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ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exercises, and sample test questions. Topics covered include factors and primes, perfect numbers, divisibility, expanded notation, repeating decimals, number systems in other bases, common factors, and common multiples. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-18

SUPPLEMENTARY and ENRICHMENT SERIES

FACTORS AND PRIMES

Teachers' Commentary

Edited by Henry W. Syer

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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FACTORS AND PRIMES

Answers to Exercises

Exercises 1.

1. (a) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

   (b) 101, 103, 107, 109, 113, 127

2. (a) 15 (b) 25 (c) 31

3. 0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60

4. 0, 7, 14, 21, 28, 35, 42, 49

5. 0, 15, 30, 45, 60, 75, 90

6.

	12	14	17	18	20	25	27
1	12	14	17	18	20	25	27
2	6	7	no.	9	10	no	no
3	4	no	no	6	no	no	9
4	3	no	no	no	5	no	no
5	no	no	no	no	4	5	no
6	2	no	no	3	no	no	no
7	no	2	no	no	no	no	no

7. (a) $12 = 3 \times 4$ or 2×6
   (b) $36 = 2 \times 18$ or 3×12 or $4 \quad 9$ or 6×6
   (c) 31 is prime.
   (d) 7 is prime.
   (e) $8 = 2 \times 4$
   (f) 11 is prime.
   (g) $35 = 5 \times 7$
   (h) 5 is prime.
   (i) $39 = 3 \times 13$
   (j) $42 = 2 \times 21$ or 3×14 or 6×7
   (k) $6 = 2 \times 3$

(l) 41 is prime.

(m) $82 = 2 \times 41$

(n) $95 = 5 \times 19$

Of course, in all these cases the two factors may be written in reverse order.

8. (a) The number, 24, is divisible by 1, 2, 3, 4, 6, 12, and 24.

   (b) 24 is a multiple of 1, 2, 3, 4, 6, 12, and 24.

   (c) 24 is a multiple of each of the sets of numbers in (a) and (b) since being a "multiple of" is the same as being "divisible by."

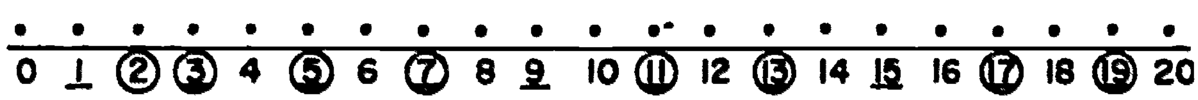
9. $12 = 2 \times 6 = 6 \times 2 = 3 \times 4 = 4 \times 3 = 2 \times 2 \times 3 = 2 \times 3 \times 2 = 3 \times 2 \times 2$

10. 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43, 59 and 61, 71 and 73. There are eight such pairs.

11. $4 = 2 + 2$; $6 = 3 + 3$; $8 = 5 + 3$; $10 = 3 + 7 = 5 + 5$; $12 = 5 + 7$; $14 = 3 + 11 = 7 + 7$; $16 = 3 + 13 = 5 + 11$; $18 = 5 + 13 = 7 + 11$; $20 = 3 + 17 = 7 + 13$; $22 = 3 + 19 = 5 + 17 = 11 + 11$.

12. Yes; 3, 5, 7. This is the only set because at least one of any set of three consecutive odd numbers is divisible by 3.

13.



etc.

   (d) Yes. The numeral 2 is the only numeral circled which is not underlined.

14. 3, 5, 7, 11, 13, 17, 19, 23, 29

<u>Exercises 2.</u>

1. (a) 1, 2, 5, 10

   (b) 1, 3, 5, 15

   (c) 1, 3, 9

   (d) 1, 2, 3, 6, 9, 18

   (e) 1, 3, 9, 27

   (f) 1, 2, 3, 4, 6, 8, 12, 24

   (g) 1, 11

2. (a) $1 \cdot 10$; $2 \cdot 5$
   (b) $1 \cdot 15$; $3 \cdot 5$
   (c) $1 \cdot 9$; $3 \cdot 3$
   (d) $1 \cdot 100$; $2 \cdot 50$; $4 \cdot 25$; $5 \cdot 20$; $10 \cdot 10$
   (e) $1 \cdot 24$; $2 \cdot 12$; $3 \cdot 8$; $4 \cdot 6$
   (f) $1 \cdot 16$; $2 \cdot 8$; $4 \cdot 4$
   (g) $1 \cdot 72$; $2 \cdot 36$; $3 \cdot 24$; $4 \cdot 18$; $6 \cdot 12$; $8 \cdot 9$
   (h) $1 \cdot 81$; $3 \cdot 27$; $9 \cdot 9$

3. (a) $2 \cdot 5$
   (b) $3 \cdot 5$
   (c) $3 \cdot 3$ or 3^2
   (d) $2 \cdot 3 \cdot 5$
   (e) $3 \cdot 3 \cdot 5$ or $3^2 \cdot 5$
   (f) $2 \cdot 5 \cdot 5$ or $2 \cdot 5^2$
   (g) 13

4. Zero is not a factor of six since there is not a number which, when multiplied by zero, gives a product of six; six is a factor of zero since the product of six and zero is zero, thus the definition is satisfied.

5. (a) 1, 4, 10, 20
   (b) 1, 4, 6, 8, 9, 18, 24, 36, 72

6. (a) $3 \cdot 5 \cdot 7$
   (b) $2 \cdot 3 \cdot 7$
   (c) $3 \cdot 5 \cdot 5$ or $3 \cdot 5^2$
   (d) $3 \cdot 2 \cdot 2 \cdot 5 \cdot 5$ or $3 \cdot 2^2 \cdot 5^2$
   (e) 2^6
   (f) $3 \cdot 5 \cdot 23$
   (g) 311 (This is a prime number.)
   (h) $2^3 \cdot 5^3$
   (i) $7 \cdot 43$
   (j) $17 \cdot 19$

7. (a) Even
   (b) Even
   (c) Even
   (d) Even
   (e) Odd
   (f) Even
   (g) Odd
   (h) Odd
   (i) Odd
   (j) Even

*8. Some of the items in the table will be:

	Factors of N	Number of Factors	Sum of Factors
18	1,2,3,6,9,18	6	39
24	1,2,3,4,6,8,12,24	8	60
28	1,2,4,7,14,28	6	56

(a) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 (the prime numbers)

(b) 4, 9, 25 (the squares of prime numbers)

(c) Three: 1, p, and p^2

(d) Four: 1, p, g, pg. The sum is $1 + p + g + pg$.

(e) The factors are: $1, 2, 2^2, 2^3, \ldots, 2^k$. There are $k + 1$ of them.

(f) The factors are: $1, 3, 3^2, 3^3, \ldots, 3^k$. There are $k + 1$ of them.

(g) If $N = p^k$, the factors are $1, p, p^2, p^3, \ldots, p^k$. There are $k + 1$ of them.

(h) In this list the only numbers having 2N for the sum of their factors are 6 and 28. These are the two smallest perfect numbers. There is a general formula for all even perfect numbers. It is

$$2^{p-1}(2^p - 1)$$

where both p and $2^p - 1$ are prime numbers. The third perfect number is 496.

Exercises 3.

1. (a) ab (b) a^2 (c) ab (d) ab; a^3b

2. 21, 22, 26, 33, 34, and so on

3. 25, 49, 121, or any other square of a prime number

4. 28, 44, 45, or any others of the form a^2b where a and b are prime

5. 24, 40, 54, 135, or any others of the form a^3b

6. Here are some possible forms: a^2b^2: 36, 100, 225; a^3b^2: 72, 200, 108

Exercises 5.

1. (a) 226843

Sum of digits 25

$\frac{25}{9} = 2$ remainder 7

Not divisible by 9

67945

Sum of digits 31

$\frac{31}{9} = 3$ remainder 4

Not divisible by 9

427536

Sum of digits 27

$\frac{27}{9} = 3$ remainder 0

Divisible by 9

45654

Sum of digits 24

$\frac{24}{9} = 2$ remainder 6

Not divisible by 9

(b) and (c) The remainder upon division by 9 is equal to the remainder when the sum of the digits is divided by 9.

2. (a) The uniqueness property of addition

(b) Yes

3. (a) Yes

(b) Same reason as number 2

4. (a) Yes (c) Yes

(b) Yes (d) Yes

5. (a) $\frac{(9 + 1)^{20}}{9}$ has a remainder of 1^{20} or 1.

(b) $\frac{(3 \times 3 + 1)^{20}}{3}$ has a remainder of 1^{20} or 1.

(c) $10^{20} = 100^{10} = (99 + 1)^{10}$ has a remainder of 1^{10} or 1.

6. $\frac{(6+1)^{20}}{6}$ has a remainder of 1.

7. (a) Divisibility by 4: If the number formed by the last two digits is divisible by 4, then the number is divisible by 4.

   (b) Divisibility by 8: If the number formed by the last three digits is divisible by 8, then the number is divisible by 8.

   (c) Divisibility by 25: If the number formed by the last two digits is divisible by 25 (e.g., 00, 25, 50, 75), then the number is divisible by 25.

8. 2, 3, 4, 6, 12 (0 is divisible by 12).

9. (a) Multiples of powers of 7. (This includes negative powers of 7, e.g., $\frac{13}{49}$ in the decimal system is .16 in the system to the base 7.)

   (b) $(.1254\ldots)_7$

10. Remainder 3

11. (a) $\frac{(9+1)^{20}-1}{9}$ 1 - 1 = 0 (Refer to Exercise 5.)

    (b) $\frac{(6+1)^{108}-1}{6}$ 1 - 1 = 0 (as above)

12. Cast 9's from the sum as you go along.

13. See text.

Exercises 6.

1.

927	sum of digits: 18	sum of digits: 9	sum of digits: 9	
865	sum of digits: 19	sum of digits: 10	sum of digits: 1	
4635			Product	9
5562				
7416				
801855	sum of digits: 27	sum of digits: 9		

2. Answered in the text

3. 810855

4. 3, 0, 0

5. Answered in the text

6. Casting out the 6's

7. $\begin{array}{r} 7543 \\ \underline{5437} \\ 2106 \end{array}$ sum of digits = 9 $\quad 2 + 1 + 0 = 3$, other digit is 6, or $1 + 0 + 6 = 7$, other digit is 2, or $2 + 1 + 6 = 9$, other digit is 0 or 9

Exercises 7b.

1. (1) (a) $758 = 7 \times 10^2 + 58$ which has remainder $7 + 3 = 10$ when divided by 11 since 100 has the remainder 1.

   (b) $758 = 7 \times 10^2 + 5 \times 10 + 8$ which has the remainder $8 - 5 + 7 = 10$.

   (2) (a) $7246 = 72 \times 10^2 + 46$ which has the same remainder as has $72 + 46$, that is, $6 + 2$ or 8.

   (b) $7246 = 7 \times 10^3 + 2 \times 10^2 + 4 \times 10 + 6$ which has the same remainder as has $6 - 4 + 2 - 7 = -3$, that is, 8.

   (3) (a) $81675 = 8 \times 10^4 + 16 \times 10^2 + 75$ which has the same remainder as has $8 + 5 + 9 = 22$. Hence, the remainder is 0.

   (b) $81675 = 8 \times 10^4 + 1 \times 10^3 + 6 \times 10^2 + 7 \times 10 + 5$ which has the same remainder as has $5 - 7 + 6 - 1 + 8 = 11$. Hence, the remainder is 0.

2. Since 8 is 1 more than 7, a number to the base 7 can be tested for divisibility by 8 in the same way that we can test for divisibility by 11 in the decimal system. For instance, consider $(5326)_7$.

   Using the first method, we have $(5326)_7 = (53)_7 \times (10^2)_7 + (26)_7$. Since 8 is $(11)_7$, the remainder when $(10^2)_7$ is divided by 8 is 1. Thus, the remainder when the given number is divided by 8 is the same as when $(53)_7 + (26)_7$ is divided by $(11)_7$. But $(53)_7 - (44)_7 = 6$ and $(26)_7 - (22)_7 = 4$ and, hence, the remainder is the same as when $6 + 4$ is divided by 8, that is, 2.

   Using the second method, we have $(5326) = 5 \times (10^3)_7 + 3 \times (10^2)_7 + 2 \times (10)_7 + 6$. When this is divided by 8, the remainder is the same as that for $6 - 2 + 3 - 5 = 2$.

3. 3, 9, 27, 37, 3×37, 9×37, 999

4. In the numeral system to the base 7, we can test for division by grouping in triples all the divisors of $7^3 - 1 = 342$. These divisors are:

$$2, 3, 6, 9, 18, 19, 38, 57, 114, 171, 342.$$

In the number system to the base 12, we would have the divisors of $12^3 - 1 = 1727$. These factors are 11, 157, 1727.

5. (a) Yes. The remainders when the powers of 10 are divided by 11 are -1, 1, -1, 1, ... which has a period of 2 since 11 is a divisor of $10^2 - 1$.

   (b) The divisors 3 and 9 listed in the answers to Exercise 3 are divisors of $10^1 - 1$, as well as of $10^3 - 1$. All the others have three digits in the decimal equivalent of their reciprocals and, for these, grouping the digits in threes gives a divisibility test.

6, 7. See text.

Exercises 8.

	3	7	9	11	13	17	19	21	37	101	41
1	1	1	1	1	1	1	1	1	1	1	1
10^1	1	3	1	10	10	10	10	10	10	10	10
10^2	1	2	1	1	9	15	5	16	26	100	18
10^3	1	6	1	10	12	14	12	13	1	91	16
10^4	1	4	1	1	3	4	6	4	10	1	37
10^5	1	5	1	10	4	6	3	19	26	10	1
10^6	1	1	1	1	1	9	11	1	1	100	10
10^7	1	3	1	10	10	5	15	10	10	91	18
10^8	1	2	1	1	9	16	17	16	26	1	16
10^9	1	6	1	10	12	7	18	13	1	10	37
10^{10}	1	4	1	1	3	2	9	4	10	100	1
10^{11}	1	5	1	10	4	3	14	19	26	91	10
10^{12}	1	1	1	1	1	13	7	1	1	1	18
10^{13}	1	3	1	10	10	11	13	10	10	10	16
10^{14}	1	2	1	1	9	8	16	16	26	100	37
10^{15}	1	6	1	10	12	12	8	13	1	91	1
10^{16}	1	4	1	1	3	1	4	4	10	1	10

The number of digits in the cycle in the repeating decimal is the same as the number of digits in the cycle of remainders. Since, for example, there are five remainders in the column headed by 41, one can test for divisibility by 41 by grouping the digits in fives.

Exercises 9.

1. (a) 5
   (b) 3
   (c) 3
   (d) 2
   (e) 7
   (f) 11

2. (a) $39 = 3 \times 13$
   (b) $60 = 2^2 \times 3 \times 5$
   (c) $81 = 3^4$
   (d) $98 = 2 \times 7^2$
   (e) $180 = 2^2 \times 3^2 \times 5$
   (f) $2 \times 3 \times 43$
   (g) $378 = 2 \times 3^2 \times 7$
   (h) $432 = 2^4 \times 3^3$
   (i) $576 = 2^6 \times 3^2$
   (j) $729 = 3^6$
   (k) $1098 = 2 \times 3^2 \times 61$
   (l) $2324 = 4 \times 7 \times 83$

3. See discussion above.

4. For divisibility by 5, one needs only to see that when 5 is added to a number whose units digit is 0, the sum has units digit 5; if 5 is added to a number whose units digit is 5, the sum has units digit 0. This pattern repeats to show the test for all numbers.

5. The following table can be made for multiples of 9:

Multiple of 9	9	18	27	36	45...90	99	108...
Sum of digits	9	9	9	9	9... 9	18	9...

6. A counting number will be divisible by 6 if, and only if, it is divisible by both 2 and 3. Hence, the test is that it must be even and the sum of its digits must be divisible by 3.

7. If a number is divisible by 15, it must be divisible by both 3 and 5, and conversely. Hence, the test is that its last digit must be one of 0 and 5, and the sum of its digits must be a multiple of 3.

8. (a) Since the last digit is odd, the number is odd.
   (b) The number is 390 in the decimal system and, hence, is even. Notice that the number is even since it can be written in the form $7^3 + 7^2 + 7 + 1$, which is the sum of an even number of odd numbers.
   (c) Here the number in the decimal system is 259, which is odd. Notice that it may be written $6^3 + 6^2 + 6 + 1$, which is the sum of 1 and three even numbers, hence is odd.

(d) Here the number in the decimal system is 40, which is even; or it may be shown even by the same kind of argument as in part (b).

9. If a number is written in the system to the base 7, its last digit is 0 if, and only if, it is divisible by 7, but it need not be divisible by 10. To test divisibility by 3, write the first few multiples of 3 in the system to the base 7 as follows:

Number to the base 7	3	6	12	15	21	24	30	33	36	42
Sum of the digits	3	6	3	6	3	6	3	6	12*	6

*Notice that 12 is also written in the system to the base 7. Here, when the first digit increases by 1, the second digit decreases by 4, giving a net decrease of 3. Hence, the same test for divisibility by 3 works both in the decimal system and in the system to the base 7.

10. If a number is written in the number system to the base 12 and has zero as its last digit, it must be divisible by 12, but need not be divisible by 10. It will be divisible by 3 if its last digit is one of 0, 3, 6, 9. This may be shown in the same way that we tested for divisibility by 5 in the decimal system, since the pattern 3, 6, 9, 0 repeats in the sequence of multiples of 3 written to the base 12.

11. A number written to the base 7 will be divisible by 6 if the sum of its digits is divisible by 6. This is apparent from the table given for Exercise 9 if we notice that every other sum of digits is even.

12. A number written in the decimal system will be divisible by 4 if one of the following holds:

(a) the last digit is one of 0, 4, 8 and the tens digit is even.
(b) the last digit is 2 or 6 and the tens digit is odd.

This can be seen from the pattern in which the multiples of 4 fall. Also, since any multiple of 100 is divisible by 4, we could also say that a number is divisible by 4 if the number represented by the last two digits is divisible by 4.

Exercises 10.

1. (a) {1,2,3,6}
   (b) {1,2,4,8}
   (c) {1,2,3,4,6,12}
   (d) {1,3,5,15}
   (e) {1,2,4,8,16}
   (f) {1,3,7,21}

2. (a) {1,2}
   (b) {1,2,4}
   (c) {1,3}
   (d) {1,2}
   (e) {1,3}
   (f) {1,2,4}

3. (a) {1,19}
   (b) {1,2,4,7,14,28}
   (c) {1,2,3,4,6,9,12,18,36}
   (d) {1,2,4,5,8,10,20,40}
   (e) {1,3,5,9,15,45}
   (f) {1,2,3,4,6,8,9,12,18,24,36,72}

4. (a) {1}
   (b) {1,2,4}
   (c) {1,2,4}
   (d) {1,3,9}
   (e) {1,2,4,8}
   (f) {1}

5. (a) 4
   (b) 4
   (c) 4
   (d) 9
   (e) 8
   (f) 4

6. (a) 5
   (b) 6
   (c) 12
   (d) 25
   (e) 16
   (f) 3
   (g) 12
   (h) 8
   (i) 15
   (j) 10

7. (a) 6
   (b) 29
   (c) a

8. (a) 1
   (b) 1
   (c) 1

9. (a) Yes, 1

(b) Yes, $c = 3$, $a = 3$, $b = 6$ or 9 or 12, etc.

(c) No; the greatest common factor can never be greater than the smallest member of the set of numbers used.

10. (a) No. $\{6, 10, 15\}$

(b) Yes. G. C. F. of 6 and 10 is 2.
G. C. F. of 6 and 15 is 3.
G. C. F. of 10 and 15 is 5.

11. (a) $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$

$45 = 3 \cdot 3 \cdot 5 = 3^2 \cdot 5$

(b) The greatest common factor is 3^2 or 9.

12. (a) $18 = 2 \cdot 3 \cdot 3 = 2 \cdot 3^2$

(b) $90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5$

(c) G. C. F. $= 2 \cdot 3^2 = 18$

13. (a) $24 = 2^3 \cdot 3$, $60 = 2^2 \cdot 3 \cdot 5$
G. C. F. $= 2^2 \cdot 3 = 12$

(b) $36 = 2^2 \cdot 3^2$, $90 = 2 \cdot 3^2 \cdot 5$
G. C. F. $= 2 \cdot 3^2 = 18$

(c) $72 = 2^3 \cdot 3^2$, $108 = 2^2 \cdot 3^3$
G. C. F. $= 2^2 \cdot 3^2 = 36$

(d) $25 = 5^2$, $75 = 3 \cdot 5^2$, $125 = 5^3$
G. C. F. $= 5^2 = 25$

(e) $24 = 2^3 \cdot 3$, $60 = 2^2 \cdot 3 \cdot 5$, $84 = 2^2 \cdot 3 \cdot 7$
G. C. F. $= 2^2 \cdot 3 = 12$

(f) $42 = 2 \cdot 3 \cdot 7$, $105 = 3 \cdot 5 \cdot 7$, $147 = 3 \cdot 7^2$
G. C. F. $= 3 \cdot 7 = 21$

(g) $165 = 3 \cdot 5 \cdot 11$, $234 = 2 \cdot 3^2 \cdot 13$
G. C. F. $= 3$

(h) $306 = 2 \cdot 3^2 \cdot 17$, $1173 = 3 \cdot 17 \cdot 23$
G. C. F. $= 3 \cdot 17 = 51$

(i) $2040 = 2^3 \cdot 3 \cdot 5 \cdot 17$, $2184 = 2^3 \cdot 3 \cdot 7 \cdot 13$
G. C. F. $= 2^3 \cdot 3 = 24$

14. (a) 6 $(6 \times 0 = 0)$

(b) 1

(c) 1

15. (a) Yes

(b) Yes

(c) Yes

Exercises 11.

1.

	Dividend	Divisor	Quotient	Remainder
a.			2	
b.			4	2
c.		9		
d.			7	2
e.			4	2
f.			3	2
g.		10		
h.	66			
i.		20		

There are several possible answers for j.

	Dividend	Divisor	Quotient	Remainder
j.		9	9	
		3	27	
		1	81	
		27	3	
		81	1	

2. (a) No; see a. and j.

(b) The dividend is greater than the quotient in this case.

(c) The divisor must always be greater than the remainder.

(d) Yes. $0 \div 3 = 0$ or $0 = 3 \cdot 0 + 0$.

(e) No. $3 \div 0$ is impossible because there is not a number which, when multiplied by 0, gives 3, with a remainder less than the divisor.

(f) Yes. $0 \div 3 = 0$

or $3 \div 5$ may be considered as giving a quotient of 0 with remainder 3. This might be the answer to the question: "How many \$5 shirts can you buy with \$3?"

(g) Yes. $6 \div 6 = 1$ with 0 remainder.

$6 \div 2 = 3$ with 0 remainder.

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3. (a) Yes

(b) No. It is impossible to divide by 0.

(c) Yes

(d) Yes. A remainder of 0 is a whole number.

4.

	a	b	q	R
(a)		2		1
(b)	98			
(c)			4	2

(d) There are many possible answers, as indicated here. By the commutative property, the reverse order for each of these is also an acceptable answer.

	a	b	q	R
		1	100	
		2	50	
		4	25	
		5	20	
		10	10	
(e)			16	11
(f)		25		

5. (a) No

(b) Yes, $(16 \div 2 = 8,\ 8 > 2,\ q = 8)$.

(c) Yes, ($200 \div 75 = 2$ and Remainder 50
$q = 2,\ R = 50$).

(d) No. The divisor, b, may be any whole number <u>except</u> 0. (Division by 0 is impossible.)

(e) Yes. (Counting numbers do not include 0.)

(f) Yes. (a may be 0, then $q = 0$. Or, a may be any other number. However, q is not a counting number if $a < b$.)

6. (a) {0,1,2,3}

(b) The members of the set of all remainders are the whole numbers less than 11.

(c) 25

(d) K

7. (a) (1) $92 \div 32 = 2$ and Remainder 28

(2) $32 \div 28 = 1$ and Remainder 4

(3) $28 \div 4 = 7$ and Remainder 0

4 is the divisor that results in a 0 remainder.

The G. C. F. is 4.

(b) (1) $192 \div 81 = 2$ and Remainder 30

(2) $81 \div 30 = 2$ and Remainder 21

(3) $30 \div 21 = 1$ and Remainder 9

(4) $21 \div 9 = 2$ and Remainder 3

(5) $9 \div 3 = 3$ and Remainder 0

3 is the divisor that results in a 0 remainder.

Therefore, the G. C. F. is 3.

(c) (1) $150 \div 72 = 2$ and Remainder 6

(2) $72 \div 6 = 12$ and Remainder 0

The G. C. F. is 6.

(d) (1) $836 \div 124 = 6$ and Remainder 92

(2) $124 \div 92 = 1$ and Remainder 32

(3) $92 \div 32 =$ (see answer 7(a).) The G. C. F. is 4.

(e) G. C. F. is 28.

(f) G. C. F. is 71.

Exercises 12.

1. (a) The set of multiples of 6 less than 100 is {0,6,12,18,24,30,36,42,48,54,60,66,72,78,84,90,96}.

(b) The set of multiples of 8 less than 100 is {0,8,16,24,32,40,48,56,64,72,80,88,96}.

(c) The set of multiples of 9 less than 100 is {0,9,18,27,36,45,54,63,72,81,90,99}.

(d) The set of multiples of 12 less than 100 is {0,12,24,36,48,60,72,84,96}.

2. (a) The set of common multiples of 6 and 8 less than 100 is {0,24,48,72,96}.

(b) The set of common multiples of 6 and 9 less than 100 is {0,18,36,54,72,90}.

(c) The set of common multiples of 6 and 12 less than 100 is {0,12,24,36,48,60,72,84,96}.

(d) The set of common multiples of 8 and 9 less than 100 is {0,72}.

(e) The set of common multiples of 8 and 12 less than 100 is {0,24,48,72,96}.

(f) The set of common multiples of 9 and 12 less than 100 is {0,36,72}.

3. (a) The least common multiple of 6 and 8 is 24.
(b) The least common multiple of 6 and 9 is 18.
(c) The least common multiple of 6 and 12 is 12.
(d) The least common multiple of 8 and 9 is 72.
(e) The least common multiple of 8 and 12 is 24.
(f) The least common multiple of 9 and 12 is 36.

4. (a) 10 (e) 30
(b) 12 (f) 60
(c) 30 (g) 42
(d) 12 (h) 72

5. (a) 6 (g) 26
(b) 15 (h) 77
(c) 21 (i) 39
(d) 35 (j) 143
(e) 22 (k) 30
(f) 55 (l) 667

6. (a) prime numbers
(b) The L. C. M. of two different prime numbers is equal to the product of the two numbers.

7. (a) 12 (f) 60
(b) 8 (g) 60
(c) 20 (h) 60
(d) 18 (i) 30
(e) 40 (j) 24

8. Composite Numbers

9. (a) Yes. 8 is the L. C. M. of 4 and 8.
   (b) No. The L. C. M. of 8 and 9 is 72.

10. (a) 6
    (b) 29
    (c) a

11. (a) 6
    (b) 29
    (c) a

12. (a) No. What is the L. C. M. of 2 and 3?
    (b) For two different prime numbers, a and b, the L. C. M. is the product of the two numbers, $a \cdot b$.
    (c) For three different prime numbers, a, b, and c, the L. C. M. is the product of the three numbers, $a \cdot b \cdot c$.

13. (a) 48
    (b) 112
    (c) 45
    (d) 70
    (e) 144
    (f) 60
    (g) 72
    (h) 360
    (i) 660
    (j) 720
    (k) 1000
    (l) 6480
    (m) 7083

14. (a) No
    (b) No
    (c) No

15. (a) Yes
    (b) Yes
    (c) Yes
    (d) 0
    (e) No. Zero is not a counting number.

Exercises 13.

1. (a) 1
   (b) 2
   (c) 7
   (d) 5
   (e) 12
   (f) 3
   (g) 1
   (h) 6
   (i) 3
   (j) 2
   (k) 9
   (l) 4

2. (a) 6
   (b) 24
   (c) 14
   (d) 75
   (e) 36
   (f) 105
   (g) 989
   (h) 858
   (i) 663
   (j) 5402
   (k) 2520
   (l) 31,372

3.

	Product of the numbers in Problem 1	Product of the G. C. F. and L. C. M. of the numbers in Problem 1
(a)	$2 \times 3 = 6$	$1 \times 6 = 6$
(b)	$6 \times 8 = 48$	$2 \times 24 = 48$
(c)	$7 \times 14 = 98$	$7 \times 14 = 98$
(d)	$15 \times 25 = 375$	$5 \times 75 = 375$
(e)	$12 \times 36 = 432$	$12 \times 36 = 432$
(f)	$15 \times 21 = 315$	$3 \times 105 = 315$
(g)	$23 \times 43 = 989$	$1 \times 989 = 989$
(h)	$66 \times 78 = 5148$	$6 \times 858 = 5148$
(i)	$39 \times 51 = 1989$	$3 \times 663 = 1989$
(j)	$74 \times 146 = 10,804$	$2 \times 5402 = 10,804$
(k)	$45 \times 72 \times 252 = 22,680$	$9 \times 2520 = 22,680$
(l)	$44 \times 92 \times 124 = 501,952$	$4 \times 31,372 = 155,488$

It is always true that the product of <u>two</u> numbers is equal to the product of their G. C. F. and their L. C. M. This can be seen from the following example:

Let $r = 2^3 \times 5 \times 7$, and $s = 2 \times 5^2 \times 13$. To get the G. C. F. we take the product of the primes occurring in both, raised to the smaller power: 2×5. To get the L. C. M., we take the product of the primes raised to the larger power:

$$2^3 \times 5^2 \times 7 \times 13.$$

Then $rs = 2^3 \times 5 \times 7 \times 2 \times 5^2 \times 13$ and G. C. F. times L. C. M. =

$2 \times 5 \times 2^3 \times 5^2 \times 7 \times 13$. One product is the same as the other except that the factors are rearranged.

This is not true for three or more numbers.

4. (a) {4,6,8,9,10,12,14,15,16,18,20,21,22,24,25,26,27,28,30}
   (b) {2,3,5,7,11,13,17,19,23,29,31,37,41,43,47}

5. (a) The product, $a \cdot b$

   Example (1) The G. C. F. of 3 and 5 is 1.
   The L. C. M. of 3 and 5 is 15.

   Example (2) The G. C. F. of 4 and 9 is 1.
   The L. C. M. of 4 and 9 is 36.

   (b) No. Not necessarily.
   The G. C. F. of 2, 3, and 4 is 1.
   The L. C. M. of 2, 3, and 4 is $2^2 \cdot 3$, or 12. It is not $2 \cdot 3 \cdot 4$.

6. (a) 2 is a prime number. It is the only even prime number.
   (b) All primes except 2 are odd, i.e., 3, 5, ...
   (c) One. Only the prime number 5 has an ending in 5. All other numbers ending in 5 are multiples of 5, i.e., 15, 25, 35, ...
   (d) 2 and 5
   (e) 1, 3, 7, and 9

7. The two numbers are the same. For example: The G. C. F. of 7 and 7 is 7. The L. C. M. of 7 and 7 is 7.

8. (a) 1. 1 is the least common factor of any two whole numbers.
   (b) No answer is possible for the greatest common multiple of any two whole numbers.

9. (a) (Bulbs and rows may be interchanged.)

Rows	Bulbs per row
1	112
2	56
4	28
8	14
16	7

10. (a) 3, 6, 9, 12, 15--first bell,
5, 10, 15--second bell.
They strike together again in 15 minutes.

(b) 6, 12, 18, 24, 30--first bell,
15, 30--second bell.
They strike together again in 30 minutes.

(c) 15 is the least common multiple of 3 and 5.
30 is the least common multiple of 6 and 15.

11. (a) Yes. The G. C. F. of 6 and 6 is 6.
The L. C. M. of 6 and 6 is 6.

(b) No. The G. C. F. of the members of a set of numbers can never be greater than the largest number of the set of numbers, because a factor of a number is always less than a multiple of the number unless the multiple is zero.

(c) No. The L. C. M. of the members of a set of numbers can never be less than the largest member of the set of numbers.

The least common multiple of two numbers is at least as big as the larger of the two numbers (since the L. C. M. is a multiple of the larger number). The greatest common factor of two numbers is no larger than the smaller of the two numbers (since the G. C. F. is a factor of the smaller number). If the least common multiple and greatest common factor are equal, the larger and smaller number must also be equal. Illustrate with 4 and 6, locating the L. C. M. and G. C. F. on the number line; then use 4 and 4.

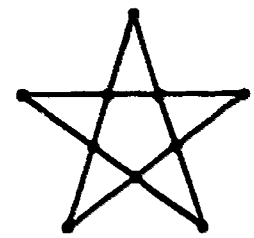
12. (a) No. It is not possible to have exactly four numbers between two odd numbers. Between any two odd primes there is always an odd number of numbers. If they are consecutive odd primes, all the numbers between would have to be composite.

(b) Yes. For example, between 23 and 29 there are exactly 5 composite numbers; 24, 25, 26, 27, 28.

13. (a) 135, 222, 783, and 1065 are all divisible by three.
(b) 222 is the only numeral divisible by six.
(c) 135 and 783 are divisible by nine.
(d) 135 and 1065 are divisible by five.
(e) 135 and 1065 are divisible by fifteen.
(f) None of the numerals is divisible by four.

14. Many different answers should be expected from the student; in a way, his reasons for his answer are more important than the answer itself. The most obvious answer is that, as far as multiplication is concerned, the prime numbers are the building blocks of all counting numbers, since any counting number can be expressed as a product of prime numbers. If we know a complete factorization of a number, then we can find all its factors. From complete factorizations the G. C. F. and L. C. M. of sets of numbers may also be found.

15. BRAINBUSTER. The pattern is a five-pointed star.



16. There is no greatest prime number.

To show there is no largest prime number, we will show that if p is any prime, there is another prime larger than p. Denote by M the product of all the primes less than or equal to p:

$$M = 2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p.$$

Then, $M + 1$ is certainly larger than p, and $M + 1$ has at least one prime factor (it may itself be a prime). But $M + 1$ does not have any of the primes, 2, 3, 5, 7, ..., p as a factor since division by any of these primes leaves a remainder of 1. Thus, all the prime factors of $M + 1$ are larger than p, and hence, p is not the largest prime. Since p was an arbitrary prime, there is no largest prime.

Exercises 14.

1.

m	1	2	3	4	5	6	7	8	9	10	11	12
n	1	3	7	15	31	63	127	255	511	1023	2047	4095

m	13	14	15	16	17	18	19	20
n	8191	16383	327677	65535	131071	262143	524287	1048575

2r

2. (a) 3 divides 3, 15, 63, 225, etc.

   (b) 7 divides 63, 511, 4095, etc.

   (c) 31 divides 31, 1023, 32767, and 1048575.

   (d) If m is divisible by b, then n is divisible by $2^b - 1$.

Note the difference between

$$3^{(2^4)} = 3^{16},$$

$$(3^2)^4 = 3^8.$$