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ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exercises, and sample test questions. Topics covered include history of numerals, the decimal system, expanded numerals and exponential notation, numerals in base seven, computation in base seven, changing from base ten to base seven, and numerals in other bases. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-15

SUPPLEMENTARY and ENRICHMENT SERIES

NUMERATION

Teachers' Commentary

Edited by Augusta L. Schurrer

SHSG

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

Prepared under the supervision of the Panel on Supplementary Publications of the School Mathematics Study Group:

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TEACHERS' COMMENTARY

NUMERATION

Introduction

For thi unit little background is needed except familiarity with the number symbols and the basic operations with numbers. The purpose of the unit is to deepen the pupil's understanding of the decimal notation for whole numbers, especially with regard to place value, and thus to help him delve a little deeper into the reasons for the procedures, which he already knows, for carrying out the addition and multiplication operations. One of the best ways to accomplish this is to consider systems of number notations using bases other than ten. Since, in using a new base, the pupil must necessarily look at the reasons for "carrying" and the other mechanical procedures in a new light, he should gain deeper insight into the decimal system. A certain amount of computation in other systems is necessary to "fix" these ideas, but such computation should not be regarded as an end in itself. Some of the pupils, however, may enjoy developing a certain proficiency in using new bases in computing.

Perhaps the most important reason for introducing ancient symbolisms for numbers is to contrast them with our decimal system, in which not only the symbol, but its position, has significance. It should be shown, as other systems are presented, that position has some significance in them also. The Roman System has a start in this direction in that XL represents a different number from LX, but the start was a very primitive one. The Babylonians also made use of position, but lacked a symbol for zero until about 200 B.C. The Babylonian symbol "≰" denoted the absence of a figure but apparently was not used in computation. The numeral zero is necessary in a positional system. In order for the pupils to appreciate the important characteristics of our system of numeration the following table may be discussed.

	Base	Place Value	Zero
Egyptian	Ten	No	No
Babylonian	Sixty	Yes	Limited meaning
Roman	Varied	No, but it has positional value	No
Decimal	Ten	Yes	Yes

Pupils should not be expected to memorize ancient symbolism. It is recommended that little time be spent on the use of the symbols themselves.

It is especially important to distinguish between a number and the symbols by which it is represented. Some of the properties usually connected with a number are really properties of its notation. The facts that, in decimal notation, the numeral for a number divisible by 5 ends in 5 or 0, and that $\frac{1}{3}$ has an unending decimal equivalent (0.333.....), are illustrations. Many of the statements we make deal with properties of the numbers themselves and are entirely independent of the notation in which they are represented. Examples of such statements are: $2 + 3 = 3 + 2$; the number eleven is a prime number; and six is greater than five. The distinction between a number and the notation in which it is expressed should be emphasized whenever there is opportunity.

An attempt has been made to use "number" and "numeral" with precise meaning in the text. For example, "numerals" are written, but "numbers" are added. A numeral is a written symbol. A number is a concept. Later in the text it may be cumbersome to the point of annoyance to speak of "adding the numbers represented by the numerals written below." In such case the expression may be elided to "adding the numbers below."

At several points, numbers are represented by collections of x's. Exercises of this kind are important, because they show the role of the base in grouping the x's, as well as the significance of the digits in the numeral for the number.

1. History of Numerals

The purpose of the historical material is to trace the continuing need for convenient symbols and for a useful way of writing expressions for numbers. The idea of "one-to-one" correspondence is introduced.

Here the emphasis is upon numerals rather than upon number. Egyptian symbolism is introduced to familiarize the pupils with one of the first important systems of notation. Do not consume an excessive amount of time in discussing the Egyptian or Babylonian systems.

The Babylonians were among the first to use place value. The base sixty system is mentioned because of later reference to it, particularly in measurement. There is evidence that the Babylonians also used symbols like  but there is no need to introduce these to the pupils. No pupil should be required to memorize ancient symbolism except in the case of Roman numerals.

The Roman system may be stressed because of its continued use. Note that the subtracting principle was a late development. It may be pointed out that computation in ancient symbolism was complex and sometimes very difficult. Because of this, various devices were used, such as the sand reckoner, counting table, and abacus. After decimal numerals became known, algorithms were devised and people were able to calculate with symbols alone. There was much opposition in Europe to the introduction and use of Hindu-Arabic numerals, especially on the part of the abacists. As the new system became accepted, the abacus and other computing devices slowly disappeared in Europe.

2. <u>The Decimal System</u>

The illustration of grouping in tens suggests a method useful in mental calculation as 37 + 62 is 3 tens + 6 tens + 7 + 2 which is 99. Note that parentheses are used to show that certain combinations are to be considered as representing a single number.

Emphasize the value represented by a digit and the value of position in decimal notation.

Emphasize the importance of the invention of a useful system which lends itself easily to calculation. The efficiency of the decimal system lies in a combination of factors.

1. Only a few symbols are needed, no matter how large or small the number expressed. Some students may observe that we use ten symbols while the Babylonians used just two and the Egyptians and the Romans each used 7. However, in the decimal system no additional symbols are ever needed as larger numbers are introduced; this is not true of the other systems.
2. Place value in which each position corresponds to a power of the base is of importance in a system used for calculation.
3. The concept of zero as a place holder is essential in the development of a place value system.

The reading and writing of numerals may be treated as a review, or if needed, as a thorough study, depending on the needs of pupils. Some pupils may know and understand this material completely. Others may have a very limited proficiency in this area.

A class discussion might deal with the following:

Suppose we used systematic names for numerals such as "two tens" for twenty, "two tens, one" for twenty-one, "ten, one" for eleven, and "ten, two" for twelve. How many different, basic words would be needed to name all counting numbers up to a trillion? (Remember that a numeral like "one hundred, three tens, six" is made up of basic words used many times in other numerals.) There are fifteen essential words; "one, two, three, ..., ten, hundred, thousand, ..., trillion."

3. Expanded Numerals and Exponential Notation

Exponents are introduced here in a situation which shows clearly their usefulness for concise notation. Furthermore, their use serves to emphasize the role of the base and of position. This role will be more fully utilized in the sections to follow.

The Celts and Mayans used twenty as a base probably because they used their toes as well as their fingers in counting. The special name sometimes used for twenty is "score." Some Eskimo tribes count by five using the fingers of one hand.

4. Numerals in Base Seven

The purpose of teaching systems of numeration in bases other than ten is not to produce facility in calculating with such systems. A study of an unfamiliar system aids in understanding a familiar one, just as the study of a foreign language aids us in understanding our own. The decimal system is so familiar that its structure and the ideas involved in its algorithms are easily overlooked. In this section attention is focused on numerals, rather than on numbers.

Questions may arise about the notation for a numeral to base seven. We do not write "15_7" because the symbol "7" does not occur in a system of numeration to this base. Replacing the numeral by the written word emphasizes this fact.

Later in the pamphlet and in succeeding work, some classes may agree to indicate the base, as different systems of numeration are introduced, by a numeral in decimal notation. They may agree that they will regard the subscript in this case as always based on ten. Thus they may write for 25_{seven} the expression 25_7; for 18_{twelve} the expression 18_{12}; and for 110_{two}, 110_2.

After the pupils have had practice in grouping by sevens, introduce counting. Pupils enjoy counting in turn, and helping each other as 30_{seven}, 40_{seven}, and like numerals arise. Have them fill in missing parts of the list on page 12 orally with perhaps a recorder at the board.

Numerals (in base seven) from 21_{seven} to 202_{seven}

	30	40	50	60	100	110	120	130	140	150	160	200
21	31	41	51	61	101	111	121	131	141	151	161	201
22	32	42	52	62	102	112	122	132	142	152	162	202
23	33	43	53	63	103	113	123	133	143	153	163	
24	34	44	54	64	104	114	124	134	144	154	164	
25	35	45	55	65	105	115	125	135	145	155	165	
26	36	46	56	66	106	116	126	136	146	156	166	

At 66_{seven} you may wish to say: "This is the number of states we had in the United States before Alaska became a state. How many states did we have after Alaska and before Hawaii? How shall we express this number in base seven numerals? We have gone as high as we can in the 'one' place and in the 'seven' place. What is one more than 66_{seven}? What do you do when you reach 99 in the decimal system?" When the pupils understand that after 66_{seven} comes 100_{seven}, ask them, "How many states are there when we include Hawaii?" Pupils may read 101_{seven} as "one, zero, one, base seven."

Have the pupils continue to count orally until they reach 202_{seven}.

It is usually helpful to keep the chart on page 13 on the board during the time this section is studied. Some teachers emphasize the meaning of exponents by writing the chart in two ways:

$seven^4$	$seven^3$	
(seven × seven × seven × seven)	(seven × seven × seven)	
$seven^2$	$seven^1$	one
(seven × seven)	(seven)	(one)

It is suggested that alternate exercises in this list be discussed and answered in class as a group undertaking. The pupils should then be ready to attempt the remaining exercises without further help.

5. Computation in Base Seven

Be sure that pupils understand the construction of the addition table for base ten.

Addition, Base Ten

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Pupils should be helped to observe the symmetry of the table with respect to the diagonal. They will notice that $8 + 6 = 6 + 8$, for example, and that this is true for any pair of numbers. This illustrates the commutative property of addition.

If pupils know the facts, no time should be wasted on the table after its characteristics have been discussed.

Addition, Base Seven

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

There is no value in memorizing this table. The process is more important than the facts. The point to be emphasized is that numbers and number properties are independent of the numerals or symbols used to represent the numbers. Commutativity holds in base seven as well as base ten because it is a property of numbers, not numerals.

Addition in base seven is undertaken to clarify addition in decimal notation. Some of the newer elementary school textbooks prefer to use the word "change" or "regroup" rather than "borrow" since the first two words seem to describe the actual process better than the last.

Point out to the pupils that in adding in base ten it is often necessary to regroup ten ones as one ten, whereas in base seven we regroup seven ones as one seven.

As pupils use the table in subtraction, they may observe that subtraction is the inverse of addition.

Some of the exercises in addition and subtraction are written in horizontal fashion in preparation for future work in algebra.

Since division is the most demanding operation, it is suggested that teachers regard the topic as optional and do only as much as they judge appropriate in class discussion. Pupils may need help in learning how to use the multiplication table to find division facts. Exercises are included for those pupils who wish to attempt them.

Working with base ten and base seven numerals, using the tables, and changing from base seven to base ten, provide many opportunities for needed drill in the four operations. Teachers may wish to observe the kinds of drill needed and to devise additional exercises of the appropriate type.

6. <u>Changing from Base Ten to Base Seven</u>

Pupils in general find it easier to change from base seven to base ten numerals than the reverse. This section gives much detail in order to provide help to pupils who may have trouble.

Ask for the highest power of seven which is contained in the number given in base ten numeration. For example, consider 634_{ten}. Is 7^4 or (2401_{ten}) contained in 634_{ten}? Is 7^3 or (343_{ten})? After we have taken as many 343_{ten}'s as possible from 634_{ten} how much remains? The next power of 7 is 7 ? How many 49_{ten}'s are contained in 291_{ten} ? Finally, how many 7's, and how many 1's are left?

A second method of changing from base ten to base seven numerals is developed in Exercises 3, 4, 5 on Page 27.

10	1958			
10	195	remainder	8	
10	19	remainder	5	The number is 1958
10	1	remainder	9	
	0	remainder	1	

7	524			
7	74	remainder	6	
7	10	remainder	4	The number is 1346 base seven
7	1	remainder	3	
	0	remainder	1	

Some pupils will see that the 6 ones are found first when 74 groups of 7 are taken away. These groups of 7 are then put together in groups of 7 sevens with 4 groups left for the numeral in the 7 place. The process is repeated to find the digits for successive places in the numeral.

7. Numerals in Other Bases

Bring out the idea that the base of the system that we use is "ten" for historical rather than mathematical reasons. Some mathematicians have suggested that a prime number such as 7 has certain advantages. The Duodecimal Society of America, 20 Carlton Place, Staten Island 4, New York supports the adoption of twelve as the best number base. Information about the duodecimal system is furnished by this society on request. Exercises in other number bases help establish an understanding of what a positional, power system of numeration is.

SAMPLE QUESTIONS

The sets of questions presented ere are not intended as a test. Teachers should construct tests carefully by combining selected items from the set of questions included in this pamphlet and questions of their own writing. Great care should be used to avoid making the test too long.

Part I. True - False

1. The 3 in 356_{seven} stands for three hundred.
2. 10^4 means $10 \times 10 \times 10 \times 10$.
3. The numeral 8 represents the same number in the ten system as in the twelve system.
4. The smaller the base, the more basic combinations there are in the multiplication table.
5. The fourtn place from the right in the decimal system has the place value 10^5.
6. In base two numerals the number after 100 is 1000.
7. We can make a symbol mean what we wish.
8. When we "carry" or "regroup" in addition the value of what is carried depends upon the base.
9. A number may be expressed in numerals with any whole number greater than one as a base.
10. In the symbol 6^3, the exponent is 3.
11. 513_{six} means $(5 \times six \times six \times six) + (1 \times six \times six) + (3 \times six)$.
12. The 1 in $10{,}000_{two}$ means 1×2^4 or sixteen.
13. The following numerals represent the same number: 183_{twelve}; 363_{eight}; 1033_{six}.
14. In base eight numerals, the number before 70 is 66.
15. Four symbols are sufficient for a numeration system with base five.
16. In the base four system $3 + 3 = 11_{four}$.
17. When we "borrow" in the twelve system as in 157 - 6E, we actually "borrow" twelve units.

18. In the Egyptian system a single symbol could be used to represent a collection of several things.

19. The Babylonians made use of place value in their numeration system.

20. The Roman numeral system had a symbol for zero.

Part II. Completion

1. In decimal numerals 14_{twelve} is ______.

2. MCXXIV in decimal numerals is ______.

3. The decimal system uses ______ different symbols.

4. In any numeration system, the smallest place value for whole numbers is ______.

5. 629,468,000,000 written in words is ______________________________.

6. The number represented by 212_{seven} is ____________ (even or odd).

7. In expanded notation $5{,}678_{ten}$ is ____________.

8. $625_{seven} + 344_{seven} =$ ________________$_{seven}$.

9. The product of 312_{four} and 32_{four} is ______________$_{four}$.

10. $110011_{two} =$ __________$_{ten}$.

11. The numeral 444_{five} represents an ______________ (even, odd number).

12. Add: $42_{five} + 14_{five} =$ __________ .

13. $13_{ten} =$ __________$_{two}$.

14. The numeral after 37_{eight} is __________$_{eight}$.

15. Write the 4-place base ten numeral which names the largest number you can represent using all of the digits 5, 6, 7, and 0.

16. Write the 4-place base ten numeral which names the smallest number you can represent using all of the digits 5, 6, 7, and 0.

17. What is the largest number you can write, using two 4's and no other symbols?

18. Write this numeral without exponents: 5^3 ________.

19. The numeral immediately before 1000_{two} is ________.

20. Subtract: $42_{five} - 14_{five} =$ __________.

Part III. Multiple-Choice

1. In which of the numerals below does 1 stand for four?

   (a) 21_{four} (d) 102_{three}

   (b) 21_{eight}

   (c) 100_{two}

2. In what base are the numerals written if $2 \times 2 = 10$?

   (a) Base two (d) Base five

   (b) Base three

   (c) Base four

3. A decimal numeral which represents an odd number is:

   (a) 461,000 (d) 9,000,000

   (b) 7629

   (c) 5634

4. If N represents an even number, the next consecutive even number can be represented by:

   (a) N (d) 2N

   (b) N + 1

   (c) N + 2

5. Which numeral represents the largest number?

   (a) 43_{five} (d) 24_{nine}

   (b) 212_{three} (e) $10_{twenty\text{-}five}$

   (c) 10110_{two}

6. Which is correct?

   (a) $5^4 = 5 + 5 + 5 + 5$ (d) $2^3 = 2 \times 3$

   (b) $4^3 = 4 \times 4 \times 4$ (e) None of the above is correct.

   (c) $5^4 = 4 \times 4 \times 4 \times 4 \times 4$

7. 6120_{nine} is how many times as large as 612_{nine}?

   (a) twelve (d) five

   (b) ten

   (c) nine

8. In which base does the numeral 53 represent an even number?

   (a) twelve (d) seven

   (b) ten (e) six

   (c) eight

ANSWERS TO SAMPLE QUESTIONS

Part I. True - False

1. False
2. True
3. True
4. False
5. False
6. False
7. True
8. True
9. True
10. True
11. False
12. True
13. False
14. False
15. False
16. False
17. True
18. True
19. True
20. False

Part II. Completion

1. 16
2. 1124
3. Ten
4. One
5. Six hundred twenty-nine billion, four hundred sixty-eight million.
6. Odd
7. $(5 \times 10^3) + (6 \times 10^2) + (7 \times 10^1) + (8 \times 1)$
8. 1302_{seven}
9. 23310_{four}
10. 51
11. even
12. 111_{five}
13. 1101_{two}
14. 40_{eight}
15. 7650
16. 5067
17. 4^4
18. 125
19. 111_{two}
20. 23_{five}

Part III. Multiple Choice

1. (c)
2. (c)
3. (b)
4. (c)
5. (e)
6. (b)
7. (c)
8. (d)

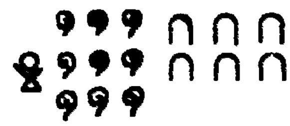
ANSWERS

Exercises 1 -- Page 4

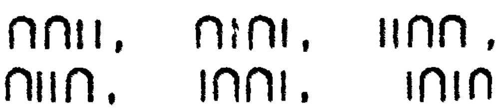
1. (a) 

   (b) 

   (c) 

   (d) 

   (e) 

2. 

3. (a) 200,105
   (b) 2052
   (c) 1029
   (d) 1,100,200

4. (a) 
   (b) 
   (c) 

5. (a) 15
   (b) 37
   (c) 55

6. (a) 29
   (b) 61
   (c) 90
   (d) 105
   (e) 666
   (f) 500,000
   (g) 1492

7. (a) XIX
   (b) LVII
   (c) DCCCLXXXVIII
   (d) MDCXC
   (e) $\overline{M}$
   (f) $\overline{XV}$

8. (a) 7
   (b) 2
   (c) 7
   (d) 10

9. (a) No. Since X represents a smaller number than C, XC means 100 - 10 or 90 while CX means 100 + 10 or 110.

   (b) Yes, since the position indicated whether the numbers were to be added or subtracted.

10. (a) Three

(b) One hundred eleven

(c) In the Roman system, the symbol shows the number of units to be added while our system shows the number of groups and each group has a different number of units. The decimal system involves place value while the Roman system does not.

11. (a)

```
 1709
  654
 2363  or  MMCCCLXIII
```

(b)

```
 2640
 1408
 4048  or  MMMMXLVIII
```

Exercises 2 -- Page 7

1. Ten; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

2. (1) units, (2) tens, (3) hundreds, (4) thousands, (5) ten thousands, (6) hundred thousands, (7) millions, (8) ten millions, (9) hundred million

3. (a) three hundred
   (b) three thousand five
   (c) seven thousand one hundred nine
   (d) fifteen thousand fifteen
   (e) two hundred thirty-four thousand
   (f) six hundred eight thousand fourteen
   (g) one hundred thousand nine
   (h) four hundred thirty thousand one
   (i) nine hundred ninety-nine thousand nine hundred ninety-nine
   (Note: Only the ten numbers are hyphenated, as "twenty-three," etc.)

4. (a) seven million thirty-six thousand two hundred ninety-eight
   (b) nine trillion three hundred billion seven hundred eight million five hundred thousand
   (c) twenty billion three hundred million four hundred thousand five hundred
   (d) nine hundred billion

5. (a) 159
   (b) 503
   (c) 6857
   (d) 3,070,013
   (e) 4,376,007,000
   (f) 20,010
   (g) 9,015,200

6. 99,999 means $(9 \times 10,000) + (9 \times 1000) + (9 \times 100) + (9 \times 10) + (9 \times 1)$ ninety-nine thousand nine hundred ninety-nine.

7. 100,000 means $(1 \times 100,000) + (0 \times 10,000) + (0 \times 1000) + (0 \times 100) + (0 \times 10) + (0 \times 1)$. One hundred thousand.

Exercises 3 -- Page 10

1. Ten to the first power, ten to the second power (or ten square), ten to the third power (or ten cube), ten to the fourth power, ten to the fifth power.

2. (a) 3^5
   (b) 2^4
   (c) 6^6
   (d) 25^3
   (e) 5^6
   (f) 4^2
   (g) 279^5
   (h) 16^1

3. (a) three
   (b) seven
   (c) two
   (d) ten
   (e) An exponent of one indicates the value of the base. In a strict sense this 5 is not a factor.
   (f) five

4. (a) $4 \times 4 \times 4$
   (b) $3 \times 3 \times 3 \times 3$
   (c) $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
   (d) $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$
   (e) $33 \times 33 \times 33 \times 33 \times 33$
   (f) $175 \times 175 \times 175 \times 175 \times 175 \times 175$

5. The exponent tells how many times the base is taken as a factor.

6. (a) $3 \times 3 \times 3 = 27$
   (b) $5 \times 5 = 25$
   (c) $4 \times 4 \times 4 \times 4 = 256$
   (d) $2 \times 2 \times 2 \times 2 \times 2 = 32$
   (e) $6 \times 6 = 36$
   (f) $7 \times 7 \times 7 = 343$
   (g) $8 \times 8 = 64$
   (h) $9 \times 9 = 81$
   (i) $10 \times 10 \times 10 = 1000$
   (j) $3 \times 3 \times 3 \times 3 = 81$
   (k) $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$
   (l) $4 \times 4 \times 4 \times 4 \times 4 = 1024$

7. (a) 4^3 means 64. 3^4 means 81.

(b) 2^9 means 512. 9^2 means 81.

8. (a) $(4 \times 10^2) + (6 \times 10^1) + (8 \times 1)$

(b) $(5 \times 10^3) + (3 \times 10^2) + (2 \times 10^1) + (4 \times 1)$

(c) $(7 \times 10^3) + (0 \times 10^2) + (6 \times 10^1) + (2 \times 1)$

(d) $(5 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (2 \times 10^1) + (6 \times 1)$

(e) $(1 \times 10^5) + (0 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (8 \times 10^1) + (0 \times 1)$

9.

10^{10}	10,000,000,000	ten billion
10^9	1,000,000,000	one billion
10^8	100,000,000	one hundred million
10^7	10,000,000	ten million
10^6	1,000,000	one million
10^5	100,000	one hundred thousand
10^4	10,000	ten thousand
10^3	1,000	one thousand
10^2	100	one hundred
10^1	10	ten

10. The exponent of the base "10" tells how many zeros are written to the right of the "1" when the numeral is written in the usual way.

11. (a) 10^3 (c) 10^6
(b) 10^5 (d) 10^8

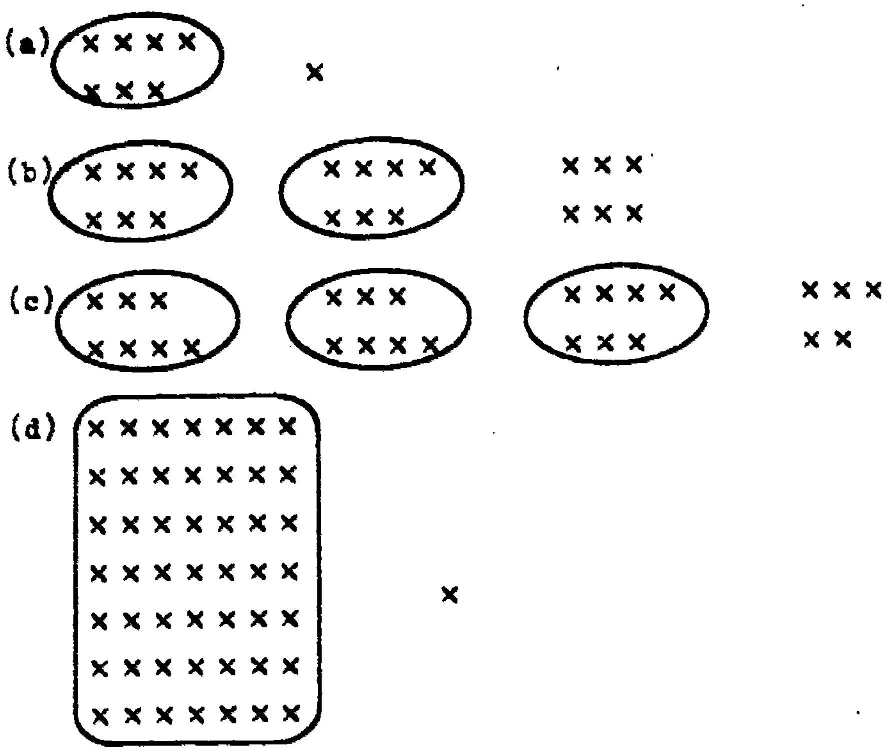
12. 10^{100}. (It may be pointed out to the pupils that 1^{100} is 1.)

13. 100, 10, 1. Some discussion might be devoted to the meaning given to 10^0. This point need not be stressed at this time, but it is useful in later work.

Exercises 4 -- Page 14

1. (a) 13_{seven} (b) 24_{seven} (c) 116_{seven}

2.



3. (a) (3 × seven) + (3 × one) = 24
   (b) (4 × seven) + (5 × one) = 33
   (c) (1 × seven × seven) = 49
   (d) (5 × seven × seven) + (2 × seven) + (4 × one) = 263

4. (a) 10_{seven}
   (b) 11_{seven}
   (c) 55_{seven}
   (d) 163_{seven}
   (e) 1000_{seven}
   (f) 1010_{seven}

5. (a) 560_{seven} The 6 means 6 sevens
   (b) 56_{seven} The 6 means 6 ones
   (c) 605_{seven} The 6 means 6 (seven × seven)'s or 6 (forty-nine)'s
   (d) 6050_{seven} The 6 means 6 (seven × seven × seven)'s or 6 (three hundred forty-three)

6. $seven^4$ or seven to the fourth power

7. The product of 9 sevens or (7^9).

* 8. 132_{seven}

Throughout the text, problems, topics, and sections which were designed for the better students are indicated by an asterisk (*).

9. 452_{seven}

10. 205_{ten}

11. Neither. They are equal.

12. (a) Yes. $30_{ten} = (3_{ten} \times \text{ten}) + (0 \times 1)$

    (b) No. When 241_{ten} is divided by 10_{ten} there is a non-zero remainder.

    (c) If the units digit is zero the number is divisible by ten; otherwise it is not divisible by ten.

*13. $30_{seven} = 21_{ten}$ is not divisible by ten.

$60_{seven} = 42_{ten}$ is not divisible by ten.

*14. (a) It has a remainder of zero when divided by seven.

    (b) Yes. $(3 \times \text{seven}) + (0 \times 1)$ is divisible by seven; remainder is 0.

*15. No. $(3 \times 7) + (1 \times 1) = 22$ is not divisible by seven.

*16. A number written in base seven is divisible by seven when the units digit is zero.

17. 24_{ten} and 68_{ten} are divisible by two. A number divisible by two is called an <u>even</u> number. A number not divisible by two is called an <u>odd</u> number.

*18. 11_{seven} is even.

No. You cannot tell merely by glancing at the numerals. You could tell by converting each numeral to base ten. There is another method which is shorter. It may seem a bit hard at first. For example,

$$\begin{aligned} 12_{seven} &= (1 \times 7) + (2 \times 1) = 1 \times (6 + 1) + (2 \times 1) \\ &= (1 \times 6) + (1 \times 1) + (2 \times 1) \\ &= (1 \times 6) + [(1 + 2) \times 1] \end{aligned}$$

The first term is divisible by two but the second term is not divisible by two; hence the sum is not divisible by two. Note that the digit in the units place of the last expression in the display $1 + 2$, is the sum of the digits of 12_{seven} and this sum is not divisible by two. This is a general rule for base seven numerals.

19. They use seven symbols and seem to have a place value system with base seven. They appear to use |, ∠, △, □, ⧅, ⊠, for 1, 2, 3, 4, 5, 6 and — for zero. ∠∠ follows ∠| .

20. See discussion problem 18. If the sum of the digits is divisible by 2, the number is even.

<u>Exercises</u> <u>5a</u> -- <u>Page</u> <u>17</u>

1. (a) 11 (b) 17

3. (a) Yes
   (b) By reading the table. 6 + 5 is shown in row 6 and column 5. 9 + 8 or 17 is the entry in row 9 and column 8.
   (c) Chart is symmetric with respect to the diagonal.
   (d) 55 different combinations; just a bit over half the total number of combinations.

5. (a) 28 different combinations. Fewer than 49 because of the commutative law of addition.
   (b) In base seven because there are fewer.
   (c) They are equal since $9 = 12_{seven}$.

<u>Exercises</u> <u>5b</u> -- <u>Page</u> <u>20</u>

1. (a) 56_{seven}
   (19 + 22 = 41)
   (b) 110_{seven}
   (41 + 15 = 56)
   (c) 300_{seven}
   (109 + 38 = 147)
   (d) 620_{seven}
   (91 + 217 = 308)
   (e) 241_{seven}
   (33 + 94 = 127)
   (f) 1266_{seven}
   (199 + 290 = 489)
   (g) 1553_{seven}
   (327 + 299 = 626)
   (h) $14{,}562_{seven}$
   (2189 + 1873 = 4062)
   (i) 6441_{seven}
   (2160 + 123 = 2283)
   (j) 1644_{seven}
   (327 + 342 = 669)
   (k) $14{,}654_{seven}$
   (1917 + 2189 = 4106)

2. (a) 2_{seven} (b) 4_{seven} (c) 4_{seven}

3. (a) 2_{seven}

(7 - 5 = 2)

(b) 36_{seven}

(47 - 20 = 27)

(c) 163_{seven}

(98 - 4 = 94)

(d) 151_{seven}

(91 - 6 = 85)

(e) 6_{seven}

(32 - 26 = 6)

(f) 506_{seven}

(323 - 72 = 251)

(g) 203_{seven}

(247 - 146 = 101)

(h) 406_{seven}

(1715 - 1513 = 202)

(i) 552_{seven}

(319 - 37 = 282)

(j) 36_{seven}

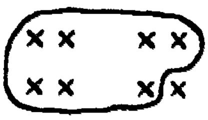
(74 - 47 = 27)

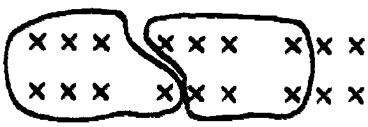
(k) 1254_{seven}

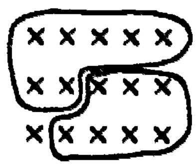
(1261 - 781 = 480)

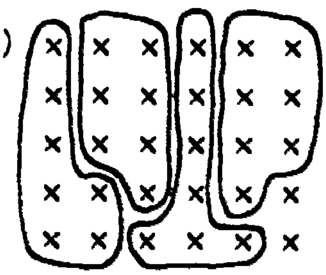
(l) 54_{seven}

(136 - 97 = 39)

4. (a) 

(b) 

(c) 

(d) 

Multiplication, Base Ten

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Exercises 5c -- Page 22

1. Study of this table should emphasize the following:
   - (a) The product of 0 and any number is zero.
   - (b) The product of 1 and any number is the number.

2. The order in multiplication does not affect the product. This is indicated by the fact that the parts of the table on opposite sides of the diagonal line are alike.

3. Multiplication, Base Seven

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Study of this table is valuable for the additional insight it affords into the understanding of multiplication. There is no value in memorizing it. The table may be used to emphasize that division is the inverse of multiplication.

4. Since multiplication combinations are needed only up to 6×6 instead of up to 9×9, multiplication is easier to learn in base seven than in base ten.

5. (a) Both parts are alike.
   (b) $3_{seven} \times 4_{seven} = 4_{seven} \times 3_{seven}$; this is an illustration of the fact that multiplication is commutative.

6. 2835; 18,675; 2,017,372; 697,226; 3,981,354.

Exercises 5d -- Page 24

1. (a) 45_{seven}
   (b) 222_{seven}
   (c) 1116_{seven}
   (d) 3325_{seven}
   (e) 3464_{seven}
   (f) $443{,}115_{seven}$
   (g) $106{,}533_{seven}$
   (h) $5{,}511\ 426_{seven}$
   (i) 125,1[illegible] seven
   (j) $1{,}660{,}101_{seven}$

*2. (a) 5_{seven}

(b) 62_{seven}

(c) 421_{seven} with a remainder of 2_{seven}.

(d) 123_{seven} with a remainder of 12_{seven}.

3. (a) $(4 \times 7 \times 7) + (0 \times 7) + (3 \times 1) = 199_{ten}$

(b) $(1 \times 10 \times 10) + (8 \times 10) + (9 \times 1) = 189_{ten}$

4. 403_{seven}

5. (a) 66_{seven}

(b) 123_{seven}

(c) 1061_{seven}

(d) 1205_{seven}

6. (a) 4_{seven} (b) 26_{seven} (c) 334_{seven}

7. Grade 7, room 123; book 7; 15 chapters; 394 pages; 32 pupils; 5 days; 55 minutes; 13 girls; 19 boys; 11 years old; 66 inches or 5 feet 6 inches tall.

<u>Exercises 6 -- Page 27</u>

1. (a) $50_{ten} = (1 \times \text{seven}^2) + (0 \times \text{seven}) + (1 \times \text{one})$
$= 101_{seven}$

(b) $145_{ten} = (2 \times \text{seven}^2) + (6 \times \text{seven}) + (5 \times \text{one})$
$= 265_{seven}$

(c) $1024_{ten} = (2 \times \text{seven}^3) + (6 \times \text{seven}^2) + (6 \times \text{seven}) + (2 \times \text{one})$
$= 2662_{seven}$

2. (a) 15_{seven}

(b) 51_{seven}

(c) 62_{seven}

(d) 104_{seven}

(e) 431_{seven}

(f) 3564_{seven}

3. Q = 195 R = 8
Q = 19 R = 5
Q = 1 R = 9
Q = 0 R = 1

4. Q = 74 R = 6
Q = 10 R = 4
Q = 1 R = 3
Q = 0 R = 1

$524_{ten} = 1346_{seven}$; the digits of the base seven numeral are the remainders which have just been obtained.

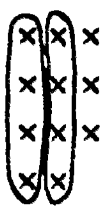
5. Divide by seven and continue to divide each quotient by seven. The digits in the numeral sought will be the remainders in order with the first remainder in the "one" place.

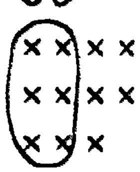
6. (a) 1161
(b) 275
(c) 654_{seven}
(d) 462_{seven}
(e) 1116_{seven}
(f) 3_{seven}
(g) 462_{seven}

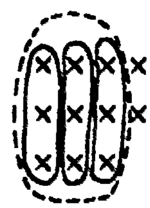
Exercises 7 -- Page 29

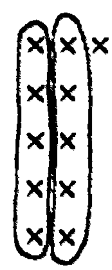
1. 

(a) 5 groups of three and 1 left over.
(b) No. Only the digits "0", "1", and "2" are used in the base three system. "5" is not one of these.
(c) (1 group of $three^2$) + (2 groups of three) + (1 left over).
(d) $16_{ten} = 121_{three}$.

2. (a)  , 11_{ten}

(b)  , 11_{ten}

(c)  , 11_{ten}

(d)  , 11_{ten}

3.

Base Ten	0	1	2	3	4	5	6	7	8	9	10
Base Five	0	1	2	3	4	10	11	12	13	14	20

Base Ten	11	12	13	14	15	16	17	18	19	20
Base Five	21	22	23	24	30	31	32	33	34	40

Base Ten	21	22	23	24	25	26	27	28	29	30
Base Five	41	42	43	44	100	101	102	103	104	110

4. (a) two
(b) two
(c) two
(d) two

5. (a) $(2 \times 36) + (4 \times 6) + (5 \times 1) = 101$
   (b) $(4 \times 25) + (1 \times 5) + (2 \times 1) = 107$
   (c) $(1 \times 27) + (0 \times 9) + (0 \times 3) + (2 \times 1) = 29$
   (d) $(1 \times 64) + (0 \times 16) + (2 \times 4) + (1 \times 1) = 73$

   Other answers are acceptable, i.e., $(2 \times 6^2) + (4 + 6^1) + (5 \times 1)$.

6.

	Base Ten	Base Six	Base Five	Base Four	Base Three
(a)	11	15	21	23	102
(b)	15	23	30	33	120
(c)	28	44	103	130	1001
(d)	36	100	121	210	1100

7. Two. The binary base. With only one symbol it would be impossible to express both zero and one.

*8. (a) 1003_{four}
   (b) 1110_{six}
   (c) 1002_{three}
   (d) 424_{five}
   (e) 111_{four}
   (f) 1045_{six}
   (g) 112_{six}
   (h) 22_{three}

9. The new system is in base four.

Base Ten	New Base	New Base Names
0	O	do
1	|	re
2	∧	mi
3	≥	fa
4	|O	re do
5	||	re re
6	|∧	re mi
7	|≥	re fa
8	∧O	mi do
9	∧|	mi re
10	∧∧	mi mi

Base Ten	New Base	New Base Names
11	∧≥	mi fa
12	≥O	fa do
13	≥|	fa re
14	≥∧	fa mi
15	≥≥	fa fa
16	|OO	re do do
17	|O|	re do re
18	|O∧	re do mi
19	|O≥	re do fa
20	||O	re re do

10.

+	O	I	∧	≥
O	O	I	∧	≥
I	I	∧	≥	IO
∧	∧	≥	IO	II
≥	≥	IO	II	I∧

×	O	I	∧	>
O	O	O	O	O
I	O	I	∧	≥
∧	O	∧	IO	I∧
≥	O	≥	I∧	∧I

Exercises 8 -- Page 33

1.

Base ten	0	1	2	3	4	5	6	7	8	9	10
Base two	0	1	10	11	100	101	110	111	1000	1001	1010

11	12	13	14	15	16	17	18	19	20
1011	1100	1101	1110	1111	10000	10001	10010	10011	10100

21	22	23	24	25	26	27	28	29
10101	10110	10111	11000	11001	11010	11011	11100	11101

30	31	32	33
11110	11111	100000	100001

2. Addition, Base two

+	0	1
0	0	1
1	1	10

There are only four addition "facts."

3. Multiplication, Base Two

×	0	1
0	0	0
1	0	1

There are only four multiplication "facts." The two tables are not alike, except that 0 + 0 and 0 × 0 both equal 0.

The binary system is very simple because there are only four addition and four multiplication "facts" to remember. Computation is simple. Writing numerals for large numbers, however, is tedious.

4. (a) $111_{two} = (1 \times two^2) + (1 \times two) + (1 \times one) = 7$

(b) $1000_{two} = (1 \times two^3) + (0 \times two^2) + (0 \times two) + (0 \times one) = (1 \times 2^3) = 8$

(c) $10101_{two} = (1 \times two^4) + (0 \times two^3) + (1 \times two^2) + (0 \times two) + (1 \times one) = (1 \times 2^4) + (1 \times 2^2) + (1 \times 1) = 21$

(d) $11000_{two} = (1 \times two^4) + (1 \times two^3) + (0 \times two^2) + (0 \times two) + (0 \times one) = (1 \times 2^4) + (1 \times 2^3) = 24$

(e) $10100_{two} = (1 \times two^4) + (0 \times two^3) + (1 \times two^2) + (0 \times two) + (0 \times one) = (1 \times 2^4) + (1 \times 2^2) = 20$

5. $22 = 1T_{twelve}$

$23 = 1E_{twelve}$

$24 = 20_{twelve}$

6. 408

7. (a) $111_{twelve} = (1 \times twelve^2) + (1 \times twelve) + (1 \times one) = (1 \times 144) + (1 \times 12) + (1 \times 1) = 157$

(b) $3T2_{twelve} = (3 \times twelve^2) + (T \times twelve) + (2 \times one) = (3 \times 144) + (10 \times 12) + (2 \times 1) = 554$

(c) $47E_{twelve} = (4 \times twelve^2) + (7 \times twelve) + (E \times one) = (4 \times 144) + (7 \times 12) + (11 \times 1) = 671$

(d) $T0E_{twelve} = (T \times twelve^2) + (0 \times twelve) + (E \times one) = (10 \times 144) + (11 \times 1) = 1451$

8. (a) 111_{two} (c) 110000_{two}

(b) 1011_{two} (d) 110110_{two}

9. (a) 10_{two} (c) 111_{two}

(b) 11_{two} (d) 11_{two}

10. (a) 100011_{two} (c) 1100_{two}

(b) 10000000_{two} (d) 1100100_{two}

11. (a) Addition 323_{twelve}; Subtraction 149_{twelve}

(b) 1005_{twelve}; $85E_{twelve}$

12. The binary system is extremely simple in computation. Numerals for large numbers are tedious to write.
The duodecimal system may be used conveniently to represent large numbers. Twelve is divisible by 1, 2, 3, 4, 6 and 12 while ten is divisible only by 1, 2, 5 and 10. The twelve system requires more computational "facts" which will increase difficulties in memorizing tables of addition and multiplication. We do use twelve in counting dozens, gross, etc., and in some of the common measures of length.

13. (a) $2E5_{twelve}$ (b) 378_{twelve}

14. Five weights; 1 oz., 2 oz., 4 oz., 8 oz., may be used to check any weight up to 15 ounces. By adding a 16 oz. weight, any weight up to 31 ounces may be checked.

15. People who work with computers often use the base eight. To change from binary to octal and back is simple with the help of the table:

Binary	Octal
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7

For example, we have

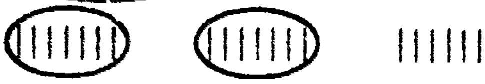
$$2000_{ten} = 011,111,010,000_{two} = 3720_{eight}$$

Note the grouping of numerals by threes in the binary numeral. The sum of the place values of digits in each group results in the octal numeral. Hence

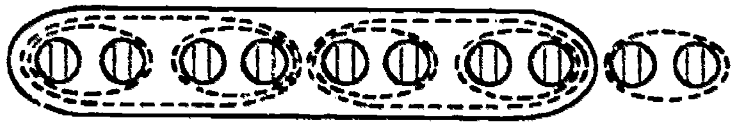
$$011 = (1 \times 2) + (1 \times 1) = 3$$
$$111 = (1 \times 4) + (1 \times 2) + (1 \times 1) = 7, \text{ etc.}$$

Exercises 9 -- Page 36

1. (a)  = 18_{twelve}

   (b)  = 26_{seven}

   (c)  = 40_{five}

   (d)  = 10100_{two}

2. (a) twelve (c) five

   (b) seven (d) two

3. (a) $111_{two} = (1 \times two^2) + (1 \times two) + (1 \times one)$
   $= (1 \times 4) + (1 \times 2) + (1 \times 1) = 7$

   (b) $321_{four} = (3 \times four^2) + (2 \times four) + (1 \times one)$
   $= (3 \times 16) + (2 \times 4) + (1 \times 1) = 57$

   (c) $2631_{seven} = (2 \times seven^3) + (6 \times seven^2) + (3 \times seven) + (1 \times one)$
   $= (2 \times 343) + (6 \times 49) + (3 \times 7) + (1 \times 1) = 1002$

   (d) $37T_{twelve} = (3 \times twelve^2) + (7 \times twelve) + (T \times one)$
   $= (3 \times 144) + (7 \times 12) + (10 \times 1) = 526$

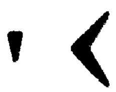
4. $1000_{ten} = 1750_{eight} = 1,111,101,000_{two}$

5. 100_{nine}, 101_{nine}, 102_{nine}, 103_{nine}, 104_{nine}.

*6. (a) 2; 61; 3601; 216,001; etc.

   (b) No. It cannot because the decimal system has a symbol for zero. If this symbol doesn't appear between the two 1's then no zero can be intended.

   (c)  = ( $\times$ $sixty^2$) + ( $\times$ sixty) + ( $\times$ one)
   $= (2 \times 3600) + (12 \times 60) + (11 \times 1)$
   $= 7931$

   (d) 70_{ten} = 

   111_{ten} = 

   4000_{ten} = 

   (e) 

*7. The base is twenty. Four score and seven = $47_{twenty} = 87_{ten}$.

*8. Since there are only five symbols the base is five.

DCBAO = $43{,}210_{five}$

$= (4 \times five^4) + (3 \times five^3) + (2 \times five^2) + (1 \times five) + (0 \times one)$

$= (4 \times 625) + (3 \times 125) + (2 \times 25) + (1 \times 5) + (0 \times 1)$

$= 2930_{ten}$

*9. 4

10. The base is twenty-six.

BE = $25_{twenty\text{-}six} = (2 \times twenty\text{-}six) + (5 \times one)$

$= (2 \times 26) + (5 \times 1) = 57_{ten}$

TWO = $(T \times twenty\text{-}six^2) + (W \times twenty\text{-}six) + (0 \times one)$

$= (19 \times twenty\text{-}six^2) + (22 \times twenty\text{-}six) + (0 \times one)$

$= (19 \times 676) + (22 \times 26) + (0 \times 1)$

$= 13{,}416$

FOUR = $(F \times twenty\text{-}six^3) + (0 \times twenty\text{-}six^2) + (U \times twenty\text{-}six) + (R \times one)$

$= (6 \times twenty\text{-}six^3) + (0 \times twenty\text{-}six^2) + (20 \times twenty\text{-}six)$
$+ (17 \times one)$

$= (6 \times 17{,}576) + (0 \times 676) + (20 \times 26) + (17 \times 1)$

$= 105{,}993_{ten}$

11. The method works for base twelve and base seven. It will also work for other bases. For bases larger than ten, add. For bases less than ten, subtract.

Example in Base Six:

44_{six} multiplying: $(4 \times 4) = 16$

$44 - 16 = 28_{ten}$

Example in Base Fifteen:

$46_{fifteen}$ multiplying: $(4 \times 5) = 20$

$46 + 20 = 66_{ten}$

Students may suggest other methods which should be checked carefully for validity.