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ABSTRACT

This is part two of a two-part SMSG algebra text for ninth-grade students. The text was written for those students whose mathematical talent is underdeveloped. Chapter topics include the real numbers, addition of real numbers, multiplication of real numbers, properties of order, and subtraction and division for real numbers. (MP)

SCHOOL MATHEMATICS STUDY GROUP

INTRODUCTION TO ALGEBRA

(Part 2)

(preliminary edition)

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Chapter 6

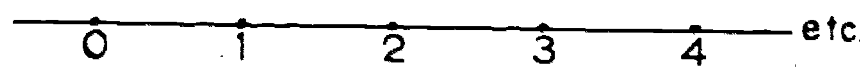
THE REAL NUMBERS

6-1. The Real Numbers.

Integers

You remember that earlier we "labeled" points on a line with names of numbers. A better way to say this is to say that we <u>associated</u> numbers with the points of a line called the number line. In this chapter, we are going to use the number line to introduce some new numbers.

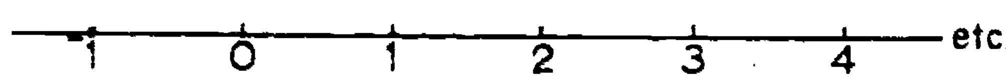
To begin with, remember that we graphed the <u>whole numbers</u> on the number line, like this:



All of these numbers are associated with points to the <u>right</u> of 0 (we mean, of course, to the right of the <u>point</u> associated with 0). Maybe you have wondered about numbers associated with points to the <u>left</u> of 0.

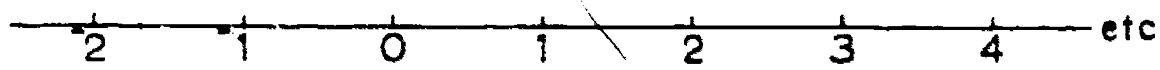
Very soon we'll see that a new kind of number is needed to solve certain problems. These new numbers will be called <u>negative numbers</u>, and we can associate them with points of the number line to the left of 0.

Let's start by noticing the interval (or the "piece" of the line) between 0 and 1. We will again use this interval as a <u>unit of measure</u>, but this time points will be marked to the left of 0. Using this interval as a unit of measure, the first point located to the left of 0 is shown below:

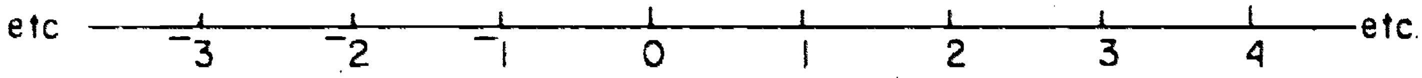


We label this point "$^{-}1$" and read it as "NEGATIVE ONE". Notice how high the "dash" in "$^{-}1$" is written. This dash is a signal to us that we are talking about a point to the left of zero.

The next point located is labeled "$^{-}2$" and is read "NEGATIVE TWO".



The next point located is labeled "$^{-}3$" and is read "NEGATIVE THREE".



We could go on and on locating points like this. That's what the "etc." means. How would you locate the point to be labeled "$^{-}7$"? How would you locate the point to be labeled "$^{-}15$"?

Using this way of labeling points on the number line, would there be a point labeled "$^{-}1,000,000$"? How would this "label" be read?

All of these "labels" we've been giving to points to the left of 0 will be used as names of numbers (that is, as numerals). We'll soon see how these numbers behave and how useful they can be.

We can now take the whole numbers, {0, 1, 2, 3, . . .}, together with the new numbers we have named and form a new set of numbers that can be shown like this:

{. . ., $^{-}3$, $^{-}2$, $^{-}1$, 0, 1, 2, 3, . . .}.

This set is called the set of integers.

Any one of the numbers in the set is called an integer. For example, 8 is an integer, $^{-}43$ is an integer, and so on. (On the other hand, $\frac{1}{2}$ is not an integer.)

In the set of integers shown above, do you understand the "three dots" after 3? They mean, of course, that we could go on forever locating integers to the right of 3 on the number line. The three dots before $^{-}3$ mean that we could go on and on locating integers to the left of $^{-}3$ on the number line.

There are some special subsets of the set of integers that can be shown like this:

$\ldots, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, \ldots$

<u>negative integers</u> | <u>zero</u> | <u>positive integers</u>

The positive integers are associated with points to the right of zero. We can show the set of positive integers like this:

$$\{1, 2, 3, \ldots\}.$$

The negative integers are associated with points to the left of zero. We can show the set of negative integers like this:

$$\{\ldots, {}^-3, {}^-2, {}^-1\}.$$

However, many times the set of negative integers is shown like this:

$$\{{}^-1, {}^-2, {}^-3, \ldots\}.$$

Zero itself is an integer, but it is neither positive nor negative.

We can make the following statement:

> The set of positive integers, the set of negative integers, and zero make up the <u>set of integers</u>.

<u>Discussion Questions 6-1a</u>

1. How do we show that a number is the coordinate of a point to the left of zero on the number line?
2. What elements make up the set of integers? How many are there?
3. What elements make up the set of positive integers? How many are there?
4. What elements make up the set of negative integers? How many are there?
5. Are there any other integers besides the positive and negative integers?

6. Is zero positive or is it negative?

Oral Exercises 6-1a

1. Name five elements of the set of positive integers.
2. Name five elements of the set of negative integers.
3. Name five elements in the set of whole numbers.
4. Name five elements in the set of counting numbers.
5. Describe $\emptyset$.

These are all subsets of the set of integers.

Problem Set 6-1a

1. (a) Which of the following sets are the same?

   W is the set of whole numbers.

   P is the set of positive integers.

   L is the set of non-negative integers.

   (Hint: "non" means "not" so "non-negative numbers" means "numbers that are not negative")

   I is the set of integers.

   N is the set of counting numbers.

   Q is the set of non-positive integers.

   S is the set of negative integers.

   (b) Which of the above are subsets of I? of Q? of L? of P?

2. Draw the graphs of the following sets:

   (a) $\{0, 3, 5, ^{-}2, ^{-}4\}$.

   (b) The set of positive integers less than 7.

   (c) The set of negative integers ≥ 5.

   (d) All integers greater than $^{-}5$ but less than 4.

   (e) The set of counting numbers less than 1.

3. Of the two points whose coordinates are given, which is to the right of the other on the number line?

   (a) 3, $^{-}4$

   (b) 5, $^{-}4$

   (c) $^{-}2$, $^{-}4$

   (d) $^{-}5$, $^{-}1$

   (e) $^{-}2$, 0

   (f) 0, $^{-}4$

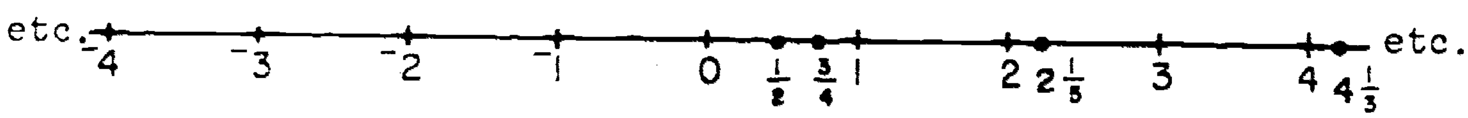
   (g) 5, 0

   (h) 0, 3

4. Translate this problem into a sentence in algebra after selecting a variable and telling what it represents. You need not find the truth set of the sentence. The first of two trains travels at the rate of 45 miles per hour and the other travels at the rate of 10 miles per hour more than $\frac{1}{3}$ of the speed of the first. How long does it take the trains to get 300 miles apart if they start from the same point and travel in opposite directions? (Hint: If you travel 30 miles per hour for two hours, how far have you gone?)

Rational Numbers

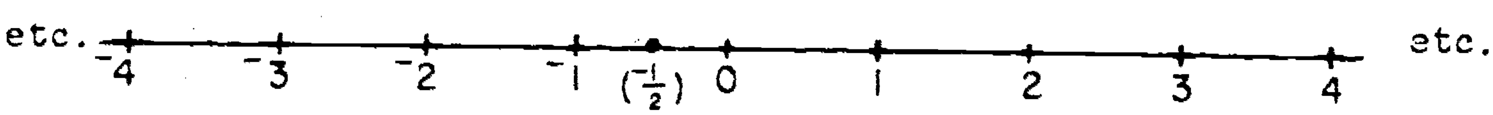
In the last section, it was pointed out that $\frac{1}{2}$ is <u>not</u> an integer. Some other examples of numbers that are <u>not</u> integers are $\frac{3}{4}$, $2\frac{1}{5}$, and $4\frac{1}{3}$. You may remember, though, that we have called such numbers as these <u>rational</u> <u>numbers</u>. They also are associated with points of the number line. We can show the graph of the four numbers mentioned in this paragraph like this:



Of course, there are many, many other rational numbers. In fact, as we saw in Chapter 1, there is an infinite number of such rational numbers.

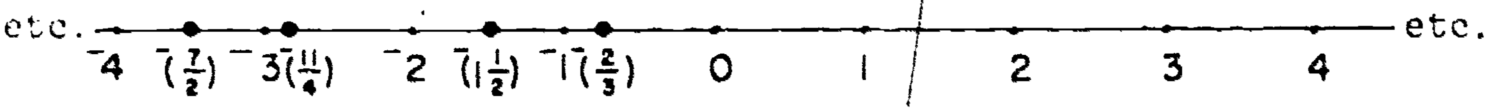
All of the rational numbers we have worked with so far have been associated with points to the right of 0 (and then, of course, 0 itself). But you were warned earlier that there are other rational numbers.

To begin with, after graphing the negative integers, it seems natural to put the label "$^{-}(\frac{1}{2})$" with a point of the number line as shown below:



"$^{-}(\frac{1}{2})$" is read "NEGATIVE ONE-HALF" and $^{-}(\frac{1}{2})$ is a rational number.

Below you see some other rational numbers graphed on the number line:



Read the name of each one of these numbers.

The rational numbers to the right of 0 are called positive rational numbers. The rational numbers to the left of 0 are called negative rational numbers. Again, we shall see that these new numbers are very useful.

You may remember that we agreed in Chapter 1 that every whole number is also a rational number. For example, we said that 3 is not only a whole number but is also a rational number. (We can give it the name "$\frac{3}{1}$".) For much the same reason, we say that every integer is also a rational number. For example, $^{-}2$ is not only an integer; it is also a rational number. Of course, it is not true that every rational number is an integer.

We can now say:

> The set of positive rational numbers, the set of negative rational numbers, and zero form the set of rational numbers.

Discussion Questions 6-1b

1. What point is associated with the number $^{-}(\frac{1}{2})$? How do we say the name of this number?
2. What sets form the set of all rational numbers? Give some examples from each set.

Oral Exercises 6-1b

If set R = {$^{-}5$, $^{-}4$, $^{-}(\frac{10}{3})$, $^{-}(\frac{3}{2})$, $^{-}1$, 0, $\frac{1}{4}$, 1, $\frac{3}{2}$, 2, $\frac{7}{3}$, 6},

1. List I, the set of integers in R.
2. List W, the set of whole numbers in R.
3. List A, the set of positive integers in R.
4. List P, the set of negative integers in R.

5. List N, the set of counting numbers in R.
6. List G, the set of positive rational numbers in R.
7. List L, the set of negative rational numbers in R.
8. List Y, the set of non-negative integers in R.

Problem Set 6-1b

1. Draw the graphs of the following sets:

   (a) $\{0, 2, {}^{-}3, {}^{-}(\frac{1}{2}), \frac{1}{2}\}$

   (b) $\{{}^{-}(\frac{2}{3}), \frac{2}{3}, \frac{5}{2}, {}^{-}(\frac{5}{2})\}$

   (c) $\{{}^{-}(\frac{3}{2}), 5, {}^{-}7, {}^{-}(\frac{11}{3})\}$

   (d) $\{{}^{-}1, {}^{-}(1 + \frac{1}{2}), (1 + \frac{1}{2})\}$

2. Of the two points whose coordinates are given, which is to the right of the other?

   (a) $5, 4$

   (b) $0, {}^{-}5$

   (c) ${}^{-}4, {}^{-}7$

   (d) ${}^{-}(\frac{1}{2}), 1$

   (e) ${}^{-}(\frac{5}{2}), {}^{-}(\frac{10}{4})$

   (f) ${}^{-}4, {}^{-}(\frac{15}{4})$

   (g) ${}^{-}(\frac{16}{3}), {}^{-}(\frac{21}{4})$

3. Translate this sentence and write it in an algebraic sentence. You need not find the answer.

   What is the length of a rectangle if the length is 4 times the width and the perimeter is 24 inches?

   Be sure you have selected and described a variable.

Irrational Numbers and The Real Numbers

So far in this chapter we have looked at the set of rational numbers. Every rational number is <u>one</u> of the following: a positive number, a negative number, or zero. Some rational numbers are integers, and some are not. Can you give examples of rational numbers that are integers? Can you give examples of rational numbers that are not integers?

You may have the feeling that every point on the number line can be associated with a rational number. Strange as it seems, this is not true. There are points on the number line that cannot be associated with rational numbers.

You may not find it easy to think of a number that is not rational. Let's begin by looking at some numbers that are rational:

$\sqrt{9} = 3$, since $(3)(3) = 9$. (We read " $\sqrt{9}$ " as "the square root of 9".)

So $\sqrt{9}$ is a rational number because it can be given the name "$\frac{3}{1}$".

$\sqrt{\frac{9}{16}} = \frac{3}{4}$, since $(\frac{3}{4})(\frac{3}{4}) = \frac{9}{16}$.

$\sqrt{\frac{9}{16}}$ names a rational number because it can also be given the name "$\frac{3}{4}$" or ".75".

What about $\sqrt{2}$? You may say, "Oh yes, $\sqrt{2}$ is a rational number because it can be given the name '1.4' ". If this is true, it means that (1.4)(1.4) is the same as 2. Let's test it.

$$\begin{array}{r} 1.4 \\ 1.4 \\ \hline 5\,6 \\ 1\,4 \\ \hline 1.9\,6 \end{array}$$

This shows that it is not true that $\sqrt{2} = 1.414$. In the same way, test to see if $1.41 = \sqrt{2}$. Test to see if $1.414 = \sqrt{2}$.

No matter how many rational numbers we test, we will never find one that is the same as $\sqrt{2}$. Therefore, $\sqrt{2}$ is not a rational number.

A number that is not rational but is associated with a point on the number line is called an irrational number. $\sqrt{2}$ is an example of an irrational number. Here are some other examples of irrational numbers:

$$\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}$$

There are many, many other numbers in the set of irrational numbers. Is $\sqrt{25}$ one of them?

We can now form a new set of numbers that will include the rational numbers and the irrational numbers. This set is called the

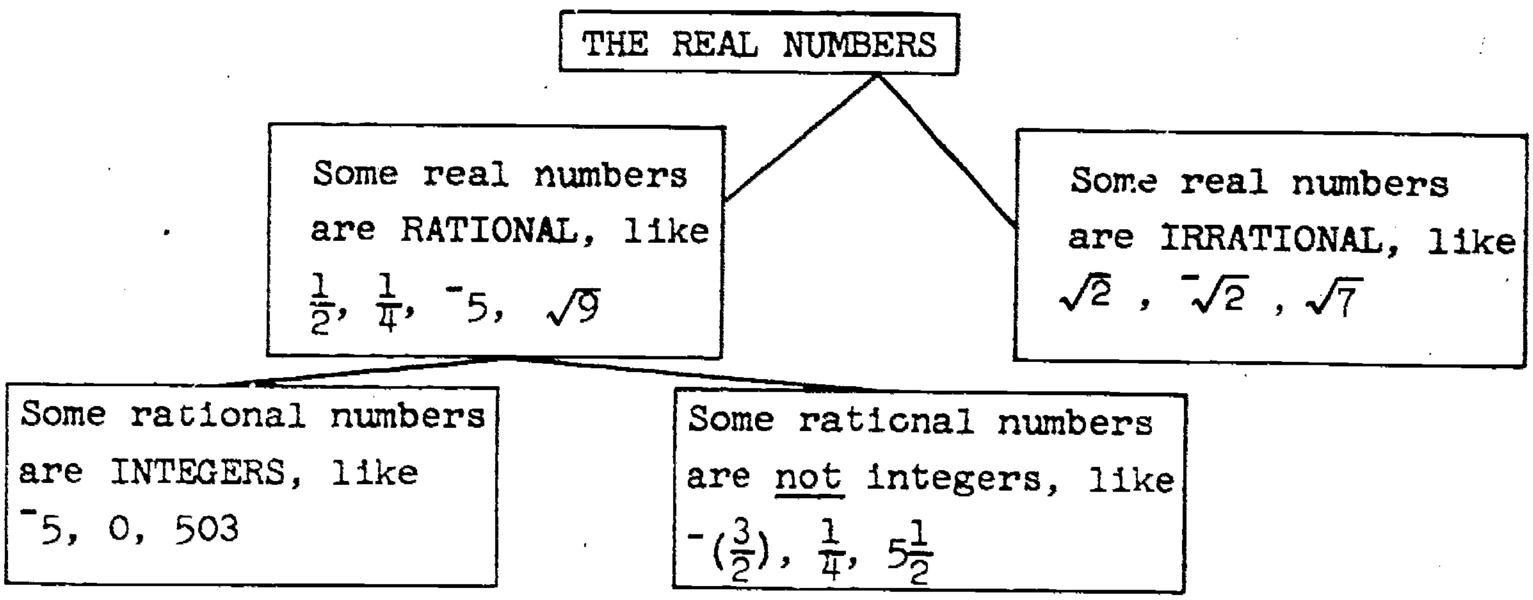
<u>Set of Real Numbers</u>.

All the rational numbers are in the set of real numbers. All the irrational numbers are in the set of real numbers. So we can say that the rational numbers form a subset of the real numbers, and the irrational numbers form another subset of the real numbers.

The points associated with the real numbers make up the whole number line, which is called the <u>real number line</u>.

We can think of the real number line as the graph of the set of real numbers.

The following diagram may help to review the kinds of numbers we have discussed: —



<u>Discussion Questions 6-1c</u>

1. Is $\sqrt{9}$ rational? $\sqrt{2}$? $\sqrt{\frac{9}{16}}$?
2. Are there any points on the number line with coordinates that are not rational numbers? If so, what are the coordinates called?
3. What is the square of $\sqrt{2}$?

4. Can you suggest a rational number whose square is 2? What rational number, that you can think of, comes closest?
5. What do we call the set of numbers that corresponds to the set of all points on the number line?
6. What are the two principal subsets of the set of real numbers?
7. Is $^{-}(\frac{3}{2})$ a rational number ? an integer? a real number?
8. Are all real numbers rational numbers? Are all rational numbers real numbers?

Problem Set 6-1c

1. (a) Is $^{-}2$ a whole number? an integer? a rational number? a real number?

   (b) Is $^{-}(\frac{10}{3})$ a whole number? an integer? a rational number? a real number?

   (c) Is $^{-}\sqrt{2}$ a whole number? an integer? a rational number? a real number?

   (d) Is 0 a whole number? an integer? a rational number? a real number? a positive number? a negative number?

2. The number π is the ratio of the circumference of a circle to its diameter. (Remember "ratio" means "the first number divided by the second number".) Thus, a circle whose diameter is of length 1, has a circumference of length π. The number π is an irrational number. Imagine such a circle resting on the number line at the point 0. If the circle is rolled on the line, without slipping, one complete revolution to the right, it will stop on a point. What is the coordinate of this point? If rolled to the left it will stop on what point? Can you locate these points, approximately, on the real number line?
3. Write the sentence in algebra which represents this problem. You need not find the answer. Be sure to select and describe what the variable represents.

   Mary is twice as old as her brother, and her brother is twice as old as their baby sister; the sum of their ages is 15 years. How old is each?

6-2. Order on the Real Number Line.

$$6 > 5.$$

This true sentence reads "6 is greater than 5". It means "6 is to the right of 5" on the number line.

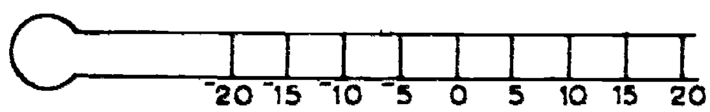
$$4 > {}^-8.$$

This true sentence reads "4 is greater than $^-8$". It means "4 is to the right of $^-8$" on the number line.

"Is to the right of" on the number line and "is greater than" describe the same order. What shall we mean by "is greater than" for any two real numbers, whether they are positive, negative, or 0?

Our answer is: "is to the right of" on the number line.

Here is a common example. Scales on thermometers use numbers above 0 and numbers below 0, as well as 0 itself. We know that the warmer the weather, the higher up the scale we read the temperature. If we place the thermometer as is shown below, we see that it looks like a model of a part of the real number line.



When we say "is greater than" ("is a higher temperature than"), we mean "is to the right of" on the thermometer scale.

On this scale, which number is greater, $^-5$ or $^-10$? 5 or $^-10$? $^-15$ or 0?

For any two real numbers a and b,

a <u>is greater than</u> b
means the same as
a <u>is to the right of</u> b on the number line.

No matter which way we want to say it, we can write:

$$a > b.$$

Oral Exercises 6-2a

1. Describe the meaning, on the number line, of "is less than" for real numbers, as we did above for ">".
2. What is the meaning, on the number line, of "$\geq$" for real numbers?
3. What is the meaning, on the number line, of "$\leq$" for real numbers?

Problem Set 6-2a

1. Determine which of the following sentences are true and which are false.

   (a) $7 < 4$

   (b) ${}^{-}(\frac{1}{2}) < 0$

   (c) ${}^{-}3 < 4$

   (d) ${}^{-}6 > {}^{-}3$

   (e) ${}^{-}3 < {}^{-}2.8$

   (f) $3.5 < {}^{-}4$

   (g) ${}^{-}4 \neq 3.5$

   (h) $3 \leq {}^{-}1$

   (i) $2 \geq {}^{-}(\frac{7}{2})$

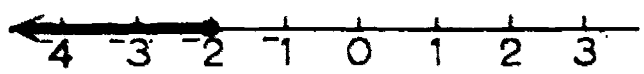
   (j) ${}^{-}5 \geq {}^{-}(\frac{10}{2})$

2. Graph the truth set of each of the following open sentences:

   For example:

   $x > {}^{-}5$

   $x \leq {}^{-}2$



   (a) $y > 2$

   (b) $y \geq {}^{-}2$

   (c) $y \neq 2$

   (d) $x \geq {}^{-}5$

   (e) $x = 3$ or $x < {}^{-}1$

   (f) $c < 2$ and $c > {}^{-}2$

3. In the blanks below use "=", "<", or ">" to make a true sentence.

   (a) $\frac{3}{5}$ ____ $\frac{6}{10}$

   (b) $\frac{3}{5}$ ____ $\frac{3}{6}$

   (c) $\frac{9}{12}$ ____ $\frac{8}{12}$

(d) $\frac{9}{12}$ ___ $\frac{4}{6}$

(e) $^{-}(\frac{3}{5})$ ___ $\frac{3}{6}$

(f) $^{-}(\frac{3}{5})$ ___ $^{-}(\frac{4}{5})$

(g) $^{-}(\frac{3}{5})$ ___ $^{-}(\frac{3}{6})$

4. (a) During a cold day the temperature rises 10 degrees from $^{-}4$. What is the final temperature?
   (b) On another day the temperature rises 5 degrees from $^{-}10$. How high does it go?
   (c) During one day the temperature rises from $^{-}15$ to 35. How much does it rise?
5. Translate the following sentences into algebraic sentences.
   (a) A number is greater than or equal to 18.
   (b) One number is greater than another.
   (c) One number is 5 greater than another.
   (d) One number less 7 is less than the same number increased by 4.
   (e) The sum of a number and 5 is less than 8 more than twice the number.
   (f) 5 less than a number is four less than twice the number.
   (g) 3 times a number is 8, or 4 less than the same number is greater than 2.

Suppose you try to guess the number of marbles in a bowl. Then the marbles are counted so that the number of marbles is known. It is easy to see that exactly <u>one</u> of the following three statements would be true:

(1) Your number <u>is greater than</u> the number of marbles.

(2) Your number <u>is less than</u> the number of marbles.

(3) Your number <u>is equal to</u> the number of marbles.

We could say that after the marbles were counted, your number was <u>compared</u> with the number of marbles.

In fact, if any two real numbers a and b are compared, exactly one of the following is true:

$$a > b$$
$$a < b$$
$$a = b.$$

The way in which the real numbers are ordered on the number line makes it easy to see why exactly one of the statements above must be true. In fact, this is a property of order for the real numbers. It is sometimes called the

<u>comparison property</u>.

Discussion Questions 6-2b

1. What is the comparison property of two real numbers a and b?
2. If two numbers are not the same, what else can be said about them?

Problem Set 6-2b

1. In the blanks below, use "=", "<" or ">" to make a true sentence in each case.

   (a) 2 ____ $^{-}3$

   (b) 2 ____ 1.6

   (c) $\frac{3}{5}$ ____ $\frac{3}{6}$

   (d) $\frac{3}{5}$ ____ $\frac{6}{10}$

   (e) $^{-}(\frac{3}{5})$ ____ $^{-}(\frac{3}{6})$

   (f) $\frac{4}{2}$ ____ $^{-}(\frac{5}{2})$

   (g) $\frac{4}{5}$ ____ $\frac{11}{10}$

   (h) $\frac{13}{15}$ ____ $\frac{2}{3}$

   (i) 2 ____ 1.5

   (j) $^{-}2$ ____ $^{-}1.5$

2. Write true sentences using "<" for the following pairs of real numbers.

   (a) 6 and 5

   (b) $^{-}3$ and 0

   (c) $^{-}(\frac{2}{3})$ and $^{-}(\frac{5}{9})$

   (d) 2 and $(1.4)^2$

   (e) π and 3

3. Write true sentences using ">" for the following pairs of real numbers.

   (a) $^{-}7$ and $^{-}5$

   (b) $^{-}8$ and 0

   (c) 8 and 0

   (d) $\frac{1}{3}$ and $.3$

   (e) 2 and $(1.41)^2$

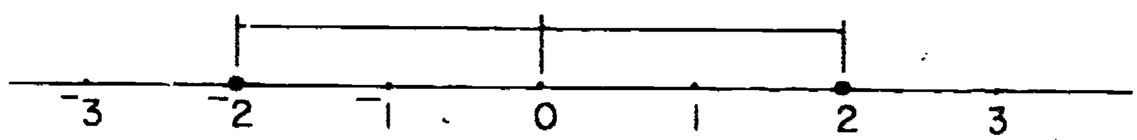
4. Write an algebraic sentence for this problem after choosing and describing a variable.

A coat was sold for \$33. This was at a discount of $\frac{1}{3}$ of the original price. What was the original price? (Hint: It wasn't \$44.)

6-3. Opposites.

We have now labeled points on the number line to the left of 0 as well as to the right of 0. This means that we can "pair off" points which are at the same distance from 0.

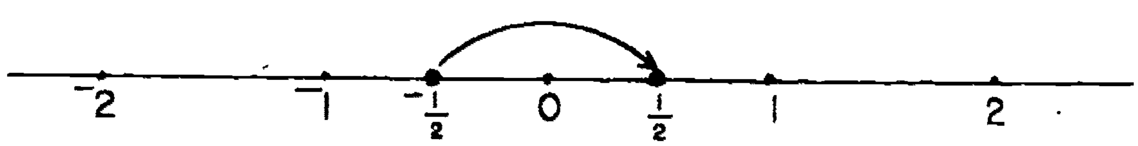
For example, think of the point associated with the number 2. It is easy to see that there is another point at the same distance from 0



This other point lies on the opposite side of 0, and the number associated with it is $^{-}2$.

Since the two points we have just located lie on opposite sides of 0, and at the same distance from 0, it seems natural to say that the numbers 2 and $^{-}2$ are opposites. Each number is the opposite of the other. That is, $^{-}2$ is the opposite of 2; also, 2 is the opposite of $^{-}2$.

In the example above, we started with the number 2. We could just as easily start with any number and find its opposite. For instance, suppose we start with the number $^{-}(\frac{1}{2})$. Then find another point at the same distance from 0 but on the opposite side. This is also easy to do; and, as in the diagram below, you see that the coordinate of this point is $\frac{1}{2}$.



Therefore, $\frac{1}{2}$ is the opposite of $^{-}(\frac{1}{2})$. This means also that $^{-}(\frac{1}{2})$ is the opposite of $\frac{1}{2}$. Or we could just say that $\frac{1}{2}$ and $^{-}(\frac{1}{2})$ are opposites.

We agree that the opposite of zero is zero.

Here are some statements about pairs of numbers that are opposites:

(1) $^{-}7$ is the opposite of 7.

(2) $12\frac{1}{4}$ is the opposite of $^{-}12\frac{1}{4}$.

(3) 73.2 is the opposite of $^{-}73.2$.

(4) $^{-}1,000,002$ is the opposite of 1,000,002.

You can see that writing "the opposite of" every time means a lot of writing. So we will agree that when we want to write "opposite of 2" we can just write "-2". If we want to write "opposite of $^{-}3$", we can write "$-{}^{-}3$" or "$-(^{-}3)$".

Notice that the dash we use to mean "opposite of" is written lower than the dash we use in writing the name of a negative number. The dash we use to mean "opposite of" looks like the "minus" sign used in subtraction. But it is important to understand that we are not using it here to mean subtraction.

Now we can write the four statements above like this:

(1) $^{-}7 = -7$.

(2) $12\frac{1}{4} = -{}^{-}12\frac{1}{4}$. Sometimes we write this $12\frac{1}{4} = -(^{-}12\frac{1}{4})$.

(3) $73.2 = -(^{-}73.2)$.

(4) $^{-}1,000,002 = -1,000,002$.

Discussion Questions 6-3a

1. What is the opposite of 2? of $^{-}2$?
2. What is the opposite of 0?
3. What is the symbol we use for "the opposite of"?
4. What does -2 mean? What does $-{}^{-}3$ mean?
5. Do we mean "subtraction" by the sign "-" in "-3"?
6. Can you recall the difference between the meaning of the upper dash "$^{-}$" and the centered dash "-"?

Problem Set 6-3a

1. Each of the numerals below names a number. For each one, give the common name for the number. For example, if you were given the numeral "$-(^{-}9)$", the common name would be "9". If you were given "-9" the common name would be "$^{-}9$" or "negative 9".

(a)	-55	(h)	$-(^{-}2\frac{1}{2})$
(b)	$-(^{-}55)$	(i)	$-2\frac{1}{2}$
(c)	-33.5	(j)	$-1,000,000,000$
(d)	-0	(k)	$-(^{-}1,000,000,000)$
(e)	$-(^{-}100)$	(l)	$-(3 + 5)$
(f)	$-\left(^{-}(\frac{3}{4})\right)$	(m)	$-(7 + 2)$
(g)	$-\frac{2}{3}$	(n)	$-(8.4 + 7.6)$

2. In looking over your answers from Problem 1, which of the following statements do you think is true?

   The opposite of a positive number is a positive number.
   The opposite of a positive number is a negative number.
   The opposite of a positive number is zero.

3. Which of these statements do you think is true?

   The opposite of a negative number is a positive number.
   The opposite of a negative number is a negative number.
   The opposite of a negative number is zero.

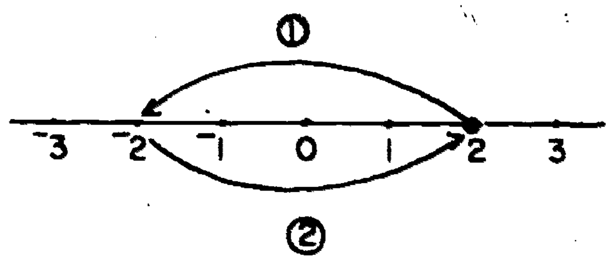
4. Is the opposite of zero a positive number, a negative number, or zero?

5. Are "-9" and "$^{-}9$" names for the same number? Write each in words.

When you see the numeral "$-(-2)$", it may look confusing. In words, we could read it as "the opposite of the opposite of 2". Here is one way we might think about it:

$-(-2)$

- start with 2
- ① find the opposite, getting $^-2$
- ② then find the opposite of $^-2$, getting 2



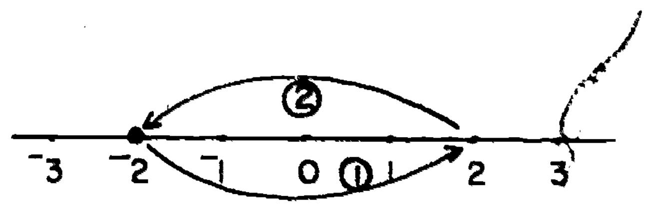
So we can say that "the opposite of the opposite of 2 is 2". Or, more briefly, we can write:

$$-(-2) = 2.$$

In the example above, we started with a positive number. We could just as well start with a negative number. For instance, how could we decide upon the common name for "$-(-^-2)$"?

$-(-^-2)$

- start with $^-2$
- ① find the opposite, getting 2
- ② then find the opposite of 2, getting $^-2$



This shows that we can write:

$$-(-^-2) = ^-2.$$

These two examples suggest that

$$-(-y) = y, \qquad \text{for any real number } y.$$

We have already seen that this statement is true when y represents 2 and when y represents $^-2$. Is it true if y represents 10? if y represents $^-10$? Is it true if y represents 0?

We cannot use every number, but with a little work on the number line, you should soon see that $-(-y) = y$ for any real number y.

Discussion Questions 6-3b

1. State this sentence in words: $-(-y) = y$ for any real number y.
2. Describe how we would find the common name for $-(-^-2)$.

Oral Exercises 6-3b

1. Each of the numerals below is the name of a number. For each one, state the common name of the number. For example, if you were given the numeral "-(-3)", the common name would be "3".

(a) $-(-20)$

(b) $-(-\,^{-}20)$

(c) $-\left(-\,^{-}\left(\frac{1}{2}\right)\right)$

(d) $-(^{-}5)$

(e) $-(-210)$

(f) $-(-\,^{-}37.5)$

Problem Set 6-3b

1. Each of the numerals below is the name of a number. For each one, write the common name of the number.

(a) $-(-40)$

(b) $-\left(-\,^{-}\left(\frac{7}{2}\right)\right)$

(c) $-\left(-\,^{-}(5 + 3)\right)$

(d) $-\left(-(5 + 3)\right)$

(e) $-(-0)$

(f) $-\left(-(-9)\right)$

2. If y represents a positive number, does -y represent a positive number, a negative number, or zero?
3. If y represents a negative number, does -y represent a positive number, a negative number, or zero?
4. If y represents zero, does -y represent a positive number, a negative number, or zero?
5. If -y is positive, is y positive, negative, or zero?
6. If -y is negative, is y positive, negative, or zero?
7. If -y is zero, is y positive, negative, or zero?
8. Write an algebraic sentence whose truth set includes the answer to this problem. You need not find the answer.

   A rectangular lawn measures 20 feet by 30 feet. A walk of uniform width is put along both ends and one side. The perimeter of the entire area of lawn and walk is then 150 feet. How wide is the walk?

   Hint: Draw a diagram.

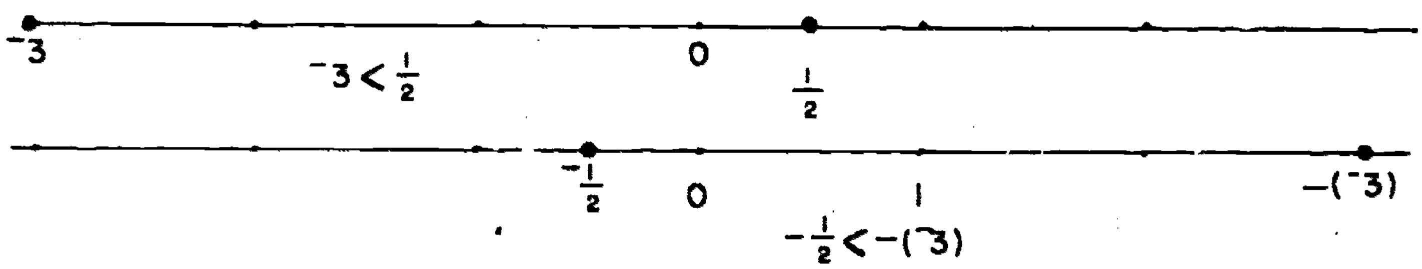
We know that the real numbers are ordered on the number line. That is, if we have any two different numbers, one will be less than the other. For example, using the pair of numbers $\frac{1}{2}$ and $^-3$, we can say:

$$^-3 < \frac{1}{2}.$$

Let's take the opposite of each one of these two numbers. The opposite of $^-3$ is 3. The opposite of $\frac{1}{2}$ is $^-(\frac{1}{2})$. Since $^-(\frac{1}{2}) < 3$, we can say:

$$-(\frac{1}{2}) < -(^-3).$$

The number lines below may help in seeing what happened in this "experiment".



Let's try the experiment again, this time using the numbers 2 and 100.

$$2 < 100,$$

but $$-100 < -2.$$

Let's try one more pair of numbers, $^-8$ and $^-2$.

$$^-8 < ^-2,$$

but $$2 < 8.$$

By this time, perhaps you see that if $a < b$, then $-b < -a$.

If you feel that you are not sure of this statement, try letting a and b be some numbers different from the ones we used above. Then see if the statement says what we want it to say.

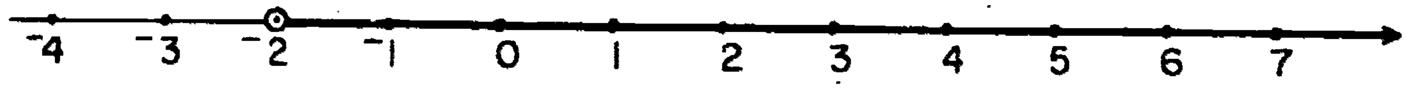
Sometimes the fact that if $a < b$, then $-b < -a$ is a big help in finding the truth sets of open sentences. Suppose you were trying to graph the truth set of:

$$-x < 2.$$

We know that $-x < 2$ and $-2 < x$ mean the same. So, we can draw the graph of:

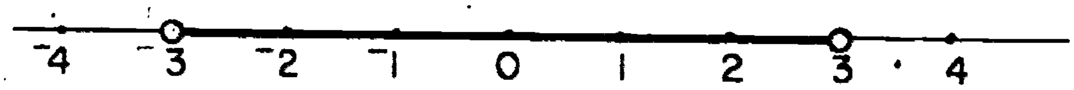
$$-2 < x.$$

Of course, the opposite of 2 is $^-2$, and it is easy to draw the graph of $^-2 < x$. The graph looks like this:



Could you show somebody why, for example, $^-3$ is not in the truth set?

Here is another problem. Below is a graph of a set of numbers.



Notice that this set of numbers does not include the number 3 or any number greater than 3. Also, the set does not include the number $^-3$ or any number less than $^-3$. So, if we let x represent a number in this set, x must be greater than $^-3$ and less than 3. We can say that this graph is the graph of the truth set of:

$$x > {}^-3 \text{ and } x < 3.$$

Discussion Questions 6-3c

1. What happens to the order relation $^-3 < \frac{1}{2}$ when we take opposites of the two numbers?
2. If $a < b$, what is the relationship of $-a$ and $-b$?
3. How do we draw the graph of $-x < 2$? What can we do to make the work easier? Is $^-3$ in the truth set?

Oral Exercises 6-3c

1. In each of the following pairs, decide which is the greater number; then take the opposites of the two numbers and again decide which is greater.

(a) 2.97, -2.97	(d) -1, 1	(g) 0, -0
(b) -12, 2	(e) -370, -121	(h) -.1, -.01
(c) -358, -762	(f) .12, .24	(i) .1, .01

2. What is the meaning of $x \neq 3$? Express this sentence using "<" and ">".

3. What is the truth set of $x \neq 3$? of $-x \neq 3$?

Problem Set 6-3c

1. Write two true sentences for the following numbers and their opposites, using the relation "<".

   Example: 2, 7

   $2 < 7$ and $-7 < -2$.

   (a) 3, -1

   (b) $\frac{3}{4}$, $^-(\frac{1}{2})$

   (c) $\frac{2}{7}$, $^-(\frac{1}{6})$

   (d) $\sqrt{2}$, $^-\pi$

   (e) π, $\frac{22}{7}$ (π is approximately 3.1416)

   (f) $3(\frac{4}{3} + 2)$, $\frac{5}{4}(20 + 8)$

   (g) $2(8 + 5)$, $^-(5 + 4)$

   (h) $-\frac{(8 + 6)}{7}$, $^-2$

2. For each of the following numbers, choose the greater of the number and its opposite.

   (a) $^-7.2$

   (b) 3

   (c) 0

   (d) $^-\sqrt{2}$

   (e) 17

   (f) -.01

   (g) -(-2)

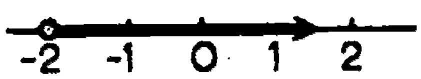
   (h) $(1 - \frac{1}{4})^2$

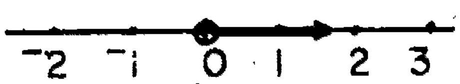
   (i) $1 - (\frac{1}{4})^2$

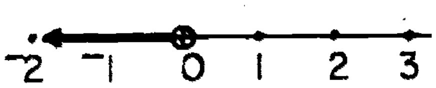
   (j) $-(\frac{1}{2} - \frac{1}{3})$

3. Write two open sentences for each of the following graphs, one involving x, and the other involving -x.

   (a) 

   (b) 

   (c) 

   (d) 

   (e) 

   (f) 

   (g) 

4. Graph the truth sets of the following open sentences:

   (a) $x > 3$

   (b) $x > {}^{-}3$

   (c) $-x > 3$ (Hint: first express the relation involving the opposites of these numbers.)

   (d) $-x > {}^{-}3$

5. Describe the truth set of each open sentence.

   (a) $-x \neq 3$ (Remember "$\neq$" means "$<$" or "$>$".)

   (b) $-x \neq {}^{-}3$

   (c) $x < 0$

   (d) $-x < 0$

   *(e) $-x \geq 0$

   *(f) $-x \leq 0$

6. Write open sentences that would help solve the following problems. Be sure to tell what the variable represents.

   (a) John's score is greater than minus 100. What is his score?

   (b) He doesn't have any money, but he is no more than $200 in debt. How much money does he have?

   (c) Paul has paid $10 on his bill, but still owes more than $25. What was the original amount of the bill?

What is the "opposite of 5"? By this time, this is an easy question. You know that the opposite of 5 is ${}^{-}5$.

Since we have agreed to use "-5" to mean "opposite of 5", we can say:

$$-5 = {}^{-}5.$$

In other words, -5 and ${}^{-}5$ are equal; that is, they are names for the same number. Maybe you have noticed this already. Of course, it is also true that -2 and ${}^{-}2$ are equal, -7 and ${}^{-}7$ are equal, $-\frac{1}{2}$ and ${}^{-}(\frac{1}{2})$ are equal, and so on.

This means that we don't really have to use two different dashes anymore. From now on, for example, we can use "-5" to mean either "opposite of 5" or "negative 5". However, "$-x$"

does not mean a negative number; it means "the opposite of x" and will be positive when x is negative, or negative when x is positive.

6-4. Absolute Value.

Think of the number 5. Its opposite is -5.
This gives us the pair of numbers 5 and -5.
Which number of the pair is greater?

Think of the number -8. Its opposite is 8.
This gives us the pair of numbers -8 and 8.
Which number of the pair is greater?

Think of the number $-\frac{1}{2}$. Its opposite is $\frac{1}{2}$.
This gives us the pair of numbers $-\frac{1}{2}$ and $\frac{1}{2}$.
Which number of the pair is greater?

When you answered the questions above, you were working with a new and useful operation in mathematics. It is called "taking the absolute value" of a number. That is, we could answer the questions above by saying:

The absolute value of 5 is 5.
The absolute value of -8 is 8.
The absolute value of $-\frac{1}{2}$ is $\frac{1}{2}$.

The absolute value of 0 is 0. The absolute value of any other real number is the greater of the number and its opposite.

What is the absolute value of 17? What is the absolute value of -100? What is the absolute value of 0?

Instead of writing "absolute value" each time, we use a new symbol. For example:

"$|5| = 5$" means "absolute value of 5 is 5".

"$|-8| = 8$" means "absolute value of -8 is 8".

"$|0| = 0$" means "absolute value of 0 is 0".

"$|n|$" means "absolute value of the number n".

Discussion Questions 6-4a

1. What is the absolute value of a non-zero number?
2. What is the absolute value of zero?
3. What does the symbol $|n|$ mean?

Oral Exercises 6-4a

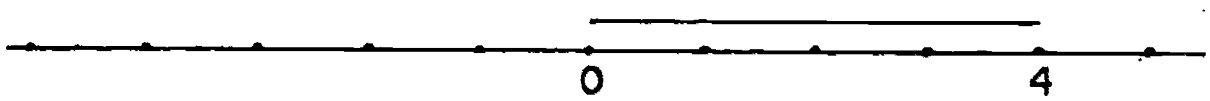
1. What is the absolute value of each of the following numbers?

-7	14×0
$-(-3)$	$-(14 + 0)$
$(6 - 4)$	$-(-(-3))$

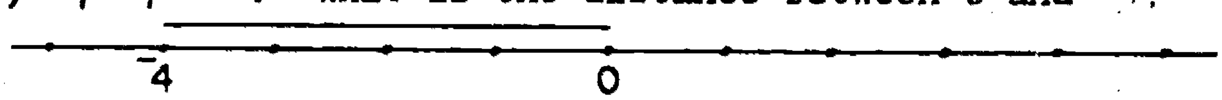
2. What is the absolute value of x if x is 3? If x is -2?
3. If x is a non-negative real number, what kind of number is $|x|$?
4. If x is a negative real number, what kind of number is $|x|$?
5. Is $|x|$ a non-negative number for every x?
6. For a negative number x, which is greater, x or $|x|$?
7. Is the absolute value of a number always a non-negative number?
8. When is the absolute value of a number not a positive number?

Using the number line sometimes helps in working with absolute values. Look at the following examples.

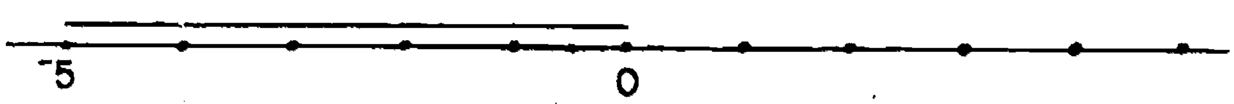
(1) $|4| = 4$. What is the distance between 0 and 4?



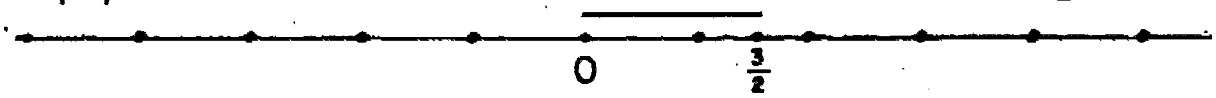
(2) $|-4| = 4$. What is the distance between 0 and -4?



(3) $|-5| = 5$. What is the distance between 0 and -5?



(4) $\left|\frac{3}{2}\right| = \frac{3}{2}$. What is the distance between 0 and $\frac{3}{2}$?



Do you see that the absolute value of a number is the distance between the number and 0 on the number line? By "distance", we mean just the "number of units". Here, the word "distance" has nothing to do with direction.

If x is 7, what is $|x|$?

If x is 12, what is $|x|$?

If x is 8,750, what is $|x|$?

If x is 0, what is $|x|$?

In these four examples, x has been either a positive number or zero. In other words, x has been a non-negative number. In each of these cases, it turned out to be true that $|x| = x$. After thinking about other non-negative numbers, you should see that we can say:

$$|x| = x, \text{ if } x \geq 0.$$

Is it always true that $|x| = x$? Let's try some negative numbers.

If x is -5, what is $|x|$? $|x| = 5$. Notice that $5 = -(-5)$.

If x is -3, what is $|x|$? $|x| = 3$. Notice that $3 = -(-3)$.

If x is -45, what is $|x|$? $|x| = 45$. Notice that $45 = -(-45)$.

In these three examples, x has been a negative number. Each time, $|x|$ has been, not x, but -x. To show this, we can say:

$$|x| = -x, \text{ if } x < 0.$$

You may have noticed that an absolute value is always a non-negative number. So it may seem strange ever to say "$|x| = -x$". But remember, if x is a negative number, -x is a positive number. Therefore,

$$|x| = x, \text{ if } x \geq 0$$

$$\text{and } |x| = -x, \text{ if } x < 0$$

is just another way of saying that $|x|$ is always a non-negative number.

As one more example, let's look at $|-20|$:

$$|-20| = 20.$$

This agrees with what was said above, since $-20 < 0$, and $|-20| = -(-20)$.

Discussion Questions 6-4b

1. Which of the following open sentences are true for all real numbers x?

   $|x| \geq 0$ $\qquad$ $-x \leq |x|$

   $x \leq |x|$ $\qquad$ $-|x| \leq x$

2. State in words those sentences in Question 1 which you decided are true.

Oral Exercises 6-4b

1. For a negative number x, which is greater, x or -x?
2. Which of the following sentences are true?

   (a) $|{}^{-}7| < 3$
   (b) $|{}^{-}2| \leq |{}^{-}3|$
   (c) $|4| < |1|$
   (d) $2 \leq |{}^{-}3|$
   (e) $|{}^{-}5| \neq |2|$
   (f) ${}^{-}3 < 17$
   (g) ${}^{-}2 < |{}^{-}3|$
   (h) $|{}^{-}2|^2 = 4$

3. State each as a simple numeral.

   (a) $|2| + |3|$
   (b) $|{}^{-}2| + |3|$
   (c) $-(|2| + |3|)$
   (d) $-(|-2| + |3|)$
   (e) $|-7| - (7-5)$
   (f) $7 - |-3|$
   (g) $|-5| \times 2$
   (h) $-(|-5| - |-2|)$
   (i) $|-3| - |2|$
   (j) $|-2| + |-3|$
   (k) $-(|-3| - 2)$
   (l) $-(|-2| + |-3|)$
   (m) $3 - |3 - 2|$
   (n) $-(|-7| - 6)$
   (o) $|-5| \times |-2|$
   (p) $-(|-2| \times 5)$
   (q) $-(|-5| \times |-2|)$

Problem Set 6-4

1. What is the truth set of each open sentence?
   (a) $|x| = 1$
   (b) $|x| = 3$
   (c) $|x| + 1 = 4$
   (d) $5 - |x| = 2$

2. Graph the truth sets of the following sentences.
   (a) $|x| < 2$
   (b) $x > -2$ and $x < 2$
   (c) $|x| > 2$
   (d) $x < -2$ or $x > 2$

3. Graph the integers less than 5 whose absolute values are greater than 2. Is -5 an element of this set? Is 0 an element of this set? Is -10 an element of this set? Is 4 an element of this set?

4. If R is the set of all real numbers, P the set of all positive real numbers, and I the set of all integers, write three numbers which are
   (a) in P but not in I,
   (b) in R but not in P,
   (c) in R but not in P or in I,
   (d) in P but not in R.

5. Compare the truth sets of the two sentences.

   $|x| = 0, \qquad |x| = -1.$

*6. Three boys, Sam, Bob, and Pete, were talking. Sam said, "Bob is older than I am." Pete said, "Bob is twice as old as I am and Sam is 3 years older than I am." Bob said, "My father is more than twice as old as all of our ages put together, and he is 45." How old was each boy? Write the sentence whose truth set will lead to the answer to this problem. It is not necessary to find the answer.

Summary

(1) Points to the left of 0 on the number line are associated with negative numbers.

(2) The real numbers are those numbers that can be associated with points of the real number line. They include rational numbers and irrational numbers.

(3) The integers form a special subset of the rational numbers.

(4) $\sqrt{2}$ is an example of an irrational number.

(5) "a is greater than b" and "a is to the right of b on the number line" have the same meaning for any two real numbers a and b.

(6) For any two real numbers a and b, exactly one of the following is true: $a > b$, $a < b$, $a = b$.

(7) The opposite of 0 is 0. The opposite of any other real number is the number which is at an equal distance from 0 on the number line.

(8) The opposite of the opposite of a number is just the number itself. That is, $-(-x) = x$.

(9) The absolute value of 0 is 0. The absolute value of any other real number n is the greater of n and -n. "Absolute value of n" is written "$|n|$".

(10) $|n|$ is the distance between 0 and n on the real number line.

(11) $|n| = n$, if $n \geq 0$.
$|n| = -n$, if $n < 0$.

Review Problem Set

1. Consider the following sets.

   R*: the set of all real numbers
   P: the set of positive real numbers
   Q: the set of negative real numbers
   R: the set of all rational numbers
   N: the set of counting numbers
   W: the set of whole numbers
   I: the set of integers
   J: the set of irrational numbers

   In each of the following pairs of sets tell which set is a subset of the other. For some pairs neither set may be a subset of the other.

   (a) I, W
   (b) N, W
   (c) N, R
   (d) R*, R
   (e) R, I
   (f) P, Q
   (g) R*, J
   (h) I, J
   (i) R*, W
   (j) P, W

2. Give two meanings of the symbol ">", one related to numbers and the other related to the position of points on the number line.

3. Place the symbol (>, <, or =) between these numbers so that a true sentence results.

   (a) 3 ____ 5
   (b) $^{-}3$ ____ 5
   (c) $^{-}3$ ____ $^{-}5$
   (d) 3 ____ $^{-}5$
   (e) $\frac{3}{4}$ ____ $\frac{26}{36}$
   (f) $^{-}(\frac{3}{4})$ ____ $\frac{26}{36}$
   (g) $^{-}(\frac{3}{4})$ ____ $^{-}(\frac{26}{36})$
   (h) $\frac{3}{4}$ ____ $\frac{27}{37}$
   (i) $^{-}(\frac{3}{4})$ ____ $^{-}(\frac{45}{56})$
   (j) $\frac{3}{4}$ ____ $\frac{96}{128}$

4. Name three positive numbers and three negative numbers that are not rational numbers. Is $\sqrt{16}$ irrational? $\sqrt{49}$? $\sqrt{5}$?

5. Graph the truth sets of these sentences.

(a) $x < 5$ (d) $|x| > 0$

(b) $-x < 5$ (e) $|x| < 0$

(c) $|x| = 4$ (f) $|x| = 0$

6. Tell what you know about the order of any two real numbers a and b. What property is involved?

7. If $x < 3$, what can we state about the order of $-x$ and -3?

8. Graph these sets.

(a) $\{^-(\frac{4}{3}),\ 0,\ ^-(^-\frac{4}{3}),\ 2,\ ^-2\}$

(b) The set of all positive integers less than 2.

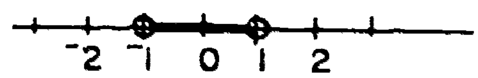
(c) The set of all counting numbers less than $^-2$.

(d) The set of all integers between $^-4$ and 2.

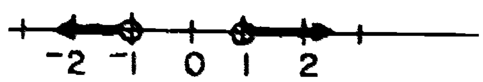
(e) The set of all numbers between $^-4$ and 2.

9. Write an open sentence for each graph.

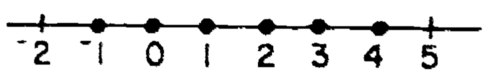
(a) 

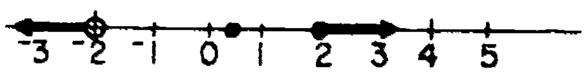
(c) 

(b) 

(d) 

10. Describe in words the sets for which the following are the graphs.

(a) 

(b) 

11. What number, added to the same number increased by $4\frac{1}{2}$, will result in a sum of 7.3? Write the sentence whose truth set includes the answer to this problem. You need not find the answer. In each case tell what the variable represents.

12. Do as in Problem 11 for this problem.

If a number is increased by 7.6 times the number, and the resulting sum is 56, what is the number?

13. Do as in Problem 11.

The product of a number and the number increased by $3\frac{1}{2}$ is 84. What is the number?

14. Do as in Problem 11.

One book has 310 pages more than another. The number of pages in the combined volumes is more than 1000 pages. How many pages are in each volume?

15. An airplane flies due east at an average speed of 200 miles per hour. Another plane leaves from the same starting point one hour later. It flies in the same direction and overtakes the first 800 miles away. What was the average speed of the second plane?

Chapter 7

ADDITION OF REAL NUMBERS

7-1. Using Real Numbers in Addition.

Ever since the first grade, you have been adding numbers--the numbers of arithmetic. Now we are ready to work with a larger set of numbers--the real numbers. Your work in adding the numbers of arithmetic should give a clue as to how we add any two real numbers.

To begin with, think of an ice cream salesman in business for ten days. On some days, he makes money; then we say that he shows a profit. On other days, he loses money; then we say that he shows a loss. On still other days, he may show neither a profit nor a loss.

At the bottom of the page are two columns. The one on the left gives, in words, the profit or loss for each one of the ten days. The column on the right shows the arithmetic used in figuring the profit or loss for two-day periods.

Notice that to find the profit or loss for a two-day period, we "put together", or add, the profit or loss for one day and the profit or loss for the other day. The numbers that we add are positive in some cases, negative in others, and zero in still others.

Monday:	Profit of $7	$7 + 5 = 12$	12 is the number showing his net income for 2 days.
Tuesday:	Profit of $5		
Wednesday:	Profit of $6	$6 + (-4) = 2$	
Thursday:	Loss of $4		
Friday:	Loss of $7	$(-7) + 4 = -3$	
Saturday:	Profit of $4		
Sunday:	Day of rest	$0 + (-3) = -3$	
Monday:	Loss of $3		
Tuesday:	Loss of $4	$(-4) + (-6) = -10$	
Wednesday:	Loss of $6		

In these examples, we have found the sum of two positive numbers, the sum of two negative numbers, and the sum of a positive number and a negative number. We have also found a sum involving zero.

Below are some indicated sums of real numbers. Decide how each one should be completed in order to make a true statement.

$\frac{1}{2} + 2 =$	$3 + (-1) =$	$(-\frac{7}{5}) + 0 =$
$0 + \frac{4}{3} =$	$(-4) + 2 =$	$(-4) + (-\frac{1}{2}) =$

Maybe you thought of the numbers above as representing profits and losses, as in the example about the ice cream salesman. You may want to keep thinking about positive and negative numbers in this way for awhile. However, you will probably find, as you study this chapter, that you will be able to add real numbers without thinking about profits and losses at all.

Discussion Questions 7-1

1. When you added two negative numbers, was the sum a positive or negative number?
2. When you added two positive numbers, was the sum a positive or a negative number?
3. When you added a positive number and a negative number, how did you decide whether the sum was a negative number or a positive number?
4. When you added zero to a real number, how do you decide whether the answer is a negative or a positive number?

Oral Exercises 7-1

1. Think of gains as positive numbers and losses as negative numbers to answer the following questions:
   (a) Harry earned \$5 yesterday and spent \$3 today. What is his financial account?
   (b) Bill had 50¢ when he went to school today. He spent 40¢ for his lunch and he was charged 25¢ for supplies. What is his situation?
   (c) A certain stock market price gained two points one day then lost 5 points the next day. What was the net change?
   (d) Miss Jones lost 6 pounds during the first week of her dieting, lost 3 pounds the second week, gained 4 pounds the third week, and gained 5 pounds the last week. What was her net gain or loss?
   (e) A football team lost 6 yards on the first play and gained 8 yards on the second play. What was the net yardage on the two plays?

2. Find the following sums by thinking of the positive numbers as profits and the negative numbers as losses.

(a) $7 + 2$	(e) $(-5) + 8$	(i) $(-6) + 0$
(b) $(-4) + (-3)$	(f) $(-8) + 5$	(j) $(-2) + (-3) + 6$
(c) $5 + (-8)$	(g) $(-5) + (-8)$	(k) $7 + (-4) + (-2)$
(d) $8 + (-5)$	(h) $8 + 0$	(l) $(-2) + (-4) + 5$

Problem Set 7-1

1. Write the common name for each of the following sums. Think of the positive numbers as profits and the negative numbers as losses.

(a) $4 + 7$	(f) $(-7) + 2$	(l) $(-5) + 0$
(b) $(-4) + 8$	(g) $2 + (-7)$	(m) $(-4) + 0 + 3$
(c) $(-2) + (-10)$	(h) $(-2) + (-7)$	(n) $8 + (-3) + (-4)$
(d) $(-2) + 7$	(i) $7 + 0$	(o) $(-3) + (-4) + 5$
(e) $7 + (-2)$	(j) $4\frac{1}{2} + (-2\frac{1}{2})$	(p) $(-2) + 4 + (-1)$
	(k) $(-6.3) + (-4.5)$	

Problem Set 7-1
(continued)

2. In each of the following what "profit" or "loss" will make the open sentence true?

(a) $a + 2 = 5$

(b) $a + (-2) = 5$

(c) $a + 2 = -5$

(d) $b + (-2) = -5$

(e) $(-4) + c = 2$

(f) $(-3) + m = (-6)$

(g) $4 + n = 4$

(h) $(-4) + n = (-4)$

(i) $m + (-1) = (-2)$

(j) $c + (-3) = 3$

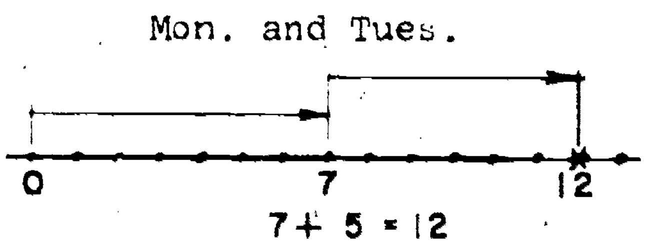
(k) $b + 4 = (-7)$

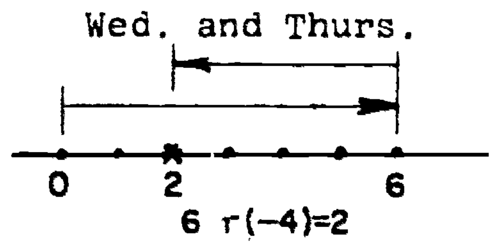
(l) $(-2\frac{1}{2}) + a = (-5)$

(m) $b + (1.5) = 1.1$

7-2. Addition and the Number Line.

In earlier chapters, we used the number line to show some facts about numbers. Now let's use the number line to show the arithmetic of the ice cream salesman's record.

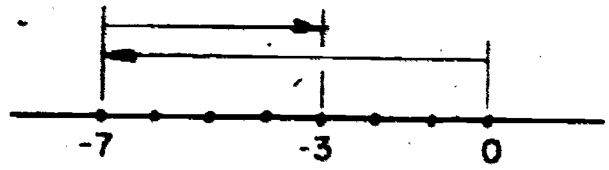




Notice that in the first example (Mon. and Tues.), we start at zero. We first move 7 units to the right. Then we move 5 more units to the right. The final position on the number line is shown by a "*". This position shows us that the result of adding 5 to 7 is 12.

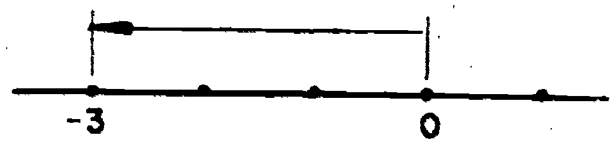
In the second example (Wed. and Thurs.), we again start at zero. First we move 6 units to the right. Then we move 4 units to the left. (Why do we move 4 units to the left instead of 4 units to the right?) What is the result?

Fri. and Sat.



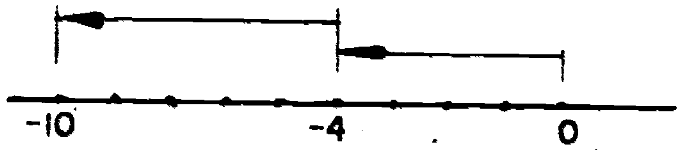
(-7)+4=-3

Sun. and Mon.



0+(-3)=-3

Tues. and Wed.



(-4)+(-6)=-10

In working with the number line, remember that we show addition of a positive number by moving to the right. From the above examples, you can see that addition of a negative number is shown by moving to the left. In which direction do we move to show addition of zero?

Below is a summary of things to remember when showing addition of two real numbers on the number line.

(1) Start at zero.

(2) From zero, move as many units to the right or left as the first number indicates. Move to the right if the first number is positive; move to the left if it is negative. In doing this, you will determine a point on the line.

(3) From this point, move as many units to the right or left as the second number indicates. This determines another point, or the "final position", on the number line.

(4) The coordinate of the "final position" point is the sum of the two numbers.

Discussion Questions 7-2

1. What is the starting point when we add on the number line?
2. In which direction do we move on the number line to indicate the addition of a positive number?
3. In which direction do we move on the number line to indicate the addition of a negative number?
4. What does the final position on the number line indicate?

Oral Exercises 7-2

1. If a thermometer registered -15° F and the temperature rises 10 degrees, what does the thermometer then register?
2. A thermometer registered 10° F at noon and dropped 6 degrees in three hours. What was the temperature at 3 PM?
3. How would you represent a drop of 3° F by a number?
4. If a thermometer registers -5° C and then rises 6°, what is the new temperature?
5. If a thermometer registers 3° C and then drops 4 degrees, what is the temperature?
6. Describe how you would find these sums on the number line.

(a) $(-5) + 2$	(f) $(-11) + 15$	(k) $4 + (-6) + (-8)$
(b) $(-5) + (-2)$	(g) $4 + 12$	(l) $(-5) + (-2) + (-7)$
(c) $5 + 2$	(h) $6 + (-7)$	(m) $(-4) + 8 + (-4)$
(d) $5 + (-2)$	(i) $6 + (-6)$	(n) $0 + (-2) + 2$
(e) $(-6) + (-7)$	(j) $(-7) + 0$	(o) $7 + (-2) + 3$

Problem Set 7-2

1. Find the following sums. Use the number line to aid you if necessary.

(a) $5 + 2$
(b) $5 + (-2)$
(c) $(-2) + 5$
(d) $(-2) + (-5)$
(e) $(-5) + 2$
(f) $(-7) + (-3)$
(g) $(-7) + 3$
(h) $6 + 0$
(i) $8 + (-10)$
(j) $(-5) + 0$
(k) $(-6) + 9$
(l) $6 + (-12)$
(m) $(-6) + 12$
(n) $0 + 0$
(o) $(-7) + 4 + 1$
(p) $8 + (-2) + (-3)$
(q) $(-5) + 9 + (-4)$

2. Think of the numbers in Problem 1 as "gains" and "losses" and then find the sums. Do your answers agree with those in Problem 1?

3. Which of the following sentences are true? Use "gains" or "losses", or the number line, to help you decide.

(a) $(-3) + 4 = 7$
(b) $(-3) + 4 > 7$
(c) $(-3) + 4 < 7$
(d) $(-2) + 5 \neq 6 + (-4)$
(e) $(-1) + 6 < 1 + (-6)$
(f) $(-1) + 6 > 1 + (-6)$
(g) $(-1) + 6 = 1 + (-6)$
(h) $6 + (-7) = 7 + (-8)$
(i) $3(2) + (-4) \neq (-5) + 6$
(j) $3(2) + (-4) > (-5) + 6$
(k) $3(2) + (-4) < (-5) + 6$
(l) $3(2) + (-4) = (-5) + 6$

4. Perform the following additions of real numbers. (Use the number line to help you if you need it.)

(a) $(-5) + 2 + 7$
(b) $(-5) + (-2) + 6$
(c) $6 + (-2) + 4$
(d) $(-7) + 4 + (-3)$
(e) $(-7) + 3 + (-4)$
(f) $7 + (-4) + 3$
(g) $7 + (-3) + 4$
(h) $(-7) + (-3) + 4$
(i) $(-7) + (-4) + 3$
(j) $(-7) + (-4) + (-3)$

5. If the domain of the variable is the set of real numbers, find the truth sets of the following open sentences.

(a) $m + 5 = 1$

(b) $a + (-2) = 4$

(c) $(-3) + a = -5$

(d) $4 + b = 2$

(e) $(-2) + m = 8$

(f) $n + (-1) = 7$

(g) $b + 4 = -3$

(h) $b + (-4\frac{1}{2}) = 1\frac{1}{2}$

(i) $a + 2.5 = (-2.5)$

(j) $a + (-3) > 1$

(k) $m + (-2) > (-2)$

(l) $(-3) + m < (-5)$

(m) $(-5) + m = (-3) + m + (-2)$

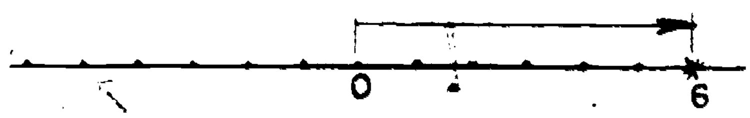
(n) $(-4) + b \neq (-5) + b + 1$

7-3. Addition Property of Zero and Addition Property of Opposites.

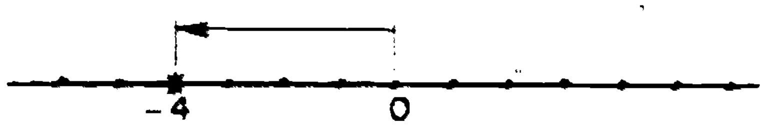
We have now seen that addition of real numbers can be shown on the number line. We move to the right to show addition of a positive number. We move to the left to show addition of a negative number. We have also seen that we move neither to the right nor to the left to show addition of zero.

Here are some examples of addition with the number zero:

6 + 0

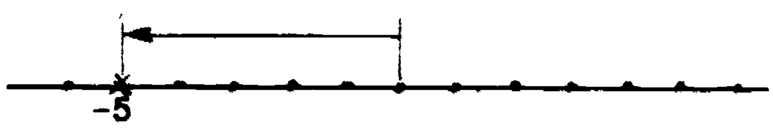


(-4) + 0



0 + 0

0 + (-5)



In each example, we added zero and another number; and each time the sum was the same as that other number. We showed this for just four cases. In fact, though, the sum of zero and any real number is the number itself. This is called the

addition property of zero.

It can be written like this:

For any real number a, $a + 0 = a$.

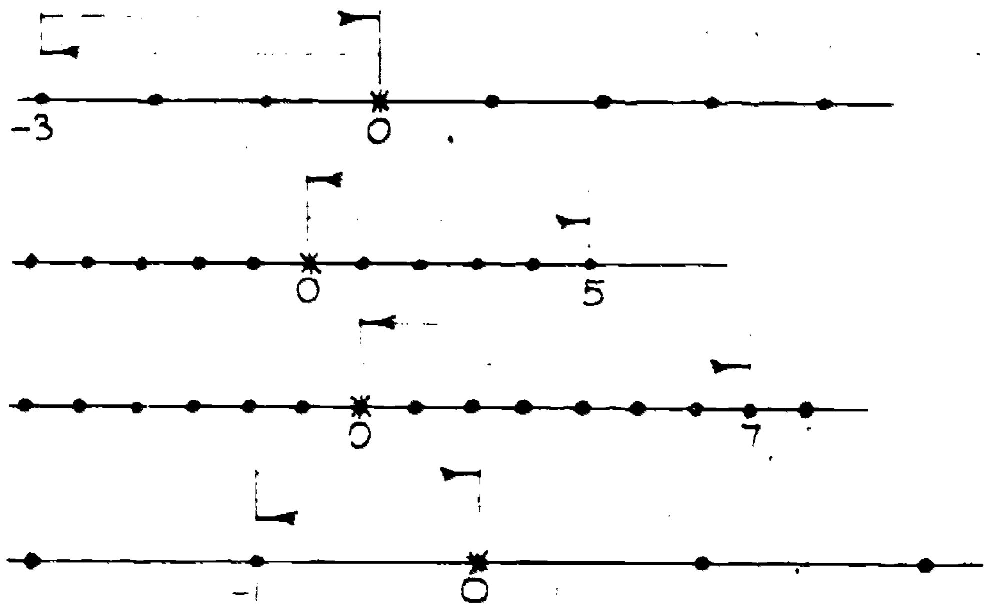
There is one case of addition of real numbers that turns out to be expecially interesting and useful. The examples below will show what we mean:

(-3) + 3

5 + (-5)

7 + (-7)

(-1) + 1



In each of these examples, we found the sum of a number and its opposite. Each time, the sum was zero. We have shown this for only four cases. However, it is true that the sum of any real number and its opposite is zero. This is called the

addition property of opposites.

It may be stated like this:

For any real number a, $a + (-a) = 0$.

Discussion Questions 7-3

1. What is the sum of any real number and zero?
2. What is the sum of any real number and its opposite?
3. State the addition property of zero for real numbers.
4. State the addition property of opposites for real numbers.

Oral Exercises 7-3

Which of the following are true sentences? Which are false?

1. $7 + (-7) = 0$
2. $8 + 8 = 0$
3. $(-4\frac{2}{3}) + 4\frac{2}{3} = 0$
4. $(-9) + 0 = -9$
5. $(-7) + 0 = 7$
6. $(+8) + (-8) = 0$
7. $(-9) + \left(-(-9)\right) = 0$
8. $(-15) + (-15) = 0$
9. $\left(-(-7)\right) + 0 = 7$
10. $7(0) = 7$
11. $\left(-(-15)\right) + 15 = 0$
12. $7(0) + (-7) = 0$
13. $7\left(7 + (-7)\right) = 0$
14. $49 + (-49) = 0$
15. Can you see any connection between Problem 13 and Problem 14?

Problem Set 7-3

For what value (or values) of the variable is each of the following open sentences true?

1. $(-14) + t = 0$
2. $8 + x = 8$
3. $r + (-4\frac{2}{3}) = (-4\frac{2}{3})$
4. $s + 0 = (-8)$
5. $9 + x = 0$
6. $(-45) + 0 = c$
7. $a + 0 = 0$
8. $a + (-3) > 0$
9. $3 + b < 0$
10. $a + (-3) > 3$
11. $2 + b < (-2)$

7-4. A Definition of Addition.

Give the common name for each of the following indicated sums. Be ready to tell which ones are examples of the addition property of zero and which ones are examples of the addition property of opposites.

(a) $7 + 10$
(b) $7 + (-10)$
(c) $10 + (-7)$
(d) $6 + 0$
(e) $(-10) + (-7)$
(f) $(-5) + 5$
(g) $(-7) + 10$
(h) $(-10) + 7$
(i) $0 + (-5)$
(j) $(-7) + 7$
(k) $0 + 4$
(l) $3 + (-3)$
(m) $(-8) + (-3)$
(n) $4 + 3$
(o) $(-11) + 7$

In which of the problems on page 226 were two positive numbers added? Was the sum in every case positive? negative? zero?

In which problems did you add two negative numbers? Was the sum in every case positive? negative? zero?

Look at Problem (e), which reads " (-10) + (-7)". What is the sum of -10 and -7? What is the absolute value of this sum?

What is the absolute value of -10? What is the absolute value of -7? When you add the absolute value of -10 and the absolute value of -7, what do you get? See if you agree with the following:

$$(-10) + (-7) = -17 \qquad \left.\begin{array}{l} |-10| = 10 \\ |-7| = 7 \\ |-17| = 17 \end{array}\right\} \qquad |-10| + |-7| = |-17|$$

In which problems did you add a positive number and a negative number? The first such problem was "7 + (-10)". What is the sum of 7 and -10? What is the absolute value of the sum?

What is the absolute value of 7? What is the absolute value of -10? What is the difference between these absolute values? See if you agree with the following:

$$7 + (-10) = -3 \qquad \left.\begin{array}{l} |7| = 7 \\ |-10| = 10 \\ |-3| = 3 \end{array}\right\} \qquad |-10| - |7| = |-3|$$

Another example of adding a positive number and a negative number is Problem (o), "(-11) + 7". Do you agree with the following:

$$(-11) + 7 = -(|-11| - |7|) \ ?$$

From the answers to the above questions, together with our work with profits and losses and with our work showing addition on the number line, you may want to use the following definition of the sum of two real numbers. Notice that the definition is given in several parts.

> The sum of a real number and zero is the number itself.
>
> The sum of a real number and its opposite is zero.
>
> The sum of two positive numbers is the sum of their absolute values.
>
> The sum of two negative numbers is the opposite of the sum of their absolute values.
>
> The sum of a positive number and a negative number, where the positive number has the greater absolute value, is found by subtracting the smaller absolute value from the greater.
>
> The sum of a positive number and a negative number, where the negative number has the greater absolute value, is found by:
>
> (a) subtracting the smaller absolute value from the greater
>
> (b) then finding the opposite of this difference.

Below are some examples. Study them in order to see if the definition says exactly what we want it to say.

Example 1 $(8) + (2) = (|8| + |2|)$
$= (8 + 2)$
$= 10$

8 and 2 are both positive. Their sum is the same as the sum of their absolute values.

Example 2 $(-8) + (-2) = -(|-8| + |-2|)$
$= -(8 + 2)$
$= -10$

-8 and -2 are both negative. Their sum is the opposite of the sum of their absolute values.

Example 3 $(8) + (-2) = (|8| - |-2|)$
$= (8 - 2)$
$= 6$

One number is positive and one is negative. The positive number has the greater absolute value. The sum is found by subtracting the smaller absolute value from the greater.

Example 4 $(-8) + (2) = -(|-8| - |2|)$
$= -(8 - 2)$
$= -6$

One number is positive and one is negative. The negative number has the greater absolute value. The smaller absolute value is subtracted from the greater. The opposite of this difference is the sum of the two numbers.

Discussion Questions 7-4

1. What must we do to any two real numbers we are adding if we apply the definition given?

2. What do we do with the absolute values of two positive numbers to obtain their sum? Of two negatives? Of a positive and a negative?

3. If you have a positive number and a negative number such that the positive one has the larger absolute value, what do you know about the sum? If the negative one has the larger absolute value, what do you know about the sum?

4. For what case of addition of two real numbers do we find the opposite of the difference of two absolute values?

Oral Exercises 7-4

Describe your steps in finding each of the following sums according to the example.

Example $(-7) + 4$ First, find the absolute values of each, they are 7 and 4.

Second, find the difference of the absolute values $7 - 4 = 3$.

Third, the number having the greater absolute value (-7) was negative so the sum is negative.

Therefore $(-7) + 4 = -3$.

Oral Exercises 7-4
(continued)

1. $7 + 4$
2. $(-7) + 4$
3. $(-7) + (-4)$
4. $7 + (-4)$
5. $7 + (-7)$
6. $(-7) + 0$
7. $11 + (-7)$
8. $(-11) + (-7)$
9. $(-11) + 7$
10. $7 + 11$
11. $(-5) + (-5)$
12. $5 + (-5)$
13. $5 + 5$
14. $5 + 0$
15. $(-5) + 0$
16. $0 + (-11)$
17. $(-14) + 3$
18. $(-11) + 5$
19. $11 + (-5)$
20. $(-11) + (-5)$

Problem Set 7-4

1. Use the definition of addition to find the following sums.

(a) $8 + 4$
(b) $(-8) + (-4)$
(c) $(-7) + 3$
(d) $7 + (-3)$
(e) $(-2) + 9$
(f) $(-3) + (-9)$
(g) $(-4) + 0$
(h) $(-5) + (-5)$
(i) $6 + (-6)$
(j) $0 + (-8)$
(k) $1.5 + (-2.0)$

2. Which of the following are true sentences? Which are false?

(a) $8 + 2 = |8| + |2|$
(b) $(-8) + (-2) = |-8| + |-2|$
(c) $(-8) + 2 = |-8| + |-2|$
(d) $8 + (-2) = |8| + |-2|$
(e) $(-8) + (-2) = -(|-8| + |-2|)$
(f) $(-7) + 2 = |7| - |2|$
(g) $7 + (-2) = |7| - |-2|$
(h) $(-7) + 2 = -(|-7| - |2|)$
(i) $(-4) + (-2) = -(|-4| + |-2|)$
(j) $(-10) + (-3) = -(|-10| + |-3|)$
(k) $(-6) + (-2) = |-6| + |-2|$
(l) $(-6) + (-2) = -|-8|$
(m) $(-6) + (-2) < |-8|$
(n) $(-6) + (-2) > -|8|$
(o) $(-6) + (-2) > |-8|$
(p) $(-9) + 4 \neq |-5|$
(q) $(-9) + 4 < |-5|$

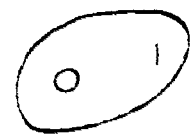
7-5. Closure and the Real Numbers.

Earlier we discussed the property of closure. The following examples show again what is meant by this property.

In the set of numbers

$$\{0, 1\},$$

there are only two elements. We might write the names of these elements "inside a loop", as shown below:



Now, with this set, think of the operation of multiplication. Using the elements of the set, we have the following possible products:

$$0 \times 0, \quad 0 \times 1, \quad 1 \times 0, \quad 1 \times 1.$$

Notice that these products are, in order, 0, 0, 0, and 1. Every product is a number in the set. We don't have to go outside the set, or "break through the loop", for any of the products. The set is closed under multiplication.

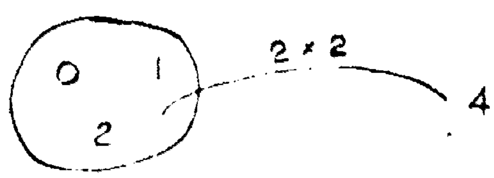
In the set

$$\{0, 1, 2\},$$

there are three elements. We have the following possible products:

$$0 \times 0,\ 0 \times 1,\ 0 \times 2,\ 1 \times 0,\ 1 \times 1,\ 1 \times 2,\ 2 \times 0,\ 2 \times 1,\ 2 \times 2.$$

These products are, in the order given above, 0, 0, 0, 0, 1, 2, 0, 2, 4. Notice that one of these products, 4, is not in our set. The product 2 × 2 takes us outside the set, or "breaks the loop." This set is not closed under the operation of multiplication.



These "homely" examples may have helped you to remember what the property of closure is. To check your understanding, see if you can answer the following questions:

> Is the set of counting numbers closed under addition? under subtraction? under multiplication? under division?
>
> Is the set of odd whole numbers closed under the operation of addition? under the operation of multiplication?

The last time we discussed the property of closure, we were not working with the entire set of real numbers. The following problem set is very important, because you will have a chance to decide what properties of closure the real numbers have.

Discussion Questions 7-5

1. What do we mean by "closure"? When is a set "closed under addition"?
2. Is the set {0, 1} closed under multiplication? Is the set {0, 1, 2} closed under multiplication? Why not?

Oral Exercises 7-5

Tell whether the following sets are closed under (a) addition (b) multiplication (c) division (d) subtraction.

1. The counting numbers
2. The odd whole numbers
3. The rational numbers which can be expressed as fractions with denominator 5
4. The rational numbers which can be expressed as fractions with numerator 3
5. The rational numbers from 0 to 1 inclusive
6. The set of all multiples of 5

Problem Set 7-5

1. Given the set $K = \{-1, -\frac{1}{2}, 0, 2\}$. Is this set closed under the operation of addition? (That is, is the sum of every element of K with any element of K including itself, also an element of K ?)

2. (a) Is the set $N = \{-3, -2\frac{1}{3}, -1, 0\}$ closed under addition?

   (b) Is the subset of N consisting of the negative numbers of N closed under addition?

3. Given the set

$$R = \{\ldots, -2, -1\frac{1}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots\}.$$

   (a) Is the set R closed under addition?

   (b) Is the subset of R consisting of all the negative elements of R closed under addition?

   (c) Is the set of all positive numbers, which are elements of R, closed under addition?

   (d) Is the set of all real numbers, which are elements of R, closed under addition?

7-6. Another Look at Addition.

Before we go on to look at some of the properties of addition, we might look back over the ways we have had of thinking about addition of real numbers. Below is a list of the three ways:

Profits and Losses. Maybe you have stopped thinking about such things as profits and losses and football games when you add real numbers. If so, this is a good sign that you are beginning to feel "at home" with addition of real numbers.

Addition on the Number Line. Maybe you have also stopped thinking about the number line in adding real numbers. However, it is important to remember that on the number line we move to the right to show addition of a positive number, we move to the left to show addition of a negative number.

Definition of Addition. The definition of addition of real numbers was given on page 228. You may not choose to use this definition every time you add real numbers. But it might be a good idea to go back and check once again to see that the definition says exactly what we want it to say.

The common name for "(-15) + (8)" is "(-7)". Use each one of the three methods above to show that this is true.

Oral Exercises 7-6

Find a common name of each of the following sums and tell how you would find each sum in terms of gain or loss, in terms of the number line, and in terms of absolute value.

1. $12 + 7$
2. $12 + (-7)$
3. $(-12) + 7$
4. $(-12) + (-7)$
5. $(-4) + 12$
6. $5 + (-12)$
7. $0 + 8$
8. $8 + (-8)$
9. $0 + (-9)$
10. $(-\frac{3}{2}) + \frac{1}{2}$
11. $1 + (-\frac{1}{2})$
12. $200 + (-201)$
13. $7 + (-1) + 1$
14. $6 + (-4) + (-2)$
15. $|-5| + |2|$
16. $|3| + |-4|$
17. $|-5| + |-6|$
18. $|5| + |6|$
19. $-(|5| + |-6|)$
20. $(|-7| - |-5|)$
21. $(|-7| - |5|)$

Problem Set 7-6

1. Find each of the following sums:
   (a) $(-3) + (-2)$
   (b) $4 + 3$
   (c) $2 + (-1)$
   (d) $6 + (-9)$
   (e) $(-10) + 5$
   (f) $(-8) + 6$
   (g) $8 + (-5)$
   (h) $(-7) + 7$
   (i) $6 + (-6)$
   (j) $6 + (-8) + (-6)$
   (k) $11 + (-5) + (-6)$
   (l) $17 + (-5) + (-11) + 8 + (-6)$

2. Check the results in Problem 1 by:
   (a) Thinking of the numbers as "profit" and "loss",
   (b) Using the number line,
   (c) Using the definition of addition.
   (d) Were the results the same in each case?

3. Which of the following sentences are true? Which are false?
   (a) $(-7) + 2 = 5$
   (b) $(-4) + 1 = 6 + (-9)$
   (c) $9 + (-2) = (-9) + 2$
   (d) $14 + (-9) = 10 + (-5)$
   (e) $19 + (-6) = (-17) + 4$
   (f) $9 + (-9) = 16 + 2$
   (g) $(-8) + 0 = (-6) + (-2)$
   (h) $10 + (-9) > (-10) + 9$
   (i) $5 + (-10) \neq (-5) + (-10)$
   (j) $5 + (-10) > (-5) + (-10)$
   (k) $5 + (-10) < (-5) + (-10)$

4. Translate the following sentences into open sentences. (Be sure to identify the variable.)

   Example: John and Mike played three hands of cards. John asked Mike what he made on each hand but Mike would not tell him. John knew Mike's total score was 85, that he made 90 points on the second hand and lost 110 points on the third hand. With this information he was able to decide what Mike's score was on the first hand. What was the score?

Problem Set 7-6
(continued)

x is Mike's score on the first hand

$x + 90 + (-110) = 85$

This could have been written

$$x + (-20) = 85$$

(a) A stock market price lost 3 points in the morning and by evening it was down a total of 7 points. What happened to the stock price during the afternoon?

(b) One summer Charles operated a fruit stand beside the highway. On Tuesday evening he began checking his money to decide if he was going to be able to make another $8 payment on a motor scooter. He had no money Monday morning. On Monday he had made a profit of $6, but on Tuesday it rained all day and he lost $2. How much profit must he make on Wednesday to be able to make his payment?

(c) At the end of the year the assets of a business firm are $101,343 and the liabilities are $113,509. What is the balance?

(d) From a submarine submerged 215 feet below the surface, a rocket is fired which rises 3,000 feet above the submarine. How far above sea level does the rocket go?

7-7. Commutative and Associative Properties.

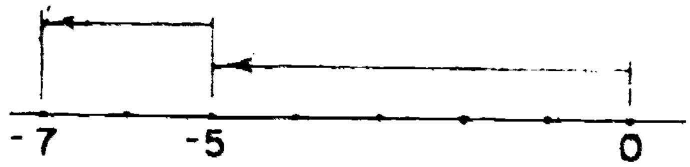
We have discussed properties of addition in earlier chapters. At that time, we were working only with the numbers of arithmetic. Now that we are working with the entire set of real numbers, we want to be sure that these properties are true for the real numbers.

One property is the commutative property of addition. For example, is it true that

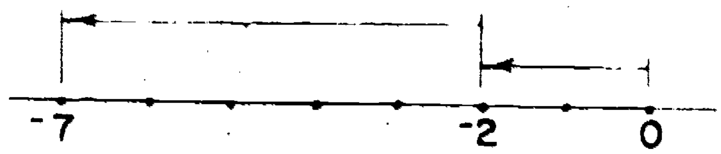
$$(-5) + (-2) = (-2) + (-5)\ ?$$

On the number line,

(-5) + (-2)
is shown like this:



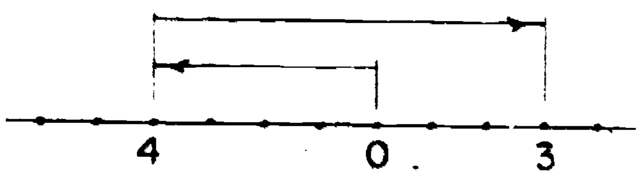
(-2) + (-5)
is shown like this:

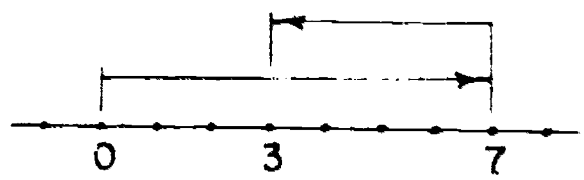


From the number line, we see that it is true that

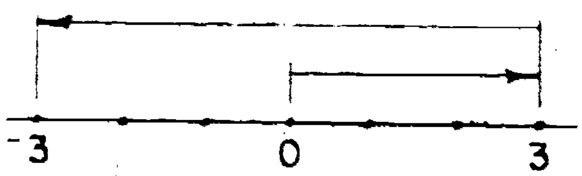
$$(-5) + (-2) = (-2) + (-5).$$

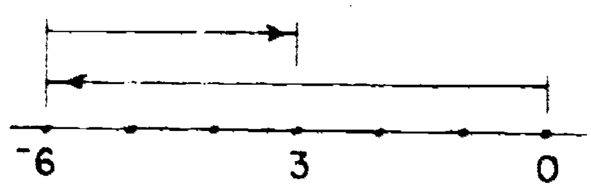
It might help to look also at the following sentences and "pictures" on the number line.





$(-4) + 7 \quad = \quad 7 + (-4)$





$3 + (-6) \quad = \quad (-6) + 3$

We have shown only three examples. However, perhaps you begin to see that for any two real numbers, the order in which the numbers are added does not affect the sum.

What we are saying is that the <u>commutative property of addition</u> is true for the set of real numbers. We can write:

<u>For any two real numbers a and b</u>, $a + b = b + a$.

Another property we might investigate at this time is the associative property of addition. We already know, for example, that

$$7 + (3 + 2) = (7 + 3) + 2.$$

Now check to see if the following statements are true:

$$(7 + (-9)) + 3 = 7 + ((-9) + 3)$$
$$(8 + (-5)) + 2 = 8 + ((-5) + 2)$$
$$(4 + 5) + (-6) = 4 + (5 + (-6))$$

These examples may help you to see that the associative property of addition is true for the set of real numbers. We can say:

<u>For any real numbers a, b, and c</u>, $(a + b) + c = a + (b + c)$.

Discussion Questions 7-7

1. Add a negative number to a positive number; now add the same positive number to the negative number. Did you get the same sum?
2. What property of addition was shown in Question 1?
3. (a) Add $(-3) + (-4)$, then add 5 to the result.
   (b) Add $5 + (-4)$, then add (-3) to the result.
   (c) Did you get the same sum in (a) and (b)?
   (d) What property of addition is illustrated here?
4. State the commutative property of addition in your own words.
5. State the associative property of addition in your own words.

Oral Exercises 7-7

Tell what property or properties of addition are illustrated by each of the following:

1. $(-5) + (-7) = (-7) + (-5)$
2. $\left(4 + (-3)\right) + 2 = 4 + \left((-3) + 2\right)$
3. $(-3) + \left((-2) + (-5)\right) = \left((-3) + (-2)\right) + (-5)$
4. $\left((-4) + (-5)\right) + 2 = 2 + \left((-4) + (-5)\right)$
5. $\left((-4) + 1\right) + (-7) = (-4) + \left((-7) + 1\right)$
6. $(m + n) + q = m + (n + q)$
7. $a + (b + c) = (b + c) + a$
8. $(x + y) + z = (y + z) + x$
9. $x + (a + b) = (a + x) + b$
10. $(a + b) + (x + y) = (a + y) + (b + x)$

Problem Set 7-7

1. Which of the following sentences are true? Which are false?
   (a) $(-5) + (-2) = (-2) + (-5)$
   (b) $\left(7 + (-2)\right) + (-6) = 7 + \left((-2) + (-6)\right)$
   (c) $(-6) + 3 = 3 + (-6)$
   (d) $(-8) + 2 = (-2) + 8$
   (e) $(-8) + \left(6 + (-4)\right) = 8 + 6 + (-4)$
   (f) $(-9) + \left((-3) + 6\right) = \left((-9) + (-3)\right) + (-6)$
   (g) $(7 + 5) + 4 = 4 + (7 + 5)$
   (h) $(14 + 5) + (-2) = 14 + \left(2 + (-5)\right)$
2. In each of the following find the real number (or numbers) which makes the open sentence true.
   (a) $(-6) + (-3) = (-3) + r$
   (b) $(-5) + y = 7 + (-5)$
   (c) $15 + \left((-9) + 2\right) = (15 + x) + 2$
   (d) $\left((-2) + 5\right) + c = (-2) + \left(5 + (-3)\right)$
   (e) $14\frac{1}{3} + x = 2\frac{1}{2} + 14\frac{1}{3}$
   (f) $(14 + 8) + (-4) = 14 + \left((-4) + b\right)$

Problem Set 7-7
(continued)

(g) $a + (-16) = (-12) + 10 + (-4)$
(h) $\big(5 + (-5)\big) + m = 5 + \big((-5) + m\big)$
(i) $\big(3.2 + (-n)\big) + 2.8 = 3.2 + \big(2.8 + (-n)\big)$
(j) $|-5| + a = |-4| + |-5|$
(k) $b + (-9) = |-6| + |-3|$

7-8. Addition Property of Equality.

"3 + 4" is the name of a number. If 8 is added to the number, we get a number which can be named "(3 + 4) + 8."

"9 - 2" is the name of a number. If 8 is added to the number, we get a number which can be named "(9 - 2) + 8."

Did you notice something strange about the pair of statements above? They were really saying the same thing. In both cases, we had the number 7, and we added 8. Of course, the result in both cases was 15. It is true that we used different names for the number 7 and the number 15. However, we were talking about the same numbers.

We can write:

$$(3 + 4) = (9 - 2)$$

so $$(3 + 4) + 8 = (9 - 2) + 8.$$

This is an example of a property of equality.

Here are some other examples:

Example 1

$$(-7) + 2 = -5$$
$$\big((-7) + 2\big) + (-2) = (-5) + (-2)$$

Example 2

$$(-3) + (-2) = (-8) + 3$$
$$\big((-3) + (-2)\big) + 2 = \big((-8) + 3\big) + 2$$

Sometimes people talk about "adding the same number to both sides." In the two examples above, we could say we "added -2 to both sides" in the first one, and "added 2 to both sides" in the second one.

"Sides" is a kind of slang word in mathematics. In the sentence "$(-7) + 2 = -5$," we might speak of "$(-7) + 2$" as being on the left side of the verb "=" and "-5" as being on the right side of the verb "=".

Here is another example:

$$(3 \times 2) + (-4) = 2.$$

This true sentence says that we have two different names for the same number. If 4 is added to the number, the sentence then reads:

$$\left((3 \times 2) + (-4)\right) + 4 = 2 + 4.$$

This is also a true sentence. You can see that the "left side" and the "right side" of the sentence both show that 4 has been added to the number we started with.

All of these examples show a property of equality called the

<u>addition property of equality</u>,

which can be written:

<u>For any real numbers a, b, and c</u>, if $a = b$, then $a + c = b + c$.

Of course, it could also be stated:

<u>For any real numbers a, b, and c</u>, if $a = b$ then $c + a = c + b$.

Discussion Questions 7-8

1. What happens when we add the same number to both sides of a true sentence?
2. What happens when we add a positive number to the right side only of a true sentence?
3. What happens when we add a negative number to the left side only of a true sentence?
4. State the addition property of equality in your own words.

Oral Exercises 7-8

1. For what values of the variables are the following sentences true?

   (a) $3 + x = x + 3$

   (b) $(-3) + 4 + (-5) + x = (-4) + x$

   (c) $x + (-8) + (-7) + (-6) = (-14) + (-7) + x$

2. In each of the following what real number must be added to the given number so that the result is x?

   (a) $x + 5$

   (b) $x + (-6)$

   (c) $x + (-11)$

   (d) $x + 8$

   (e) $9 + x$

   (f) $(-7) + x$

   (g) $(-3) + 4 + x$

   (h) $x + 3 + (-8)$

   (i) $(-5) + x + (-7)$

   (j) $12 + x + (-24)$

3. In each of the following sentences assume that both "sides" are names for the same number. Now tell what number could be added to both sides so that the new sentence will have the variable alone on one side.

   Example: $x + 4 = 5 + (-2)$

   We would say "Add (-4) to both sides," because this would give us

   $$x + 4 + (-4) = 5 + (-2) + (-4),$$

   $$x + 0 = 5 + (-2) + (-4)$$

   or $$x = 5 + (-2) + (-4).$$

   Here x is alone on the left side.

Oral Exercises 7-8
(continued)

(a) $x + 3 = 10$
(b) $(-4) + x = 11$
(c) $x + (-16) = 16$
(d) $x + (-8) = -8$
(e) $(-5) + m + (-4) = 12$
(f) $-18 = b + (-3)$
(g) $16 = 16 + a$
(h) $28 + a + (-12) = -7$
(i) $(-6) + (-5) = x + 5$
(j) $(-14) + x = 14 + (-6)$
(k) $(-4) + y + (-3) = -15$
(l) $y + (-5) + 5 = -8$

Problem Set 7-8

1. Which of these sentences are true? Which are false?
   (a) $\left((-7) + 2\right) + (-2) = (-5) + (-2)$
   (b) $\left((-14) + (-3)\right) + 3 = \left((-20) + 3\right) + 3$
   (c) $\left((-6) + 9\right) + (-9) = (-6) + (-9)$
   (d) $\left(2 + (-10)\right) + 6 = (-8) + 6$
   (e) $\left((-8) + 5\right) + \left(5 + (-10)\right) = (-3) + (-5)$
   (f) $\left((-15) + 23\right) + (-23) = \left(8 + (-16)\right) (-23)$
   (g) $\left((-2\frac{1}{3}) + 7\frac{2}{3}\right) + 2\frac{1}{3} = 5\frac{1}{3} + 2\frac{1}{3}$
   (h) $(3.6 + 4.2) + (-3.6) = 7.8 + (-3.6)$

2. In each of the following sentences assume that both sides are names for the same number. Determine what number we could add to both sides so that when the sums are simplified one side consists of only the variable or a product involving the variable.
   (a) $3x + 4 = 10$
   (b) $(-6) + x + 4 = 12$
   (c) $5 + 3y = 12 + (-2)$
   (d) $4 + 8 + (-7) = 7a + (-2)$
   (e) $5a + (-2) + (-7) = (-8) + (-4)$
   (f) $6 + b = (-11) + (-5)$
   (g) $4 + 11x + (-6) = (-8) + 8$
   (h) $(-30) + (-10) + y = (-40)$
   (i) $-\left(3(4 + 2)\right) + 3m = 0$
   (j) $\frac{3}{4} + (-8) = \frac{4}{3}y$

Problem Set 7-8
(continued)

3. What real number will make each open sentence true?

(a) $((-5) + (-2)) + 4 = (-7) + x$

(b) $(14 + (-3)) + 3 = y + 3$

(c) $((-7) + 4) + (-4) = m + (-4)$

(d) $(4 + (-11)) + 11 = n + 11$

(e) $(3\frac{3}{4} + 2\frac{1}{4}) + (-2\frac{1}{4}) = a + 2\frac{1}{4}$

(f) $(1.5 + (-.5)) + 1 = b + 1$

(g) $(m + (-8)) + 8 = 3 + 8$

(h) $7.2 + (-5.2) + (-5.2) = m + 5.2$

(i) $(-5) + 3y + 5 = 9$

(j) $7 + (-14) + 2x = (-4) + 12$

7-9. Truth Sets of Open Sentences.

Earlier we worked with open sentences and found their truth sets. We had to work with very simple open sentences, since we could only "guess" the truth numbers.

You may have wondered how you could find the truth set of an open sentence if you couldn't guess the truth numbers. Now that we have some important properties of real numbers, we can use them to find truth sets. No longer do we have to guess!

As a first example, what is the truth set of

$$x + 4 = -2\ ?$$

Maybe this open sentence has no truth number at all. Then the truth set would be the empty set. But, if there is a number x that makes the sentence true, then "$x + 4$" and "-2" are names for the same number. Then we can write:

$(x + 4) + (-4) = (-2) + (-4)$ Addition Property of Equality. If you're wondering why we added -4, see what happens in the next steps.

$x + \left(4 + (-4)\right) = (-2) + (-4)$ — Associative Property of Addition--an old friend!

$x + 0 = (-2) + (-4)$ — $4 + (-4) = 0$, by the Addition Property of Opposites. Now do you see why we added -4?

$x = (-2) + (-4)$ — Addition Property of Zero

$x = -6$ — "-6" is the common name for "(-2) + (-4)".

So, if there is a number x that makes "$x + 4 = -2$" true, then it must also make "$x = -6$" true, since we have used only properties that are true for all real numbers.

It is not hard to see that -6 is a truth number of "$x = -6$".

Is -6 a truth number of "$x + 4 = -2$"?

"$(-6) + 4 = 2$" is a true sentence. So, -6 is a truth number.

Therefore, the truth set of "$x + 4 = -2$" is

$$\{-6\} .$$

Example 2 Find the truth set of $x + \frac{3}{5} = -2$.

The truth set is found below. See if you can give the reason for each step in finding the truth set.

If there is a number x that makes the sentence true, then

$$(x + \tfrac{3}{5}) + (-\tfrac{3}{5}) = -2 + (-\tfrac{3}{5}) \quad \text{. Why did we add } -\tfrac{3}{5}\text{?}$$

$$x + \left(\tfrac{3}{5} + (-\tfrac{3}{5})\right) = -2 + (-\tfrac{3}{5})$$

$$x + 0 = -2 + (-\tfrac{3}{5})$$

$$x = -2 + (-\tfrac{3}{5})$$

$$x = -\tfrac{13}{5}$$

So, if there is a number x that makes "$x + \frac{3}{5} = -2$" true, then "$x = -\frac{13}{5}$" is true for the same x.

$-\frac{13}{5}$ is certainly a truth number of "$x = -\frac{13}{5}$". Is it a truth number of "$x + \frac{3}{5} = -2$"?

"$(-\frac{13}{5}) + \frac{3}{5} = -2$" is a true sentence since both the left side and the right side name the number, -2.

Therefore, the truth set of "$x + (\frac{3}{5}) = -2$" is

$$\{-\frac{13}{5}\} .$$

Example 3 Find the truth set of $4 + (-2) = x + (-5)$.

If there is a number x that makes this sentence true, then

$$\begin{aligned} \big(4 + (-2)\big) + 5 &= \big(x + (-5)\big) + 5. \\ \big(4 + (-2)\big) + 5 &= x + \big((-5) + 5\big) \\ \big(4 + (-2)\big) + 5 &= x + 0 \\ \big(4 + (-2)\big) + 5 &= x \\ (2) + 5 &= x \\ 7 &= x \end{aligned}$$

Is 7 a truth number of "$4 + (-2) = x + (-5)$"?

If x is 7, the right side of the sentence is the numeral "$7 + (-5)$", whose common name is "2".

The left side of the sentence is the numeral "$4 + (-2)$", whose common name is "2".

Therefore, the truth set of "$4 + (-2) = x + (-5)$" is

$$\{7\} .$$

Discussion Questions 7-9

1. Which of the following properties of addition are true for all real numbers?
   (a) $a + b = b + a$
   (b) $(a + b) + c = a + (b + c)$
   (c) $a + 0 = a$
   (d) $a + (-a) = 0$
   (e) If $a = b$, $a + c = b + c$.
2. Give a numerical example for each of the properties in Question 1.

Discussion Questions 7-9
(continued)

3. Give the name of each of the properties that are illustrated in Question 1.
4. In finding the truth set of the sentence "$x + 4 = -2$" why did we "add (-4) to both sides" of the sentence? What property of equality permitted us to do this?
5. How can we be sure that $-\frac{13}{5}$ is a truth number of the sentence "$x + \frac{3}{5} = -2$"?

Problem Set 7-9

1. Find each of the following sums, using the easiest way. At each step name the property that you used.
   (a) $(-3) + 7 + 5 + 3 + (-5)$
   (b) $14 + 6 + (-7) + 4 + 3$
   (c) $5 + (-8) + 6 + (-3) + 2$
   (d) $(-9) + 5 + 6 + (-3)$
   (e) $11 + (-17) + 9 + (-3) + 4$
   (f) $c + 2 + (-c) + 5$
   (g) $2r + 4 + (-2r) + (-4)$
   (h) $15r + 6 + (-10r) + (-3) + (-5r)$
2. Find the truth number of the following open sentences.
   (a) $(-5) + a = (-5) + 4$
   (b) $a + (-3) = 6 + \left((-2) + (-1)\right)$
   (c) $b + (-6) = \left((-3) + 2\right) + (-6)$
   (d) $5 + (-4) = m + (-4)$
   (e) $(-7) + (-5) = \left((-4) + (-3)\right) + n$
   (f) $5\frac{1}{2} + (-1\frac{1}{2}) = 5\frac{1}{2} + b$
   (g) $(-2.6) + c = (-1.6) + (-1.0) + 3.1$
   (h) $3.5 + c = \left(5.0 + (-1.5)\right) + c$
   (i) $(-5) + 3x + (-8) = 15 + (-20) + (1)$
   (j) $15 + (-12) + (-3) = (-16) + 5y + (-4)$
   (k) $11m + 17 + (-20) = 24 + (-5)$
   (l) $(-4\frac{1}{2}) + 12 + 4y = 18\frac{1}{2}$

7-10. Additive Inverse.

Can you complete each of the following so as to have a true sentence?

$$4 + (-4) =$$

$$(-8) + 8 =$$

$$x + (-x) =$$

Can you answer the following questions:

What number added to -7 gives a sum of zero?
What number makes the open sentence "$x + 9 = 0$" true?

These questions were probably not very hard. We have already learned that the sum of a number and its opposite is zero.

In mathematics, there is another important name we can start using when talking about statements like "$4 + (-4) = 0$".

-4 is called an additive inverse of 4, since $4 + (-4) = 0$.
4 is called an additive inverse of -4, since $(-4) + 4 = 0$.
-7 is called an additive inverse of 7, since $7 + (-7) = 0$.
7 is an additive inverse of -7. Why?
Give an additive inverse of 25.

If we have two real numbers x and y whose sum is zero, like this:

$$x + y = 0 ,$$

then y is the additive inverse of x
and x is the additive inverse of y.

Discussion Questions 7-10

1. What number added to (-5) gives the sum zero?
2. What number added to 6 gives the sum zero?
3. What is the additive inverse of (-7)?
4. What is the additive inverse of 10?

Discussion Questions 7-10
(continued)

5. In the sentence $x + 5 = 0$
   (a) Choose an additive inverse that will help you find the truth set.
   (b) Think of the addition property of equality and the additive inverse to find the truth set of this sentence.
   (c) What is the truth set of the sentence?
6. If the sum of two numbers is zero, what can we say about the numbers?

Oral Exercises 7-10

1. Give the additive inverse of each of the following.

(a)	4	(g)	$(-7) + 2$	(m)	$x + 5$
(b)	-9	(h)	$3 + 5$	(n)	$(-6) + 3m$
(c)	25	(i)	$4 + (-6)$	(o)	$(-4) + 2y + 5$
(d)	-12	(j)	$3m$	(p)	$a + b$
(e)	x	(k)	$-5k$	(q)	$3m + n$
(f)	$-x$	(l)	$(-11) + (-4)$	(r)	$4y - x + 2$

Problem Set 7-10

1. Find the truth set of each of these open sentences.
   (a) $x + (-6) = (-8)$
   (b) $(-14) = x + 3$
   (c) $(-4) + x = (-9) + (-2)$
   (d) $7 + (-3) = 11 + x$
   (e) $(-7) + (-5) + x = (-6) + 4 + (-6)$
   (f) $7 + x + (-5) = (-4) + 1 + (-9)$
   (g) $9 + 2 + (-4) = (-7) + (-3) + x$
   (h) $x + (-3) > (-2)$
   (i) $(-5) + 2 < x + (-6)$

Problem Set 7-10
(continued)

(j) $(-8) + (-2) + x > (-10) + 2 + 3$

(k) $3m + 5 = 12$

(l) $(-4) + 4x + (-2) = 11$

(m) $(-8) + 2 = (-7) + 3k + (-8)$

(n) $(-12) + (3) + m = 3m + (-15)$

(o) $(-5) + 4x = 3x + 7$

(p) $6 + 2y + (-7) = -4y + 2$

(q) $8 + (-7) + 3m = 7m + (-12)$

2. Translate these sentences into algebra.
   (a) A number added to its additive inverse has the sum zero.
   (b) The sum of a number and the additive inverse of 5 is 15.
   (c) Three times a number, increased by the additive inverse of $\frac{1}{2}$ is the additive inverse of the number.
   (d) The sum of a number and negative 3 multiplied by the sum of the same number and 3 is equal to the square of the number decreased by 9.
   (e) Five less than a certain number is the same as the sum of the number and the additive inverse of 5.
   (f) Is the sentence in (a) true for every number 3 or greater? Is the sentence in (e) true for every number 5 or greater?

3. John had four times as many pennies as nickels and twice as many dimes as nickels. Then when his uncle gave him 7 cents he had a total of 94 cents. How many of each coin did he finally have?

4. The largest angle of a triangle is 20° more than twice the smallest. The third angle is 70°. The sum of the angles of a triangle is 180°. How large is each angle?

5. A rectangle is 6 times as long as it is wide. Its perimeter is 112 inches. What are its dimensions?

7-11. Proving Theorems.

When you were answering questions about additive inverses, you probably never thought that a number could have more than one additive inverse. For example, it is easy to see that -4 is an additive inverse of 4. But suppose somebody claimed there was another number, different from -4, that is also an additive inverse of 4.

Could you prove to this person that he is wrong? Could you prove to him that -4 is the only additive inverse of 4?

You could, of course, ask this other person to name this other additive inverse of 4. However, you don't really need to. We can let "z" be that "other additive inverse of 4" that he claims to have.

If it is true that z is an additive inverse of 4, then

$4 + z = 0$	This sentence must be true if z is an additive inverse of 4.
$(-4) + (4 + z) = (-4) + 0$	Here we are using the Addition Property of Equality. If you are wondering why we added -4 to each side, see if you can discover why from the next step.
$\left((-4) + 4\right) + z = (-4) + 0$	Here we have used the Associative Property of Addition.
$0 + z = (-4) + 0$	Here we have used the Addition Property of Opposites. Do you see now why we added -4 back in the second step? It was the only way we could get that "0" on the left side.
$z = -4$	Addition Property of Zero.

You may have that "So what?" feeling; but look at the last line, "z = -4". It shows that that "other" additive inverse of 4 turned out to be not another one at all. It is equal to, or the same as, -4. So, no matter what anybody claims, we have proved that 4 has one and only one additive inverse, namely -4.

Could we go through the same kind of argument to show that 5 has only one additive inverse, that 20 has only one additive inverse, and so on? We could, but we don't have to do it for one number at a time. Instead, we can talk about <u>any</u> real number x.

We know that if x is <u>any</u> real number, then -x is an additive inverse of x, because

$$x + (-x) = 0.$$

If there is some other additive inverse of x, we can call it "z" and write:

$$x + z = 0.$$
$$(-x) + (x + z) = (-x) + 0$$
$$\big((-x) + x\big) + z = (-x) + 0$$
$$0 + z = (-x) + 0$$
$$z = -x$$

If you are puzzled about some of the steps above, compare this with the proof that 4 has only one additive inverse. The reasons are written out there, and the reasons here are the same.

In the final line, you see

$$z = -x .$$

This shows that if z is an additive inverse of x, then z is the same as, or is equal to, -x. In other words, there is no <u>other</u> additive inverse of x, except -x.

We can state it this way:

> Any real number x has one and only one additive inverse. This additive inverse is -x.

We have proved this statement to be true. We proved it to be true by using properties that are true for all real numbers. When a statement is proved in this way, it is often called a <u>theorem</u>.

Summary

In this chapter, we have discussed addition of real numbers. If a and b are any two real numbers, their sum may be written

$$a + b.$$

However, it is important to remember that (a + b) is itself a number. This is true because the real numbers have the property of closure under addition.

Besides the property of closure, we also discussed the following properties:

Commutative Property of Addition
For any two real numbers a and b, $a + b = b + a.$

Associative Property of Addition
For any real numbers a, b, and c, $(a + b) + c = a + (b + c).$

Addition Property of Opposites
For every real number a, $a + (-a) = 0$

Addition Property of Zero
For every real number a, $a + 0 = a.$

Addition Property of Equality
For any real numbers a, b, and c, if $a = b$ then $a + c = b + c.$

It is important to remember that the addition property of equality can be used in finding the truth sets of certain open sentences.

Review Problem Set

1. Perform the following additions on the number line.

(a) $4 + 3$ (c) $7 + 0$ (e) $1.5 + 2.5$
(b) $2 + 6$ (d) $0 + 4$ (f) $6 + 0$

2. In each of the following find the sum by using the <u>definition of addition</u>. Check your answers by using any convenient method.

(a) $3 + (-5)$
(b) $(-5) + (-11)$
(c) $0 + (-15)$
(d) $\sqrt{2} + (-\sqrt{2})$
(e) $18 + (-14)$
(f) $(-\pi) + \pi$
(g) $(-\frac{2}{3}) + \frac{1}{2}$
(h) $(-35) + (-65)$
(i) $12 + 7$
(j) $(-6) + 10$
(k) $1 + (-\frac{3}{2})$
(l) $200 + (-201)$

3. Find the following sums.

(a) $3 + (6 - 2)$
(b) $8 - (4 + 1)$
(c) $(8 - 4) + 1$
(d) $4 \times 2 + 5$
(e) $4 \times (2 + 5)$
(f) $24 - 3 \times 5$
(g) $(24 - 3) \times 5$
(h) $5 \times 0 + 6$
(i) $5 \times (0 + 6)$
(j) $5 + (-5) + (-10)$
(k) $6 + (-4) + (-1)$
(l) $(-7) + (-9) + 10$
(m) $8 + (-6) + (-8)$
(n) $|-9| + |4| + (-|-6|)$
(o) $|-4| + |8| + 4 + (-4)$

4. Which of the following are true sentences? Which are false?

(a) $(4 + 6) + 5 = 4 + (6 + 5)$
(b) $(3 + 2) + m = 3 + (m + 2)$
(c) $6(\frac{1}{4} + \frac{3}{4}) = 6$
(d) $6(1.5 + 3.5) = 6(1.5) + (3.5)$
(e) $2(r + 3) = 2r + 3$
(f) $4r(2y + z) = 8ry + 4rz$
(g) $(-4) + 0 = 4$
(h) $-(|-1.5| - |0|) = -1.5$
(i) $(-3) + 5 = 5 + (-3)$

Review Problem Set
(continued)

(j) $\big(4 + (-6)\big) + 6 = 4 + \big((-6) + 6\big)$

(k) $(-5) + \big(-(-5)\big) = -10$

(l) $(-7) + \big((-5) + (-3)\big) = \big((-7) + (-5)\big) + 3$

(m) $-\big(6 + (-2)\big) = (-6) + (-2)$

(n) $(-7) + (-9) = -(7 + 9)$

(o) $(-3) + 7 = -\big(3 + (-7)\big)$

5. For each of the following find the real number value (or values) of the variable which makes the given sentence true.

(a) $x + 2 = 7$

(b) $3 + x = 0$

(c) $3 + y = -7$

(d) $(-2) + a = 0$

(e) $a + 5 = 0$

(f) $3 + 5 + y = 0$

(g) $b + (-7) = 3$

(h) $x + (-\frac{1}{2}) = 0$

(i) $(-\frac{5}{6}) + x = -\frac{5}{6}$

(j) $c + (-3) = -7$

(k) $y + \frac{2}{3} = -\frac{5}{6}$

(l) $\big(y + (-2)\big) + 2 = 3$

(m) $(3 + x) + (-3) = -1$

(n) $\frac{1}{2}b + (-4) = 6$

(o) $6 + x + (-4) = 5 + (-2)$

(p) $5 + x + (-7) = 4$

(q) $7 + m + (-3) = m + 9 + (-5)$

(r) $\big(10 + (-6)\big) + b = (-10) + (6 + b)$

(s) $|-5| + |6| + a = 0$

(t) $b + |-4| + |-3| = |-7|$

Review Problem Set
(continued)

6. Find the truth set of each of the following open sentences.
   (a) $m + 7 = 12$
   (b) $a + (-5) = 8$
   (c) $a + (-4) + (-5) = (-9) + 3$
   (d) $(-6) + 7 = (-8) + x$
   (e) $(-1) + 2 + (-3) = 4 + x + (-5)$
   (f) $(-2) + x + (-3) = x + (-\frac{5}{2})$
   (g) $x + (-3) = |-4| + (-3)$
   (h) $(-\frac{4}{3}) + (x + \frac{1}{2}) = x + (x + \frac{1}{2})$
   (i) $x + (-3) = |-5| + |-3|$
   (j) $|-4| + b = |-6| + |4|$
   (k) $|-5| + a = a + |9| + |-2|$
   (l) $|-8| + m = (-9) + |-1| + m$
   (m) $x + 2 + x = (-3) + x$

7. Tell why each of the following sentences is true. Name the property, or properties, that are illustrated by each sentence, whether associative, commutative, addition property of 0, or addition property of opposites.
   (a) $3 + ((-3) + 4) = 0 + 4$
   (b) $(5 + (-3)) + 7 = ((-3) + 5) + 7$
   (c) $(7 + (-7)) + 6 = 6$
   (d) $|-1| + |-3| + (-3) = 1$
   (e) $(-2) + (3 + (-4)) = ((-2) + 3) + (-4)$
   (f) $(-|-5|) + 6 = 6 + (-5)$
   (g) $((-2) + 6) + (-8) = (-2) + (6 + (-8))$
   (h) $8 + |-5| + a = |-5| + 8 + a$
   (i) $|-6| + (-6) + 0 = 0$
   (j) $a + 4 + (-a) = 4$

Review Problem Set
(continued)

8. Use the commutative and associative properties to obtain the following sums in an easy way.

   (a) $(-\frac{1}{2}) + 7 + (-2) + (-\frac{3}{2}) + 2$

   (b) $\frac{5}{3} + (-3) + 6 + \frac{1}{3} + (-2)$

   (c) $125 + (-17) + (-13) + (-25)$

   (d) $(-3) + 8 + 11 + (-5) + (-3) + 12 + (-4)$

   (e) $\frac{2}{3} + \frac{3}{2} + (-\frac{5}{3}) + (-\frac{1}{2}) + |-2|$

   (f) $|-5| + 21 + (-5) + (-8) + (-7)$

   (g) $(-9) + |-7| + 12 + |-2| + 7$

   (h) $-|-10| + (-15) + 15 + (-3) + (-|-6|)$

9. Write open sentences for the following: (Be sure to identify the variable)

   (a) Jim learned that on a certain day the low tide registered 0.6 feet below sea level and that it rose 5.1 feet during a six hour period. How far above sea level did the tide register after it rose to the high tide?

   (b) Dave shot at a target and hit 10 inches above the center on the first shot. The second shot hit 3 inches below the first shot. How far above the center was the second shot?

   (c) A submarine that was cruising at 254 feet below sea level rose 78 feet. How far below sea level was it after it rose to the new position?

   (d) A man left a \$50,000 estate. His will stated that his son was to receive twice as much as his daughter and his widow was to receive as much as both together received. How much did each receive?

   (e) Mr. Johnson owed the bank \$200 then had to borrow a small amount again. How much did he owe the bank?

Chapter 8

MULTIPLICATION OF REAL NUMBERS

8-1. Products.

In the last chapter, it was found that the sum of any two real numbers is another real number. Also, addition was defined so that addition of real numbers "behaves" as addition of the numbers of arithmetic behaves. That is, it has the same properties.

In this chapter, we shall try to decide how any two real numbers are multiplied. Again, it would be easier for us if multiplication of real numbers behaved like multiplication of the numbers of arithmetic. That is, we would like the following properties to be true for any real numbers a, b, and c:

$ab = ba$	commutative property of multiplication
$(ab)c = a(bc)$	associative property of multiplication
$(a)(1) = a$	multiplication property of one
$(a)(0) = 0$	multiplication property of zero
$a(b + c) = ab + ac$	distributive property

Notice that we would like these properties to be true for real numbers; that doesn't make them true. The fact that we want them to be true will help us decide how multiplication of real numbers shall be defined.

Here are some possible products of real numbers:

$$(2)(3), \quad (3)(0), \quad (0)(0), \quad (-3)(0), \quad (3)(-2), \quad (-2)(-3).$$

The first three examples use only numbers that are not negative.

As for the fourth one, $(-3)(0)$, if we want the multiplication property of zero to be true, we must be able to say "$(-3)(0) = 0$". So we can make the following definition:

$$(-3)(0) = 0.$$

Of course, if we want the commutative property of multiplication to hold, we must also say:

$$(0)(-3) = 0.$$

In fact, we make the following definition for <u>any real number</u> a:

$$(a)(0) = 0.$$
$$(0)(a) = 0.$$

Discussion Questions 8-1a

1. What have you learned in a previous chapter about the product of two positive numbers?
2. What have you learned about the product of a positive number and zero?
3. What have you learned about the product (0)(0)?
4. What do you know about the product of a negative number and zero?
5. What do you know about the product of zero and a negative number?
6. What is the product of any real number and zero?

Oral Exercises 8-1a

1. Give the common name for each of the following:

(a) $(7)(0)$

(b) $(0)(7)$

(c) $(0)(0)$

(d) $(-3)(0)$

(e) $(0)(-3)$

(f) $(0)(-\frac{1}{3})$

(g) $(0)(-\frac{19}{24})$

(h) $(-\frac{34}{45})(0)$

(i) $(-2.5)(0)$

(j) $(-92.75)(0)$

(k) $(0)(-1.16)$

(l) $(m)(0)$

(m) $(0)(-n)$

(n) $(0)(-\frac{4}{5}n)$

(o) $(-x)(0)$

(p) $(-2.65a)(0)$

(q) $(-5.2)(1.0)$

(r) $(0)(4 + 5)$

(s) $0((-3) + 7)$

(t) $0(8 + (-10))$

(u) $0((-9) + (-3))$

(v) $0((-6) + 6)$

Oral Exercises 8-1a (continued)

2. Which of the following sentences are true? Which are false?

(a) $(0)(5) = 0$

(b) $(5)(0) \neq 0$

(c) $(-7)(0) < 0$

(d) $(0)(-2) > 0$

(e) $(0)(0) = 0$

(f) $(-5)(0) = 0$

(g) $(2.14)(0) > 0$

(h) $(0)(-7.9) < 0$

(i) $(0)(a) = 0$

(j) $(-a)(0) = 0$

(k) $(0)(-n) < 0$

(l) $(0)(m + (-m)) = 0$

(m) $(0)(m + m) > 0$

(n) $((-a) + (-a))(0) < 0$

(o) $((-x) + y)(0) \neq 0$

Next, consider the product $(3)(-2)$.

We can write:

$0 = (3)(0)$	We have already agreed that the product of zero and any real number is zero.
$0 = (3)(2 + (-2))$	If the first statement is true, so is this. "$2 + (-2)$" is just another name for zero.
$0 = (3)(2) + (3)(-2)$	This is what we <u>want</u> to be able to say, since we want the distributive property to hold for all real numbers.
$0 = (6) + (3)(-2)$	Here we have just used the common name "6" for "$(3)(2)$".

The sentences above show things we would like to be able to say. But we see that we can say them only if $(3)(-2)$ is a number that can be added to 6 to get zero. In other words, we must say that $(3)(-2)$ is an <u>additive inverse</u> of 6.

We have already agreed that 6 has only one additive inverse, namely -6. So we will want $(3)(-2)$ to be -6. Since we want the commutative property of multiplication to hold, we also want $(-2)(3)$ to be -6.

We want the product (2)(-7) to be -14. To see why, we could write statements very much like the ones we wrote for (3)(-2). The statements are written below. See if you can give the reasons.

$$0 = (2)(0)$$
$$0 = (2)\big(7 + (-7)\big)$$
$$0 = (2)(7) + (2)(-7)$$
$$0 = 14 + (2)(-7)$$

We see once again that if we want certain properties to hold, then (2)(-7) must be the additive inverse of 14. That is why we want (2)(-7) to be -14. We also want (-7)(2) to be -14.

The things we have said seem to agree with multiplication of numbers of arithmetic in another way. For example,

(3)(2)	=	6	Notice that as the number we multiply 3 by
(3)(1)	=	3	gets smaller, the product gets smaller.
(3)(0)	=	0	This makes it seem natural to define (3)(-1)
(3)(-1)	=	-3	as -3 and (3)(-2) as -6.
(3)(-2)	=	-6	

Following the same line of thinking as above, suggest a common name for each of the following products:

$$(7)(-10), \quad (-\tfrac{1}{2})(42), \quad (1)(-1), \quad (-5)(5).$$

How would you complete this sentence?

The product of a positive number and a negative number is a ______________ number.

Discussion Questions 8-1b

1. What is the sum of 2 and (-2)?
2. What is the additive inverse of the product (3)(2)?
3. When you multiply a positive number by a negative number, the product is what kind of number?
4. What is the sum of a and (-a)?
5. What is the product of 3 and (-a)?

Oral Exercises 8-1b

1. Give the common name for each of the following:
   (a) 6(3)
   (b) 9(0)
   (c) 3(6)
   (d) 0(8)
   (e) 4(-5)
   (f) 8(-3)
   (g) $6(-\frac{1}{2})$
   (h) $\frac{5}{4}(-\frac{1}{3})$
   (i) (-1.5)(8)
   (j) 9(-0.4)
   (k) (-1.6)(.2)
   (l) 0(7.83)
   (m) 0(a)
   (n) 2(m)
   (o) 1(-m)
   (p) a(0)
   (q) (-m)(1)

Problem Set 8-1b

1. Which of the following sentences are true? Which are false?
   (a) (5)(4) = (20)
   (b) (5)(0) = 5
   (c) (0)(5) = 0
   (d) 4(-4) = (16)
   (e) $7(-\frac{2}{3}) = (-\frac{14}{3})$
   (f) 5(-6) = (-30)
   (g) $3(-\frac{7}{8}) = (\frac{7}{24})$
   (h) 2(-2.4) = (2.8)
   (i) 8(-1.1) = (-8.8)
   (j) $1(\frac{3}{8}) = \frac{4}{8}$
   (k) -2(a) = -2a
   (l) m(-1) = -m
   (m) 0(-b) = -b
   (n) 3(-4) = (-4) + 3
   (o) 2(-8) = (-9) + (-7)
   (p) 2(-7) = (-7) + (-7)
   (q) 6(-6) = 0

We have not yet looked at a product of two negative numbers. Let's start with $(-2)(-3)$.

$0 = (-2)(0)$	We have already agreed that the product of zero and any real number is zero.
$0 = (-2)\big(3 + (-3)\big)$	This is the same as the first sentence, except "3 + (-3)" has been used as the name for zero.
$0 = (-2)(3) + (-2)(-3)$	This is what we want to be able to say, since we want the distributive property to hold for all real numbers.
$0 = (-6) + (-2)(-3)$	Here we have used the common name "-6" for "(-2)(3)".

These sentences show that if we want the distributive property to hold for all real numbers, then it turns out that $(-2)(-3)$ must be a number that can be added to -6 to get a sum of zero. In other words, $(-2)(-3)$ must be the additive inverse of -6.

The only additive inverse of -6 is 6. So we see that we want $(-2)(-3)$ to be 6.

The same kind of argument could be given for the product of any two negative numbers. As another example, consider $(-7)(-5)$.

$0 = (-7)(0)$	Why?
$0 = (-7)\big(5 + (-5)\big)$	What name has been used for zero?
$0 = (-7)(5) + (-7)(-5)$	We <u>want</u> this to be true.
$0 = (-35) + (-7)(-5)$	

This argument shows why we want $(-7)(-5)$ to be 35.

The things we have said here seem to agree with what has been done before. For example,

$(-3)(2) = -6$	Notice that as the number we multiply -3
$(-3)(1) = -3$	by gets smaller, the product gets <u>larger</u>.
$(-3)(0) = 0$	This makes it seem natural to define
$(-3)(-1) = 3$	$(-3)(-1)$ as 3 and $(-3)(-2)$ as 6.
$(-3)(-2) = 6$	

Using the examples above as a guide, give a common name for these products:

$$(-2)(-8), \quad (-5)(-5), \quad (-8)(-\tfrac{1}{2}), \quad (-1)(-1).$$

How would you complete this sentence?

The product of a negative number and another negative number is a ____________ number.

Discussion Questions 8-1c

1. What is the product of (-3) and (-4)?
2. When you multiply one negative number by another negative number, the product is what kind of number?
3. Tell in your own words how to multiply one positive number by another positive number.
4. Tell in your own words how to multiply one negative number by another negative number.
5. Tell in your own words how to multiply a negative number by a positive number.

Oral Exercises 8-1c

1. Give the common name for each of the following.

(a) (6)(5)
(b) (-4)(3)
(c) (-3)(4)
(d) (-3)(-4)
(e) (3)(4)
(f) (0)(6)
(g) (8)(0)
(h) (-1)(7)
(i) (-1)(-1)
(j) (-1)(0)
(k) (-3)(b)
(l) (-7)(-b)
(m) (-4)(b)(0)
(n) (0)(-1)(-5)

Problem Set 8-1c

1. Give the common name for each of the following.

(a) $5(8)$
(b) $6(-4)$
(c) $7(0)$
(d) $8(-5)$
(e) $(-6)(4)$
(f) $(3)(-3)$
(g) $(-3)(-3)$
(h) $(-5)(0)$
(i) $(-2)(-7)$
(j) $(-8)(-9)$
(k) $(-3)(-4)$
(l) $(-6)(-\frac{1}{2})$
(m) $(-12)(-\frac{1}{3})$
(n) $(-9)(-0)$
(o) $(-\frac{1}{4})(-\frac{1}{3})$
(p) $(-.5)(-8)$
(q) $(-.7)(-5)$
(r) $(-.6)(-.3)$
(s) $(-4)(-.2)$

2. Which of the following sentences are true? Which are false?

(a) $5(10) = 50$
(b) $(6)(0) = 6$
(c) $1(-5) = 5$
(d) $3(-4) = -12$
(e) $(8)(0) = 0$
(f) $(-9)(0) = 0$
(g) $(-2)(-3) = 6$
(h) $(-5)(-4) = -20$
(i) $(-1)(-10) = 10$
(j) $(-1)(1) = 0$
(k) $(-0)(-5) = 5$
(l) $(-8)(-7) = 56$
(m) $(-\frac{1}{3})(-\frac{1}{5}) = (-\frac{1}{15})$
(n) $(-2\frac{1}{2})(-2) = 5$
(o) $(-\frac{3}{4})(-\frac{2}{5}) = \frac{5}{20}$
(p) $(-1.5)(-3) = 4.5$
(q) $(-.6)(-.4) = (-.24)$
(r) $(-4)(-3) = (-4) + (-3)$
(s) $(-5)(a) = 5a$

You remember that we were able to give a definition for the sum of any two real numbers. We are now ready to give a definition for the product of any two real numbers. The following examples will guide us in making a definition.

(5)(8)
- The absolute value of 8 is 8.
- The absolute value of 5 is 5.

The product of the absolute values is (5)(8), or 40.
(5)(8) is the same as the product of the absolute values.

(5)(-8)
- The absolute value of -8 is 8.
- The absolute value of 5 is 5.

The product of the absolute values is (5)(8), or 40.
(5)(-8) is the <u>opposite</u> of this product, or -40.

(-5)(-8)
- The absolute value of -8 is 8.
- The absolute value of -5 is 5.

The product of the absolute values is (5)(8), or 40.
(-5)(-8) is the same as this product.

Now, here is a definition for the product of any two real numbers:

The product of any real number and zero is zero.

The product of two positive numbers is the product of their absolute values.

The product of two negative numbers is the product of their absolute values.

The product of a positive number and a negative number is the opposite of the product of their absolute values.

Use the following products to show that the definition says exactly what we want it to say:

$(12)(0), \quad (16)(\frac{1}{4}), \quad (-12)(-3), \quad (-12)(\frac{1}{2}).$

Discussion Questions 8-1d

1. What is the absolute value of 8?
2. What is the absolute value of (-8)?
3. What is the absolute value of a?
4. What is the absolute value of -a?
5. What is the absolute value of any real number a?
6. What is the product of (12)(0)?
7. What is the product of (0)(a)?
8. State in your own words: $(a)(b) = |a| \cdot |b|$
9. State in your own words: $(-a)(-b) = |-a| \cdot |-b|$
10. State in your own words: $(-a)(b) = -(|-a| \cdot |b|)$
11. State in your own words: $(a)(-b) = -(|a| \cdot |-b|)$

Problem Set 8-1d

1. Find the common name for each of the following:
   (a) $2 \cdot |-5|$
   (b) $3 \cdot |4|$
   (c) $|5| \cdot |-4|$
   (d) $|-3| \cdot |-2|$
   (e) $|15| \cdot |8|$
   (f) $0 \cdot |-9|$
   (g) $|-1| \cdot |-7|$
   (h) $0 \cdot |4|$

2. Which of the following sentences are true? Which are false?
   (a) $2 \cdot |4| = 2 \cdot 4$
   (b) $|3| \cdot |-4| = -(3 \times 4)$
   (c) $(-5)(6) = -|5 \times 6|$
   (d) $|2| \cdot |6| = -(2 \times 6)$
   (e) $|-5| \cdot 4 = -20$
   (f) $|-2| \cdot |-6| = -(2 \cdot 6)$
   (g) $(8)(3) = |8 \times 3|$
   (h) $(7)(-4) = -(|7| \cdot |-4|)$
   (i) $0(-5) = 0 \cdot |5|$
   (j) $(-5)(-5) = |-5| \cdot |-5|$
   (k) $(-4)(-6) = -(|-4| \cdot |-6|)$
   (l) $(-3.5)(2) = -(|-3.5| \cdot |2|)$
   (m) $(-\frac{9}{2})(-\frac{1}{4}) = -(|-\frac{9}{2}| \, |-\frac{1}{4}|)$
   (n) $|4| \cdot |-3| = 1$
   (o) $|-6| \cdot |-5| = 20 + 10$
   (p) $|-5| \cdot |4| = (-25) + 5$
   (q) $|3| \cdot |-8| = 30 + (-6)$
   (r) $(-1)|-a| = a$
   (s) $|-3| \cdot |b| = -3b$
   (t) $|-6| \cdot |-m| = 6m$
   (u) $(-5)|-b| = -5b$

Problem Set 8-1d
(continued)

3. Given the set $S = \{-3, -2, -1, 0, 1, 2, 3\}$.
   (a) Find the set of all possible products of pairs of elements of the set S.
   (b) Is the set S closed under the operation of multiplication?
   (c) Is the set of all of the integers closed under the operation of multiplication? Can you think of two integers whose product is not an integer?
4. Given the set $R = \{-2, -1, -\frac{1}{2}, 0\}$.
   (a) Find the set of all possible products of pairs of elements of the set R.
   (b) Is the set R closed under the operation of multiplication?
   (c) Is the set of all of the negative real numbers closed under the operation of multiplication? Can you think of two negative numbers whose product is a negative number?
5. Given the set $A = \{\ldots -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$.
   (a) Find the set of all possible products of pairs of elements of the set A.
   (b) Is the set A closed under the operation of multiplication?
   (c) Is the set of all of the real numbers closed under the operation of multiplication? Can you think of two real numbers whose product is not a real number?

8-2. Properties of Multiplication.

We have discussed properties many times. And we have seen that properties of numbers are important in learning how numbers behave. Now that we have defined multiplication of real numbers, we must be sure that the properties of multiplication that we listed for the numbers of arithmetic are true for the entire set of real numbers.

To begin with, let's consider the multiplication property of one. Is it true that $a \cdot 1 = a$, for any real number a?

We already know that $a \cdot 1 = a$ if a is a positive number or zero, since such numbers are numbers of arithmetic. You may be willing to believe that it is true if a is a negative number. But it can actually be proved to be true, just from the definition we have made for multiplication.

For example, suppose we want to prove that $(-4)(1) = -4$. (You might imagine you were trying to show somebody who didn't believe it--there's nearly always a doubter in the crowd.)

By definition, the product $(-4)(1)$ is the opposite of the product of the absolute values of -4 and 1.

The absolute value of -4 is 4.

The absolute value of 1 is 1.

The product of these absolute values is $4 \cdot 1$, or 4.

So the product $(-4)(1)$ is the opposite of 4, or -4.

That is, $(-4)(1) = -4$.

Of course, we have worked with just one case, $(-4)(1)$. But we can prove that $a \cdot 1 = a$, for any negative number a.

Since a is negative, by definition the product $a \cdot 1$ is the opposite of the product of the absolute values of a and 1.

The absolute value of a may be written $|a|$.

The absolute value of 1 is 1.

The product of these absolute values is $1 \cdot |a|$.
But $|a|$ is a number of arithmetic. So we know $1 \cdot |a| = |a|$. Then the product of the absolute values is $|a|$.

The product $a \cdot 1$ is the opposite of $|a|$, or $-|a|$.
Since a is negative, $|a| = -a$. That is, a and $|a|$ are opposites. So, $a = -|a|$.

Therefore, the product $a \cdot 1 = a$.

This shows that the multiplication property of one is true for negative numbers as well as for positive numbers and zero. So we can write:

For any real number a, $a \cdot 1 = a$.

Below are some other properties which are true for the entire set of real numbers. Each one of these properties could be proved to even the most stubborn doubter. They could be proved from our definition of multiplication, just as we proved the multiplication property of one. However, the proofs are long in most cases; so we just list them here with some examples.

Commutative Property of Multiplication

For any real numbers a and b, $ab = ba$.

Examples:

$$(5)(-4) = (-4)(5)$$
$$(0)(86) = (86)(0)$$
$$(-7)(-5) = (-5)(-7)$$

Associative Property of Multiplication

For any real numbers a, b, and c, $(ab)c = a(bc)$.

Examples:

$$\big((3)(2)\big)(-4) = (3)\big((2)(-4)\big)$$
$$\big((7)(-3)\big)(-2) = (7)\big((-3)(-2)\big)$$
$$\big((-2)(-3)\big)(-5) = (-2)\big((-3)(-5)\big)$$

Distributive Property

For any real numbers a, b, and c, $a(b + c) = ab + ac$.

Examples:

$$5\big(2 + (-3)\big) = 5(2) + 5(-3)$$
$$5\big((-2) + (-3)\big) = 5(-2) + 5(-3)$$
$$(-5)\big((-2) + (-3)\big) = (-5)(-2) + (-5)(-3)$$

Discussion Questions 8-2a

1. Is $a \cdot 1 = a$ a true sentence if a is a positive number?
2. Is $a \cdot 1 = a$ a true sentence if a is zero?
3. What is the absolute value of (-4) ?
4. What is the absolute value of 1 ?
5. What is the product of the absolute values of (-4) and 1 ?
6. What is the opposite of the product obtained in Question 5?
7. What is the product of any real number and one?
8. Is the sentence "$5(-4) = (-4)(5)$" true?
9. Is "$ab = ba$" true for any real numbers a and b?
10. Is "$\left((7)(-3)\right)(-2) = (7)\left((-3)(-2)\right)$" a true sentence?
11. Is "$(ab)c = a(bc)$" true for any real numbers a, b and c?
12. Is "$5\left(2 + (-3)\right) = 5(2) + 5(-3)$" true?
13. Is "$a(b + c) = ab + ac$" true for any real numbers a, b and c?

Oral Exercises 8-2a

1. Give the common name for each of the following.

(a) $(4)(1)$

(b) $(1)(-5)$

(c) $(1)(12)$

(d) $(1)(-10)$

(e) $(-9)(1)$

(f) $(-7)(2)(1)$

(g) $(3)(1)(6)$

(h) $(5)(-3)(1)$

(i) $(-6)(1)(-4)$

(j) $(-3)(2)(1)(0)$

(k) $(1.5)(-2)(1)$

(l) $(-2)(1)(-\frac{1}{2})$

(m) $(7.8)(-4.5)(1)(0)$

(n) $(-b)(1)(3)$

(o) $(-a)(1)(-b)(0)$

(p) $\frac{1}{2}\left(6 + (-4)\right)$

(q) $(-\frac{2}{3})(5 + 4)$

(r) $\left((-8) + (-10)\right)\frac{1}{3}$

(s) $\left((-9) + (-3)\right)(-\frac{2}{3})$

Problem Set 8-2a

1. Tell what property or properties (associative or commutative property of multiplication, distributive property, multiplication property of one or multiplication property of zero) are illustrated in each of the following true sentences.

   (a) $(2)(3) = (3)(2)$
   (b) $(-2)(5) = (5)(-2)$
   (c) $\big((-4)(5)\big)(-3) = (-4)\big((5)(-3)\big)$
   (d) $\big((-3)(a)\big)(-2) = (-3)\big((-2)(a)\big)$
   (e) $(a)(3) = (1)(a)(3)$
   (f) $(-2)(a)(1) = (-2)(a)$
   (g) $(-4)(b)(1) = (-4)(b)$
   (h) $(3)(-m)(0) = (3)(-m)(0)$
   (i) $(2)(-b)(0) = (0)(-b)(2)$
   (j) $(-3)(0)(1) = (-3)(1)(0)$
   (k) $2\big(3 + (-4)\big) = (2)(3) + (2)(-4)$
   (l) $(-3)\big((-1) + 2\big) = (-1)(-3) + 2(-3)$
   (m) $(-5)\big(a + (-b)\big) = \big(a + (-b)\big)(-5)$

2. Find the common name for each of the following by using the properties of multiplication.

   (a) $(-\frac{1}{2})(-4)$
   (b) $(-\frac{1}{2})(2)(-5)$
   (c) $(-\frac{3}{2})(-4)(\frac{2}{3})(1)$
   (d) $(-\frac{1}{3})(3)(7)$
   (e) $(-\frac{1}{3})(-3)(-7)$
   (f) $(4)(-6)(1)(\frac{7}{4})$
   (g) $|-3|(-2)(4)$
   (h) $|-3||-2|(-6)$
   (i) $(-3)|-2|(4)$
   (j) $(-2)^2(3)$
   (k) $\big((-2)(3)\big)^2$
   (l) $(|-3|)^2$
   (m) $\big((-3) + 2\big)^2$
   (n) $(-3)^2 + (2)^2$
   (o) $(3)(-2.1)(0)(7.3)(1)$
   (p) $(-4)(b)(1)(2c)(0)$
   (q) $9\big(4 + (-2)\big)$
   (r) $(-3)\big((-7) + 4\big)$
   (s) $\frac{2}{3}\big((-6) + (-3)\big)$
   (t) $\big((-\frac{5}{3}) + (-\frac{1}{3})\big)(-\frac{1}{2})$
   (u) $\big((-9.25) + 6.5\big)(0)$

Problem Set 8-2a
(continued)

3. Write each of the following indicated products as an indicated sum:

(a) $2(3 + (-2))$
(b) $(-3)((-4) + (-6))$
(c) $4((1) + (-5))$
(d) $(1)((-5) + (-10))$
(e) $(1)(0 + (-1))$
(f) $(1)((-1) + 1)$
(g) $5((-6) + (-2))$
(h) $(-6)(10 + (-3))$
(i) $(6 + (-2))(4)$
(j) $((-8) + (-2))(-7)$
(k) $((1.2) + (-1.1))6$
(l) $((-\frac{2}{3}) + (\frac{3}{4}))(-\frac{2}{5})$

4. Write the following indicated sums as indicated products:

(a) $2(5) + 2(4)$
(b) $(-3)(7) + (-3)(3)$
(c) $(4)(-6) + 4(-9)$
(d) $(-7)(-5) + (-7)(-4)$
(e) $(-8)(-3) + (5)(-3)$
(f) $(7)(-9) + (-6)(7)$
(g) $(1)(4) + (1)(6)$
(h) $(1)(-3) + (1)(9)$
(i) $(-2)(a) + (-2)(b)$
(j) $(3)(a) + (4)(y)$

What is the product of 5 and -1? It is easy to see that the product is -5. We could say that when 5 is multiplied by -1, the product is the <u>opposite of 5</u>.

Check the following true sentences:

$$(-1)(7) = -7$$
$$(-1)(3) = -3$$
$$(-1)(-8) = 8$$
$$(-1)(-2) = 2$$

Notice that in each of these cases, a number was multiplied by -1; the product turned out to be the <u>opposite</u> of that number. In the last sentence, for example, -2 was multiplied by -1; the product, 2, is the opposite of -2.

From these examples, you might have a hunch that if any real number is multiplied by -1, the product is the opposite of that number. We can use the distributive property to prove that this is really the case.

We are trying to prove that $(-1)a = -a$.

This mean that we must show that $(-1)a$ is the opposite of a, that is, that

$$a + (-1)a = 0.$$

$a + (-1)a = 1(a) + (-1)a$	Here we have used "1(a)" for "a". We have the right to do this, because of the multiplication property of 1.
$= \big(1 + (-1)\big)a$	Here the distributive property has been used.
$= (0)a$	$1 + (-1)$ is the same as 0, since the sum of a number and its opposite is zero.
$= 0$	We have agreed that the product of any number and zero is zero.

We have now shown that, for any real number a, if $(-1)a$ is added to a, the sum is zero. In other words, $(-1)a$ is the additive inverse of a, or $-a$ More briefly, we can say:

For any real number a, $(-1)a = -a$.

Discussion Questions 8-2b

1. What is the product of 5 and (-1)?
2. What is the product of (-2) and (-1)?
3. What is the product of a and 1?
4. What is the product of a and (-1)?
5. What is the product of any real number and (-1)?
6. What is the additive inverse of a?

Oral Exercises 8-2b

1. Give the common name for each of the following:

(a) $(1)(5)$
(b) $(1)(-6)$
(c) $6(-1)$
(d) $(-1)(3)$
(e) $(-4)(-1)$
(f) $(-1)(0)$
(g) $(0)(-1)$
(h) $(-1)(b)$
(i) $(m)(-1)$
(j) $(-1)(-n)$
(k) $(-t)(-1)$
(l) $(-1)(5)(6)$
(m) $(-7)(5)(-1)$
(n) $(-4)(-1)(3)$
(o) $(-\frac{3}{4})(\frac{1}{2})(-1)$
(p) $(-\frac{7}{3})(-\frac{7}{8})(-1)$
(q) $(\frac{212}{325})(\frac{91}{245})(0)(-1)$
(r) $(-1.5)(-1)(2)$
(s) $(-3.2)(-2)(-1)$
(t) $(-5.1)(a)(-1)$
(u) $(2.3)(-1)(b)$

8-3. Using the Multiplication Properties.

Now that we know how to multiply any two real numbers and we have some important properties of multiplication, we can work with many kinds of open phrases in algebra.

Here are some examples:

Example 1. Use the distributive property to multiply $(-6)\big(2 + (-x)\big)$.

$$(-6)\big(2 + (-x)\big) = (-6)(2) + (-6)(-x)$$
$$= -12 + 6x$$

Do you see why $(-6)(-x)$ is the same as $6x$?
It could be written out this way:

$$(-6)(-x) = (-6)\big((-1)(x)\big)$$
$$= \big((-6)(-1)\big)(x)$$
$$= (6)(x)$$

Example 2. Use the distributive property to write $xy + x$ as a product.

$xy + x = x(y + 1)$

Do you see that the distributive property tells us this sentence is true for any real numbers x and y?

We have written $xy + x$ as the product $x(y + 1)$, which may be thought of as the product of the number x and the number $(y + 1)$.

Example 3. Use the distributive property to "simplify" the open phrase "$5x + (-2)x$."

$$5x + (-2)x = \left(5 + (-2)\right)x$$
$$= (3)x$$
$$\text{or } 3x$$

You will probably agree that the open phrase "3x" seems "simpler" than "$5x + (-2)x$."

Problem Set 8-3a

1. Use the distributive property, as in Example 1, to multiply the following:
   (a) $5(3 + a)$
   (b) $4\left((-3) + b\right)$
   (c) $(-2)\left((-3) + (-b)\right)$
   (d) $(4)\left((-2) + (-c)\right)$
   (e) $(-3)(a + b)$
   (f) $(-2)\left(a + (-b)\right)$
   (g) $(-5)\left((-m) + (-n)\right)$
   (h) $(-4)\left((-2.1a) + 1.2b\right)$
   (i) $m\left((-a) + (-c)\right)$

Problem Set 8-3a
(continued)

2. Use the distributive property to write the following indicated sums as indicated products as in Example 2:

(a) $2a + 2b$

(b) $(-5)(a) + (-5)(m)$

(c) $(-2)(b) + (-2)(c)$

(d) $(-3)(c) + (-4)(c)$

(e) $(-1)(m) + (-1)(-n)$

(f) $(-2)(m) + (-1)(m)$

(g) $ab + am$

(h) $xy + ry$

(i) $mr + m$

(j) $(-\frac{2}{3})(m) + (-\frac{2}{3})(n)$

(k) $1.5a + 1.5b$

(l) $2m + (-4)n$ (Hint: $(-4)n$ may be written as $(-2)(2)n$.)

(m) $6a + 9b$

(n) $9x + (-12y)$

(o) $(-10m) + (-15n)$

(p) $6a + (-15b)$

(q) $9ax + 12ay$

(r) $(-4ac) + (-6ab)$

(s) $m + m^2$ (Hint: m^2 can be written $(m)(m)$)

(t) $a^2 - 2a$

(u) $(-5m) + 6n$

In Example 3 of the section before this one, the distributive property was used to write "$5x + (-2)x$" as "$3x$."

In an expression like "$5x + (-2)x$," "$5x$" and "$(-2)x$" are called <u>terms</u> of the phrase. When we simplify "$5x + (-2)x$" to "$3x$," we often say that we are <u>collecting terms</u>.

Below you will find some other examples of simplifying by collecting terms.

<u>Example 1</u>. Simplify "$(-4)n + 20n$."

$$(-4)n + 20n = ((-4) + 20)n$$
$$= 16n$$

<u>Example 2</u>. Simplify "$8x + 3y$."

Do you see why the terms in this phrase cannot be collected? We can say that the phrase "$8x + 3y$" is already in simplest form.

Example 3. Simplify "$10x + (-2)y + 7x + 14y$."

Using the associative and commutative properties, we can rearrange the terms, like this:

$$10x + 7x + (-2)y + 14y.$$

Then, using the distributive property, we have

$$(10 + 7)x + ((-2) + 14)y, \quad \text{or}$$

$$17x + 12y.$$

In this example, the work could also have been shown like this:

$$\begin{array}{rr} 10x + & (-2)y \\ 7x + & 14y \\ \hline 17x + & 12y \end{array}$$

Example 4. Simplify "$4r + (-2z) + (-7)r + 8z$."

Two ways of showing the simplification are given below:

$$\begin{array}{l} 4r + (-2z) + (-7r) + 8z \\ (4r + (-7)r) + ((-2)z + 8z) \\ (4 + (-7))r + ((-2) + 8)z \\ (-3)r + 6z \end{array} \qquad \begin{array}{rr} 4r + & (-2)z \\ (-7)r + & 8z \\ \hline (-3)r + & 6z \end{array}$$

No matter which way you show the work, it is important to understand the properties that give you the right to collect terms as you do.

Discussion Questions 8-3b

1. What do we call an expression such as "$5x + (-2)x$"?
2. What do we mean by a term of "$5x + (-2)x$"?
3. When we write "$5x + (-2x)$" as "$3x$", what do we call the process?
4. What gives us the right to write "$5x + (-2x)$" as "$3x$"?

Problem Set 8-3b

1. Simplify the following expressions by collecting terms as in the examples.

   (a) 6m + 4m
   (b) 7a + (-3)a
   (c) 9a + (-15)a
   (d) (-5)y + 14y
   (e) (-3)m + (-6)m
   (f) 4a + 3b
   (g) (4a + a) (Hint: a may be written as (1)a.)
   (h) (-5)x + 2y
   (i) (-6)a + a
   (j) (-a) + (-4)a (Hint: (-a) may be written as (-1)a.)
   (k) 2t + (-4)w + 3t + (-2)w
   (l) (-5)a + (-2)b + 6a + 5b
   (m) (-2)m + 6b + 2m + (-2)b
   (n) y + 6x + (-7)x + (-y)
   (o) 8a + (-5)a + 2a + (-3)b
   (p) (-2)m + (-3)n + 6m + m
   (q) (-4)a + 5a + (-3)a + 7a
   (r) (-3)y + 2x + 4z + (-3)w

We have already seen how the properties of multiplication allow us to simplify phrases by collecting terms. There are other ways to simplify phrases, however.

For example, consider the phrase "(3x)(2x)."

$(3x)(2x) = 3 \cdot x \cdot 2 \cdot x$	The associative property tells us that we may associate in any way; so the symbols of grouping may be omitted.
$= 3 \cdot 2 \cdot x \cdot x$	The commutative property gives us the right to change the order of the numbers.
$= (3 \cdot 2) \cdot (x \cdot x)$	The associative property allows us to group the numbers in this way.
$= 6x^2$	Here we have just used the common name "6" for "3·2" and the simpler name "x^2" for "x·x".

All of the steps above are important, because they show how the properties of multiplication allow us to make the simplification. It is not necessary to write all the steps each time, however. It is possible to do most of the work in you head and just write:

$$(3x)(2x) = 6x^2.$$

Here are two other examples:

<u>Example 1</u>. Simplify (3xy)(-7ay)."

$$\begin{aligned}(3xy)(-7ay) &= 3 \cdot x \cdot y \cdot (-7) \cdot a \cdot y \\ &= 3 \cdot (-7) \cdot a \cdot x \cdot y \cdot y \\ &= \big(3 \cdot (-7)\big) \cdot a \cdot x \cdot (y \cdot y) \\ &= (-21)axy^2\end{aligned}$$

<u>Example 2</u>. Simplify "(-2ax)(-7ax)."

$$(-2ax)(-7ax) = 14a^2x^2$$

Discussion Questions 8-3c

1. What property or properties are used in multiplying $(2x)(3x)$?
2. Can you write "$3\cdot2\cdot a\cdot a$" in a shorter form?

Oral Exercises 8-3c

1. Simplify each of the following phrases:

(a) $(2)(2m)$
(b) $(-3)(4xy)$
(c) $(6)(-2a)$
(d) $(-2)(-4st)$
(e) $(3a)(3b)$
(f) $(-2m)(8n)$
(g) $(4x)(-7y)$
(h) $(-5a)(-3b)$
(i) $(-1)(m)$
(j) $(-b)(4c)$ (Hint: $(-b)$ can be written $(-1)b$.)
(k) $(6a)(-c)$
(l) $(-5m)(-n)$
(m) $(0)(-5xy^2)$
(n) $(6am^2)(2m)(0)$
(o) $(2ax)(-by)$
(p) $(-4am)(-3am)$

Problem Set 8-3c

1. Simplify each of the following phrases as in Examples 1 and 2.

(a) $(2a)(4am)$
(b) $(-4y)(3y)$
(c) $(3b)(-2ab)$
(d) $(-4cd)(-6d)$
(e) $(-3c^2)(d)$
(f) $(-x)(6ay)$
(g) $(-c)(-4abd)$
(h) $(2ax)(-6by)$
(i) $(-3mn)(-7amn)$
(j) $(abc)(abxy)$
(k) $(am)(-9mx^2)$
(l) $(-9a^2m)(-8mn^2)$
(m) $(\frac{4}{3}m^2)(-\frac{3}{4}n^2)$
(n) $(-\frac{2}{3}ab^2)(-\frac{4}{5}ac^2)$
(o) $(-1.5c)(-3b)$
(p) $(-4.75d)(3.12cdm^2)(0)$

Now we can use the distributive property to work with more complicated phrases. Study the following example:

By the distributive property,

$$\begin{aligned}(-3a)\big(2a + (-5c)\big) &= (-3a)(2a) + (-3a)(-5c)\\ &= -6a^2 + 15ac\end{aligned}$$

In working with some phrases, it is useful to remember that, for any real number a,

$$-a = (-1)a.$$

Following are two examples.

Example 1.

$$\begin{aligned}-(x + 2) &= (-1)(x + 2)\\ &= (-1)(x) + (-1)(2)\\ &= -x + (-2)\end{aligned}$$

Example 2.

$$\begin{aligned}-\big(x^2 + (-7x) + (-6)\big) &= (-1)\big(x^2 + (-7x) + (-6)\big)\\ &= (-x^2) + 7x + 6.\end{aligned}$$

Problem Set 8-3d

Use the distributive property to simplify the following:

1. $3x(2x + 4z)$
2. $6x\big((-3a) + 2b\big)$
3. $(-2m)\big(m + (-3n)\big)$
4. $(-3m)\big((-x) + (-y)\big)$
5. $(-5m)\big((-4a) + (-1)c\big)$
6. $a\big(2 + (-3)\big)$
7. $a\big(b + (-2c)\big)$
8. $(-m)\big((-c) + (-d)\big)$
9. $-\big((-b) + c\big)$
10. $-(4x + 3y)$
11. $-\big((-2x) + (3z)\big)$
12. $-\big((-4c) + (-6d)\big)$
13. $(-\frac{2}{3}c)(6a + 9b)$
14. $\frac{5}{2}a\big((-4m) + 6n\big)$
15. $(-1.5m)\big((-2)mx + (-3)my\big)$
16. $(-1.4c)\big((-3)ab + 3ab\big)$
17. $-(a + b + c)$
18. $-\big(b + (-c) + m\big)$
19. $-\big(b + (-d) + (-t)\big)$
20. $(-2m)\big((-2)x + 3y + (-4)z\big)$

Sometimes the distributive property is used more than once in working with a phrase. Below are two examples.

Example 1. Multiply "$(x + 3)(x + 2)$."

$(x + 3)(x + 2)$	$= (x + 3)x + (x + 3)2$	by the distributive property
	$= x^2 + 3x + 2x + 6$	by the distributive property
	$= x^2 + (3 + 2)x + 6$	by the distributive property
	$= x^2 + 5x + 6$	

Example 2. $\big(a + (-7)\big)(a + 3) = \big(a + (-7)\big)a + \big(a + (-7)\big)3$

$$= a^2 + (-7)a + 3a + (-21)$$

$$= a^2 + \big((-7) + 3\big)a + (-21)$$

$$= a^2 + (-4)a + (-21)$$

Problem Set 8-3e

Use the distributive property to simplify the following phrases.

1. $(a + 3)(a + 2)$
2. $\big(a + (-2)\big)\big(a + (-3)\big)$
3. $\big(a + (-2)\big)\big(a + (-3)\big)$
4. $(b + 4)(b + 6)$
5. $\big(c + (-5)\big)(c + 7)$
6. $(m + 8)\big(m + (-1)\big)$
7. $\big(m + (-5)\big)\big(m + (-4)\big)$
8. $(m + 1)(m + 1)$
9. $\big(t + (-1)\big)(t + 1)$
10. $(x + 3)(x + 3)$
11. $\big(a + (-5)\big)\cdot(a + 5)$
12. $\big(m + (-3)\big)\big(m + (-3)\big)$
13. $(a + 5)(a + 5)$
14. $\big(a + (-5)\big)\big(a + (-5)\big)$
15. $(b + 2)(b + 2)$
16. $\big(4 + (-b)\big)(2 + b)$
17. $\big(6 + (-a)\big)\big(3 + (-a)\big)$
18. $\big(5 + (-a)\big)(6 + a)$
19. $(2a + 3)(4a + 5)$
20. $(4m + 2n)(3m + n)$
21. $(a + b)(c + d)$

8-4. Multiplicative Inverse.

Find the common name for these products:

$$4 \cdot \frac{1}{4}$$

$$9 \cdot \frac{1}{9}$$

$$\frac{1}{7} \cdot 7$$

$$\frac{2}{3} \cdot \frac{3}{2}$$

In each case, the common name for the product is "1".

Find the truth set of the following open sentences:

$$14a = 1$$

$$2n = 1$$

$$\frac{1}{5}(t) = 1$$

$$\frac{5}{2}(a) = 1$$

In each case, the truth set has only one element.

The examples above used only numbers of arithmetic. However, we can also work with the set of real numbers and use some negative numbers. For example, consider "$(-4)(-\frac{1}{4})$".

The absolute value of -4 is 4. The absolute value of $-\frac{1}{4}$ is $\frac{1}{4}$. The product of 4 and $\frac{1}{4}$ is 1, as we just saw.

Since -4 and $-\frac{1}{4}$ are both negative numbers, their product is the same as the product of their absolute values. That is,

$$(-4)(-\frac{1}{4}) = 1.$$

Find the truth set of these open sentences:

$$(-2)n = 1$$

$$(-\frac{1}{5})t = 1 .$$

Again, the truth set has only one element. The following oral exercises will give you a chance to work with some other numbers.

Oral Exercises 8-4

1. Find the common name for each of the following:

   (a) $5 \cdot \frac{1}{5}$
   (b) $3 \cdot \frac{1}{3}$
   (c) $\frac{1}{4} \cdot 4$
   (d) $\frac{1}{8} \cdot 8$
   (e) $\frac{3}{4} \cdot \frac{4}{3}$
   (f) $(-\frac{1}{6})(-6)$
   (g) $(-7)(-\frac{1}{7})$
   (h) $(-\frac{5}{6})(-\frac{6}{5})$
   (i) $(-2\frac{1}{2})(-\frac{2}{5})$

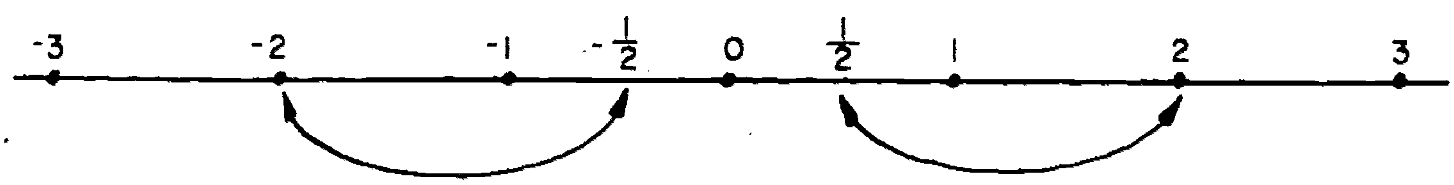
2. Give the truth set of each of the following sentences:

   (a) $4n = 1$
   (b) $(-3)n = 1$
   (c) $(-\frac{1}{3})a = 1$
   (d) $\frac{1}{5}b = 1$
   (e) $\frac{4}{5}m = 1$
   (f) $(-\frac{4}{7})m = 1$
   (g) $\frac{5}{2}x = 1$
   (h) $(-\frac{4}{3})y = 1$
   (i) $(2\frac{1}{4})b = 1$
   (j) $(0)b = 1$

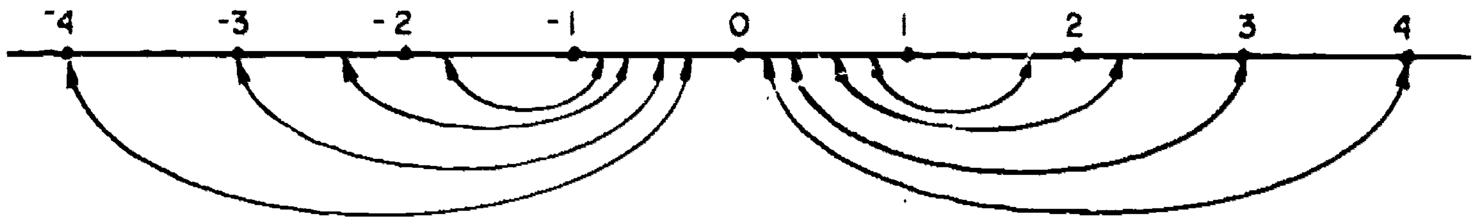
We have been "pairing off" numbers whose product is one. For example,

$\frac{1}{2}$ was paired with 2, because $(\frac{1}{2})(2) = 1$.
-2 was paired with $-\frac{1}{2}$, because $(-2)(-\frac{1}{2}) = 1$.

On the number line, these pairings can be pictured like this:



Below are pictured some other pairings of numbers whose product is one:



A positive number is paired with a positive number. A negative number is paired with a negative number. It would be impossible to find a pair, one positive and one negative, whose product is one. Why is this impossible?

Notice once again that the product of the numbers in each pair is one. There is a special name to describe the numbers in such a pair. The name is multiplicative inverse. For example,

$\frac{1}{2}$ is the multiplicative inverse of 2, because $(2)(\frac{1}{2}) = 1$.

2 is the multiplicative inverse of $\frac{1}{2}$, because $(\frac{1}{2})(2) = 1$.

$-\frac{1}{4}$ is the multiplicative inverse of -4, because $(-4)(-\frac{1}{4}) = 1$.

-4 is the multiplicative inverse of $-\frac{1}{4}$, because $(-\frac{1}{4})(-4) = 1$.

In fact, for any pair of numbers c and d such that

$$(c)(d) = 1,$$

d is the multiplicative inverse of c, and
c is the multiplicative inverse of d.

You may have noticed that at no time have we paired zero with a number. If there were a number x to pair zero with, then it would have to be true that

$$(0)(x) = 1,$$

but there is no such number x. Why?

We can say then:

The multiplicative inverse of a positive number is a positive number.

The multiplicative inverse of a negative number is a negative number.

Zero has no multiplicative inverse.

Discussion Questions 8-4

1. What is the multiplicative inverse of 2?
2. What is the multiplicative inverse of $\frac{1}{2}$?
3. What is the multiplicative inverse of (-4)?
4. What is the multiplicative inverse of $(-\frac{1}{4})$?
5. What is the multiplicative inverse of (-2)?
6. What is the multiplicative inverse of $(-\frac{1}{2})$?
7. What is the product of (-2) and $(-\frac{1}{2})$?
8. What is the product of $(-\frac{1}{4})$ and (-4)?

Oral Exercises 8-4

1. What is the multiplicative inverse of 1?
2. What is the multiplicative inverse of (-1)?
3. What is the multiplicative inverse of a?
4. What is the multiplicative inverse of $(-a)$?
5. What is the multiplicative inverse of $\frac{1}{b}$?
6. What is the multiplicative inverse of $(-\frac{1}{b})$?
7. Is it possible for the multiplicative inverse of a to be any number other than $\frac{1}{a}$?
8. Is it possible for a real number to have more than <u>one</u> multiplicative inverse?
9. What is the multiplicative inverse of zero?
10. Why does zero have no multiplicative inverse?
11. Is the multiplicative inverse of a positive real number always a positive real number?
12. Is the additive inverse of a positive real number always a positive real number?
13. How would you write the multiplicative inverse of any number?
14. What is the additive inverse of a?
15. What is the additive inverse of $(-a)$?
16. Does every real number have a multiplicative inverse? an additive inverse?

Problem Set 8-4

Which of the following sentences are true? Which are false?

1. $4(\frac{1}{4}) = 1$
2. $(-3)(\frac{1}{3}) = 1$
3. $(-5)(-\frac{1}{5}) = 1$
4. $(6)(-\frac{1}{6}) = 1$
5. $(\frac{2}{3})(\frac{3}{2}) = 1$
6. $(-\frac{1}{3})(3) = 1$
7. $(3\frac{1}{2})(-\frac{2}{7}) = 1$
8. $(-2\frac{1}{3})(-\frac{3}{7}) = 1$
9. $(-\frac{5}{1})(5) = 1$
10. $(-\frac{5}{1})(-\frac{1}{5}) = 1$
11. $(-\frac{4}{1})(0) = 1$
12. $(\frac{17}{1})(\frac{1}{17}) = 1$
13. $(\frac{1}{a})(a) = 1$
14. $(-a)(\frac{1}{a}) = 1$
15. $(-\frac{1}{a})(-a) = 1$
16. $(\frac{1}{8})(1) = 8$
17. $(\frac{1}{a})(1) = a$

8-5. Multiplication Property of Equality.

"(2)(6)" is the name of a number. If the number is multiplied by 2, we get a new number whose name may be written "$(2)\big((2)(6)\big)$."

"(4)(3)" is the name of a number. If the number is multiplied by 2, we get a new number whose name may be written "$(2)\big((4)(3)\big)$."

In both cases, we were really doing the same thing. We started with the number 12. We multiplied by 2. So, of course, both times we got the number 24. In other words,

since $(2)(6) = (4)(3)$ is true,
then $(2)\big((2)(6)\big) = (2)\big((4)(3)\big)$ is true.

This is an example of the multiplication property of equality.

Here are some other examples:

Example 1. Since $(2)(-3) = -6$,
then $(\frac{1}{2})\big((2)(-3)\big) = (\frac{1}{2})(-6)$.

Example 2. Since $(-3)(8) = (6)(-4)$,
then $(-\frac{1}{3})\big((-3)(8)\big) = (-\frac{1}{3})\big((6)(-4)\big)$.

The multiplication property of equality may be written like this:

<u>For real numbers a, b, and c,</u> if $a = b$, then $ac = bc$.

Of course, it may also be written:

<u>For real numbers a, b, and c,</u> if $a = b$, then $ca = cb$.

Do you remember that we also have an addition property of equality? How is it stated? We found it useful in finding truth sets of open sentences. The multiplication property of equality is useful in this way, too.

<u>Example</u>: Find the truth set of

$$-2x = 10$$

If there is an x such that

$-2x = 10$ is true,

then the same x makes

$$(-\tfrac{1}{2})(-2x) = (-\tfrac{1}{2})(10)$$

or

$$\left((-\tfrac{1}{2})(-2)\right)x = (-\tfrac{1}{2})(10)$$

or

$x = -5$ true also.

Here we have used the multiplication property of equality.

<u>Discussion Questions 8-5</u>

1. Is (2)(6) the name of the same number as (4)(3)?
2. If each of these products, (2)(6) and (4)(3), is multiplied by 2, what is true about the results?
3. What property is illustrated in Question 2?
4. If the sentence $a = b$ is true, what is the result if a and b are each multiplied by c?

Oral Exercises 8-5

Use the multiplicative inverse and the multiplication property of equality to give the truth sets of each of the following:

1. $2m = 4$
2. $3n = -6$
3. $4n = 20$
4. $(3)b = 17$
5. $(-5)a = 15$
6. $(-3)a = 19$
7. $(-2)a = -10$
8. $(-4)a = -9$
9. $\frac{1}{2}x = 4$
10. $(-\frac{1}{3})x = 2$
11. $(-\frac{1}{2})x = 1$
12. $(-\frac{1}{3})y = -3$
13. $(-\frac{2}{3})c = 0$
14. $(-\frac{5}{6})c = 1$
15. $(-\frac{3}{4})b = 12$

Problem Set 8-5

Use the multiplicative inverse and the multiplication property of equality to find the truth set of each of the following open sentences.

1. $2a = 12$
2. $(-3)a = 15$
3. $5a = -25$
4. $(-8)a = -16$
5. $(-6)x = 0$
6. $-m = 7$
7. $\frac{1}{2}b = 4$
8. $(-\frac{1}{4})b = 8$
9. $\frac{1}{3}b = -3$
10. $(-\frac{1}{2})m = -5$
11. $4 = \frac{2}{3}m$
12. $(-\frac{5}{6})m = 10$
13. $4n = 7$
14. $(-3)n = 17$
15. $(-12)b = -30$
16. $(-3)c = \frac{3}{4}$
17. $\frac{7}{2} = 9c$
18. $\frac{2}{3} = \frac{4}{5}b$
19. $(-\frac{5}{6})c = \frac{10}{11}$
20. $(-\frac{4}{3})b = (-\frac{9}{10})$

8-6. Solutions of Open Sentences.

Find the truth set of

$$2x + 5 = 27.$$

Perhaps you could do this by "guessing". However, we shall soon find cases where guessing the truth set would be very, very difficult. So let's try finding the truth set in another way.

The phrase "solve the open sentence" is often used instead of "find the truth set of the open sentence." Here, then, is how we would solve the open sentence "$2x + 5 = 27$".

If there is an x that makes

$$2x + 5 = 27 \quad \text{true,}$$

then the same x makes

$$(2x + 5) + (-5) = 27 + (-5)$$

or

$$2x + \big((5) + (-5)\big) = 27 + (-5)$$

or

$$2x = 22 \quad \text{true, also.}$$

Here we have used the addition property of equality. We added (-5). (Of course, we could say that (-5) was added "to both sides".)

Now if there is an x that makes

$$2x = 22 \quad \text{true,}$$

then the same x makes

$$\tfrac{1}{2}(2x) = \tfrac{1}{2}(22)$$

or

$$(\tfrac{1}{2}\cdot 2)(x) = \tfrac{1}{2}(22)$$

or

$$x = 11 \quad \text{true, also.}$$

Here we have used the multiplication property of equality. We multiplied by $\frac{1}{2}$. We could say that we multiplied "both sides" by $\frac{1}{2}$.

It is very easy to see that the number 11 makes "$x = 11$" true. We could now check to see if the number 11 makes the original sentence, "$2x + 5 = 27$" true also. Actually, however, provided we have made no mistakes in arithmetic, it is not necessary to do this. Let's see why.

We have just shown that if there is a number x that makes "$2x + 5 = 27$" true, then it also makes each of the following sentences true:

$(2x + 5) + (-5) = 27 + (-5)$ addition property of equality

$2x + \big((5) + (-5)\big) = 27 + (-5)$ associative property of addition

$2x = 22$

$\frac{1}{2}(2x) = \frac{1}{2}(22)$ multiplication property of equality

$(\frac{1}{2}\cdot 2)(x) = \frac{1}{2}(22)$ associative property of multiplication

$x = 11$

However, we could just as easily "go the other way". That is, we could start with the sentence "$x = 11$". If there is an x that makes "$x = 11$" true, then it also makes each of the following sentences true:

$2(x) = 2(11)$ multiplication property of equality

$2x = 22$

$2x + 5 = 22 + 5$ addition property of equality

$2x + 5 = 27$

In other words, any x that makes "$2x + 5 = 27$" true also makes "$x = 11$" true; and any x that makes "$x = 11$" true also makes "$2x + 5 = 27$" true. The two sentences have the same truth set. Since the truth set of "$x = 11$" is easy to see, we can find it instead of the truth set of "$2x + 5 = 27$".

Two open sentences with the same truth set are called

equivalent sentences.

In the example above, "$2x + 5 = 27$" and "$x = 11$" are equivalent sentences. They have the same truth set. Since {11} is the truth set of "$x = 11$", it is also the truth set of "$2x + 5 = 27$".

It is easy to form equivalent sentences. If the same real number is added to both sides of an open sentence, or if both sides are multiplied by the same real number (except zero), the result is an open sentence equivalent to the original one.

Here is another example. Solve

$$3x + 7 = x + 15.$$

$3x + 7 = x + 15$

is equivalent to

$(3x + 7) + (-7) = (x + 15) + (-7)$ — addition property of equality (We added (-7).)

or

$3x + \big((7) + (-7)\big) = x + \big((15) + (-7)\big)$

or

$3x = x + 8.$

This is equivalent to

$3x + (-x) = x + 8 + (-x)$ — addition property of equality (We added (-x).)

or

$2x = 8.$

This is equivalent to

$\frac{1}{2}(2x) = \frac{1}{2}(8)$ — multiplication property of equality (We multiplied by $\frac{1}{2}$.)

or

$x = 4.$

Thus, we have a list of equivalent sentences. They all have the same truth set. It is easy to see that the truth set of "$x = 4$" is {4}. If we have not made a mistake in arithmetic somewhere down the line, {4} is also the truth set of "$3x + 7 = x + 15$". Being human, we do sometimes make mistakes in arithmetic. So it is still a good idea to check to see if {4} is the truth set of the original sentence. We'll leave this for you to do.

Here is a final example. Solve

$$7 + 3x + (-5) + 9x = 37 + 5x.$$

See if you can give the reason for each of the following steps.

$$7 + 3x + (-5) + 9x = 37 + 5x$$
$$12x + 2 = 37 + 5x$$
$$12x + \big((2) + (-2)\big) = 5x + \big((37) + (-2)\big)$$
$$12x = 5x + 35$$
$$12x + (-5) = 35 + \big((5x) + (-5x)\big)$$
$$7x = 35$$
$$\frac{1}{7}(7x) = \frac{1}{7}(35)$$
$$x = 5$$

Therefore, "$x = 5$" and "$7 + 3x + (-5) + 9x = 37 + 5x$" are equivalent sentences. They have the same truth set. $\{5\}$ is the truth set of "$x = 5$." We sometimes say that 5 is a solution of the sentence "$x = 5$".

As a guard against mistakes in arithmetic, check to see if 5 is a solution of "$7 + 3x + (-5) + 9x = 37 + 5x$".

From these examples, you see that solving an equation is something like a game. The rules of the game are just the properties of real numbers. We usually try to use the properties to get an equivalent sentence in which the variable stands "alone" as in "$x = 5$" and "$x = 11$" and "$x = 4$".

Problem Set 8-6

Find the truth sets of each of the following open sentences.

1. $5x + (-4x) = 7$
2. $6c + 3c = 16 + 2$
3. $8r + (-2r) + 4 = 16$
4. $17 + (-5) = 4m$
5. $19 + (-4) = (-3m) + 5m$
6. $3m + (-6) + 2m = 9$
7. $2m + 5m + 16 = (-5)$
8. $(-4m) + (-6m) + (-10) = 15$
9. $a + (-6a) = 7 + (-2a)$
10. $12n + (-27) = 5n + (-4)$
11. $(-5b) + (-14) = 6 + (-10b)$
12. $(-8a) + (-12) = (-5a) + 5$
13. $6m + 9 = 5m + (-8)$
14. $10m + 8 = 9m + 9$
15. $17x = (-11) + 18x + (-12)$
16. $(-5) + 3x + \frac{1}{3}x = 5$
17. $(-6\frac{2}{3}) + 4x + 9\frac{4}{5} = 4x + 7\frac{1}{2}$
18. $(-a) + 2a + (\frac{27}{4}) = (-a) + \frac{13}{4}$
19. $(-5a) + (-\frac{21}{4}) = (-3a) + 5\frac{1}{4}$
20. $(-3y) + \frac{17}{3} + 2y = 5 + (-y) + \frac{2}{3}$

*21. $\frac{5}{8} + \frac{1}{4}|y| = \frac{7}{8} - \frac{4}{16}$

*22. $\frac{9}{16} + |y| = -\frac{3}{16}$

8-7. Products and the Number Zero.

What is the common name for each of these products:

$$(7)(0)$$
$$(0)(-3)$$
$$(0)(0)$$
$$(\tfrac{1}{2})(0) \text{ ?}$$

In each case, the common name for the product is "0". In fact, these are just examples of the multiplication property of zero.

If we have a product (a)(b) of real numbers a and b, we can say the following things about this product:

If $a = 0$, $ab = 0$.	This is true since $(0)(b) = 0$, no matter what number b represents.
If $b = 0$, $ab = 0$.	This is true since $(a)(0) = 0$, no matter what number a represents.
If $a = 0$ and $b = 0$, $ab = 0$.	This is true, since $(0)(0) = 0$.

The sentences above show the only conditions that will make "ab = 0" a true sentence.

Example 1. Find the truth set of "$(7)(x) = 0$".

If $x = 0$, the sentence reads "$(7)(0) = 0$", which is true. Is there any other number that will make the sentence true? The truth set is

$$\{0\} .$$

Example 2. Find the truth set of "$(y)(8) = 0$".

If $y = 0$, the sentence reads "$(0)(8) = 0$", which is true. Is there any other number that will make the sentence true? The truth set is

$$\{0\}$$

Example 3. Find the truth set of "$8(x - 3) = 0$".

Notice that, on the left side, we have the product of the number 8 and the number $x - 3$. The product will be zero if either of these numbers is zero. 8, of course, cannot be zero. However, if $x = 3$, $(x - 3)$ is the number zero; and the sentence reads "$8(3 - 3) = 0$", which is true. Is there any other number that will make the sentence true? The truth set is

$$\{3\} .$$

Example 4. Solve "$(x + 5)(x + 2) = 0$".

On the left side, we have the product of the numbers $(x + 5)$ and $(x + 2)$. If x is (-5), the sentence reads "$(0)(-3) = 0$", which is true. If x is (-2), the sentence reads "$(3)(0) = 0$", which is true. The truth set is:

$$\{-2, -5\} .$$

Discussion Questions 8-7

1. What is the common name for $(7)(0)$?
2. What is the common name for $(\frac{1}{2})(0)$?
3. What is the product of any real number and zero?
4. What do we know about one of the numbers if the product of two numbers is zero?
5. If $ab = 0$, what must be true?
6. If a and b are real numbers such that $ab = 0$, is it possible that both a and b might be zero?

Oral Exercises 8-7

Which of the following sentences are true?

1. $7(0) = 0$
2. $(0)(m) = 1$
3. $(-5)(0) = -5$
4. $(0)(-7) = 0$
5. $(0)(-m) = -m$
6. $(b)(-0) = b$
7. $(0)(0) = 0$
8. $(\frac{2}{3})(0) = \frac{2}{3}$
9. $(-\frac{5}{4})(0) = 0$
10. $(2.5)(0) = 2.5$
11. $(6.325)(0) = 0$

Problem Set 8-7

1. Find the truth set of each of the following open sentences:

   (a) $(5)(b) = 0$

   (b) $(-7)(a) = 0$

   (c) $(\frac{2}{3})(m) = 0$

   (d) $(b)(-\frac{5}{6}) = 0$

   (e) $(3.2)(c) = 0$

   (f) $(-4.34)(b) = 0$

2. Find the truth set of each of the following open sentences:

   (a) $(x - 4) = 0$

   (b) $5(x - 3) = 0$

   (c) $-2(a - 7) = 0$

   (d) $5(m + 2) = 0$

   (e) $8(b + 6) = 0$

   (f) $-6(a + 1) = 0$

   (g) $(n + \frac{2}{3}) = 0$

   (h) $-2(n + .8) = 0$

   (i) $-1(m - 1) = 0$

   (j) $-9(b - \frac{5}{4}) = 0$

   (k) $7(\frac{7}{8} + a) = 0$

   (l) $3(\frac{5}{6} - a) = 0$

   (m) $-\frac{4}{5}(\frac{3}{4} + m) = 0$

   (n) $(.91 - r) = 0$

Problem Set 8-7
(continued)

3. Find the truth set of each of the following:
   (a) $(x + 4)(x + 3) = 0$
   (b) $(m + 2)(m + 6) = 0$
   (c) $(a - 5)(a + 6) = 0$
   (d) $(b + 10)(b - 5) = 0$
   (e) $(n - 3)(n - 5) = 0$
   (f) $(n + 9)(n - 9) = 0$
   (g) $(a + \frac{2}{3})(a - \frac{1}{8}) = 0$
   (h) $(m - \frac{1}{2})(m - \frac{5}{6}) = 0$
   (i) $(b - \frac{3}{4})(b + \frac{9}{12}) = 0$
   (j) $(a + 3.4)(a + 2.18) = 0$
   (k) $(c + 5.15)(c - 3.12) = 0$
   (l) $(y - 1.75)(y - 2.25) = 0$
   (m) $n(n + 2) = 0$
   (n) $a(a - 3) = 0$
   (o) $b(b + \frac{4}{5}) = 0$
   (p) $c\left(c + (-\frac{7}{3})\right) = 0$
   (q) $(m - 1)m = 0$
   (r) $n(n + 1) = 0$
   *(s) $(w + 6)(2w + 5) = 0$
   *(t) $(m + 5)(\frac{1}{2}m - 4) = 0$
   *(u) $(1.5n - 6)(.5n - 2) = 0$
   *(v) $(\frac{2}{3}a - 1)(\frac{3}{4}a - \frac{3}{8}) = 0$
   *(w) $(\frac{5}{8}m + 1)(\frac{1}{12}m + 2) = 0$

Problem Set 8-7
(continued)

4. Translate the following into open sentences and find their truth sets:

   (a) Mr. Johnson bought 30 feet of wire and later bought 55 more feet of the same kind of wire at the same price per foot. He found that he paid $4.20 more than his neighbor paid for 25 feet of the same kind of wire at the same price per foot. What was the cost per foot of the wire?

   (b) Four times an integer is ten more than twice the successor of that integer. What is the integer?

   (c) In a stock car race, one driver, starting with the first group of cars drove for 5 hours at a certain speed and was then 120 miles from the finish line. Another driver, who set out with a later heat, had traveled at the same rate as the first driver for 3 hours and was 250 miles from the finish. How fast were these men driving?

   (d) The perimeter of a triangle is 44 inches. The second side is three inches more than twice the length of the third side, and the first side is five inches longer than the third side. Find the lengths of the three sides of this triangle.

   (e) If an integer and its successor are added, the result is one more than twice that integer. What is the integer?

   (f) In a farmer's yard were some pigs and chickens, and no other creatures except the farmer himself. There were, in fact, sixteen more chickens than pigs. Observing this fact, and further observing that there were 74 feet in the yard, not counting his own, the farmer exclaimed happily to himself--for he was a mathematician as well as a farmer, and was given to talking to himself--"Now I can tell how many of each kind of creature there are in my yard." How many were there? (Hint: Pigs have 4 feet, chickens 2 feet.)

Problem Set 8-7
(continued)

(g) At the target shooting booth at a fair, Montmorency was paid 10¢ for each time he hit the target, and was charged 5¢ each time he missed. If he lost 25¢ at the booth and made ten more misses than hits, how many hits did he make? ____________________

Summary

In this chapter, we have discussed multiplication of real numbers. The product of two real numbers a and b may be written in any of the following ways:

$$ab, \quad (a)(b), \quad a \cdot b.$$

We were able to give a definition for the product of any two real numbers. Furthermore, we found that for any two real numbers a and b, the product ab is also a real number. This is the property of closure under multiplication.

Here are some other properties of real numbers:

Multiplication Property of One
For any real number a, $(a)(1) = a$.

Multiplication Property of Zero
For any real number a, $(a)(0) = 0$.

Commutative Property of Multiplication
For any real numbers a and b, $ab = ba$.

Associative Property of Multiplication
For any real numbers a, b, and c, $(ab)c = a(bc)$.

Distributive Property
For any real numbers a, b, and c, $a(b + c) = ab + ac$.

Multiplication Property of Equality
For real numbers a, b, and c, if $a = b$,
then $ac = bc$.

For any real number a, $(-1)a = -a$.

If $ab = 1$, b is called the multiplicative inverse of a, and a is called the multiplicative inverse of b. Every real number, except zero, has a multiplicative inverse. A number and its multiplicative inverse are either both positive or both negative.

Two open sentences with the same truth set are called equivalent sentences.

The sentence "$ab = 0$" is: true if $a = 0$,
true if $b = 0$,
true if $a = 0$ and $b = 0$,
false under all other conditions.

Review Problem Set

1. Find the common name for each of the following sums:

   (a) $7 + (-10)$
   (b) $12 + 12$
   (c) $(-10) + (-15)$
   (d) $6 + (-16)$
   (e) $a + (-a)$
   (f) $(-2a) + 2a$

2. Find the common names for the following products:
   (a) $(2)(4)$
   (b) $(2)(-8)$
   (c) $(-6)(-5)$
   (d) $(-4)(6m)$
   (e) $(-8m)(-3a)$
   (f) $(-2m)(2.5m)$
   (g) $(-\frac{2}{3}m)(-\frac{3}{4}y)$
   (h) $(-1)(\frac{4}{5}x)(-\frac{2}{3}y)$
   (i) $(-3)(-4a)(2b)$
   (j) $(-2m)(12b)(\frac{1}{2})$
   (k) $(8a)(7\frac{1}{2})(-\frac{3}{4}a)$
   (l) $(19m)(-1.75n)(-2.25a)(0)$
   (m) $(2x)(-4a)(-6am)(-\frac{5}{4})$

3. Use the distributive property to change the form of each of the following:
   (a) $2(a + 2b)$
   (b) $4\big((-c) + 4d\big)$
   (c) $-6\big((-7c) + 6d\big)$
   (d) $8a\big((-4m) + (-3n)\big)$
   (e) $(-7a)\big((-3b) + (-4c)\big)$
   (f) $(-4a)\big((-4a) + (\frac{1}{2}b)\big)$
   (g) $(-1)\big((-3a) + 6b\big)$
   (h) $(a)\big(4m + (-2n)\big)$
   (i) $(-b)\big((-2c) + (-3d)\big)$
   (j) $(-m)\big((-5m) + 10n\big)$

Review Problem Set
(continued)

(k) $(-2c)\big((-5c) + (-10d)\big)$
(l) $\big((-5m) + 8b\big)(-3b)$
(m) $\big(4a + (-4c)\big)(-d)$
(n) $(a + 3)(a + 4)$
(o) $\big(a + (-3)\big)(a + 4)$
(p) $(m + 6)\big(m + (-6)\big)$
(q) $\big(y + (-2)\big)\big(y + (-9)\big)$
(r) $(y + \frac{1}{2})(y + \frac{3}{4})$
(s) $\big(z + (-5)\big)(z + 5)$
(t) $\big(w + (-\frac{3}{4})\big)\big(w + (-\frac{3}{4})\big)$
(u) $(a + 1.5)(a + 1.5)$
(v) $\big(b + (-2.1)\big)(b + 2.1)$
(w) $(-2y)\big((-3)y + 2z + 4\big)$
*(x) $\big((-3y) + 4\big)\big(2y + (-3)z + 4\big)$
*(y) $\big((-2m) + (-1)\big)\big((-4)m + (-n) + 1\big)$

4. Use the distributive property to collect terms in each of the following:

(a) $3x + 10x$
(b) $(-9)a + (-4)a$
(c) $11k + (-2)k$
(d) $(-27)b + 30b$
(e) $17n + (-16)n$
(f) $x + 8x$
(g) $(-15)a + a$
(h) $\frac{7}{8}a + \frac{9}{8}a$
(i) $5p + 4p + 8p$
(j) $7x + (-10)x + 3x$
(k) $12a + 5c + (-2)c$
(l) $6a + 4b + c$
(m) $9p + 4q + (-3)p + 7q$
(n) $6p + (-4r) + (-8p) + (-2r)$

Review Problem Set
(continued)

(o) $3a + (-9b) + 5a + (-8a)$
(p) $(-5m) + (-6c) + (3m) + (6c) + (2m)$
(q) $r + 2t + (-n) + 5s$
(r) $\frac{5}{2}a + \frac{5}{6}b + (-\frac{3}{2}a) + \frac{1}{3}b$
(s) $(-\frac{7}{3})m + \frac{2}{3}n + (-\frac{5}{3})m + \frac{1}{3}n + a$
(t) $\frac{14}{3}a + 4b + \frac{7}{3}a + (-4)b$

5. Use the multiplication property of <u>one</u> to simplify each of the following:

(a) $\frac{3}{4} + \frac{4}{5}$

(b) $\frac{5}{6} + \frac{7}{8}$

(c) $\frac{3}{2} + \frac{3}{5}$

(d) $\dfrac{\frac{5}{8}}{\frac{2}{3}}$

(e) $\dfrac{\frac{9}{5}}{\frac{2}{3}}$

(f) $\dfrac{3\frac{1}{4}}{2\frac{2}{5}}$

6. Use the distributive property to write the following indicated sums as indicated products:

(a) $3a + 3b$
(b) $(-5)c + (-5)d$
(c) $10m + 5n$
(d) $(-10a) + (-15b)$
(e) $mn + nay$
(f) $-mx + 2my$
(g) $2bx + 4by$
(h) $4am + 6an$
(i) $(-6)bx + (-9)bw$
(j) $\frac{2}{3}at + \frac{4}{3}bt$
(k) $(-\frac{5}{8})b + (-\frac{5}{8})c$
(l) $2.5m + 5.0n$

Review Problem Set
(continued)

7. Which of the following sentences are true for all values of the variables:

   (a) $(2)(5 + 3) = (5 + 3)(2)$
   (b) $a\big(b + (-c)\big) = \big((-c) + b\big)a$
   (c) $2a + (-b) + 3c = (-b) + 3c + 2a$
   (d) $5m + (-5)n = 5(m + n)$
   (e) $(-1)(2) = 2$
   (f) $(-1)(-3) = 3$
   (g) $(-1)(c) = c$
   (h) $(-1)(0)(-4) = 4$
   (i) $(b)(-1)(0) = 0$
   (j) $(-m)(-1) = m$
   (k) $(0)(m) = m$
   (l) $(-2.7)(-1)(3.95)(0) = 6.65$
   (m) $(3.25)(-1)(-t) = 3.25t$
   (n) $(-5)(1) > 5$
   (o) $(6)(-3) \neq 18$
   (p) $6(-3) > 18$
   (q) $6(-3) < 18$
   (r) $(0)m < 0$
   (s) $a(-1) \neq 1$
   (t) $2a + (-5a) \neq 3a$
   (u) $10m + (-12n) = 2\big(5m + (-6n)\big)$

8. Which sentences are true for the given value of the variable?

   (a) $2m = -10; \quad m = 5$
   (b) $3m = 1; \quad m = 0$
   (c) $\frac{1}{3}a = 3; \quad a = 9$
   (d) $\frac{3}{4}a = 0; \quad a = 0$
   (e) $-\frac{2}{3}a = 1; \quad a = \frac{3}{2}$
   (f) $\frac{5}{6}a = 1; \quad a = \frac{6}{5}$

Review Problem Set
(continued)

(g) $2a + 4 = (-2)$; $a = (-1)$

(h) $\frac{3}{2}m + 6 = (-6)$; $m = \frac{2}{3}$

(i) $-\frac{7}{8}m + (-3) = (-4)$; $a = (-\frac{7}{8})$

(j) $(-2a)(-3) \neq 6$; $a = 1$

(k) $2(-2.4m) > (-9.6)$; $m = (-4.8)$

(l) $4|x| < 0$; $x = -2$

(m) $|m| + (-2) = 0$; $m = -2$

9. Find the truth set of each of the following sentences:

(a) $6x + 9x = 30$

(b) $12y + (-5y) = 35$

(c) $(-3)a + (-7)a = 40$

(d) $x + 5x = 3 + 6x$

(e) $3y + 8y = -99$

(f) $-15z + 12z = 24$

(g) $14x + (-14)x = 15$

(h) $(-3)a + 3a = 0$

(i) $13k + (-14)k + 9k = 0$

(j) $x + 2x + 3x = 42$

(k) $a + 2a = 3$

(l) $(-8)y + 9y \geq 5$

(m) $7a + (3)a < 10$

(n) $|x| + 3|x| < 0$

(o) $-|m| + 2|m| > 5$

(p) $4y + 3 = 3y + 5 + y + (-2)$

(q) $12x + (-6) = 7x + 24$

(r) $8x + (-3)x + 2 = 7x + 8$ (Collect terms first.)

(s) $6z + 9 + (-4)z = 18 + 2z$

(t) $12n + 5n + (-4) = 3n + (-4) + 2n$

(u) $15 = 6x + (-8) + 17x$

(v) $5y + 8 = 7y + 3 + (-2y) + 5$

Review Problem Set
(continued)

10. Write open sentences for each of the following, then find the truth set.

(a) The sum of twice a number and 5 is 47. What is the number?

(b) A farmer had his wheat hauled to an elevator in two trucks, with the same capacity, which carried a full load each trip. He stored 490 bushels in the elevator. One truck made 3 trips, the other made 4. (1) How much grain did each truck hold? (2) How many bushels did each truck haul altogether?

(c) Mrs. Abbott tried to remember how much she paid per can for two cans of peaches she bought when she shopped for groceries. She could only remember that she had bought a can of coffee which cost 83¢ and that she received 4¢ change from $1.50 which she gave the clerk. How much did Mrs. Abbott pay for each can of peaches? (Disregard the sales tax.)

(d) One angle of a triangle is twice as large as the second. The third angle contains 12° more than the smaller of the first two angles. How many degrees are in each angle?

(e) A freight train and a passenger train, running on a double track, left Washington, D. C. for New York. The freight train left at 6:00 AM and traveled at an average speed of 40 miles per hour. The passenger train left at 7:00 AM and averaged 60 miles per hour. (1) How long will it take the passenger train to overtake the freight?

(Hint: If t is the number of hours that the passenger train ran before overtaking the freight train then $(t + 1)$ is the time that the freight train ran before it was overtaken by the passenger train.)

(2) At what time did the passenger train overtake the freight? (3) How far were they from Washington when the passenger train overtook the freight?

Review Problem Set
(continued)

(f) The length of a rectangular flower bed is 8 feet more than twice the width. Its perimeter is 196 feet. What are its dimensions?

11. Draw the graphs of the truth sets of the following sentences.

(a) $|x| > 5$
(b) $|y| < 0$
(c) $x < 2$ and $x > -1$
(d) $x + 1 = 5$ or $x + 1 = 4$
(e) $x \leq 3$ and $x > 0$
(f) $x \neq 5$

Chapter 9

PROPERTIES OF ORDER

9-1. The Order Relation for Real Numbers.

Suppose you were asked to name the letters of the alphabet. Would you be apt to say, "c, f, y, s, q, t, k, etc."? This is very unlikely. In all probability you would begin, "a, b, c, d, e, f, g, etc." So would everybody else.

There must be a reason for this. It is because you are in the habit of using what we call an order relation. Whenever you look up a word in the dictionary, or a topic in the index of a book such as this one, you will be taking advantage of an order relation; that is, alphabetical order. The concept of order occurs in a variety of situations. For example a baseball team has a certain batting order. Can you think of other instances?

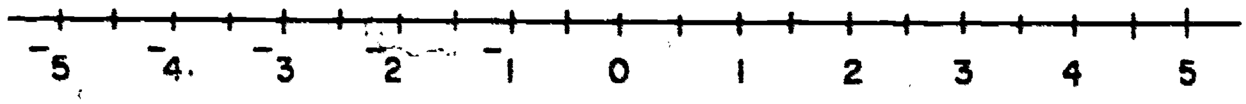
As you have already seen, the idea of order plays an important part in the study of numbers. You will remember the type of sentence which we wrote, for example, as

$$4 < 7,$$

where the symbol "<", meaning "is less than", shows a relation between 4 and 7. We call a relation of this type an order relation. The relation "is greater than", for which we use the symbol ">", is also an order relation for numbers. To avoid confusion, however, we shall concentrate mainly on the first relation, "is less than", in this chapter. It will be easy to show that the ideas and properties which apply to one of these relations can also be applied to the other.

In Chapter 6 the relation of order was extended from the numbers of arithmetic to the real numbers, which include the negative numbers as well as the positive numbers and zero. This was done by using the number line. We agreed that:

"is less than" for real numbers means
"is to the left of" on the real number line.



Thus, by referring to the above line we see that -3 is less than -1, -4 is less than 2, etc. It should be clear that the following sentences are also true.

$$-2 < 2. \qquad -1 < 0. \qquad -4 < -3.$$

Write down any real number at all! Feel free to choose a negative number, or a number in the form of a fraction, whatever you wish. Without saying what your number is, ask a friend to write any real number. Call your number a, and your friend's number b. Now look at both numbers. Then state which of the following is a true sentence:

$$a < b$$
$$a = b$$
$$b < a$$

You have undoubtedly found that one, and only one, of the sentences is true, and that the other two sentences are false. Now no matter how many times you repeat the experiment with different numbers, you will always find that the same situation occurs. It will always happen that <u>one</u> of the sentences (not the same one every time, of course) is true, and that the other two are false. Try the experiment several times. Make a record each time as to which sentence is true and which two sentences are false.

The fact that it always happens that for any real number a and any real number b one of the sentences

$$a < b$$
$$a = b$$
$$b < a$$

is true and the other two are false is an important property which is called the

<u>comparison property</u>.

We shall now discover another property. Suppose we are given any three different real numbers a, b, and c. Assume it is known that

$$a < b.$$

Suppose we also know that

$$b < c.$$

What is the relation between a and c? You will probably say immediately that

$$a < c.$$

Although you may have had no difficulty in giving the right answer in this case, what if you were asked to state in words the reason why you felt sure that a was less than c? Your answer might be something like this.

> "If one number is less than a second number, and if the second number is less than a third number, then the first number must be less than the third."

This statement describes a property of the order relation "is less than". It is called the

transitive property.

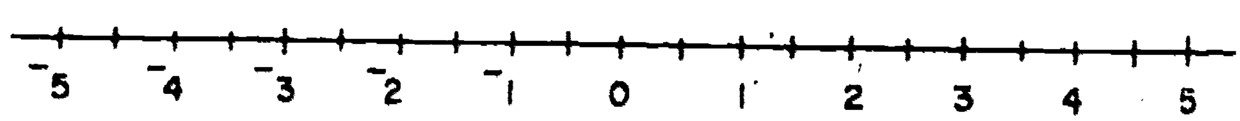
This property can be very clearly illustrated by the use of the number line. Can you show that the order relation "is greater than" also has the transitive property?

Another property of order which was obtained in Chapter 6 connects the order relation with the operation of taking opposites. This property, which you have already studied, states that

if a and b are any real numbers, and if $a < b$, then

$$-b < -a$$

Some examples using the number line will help to make the meaning of this property completely clear.



We see that $2 < 4$, and it is also true that $-4 < -2$
Now let $a = -3$ and let $b = 1$. In the first place it is evident that

$$a < b.$$

The opposite of a, namely $-a$, we see to be 3. The opposite of b, namely $-b$, is -1. It is clear, then, that since -1 is to the left of 3, we must say that

$$-b < -a.$$

Discussion Questions 9-1

1. What symbol do we use to show the order relation between two numbers?
2. What meaning on the number line do we associate with " is less than"?
3. If you think of any two numbers a and b, what does the comparison property tell you about the numbers?
4. If the relationship of a and b is given by "$a < b$", what is the relationship of $-a$ and $-b$? Illustrate this with two different numbers for a and two different numbers for b, not all of which are positive.

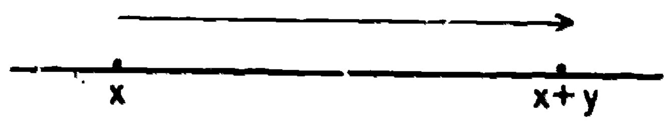
Oral Exercises 9-1

1. If one number is less than a second number and the second number is less than a third number, what can you then say? What property is this? State this property using the variables x, y, and z.
2. Use the numbers -5, -1, and a to illustrate the transitive property of order.
3. Try to list all the things you know about the order of the numbers -7, b, and 3 and their opposites.

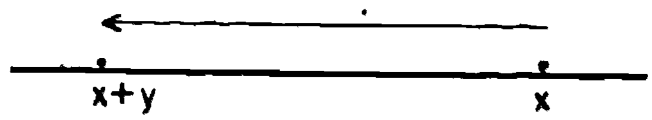
9-2. Addition Property of Order.

We have just seen that there is a definite connection between the order relation and the operation of taking opposites. We now wish to find out whether there is a connection between order and the operation of addition.

As before, it will be helpful to use the number line. We first locate a definite number x on the line. Then we wish to locate the number $x + y$. Remember that if y is positive, we move to the right.

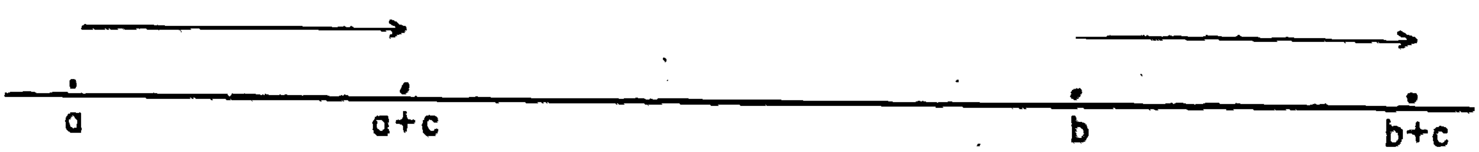


Thus, we see that $x < x + y$ if y is positive.
On the other hand if y is negative, we move to the left,



and we see that $x + y < x$ if y is negative.

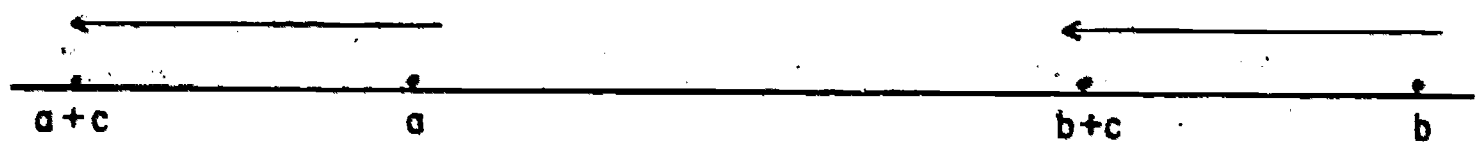
Now let's select two numbers a and b on the line with $a < b$. If we add the same number c to a and to b, we move to the right of a and to the right of b if c is positive. The distance we move in each case is equal to c.



From the above line we see that

$$\text{if } a < b, \text{ then } a + c < b + c.$$

Now with a and b in the same positions, let c be a negative number and add c to both a and to b. This time, of course, we move to the left.



Once again it is clear that $a + c < b + c$. In other words, it is true that if $a < b$, then $a + c < b + c$ whether c is positive or negative. What about the case where c is zero?

The situation which we have observed above describes a property called the

<u>addition property of order</u>.

It states that if a, b, and c, are real numbers and

if $a < b$, then $a + c < b + c$.

The following table will help us to understand the property. Try experimenting with further examples.

a	b	c	a + c	b + c
-3	1	2	-1	3
-3	1	-2	-5	-1
-8	-4	8	0	4
$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0

You should check the table to see if $a < b$. Are the numbers under "a + c", and "b + c" correct in each case? Use the number line to show that in every instance

$$a + c < b + c.$$

By using the number line and a discussion similar to the one above can you show that the addition property of order applies also to the relation "is greater than"? This property can be stated as follows:

If a, b, and c, are any real numbers,
and if $a > b$, then $a + c > b + c$.

Illustrate this property by means of a table like the one above!

Discussion Questions 9-2a

1. If we wish to show addition on the number line, in which direction do we move to add a positive number to a number x? In which direction do we move to add a negative number?

2. If we move a distance c to the right of a point a and a distance c to the right of a point b, will the final points have the same relative position as a and b had? How can we express this in a mathematical sentence?

3. State the addition property of order. Does it agree with the last sentence you wrote in Question 2?

4. Is the sentence "If $a < b$, then $a + c < b + c$" true when $c = 0$? Why? Can c be negative and the sentence still be true?

5. If a is -3 and b is 1, state an order relationship involving a and b. If c is 2, state and order relationship involving $(a + c)$ and $(b + c)$.

Oral Exercises 9-2a

1. Which of the following sentences are true?

   (a) If $m < n$, then $m + 2 < n + 2$.

   (b) If $-2 < 5$, then $-2 + n < 5 + n$.

   (c) If $-5 < a$, then $a > -5$.

   (d) If $a < b$ and $c < d$, then $a < d$.

   (e) If a and b are two different numbers, then exactly one of the following is true: $a < b$, or $b < a$.

   (f) If $m < n$ and $p < m$, then $p < n$.

   (g) If $-a < b$, then $-b < a$.

   (h) If $-a + 5 > b + 5$, then $-b > a$.

   (i) If $m < -n$, then $-m < n$.

2. (a) Does the relation indicated by the symbol "=" have the transitive property? Give an example.

   (b) Does the relation indicated by the symbol ">" have the transitive property? Give an example.

   (c) Does the relation indicated by the symbol "≠" have the transitive property? Give an example.

   (d) Can you restate the comparison property so that it describes the relationship of two different numbers?

3. In each of the following indicate which symbol, "<" or ">", should be put in the place occupied by the question mark, so that the resulting sentence is true.

   (a) $-3 \; ? \; -5$

   (b) If $m < n$ and $n \; ? \; x$, then $m < x$.

   (c) If $-3 < a$, then $3 \; ? \; -a$.

   (d) If $m < n$, then $-n \; ? \; -m$.

   (e) If $a + 5 < b + 5$, then $a \; ? \; b$.

   (f) If $a + (-5) < b + (-5)$, then $-a \; ? \; -b$.

   (g) If $a < b$, then $a + 5 < b + 5$ and $a + 5 + (-5) \; ? \; b + 5 + (-5)$.

   (h) If $x + 5 < 2$, then $x + 5 + (-5) \; ? \; 2 + (-5)$.

   (i) If $5 + m \; ? \; 3m$, then $5 + m + (-m) < 3m + (-m)$.

   (j) If $(-a) < b + 5$ and $c < (-a)$, then $c + (-5) \; ? \; b$.

Problem Set 9-2a

1. Write each of the following sentences with either ">" or "<" replacing the question mark so that the resulting sentence is true.

   (a) If $(-5) \; ? \; a$, then $5 > (-a)$.

   (b) If $b < 4$, then $b + 5 \; ? \; 4 + 5$.

(c) If $b + 5 < (-7)$, then $b + 5 + (-5) \ ? \ (-7) + (-5)$.

(d) If $b < 3$, then $-(b + 2) \ ? \ -(3 + 2)$.

(e) If $a < b$ and $b < c$, then $a + 2 \ ? \ c$.

(f) If $b + 5 < (-7)$, then $b \ ? \ (-12)$. (see part c)

(g) If $a < -3$ and $0 < b$, then $a + 2 \ ? \ b$.

(h) If $a = b$, then $a + c \ ? \ b$.

(i) If $3 < x$, and if $3 + m = x$, then $0 \ ? \ m$.

(j) $x \ ? \ a$ if $x < b$ and $b < a$.

2. Which of the following sentences are true? Which are false? (Hint: the addition property of order, if wisely used, may help.)

(a) $3 + 4 < \frac{12}{4} + 4$

(b) $(-6) + 7 < (-3) + 7$

(c) $5 + (-\frac{24}{25}) < 3\frac{1}{8} + (-\frac{24}{25})$

(d) $\frac{15}{16} + (-\frac{3}{8}) < (-\frac{3}{8}) + \frac{30}{31}$

(e) $3 + (-12) + 47 < 47 + (-18) + 9$

(f) $18 + (-432) + (-79) < 24 + (-432) + (-79)$

(g) $(-273) + 1\frac{3}{8} + (-382) < 1\frac{3}{8} + (-114) + (-382)$

(h) $(-\frac{5}{3})(\frac{6}{5}) + (-5) < (-\frac{5}{2}) + (-5)$

(i) $(-5.3) + (-2)(-\frac{4}{3}) < (0.4) + \frac{8}{3}$

(j) $(\frac{5}{2})(-\frac{3}{4}) + 2 \leq (-\frac{15}{8}) + 2$

*3. A proof of an order property is shown below. Supply a reason for each step.

If a, b, c and d are real numbers such that $a < b$ and $c < d$, then $a + c < b + d$.

Since $a < b$, we know that $a + c < b + c$. Why?

Since $c < d$, we know that $b + c < b + d$. Why?

Therefore, $a + c < b + d$. Why?

We can use the addition property of order in problems involving truth sets for open sentences. For example suppose we were asked to find the truth set of the sentence

$$x + 5 < 8.$$

If there is a real number x for which our sentence is true, then for this x the following sentence is also true:

$$x + 5 + (-5) < 8 + (-5).$$

This follows from the addition property of order. This sentence can be written

$$x < 3.$$

The truth set for the sentence "$x < 3$" can be described quite easily. Can you see that it is the set of all real numbers less than 3? Draw the graph of this set on the number line.

We have shown that any number x which makes the sentence "$x + 5 < 8$" true also makes the sentence "$x < 3$" true. Now we must check to see that any real number x which makes "$x < 3$" true also makes the sentence "$x + 5 < 8$" true. Once again we can use the addition property. If there is a real number x for which

$$x < 3,$$

then for this x $\quad x + 5 < 3 + 5$

or $\quad x + 5 < 8.$

The "check" which we used for the above problem is not strictly necessary. In Chapter 8 we learned that if a real number is added to both sides of an open sentence, where the open sentence expresses equality, then the resulting open sentence will be equivalent to the first one. In the same way it can be shown that this is true for open sentences which involve the relation of inequality. Thus, the two sentences

$$x + 5 < 8$$

and

$$x < 3$$

are <u>equivalent</u>. As you recall, this means that they have the <u>same</u> truth set. Therefore, if we can determine the truth set of "$x < 3$", we can be sure that this is also the truth set of "$x + 5 < 8$".

Can you see that the truth set of

$$x < 10$$

is the same as the truth set of

$$x + 7 < 17?$$

What is this truth set?

As another example, let us find the truth set of the sentence

$$5x + 9 < 4x + 3.$$

If there is a real number x for which this is true, then

$$5x + 9 + (-4x) + (-9) < 4x + 3 + (-4x) + (-9) \quad \text{(Why?)}$$
$$\big(5x + (-4x)\big) + \big(9 + (-9)\big) < \big(4x + (-4x)\big) + \big(3 + (-9)\big) \quad \text{(Why?)}$$
$$\big(5 + (-4)\big)x + 0 < \big(4 + (-4)\big)x + (-6) \quad \text{(Why?)}$$
$$x < (-6)$$

What is the truth set of "$x < (-6)$"? Is it the same as the truth set of

$$5x + 9 < 4x + 3?$$

Draw the graph

<u>Discussion Questions 9-2b</u>

1. What number could we "add to both sides of the sentence" $x + 5 < 8$ which would help us find the truth set of the sentence? What is the truth set of the sentence?

2. Are the sentences "$x < 3$" and "$x + 5 < 8$" the same sentence? Are they equivalent sentences? What do we mean

by equivalent sentences? How can we tell if two sentences are equivalent?

3. When finding the truth set of the sentence $5x + 9 < 4x + 3$, why do we add $(-4x)$ to both sides of the sentence? Why do we add (-9) to both sides of the sentence? What properties are we using when we do this? What is a simpler name for $\left(5x + (-4x)\right)$? What property do we use to obtain it?

4. How would you draw the graph of the truth set of the sentence "$5x + 9 < 4x + 3$"?

Oral Exercises 9-2b

1. Do you agree with the following statements?

   (a) If we are finding the truth set of $3x + 11 < 12$, the first step might be to add $(-3x)$ to both sides.

   (b) One way to find the truth set of the sentence $(-5) + 3x + 4 < 2x + 8$ would be to write the equivalent sentence $(-5) + 5 + 3x + (-2x) + 4 + (-4) < 2x + (-2x) + 8 + 5 + (-4)$.

   (c) One step in finding the truth set of the sentence $(-5) + 3x + 4 < 2x + 8$ would be to write the sentence $3x + (-1) < 2x + 8$.

   (d) The only reason for "checking" the truth set of a sentence after using the order properties is that the original sentence and the final sentence may not be equivalent.

2. For each of the following sentences use the addition property of order to obtain an equivalent sentence having the variable alone on one side:

   (a) $3 + x < (-4) + 2$

   (b) $2n + (-8) < (-27)$

   (c) $(-8) + 12 < (-3n) + 4$

(d) $7 + (-\frac{3}{2}) + 2 < \frac{8}{3} + 2x$

(e) $.8 + 14 + (-\frac{2}{3}) < 3y + (-\frac{1}{2}) + y$

3. For each of the following sentences the simplest equivalent sentence would be one which would have the variable alone on one side and a specified number on the other side. Tell how you would use the addition property of order to achieve this in each case.

(a) $3 + 2x < x + (-2)$

(b) $(-7) + 4y + 3 < 2 + 3y + (-3)$

(c) $(-4n) + (-3) + 17 < (-12) + (-3n) + 7 + (-2)$

(d) $\frac{y}{2} + \frac{4}{3} < 1\frac{3}{4} + (-\frac{y}{2})$

(e) $.7 + 3.2y + (-.4) < (-.4) + .7 + 2.2y$

Problem Set 9-2b

1. Find the truth set of each of the following sentences.

(a) $(-5) + x + 4 < 7$

(b) $3x < 2x + 2$

(c) $\frac{1}{2} + n + (-\frac{3}{2}) < \frac{4}{3} + (-\frac{7}{3})$

(d) $2y + 2 < 3y + 2$

(e) $\frac{3}{5} + (-\frac{3}{10}) < x + (-\frac{4}{5})$

(f) $\frac{4}{3} + 2x < 3x$

(g) $3x + (-4) + \frac{1}{2} \leq (-8) + 2x + \frac{1}{2}$

(h) $\frac{5}{3} + x \leq (-\frac{2}{3}) + 2x$

(i) $3y + |-3| < 3y + (-2)$

(j) $(-x) + 7 < \frac{1}{2} + (-3)$

(k) $(-x) + 4 < (-3) + |-3|$

(l) $(-5) + (-x) < \frac{2}{3} + \left|-\frac{4}{3}\right|$

(m) $(-2) + 2x < (-3) + 3x + 5$

(n) $2x + \left|-\frac{3}{2}\right| \geq (-\frac{3}{4}) + \frac{5}{4}$

(o) $5x + 2 = 2x + (-7)$

(p) $(-7x) + |-5| = 4x + (-6)$

(q) $(-4) + (-2x) + (-7) < 3x + 4$

(r) $\frac{1}{3} + y + 2 \leq \frac{5}{2} + 2 + y$

(s) $(-5x) + (-4) = (-8x) + (-16)$

(t) $|-5|x + 2.5 < |-7.5| + 4x$

2. Draw the graphs of the truth sets of parts (a), (d), (g), (i), (p), (r) in Problem 1.

3. Which of the following sentences are true for all values of the variable?

(a) If $b < 0$, then $3 + b < b$.

(b) If $b < 0$, then $3 + b < 3$.

(c) If $b < 0$, then $3 + b + (-4) < 3 + (-4)$.

(d) If $x < 2$, then $2x < 4$.

(e) If $x < 2$, then $2x + 5 < 4 + (-5)$.

(f) $|3 + 4| \leq |3| + |4|$.

(g) $|(-3) + 4| \leq |-3| + |4|$.

(h) $|(-3) + (-4)| \leq |-3| + |-4|$.

(i) $|a| + |b| \leq |a + b|$.

(j) $|a + b| \leq |a| + |b|$.

4. When Joe and Moe were planning to buy a sailboat, they asked a salesman about the cost of a new type of boat that was being designed. The salesman replied, "It won't cost more than \$380." If Joe and Moe had agreed that Joe was to contribute \$130 more than Moe when the boat was purchased,

how much would Moe have to pay?

5. Three more than six times a number is greater than seven increased by four times the number. What is the number?

6. A teacher says, "If I had three times as many students in my class as I do have, I would have at least 26 more than I now have". How many students does he have in his class?

7. A student has test grades of 82 and 91. What must he score on a third test to have an average of 90 or higher?

8. Bill is 5 years older than Norma and the sum of their ages is less than 23. How old is Norma?

There is a strong connection between the relation "is less than" and the idea of equality. This connection can be seen if we consider a few examples. Let $a = 2$ and let $b = 7$. It is clear that

$$a < b$$

since the number 2 is located to the left of the number 7 on the number line. But suppose you were asked to give another reason why it is clear that 2 is less than 7 without using the number line. One good reason might be the following: We could say that 2 is less than 7 <u>because there is a positive number which we have to add to 2 in order to get 7</u>. What number is this? Evidently the number is 5, since

$$2 + 5 = 7.$$

Notice that the number 5, which we added, is a positive number. Thus, we see that there is a close connection between the open sentence

$$2 < 7$$

and the sentence $2 + 5 = 7$.

Both of these sentences bring out the order relation between 2 and 7. This connection can be more firmly extablished by studying further examples.

Sentence involving "<"	Sentence involving "="
$2 < 6$	$2 + 4 = 6$
$(-2) < 6$	$(-2) + 8 = 6$
$(-10) < (-3)$	$(-10) + 7 = (-3)$
$\frac{1}{2} < 4$	$\frac{1}{2} + 3\frac{1}{2} = 4$

If you study the sentences on the right, you will see that in each case the number which is added in order to make the two sides equal is always positive.

We can describe this connection between "is less than" and "is equal to" in two statements

(1) If a real number a is less than a real number b, then there is a positive real number c such that

$$a + c = b.$$

(2) For real numbers a, b, and c, if c is positive and

$$\text{if} \quad a + c = b,$$
$$\text{then} \quad a < b.$$

If $a = (-4)$, and if $b = 5$, then $a < b$. What, in this case, is c?

Suppose $a + 3 = b$. What can you say about the relation between a and b?

Discussion Questions 9-2c

1. If $a < b$, what kind of number must be added to a to obtain b as a sum?
2. If $a + 3 = b$, what is the relation between a and b?

Oral Exercises 9-2c

1. Write a statement of equality for each of the following order relations.

   (a) $3 < 7$
   (b) $-2 < 4$
   (c) $-4 < -5$
   (d) $-\frac{12}{5} < -\frac{9}{5}$
   (e) $.99 < .999$
   (f) $-.4000 < -.39999$
   (g) $m < n$
   (h) $m + k < x$
   (i) $-(m) < -y$
   (j) $3x < 4$

2. State an order relation for each of the following sentences.

   (a) $2 + 4 = 6$
   (b) $-7 + 12 = 5$
   (c) $x + 3 = 8$
   (d) $2x + 11 = x$
   (e) $\frac{2}{3} + \frac{3}{2} = 1$
   (f) $x + 2 = x + 5$
   (g) $-7 + (x + 3) = x + (-4)$
   (h) $3x = 12$
   (i) $5(x + 6) = 18$

9-3 Multiplication Property of Order.

We have seen that whenever a number a is less than a number b, then

$$a + c < b + c$$

no matter what the number c is. That is, it makes no difference whether c is positive, negative, or zero.

Now we ask the following question. Suppose once again that

$$a < b.$$

What happens to the order relation when we multiply both of these numbers by a real number c? In other words, will the product ca be always less than the product cb regardless of what the number c is?

To begin with let's consider an example, letting $a = 5$ and $b = 8$. It is certainly true that

$$5 < 8.$$

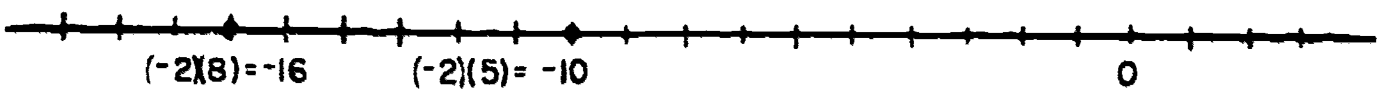
Let $c = 2$. Is the following sentence true?

$$2(5) < 2(8)$$

The answer is easy, since we know that 10 is less than 16. However suppose we let $c = -2$. What about the following sentence?

$$(-2)(5) < -2(8)$$

A glance at the number line



shows us that $(-2)(8)$ is less than $(-2)(5)$, and our sentence above is false. Suppose we try another experiment, this time letting both a and b be negative numbers. Let $a = -7$ and let $b = -2$. Again $a < b$. If c is a positive number, say 3, what is the value of ca? What is the value of cb? Is the following sentence true?

$$3(-7) < 3(-2)$$

Finally, let c have the value -3. What is the relation between

$$(-3)(-7) \text{ and } (-3)(-2)?$$

In Chapter 8 we learned that the product on the left is 21 and the product on the right is 6. It is clear that the sentence

$$21 < 6$$

is a false sentence, and that

$$6 < 21 \quad \text{is true}$$

If we review the previous examples carefully, we should see that in any situation in which one real number is less than another, that is, if

$$a < b,$$

then $ca < cb$ if c is a positive number,

but $cb < ca$ if c is a negative number.

This is called the <u>multiplication property of order</u>. The following table will be helpful

a	b	c	ca	cb	
-3	1	-5	15	-5	cb < ca
-3	1	5	-15	5	ca < cb

You should check each product in the above table, and also the order relations on the right.

If we were to use the other order relation "is greater than", with the symbol ">", it would be possible to state the above property in a different way. We could say that if

$$b > a$$

then if we multiply both sides by a negative number c, we will find that

$$ca > cb.$$

This could be summed up by saying that if two unequal numbers are each multiplied by the same <u>positive</u> number, then the order relation of the products remains the same. If both numbers are multiplied by the same negative number, then the order of the products is reversed.

The multiplication property of order is useful in finding truth sets for certain types of open sentences, such as

$$2x < 8.$$

If there is a real number x for which the above sentence is true, then for this x we have:

$$\frac{1}{2}(2x) < \frac{1}{2}(8)$$

$$\left(\frac{1}{2}(2)\right)x < \frac{1}{2}(8)$$

$$x < 4$$

What is the truth set of $x < 4$? Is this the same as the truth set of

$$2x < 8?$$

Before answering the second question we will state a property which is based on a similar property for equations. If the two sides of an open sentence involving inequalities are each multiplied by the same non-zero real number using the multiplication property of order then the resulting open sentence has the same truth set as the first sentence.

Let us consider a case where the number we are multiplying by is negative.

$$2x < 10$$
$$(-4)(10) < (-4)2x$$
$$-40 < -8x$$

All three of the above open sentences have the same truth set. In other words, they are equivalent.

Discussion Questions 9-3

1. Since $5 < 8$, what relation exists between $2(5)$ and $2(8)$? What relation exists between $(-2)(5)$ and $(-2)(8)$?
2. Is the sentence "$5a < 8a$" true for every a?
3. State the multiplication property of order.
4. By what number would we multiply both sides of "$2x < 8$" to obtain a simpler equivalent sentence?
5. State the true sentence that results from multiplying both sides of the sentence "$2x < 10$" by (-4). Does this sentence have the same truth set as the original sentence?

Oral Exercises 9-3

1. Write the true sentence that results from multiplying both sides of each of the following sentences by the given number, according to the multiplication property of order.

   (a) $2 < 5$, multiply by 4

   (b) $(-2) < (-1)$, multiply by 2

(c) $4 < 7$, multiply by (-1)

(d) $(-1) < 0$, multiply by (-3)

(e) $(-11) < (-3)$, multiply by (0)

(f) $a < b$, multiply by 3

(g) $-3a < 2a$, multiply by (-2)

(h) $a + b < 5$, multiply by 4

(i) $-3x < 4$, multiply by $(-\frac{1}{3})$

(j) $-2 < 4$, multiply by b.

2. In order to find the truth set of each of the following sentences we need a simpler equivalent sentence. By what number would you multiply in each case to obtain this simpler sentence?

(a) $3x < 12$

(b) $\frac{x}{3} < 7$

(c) $-\frac{1}{2}x < 14$

(d) $15 < -3x$

(e) $-12 < 4x$

(f) $7 < \frac{2}{3}x$

(g) $-\frac{5}{8}x < \frac{2}{3}$

(h) $x + 5 < 12$

(i) $2x + 4 < 8$ (Hint: First, add what number?)

(j) $(-6) + \frac{1}{4} < 3 + 3x$

Problem Set 9-3

1. Find the truth set of each of the following sentences. Try some of the numbers in the set to see if they make the sentence true.

(a) $4x < 12$

(b) $\frac{1}{3}x < \frac{1}{2}$

(c) $-2x < -\frac{3}{8}$

(d) $\frac{4}{7} \leq 4x$

(e) $-12 < \frac{2}{3}x$

(f) $-\frac{x}{16} < \frac{1}{2}$

(g) $x + 2 < 3$ (i) $3x = 0$

(h) $(-4) + 7 < -4x + 3x$ (j) $x + 11 = x$

2. Draw the graphs of the truth sets in (a), (d) and (i) of Problem 1.

3. Find the truth sets of the following sentences.

   (a) $x + 5 < 2$

   (b) $(-3x) + (-3) < 8$

   (c) $(-17) + 12 \leq 2x + 4$

   (d) $3x + 2 + (-2x) = x + 5$

   (e) $\frac{1}{2}x + (-7) = -\frac{3}{2}x + 4$

   (f) $x + (-5) < 2x + 7$

   (g) $4.2 + 7x < (-3x) + 7.3$

   (h) $17x + (-7) < 3 + 12x + (-2x) + 7x$

   (i) $(-12x) + \frac{3}{4} = 5x + \frac{1}{2}$

   (j) $(-11) + (-7x) < 4x + 12 + (-4)$

9-4. Summary of the Fundamental Properties of Real Numbers.

Suppose you were asked at this time to give a careful description of just what we mean when we say, "The Real Numbers." It might be very difficult to do. We could say something like, "These are the numbers people use every day," or "They are things like 1, 2, 3, 4, 5," or "They are things we count with," and so forth.

Actually, none of these descriptions would be very accurate, although they do tell us something. To present a better picture of the real numbers, we would probably need to describe many of the properties which we have been studying. The following list could be thought of as a better description of the real number system. It should also provide a very helpful review.

The real number system is a set of elements called real numbers. The system has two operations: addition, with the symbol "+", and multiplication with the symbol "·". The system also has an order relation "is less than" with the symbol "<". The two operations and the relation have the following properties.

1. For any real numbers a and b, $a + b$ is a real number. (closure for addition)

2. For any real numbers a and b, $a + b = b + a$. (commutative property of addition)

3. For any real numbers a, b, and c, $(a + b) + c = a + (b + c)$. (associative property of addition)

4. There is a special real number 0 such that, for any real number a, $a + 0 = a$. (identity element of addition)

5. For every real number a there is a real number $-a$ such that $a + (-a) = 0$. (inverse elements of addition)

6. For any real numbers a and b, $a \cdot b$ is a real number. (closure for multiplication)

7. For any real numbers a and b, $a \cdot b = b \cdot a$. (commutative property of multiplication)

8. For any real numbers a, b, and c, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (associative property of multiplication)

9. There is a special real number 1 such that, for any real number a, $a \cdot 1 = a$. (identity element of multiplication)

10. For every real number a different from 0, There is a real number $\frac{1}{a}$ such that $a \cdot (\frac{1}{a}) = 1$. (inverses of multiplication)

11. For any real numbers a, b, and c, $a(b + c) = a \cdot b + a \cdot c$. (distributive property)

12. For any real numbers a and b, exactly one of the following is true: $a < b$, $a = b$, $b < a$. (comparison property of order)

13. For any real numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$. (transitive property of order)

14. For any real numbers a, b, and c, if $a < b$, then $a + c < b + c$. (addition property of order)

15. For any real numbers a, b, and c, if
    $a < b$ and c is positive
    then $c \cdot a < c \cdot b$
    if $a < b$ and c is negative
    then $c \cdot b < c \cdot a$ (multiplication property of order)

Some of the properties which we have studied are not included in this list. We have left these out for a definite reason, It is because these other properties can be proved by using the ones which appear in the list. We shall include below a set of additional properties. We hope that you will try to discover how these can be shown to be true for real numbers.

Since the first set of properties is all that we need in order to prove, or discover, the others, this first set, as we said before, can be thought of as a description of The Real Number System.

Additional Properties

16. Any real number x has just one additive inverse, namely $-x$.

17. For any real numbers a and b,

$$-(a + b) = (-a) + (-b).$$

18. For real numbers a, b, and c, if $a + c = b + c$, then $a = b$.

19. For any real number a, $a \cdot 0 = 0$.

20. For any real number a, $(-1)a = -a$.

21. For any real numbers a and b, $(-a)b = -(ab)$ and $(-a)(-b) = ab$.

22. Any real number x different from 0 has just one multiplicative inverse, namely $\frac{1}{x}$.

23. The number 0 has no reciprocal.

24. The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.

25. The reciprocal of the reciprocal of a non-zero real number a, is a.

26. For any non-zero real numbers a and b,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

27. For real numbers a and b, if $ab = 0$ then $a = 0$ or $b = 0$.

28. For real numbers a, b, and c with $c \neq 0$, if $ac = bc$, then $a = b$.

29. For any real numbers a and b, if $a < b$, then $-b < -a$.

30. If a and b are real numbers such that $a < b$, then there is a positive number c such that $b = a + c$.

31. If a and b are positive real numbers, and if $a < b$, then $\frac{1}{b} < \frac{1}{a}$.

As an illustration of how a proof might be carried out, let us consider Number 18 from the list of additional properties. "For real numbers a, b, and c, if $a + c = b + c$, then $a = b$." We assume that a, b, and c, are given and that

$$a + c = b + c.$$

We can now use fundamental property Number 5, which, with a change of letter, can be stated,

> For every real number c there is a real number $-c$ such that $c + (-c) = 0$.

Thus, we can now say that

$$(a + c) + (-c) = (b + c) + (-c)$$

since the right and left sides of the equation name the same number.

Also $(a + c) + (-c) = a + \big(c + (-c)\big)$

and $(b + c) + (-c) = b + \big(c + (-c)\big)$ by property Number 3

then $a + 0 = b + 0$

Now by property Number 4 we get

$$a = b,$$

which is the statement we were trying to prove.

Review Problem Set.

1. Which of the following sentences are true for all values of the variables?

   (a) If $a + 1 = b$, then $b > a$.

   (b) If $a + (-1) = b$, then $b + 1 = a$.

   (c) If $a + (-1) = b$, then $a < b$.

   (d) If $(a + c) + 2 = b + c$, then $a + c < b + c$.

   (e) If $(a + c) + (-2) = b + c$, then $a + c = (b + c) + 2$.

   (f) If $(a + c) + (-2) = b + c$, then $b + c < a + c$.

   (g) If $a < (-2)$ then there is a positive number d such that $(-2) + d = a$.

   (h) If $(-2) < a$, then there is a number d such that $ad < -2d$.

   (i) If $3 < 5$, then $3a < 5a$ for every number a.

   (j) If b is a positive number, then $-bc$ is a negative number.

2. Find the truth set of each of the following sentences.

   (a) $(-4) + 7 < (-2)x + (-5)$

   (b) $4x + (-3) > 5 + (-2x)$

   (c) $\frac{2}{3} + (-\frac{5}{6}) < (-\frac{1}{6}) + (-3)x$

   (d) $\frac{1}{2}x + (-2) < (-5) + \frac{5}{2}x$

   (e) $\frac{7}{3}x + 5 + (-2x) = (-x) + (-7) + (-\frac{2}{3}x)$

   (f) $m + (-\frac{1}{2}) \leq \frac{2}{3}m + 7$

   (g) $-5n = 4n + 7 + (-6n)$

   (h) $2x < 3 + (-2)(-\frac{4}{3})$

   (i) $4x + 7 + (-2x) > (-2) + 5 + (-3x)$

   (j) $-(2 + x) < 3 + (-7)$

3. Draw the graphs of the truth sets of parts (a) and (b) of Problem 2.

4. If a rectangle has area 12 square inches and one side has length less than 6 inches, what is the length of the adjacent side?

5. If a rectangle has area 12 square inches and one side has length between 4 and 6 inches, what is the length of the adjacent side?

6. If $x \neq 0$, then x is either negative or positive. If x is positive, then what kind of number is x^2? If x is negative, what about x^2? State a general result about x^2 if $x \neq 0$. What is a general result about x^2 for <u>any</u> real number x?

7. Each of the following expressions is either an indicated sum or an indicated product. Write each indicated sum as an indicated product and write each indicated product as an indicated sum, by using the distributive property.

   (a) $3a\big(a + (-2ab) + b\big)$

   (b) $4x^2 + 2xy + (-2x)$

   (c) $(-4a)\big((-3) + (-2a) + x\big)$

   (d) $a(x + 2) + (-4)(x + 2)$

   (e) $(x + 2)\big(x + (-1)\big)$

   (f) $(3x + 1)(2x + 2)$

   (g) $\big(7y + (-2)\big)\big((-2y) + 4\big)$

   (h) $4mx + 2m + 6am$

   (i) $\big(2r + (-4)\big)(3t + 1)$

   (j) $(x + 2m)^2$

8. Simplify these expressions using the distributive property.

   (a) $3x + (-4)x + (-10x)$

   (b) $7a + 2b$

   (c) $(-4y) + (-8y) + 12y$

   (d) $2rst + (-6stm)$

   (e) $3x + 4y + (-2x) + (-7y)$

9. Two cars start from the same point at the same time and travel in the same direction at average speeds of 34 and 45 miles per hour respectively. In how many hours will they be 35 miles apart?

10. Henry and Charles were opposing candidates in a class election. Henry received 30 votes more than Charles, and 516 members of the class voted. How many votes did Charles get?

11. A man left $10,500 for his widow, a son and a daughter. The widow received $5,000 and the daughter received twice as much as the son. How much did the son get?

12. Each of the following expressions is written as an opposite. In each case write an equivalent expression which is an indicated sum, as shown in the example.

Example: $-\big(3a + (-4) + 2b\big)$

This can be written as

$$(-3a) + 4 + (-2b)$$

(a) $-\big(3a + (-2b)\big)$

(b) $-\big(-2x + 3a + (-7)\big)$

(c) $-\big(2a + (-3) + 2a + 3\big)$

(d) $-\Big(-\big(3a + (-2b)\big)\Big)$

(e) $-(2x+1)(x-1)$ (Hint: Find the product first.)

*13. Prove the following properties of real numbers.

(a) $-(a + b) = (-a) + (-b)$ (Hint: Add $(a + b)$ to $(-a) + (-b)$. What does your result show?)

(b) For real numbers a, b, and c, if $a + c = b + c$ then $a = b$ (Hint: Use the addition property of equality.)

(c) For any non-zero real numbers a and b $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$

(Hint: Multiply $\frac{1}{a} \cdot \frac{1}{b}$ by ab. What does your result show?)

Chapter 10

SUBTRACTION AND DIVISION FOR REAL NUMBERS

10-1. The Meaning of Subtraction.

In arithmetic we did a great deal of subtracting. In this process, however we always subtracted a positive number from an equal or larger positive number. Now that we are working with real numbers, which include negative numbers, we will want a method of subtraction for all of these numbers as well. That is, we shall be interested in finding out how to subtract any real number from any real number. This process will have to apply to the subtraction of a larger number from a smaller number as well as subtraction involving negative numbers.

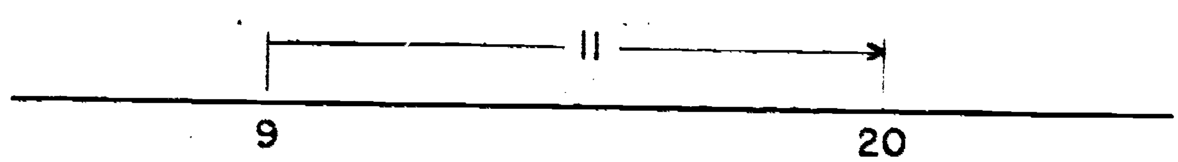
How can we find a rule which will work in all cases? We will begin by studying the process with which you are already familiar. Perhaps we can find out more about this process by using the number line. Let's start, then, with the following simple example:

"Subtract 9 from 20."

We can write this operation in either of the following ways:

$$20 - 9 \quad \text{or} \quad \begin{array}{r} 20 \\ \underline{-\ 9} \end{array}$$

We read "20 - 9" as "20 minus 9". Notice that "20 - 9" is a numeral; it represents a number. What number? The number which must be added to 9 to get 20. To find our answer, or "difference", let us use the number line.

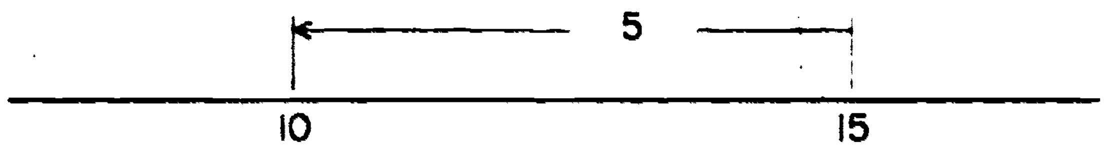


We locate 9 on the number line, and also 20. How far and in what direction do we move to get from 9 to 20? The answer is: 11 units to the right!

Now let's see what happens when we try to

"Subtract 15 from 10."

We can use the same approach. We ask, "What number must be added to 15 to get 10?"



After locating 15 and then 10 we wish to find out how far and in what direction we move to get from 15 to 10. The answer is that we must move 5 units to the left from 15 to get to 10. As we learned before, the process of moving 5 units to the left is associated with the operation of adding the opposite of 5. Thus,

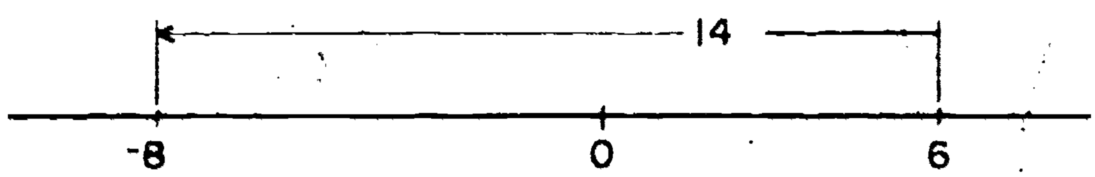
$$15 + (-5) = 10$$

Our answer, then, is (-5). In other words the subtraction statement becomes

$$10 - 15 = (-5).$$

As a further example, let us

"Subtract 6 from -8."



to get from 6 to (-8) we must move 14 units to the left. That is,

$$6 + (-14) = (-8).$$

The answer is (-14). The subtraction statement is

$$(-8) - 6 = (-14).$$

Notice that we are using the symbol "-" to mean two different things. We use the dash <u>between</u> the numerals to indicate subtraction. The dash <u>with</u> the numeral means "the opposite of". Thus, we read the expression

$$(-10) - (-7)$$

as, "The opposite of ten minus the opposite of seven." As a subtraction problem, we can state this as

"Subtract (-7) from (-10)."

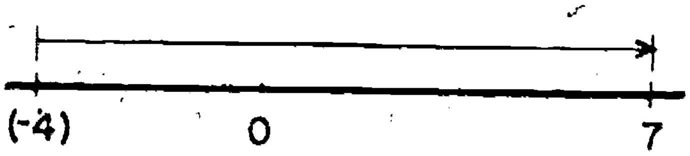
The number line picture of the situation is

We must move three units to the left to get from (-7) to (-10).

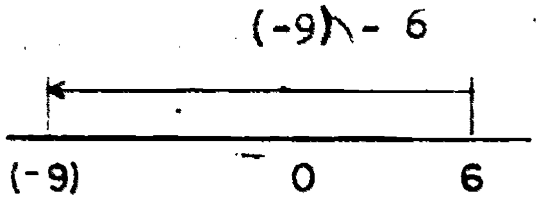
$$(-7) + (-3) = (-10)$$

(-3) is the answer. That is (-10) - (-7) = (-3). Can you supply the missing numbers indicated by "?" in the following examples?

7 - (-4)



(-4) + (?) = 7 so 7 - (-4) = (?).



6 + (?) = (-9) so (-9) - 6 = (?).

In all of the examples above we have subtracted one real number from a second real number. That is, each problem has been of the type a - b. We locate a and b on the number line. Then we ask how far we have to move from b to get to a, and in what direction. If we move to the right, the answer to our subtraction problem is positive. If we move to the left, then the answer is a negative number. If we do not move, the answer is zero.

Discussion Questions 10-1a

1. Express the instruction "Subtract 15 from 10" as a question about addition.
2. In what direction do we move on the number line to add a positive number? To add a negative number?
3. Give two different meanings of the symbol "-" and tell when each meaning applies.
4. Tell what it means on the number line to subtract one number from a second number. From which number do we start? Does the direction of movement have any importance?

Oral Exercises 10-1a

1. State each of the following subtractions as a question about addition.

(a) 5 - 4 = (?)
(b) -5 - 4 = (?)
(c) -5 - (-4) = (?)
(d) 5 - (-4) = (?)
(e) 4 - 5 = (?)
(f) -4 - 5 = (?)
(g) -4 - (-5) = (?)
(h) 4 - (-5) = (?)
(i) 4 - (4 - 5) = (?)
(j) -4 - (4 - (-5)) = (?)

2. In each of the parts of Exercise 1, replace the symbol "?" by the common name for the indicated difference.

3. (a) Is subtraction closed over the set of positive integers?

   (b) Is subtraction closed over the set of non negative real numbers?

   (c) Is subtraction closed over the set of all real numbers?

Problem Set 10-1a

1. Draw diagrams on the number line showing the following differences and write the answers.

   (a) $4 - 3$

   (b) $3 - 4$

   (c) $4 - (-3)$

   (d) $-3 - (-4)$

   (e) $-4 - 3$

2. Refer to the number line, or to the equivalent statement about addition, if necessary, to find the common name for each of the following indicated differences:

   (a) $-18 - 5$

   (b) $4 - 20$

   (c) $18 - (-2)$

   (d) $-49 - (-8)$

   (e) $-30 - 30$

   (f) $30 - (-30)$

   (g) $-30 - (-30)$

   (h) $30 - 30$

   (i) $-108 - (-8)$

   (j) $27 - 49$

3. Complete this statement: To find the difference $a - b$ on the number line start at . . . .

Suppose we now think of subtraction in a somewhat different way. You buy something at the store which costs 83 cents and give the clerk one dollar. How does she count out your change? She first says, "83." Then she puts down two cents and says, "85". She then puts down one nickel and says, "90," and finally she puts down a dime and says, "One dollar, thank you!"

The clerk wants to find the difference between 83 and 100. That is, her problem is really one of subtraction. But what does she actually do? She finds what amount needs to be added to 83 to get 100. In other words, she finds her subtraction answer by adding! She has mentally changed the wording of the problem from the question, "One hundred minus eighty-three equals what?" to "Eighty-three plus what equals one hundred?" The sentence

$$100 - 83 = x$$

has become

$$83 + x = 100.$$

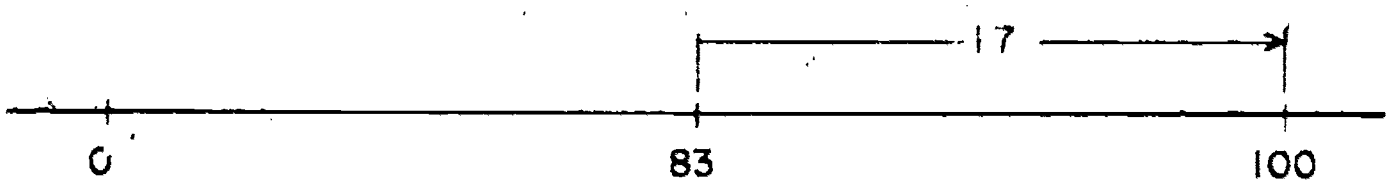
Now, how do we find the truth number of the sentence "$83 + x = 100$"? We can use the addition property as follows:

$$\begin{aligned} 83 + x &= 100 \\ 83 + x + (-83) &= 100 + (-83) \\ x &= 100 + (-83) \\ x &= 17 \end{aligned}$$

The truth set of our sentence is $\{17\}$. The number 17 is also the answer to our subtraction problem. If we had put the problem

$$100 - 83$$

on the number line as we did in the previous examples, we would see that the "move instruction" would give us the same result.



We have seen how to obtain the answers to subtraction problems by three different methods. We used the number line; we counted out the difference, as with the change example; and for the same example we found the truth value for an open sentence. In every instance the process appears to involve some form of addition.

Now let us look particularly at the "truth value" approach. Notice that one of our equivalent open sentences is the following:

$$x = 100 + (-83).$$

The above sentence tells us that our answer in this case is obtained by adding the opposite of 83 to 100. In other words, we can work this subtraction problem by means of two operations, addition, and finding the opposite of a number. Moreover, we have learned how to use these two operations on all real numbers, negative as well as positive. Therefore, it would seem natural for us to describe the subtraction process for all real numbers in this particular way, that is, by means of addition and finding opposites.

We shall give a description of the subtraction process for all real numbers. It will be interesting to see whether this process gives the same results as those which were obtained by means of the number line. Our description may be stated as follows:

> To subtract a real number b from a real number a, add the opposite of b to a. Thus, for real numbers a and b
>
> $$a - b = a + (-b).$$

The following table is based on problems which were solved on the number line.

subtraction problem	number line result	addition of opposites
20 - 9	11	20 + (-9)
10 - 15	(-5)	10 + (-15)
(-8) - 6	(-14)	(-8) + (-6)
(-10) - (-7)	(-3)	(-10) + 7
7 - (-4)	11	7 + 4

Check to see if the numerals in the third column represent the same numbers as those in the second column. Use addition as defined in Chapter 7.

Discussion Questions 10-1b

1. How would you determine the change for a purchase of \$1.63 if payment was made by a \$3.00 bill?
2. Can every subtraction be done by performing addition only? How could you find the difference "300 - 163" by addition?
3. How could you write the question "300 - 163 = ?" as an open sentence involving addition?
4. Subtraction of real numbers can be thought of as involving what two operations?
5. State the definition of subtraction for real numbers.

Oral Exercises 10-1b

1. State the following subtractions in terms of adding an opposite.

(a) 5 - 4	(f) 4a - 3a	(k) $\pi - (-\pi)$
(b) 11 - 12	(g) -2x - (-2)	(ℓ) 8k - (-11k)
(c) -4 - 8	(h) 7y - (-2y)	(m) 6x - 2x
(d) -11 - (-5)	(i) (8 - 12) - 2	(n) 0 - (-3m)
(e) 24 - (-8)	(j) 2 - (8 - 12)	(o) $6\sqrt{2} - 9\sqrt{2}$

Problem Set 10-1b

1. Write each of the following differences in terms of the addition of opposites, and determine the common name.

(a) $15 - 25$

(b) $132 - (-18)$

(c) $-12 - (-24)$

(d) $-7b - 12b$

(e) $-3x - (-4x)$

(f) $\frac{3}{4} - (-\frac{1}{2})$

(g) $7m - (m + 12)$

(h) $-4x - (2x - b)$

(i) $\frac{3}{4}x - \frac{1}{5}x$

(j) $7.4m - (12 - 3.5m)$

2. Find a simpler expression for each of the following.

(a) $2x - 3x + 5x$

(b) $4m - (-3m) + 7m$

(c) $5 - 2m + 6m - 8$

(d) $(2y + 1) - (2y - 1)$

(e) $a + b - \left(-(a + b)\right)$

(f) $6 - |-6| - (-6)$

(g) $-\frac{3}{4} - \frac{1}{20} - (-\frac{4}{5})$

(h) $5 - (3a - 2b - 5)$

(i) $(x + 1)^2 - (x^2 + 2x + 1)$

(j) $\frac{3 + 5}{2 \times 4} - \left(-(-1)\right)$

3. Find the truth sets of these sentences.

(Hint: First change subtractions to addition of the opposite and then proceed as before.)

(a) $x - 5 = -4$

(b) $x - 7 = 6 - 11$

(c) $2x - 3 = 4 - 9$

(d) $3x - |-8| = 5 - 2x$

(e) $4 - 7m - 3 = 2 - m$

(f) $11 - 18 = 4 - 2m$

(g) $4 - (y + 5) = y - 2$

(h) $-18 = 5 - (2y - 4)$

(i) $4 - \frac{2}{3} + x = 5 - \frac{1}{2}x$

(j) $7.3 + 2 - (x - .5) = 2x$

(k) $x - 5 < 4$

(ℓ) $3x - 7 < 4 - 11$

(m) $3x - 1 < 3x - 4$

(n) $14x - 5 < 2x - 3 - (-12x)$

(o) $18 - 3x = x + 18 - 4x$

4. Write a phrase for each of the following:

   (a) Subtract -8 from 15.

   (b) From -25 subtract -4.

   (c) What number is 6 less than -9?

   (d) From 22 take away -30.

   (e) -12 is how much greater than -17?

   (f) How much greater is 8 than -5?

   (g) What is 5 less 10?

   (h) What is the distance on the number line from 11 to 4?

   (i) What is the difference between -5 and -8?

   (j) What number added to -8 gives 7?

5. A marksman hears the bullet hit the target 2 seconds after he fires. He knows the speed of the bullet is 3300 feet per second and the speed of sound is 1100 feet per second. How far away was the target.

   (Hint: Let t be the number of seconds it takes for the bullet to reach the target.)

6. A service station manager wishes to mix "regular" gas at 30¢ per gallon with "ethyl" gas at 35¢ per gallon to fill a 500 gallon tank with gas selling for 32¢ per gallon. How much of each type of gas need he put in the tank?

7. Two men start walking from the same place in the same direction. The second man starts one hour later than the first and walks 3 miles per hour. The first man walks 2 miles per hour. How long will the second man have walked when he catches up to the first?

10-2. Properties of Subtraction.

In previous chapters we have discussed the commutative and associative properties as applied to addition and multiplication. Do you think that subtraction is commutative? To answer this we must determine whether the following sentences are true:

$$5 - 2 = 2 - 5$$
$$6 - 9 = 9 - 6$$
$$(-4) - 2 = 2 - (-4)$$

From our definition of the subtraction process we see that the left side of our first sentence represents the number 3, but the right side of the same sentence represents the number (-3). Therefore, the sentence is false. What can you say about the other two sentences?

Is subtraction associative? Consider the following sentence:

$$10 - (7 - 2) = (10 - 7) - 2.$$

On the left side the phrase inside the parentheses represents the number 5. Thus, the left side has the same value as

$10 - 5$, which is 5.

The phrase in parentheses on the right represents the number 3. The right side has the same value as

$3 - 2$, which is 1.

Is the original sentence true?

Since we have shown that

$$10 - (7 - 2) \neq (10 - 7) - 2,$$

it is clear that the placement of parentheses, that is, the grouping of terms does make a difference.

We found that with addition the associative property allows us to group terms in any way we wish. Thus, an expression like

$$10 + 7 + 2$$

has a perfectly clear meaning. Even though addition is a binary operation, we know perfectly well that

$$10 + 7 + 2 = 19$$

with or without parentheses.

However, if we had an expression such as

$$10 - 7 - 2,$$

how would we interpret it? From the discussion above we see that it might be equal to 5 or to 1, depending on how the terms are grouped. To avoid this confusion we shall agree to a rule, or convention. We shall say that an expression like

$$10 - 7 - 2$$

shall always mean

$$(10 - 7) - 2.$$

From our definition of subtraction this enables us to state that

$$\begin{aligned} 10 - 7 - 2 &= (10 - 7) - 2 \\ &= 10 + (-7) + (-2) \\ &= 1. \end{aligned}$$

<u>Discussion Questions 10-2a</u>

1. Give an example illustrating whether or not subtraction is commutative.
2. How can a phrase involving an indicated subtraction be rewritten so that the commutative and associative properties of addition apply?
3. Which of the following sentences are true?
   (a) $10 - 7 - 2 = (10 - 7) - 2$
   (b) $10 - 7 - 2 = 10 - (7 - 2)$
   (c) $10 - 7 - 2 = 10 + (-7) + (-2)$

<u>Oral Exercises 10-2a</u>

1. Which of the following sentences are true?
   (a) $8 - 5 - 4 = 8 + (-5) + (-4)$
   (b) $8 - 5 - 4 = (8 - 5) - 4$
   (c) $8 - 5 - 4 = \left(8 + (-5)\right) + (-4)$

(d) $8 - 5 - 4 = 8 - (5 - 4)$

(e) $8 - 5 - 4 = 8 + \left((-5) + (-4)\right)$

(f) $8 - 5 - 4 = 8 + \left((-5) + 4\right)$

2. State a common name for each of the following expressions.

(a) $15 - 9 - 12$

(b) $- 4 - 8 - 2$

(c) $- 11 + 8 - 5$

(d) $5 - (4 - 2)$

(e) $15 - 17 + 4$

(f) $(7 - 2) - 8$

(g) $3x - 12x - 5x$

(h) $(8a - 5a) - 4a$

(i) $- 11 - 18 - 7 - 6$

(j) $17 + 12 - 8a - 29 - 2a$

Problem Set 10-2a.

1. Write a common name for each of the following expressions:

(a) $24 - 35 - 47 + 6$

(b) $-3a - 4a - 18a + 22a$

(c) $14x - 11x + \frac{3}{2}x - \frac{7}{2}x$

(d) $\frac{2}{3}y - 2x - 7y + \frac{1}{2}x$

(e) $3(x - 2) - (-2)(x + 3)$

(f) $3x\left((-2x) + (-3) + (-2y)\right)$

(g) $3x(-2x - 3 - 2y)$

(h) $-6.7x - 3.3y - 7.2x + y$

(i) $(-1)\left((-7m) + 2x - 4\right)$

(j) $-\left((-7m) + 2x - 4\right)$

(k) $(a + 2b) - (3a - b) - (a - 2b)$

2. Determine which of the following sentences are true for all values of the variables.

(a) $-a - b - c = (-1)(a + b + c)$

(b) $3a - 2b - c = 3a + (-2b) + (-c)$

(c) $3x + (-7y) + 4 = 3x - 7y + 4$

(d) $12m - (2n + 2) = (12m - 2n) + 2$

(e) $-(3a - x - 2y) = 3a + x + 2y$

(f) $-3m - (2m + 3 - 5n) = 5n - 3 - (3m + 2m)$

It should be clear that once we have changed a statement about subtraction into one involving addition, all the properties of addition will hold.

The sentence

$$7 - 4 = 4 - 7$$

is certainly false. On the other hand, the sentence

$$7 + (-4) = (-4) + 7$$

is definitely a true sentence. Both sides represent the number 3. The following example illustrates how the commutative and associative properties may be used in a problem involving subtraction. Suppose we are given the expression

$$(\frac{6}{5} + 2) - \frac{1}{5}$$

and are asked to represent the same number in a simpler form. That is, suppose we are asked to write the <u>common name</u> for this number. In the future we shall frequently use the word "simplify" to indicate the same process. Using our definition of subtraction we see that

$$(\frac{6}{5} + 2) - \frac{1}{5} = (\frac{6}{5} + 2) + (-\frac{1}{5})$$

By the commutative law for addition we can reverse the numerals in the first parentheses. This gives us

$$(2 + \frac{6}{5}) + (-\frac{1}{5}).$$

By the associative property this can be changed to

$$2 + \left(\frac{6}{5} + (-\frac{1}{5})\right).$$

Our expression now becomes

$$2 + 1$$

which we see is equal to 3.

Before going on with further examples, we should review carefully what it means to find the opposite of a number, when that number is expressed as a numerical phrase. How could we write the opposite of the number represented by the phrase

$$(7 + 2).$$

In Chapter 8 we learned that the opposite of a number could be found by multiplying that number by (-1), that is,

$$(-1)(a) = (-a).$$

From this we see that

$$\begin{aligned} -(7 + 2) &= (-1)(7 + 2) \\ &= (-7) + (-2) \\ &= -7 - 2 \end{aligned}$$

In the same way it follows that

$$\begin{aligned} -(a + b + c) &= (-1)(a + b + c) \\ &= (-a) + (-b) + (-c) \end{aligned}$$

Here we see that the opposite of a sum is the sum of the opposites. By the definition of subtraction this last expression could be written

$$-a - b - c.$$

See if you can state what properties are being used when we <u>simplify</u> the expression $-3x + 5x - 8x$ as follows:

$$\begin{aligned} -3x + 5x - 8x &= (-3x) + 5x + (-8x) \\ &= (-3)x + 5x + (-8)x \\ &= \big((-3) + 5 + (-8)\big)x \\ &= \big(2 + (-8)\big)x \\ &= (-6x). \end{aligned}$$

As a further example, we will simplify the following:

$$\begin{aligned}(5y - 3) - (6y - 8) &= (5y - 3) + \big(-(6y) + 8\big)\\ &= (5y - 3) + \big((-6y) + 8\big)\\ &= 5y + (-3) + (-6y) + 8\\ &= 8 + (-3) + 5y + (-6y)\\ &= 5 + \big(5 + (-6)\big)y\\ &= 5 + (-1)y\\ &= 5 + (-y)\\ &= 5 - y\end{aligned}$$

In each of these two examples every step has been listed. You may eventually be able to do several steps at one time. Perhaps you can discover other ways in which the work can be shortened.

Let us see how the distributive property works in a problem involving subtraction.

Example: Simplify

$$(-3)(2x - 5).$$

By our definition of subtraction, we can write

$$\begin{aligned}(-3)(2x - 5) &= (-3)\big(2x + (-5)\big)\\ &= (-3)(2x) + (-3)(-5)\\ &= (-3)(2)x + 15\\ &= -6x + 15.\end{aligned}$$

Discussion Questions 10-2b

1. How would you simplify $(\frac{6}{5} + 2) - \frac{1}{5}$ in the easiest way?
2. When the instruction "simplify" is given, what can it be taken to mean?
3. What is another way of writing "the opposite of a" besides "$-a$"?
4. How would you write $-(7 + 2)$ as an indicated sum?
5. Complete this sentence: The opposite of the sum of several numbers is the . . . .
6. How would you write $(-3)(2x - 5)$ as an indicated sum?

Oral Exercises 10-2b

1. State the opposite of each of the following expressions:

   (a) b

   (b) $-c$

   (c) $-3c$

   (d) $11x$

   (e) $3x + 2$

   (f) $-2y + 4x$

   (g) $2m + 5n + (-\frac{3}{4})$

   (h) $7x + (-3y) - 4$

   (i) $-x^2 - 7x + 3.2$

   (j) $-\big(-(2y - 8x)\big)$

2. Simplify:

   (a) $(-1)\big(-7x + (-3y) + 4\big)$

   (b) $(-1)(-7x - 3y + 4)$

   (c) $-(-7x - 3y + 4)$

   (d) $3x + 7x + (-2y) + (-4y)$

   (e) $3x + 7x - 2y -4y$

   (f) $\frac{3}{4}x - 2y - \frac{5}{4}x + 2y$

   (g) $3x(2x + 3)$

   (h) $7m(3m - 2)$

   (i) $-4y(-6y + 5)$

   (j) $4x^2 - 7x - 2x(x - 3)$

Problem Set 10-2b

1. Simplify each of the following expressions:

   (a) $-(-2x + 3y -2)$

   (b) $(-1)(4m - 2n + 3)$

   (c) $-\big(-(2x - y)\big)$

   (d) $3x(x + 2) + 5x$

   (e) $-4m(m - 3) - 12m$

(f) $2m^2 - 6m(m - 1) - m$

(g) $-(-(-a)) + (-a) - a$

(h) $3x - 7x^2 - 4x(x - 5)$

(i) $(x + 1)(x + 2) + 2x$

(j) $(x - 1)(x - 2) - 2x$

2. Which of the following sentences are true for all values of the variables?

(a) $-a(b - c) = -ab + ac$

(b) $-(3a - 2b - c) = 3a + 2b + c$

(c) $-\frac{3x - 2y}{2} = \frac{-3x - 2y}{2}$

(d) $(9x - 2)(x - 5) = 9x^2 - 47x + 10$

(e) $-(2a + b) \leq 2a + b$

3. Find the truth set of each of the following sentences:

(a) $3x + (-|-5|) = 7$

(b) $-(2x - 4) = 8$

(c) $m - (2m + 2) = -5$

(d) $-3m < 6$

(e) $-2(x - 4) = 8$

(f) $|-x| = -3$

(g) $2 - (3x + 4) = 5x - (4 - 2x)$

(h) $\frac{1}{12}x + 7 < (-\frac{1}{2})(-\frac{1}{6}x)$

(i) $8.67x = 0$

(j) $(x - 2)(x + 5) = 0$

4. Write simpler expressions for each of the following:

 (a) $(a^2 - 2ab + b^2) - (3a^2 - 2ab - b^2)$

 (b) $(3x - 2y + m) - (6x - 7y + m)$

 (c) $(7a + \frac{3}{4}b - 5) - (6 - 2a + \frac{1}{2}b)$

 (d) $-(2k + 6k^2 - 4) - 3(11 - k + k^2)$

 (e) $-5n(n - 4) + 3(2n - 1)(n + 1)$

5. Translate each of the following into open sentences or phrases.

 (a) John's age 8 years ago.

 (b) A man is 6 times as old as his son.

 (c) Five times a certain distance is 36 miles.

 (d) A rectangle is 2 feet move than twice as long as it is wide.

 (e) The number of feet in 3y yards.

 (f) The value of a certain number of pounds of candy at $1.10 per pound.

 (g) The total value of a certain number of gallons of gasoline at 30¢ per gallon and of 40 gallons of gasoline more than that of gasoline worth 35¢ per gallon.

 (h) The number of cents in 2d dollars.

 (i) I need 15 dollars more than twice the number of dollars I have.

 (j) I need more than twice as much money as I now have.

6. Write the expression that shows the form for the following exercise. Then use the properties of operations and numbers to write this form in the simplest way.

 Take a number, multiply by 7, add 12, subtract 4, add your original number, divide by 8, multiply by 2 subtract 4, multiply by $\frac{1}{2}$.

What answer would you have if you started with 2? with 11? with -3?

7. John has $1.65 in his pocket, all in nickels, dimes, and quarters. He has one more quarter than he has dimes, and the number of nickels he has is one more than twice the number of dimes. How many quarters has he?

8. A milkman has a tray of pint and half-pint bottles. There are 6 times as many pint bottles as half-pint bottles. The total amount of milk contained in the bottles is 39 quarts. How many half-pint bottles are there? Can you show two ways to do this one?

10-3. The Meaning of Division.

Division is a familiar process from arithmetic. As we did with subtraction, we must now figure out a method of division which will work for all real numbers, the negative as well as the positive numbers. Let's begin with a simple problem. You will already know the answer.

"Divide 15 by 3."

We can write this operation in any of the following ways:

$$15 \div 3 \quad \text{or} \quad 3\overline{)15} \quad \text{or} \quad \frac{15}{3}$$

To arrive at an answer we could ask ourselves one of two questions

(1) "How many 3's are there in 15?"
(2) "What number do we multiply by 3 to get 15?"

The second question is the one we will find more useful in this work. In either case the answer is clearly 5.
But suppose we had the following problem:

"Divide (-20) by 4."

The question this time is, "What number do we multiply by 4 to get (-20)?" This may take a little more thought, but it should occur to us from our recent study of the multiplication of real numbers that

$$(4)(-5) = (-20).$$

What is our answer this time? Now suppose we were asked to

divide (-21) by (-7).

since

$$(-7)(3) = (-21),$$

do you see that the answer, or "quotient", in this case is 3?

Discussion Questions 10-3a

1. What are two questions that show the meaning of the symbol $\frac{15}{3}$?
2. What is the meaning of "Divide (-20) by 4"?

Oral Exercises 10-3a

1. For each of the following, state a question that has to do with multiplication, then answer it.

   (a) $\frac{12}{4}$
   (b) $\frac{-12}{4}$
   (c) $\frac{12}{-4}$
   (d) $\frac{-12}{-4}$
   (e) $\frac{4a}{4}$
   (f) $\frac{12m}{3m}$
   (g) $\frac{27x^2}{9}$
   (h) $\frac{x(x + 3)}{x}$
   (i) $\frac{(x + 2)(x - 2)}{(x - 2)}$
   (j) $\frac{3x(x + 4)}{x}$

2. Each of the following phrases or questions can be represented by $\frac{a}{b}$ or $\frac{b}{a}$. Decide which it will be, then state each as a question of the type "What number multiplied by a (or b) gives the product b (or a).

   (a) b divided by a.

   (b) What number multiplied by b gives a?

(c) a times the multiplicative inverse of b.

(d) The quotient of a by b

(e) The ratio of b to a

(f) The number expressed by a fraction whose numerator is a and whose denominator is b.

(g) The result of division where the divisor is b and the dividend is a.

Problem Set 10-3a

Find a simpler expression for each of the following:
(Mentally use the method of changing each of the following to a question about multiplication)

1. $\frac{4}{-1}$

2. $\frac{6a}{3}$

3. $\frac{-30}{6}$

4. $\frac{49}{-7}$

5. $\frac{20m}{-4}$

6. $\frac{-21x}{x}$

7. $\frac{-35a^2}{5a}$

8. $\frac{(a + b)^2}{a + b}$

9. $\frac{-45ax^2}{-9x}$

10. $\frac{3(a + b)}{3}$

11. $\frac{3x(x + 1)^2}{x(x + 1)}$

12. $\frac{6a^2(a - 1)}{-3a(a - 1)}$

13. $\frac{(m + n)(m - n)}{m - n}$

14. $\frac{4x(x - 5)^2}{-2(x - 5)}$

15. $\frac{m^2(m - n)^2}{m(m - n)}$

16. $\frac{-15a^2x^2m^2}{5ax^2}$

*17. $\frac{9ax^2(3 - x)}{3ax(x - 3)}$

*18. $\frac{-7ax^3(x + 2)}{by(x + 1)}$

You will recall that we defined subtraction of a real number as addition of the opposite of the number. Subtraction, that is, was defined in terms of addition.

Since division is related to multiplication in much the same way as subtraction is related to addition, we might expect to define division in terms of multiplication. Let us see how this can be done.

In dividing 15 by 3 we asked the question, "What number multiplied by 3 gives 15?" If we call the answer we are looking for by the name x, then we can write

$$(3)(x) = 15$$

or

$$3x = 15.$$

How can we find the truth value of this sentence? Using the multiplication property of equality we can multiply both sides by $\frac{1}{3}$. This will give us the equivalent sentence

$$(\frac{1}{3})3x = (\frac{1}{3})15$$

or

$$x = 5.$$

You will remember that $\frac{1}{3}$ is what we called the multiplicative inverse of 3. In this chapter we will use the word "reciprocal" to stand for multiplicative inverse. From the above example we see that our answer, or "quotient", 5 was obtained when we <u>multiplied</u> 15 by the <u>reciprocal</u> of 3.

Let's ask the following questions:

Is $(-15) \div 5$ the same as $(-15)(\frac{1}{5})$?

Is $(-21) \div (-7)$ the same as $(-21)(-\frac{1}{7})$?

It seems natural, then to define division on this basis. The definition follows:

> For any real numbers a and b, with b not equal to zero, "a divided by b" means "a multiplied by the reciprocal of b".

The symbol which we will use for "a divided by b" will be the familiar fractional form

$$\frac{a}{b},$$

which means

$$a\cdot\frac{1}{b}$$

As in arithmetic, we shall call "a" the numerator and "b" the denominator. According to our definition we see that

$\frac{10}{2}$ means $(10)(\frac{1}{2})$ which is equal to 5.

$\frac{3}{\frac{1}{5}}$ means $3\cdot\frac{1}{\frac{1}{5}}$, or $3(5)$ which is equal to 15.

$\frac{\frac{2}{3}}{\frac{3}{4}}$ means $\frac{2}{3}(\frac{1}{\frac{3}{4}}) = \frac{2}{3}\cdot\frac{4}{3}$ which is equal to $\frac{8}{9}$.

You will recall from Chapter 8 that

the multiplicative inverse or reciprocal of 2 is $\frac{1}{2}$;

the multiplicative inverse or reciprocal of $\frac{1}{5}$ is $\frac{1}{\frac{1}{5}}$

which is the same as 5;

and the multiplicative inverse or reciprocal of $\frac{3}{4}$ is $\frac{1}{\frac{3}{4}}$

which is the same as $\frac{4}{3}$.

Notice in the definition that we have stated that b is never zero. You must be careful that you never try to divide by zero! Our definition gives us further properties of division by real numbers which will be useful. You should try several examples to convince yourself that the following statements are true.

1) For any real number a

$$\frac{a}{1} = a\cdot(\frac{1}{1}) = a\cdot 1 = a$$

2) For any non-zero real number a

$$\frac{a}{a} = a\cdot\frac{1}{a} = 1$$

3) Suppose a, b, and c are real numbers and $b \neq 0$.

If $\frac{a}{b} = c$, then $a = cb$. If $a = cb$, then $\frac{a}{b} = c$.

4) $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$. This follows from the properties of multiplication.

Discussion Questions 10-3b

1. State the open sentence suggested by the symbol $\frac{15}{3}$.
2. What will we multiply by to obtain a simpler sentence equivalent to $3x = 15$?
3. What is another name for "the multiplicative inverse" of a number?
4. Is $(-15) \div 5$ the same as $(-15) \times \frac{1}{5}$?
5. State the definition of division.
6. What is the reciprocal of b?
7. Does every real number have a reciprocal?

Oral Exercises 10-3b

1. Which of the following sentences are true for all values of the variables? If there is only one exception, state it. Give an example for those which you think are true.

(a) $\frac{1}{b} \cdot b = 1$

(b) $\frac{a}{a} = 0$

(c) $\frac{1}{a}(b + c) = \frac{b}{a} + \frac{c}{a}$

(d) If $-3x = 15$ is true, then $(-\frac{1}{3})(-3x) = (-\frac{1}{3})(15)$ is true.

(e) If $\frac{a}{b} = c$, then $a = cb$.

2. Tell what number you would multiply by to obtain an equivalent sentence whose truth set is easily found.

(a) $15x = 5$

(b) $-3y = -12$

(c) $4m = m + 2$

(d) $\frac{1}{3}w = 28$

(e) $-\frac{2}{3}m = 14$

(f) $\frac{9}{10} + 4 = \frac{3}{8}x - 2$

(g) $4 - \frac{4}{5}y = 6$

(h) $-.9x = 12$

(i) $y = -8 + 4 - 7$

(j) $2k - 5 = k + 2$

3. Tell how each of the following indicated divisions could be expressed as a multiplication, then give its simplest name

(a) $\frac{1}{\frac{1}{5}}$

(b) $\frac{6}{\frac{3}{4}}$

(c) $\frac{-7}{\frac{2}{3}}$

(d) $\frac{x}{\frac{1}{x}}$

(e) $\frac{-3y}{\frac{2}{-3y}}$

Problem Set 10-3b

1. Simplify the following indicated divisions.

(a) $\frac{\frac{3}{1}}{3}$

(b) $\frac{-8}{\frac{3}{4}}$

(c) $\frac{5x}{\frac{9}{10}}$

(d) $\frac{3a - 1}{\frac{1}{3}}$

(e) $\frac{-7y}{\frac{1}{2a}}$

(f) $\frac{2m^2}{\frac{m}{-3}}$

(g) $\frac{-7x}{\frac{1}{x - 1}}$

(h) $\frac{14y + 7}{\frac{7}{2y + 1}}$

(i) $\frac{\frac{x}{5}}{\frac{10}{x + 2}}$

(j) $\frac{(x - 1)^2}{\frac{(x - 1)}{x}}$

2. Find the truth set of each of the following sentences.

(a) $6y = 42$

(b) $2x = 70$

(c) $-5m = -20 + 5$

(d) $3x = 7 + 7x$

(e) $\frac{1}{5}y = 5 - 10$

(f) $\frac{a}{3} - 2 = -21 + 3$

(g) $4 - \frac{3}{4}b = 1 - \frac{2}{3}$

(h) $3m + \frac{2}{3} - 5 = \frac{1}{2}m - 4$

(i) $\frac{2}{3}k \leq 3k - 2$

(j) $-\frac{5}{8}m = \frac{3}{7} + \left|-\frac{1}{7}\right| - \frac{m}{8} + \frac{m}{2}$

3. If it takes $\frac{2}{3}$ of a pound of sugar to make one cake, how many pounds of sugar are needed for 35 cakes for a banquet?

4. If six times a number is decreased by 5, the result is -37. Find the number.

5. If two-thirds of a number is added to 32, the result is 38. What is the number?

6. Find two consecutive odd positive integers whose sum is less than or equal to 83.

10-4. Common Names.

In Chapter 2 we referred to some special names for rational numbers, which we called "common names". The common name is, in a sense, the simplest name for a number. For example:

a common name for $\frac{20}{5}$ is 4

a common name for $\frac{14}{21}$ is $\frac{2}{3}$

How do we obtain these common names? We use the property of 1. Remember, the property of 1 tells us that if a is any real number, then

$$(a)(1) = a.$$

Using this property and also the property that for any non-zero real number a

$$\frac{a}{a} = 1$$

we can see that

$$\frac{20}{5} = \frac{4\cdot 5}{5} = 4\cdot\frac{5}{5} = 4(1) = 4.$$

By means of an additional property we can also show that

$$\frac{14}{21} = \frac{2\cdot 7}{3\cdot 7} = \frac{2}{3}\cdot\frac{7}{7} = \frac{2}{3}(1) = \frac{2}{3}.$$

The step in the above process is which we wrote

$$\frac{2\cdot 7}{3\cdot 7} = \frac{2}{3}\cdot\frac{7}{7}$$

represents an operation which is familiar from our study of fractions in arithmetic. By means of our definition of division it is now possible for us to "prove" that this operation will work in all cases provided that the denominators are not zero.

The theorem, which we will now prove, states that

for any real numbers a, b, c, d,
if $b \neq 0$, and if $d \neq 0$, then

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}.$$

To prove this, we first see that from our definition of division

$$\frac{a}{b} = a\cdot\frac{1}{b} \quad \text{and} \quad \frac{c}{d} = c\cdot\frac{1}{d}$$

and therefore

$$\frac{a}{b}\cdot\frac{c}{d} = (a\cdot\frac{1}{b})(c\cdot\frac{1}{d})$$

$$= (ac)(\frac{1}{b}\cdot\frac{1}{d})$$ (associative and commutative properties of multiplication)

At this point it is necessary to remember that the product of two reciprocals is equal to the reciprocal of the product. This means that in our example

$$\frac{1}{b}\cdot\frac{1}{d} = \frac{1}{bd}$$ (If we multiply $bd(\frac{1}{b}\cdot\frac{1}{d})$, we obtain the product 1, showing $(\frac{1}{b}\cdot\frac{1}{d})$ to be the reciprocal of bd.)

Now we see that

$$\frac{a}{b}\cdot\frac{c}{d} = (ac)(\frac{1}{b}\cdot\frac{1}{d})$$

$$= (ac)(\frac{1}{bd})$$

$$= \frac{ac}{bd}.$$ (using, once more, the definition of division)

Thus, we have "proved" our theorem.

In arithmetic you were told, "To multiply two fractions, we multiply the numerators to get the new numerator and we multiply the denominators to get the new denominator."
The theorem, which states that

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd},$$

shows exactly why your teacher could say this.
The following examples will further illustrate the process of finding the common name.

Example 1: Simplify $\frac{3y - 3}{2y - 2}$

Note: When we write this phrase, we must assume that the domain of the variable y cannot include the number 1. Can you see why?

$$\frac{3y - 3}{2y - 2} = \frac{3(y - 1)}{2(y - 1)}$$ by the distributive property

$$= \frac{3}{2}(1)$$ since $\frac{y - 1}{y - 1} = 1$

$$= \frac{3}{2}$$

Example 2: Simplify $\frac{(2x + 5) - (5 - 2x)}{8}$

$$\frac{(2x + 5) - (5 - 2x)}{8} = \frac{(2x + 5) + \left(-(5 - 2x)\right)}{8} \quad \text{(Why?)}$$

$$= \frac{2x + 2x + 5 + (-5)}{8}$$

$$= \frac{(2 + 2)x}{8} \quad \text{(Why?)}$$

$$= \frac{4x}{8}$$

$$= \frac{4 \cdot x}{4 \cdot 2}$$

$$= \frac{4}{4} \cdot \frac{x}{2} \quad \text{(Why?)}$$

$$= \frac{x}{2}$$

Discussion Questions 10-4

1. What is the meaning of "$\frac{a}{b}$" according to the definition of division?

2. Why is it true that $\frac{a}{b} \cdot \frac{c}{d} = (a \cdot \frac{1}{b})(c \cdot \frac{1}{d})$?

3. Why is it true that $(a \cdot \frac{1}{b})(c \cdot \frac{1}{d}) = ac(\frac{1}{b} \cdot \frac{1}{d})$?

4. What happens if we multiply $bd(\frac{1}{b} \cdot \frac{1}{d})$? What does this show about the relationship between bd and $(\frac{1}{b} \cdot \frac{1}{d})$?

5. Why is it true that $ac(\frac{1}{b} \cdot \frac{1}{d}) = ac(\frac{1}{bd})$?

6. Why is it true that $ac(\frac{1}{bd}) = \frac{ac}{bd}$?

7. Give in your own words the statement of the theorem about the multiplication of two fractions.

8. Why do we not permit y to have the value 1 in the expression $\frac{3y - 3}{2y - 2}$?

Oral Exercises 10-4

1. Tell how each of the following fractions can be written as the product of a simpler fraction and a numeral for 1.

Example: $\frac{4(y + 2)}{5(y + 2)}$ can be written $\frac{4}{5}\left(\frac{y + 2}{y + 2}\right)$, where $\frac{y + 2}{y + 2}$ is a numeral for 1.

(a) $\frac{2}{4}$

(b) $\frac{14}{16}$

(c) $\frac{3x}{3y}$

(d) $\frac{-x}{x^2}$

(e) $\frac{3(x + 2)}{(x - 1)3}$

(f) $\frac{2x + 4}{6x + 12}$

(g) $\frac{2x - 4}{3x - 6}$

(h) $\frac{x + 5}{y + 5}$

(i) $\frac{3x + 4}{3x + 5}$

(j) $\frac{(x + 2)(x + 3)}{x + 3}$

(k) $\frac{3x^2y(m - n)}{9xy^2(m - n)^2}$

2. State a simpler fraction for each part of Exercise 1.

Problem Set 10-4

1. Simplify each of the following expressions.

(a) $\frac{3x}{x^2}$

(b) $\frac{7(x + 2)}{x + 2}$

(c) $\frac{-9xy}{3xm}$

(d) $\frac{3x + 3}{5x + 5}$

(e) $\frac{4x^2 - 4x}{3x - 3}$

(f) $\frac{2x + 4}{2x + 1}$

(g) $\frac{(x + 2)(x - 3)^2}{(x + 2)(x - 3)}$

(h) $(x - 2)(x + 4)\frac{9x^2}{(3x)^2}$

(i) $\frac{x}{b}\cdot\frac{b + 1}{x}$

(j) $\frac{8(x + 1) - (8 - 4x)}{2y - 6 - 6(-y - 1)}$

2. What must we assume about the variables in every part of Problem 1?

3. Find the truth set of each of the following sentences:

(a) $\frac{9x^2}{3x} = 6$

(b) $\frac{3x(x - 1)}{x - 1} = 3$

(c) $\frac{(2x - 5) + (4x - 1)}{4x - 4} < 5$

*(d) $\frac{2x + 2 - (3x - 5)}{21x - 3x^2} = 3x$

*(e) $\frac{7(x - \frac{2}{7}) + 3x - 4}{8x - 2(4 - 4x)} = -\frac{1}{2}$

4. John is three times as old as Dick, and three years ago the sum of their ages was 22 years. How old is each now?

5. Mr. Brown is employed at an initial salary of $3600, with annual increments of $300, while Mr. Jones starts at the same time at an initial salary of $4500, with annual increments of $200. After how many years will both men be earning the same salary?

6. Bob is twice as old as Bill. Three years from now the sum of their ages will be 30 years. How old is each boy now?

7. A table is three times as long as it is wide. If it were 3 feet shorter and 3 feet wider it would be a square. How long and how wide is it?

10-5. Fractions.

When we are asked to simplify a given expression it is important that we understand exactly what is meant. "Simplify", we recall, means "find a common name for". To do this properly we shall have to agree to certain basic ideas as to just what a "common name" should really be. There are three important ideas, or conventions, which we will now discuss.

1) A common name contains no indicated division if it can be avoided. For example:

$\frac{15}{3}$ should be "simplified" to 5.

2) If a common name must contain an indicated division, then the resulting expression should be written in "lowest terms". By this we mean that a fraction such as

$\frac{6}{9}$ should be changed to $\frac{2}{3}$ if we want the common name.

Note: In Chapter 1 we described a "fraction" as a numeral which indicates the quotient of two numbers. Thus, a fraction involves two numerals, a numerator and a denominator. When there is no chance for confusion, we shall often use the word "fraction" to mean the number itself.

3) We have learned that the fraction

$$\frac{-a}{b}$$

may be written as

$$\frac{a}{-b} \text{ or } -\frac{a}{b}.$$

We will call the third form, $-\frac{a}{b}$, the common name.

For example, we write $\frac{-2}{3}$ or $\frac{2}{-3}$ as $-\frac{2}{3}$.

The theorem which ends with the sentence

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$$

tells us how to write the indicated product of two fractions as one fraction. We have used this theorem in applying the property of 1 to the fraction

$$\frac{14}{21}.$$

In this case we used the theorem in an "opposite" sense by splitting one fraction into two, that is

$$\frac{14}{21} = \frac{2\cdot7}{3\cdot7} = \frac{2}{3}\cdot\frac{7}{7}.$$

A direct application can be found in the following example.
Simplify $\frac{x}{3}\cdot\frac{5}{6}$

$$\frac{x}{3}\cdot\frac{5}{6} = \frac{x\cdot 5}{3\cdot 6}$$
$$= \frac{5x}{18}$$

Example: Simplify $\frac{3}{4}\cdot\frac{(x + 2)}{5}$

$$\frac{3}{4}\cdot\frac{(x + 2)}{5} = \frac{3\cdot(x + 2)}{4\cdot 5}$$
$$= \frac{3x + 6}{20}$$

Sometimes we use this theorem "both ways" in the same problem.
Example: Simplify $\frac{3}{2}\cdot\frac{14}{9}$

$$\frac{3}{2}\cdot\frac{14}{9} = \frac{3\cdot 14}{2\cdot 9}$$
$$= \frac{3\cdot(2\cdot 7)}{2\cdot(3\cdot 3)}$$
$$= \frac{7\cdot(2\cdot 3)}{3\cdot(2\cdot 3)} \qquad \text{Why?}$$
$$= \frac{7}{3}\cdot\left(\frac{6}{6}\right)$$
$$= \frac{7}{3}$$

Discussion Questions 10-5a

1. What is a shorter way of saying "find a common name for. . ."?
2. State two things to consider to be sure an expression has been "simplified".
3. What is the common name of $\frac{-a}{b}$? of $\frac{a}{-b}$? of $-\frac{a}{b}$?
4. What theorem shows us how to "simplify" $\frac{3}{4}\cdot\frac{x + 2}{5}$?

Oral Exercises 10-5a

Simplify the following expressions:

1. $\frac{-1}{7}$
2. $\frac{3}{9}$
3. $\frac{-3}{5}$
4. $\frac{4x}{-3}$
5. $\frac{1}{3}\cdot\frac{7x}{2}$
6. $\frac{2x}{3}\cdot\frac{x+1}{2}$
7. $\frac{3}{x}\cdot\frac{x(x-1)}{3}$
8. $\frac{-2}{x-1}\cdot\frac{(x-3)(x-1)}{2x}$
9. $\frac{24x}{-2(x-1)}$
10. $\frac{-(x+1)}{-3(x+1)}$
11. $\frac{x-1}{1-x}$
12. $\frac{3(a+b)}{3(-a-b)}$
13. $\frac{30a^2(a-2)}{2a}\cdot\frac{4}{5a(a-2)}$

Problem Set 10-5a

1. Simplify each of the following expressions:

(a) $\frac{2}{9}\cdot\frac{18}{35}\cdot\frac{7}{4}$

(b) $\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a^2}$

(c) $\frac{6x}{11y}\cdot\frac{7y}{5x}$

(d) $\frac{9b}{7c}\cdot\frac{28c}{81b}$

(e) $\frac{7ab}{5cd}\cdot\frac{15c^2d^2}{28a^2b^2}$

(f) $\frac{2r^2s^2}{9t^2u^2}\cdot\frac{45tu}{14rs}$

(g) $\frac{(x-y)(x+y)}{14}\cdot\frac{7}{x-y}$

*(h) $\frac{2u+2v}{5}\cdot\frac{10}{u+v}$ (Hint: Use the distributive property.)

*(i) $\frac{2c-3d}{c}\cdot\frac{c^2}{10c-15d}$

*(j) $\frac{7m^2+2m}{m}$

2. Simplify each of the following expressions:

(a) $\dfrac{\frac{3}{7}}{\frac{11}{14}}$

(b) $\dfrac{\frac{a}{b}}{\frac{a}{b^2}}$

(c) $\dfrac{\frac{3b}{4c^2}}{\frac{15b^2}{16c^2}}$

(d) $\dfrac{\frac{4a^2}{7}}{8a}$

*(e) $\dfrac{\frac{a-b}{4}}{\frac{a-b}{2}}$

*(f) $\dfrac{\frac{(m-n)(m+n)}{mn}}{m-n}$

3. Find the truth sets of the following sentences:

(a) $\dfrac{\frac{a}{4}}{\frac{1}{4}} = 8$

(b) $\dfrac{8x^2}{2x} + (-4) = 7 + x$

(c) $\dfrac{5(x-2)}{(x-2)} = 12x - 6$

(d) $\dfrac{\frac{3m^2}{5}}{\frac{m}{2}} + 5 = \dfrac{3m}{5}$

(e) $(x - 7)(x + 2) = 0$

(f) $\dfrac{3x + |-6|}{3} = 4$

(g) $|x| = 7$

(h) $\dfrac{4(x-3)(x+3)}{x-3} < 3\left(\frac{4}{3}x + \frac{12}{3}\right)$

(i) $2x - 4 - \frac{1}{3}x \leq 9x + 2$

(j) $18x - 4 + 7x - 11 = 32x + (-11)$

4. A passenger train averages 20 miles per hour more than a freight train. At the end of 5 hours the passenger train has traveled 100 miles farther than the freight train. How fast did the freight train travel?

5. On a 20% discount sale, a chair is marked $30.00. What was the price of the chair before the sale?

6. One-half of a number is 3 more than one-sixth of the same number. What is the number?

As we have seen, a product of two fractions can always be written as one fraction. Thus, in phrases which involve the product of several fractions we can always simplify the phrase to just one fraction. Can you state which property of multiplication makes this possible?

There are some phrases which indicate addition, subtraction, or division of fractions. We shall soon see that in all of these cases we may always find a "simplified" phrase which involves only one indicated division. That is, we may simplify the phrase to one which consists of a single division.

The property which we chiefly use in simplifying the sum or difference of two fractions is the property of 1. You will also recognize many of the other properties which are being used in the following simplification.

$$\frac{x}{3} + \frac{y}{5} = \frac{x}{3}(1) + \frac{y}{5}(1) \qquad \text{(property of 1)}$$

$$= (\frac{x}{3})(\frac{5}{5}) + (\frac{y}{5})(\frac{3}{3}) \qquad (\frac{a}{a} = 1)$$

$$= \frac{5x}{15} + \frac{3y}{15} \qquad (\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd})$$

$$= 5x(\frac{1}{15}) + 3y(\frac{1}{15}) \qquad \text{(definition of division)}$$

$$= (5x + 3y)(\frac{1}{15}) \qquad \text{(distributive property)}$$

$$= \frac{5x + 3y}{15} \qquad \text{(definition of division)}$$

Can you supply the reasons for the steps of the following simplification?

$$\frac{5x}{9} - \frac{2x}{3} = \frac{5x}{9} + (-\frac{2x}{3})$$

$$= \frac{5x}{9} + (\frac{-2x}{3})$$

$$= \frac{5x}{9} + (\frac{-2x}{3})(1)$$

$$= \frac{5x}{9} + \frac{-2x}{3}\cdot\frac{3}{3}$$

$$= \frac{5x}{9} + \frac{-6x}{9}$$

(continued on next page)

$$= 5x\left(\frac{1}{9}\right) + (-6x)\left(\frac{1}{9}\right)$$

$$= \left(5x + (-6x)\right)\frac{1}{9}$$

$$= -x\cdot\frac{1}{9} = \frac{-x}{9} = -\frac{x}{9}$$

If we were asked to find the truth set for an open sentence containing fractions, such as

$$\frac{2x}{3} = \frac{x}{3} + 2$$

we could use the multiplication property as follows:

$$\frac{2x}{3} = \frac{x}{3} + 2$$

may be written as

$$3\left(\frac{2x}{3}\right) = 3\left(\frac{x}{3} + 2\right)$$

where both sides have been multiplied by 3. This becomes

$$\frac{3(2x)}{3} = \frac{3(x)}{3} + 3(2)$$

and then we have

$$\frac{3}{3}(2x) = \frac{3}{3}(x) + 6$$

then we have

$$2x = x + 6 \quad \text{since} \quad \frac{3}{3} = 1.$$

The truth value for this sentence can easily be found by adding $(-x)$ to both sides. Do you see that the truth value is 6?

As another example, consider the sentence

$$\frac{2x}{3} + \frac{1}{3} = \frac{x}{4} + 2.$$

To make our sentence easier to work with, we can multiply both sides by 12. We then have

$$12\left(\frac{2x}{3} + \frac{1}{3}\right) = 12\left(\frac{x}{4} + 2\right)$$

which becomes

$$12\left(\frac{2x}{3}\right) + 12\left(\frac{1}{3}\right) = 12\left(\frac{x}{4}\right) + 12(2) \quad \text{or}$$

$$\frac{3\cdot4\cdot2x}{3} + \frac{3\cdot4}{3} = \frac{3\cdot4\cdot x}{4} + 24$$

As before we now have

$$\left(\frac{3}{3}\right)8x + \left(\frac{3}{3}\right) = \left(\frac{4}{4}\right)3x + 24$$

or

$$8x + 4 = 3x + 24.$$

The truth value of this sentence may now be found quite easily. Do you see that it is 4?

In using the multiplication property we multiplied both sides by the number 12. Can you see why this is a good number to use? We know that 12 can be written as $3\cdot4$.

For the problem of finding the truth set of

$$\frac{3}{7} + \frac{5}{4}y = \frac{3}{14} + \frac{3}{2}y + \frac{1}{7}$$

the term $\frac{5}{4}y$ may be written as $\frac{5}{4}\cdot\frac{y}{1}$, or $\frac{5y}{4}$. Likewise $\frac{3}{2}y = \frac{3y}{2}$. We can then multiply all terms by 28. This gives us

$$28\left(\frac{3}{7}\right) + \frac{5y}{4} = 28\left(\frac{3}{14}\right) + \frac{3y}{2} + \frac{1}{7}$$

or

$$\frac{4\cdot7\cdot3}{7} + \frac{4\cdot7\cdot5\cdot y}{4} = \frac{4\cdot7\cdot3}{2\cdot7} + \frac{2\cdot2\cdot7\cdot3\cdot y}{2} + \frac{4\cdot7\cdot1}{2}$$

Can you complete the steps?

Discussion Questions 10-5b

1. What is the greatest number of indicated divisions that can occur in an expression which is in simplest form?

2. Before the expression $\frac{x}{3} + \frac{y}{5}$ can be simplified what are the forms of 1 that would be used to help in the simplification?

3. Why is $\frac{5x}{15}$ the same as $5x(\frac{1}{15})$?

4. What property can be applied to "$5x(\frac{1}{15}) + 3y(\frac{1}{15})$" to simplify the expression?

Oral Exercises 10-5b

1. State what form of 1 would be used to make the indicated change in each of the following fractions. Then name the new numerator.

(a) $\frac{3x}{4} = \frac{?}{8}$

(b) $\frac{\frac{3}{4}x}{7} = \frac{?}{28}$

(c) $\frac{3}{x} = \frac{?}{3x}$

(d) $\frac{5a}{-y} = \frac{?}{ay}$

(e) $\frac{7a}{x + y} = \frac{?}{3(x + y)}$

(f) $\frac{5}{a - b} = \frac{?}{(a - b)(a + b)}$

(g) $\frac{6a}{3a(m + n)} = \frac{?}{3a(m + n)^2}$

(h) $\frac{b(x - y)}{a(x + y)} = \frac{?}{ab(x + y)(x - y)}$

(i) $\frac{4}{2 - x} = \frac{?}{x - 2}$

(j) $\frac{2 - x}{3 - x} = \frac{?}{x - 3}$

2. Simplify each of the following sums:

(a) $\frac{x}{4} + \frac{3x}{4}$

(b) $\frac{3a}{6} - \frac{a}{6}$

(c) $\frac{4m}{5} + \frac{n}{5}$

(d) $\frac{5x + 2}{4} + \frac{7x - 1}{4}$

(e) $\frac{4y - 8}{11} - \frac{8y + 2}{11}$

(f) $\frac{\frac{3}{4}t + 2}{6} - \frac{\frac{3}{4}t - 11}{6}$

(g) $\frac{7}{8x} + \frac{3}{8x} + \frac{y}{8x}$

(h) $\frac{-11}{x + y} + \frac{12}{x + y} - \frac{3y}{x + y}$

(i) $x + \frac{x}{4} - \frac{x}{2}$

(j) $\frac{3}{x} - \frac{2}{3x} + \frac{1}{x}$

Problem Set 10-5b

1. Simplify the following expressions:

(a) $\frac{a}{a + b} + \frac{b}{a + b}$

(b) $\frac{3x}{4} + \frac{2y}{4}$

(c) $a^2 - b^2 - (a^2 - b^2)$

(d) $\frac{3x}{7} + \frac{y}{14}$

(e) $\frac{4a}{36} - \frac{a}{4}$

(f) $x + \frac{1}{x}$

(g) $a + b + \frac{1}{a + b}$

(h) $\frac{7}{a + b} + b + \frac{5}{a}$

(i) $\frac{\frac{3}{4}}{\frac{2x}{5}}$

(j) $\frac{3}{x + 1} - \frac{4}{x - 1}$

2. Find the truth set of each of the following sentences:

(a) $\frac{x}{5} + 3 = \frac{2}{3}$

(b) $\frac{7}{8} - \frac{1}{4}z = \frac{1}{2}$

(c) $\frac{3}{5}x - \frac{1}{2} = \frac{8}{15}x$

(d) $\frac{y}{2} + \frac{1}{3}y = \frac{2y}{5}$

(e) $\frac{x}{7} + \frac{1}{2} > \frac{9}{14}$

(f) $\frac{3}{4}x - \frac{2}{3} = \frac{1}{4} + \frac{5}{6}x$

(g) $\frac{3}{7} + \frac{5}{4}y = \frac{3}{14} + \frac{3}{2}y + \frac{1}{7}$

(h) $\frac{1}{3}z + \frac{1}{3} = 2 + \frac{1}{2}z$

(i) $4y + 7 = 4y - 3$

(j) $-\frac{3}{4} + \frac{4}{5}u = \frac{11}{8} + u - \frac{5}{8}$

3. Mary bought 15 three-cent stamps and some four-cent stamps. If she paid \$1.80 for all the stamps, was she charged the correct amount?

4. John has 50 coins which are nickels, pennies, and dimes. He has four more dimes than pennies, and six more nickels than dimes. How many of each kind of coin has he? How much money does he have?

5. John, who is saving his money for a bicycle, said, "When I have one dollar more than three times the amount I now have, I will have enough money for my bicycle." If the bicycle costs \$76, how much money does John have now?

We will now see how to simplify the quotient of two fractions. Several methods are possible for doing this. Let us look at the following example, which will be worked in three ways. Simplify

$$\frac{\frac{5}{3}}{\frac{7}{2}}$$

Method 1: We shall use the property of 1, in which we let $1 = \frac{6}{6}$. The reason for using $\frac{6}{6}$ will be made clear as the work goes on.

$$\frac{\frac{5}{3}}{\frac{7}{2}} = \frac{\frac{5}{3}}{\frac{7}{2}} \cdot \frac{6}{6} \qquad \text{(Multiplication property of 1)}$$

$$= \frac{\frac{5}{3} \cdot \frac{6}{1}}{\frac{7}{2} \cdot \frac{6}{1}} \qquad \text{(Multiplication property of 1)}$$

$$= \frac{\frac{5 \cdot 6}{3}}{\frac{7 \cdot 6}{2}}$$

(continued on next page)

$$= \frac{\frac{5\cdot 2\cdot 3}{3}}{\frac{7\cdot 3\cdot 2}{2}}$$

$$= \frac{(10)(1)}{(21)(1)}$$

$$= \frac{10}{21}$$

Method 2: We shall also use the property of 1, but this time we shall let

$$1 = \frac{\frac{2}{7}}{\frac{2}{7}}$$

where we choose $\frac{2}{7}$ because it is the "reciprocal" of $\frac{7}{2}$.

$$\frac{\frac{5}{3}}{\frac{7}{2}} = \frac{\frac{5}{3}}{\frac{7}{2}} \cdot \frac{\frac{2}{7}}{\frac{2}{7}}$$

$$= \frac{\frac{5}{3}\cdot\frac{2}{7}}{\frac{7}{2}\cdot\frac{2}{7}}$$

$$= \frac{\frac{10}{21}}{1}$$

$$= \frac{10}{21}$$

Method 3: In this method we apply the definition of division directly. From the definition we see that

$$\frac{\frac{5}{3}}{\frac{7}{2}} = \frac{5}{3}\cdot\frac{2}{7}$$ ($\frac{2}{7}$ is the "reciprocal" of $\frac{7}{2}$)

$$= \frac{10}{21}$$

You may use any of these methods for simplifying the quotient of two fractions, as long as you do not receive specific instructions to use one particular method for a given exercise. It is very important that you understand each step in whatever process you are using.

Discussion Questions 10-5c

1. In Method 1 why was $\frac{6}{6}$ used as the form of 1?

2. In Method 2 what happened as a result of using $\frac{\frac{2}{7}}{\frac{2}{7}}$ for 1?

3. How was Method 3 different from the first two methods?

Oral Exercises 10-5c

1. State two forms of 1 that could be used to simplify each of the following expressions:

(a) $\frac{\frac{1}{2}}{\frac{3}{4}}$

(b) $\frac{\frac{a}{b}}{\frac{c}{d}}$

(c) $\frac{\frac{3x}{2}}{\frac{y}{3}}$

(d) $\frac{\frac{1}{3x}}{\frac{2}{5x}}$

(e) $\frac{\frac{4x}{3y}}{\frac{3y}{4x}}$

(f) $\frac{\frac{-7a^2}{4b}}{\frac{-2a}{7b}}$

(g) $\frac{\frac{2r + t}{3m}}{\frac{-7}{6m}}$

(h) $\frac{\frac{x + 2}{3}}{\frac{x - 1}{6}}$

(i) $\frac{\frac{4(x - 2)}{3x}}{\frac{x - 2}{x}}$

(j) $\frac{\frac{(x - 1)(x - 2)}{(x - 3)}}{\frac{3x}{(x - 1)}}$

2. Tell how you could simplify each of the expressions in Problem 1 by using the idea of "reciprocal".

Problem Set 10-5c

1. Simplify each of the following expressions or perform the indicated operations:

(a) $\dfrac{\frac{3a}{4}}{\frac{a}{3}}$

(b) $\frac{3a}{4} + \frac{a}{3} - 2a$

(c) $(-3a^2b)(2bx)$

(d) $-(2a - 3b + 2c - a + b)$

(e) $\dfrac{\frac{x+1}{6}}{\frac{(x+1)^2}{3}}$

(f) $\dfrac{-28a^2bx}{-12ab^2x}$

(g) $(x + 2)(x - 2)$

(h) $\dfrac{\frac{(y-2)(y-3)}{9y}}{\frac{3xy(y-3)}{81xy^2}}$

(i) $(\frac{3}{8}x - 2y + 4) - (\frac{1}{2}x - \frac{7}{5}y - 8)$

(j) $\dfrac{x + 5}{2x + 5}$

(k) $\dfrac{\frac{4x}{3} - \frac{1}{5}}{\frac{7}{15}}$

(ℓ) $\dfrac{\frac{4a}{3} + 2a}{\frac{1}{2} + \frac{a}{3}}$

*(m) $\dfrac{\frac{5y}{4} - \frac{2y}{3}}{\frac{4}{15} + \frac{y}{5}}$

*(n) $\dfrac{\frac{2}{3x} + \frac{3}{x}}{\frac{7}{3x^2}}$

*(o) $\dfrac{\frac{3-x}{x-2} + \frac{x+4}{x-3}}{\frac{4}{x-3} - \frac{x}{x-2}}$

2. Find the truth set of each of the following sentences:

(a) $(x - 4)(x + 8) = 0$

(b) $\dfrac{\frac{(x - 5)^2}{2}}{\frac{x - 5}{4}} = 0$

(c) $-7x + 4 - 5x = -8 - 2x + 2$

(d) $\frac{x}{5} + \frac{1}{3} - \frac{x}{3} = \frac{3}{5}$

(e) $\dfrac{\frac{7x}{-3}}{\frac{4x}{2}} + \frac{x}{3} - 5 = 7$

(f) $|-3|\, x = \big|\,|4| - |-8|\,\big|$

(g) $|-4| + x - (-7) < 8 - (-x) - (-|8|)$

(h) $\dfrac{\frac{3x(x - 2)}{4}}{12x} + 3 - x = 9$

(i) $\frac{3(x + 4)}{4(x + 4)} = \frac{3}{4}$

(j) $x(x - 3)(x + 2) = 0$

3. Draw the graphs of the truth sets of parts (f), (g), (i), (j), of Problem 2

You can probably see how to obtain the answers to the following problems without using a variable. Use this as a check of your work after you solve them by using a variable, translating the problem to a mathematical sentence and finding the truth set of the sentence.

4. The sum of three successive positive integers is 1080. Find the integers.

5. The sum of two successive positive integers is <u>less</u> than 25. Find the integers.

6. Find two consecutive even integers whose sum is 46.

7. The sum of a whole number and its successor is 45. What are the numbers?

8. The sum of two consecutive odd numbers is 75. What are the numbers?

9. One number is 5 times another. The sum of the two numbers is 15 more than 4 times the first. What are the numbers?

10. Two trains leave New York at the same time; one travels north at 60 m.p.h. and the other south at 40 m.p.h. After how many hours will they be 125 miles apart?

Summary.

In this chapter, we have introduced new definitions of subtraction and division of real numbers. These definitions enable us to

1) Perform the operation of subtraction by means of addition and the finding of opposites.

2) Perform the operation of division through multiplication by reciprocals.

We have shown that subtraction is neither associative nor commutative. However, a rule has been established for dealing with expressions of the form

$$a - b - c.$$

we say that

$$\begin{aligned} a - b - c &= (a - b) - c \\ &= a + (-b) + (-c). \end{aligned}$$

We have shown that phrases which take the form of addition, subtraction, multiplication, and division of fractions may be simplified by means of the property of 1, and the definition of division.

We have agreed that the "common name" of a phrase shall contain no indicated division if it can be avoided. We further agree that all fractions should be expressed in "lowest" terms, and that of the possible three forms of a fraction

$$\frac{-a}{b}, \quad \frac{a}{-b}, \quad -\frac{a}{b}$$

the common name will be $-\frac{a}{b}$.

Review Problem Set

1. Find the reciprocals of the following numbers.

(a) $\frac{3}{4}$ (f) -1 (k) $-\frac{1}{2}$

(b) 0.3 (g) $\sqrt{2}$ (ℓ) $\frac{5}{6}$

(c) -0.3 (h) $a^2 + 1$ (m) $\frac{1}{100}$

(d) $.33$ (i) $\frac{1}{x^2 + 4}$ (n) -6.8

(e) 1 (j) $y^2 + 1$ (o) $.45$

2. For what values of a do the following expressions have no reciprocals? (Hint: What number has no reciprocals?)

(a) $a - 1$

(b) $a + 1$

(c) $a^2 - 1$

(d) $a(a + 1)$

(e) $\frac{a}{a + 1}$

(f) $a^2 + 1$

(g) $\frac{1}{a^2 + 1}$

(h) $\frac{(a + 1)^2}{3(a + 1)}$

(i) $\frac{(a - 3)(a + 3)}{(a - 3)(a + 3)}$

*3. Consider the sentence

$$(a - 3)(a + 1) = a - 3,$$

which has the truth set $\{0,3\}$. Check this. If both sides of the sentence ase multiplied by the reciprocal of $(a - 3)$, that is, by $\frac{1}{a - 3}$, and some properties of real numbers are used (which properties?), we obtain

$$a + 1 = 1.$$

If we let a be 3 in this last sentence we get $3 + 1 = 1$, and which is clearly a false sentence. Why doesn't this new sentence have the same truth set as the original sentence?

4. Obtain the simplest expression for each of the following indicated subtractions:

(a) $(19x^2 + 12x - 15) - (20x^2 - 3x - 1)$

(b) $(8a - 13) - (7a + 12)$

(c) $(14a^2 - 5a + 1) - (6a^2 - 9)$

(d) $(3n + 12p - 8a) - (5a - 7n - p)$

(e) $(7x^2 - 7) - (3x + 9)$

(f) $(a^2 - b^2) - (a^2 - 2ab + b^2)$

(g) From $11a + 13b - 7c$ subtract $8a - 5b - 4c$.

(h) What is the result of subtracting $-3x^2 + 5x - 7$ from $-3x + 12$?

(i) What must be added to $3s - 4t + 7u$ to obtain $-9s - 3u$?

5. Consider three pairs of numbers: (a) $a = 2$, $b = 3$; (b) $a = 4$, $b = -5$; (c) $a = -4$, $b = -7$. Does the sentence $\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab}$ hold true in all three cases?

6. Is the sentence $\frac{1}{a} > \frac{1}{b}$ true in all three cases of Problem 5? Plot the reciprocals on the number line.

7. Is it true that if $a > b$ and a, b are positive, then $\frac{1}{b} > \frac{1}{a}$? Try this for some particular values of a and b.

8. Is it true that if $a > b$ and a, b are negative, then $\frac{1}{b} > \frac{1}{a}$? Substitute some particular values of a and b.

9. Could you tell immediately which reciprocal is greater than another if one of the numbers is positive and the other negative?

10. If $a > b$, what can you say of $a - b$? Complete this statement: If a is to the right of b, then the difference $a - b$ is ________________

11. If $(a - b)$ is a positive number, which of the statements, $a < b$, $a = b$, $a > b$, is true? What if $(a - b)$ is a negative number? What if $(a - b)$ is zero?

12. If a, b, and c are real numbers and $a > b$, what can we say about the order of $a - c$ and $b - c$?

13. Simplify these expressions using the distributive property where necessary.

    (a) $a^2 + 3a^2$

    (b) $\pi - (-\pi)$

    (c) $8k - (-11k)$

    (d) $6\sqrt{2} - 9\sqrt{2}$

    (e) $6x - 2x$

    (f) $9x^2 + (-4x^2)$

    (g) $0 - 5a$

    (h) $(-25pq) - pq$

    (i) $(-12y) - 4y$

    (j) $(-3b) - (-3b)$

    (k) $(-4y) - 0$

    (ℓ) $0 - (-3m)$

14. The temperature drops 15° from an initial temperature of 4° above zero. Express this statement as a subtraction of real numbers and find the resulting temperature.

15. A submarine has been cruising at 50 feet below the surface. It then goes 30 feet deeper. Express this change as a subtraction of real numbers and find the resulting depth.

16. -16 is 25 less than some number. Find the number.

17. If the time at 12 o'clock midnight is considered as the starting time, that is, at 12 o'clock midnight $t = 0$, what is the time interval from 11 o'clock P.M. to 2 o'clock o'clock A.M.? From 6 o'clock A.M. to 4 o'clock A.M. the next day?

18. From a point marked 0 on a straight road, John and Rudy ride bicycles. John rides 10 miles per hour and Rudy rides 12 miles per hour. Find the distance between them after 3 hours if

   (a) They start from the 0 mark at the same time and John goes east and Rudy goes west.

   (b) John is 5 miles east and Rudy is 6 miles west of the 0 mark when they start and they both go east.

   (c) John starts from the 0 mark and goes east. Rudy starts from the 0 mark 15 minutes later and goes west.

   (d) Both start at the same time. John starts from the 0 mark and goes west and Rudy starts 6 miles west of the 0 mark and also goes west.

19. If b is the multiplicative inverse (reciprocal) of a,

   (a) What values of b do we obtain if a is larger than 1?

   (b) What values of b do we obtain if a is between 0 and 1?

   (c) What is b if a is 1?

   (d) What is b if a is -1?

   (e) What values of b do we obtain if a is less than -1?

   (f) What values of b do we obtain if $a < 0$ and $a > -1$?

   (g) What kind of number is b if a is positive?

   (h) What kind of number is b if a is negative?

   (i) What is b if a is 0?

   (j) If b is the reciprocal of a, what can you say about a?

20. (a) What is the value of $87 \times (-9) \times 0 \times \frac{2}{3} \times 642$?

(b) Is $8 \cdot 17 = 0$ a true sentence? Why?

(c) If $n \cdot 50 = 0$, what can you say about n?

(d) If $p \cdot 0 = 0$, what can you say about p?

(e) If $p \cdot q = 0$, what can you say about p or q?

(f) If $p \cdot q = 0$, and we know that $p > 10$, what can we say about q?

(g) If $(x - 5) \cdot 7 = 0$, what must be true about 7 and $(x - 5)$? Can 7 be equal to 0? What about $(x - 5)$ then?

(h) Explain how we know that the only value of y which will make $9 \times y \times 17 \times 3 = 0$ a true sentence is 0.

(i) How can we, without just guessing, determine the truth set of the equation $(x - 8)(x - 3) = 0$?

21. Find the truth set of each of the following equations:

(a) $(x - 20)(x - 100) = 0$

(b) $(x + 6)(x + 9) = 0$

(c) $x(x - 4) = 0$

(d) $4(x + 34) = 0$

(e) $(x - 1)(x - 2)(x - 3) = 0$

(f) $2(x - \frac{1}{2})(x + \frac{3}{4}) = 0$

(g) $(3x - 5)(2x + 1) = 0$

(h) $9|x - 6| = 0$

22. Simplify each of the following expressions:

(a) $\dfrac{\frac{2}{3} + \frac{1}{6}}{\frac{5}{6}}$

(b) $\dfrac{2}{\frac{1}{8} + \frac{5}{12}}$

(c) $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{3} - \frac{1}{4}}$

(d) $\dfrac{\frac{-3ax}{17}}{\frac{5x}{8}}$

*(e) $\dfrac{\frac{3}{a} - \frac{5}{2a}}{\frac{3}{b} + \frac{5}{2b}}$

(f) $\dfrac{\frac{a - b}{2}}{\frac{a - b}{-4}}$

*(g) $\dfrac{\frac{6x}{5} - \frac{x}{4}}{\frac{x}{2} + \frac{2x}{5}}$

*(h) $\dfrac{\frac{x + 8}{9}}{\frac{3}{x + 2}}$

(i) $\dfrac{\frac{y}{y + 1}}{\frac{y - 1}{3y}}$

(j) $(\frac{3}{8} - \frac{1}{2}) + (\frac{5}{4} + \frac{1}{16})$

(k) $(\frac{1}{9} + \frac{2}{3})(\frac{1}{3} - \frac{5}{6})$

*(ℓ) $\dfrac{a + 6}{a - 6} \cdot \dfrac{a + 2}{a - 2}$

(m) $\dfrac{a^2 - 3}{9} \div \dfrac{a^2 - 3}{9}$

*(n) $\dfrac{1 - \frac{2}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$

*(o) $\dfrac{\frac{a}{b} + \frac{b}{a}}{\frac{1}{a} + \frac{1}{b}}$

*(p) $\dfrac{x - 2}{3 - x} \cdot \dfrac{x - 3}{2 - x}$

*(q) $\dfrac{x - 2}{3 - x} \div \dfrac{2 - x}{x - 3}$

*(r) $\dfrac{\frac{3}{x - 1} + 1}{\frac{-5}{x + 1} - 1}$

*(s) $\dfrac{\frac{2}{a - 1} + \frac{1}{a + 1}}{\frac{1}{a - 1} + \frac{2}{a + 1}}$

*23. Find the truth set of each of the following sentences:

(a) $\dfrac{\frac{a-1}{2}}{\frac{3}{4}} = 1$

(b) $\dfrac{2x}{\frac{1}{6} + \frac{5}{12}} = 1$

(c) $\dfrac{2x}{\frac{1}{8} + \frac{5}{12}} = 0$

(d) $\dfrac{\frac{6x}{5} - \frac{x}{4}}{\frac{1}{2} + \frac{4}{5}} = x$

24. Write the first step in using the distributive property to expand $(3x + 5)(2x - 3)$.

25. Why is the following sentence true?

$$\left(25 + (-10)\right) + (-25) = \left(25 + (-25)\right) + (-10).$$

26. Show that $\frac{3}{8} < \frac{9}{20}$ and $\frac{9}{20} < \frac{7}{15}$ are true sentences. Then tell why you know immediately that $\frac{3}{8} < \frac{7}{15}$ is true.

27. Each edge of a square is made twice as long. How much has the perimeter been increased? How much has the area been increased?

28. $A = \{0, |-1|\}$ and $B = \{-1, 0, 1\}$.

(a) Under which of the operations: addition, subtraction, multiplication, division, is set A closed? set B?

(b) If C is the set of numbers obtained by squaring elements belonging either to set A or set B, enumerate set C. Is it a subset of A? of B?

29. Given the fraction $\frac{3x + 5}{2x - 7}$; what is the only value of x for which this is not a real number?

30. (a) A positive rational number is equal to $\frac{2}{3}$. If its numerator is less than 24, what can be said of its denominator?

(b) If the denominator is less than 24, what can be said of its numerator?

31. The product of two numbers is 2. If one of the numbers is less than 3, what is the other? If one is less than -3, what is the other?

32. Does division have the associative property? That is, is $(a \div b) \div c = a \div (b \div c)$? Give reasons for your answer.

33. Is division commutative? Give reasons for your answer.

34. If $x = a + \frac{1}{a}$, and $a = \frac{1}{2}$, what is the value of $ax + a^2$?

35. If a is between p and q, is $\frac{1}{a}$ between $\frac{1}{p}$ and $\frac{1}{q}$? Explain.

36. Translate each of the following phrases into open phrases. Describe the variable carefully where necessary.

    (a) The number of feet in 6y yards

    (b) The number of inches in 2f feet

    (c) The number of pints in 4k quarts

    (d) A girl's age 10 years ago

    (e) The number of ounces in k pounds and t ounces

    (f) The number of square inches in f square feet

    (g) The number of cents in d dollars and k quarters

    (h) The number of cents in d dollars, k quarters, t dimes and n nickels

    (i) The successor of a whole number

    (j) The reciprocal of a number (Two numbers are reciprocals if their product is one.)

    (k) The number of feet traveled in k miles

    (ℓ) Twice the number of feet traveled in k miles

    (m) The second side of a triangle is three inches longer than the first side

(n) The number of inches in the third side is one-half the sum of the number of inches in each of the other two sides (as described in part m)

(o) Choose a number.
Add five to it to get a second number.
Subtract five from it to get a third number.
Average the second and third numbers.

(p) Choose a number.
Multiply it by five.
Add to this product the number with which you started.

37. Write meaningful English sentences which correspond to the following open sentences.

(a) $x < 80$

(b) $y = 3600$

(c) $z > 100{,}000{,}000$

(d) $u + v + w = 180$

(e) $z(z + 18) = 360$

(f) $x(3x) \leq 300$

(g) $(x + 1)^2 > x(x + 2)$

(h) $30(20.00) \leq (30 - x)(24.00)$

(i) $3a = 4b$

(j) $n > 13$ and $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) < 90$

38. Write open phrases corresponding to the following word phrases, being careful to describe what number the variable represents.

(a) A number diminished by 3

(b) Temperature rising 20 degrees

(c) Cost of n pencils at 5 cents each

(d) The amount of money in my pocket: x dimes, y nickels, and 6 pennies

(e) A number increased by twice the number

(f) A number increased by twice another number

(g) The number of days in w weeks.

(h) Cost of purchases: x melons at 29 cents each and y pounds of hamburger at 59 cents a pound

(i) Area of a rectangle having one side 3 inches longer than another

(j) One million more than twice the population of any city in Kansas

(k) Annual salary equivalent to x dollars per month

(ℓ) Arthur's allowance, which is one dollar more than twice Betty's

(m) The distance traveled in h hours at an average speed of 40 m.p.h.

(n) The real estate tax on property having a valuation of y dollars, the tax rate being $25.00 per $1000 valuation

(o) Donald's weight, which is 40 pounds more than Earl's.

(p) Speed of a car which is one mile per hour less than that of a following car

(q) Cost of x pounds of steak at $1.59 per pound

(r) Catherine's earnings for z hours at 75 cents an hour

(s) Cost of g gallons of gasoline at 33.2 cents a gallon

39. Write open sentences corresponding to the following word sentences, and carefully describe the variable used.

(a) Mary, who is 16, is 4 years older than her sister.

(b) Bill bought bananas at 9 cents each and paid 54 cents.

(c) If a number is added to twice the number, the sum is less than 39.

(d) Arthur's allowance is one dollar more than twice Betty's but is two dollars less than 3 times Betty's.

(e) The distance from Dodge City to Oklahoma City, 260 miles, was traveled in t hours at an average speed of 40 miles an hour.

(f) The auto trip from St. Louis to Memphis, 300 miles, was made in t hours at a maximum speed of 50 miles an hour.

(g) Pike's Peak is more than 14,000 feet above mean sea level.

(h) A book, 1.4 inches thick, has n sheets; each sheet is 0.003 inches thick, and each cover is $\frac{1}{10}$ inches thick.

(i) Three million is over one million more than twice the population of any city in Colorado.

(j) A square of side x has a smaller area than has a rectangle of sides $(x - 1)$ and $(x + 1)$.

(k) The tax on real estate is calculated at \$24.00 per \$1,000 valuation. The tax assessment on property valued at y dollars is \$348.00.

(ℓ) Donald's weight, 152 pounds, is at least 40 pounds more than Earl's.

(m) The sum of a natural number and its successor is 575.

(n) The sum of a natural number and its successor is 576.

(o) The sum of two mumbers, the second greater than the first by 1, is 576.

(p) A board 16 feet long is cut in two pieces such that one piece is one foot longer than twice the other.

(q) Catherine earns \$2.25 baby-sitting for 3 hours at x cents an hour.

(r) A familiar formula for making coffee is: "Use one tablespoon of coffee for each cup of water, and add one tablespoon of coffee for the pot." Use C for the number of cups of water, and T for the number of tablespoons of coffee.

(s) In 4 years Mary will be twice as old as she was 6 years ago.

(t) A two-digit number is 7 more than 3 times the sum of the digits.

(u) A number is increased by 17 and the sum is multiplied by 3. The resulting product is 192.

(v) If 17 is added to a number and the sum is multiplied by 3, the resulting product is less than 192.

40. Mr. York is reducing. During each month for the past 8 months he has lost 5 pounds. His weight is now 175 pounds. What was his weight m months ago if $m < 8$? Write an open sentence stating that m months ago his weight was 200 pounds.

41. In a "guess a number" game Betty is asked to pick a natural number less than or equal to 7.

(a) With x for the number, write the inequalities which indicate the restrictions on the number.

(b) If Betty picks a natural number less than or equal to 7 and Paul picks a natural number less than or equal to 5, what can we say about the sum of Betty's number and 3 times Paul's number?

(c) If Betty picks a natural number less than or equal to 7 and Paul picks a whole number less than or equal to 5, what can we say about the sum of Betty's number and three times Paul's number?

42. (a) At an auto parking lot, the charge is 35 cents for the first hour, or fraction of an hour, and 20 cents for each succeeding (whole or partial) one-hour period. What is the parking fee for 4 hours of parking?

(b) If t is the number of one-hour periods parked after the initial hour, write an open phrase for the parking fee.

(c) With the same charge for parking as in the preceding problem, if h is the total number of one-hour periods parked, write an open phrase for the parking fee.

43. (a) Two water-pipes are bringing water into a reservoir. One pipe has a capacity of 100 gallons per minute, and the second 40 gallons per minute. If water flows from the first pipe for x minutes and from the second for y minutes, write an open phrase for the total flow in gallons.

(b) In part (a), if the flow from the first pipe is stopped after two hours, write the expression for the total flow in gallons in y minutes, where y is greater than 120.

*(c) With the same data, write an open sentence stating that the total flow is 20,000 gallons. Find five or more pairs of numbers for the variables which yield true numerical sentences.

44. The reading on a Fahrenheit thermometer is 32° more than 1.8 times the reading on a Centigrade thermometer, if the temperature is less than 50° Fahrenheit, what is the temperature Centigrade?

*45. David, obeying the speed limits all the way, drove 200 miles without ever exceeding r miles per hour. How long did it take him? On the other hand, he never went less than 20 miles per hour. How long did it take him? Are you sure he made the trip in a day? Can you be sure he made it in less than 4 hours?

*46. If you fly from New York to Los Angeles, you gain three hours. If the flying time is h hours, when do you have to leave New York in order to arrive in Los Angeles before noon?

47. Find a number such that one-third of it added to three-fourths of it is equal to or greater than 26.

48. The amount of $205 is to be divided among Tom, Dick and Harry. Dick is to have $15 more than Harry and Tom is to have twice as much as Dick. How must the money be divided?

49. An amount between $205 and $225 inclusive is to be divided among three brothers, Tom, Dick and Harry. Dick is to have $15 more than Harry, and Tom is to have twice as much as Dick. How much money can Harry expect?

50. A ball player reasons: Only if I hit .300 or better can I expect a raise in pay next year. If I hit less than .270 I will lose my job. Write a sentence to express how he may retain his job with no pay raise. Also write a sentence to express how he may recieve a pay increase.

51. A student has test grades of 75 and 82. What must he score on a third test to have an average of 88 or higher? If 100 is the highest score possible on the third test, how high an average can he achieve?

2. Ray bought a used car, paying $175 for it. His friend, Stan said, "I don't know just how fast your car will go, but if you'll let me work on it I'll guarantee it will go at least ten miles an hour faster". After Stan worked on the car Ray found it would now go at a top speed of 73 miles per hour. What was the original top speed?

53. If $\frac{1}{4}$ of a number increased by $\frac{1}{8}$ of the number is less than the number diminished by 25, what is the number?

54. Last year's tennis balls cost d dollars a dozen. This year the price is c cents per dozen higher than last year. What will half a dozen balls cost at the present price?

55. A man distributes $24 between his children in amounts proportioned to their ages. The older is 7, and the younger 3. How much should each receive?

56. In a class of 10 pupils the average grade was 72. The students with the two highest grades, 94 and 98, were transferred to another class, and the teacher decided to find the average of the grades of the 8 remaining students. What was the new average?

57. Given the set S of all the even integers, (positive and negative and zero) which of five operations, (1) addition (2) subtraction (3) multiplication (4) division (5) average, applied to every pair of the elements of S, will give only elements of S? Describe your conclusion in terms of "closure".

58. A haberdasher sold two shirts for $3.75 each. On the first he lost 25% of the cost and on the second he gained 25% of the cost. How much did he pay for each shirt and did he gain or lose from the sale?

59. A boy has 95 cents in nickels and dimes. If he has 12 coins, how many of each coin does he have?

60. William has 5 hours at his disposal. How far can he ride his bike into the spooky woods if he goes in at the rate of 4 miles per hour and returns at the rate of 15 miles per hour?

61. A plane which flies at an average speed of 200 m.p.h. (when no wind is blowing) is held back by a head wind and takes $3\frac{1}{2}$ hours to complete a flight of 630 miles. What is the average speed of the wind?

CHALLENGE PROBLEMS

1. Let us write ">" for the phrase "is further from 0 than" on the real number line. Does ">" have the comparison property, that is, if a and b are different real numbers, is it true that $a > b$ or $b > a$ but not both? Does ">" have a transitive property? For which subset of the set of real numbers do ">" and ">" have the same meaning?

2. Prove that the absolute value of the product ab is the product $|a|\cdot|b|$ of the absolute values.

3. Prove: For each non-zero real number a there is only one multiplicative inverse of a.

   Hint: Assume that a multiplicative inverse of a is b; that is, $ab = 1$. If there is <u>another</u> inverse under multiplication, say x, then $ax = 1$.

4. Prove: The number 0 has no reciprocal.

   Hint: Assume 0 does have a reciprocal and see what happens when you apply the definition of reciprocal.

5. Prove: The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.

   Hint: The product of a number and its reciprocal is 1.

6. Prove: The reciprocal of the reciprocal of a non-zero real number a is a.

   Hint: Consider the product $\left(\frac{1}{a}\right)\left(\frac{1}{\frac{1}{a}}\right)$ and the product $a\left(\frac{1}{a}\right)$. Compare the results.

7. How can we interpret $|8 - 2|$ on the number line? What about $|2 - 8|$? What does $(2 - 8)$ tell about a distance on the number line that $|2 - 8|$ does not tell? If we were interested in expressing the distance <u>between</u> a and b on the number line and did not care about direction on the line, would $b - a$ serve our purpose? How can we express the distance <u>from</u> a <u>to</u> b as a difference?

8. For each of the following pairs of expressions, fill in the symbol ">", "=", or "<", which will make a true sentence.

(a) $|9 - 2| \; ? \; |9| - |2|$

(b) $|2 - 9| \; ? \; |2| - |9|$

(c) $|9 - (-2)| \; ? \; |9| - |-2|$

(d) $|(-2) - 9| \; ? \; |-2| - |9|$

(e) $|(-9) - 2| \; ? \; |-9| - |2|$

(f) $|2 - (-9)| \; ? \; |2| - |-9|$

(g) $|(-9) - (-2)| \; ? \; |-9| - |-2|$

(h) $|(-2) - (-9)| \; ? \; |-2| - |-9|$

9. Write a symbol between $|a - b|$ and $|a| - |b|$ which will make a true sentence for all real numbers a and b. Do the same for $|a - b|$ and $|b| - |a|$. For $|a - b|$ and $\big||a| - |b|\big|$.

10. Describe the resulting sentences in Problem 9 in terms of distances on the number line.

11. What are the two numbers x on the number line such that

$$|x - 4| = 1,$$

that is, the two numbers x such that the distance between x and 4 is 1?

12. What is the truth set of the sentence

$$|x - 4| < 1,$$

that is, the set of numbers x such that the distance between x and 4 is less than 1? Draw the graph of this set on the number line.

13. Draw the graph of the truth set of the compound open sentence

$$x > 3 \text{ and } x < 5$$

on the number line. Is this set the same as the truth set of $|x - 4| < 1$? (We usually write "$3 < x < 5$" for the sentence "$x > 3$ and $x < 5$".)

14. Find the truth set of each of the following equations; draw the graph of each of these sets:

(a) $|x - 6| = 8$

(b) $y + |-6| = 10$

(c) $|10 - a| = 2$

(d) $|x| > 3$

(e) $|v| > -3$

(f) $|y| + 12 = 13$

(g) $|y - 8| < 4$ (Read this: The distance between y and 8 is less than 4.)

(h) $|z| + 12 = 6$

(i) $|x - (-19)| = 3$

(j) $|y + 5| = 9$

15. Prove for any non-zero real numbers a and b that

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

16. Prove $\frac{1}{(-a)} = -(\frac{1}{a})$, if $a \neq 0$.

Hint: We have proved $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.

Now consider the reciprocal of -1. How could it be used to help in this proof?

17. If $a < b$ and a and b are both positive real numbers prove that $\frac{1}{a} > \frac{1}{b}$

Hint: Multiply the inequality "$a < b$" by $(\frac{1}{a} \cdot \frac{1}{b})$.

18. Prove that if $a < b$, where a and b are both negative, then $\frac{1}{a} > \frac{1}{b}$.

19. Prove that if $a < 0$ and $b > 0$ then $\frac{1}{a} < \frac{1}{b}$.

20\. Prove: If $b \neq 0$ then $a = cb$ if and only if $\frac{a}{b} = c$

Hint: Two things must be proved.

1. If $b \neq 0$ and $\frac{a}{b} = c$ then $a = cb$

2. If $b \neq 0$ and $a = cb$ then $\frac{a}{b} = c$

Consider using the multiplication property of equality.

21\. Prove: For all real numbers a, b, c, d if $b \neq 0, d \neq 0$ then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Hint: Begin with $\frac{a}{b} \cdot \frac{c}{d}$ and apply the definition of division and the associative and commutative properites of multiplication.

22\. Prove that $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$ for real numbers a, b, and c $(c \neq 0)$.

Hint: Consider the definition of division, then the distributive property.

23\. Prove that $\frac{a}{c} + \frac{b}{d} = \frac{ad + bc}{cd}$ for real numbers a, b, c, and d, $(c \neq 0,\ d \neq 0)$.

Hint: Use the multiplication property of 1, choosing the form of 1 carefully.

24\. (a) A procedure sometimes used to save time in averaging large numbers is to guess at an average, subtract the guess from each number, average the differences, and add that average to your guess. Thus, if the numbers to be averaged--say your test scores--are 78, 80, 76, 72, 85, 70, 90, a reasonable guess for your average might be 80. We find how far each of our numbers is from 80.

78 - 80 = -2	
80 - 80 = 0	
76 - 80 = -4	
72 - 80 = -8	
85 - 80 = 5	
70 - 80 = -10	
90 - 80 = 10	

The sum of the differences is -9. The average of the differences is $-\frac{9}{7}$ Adding this to 80 give $78\frac{5}{7}$ for the desired average. Can you explain why this works?

(b) The weights of a university football squad were posted as: 195, 205, 212, 201, 198, 232, 189, 178, 196, 204, 182. Find the average weight for the team by the method of differences explained in part a.

25. Given the set {1, -1, j, -j} and the following multiplication table.

x	1	-1	j	-j
1	1	-1	j	-j
-1	-1	1	-j	j
j	j	-j	-1	1
-j	-j	j	1	-1

(a) Is the set closed under multiplication?

(b) Verify that this multiplication is commutative for the cases (-1, j), (j, -j) and (-1, -j).

(c) Verify that this multiplication is associative for the cases (-1, j, -j) and (1, -1, j).

(d) Is it true that $a \times 1 = a$, where a is any element of {1, -1, j, -j}.

(e) Find the reciprocal of each element in this set.

If x is an unspecified member of the set, find the truth sets of the following (make use of question (e)).

(f) $j \times x = 1$.

(g) $-j \times x = j$.

(h) $j^2 \times x = -1$.

(i) $j^3 \times x = -j$.

26. We saw earlier that a fraction, such as $\frac{2}{3}$, is often called the ratio of 2 to 3, or the ratio $\frac{2}{3}$. We also call a sentence in the form

$$\frac{a}{b} = \frac{c}{d}$$

a proportion. It is read "a, b, c, d are in proportion." These words are convenient when we are using division to show the relative size of two numbers. Since a ratio is a fraction, and a proportion is a simple sentence involving two fractions, these two words are just names for things with which we are already familiar.

Example: Two partners in a firm are to divide the profits in the ratio $\frac{3}{5}$. If the man receiving the larger share receives \$8550, how much does the other partner receive?

If the smaller share is p dollars, the $\frac{p}{8550} = \frac{3}{5}$.

If there is a number p such that the sentence is true, then

$$p = \frac{3}{5} \cdot 8550$$

$$p = 5130$$

If $p = 5130$, then $\frac{p}{8550} = \frac{5130}{8550} = \frac{3}{5} \times \frac{1710}{1710} = \frac{3}{5}$.

Hence, the smaller share is \$5130.

Notice how saying that the shares are in the ratio $\frac{3}{5}$ leads naturally to writing the proportion $\frac{p}{8550} = \frac{3}{5}$.

(a) In a certain school the ratio of boys to girls was $\frac{7}{6}$. If there were 2600 students in the school, how many girls were there?

(b) In a shipment of 800 radios, $\frac{1}{20}$ of the radios were defective. What is the ratio of defective radios to non-defective radios in the shipment?

(c) The ratio of faculty to students in a college is $\frac{2}{19}$. If there are 1197 students, how many faculty members are there?

(d) If two numbers are in the ratio $\frac{5}{9}$, explain why we may represent those numbers as $5x$ and $9x$. What are the numbers if $x = 7$? If $x = 100$? If their sum is 210?

(e) Prove that if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

(f) Prove that if $ad = bc$ and $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$.

(g) Show, using the properties of one, that the proportion $\frac{6}{15} = \frac{2}{5}$ is true.

(h) Assuming that the proportion $\frac{x}{y} = \frac{p}{q}$ is true, use parts (e) and (f) to find seven other true proportions. For example, if $\frac{x}{y} = \frac{p}{q}$, then $xq = yp$ or $qx = yp$. Hence, by part (f) $\frac{q}{y} = \frac{p}{x}$.

27. A man travels 360 miles due west at a rate of 3 minutes per mile and returns by plane at a rate of 3 miles per minute. What was his total traveling time? What was his average rate of speed for the entire trip?

28. A set of ten numbers has a sum t. If each number is increased by 4, then multiplied by 3, and then decreased by 4 the new total will be how much? If you had twenty numbers instead of ten and the same conditions, what would be the new total?

GLOSSARY

ABSOLUTE VALUE - The absolute value of 0 is 0. The absolute value of any other real number n is the greater of n and $-n$. "Absolute value of n" is written "$|n|$".

ADDITION PROPERTY OF EQUALITY - For any real numbers a, b, and c, if $a = b$, then $a + c = b + c$.

ADDITION PROPERTY OF OPPOSITES - For every real number a, $a + (-a) = 0$.

ADDITION PROPERTY OF ORDER - For any real numbers a, b, and c, if $a < b$, then $a + c < b + c$.

ADDITION PROPERTY OF ZERO - For every number a, $a + 0 = a$.

ADDITIVE INVERSE - If we have two real numbers x and y whose sum is zero, like this: $x + y = 0$, then y is the additive inverse of x and x is the additive inverse of y. (Any real number x has one and only one additive inverse. This additive inverse is $-x$. See INVERSE ELEMENTS OF ADDITION).

ASSOCIATIVE PROPERTY OF ADDITION - For every number a, every number b, and every number c, $(a + b) + c = a + (b + c)$.

ASSOCIATIVE PROPERTY OF MULTIPLICATION - For every number a, every number b, and every number c, $a(bc) = (ab)c$.

BINARY OPERATION - An operation that is applied to only two numbers at a time.

CLOSURE PROPERTY OF ADDITION - For every number a, and every number b, $a + b$ is an element of the set containing a and b.

CLOSURE PROPERTY OF MULTIPLICATION - For every number a, and every number b, ab is an element of the set containing a and b.

COMMON NAME - The simplest name for a number.

COMMUTATIVE PROPERTY OF ADDITION - For every number a and every number b, $a + b = b + a$.

COMMUTATIVE PROPERTY OF MULTIPLICATION - For every number a and every number b, $ab = ba$.

COMPARISON PROPERTY - For any real numbers a and b, exactly one of the following is true: $a < b$, $a = b$, $b < a$.

COORDINATE - The number that is associated with a particular point on the number line.

CORRESPONDENCE - The pairing of the elements of one set with the elements of another set.

COUNTING NUMBER - An element of the set $\{1, 2, 3, 4, 5, \ldots\}$.

DISTRIBUTIVE PROPERTY - For every number a, every number b, and every number c, $a(b + c) = ab + ac$.

DIVISION - For any real numbers a and b, with b not equal to zero, "a divided by b" means "a multiplied by the reciprocal of b".

DOMAIN - The set of numbers from which the value of the variable may be chosen.

ELEMENTS - The objects in a set.

EMPTY SET - A set which has no elements, sometimes called the null set.

EQUIVALENT SENTENCES - Two open sentences with the same truth set are called equivalent sentences.

EVEN NUMBER - An element of the set $\{0, 2, 4, 6, \ldots\}$.

FINITE SET - A set whose elements can be counted with the counting coming to an end.

FRACTION - A symbol which represents the quotient of two numbers.

GRAPH OF A TRUTH SET - The set of points whose coordinates make an open sentence true.

IDENTITY ELEMENT OF ADDITION - There is a special real number 0 such that, for any real number a, $a + 0 = a$.

IDENTITY ELEMENT OF MULTIPLICATION - There is a real number 1 such that, for any real number a, $a \cdot 1 = a$.

INFINITE SET - A set whose elements cannot be counted, that is, with the counting coming to an end.

INTEGERS - The set $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}$. Any one of the numbers in this set is called an integer.
The set of positive integers, the set of negative integers, and zero make up the set of integers.

INVERSE ELEMENTS OF ADDITION - For every real number a there is a real number $-a$ such that $a + (-a) = 0$.

INVERSE ELEMENTS OF MULTIPLICATION - For every real number a different from 0, there is a real number $\frac{1}{a}$ such that $a \cdot (\frac{1}{a}) = 1$.

IRRATIONAL NUMBER - There are points on the number line that cannot be associated with rational numbers. A number that is not rational but is associated with a point on the number line is called an irrational number.
$\sqrt{2}$, $\sqrt{3}$, and π are examples of an irrational number.

MULTIPLE OF A NUMBER - A number obtained by multiplying the given number by a whole number.

MULTIPLICATION PROPERTY OF EQUALITY - For real numbers a, b, and c, if $a = b$, then $ac = bc$.

MULTIPLICATION PROPERTY OF ONE - For every number a, $a(1) = a$.

MULTIPLICATION PROPERTY OF ORDER - For any real numbers a, b, and c, if $a < b$ and c is positive, then $c \cdot a < c \cdot b$, if $a < b$ and c is negative, then $c \cdot b < c \cdot a$.

MULTIPLICATION PROPERTY OF ZERO - For every number a, $a(0) = 0$.

MULTIPLICATIVE INVERSE - For any pair of numbers c and d such that $(c)(d) = 1$, d is the multiplicative inverse of c, and c is the multiplicative inverse of d.
Note: Zero has no multiplicative inverse. (See INVERSE ELEMENTS OF MULTIPLICATION.)

NEGATIVE NUMBERS - Points to the left of 0 on the number line are associated with negative numbers.

NULL SET - The empty set.

NUMBER LINE - A line whose points have been labeled with numbers.

NUMERAL - A name for a number.

ODD NUMBER - An element of the set {1, 3, 5, 7, 9, ...}, obtained by adding 1 to each element of the set of even numbers.

OPEN PHRASE - A mathematical phrase which contains one or more variables.

OPEN SENTENCE - A mathematical sentence which contains one or more variables.

OPPOSITES - The opposite of 0 is 0. The opposite of any other real number is the number which is at an equal distance on the other side from 0 on the number line.

PARENTHESES - ( ) - Symbols to show that the phrase inside is the name for one number.

RATIONAL NUMBER - A number which may be represented by a fraction which is the quotient of two whole numbers, excluding division by zero.

REAL NUMBER LINE - The points associated with the real numbers make up the whole number line, which is called the real number line. The kinds of numbers associated with the real number line are rational numbers (which includes integers and fractions) and the irrational numbers.

REAL NUMBER SYSTEM - The real numbers are those numbers that can be associated with points of the real number line. They include rational numbers and irrational numbers. The real number system is a set of elements called real numbers. The system has two operations: addition and multiplication. The system also has an order relation "is less than" with the symbol "<".

SET - A collection of objects.

SYMBOLS

...	and so forth, following the indicated pattern
∅	the empty set
{ }	notation for a set
( )	notation for a single number
=	is equal
≠	is not equal
>	is greater than
<	is less than
≥	is greater than or equal to
≤	is less than or equal to
| |	means absolute value of

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