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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include the decimal system, exponential notation, base seven, and the binary and duodecimal systems. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-14

SUPPLEMENTARY and ENRICHMENT SERIES

NUMERATION

Edited by Augusta L. Schurrer

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

Prepared under the supervision of the Panel on Supplementary Publications of the School Mathematics Study Group:

INTRODUCTION

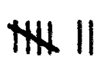
As you read this pamphlet we hope that you will come to appreciate and understand more about our decimal place value system of numeration. To help you do this we will first look at some of the ways in which ancient peoples wrote their numerals. Then we shall see how we would express our numbers if instead of thinking decimally, that is in groups of ten, we decided to think about sets of objects in groups of seven or by the dozen. You may be surprised to learn that it's easier for our high speed computers to think in terms of pairs or groups of two. There's even a special section, Appendix I, about some of the problems we would have to solve if we wished to build a simple computing machine.

You do not need to know a great deal of mathematics to read this booklet. If you can add, subtract, multiply, and divide, you're ready to begin. The problems will help you discover how well you understand the ideas you are reading about. Those with stars will require some extra thought.

CONTENTS

NUMERATION

1. Caveman's Numerals

In primitive times, boys and girls of your age were probably aware of simple numbers in counting, as in counting "one deer" or "two arrows." Primitive peoples learned to keep records of numbers. Sometimes they tied knots in a rope, or used a pile of pebbles, or cut marks in sticks to count objects. A boy counting sheep would have  pebbles, or he might make notches in a stick, as . One pebble, or one mark in the stick would represent a single sheep. He could tell several days later that a sheep was missing if there was not a sheep for each pebble, or for each mark on the stick. You make the same kind of record when you count votes in a class election, one mark for each vote, as . When people began to make marks for numbers, by making scratches on a stone or in the dirt, or by cutting notches in a stick, they were writing the first numerals. Numerals are symbols for numbers. Thus the numeral "7" is a symbol for the number seven. Numeration is the study of how symbols are written to represent numbers.

As centuries passed, early people used sounds, or names, for numbers. Today we have a standard set of names for numbers. Man now has both symbols (1,2,3,...) and words (one, two, three, ...) which may be used to represent numbers.

Ancient Number Systems. One of the earliest systems of writing numerals of which we have any record is the Egyptian. Their hieroglyphic, or picture, numerals have been traced as far back as 3300 B.C. Thus, about 5000 years ago, Egyptians had developed a system with which they could express numbers up to millions. Egyptian symbols are shown in the following table.

Our Number	Egyptian Symbol	Object Represented
1	𓏺	stroke or vertical staff
10	𓎆	heel bone
100	𓍢	coiled rope or scroll
1,000	𓆼	lotus flower
10,000	𓂭	pointing finger
100,000	𓆐	burbot fish (or polliwog)
1,000,000	𓁨	astonished man

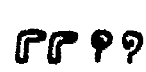
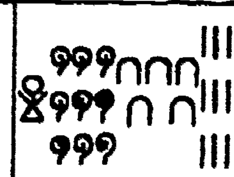
These symbols were carved on wood or stone. The Egyptian system was an improvement over the caveman's system because it used these ideas:

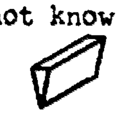
1. A single symbol could be used to represent the number of objects in a collection. For example, the heel bone represented the number ten.

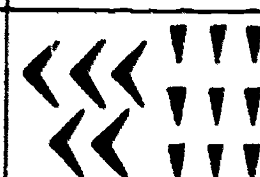
2. Symbols were repeated to show other numbers. The group of symbols  meant 100 + 100 + 100 or 300.

3. This system was based on groups of ten. Ten strokes make a heelbone, ten heelbones make a scroll, and so on.

The following table shows how Egyptians wrote numerals:

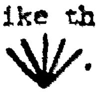
Our numerals	4	11	23	20,200	1959
Egyptian Numerals	||||	∩|	∩∩|||		

About 4000 years ago, around 2000 B.C., the Babylonians lived in the part of Asia we now call the Middle East. They did their writing with a piece of wood on clay tablets. These tablets are called cuneiform tablets. Clay was used because they did not know how to make paper. The pieces of wood were wedge-shaped at the ends as . A drawing instrument of this type is called a stylus. With the stylus a mark  was made on the clay to represent the number "one." By turning the sytlus, they made this symbol  for "ten." They combined these symbols to write numerals up to 59 as shown in the table below:

Our numerals	5	13	32	59
Babylonian Numerals				

Later in this pamphlet you will learn how the Babylonians wrote numerals, or symbols, for numbers greater than 59.

The Roman system was used for hundreds of years. There are still a few places at the present time where these numerals are used. Dates on corner-stones and chapter numbers in books are often written in Roman numerals.

Historians believe that the Roman numerals came from pictures of fingers, like this: |, ||, |||, and ||||. The Romans then used a hand for five, . Gradually some of the marks were omitted, and they wrote V for five. Two fives put together made the symbol for ten, X. The other symbols were letters of their alphabet. The following table shows the other letters used by the Romans:

Our Numeral	1	5	10	50	100	500	1000
Roman Numeral	I	V	X	L	C	D	M

In early times the Romans repeated symbols to represent larger numbers in the same way that the Egyptians had done many years before. Later, the Romans made use of subtraction to shorten some of these numerals.

The values of the Roman symbols are added when a symbol representing a larger quantity is placed to the left in the numeral.

MDCLXVI = 1,000 + 500 + 100 + 50 + 10 + 5 + 1 = 1666

DLXI = 500 + 50 + 10 + 1 = 561

When a symbol representing a smaller value is written to the left of a symbol representing a larger value, the smaller value is subtracted from the larger.

IX = 10 - 1 = 9

XC = 100 - 10 = 90

The Romans had restrictions on subtracting.

1. V, L and D (the symbols which we represent by numerals starting with 5) are never subtracted.
2. A number may be subtracted only in the following cases:

   I can be subtracted from V and X only.

   X can be subtracted from L and C only.

   C can be subtracted from D and M only.

Addition and subtraction can both be used to build a numeral. First, any number whose symbol is placed to show subtraction is subtracted from the number to its right; second, the values found by subtraction are added to all other numbers indicated in the numeral.

CIX = 100 + (10 - 1) = 100 + 9 = 109.

MCMLX = 1,000 + (1,000 - 100) + 50 + 10

= 1,000 + 900 + 50 + 10 = 1960.

Sometimes the Romans wrote a bar over a numeral. This multiplied the value of the symbol by 1,000.

$\overline{X}$ = 10,000, $\overline{C}$ = 100,000, and $\overline{XXII}$ = 22,000.

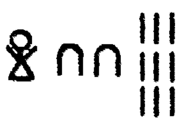
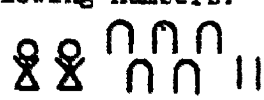
There were many other numeration systems used throughout history, the Korean, Chinese, Japanese, and Indian systems in Asia; the Mayan, Incan, and Aztec systems of the Americas; the Hebrew, Greek, and Arabian systems in the

Mediterranean regions. You might enjoy looking up one of these. If you do, you will find the study of these various systems very interesting. We do not have the time needed to discuss all of them in this pamphlet.

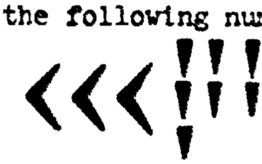
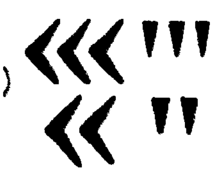
Exercises 1

1. Represent the following numbers using Egyptian numerals:
   (a) 19
   (b) 53
   (c) 666
   (d) 1960
   (e) 1,003,214

2. The Egyptians usually followed a pattern in writing large numbers. However, the meanings of their symbols were not changed if the order in a numeral was changed. In what different ways can twenty-two be written in Egyptian notation?

3. Write our numerals for each of the following numbers:
   (a) 
   (b) 
   (c) 
   (d) 

4. Express the following numbers in Babylonian numerals:
   (a) 9
   (b) 22
   (c) 51

5. Write our numerals for each of the following numbers:
   (a) 
   (b) 
   (c) 

6. Write our numerals for each of the following numbers:
   (a) XXIX
   (b) LXI
   (c) XC
   (d) CV
   (e) DCLXVI
   (f) $\overline{D}$
   (g) MCDXCII

7. Represent the following numbers in Roman numerals:
   (a) 19
   (b) 57
   (c) 888
   (d) 1690
   (e) 1,000,000
   (f) 15,000

8. (a) How many different symbols were used in writing Egyptian numerals?
   (b) How many different symbols were used in writing the Babylonian numerals for numbers up to 59?
   (c) How many different symbols did the Romans use in writing numerals?
   (d) How many different symbols are there in our system?

9. (a) Do XC and CX have the same meaning when written in Roman notation? Explain, using our numeral or numerals.
   (b) Was the position of a symbol in a numeral important in the later Roman system? If so, what does the position of the numeral show?

10. (a) What number is represented by III in the Roman system?
    (b) What number is represented by 111 in our system?
    (c) Can you explain why your answers are different for parts (a) and (b)?

11. Express each of these pairs of numbers in our notation, then add the results and change the answer to Roman numerals.
    (a) MDCCIX and DCLIV
    (b) MMDCXL and MCDVIII

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2. The Decimal System - History and Description

All of the early numeration systems are an improvement over matching notches or pebbles. It is fairly easy to represent a number in any of them. It is difficult to use them to add and multiply. Some instruments, like the abacus, were used by ancient peoples to do arithmetic problems.

The way we write numerals was developed in India. Arab scholars learned about these numerals and carried them to Europe. Because of this, our numerals are called Hindu-Arabic numerals. It is interesting to note that most Arabs have never used these symbols. Because our system uses groups of ten, it is called a decimal system. The word decimal comes from the Latin word "decem," which means "ten."

The decimal system is used in most of the world today because it is a better system than the other number systems discussed in the previous section. Therefore, it is important that you understand the system and know how to read and write numerals in this system.

Long ago man learned that it was easier to count large numbers of objects by grouping the objects. We use the same idea today when we use a dime to represent a group of ten pennies, and a dollar to represent a group of ten dimes. Because we have ten fingers it is natural for us to count by tens. We use ten symbols for our numerals. These symbols, which are called digits, are 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. The word digit refers to our fingers and to these ten number symbols. With these ten symbols we can represent any number as large or as small as we wish.

The decimal system uses the idea of place value to represent the size of a group. The size of the group represented by a symbol depends upon the position of the symbol or digit in a numeral. The symbol tells us how many of that group we have. In the numeral 123, the "1" represents one group of one hundred; the "2" represents two groups of ten, or twenty; and the "3" represents three ones, or three. This clever idea of place value makes the decimal system the most convenient system in the world.

Since we group by tens in the decimal system, we say its base is ten. Because of this, each successive (or next) place to the left represents a group ten times that of the preceding place. The first place tells us how many groups of one. The second place tells us how many groups of ten, or ten times one (10×1). The third place tells us how many groups of ten times ten (10×10), or one hundred; the next, ten times ten times ten ($10 \times 10 \times 10$), or one thousand, and so on. By using a base and the ideas of place value, it is possible to write any number in the decimal system using only the ten basic symbols. There is no limit to the size of numbers which can be represented by the decimal system.

To understand the meaning of the number represented by a numeral such as 123 we add the numbers represented by each symbol. Thus 123 means $(1 \times 100) + (2 \times 10) + (3 \times 1)$, or $100 + 20 + 3$. The same number is represented by $100 + 20 + 3$ and by 123. When we write a numeral such as 123 we are using number symbols, the idea of place value, and base ten.

One advantage of our decimal system is that it has a symbol for zero. Zero is used to fill places which would otherwise be empty and might lead to misunderstanding. In writing the numeral for three hundred seven, we write 307. Without a symbol for zero we might find it necessary to write 3-7. The meaning of 3-7 or 3 7 might be confused. The origin of the idea of zero is uncertain, but the Hindus were using a symbol for zero about 600 A.D., or possibly earlier.

The clever use of place value and the symbol for zero makes the decimal system one of the most efficient systems in the world. Pierre Simon Laplace (1749 - 1827), a famous French mathematician, called the decimal system one of the world's most useful inventions.

Reading and Writing Decimal Numerals. Starting with the first place on the right, each place in the decimal system has a name. The first is the units place, the second the tens place, the third the hundreds place, the fourth the thousands place, and so on. The places continue indefinitely. Do not confuse the names of our ten symbols with the names of the places. Long numerals are easier to read if the digits are separated at regular intervals. Starting at the right, every group of three digits is separated by a comma. These groups also have names as shown by the following table for the first four groups of digits.

Group Name	Billion			Million			Thousand			Unit		
Place Name	Hundred Billion	Ten Billion	Billion	Hundred Million	Ten Million	Million	Hundred Thousand	Ten Thousand	Thousand	Hundred	Ten	Unit
Digits	5	4	5,	4	6	5,	7	3	8,	9	2	1

The names of the digits, the concept of place value and the group name are all used to read a numeral. To read the numeral shown in the table we start with the group on the left, reading the number represented by the first group of digits as one numeral. This is followed by the name of the group, as "five hundred forty-five billion." Then we read each of the following groups, using the name for each group as shown in the table, except that we do not use the name "unit" in reading the last group on the right. The whole numeral shown in the table is correctly read as "five hundred forty-five billion, four hundred sixty-five million, seven hundred thirty-eight thousand, nine hundred twenty-one." The word "and" is not used in reading numerals for whole numbers.

Although we have only ten symbols, we use these symbols again and again. They are used in different positions in numerals to express different numbers. Similarly, in reading numerals we use only a few basic words. We use the names for the symbols, "one, two, three, four," and so on. Then we have the words "ten, eleven, twelve, hundred, thousand," and so on. For other names we use combinations of names, as in "thirteen" which is "three and ten," or in "one hundred twenty-five" which is "one hundred, two tens, and five ones."

Exercise 2

1. How many symbols are used in the decimal system of notation? Write the symbols.

2. Write the names for the first nine places in the decimal system. Begin with the smallest place and keep them in order, as "one, ten, ?, ?, ...

3. Practice reading the following numerals orally, or write the numerals in words.

(a) 300 (d) 15,015 (g) 100,009

(b) 3005 (e) 234,000 (h) 430,001

(c) 7109 (f) 608,014 (i) 999,999

4. Practice reading the following numerals orally, or write the numerals in words.

(a) 7,036,298 (c) 20,300,400,500

(b) 9,300,708,500,000 (d) 900,000,000,000

5. Write the following using decimal numerals:

(a) one hundred fifty-nine.

(b) five hundred three.

(c) six thousand, eight hundred fifty-seven.

(d) three million, seventy thousand, thirteen.

(e) four billion, three hundred seventy-six million, seven thousand.

(f) twenty thousand, ten.

(g) nine million, fifteen thousand, two hundred.

6. (a) Write the numeral representing the largest five place number in the decimal system.

(b) Explain what this numeral represents just as $(3 \times 10) + (5 \times 1)$ explains the meaning of 35.

(c) Write the numeral in words.

7. (a) Write the numeral representing the smallest 6-place number in the decimal system.

(b) Explain the meaning of this numeral.

(c) Write the numeral in words.

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3. Expanded Numerals and Exponential Notation

We say that the decimal system of writing numerals has a base ten. Starting at the units place, each place to the left has a value ten times as large as the place to its right. The first six places from the right to the left are shown below:

hundred thousand	ten thousand	thousand	hundred	ten	one
(10x10x10x10x10)	(10x10x10x10)	(10x10x10)	(10x10)	(10)	(1)

Often we write these values more briefly, by using a small numeral to the right and above the 10. This numeral shows how many 10's are multiplied together.

Numbers that are multipli.. together are called <u>factors</u>. In this way, the values of the places are written and read as follows:

$(10 \times 10 \times 10 \times 10 \times 10)$	10^5	"ten to the fifth power"
$(10 \times 10 \times 10 \times 10)$	10^4	"ten to the fourth power"
$(10 \times 10 \times 10)$	10^3	"ten to the third power"
(10×10)	10^2	"ten to the second power"
(10)	10^1	"ten to the first power"
(1)	1	"one"

In an expression as 10^2, the number 10 is called the <u>base</u> and the number 2 is called the <u>exponent</u>. The exponent tells how many times the base is taken as a factor in a product. 10^2 indicates (10×10) or 100. A number such as 10^2 is called a <u>power</u> of ten, and in this case it is the second power of ten. The exponent is sometimes omitted for the first power of ten; we usually write 10, instead of 10^1. All other exponents are always written. Another way to write $(4 \times 4 \times 4)$ is 4^3, where 4 is the base, and 3 is the exponent.

How can we write the meaning of "352" with exponents?

$$352 = (3 \times 10 \times 10) + (5 \times 10) + (2 \times 1)$$
$$= (3 \times 10^2) + (5 \times 10^1) + (2 \times 1).$$

This is called <u>expanded notation</u>. Writing numerals in expanded notation helps explain the meaning of the whole numeral.

<u>History</u>. Probably the reason that we use a numeration system with ten as a base is that people have ten fingers. This accounts for the fact that the ten symbols are called "digits" when they are used as numerals. We use the term "digits" when we wish to refer to the symbols apart from the longer numerals in which they are used.

The Celts, who lived in Europe more than 2000 years ago, used twenty as a base, and so did the Mayans in Central America. Can you think of a reason for this? What special name do we sometimes use for twenty? Some Eskimo tribes group and count by fives. Can you think of a good reason for this? Later we shall see how systems work when we use bases other than ten.

Exercises 3

1. Write the following in words: 10^1 as "ten to the first power," and so on up to 10^5.

2. Write each of the following using exponents.
   (a) $3 \times 3 \times 3 \times 3 \times 3$
   (b) $2 \times 2 \times 2 \times 2$
   (c) $6 \times 6 \times 6 \times 6 \times 6 \times 6$
   (d) $25 \times 25 \times 25$
   (e) $5 \times 5 \times 5 \times 5 \times 5 \times 5$
   (f) 4×4
   (g) $279 \times 279 \times 279 \times 279 \times 279$
   (h) 16

3. How many fives are used as factors in each of the following?
   (a) 5^3
   (b) 5^7
   (c) 5^2
   (d) 5^{10}
   (e) 5^1
   (f) 5^5

4. Write each of the following without exponents as $2^3 = 2 \times 2 \times 2$.
   (a) 4^3
   (b) 3^4
   (c) 2^8
   (d) 10^7
   (e) 33^5
   (f) 175^6

5. What does an exponent tell?

6. Write each of the following expressions as shown in the example:
   4^3 means $4 \times 4 \times 4 = 64$.
   (a) 3^3
   (b) 5^2
   (c) 4^4
   (d) 2^5
   (e) 6^2
   (f) 7^3
   (g) 8^2
   (h) 9^2
   (i) 10^3
   (j) 3^4
   (k) 2^6
   (l) 4^5

7. Which numeral represents the larger number?
   (a) 4^3 or 3^4
   (b) 2^9 or 9^2

8. Write the following numerals in expanded notation as shown in the example: $210 = (2 \times 10^2) + (1 \times 10^1) + (0 \times 1)$
   (a) 468
   (b) 5324
   (c) 7062
   (d) 59,126
   (e) 109,180

9. Complete the table shown below for the powers of ten, starting with 10^{10} and working down. The next expression will be 10^9, and so on. Write the numeral represented by each expression, and then write each numeral in words. Continue until you reach 10^1.

Power	Numeral	Words
10^{10}	10,000,000,000	ten billion
10^9	1,000,000,000	one billion
10^8	?	?

10. In the table of problem 9, what is the relation between the exponent of a power of 10 and the zeros when that number is expressed in decimal notation?

11. Write the following numerals with exponents:
    (a) 1000
    (b) 100,000
    (c) 1,000,000
    (d) one hundred million

12. A mathematician was talking to a group of arithmetic students one day. They talked about a large number which they decided to call a "googol." A googol is 1 followed by 100 zeros. Write this with exponents.

13. BRAINBUSTER. What is the meaning of 10^2? of 10^1? What do you think should be the meaning of 10^0?

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4. Numerals in Base Seven

You have known and used decimal numerals for a long time, and you may think you understand all about them. Some of their characteristics, however, may have escaped your notice simply because the numerals are familiar to you. In this section you will study a system of notation with a different base. This will increase your understanding of decimal numerals.

Suppose we found people living on Mars with seven fingers. Instead of counting by tens, a Martian might count by sevens. Let us see how to write numerals in base seven notation. This time we plan to work with groups of seven. Look at the x's below and notice how they are grouped in sevens with some x's left over.

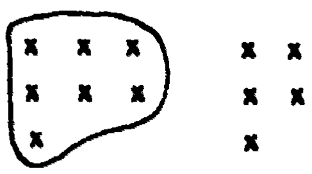


Figure 4-a.

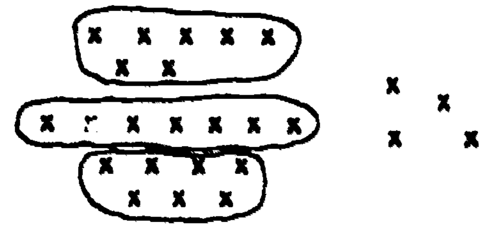


Figure 4-b.

In Figure 4-a we see one group of seven and five more. The numeral is written 15_{seven}. In this numeral, the 1 shows that there is one group of seven, and the 5 means that there are five ones.

In Figure 4-b, how many groups of seven are there? How many x's are left outside the groups of seven? The numeral representing this number of x's is 34_{seven}. The 3 stands for three groups of seven, and the 4 represents four single x's or four ones. The "lowered" seven merely shows that we are working in base seven.

When we group in sevens the number of individual objects left can only be zero, one, two, three, four, five, or six. Symbols are needed to represent those numbers. Suppose we use the familiar 0, 1, 2, 3, 4, 5 and 6 for these rather than invent new symbols. As you will discover, no other symbols are needed for the base seven system.

If the x's are marks for days, we may think of 15_{seven} as a way of writing 1 week and five days. In our decimal system we name this number of days "twelve" and write it "12" to show one group of ten and two more. We do not write the base name in our numerals since we all know what the base is.

We should not use the name "fifteen" for 15_{seven} because fifteen is 1 ten and 5 more. We shall simply read 15_{seven} as "one, five, base seven."

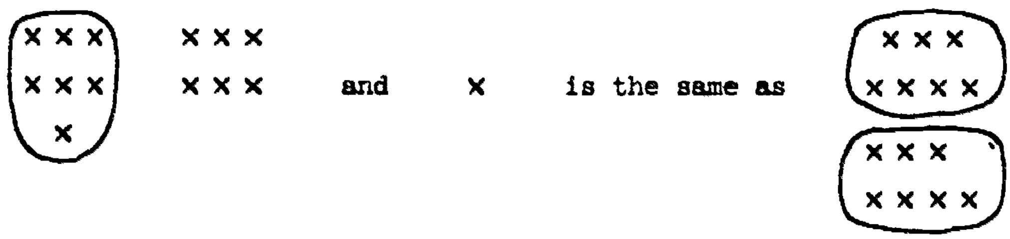
You know how to count in base ten and how to write the numerals in succession. Notice that one, two, three, four, five, six, seven, eight, and nine are represented by single symbols. How is the base number "ten" represented? This representation, 10, means one group of ten and zero more.

With this idea in mind, think about counting in base seven. Try it yourself and compare with the following table, filling in the numerals from 21_{seven} to 63_{seven}. In this table the "lowered" seven is omitted.

Counting in Base Seven

Number	Symbol	Number	Symbol
one	1	one, four	14
two	2	one, five	15
three	3	one, six	16
four	4	two, zero	20
five	5	two, one	21
six	6	----------	--
one, zero	10	six, three	63
one, one	11	six, four	64
one, two	12	six, five	65
one, three	13	six, six	66

How did you get the numeral following 16_{seven}? You probably thought something like this:



which is 2 groups of seven x's and 0 x's left over.

What would the next numeral after 66_{seven} be? Here you would have 6 sevens and 6 ones plus another one. This equals 6 sevens and another seven, that is, seven sevens. How could we represent $(\text{seven})^2$ without using a new symbol? We introduce a new group, the $(\text{seven})^2$ group. This number would then be represented by 100_{seven}. What does the numeral really mean? Go on from this point and write a few more numerals. What would be the next numeral after 666_{seven}?

Now you are ready to write a list of place values for base seven. Can you do this for yourself by studying the decimal place values on page 8 and thinking about the meaning of 100_{seven}?

Place Values in Base Seven

$(\text{seven})^5$	$(\text{seven})^4$	$(\text{seven})^3$	$(\text{seven})^2$	$(\text{seven})^1$	(one)

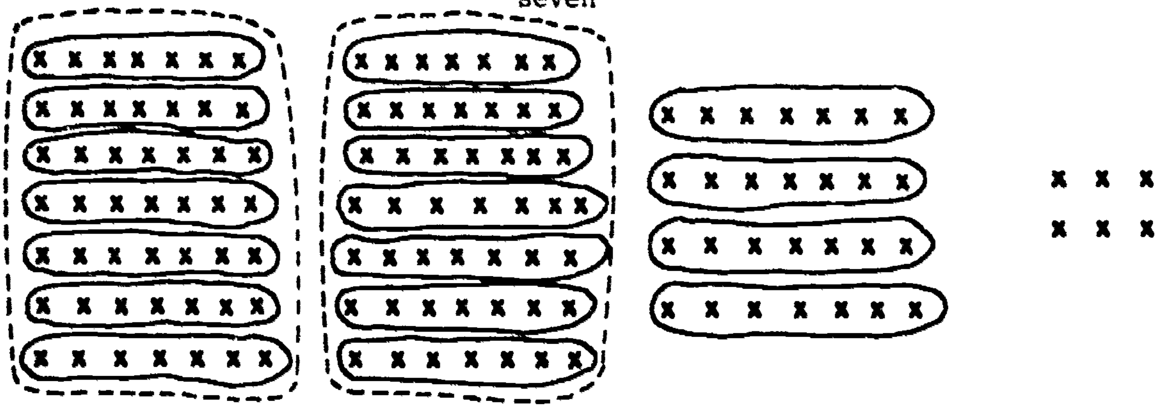
Notice that each place represents seven times the value of the next place to the right. The first place on the right is the <u>one place</u> in both the decimal and the seven systems. The value of the second place is the base times one. In this case what is it? The value of the third place from the right is (seven × seven), and of the next place (seven × seven × seven).

What is the decimal name for (seven × seven)? We need to use this (forty-nine) when we change from base seven to base ten. Show that the decimal numeral for $(\text{seven})^3$ is 343. What is the decimal numeral for $(\text{seven})^4$?

Using the chart above, we see that

$246_{\text{seven}} = (2 \times \text{seven} \times \text{seven}) + (4 \times \text{seven}) + (6 \times \text{one})$.

The diagram shows the actual grouping represented by the digits and the place values in the numeral 246_{seven}:



(2 × seven × seven) (4 × seven) (6 × one)

If we wish to write the number of x's above in the decimal system of notation we may write:

$$\begin{aligned} 246_{\text{seven}} &= (2 \times 7 \times 7) + (4 \times 7) + (6 \times 1) \\ &= (2 \times 49) \quad + (4 \times 7) + (6 \times 1) \\ &= \quad 98 \quad + \quad 28 \quad + \quad 6 \\ &= 132_{\text{ten}} \end{aligned}$$

Regroup the x's above to show that there are 1 (ten × ten) group, 3 (ten) groups, and 2 more. This should help you understand the $246_{\text{seven}} = 132_{\text{ten}}$.

Exercises 4

1. Group the x's below and express the number of x's in base seven notation:

(a)
× × × × ×
× × × × ×

(b)
× × × × ×
× × × × ×
× × × ×
× × × ×

(c)
× × × × × × × × ×
× × × × × × × × ×
× × × × × × × × ×
× × × × × × × × ×
× × × × × × × × ×
× × × × × × × × ×
× × × × × × × ×

2. Draw x's and group them to show the meaning of the following numerals.

(a) 11_{seven} (c) 35_{seven}

(b) 26_{seven} (d) 101_{seven}

3. Write each of the following numerals in expanded form and then in decimal notation.

(a) 33_{seven} (c) 100_{seven}

(b) 45_{seven} (d) 524_{seven}

4. Write the next consecutive numeral after each of the following numerals.

(a) 6_{seven} (c) 54_{seven} (e) 666_{seven}

(b) 10_{seven} (d) 162_{seven} (f) 1006_{seven}

5. What is the value of the "6" in each of the following numerals?

(a) 560_{seven} (c) 605_{seven}

(b) 56_{seven} (d) 6050_{seven}

6. In the base seven system write the value of the fifth place counting left from the units place.

7. In the base seven system, what is the value of the tenth place from the right?

* 8. What numeral in the seven system represents the number named by six dozen?

9. Which number is greater, 452_{seven} or 432_{seven}?

10. Which number is greater, 250_{seven} or 205_{ten}?

11. Which is less, 2125_{seven} or 754_{ten}?

12. A number is divisible by ten if a remainder of zero is obtained when the number is divided by ten.

(a) Is 30_{ten} divisible by ten? Why?

(b) Is 241_{ten} divisible by ten? Why?

(c) How can you tell by glancing at a base ten numeral whether the number is divisible by ten?

*13. Is 30_{seven} divisible by ten? Explain how you arrived at your answer.

Is 60_{seven} divisible by ten?

* 14. (a) What would the phrase "a counting number is divisible by seven" mean?

(b) Is 30_{seven} divisible by seven? Why?

* 15. Is 31_{seven} divisible by seven? Explain your answer.

* 16. State a rule for determining when a number written in base seven is divisible by seven.

You should see from problems 11-15 that the way we determine whether a number is divisible by ten depends on the system in which it is written. The rule for divisibility by ten in the decimal system is similar to the rule for divisibility by seven in the base seven system.

17. Which of the numbers 24_{ten}, 31_{ten}, and 68_{ten} are divisible by two? How do you tell? What do you call a number which is divisible by two? What do you call a number not divisible by two?

* 18. Is 11_{seven} an even number or an odd number? Can you tell simply by glancing at the following which represent even or odd numbers?

$$12_{seven},\ 13_{seven},\ 14_{seven},\ 25_{seven},\ 66_{seven}$$

What could you do to tell?

Here again a rule for divisibility in base ten will not work for base seven. Rules for divisibility seem to depend on the base with which we are working.

19. BRAINBUSTER. On planet X-101 the pages in books are numbered in order as follows: 1, ∠, ∠ ⌋, ⧄, ⊠, 1-, 11, 1∠, 1△, 1□, 1⧄, 1⊠, ∠-, ∠1, and so forth. What seems to be the base of the numeration system these people use? Why? How would the numeral after ∠1 be written? Which symbol corresponds to our zero? Write numerals for numbers from □- to ⧄△.

20. BRAINBUSTER. Find a rule for determining when a number expressed in base seven is divisible by two.

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5. Computation in Base Seven

Addition. In the decimal, or base ten, system there are 100 "basic" addition combinations. By this time you know all of them. The combinations can be arranged in a convenient table. Part of the table is given below:

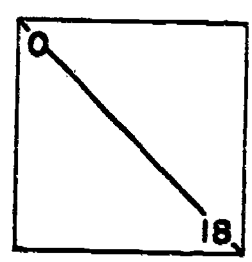
Addition, Base Ten

+	0	1	2	3	4	5	6	7	8	9
0	0									
1	1	2								
2				5						
3	3	4	5	6						
4	4	5	6	7	8					
5	5	6	7	8	9	10	11			
6										
7										
8										
9									17	

The numbers represented in the horizontal row above the line at the top of the table are added to the numbers in the vertical row under the "+" sign at the left. The sum of each pair of numbers is written in the table. The sum 2 + 3 is 5, as pointed out by the arrows.

Exercises 5-a

1. Find the sums.
   (a) 6 + 5 (b) 9 + 8
2. Use cross ruled paper and complete the addition table above (you will use it later).
3. Draw a diagonal line from the upper left corner to the lower right corner of the chart as shown at the right.



   (a) Is 3 + 4 the same as 4 + 3?
   (b) How could the answer to part (a) be determined from the chart?
   (c) What do you notice about the two parts of the chart?
   (d) What does this tell you about the number of different combinations which must be mastered? Be sure you can recall any of these combinations whenever you need them.

4. Make a chart to show the basic sums when the numbers are written in base seven notation. Four sums are supplied to help you.

+	0	1	2	3	4	5	6
0							
1				4			
2					6		
3						11	
4							13
5							
6							

5. (a) How many different number combinations are there in the base seven table? Why?

   (b) Which would be easier, to learn the necessary addition combinations in base seven or in base ten? Why?

   (c) Find $4_{ten} + 5_{ten}$ and $4_{seven} + 5_{seven}$ from the tables. Are the results equal; that is, do they represent the same number?

The answer to problem 5c is an illustration of the fact that a number is an idea independent of the numerals used to write its name. Actually, 9_{ten} and 12_{seven} are two different names for the same number.

Do not try to memorize the addition combinations for base seven. The value in making the table lies in the help it gives you in understanding operations with numbers.

The table that you completed in problem 4 of the last set of exercises shows the sums of pairs of numbers from zero to six. Actually, little more is needed to enable us to add larger numbers. In order to see what else is needed, let us consider how we add in base ten. What are the steps in your thinking when you add numbers like twenty-five and forty-eight in the decimal notation?

$$\begin{array}{rcll} 25 &=& 2 \text{ tens} + 5 \text{ ones} = \longrightarrow & 25 \\ \underline{48} &=& \underline{4 \text{ tens} + 8 \text{ ones}} = \longrightarrow & \underline{48} \\ & & 6 \text{ tens} + 13 \text{ ones} = 7 \text{ tens} + 3 \text{ ones} & = 73 \end{array}$$

Try adding in base seven: $14_{seven} + 35_{seven}$

$$\begin{array}{l} 1 \text{ seven} + 4 \text{ ones} \\ \underline{3 \text{ sevens} + 5 \text{ ones}} \\ 4 \text{ sevens} + 12 \text{ ones} = 5 \text{ sevens} + 2 \text{ ones} = 52_{seven}. \end{array}$$

(You may look up the sums 5 + 4 and 3 + 1 in the base seven addition table.)

How are the two examples alike? How are they different? When is it necessary to "carry" (or regroup) in the ten system? When is it necessary to "carry" (or regroup) in the seven system?

Try your skill in addition on the following problems. Use the addition table for the basic sums.

42_{seven}	65_{seven}	32_{seven}	254_{seven}	435_{seven}	524_{seven}
13_{seven}	11_{seven}	25_{seven}	105_{seven}	625_{seven}	564_{seven}

The answers in order are 55_{seven}, 106_{seven}, 60_{seven}, 362_{seven}, 1363_{seven}, and 1421_{seven}.

Subtraction. How did you learn to subtract in base ten? You probably used subtraction combinations such as 14 - 5 until you were thoroughly familiar with them. You know the answer to this problem but suppose, for the moment, that you did not. Could you get the answer from the addition table? You really want to ask the following question "What is the number which, when added to 5, yields 14?" Since the fifth row of the base ten addition table gives the results of adding various numbers to 5, we should look for 14 in that row. Where do you find the answer to 14 - 5? Did you answer "the last column"? Use the base ten addition table to find

9 - 2, 8 - 5, 12 - 7, 17 - 9.

The idea discussed above is used in every subtraction problem. One other idea is needed in many problems, the idea of "borrowing" or "regrouping." This last idea is illustrated below for base ten to find 761 - 283:

7 hundreds + 6 tens + 1 one	=	6 hundreds + 15 tens + 11 ones	= 761
2 hundreds + 8 tens + 3 ones	=	2 hundreds + 8 tens + 3 ones	= 283
		4 hundreds + 7 tens + 8 ones	= 478

Now let us try subtraction in base seven. How would you find $6_{seven} - 2_{seven}$? Find $13_{seven} - 6_{seven}$. How did you use the addition table for base seven? Find answers to the following subtraction examples:

15_{seven}	12_{seven}	11_{seven}	14_{seven}	13_{seven}
6_{seven}	4_{seven}	6_{seven}	5_{seven}	4_{seven}

The answers to these problems are 6_{seven}, 5_{seven}, 2_{seven}, 6_{seven}, and 6_{seven}.

Let us work a harder subtraction problem in base seven comparing the procedure with that used above:

$$\begin{array}{rcl} 43_{seven} & = 4 \text{ sevens} + 3 \text{ ones} = & 3 \text{ sevens} + 13 \text{ ones} = 43_{seven} \\ \underline{16_{seven}} & = \underline{1 \text{ seven} + 6 \text{ ones}} = & \underline{1 \text{ seven} + 6 \text{ ones}} = \underline{16_{seven}} \\ & & 2 \text{ sevens} + 4 \text{ ones} = 24_{seven} \end{array}$$

Be sure to note that "13 ones" above is in the seven system and is "one seven, three ones." If you wish to find the number you add to 6_{seven} to get 13_{seven}, how can you use the table to help you? Some of you may think of the number without referring to the table.

Practice on these subtraction examples:

$$\begin{array}{ccccc} 56_{seven} & 61_{seven} & 34_{seven} & 452_{seven} & 503_{seven} \\ \underline{14_{seven}} & \underline{35_{seven}} & \underline{26_{seven}} & \underline{263_{seven}} & \underline{140_{seven}} \end{array}$$

The answers are 42_{seven}, 23_{seven}, 5_{seven}, 156_{seven} and 333_{seven}.

Exercises 5-b

1. Each of the following examples is written in base seven. Add. Check by changing the numerals to decimal notation and adding in base ten as in the example:

Base Seven		Base Ten
16_{seven}	=	13
$\underline{23_{seven}}$	=	$\underline{17}$
42_{seven}		30

Does $42_{seven} = 30$?

(a) $\begin{array}{r} 25_{seven} \\ \underline{31_{seven}} \end{array}$ (b) $\begin{array}{r} 56_{seven} \\ \underline{21_{seven}} \end{array}$ (c) $\begin{array}{r} 214_{seven} \\ \underline{53_{seven}} \end{array}$

(d) $160_{seven} + 430_{seven}$ (e) $45_{seven} + 163_{seven}$

(f) $403_{seven} + 563_{seven}$ (g) $645_{seven} + 605_{seven}$

(h) $6245_{seven} + 5314_{seven}$ (i) $6204_{seven} + 234_{seven}$

(j) $645_{seven} + 666_{seven}$ (k) $5406_{seven} + 6245_{seven}$

2. Use the base seven addition table to find:

(a) $6_{seven} - 4_{seven}$

(b) $11_{seven} - 4_{seven}$

(c) $12_{seven} - 5_{seven}$

3. Each of the following examples is written in base seven. Subtract. Check by changing to decimal numerals.

(a) $\begin{array}{r} 10_{seven} \\ \underline{5_{seven}} \end{array}$ (b) $\begin{array}{r} 65_{seven} \\ \underline{26_{seven}} \end{array}$ (c) $\begin{array}{r} 200_{seven} \\ \underline{4_{seven}} \end{array}$ (d) $\begin{array}{r} 160_{seven} \\ \underline{6_{seven}} \end{array}$

(e) $44_{seven} - 35_{seven}$

(f) $641_{seven} - 132_{seven}$

(g) $502_{seven} - 266_{seven}$

(h) $5000_{seven} - 4261_{seven}$

(i) $634_{seven} - 52_{seven}$

(j) $134_{seven} - 65_{seven}$

(k) $3451_{seven} - 2164_{seven}$

(l) $253_{seven} - 166_{seven}$

4. Show by grouping x's that:

(a) 4 twos = 11_{seven}

(b) 6 threes = 24_{seven}

(c) 3 fives = 21_{seven}

(d) 5 sixes = 42_{seven}

<u>Multiplication</u>. In order to multiply, we may use a table of basic facts. Complete the following table in decimal numerals and be sure you know and can recall instantly the product of any two numbers from zero to nine.

Multiplication, Base Ten

x	0	1	2	3	4	5	6	7	8	9
0	0	0								
1	0	1								
2			4	6						
3				9	12					
4										
5										
6										
7										
8										
9										

Exercises 5-c

1. Refer to the preceding table.
   (a) Explain the row of zeros and the column of zeros.
   (b) Which row in the table is exactly like the row at the top? Why?

2. Imagine a diagonal line drawn from the × sign in the table to the lower right corner. What can you say about the two triangular parts of the table on each side of the line?

3. Complete the multiplication table below for base seven. Suggestion: To find $4_{seven} \times 3_{seven}$ you could write three x's four times and regroup to show the base seven numeral. Better still, you might think of this as $3_{seven} + 3_{seven} + 3_{seven} + 3_{seven}$.

Multiplication Base Seven

×	0	1	2	3	4	5	6
0							
1							
2					11	13	
3							
4				15			
5							
6							51

4. There are fewer entries in the base seven table than in the table for base ten. What does this fact tell you about the ease of learning multiplication in base seven?

5. Imagine the diagonal line drawn from the "×" sign to the lower right-hand corner of the last table.
   (a) How are the entries above the diagonal line related to those below it?
   (b) What fact does the observation of part (a) tell you about $3_{seven} \times 4_{seven}$?

There is no value in memorizing the table for base seven. The value of this table lies in your understanding of it.

Multiply the following numbers in base ten numerals:

$$\begin{array}{r} 45 \\ \times\ \underline{63} \end{array} \qquad \begin{array}{r} 249 \\ \times\ \underline{75} \end{array} \qquad \begin{array}{r} 4627 \\ \times\ \underline{436} \end{array} \qquad \begin{array}{r} 7834 \\ \times\ \underline{89} \end{array} \qquad \begin{array}{r} 51043 \\ \times\ \underline{78} \end{array}$$

You know about carrying (or regrouping) in addition, and you have had experience in multiplication in base ten. Use the base seven multiplication table to find the following products.

$$\begin{array}{r} 52_{seven} \\ \times\ 3_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 34_{seven} \\ \times\ 6_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 421_{seven} \\ \times\ 4_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 621_{seven} \\ \times\ 2_{seven} \\ \hline \end{array} \qquad \begin{array}{r} 604_{seven} \\ \times\ 35_{seven} \\ \hline \end{array}$$

The answers are 216_{seven}, 303_{seven}, 2314_{seven}, 1542_{seven}, 31406_{seven}.

Check the multiplication shown at the right and then answer the following questions. How do you get the entry 123 on the third line? How do you get the entry 201 on the fourth line? Why is the 1 on line 4 placed under the 2 on line 3? Why is the 0 on line 4 placed under the 1 on line 3? If you do not know why the entries on lines 3 and 4 are added to get the answer, you will study this more thoroughly later.

$$\begin{array}{r} 45_{seven} \\ \times\ 32_{seven} \\ \hline 123\phantom{_{seven}} \\ 201\phantom{0_{seven}} \\ \hline 2133_{seven} \end{array}$$

One way to check your work is to change the base seven numerals to base ten numerals as shown here:

$$\begin{array}{lll} 604_{seven} & = (6 \times 49) + (0 \times 7) + (4) = 294 + 4 = & 298_{ten} \\ \times\ 35_{seven} & = (3 \times 7) + (5) \qquad\quad = 21 + 5 = & \times\ 26_{ten} \\ \hline 4226 & & 1788 \\ 2415 & & 596 \\ \hline 31406_{seven} & = (3 \times 2401) + (1 \times 343) + (4 \times 49) + (0 \times 7) + (6) = & 7748_{ten} \end{array}$$

Division. Division is left as an exercise for you. You may find that it is not easy. Working in base seven should help you understand why some boys and girls have trouble with division in base ten. Here are two examples you may wish to examine. All the numerals within the examples are written in base seven. How can you use the multiplication table here?

Division in Base Seven

$$6_{seven} \overline{)\ 4053_{seven}} \quad \text{quotient } 454_{seven}$$

$$\begin{array}{r} 33 \\ \hline 45 \\ 42 \\ \hline 33 \\ 33 \\ \hline \end{array}$$

$$46_{seven} \overline{)\ 126125_{seven}} \quad \text{quotient } 2015_{seven}$$

$$\begin{array}{r} 125 \\ \hline 112 \\ 46 \\ \hline 335 \\ 332 \\ \hline 3 \end{array}$$

Exercises 5-d

1. Multiply the following numbers in base seven numerals and check your results in base 10.

   (a) $14_{seven} \times 3_{seven}$

   (b) $6_{seven} \times 25_{seven}$

   (c) $63_{seven} \times 12_{seven}$

   (d) $5_{seven} \times 461_{seven}$

   (e) $56_{seven} \times 43_{seven}$

   (f) $654_{seven} \times 453_{seven}$

   (g) $3046_{seven} \times 24_{seven}$

   (h) $5643_{seven} \times 652_{seven}$

   (i) $250_{seven} \times 341_{seven}$

   (j) $26403_{seven} \times 45_{seven}$

* 2. Divide. All numerals in this exercise are in base seven.

   (a) $6_{seven}\overline{)\,42_{seven}}$

   (b) $5_{seven}\overline{)\,433_{seven}}$

   (c) $4_{seven}\overline{)\,2316_{seven}}$

   (d) $21_{seven}\overline{)\,2625_{seven}}$

3. Write in expanded form:

   (a) 403_{seven}

   (b) 189_{ten}

4. Which of the numerals in Exercise 3 represents the larger number?

5. Add the following:

   (a) $52_{seven} + 14_{seven}$

   (b) 65_{seven}
$\underline{25}_{seven}$

   (c) 434_{seven}
$\underline{324}_{seven}$

   (d) $601_{seven} + 304_{seven}$

6. Subtract the following:

   (a) 13_{seven}
$\underline{6}_{seven}$

   (b) $30_{seven} - 1_{seven}$

   (c) 402_{seven}
$\underline{35}_{seven}$

7. Rewrite the following paragraph replacing the base seven numerals with base ten numerals.

   Louise takes grade 10_{seven} mathematics in room 234_{seven}. The book she uses is called Junior High School Mathematics 10_{seven}. It has 21_{seven} chapters and 1102_{seven} pages. There are 44_{seven} pupils in the class which meets 5_{seven} times each week for 106_{seven} minutes daily. 16_{seven} of the pupils are girls and 25_{seven} are boys. The youngest pupil in the class is 14_{seven} years old and the tallest is 123_{seven} inches tall.

6. <u>Changing from Base Ten to Base Seven</u>

You have learned how to change a number written in base seven numerals to base ten numerals. It is also easy to change from base ten to base seven. Let us see how this is done.

In base seven, the values of the places are: one, $seven^1$, $seven^2$, $seven^3$, and so on. That is, the place values are <u>one</u> and <u>the powers of seven</u>.

$$seven^1 = 7_{ten}$$
$$seven^2 = (7 \times 7) \text{ or } 49_{ten}$$
$$seven^3 = (7 \times 7 \times 7) \text{ or } 343_{ten}$$

Suppose you wished to change 12_{ten} to base seven numerals. This time we shall <u>think</u> of groups of powers of seven instead of actually grouping marks. What is the largest power of seven which is containe` in 12_{ten}? Is $seven^1$ the largest? How about $seven^2$ (forty-nine) or $seven^3$ (three hundred forty-three)? We can see that only $seven^1$ is small enough to be contained in 12_{ten}.

When we divide 12 by 7 we have

$$\begin{array}{r} 1 \\ 7\overline{)12} \\ \underline{7} \\ 5 \end{array}.$$

What does the 1 on top mean? What does the 5 mean? They tell us that 12_{ten} contains 1 seven with 5 units left over, or that $12_{ten} = (1 \times seven) + (5 \times one)$. Thus $12_{ten} = 15_{seven}$.

Be sure you know which place in a base seven numeral has the value $seven^2$, the value $seven^3$, the value $seven^4$, and so on.

How is 54_{ten} regrouped for base seven numerals? What is the largest power of seven which is contained in 54_{ten}?

In 54_{ten} we have __?__ $\times\ seven^2$ + __?__ $\times$ seven + __?__ $\times$ one.

$$\begin{array}{r} 1 \\ 49\overline{)54} \\ \underline{49} \\ 5 \end{array}$$

We have (__1__ $\times\ seven^2$) + (0 $\times$ seven) + (__5__ $\times$ one).
Then $54_{ten} = 105_{seven}$.

Suppose the problem is to change 524_{ten} to base seven numerals. Since 524_{ten} is larger than 343 ($seven^3$), find how many 343's there are.

$$\begin{array}{r} 1 \\ 343\overline{)524} \\ \underline{343} \\ 181 \end{array}$$

Thus 524 contains one $seven^3$ with 181 remaining, or $524 = (1 \times seven^3) + 181$, and there will be a "1" in the $seven^3$ place.

Now find how many 49's ($seven^2$) there are in the remaining 181.

$$\begin{array}{r} 3 \\ 49\overline{)181} \\ \underline{147} \\ 34 \end{array}$$

Thus 181 contains three 49's with 34 remaining or $181 = (3 \times seven^2) + 34$, and there will be a "3" in the $seven^2$ place.

How many sevens are there in the remaining 34?

$$\begin{array}{r} 4 \\ 7\overline{)34} \\ \underline{28} \\ 6 \end{array}$$

Thus 34 contains 4 seven's with 6 remaining, or $34 = (4 \times seven) + 6$, and there will be a "4" in the sevens place.

What will be in the units place? We have:

$524_{ten} = (1 \times seven^3) + (3 \times seven^2) + (4 \times seven) + (6 \times one)$

$524_{ten} = 1346_{seven}$.

Cover the answers below until you have made the changes for yourself.

$10_{ten} = (1 \times seven) + (3 \times one) = 13_{seven}$

$46_{ten} = (6 \times seven) + (4 \times one) = 64_{seven}$

$162_{ten} = (3 \times seven^2) + (2 \times seven) + (1 \times one) = 321_{seven}$

$1738_{ten} = (5 \times seven^3) + (0 \times seven^2) + (3 \times seven) + (2 \times one)$
$= 5032_{seven}$.

In changing base ten numerals to base seven we first select the <u>largest</u> place value of base seven (that is, power of seven) contained in the number. We divide the number by this power of seven and find the quotient and remainder. The <u>quotient</u> is the first digit in the base seven numeral. We divide the remainder by the next smaller power of seven and this quotient is the second digit. We continue to divide remainders by each succeeding, smaller power of seven to determine all the remaining digits in the base seven numeral.

Exercises 6

1. Show that:

   (a) $50_{ten} = 101_{seven}$ (c) $1024_{ten} = 2662_{seven}$

   (b) $145_{ten} = 265_{seven}$

2. Change the following base ten numerals to base seven numerals:

   (a) 12 (c) 44 (e) 218

   (b) 36 (d) 53 (f) 1320

Problems 3, 4, and 5 will help you discover another method for changing base ten numerals to base seven.

3. Divide 1958_{ten} by ten. What is the quotient? What is the remainder? Divide the quotient by ten. What is the new quotient? The new remainder? Continue in the same way, dividing each quotient by ten until you get a quotient of zero. How are the successive remainders related to the original number? Try the same process with $123{,}456{,}789_{ten}$. Try it with any other number.

4. Divide 524_{ten} by seven. What is the quotient? The remainder? Divide the quotient by seven and continue as in Exercise 3, except that this time divide by seven instead of ten. Now write 524_{ten} as a base seven numeral and compare this with the remainders which you have obtained.

5. Can you now discover another method for changing from base ten to base seven numerals?

6. In each of the examples below there are some missing numerals. Supply the numerals which will make the examples correct. Remember that if no base name is given, then the base is ten.

   (a) Addition:
   $$\begin{array}{r} 675 \\ 486 \\ \hline ??? \end{array}$$

   (b) Addition:
   $$\begin{array}{r} 894 \\ ??? \\ \hline 1169 \end{array}$$

   (c) Addition:
   $$\begin{array}{r} 432_{seven} \\ ??? \\ \hline 1416_{seven} \end{array}$$

   (d) Addition:
   $$\begin{array}{r} 2305_{seven} \\ ??? \\ \hline 3100_{seven} \end{array}$$

   (e) Addition:
   $$\begin{array}{r} 264_{seven} \\ 352_{seven} \\ 140_{seven} \\ \hline \end{array}$$

   (f) Multiplication:
   $$\begin{array}{r} 514_{seven} \\ \times \quad ? \\ \hline 2145_{seven} \end{array}$$

   (g) Multiplication:
   $$\begin{array}{r} ??? \\ \times\ 54_{seven} \\ \hline 36201_{seven} \end{array}$$

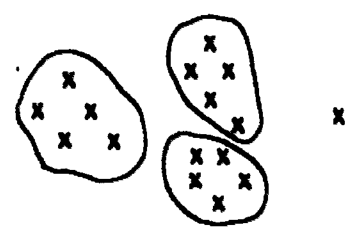
7. Numerals in Other Bases

You have studied base seven numerals, so you now know that it is possible to express numbers in systems different from the decimal scale. Many persons think that the decimal system is used because the base ten is superior to other bases, or because the number ten has special properties. Earlier it was indicated that we probably use ten as a base because man has ten fingers. It was only natural for primitive people to count by making comparisons with their fingers. If man had had six or eight fingers, he might have learned to count by sixes or eights.

Our familiar decimal system of notation is superior to the Egyptian, Babylonian, and others because it uses the idea of place value and has a zero symbol, not because its base is ten. The Egyptian system was a tens system, but it lacked efficiency for other reasons.

Bases Five and Six. Our decimal system uses ten symbols. In the seven system you used only seven symbols, 0, 1, 2, 3, 4, 5, and 6. How many symbols would Eskimos use counting in base five? How many symbols would base six require? A little thought on the preceding questions should lead you to the correct answers. Can you suggest how many symbols are needed for base twenty?

The x's at the right are grouped in sets of five. How many groups of five are there? How many ones are left?



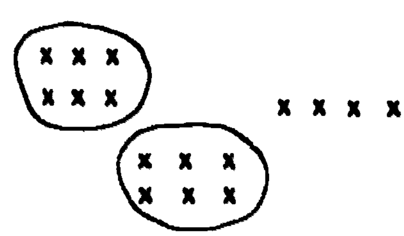
The decimal numeral for the number of x's in this diagram is 16. Using the symbols 0, 1, 2, 3, and 4, how would 16_{ten} be represented in base five numerals? An Eskimo, counting in base five, would think:

there are 3 groups of five and 1 more,

$$16_{ten} = (3 \times \text{five}) + (1 \times \text{one})$$

$$16_{ten} = 31_{five}.$$

In the drawing at the right sixteen x's are grouped by sixes. How many How many groups of six are there? Are there any x's left? How would you write 16_{ten} in base six numerals?



There are 2 groups of six and 4 more,

$$16_{ten} = (2 \times \text{six}) + (4 \times \text{one})$$

$$16_{ten} = 24_{six}.$$

Write sixteen x's. Enclose them in groups of four x's. Can you write the numeral 16_{ten} in base four numerals? How many groups of four are there? Remember, you cannot use the symbol "4" in base four. A table of the powers of four in decimal numerals is shown below.

($four^3$)	($four^2$)	($four^1$)	(one)
(4 x 4 x 4)	(4 x 4)	(4)	(1)
(64)	(16)	(4)	(1)

To express sixteen in base four we need

(1 group of $four^2$) + (0 groups of four) + (0 ones). That is,

$$16_{ten} = 100_{four}.$$

Exercises 7

1. Draw sixteen x's. Group the x's in sets of three.
   (a) There are ______ groups of three and ______ left over.
   (b) Are your answers to part (a) both digits in the base three system? Why not?
   (c) In sixteen x's there are (______ groups of $three^2$) + (______ groups of three) + (______ left over).
   (d) 16_{ten} = ______ three.

2. Draw groups of as many x's as indicated by the numbers represented below. Then write the decimal numerals for these numbers.
   (a) 23_{four}
   (b) 15_{six}
   (c) 102_{three}
   (d) 21_{five}

3. In base five notation represent the numbers from one through thirty. Start a table as shown below:

Base ten	0	1	2	3	4	5	6	7
Base five	0	1	?	?	?	?	?	?

4. (a) How many threes are there in 20_{three}?
   (b) How many fours are there in 20_{four}?
   (c) How many fives are there in 20_{five}?
   (d) How many sixes are there in 20_{six}?

5. Write the following in expanded notation. Then write the base ten numeral for each as shown in the example.

Example: $102_{five} = (1 \times 25) + (0 \times 5) + (2 \times 1) = 27$

(a) 245_{six}

(b) 412_{five}

(c) 1002_{three}

(d) 1021_{four}

6. Write the following decimal numerals in bases six, five, four, and three. Remember the values of the powers for each of these bases. Note the example:

$$7_{ten} = 11_{six} = 12_{five} = 13_{four} = 21_{three}.$$

(a) 11_{ten}

(b) 15_{ten}

(c) 28_{ten}

(d) 36_{ten}

7. What is the smallest whole number which can be used as a base for a system of notation?

* 8. Do the following computations:

(a) Add: $132_{four} + 211_{four}$

(b) Add: $15_{six} + 231_{six} + 420_{six}$

(c) Subtract: $1211_{three} - 202_{three}$

(d) Subtract: $1423_{five} - 444_{five}$

(e) Multiply: $13_{four} \times 3_{four}$

(f) Multiply: $121_{six} \times 5_{six}$

(g) $4_{six}\overline{)452_{six}}$

(h) $2_{three}\overline{)121_{three}}$

9. BRAINBUSTER. Make up a place value system where the following symbols are used:

Symbol	Decimal Value	Name
0	0	do
1	1	re
Λ	2	mi
≥	3	fa
10	4	re do

Write the numerals for numbers from zero to twenty in this system. Write the names in words using "do, re," etc.

10. BRAINBUSTER. Using the symbols and scale from the first Brainbuster, complete the addition and multiplication tables shown below.

+	0	1	∧	≥
0				
1				
∧				
≥				

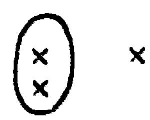
×	0	1	∧	≥
0				
1				
∧				
≥				

8. The Binary and Duodecimal Systems

There are two other bases of special interest. The base two, or binary, system is used by some modern, high speed computing machines. These computers, sometimes incorrectly called "electronic brains," use the base two as we use base ten. The twelve, or duodecimal, system is considered by some people to be a better base for a system of notation than ten.

Binary System. Historians tell of primitive people who used the binary system. Some Australian tribes still count by pairs, "one, two, two and one, two twos, two twos and one," and so on.

The binary system groups by pairs as is done with the three x's at the right. How many groups of two are shown? How many single x's are left? Three x's means 1 group of two and 1 one. In binary notation the numeral 3_{ten} is written 11_{two}.



Counting in the binary system starts as follows:

Decimal numerals	1	2	3	4	5	6	7	8	9	10
Binary numerals	1	10	11	100	101	110	111	?	?	?

How many symbols are needed for base two numerals? Notice that the numeral 101_{two} represents the number of fingers on one hand. What does 111_{two} mean?

$$111_{two} = (1 \times two^2) + (1 \times two^1) + (1 \times one) = 4 + 2 + 1 = 7_{ten}.$$

How would you write 8_{ten} in binary notation? How would you write 10_{ten} in binary notation? Compare this numeral with 101_{two}.

Modern high speed computers are electrically operated. A simple electric switch has only two positions, open (on) or closed (off). Computers operate on this principle. Because there are only two positions for each place, the computers use the binary system of notation.

We will use the drawing at the right to represent a computer. The four circles represent four lights on a panel, and each light represents one place in the binary system. When the current is flowing the light is on, shown in Figure 8b as

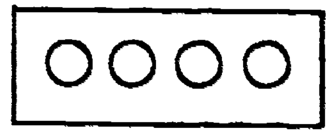


Figure 8a

◯. A ◯ is represented by the symbol "1". When the current does not flow, the light is off, shown by ◯

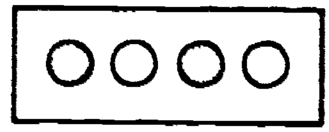


Figure 8b

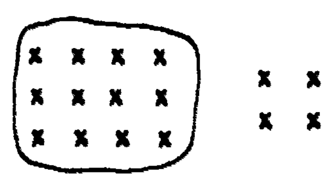
in Figure 8b. This is represented by the symbol "0". The panel in Figure 8b represents the binary numeral 1010_{two}. What decimal numeral corresponds to this numeral? The table at the right shows the place values for the first five places in base two numerals.

two^4	two^3	two^2	two^1	one
2x2x2x2	2x2x2	2x2	2	1
16	8	4	2	1

$$\begin{aligned} 1010_{two} &= (1 \times two^3) + (0 \times two^2) + (1 \times two^1) + (0 \times one) \\ &= (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1) \\ &= 10_{ten}. \end{aligned}$$

<u>Duodecimal</u> <u>System</u>. In the twelve, or duodecimal, system, we group by twelves. We frequently count in dozens, as with a dozen eggs, a dozen rolls, or a dozen pencils. Twelve dozen (12 × 12) is called a gross. Schools sometimes buy pencils by the gross.

The sixteen x's shown at the right are grouped as one group of twelve with four x's left. Written as a base twelve numeral,



$$16_{ten} = (1 \times twelve) + (4 \times one) = 14_{twelve}.$$

Draw twenty-five x's on a sheet of paper. Draw circles around groups of twelve. How many groups of twelve are there? Are any x's left over? How would you write 25_{ten} in duodecimal notation? Can you see why it is written 21_{twelve}?

$$25_{ten} = (2 \times twelve) + (1 \times one) = 21_{twelve}.$$

To write numerals in base twelve it is necessary to invent new symbols in addition to using the ten symbols from the decimal system. How many new symbols are needed? Base twelve requires twelve symbols, two more than the decimal system. We can use "T" for ten and "E" for eleven as shown in the table below:

Base ten	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Base twelve	0	1	2	3	4	5	6	7	8	9	T	E	10	11	12	?	?

Notice that "T" is another way of writing 10_{ten} and "E" is another way of writing 11_{ten}. Why is 12_{ten} written as 10_{twelve}? To write 195_{twelve} in expanded notation,

$$\begin{aligned}195_{twelve} &= (1 \times \text{twelve}^2) + (9 \times \text{twelve}^1) + (5 \times \text{one}) \\ &= (1 \times 144) + (9 \times 12) + (5 \times 1) \\ &= 257_{ten}.\end{aligned}$$

Exercises 8

1. Make a counting chart in base two for the numbers from zero to thirty-three.

Base Ten	1	2	3	4	. . .	33
Base Two	1	10	11		. . .	

2. Copy and complete the addition chart for base two shown at the right. How many addition facts are there?

Addition, Base Two

+	0	1
0		
1		

3. Using the same form as in Exercise 2, make a multiplication chart for base two. How many multiplication facts are there? How do the tables compare? Does this make working with the binary system difficult or easy? Explain your answer.

4. Write the following binary numerals in expanded notation and then in base ten notation.
   (a) 111_{two}
   (b) 1000_{two}
   (c) 10101_{two}
   (d) 11000_{two}
   (e) 10100_{two}

5. Write in duodecimal notation the three numbers following twenty-one.

6. To write $2T0_{twelve}$ in expanded notation, we have:

$$2T0_{twelve} = (2 \times twelve^2) + (T \times twelve) + (0 \times one).$$

To write $2T0_{twelve}$ as a decimal numeral we would first write the latter as

$$(2 \times 144) + (10 \times 12) + (0 \times 1).$$

What is the decimal numeral for $2T0_{twelve}$?

7. Write the following numerals in expanded notation and then in base ten notation.

(a) 111_{twelve}

(b) $3T2_{twelve}$

(c) $47E_{twelve}$

(d) $T0E_{twelve}$

8. Add these numbers which are expressed in binary notation. Check by expressing the numerals in the exercises, and in your answers, in decimal notation and adding the usual way.

(a) 101_{two}
$\underline{10}_{two}$

(b) 110_{two}
$\underline{101}_{two}$

(c) 10110_{two}
$\underline{11010}_{two}$

(d) 10111_{two}
$\underline{11111}_{two}$

9. Subtract these base two numbers. Check your answers as you did in problem 8.

(a) 111_{two}
$\underline{101}_{two}$

(b) 110_{two}
$\underline{11}_{two}$

(c) 1011_{two}
$\underline{100}_{two}$

(d) 11001_{two}
$\underline{10110}_{two}$

10. When people operate certain kinds of high speed computing machines, it is necessary to express numbers in the binary system. Change the following decimal numerals to base two notation:

(a) 35

(b) 128

(c) 12

(d) 100

11. Add and subtract the following duodecimal numerals. Check by expressing the numbers in decimal notation and adding and subtracting the usual way.

(a) 236_{twelve}
$\underline{T9}_{twelve}$

(b) $T32_{twelve}$
$\underline{193}_{twelve}$

12. What advantages and disadvantages, if any, do the binary and duodecimal systems have as compared with the decimal system?

13. Write the following in duodecimal notation.

(a) 425_{ten} (b) 524_{ten}

14. BRAINBUSTER. An inspector of weights and measures carries a set of weights which he uses to check the accuracy of scales. Various weights are placed on a scale to check accuracy in weighing any amount from 1 to 16 ounces. Several checks have to be made, because a scale which accurately measures 5 ounces may, for various reasons, be inaccurate for weighings of 11 ounces and more.

What is the smallest number of weights the inspector may have in his set, and what must their weights be, to check the accuracy of scales from 1 ounce to 15 ounces? From 1 ounce to 31 ounces?

15. BRAINBUSTER. People who work with high speed computers sometimes find it easier to express numbers in the octal, or base eight system rather than the binary system. Conversions from one system to the other can be done very quickly. Can you discover the method used?

Make a table of numerals as shown below:

Base ten	Base eight	Base two
1	1	1
2	2	10
5	5	101
7	?	?
15	?	?
16	?	?
32	?	?
64	?	?
256	?	?

Compare the powers of eight and two up to 256. Study the powers and the table above. $101,011,010_{two} = 532_{eight}$. Can you see why?

9. Summary

The decimal system has resulted from efforts of men over thousands of years to develop a workable system of notation (writing numerals). It is not a perfect system, but it has advantages other systems have not had. In this pamphlet you have studied some of the ancient systems, which, in their time, represented tremendous achievements in man's progress. You have also studied other systems in different bases. You have studied these numeration systems to gain a better understanding of your own system.

In learning about other systems of notation you have learned that a number may be expressed in different numerals. For example, twelve may be written as ∩||, XII, 12_{ten}, 15_{seven}, or 1100_{two}, and so on. These numerals are not the same, yet they represent the same number. The symbols we use are not in themselves numbers. "XII" is not twelve things, nor is "10_{twelve}." They are only different numerals, or symbols for twelve.

Sometimes we confuse numbers and numerals. A number is an idea while a numeral is a symbol for the idea. We may write "2" on the blackboard to represent a set of two objects, as two students, or two books. If we erase the "2" we remove the numeral, but we do not destroy the number. In the same way, the word "pencil" is not the same as the object you hold in your hand when you are writing on paper.

The Egyptians might not have known that their system of notation was based on ten. To know this they would have had to know that it is possible to use other bases for a number system. You know this now, and you know that it is possible to use any whole number greater than one as a base. Some of these numeration systems are used. The binary system is used by electric computers. You should understand that a high speed computer is not a "brain." Rather it is a high speed slave that does only what it is told to do. High speed calculations with computers are possible because the machines operate at the speed of the flow of electricity and use large "memories" of stored information. Man was able to invent modern high speed computers because he had invented the system of expressing numbers used in computer operation.

Exercises 9

1. Group twenty tally marks (////////////////////) to show place value for each of the number bases listed. Then write the numeral which represents twenty for each of the bases listed.

   (a) twelve (c) five

   (b) seven (d) two

2. The numerals shown below represent fifteen in various number bases. Supply the missing base for each numeral.

(a) $13_{\underline{\ ?\ }}$ (c) $30_{\underline{\ ?\ }}$

(b) $21_{\underline{\ ?\ }}$ (d) $1111_{\underline{\ ?\ }}$

3. Write the following in expanded notation and in decimal notation:

(a) 111_{two} (c) 2631_{seven}

(b) 321_{four} (d) $37T_{twelve}$

4. Write "one thousand" as a numeral in base eight and also in base two.

5. Write the next five numerals following 88_{nine}.

* 6. Babylonians used the symbols 𒁹 and 𒌋 for one and ten. By repeating these symbols they wrote fifty-nine as 𒌋𒌋𒌋𒌋𒌋 𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹

To write numbers larger than fifty-nine the Babylonians used the same symbols shown above, but they used the place value idea. Their number base was very large. It was sixty. As in our decimal system, the first place represented ones, so 𒌋𒌋𒌋𒌋𒌋 𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹 meant (59 × 1).

The second place had a value of sixty, so 𒌋𒁹𒁹 𒁹𒁹 meant

(𒌋𒁹𒁹 × sixty) + (𒁹𒁹 × one) = (12 × 60) + (2 × 1)

= 720 + 2

= 722

The first three place values in the Babylonian system were:

	sixty^2	sixty	one
Meaning in base ten	60 × 60	60	1

(a) The early Babylonians did not have a symbol for zero. They sometimes left empty spaces where we write zeros, as 𒁹 𒁹 . This meant the reader had to guess from the content of the reading whether the numeral 𒁹 𒁹 meant (𒁹 + 𒁹) × one

or (𒁹 × sixty) + (𒁹 × one)

or (𒁹 × sixty^2) + (𒁹 × one).

Write three decimal numerals for the Babylonian numeral 𒁹 𒁹 .

(b) Does the decimal numeral "11" have more than one possible meaning? Why?

(c) Find the decimal value for 𒁹𒁹 𒌋𒁹𒁹 𒌋𒁹

= (𒁹𒁹 × __?__) + (𒌋𒁹𒁹 × __?__) + (𒌋𒁹 × __?__)

= ? + ? + ?

= ?

(d) Write the decimal numerals 70, 111, and 4000 in Babylonian notation.

(e) Write 𒌋𒁹 𒌋𒁹+𒁹𒁹𒁹 𒌋𒌋 as a Babylonian numeral.

* 7. In Lincoln's Gettysburg Address, the term "four score and seven" is used. What number base did he use? Write the decimal numeral for this number.

* 8. A positional number system uses the symbols 0, A, B, C, and D to represent the numbers from zero to four. If these are the only symbols used in the system, write the decimal numeral for DCBA0.

* 9. A base ten number consists of three digits, 9, 5, and another in that order. If these digits are reversed and then subtracted from the original number, an answer will be obtained consisting of the same digits arranged in a different order still. What is that digit?

10. BRAINBUSTER. Suppose a place value number system uses the capital letters of the alphabet (A, B, C, . . . Y, Z) as symbols for numerals. The letter "O" is removed from its regular position between N and P and is used as the symbol for zero. What is the base for this system of notation? What decimal numeral is represented by "BE" in this base? by "TWO"? by "FOUR"?

11. BRAINBUSTER. There are a number of ways to change numerals written in other number bases to base ten notation. A student suggested this method:

Example A: To change 46_{twelve} to base ten notation. Because there are 2 more symbols in base twelve, multiply (2×4) and add the result to 46_{ten}.

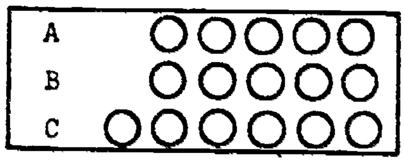
Does this method work for 46_{twelve}? Does it work for any two digit number written in base twelve?

Example B: To change 46_{seven} to base ten notation. Because there are three fewer symbols, multiply (3×4) and subtract from 46_{ten}. Does the method work for 46_{seven}? Does it work for any two digit number written in base seven?

4

Appendix I

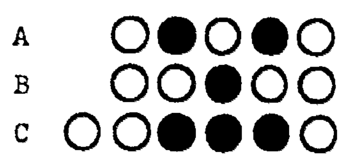
Suppose we wish to design a machine to add any two numbers which can be written in the binary scale with no more than 5 digits. We will need three rows of lights.



The numbers to be added will be entered into the machine by turning on appropriate lights in Rows A and B. The sum is to be shown in Row C. The switches which operate the lights in Row C are to be wired in such a way that they are activated by the lights in Rows A and B. Our task is to describe just when a light in Row C is to be on and when it is to be off.

Consider a few simple addition problems, and observe how they appear both in binary numerals and in lights. Suppose we wish to add 1010_{two} and 100_{two}.

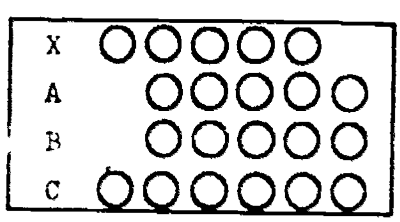
$$\begin{array}{r} 1010_{two} \\ 100_{two} \\ \hline 1110_{two} \end{array}$$



From this problem we can recognize two requirements for the wiring of our machine.

1. In a given column, if the lights in both Rows A and B are off then the light in that column in Row C should also be off.
2. In a given column, if <u>one</u> of the two lights in Rows A and B is on then the light in that column in Row C should turn on.

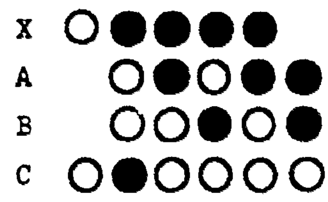
Now consider what happens when we add 1011_{two} and 101_{two}. Here we need to "carry," and our machine is not equipped to do this. It doesn't know what to do if <u>two</u> lights in a given column are on. We need one more row of lights; a "carrying row." The machine must look like this:



with Row X used only for carrying.

The second addition looks like this:

$$\begin{array}{r} 1011_{two} \\ \underline{101_{two}} \\ 10000_{two} \end{array}$$

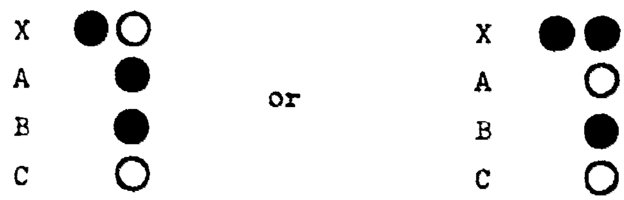


When we turn on lights in Rows A and B as shown, the indicated lights in Rows C and X must turn on automatically. Thus we see a third requirement for our wiring.

3. If, in a given column, any two lights in Rows A, B, or X are on then the light in Row C will remain off and the light in Row X in the column immediately to the left will turn on.

Requirement 3 says that if we have

then we must have



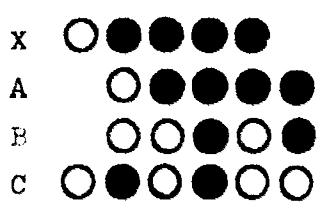
Finally, we have

4. If, in a given column, all the lights in Rows A, B, and X are on, then the light in Row C must go on and also the light in Row X of the column immediately to the left is to go on.

The addition

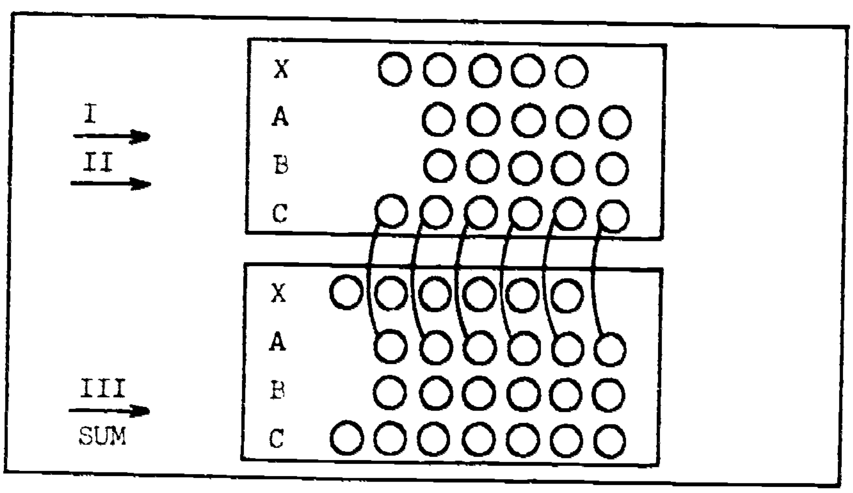
$$\begin{array}{r} 1111_{two} \\ \underline{101_{two}} \\ 10100_{two} \end{array}$$

would appear on the machine as



and the sum 10100_{two} would appear in Row C of lights almost as soon as we had entered the numbers 1111_{two} and 101_{two} in Rows A and B. We have designed an adding machine in the sense that requirements 1, 2, 3 and 4 <u>can</u> be built into a box with the required rows of lights by means of switches and relays. The important thing at this stage is not the electrical problem itself, but the knowledge that the binary system of notation makes it possible to reduce computational problems to purely mechanical and electrical construction in this rather simple way.

As an extension of the discussion, you might ask how we might use <u>two</u> of these adding machines to build a new machine for adding three numbers together. If we use the notation  to indicate two lights wired so that they are always lighted or unlighted together, then such a machine could be diagrammed as follows:



If each of the smaller boxes within the larger box is an adding machine of the type described above then the large box is a computer which will add triples of numbers. If the binary numerals of the numbers are entered in Rows I, II and III, the sum appears in the bottom Row C.

The use of binary notation in high speed computers is, of course, well known. The binary system is used for computers since there are only two digits, and an electric mechanism is either "on" or "off." Such an arrangement is called a flip-flop mechanism. A number of pamphlets distributed by IBM, Remington Rand, and similar sources may be obtained by request and used for supplementary reading and study. "<u>Yes</u> <u>No</u> - <u>One</u> <u>Zero</u>" published by Esso Standard Oil Co., 15 West 51st., New York 19, New York is available for the asking only in states served by Esso.

Appendix II

It should be of interest that the sum 11001 + 110 looks the same in the binary system, decimal system, and, in fact, all positional number systems. The meaning, however, is quite different.

The base two has the disadvantage that, while only two different digits are used, many more places are needed to express numbers in binary notation than in decimal, e.g.,

$$2000_{ten} = 11{,}111{,}010{,}000_{two}$$

Readers may be interested in the remainder method for changing a number from one base to another. This method of changing 25_{ten} to binary notation rests on repeated division by 2, to identify the powers of 2 whose sum is 25. To change 25_{ten} to base three, repeat division by 3; to base four, divide by 4; and so on.

The division is shown below for changing 25_{ten} to binary notation, followed by an interpretation of the results of the division at each stage. It will be noted that the remainders in reverse order indicate the digits in binary numerals. Recall that $2^0 = 1$.

2)25	$25 = 25 \times 2^0$
2)12 , R 1	$25 = 12 \times 2^1 + 1 \times 2^0$
2)6 , R 0	$25 = 6 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
2)3 , R 0	$25 = 3 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
2)1 , R 1	$25 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
2)0 , R 1	$25 = 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

Appendix III

Here is a set of cards which can be used in a number trick.

1			
1	9	17	25
3	11	19	27
5	13	21	29
7	15	23	31

2			
2	10	18	26
3	11	19	27
6	14	22	30
7	15	23	31

4			
4	12	20	28
5	13	21	29
6	14	22	30
7	15	23	31

8			
8	12	24	28
9	13	25	29
10	14	26	30
11	15	27	31

16			
16	20	24	28
17	21	25	29
18	22	26	30
19	23	27	31

Using the first four cards, tell a person to choose a number between 1 and 15, to pick out the cards containing that number and to give them to you. By adding the numbers at the top of the cards he gives you, you can tell him the number he chose. Note that the numerals at the top of the cards represent the powers of two.

By using all five cards, you can pick out numerals from 1 to 31. The trick is based on the application of the binary numerals.