{x^2 - 1}{x - 1}$$

$x = 1$ is not in the domain. Hence, $H(x)$ is not continuous at $x = 1$.

Also,

$$G(x) = \begin{cases} x + 1, & x \neq 1 \\ 3/2, & x = 1 \end{cases}$$

and

$$G(1) \neq L$$

since $L = 2$ and $G(1) = 3/2$.

Hence, it also is not continuous at $x = 1$. The idea of function limits and the related idea of continuous functions are important to understanding the basic calculus concepts which follow.

28. Since:

$$\lim_{x \to 2} f(x) = 5 \qquad f(x) = 2x + 1$$

If:

1. For every $\varepsilon > 0$

2. There is a $\delta > 0$

such that:

3. When x is in $2 - \delta, 2 + \delta$,

4. $f(x)$ is in $5 - \varepsilon, 5 + \varepsilon$.

The limit can now be proved from the definition.

From definition part 4., $f(x)$ is in $5 - \varepsilon, 5 + \varepsilon$, hence,

29. 1. $5 - \varepsilon < f(x) < 5 + \varepsilon$ ($f(x)$ is in $L-\varepsilon, L+\varepsilon$)

or: since $f(x) = 2x + 1$

2. $5 - \varepsilon < 2x + 1 < 5 + \varepsilon$.

Solving for x;

$$\frac{5 - \varepsilon - 1}{2} < x < \frac{5 + \varepsilon - 1}{2}$$

Simplify:

3. $2 - \frac{\varepsilon}{2} < x < 2 + \frac{\varepsilon}{2}$

Note here that when x is in $(2 - \varepsilon/2, 2 + \varepsilon/2)$, $f(x)$ is in $(5 - \varepsilon, 5 + \varepsilon)$.

4. This is equivalent to parts 3., 4. of the definition if δ is chosen $\leq \varepsilon/2$.

Hence:

1. For every $\varepsilon > 0$

2. Choose $\delta < \varepsilon/2$

And

3. When x is in 2 - δ, 2 + δ

4. f(x) will be in 5 - $\varepsilon/2$, 5 + $\varepsilon/2$.

30. The limit of a function also includes two somewhat different conditions. Suppose

$$f(x) = \frac{1}{x}$$

then

$$\lim_{x \to 0} \frac{1}{x}$$

depends on whether x approaches zero from positive or negative values of x.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

and

$$\lim_{x \to 0^+} \frac{1}{x} = 0$$

and

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

DERIVATIVES

The concept of Derivative of a Function is one of the major concepts of this course. You should direct your efforts to mastering this idea in the two categories of a. theory, b. technique. It is possible to do well in part b. without actually understanding the meaning of the derivative (part a.).

Part b. is revealed essentially in working problems. To be really proficient you must understand the ideas behind the problems. "Getting the answer" here is of secondary importance. A "correct answer" simply suggests that your technique in following math rules is probably correct but doesn't necessarily imply that you understand the ideas or concepts.

PROGRAM

1.Lecture: Educational Media Center. See E.M.C. Directory for Dial Access and slide location.

Subject: Derivative Concept.

Be prepared to go over all or parts of this lecture more than once. Use the slide text included at the end of this section for detail study. Take notes on parts you can't follow and ask your instructor about it.

2. Read: Derivative -- Definition and Tangent Lines.

Refer to Reading Program # 2 (Derivative)

Relate your reading to the lecture on this subject. This isn't repetitious reading; it is supplementary. Give special attention to the definition of the derivative and the math motation used. You must not only know the definition -- it must be meaningful to you. Study the method used to find the derivative of a function. Write out your system for doing this.

3. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Your major objective here is learning the technique of finding the derivative. This includes knowing the correct mathematical operations to perform and how to perform them. Of secondary importance is the technique of finding the equation defining tangent lines.

Example 1.

Find the derivative of the function

$$F(x) = x^{-2}$$

Finding the derivative of functions is necessary and in most cases a fairly simple task. These problems are designed as exercises to help you understand how the derivative of a function is formed from a function.

Keep in mind there are always two functions involved in this process:

1. The given function $f(x)$

2. The derivative of the given function

$f'(x)$,

It is the relationship between these two functions that forms the basis for much of our future effort.

Since

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This must be evaluated for F(a), hence:

$$F'(a) = \lim_{h \to 0} \frac{F(a + h) - F(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a + h)^{-2} - a^{-2}}{h}$$

Simplify this algebraically and find the limit if it exists.

Example 2.

We are to find the equation for the tangent and normal line to the graph of a function defined by

$$f(x) = x^2 - 3x + 2 \quad \text{at} \quad (2, f(2)).$$

To see what you are doing you should sketch the function given. It helps to write this in the form

$$f(x) = x^2 - 3x + 9/4 - 1/4$$

obtained by "completing the square".

Then

$$f(x) = (x - 3/2)^2 - 1/4$$

and the vertex of the parabola is at point (3/2,-1/4). Additional points can be plotted and the point (2,f(2)) located. Now find f'(a) where a = 2, and proceed with the problem.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{((a+h)^2 - 3(a+h) + 2 - (a^2-3a+2))}{h}$$

$$f'(a) = 2a - 3$$

or

$$f'(2) = 4 - 3 = 1$$

The slope of the tangent line is 1. The slope of the normal is the negative reciprocal, or

$$-\frac{1}{f'(2)} = -1$$

Now complete the problem with the appropriate graph.

4.Problems: Refer to Assignment Program # 4. (Derivative)

5. Read: Continuity of a Function -- Differentiation Formulas.

Refer to Reading Program # 5.

Remember:

1. Theorem 5.5. If the function f is differentiable at a, then f is continuous at a. (The converse is not true.) What is the converse?

2. If $\lim_{x \to a} f(x) = f(a)$

   then $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

3. The derivative of a constant is zero.

   $Dk = 0$

4. The derivative of x is 1.

   $Dx = 1$

5. The derivative of the power function

$Dx^n = nx^{n-1}$

6. The derivative of a constant k times a function f(x)

$Dkf(x) = kDf(x)$

7. The derivative of the sum of two functions is the sum of their derivatives.

$F(f + f) = Df + Dg.$

8. The derivative of the product of two functions (fg)a

$D(fg) = fDg = gDf$

9. The derivative of the quotient of two functions

$D(\frac{f}{g}) = \frac{gDf - fDg}{g^2} \qquad g \neq 0$

6. Problem Study: Educational Media Center. See Directory for Dial Access and Slide location.

Examples 3, 4, 5.

Example 3.

We are to differentiate

$G(t) = (3t^2 + 1)^2$

Write this in the form

$G(t) = (3t^2 + 1) \cdot (3t^2 + 1)$

and use the product formula. (No. 8)

Work out the details and verify your work.

Example 4.

Given: $G(x) = (x^2 + \frac{1}{x^2})^2$

Find G'(x).

It might be easier to write this as

$$G(x) = \left(\frac{x^4 + 1}{x^2}\right)^2$$

and use the product formula, and quotient formula.

Example 5.

Find the derivative of

$$G(x) = |x^3 - 1|$$

Since there is no formula for differentiating an absolute value, this is first removed by use of the definition for absolute values. Write the function as,

$$G(x) = x^3 - 1 \quad \text{for} \quad (x^3 - 1) > 0$$

$$G(x) = -(x^3 - 1) \quad \text{for} \quad (x^3 - 1) < 0$$

$$G(x) = 0 \quad \text{for} \quad x^3 - 1 = 0$$

The derivative of each of these can be easily found along with the domain. Does the derivative exist at x = 1?

7.Problems: Refer to Assignment Program # 7.

Differentiate

1. $f(x) = x^3 + 3x^3 - 6$
2. $F(x) = (x^2 - 2x + 1)^k$
3. $G(x) = (x^2 - 1)(x^9 + 2x + 1)$
4. $H(x) = \frac{x^2 - 1}{x - 1}$
5. $h(x) = \frac{x}{x^3 - 1}$
6. $F(x) = (x - \frac{1}{x})$
7. $f(x) = \frac{1}{x^2}$

8. $g(x) = x^{-3} - 3x^{-2}$

9. $F(t) = t^0$

10. $f(t) = t^{-1} + \frac{1}{t^{-1}}$

8. Lecture: Educational Media Center. See E.M.C. Directory.

Subject: The Chain Rule.

9. Read: The Chain Rule. Refer to Reading Program # 9.

Remember:

1. $D(f \circ g)(a) = Df'(g(a))\ Dg(a)$

$Df^r = r\ f^{r-1}\ Df$ (r rational)

10. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide Location.

Examples 6,7.

Consider first a previous problem:

Example 6.

Given: $G(t) = (3t^2 + 1)^2$

Find: $G'(t)$

Use the differentiation formula

$$Df^r = rf^{r-1}\ Df$$

letting

$$f = 3t^2 + 1 \quad \text{and} \quad r = 2,$$

then immediately,

$$G'(t) = 2(3t^2 + 1)\ D(3t^2 + 1)$$

or,

$$G'(t) = 2(3t^2 + 1)6t = 12t(3t^2 + 1)$$

Example 7.

Given: $h(z) = \sqrt{\frac{1 - z}{1 + z}}$

Find: $h'(z)$

Recall:

$$Df^r = rf^{r-1}\, Df$$

Suppose the problem is written;

$$h(z) = \left(\frac{1 - z}{1 + z}\right)^{\frac{1}{2}}$$

and

f is given by $\frac{1 - z}{1 + z}$.

You can now apply the above formula directly but you will need the quotient formula to finish the problem.

Example 8.

Given: $F(y) = \left(y - \frac{1}{y}\right)^{3/2}$

Find: $f'(y) = 3/2\left(y - \frac{1}{y}\right)^{\frac{1}{2}} F\left(y - \frac{1}{y}\right)$

11. Problems: Refer to Assignment Program # 11.

12. Read: Implicit Differentiation and Higher Derivatives. See Reading Program # 12.

13. Problem Study: Educational Media Center. See E.M.C. Directory. Examples 9, 10.

Example 9.

Given: $xy^2 - y + 6x = 0$

Find: y' Use implicit differentiation.

Suppose $xy^2 - y + 6x = 0$ is solved for y. Then $y = f(x)$. Actually in this case it is possible to solve for y, using the quadratic solution;

$$y = \frac{-1 \pm \sqrt{1 - 4\cdot x\cdot 6x}}{2x} = \frac{-1 \pm \sqrt{1 - 24x^2}}{2x}$$

Hence, $y = f(x) = \dfrac{-1 \pm \sqrt{1 - 24x^2}}{2x}$

In many implicit functions it is not possible to solve explicitly for y in terms of x. However, assume it is and that

$$y = f(x).$$

Then to emphasize this, write

$$xy^2 - y + 6x = 0$$

as

$$xf(x^2) - f(x) + 6x = 0$$

Now apply the Chain Rule to each term

$$xDf^2(x) + f^2(x)Dx - Df(x) + D6x = 0$$

or

$$x \cdot 2f(x) \cdot f'(x) + f^2(x) \cdot 1 - f'(x) + 6 = 0$$

Solve for $f'(x)$

$$f'(x)(2xf(x) - 1) = -6 - f^2(x)$$

$$f'(x) = -\frac{6 + f^2(x)}{2xf(x) - 1}$$

Since $y = f(x)$

$$y' = -\frac{6 + y^2}{2xy - 1}$$

Example 10.

Given: $x + x^2y^2 - y = 1$ (1,1)

Find: Equation of the tangent line at the given point to the given curve.

We must find y'(1,1) to get the slope of the required tangent line. Then use the point slope to find the required line.

14.Problems: Refer to Assignment Program # 14.

15. Read: Notation for Derivative. Refer to Reading Program # 15.

Remember:

1. Chain Rule in Leibnitz notation.
2. Product formula in Leibnitz notation.
3. Differential y ($dy = f'(a)\,dx$).
4. Geometric interpretation of dy, Δy.

16.Lecture: Repeat PROGRAMS 1, 4.

17.Problem Review: Refer to Assignment Program # 17.

The purpose in working problems is to reinforce your understanding of the concepts involved and to provide practice in applying this knowledge. Keep in mind these objectives as you are working the problems. If you blindly follow a rule to "get an answer" your proficiency is greatly diminished.

READING PROGRAM -- Derivatives.

All references are to the textbook, Johnson and Kioke-meister, unless otherwise noted.

Program # 2. Derivatives -- Definition and Tangent Lines.
Pages 106 - 111.

5. Continuity of a Function
Differentiation Formulas
Pages 113 - 119.

9. The Chain Rule
Pages 120 - 124.

#12. Implicit Differentiation and Higher Derivatives.
Pages 125 - 128.

#15. Notation for Derivatives
Pages 128 - 132.

ASSIGNMENT PROGRAM -- Derivative.

All page references are to the textbook, Johnson and Kiokemeister, unless otherwise indicated.

Hand in all assigned problems. They will be corrected and returned to you.

Program # 4. Page 112. Problems I, 1 - 9.

\# 7. Problems 1 - 10, Program Manual
Page 119. I Problems 1 - 10.
Page 120 II Problems 1, 5.

\#11. Page 124. I Problems 1 - 9, 11, 13, 15.

\#14. Page 128 I Problems 1, 3, 5, 7, 9, 11.

\#17. Page 133. I Problems 1, 2, 5, 7, 10.

Slide 1. Title The Derivative of a Function

The mathematical concept discussed here is called the derivative. And, since it is derived from a function it is generally referred to as the derivative of a function.

2. It is especially important that the function concept be understood as a set of distinct ordered pairs having no two first components the same.

3. All functions will having a defining equation. For instance in the function defined by the equation

$$U(x) = 3x^2$$

ordered pairs may be derived as follows:

$$U(1) = 3$$

giving the ordered pair (1,3)

$$U(2) = 12$$

giving the ordered pair (2,12) and so forth. Observe also the notation:

$$U(z + a) = 3(z + a)^2$$

which has the ordered pair

$$((z + a), 3(z + a)^2)$$

Without the defining equation of a function there is no way to find the derivative of the function. This doesn't mean that if a function is defined by an equation that the function has a derivative. Many functions have no derivatives.

The derivative of a function is itself a function. It is to the relation between these two functions that we direct out attention. Most math concepts are abstractions. That is, they are ideas viewed apart from the concrete.

4. For instance, the concept or abstraction of roundness, is inherent in these concrete examples. If the idea of roundness is abstracted from these items it can then be applied to any item having this property. It is important to be able to distinguish between the abstraction and the concrete examples. That is, we would not say "roundness is an orange" but rather, an orange is a concrete example of the quality of roundness.

5. Most math concepts are defined in mathematical notation or theorems. However, unless one understands the idea involved first, the definition provides little help. Roundness can be defined in a mathematical sense by the function defined by $x^2 + y^2 + z^2 = r^2$, but this provides limited help in understanding the concept of roundness unless

6. as shown here a concrete example of the graph of this function is used with it.

7. The definition of the derivative $f'(x)$ of the function $f(x)$ is defined by this mathematical notation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

This describes the concept mathematically but gives little aid to any depth of understanding.

Several concrete situations are next presented interpreting this definition in explicit manner.

8. Title Velocity and the Derivative

Velocity and speed have similar meaning and will, at first, be used interchangeably although velocity has the quality of direction of motion not usually associated with speed.

9. Suppose a car starts in motion from the rest position shown here.

10. It increases in speed.

11. And finally moves out of view increasing in speed as it does so.

12. Of particular interest are two qualities exhibited by the car.

1. position of the car on the road and
2. speed of the car.

These qualities are different but inseparable.

13. Suppose the car could be timed as it reached certain marks on the road. For instance the car starts in motion at the zero mark.

14. Reaches the 36' mark 3 seconds later.

15. The 49' mark at 3 1/2 seconds from start.

16. The 64' mark at 4 seconds from start.

17. And finally crosses the 100' mark in 5 seconds from start. The ordered pairs of time and position found from this are: (0,0), (3,36), (3 1/2, 49), (4,64), (5,100) and form a function defining time and position of the car. The first components of these ordered pairs are the domain of the function and the second components are the range.

18. Let t be the independent variable and L(t) the dependent variable then it might be assumed that the equation $L(t) = 4t^2$ having the domain (t = 0, 3, 3 1/2, 4, 5) defines the function as measured. Assume this equation is valid for all values of (t) from zero to 5 inclusive.

The position of the car is then described for every t in the closed interval [0,5]. This equation is a mathematical description of the quality of <u>position</u> of the car at each instant but says nothing explicitly about the quality of <u>speed</u> of the car at each instant.

Whatever the quality of speed is, this function does not explicityly define it.

19. Consider the two positions (3,36) and (5,100): These are ordered pairs from the position function, but they also say <u>something</u> about speed, since if the car moves from 36' to 100' in two seconds then this implies a speed of

$$\frac{100 - 36}{2} = 32'/\text{sec.}$$

The car is in the condition of speeding up, hence it can't be said its speed is actually 32'/sec. during this 2 second interval, but is either slower or faster than this amount and is correct at only one instant in this interval of time. To improve the description a smaller interval of time could be used, say from 3 to 3 1/2 seconds or 36' to 49'.

20. This gives

$$\frac{49 - 36}{1/2} = 26'/\text{sec.}$$

as the speed. Again, since the car is increasing in speed over this interval it is accurate at only one instant. The remaining time it is either slower or faster than this amount.

21. Instead of taking a fixed interval suppose the interval is made variable by letting the letter h represent a positive number. Then from the defining equation for position, the expression for speed could be written

22. $$\frac{L(3+h) - L(3)}{(3+h) - 3} \quad \text{or just} \quad \frac{L(3+h) - L(3)}{h}$$

The interval can now be made small by making h small approaching zero.

23. Use the defining position equation to find $L(3 + h)$

24. $L(3 + h) = 4(3 + h)^2 = 36 + 24h + 4h^2$

25. Speed $= \dfrac{24h + 4h^2}{h}$ for any h.

By making h small the interval is made small until as $h \to 0$ a mathematical expression for speed at one point of position is actually found.

26. Simplifying: by cancelling h from numerator and denominator

$$\text{Speed} = \lim_{h \to 0} \frac{24h + 4h^2}{h} = 24'/\text{sec. at } t = 3$$

This is a precise mathematical representation of the quality of speed for the car, but it is given for only one instant, (t = 3) and says nothing about the speed at any other point.

27. Instead of making the position at 36' fixed suppose it is a general position, called L(t), then let the other position be L(t + h). Position L(t) occurs at t seconds and L(t + h) occurs at (t + h) seconds, hence

28. The expression becomes:

$$\frac{L(t + h) - L(t)}{(t + h) - t}$$

The velocity is then defined as shown. This is very general since from the position equation

$$L(t) = 4t^2$$

L(t + h) and L(t) can be brought arbitrarily close together by making h as small as necessary. As h is squeezed toward zero a mathematical expression describing the quality of speed is obtained for any position in our consideration.

29. This can be said mathematically by the expression

$$\lim_{h \to 0} \frac{L(t+h) - L(t)}{h}$$

But now since the speed is precisely determined by the position function its direction is also determined and therefore it is advisable to use velocity instead of speed. The only information used in finding this expression is that given by the position function

30. Compare this with the definition of derivative $f'(x)$ for the function $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Since these two expressions are identical in form, it can be assumed that the expression for speed, if the limit exists is actually the derivative of the position function,

$$L(t) = 4t^2$$

and can be called

31. $$L'(t) = \lim_{h \to 0} \frac{L(t+h) - L(t)}{h} = \text{velocity}$$

32. To evaluate this expression; $L(t+h) - L(t)$, must be found from the position function $L(t) = 4t^2$.

33. The $\lim_{h \to 0} \frac{L(t+h) - L(t)}{h}$ becomes:

$$\lim_{h \to 0} \frac{4(t+h)^2 - 4t^2}{h}$$

This may be simplified algebraically to

34. Line 5.

$$\lim_{h \to 0} \frac{4t^2 + 8th + 4h^2 - 4t^2}{h}$$

The first and last terms in the numerator may be cancelled. Divide an h from the denominator into the remaining numerator terms. Then as $h \to 0$ this expression simplifies to $L'(t) = 8t$.

35. The two qualities of position and speed associated with the car in motion are now precisely described mathematically by the two functions one of which is the derivative of the other. For any position of the car say at $t = 3$ and $L(3) = 36$ the velocity of the car is 24 feet per second.

This is a concrete example of the derivative of a function and must not be construed as a definition of the derivative. To say the derivative is a velocity would be like saying roundness is an orange. Velocity is an example of a derivative of a position function.

36. The idea of rate will be used in the next concrete interpretation. The dictionary definition of rate is the amount of something in relation to units of something else. Applied to this example rateis the amount of change of position in relation to unit change in time. It is from this that rate is given the meaning of velocity. However, rate can be applied in other ways as will be shown in the following example.

37. THE DERIVATIVE INTERPRETED GRAPHICALLY

38. The derivative may also be interpreted graphically. In doing this the Cantor-Dedekind axiom is used. This axiom assumes all real numbers can be placed in one-to-one correspondence with the points on an infinite straight line. The real numbers are in this case given the meaning of length or position on a line. Any real positive or negative number can then be positioned from zero in the appropriate

direction of the line. If two such lines are placed perpendicular to each other a plane is determined such that any two numbers will determine a unique position of the plane.

39. These two lines, called coordinates axes, are shown perpendicular to each other. It is assumed each line is infinite in length and contains all real numbers. Each position on the black axis is assumed to extend along an intersecting black line, and each position on the red axis is assumed to extend along an intersecting red line. The zero position is called the origin.

40. Some of the ordered pairs defined by the equation $L(t) = 4t^2$ are given here: all ordered pairs are elements of the function set. All first components of these elements are a set called the domain--these are the black numbers. All second components are a set called the range of the function--these are the red numbers.

For any ordered pair the first component is positioned along the black axis; the second along the red axis. For instance, the ordered pair (1,4) is positioned by the intersection of the black and red lines at the 1, and 4 position on the axes.

It is quickly apparent that such numbers as 64 cannot be found on the red scale as shown.

41. Hence the scale is changed as shown here.

42. The ordered pairs are positioned on the cartesian plane, labelled as (1,L(1)), (2,L(2)) etc. in which L(1) = 4 and L(2) = 16.

43. The assumption is made that if all ordered pairs of the defining equation were so positioned they would form the curved line

as shown. This line is a two space visualization of the entire set of ordered pairs defined by $L(t) = 4t^2$ in the domain $-4 \leq t \leq 4$. It is visually distorted because the scale was changed in one direction and not in the other. However, the relative shape of the graph is visible.

Note the two positions (3,L(3)) and (4,L(4)).

44. L(3) and L(4) are represented by positions on the range axis and 3 and 4 by corresponding positions on the domain axis. The change in position from L(3) to L(4) on the range compared to the corresponding change from 3 to 4 on the domain is described by the general idea of rate.

45. In this case rate as the amount of something in relation to units of something else becomes: The amount of change of range per unit change in domain.

For the two points considered:

46. The rate is expressed by

$$\frac{L(4) - L(3)}{4 - 3} \quad \text{or} \quad \frac{64 - 36}{1} = 28$$

Or, the amount of change in position of the range for a unit change in domain is 28"/inch.

The equation $L(t) = 4t^2$ describes each point on the graph but just as with the car it does not express how fast the range is changing as the domain changes.

The expression for rate shown describes this but not precisely, since it assumes this rate occurs over the entire domain 3 to 4.

47. If a smaller interval is chosen such as from 3 to 3 1/2 then from L(3) to L(3 1/2) the rate is

$$\frac{49 - 36}{1/2} = 26''/\text{inch}$$

The rate of change of range compared to domain is not constant, when compared to the rate over the previous interval.

48. For an interval change of 1/4 the range changes at the rate of 25"/inch, indicating again that the rate is not constant even over the smaller previous interval.

49. Attempting to find the rate as change in range compared to a corresponding change in domain in an interval approaching zero requires again making the interval variable. That is, consider

$$(3, L(3)) \text{ and } ((3 + h), L(3 + h))$$

where h is a small positive number.

50. The rate is expressed in line 1 as

$$\frac{L(3 + h) - L(3)}{h}$$

and when the defining equation is used to simplify this it becomes

$$\frac{24h + 4h^2}{h}$$

as given in line 2.

51. If h approaches zero then the rate is 25"/inch. This gives the rate of change of range compared to the domain precisely but only at one position where $t = 3$ and $L(3) = 36$.

52. To find this rate for any other position, assume the general positions at P_1 and P. P_1 is positioned by the ordered pair

$$(t_1, L(t_1))$$

and P by

$$((t_1 + h), L(t_1 + h)).$$

53. The rate is then given in line 1. as

$$\frac{L(t_1 + h) - L(t_1)}{h}$$

If h is permitted to approach zero this form is again identical to the definition of the derivative and on simplifying can be called $L'(t)$, as the derivative of $L(t)$ or,

$$L'(t) = 8t$$

which in this case expresses the rate as change in range compared to the change in domain at each position of domain t on the graph.

If $t = 3$ the range is changing at a rate 24 times the domain change.

54. THE FUNCTION DERIVATIVE AS THE SLOPE OF A TANGENT LINE

55. The function defined by

$$L(t) = 4t^2 \qquad -4 \leq t \leq 4$$

is shown in graphic form where each position on the graph expresses an ordered pair of the function.

For the particular ordered pairs shown the rate is given by:

56. $$\frac{L(4) - L(3)}{4 - 3}$$

and express the change in range for a corresponding change in domain.

57. A line S (called a secant line) drawn through these points has a slope (m), which is identical to the rate. Rate, then, as change in range compared to change in domain is equal to the slope of the line through the points considered.

58. Consider the secant line S and the angle θ (theta) it makes with the positive direction of the x axis. This is called the angle of inclination of line S.

The tangent of θ is also expressible in identical form to the slope and rate.

59. Hence the rate, slope and tan θ are all equal. If the two points are expressed in general form using (t,L(t)) and ((t + h), L(t + h)) and rate is expressed as:

$$\lim_{h \to 0} \frac{L(t+h) - L(t)}{h}$$

then the secant line through these points becomes the line tangent to the curve, having a slope equal to its tangent of inclination which is equal to the derivative of the function. This is an important relation and one which should be understood in each meaning.

At any point (t, L(t)) on the graph of the function $L(t) = 4t^2$ the slope of the line tangent to this curve at this point has the same value as the derivative L'(t) of the function.

The temptation exists to define the derivative as being the slope of the tangent line. This is another example of calling roundness an orange. The derivative is defined as statement 2. which equals all the items in statement 1.

Since $L'(t) = 8t$ the slope can be expressed at any value t. For instance, if $t = 3$,, $L'(3) = 24$ and $\tan\theta = 24 = m$. In the final example, presented next, a function is generated and its derivative is given still another meaning.

60. THE BOX FUNCTION

61. From a sheet 6" x 8" in size a box is formed by cutting out the corners and folding up the remaining strips to form the sides. Each corner cut out is a square "a" inches on a side. This becomes the height of the box when the box is formed.

From the dimension given the length is (8 - 2a), the width is (6 - 2a) and the height of the box is "a". The volume is given by the product of these three dimensions. When multiplied together and simplified the volume expressed in equation form is

$$V(a) = 48a - 28a^2 + 4a^3$$

This has meaning only when "a" is greater than or equal to zero or less than or equal to 3 since the box will exist only for these numbers. Hence, the function domain is

$$0 \leq a \leq 3$$

62. When "a" equals zero the plate is defined having a volume of zero. This is the first ordered pair shown in the left column. Boxes corresponding to "a" = .5, 1.5 and 3.0 are shown with their

corresponding ordered pairs.

63. Assume the cylinder shown holds 30 cubic inches of water which i is permitted to drain into the box as it changes size for different values of "a".

64. For "a" = 1/4 inch the volume indicated is about 10.3 inches.

65. For "a" = 1/2 the volume indicated is 17.5 cubic inches. Although "a" doubled V(a) the volume did not.

66. For "a" = 1 inch the volume indicated is 24 cubic inches. Again doubling "a" did not double the volume.

67. How does the volume change as dimension "a" changes?

68. Applying the idea of rate as the amount of something in relation to units of something else the expression becomes:

69. Rate as the amount of change of volume in relation to unit change in dimension "a".

For instance, the change in volume if dimension "a" changes from 1/2 to 1" is 24 - 17.5 or 6 1/2 cubic inches.

70. Rate can then be expressed using the function notation 17.5 = V(1/2) and 24 = V(1) as:

$$\frac{V(1) - V(1/2)}{1 - 1/2} \text{ or cubic inches per inch.}$$

This implies that the change of volume is constant over this interval of 1/2 inch change in dimension "a".

There is no way of knowing if this is true unless the expression which describes the rate is over an interval of dimension "a" so small as to approach zero.

71. To do this, dimension "a" = 1/2 is left in the general form of "a" with corresponding volume V(a) as shown in red shading. If h is a small positive number the interval from "a" to a + h has a corresponding volume change from V(a) to V(a + h). Change in dimension "a" of an amount h induces a corresponding change in the volume of V(a + h) - V(a) and the rate

72. is expressed as

$$\frac{V(a + h) - V(a)}{h}$$

for any interval h.

73. To make the interval small let h approach zero. The expression then becomes

$$\lim_{h \to 0} \frac{V(a+h) - V(a)}{h}$$

which is again the same expression as the definition for the derivative of a function and hence can be called V'(a). The meaning assigned to this expression is the change in volume as dimension "a" changes.

To evaluate this limit V(a + h) must be expressed from the defining equation

$$V(a) = 48a - 26a^2 + 4a^3$$

74. In line 1. The equation is given.

In line 2. The mathematical notation for the derivative V'(a) is given.

Line 4. expresses this as

$$V'(a) = \lim_{h \to 0} \frac{V(a+h) - V(a)}{(a+h) - a}$$

75. To evaluate line 4. $V(a + h) - V(a)$ must be evaluated in terms of the given function. First find $V(a + h)$.

76. From line 1.

$$V(a+h) = 48(a+h) - 28(a+h)^2 + 4(a+h)^3$$

When expanded and arranged in descending powers of h this is written

$$V(a+h) = 48 - 28a + 4a^3 + h(48 - 56a + 12a^2) +$$
$$h^2(-28 + 12a) + 4h^3$$

77. The terms shaded red in line 5. are identically line 1. or just $V(a)$. Since in line 4. this amount shaded red is subtracted in the numerator, it is then equivalent to the remainder of line 5. and each term of this remainder has h as a multiplier. Cancel this with the h in the denominator of line 4., shaded in blue. Then, line 6.

$$V'(a) = \lim_{h \to 0} (48 - 56a + 12a^2 + h(-28 + 12a) + 4h^2)$$

78. As h 0 the terms shaded in blue approach zero and are dropped. The remaining terms are the derivative of the function $V(a)$. That is in line 7.

$$V'(a) = 4a - 56a + 12a^2$$

This equation defines a function as a set of ordered pairs $(a, V'(a))$. For every a in the domain, $V'(a)$ expresses the rate as change in $V(a)$ compared to "a" at any value "a".

79. The function V'(a) is the derivative of the function V(a). Using line 7.

$$V'(1/2) = 48 - 56\ 1/2 + 12 \cdot (1/2)^2 = 23$$

Hence when a = 1/2 inch the volume is changing at the rate of 23 cubic inches per inch.

80. The process of finding the derivative of a function is shown here for the function

$$U(x) = 3x^2 = x$$

It should be observed and recorded.

THE CHAIN RULE

Slide 1. The Chain Rule

2. Theorem: The Chain Rule

If:

1. f(x) is a function of x and x(t) is a function of t.

2. f and x are differentiable functions.

Then:

$$D_t f(x) = D_x f(x) \cdot D_t x(t)$$

Or in Leibnitz notation:

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

In this lecture the idea of composite functions is portrayed geometrically and from this the Chain Rule is deduced showing how such functions are differentiated.

3. A specific function $f(x) = x^2 + 1$ is shown in graphic form over a small portion of its domain. The independent variable is x.

4. Another function is added showing $x(t) = \sqrt{t}$. x is now the dependent variable and t is the independent variable. The implied domain for $x(t) = \sqrt{t}$ is, of course, $t \geq 0$. For any such value of t, x(t) is defined, producing an ordered pair of this function. For instance, if $t = 4$,

$x(4) = 2$ and the ordered pair $(t, x(t))$ is $(4,2)$. If this value of x is then applied to the other function f(x) is determined or when $t = 4$, $f(x) = 5$ producing the ordered pair $(4,5)$. All ordered pairs produced in this manner define a function called a composite function.

This is portrayed better by tipping the graph as shown here.

5. The x coordinate axes are now juxtaposed. And since the scales are the same they can be placed together.

6. The action is from t to $x(t)$ and then from $x(t)$ to $f(x)$ in determining ordered pairs of the composite function $f(x(t))$.

7. In the function $x(t) = \sqrt{t}$ the value t_o in the domain produces the range value $x(t_o)$,

8. and value t produces $x(t)$.

9. Apply these values $x(t_o)$ and $x(t)$ to the domain of the function $f(x) = x^2 + 1$ to produce the corresponding range values $f(x(t_o))$ and $f(x(t))$. As a composite function the change in domain from t_o to t produces a change in range from $f(x(t_o))$ to $f(x(t))$. This may be expressed as a rate in quotient form.

10. The quotient

$$\frac{f(x(t)) - f(x(t_o))}{t - t_o}$$

expresses the rate as change in domain of the function $x(t) = \sqrt{t}$ compared to the corresponding change in range of the function $f(x) = x^2 + 1$.

11. Multiply and divide this quotient by $x(t) - x(t_o)$. Notice the different notation used for $x(t)$ and $x(t_o)$ when these values are used with the function $f(x)$. The values are the same; just the notation is altered to accomodate the two functions.

It is assumed that all values are well defined in the quotients.

The rate is still expressed by the product quotient over the domain from t_o to t.

The instantaneous rate of change requires taking the limit by letting t approach t_o.

12. This limit defines the rate as change in f compared to t or simply the derivative of f with respect to t.

Since the limit of the product of the two quotients is the product of the limits of the two quotients, this may be written in the form;

13.
$$\lim_{t \to t_o} \frac{f(x(t)) - f(x(t_o))}{t - t_o} =$$
$$\text{limit } \frac{f(x(t)) - f(x(t_o))}{x - x_o} \cdot \text{limit } \frac{x(t) - x(t_o)}{t - t_o}$$

As t approaches t_o, x approaches x_o so the respective limits become;

14. $$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

15. The conclusion for the Chain Rule theorem is established.

APPLICATION OF THE DERIVATIVE

PROGRAM

PROGRAM

1.Lecture: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

2.Read: Refer to Reading Program #2(Application of the Derivative)

3.Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Examples 1, 2.

Example 1.

Verify Rolle's Theorem by finding the values of x for which F(x) and F'(x) vanish.

$$F(x) = 3x - x^3$$

Recall Rolle's Theorem:

If: F(x) = F(b) = 0, F(x) is continuous

Then: for some x_o such that $a < x_o < b$

$$F'(x_o) =$$

Find those values of a and b such that F(a) = F(b) = 0. Then find x_o such that $F'(x_o) = 0$.

This problem simply verifies Rolle's theorem.

Example 2.

Verify that the Mean Value Theorem holds, or give a reason why it does not, for:

$$g(x) = \frac{x - 1}{x} \quad a = 1, \quad b = 3$$

Note that the hypothesis for the Mean Value Theorem is satisfied. That is, $g(x)$ is continuous in [1,3], and $g'(x) = \frac{1}{x^2}$ exists in (1,3)· hence the theorem does apply.

So

$$\frac{g(b) - g(a)}{b - a} = g'(x_0) \qquad a < x_0 < b$$

You can now find x_0 to verify the theorem.

4. Problems: Refer to Assignment Program # 4.

5. Lecture: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Subject: Extrema of a Function.

6. Read: Application of the Derivative. See Reading Program # 6.

7. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Examples 3, 4, 5.

Example 3.

In what interval (domain) is this function strictly increasing and strictly decreasing?

$$G(x) = 4 - 4x - x^2$$

Find

$$G'(x) = -4 - 2x$$

Where $G'(x) > 0$, the function $G(x)$ is increasing. Also, where $G'(x) < 0$ the function $G(x)$ is decreasing.

Example 4.

Find the extrema of

$$g(x) = 4 - x^2$$

and sketch the graph.

Find the critical points c from $g(x) = 0$. Then $g(c)$ is the extremum. Determine where the function is increasing, where it is decreasing and the zeros of the function. This will aid in graphing.

Example 5.

Find the extrema and graph:

$$G(x) = 2x^3 - 3x^2$$

$$G'(x) = 6x^2 - 6x$$

Critical points: $C_1 = 0$, $C_2 = 1$. $G(C_1)$ and $G(C_2)$ are the extrema.

8.Problems: Refer to Assignment Program # 8.

9.Lecture: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Subject: Concavity of a Function and Second Derivative Test.

10.Read: Refer to Reading Program # 10.

11.Problem Study: Educational Media Center. See E.M.C.Directory for Dial Access and Slide location.

Examples 6, 7, 8.

Example 6.

Find the extrema of the function using the second derivative test:

$$F(x) = x^3 + 4x^2 - 3x - 9$$

First find F'(x) and the critical points. Then find F"(x) and test the critical points. Compute the extrema from F(x).

Example 7.

Find the extrema of

$$F(x) = x\sqrt{x + 3}$$

Use whatever test is most convenient.

Example 8.

Find the points of inflection of

$$y = \frac{1}{x^2 + 2}$$

and sketch the graph showing tangent lines at the point of inflection.

12.Problems: Refer to Assignment Program # 12.

13. Read: Refer to Reading Program # 13.

14. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and slide location.

Examples 9, 10.

Example 9.

An open box is made from a sheet of metal 10" x 14" by cutting out corners and folding up the sides to form the box. What size box will have the largest volume?

Example 10.

What point on the graph of $y^2 = 4x$ is nearest the point (s,1)?

15. Problems: Refer to Assignment Program # 15.

16. Read: Refer to Reading Program # 16

17. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Example 11.

The position function of a point moving on a straight line is given by:

$$s(t) = t^3 - 3t^2 - 24t.$$

Describe the motion of the point.

18. Problems: Refer to Assignment Program # 18.

READING PROGRAM -- Application of the Derivative.

All page numbers refer to the textbook, Johnson and Kiokemeister, unless otherwise noted.

Program # 2. Extrema of a Function. Pages 136 - 141.

6. Monotonic Functions, Extrema and First Derivative Test. Pages 142 - 150.

#10. Concavity and Second Derivative Test. Pages 151-156.

#13. Applications on the Theory of Extrema. Pages 158-164.

#16. Velocity and Acceleration. Pages 162 - 172

ASSIGNMENT PROGRAM -- Applications of the Derivative.

All page numbers refer to the textbook, Johnson and Kiokemeister.

PROGRAM # 4. Page 141. I Problems 13 - 20.

8. Page 150. I Problems 1, 7, 8, 11, 12, 17, 24, 29.

#12. Pages 156, 157. I Problems 3,6,11,14,20, 26.

#15. Pages 164, 165. I Problems 1,5,8,11,12.

#18. Page 172. I Problems 1,5,8,10.

THE MEAN VALUE THEOREM

Slide 1. One of the applications of the derivative of a function is its use in deriving a very important theorem called The Mean Value Theorem. This lecture develops the essential background in the form of three theorems and then uses this to establish The Mean Value Theorem.

2. The Mean Value Theorem

If: 1. f is a continuous function in a closed interval [a,b]

2. f' is defined in the open interval (a,b)

Then: there exists a number x_o in (a,b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_o)$$

It isn't possible at this time to explain why this theorem is important but as the calculus is developed the repeated application of this theorem will testify to its usefulness.

3. The form If....Then.... will be used to simplify the presentation of what is assumed as hypothesis or premise and what conclusion may be derived from this.

The premise is precisely stated. No more nor less than what is needed to deduce the conclusion is assumed.

Consider now The Extreme Value Theorem.

If: a function f is continuous on a closed interval [a,b]
Then: the function f has

1. A minimum value called small m on [a,b] .
2. A maximum value called capital M on [a,b] .

The minimum and maximum values refer, of course, to the second component of the ordered pair defined by the equation. That is, for some value of x (call it x_1), which is the first component, the corresponding second component is the smallest (m) or largest (M) in the interval considered. The ordered pairs are (x_1,m) and (x_2,M). An effort should always be made to think in terms of ordered pairs when considering functions.

The importance of the hypothesis is shown for the function f(x) = x, defined, not on a closed interval as required by hypothesis, but on the half open interval $1 \leq x < 2$, making it impossible to tell what the largest or maximum value M is. If it isn't clear why no largest value is so determined, try finding it. The minimum value is obvious, m equals 1.

4. A second case is considered in which the function is defined by

$$f(x) = \frac{1}{(x-1)^2}.$$

The graph indicates an asymptote at x = 1; hence no maximum value M is obtainable in the <u>closed</u> interval [0,2] since the function is not continuous.

5. In case three the function is continuous on the closed interval [a,b] . The values for small and capital M are shown.

6. In the second background theorem two more premises are added to the hypothesis. We retain the first premise:

If: 1. The function f is continuous on the interval capital I. (Notice the notation $f \varepsilon C$ which literally means f is contained in the set of continuous functions.)

(And add:) 2. f' exists in interval I

3. $f(x_o)$ is a minimum or maximum in I.

Then:

$$f'(x_o) = 0.$$

7. The function f is sketched in red. The interval I is also indicated, along with two values in I called x_o at which the smallest and largest second components occur. Suppose $f(x_o)$ is a minimum,

8. Then in the proof, for a suitably small value of h, $f(x_o + h) \geq f(x_o)$ or $f(x_o + h) - f(x_o) \geq 0$ for any h. h must not be so large it exceeds the domain specified. for $h > 0$

$$\frac{f(x_o + h - f(x_o)}{h} \geq 0$$

The dotted line indicates the value of $x_o + h$ and also the vertical height $f(x_o + h)$.

9. If $h < 0$ then the quotient is reversed in sign giving,

$$\frac{f(x_o + h) - f(x_o)}{h} \leq 0$$

The left dotted line indicates $(x_o + h)$ for $h < 0$ and the vertical height is $f(x_o + h)$.

10. Consider what happens if h is made small approaching zero.

This is equivalent to taking the limit as shown. By hypothesis the limit does exist and is $f'(x_o)$. Hence as $h \to 0$ the two inequalities must become equal but the only point at which they can be equal is zero.

11. Hence, if a function is continuous and its derivative exists in some interval I and if $f'(x_o)$ is a maximum or minimum in I, then $f'(x_o) = 0$.

12. The third background theorem is called Rolle's theorem. The hypothesis is in three parts:

If: 1. A function f is continuous in a closed interval [a,b]

2. $f'(x)$ is defined in the open interval (a,b)

3. $f(a) = f(b) = 0$,

Then: $f'(x_o) = 0$ for at least one x_o in the open interval (a,b).

The first two parts of this hypothesis are the same as for the previous theorem. Adding the third part permits us to deduce the conclusion.

13. If f(a) = f(b) = 0 for a continuous function then one of these conditions must always be present:

Case 1. The function can be a straight line from a to b as shown.

Case 2. It can be partially positive and partially negative.

Case 3. Or all positive or all negative.

14. Note that Rolle's theorem includes the hypothesis for the two previous theorems. Hence, in each case these theorems can be applied:

Case 1. Since f(x) = 0 for every x then f'(x) = 0 for all x in (a,b).

Case 2. f(x) is positive someplace between a and b. Then by the extreme value theorem a maximum value, call it x_o, exists and by the second theorem $f'(x_o) = 0$ since x_o is an interior point.

Case 3. The same reasoning applies where the function value becomes negative, excepting now the minimum value is $f(x_o)$ and $f'(x_o) = 0$.

It is possible there may be several such values x_o.

These will be called critical points.

In establishing Rolle's theorem the three cases were possible because of the third premise:

$$f(a) = f(b) = 0.$$

<u>The Mean Value Theorem</u> may now be deduced from Rolle's theorem.

15. Title; <u>The Mean Value Theorem</u>

16. The function f is continuous in the closed interval [a,b] and the derivative exists in the open interval (a,b).

17. The secant line (green) drawn through the two points (a,f(a)) and (b,f(b)) has a slope given by,

$$\text{slope} = \frac{f(b) - f(a)}{b - a}$$

This is the form for the slope of a line through two points.

18. Another line L parallel to this is moved outward...

19. Further......

20. Until it is just tangent to the given secant line. Assume the x value of the tangent line is x_o. Then the slope of the tangent line is $f'(x_o)$ and is equal to the slope of the secant line, hence,

$$f'(x_o) = \frac{f(b) - f(a)}{b - a}$$

which is to be proved.

21. In order to prove this, the function f is reconstructed to comply with the premise of Rolle's theorem requiring $f(a) = f(b) = 0$.

First $f(x)$ is lowered for each x in [a,b] an amount $f(a)$ to produce the dotted curve $f(x) - f(a)$. The slope m_2 is the same as the secant line and has the equation,

22. $$y = m_2(x - a)$$

valid for every x in [a,b] . Next, the dotted curve is lowered at each x in the domain an amount equal to the y value of the green dotted line at that point. Then,

23. the solid red curve is formed, defined by $f(x) - f(a) - y(x)$ as a result of these operations.

24. Call this function capital

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

where the last term $y(x)$ has been replaced by its equal $m_2(x - a)$ and m_2 is the slope of the secant line. The values $F(a)$ and $F(b)$ must be found.

25. This function capital $F(x)$ satisfies the premise of Rolle's theorem. That is, $F(a) = 0$ and $F(b) = 0$.

26. The derivative of the function capital F(x) is found to be simply:

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

27. By Rolle's theorem this must equal zero for some x, call it x_o, in the interval (a,b).

Hence,

$$f'(x_o) = \frac{f(b) - f(a)}{b - a}$$

which was to be proved.

28. <u>The Mean Value Theorem</u>

Slide 29. Extrema of a Function.

The concept of function has been involved in every basic mathematical idea discussed so far. The concept of the derivative of a function was again a function related to its primitive through the operation of differentiation. Since most of the action in science is involved with functions it isn't surprising that effort is made to expose the characteristics of various functions. It is not generally obvious why certain qualities such as extrema of functions are important, but it should become so as applications reveal this.

30. Consider the function defined by the equation given at the top of the slide, that is;

$$f(x) = x^3 - 5x^2 + 6x.$$

In column 1. at the extreme left, values of x are given between x = 0 and x = 3. In the second column the corresponding values of f(x) are shown. For instance, when x = 0, f(x) = 0. When x = .2, f(.2) = 1. When x = .4, f(.4) = 1.664. Note that f(x) is increasing in value until x equals .8. Then for further increases in the value of x, the value of f(x) decreases to -0.448 at x = 2.8. And finally f(x) again increases to zero.

31. Using these ordered pairs of the domain and range the graph of this function is constructed as shown here, revealing the characteristics of the function graphically. For instance, the zeros of the function at x = 0, 2, and 3 are shown. It can be observed that the values of the

function are positive between $x = 0$ and $x = 2$ and negative between $x = 2$ and $x = 3$.

A function may also have the quality of increasing or decreasing in a given domain.

32. Observe the domain values a, b and c as shown in red on the x axis. Also the x values x_1 and x_2. Notice that x_2 is greater than x_1.

The function f is said to be increasing from x = A to B, if for every x_1, x_2 in this domain,

$f(x_2) \geq f(x_1)$ when $x_2 > x_1$ and strictly increasing if:

$$f(x_2) > f(x_1) \quad \text{when } x_2 > x_1.$$

Also the function f is decreasing from x = b to c if for every x_1 and x_2 in this domain, $f(x_2) \leq f(x_1)$ strictly decreasing if

$$f(x_2) < f(x_1) \quad \text{when } x_2 > x_1$$

If the function is increasing or decreasing it is said to be monotonic, or if the function is strictly increasing or strictly decreasing then the function is strictly monotonic. This function appears to be strictly monotonic.

The derivative of a function is useful in establishing the domain where a function is increasing or decreasing.

For instance, for the interval $[x_2, x_1]$ the mean value theorem is :

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(d) \qquad x_1 < d < x_2$$

33. If: $f'(d) > 0$, and $x_2 > x_1$

Then both numerator and denominator of this quotient must be greater than zero. That is, $f(x_2) - f(x_1)$ must be positive, since $x_2 - x_1$ is positive, hence;

$$f(x_2) > f(x_1)$$

which is the condition for a strictly increasing function. Also,

If: $f'(d) < 0$ and $x_2 > x_1$ or $x_2 - x_1 > 0$

Then: The denominator is positive so the numerator must be negative, implying that

$$f(x_2) < f(x_1)$$

which is the condition for a strictly decreasing function.

You may therefore use the derivative in this manner to predict where a function is increasing and where it is decreasing. That is, values of the domain which make the derivative $f'(x)$ positive are those values of the domain where the function $f(x)$ is increasing, and domain values where $f'(x)$ is negative are those values where $f(x)$ is decreasing.

34. In the second line, for the function $f(x)$, the derivative $f'(x)$ is given as

$$f'(x) = 3x^2 - 10x + 6.$$

For the same domain values given in column 1. the values of $f'(x)$ are given in column 3. Note when $x = 0$ $f'(0) = 6$, and when $x = .2$, $f'(.2) = 4.12$. When $x = .4$, $f'(.4) = 2.48$. When $x = .6$, $f'(.6) = 1.08$ and then when $x = .8$ the sign changes and $f'(.8) = -.08$. The derivative of a function may be interpreted as the slope of the tangent line to $f(x)$ at x. Giving $f'(x)$ this meaning, several values of x and $f'(x)$ are next shown as tangent lines.

35. For $x = .4$ $f'(.4) = 2.48$. The black line shown tangent to the red curve has the slope $m = 2.48 = \tan \alpha$ where α is the angle of inclination, or the angle this line makes with the positive direction of the x axis (about 68°). Of most importance is the positive quality of $f'(.4)$ indicating an increasing function.

36. For $x = .6$, $f'(16) = 1.08$. The slope is still positive (about 45°).

37. At $x = .8$, $f'(.8) = -.08$. The slope is now negative, indicating a decreasing function

38. At $x = 1.0$ $f'(1) = -1$.

39. At $x = 1.2$ $f'(1.2) = -1.68$.

40. A composite of these values roughly traces the curve and reveals another characteristic of functions. That is,

where its maximum value is located. Since the maximum value must occur where f(x) changes from increasing to decreasing, it must be where the slope f'(x) changes from positive to negative. Hence, the maximum value of f(x) occurs at f'(x) = 0; which is some value of x between x = .6 and x = .8. An obvious motivation is to find those values of

$$f'(x) = 0.$$

Or,

$$3x^2 - 10x + 6 = 0.$$

41. Solving:

$$3x^2 - 10x + 6 = 0.$$

$x = C_1 = .784$ and $C_2 = 2.55$.

These are called critical values and are the x values where the horizontal slope occurs. And where a horizontal slope occurs, a maximum value of the function occurs, or possibly a minimum value as shown at C_2.

These maximum 'd minimum values of the function are called relative extrema since they occur in the restricted domain, [0,3].

42. If the graph were not shown it would still be possible to distinguish which value of C produces the relative maximum and which produces the relative minimum; by observing that for the relative maximum the slope is positive for x less than C_1 and negative for x greater than C_1.

Hence $f(C_1)$ is a relative maximum.

The relative minimum occurs at C_2 where the slope changes from negative for x less than C_2 to positive for x greater than C_2.

43. This is compiled into the first derivative test for relative extrema of a function:

1. Solve $f'(x) = 0$ for critical values C.
2. If for: $x < C$ $f'(x) > 0$

   $x > C$ $f'(x) < 0$
3. Then: $f(C)$ is a relative maximum.

   or:
4. If for: $x < C$ $f'(x) < 0$

   $x > C$ $f'(x) > 0$
5. Then: $f(C)$ is a relative minimum.

In the examples given in the lab. program special cases of this test will be shown.

44. <u>Concavity of a Function</u>

The term, concavity of a function, refers to the pictorial image of the graph as curving downwards or curving upward. This idea is useful in probing the nature of a function $f(x)$ in much the same manner as the derivative $f'(x)$ was useful in the investigation of extrema of a function.

45. Consider again the function defined by

$f(x) = x^3 - 5x^2 + 6x$

and its first and second derivatives

$f'(x) = 3x^2 - 10x + 6$

and, $f''(x) = 6x - 10.$

In column 1 values of the domain are given with the corresponding values of the function, f(x), given in column 2, and the corresponding values of the function f'(x) given in column 3, and the corresponding values of the function f"(x) given in column 4.

46. In column 3, where the derivative f'(x) is positive such as 6,4.12, 2.48, 1.08 the function f(x) (column 2) is increasing as x increases as shown by the red arrow pointing upward. Where the derivative is negative in column 3 the function is decreasing as shown in column 2 by the arrow pointing downward. Where f'(x) is again positive f(x) is increasing.

The values of the function f'(x), given in column 3, have their corresponding derivative values given in column 4. For instance, when x = 0, f'(0) = 6 and f"(0) = -10. Since a negative derivative implies decreasing function it is assumed that where column 4 is negative, f'(x) in column 3 is decreasing.

47. Where column 4 is negative the blue arrow 1 indicates a decreasing function f'(x), and where column 4 is positive the blue arrow 2 indicates an increasing function f'(x).

48. This information is shown here graphically. The values of x and f(x) may be read from the graph. The corresponding first and second derivatives are given in red and blue respectively.

Beginning with x = .2 the first derivative value is 4.12 and the second derivative value is -8.8. For x = 4, the values become 2.48 and -7.6. Note that the slope of the tangent lines is decreasing, that is, it is getting less steep as indicated by observation. This is consistent with the negative second derivative. That is, the the negative second derivative means the first derivative is a decreasing function.

The condition of decreasing slope holds until abou x = 1.6. The blue second derivative values are negative as shown throu out this interval. The nature of the decreasing slope produces the concave downward quality of the graph and is detected by the negative second derivative. That is, wherever the second derivative value is negative the graph will be curving downward, or will be concave downward.

Between x = 1.6 and 1.8 the second derivative values (blue) change from negative to positive, indicating now that the values of the slope of the tangents are increasing as shown. That is, at x = 1.8, the first derivative value (in red) is -2.28, and increases to -2.0 for x = 2.0 and to -1.48 at x = 2.2, etc. This is consistent with the positive value of the second derivative (blue) in this x interval. The nature of the increasing slopes here produces the concave upward quality of the graph, and may be detected by the positive values of the second derivative

An important point (ordered pair) is where the graph changes from concave upward to concave downward. This is at the point where f"(x) is neither positive nor negative, but where f"(x) = 0, or in this , = 6x - 10, or x = 1 2/3. This value of x is given the descriptive term, "point of inflection". It is found by equating the second derivative to zero and solving for x. This point is useful in graphing, but of greater importance is the detection of concavity by use of the second derivative.

49. For instance, at the critical point found by solving f'(x) = 0, or x = .784 the second derivative value is negative indicating a concave downward nature of the graph. But this implies that the graph is below the tangent line at this point, and hence at this critical point a maximum extremum is indicated.

At the other critical point, x = 2.55, the second derivative (blue values) is positive, which indicates the concave upward nature of the graph, implying all the graph is above the tangent line to the curve at this critical point. Hence, a minimum extremum is indicated.

This is compiled into a statement called:

The Second Derivative Test for Extrema of a Function.

50. If c is a critical number for the function f and f' is defined in some interval about c, then:

1. f(c) is a relative maximum if f"(c) < 0.
2. f(c) is a relative minimum if f"(c) > 0.

Analytic proof of this theorem is not difficult and will establish the intuitive approach taken.

51. First, consider the analytic detection of concavity of a function at any value of x = a by use of the second derivative of the function.

The function f(x) is revealed graphically by the red curve. The point "a" in the domain at about 1.2 on the x axis locates the point (a,f(a) on the graph.

The black line T drawn tangent to the curve at this point reveals the nature of the concavity. If the graph f(x) is below the tangent line T the graph is said to concave downward. If the graph were above this tangent line it would be concave upward.

52. Analytically this can be determined by observing the directed distance called L(x) and shown as a blue dimension line. If the value (length) T(x) were known then, if T(x) were subtracted from f(x), the value L(x) would be defined. That is:

53. $L(x) = f(x) - T(x)$.

This is defined for every x considered in the neighborhood of "a". $L(x)$ is zero at "a", but if $L(x)$ is negative then this implies that $f(x)$ is below $T(x)$ for every x in the neighborhood about a, and hence means the curve is concave downward.

54. $f(x)$ is known and $T(x)$ can be found by using the point $(a, f(a))$ and the line slope equal to $f'(a)$. Hence, for the tangent line, $T(x) - f(a) = f'(a)(x - a)$ defines every point on this line for every x considered. Simplifying:

1. $T(x) = f(a) + f'(a)(x - a)$.

Then:

$L(x) = f(x) - f(a) - f'(a)(x - a)$.

Apply the mean value theorem to the first two terms on the right side of 3. That is,

55. 2. $f(x) - f(a) = f'(b)(x - a)$ (where "b" lies between x and a).

Hence:

3. $L(x) = f'(b)(x - a) - f'(a)(x - a)$

or 4. $L(x) = (f'(b) - f'(a))(x - a)$.

Now, since detection of concavity is to be by use of the second derivative, suppose

56. $f''(x) < 0$ in the domain under consideration.

Then this means that for $x < a$ $f'(x) > f'(a)$ since the negative derivative of a function means the functic is decreasing, for every x in the domain considered. As the domain increases the derivative value decreases. This is true for any $x < a$, so, if b is in this domain then for $b < a \quad f'(b) > f'(a)$.

Also for $x > a$ since $f''(x) < 0$ then $f'(x) < f'(a)$. And if $x = b$ is in this domain, then $f'(b) < f'(a)$. Apply this to

57. 5. $L(x) = (f'(b) - f'(a))(x - a)$

If: $x < a$, or $(x - a) < 0$ and, $f'(b) > f'(a)$

or $f'(b) - f'(a) > 0$

Then: $L(x)$ is negative.

If: $x > a$, or $(x - a) > 0$ and $f'(b) < f'(a)$

or $f'(b) - f'(a) < 0$

and

$L(x)$ is again negative.

This essentially establishes the second derivative test for extrema since if "a" is a critical point "c", then the same results follow and concavity downward at a critical point implies a maximum extremum.

In exactly the same manner the concave upward case can be verified.

ANTIDERIVATIVE

PROGRAM

PROGRAM

1.Lecture: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Subject: Develops the recovery of the position function from the velocity function and relates this to the antiderivative as the area function.

2.Read: Refer to Reading Program # 2. (Antiderivative)

3.Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide Location.

Example 1.

Given the velocity function of a moving object

$$V(t) = \frac{1}{2}t$$

Estimate the distance the object travels in three seconds by using 1/2 second subintervals and assume the maximum velocity in each subinterval.

Example 2.

Given the same function defining velocity, estimate the distance using 1/2 second subintervals but estimate distance in each subinterval by using the minimum velocity in each subinterval.

Example 3.

Given the function defined by

$$f(x) = \frac{1}{x}$$

Compute I (P) and C (P) for the regular partition of [1/2,2] into 6 subintervals.

4.Problems: Refer to Assignment Program # 4.

5.Read: Refer to Reading Program # 5.

6.Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide Location.

Subject: Sigma Notation

Example 4.

Prove by the Mathematical Induction Theorem that:

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$$

Example 5.

Evaluate:

$$\sum_{i=1}^{n} (ai + b)^2$$

7. Lecture: Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Subject: Theory of the Integral.

8. Read: Refer to Reading Program # 8.

9. Problem Study: Educational Media Center. See Directory for Dial Access and Slide Location.

Example 6.

Find the area of the region bounded by the graph of $f(x) = x^2$ and the lines $x = 1$, $x = 3$, $7 = 0$.

<u>Example 7.</u>

Evaluate: $\int_{1}^{2} x^3 \, dx$

<u>Example 8.</u>

Evaluate: $\int_{0}^{3} (z + 1)^2 \, dz$

<u>Example 9.</u>

Evaluate: $\int_{-5}^{5} \frac{x^2}{2} \, dx$

10. Problems: Refer to Assignment Program # 10.

11. Read: Refer to Reading Program # 11.

12. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide Location.

Example 10.

Evaluate: $\int_0^2 (x^3 + x^2)\,dx$

Example 11.

Evaluate: $\int_0^4 (\sqrt{x} + 1)^2\,dx$

Example 12.

Evaluate: $\int_1^2 x(\sqrt{x} + 1)\, dx$

13. Problems: Refer to Assignment Program # 13.

14. Read: Refer to Reading Program # 14.

15. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide Location.

The following equation evolved from the lecture on "Theory of the Integral" and the Fundamental Theorem of Calculus.

$$\lim_{||\Delta x|| \to 0} \sum_{i=1}^{n} G(x_i)\Delta x_i = \int_a^b G'(x)dx = G(b) - G(a)$$

In the limit of the sum Δx_i has a precise meaning necessary to comprehension of the sum. However, in the integral dx has no meaning. It is simply vestigial of Δx.

Finding antiderivatives has been a matter of recalling what derivative produced the function. For instance, since,

$$Dx^2 = 2x$$

obtained from the derivative formula

$$Dx^n = nx^{n-1}$$

The reverse process of finding the antiderivative

(1) $$\int x^n = \frac{x^{n+1}}{n + 1}$$

seems reasonable. That is,

$$\int x^4 = \frac{x^5}{5}$$

Suppose, however, that

(2) $D\ F^r$ is considered, then

(3) $D\ F^r = r\ F^{r-1}\ D\ F = rF^{r-1}\ F'$

by the chain rule for derivatives. Also since

(4) $\int D\ F^r = F^r = \int r\ F^{r-1}\ F'$

it is clear that any expression that is of the form

(5) $r\ F^{r-1}\ F'$

must have the antiderivative F^r.

<u>Example 13</u>.

Suppose we wish to find

$$\int_0^2 (1 + x^3)^{\frac{1}{2}}\ x^2 \cdot dx$$

Recall first that dx has no particular meaning and consider the problem just

$$\int_0^2 (1 + 2x^3)^{\frac{1}{2}}\ x^2$$

From (4) assume $F = 1 + 2x^3$

then $F' = 6x^2$

$r - 1 = \frac{1}{2}$ and $r = 3/2$

then if

$$\int_0^2 (1 + 2x^3)^{1/2} x^2 \, dx$$

is written

$$\frac{1}{6} \cdot \frac{2}{3} \int_0^2 \frac{3}{2} (1 + 2x^3)^{1/2} 6x^2$$

Note that $\frac{1}{6} \cdot 6$ and $\frac{2}{3} \cdot \frac{3}{2}$ has in no way changed the identity of the problem. However, the problem is now exactly of the form

$$\int_0^2 r\, F^{r-1}\, F'$$

and hence equals

$$F^r \Big|_0^2 = (1 + 2x^3)^{3/2} \Big| = (1 + 2 \cdot 2^3)^{3/2} - 1$$

Probably you will find this form preferable to that given in the text; which can then be mastered after this method is understood.

Example 14.

Evaluate the integral:

$$\int_{-3}^{-1} \frac{1}{(4x-1)^2} \, dx$$

Example 15.

Evaluate

$$\int_1^2 \frac{1}{x^2}\left(1 - \frac{1}{x}\right)^{\frac{1}{2}} dx$$

Example 16.

Evaluate

$$\int \frac{t^2}{(1 + 4t^3)^2} dt$$

16. Problems: Refer to Assignment Program # 16.

17. Problem Study: Educational Media Center: See Directory for Dial Access and Slide Location.

Example 17.

Sketch the graph of f and g in the given interval on the same coordinate system.

$$f(t) = (1 - t) \qquad g(x) = \int_0^x (1 - t)\, dt$$

Example 18.

Evaluate:

$$\int_0^1 x(x^2 + a^2)^n \, dx$$

Example 19.

Find the limit if it exists.

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx$$

18. Problems: Refer to Assignment Program # 18.

READING PROGRAM -- Antiderivative

All page numbers refer to the textbook, Johnson and Kiokemeister, unless otherwise indicated.

PROGRAM # 2. Completeness Property - Intermediate Value Theorem. Pages 185 - 194

5. Sigma Notation. Pages 195 - 197

8. Upper and Lower Integrals, Integrals, Fundamental Theorem of Calculus. Pages 198 - 213

#11. Integration Formulas. Pages 214 - 216

#14. Change of Variable -- Integration. Pages 217 - 219 Compare this method to the method given in Program # 15.

ASSIGNMENT PROGRAM -- Antiderivative

All page numbers refer to the textbook, Johnson and Kiokemeister, unless otherwise indicated.

PROGRAM # 4.	Page 194.	I Problems 1, 3, 5.
	Page 204.	I Problems 1, 3, 5, 8.
#10.	Page 213.	I Problems 1 - 12.
#13.	Page 216.	I Problems 1 - 10.
#16.	Page 219.	I Problems 1 - 16.
#18.	Page 222.	I Problems 3,5,8,11,13,14,19,21,22.

Slide 1. Title The Integral Concept (Antiderivative)
--Gorman R. Nelson

All basic mathematical ideas contained in Calculus are related in a precise and natural way. For instance, from the concept of a function the idea of the derivative evolved and from this, motivation developed for the limit concept of a function.

It would be difficult, if not impossible, to understand these concepts as isolated ideas, since the natural relation between them is essential to comprehension.

Another basic idea in this relation is called the anti-derivative of a function. This is first examined intuitively and finally in a more mathematically vigorous manner.

2. The derivative concept was developed as a mathematical description of the quality of velocity when only the quality of position was known. In this slide a car was assumed to have a position defined by the equation

$$L(t) = 4t^2$$

From this function another function was derived describing the velocity at any time t. The derived equation

$$L'(t) = 8t$$

conformed precisely to the definition of a derivative and hence was called the derivative of the position function. The mathematical process of differentiation was developed as the

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

and shown to have meaning as the rate of change of f compared to x.

3. The velocity function evolved naturally from the position function. Suppose as shown here the velocity function

$$L'(t) = 8t$$

is known. Can the position function be retreived naturally from this?

Since distance is the product of velocity and time, then the distance from the zero position defines position at time t. Since the car is increasing in velocity it is assumed that velocity can only be approximated for any interval of time. It also seems natural that the smaller the interval of time considered the closer the approximation becomes. For instance, if the car starts as shown at zero position then 1 second later its velocity as L'(1) feet per second. If this value is assumed for the entire interval then the distance in the first second is $L'(1)\cdot 1$ feet. Of course this is an approximation since the velocity is not constant in this time interval, and the maximum velocity was chosen to compute the distance.

4. In the interval of time from 1 to 2 seconds suppose the velocity is assumed constant at L'(2) feet per second, then the distance in this interval is $L'(2) \cdot 1$ feet.

5. In the time interval from 2 to 3 seconds the distance is approximated by taking the velocity at 3 seconds or L'(3) feet per second. The distance is then $L'(3) \cdot 1$ feet. Estimating the distance traveled throughout all five time intervals and adding these gives an approximation to the

distance traveled and the position L(t) from the zero position.

Hence L(t) is approximated

$$L(t) = L'(1)\cdot 1 + L'(2)\cdot 1 + L'(3)\cdot 1 + L'(4)\cdot 1 + L'(5)\cdot 1$$

6. Evaluating each product and adding gives the distance from zero to be 120 feet. This is a rather poor approximation to the known 100 feet, and obviously occurs because the velocity is not constant in each interval and the maximum value in each interval was chosen to compute the distance. It seems natural to suppose that a better approximation can be made by taking smaller intervals.

7. The intervals of time are here shortened to ½ second and the distance is approximated by the sum

$$L(t) = L'(\tfrac{1}{2})\cdot\tfrac{1}{2} + L'(1)\cdot\tfrac{1}{2} + L'(3/2)\cdot\tfrac{1}{2} + L'(2)\cdot\tfrac{1}{2} + L'(5/2)\cdot\tfrac{1}{2}$$
$$L'(3)\cdot\tfrac{1}{2} + L'(7/2)\cdot\tfrac{1}{2} + L'(4)\cdot\tfrac{1}{2} + L'(9/2)\cdot\tfrac{1}{2} + L'(5)\cdot\tfrac{1}{2}$$

Again the maximum velocity is chosen in each interval in computing distance.

An important transition in meaning of these terms can be made here by observing that each term is a product of L'(t) and a constant. For instance, consider the term

$$L'(5/2)\cdot\tfrac{1}{2}$$

8. Assume this is the graph of L'(t) = 8t. The value of L'(5/2) is then the length of the dotted line from the base line at 5/2 to the graph, and has the value determined by

$L'(5/2) = 8 \cdot 5/2$. This is multiplied by ½, which is the distance between intervals. It is apparent that this term which expresses distance as velocity multiplied by time can also be interpreted as area. And so it is with every term in this sum.

9. The first term $L'(½) \cdot ½$ becomes area A_1, the second term $L'(1) \cdot ½$ becomes A_2 and so on to A_{10} which is $L'(5) \cdot ½$. Evaluating all terms gives the sum 2 + 4 + etc. to + 20. This sum is 110 feet --again an overestimate to the correct value of 100 feet but better than the first approximation of 120. However, this approximation now has a double meaning. It can also be interpreted as the red area.

10. Suppose the subintervals of time are constructed mathematically by dividing the entire interval of time into n equal parts. That is, subtract from the time at the finish of the run, the time at start and divide by the number of subintervals desired. If a is the time at start and b the time at finish, then

$$\frac{b - a}{n}$$

is the time in each of n subintervals. Or if each division line defining the subintervals is marked as t_0 for the beginning and then t_1, t_2, t_3, etc. up to the final time t_n, then the length of time for each subinterval

$$\frac{t_n - t_0}{n}$$

of for any specific subinterval $t_i - t_{i-1}$.

The subintervals of time are all equal and are determined by the integer n. For instance, if n = 5, then the first case of 1 second subintervals is given. If n = 10, then the second approximation of ½ second subintervals is given. The subintervals of time depend on the values of n, a and b.

11. With the starting time at zero, a = 0. If b is assumed to be the finish time then each subinterval becomes just $\frac{b}{n}$.

The progressive values of time on the base are from t_0 which is the starting time, to the first time interval which is t_1 or $\frac{b}{n}$, then t_2 or $\frac{2b}{n}$ etc. Each subinterval of time is the same but the time increases by this amount across the base, up to the final time at $\frac{nb}{n}$ or t_n.

On this base the sum of products is constructed in which each term is a product of velocity and time but represented by an area. The interval from t_9 to t_{10} is $\frac{b}{n}$ and the area A_{10} as one of the terms is

$$L'\left(\frac{10b}{n}\right) \cdot \frac{b}{n}.$$

12. The sum of all terms becomes:

$$L(b) = L'\left(\frac{b}{n}\right)\cdot\frac{b}{n} + L'\left(\frac{2b}{n}\right)\cdot\frac{b}{n} + L'\left(\frac{3b}{n}\right)\cdot\frac{b}{n} + \cdots + L'\left(\frac{nb}{n}\right)\cdot\frac{b}{n}$$

The form of this sum is now examined for increasingly large values of n.

13. 1. $L(t) = L'\left(\frac{b}{n}\right)\cdot\frac{b}{n} + L'\left(\frac{2b}{n}\right)\cdot\frac{b}{n} + \cdots + L'\left(\frac{nb}{n}\right)\cdot\frac{b}{n}$

Note: If n = 5 and b = 5

$L(5) = L'(1)\cdot 1 + L'(2)\cdot 1 + \cdots + L'5.1$

which was the first estimate found =120

If $n = 10$ and $b = 5$

$L(5) = L'(\frac{1}{2})\cdot\frac{1}{2} + L'(1)\cdot\frac{1}{2} + \cdots + L'(5)\cdot\frac{1}{2}$

which was the second estimate found = 110.

Suppose the sum is simplified by replacing the function notation L' by the given function. That is $L'(t) = 8t$. Then

2. $L(b) = 8(\frac{b}{n})\cdot\frac{b}{n} + 8(\frac{2b}{n})\cdot\frac{b}{n} + 8(\frac{3b}{n})\cdot\frac{b}{n} + \cdots + 8(\frac{nb}{n})\cdot\frac{b}{n}$

or

14. 3. $L(b) = 8\frac{b^2}{n^2}\,[1 + 2 + 3 + \cdots + n]$

The sum of the first n integers is

15. 4. $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}(n)\,(n + 1)$

Hence,

5. $L(b) \doteq 8\frac{b^2}{n^2}\cdot\frac{1}{2}(n^2 + n) = 4b^2 + 4\frac{b}{n}$

From this it is immediately apparent that the limit of this expression as n becomes infinite reduces the term $\frac{4b}{n}$ to zero. That is, as the size of the subintervals become smaller the sum of the products approaches just $4b^2$ as given in line 6.

Since b is any arbitrary time t this equation can then be written

$L(t) = 4t^2$

For $b = t = 5$ this is

$L(5) = 4\cdot 5^2 = 100$

which is the exact distance or position which the car was known to have. But in addition to this recovery, the transition to a geometric interpretation is complete.

16. That is, the position function

$$L(t) = 4t^2$$

recovered by the process of taking the sum of the products as the number of such products increases to infinity can be identified with the area shown here shaded in red. This area may also be computed rather easily by the triangle formula.

$$A = \tfrac{1}{2} \text{ base} \cdot \text{altitude} = \tfrac{1}{2}b \cdot h$$

where h is found to be $L'(b) = 8b$, hence,

$$A = \tfrac{1}{2}b \cdot 8b = 4b^2$$

17. The process involved in retrieving the position function from the velocity function suggests a means for reversing the process of differentiation indicated by

$$D\,L(t) = L'(t).$$

The notation used for the reverse process is an elongated ∫ as shown in line 2. This is read "the antiderivative of $L'(t) = L(t)$". The operation is called antidifferentiation, or integration and the function obtained from the process is called the antiderivative.

In many cases the antiderivative of elementary functions is quite obvious.

For instance, if $f'(x) = 2x$

then $f(x) = x^2 + c$ (c a constant)

since the derivative gives $f'(x)$

18. The antiderivative of a function as related to the area bounded by the graph of the function and the axis of its independent variable is examined now for several functions and in greater detail. First consider the function

$$f'(x) = \tfrac{1}{2}x$$

19. The area shaded in blue is a triangle and has the area of ½ the base times the altitude. For the dimension shown this is just

$$A(x) = \tfrac{1}{4}x^2$$

20. That is: ½ the base x times the altitude $\tfrac{1}{2}x = \tfrac{1}{4}x^2$

In line 1. the derivative of the area function is equal to the function f'(x), hence in line 2. this is expressed in mathematical notation as the antiderivative of f'(x) from 0 to x equals f(x), by use of the symbol ∫. This is read as "the antiderivative of f'(x) evaluated from 0 to x is f(x)."

21. Suppose

$$g(x) = 1/3\ x^2$$

has the graph shown.

22. Then the antiderivative of g(x) should be the area function A(x), or using mathematical notation

$$\int g(x) = A(x)$$

Finding A(x) for this area has complications not found in the previous example where A(x) had the elemental form of a triangle. However, the process of finding the area suggested in the beginning by taking rectangles of variable width can be tried.

23. Suppose the estimate is made first using heights of rectangles which are maximum in any interval as shown here, having n equal divisions.

24. The function domain which is the base of the area is given by b-a or since a = 0, just b. This divided by the number of intervals provides n subintervals having bases, all equal, of $\frac{b}{n}$.

25. The value of x at each division point along the x axis is given in terms of b and n. The first division is $\frac{b}{n}$, the second is $\frac{2b}{n}$, the third is $\frac{3b}{n}$ etc. up to the last which is $\frac{nb}{n}$ or just b. The area of the rectangle shaded blue is is given by taking the height which is

$$\text{height} = g(9\tfrac{b}{n})$$

times $\quad \text{width} = \frac{b}{n}$

or

$$A_9 = 1/3\ (\tfrac{9b}{n})^2 \cdot \tfrac{b}{n}$$

26. The first and all following rectangles are formed from

the prc·luct of width $(\frac{b}{n})$ and the maximum height in the interval as given by the function

$$A_1 = \frac{b}{n} \cdot \frac{1}{3} \cdot (\frac{b}{n})^2$$

$$A_2 = \frac{b}{n} \cdot \frac{1}{3} \cdot (\frac{2b}{n})^2$$

$$A_3 = \frac{b}{n} \cdot \frac{1}{3} \cdot (\frac{3b}{n})^2$$

etc. $\vdots$

$$A_n = \frac{b}{n} \cdot \frac{1}{3} \cdot (\frac{nb}{n})^2$$

Each product is of the form $g(x_i)(x_i - x_{i-1})$

Note also that in each product the common factors are

$$\frac{1}{3}, \frac{b}{n} \text{ and } (\frac{b}{n})^2 .$$

These can be removed from each term leaving only the squared integers from 1 to n.

27. At the bottom of the slide this is given as a sum of all A's from 1 to n or

$$\sum_{i=1}^{n} A_i = \frac{1}{3}(\frac{b}{n})^3[1^2 + 2^2 + 3^2 + \ldots 9^2 \ldots + n^2]$$

where each A_i is a product of a value of the function and a small interval of the domain.

28. In line 1. the sigma notation is used to indicate the sum of the first n rectangles of area A. The symbol Σ is the Greek letter sigma and means in mathematical use the summation of A_i for all values of i from 1 to n. as shown. In line 2. the notation implies the sum of the first i^2

integers where i has values from 1 to n,

or

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + (n-1)^2 + n^2$$

29. By using the Theorem on Mathematical Induction this sum can be shown to equal

$$\frac{1}{6} \cdot n \cdot (n+1)\ (2n+1)$$

as given in line 3. This isn't obvious but time won't be taken here to establish this equality. This will be proved in a later problem.

30. Recall that the sum of the rectangles was found to be

$$\sum_{i=1}^{n} A_i = \frac{1}{3}\left(\frac{b}{n}\right)^3 \cdot \frac{1}{6}n(n+1)(2n+1)$$

Replacing the sum of the first n integers squared by its equal, line 4., provides the sum of the areas of the rectangles. Or

$$\sum_{i=1}^{n} A_i = \frac{1}{3}\left(\frac{b}{n}\right)^3 \cdot \frac{1}{6}\ n(n+1)(2n+1)$$

31. This can be simplified by cancellation of the n's. Then as the number of rectangles is greatly increased, that is, as $n \to \infty$. This can be written; shown in line 5. as,

$$\text{Limit}_{n\to\infty} \sum_{i=1}^{n} A_i = \text{limit}_{n\to\infty} \frac{1}{18}b^3(2 + \frac{3}{n} + \frac{1}{n}2)$$

or, using more explicit notation

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)(x_i - x_{i-1}) = \lim_{n\to\infty} \frac{1}{18} b^3 \left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

32. As n becomes infinite the terms $\frac{3}{n}$ and $\frac{1}{n^2}$ approach zero leaving the sum

$$A = \frac{1}{9} b^3$$

This is presumed to be the area under the curve. That is

$$A(b) = \frac{1}{9} b^3$$

or $$A(x) = \frac{1}{9} x^3$$

33. The area $A(x)$ shaded blue is $\frac{1}{9} x^3$ as shown in line 1. Note that again the derivative of bounded area is precisely $g(x)$, or as shown in line 3..

The antiderivative of $g(x)$ from 0 to b, is equal to $A(b)$.

34. In every case considered so far the maximum value of the function was used in estimating the intervals of distance or area. What effect is produced by using the minimum values of the function in each subinterval in retrieving its antiderivative?

Consider the graph of the function $G(x)$ as shown, having vertical boundaries at $x = a$ and $x = b$.

35. The base of the area from a to b is partitioned into n equal parts. Each division line is identified by x with a subscript as shown, such as x_0, x_1 etc.

36. Vertical lines erected on this base form the sides of the sub areas.

37. The dot shaded sub areas are shown in which each is the product of a subinterval of the base such as $x_4 - x_3$ or in general $(x_{i-1} - x_i)$ and the maximum value of the function in this subinterval. Since the function is monotone increasing this is always the right side of the subinterval. The area of the second rectangle is $G(x_2) \cdot (x_2 - x_1)$ etc. The sum of all such areas is given in the form

$$S_u = \sum_{i=1}^{n} G(x_i)(x_i - x_{i-1})$$

Notice again the form of the product of each sub area.

This is called the upper sum, S_u, since it is an upper bound of the area under the curve.

38. In similar manner the blue shaded areas are shown here in which each sub area is the product of the minimum value of G(x) in each subinterval, and the subinterval. The second rectangle here would have an area $G(x_1)(x_2 - x_1)$. The sum of all such areas is

$$S_L = \sum_{i=1}^{n} G(x_{i-1})(x_i - x_{i-1})$$

called a lower sum since the estimated area is a lower bound of the actual area under the curve.

39. By super-imposing S_L on S_u the difference may be observed as the clear dotted area. What happens to this differ-

ence as the subintervals are increased in number is of immediate concern.

The subintervals $(x_i - x_{i-1})$ are all equal and common to the corresponding products so these are replaced by the simpler notation

$$\Delta x_i = (x_i - x_{i-1})$$

40. S_u and S_L are shown with this notation in line 1. $S_u - S_L$ the clear dotted shaded area is just

$$\sum_{i=1}^{n} [G(x_i) - G(x_{i-1})] \Delta x_i$$

as given in line 2.

41. In line 3. an interesting consequence of taking the sum is observed.

$$\sum_{i=1}^{n} [G(x_i)-G(x_{i-1})] = [G(x_i)-G(x_0)] + [G(x_2)-G(x_1)] + [G(x_3) - G(x_2)] +\cdots+ [G(x_{n-1}) - G(x_n)]$$

Notice that each value of G_i, excepting the first and last, has a positive and negative term and so vanish. That is, from $G(x_i)$ in the first term, $G(x_1)$ in the second is subtracted. In fact, all terms vanish except $G(x_n)$ and $-G(x_0)$. Hence in line 4

$$S_u - S_L = [G(x_n) - G(x_0)] \Delta x$$

and since each subinterval

$$\Delta x = \frac{b - a}{n},$$

this can be written in line 5 as,

$$S_u - S_L = [G(b) - G(a)] \left(\frac{b - a}{n}\right)$$

Remember this expression for the difference, $S_u - S_L$ and;

42. observe on this drawing that G(b) - G(a) is the height of the dot shaded column and the base is simply $\frac{b-a}{n}$. As n becomes infinite the base goes to zero and the difference $S_u - S_L$ vanishes.

It appears that the same area is found whether S_u or S_L is used in the process. Also the same form of products was used as in all previous cases.

43. Suppose the same interval [a,b] is used but the partition is <u>not</u> required to form equal subintervals. That is, $(x_1 - x_0)$ need not equal $(x_2 - x_1)$ etc. The partitioning points are still called $x_0, x_1, x_2, x_{i-1}, x_i \cdots x_n$. Each subinterval is given the notation Δx_i, where the subscript identifies its position in [a,b]. That is, $\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1$, etc.

Using this and previous notation S_u may be given in mathematical notation using the same form of a sum of products of a function and a subinterval of its domain.

$$S_u = \sum_{i=1}^{n} G(x_i)\ \Delta x_i$$

44. and in like manner

$$S_L = \sum_{i=1}^{n} G(x_{i-1})\ \Delta x_i$$

45. In forming each sum S_u and S_L the maximum or minimum value of the function is used respectively.

Suppose any other value of x in each subinterval is chosen to form the sums of products. That is, in the first sub area suppose any other value of $x = \xi_1$ is chosen instead of x_0 or x_1 to form the sub area. Then if this is done for each sub area, since $G(x_{i-1}) \leq G(\xi i) \leq G(x_i)$ it may be deduced that

46. $$S_L \leq S_\xi \leq S_u$$

47. Retain this for future use and return to the difference areas as shown here in clear dot shading.

$$S_u - S_L = \sum_{i=1}^{n} [G(x_i) - G(x_{i-1})] \Delta x_i$$

The largest of these subintervals Δx_i is called the "norm" for this partition. It appears to be the first subinterval $x_i - x_o = \Delta x_1$. If this value is substituted for each Δx_i then

48. in line 3.

$$S_u - S_L \leq \sum_{i=1}^{n} [G(x_i) - G(x_{i-1})] \; ||\Delta x_i||$$

The inequality exists because the norm is larger or equal to every other Δx_i. The notation used to identify the norm are double parallel bars.

The summation of

$$\sum_{i=1}^{n} [G(x_i) - G(x_{i-1})]$$

may be telescoped to

$$G(x_n) - G(x_0)$$

or, in line 4.

49. $$S_u - S_L \leq [G(b) - G(a)] \, ||\Delta x||$$

If the norm is chosen so

$$||\Delta x|| < \frac{\varepsilon}{G(b) - G(a)}$$

then

$$S_u - S_L < \varepsilon \text{ for every } \varepsilon, \text{ and}$$

hence

$$S_u = S_L.$$

Since $S_L < S_\xi < S_u$ then

$$S_u = S_L = S_\xi = S.$$

And if this limit exists for <u>any</u> partitioning then,

$$\lim_{||\Delta x|| \to 0} \sum_{i=1}^{n} G(\xi_i) \, \Delta x_i = S = \int_a^b G(x) \, dx$$

Observe the form of the sum of the products, that is, $G(\xi_i)$ times Δx_i

The Fundamental Theorem of Calculus may be deduced from this statement. All efforts so far have been directed to evaluating sums such as:

$$\sum_{i=1}^{n} G(\xi_i) \, \Delta x_i$$

Such sums when they exist were recognized as the anti-derivative of G and could geometrically be identified with the area bounded by the function graph, the vertical lines on the base and the axis of the independent variable. Finding this area proves to be rather formidable for all but the most elementary functions. The process of finding antiderivative is called antidifferentiation and is represented by

$$\int_a^b G(x)\ dx$$

50. The Fundamental Theorem of Calculus

51. Evaluation of such sums is greatly simplified by use of the

Fundamental Theorem of Calculus.

Observe in line 1. the telescoping sum

$$\sum_{i=1}^{n} [Gx_i - Gx_{i-1}] = G(b) - G(a)$$

Refer to slide 41. for explanation of this if it is not recalled.

In line 2. the important Mean Value Theorem is applied to each of the subintervals implied in line 1. That is, in line 1. suppose $i = 2$, then $G(x_2) - G(x_1)$ implies an interval $x_2 - x_1$ and so on for all the sums considered.

In each of these subintervals $(x_i - x_{i-1})$ the mean value theorem gives line 2. That is,

$$\frac{G(x_i) - G(x_{i-1})}{x_i - x_{i-1}} = G'(\xi_i) \qquad (x_{i-1} < \xi_i < x_i)$$

If each $(x_i - x_{i-1})$ is given the notation Δx_i then this is written in line 3. as

$$G(x_i) - G(x_{i-1}) = G'(\xi_i)\ \Delta x_i$$

52. Equation 4 is recalled from slide 49. For some value of the norm $||\Delta x|| > 0$, this can be written as given in line 5. where,

$$| \Sigma\ G(\xi_i)\ \Delta x_i - \int_a^b G'(x)\ dx| < \varepsilon$$

for all $\delta < ||\ \Delta x\ ||$

This says simply that the difference between the left side of equation 4. and the right side differs by an amount less than ε for some norm $||\ \Delta x\ ||$. And for every $\varepsilon > 0$ there is a norm for which this difference holds.

Two substitutions are now made for $\Sigma\ G'(\xi_i)\ \Delta x_i$ in line 5. First from line 3. $G'(\xi_i)\ \Delta x_i$ is replaced by $G(x_i) - G(x_{i-1})$ or

$$\sum_{i=1}^{n} G'(\xi_i)\ \Delta x_i = \sum_{i=1}^{n} [G(x_i) - G(x_{i-1})]$$

which is then replaced by its equal from line 1. to give in line 6.

53. $$|[G(b) - G(a)] - \int_a^b G'(x)\ dx| < \varepsilon$$

Since this inequality is true for all ε as deduced from line 4. it may be deduced finally that

54. $$\int_a^b G'(x)\ dx = G(b) - G(a)$$

This equation implies the Fundamental Theorem of Calculus and permits an easy evaluation of the definite integral when the antiderivative of the function is known.

The form

$$G'(x)\ dx$$

should serve as a reminder of the form of the products which produced this sum. Otherwise the expression dx has no significant meaning.

This theorem must be known for future use.

The process, in summary, evolved in rather precise steps. These should be carefully observed.

1. The idea began with a function

$$L'(t) = 8t$$

The prime was retained only to suggest that this was a derivative of some function. In this case representing velocity of a car.

2. The given function L(t) was obtained by differentiation. Some process, called antidifferentiation, was needed to reverse the process and obtained the position function L(t).

3. This suggested a division of the total time of travel in subintervals. Note; the subdivision occurs on the domain of the function. In these subintervals, distance was computed as the product of the function defining velocity. and the subintervals under consideration. The total distance was approximated by adding these products.

4. Increasing the number of subintervals improved the approximation: hence, finally, by taking the

$$\lim_{n \to \infty} \sum_{i=1}^{n} L'(t_i)\ (t_i - t_{i-1})$$

the function L(t) was recovered.

5. This process was called antidifferentiation and represented by the symbol given in slide 17.

The ideas involved in this process are considered in greater detail in what follows.

PROGRAM -- APPLICATIONS OF THE INTEGRAL

PROGRAM

1. Read — Areas. See Reading Program # 1. Application of the Integral.

2. Problem Study: — Educational Media Center. See E.M.C. Directory for Dial Access and Slide location.

Example 1.

Find the area of the region bounded by the curves

$$y = x^3,\ y = 0,\ x = 1,\ x = 3$$

Example 2.

Find the area bounded by

$$y = \sqrt{x + 4},\ y = 0,\ x = 0$$

Example 3.

Find the area bound by

$y^2 = 4x, \quad x = 1$

3. Problems: See Assignment Program # 3.

4. Read: See Reading Program # 4.

5. Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide Location.

Example 4.

Find the volume obtained by rotating the region about the x axis.

$y = x^2, \; y = 0, \quad x = 2$

Example 5.

Find the volume obtained by rotating

$y = \quad , \; x = 1, \; x = 3, \; y = 0$

6.Problems: See Assignment Program # 6.

7.Read: See Reading Program # 7.

8.Problem Study: Educational Media Center. See E.M.C. Directory for Dial Access and Slide Location.

Example 6.

Find the work done in stretching a spring from its natural length of 12" to 18" if 4 pounds of force is needed to stretch it 1".

9.Problems: See Assignment Program # 9.

READING PROGRAM -- Application of the Integral.

Program # 1. Areas. Pages 246 - 253.

4. Volume. Pages 253 - 259.

7. Work. Pages 260 - 265.

ASSIGNMENT PROGRAM -- Application of the Integral.

Program # 3. Page 252. I Problems: 1, 2, 3, 4, 5, 7, 9.

6. Page 258. I Problems: 1, 3, 4, 9.

9. Page 264. I Problems: 1 - 6

All page numbers in the Reading Program and the Assignment Program refer to the textbook, Johnson and Kiokemeister.