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ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes comm[illegible]taries on the sections of the student's booklet, answers to the exer[illegible]es, and sample test questions. Topics covered include addition, multiplication, operations, closure, identity element, mathematical systems, mathematical systems without numbers, the counting numbers, whole numbers, and modular arithmetic. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-20

SUPPLEMENTARY and ENRICHMENT SERIES

MATHEMATICAL SYSTEMS

Teachers' Commentary

Edited by Henry W. Syer

MSG

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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MATHEMATICAL SYSTEMS

COMMENTARY FOR TEACHERS

In this booklet it is particularly important that teachers have clearly in mind both the objectives of the booklet and the suggested method of approach to be used with it.

The main objective is to lead the students to achieve some appreciation of the nature of mathematical systems. It is neither intended nor desirable that the students memorize the various tables introduced here, or drill for mastery of the operations introduced here.

It is especially important that the teacher read this booklet through very carefully before planning his presentation, and give considerable thought to some introductory motivation, and even more to how to lead the students to discover the various relationships and properties which appear in the booklet for themselves in advance of the reading of the text. The text itself attempts to suggest problems and processes for doing this as does this teacher's guide. However, these can be effective only if carefully planned for by the teachers. The process of discovering, of perceiving for one's self is a vital step in achieving our major objective: an appreciation of the nature of some types of mathematical systems. This is close to an appreciation of the nature of modern mathematics and of the work of mathematicians.

One of the most important activities of modern mathematicians is the search for common attributes of properties often found in apparently diverse situations or systems. Sometimes these common elements are deliberately built into new systems which are constructed as generalizations or abstractions of old systems, as when the number system is extended from the system of counting numbers to the whole numbers, to the rational numbers, etc., etc. Sometimes these common elements are observed in systems less clearly related at first glance, as when the changes of position of a rectangle into itself are conceived of as forming an algebraic system with a "multiplication" table, which is discussed in the booklet.

Frequently, the systems developed out of the intellectual curiosity of mathematicians and their search for patterns in diverse abstract situations have been exactly the tools needed and seized upon by scientists in their attack on the problems of our physical world. The theory of groups, which actually has as its logical beginnings the properties discussed in this booklet, had its chronological beginnings in the early 19th Century in problems relating to the solution of equations. Matrices, some of which form groups and give further examples of the principles of this booklet were invented largely by the Englishman Arthur Cayley a little later. Within our generation the German physicist Werner Heisenberg has used matrices in the formulation of the quantum mechanics which is highly important in modern physics. Analogous stories relate the development of radio by Marconi to the differential equations of Maxwell, and point out that the outgrowths of Einstein's relativity theory owe much to his use of the tensor calculus developed by the Italian geometers Ricci and Levi-Civita. All of these stories have the same theme, namely, that both mathematicians and scientists are always seeking unifying principles or patterns. Frequently mathematics, developed solely for the intrinsic interest of its properties and structure, was later found to fit the needs of science, but for both science and mathematics we need to develop students who can see and understand patterns and structure.

In this booklet we are studying mathematical systems involving sets of elements and binary operations. Such systems which have certain simple additional properties are called groups and their study is a major branch of so-called "modern algebra." We shall not use all of these technical terms. However, other substantial objectives incidental to the major concern of this booklet and appropriate for secondary school students are:

1. Increased understanding of the nature and occurrence of the commutative, associative, and distributive properties, as well as the concepts of closure, identity element, inverse of an element.

2. Increased understanding of the inverse of an operation and its relationship to inverse and identity elements.

Additional discussions of these ideas and problem materials may be found in the books listed in the bibliography at the end of this booklet.

Aside from the general considerations mentioned above, there are very specific applications of the modular systems with which this booklet is chiefly concerned. The applications to days of the week, hours of the day, days in the month are obvious and immediate. Not quite so obvious are applications to two-way switches (mod 2) which are most common, but also to n-way switches for a number of small values of n. These are used increasingly in modern computing and in industry. The recognition that all are aspects of one system -- modular arithmetic -- gives insight not only to mathematics but to various applications as well. This in turn is an example of periodicity -- a repetitive pattern -- that occurs so often within and outside of mathematics.

The teacher should be especially cautioned in the use of the exercises in this booklet. There are altogether too many for use in one class. To give all would lay too much stress on techniques and make a chore out of what should be an interesting development. Many exercises are given so that the teacher may use different sets in different classes and have some left over for review at the end.

1. A New Kind of Addition

Several of the sections, including the first, discuss the properties of what is referred to as modular arithmetic. The face of a clock is used to illustrate modular addition. The following quote provides background for the basic notion of this idea:

"In number theory we are often concerned with properties which are true for a whole class of integers differing from each other by multiples of a certain integer. Take, for instance, the fact that the square of an odd integer when divided by 8 leaves 1 for a remainder. Here we have a property holding for all odd numbers; that is, for a class of numbers differing from each other by multiples of 2. As another example, we see that when the last digit of a number, in decimal notation, is 6, then the last digit of its square will also be 6. Thus, in this simple example, we deal again with a property shared by integers differing by a multiple of an integer; namely, 10.

"The consideration of properties holding for all integers differing from each other by a multiple of a certain integer leads in a natural way to the notion of congruence. Two integers a and b whose difference a - b is divisible by a given number m (not 0) are said to be congruent for the modulus m or simply congruent modulo m. Gauss, who introduced the notion of congruence, proposed the notation

$$a \equiv b \pmod{m}$$

to designate the congruence of a and b modulo m."[1]

Some textbooks use the following definition:

If $a = km + b$, then $a \equiv b \pmod{m}$. The $\equiv$ sign is read, "Is equivalent to" or "is congruent to".

We emphasize that there is no need for the pupil to become familiar with the terms used in the above discussion, including "modular arithmetic".

In solving problems using replacements, encourage the pupil to make a list of possible replacements first. For (mod 5) the set of possible replacements would be {0,1,2,3,4}; for (mod 8), the set would be {0,1,2,3,4,5,6,7}. These are examples of finite systems.

---

[1]Uspensky and Heaslet, Elementary Number Theory, McGraw-Hill, 1939, page 126.

For the teacher's information each element of the set can be considered as an <u>equivalence</u> class, thus, numbers are put in <u>equivalence</u> classes.

Without a doubt, some pupils will wonder why the symbol "≡" ("is congruent to") is used instead of "=". This is an excellent opportunity to point out that the = sign is used when we have two names for the same thing; thus $3 + 2 = 4 + 1$ since these are two names for the same number, five. In the case of modular arithmetic, when we say "Five is congruent to one, (mod 4)", the "five" and the "one" are <u>not</u> names of the same thing, thus it is necessary to introduce another symbol to describe this relationship.

<u>Answers to Exercises 1</u>

1.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(a) 0
(b) 2
(c) 0
(d) 1

2.

(mod 3)

+	0	1	2
0	0	1	2
1	[illegible]	2	0
2	2	0	1

(mod 5)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

3. (a) 0
(b) 1
(c) 1
(d) 2

The teacher may want to let the students try exercises (mod 4) before taking up other moduli.

4\. (mod 6)

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(mod 7)

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

(a) 2
(b) 4
(c) 2
(d) 2

The pupils may use the tables made in Problem 3 above or make sketches of clocks.

5\. (a) 1
(b) 1
(c) 1
(d) 2
(e) 4
(f) 4
(g) 1
(h) 0

6\. $23 = 5(4) + 3$. The hand will go around four times and stop at 3.

7\. Seven hours after eight o'clock is five o'clock. This is addition (mod 12).

8\. Nine days after the 27th of March is the fifth of April. This is addition (mod 31) since there are 31 days in March.

2\. A New Kind of Multiplication

This section does for multiplication what the first section did for addition. It not only gives other examples of operations for use in the next section but also prepares for modular arithmetic in a later section. The transition from getting a multiplication table by adding to getting it by dividing and taking the remainder should be made on the initiative of the students as a means of making computation easier. It is hoped that this could be discovered by some of the students themselves. Problems 5 and 6 are designed to encourage this transition. This is certainly one place where to push a transition too rapidly can lead to trouble but where discovery in the students' own good time can be an enjoyable experience for all concerned.

Answers to Exercises 2

1. (a) (mod 5)

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(b) (mod 7)

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(c) (mod 6)

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

2. (a) 1 (d) 1

(b) 0 (e) 0

(c) 3

3. (a) 6 (c) 6

(b) 1 (d) 0

4. (a) $(7)(10) \equiv ?\ (\text{mod } 31)$

$70 \equiv 8\ (\text{mod } 31)$

$4 + 8 = 12$; hence February 12 is the date 10 weeks after December 4th.

(b) $(2)(365) \equiv ?\ (\text{mod } 7)$

$730 \equiv 2\ (\text{mod } 7)$

Thursday was the day of the week for August 6, 1959.

5. (mod 5)

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

6. The Table is identical with the multiplication Table (mod 5). Dividing a whole number by 5 and retaining the remainder yields the same results as those obtained by subtracting the greatest multiple of five contained in a given number and retaining the remainder. It may be easier to divide and retain the remainder.

*7. (a) $x = 2$ (d) $x = 3$

(b) $x = 4$ (e) $x = 0$

(c) $x = 1$

*8. (a) impossible

(b) impossible

(c) $x = 1,\ x = 3,\ x = 5$

(d) impossible

(e) $x = 0,\ x = 2,\ x = 4$

3. What Is an Operation?

Skills and Understandings

1. To recognize a binary operation described by a table.
2. To recognize a binary operation described in words.
3. To find, from a table, the result of putting two elements together in a binary operation described by the table.
4. To find, by computation, the result of putting two elements together in a binary operation described in words.
5. To tell, from the table for a binary operation, whether or not the operation is commutative.
6. To know that:
   (a) In order to show that a binary operation is associative, it is necessary to show that an equation [e.g. $a * (b * c) = (a * b) * c$] holds for every triple of elements a, b, c.
   (b) In order to show that a binary operation is not associative, it is sufficient to find one triple of elements a, b, c for which the equation does not hold [e.g. $a * (b * c) \neq (a * b) * c$].

Teaching Suggestions

To be given a binary operation, we must be given a set of elements and a way of combining any two elements to get a definite thing. The "definite thing" may or may not belong to the original set of elements. The two elements we combine may be the same element taken twice. If the operation is given to us by a table, the set is composed of those elements which appear in the left-hand column and in the top row (the same elements must appear in both places). For example, the set for the operation of Table (c) is $\{0,1,2,3\}$; that for the operation of Table (d) is $\{1,2,3\}$. In Table (d), all the entries in the table belong to the set $\{1,2,3\}$; in Table (c) many of the entries in the table do not belong to the set $\{0,1,2,3\}$. This point is discussed more fully in the next section on closure.

Bring out by class discussion that the entries in the tables (the results of putting two elements together) could be anything at all. As later examples will show, they do not have to be numbers.

Practice reading the tables. Stress that, in evaluating $1 \square 3$, the "1" is to be found in the left column, and the "3" in the top row. Point out that $1 \square 3 = 5$ and $3 \square 1 = 7$, so it is necessary to be careful about the order in which elements are written.

Some examples for class discussion are given below.

Example 1: Set: The counting numbers.
Rule of Procedure: Given any two elements, take twice the first and add three times the second. This is an operation, but it is not commutative and it is not associative.

Example 2: Set: The counting numbers.
Rule of Procedure: Given any two elements, take twice one of them and add three times the other.

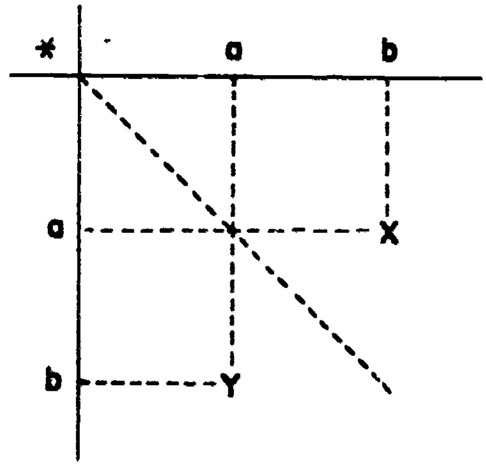
This rule does not define an operation since a "definite thing" is not always determined. For instance, in combining 2 and 3, we are allowed to form either $2 \cdot 2 + 3 \cdot 3 = 13$, or $2 \cdot 3 + 3 \cdot 2 = 12$. The result of an operation applied to two elements must be unique, that is, there can be one and only one answer. It could also be seen from a table that this rule does not describe an operation. The table would have more than one entry in some places (everywhere except on the diagonal from upper left to lower right).

Example 3: Set: The whole numbers.
Rule of Procedure: Given any two elements, divide the first by the second.

This rule does not define an operation, since division by zero is impossible. The elements 2 and 0 cannot be put together in that order. Notice that, in the order 0 and 2, they can be combined (the result is zero). It could also be seen from a table that this rule does not describe an operation. The table would have some of the spaces blank (the column with "0" at the top would be blank).

Discussion of Exercises 3

2. In the schematic diagram at the right, $a * b$ is to be entered at position X, and $b * a$ is to be entered at position Y. These two positions are symmetrically located with respect to the diagonal from upper left to lower right. (The elements are arranged in the same order in the top row and left column.) If an operation is commutative, its table will be symmetric about this diagonal, and conversely.



3. Bring out by discussion that, to prove an operation is associative, requires testing every triple; one example would be sufficient to prove an operation is not associative. To prove associativity for an operation by examining all the cases is almost always a long process. Each student should check 2 or 3 cases and if all of them are satisfactory, the following statement can be made: "This operation appears to be associative, but we are not really sure."

4. In making a table for each of these operations, arrange the elements of the set in the same order in the top row and left column. Compute and fill in as many entries in the table as needed to see the pattern. Associativity can be decided from known properties of the counting numbers.

   (a) This operation is not completely described. If the two given numbers are equal, there is no smaller one, of course. Bring out, by class discussion, that, if the two numbers are the same, the result of the operation should be defined as that same number.

5. and 6. The students will need help in beginning these problems. The successive steps are as follows:

(1) Choose a set (each pupil may have a different set, but it is better not to have too many elements in each set so the problem will not be too long). Suppose the set {1,2,3} is chosen.

(2) Make the framework for a table as shown at the right. The elements in the set which was chosen in (1) will appear in the left column and top row. Arrange them in the same order.

*	1	2	3
1			
2			
3			

(3) Choose a symbol, such as *, for the operation and put it in the left-hand corner of the framework; make up a name to go with it, such as "star".

(4) Fill in the table. Emphasize that the names of any objects whatever may be placed in the body of the table -- it is not necessary that these objects be elements of the set chosen in (1). If the operation is to be commutative, the table must be symmetric about the diagonal from upper left to lower right. If the operation is not to be commutative, the table must not be symmetric.

7. Here, a way to write the information is to arrange the elements and the corresponding results of the operation in two rows or columns. Usually some symbol (such as "x") is used to denote an element of the set and a different symbol (such as "y" or, in this case, "x^3") denotes the corresponding result of the operation. The table is given in two columns in the answers. It could also be written in two rows as shown below.

x	0	1	2	3	4	5	6	7	8	9	10
x^3	0	1	8	27	64	125	216	343	512	729	1000

Notice that a unary operation requires only a one-dimensional table, a binary operation requires a two-dimensional table, and a ternary operation would require a three-dimensional table.

Answers to Exercises 3

1. (a) 1
   (b) 6
   (c) 8
   (d) 7
   (e) 2
   (f) 3
   (g) 1
   (h) 1
   (i) 8
   (j) Not possible; 1 □ 2 = 4 and 1 □ 4 is not defined since 4 does not appear in the top row.
   (k) 3
   (l) 3

2. (a), (b), (d), (e). The table must be symmetric about the diagonal from upper left to lower right. See discussion, Exercises 2.

3. There is no short-cut method; to prove associativity each triple of elements must be combined in the two ways and the corresponding results must be equal. The operations of Tables (a) and (d), (e) are associative; those of Tables (b) and (c) are not. See discussion, Exercises 3.

4. See discussion, Exercises 3. The operation symbols are omitted in the following tables:

(a)

	26	27	28	...	74
26	26	26	26	...	26
27	26	27	27	...	27
28	26	27	28	...	28
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
74	26	27	28	...	74

Commutative: Yes
Associative: Yes

(b)

	501	502	503	...	535
501	501	502	503	...	535
502	502	502	503	...	535
503	503	503	503	...	535
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
535	535	535	535	...	535

Commutative: Yes
Associative: Yes

(c)

	2	3	5	7	11	...
2	2	3	5	7	11	...
3	3	3	5	7	11	...
5	5	5	5	7	11	...
7	7	7	7	7	11	...
11	11	11	11	11	11	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(d)

	40	42	44	...	60
40	40	40	40	...	40
42	42	42	42	...	42
44	44	44	44	...	44
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
60	60	60	60	...	60

Commutative: No

Associative: Yes

(e)

	1	2	3	...	49
1	3	4	5	...	51
2	5	6	7	...	53
3	7	8	9	...	54
.	.	.	.	...	.
.	.	.	.	...	.
49	99	100	101	...	147

Commutative: No

Associative: No

(Try the triple 1, 2, 3.)

(f)

	1	2	3	4	5	6	...
1	1	1	1	1	1	1	...
2	1	2	1	2	1	2	...
3	1	1	3	1	1	3	...
4	1	2	1	4	1	2	...
5	1	1	1	1	5	1	...
6	1	2	3	2	1	6	...
.	.	.	.	.	.	.	...
.	.	.	.	.	.	.	...
.	.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(g)

	1	2	3	4	5	...
1	1	2	3	4	5	...
2	2	2	6	4	10	...
3	3	6	3	12	15	...
4	4	4	12	4	20	...
5	5	10	15	20	5	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(h)

	1	2	3	4	...
1	1	1	1	1	...
2	2	4	8	16	...
3	3	9	27	81	...
4	4	16	64	256	...
.	.	.	.	.	...
.	.	.	.	.	...
.	.	.	.	.	...

Commutative: No

Associative: No

(Try the triple 2, 1, 3: $(2^1)^3 = 8 \neq 2 = 2^{(1^3)}$.)

5. Many answers are possible, of course. The only requirement is that the table be symmetric about the diagonal from upper left to lower right (and that each place in the table be filled uniquely so that the table does describe an operation). See discussion, Exercises 2.

*	1	2	3
1	X	Y	Z
2	Y	P	Q
3	Z	Q	R

6. Many answers are possible, of course. The only requirement is that the table must not be symmetric about the diagonal from upper left to lower right (and that each place in the table be filled uniquely so that the table does describe an operation). See discussion, Exercises 3.

*	1	2	3
1	X	P	Z
2	P	R	Y
3	Q	P	Q

*7.

x	x^3
0	0
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729
10	1000

See discussion, Exercises 3.

4. Closure

Skills and Understandings

1. To recognize, from the table describing a binary operation, whether or not a set is closed under the operation.

2. To find whether or not a set is closed under a binary operation described in words.

Teaching Suggestions

The discussion here should prepare the pupil for consideration of more general systems where the elements may not be numbers.

Bring out, by class discussion, that closure involves two things: (1) It must be possible to put any two (not necessarily different) elements of the set together and (2) the result obtained must always be an element of the set. Material for class discussion is provided by the various parts of Problem 4 of Exercises 3.

As with associativity (see discussion of Problem 3, Exercises 3), to prove a set is closed under an operation, all cases must be considered; a single counter example would prove that the set is not closed under the operation.

It has been found in some classes that the pupils have difficulty because they expect the concept of closure to be much more difficult than it really is. Perhaps they should be reassured on this point.

The chief purpose of Examples 5 and 6 is to contribute to the understanding of closure by showing what a set must contain if it is to be closed. This is in a way also a preparation for the discussion of the existence of an inverse. Incidentally, the idea of a generator is an important mathematical concept; e.g., all the counting numbers are generated by the single number 1 under addition. This is the principle of mathematical induction: A statement is true for all counting numbers if, first, it is true for the number 1 and, second, whenever it is true for a counting number k it is also true for $k + 1$. In a way we "generate" the truth of the statement for all counting numbers by starting with 1 and proceeding step by step. Some teachers may feel that these two examples are too hard. If they are omitted the following Problems in Exercises 4 should also be omitted: 3, 4, 5, 6; also Problem 7 in Exercises 6 should be omitted.

Discussion of Exercises 4

1. Each table determines a set (the set of elements in the left column and top row), and describes completely the corresponding operation. For a set to be closed under the corresponding operation, each entry in the body of the table must be an element of the set. In Tables (a), (d), and (e) this is true; in (b) and (c), it is not.

*7. From the definition of commutativity in Section 3, it must be possible to put any two elements of the set together in either order and the same result must be obtained, but the result of the operation is not required to be an element of the set. In fact, Table (b) of Section 3 gives an example of a commutative operation, and the set on which the operation is defined is not closed under the operation.

*8. From the definition of associativity in Section 3, it must be possible to put any three elements of the set together in the two ways specified and the same result must be obtained. This means that the set on which the operation is defined must be closed under the operation since, if we can combine a, b, c as $(a + b) + c$, then certainly $a + b$ must be an element of the set on which the operation is defined; otherwise we cannot proceed with $(a + b) + c$. That is, the set is closed under the operation.

9. and 10. The pupils may need help in beginning these problems. The set of elements has been chosen, but each pupil should choose a symbol for his operation and fill in the entries in the table. See discussion of Problems 5 and 6, Exercises 3.

Answers to Exercises 4

1. The sets of (a) and (d) are closed under the corresponding operations (all the entries in the table appear in the left column and in the top row); those of (b) and (c) are not closed (some entries in tables (b) and (c) do not appear in the left column and in the top row). See discussion, Exercises 4.

2. (a) Closed
   (b) Closed
   (c) Closed
   (d) Not closed
   (e) Closed
   (f) Not closed: $15 - 35$ cannot be performed
   (g) Closed
   (h) Closed
   (i) Not closed: $3 + 7$ is not a prime
   *(j) Not closed: $3 + 3 = 11$ (base 5)

3. (a) $\{2,4,6,\ldots,2k,\ldots\}$ where k is a counting number.

(b) $\{2,2^2,2^3,\ldots,2^k,\ldots\}$ where k is a counting number.

4. (a) $\{7,14,21,\ldots,7k,\ldots\}$ where k is a counting number.

(b) $\{7,7^2,7^3,\ldots,7^k,\ldots\}$ where k is a counting number.

5. (a) $1 \odot 1 = 3$, $(1 \odot 1) \odot 1 = 3 \odot 1 = 2$.
$[(1 \odot 1) \odot 1] \odot 1 = 2 \odot 1 = 1$.
If we continue the operation $\odot$, we generate the same set again. Hence the set $\{1,2,3\}$ is the sub-set of S generated by 1 under the operation $\odot$.

(b) $2 \odot 2 = 2$, $(2 \odot 2) \odot 2 = 2$.
$[(2 \odot 2) \odot 2] \odot 2 = 2 \odot 2 = 2$.
It is clear that the subset of S generated by 2 under the operation $\odot$ is the subset $\{2\}$.

*6. $\{3, (3 \div 3), (3 \div 3) \div 3, [(3 \div 3) \div 3] \div 3, \ldots\}$ or $\{3,1,\frac{1}{3},\frac{1}{9},\ldots\}$
Yes; 3 and $\frac{1}{3}$ are in the subset of rationals generated by 3 under division. No; $3 \div \frac{1}{3}$ or 9 is not in this subset. Therefore the set is not closed under division and hence it cannot be associative. See the discussion on Problem 8.

7. No; see discussion, Exercises 4.

8. Yes; see discussion, Exercises 4.

9. Many answers are possible, of course. The only requirement is that each entry in the table belong to the set $\{0,43,100\}$ and that each place in the table be filled uniquely so that the table does describe an operation. See discussion, Exercises 4.

*	0	43	100
0	0	0	0
43	43	0	43
100	0	43	0

10. Many answers are possible, of course. The only requirement is that at least one entry in the table must not be an element of the set $\{0,43,100\}$ (and that each place in the table be filled uniquely so the table does describe an operation). See discussion, Exercises 3.

*	0	43	100
0	0	0	43
43	43	1	0
100	2	0	43

5. Identity Element; Inverse of an Element

Skills and Understandings

1. To determine from a table whether there is an identity element for the operation, and if so, what it is.
2. To realize that an element cannot have an inverse unless there is an identity element.
3. To determine from a table which elements have inverses.
4. To find the inverse of an element, if the element has an inverse.

Teaching Suggestions

Let the students experiment with several tables finding identity elements and inverses of elements. Try to lead them to discover that there is an identity element for an operation if, in the table, (1) there is a column exactly like the left column, and (2) there is a row exactly like the top row. The element associated with both will be the same, and will be the identity, because if $ax = x$ and $yb = y$ for all x and y in the set, we may replace x by b and y by a to get $ab = b = a$.

*	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

In the figure the last row and the last column fit the above conditions. 5 is the identity element.

*	3	4	5	1	2
1	4	5	1	2	3
2	5	1	2	3	4
4	2	3	4	5	1
5	3	4	5	1	2
3	1	2	3	4	5

The third column and the fourth row fit the conditions. 5 is the identity.

Lead the students to discover that an element has an inverse if the identity appears in the same relative position in the row as in the column associated with this element when the top row and left column are in the same order.

For example: In the first table the second element in the third row and the second element in the third column are both the identity element 5. This means that 3 has an inverse. Since 3 was associated with 2 both times to get the identity 5, then 2 and 3 must be inverses. The pairs 1 and 4, and 5 and 5 are seen to be inverses in a similar way.

Notice that the second table has the same elements and the same operation as the first, but that the order of the elements in the left column is different from that in the top row. It is not possible now to use our usual check of symmetry about the diagonal for commutativity. The method of finding the inverse of an element discussed above does not work out either.

The above may be pointed out to the students if you wish. None of the other tables in the chapter will have its top row and left column in different order.

The teacher should be warned that there is some difficulty about division and subtraction in a non-commutative system. For multiplication b is called the inverse of a if $ab = ba = 1$. This can happen in a non-commutative system. This is such an example, where each element is its own inverse. But the symbol $\frac{2}{3}$ is ambiguous since $3 \cdot x = 2$ has the solution $x = 1$ and $x \cdot 3 = 2$ has the solution $x = 2$. Actually what is usually done for such systems is to multiply by the inverse and not divide at all. For instance we would either have the product $\frac{1}{3} \cdot 2 = 3 \cdot 2 = 3$ or the product $2 \cdot \frac{1}{3} = 2 \cdot 3 = 2$.

·	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

An analagous situation exists for subtraction when addition is not commutative. This can be illustrated in terms of the above example if we replace · by + and $\frac{1}{3}$ by -3.

However, it was felt that such considerations as these were much too complex for inclusion in the text and hence when quest. ns of division or subtraction arise, we restrict the systems to commutat! systems.

Answers to Exercises 5a

1. (a) In table (a), the identity is 5.
   In table (d), the identity is 2.

   (b) In table (a), the inverse of 1 is 4; of 2 is 3; of 5 is 5.
   In table (b), no element has an inverse.
   In table (c), no element has an inverse.
   In table (d), the inverse of 1 is 3; of 2 is 2.

   Each member of the sets for tables (a) and (d) has an inverse. The operations described by tables (b) and (c) do not have identities so no inverses can exist.

2. (a)

Operation	Identity
(a)	74
(b)	501
(c)	2
(d)	None
(e)	None
(f)	None
(g)	1
(h)	None

   (b) The only inverses are those listed below.

(a)	74 is the inverse of 74.
(b)	501 is the inverse of 501.
(c)	2 is the inverse of 2.
(g)	1 is the inverse of 1.

   (c) None.

3. No; if there are two identities (P and Q) for a given operation, then consider the result when P is combined with Q. Since Q is an identity, the result must be P. But since P is also an identity, the result must be Q. Thus, P and Q must be the same element since each equals the result of combining P and Q.

Answers to Exercises 5b

1. (a) $1x \equiv 1 \pmod{6}$, $x = 1$

$2x \equiv 1 \pmod{6}$, not possible

$3x \equiv 1 \pmod{6}$, not possible

$4x \equiv 1 \pmod{6}$, not possible

$5x \equiv 1 \pmod{6}$, $x = 5$

(b) 1, 5. Each is its own inverse.

2. (mod 5)

b	a	multiplicative inverse of a	b ÷ a	b · (multiplicative inverse of a)
1	2	3	$1 \div 2 \equiv 3$	$1 \cdot 3 \equiv 3$
2	2	3	$2 \div 2 \equiv 1$	$2 \cdot 3 \equiv 1$
3	2	3	$3 \div 2 \equiv 4$	$3 \cdot 3 \equiv 4$
2	3	2	$2 \div 3 \equiv 4$	$2 \cdot 2 \equiv 4$
3	3	2	$3 \div 3 \equiv 1$	$3 \cdot 2 \equiv 1$
4	3	2	$4 \div 3 \equiv 3$	$4 \cdot 2 \equiv 3$
1	4	4	$1 \div 4 \equiv 4$	$1 \cdot 4 \equiv 4$
2	4	4	$2 \div 4 \equiv 3$	$2 \cdot 4 \equiv 3$
3	4	4	$3 \div 4 \equiv 2$	$3 \cdot 4 \equiv 2$
4	4	4	$4 \div 4 \equiv 1$	$4 \cdot 4 \equiv 1$

3. (mod 5)

b	a	additive inverse of a	b - a	b + (additive inverse of a)
0	1	4	0 - 1 ≡ 4	0 + 4 ≡ 4
2	1	4	2 - 1 ≡ 1	2 + 4 ≡ 1
4	1	4	4 - 1 ≡ 3	4 + 4 ≡ 3
1	2	3	1 - 2 ≡ 4	1 + 3 ≡ 4
2	2	3	2 - 2 ≡ 0	2 + 3 ≡ 0
3	2	3	3 - 2 ≡ 1	3 + 3 ≡ 1
2	4	1	2 - 4 ≡ 3	2 + 1 ≡ 3
3	4	1	3 - 4 ≡ 4	3 + 1 ≡ 4
4	4	1	4 - 4 ≡ 0	4 + 1 ≡ 0

4. (a) no
   (b) no
   (c) no
   (d) yes, except division by zero

5. (a) {0,1,2,3,4,5}
   (b) {1,5}, {5}
   (c) {2,4}, {1,5}, {5}

6. (a) {A,B}, {C,D}, {A,D}
   (b) yes, D
   (c) {C,D}
   (d) {C,D}

If you wish, you might bring up the general problem of defining an operation which is inverse to a given operation ✱ defined on a set. If there is an identity element e for ✱, if every element of the set has an inverse element in the set, and if ✱ is associative, then

(the inverse of b) ✱ a

could be written a ✱' b. Then ✱' will be the inverse operation for ✱.

Hence

$$a \ast' b = (\text{the inverse of } b) \ast a.$$

For example: Suppose a and b are rational numbers, $b \neq 0$, and $\ast$ is the multiplication operation, then $\ast'$ is division (the inverse operation) and $\frac{1}{b}$ is the inverse of b.

Hence:

$$a \div b = \frac{1}{b} \times a.$$

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6. What Is a Mathematical System?

Here the mathematical system is given an informal definition and is followed by discussion in terms of previous examples and some new ones. Here the teacher should not try to be too formal.

Teaching Suggestions

In Section 3, it was pointed out that a table can list a set and describe an operation defined on that set. Thus, a table really describes a mathematical system, and not merely an operation. Illustrate by discussing tables (a) - (e) of Section 3, and by showing that each table does describe a mathematical system (a set and one or more operations defined on that set -- in each case, it will be one operation).

In Example 1, Part (c) (egg-timer arithmetic), remind the pupils of the symmetry test for commutativity discovered in Problem 2 of Exercises 3. The table for egg-timer arithmetic is symmetric, so the operation is commutative.

Have the class decide on a word for the operation in Table (c) of this section. ("Twiddle" is sometimes used.)

Answers to Exercises 6

1. Each one of Tables (a), (b), (c) describes a mathematical system.
   For Table (a), the set is {A,B}; the operation is o.
   For Table (b), the set is {P,Q,R,S}; the operation is $\ast$.
   For Table (c), the set is {△, □, ○, ╲}; the operation is ~.

2. (a) A (e) Q (i) Δ
(b) O (f) S (j) B
(c) O (g) P (k) A
(d) B (h) \ (l) R

3. The operation o is not commutative, since Table (a) is not symmetric. The operations * and ~ are both commutative, since both Tables (b) and (c) are symmetric.

4. There is no identity element for the operation o.
There is no element e, such that both of the equations A o e = A and B o e = B are correct.
The element R is the identity element for the operation *. The row of Table (b) with "R" in the left column is the same as the top row, and the column with "R" at the top is the same as the left column.
The element Δ is the identity element for the operation ~. The first row and column of Table (c) are the same as the top row and left column respectively.

5. (a) S (e) Q (i) \
(b) S (f) Q (j) \
(c) R (g) \
(d) R (h) \

6. Each of the operations * and ~ seems to be associative since, in each of the cases we have tried, the corresponding expressions are equal. To prove the operations are associative, we would have to examine all cases and show that the corresponding expressions are equal. To prove an operation is not associative, a person would have to find one example where the corresponding expressions are not equal.

7. BRAINBUSTER. (a) The element 2 cannot be combined with 2 by the operation * (that is, 2 * 2 is not defined).

(b) 2 * 1 is not uniquely defined. Many results are possible when 2 and 1 are combined.

(c) The set given by this table is {1,2,3,4}. But it is not possible to combine every pair of elements (e.g. 3 and 3). We do not have an operation <u>defined on the set</u>.

7. Mathematical Systems Without Numbers

Skills and Understandings

1. To recognize a mathematical system when it is described in words.
2. For systems without numbers: To recognize the elements of the set; to recognize the operation; to recognize an identity element; to recognize the inverse of an element.

Teaching Suggestions

Each pupil should have his own rectangle to manipulate, such as, a 3" × 5" card. Do not use square cards. Be sure that each pupil labels his rectangle correctly so that comparisons between different pupils are possible. Check especially that each corner of the card is labeled with the same letter on both sides. Stress that the card is used only to represent a geometric figure -- a closed rectangular region.

It cannot be repeated too often that the changes of position of a rectangle are the elements of the set in the mathematical system discussed in this section. One of these changes is something that is "done"; that is, it is a physical activity, but it is an element of the set -- it is not the operation of the system. The operation of the system is much more elusive. Any operation defined on the set must be a way of combining any two of these physical activities (changes) to get a definite thing. The particular operation we have chosen combines two of these changes by doing the first one and then the other. The result (definite thing) obtained is one of the changes, but the operation is the way of combining them, that is: First do ..., and then do ... .

Discussion of Exercises 7

3. In proving associativity, "all cases" must be considered. There is one case for each triple of (not necessarily different) elements of the set on which the operation is defined. For the operation ANTH, there are 4 elements in the set, so there will be $4 \cdot 4 \cdot 4 = 64$ triples; that is, 64 cases must be considered to prove the associative property.

5. and *6. For ease in grading written work it is essential that all students use the same notation in these exercises. One possible notation is described in the answers.

Answers to Exercises 7

1.

ANTH	I	V	H	R
I	I	V	H	R
V	V	I	R	H
H	H	R	I	V
R	R	H	V	I

2. (a) V
   (b) V
   (c) V
   (d) V
   (e) I
   (f) I
   (g) I
   (h) I
   (i) I

3. (a) Yes
   (b) Yes
   (c) Yes, the operation is associative. A proof would require that 64 cases be checked. Each pupil should check two or three; do not attempt to check all cases. See discussion, Exercises 6.
   (d) Yes. I is the identity.
   (e) Yes. Each element is its own inverse.

4. (a)

ANTH	I	F
I	I	F
F	F	I

   (b) Yes
   (c) Yes
   (d) Yes. All cases can be checked (there are 8 cases in all). See discussion of Problem 3, Exercises 6.
   (e) Yes. I is the identity element.
   (f) Yes. Each element is its own inverse.

5.

ANTH	I	R	S	T	U	V
I	I	R	S	T	U	V
R	R	S	I	U	V	T
S	S	I	R	V	T	U
T	T	V	U	I	S	R
U	U	T	V	R	I	S
V	V	U	T	S	R	I

The operation is not commutative (R ANTH T $\neq$ T ANTH R) I is the identity element. Each of I, T, U, V is its own inverse element; R and S are inverses of each other.

*6. Notation:

I: Leave the square in place.

R_1: Rotate clockwise $\frac{1}{4}$ of the way around.

R_2: Rotate clockwise $\frac{1}{2}$ of the way around.

R_3: Rotate clockwise $\frac{3}{4}$ of the way around.

H: Flip the square over, using a horizontal axis.

V: Flip the square over, using a vertical axis.

D_1: Flip the square over, using an axis from upper left to lower right.

D_2: Flip the square over, using an axis from lower left to upper right.

Note: It was suggested that a square card not be used. This problem is included to show why such a suggestion was made.

ANTH	I	R_1	R_2	R_3	H	V	D_1	D_2
I	I	R_1	R_2	R_3	H	V	D_1	D_2
R_1	R_1	R_2	R_3	I	D_2	D_1	H	V
R_2	R_2	R_3	I	R_1	V	H	D_2	D_1
R_3	R_3	I	R_1	R_2	D_1	D_2	V	H
H	H	D_1	V	D_2	I	R_2	R_1	R_3
V	V	D_2	H	D_1	R_2	I	R_3	R_1
D_1	D_1	V	D_2	H	R_3	R_1	I	R_2
D_2	D_2	H	D_1	V	R_1	R_3	R_2	I

I is the identity element. The operation is not commutative (R_1 ANTH $H \neq H$ ANTH R_1).

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8. The Counting Numbers and the Whole Numbers

This section has problems which lead the pupils to conclude that the counting numbers and the whole numbers each form a mathematical system. It is pointed out that the distributive property with which the pupil is familiar comes from the abstract discussion of this property. The pupils should not be expected to duplicate the abstract definition.

One of the objectives of the section is to show a way to pull together the concept of systems.

Some of the sets of numbers considered in ordinary arithmetic are: the rational numbers, the whole numbers, the counting numbers, the even numbers, etc.

Discussion of Exercises 8

4. One possible model of the mathematical system in this exercise is as follows: Let A = {1,2}, B = {1,2,3}, C = {1,2,4}, D = {1,2,3,4}. Then, from the tables in the problem, the operation $*$ is intersection and the operation o is union. Each of these operations distributes over the other.

Answers to Exercises 8

1. (a) Since the sum of two counting numbers is always another counting number and the product of two counting numbers is always a counting number, the set is closed under addition and multiplication.

   (b) Both the commutative property and the associative property hold for addition and multiplication.

   Examples: Commutative: $2 + 3 = 3 + 2$;
   $4 \times 6 = 6 \times 4$

   Associative: $3 + (4 + 7) = (3 + 4) + 7$;
   $3 \times (6 \times 8) = (3 \times 6) \times 3$.

   (c) There is no identity element for addition.
   The identity element for multiplication is 1; for every counting number n, $n \cdot 1 = n = 1 \cdot n$.

   (d) The counting numbers are not closed under subtraction or division.

2. (a) The set of whole numbers is closed under addition and multiplication.

   (b) Both operations are commutative and associative.

   (c) There is an identity element for addition. It is zero; for any whole number n, $n + 0 = n = 0 + n$. The number 1 is the identity element for multiplication.
   The answers are the same as for 1 (a), (b), (c) except that there is an identity element for addition in the whole number system and not in the counting number system.

3. (a) Three examples are: $2(3 + 4) = (2 \cdot 3) + (2 \cdot 4)$;
   $5(7 + 10) = (5 \cdot 7) + (5 \cdot 10)$;
   $1(1 + 1) = (1 \cdot 1) + (1 \cdot 1)$.

   (b) Addition does not distribute over multiplication; for example, $2 + (3 \cdot 4) = 14 \neq 30 = (2 + 3) \cdot (2 + 4)$.

4. See discussion, Exercises 7.

(a) Yes, here are 3 illustrations that $*$ distributes over o:

$A * (B \circ C) = A = (A * B) \circ (A * C)$
$B * (B \circ B) = B = (B * B) \circ (B * B)$
$C * (B \circ D) = C = (C * B) \circ (C * D)$

(b) Yes, here are 3 illustrations that o distributes over $*$.

$A \circ (B * C) = A = (A \circ B) * (A \circ C)$
$B \circ (B * B) = B = (B \circ B) * (B \circ B)$
$C \circ (B * D) = D = (C \circ B) * (C \circ D)$

5. (a) Closed; commutative; associative; 1 is the identity; only the number 1 has an inverse.

(b) Closed; commutative; associative; no identity; no inverses.

(c) Closed; commutative; associative; 0 is the identity; only the number 0 has an inverse.

(d) Closed; commutative; associative; no identity; no inverses.

(e) Closed; commutative; associative; 0 is the identity; only the number 0 has an inverse.

(f) Not closed; commutative; not associative; no identity; no inverse

6. (a) Both sets are closed under the operations. Both operations are commutative and associative. Both systems involve the same set.

(b) The system 5(a) has an identity and 5(b) does not. Also, the sets are different in these two systems.

*7. Many results are possible, of course.

*8. (a) Yes. We are asked to consider the two expressions $a * (b \circ c)$ and $(a * b) \circ (a * c)$, and find whether or not they are always equal. For example, using $a = 8$, $b = 12$, $c = 15$,
$8 * (12 \circ 15) = 8 * 60 = 4$.
$(8 * 12) \circ (8 * 15) = 4 \circ 1 = 4$.

(b) Yes. We are asked to consider the two expressions $a \circ (b * c)$ and $(a \circ b) * (a \circ c)$, and find whether or not they are always equal. For example using $a = 8$, $b = 12$, $c = 15$,
$8 \circ (12 * 15) = 8 \circ 3 = 24$.
$(8 \circ 12) * (8 \circ 15) = 24 * 120 = 24$.

9. Modular Arithmetic

In this section, the number line is used to provide a picture of how equivalence classes of whole numbers can be developed. At this time it may be wise to re-read the first paragraphs of Section 1. We use the term "multiple" to mean "multiple by a whole number".

Problems which may be used for motivation to explain the meaning of modular systems include the ordinary 12-hour clock, the days of the week, and the months of the year. For example, "Today is Tuesday; what day will it be six days from now?" Answer: Monday; this is (mod 7). "It is 4:30 o'clock. What time will it be 10 hours from now?" Answer: 2:30; this is (mod 12).

Modular arithmetic may be thought of as a mathematical system with two operations. Section 1 discussed modular addition and Section 2 discussed modular multiplication. The two operations together allow us to use the distributive property; thus, the whole numbers form a system under modular addition and multiplication. In modular arithmetic only a finite number of symbols is needed because infinitely many whole numbers are represented by each symbol.

Other interesting highlights are:

A product of non-zero factors may be zero in some systems.

There may be many replacements for x in a number sentence to make it true.

Answers to Exercises 9

1. (mod 5)

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(mod 8)

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

(Encourage the pupils to look for patterns and to use what they have previously learned about systems to make the tables.)

2. (a) (mod 5): Yes; (mod 8): Yes

(b) (mod 5): Yes; (mod 8): Yes

(c) (mod 5): Yes; (mod 8): Yes

(d) (mod 5): 1; (mod 8): 1

(e) (mod 5): 1 and 4 are their own inverses; 2 and 3 are inverses of each other; 0 has no inverse.

(mod 8): Only 1, 3, 5, 7 are inverses; each is its own inverse.

(f) (mod 5): Yes; (mod 8): No. $2 \times 4 \equiv 0 \pmod{8}$, $4 \times 2 \equiv 0 \pmod{8}$, $4 \times 4 \equiv 0 \pmod{8}$, $4 \times 6 \equiv 0 \pmod{8}$, $6 \times 4 \equiv 0 \pmod{8}$.

3. (a) 3
(b) 2
(c) 6, 8, 12, 24
(d) 4, 8

4. (a) 2
(b) 0
(c) 5
(d) 0
(e) 4
(f) 1
*(g) 1. Any power of 6 ends in 6.

5. (a) 4
(b) 2
(c) 1
(d) 3

6. (a) 4, 4
(b) 1, 1
(c) 3, 3
(d) Yes

7. (a) 0, 0
(b) 2, 0
(c) 0, 1
(d) No

8. (a) 6
(b) 3, 7
(c) 0, 4
(d) 2
(e) 0
(f) 0
(g) 9
(h) Not defined in this system.

9. (a) 4; What number added to 3 gives 7?
(b) 4
(c) 7
*(d) 7

10.

-	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

The set is closed under subtraction (mod 5).

11. (a) 3, 8, 13 and others (add 5)
    (b) 3, 7, 11 and others (add 4)
    (c) 0 and all multiples of 5 of the form 5K, K is a counting number.
    (d) Any even number
    (e) 3, or any odd number greater than 3
    (f) 1, 3, 5 and so on (all odd numbers)

12. (d) Any even number
    (f) 1, 3, 5, 7, 9, 11, 13 and so on (all odd numbers)

Sample Questions

Part I. True - False

T 1. Operations can be described by tables.

T 2. A symbol can be made to mean anything providing we define it.

F 3. The identity for multiplication in ordinary arithmetic is zero.

F 4. The identity for addition in ordinary arithmetic is one.

T 5. The additive inverse of 2 in the (mod 4) system is 2.

T 6. In ordinary arithmetic, with the set composed of all the rational numbers except zero, the inverse of division is multiplication.

F 7. All mathematical systems are sets of numbers.

F 8. In (mod 5) arithmetic, $0/3 \equiv 2$ (mod 5).

T 9. The set {0,1,2,3} is closed under subtraction (mod 4).

Part II. Computation

	Answers:
Find the sums:	
1. (9 + 2) (mod 12)	11
2. (5 + 4 + 3) (mod 6)	0
Find the differences:	
3. (5 - 2) (mod 6)	3
4. (3 - 5) (mod 7)	5
Find the products:	
5. [(3 + 7) × 6] (mod 9)	6
6. 3^2 (mod 8)	1
Find the quotients:	
7. $\frac{2}{3}$ (mod 5)	4
8. $\frac{0}{7}$ (mod 11)	0

Part III. Multiple Choice

The table below describes a mathematical system. It is to be used in answering questions 1, 2, and 3 below.

o	A	B	C	D
A	C	D	A	B
B	D	A	B	C
C	A	B	C	D
D	B	C	D	A

1. Which one of the following statements is true? (Answers are starred).

   A. The set {A,B,C,D} is not closed with respect to the operation o.

   * B. The operation o is commutative.

   C. The operation o does not have an identity element.

   D. The operation o is not associative.

   E. None of the above.

2. The identity for the operation o is:

   A. D

   B. B

   * C. C

   D. Both A and B

   E. None of the above.

3. In the mathematical system:

   A. Only B has an inverse.

   B. Only D has an inverse.

   C. Only A and C have inverses.

   D. None of the elements has an inverse.

   *E. All the elements have inverses.

4. For what modulus m is $2 - 5 \equiv 4 \pmod{m}$ true?

A. Mod 9

B. Mod 6

C. Mod 8

* D. Mod 7

E. None of the above.

5. For the system consisting of the set of odd numbers and the operation of multiplication:

A. The system is not closed.

B. The system is not commutative.

C. The system has no identity element.

* D. None of the above is correct.

E. All of the above are correct.

6. For the system consisting of the set of even numbers and the operation of addition:

A. The system is not closed.

* B. The system has an identity element.

C. The system has an inverse for addition for each element.

D. All of the above are correct.

E. None of the above is correct.

7. A mathematical system consists of several things. Which of the following is <u>always</u> <u>necessary</u> in a mathematical system?

A. Numbers

B. An identity element

C. The commutative property

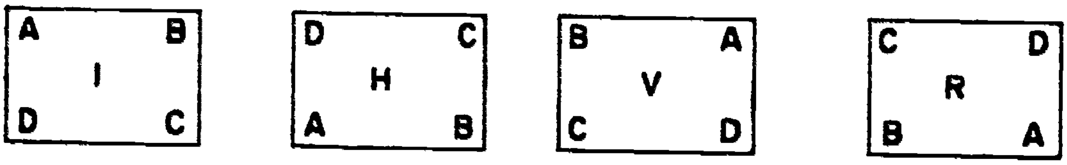
* D. One or more operations

E. None of the above

4

Use the mathematical system as described below in answering Questions 8, 9, and 10. The set of elements in our system is the set of changes of a rectangle.

The elements are



I means leave alone.

H means flip on the horizontal axis.

V means flip on the vertical axis.

R means turn halfway around its center.

The following is an illustration of our operation * ;

V * H means do change V and then do change H.
Thus V * H = R.

8. H * H is:

A. H

* B. I

C. R

D. V

E. None of the above.

9. I * R is:

* A. R

B. V

C. I

D. H

E. R * H

10. (H * V) * V is:

A. I

B. V * V

* C. H * I

D. V

E. None of the above.

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