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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include a new kind of addition and multiplication, operations, closure, identity, mathematical systems without numbers, and modular arithmetic. (MP)

SCHOOL MATHEMATICS STUDY GROUP

SP-19

SUPPLEMENTARY and ENRICHMENT SERIES

MATHEMATICAL SYSTEMS

Edited by Henry W. Syer

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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FOREWORD

This booklet introduces simple and fundamental ideas of mathematics. Basic properties of modular number systems are used to discuss axioms for all number systems and then these axioms are investigated for some non-numerical examples. This material would be most appropriate for the junior high school level, but could be used at higher grade levels. A separate teachers commentary with answers is available.

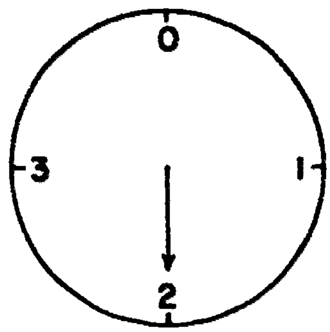
As background the reader needs only some facility in the arithmetic of the positive whole numbers and zero. The most important attitude to bring to the booklet is the willingness to play with new ideas; to be open-minded about mathematics that seems, at first, to contradict old arithmetic; and to enjoy the learning of games with new rules.

This material was originally published as part of the texts for Junior High School Mathematics by SMSG.

CONTENTS

MATHEMATICAL SYSTEMS

1. A New Kind of Addition



The sketch above represents the face of a four-minute clock. Zero is the starting point and, also, the end-point of a rotation of the hand.

With the model we might start at 0 and move to a certain position (numeral) and then move on to another position just like the moving hand of a clock. For example, we may start with 0 and move $\frac{2}{4}$ of the distance around the face. We would stop at 2. If we follow this by a $\frac{1}{4}$ rotation (moving like the hand of a clock), we would stop at 3. Another time after a rotation of $\frac{2}{4}$ from 0 we could follow with a $\frac{3}{4}$ rotation. This would bring us to 1. The first example could be written 2 + 1 gives 3 where the 2 indicates $\frac{2}{4}$ of a rotation from 0, the + means to follow this by another rotation (like the hand of a clock), and the 1 means $\frac{1}{4}$ rotation, thus we arrived at the position marked 3 (or $\frac{3}{4}$ of a rotation from 0). The second example could be 2 + 3 gives 1 where the 2 and + still mean the same as in the first example and the 3 means a rotation of $\frac{3}{4}$. A common way to write this is:

$$2 + 3 \equiv 1 \pmod 4$$

which is read:

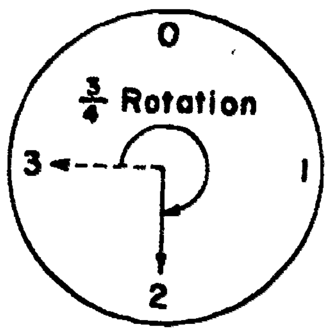
Two plus three is congruent to one, modulo 4,

or

Two plus three is congruent to one (mod 4).

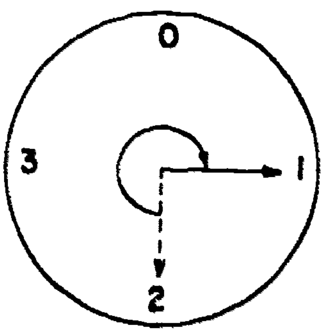
The (mod 4) means that there are four numerals: 0, 1, 2, 3 on the face of the clock. The + sign means what we described above; this is our new type of addition. The $\equiv$ between 2 + 3 and the 1 indicates that 2 + 3 and 1 are the same (that is, "equivalent") <u>on</u> <u>this</u> <u>clock</u>. We call this briefly "addition (mod 4)." Of course there are other possible notations which could be used but this is the usual one. The expression "(mod 4)" is derived from the fact that sometimes 4 is called "the modulus" which indicates how many single steps are taken before repeating the pattern.

<u>Example</u> <u>1</u>: Find 3 + 3 (mod 4).

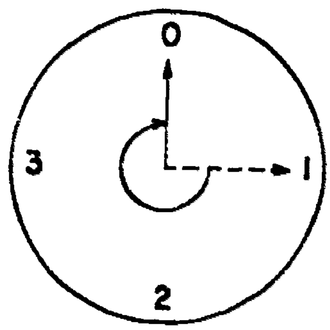


$3 + 3 \equiv 2 \pmod 4$

<u>Example</u> <u>2</u>: Find (2 + 3) + 3 (mod 4).



$2 + 3 \equiv 1 \pmod 4$



$1 + 3 \equiv 0 \pmod 4$

$(2 + 3) + 3 \equiv 1 + 3 \equiv 0 \pmod 4$

The following table illustrates some of the addition facts in the (mod 4) system.

(Mod 4)

+	0	1	2	3 ↓
0	0	1	2	
1			3	0
→ 2				1
3		0		

We read a table of this sort by following across horizontally from any entry in the left column, for instance 2, to the position below some entry in the top row, such as 3 (see arrows). The entry in this position in the table is then taken as the result of combining the element in the left column with the element in the top row (in that order). In the case above we write $2 + 3 \equiv 1 \pmod 4$. Use the table to check that $3 + 1 \equiv 0 \pmod 4$.

Example 3: Complete the following number sentences to make them true statements.

(a) $3 + 4 \equiv ? \pmod 5$

The mod 5 system represented by the face of a clock should have five positions; namely, 0, 1, 2, 3, and 4. If you draw this clock you will see that $3 + 4 \equiv 2 \pmod 5$ since the 3 means a rotation of $\frac{3}{5}$ from 0. This is followed by a $\frac{4}{5}$ rotation which ends at 2.

(b) $2 + 3 \equiv ? \pmod 5$

$2 + 3 \equiv 0 \pmod 5$. This is a $\frac{2}{5}$ rotation from 0 followed by a $\frac{3}{5}$ rotation which brings us to 0.

(c) $4 + 3 \equiv ? \pmod 6$

In the mod 6 system, the positions on the face of the clock are marked 0, 1, 2, 3, 4, and 5. If you draw this clock you will see that $4 + 3 \equiv 1 \pmod 6$.

Exercises 1

1. Copy and complete the table for addition (mod 4). Use it to complete the following number sentences:

   (a) $1 + 3 \equiv ? \pmod 4$ (c) $2 + 2 \equiv ? \pmod 4$

   (b) $3 + 3 \equiv ? \pmod 4$ (d) $2 + 3 \equiv ? \pmod 4$

2. Make a table for addition (mod 3) and for addition (mod 5).

3. Use the tables in Problem 2 to find the answers to the following:

   (a) $1 + 2 \equiv ? \pmod 3$ (c) $2 + 2 \equiv ? \pmod 3$

   (b) $3 + 3 \equiv ? \pmod 5$ (d) $4 + 3 \equiv ? \pmod 5$

4. Make whatever tables you need to complete the following number sentences.

   (a) $5 + 3 \equiv ?\ (\text{mod } 6)$

   (b) $5 + 5 \equiv ?\ (\text{mod } 6)$

   (c) $3 + 6 \equiv ?\ (\text{mod } 7)$

   (d) $4 + 5 \equiv ?\ (\text{mod } 7)$

   Note: Be sure to keep all the tables you have made. You will find use for them later in this pamphlet.

5. Find a replacement for x to make each of the following number sentences a true statement.

   (a) $4 + x \equiv 0\ (\text{mod } 5)$

   (b) $x + 1 \equiv 2\ (\text{mod } 3)$

   (c) $1 + x \equiv 2\ (\text{mod } 3)$

   (d) $2 + x \equiv 4\ (\text{mod } 5)$

   (e) $3 + x \equiv 2\ (\text{mod } 5)$

   (f) $x + 4 \equiv 3\ (\text{mod } 5)$

   (g) $x + 2 \equiv 0\ (\text{mod } 3)$

   (h) $4 + x \equiv 4\ (\text{mod } 5)$

6. You have a five-minute clock. How many complete revolutions would the hand make if you were using it to tell when 23 minutes had passed? Where would the hand be at the end of the 23 minute interval? (Assume that the hand started from the 0 position.)

7. Seven hours after eight o'clock is what time? What new kind of addition did you use here?

8. Nine days after the 27th of March is what date? What new kind of addition did you use here?

---

2. A New Kind of Multiplication

Before considering a new multiplication let us look at a part of a multiplication table for the whole numbers. Here it is:

x	0	1	2	3	4	5	↓ 6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	8	10	12
3	0	3	6	9	12	15	18
4	0	4	8	12	16	20	24
→ 5	0	5	10	15	20	25	30
6	0	6	12	18	24	30	36

If we had this table and forgot what 5 times 6 is equal to, we could look in the row labeled 5 and the column labeled 6 and find the answer, 30, in the 5-row and 6-column. (see arrows above). Of course it is easier to memorize the table since we use it so frequently, but if we had not memorized it, it might be a very convenient thing to have in our pocket for easy reference.

How would you make such a table if you didn't know it already? This would be quite easy if you could add. The first line is very easy -- you write a row of zeros. For the second line you merely have to know how to count. For the third line you add 2 each time; for the fourth line add 3 each time, and so forth.

Now, if we use the same method, we can get a multiplication table (mod 4). First block it out, filling in the first and second rows and columns:

(Mod 4)

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
→ 2	0	2		
3	0	3		

We have just four blanks to fill in. To get the 2-row (indicated by the arrow above), we add twos. Thus the third entry (which is 2×2) is $2 + 2 \equiv 0 \pmod 4$. Then our first three entries will look like this:

2 | 0 2 0

To get the fourth entry, we add 2 to the third entry. Since $0 + 2 \equiv 2 \pmod 4$, the complete 2-row is now

2 | 0 2 0 2.

For the last row we will have to add threes. Here $3 + 3 \equiv 2 \pmod 4$ and $3 + 2 \equiv 1 \pmod 4$. So the complete table is:

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Now consider one way in which this table could be used. Suppose a lamp has a four-way switch so that it can be turned to one of four positions: off, low, medium, high. We might let numbers correspond to these positions as follows:

off	low	medium	high
0	1	2	3

If the light were at medium and we flicked the switch three times, the light would be at the low position since $2 + 3 \equiv 1 \pmod 4$. Suppose the light were off and three people flicked the switch three times each; what would be the final position of the light? The answer would be "low" since $3 \cdot 3 \equiv 1 \pmod 4$ and the number 1 corresponds to "low".

Consider an application of another multiplication table. A jug of juice lasts three days in the Willcox family. One Saturday, Mrs. Willcox bought six jugs which the family started using on the following day. What day of the week would it be necessary for her to purchase juice again? Of course it would be possible to count on one's fingers so to speak: 3 days after Saturday is Tuesday, 3 days after Tuesday is Friday, etc. But it is much easier if we notice that since there are seven days in the week, this is connected with multiplication (mod 7). We could let the number 0 correspond to Saturday since this is the day we start with and so on as follows:

Sat.	Sun.	Mon.	Tues.	Wed.	Thur.	Fri.
0	1	2	3	4	5	6

Now we need to find what $6 \cdot 3$ is (mod 7). We do not need the complete multiplication table since we are trying to find a multiple of 6. So we compute the 6-row in the usual way by adding sixes, using the addition table (mod 7) which we constructed for Problem 4 of the previous set of exercises.

(Mod 7)

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
.	.	.	.	.	.	.	.
6	0	6	5	4	3	2	1

This means that $6 \cdot 3 \equiv 4 \pmod 7$ and since Wednesday corresponds to 4, it follows that Mrs. Willcox would next have to buy juice on a Wednesday. Actually we did not really need to construct all the 6-row.

Exercises 2

1. (a) Make a table for multiplication (mod 5).
   (b) Make a table for multiplication (mod 7).
   (c) Make a table for multiplication (mod 6).

   Note: Keep these tables for future use.

2. Complete the following number sentences to make them true statements. You may find the tables you constructed in Problem 1 useful.
   (a) $3 \times 2 \equiv ? \pmod 5$
   (b) $3 \times 4 \equiv ? \pmod 6$
   (c) $6 \times 4 \equiv ? \pmod 7$
   (d) $1 + (3 \times 4) \equiv ? \pmod 6$
   (e) $5 + (6 \times 5) \equiv ? \pmod 7$

3. Find a replacement for x to make each of the following number sentences a true statement: (Draw the clocks if you need them to find the answer.)
   (a) $5 \cdot 10 \equiv x \pmod{11}$
   (b) $7 \cdot 13 \equiv x \pmod{15}$
   (c) $(3 \cdot 4) + 2 \equiv x \pmod 8$
   (d) $(4 \cdot 7) + 11 \equiv x \pmod{13}$

4. (a) Find the date of 10 weeks after December fourth.
   (b) In 1957, August sixth was a Tuesday. What day of the week was August sixth in 1959?

5. Form a table of remainders after division by 5, where the entry in any row and column is the remainder after the product is divided by 5. For instance, since the remainder is 1 when $2 \cdot 3$ is divided by 5, we will have a 1 in the 2-row and 3-column (see arrows). We have written in a few entries to show how it goes. For instance, to get the entry in the 2-row and 4-column we multiply 2 by 4 to get 8 and since the remainder when 8 is divided by 5 is 3, we put 3 in the 2-row and 4-column. Complete the table:

	0	1	2	↓ 3	4
0	0	0	0	0	0
1	0	1	2	3	4
→ 2	0	2		1	3
3	0	3		4	
4	0	4	3		

6. Do you notice any relationship between the table of Problem 5 and another you have found? Can you give any reason for this? How can this be used to make the solution of some of the problems simpler?

* 7. Use the multiplication table (mod 5) to find the replacement for x to make each of the following number sentences a true statement:

(a) $3x \equiv 1 \pmod 5$

(b) $3x \equiv 2 \pmod 5$

(c) $3x \equiv 3 \pmod 5$

(d) $3x \equiv 4 \pmod 5$

(e) $3x \equiv 0 \pmod 5$

*8. If it were (mod 6) instead of (mod 5) in the previous problem, would you be able to find x in each case? If not, which equivalences would give some value of x?

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3. <u>What is an Operation?</u>

We are familiar with the operations of ordinary arithmetic -- addition, multiplication, subtraction and division of numbers. In Section 1, a different operation was discussed. We made a table for the new type of addition of the numbers 0, 1, 2, 3. This operation is completely described by the table that you made in Problem 1 of Exercises 1. That is, there are no numbers to which the operation is applied except those indicated and the results of the operation on all pairs of these numbers are given. The table tells what numbers can be put together. For instance, the table tells us that the number 5 cannot be combined with any number in the new type of addition since "5" does not appear in the left column nor in the top row. It also tells us that $2 + 3 \equiv 1 \pmod 4$. Study the following tables:

---

* will be used on exercises to indicate slightly greater difficulty.

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(d)

⊙	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(e)

Δ	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(c)

□	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

So far the only operations we have had have been called multiplication or addition. Here in (c), (d) and (e) we have different operations and so we use different symbols: □ , ⊙ , and Δ.

From each one of these tables we can find a certain set (the set of elements in the left column and top row) and we can put any two elements of this set together to get one and only one thing. For instance, in Table (a), the set is {1,2,3,4,5} since these are the numbers which appear in the left column and top row. These are the only numbers which can be put together by Table (a). In Table (b), the set is {3,5,7,9}. What set is given by Table (c)? by Table (d)? by Table (e)?

Here are some examples from the tables:

3 + 5 = 3 in Table (a),

3 + 5 = 8 in Table (b),

2 □ 1 = 5 and 2 □ 2 = 6 in Table (c). Read "2 square 1 equals 5".

1 ⊙ 1 = 3 in Table (d). Read "1 circle-dot 1 equals 3."

5 Δ 2 = 3 in Table (e). Read "5 triangle 2 equals 3."

In each case we had a set of elements: in (a) the set is {1,2,3,4,5}; in (d) it was {1,2,3}. We also had an operation: in (a) it was +; in (d) it was ⊙. Finally we had the result of combining any two elements by means of the operation; in (a) the results were 1, 2, 3, 4, or 5; in (c) they were 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. All of these operations are called <u>binary operations</u> because they are applied to two elements to get a third. So far the elements have been numbers but we shall see later that they do not need to be.

The two elements which we combine may be the same and the result of the operation may or may not be an element of the set but it must be something definite -- not one of several possible things.

You are already familiar with some operations defined on the set of whole numbers.

Any two whole numbers can be added. Addition of 8 and 2 gives 10.

Any two whole numbers can be multiplied. Multiplication of 8 and 2 gives 16.

Addition and multiplication are two different operations defined on the set of whole numbers.

In discussing subtraction, for instance with whole numbers, it is convenient to look ahead to later work in mathematics. The expression "6 - 9" is not the name of anything you have used in this pamphlet. That is, it is not now possible for us to combine 6 and 9 (in that order) by subtraction and get "a definite thing" and you may wonder whether or not subtraction of whole numbers is an operation. In fact, you may already know that there <u>is</u> "a definite thing" (in fact, a number) which is called "6 - 9". With this in mind, we will consider subtraction a binary operation defined on the whole numbers (or rational numbers, etc.), even though we are not yet acquainted with all the results obtained from subtraction.

When an operation is described by a table, the elements of the set are written in the same order in the top row (left to right) and in the left column (top to bottom). Keeping the order the same will make some of our later work easier.

We must also be careful about the order in which two elements are combined. For example,

$$2 \square 1 = 5, \quad \text{but} \quad 1 \square 2 = 4.$$

For this reason, we must remember that when the procedure for reading a table was explained, it was decided to write the element in the left column first and the element in the top row second with the symbol for the operation between them. We must examine each new operation to see if it is commutative and associative.

An operation $*$ defined on a set is called <u>commutative</u> if, for any elements, a, b, of the set, $a * b = b * a$.

An operation $*$ defined on a set is called <u>associative</u> if any elements, a, b, c, of the set can be combined as $(a * b) * c$, and also as $a * (b * c)$, and the two results are the same: $(a * b) * c = a * (b * c)$.

<u>Exercises 3</u>

1. Use the tables in the text to answer the following questions.

   (a) $3 + 3 = ?$ if we use Table (a).

   (b) $3 + 3 = ?$ if we use Table (b).

   (c) $3 \square 2 = ?$

   (d) $2 \square 3 = ?$

   (e) $2 \odot 2 = ?$

   (f) $1 \odot 1 = ?$

   (g) $(2 \odot 3) \odot 3 = ?$

   (h) $2 \odot (3 \odot 3) = ?$

   (i) $(1 \square 1) \square 2 = ?$

   (j) $1 \square (1 \square 2) = ?$

   (k) $2 \triangle (3 \triangle 4) = ?$

   (l) $(2 \triangle 3) \triangle 4 = ?$

2. (a) Which of the binary operations described in the tables in this section are commutative?

   (b) Is there an easy way to tell if an operation is commutative when you examine the table for the operation? What is it?

3. How can you tell if a operation is associative by examining a table for the operation? Do you think the operations described in the tables in this section are associative?

4. Are the following binary operations commutative? Make at least a partial table for each operation. Which ones do you think are associative?

(a) Set: All counting numbers between 25 and 75.
Operation: Choose the smaller number.
Example: 28 combined with 36 produces 28.

(b) Set: All counting numbers between 500 and 536.
Operation: Choose the larger number, or, if the two numbers are equal, choose either.
Example: 520 combined with 509 produces 520.

(c) Set: The prime numbers.
Operation: Choose the larger number, or, if the two numbers are equal, choose either.

(d) Set: All even numbers between 39 and 61.
Operation: Choose the first number.
Example: 52 combined with 46 produces 52.
46 combined with 52 produces 46.

(e) Set: All counting numbers less than 50.
Operation: Multiply the first by 2 and then add the second.
Example: 3 combined with 5 produces 11, since $2 \cdot 3 + 5 = 11$.

(f) Set: All counting numbers.
Operation: Find the greatest common factor.
Example: 12 combined with 18 produces 6.

(g) Set: All counting numbers.
Operation: Find the least common multiple.
Example: 12 combined with 18 produces 36.

(h) Set: All counting numbers.
Operation: Raise the first number to a power whose exponent is the second number.
Example: 5 combined with 3 produces 5^3.

5. Make a table for an operation that has the commutative property.

6. Make up a table for an operation that does not have the commutative property.

We have been discussing binary operations. The word "binary" indicates that two elements are combined to produce a result. There are other kinds of operations. A result might be produced from a single element, or by combining three or more elements. When we have a set and, from any one element of the set, we can determine a definite thing, we say there is a "unary operation" defined on the set. If we combined three elements to produce a fourth we would call it a "ternary operation". One example of a ternary operation would be finding the greatest common factor (G. C. F.) of three counting numbers: e.g. the G. C. F. of 6, 8, and 10 is 2.

*7. Try to show a way of describing the following unary operation by some kind of a table.

Set: All the whole numbers from 0 to 10.

Unary Operation: Cube the number.

Example: If we perform the operation on 5 we get $5^3 = 125$.

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4. <u>Closure</u>

(a)

*	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(c)

□	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

(b)

*	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(d)

⊙	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

Study the Tables (a) and (b). In Table (a) the results of performing the operation are the numbers which were combined by the operation, namely 1, 2, 3, 4, 5 over again. But in Table (b) the results of performing the operation were different numbers from those combined (6, 8, 10, etc. instead of 3, 5, 7, 9). We have seen this kind of difference before, and we have a name for it. We have said that the set of whole numbers is "closed under addition" because if any two whole numbers are combined by adding them, the result is a whole number. In the same way the set {1,2,3,4,5} in Table (a) is closed under the new type of addition there since the results of the operation are again in the same set.

However, the set of odd numbers is not closed under addition since the result of adding two odd numbers is not an odd number. In the same way, in Table (b) the set {3,5,7,9} is not closed under the operation of addition given there since the result is not one of the set {3,5,7,9}.

Example 1:

The set of whole numbers: {1,2,3,4} is not closed under multiplication because $2 \cdot 3 = 6$ which is not one of the set. Of course $1 \cdot 2 = 2$ is in the set but for a set to be closed the result must be in the set no matter what numbers of the set are combined.

Example 2:

The set of all whole numbers is closed under multiplication because the product of any two whole numbers is a whole number again.

Example 3:

The set of whole numbers is not closed under subtraction. For example, consider the two whole numbers 6 and 9. There are two different ways we can put these two numbers together using subtraction: $9 - 6$ and $6 - 9$. The first numeral, "9 - 6", is a name for the whole number 3, but the numeral "6 - 9" is not the name of any whole number. Thus, subtracting two whole numbers does not always give a whole number.

Example 4:

The set of counting numbers is not closed under division. It is true that $\frac{8}{2} = 8 \div 2$ is a counting number, but there is no counting number $\frac{9}{2}$. Can you give some other illustrations of closure, that is, sets closed under an operation and sets not closed under an operation?

Example 5:

What can we say about a set S of counting numbers which is closed under addition and which contains the number 3? What other numbers must it contain? Since 3 is in S, $3 + 3$, or 6, must also be in S. Since $(3 + 3)$ and 3 are members of S, $(3 + 3) + 3 = 6 + 3 = 9$ must be in S. Since $(3 + 3 + 3)$ and 3 are in S, $(3 + 3 + 3) + 3 = 9 + 3 = 12$ must also be in S. We can continue adding 3 to the resulting numbers to see that $3k$ must be in S for any counting number k. Thus S must contain all of the multiples of 3. What is the smallest set S of counting numbers containing 3 and closed under addition? As we have seen S must contain all multiples of 3. What if S contains only these numbers:

$$S = \{3,6,9,12,\ldots\}.$$

Is S closed under addition? Is the sum of any two multiples of 3 a multiple of 3? If k and m are counting numbers, is $3k + 3m$ a multiple of 3? The answer is yes, of course, since, by the distributive property of multiplication over addition,

$$3k + 3m = 3(k + m).$$

Thus, $S = \{3, 6, 9, 12, \ldots\}$ is closed under addition, and is the smallest set closed under addition, which contains 3. We call S the set generated by 3 under addition.

Example 6:

We could ask the same questions about multiplication which we asked about addition in Example 5. What is the smallest set closed under multiplication and containing 3? Such a set certainly must contain

$$3,$$
$$3 \cdot 3 = 3^2,$$
$$3^2 \cdot 3 = (3 \cdot 3) \cdot 3 = 3^3,$$
$$3^3 \cdot 3 = [(3 \cdot 3) \cdot 3] \cdot 3 = 3^4,$$

and so on.

That is, every number 3^k, where k is a counting number, must be in the set. Is the set $T = \{3,3^2,3^3,\ldots\}$ closed under multiplication? If n and m are counting numbers, is $3^n \cdot 3^m$ a member of T?

Write

$$3^n = \overbrace{3 \cdot 3 \cdot \ldots \cdot 3}^{n \text{ factors}},$$

$$3^m = \overbrace{3 \cdot 3 \cdot \ldots \cdot 3}^{m \text{ factors}},$$

$$3^n \cdot 3^m = \overbrace{3 \cdot 3 \cdot \ldots \cdot 3}^{n} \cdot \overbrace{3 \cdot 3 \cdot \ldots \cdot 3}^{m} = \overbrace{3 \cdot 3 \cdot \ldots \cdot 3}^{n + m} = 3^{n + m}.$$

Thus $3^n \cdot 3^m = 3^{n + m}$ is a power of 3 also, so if 3^n and 3^m are in T so is their product. Thus

$$T = \{3, 3^2, 3^3, 3^4, \ldots\} \quad .$$

is the smallest set closed under multiplication and containing 3. We call T the set generated by 3 under multiplication.

Following Examples 5 and 6 we say that the set generated by an element a under an operation $*$ is the set

$$\{a,\ a * a,\ (a * a) * a,\ [(a * a) * a] * a,\ \ldots\}.$$

Exercises 4

1. Study again Tables (a) - (d) in this section. Which tables determine a set that is closed under the operation? Which tables determine a set that is not closed under the operation? How do you know?

2. Which of the sets below are closed under the corresponding operations?

   (a) The set of even numbers under addition.

   (b) The set of even numbers under multiplication.

   (c) The set of odd numbers under multiplication.

   (d) The set of odd numbers under addition.

   (e) The set of multiples of 5 under addition.

   (f) The set of multiples of 5 under subtraction.

   (g) The set $\{1,2,3,4\}$ under multiplication (mod 5).

   (h) The set of counting numbers less than 50 under the operation of choosing the smaller number.

   (i) The set of prime numbers under addition.

   (j) The set of numbers whose numerals in base five end in "3" under addition.

3. Find the smallest set of counting numbers.

   (a) closed under addition and containing 2.

   (b) closed under multiplication and containing 2.

4. (a) Find the set generated by 7 under addition.

   (b) Find the set generated by 7 under multiplication.

5. Let S be the set determined by Table (d) in this section.
   Find the subset of S which is generated by 1 under ⊙.
   Find the subset of S which is generated by 2 under ⊙.

* 6. What subset of the set of rational numbers is generated by 3? Is this set closed under division? (Is 3 in the set? Is $\frac{1}{3}$ in the set? Is $3 \div \frac{1}{3}$ in the set?) Does $(3 \div 3) \div 3 = 3 \div (3 \div 3)$? Is the division operation associative?

* 7. If an operation defined on a set is commutative, must the set be closed under the operation?

* 8. If an operation defined on a set is associative, must the set be closed under the operation?

* 9. Make up a table for an operation defined on the set {0,43,100} so that the set is closed under the operation.

*10. Make up a table for an operation defined on the set {0,43,100} so that the set is not closed under the operation.

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5. Identity Element: Inverse of an Element

The product of any number and 1 (in either order) is the same number. For instance

$$2 \times 1 = 2, \quad 1 \times 2 = 2, \quad 156 \times 1 = 156, \quad 1 \times 156 = 156.$$

For any number n in the arithmetic of rational numbers,

$$n \cdot 1 = n \text{ and } 1 \cdot n = n.$$

The sum of 0 and any number (in either order) gives that same number; that is the sum of any number and 0 is the number. For instance

$$2 + 0 = 2, \quad 0 + 2 = 2, \quad 468 + 0 = 468, \quad 0 + 468 = 468.$$

For any number n in ordinary arithmetic, $n + 0 = n$ and $0 + n = n$.

One is the identity for multiplication in ordinary arithmetic.
Zero is the identity for addition in ordinary arithmetic.

Suppose we let $*$ stand for a binary operation. Some possibilities for $*$ are the following:

1. If $*$ means addition of rational numbers, 0 is an identity element because $0 * a = a = a * 0$ for any rational number, a.

2. If $*$ means multiplication of rational numbers, 1 is an identity element because $1 * a = a = a * 1$ for any rational number.

3. If $*$ means the greater of two counting numbers, then $1 * 2 = 2$ because 2 is greater than 1; $1 * 3 = 3$ because 3 is greater than 1; $1 * 4 = 4$ since 4 is greater than 1, etc. In fact

$$1 * a = a = a * 1$$

no matter what counting number a is. So 1 is the identity for this meaning of the operation $*$.

We could state this formally as follows: If $*$ stands for a binary operation on a set of elements and if there is some element, call it e, which has the property that

$$e * a = a * e = a$$

for every element a of the set, then e is called an identity element of the operation $*$.

As another example consider the following table for an operation which we might call #.

#	A	B	C	D
A	B	C	D	A
B	C	D	A	B
C	D	A	B	C
D	A	B	C	D

Is there an identity element for #? Could it be A? Is A # B = B? (Read "A sharp B equals B"). Since, from the table A # B = C, the answer to the question is "no" and we see that A cannot be the identity. Neither can B be the identity since A # B is not A. However, D is an identity for #, since

$$A \# D = D \# A = A,$$
$$B \# D = D \# B = B,$$
$$C \# D = D \# C = C,$$
$$D \# D = D.$$

Compare the column under D with the column under the #. Compare the row to the right of D with the row to the right of the #. What do you notice? Does this suggest a way to look for an identity element when you are given a table for the operation?

If we have an identity element then we may also have for each given element what is called an inverse element. If the operation is multiplication for rational numbers, the identity is 1 and we call two rational numbers a and b inverses of each other if their product is 1, that is, if each is the reciprocal of the other.

Suppose the operation is addition (mod 4). Here 0 is the identity element and we call two numbers inverses if their sum is 0, that is, if combining the two numbers by the operation gives 0. To find inverses (mod 4) for addition form the table:

0 is the identity

$2 + 2 \equiv 0 \pmod 4$

$3 + 1 \equiv 0 \pmod 4$

$1 + 3 \equiv 0 \pmod 4$

(mod 4)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Here 0 is its own inverse, 2 is its own inverse, and 3 and 1 are inverses of each other.

<u>Definition</u>. Two elements a and b are <u>inverses</u> (or either one is the <u>inverse</u> of the other) under a binary operation $*$ with identity element e if $a * b = e$ and $b * a = e$.

Write again the table for # which we had in the beginning of this section.

#	A	B	C	D
A	B	C	D	A
B	C	D	A	B
C	D	A	B	C
D	A	B	C	D

Remember that we showed that D is the identity element for this table.

Can you find an element of the set {A,B,C,D} which will make the statement A # ___ = D true? It is C: A # C = D. A and C are inverses of each other under # . Can you find any other elements with inverses under #?

Exercises 5a

1. Study Tables (a) - (d) in Section 3.

   (a) Which tables describe operations having an identity and what is the identity?

   (b) Pick out pairs of elements which are inverses of each other under these operations. Does each member of the set have an inverse?

2. For each of the operations of Problem 4, Exercises 3;

   (a) Does the operation have an identity and, if so, what is it?

   (b) Pick out pairs of elements which are inverses of each other unde. these operations.

   (c) For which operations does each element have an inverse?

*3. Can there be more than one identity element for a given binary operation?

If the operation is multiplication we call inverses multiplicative inverses. (The multiplicative inverse of a number is its reciprocal.) Consider the set of counting numbers with multiplication as the operation. What elements have multiplicative inverses? Does 5 have a multiplicative inverse in the set of counting numbers? Is $\frac{1}{5}$ a counting number? The element 5 has no multiplicative inverse in this set. Does 1 have a multiplicative inverse in this set? Yes, it does, for $1 \cdot 1 = 1$. It is the only element of this set which has a multiplicative inverse, and it is its own multiplicative inverse. Of course, if we expand the set under consideration to include all of the rational numbers except zero then each element has a multiplicative inverse. The numbers in the pairs 5 and $\frac{1}{5}$, 1 and 1, $\frac{4}{9}$ and $\frac{9}{4}$ are multiplicative inverses of each other. Does 0 have a multiplicative inverse? Is there any number b such that $0 \cdot b = 1$?

Recall the (mod 5) multiplication.

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

How would we decide what elements of the set {0,1,2,3,4} have (multiplicative) inverses in this mathematical system? The identity for multiplication (mod 5) is 1. We would be looking for products which are the identity, so we should look for ones in the table. There are 4 ones in the table. They tell us that $1 \cdot 1 \equiv 1$ (mod 5), $2 \cdot 3 \equiv 1$ (mod 5), $3 \cdot \_\_\_ \equiv 1$ (mod 5), and $4 \cdot \_\_\_ \equiv 1$ (mod 5). (You supply the missing numbers.) Thus the multiplicative inverse of 2 in (mod 5) is 3. What is the multiplicative inverse of 3 in (mod 5)? of 4?

Do you see any connection between multiplicative inverses and the property of closure under division? Suppose you are given a set S of numbers which is closed under multiplication and suppose a is an element of S. How would you know whether it is possible to "divide by a in S"? That is, when is it possible to divide any element of S (including a) by a and obtain another element of S?

First of all, S must contain 1, since $a \div a = 1$. For instance if S were a set of rational numbers closed under multiplication, and if $\frac{1}{2}$ were in the set, then $\frac{1}{2} \div \frac{1}{2} = 1$ would also have to be in the set.

Second, since 1 is in S, S must contain $1 \div a = \frac{1}{a}$, no matter what element of S <u>a</u> stands for. This means that if 2 is in S, then $\frac{1}{2}$ must also be in S. If $\frac{1}{2}$ is in S, then 2 must be in S. S cannot contain zero since $1 \div 0$ has no meaning.

Third, if b is any element of S and $\frac{1}{a}$ is in S, then

$$b \cdot \frac{1}{a} = b \div a = \frac{b}{a}$$

is an element of S. For instance, if 2 is in S and $\frac{1}{3}$ is in S, then $2 \cdot \frac{1}{3} = \frac{2}{3}$ is also in S.

If S is to be closed under division it must be possible to divide any element of S by any element of S. Thus S must contain 1, every element of S must have a multiplicative inverse in S, and we must be able to divide every element by every other element.

If the system were not commutative, $b \cdot \frac{1}{a}$ might not be equal to $\frac{1}{a} \cdot b$ which means that $\frac{b}{a}$ might mean two different things. So in this booklet we consider division only when multiplication is commutative.

We can summarize what we have learned: Let S be a set of numbers closed under multiplication where multiplication is commutative. Then:

> If S is closed under division, S contains the number 1, and <u>every</u> element of S has a multiplicative inverse in S. (If a is in S then $\frac{1}{a}$ is also in S.)

Also, the other way around,

> If S contains 1 and if <u>every</u> element of S has its multiplicative inverse in S, then S is closed under division.

Perhaps you can now see another reason why we call division the inverse operation for multiplication:

> Dividing by a number a is the same as multiplying by the multiplicative inverse of a.

For instance, if S is the set of rational numbers with zero excluded, $\frac{1}{2}$ is the multiplicative inverse (reciprocal) of 2 and hence multiplying by $\frac{1}{2}$ always gives the same result as dividing by 2. If S were the set of numbers 0, 1, 2, 3, 4 and multiplication were (mod 5) as in the table above, then, since 3 is the inverse of 2, multiplying by 3 gives the same result as dividing by 2, that is, the values of x in the two following equivalences are the same:

$$1 \equiv 2x \pmod 5; \quad 1 \cdot 3 \equiv x \pmod 5.$$

In the first case $x \equiv 1 \div 2 \pmod 5$ and in the second $x \equiv 1 \cdot 3 \pmod 5$.

Everything we have said about division and multiplicative inverses can be said about subtraction and additive inverses. Here again we consider subtraction only when addition is commutative in the system. Consider first (mod 4) subtraction. What is $3 - 1 \pmod 4$? Since subtraction is the inverse operation for addition, to find $3 - 1 \pmod 4$ we must find the missing number in the sentence $? + 1 \equiv 3 \pmod 4$.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

To find the answer from the Table notice that, since the table gives sums, the 3 will be inside the table, and since 1 is the number which is added it will appear at the top of the table. If we look down the 1-column until we find a 3, we see that it is in the 2-row. So 2 is the number which, when you add 1 to it, you get 3. Since the system is commutative, the answer to $1 + ? \equiv 3 \pmod 4$ is also 2; that is, if we look along the 1-row until we see a 3, it will be in the 2-column. If the system were not commutative $3 - 1$ would have two meanings, which would be awkward. What is $2 - 3 \pmod 4$? What number must we add to 3 in (mod 4) to obtain 2? We see from the table that $3 + 3 \equiv 2 \pmod 4$, so $2 - 3 \equiv 3 \pmod 4$. What is $1 - 3 \pmod 4$?

Now let us ask another kind of question. Is there any number which we can add to 2 to obtain $2 - 3 \pmod 4$? Now $2 - 3 \equiv 3 \pmod 4$ since $2 \equiv 3 + 3 \pmod 4$. If we look at the table we see that $2 + 1 \equiv 3 \pmod 4$ and hence

$$2 - 3 \equiv 2 + 1 \pmod 4.$$

This means that if we subtract 3 from 2 we have the same result as if we add 1 to 2. In other words, adding 1 to 2 gives the same result as subtracting its inverse, 3, from 2.

In the same way you should show that

$$1 - 3 \equiv 1 + 1 \pmod 4$$
$$3 - 1 \equiv 3 + 3 \pmod 4.$$

From the first two of these examples it appears that, in (mod 4), subtracting 3 produces the same result as adding 1. Is this always true in this system? Is $0 - 3 \equiv 0 + 1 \pmod 4$? Is $3 - 3 \equiv 3 + 1 \pmod 4$?

What is the relationship between 1 and 3 in (mod 4)? Since $1 + 3 \equiv 0 \pmod 4$, and 0 is the identity under addition in (mod 4), what do we say about 1 and 3? They are additive inverses of each other.

Perhaps you can guess a general principle from this example. We observe that:

> Subtracting a number produces the same result as adding the additive inverse of the number.

This principle will be true in any commutative system where we call an operation "addition" and where the elements have inverses. Also, similar to a property which we have observed for multiplication, we have:

> A set which is closed under addition (where addition is commutative) will be closed under subtraction if it contains 0 and contains the additive inverse of each of its members.

Notice that in addition (mod n) we have our first examples of sets which are closed under subtraction. Nowhere in our study of the counting numbers, the whole numbers, and the rational numbers have we had additive inverses, except that in all these systems the number zero is its own additive inverse.

In the following exercises you will be given the chance to test these general principles further.

Exercises 5b

1. (a) Use the multiplication table for (mod 6) to find, wherever possible, a replacement for x to make each of the following number sentences a true statement:

   $1 \cdot x \equiv 1 \pmod 6$ $4x \equiv 1 \pmod 6$

   $2x \equiv 1 \pmod 6$ $5x \equiv 1 \pmod 6$

   $3x \equiv 1 \pmod 6$

   (b) Which elements of the set $\{0,1,2,3,4,5\}$ have multiplicative inverses in (mod 6)?

2. Remember that division is defined as the inverse operation for multiplication. Thus, in the arithmetic of rational numbers, the question "Six divided by two is what?" means, really, "Six is obtained by multiplying two by what?" We can define division (mod n) in t[illegible] way.

$$6 \div 4 \equiv ? \pmod 5 \quad \text{means} \quad (4)(?) \equiv 6 \pmod 5$$

Copy and complete the following table using multiplication and division (mod 5).

(Mod 5)

b	a	multiplicative inverse of a	b ÷ a	b · (multiplicative inverse of a)
1	2	3	$1 \div 2 \equiv 3$	$1 \cdot 3 \equiv 3$
2	2	3	$2 \div 2 \equiv 1$	$2 \cdot 3 \equiv 1$
3	2	3	$3 \div 2 \equiv$	$3 \cdot 3 \equiv$
2	3			
3	3			
4	3			
1	4			
2	4			
3	4			
4	4			

3. Here is a table for addition (mod 5).

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Copy and complete the following table.

(Mod 5)

b	a	additive inverse of a	b - a	b + (additive inverse of a)
0	1	4	0 - 1 = 4	0 + 4 = 4
2	1		2 - 1 =	2 + 4 =
4	1			
1	2			
2	2			
3	2			
2	4			
3	4			
4	4			

4. In the arithmetic of rational numbers which of the following sets is closed under division?

(a) $\{1, 2, \frac{1}{2}\}$

(b) $\{1, 2, 2^2, 2^3, \ldots\}$

(c) The non-zero counting numbers.

(d) The rational numbers.

5. (a) Which of the following sets is closed under multiplication (mod 6)?

{0,1,2,3,4,5} {2,4} {0,1,5} {1,5} {5}

(b) Which of the sets in (a) contain a multiplicative inverse (mod 6) for each of its elements?

(c) Which of the sets in (a) is closed under division (mod 6)?

6. (a) Which of the sets {A,B}, {C,D}, {B,C,D}, {A,D} is closed under the operation ✱ defined by the table below?

✱	A	B	C	D
A	A	A	A	A
B	A	A	B	B
C	A	B	D	C
D	A	B	C	D

For instance, {A,D} is closed under the operation because if we pick out that part of the table we have the little table

✱	A	D
A	A	A
D	A	D

which contains only A's and D's. On the other hand the set {A,C} is not closed since its little table would be

✱	A	C
A	A	A
C	A	D

Here the table contains a D, which is not one of the set {A,C}.

(b) Is there an identity for ✱ ? If so, what is it?

(c) Which of the sets in (a) has an inverse under ✱ for each of its elements?

✱ (d) Which of the sets in (a) is closed under the inverse operation for ✱ ? (You might use the symbols $\frac{*}{*}$ for this operation, so that a $\frac{*}{*}$ b = ? means b ✱ ? = a.)

6. What Is a Mathematical System?

The idea of a set has been a very convenient one in mathematics but there is really not a great deal that can be done with just a set of elements. It is much more interesting if something can be done with the elements (for instance, if the elements are numbers, they can be added or multiplied). If we have a set and an operation defined on the set, it is interesting to find out how the operation behaves. Is it commutative? associative? Is there an identity element? Does each element have an inverse?

We have seen that different operations may "behave alike" in some ways (both commutative, for instance). This suggests that we study sets with operations defined on them to see what different possibilities there are. It is too hard for us to list all the possibilities, but some examples will be given in this section and the next. These are examples of mathematical systems.

Definition. A mathematical system is a set of elements together with one or more binary operations defined on the set.

The elements do not have to be numbers. They may be any objects whatever. Some of the examples below are concerned with letters or geometric figures instead of numbers.

Example 1: Let's look at egg-timer arithmetic -- arithmetic (mod 3)

(a) There is a set of elements, the set of numbers $\{0,1,2\}$.

(b) There is an operation + (mod 3), defined on the set $\{0,1,2\}$.

Mod 3

+	1	2	0
1	2	0	1
2	0	1	2
0	1	2	0

Therefore, egg-timer arithmetic is a mathematical system. Does this system have any interesting properties?

(c) The operation, + (mod 3), has the commutative property. Can you tell by the table? If so, how? We can check some special cases, too. $1 + 2 \equiv 0 \pmod 3$ and $2 + 1 \equiv 0 \pmod 3$, so $1 + 2 \equiv 2 + 1 \pmod 3$.

(d) There is an identity for the operation + (mod 3) (the number 0).

(e) Each element of the set has an inverse for the operation + (mod 3).

Study the following tables.

(a)

O	A	B
A	A	B
B	A	B

(c)

~	△	□	○	╲
△	△	□	○	╲
□	□	○	╲	△
○	○	╲	△	□
╲	╲	△	□	○

(b)

*	P	Q	R	S
P	R	S	P	Q
Q	S	R	Q	P
R	P	Q	R	S
S	Q	P	S	R

Exercises 6

1. Which one, or ones, of the Tables (a), (b), (c) describes a mathematical system? Show that your answer is correct.

2. Use the tables above to complete the following statements correctly.

   (a) B O A = ?
   (b) △ ~ ○ = ?
   (c) ╲ ~ ╲ = ?
   (d) A O B = ?
   (e) Q * R = ?
   (f) R * S = ?
   (g) P * R = ?
   (h) □ ~ ○ = ?
   (i) ╲ ~ □ = ?
   (j) B O B = ?
   (k) A O A = ?
   (l) S * S = ?

3. Which one, or ones, of the binary operations O, * , ~ is commutative? Show that your answer is correct.

4. Which one, or ones, of the binary operations O, * , ~ has an identity element? What is it in each case?

5. Use the tables above to complete the following statements correctly.

(a) $P * (Q * R) = ?$

(b) $(P * Q) * R = ?$

(c) $P * (Q * S) = ?$

(d) $(P * Q) * S = ?$

(e) $(R * P) * S = ?$

(f) $R * (P * S) = ?$

(g) $\triangle \sim (\triangle \sim \backslash) = ?$

(h) $(\triangle \sim \triangle) \sim \backslash = ?$

(i) $(\bigcirc \sim \square) \sim \triangle = ?$

(j) $\bigcirc \sim (\square \sim \triangle) = ?$

6. Does either of the operations described by Table (b) or Table (c) seem to be associative? Why? How could you prove your statement? What would another person have to do to prove you wrong?

7. (a) In Table (c) what set is generated by the element □?
   (b) In Table (b) what set is generated by the element P?

8. BRAINBUSTER. For each of the following tables, tell why it does <u>not</u> describe a mathematical system.

(a)

*	1	2
1	1	1
2	1	

(c)

*	1	2
3	1	4
4	2	3

(b)

*	1	2
1	the product of 3 and 6	the sum of 2 and 4
2	a number between 3 and 8	0

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7. <u>Mathematical Systems Without Numbers</u>

In the last section there were some examples of mathematical systems without numbers in them. Suppose we want to invent one. What do we need?

We must have a set of things. Then, we need some kind of a binary operation -- something that can be done with any two elements of our set. We have found that the properties of closure, commutativity, associativity, etc. are very helpful in simplifying expressions. It would be nice to have some of these properties.

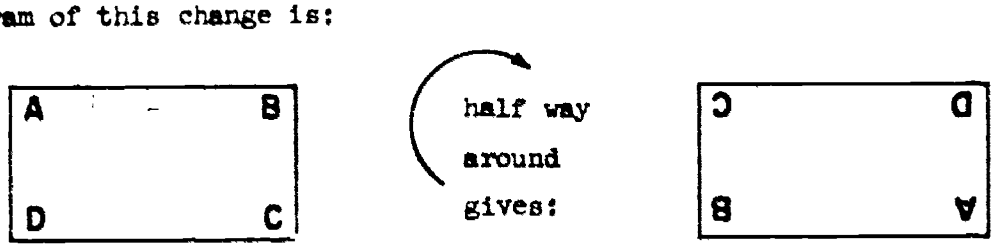
Let's start wi h a card. Any rectangular shaped card will do. We will use it to represent a closed rectangular region. Lay the card on your desk and label the corners as in the sketch.

Now pick the card up and write the letter "A" on the other side (the side that was touching the desk) behind the "A" you have already written. Be sure the two letters "A" are back-to-back so they are labels for the same corner of the card. Similarly, label the corners B, C, and D on the other side of the card (be sure they're back-to-back with the B, C, and D you have already written.)

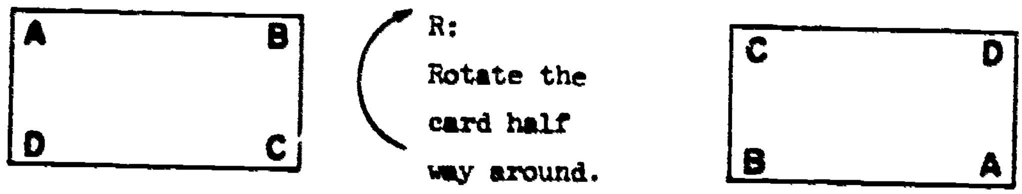
What set shall we take? Instead of numbers, let us take elements which have something to do with the card. Start with the card in the center of your desk and with the long sides of the card parallel to the front of your desk. Now move the card -- pick it up, turn it over or around in any way -- and put it back in the center of your desk with the long sides parallel to the front of your desk. The card looks just the same as it did before, but the corners may be labeled differently (a corner that started at the top may now be at the bottom, for instance). The position of the card has been changed, but the closed rectangular region looks as it did in the beginning. (The "picture" stays the same. Individual points may be moved.) The elements of our set will be these changes of position. We will take all the changes of position that make the closed rectangular region look as it did in the beginning. (Long sides parallel to the front of the desk.) How many of these changes are there?

We may start with the card in some position which we will call the standard position. Suppose it looks like the figure below.

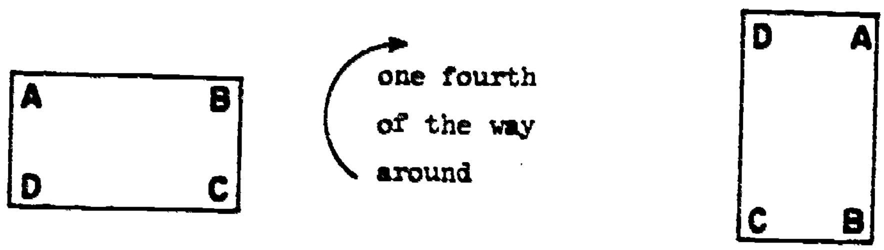
Leaving the card on your desk, rotate it half way around its center. A diagram of this change is:



Since the letters "A", "B", etc. are only used as a convenience to label the different corners of the card, we will not bother to write them upside down. The diagram below represents this change of position, and we will call the change "R" (for rotation).

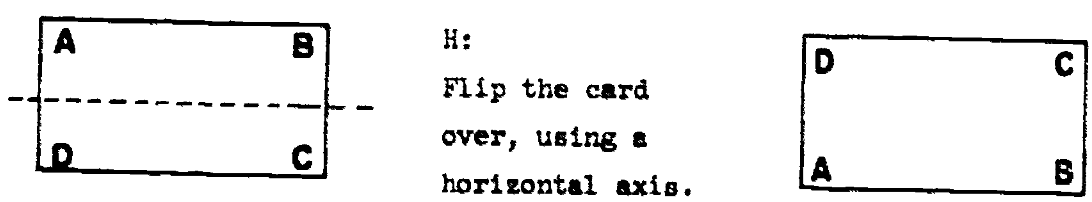


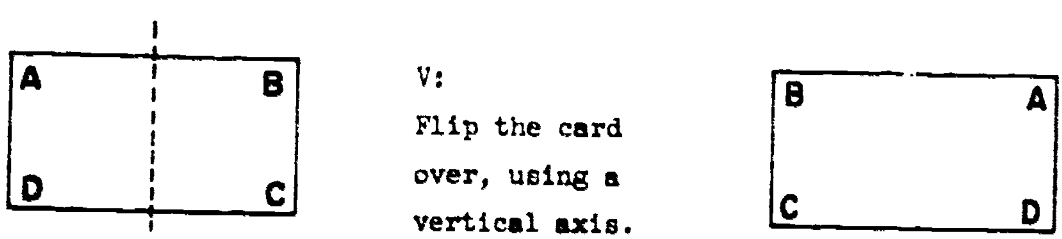
What would happen if the card were rotated one fourth of the way around?



Does the card look the same before and after the change? No, this change of position cannot be in our set, since the two pictures are quite different.

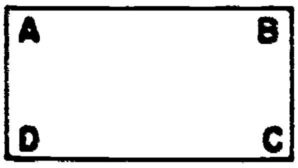
Are there other changes of position of the closed rectangular region which make it look the way it did in the beginning? Yes, we can flip the card over in two different ways as shown by the diagrams below:





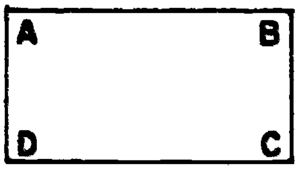
Now you know why you had to label both sides of the card so carefully. Remember, the card only represents a geometric figure for us. Turning over a card makes it different -- you see the other side; but turning over the closed rectangular region would not make it different (of course, some of the individual points would be in different positions, but the whole geometric figure would look just the same).

There is one more change of position which we must consider. It is the change which leaves the card alone (or puts each individual point back in place). Let us call it "I".

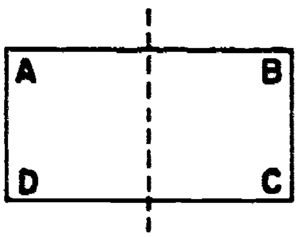


I:
Leave the card
in place.

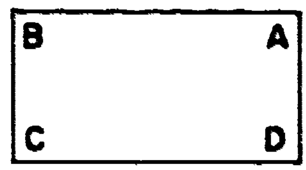
Now we have our set of elements; it is {I, V, H, R}. Let us summarize what they are for easy reference:

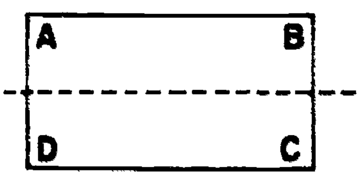


Element I:
Leave the card
in place.

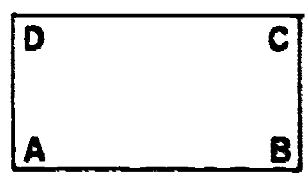


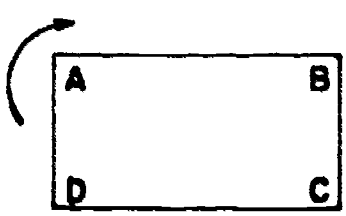
Element V:
Flip the card
over using
a vertical axis.





Element H:
Flip the card
over using a
horizontal axis.





Element R:
Rotate the card
halfway around in
the direction
indicated.

Recall the definition of a mathematical system. There were two requirements:

(a) A set of elements.

(b) One or more binary operations defined on the set of elements.

Our set {I,V,H,R} satisfies the first condition. Now we need to satisfy the second condition; we need an operation. What operation shall we use? How can we "combine any two elements of our set" to get a "definite thing"? If the set is to be closed under the operation, the "definite thing" which is the result of the operation should be one of the elements again.

Here is a way of combining any two elements of our set. We will do one of the changes AND THEN do the other one. We will use the symbol "ANTH" for this operation (perhaps you can think of a better one). Thus "H ANTH V" means flip the card over, using a horizontal axis, and then flip the card over, using a vertical axis. Start with the card in the standard position and do these changes to it. What is the final position of the card? Is the result of these two changes the same as the change R? What does "V ANTH H" mean? Try it with your card. Now we can fill in the table for our operation. Some of the entries are given in the table at the right.

ANTH	I	V	H	R
I	I	V		
V			R	H
H	H	R		
R			V	

Exercises 7

1. Check the entries that are given in the table above and find the others. Use your card.

2. From your table for the operation ANTH, or by actually moving a card, fill in each of the blanks to make the equations correct.

   (a) R ANTH H = ?
   (b) R ANTH ? = H
   (c) ? ANTH R = H
   (d) ? ANTH H = R
   (e) (R ANTH H) ANTH V = ?
   (f) R ANTH (H ANTH V) = ?
   (g) (R ANTH H) ANTH ? = V
   (h) (R ANTH ?) ANTH V = H
   (i) (? ANTH H) ANTH V = R

3. Examine the table for the operation ANTH.

(a) Is the set closed under the operation?

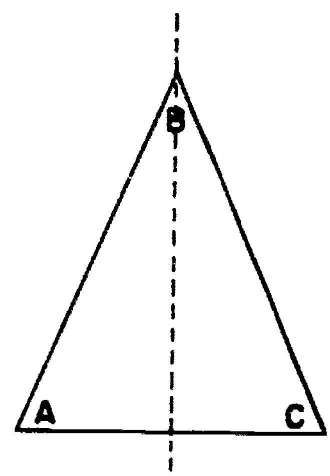
(b) Is the operation commutative?

(c) Do you think the operation is associative? Use the operation table to check several examples.

(d) Is there an identity element for the operation ANTH?

(e) Does each element of the set have an inverse under the operation ANTH?

4. Here is another system of changes. Cut a triangular card with two equal sides. Label the corners as in the sketch (both sides, back-to-back). The set for the system will consist of two changes. The first change, called I, will be: Leave the card in place. The second change, called F, will be: Flip the card over, using the vertical axis. F ANTH I will mean: Flip the card over, using the vertical axis, and then leave the card in place. How will the card look -- as if it has been left in place, I, or as if the change F had been done? What does I ANTH F mean? Does F ANTH I = F or does F ANTH I = I?



(a) Complete the table below:

ANTH	I	F
I		
F		

(b) Is the set closed under this operation?

(c) Is the operation commutative?

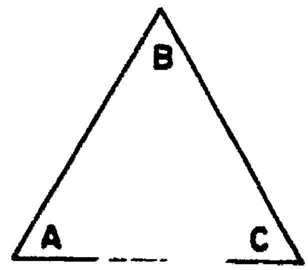
(d) Is the operation associative? Are you sure?

(e) Is there an identity for the operation?

(f) Does each element of the set have an inverse under the operation?

5. Make a triangular card with three equal sides and label the corners as in the sketch (both sides, back-to-back). The set for this system will be made up of these six changes.

I: Leave the card in place.



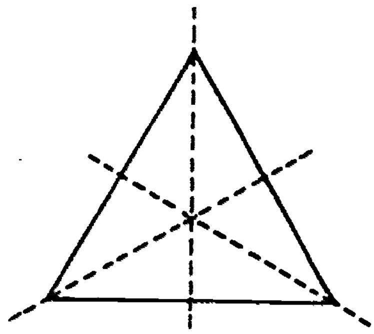
R: Rotate the card clockwise $\frac{1}{3}$ of the way around.

S: Rotate the card clockwise $\frac{2}{3}$ of the way around.

T: Flip the card over, using a vertical axis.

U: Flip the card over, using an axis through the lower right vertex.

V: Flip the card over, using an axis through the lower left vertex.



Three of these will be rotations about the center (leave in place and two others). The other three will be flips about the axes. (Caution: the axes are stationary; they do not rotate with the card. For example, the vertical axis remains vertical -- it would go through a different corner of the card after rotating the card one third of the way around its center.) Make a table for these changes. Examine the table. Is this operation commutative? Is there an identity change? Does each change have an inverse?

* 6. Try making a table of changes for a square card. There are eight changes (that is, eight elements). What are they? Is there an identity element? Is the operation ANTH commutative?

8. The Counting Numbers and the Whole Numbers

The mathematical systems that we have studied so far in this booklet are composed of a set and one operation. Examples are modular addition or multiplication and the changes of a rectangular or triangular card. A mathematical system given by a set and two operations would appear to be more complicated than these examples. However, as you may have guessed, ordinary arithmetic is also a mathematical system and we know that we can do more than one operation using the same set of numbers -- for examples, we can add and multiply.

To be definite, let us choose the set of rational numbers. This set together with the two operations of addition and multiplication forms a mathematical system.

Are there properties of this system which are entirely different from those we have considered in systems with only one operation? Yes, you are familiar with the fact that $2 \cdot (3 + 5) = (2 \cdot 3) + (2 \cdot 5)$. This is an illustration of the distributive property. More precisely, it illustrates that multiplication distributes over addition. The distributive property is also of interest in other mathematical systems.

Definition. Suppose we have a set and two binary operations, $*$ and $\circ$, defined on the set. The operation $*$ distributes over the operation $\circ$ if

$$a * (b \circ c) = (a * b) \circ (a * c)$$

for any elements a, b, c, of the set. (And we can perform all these operations.)

In a mathematical system with two operations, there are the properties which we previously discussed for each of these operations separately. The only property which is concerned with both operations together is the distributive property.

Exercises 8

1. Consider the set of counting numbers.

   (a) Is the set closed under addition? under multiplication? Explain.

   (b) Do the commutative and associative properties hold for addition? for multiplication? Give an example of each.

   (c) What is the identity element for addition? for multiplication?

   (d) Is the set of counting numbers closed under subtraction? under division? Explain.

   The answers to (a), (b), and (c) tell us some of the properties of the mathematical system composed of the set of counting numbers and the operations of addition and multiplication.

2. Answer the questions of Problem 1 (a), (b), (c) for the set of whole numbers. Are your answers the same as for the counting numbers?

3. (a) For the system of whole numbers, write three number sentences illustrating that multiplication distributes over addition.

   (b) Does addition distribute over multiplication? Try some examples.

4. The two tables below describe a mathematical system composed of the set {A,B,C,D} and the two operations * and o.

*	A	B	C	D
A	A	A	A	A
B	A	B	A	B
C	A	A	C	C
D	A	B	C	D

o	A	B	C	D
A	A	B	C	D
B	B	B	D	D
C	C	D	C	D
D	D	D	D	D

   (a) Do you think * distributes over o? Try several examples.

   (b) Do you think o distributes over * ? Try several examples.

5. Answer these questions for each of the following systems. Is the set closed under the operation? Is the operation commutative? associative? Is there an identity? What elements have inverses?

   (a) The system whose set is the set of odd numbers and whose operation is multiplication.

   (b) The system whose set is made up of zero and the multiples of 3 and whose operation is multiplication.

   (c) The system whose set is made up of zero and the multiples of 3 and whose operation is addition.

   (d) The system whose set is made up of the rational numbers between 0 and 1 (not including 0 and 1) and whose operation is multiplication.

   (e) The system whose set is made up of the even numbers and whose operation is addition. (Zero is an even number.)

   (f) The system whose set is made up of the rational numbers between 0 and 1 and whose operation is addition.

6. (a) In what ways are the systems of 5(b) and 5(c) the same?

   (b) In what ways are the systems of 5(a) and 5(b) different?

*7. Make up a mathematical system of your own that is composed of a set and two operations defined on the set. Make at least partial tables for the operations in your system. List the properties of your system.

*8. Here is a mathematical system composed of a set and two operations defined on that set.

   Set: All counting numbers
   Operation $*$: Find the greatest common factor.
   Operation o: Find the least common multiple.

   (a) Does the operation $*$ seem to distribute over the operation o? Try several examples.

   (b) Does the operation o seem to distribute over the operation $*$? Try several examples.

9. **Modular Arithmetic**

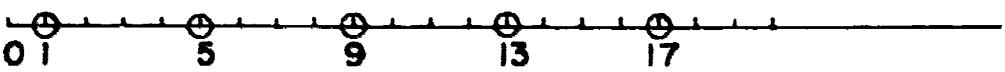
In section 1 we studied a new addition done by rotating the hand of a clock. Using a four-minute clock, we said that $2 + 3 = 1 \pmod 4$. The tables which we made described the mathematical system (mod 4). In Section 1 we studied a new multiplication using the same clock.

Modular systems are the result of classifying whole numbers in a certain way. For example, we could classify whole numbers as even or odd. In this case, the even numbers: 0, 2, 4, 6, ... are put in the same family and the family is named by its smallest member: 0. Thus the class of all even numbers is 0 (mod 2). Starting from 1, the odd numbers: 1, 3, 5, 7, ... are put in the same family which we call 1 (mod 2). For the odds and evens, we then have two classes, 0 (mod 2) and 1 (mod 2). The number 5 belongs to the class 1 (mod 2), 8 belongs to the class 0 (mod 2).

If we put every fourth whole number in the same class, we have the (mod 4) system. Here is a sketch of some of the numbers belonging to the class 0 (mod 4).

Every fourth whole number starting with 0 belongs to the same class. Thus, numbers which are multiples of 4 belong to the class 0 (mod 4).

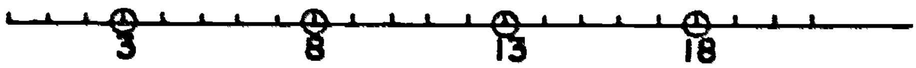
Here is a sketch showing some of the numbers which belong to the class 1 (mod 4).



Every fourth whole number starting with 1 belongs to the same class, that is, 1 (mod 4). Thus the numbers which are 1 plus a multiple of 4 belong to this class.

The two sketches below show respectively some of the numbers which belong to the class 0 (mod 5) and the class 3 (mod 5).

The numbers belonging to the class 0 (mod 5) are multiples of 5.



The numbers belonging to the class 3 (mod 5) are 3 plus multiples of 5.

Our first problems in a modular system used the operation of addition. When we changed the operation to multiplication, we got a different mathematical system. With both operations, modular arithmetic is more like ordinary arithmetic than it was with just one operation.

For each of the modular systems we can state the number of elements in the set. For instance, there are four elements if it is (mod 4), seven elements if it is (mod 7), and so forth. Such a set is called a <u>finite set</u> and the system is called a <u>finite system</u>. The modular systems and the systems of Section 7 are finite systems. On the other hand, the set of rational numbers considered in Section 8 is so large that it contains more elements than any number you could name. Such a set is called an <u>infinite set</u> and the system is called an <u>infinite system</u>.

<u>Exercises 9</u>

1. Write the multiplication table (mod 8) and recall or write again the multiplication table (mod 5) which you found in Exercises 2.

2. Answer each of the following questions about the mathematical systems of multiplication (mod 5) and (mod 8).

   (a) Is the set closed under the operation?

   (b) Is the operation commutative?

   (c) Do you think the operation is associative?

   (d) What is the identity element?

   (e) Which elements have inverses and what are the pairs of inverse elements?

   (f) Is it true that if a product is zero at least one of the factors is zero?

3. Complete each of the following number sentences to make it a true statement.

   (a) $2 \times 4 \equiv ?$ (mod 5) (c) $5^2 \equiv 1$ (mod ?)

   (b) $4 \times 3 \equiv ?$ (mod 5) (d) $2^3 \equiv 0$ (mod ?)

4. **Find the products.**

(a) $2 \times 3 \equiv ? \pmod 4$

(b) $2 \times 3 \equiv ? \pmod 6$

(c) $5 \times 8 \equiv ? \pmod 7$

(d) $3 \times 4 \times 6 \equiv ? \pmod 9$

(e) $4^3 \equiv ? \pmod 5$

(f) $6^2 \equiv ? \pmod 5$

*(g) $6^{256} \equiv ? \pmod 5$

5. Find the sums.

(a) $1 + 3 \equiv ? \pmod 5$

(b) $4 + 3 \equiv ? \pmod 5$

(c) $2 + 4 \equiv ? \pmod 5$

(d) $4 + 4 \equiv ? \pmod 5$

6. (a) Find the values of $3(2 + 1) \pmod 5$ and $(3 \cdot 2) + (3 \cdot 1) \pmod 5$.

(b) Find the values of $4(3 + 1) \pmod 5$ and $(4 \cdot 3) + (4 \cdot 1) \pmod 5$.

(c) Find the values of $(3 \quad 2) + (3 \cdot 4) \pmod 5$ and $3(2 + 4) \pmod 5$.

(d) In the examples of this problem is multiplication distributive over addition?

7. (a) Find the values of $3 + (2 \cdot 1) \pmod 5$ and $(3 + 2) \cdot (3 + 1) \pmod 5$.

(b) Find the values of $4 + (3 \cdot 1) \pmod 5$ and $(4 + 3) \cdot (4 + 1) \pmod 5$.

(c) Find the values of $(3 + 2) \cdot (3 + 4) \pmod 5$ and $3 + (2 \cdot 4) \pmod 5$.

(d) In the examples of this problem is addition distributive over multiplication?

Remember that division is defined after we know about multiplication. Thus, in ordinary arithmetic, the question "Six divided by 2 is what?" means, really "Six is obtained by multiplying 2 by what?" An operation that begins with one of the members and the "answer" to another binary operation and asks for the other number, is called an inverse operation. Division is the inverse of the multiplication operation.

*8. Find the quotients.

(a) $2 \div 3 \equiv ? \pmod 8$

(b) $6 \div 2 \equiv ? \pmod 8$

(c) $0 \div 2 \equiv ? \pmod 8$

(d) $3 \div 4 \equiv ? \pmod 5$

(e) $0 \div 2 \equiv ? \pmod 5$

(f) $0 \div 4 \equiv ? \pmod 5$

(g) $7 \div 3 \equiv ? \pmod{10}$

*(h) $7 \div 6 \equiv ? \pmod 8$

9. Find the following; remember that subtraction is the inverse operation of addition.

(a) $7 - 3 \pmod 8$
(b) $3 - 4 \pmod 5$
(c) $3 - 4 \pmod 8$
* (d) $4 - 9 \pmod{12}$

10. Make a table for subtraction (mod 5). Is the set closed under the operation?

11. Find a replacement for x which will make each of the following number sentences a true statement. Explain.

(a) $2x \equiv 1 \pmod 5$
(b) $3x \equiv 1 \pmod 4$
(c) $3x \equiv 0 \pmod 5$
(d) $3x \equiv 0 \pmod 6$
(e) $x \cdot x \equiv 1 \pmod 8$
(f) $4x \equiv 4 \pmod 8$

12. In Problem 11 (d) and (f), find at least one other replacement for x which makes the number sentence a true statement.

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10. Summary and Review

A binary operation defined on a set is a rule of combination by means of which any two elements of the set may be combined to determine one definite thing.

A mathematical system is a set together with one or more binary operations defined on that set.

A set is closed under a binary operation if every two elements of the set can be combined by the operation and the result is always an element of the set.

An identity element for a binary operation defined on a set is an element of the set which does not change any element with which it is combined.

Two elements are inverses of each other under a certain binary operation if the result of this operation on the two elements is an identity element for that operation.

A binary operation is commutative if, for any two elements, the same result is obtained by combining them first in one order, and then in the other.

A binary operation $*$ is associative if, for any three elements, the result of combining the first with the combination of the second and third is the same as the result of combining the combination of the first and second with the third.

$$a * (b * c) = (a * b) * c.$$

The binary operation $*$ distributes over the binary operation o provided

$$a * (b \circ c) = (a * b) \circ (a * c)$$

for all elements a, b, c.

A set S is generated by an element b under the operation $*$ if

$$S = \{b, (b * b), (b * b) * b, [(b * b) * b] * b, \ldots\}$$