DOCUMENT RESUME

ED 175 664 SE 028 652

AUTHOR Kalman, Karl, Ed.
TITLE Supplementary and Enrichment Series: The Complex Number System. Teachers' Commentary. SP-5.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 64
NOTE 62p.; For related documents, see SE 028 648-675; Contains occasional light and broken type

EDRS PRICE MF01/PC03 Plus Postage.
DESCRIPTORS *Algebra; Curriculum; *Curriculum Guides; Enrichment; Graphs; *Instruction; Mathematics Education; *Number Systems; Secondary Education; *Secondary School Mathematics
IDENTIFIERS *Complex Numbers; *School Mathematics Study Group

ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exercises, and sample test questions. Topics covered include complex numbers, operations, standard form, equations, graphs and conjugates. (MP)

SP-5

SUPPLEMENTARY and ENRICHMENT SERIES

THE COMPLEX NUMBER SYSTEM

Teachers' Commentary

Edited by Karl Kalman

SMSG

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Printed in the United States of America

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

CONTENTS

COMMENTARY FOR TEACHERS

THE COMPLEX NUMBER SYSTEM

Introduction.

The complex number system is one of the supreme achievements of the human intellect. Compelling reasons for extending the real number system are easy to find. In the context of the real number system the theory of quadratic equations is most unsatisfactory, for some quadratic equations with real coefficients have real solutions, while others have no real solutions. The desire to remedy this situation is surely reasonable and modest. What is remarkable is the fact that this modest aim, once attained, yields a system so rich that no further extensions are necessary to capture the roots of any algebraic equation of whatever degree. However, the solution of algebraic equations is only one of the achievements of the complex number system. It is surely lamentable that we are unable, at this level of the students' development, to indicate the profusion of important and beautiful results to be found in the theory of functions of a complex variable. We can only state--with all the enthusiasm we can muster--that this field of mathematics (and others closely related to it) is probably the most intensively cultivated at the present time, and that its applications in the sciences and engineering seem to grow daily.

The extension of the real number system to the complex number system can be regarded as the solution of a problem--the problem of constructing a number system with certain properties. The solution of any problem generally proceeds in three stages (the solution of an equation is typical): 1. statement of the problem; 2. identification of a possible solution, assuming that a solution exists; 3. verification that the possible solution actually is a solution. Accordingly, in Section 1 we state the properties that the system is required to have; in Sections 2, 3, 4 we identify the system by finding its elements and the rules for operating with them, assuming that such a system exists; and in Section 11 we verify that the system constructed with these elements and rules of operation has the required properties.

In the complex number system, classical algebra--the theory of equations--finds its proper setting. The role of the complex number system in the theory of equations is discussed in Sections 5 and 8.

The connection between the complex number system and geometry is of great importance for geometry and analysis as well as for algebra. This connection is introduced in Sections 6 and 7 and further explored in Chapter 12 of SMSG Intermediate Mathematics.

1. Comments on the Introduction to Complex Numbers.

In Section 1 we review the inadequacy of the real number system with respect to the solution of quadratic equations and announce our intention to attempt a remedy by extending the real number system. We state that we will find a system in which every quadratic equation with real coefficients has a solution if we seek one in which the equation $x^2 + 1 = 0$ has a solution. This is so, of course, because every quadratic equation

$$ax^2 + bx + c = 0$$

with negative discriminant can be transformed into the equivalent equation

$$\left(\frac{x + \frac{b}{2a}}{\sqrt{\frac{4ac - b^2}{4a^2}}}\right)^2 = -1 .$$

This is not discussed in the text until Section 5, but a brief informal class discussion might be appropriate at this time.

The properties C-1, C-2, and C-3 which we require our new number system to posses are just explicit statements of the simple and natural requirements that the system have all the algebraic properties of the real number system, include the real number system, and contain a solution of the equation $x^2 + 1 = 0$. In Section 2 we impose a fourth requirement--also simple, but not so natural.

It should be observed that in extensions of the number system the extended system was required to have many of the order properties of the original system, but this is not done here. It is not done because it cannot be done. If the complex number system had the order properties of the real number system, the theorem that the square of every number is non-negative would have to hold, but this contradicts $i^2 = -1$.

Problems 1 and 4 of Exercises 2 can be assigned after Section 1, if desired: Problem 1 reviews the reasons for previous extensions of the number system; Problem 4 is intended to stimulate discussion of the fact, mentioned above, that the order properties of a number system may not be preserved when the system is extended.

---

2. Complex Numbers.

In the preceding section we stated a problem which we tacitly assumed had a unique solution. It does not--as we will see later. An additional condition is needed to make the problem definite, that is, to insure that it has a unique solution.

To expose this difficulty let us consider it in a more familiar setting. Suppose that our number system is the system of rational numbers and that we wish to extend it to a system in which the equation $x^2 = 2$ has a solution. Explicitly, we seek a system which has Properties C-1 and Properties C-2 with the word "real" replaced by "rational" wherever it occurs; and which has the third property--corresponding to C-3--that it contains a number $\sqrt{2}$, such that $(\sqrt{2})^2 = 2$. Let us call these Properties S-1, S-2, and S-3, respectively.

We know that the system of real numbers has these properties, but looking ahead, so does the system of complex numbers. Our problem does not have a unique solution; it has at least two solutions, and possibly more.

It would seem foolish to extend the system of rational numbers to the system of complex numbers just to achieve Properties S-1, S-2 and S-3; the system of complex numbers is too large--it contains a number system (the real numbers) which already has all the properties we require. Pursuing this objection, the system of real numbers might be larger than we require. It seems natural to add to our conditions the requirement S-4 that the system be as small as possible. With this condition added, our problem has a unique solution S: The elements of S are those real numbers which can be written in the form $a + b\sqrt{2}$, where a and b are rational numbers; and the operations in S are addition and multiplication of real numbers.

It is obvious that S has Properties S-2 and S-3. That it has all the Properties S-1 except (i), (iv) and (vii) follows immediately from the fact that the system of real numbers has these properties, and from S-2.. It can be verified by calculation that the sum, product, opposite and reciprocal of real numbers which can be expressed in the form $a + b\sqrt{2}$, a and b rational, can also be expressed in the form, so that S has properties S-1(i), (iv) and (vii). Thus, S is a solution of our problem. Notice that in this argument the only statements whose proofs were not immediate are those asserting that the sum, product, additive inverse and multiplicative inverse of numbers in S are in S.

It is easy to see that S is the smallest system which solves our problem. Consider any other set of real numbers which, with addition and multiplication of real numbers as operations, forms a system S' which is a solution of the problem. Then S' contains all rational numbers and $\sqrt{2}$, and is closed with respect to addition and multiplication. Hence, it must contain all real numbers which can be expressed in the form $a + b\sqrt{2}$, a and b rational--that is, it must contain S.

We summarize the salient features of this discussion: The properties we have required do not determine a unique number system; The natural additional condition to impose to determine a unique system is that the system be the smallest possible one having the given properties; This additional condition is logically equivalent to the condition that every number in the system be expressible in a certain form; The essential part of the proof of the equivalence of the two conditions is the proof that the sum, product, additive and multiplicative inverses of numbers which can be expressed in the stated form can also be expressed in that form.

The problem of extending the system of real numbers to the system of complex numbers is entirely analogous to the problem we have just discussed. Each of the summary statements we have just made holds also for the extension from the real numbers to the complex numbers.

We could have presented a discussion analogous to that given here in the text. Such a discussion, however, would have been an extensive and sophisticated preliminary to a program whose first objective is the introduction of complex numbers and the rules for calculating with them. Instead we have adopted a middle course.

In Section 2 we add Property C-4 to our requirements instead of the more natural condition that the system be the smallest possible system having Properties C-1, C-2, and C-3. The connection between these two conditions is suggested through brief discussion. However, in the discussion of addition, multiplication, additive inverse and multiplicative inverse in Section 3, 4 and 5 we make no essential use of Property C-4; we use it only as a guide. Thus, at the end of Section 5 one can look back and see that Property C-4 is not necessary, but that to find a system having Properties C-1, C-2, and C-3 it is sufficient to consider the system with Property C-4. Better students should be encouraged to do this, and all students should be aware of the need to check at each stage the compatibility of Property C-4 with the other properties of the system.

We still have to present an example of a system larger than the system of complex numbers which has Properties C-1, C-2, and C-3. The simplest example is the following. Let H contain the complex numbers, an element j which is not a complex number, and all expressions of the form

$$\frac{a_o j^n + a_1 j^{n-1} + \dots + a_{n-1} j + a_n}{b_o j^m + b_1 j^{m-1} + \dots + b_{m-1} j + b_m}$$

where n and m are non-negative integers, $a_o, a_1, \dots, a_n$ and $b_o, b_1, \dots, b_m$ are any complex numbers, and $a_o \neq 0$, $b_o \neq 0$. Thus, H is the set of all quotients of polynomials in j with complex coefficients. Addition and multiplication are defined according to the usual rules for operating with polynomials. Then H has the desired properties.

Problem 2 of Exercises 2 is intended to point out that in previous extensions of the number system the system sought was the smallest one having the desired properties. Problem *5 provides an opportunity for the student to carry through for himself the discussion presented above.

Exercises 2. Answers.

1. (a) The system of integers has an additive identity element, and each integer has an additive inverse.

   (b) In the rational number system each element except zero has a multiplicative inverse.

   (c) In the real number system every non-negative number has two real even roots, and every negative number has one real odd root.

   (d) The complex number system contains an element i which has the property $i^2 = -1$.

2. (a) System of integers.
   (b) Rational number system.
   (c) Rational number system.

3. (a) $1 + 0i$
   (b) $0 + 0i$
   (c) $-1 + 0i$
   (d) $0 + (1)i$
   (e) $3 + 0i$
   (f) $0 + 2i$
   (f) $-1 + 0i$

4. (a) The natural number system has the Well Order property. Every subset has a least element.

   (b) The real number system has an order relation. No order relation has been defined for the complex number system.

5. (a) If $\sqrt{3}$ were in S we could write

$$\sqrt{3} = a + b\sqrt{2}$$

where a and b are rational. If we square both sides of this equation we get

$$3 = a^2 + 2ab\sqrt{2} + 2b^2$$

or

$$\frac{3 - a^2 - 2b^2}{2ab} = \sqrt{2} .$$

Since a and b are rational, the left side of the last equation is rational, and the equation says that $\sqrt{2}$ is rational. Since we know this is false, the assumption that $\sqrt{3}$ belongs to S must be false.

(b) $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$, and if a, b, c, d are rational, so are $a + c$ and $b + d$, since the rational numbers are closed with respect to addition.

$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2}$, and if a, b, c, d are rational, so are $ac + 2bd$ and $bc + ad$, since the rational numbers are closed with respect to addition and multiplication.

(c) The additive inverse of $a + b\sqrt{2}$ in the real number system is $-(a + b\sqrt{2})$. But

$$-(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2}$$

and if a and b are rational, so are $-a$ and $-b$.

The additive identity in S is $0 = 0 + 0\sqrt{2}$. If $a + b\sqrt{2}$ is not zero, it has a multiplicative inverse $\frac{1}{a + b\sqrt{2}}$ in the real number system. But

$$\frac{1}{a + b\sqrt{2}} = \left(\frac{a}{a^2 - 2b^2}\right) + \left(\frac{-b}{a^2 - 2b^2}\right)\sqrt{2}$$

and if a and b are rational, so are $\frac{a}{a^2 - 2b^2}$ and $\frac{-b}{a^2 - 2b^2}$, since the rational number system is closed with respect to addition, multiplication, subtraction and division.

(d) Property (i) of C-1 was established in part (b) of this problem.

Property (ii) is established by observing that addition in S is addition of real numbers and addition of real numbers is associative and commutative. To be more explicit, addition is commutative since $x + y = y + x$ if x and y are any real numbers, and hence, in particular, if $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$.

Property (iii) is established by observing that $0 = 0 + 0\sqrt{2}$ is in S, and $x + 0 = x$ for any real number. Thus, in particular, if $x = a + b\sqrt{2}$, $x + 0 = x$, and 0 is an additive identity in S. 0 is the only additive identity in S since any other additive identity c in S would be a real number which satisfied $x + c = x$ for all x in S. But, taking $x = 0$, this becomes $0 + c = 0$ or $c = 0$. Property (vii) is established in a similar way.

Since 0 is the additive identity in S, an additive inverse of a number $a + b\sqrt{2}$ in S is a solution x in S of the equation $x + (a + b\sqrt{2}) = 0$. There is one and only one real number $-(a + b\sqrt{2})$ which satisfies this equation. We showed in part (c) that $-(a + b\sqrt{2})$ is in S, and since this is the only real number which satisfies the equation, it is the only number in S which satisfies the equation. This established property (iv). Property (vii) is established in a similar way.

(e) S has the stated properties. Let S' be another part of the real number system with the stated properties, and let a and b be any rational numbers. Then a, b and $\sqrt{2}$ are in S'. Since S' is closed with respect to addition, it contains $a + b\sqrt{2}$. Thus, every number in S is in S', and S' contains the system S.

---

3. Addition, Multiplication and Subtraction.

In this section we begin the discussion of operations with complex numbers. It should be emphasized that our objective is to perform operations with complex numbers in terms of operations with real numbers. The discussion of addition and multiplication is straightforward, but that of subtraction deserves some comment.

Subtraction is, as usual, defined as the inverse of addition. We show that the equation

$$z_1 + z = z_2$$

has at least one solution $z = z_2 + (-z)$. Notice, however, that in order to define $z_2 - z_1$ as the solution of this equation, and to assert

$$z_2 - z_1 = z_2 + (-z_1),$$

it is essential to show that the equation has at most one solution--a unique solution. The teacher may find it desirable to present the proof of uniqueness to the class.

The additive inverse $-z$ of z is defined by the equation

$$z + (-z) = 0.$$

According to Property C-4, $-(a + bi) = x + yi$ where x and y are real. Substituting in the equation defining $-(a + bi)$ we obtain two real equations in x and y which have the solution $x = -a$ and $y = -b$. We therefore conclude that

$$-(a + bi) = -a + (-b)i.$$

Notice, however, that here we have been using Property C-4 only as a guide. To prove the last equation it is only necessary to verify that

$$[a + bi] + [(-a)+(-b)i] = 0$$

and this is done without using C-4.

Exercises 3 provide practice in addition, multiplication and subtraction of complex numbers.

Exercises 3. Answers.

1. (a) $4 + 9i$
   (b) $4 + 0i$
   (c) $3 + 7i$
   (d) $(4 + \pi) + \pi i$
   (e) $(\sqrt{2} + 1) + 5i$
   (f) $-1 + 7i$
   (g) $8 + (1)i$
   (h) $0 + 7i$
   (i) $15 + (1)i$
   (j) $(3 + \sqrt{2}) + (9 + \sqrt{3})i.$

2. (a) $5i$
   (b) yi
   (c) $\sqrt{3}\,i$
   (d) $5i$

   Yes, any real number might have been added to the answer given here.

3. (a) $-13 + 26i$
   (b) $24 + (-10)i$
   (c) $5 + 5i$
   (d) $-5 + 3i$
   (e) $2 + 2\sqrt{2}\,i$
   (f) $(8 - \sqrt{6}) + (8\sqrt{3} + \sqrt{2})i$
   (g) $-7 + 24i$
   (h) $2 + 0i$
   (i) $-18 + 0i$
   (j) $14 + (-84)i$
   (k) $70 + 40i$
   (l) $-106 - 83i$
   (m) $92 - 18i$
   (n) $(cx - dy)+(cy + dx)i$, if c, d, x, y are real numbers.
   (o) $(x^2 - xy)+(xy - y^2)$, if x, y are real numbers.

4. (a) $-3 + 0i$
(b) $0 + (-1)i$
(c) $-1 + (-1)i$
(d) $-2 + (-3)i$
(e) $-5 + 4i$
(f) $4 + 3i$
(g) $-a + bi$, if a, b are real numbers.
(h) $-x + (-y)i$, if x, y are real numbers.

5. (a) $5 + 8i$
(b) $-2 + 2i$
(c) $0 + 10i$
(d) $-1 + 0i$
(e) $(\sqrt{3} - 2) + (1 - \sqrt{2})i$
(f) $1 + i$
(g) $\pi + (-\pi)i$
(h) $0 + 6i$
(i) $1 + (-3)i$

6. (a) $i^3 = i^2 \cdot i = (-1)i = 0 + (-1)i$
(b) $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 + 0i$
(c) $i^9 = (i^2)^4 \cdot i = (-1)^4 i = 0 + (1)i$
(d) $i^{15} = (i^2)^7 i = (-1)^7 i = 0 + (-1)i$
(e) $i^{4n+1} = (i^2)^{2n} \cdot i = [(-1)^2]^n i = 0 + (1)i$
(f) $i^{79} = (i^2)^{39} \cdot i = (-1)^{39} i = 0 + (-1)i$

7. General rule: The values of the powers of i recur in cycles of 4.

To explain the general rule first note that

$$i^1 = i,$$
$$i^2 = -1,$$
$$i^3 = i^2 \cdot i = (-1)i = -i,$$
$$i^4 = (i^2)^2 = (-1)^2 = 1.$$

Making use of the first four powers we have

$$i^5 = i^4 \cdot i = (1)i = i,$$
$$i^6 = i^4 \cdot i^2 = (1)(-1) = -1,$$
$$i^7 = i^4 \cdot i^3 = (1)(-i) = -i,$$
$$i^8 = i^4 \cdot i^4 = (1)(1) = 1.$$

In general, if n and m are natural numbers such that $n = 4m$, we have

$$i^n = i^{4m} = (i^4)^m = 1^m = 1.$$

Thus,
$$i^{4m+1} = i^{4m} \cdot i = (1)i = i,$$
$$i^{4m+2} = i^{4m} \cdot i^2 = (1)(-1) = -1,$$
$$i^{4m+3} = i^{4m} \cdot i^3 = (1)(-i) = -i,$$
$$i^{4m+4} = i^{4m} \cdot i^4 = (1)(1) = 1.$$

These possibilities are all there are, for if n is a natural number and we divide it by 4, the only non-negative remainders less than 4 which we can get are 0, 1, 2, 3.

8. (a) $1 + (-1)i$
   (b) $0 + (-1)i$
   (c) $0 + 107i$
   (d) $-7 + 84i$
   (e) $-1 + (-1)i$
   (f) $11 + 20i$
   (g) $2abc + [-a^3 - b^3 - c^3 - (b + c)(c + a)(a + b)]i$
   (h) $-1 + 0i$
   (i) $-10 + 0i$

9. $$2\left(\frac{3 + \sqrt{7}i}{4}\right)^2 - 3\left(\frac{3 + \sqrt{7}i}{4}\right) + 2$$
$$= \frac{2 + 6\sqrt{7}i}{8} - \frac{9 + 3\sqrt{7}i}{4} + 2$$
$$= \frac{2 + 6\sqrt{7}i - 18 - 6\sqrt{7}i}{8} + 2$$
$$= -2 + 2 = 0$$

---

4. Standard Form of Complex Numbers.

Section 4 is devoted to proving Theorem 4 and to defining some important terms. Theorem 4 asserts that each complex number z may be written in the form $a + bi$ (a and b real) in only one way. (C-4 asserts that z may be written in this form in at least one way.) This theorem justifies the definite article in the expression "the standard form" used to describe this way of writing complex numbers. (One advantage of Theorem 4 is that it shows us we can have only one answer for exercises like those in Section 3 where the student is asked to express certain complex numbers in what we now call "standard form".) The double-barrelled way in which Theorem 4 is stated gives the teacher an opportunity to refresh the students' minds on the distinction between "if" and "only if", a distinction which cannot be over-emphasized. However, the statement containing "only if" is the only part that requires a proof.

Any tendency to regard Theorem 4 as obvious can be overcome by emphasizing that the requirement in the hypothesis that a, b, c, d be real is essential; without this requirement the conclusion is false. Example 4a demonstrates this.

It is worth observing that the proof of Theorem 4 can be based on the following special case of the theorem: If a and b are real, then $a + bi = 0 (= 0 + 0i)$ if and only if $a = 0$ and $b = 0$. Let us suppose this has been proved and show how the general case follows from it. Let a, b, c, d be real. Then

$$a + bi = c + di$$

if and only if

$$(a - c) + (b - d)i = 0.$$

The equation in the last line holds if and only if $a - c = 0$ and $b - d = 0$. This proves Theorem 4.

A word (or two) about the terms defined in Section 4 may be in order. "Standard form" should cause no trouble; though one must emphasize that the a and b appearing in the standard form are real numbers. (Throughout the rest of the chapter we sometimes say "a + bi, in standard form" and sometimes "a + bi, where a and b are real numbers"; these expressions have identical meanings.) "Real part" is straightforward and should cause no trouble. Mathematicians have used the expression "imaginary part" as defined in the text for many years: The imaginary part of a complex number is a real number. This terminology may be unfortunate, but it is standard. Writers of many elementary books have departed from the mathematicians' usage, saying that bi is the "imaginary part" of a + bi. Students reading other books will notice that they are not all in agreement. (This experience is a valuable part of anyone's education.) A student who goes on in mathematics has to learn sooner or later that in advanced work b is called the imaginary part of a + bi. Since it seems a shame to teach him something he must later unlearn, we stick to the mathematicians' standard terminology: The imaginary part of a complex number is a real number.

Observe that 0 is both real and pure imaginary, but that it is not imaginary. This may be momentarily disconcerting; but it should be so only momentarily. One has only to remember that everyday connotations and relations of words and phrases are irrelevent to their technical use: A technical term means only what its definition says it means.

Problems 1-5 of Exercises 4 are practice problems. Problem 6 refers to the special case $c = d = 0$ of Theorem 4 discussed above, and emphasizes again the necessity of the condition that a and b be real. Problem 7 generalizes Theorem 4: Theorem 4 is the special case obtained by setting $z_1 = 1$, $z_2 = i$.

Exercises 4. Answers.

1.

	Real part	Imaginary part
(a)	0	2
(b)	0	0
(c)	0	1
(d)	5	-1
(e)	2x	3
(f)	a	-2
(g)	1	$-2\sqrt{2}$
(h)	-2	$-2\sqrt{3}$
(i)	-3	1
(j)	2	0
(k)	0	3
(l)	1	2

2. (a) -3
   (b) 0
   (c) -5
   (d) -5.

   There is only one way in each case.

3. (a) $x = 3$, $y = -6$
   (b) $x = 3$, $y = 0$
   (c) $x = 0$, $y = -4$
   (d) $x = \frac{1}{2}$, $y = -3$
   (e) $x = -\frac{4}{3}$, $y = 2$
   (f) $x = 4$, $y = 2$
   (g) $x = 2$, $y = 6$
   (h) $x = 0$, $y = 0$
   (i) $x = \pm 1$, $y = 0$
   (j) $x = 0$, $y = -1$.

4. (a) $8 + 3i$
   (b) $-2 + 0i$
   (c) $6 + 12i$
   (d) $4 + 8i$
   (e) $11 + (-16)i$
   (f) $10 + (-11)i$
   (g) $18 + 14i$
   (h) $(a^2 + 2ab + b^2 + c^2) + 0i$
   (i) $(x^3 - 3xy^2) + (3x^2y - y^3)i$.

5. Let $z^2 = (x + yi)^2 = 8 + 6i$.

Then $(x^2 - y^2) + 2xyi = 8 + 6i$.

Since x and y are real, we must have

$$\text{(i)} \quad x^2 - y^2 = 8,$$
$$\text{(ii)} \quad 2xy = 6.$$

Squaring both members of the last two equations, we obtain

$$\text{(iii)} \quad x^4 - 2x^2y^2 + y^4 = 64,$$
$$\text{(iv)} \quad 4x^2y^2 = 36.$$

Adding the last two equations, we get

$$\text{(v)} \quad (x^2 + y^2)^2 = 100.$$

Since $x^2 + y^2$ must be positive, it follows that

$$\text{(vi)} \quad x^2 + y^2 = +10.$$

Adding (i) and (vi), we get

$$2x^2 = 18, \qquad 2y^2 = 2.$$

Hence,

$$x = \pm 3, \qquad y = \pm 1.$$

From (ii) x and y have the same sign so $\begin{cases} x = 3 \\ y = 1 \end{cases}$ and $\begin{cases} x = -3 \\ y = -1 \end{cases}$.

Note: In a sense the problem appears to be that of finding the square root of the complex number $8 + 6i$; however, we have not defined the symbol $\sqrt{\ }$ for complex numbers.

6. Let $a = x + yi$ and $b = u + vi$ where x, y, u, v are all real.

(a) If $a = 0$ and $b = 0$, then $a + bi$ and $a - bi$ are both zero.

(b) Suppose $a + bi = 0$, then

$$x + yi + (u + vi)i = 0$$

or

$$(x - v) + (y + u)i = 0.$$

By Theorem 4, we have

$$x - v = 0 \text{ and } y + u = 0$$

or,

$$\text{(1)} \quad x = v \text{ and } y = -u$$

i.e., if $a = v - ui$ and $b = u + vi$, $a + bi = 0$ with neither a nor b zero.

Since $a - bi = 0$ also, we have $x + v + (y - u)i = 0$, or

$$(2) \quad x = -v \quad \text{and} \quad y = u.$$

Both (1) and (2) can be satisfied only if $x = u = 0$ and $y = v = 0$. In this case $a = 0$ and $b = 0$.

7. Let $z = x + yi$ and $z_1 = x_1 + iy_1$, $y_1 \neq 0$. Then

$$z = a + bz_1$$

if and only if

$$x = a + bx_1, \qquad y = by_1$$

that is, if and only if

$$b = \frac{y}{y_1}, \qquad a = x - \frac{yx_1}{y_1} .$$

---

5. Division.

The discussion of division in this section parallels that of subtraction in Section 3. The comments made about subtraction hold also, with obvious modifications, for division. Once again it should be emphasized that our objective is to express calculations with complex numbers in terms of calculations with real numbers.

The central problem of this section is to express the multiplicative inverse $\frac{1}{z}$ if $z = a + bi$ in terms of a and b. Since $\frac{1}{z}$ is defined by the equation

$$\frac{1}{z} \cdot z = 1,$$

and since, by the Property C-4, $\frac{1}{z} = x + yi$, x and y real, the problem reduces to solving the equation

$$(x + yi)(a + bi) = 1$$

for real values of x and y. This equation can be transformed into the equation

$$(ax - by) + (bx + ay)i = 1.$$

Now, if x and y are real, then the expressions in parentheses are real; here we are using Property C-2. Hence, by the theorem on standard form, the equation above is satisfied if and only if

$$ax - by = 1,$$

and

$$bx + ay = 0.$$

The problem has thus been reduced to that of solving a pair of linear equations with real coefficients for the real unknowns x and y. The solution of this system proceeds in the familiar way, and we conclude

$$\frac{1}{a + bi} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i .$$

To find this result we used Property C-4. However, to establish the result we have only to verify that

$$\left(\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i\right)(a + bi) = 1,$$

and this verification makes no use of Property C-4.

Now looking back over the discussion in Sections 3, 4 and 5 we see that, as promised in Section 2, we have proved that the sum, product, and additive and multiplicative inverses of numbers given in the form $a + bi$ can again be expressed in this form. Thus, if we had required that the system we sought be the smallest possible system having Properties C-1, C-2, and C-3, we could have established Property C-4 as a theorem.

Of equal importance is the fact that we have achieved our objective of expressing all operations with complex numbers in terms of operations with real numbers.

Exercises 5 either provide practice in operations with complex fractions, or require the proof of statements made in the text without proof.

Exercises 5. Answers.

1. (a) $1 + 0i$ (e) $\frac{1}{2} + (-\frac{1}{2})i$
   (b) $\frac{1}{5} + 0i$ (f) $\frac{2}{13} + (-\frac{3}{13})i$
   (c) $0 + (-1)i$ (g) None
   (d) $0 + (1)i$ (h) $\frac{4}{25} + \frac{3}{25}i$.

2. Zero does not have a multiplicative inverse.

3. 1, -1

4. i, -i

5. (a) $\frac{2}{5} + (-\frac{1}{5})i$

(b) $0 + (-\frac{3}{2})i$

(c) $-\frac{5}{29} + (-\frac{2}{29})i$

(d) $\frac{5}{2} + (-\frac{13}{2})i$

(e) $\frac{1}{5} + \frac{3}{5}i$

(f) $\frac{23}{29} + (-\frac{14}{29})i$

(g) $-\frac{3}{25} + \frac{46}{25}i$

(h) $\frac{21}{65} + \frac{12}{65}i$

(i) $-\frac{25}{34} + (-\frac{15}{34})i$

(j) $-\frac{1}{3} + \frac{2\sqrt{2}}{3}i$

(k) $\frac{\sqrt{2} + \sqrt{6}}{3} + \frac{\sqrt{3} - 2}{3}i$

(l) $\frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$

(m) $\frac{2a^2 - 2b^2}{4a^2 + b^2} + \frac{5ab}{4a^2 + b^2}i$

(n) $\frac{m^2 - n^2}{m^2 + n^2} + \frac{2mn}{m^2 + n^2}i$

(o) $\frac{3x^2 - 2y^2}{x^2 + y^2} + \frac{5xy}{x^2 + y^2}i$

6. Let z_3 and z_4 be two solutions of the equation $z_1z = z_2$, so that

$$z_1z_3 = z_2, \quad \text{and} \quad z_1z_4 = z_2.$$

Multiply both members of each of the last two equations by $\frac{1}{z_1}$. Then

$$\frac{1}{z_1}z_2 = \frac{1}{z_1}(z_1z_3) = (\frac{1}{z_1}z_1)z_3 = 1\cdot z_3 = z_3;$$

$$\frac{1}{z_1}z_2 = \frac{1}{z_1}(z_1z_4) = (\frac{1}{z_1}z_1)z_4 = 1\cdot z_4 = z_4.$$

Therefore, $z_3 = z_4$.

<u>Alternate solution for 6</u>:

Suppose u and z are solutions of the equation. Then

$$z_1 u = z_2$$

$$z_1 z = z_2$$

and $$z_1(u - z) = 0.$$

By (5f) this can happen only if one of the factors is zero. $z_1 \neq 0$, $\therefore u - z = 0$ or $u = z$.

7. Let $z = a + bi$. (Note that $a^2 + b^2 \neq 0$.)

Then $\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$.

Thus, the real part of $\frac{1}{z} = \frac{a}{a^2 + b^2} = \frac{1}{2}$,

$$2a = a^2 + b^2.$$

(a) If $b = 0$, then $a = 2$ (since a and b cannot both be zero); and $z = a + bi = 2 + 0i$.

(b) If $b = \frac{1}{2}$, then $2a = a^2 + \frac{1}{4}$,

$$4a^2 - 8a + 1 = 0;$$

and

$$a = 1 + \frac{\sqrt{3}}{2},$$

or

$$a = 1 - \frac{\sqrt{3}}{2}.$$

So there are two possible numbers z:

$$z_1 = (1 + \frac{\sqrt{3}}{2}) + \frac{1}{2}i; \quad z_2 = (1 - \frac{\sqrt{3}}{2}) + \frac{1}{2}i.$$

(c) If $b = 1$, then $2a = a^2 + 1$,

$$a^2 - 2a + 1 = 0,$$

$$a = 1.$$

Hence, $$z = 1 + i.$$

8. The "if" part of the proof follows immediately from the fact that $0 \cdot z = 0$. To prove the "only if" part of the statement suppose that $z_1 z_2 = 0$. If $z_1 \neq 0$, then C contains a number $\frac{1}{z_1}$. Multiplying by $\frac{1}{z_1}$, we get $z_2 = 0 \cdot \frac{1}{z_1} = 0$.

Thus, if $z_1 z_2 = 0$, and $z_1 \neq 0$, we have $z_2 = 0$. Similarly, if $z_1 z_2 = 0$ and $z_2 \neq 0$, then we have $z_1 = 0$. Therefore, $z_1 z_2 = 0$ implies that either $z_1 = 0$ or $z_2 = 0$, or possibly both (since $0 \cdot 0 = 0$).

9. Let w_o be the unique solution of $z_2 w = z_1$; then $w_o = \frac{z_1}{z_2}$ and $z_2 w_o = z_1$. Similarly, let w_1 be the unique solution of $z_4 w = z_3$; then $w_1 = \frac{z_3}{z_4}$ and $z_4 w_1 = z_3$. Also, let w_2 be the unique solution of $(z_2 z_4)w = z_1 z_3$; then $w_2 = \frac{z_1 z_3}{z_2 z_4}$. We must show that $w_o w_1 = w_2$. From $z_2 w_o = z_1$ and $z_4 w_1 = z_3$ we get $(z_2 w_o)(z_4 w_1) = z_1 z_3$ or $(z_2 z_4)(w_o w_1) = z_1 z_3$. Thus, $w_o w_1$ satisfies $(z_2 z_4)w = z_1 z_3$. But w_2 is the <u>only</u> solution of this equation. Hence, $w_o w_1 = w_2$.

10. Let $w_o = \frac{z_1}{z_2}$ and $w_1 = \frac{z_3}{z_4}$, and let w_3 be the unique solution of $(z_2 z_4)w = z_1 z_4 + z_2 z_3$. To show $w_o + w_1 = w_3$. From $z_2 w_o = z_1$ and $z_4 z_1 = z_3$ we get $z_4(z_2 z_o) = z_1 z_4$ and $z_2(z_4 w_1) = z_2 z_3$; so, adding, $z_2 z_4(w_o + w_1) = z_1 z_4 + z_2 z_3$. Thus, $w_o + w_1$ satisfies the equation whose only root is w_3. Hence, $w_o + w_1 = w_3$.

11. (a) $\frac{6}{5} + 0i$ (c) $8 + 0i$

(b) $-\frac{7}{50} + (-\frac{12}{25})i$ (d) $-\frac{1}{2} + 0i$

(e) $\frac{2a^4 - 12a^2b^2 + 2b^4}{(a^2 + b^2)^2} + 0i$

*12. Whether or not a and b are real numbers, provided that $a^2 + b^2 \neq 0$, we can multiply the factors $a + bi$, $\frac{a - bi}{a^2 + b^2}$ and get $(a + bi)\frac{a - bi}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} + \frac{ab - ab}{a^2 + b^2}i = 1$ (for there is nothing in the proof of Theorem 3b which requires a and b to be real). Thus, $\frac{a - bi}{a^2 + b^2}$ is <u>an</u> inverse of $a + bi$, if $a^2 + b^2 \neq 0$. But we know already that no complex number can have more than one inverse, for if it did, Property C-1(vii) (as stated in the text) would be false.

6. Quadratic Equations.

Section 6 extends the theory of quadratic equations with real coefficients by treating the case of a negative discriminant. Since the quadratic formula involves the expression $\sqrt{b^2 - 4ac}$ and we are interested in the case $b^2 - 4ac < 0$, we are obliged to precede our discussion of the formula by a definition of $\sqrt{r}$ for r real and negative. Hence, we begin with the examples $z^2 = -1$ and $z^2 = r$, $r < 0$, and lead up to the extended definition of $\sqrt{r}$ (Definition 6). With the definition of $\sqrt{r}$ available, we summarize our results on the special quadratics (those having no first degree term) in Theorem 6a, a result we need in the proof of Theorem 6b. Theorem 6b is proved by the usual process of completing the square, and then using Theorem 6a to solve

$$(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} .$$

Since $\sqrt{4a^2} = 2|a|$, the square roots of the right member are

$$\frac{\sqrt{b^2 - 4ac}}{2|a|}, \quad \frac{-\sqrt{b^2 - 4ac}}{2|a|} .$$

One of these is $\frac{\sqrt{b^2 - 4ac}}{2a}$, the other is $\frac{-\sqrt{b^2 - 4ac}}{2a}$ (which is which depends on whether $a > 0$ or $a < 0$). Theorem 6b solves the problem of finding the solutions of the general quadratic equation with real coefficients. We find that every quadratic with real coefficients is one of three types: (1) It has one root--which is real--if its discriminant is zero; (2) It has two (different) real roots if its discriminant is positive; (3) It has two (different) non-real complex roots if its discriminant is negative.

Exercises 6, Problems 3 provide practice in calculating with the square root symbol. It should be emphasized that when a variable appears in the radicand, it is in general necessary to distinguish several cases. One reason for this is that the statement $\sqrt{r}\sqrt{s} = \sqrt{rs}$ which holds for $r \geq 0$, $s \geq 0$ is not true in general. Problem 5 requires a proof of the extension of this statement to the case in which r and s are not both negative; Problem 4 is intended to show why the statement is not true when r and s are both negative.

Problems 6-17 provide practice in the solution of quadratic equations. Problem 16 deserves particular comment. Although we have established the "quadratic formula" only for the case of real coefficients, it continues to hold when the coefficients are complex provided the discriminant is real; in this case the formula can be established exactly as it was for the case of real coefficients. Thus, the quadratic equation

$$z^2 + \beta z + \alpha = 0$$

with complex coefficients β, α can be solved by means of the quadratic formula if

$$\beta^2 - 4\alpha = r,$$

where r is a real number. We can construct quadratic equations for which this is true by choosing the complex number β and the real number r arbitrarily, and determining α from

$$\alpha = \frac{\beta^2 - r}{4}$$

The equation of Problem 16 is determined by choosing $\beta = -i$, $r = -9$. Quadratic equations with complex coefficients without the restriction that the discriminant be real are considered in Chapter 12 of SMSG Intermediate Mathematics.

Problems 18-24 provide an opportunity for the student to investigate by himself questions which will be discussed in detail in Section 9. We mention in particular Problems 19 and 20, which state important results of algebra; these will be stated more generally in Section 9. The approach suggested in the hint for Problem 22 could be used for the solution of quadratic equations with complex coefficients in general, but the method is too cumbersome to be useful. Some students might be interested in pursuing this point, however,

<u>Exercises 6</u>. <u>Answers</u>.

1. (a) $0 + 7i$
   (b) $0 + (-13\sqrt{5})i$
   (c) $0 + (4\sqrt{2})i$
   (d) $-2\sqrt{5} + 0i$
   (e) $0 + (-4\sqrt{3})i$
   (f) $0 + \frac{1}{2}i$
   (g) $\frac{3\sqrt{2}}{4} + 0i$
   (h) $\frac{\sqrt{6}}{3}$ $0i$

2. (a) $0 + 2i$ (e) $|c| + 0i$
(b) $2 + 0i$ (f) $0 + |c|i$
(c) $0 + 2i$ (g) $0 + |c|i$
(d) $|c| + 0i$

3. (a) $0 + (a + b)i$ (e) $0 + (-a\sqrt{2})i$
(b) $-2a^2\sqrt{b} + 0i$ (f) $-a^2 + 0i$
(c) $-(a + \sqrt{ab}) + 0i$ (g) $0 + 2(a + b)i$
(d) $\frac{5\sqrt{a}}{3} + 0i$

4. Proof that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ if a and b are non-negative real numbers: By the definition of the square root of a non-negative number we know that

$$(\sqrt{a})^2 = a,$$

$$(\sqrt{b})^2 = b.$$

Thus, $$(\sqrt{a}\sqrt{b})^2 = ab,$$

and we know that $(\sqrt{a}\sqrt{b})$ is a square root of ab. Since $\sqrt{a}$ and $\sqrt{b}$ are both non-negative by definition, it follows that $(\sqrt{a}\sqrt{b})$ is non-negative. Hence, $(\sqrt{a}\sqrt{b})$ must be the square root of ab; that is,

$$\sqrt{ab} = \sqrt{a}\sqrt{b}.$$

Now, if a and b are negative, then

$$\sqrt{a} = i\sqrt{-a},$$

$$\sqrt{b} = i\sqrt{-b};$$

and $$(\sqrt{a}\sqrt{b}) = (i\sqrt{-a})(i\sqrt{-b}) = -\sqrt{ab}.$$

Again $(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab$
but as we have just seen

$\sqrt{a} \cdot \sqrt{b} = -\sqrt{ab}$, a negative number which cannot be the square root of ab.

5. $r < 0$ and $s > 0$, then $\sqrt{r}\sqrt{s} = i\sqrt{-r}\sqrt{s} = i\sqrt{-rs}$;
also, $\sqrt{rs} = i\sqrt{(-r)(s)} = i\sqrt{-rs}$.

6. $0 + i$, $0 + (-1)i$

7. $\frac{-1 + \sqrt{5}}{2} + 0i$, $\frac{-1 - \sqrt{5}}{2} + 0i$

8. $-1 + (1)i$, $-1 + (-1)i$

9. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{1}{2} + (-\frac{\sqrt{3}}{2})i$

10. $-\frac{1}{3} + \frac{\sqrt{11}}{3}i$, $-\frac{1}{3} + (-\frac{\sqrt{11}}{3})i$

11. $-2 + 2i$, $-2 + (-2)i$

12. $2 + 2i$, $2 + (-2)i$

13. $-\frac{1}{4} + \frac{\sqrt{7}}{4}i$, $-\frac{1}{4} + (-\frac{\sqrt{7}}{4})i$

14. If $a \geq -\frac{1}{2}$; $(2 + 2\sqrt{1 + 2a}) + 0i$, $(2 - 2\sqrt{1 + 2a}) + 0i$.

If $a < -\frac{1}{2}$; $2 + 2\sqrt{-(1 + 2a)}i$, $2 + [-2\sqrt{-(1 + 2a)}]i$

15. $-\frac{1}{2m} + \frac{\sqrt{3}}{2m}i$, $-\frac{1}{2m} + (-\frac{\sqrt{3}}{2m})i$

16. $0 + 2i$, $0 + (-1)i$

17. If $\frac{c}{a} \geq 0$: $0 + \sqrt{\frac{c}{a}}i$, $0 + (-\sqrt{\frac{c}{a}})i$

If $\frac{c}{a} < 0$: $\sqrt{-\frac{c}{a}} + 0i$, $-\sqrt{-\frac{c}{a}} + 0i$

18. $z^3 - 8 = (z - 2)(z^2 + 2z + 4)$

$z^3 - 8 = 0$ if and only if $z - 2 = 0$ or $z^2 + 2z + 4 = 0$.

The solutions are 2, $-1 + \sqrt{3}i$, $-1 + (-\sqrt{3})i$.

19. Using Theorem 6b we obtain the following solutions for the given equation:

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$z_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Thus,

$$z_1 + z_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - b}{2a} = -\frac{b}{a}$$

and

$$z_1 z_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}.$$

20. $az^2 + bz + c = a(z^2 + \frac{b}{a}z + \frac{c}{a})$.

By making use of the results of Problem 19, the right side can be written as

$$a[z^2 - (z_1 + z_2)z + z_1z_2].$$

Hence,

$$az^2 + bz + c = a(z - z_1)(z - z_2),$$

or alternatively, multiplying out the right side of the last equation, the left may be obtained directly.

21. (a) $z^2 - 2z + 2 = 0$

(b) $z^2 - (2 + 2i)z - 1 + 2i = 0$

(c) $z^2 = 0$

(d) $z^2-[(a_1+a_2)+(b_1+b_2)i]z+[(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i] = 0$

*22. Let $z = x + yi$, where x and y are real.
Then $z^2 = x^2 - y^2 + 2xyi$.
But $z^2 = i$, so

$$\text{(i)}\quad x^2 - y^2 = 0,$$

$$\text{(ii)}\quad 2xy = 1.$$

Squaring both sides of (i) and (ii) and adding, we have

$$\text{(iii)}\quad (x^2 + y^2)^2 = 1.$$

Since $x^2 + y^2 > 0$, taking square roots of both members of (iii) we have

$$\text{(iv)}\quad x^2 + y^2 = 1.$$

Adding (i) and (8v) we obtain

$$2x^2 = 1.$$

From whence

$$x = \pm\frac{\sqrt{2}}{2}.$$

From (ii) the corresponding values of y are

$$y = \pm\frac{\sqrt{2}}{2}.$$ (Note that from (ii) x and y have the same sign.)

Therefore, $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2} + (-\frac{\sqrt{2}}{2})i.$

*23. Employing the method displayed in the solution of Problem *22 we obtain

$$z = \frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right)i, \quad -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i.$$

*24. Extending the idea of Problem 20, we have

$$[z - (1 + 2i)][z - (1 - i)][z - (1 + i)] = 0,$$

or, multiplying out the left member, we obtain

$$z^3 - (3 + 2i)z^2 + (4 + 4i)z - (2 + 4i) = 0.$$

There is no quadratic equation having all three solutions, for the formula in Problem 20 shows that no quadratic equation may have more than two solutions: If $az^2 + bz + c = a(z - z_1)(z - z_2) = 0$, $a \neq 0$, then either $z - z_1 = 0$ or $z - z_2 = 0$; i.e., $z = z_1$ or $z = z_2$. Moreover, no quadratic expression such as $az^2 + bz + c$ can be written as a product of three first degree factors, say $(z - z_1)(z - z_2)(z - z_3)$, times a constant: For any such product produces a z^3 terms and no quadratic can have such a term.

---

7. Graphical Representation--Absolute Value.

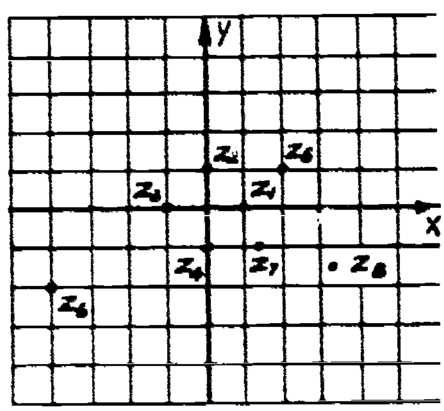
The representation of complex numbers by points in the plane had a great effect historically on the acceptance of the complex number system by mathematicians. This geometric representation overcame the feeling that the complex number system was not concrete; the employment of the complex number system in the solution of geometric problems, which it permitted, promoted an appreciation of the usefulness of the system. The discussion in Section 7, and its continuation in Section 8, may be expected to have a similar effect upon students.

The discussion in the text calls for little comment. We mention only that the notion of absolute value is a purely algebraic one, even though its definition is geometrically motivated; all of the properties of absolute value can be established algebraically. In particular, the relations $|z_1z_2| = |z_1||z_2|$, $|z_1 + z_2| \leq |z_1| + |z_2|$ can be established algebraically. It is remarkable that although the geometric interpretation of the first relation is obscure and that of the second very clear, the algebraic proof is not presented in the text. The interested teacher can find such a proof in almost any text on the theory of functions of a complex variable. (See, for example, R.V. Churchill, Introduction to Complex Variables and Applications.)

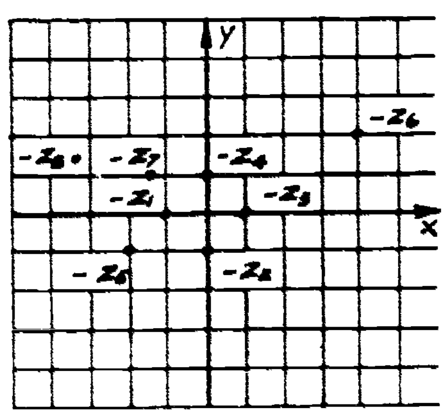
Exercises 7, Problems 1-4 provide practice in the graphical representation of complex numbers and the graphical interpretation of addition and subtraction. Problems 5-7 involve the calculation of absolute values. Problems 8-10 require the proof of statements made in the text without proof. Problems 11-12 refer to the geometric interpretation of operations with complex numbers in special cases.

Exercises 7. Answers.

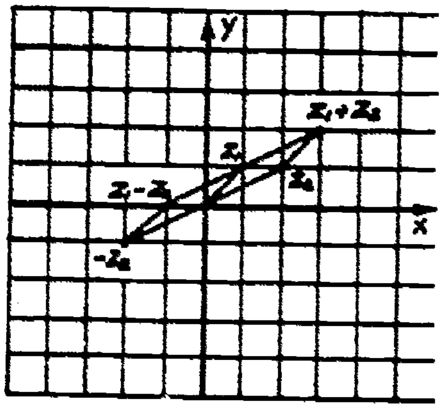
1.



2.



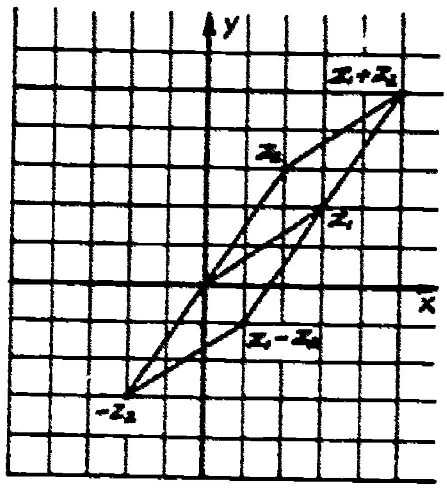
3. (a)



$z_1 + z_2 = 3 + 2i$

$z_1 - z_2 = -1 + 0 \cdot i$

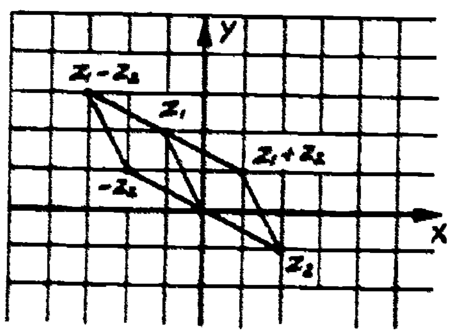
(b)



$z_1 + z_2 = 5 + 5i$

$z_1 - z_2 = 1 - i$

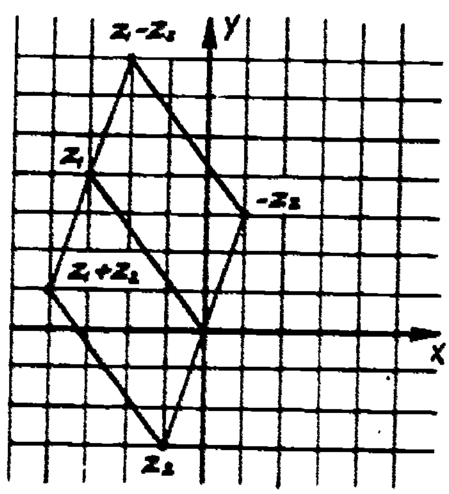
(c)



$z_1 + z_2 = 1 + i$

$z_1 - z_2 = 3 + 3i$

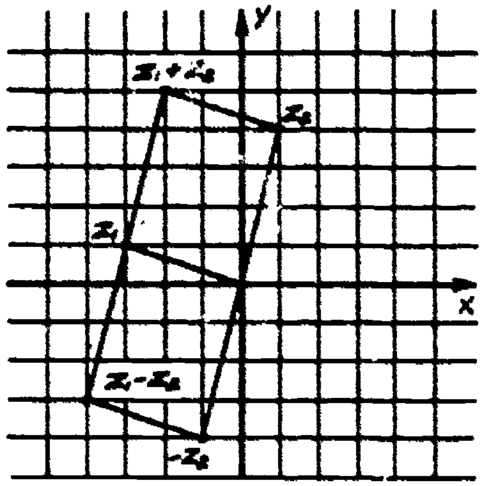
(d)



$z_1 + z_2 = -4 + i$

$z_1 - z_2 = -2 + 7i$

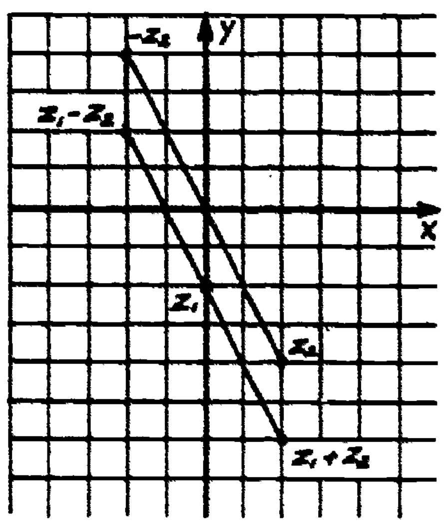
3. (e)



$z_1 + z_2 = -2 + 5i$

$z_1 - z_2 = -4 - 3i$

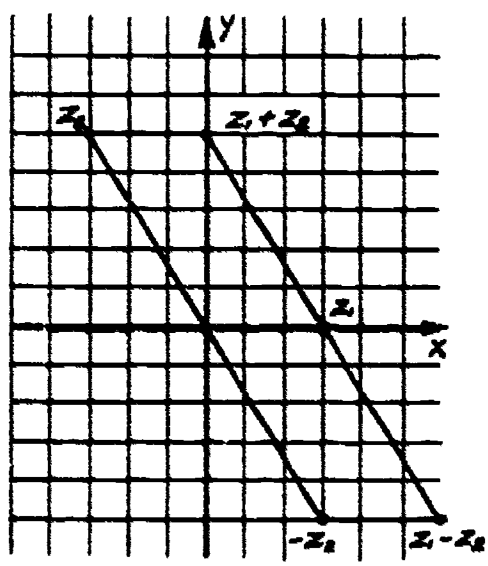
(f)



$z_1 + z_2 = 2 - 6i$

$z_1 - z_2 = -2 + 2i$

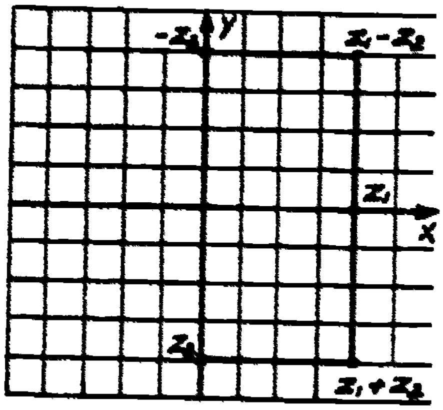
(g)



$z_1 + z_2 = 5i$

$z_1 - z_2 = 6 - 5i$

3. (h)



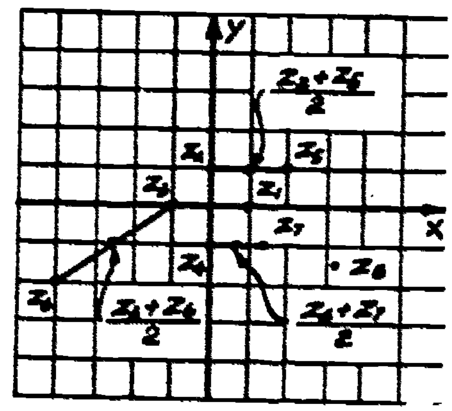
$z_1 + z_2 = 4 - 4i$

$z_1 - z_2 = 4 + 4i$

4. $\dfrac{z_2 + z_5}{2} = \dfrac{i + (2 + i)}{2} = 1 + i.$

$\dfrac{z_3 + z_6}{2} = \dfrac{-1 + (-4 - 2i)}{2} = -\dfrac{5}{2} + (-1)i$

$\dfrac{z_4 + z_7}{2} = \dfrac{-i + (\sqrt{2} - i)}{2} = \dfrac{\sqrt{2}}{2} + (-1)i$



5. (a) 5
(b) 2
(c) 0
(d) $\sqrt{2}$
(e) $\sqrt{\pi^2 + 2}$

6. Let $z = x + yi$

then $\dfrac{z}{|z|} = \dfrac{x + yi}{\sqrt{x^2 + y^2}}$,

and $\left|\dfrac{z}{|z|}\right| = \sqrt{\dfrac{x^2}{x^2 + y^2} + \dfrac{y^2}{x^2 + y^2}} = \sqrt{\dfrac{x^2 + y^2}{x^2 + y^2}} = 1.$

7. (a) The single point $(1,0)$.

(b) Let $z = x + yi$, x and y real.

Then $x + yi = \sqrt{x^2 + y^2}$.

Hence, $y = 0$, and $x = \sqrt{x^2}$.

Therefore, the set of points is the non-negative x-axis.

(c) Since z cannot be zero, the given equation may be transformed into the equation $|z| = 1$, and this is the equation of the unit circle.

8. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

Then $|z_1 z_2| = \left|(x_1 + y_1 i)(x_2 + y_2 i)\right|$

$$= \left|(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i\right|$$

$$= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_2x_2y_1 + x_2^2y_1^2}$$

$$= \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}$$

$$= \sqrt{x_1^2 + y_1^2}\ \sqrt{x_2^2 + y_2^2}$$

$$= |z_1|\cdot|z_2| \ .$$

9. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

Then $\dfrac{z_1}{z_2} = \dfrac{(x_1 + y_1 i)(x_2 - y_2 i)}{x_2^2 + y_2^2} = \dfrac{(x_1x_2+y_1y_2)+(x_2y_1-x_1y_2)i}{x_2^2 + y_2^2}$

and $\left|\dfrac{z_1}{z_2}\right| = \dfrac{\sqrt{x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2+x_2^2y_1^2 - 2x_1x_2y_1y_2 + x_1^2+y_2^2}}{x_2^2 + y_2^2}$

$$= \frac{\sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}}{x_2^2 + y_2^2}$$

$$= \frac{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}}{x_2^2 + y_2^2}$$

$$= \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}}$$

$$= \frac{|z_1|}{|z_2|} \ .$$

10. Using the fact that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side, we have

$$|z_1 - z_2| + |z_2| \geq |z_1| \quad \text{and} \quad |z_1 - z_2| + |z_1| \geq |z_2|$$

or

$$|z_1 - z_2| \geq |z_1| - |z_2| \quad \text{and} \quad |z_1 - z_2| \geq |z_2| - |z_1|.$$

From this we conclude

$$|z_1 - z_2| \geq \left||z_1| - |z_2|\right|.$$

11. If 0, $z_1 = a + bi$, and $z_2 = c + di$ are collinear, then the slope of the segment joining 0 and z_1 is the same as the slope of the segment joining 0 and z_2. Thus,

$$\text{(i)} \quad \frac{b}{a} = \frac{d}{c}.$$

If $z_3 = z_1 + z_2$,
then $z_3 = (a + c) + (b + d)i$.
The slope of the segment joining 0 and z_3 is

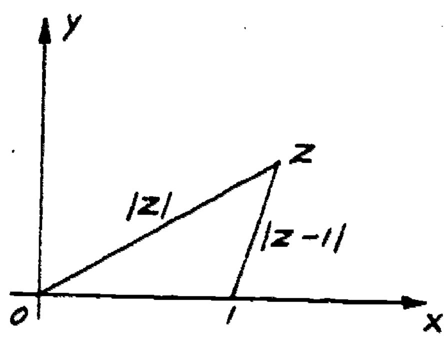
$$\text{(ii)} \quad \frac{b + d}{a + c}.$$

But (ii) is equal to both members of (i); that is,

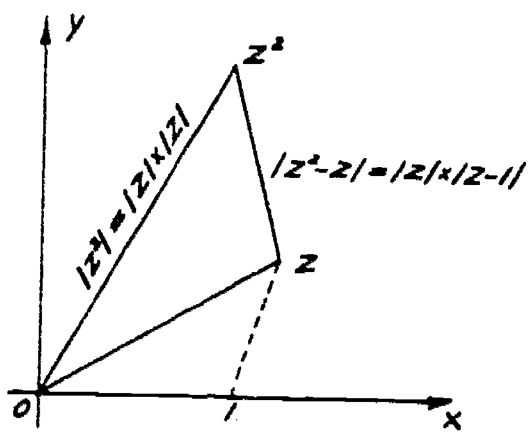
$$\frac{b}{a} = \frac{d}{c} \rightarrow \frac{b}{d} = \frac{a}{c} \rightarrow \frac{b + d}{d} = \frac{a + c}{c} \rightarrow \frac{b + d}{a + c} = \frac{d}{c}.$$

Hence, the slope of the segment joining 0 and z_3 is the same as the slopes of the segments joining 0 and the points z_1 and z_2 respectively, and since all three segments pass through 0, the points 0, z_1, z_2 and z_3 are collinear.

12. The triangle with vertices 0, 1, z is shown in the figure at the right. The lengths of the sides of the triangle are 1, $|z|$, $|z - 1|$.



If we multiply each of these lengths by $|z|$, we obtain $|z|\cdot 1$, $|z|\cdot|z|$, $|z||z - 1| = |z^2 - z|$. These are the lengths of the sides of a triangle whose vertices are 0, z, z^2 as the second figure clearly shows.



The two triangles are similar because corresponding sides are proportional.

To obtain a geometric construction for z^2, one must choose a unit of length on the x-axis, draw a triangle with vertices 0, 1, z, and then construct a second triangle similar to the first one by making each side of the second triangle $|z|$ times as long as the sides of the first. The vertex of the second triangle which corresponds to z of the first triangle is z^2.

---

8. Complex Conjugate.

The introduction of the notion of complex conjugates has several important consequences. It makes possible: the simplification of computations involving absolute values and multiplicative inverses; the algebraic representation of the geometric operation of reflection in a line; the algebraic formulation and manipulation of statements involving the real and imaginary parts of complex numbers; and the algebraic representation of all geometric relations in terms of complex numbers.

In connection with the last of these features it should be observed that only geometric conditions which are satisfied by a finite number of points can be expressed in terms of the complex variable z alone, since an equation in z has only a finite number of solutions. The solution set of an equation in z alone is, in general, a finite set of points; the solution set of an equation in z and $\bar{z}$ is, in general, a curve.

The examples and exercises of Section 8 illustrate the statements made above. In particular, Problems 2, 9 and 11 are concerned with computations involving absolute value and multiplicative inverse; Problems 6 and 14 are concerned with reflection in lines; Problems 7, 8 and 10 are concerned with the algebraic formulation of statements about the real and imaginary parts of complex numbers; and Problems 3, 4, 12, 13, and 15 are concerned with the complex algebraic formulation of geometric conditions. Problem 1 provides practice in computing conjugates, and Problem 5 requires the proof of statements made in the text without proof.

Exercises 8. Answers.

1. (a) $2 + (-3)i$
   (b) $-3 + (-2)i$
   (c) $1 + (1)i$
   (d) $-5 + 0i$
   (e) $0 + 2i$
   (f) $1 + (1)i$
   (g) $0 + \pi i$
   (h) $3 + 0i$
   (i) $-\sqrt{3} + 3i$

2. (a) $\frac{3}{2} + (-\frac{1}{2})i$

(b) $\frac{1}{10} + \frac{3}{10}i$

(c) $-\frac{1}{13} + (-\frac{5}{13})i$

(d) $\frac{7}{29} + \frac{26}{29}i$

(e) $-\frac{3}{25} + \frac{46}{25}i$

(f) $\frac{1}{2} + (-\frac{3}{4})i$

(g) $-\frac{9}{25} + (-\frac{38}{25})i$

(h) $-3 + (-\frac{3}{2})i$

(i) $\frac{15}{14} + (-\frac{5\sqrt{5}}{14})i$

(j) $(\frac{15 + \sqrt{6}}{28}) + (\frac{3\sqrt{3} - 5\sqrt{2}}{28})i$

(k) $(\frac{3 + \sqrt{35}}{8}) + (\frac{\sqrt{15} - \sqrt{21}}{8})i$

(l) $\frac{1}{2} + \frac{1}{2}i$

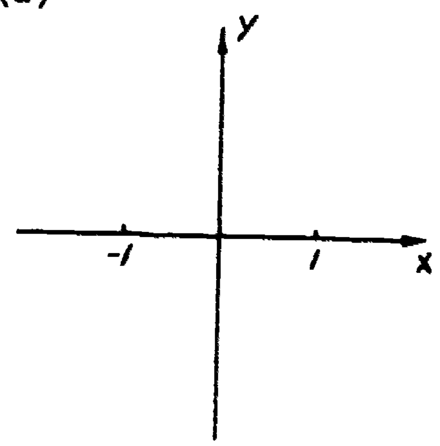
(m) $(\frac{2a^2 + 3b^2}{4a^2 + 9b^2}) + (\frac{-ab}{4a^2 + 9b^2})i$

(n) $(\frac{2x^2 - y^2}{4x^2 + y^2}) + (\frac{3xy}{4x^2 + y^2})i$

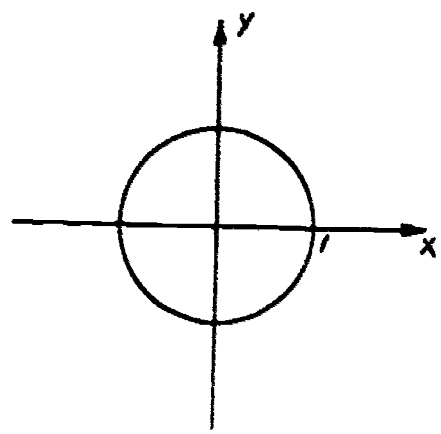
(o) $-\frac{2}{13} + (-\frac{3}{13})i$

(p) $\frac{1}{5} + 0i$

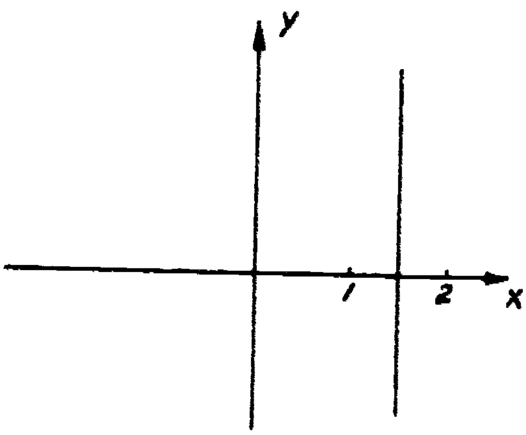
3. (a)



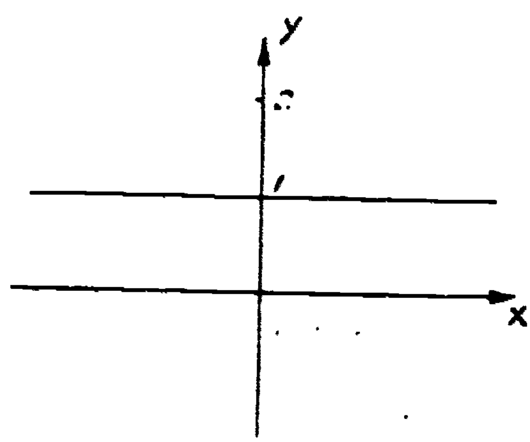
(b)



4. (a)



(b)



(c) There is no complex number z which satisfies the given equation. Hence, the set is empty.

5. (a) $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i.$

$$\begin{aligned}\overline{z_1 + z_2} &= (x_1 + x_2) - (y_1 + y_2)i \\ &= (x_1 - y_1 i) + (x_2 - y_2 i) \\ &= (x_1 - y_1 i) + (x_2 - y_2 i) \\ &= \overline{z}_1 + \overline{z}_2.\end{aligned}$$

(b) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$

$$\overline{z_1 \cdot z_2} = (x_1 x_2 - y_1 y_2) - (x_1 y_2 + x_2 y_1)i$$

But the expression in the right member is equal to the following:

$$\begin{aligned}\overline{z_1 \cdot z_2} &= (x_1 - y_1 i)(x_2 - y_2 i) \\ &= (x_1 x_2 - y_1 y_2) -(x_1 x_2 + x_2 y_1)i.\end{aligned}$$

Hence,

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2} .$$

(c) $\overline{(-z_2)} = \overline{(-x_2 - y_2 i)} = -x_2 + y_2 i$;

$-(\overline{z_2}) = -(\overline{x_2 + y_2 i}) = -(x_2 - y_2 i) = -x_2 + y_2 i$;

Hence, $\overline{(-z_2)} = -(\overline{z_2})$.

Since $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, we can now write

$$\overline{z_1 - z_2} = \overline{z_1 + (-z_2)} = \overline{z_1} + \overline{(-z_2)} = \overline{z_1} - \overline{z_2}.$$

(d) $\overline{\dfrac{1}{z_2}} = \overline{\dfrac{x_2 - y_2 i}{x_2^2 + y_2^2}} = \dfrac{x_2 + y_2 i}{x_2^2 + y_2^2}$;

$\dfrac{1}{\overline{z_2}} = \dfrac{1}{x_2 - y_2 i} = \dfrac{x_2 + y_2 i}{x^2 + y^2}$;

hence, $\overline{\dfrac{1}{z_2}} = \dfrac{1}{\overline{z_2}}$.

Since $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$, we can now write

$$\overline{\frac{z_1}{z_2}} = \overline{z_1}\,\overline{\frac{1}{z_2}} = \overline{z_1}\frac{1}{\overline{z_2}} = \frac{\overline{z_1}}{\overline{z_2}} .$$

6. The reflection of any point w in the y-axis is $-\overline{w}$. Hence, the reflection of $z^3 - (3 + 2i)z^2 + 5iz - 7$ is

$$-[\overline{z^3 - (3 + 2i)z^2 + 5iz - 7}] = -[\overline{(z^3)} - \overline{(3 + 2i)}\overline{(z^2)} + \overline{5i}(\overline{z}) - 7]$$
$$= -[(\overline{z})^3 - (3 - 2i)(\overline{z})^2 - 5i(\overline{z}) - 7]$$
$$= -\overline{z}^3 + (3 - 2i)\overline{z}^2 + 5i\overline{z} + 7.$$

7. If $z^2 = \overline{z}^2$, then $0 = z^2 - \overline{z}^2 = (z + \overline{z})(z - \overline{z})$, so either $z + \overline{z} = 0$ or $z - \overline{z} = 0$. In the first case z is pure imaginary, in the second case z is real.

8. A number w is pure imaginary if and only if $w = -\overline{w}$. Thus, $z_1\overline{z}_2$ is pure imaginary if and only if

$$z_1\overline{z_2} = -\overline{(z_1\overline{z}_2)}$$
$$z_1\overline{z_2} = -\overline{z_1}\ \overline{\overline{z_2}}$$
$$z_1\overline{z_2} = -\overline{z_1}z_2 .$$

Dividing this last equation by $z_2\overline{z_2}$ we obtain

$$\frac{z_1}{z_2} = -\frac{\overline{z_1}}{\overline{z_2}},$$

and

$$\frac{z_1}{z_2} = -\overline{\left(\frac{z_1}{z_2}\right)},$$

which holds if and only if $\frac{z_1}{z_2}$ is pure imaginary.

9. $$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2}) = (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$
$$= z_1\overline{z_1} + z_2\overline{z_2} - \overline{z_1}z_2 - z_1\overline{z_2}$$
$$= |z_1|^2 + |z_2|^2 - \overline{z_1}z_2 - z_1\overline{z_2}.$$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$
$$= z_1\overline{z_1} + z_2\overline{z_2} + \overline{z_1}z_2 + z_1\overline{z_2} .$$
$$= |z_1|^2 + |z_2|^2 + \overline{z_1}z_2 + z_1\overline{z_2}.$$

Thus, $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

10. Let $z_1 = x_1 + y_1 i$ and let $z_2 = x_2 + y_2 i$.

$z_1 + z_2$ is real if $y_1 + y_2 = 0$, and

$z_1 z_2$ is real if $x_1 y_2 + x_2 y_1 = 0$.

But if $y_1 + y_2 = 0$, then either $y_1 = y_2 = 0$, or $y_2 \neq 0$ and $y_1 = -y_2$. In the first case z_1 and z_2 are both real, and in the second case we have $x_1(y_2) + x_2(-y_2) = 0$, or $x_1 = x_2$. So in the second case $z_1 = \overline{z_2}$.

11. It is sufficient to show that

$$\left|\frac{z_1}{z_2}\right|^2 = \frac{|z_1|^2}{|z_2|^2}.$$

But $\left|\frac{z_1}{z_2}\right|^2 = \left(\overline{\frac{z_1}{z_2}}\right)\left(\frac{z_1}{z_2}\right) = \left(\frac{\overline{z_1}}{\overline{z_2}}\right)\left(\frac{z_1}{z_2}\right) = \frac{\overline{z_1}z_1}{\overline{z_2}z_2} = \frac{|z_1|^2}{|z_2|^2}$.

12. If $y = 3x + 2$ then since $x = \frac{1}{2}(\overline{z} + z)$, $y = \frac{1}{2}(\overline{z} - z)$ we have

$$\frac{1}{2}(\overline{z} - z) = 3 \cdot \frac{1}{2}(\overline{z} + z) + 2,$$

or simplifying

$$(-3 + i)\overline{z} + (-3 - i)z = 4,$$

which may also be written

$$(-3 + i)\overline{z} + \overline{(-3 + i)}z = 4.$$

13. Let $z = x + yi$ and $K = A + Bi$ where x, y and A,B are real. Substituting in

$$K\overline{z} + \overline{K}z = C$$

we get

$$(A + Bi)\overline{(x + yi)} + \overline{(A + Bi)}(x + yi) = C$$
$$(A + Bi)(x - yi) + (A - Bi)(x + yi) = C$$

$$[(Ax + By) + (Bx - Ay)i] + [(Ax + By) + (-Bx + Ay)i] = C$$

$$2(Ax + By) = C.$$

If $B \neq 0$, then

$$y = \frac{C - 2Ax}{B}$$

which is the equation of a straight line. If $B = 0$, then

$$x = \frac{C}{2A}$$

which is the equation of a straight line parallel to the y-axis.

14. The points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are symmetric with respect to the line $y = x$ if and only if $y = x$ is the perpendicular bisector of the segment joining z_1 and z_2. This is equivalent to two conditions: the midpoint of the segment joining z_1 and z_2 is on the line $y = x$; the segment joining z_1 and z_2 is perpendicular to the line $y = x$. The first of these conditions is algebraically

$$\frac{x_1 + x_2}{2} = \frac{y_1 + y_2}{2} ;$$

the second condition is

$$\frac{y_2 - y_1}{x_2 - x_1} = -1.$$

Thus, for symmetry with respect to $y = x$ the following pair of equations must be satisfied:

$$x_1 + x_2 = y_1 + y_2$$

$$y_2 - y_1 = x_1 - x_2.$$

Multiplying the second equation by i and adding the result to the first we obtain

$$x_1 + x_2 + i(y_2 - y_1) = y_1 + y_2 + i(x_1 - x_2)$$

$$(x_1 - iy_1) + (x_4 + iy_2) = (ix_1 + y_1) - (ix_2 - y_2)$$

$$(x_1 - iy_1) + (x_2 + iy_2) = i(x_1 - iy_1) - i(x_2 - iy_2)$$

$$\overline{z_1} + z_2 = i\overline{z_1} - iz_2$$

$$(1 - i)\overline{z_1} + (1 + i)z_2 = 0$$

which was to be proved.

15. Let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$. (We assume $z_1 \neq 0$, $z_2 \neq 0$ since otherwise the problem has no geometric meaning.) Then

$$z_1\overline{z_2} = (x_1 + y_1 i)(\overline{x_2 + y_2 i}) = (x_1 + y_1 i)(x_2 - y_2 i)$$
$$= (x_1 x_2 + y_1 y_2) + (y_1 x_2 - x_1 y_2)i,$$

so that if $z_1\overline{z_2}$ is real

$$y_1 y_2 - x_1 x_2 = 0.$$

If $x_1 = 0$ then since $y_1 \neq 0$ it follows from this equation that $x_2 = 0$, so that both z_1 and z_2 are on the y-axis and the segments joining them to the origin are parallel. The same conclusion is obtained in the same way if $x_2 = 0$. In the general case $x_1 \neq 0$ and $x_2 \neq 0$ so that we may divide our last equation by x_1x_2 to obtain

$$\frac{y_1}{x_1} - \frac{y_2}{x_2} = 0$$

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} .$$

Thus, the slopes of the segments joining z_1 and z_2 to the origin are equal and the segments are parallel. In every case therefore if $z_1\overline{z_2}$ is real, the segments z_1 and z_2 to the origin are parallel.

---

9. Polynomial Equations.

In this section we discuss the ultimate significance of the system of complex numbers for algebra. We state without proof the Fundamental Theorem of Algebra, and consider simple examples in which it applies.

Properly speaking, the Fundamental Theorem of Algebra states that every polynomial equation of positive degree with complex coefficients has at least one complex solution. The theorem we have stated as the Fundamental Theorem is obtained by combining the preceding statement with the Factor Theorem which asserts that if r is a solution of the polynomial equation $P(z) = 0$, then $z - r$ is a factor of $P(z)$. According to the Fundamental Theorem if $P(z)$ is a polynomial of degree $n > 0$ then the equation $P(z) = 0$ has a complex solution r_1. By the Factor Theorem then, $P(z) = (z - r_1)P_1(z)$ where $P_1(z)$ is a polynomial of degree $n - 1$. If $n - 1 > 0$, then applying the same argument to $P_1(z)$ we conclude that $P_1(z) = (z - r_2)P_2(z)$ or $P(z)=(z-r_1)(z-r_2)P_2(z)$. Continuing in this way we obtain the theorem stated in the text.

The teacher may wish to present the preceding discussion and a proof of the Factor Theorem to the class. The following simple proof of the Factor Theorem is based on the factoring identity

$$z^k - r^k = (z - r)(z^{k-1} + z^{k-2}r + z^{k-3}r^2 + \dots + r^{k-1}).$$

Let

$$P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$$

be a polynomial and let r be a solution of $P(z) = 0$; that is, $P(r) = 0$. Then,

$$P(z) = P(z) - P(r)$$

$$= (a_o z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_o) - (a_o r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_o)$$

$$= a_o(z^n - r^n) + a_1(z^{n-1} - r^{n-1}) + \dots + a_{n-1}(z - r)$$

$$= a_o(z-r)(z^{n-1} + z^{n-2} r + \dots + r^{n-1}) + a_1(z-r)(z^{n-2} + z^{n-3} r + \dots + r^{n-2}) + \dots + a_{n-1}(z - r)$$

$$= (z-r)[a_o(z^{n-1} + z^{n-2} r + \dots + r^{n-1}) + a_n(z^{n-2} + z^{n-3} r + \dots + r^{n-2}) + \dots + a_{n-1}]$$

$$= (z - r)Q(z).$$

<u>Exercises 9</u>. <u>Answers</u>.

1. (a) 1, multiplicity 1
   -2, multiplicity 3

   (b) 0, multiplicity r
   $-\frac{1}{2}$, multiplicity 2
   3, multiplicity 1

   (c) $3 - 2i$, multiplicity 2
   -1, multiplicity 5

2. (a) Since $z^5 + z^4 + 3z^3 = z^3[z - (\frac{-1 - \sqrt{11}i}{2})][z - (\frac{-1 + \sqrt{11}i}{2})]$, we have the following zeros:
   0, multiplicity 3
   $\frac{-1 - \sqrt{11}i}{2}$, multiplicity 1
   $\frac{-1 + \sqrt{11}i}{2}$, multiplicity 1

   (b) Since $z^4 + 2z^2 + 1 = (z + i)^2(z - i)^2$, we have the following zeros:
   $-i$, multiplicity 2
   i, multiplicity 2

   (c) Since $z^3 + 3z^2 + 3z + 1 = (z + 1)^3$, we have
   -1, multiplicity 3

3. (a) Example 1: $(z - 1)(z - 2) = 0$.
Example 2: $a(z - 1)(z - 2) = 0$, where a is real, non-zero, and not equal to 1.

(b) Example 1: $(z - 1)(z - 2) = 0$. This equation is of degree 2 and each zero is of multiplicity 1.

Example 2: $(z - 1)(z - 2)^2 = 0$. This equation is of degree 3 and has zeros which are of multiplicity 1 and 2, respectively.

4. An equation of degree 4 can have either <u>one</u>, <u>two</u>, <u>three</u>, or <u>four</u> solutions. The number which it has depends on the multiplicity of the zeros of the polynomial associated with the equation. The following examples are illustrative.

One solution: $(z - 1)^4 = 0$.

The polynomial $(z - 1)^4$ has the zero 1 of multiplicity four. Hence, the solution of the equation is the single value $z = 1$.

Two solutions: $(z - 1)(z - 2)^3 = 0$.

The zeros of the polynomial $(z - 1)(z - 2)^3$ are 1 (multiplicity one) and 2 (multiplicity three). The solutions of the equation are $z = 1,2$. Another example is $(z^2 + 1)^2$. Note that here we have two pairs of conjugate complex numbers.

Three solutions: $(z - 1)(z - 2)(z - 3)^2 = 0$.

The zeros of the polynomial are 1 (multiplicity one), 2 (multiplicity one), and 3 (multiplicity two). The solutions of the equation are $z = 1,2,3$.

Four solutions: $(z - 1)(z - 2)(z - 3)(z - 4) = 0$.

The zeros of the polynomial are 1,2,3,4; each is of multiplicity one. The solutions of the equation are $z = 1,2,3,4$.

5. $z^3 + 1 = (z + 1)(z^2 - z + 1) = 0$. Hence, $z = -1$ is one solution. To obtain the remaining solution, put

$$z^2 - z + 1 = 0.$$

Then

$$z = \frac{1 \pm \sqrt{3}i}{2}.$$

The solutions of the given equation are

$$-1, \quad \frac{1+\sqrt{3}i}{2}, \quad \frac{1-\sqrt{3}i}{2}.$$

6. (a) Since $z = 4$ is one solution, $(z - 4)$ is a factor of the polynomial in the left member of the given equation. Dividing the polynomial in the left member by $(z - 4)$, we find that the given equation can be rewritten in the form

$$(z - 4)(3z^2 - 8z + 4) = 0.$$

Factoring again, we have

$$(z - 4)(3z - 2)(z - 2) = 0.$$

The solutions of the equation are $4, \frac{2}{3}, 2$.

(b) $2, \quad 1 + i, \quad 1 - i$

7. (a) $-1, -2, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$.

(b) $4, 1, -1 + \sqrt{2}i, \quad -1 - \sqrt{2}i$.

8. (a) $(z - 1)(z + 2i)$ or $z^2 + (2i - 1)z - 2i$.
The polynomial is of degree 2.

(b) In order for the polynomial to have real coefficients it must have the conjugate of $-2i$ as a zero because it has $-2i$ as a zero. Hence, the polynomial must be of degree 3; the required polynomial is $(z - 1)(z + 2i)(z - 2i)$ or $z^3 - z^2 + 4z - 4$.

(c) The polynomial of lowest possible degree must contain the square of a polynomial of degree 2 which has both $-2i$ and $2i$ for its zeros. Thus, the required polynomial is of degree 5; it is

$$(z-1)[(z+2i)(z-2i)]^2 \quad \text{or} \quad z^5 - z^4 + 8z^3 - 8z^2 + 16z - 16.$$

9. Since $3 + \sqrt{2}i$ is a solution of the equation, so is $3 - \sqrt{2}i$. Thus,

$$\{[z - (3 + \sqrt{2}i)][z - (3 - \sqrt{2}i)]\} = (z^2 - 6z + 11)$$

is a factor of the polynomial in the left member of the given equation. By long division it can be found that the other factor is $(z^2 - 9)$. Hence, the solutions of the equation are

$$3 + \sqrt{2}i, \quad 3 - \sqrt{2}i, \quad 3, \quad -3.$$

10. $1 - \sqrt{5}i, \quad 1 + \sqrt{5}i, \quad \sqrt{2}, \quad -\sqrt{2}.$

11. (a) $(z - r_1)(z - r_2)(z - r_3) =$

$$z^3 + [(-r_1)+(-r_2)+(-r_3)]z^2 + [(-r_1)(-r_2)+(-r_1)(-r_3)+(-r_2)(-r_3)]z + [(-r_1)(-r_2)(-r_3)]$$

$$= z^3 - (r_1 + r_2 + r_3)z^2 + (r_1r_2 + r_1r_3 + r_2r_3)z - (r_1r_2r_3).$$

(b) $(z - r_1)(z - r_2)(z - r_3)(z - r_4)$

$$= z^4 - (r_1+r_2+r_3+r_4)z^3+(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)z^2 - (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)z + (r_1r_2r_3r_4).$$

(c) $(z - r_1)(z - r_2)\ldots(z - r_7)$

$$= z^7 - (r_1 + r_2 + \ldots + r_7)z^6 + (r_1r_2 + r_1r_3 + \ldots + r_6r_7)z^5 - (r_1r_2r_3 + r_1r_2r_4 + \ldots + r_5r_6r_7 + \ldots + (-1)^7(r_1r_2 \ldots r_7).$$

10. Answers to Miscellaneous Exercises.

1. $-(2 - 3i) = -2 + 3i$

$\overline{(2 - 3i)} = 2 + 3i$

$|2 - 3i| = \sqrt{4 + 9} = \sqrt{13}$

$|\overline{2 - 3i}| = |2 - 3i| = \sqrt{13}$

$$\frac{1}{2 - 3i} = \frac{\overline{2 - 3i}}{|2 - 3i|^2} = \frac{2 + 3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

$|2 - 3i|^2 = (\sqrt{13})^2 = 13$

$|(2 - 3i)^2| = |2 - 3i|^2 = 13$

$$\frac{4 + 5i}{2 - 3i} = (4 + 5i)\frac{1}{2 - 3i} = (4 + 5i)(\frac{2}{13} + \frac{3}{13}i)$$
$$= -\frac{7}{13} + \frac{22}{13}i.$$

2. $[z - (c + di)][z - (c - di)] = 0$

$z^2 - [(c + di) + (c - di)]z + (c + di)(c - di) = 0$

$z^2 - 2cz + (c^2 + d^2) = 0.$

3. It is closed with respect to multiplication, but not with respect to addition since $1 + 1$ is not in the set.

4.
$$y^2 \geq 0$$
$$x^2 + y^2 \geq x^2$$
$$\sqrt{x^2 + y^2} \geq \sqrt{x^2}$$
$$|x + iy| \geq |x| \geq x$$
$$|z| \geq x$$

5. (a) Circle of radius 3 with center at (2,0).

(b) Set of points exterior to circle of radius 3 with center at (-2,0).

(c) Set of points interior to circle of radius 4 with center at (0,2).

(d) Set of points interior to, or on, circle of radius 5 with center at z_0.

6.
$$|x + yi - (2 + 3i)| = 5$$
$$|(x - 2) + (y - 3)i| = 5$$
$$\sqrt{(x - 2)^2 + (y - 3)^2} = 5$$
$$(x - 2)^2 + (y - 3)^2 = 25$$
$$x^2 + y^2 - 4x - 6y - 12 = 0.$$

The set of points satisfying the given equation is the circle of radius 5 with center at (2,3).

7. (a) The distance from the origin of z_1 is less than that of z_2.

(b) z is on the circle of radius 5 with center at the origin.

(c) z_1 and z_2 are symmetric with respect to the origin.

(d) z_1 and z_2 are symmetric with respect to the y-axis.

(e) z_1 and z_2 are symmetric with respect to the x-axis.

8. If $z = x + yi$ the stated conditions become

$$x = y, \quad \sqrt{x^2 + y^2} = 1.$$

The solutions of this pair of equations are $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ and $x = -\frac{1}{\sqrt{2}}$, $y = -\frac{1}{\sqrt{2}}$. The solutions of the problem are therefore $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

9. If the coefficients are real and $3 + 2i$ is a solution, then $3 - 2i$ must also be a solution. If the equation is quadratic, it can have no more than these two solutions. Thus, the equation must be

$$a[z - (3 + 2i)][z - (3 - 2i)] = 0$$

or

$$az^2 - 6ax + 13a = 0$$

where $a \neq 0$ is any real number.

10. We show generally that if $z = x + yi$ is any complex number (not zero) the quadrilateral with vertices z, iz, i^2z, i^3z is a square. The midpoints of the diagonals of this quadrilateral are

$$\frac{z + i^2z}{2} = \frac{z - z}{2} = 0$$

$$\frac{iz + i^3z}{2} = \frac{iz - iz}{2} = 0$$

so that the diagonals bisect each other at the origin. Thus, the quadrilateral is a parallelogram. The slope of the segment joining the origin to $z = x + yi$ is $\frac{y}{x}$; the slope of the segment joining the origin to $iz = i(x + yi) = -y + xi$ is $-\frac{x}{y}$. Since these slopes are negative reciprocals, the diagonals are perpendicular. Thus, the parallelogram is a rhombus. Finally, each diagonal is equal to $2|z|$ and hence, the rhombus, having equal diagonals, is a square.

11. If z_o is a solution, then $\overline{z}_o$ is also a solution, since the coefficients are real. By the Fundamental Theorem

$$\begin{aligned} az^2 + bz + c &= a(z - z_o)(z - \overline{z}_o) \\ &= a[z^2 - (z_o + \overline{z}_o)z + z_o\overline{z}_o] \\ &= az^2 - a(z_o + \overline{z}_o)z + az_o\overline{z}_o. \end{aligned}$$

Equating coefficients we obtain

$$b = -a(a_o + \bar{z}_o), \quad c = az_o\bar{z}_o$$

or

$$z_o + \bar{z}_o = -\frac{b}{a}, \quad z_o\bar{z}_o = \frac{c}{a}.$$

The curve $z + \bar{z} = -\frac{b}{a}$ is the straight line $x = -\frac{b}{2a}$. The curve $z\bar{z} = \frac{c}{a}$ is the circle $x^2 + y^2 = \frac{c}{a}$. Since z lies on both curves it is one of the points of intersection of these two curves (the other is $\bar{z}_o$). Thus, to construct the roots of the quadratic equation $az^2 + bz + c = 0$ $(b^2 - 4ac < 0)$ draw a circle of radius $\sqrt{\frac{c}{a}}$ about the origin and draw the straight line parallel to the y-axis through $(-\frac{b}{2a}, 0)$. The solutions of the equation are the points of intersection of these curves.

12. $a(z^2 - z + 4) = 0 \qquad a$ real, $a \neq 0$.

13. If $z = x + yi$ then $z^2 = x^2 - y^2 + 2xyi$ so that the real part of z^2 is 0 if and only if $x^2 - y^2 = 0$. Since $x^2 - y^2 = (x + y)(x - y)$, $x^2 - y^2 = 0$ if and only if $x + y = 0$ or $x - y = 0$. Thus, the set of points satisfying the given condition is the set of points on the lines of slope 1 and -1 through the origin.

We have $(\frac{1}{z})^2 = (\frac{\bar{z}}{|z|^2})^2 = \frac{\bar{z}^2}{|z|^4}$. If the real part of z^2 is zero, then the real part of $\bar{z}^2$ is zero, since z^2 and $\bar{z}^2 = \overline{z^2}$ are conjugates. Since $(\frac{1}{z})^2 = \frac{1}{|z|^4}\bar{z}^2$, and $\frac{1}{|z|^4}$ is real, it follows that the real part of $(\frac{1}{z})^2$ is zero.

14. The discriminant of the equation is

$$(1 + r)^2 - 4 \cdot 2 \cdot r = r^2 - 6r + 1.$$

The equation has only one real root when the discriminant is 0; that is, when r is one of the zeros--$3 - 2\sqrt{2}$, $3 + 2\sqrt{2}$--of the discriminant. The equation has complex roots when the discriminant is negative. For very large values of r the discriminant is positive, so that it will be negative if and only if r is between its zeros--$3 - 2\sqrt{2} < r < 3 + 2\sqrt{2}$.

15. If $a = 1$, $b = i$, then $a + bi = 1 + i \cdot i = 0$; $\overline{a+bi} = \overline{0} = 0$; $a - bi - i \cdot i = 2$. Thus, $\overline{a + bi} \neq a - bi$ in this case.

16. The set of points equidistant from z_1 and z_2 is the set of points z which satisfy the equation

$$|z - z_1| = |z - z_2|.$$

Squaring this equation we have

$$|z - z_1|^2 = |z - z_2|^2$$

from which we get

$$(z - z_1)\overline{(z - z_1)} = (z - z_2)\overline{(z - z_2)}$$

$$(z - z_1)(\overline{z} - \overline{z}_1) = (z - z_2)(\overline{z} - \overline{z}_2)$$

$$z\overline{z} - \overline{z}_1 z + z_1\overline{z} + z_1\overline{z}_1 = z\overline{z} - \overline{z}_2 z - z_2\overline{z} + z_2\overline{z}_2$$

$$(\overline{z}_1 - \overline{z}_2)z + (z_1 - z_2)\overline{z} = z_1\overline{z}_1 - z_2\overline{z_2}.$$

The last equation is the equation of the perpendicular bisector of the segment.

17. The point z belongs to the set if and only if $|z - \overline{z}_o| < |z - z_o|$; that is, if and only if the distance from z to $\overline{z_o}$ is less than the distance from z to z_o. This will be true if and only if the point z lies on the same side as z_o of the perpendicular bisector of the segment joining z_o and $\overline{z_o}$. This perpendicular bisector is the x-axis. Thus, the set is the set of all points z which lie on the same side of the x-axis as $\overline{z}_o$. This can also be established by calculation.

18. Let $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c and d are real.

$$(1) \quad \frac{z_1}{z_2} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} \cdot i.$$

Hence, $\frac{z_1}{z_2}$ is real <u>if and only if</u> (2) $bc - ad = 0$.

It can be shown that $bc - ad = 0$ <u>if and only if</u> z_1 and z_2 are on a straight line through the origin. To establish this we must prove two if-then statements.

(a) If z_1 and z_2 are on a straight line through the origin then $bc - ad = 0$, <u>and</u>

(b) If $bc - ad = 0$ then z_1 and z_2 are on a straight line through the origin.

Proof of (a): If the line is the y-axis, then $a = 0$ and $c = 0$ and we have at once $b \cdot 0 - 0 \cdot d = 0$. If the line is not the y-axis, then the slope of this line joining the origin to z_1 is equal to the slope of the line joining the origin to z_2, i.e., $\frac{b}{a} = \frac{d}{c}$.

Hence, $bc = ad$ or $bc - ad = 0$.

Proof of (b): We have (2) $bc = ad$. If z_1 is on the y-axis then $a = 0$ and by (2) $bc = 0$. But $b \neq 0$ because $z = a + bi \neq 0$ by hypothesis. Hence, $c = 0$ and z_2 is also on the y-axis. This proves that z_2 is on the y-axis if z_1 is and the two points are on a straight line through the origin.

If z_1 is not on the y-axis, then $a \neq 0$. From this we see that $c \neq 0$ because if $c = 0$ and $a \neq 0$ we must conclude from (2) that $d = 0$ and this would mean that $z_2 = c + di = 0$ in violation of our hypothesis that z_2 is a non-zero complex number. Hence, $ac \neq 0$ and we may divide both members of (2) by ac to obtain

$$\frac{b}{a} = \frac{d}{c}$$

which is precisely the condition that z_1 and z_2 lie on a straight line through the origin.

We summarize our argument

$\frac{z_1}{z_2}$ is real if and only if $bc - ad = 0$ and $bc - ad = 0$ if and only if z_1 and z_2 lie on a straight line through the origin. Therefore, $\frac{z_1}{z_2}$ is real if and only if z_1 and z_2 lie on a straight line through the origin.

19. $z^4 = -1$ or $z^4 + 1 = 0$.

$(z^2 + i)(z^2 - i) = 0$

Hence, $z^2 = -i$ or $z^2 = i$. The solution set is evident by the union of the solution sets of the equations solved in Problems 22 and 23 of Exercise 6, namely,

$$\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \quad -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \quad -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \quad \text{and} \quad \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.$$

20. It will be sufficient to show that the law of trichotomy is inconsistent with O_4 for the element i. Certainly, $i \neq 0$. Then either $i > 0$ or $i < 0$. In either case, by O_4 we have $i^2 > 0$ and we are confronted by the contradiction $-1 > 0$.

21. If x and y are real, it is evident that the conjugate of $x + yi$ is $x - yi$. Moreover, it can be shown that if $\overline{x + yi} = x - yi$, then x and y are real. Let $x = a + bi$ and $y = c + di$ where a, b, c and d are real.

$$x + yi = (a - d) + (b + d)i$$

$$\overline{x + yi} = (a - d) - (b + c)i$$

$$x - yi = (a + d) + (b - c)i$$

Since $\overline{x + yi} = x - yi$ we have

$$(a - d) - (b + c)i = (a + d) + (b - c)i$$

According to Theorem 5-4

$$a - d = a + d$$

and

$$-(b + c) = b - c.$$

From these equations we conclude that $d = 0$ and $b = 0$.

$\therefore$ $x = a$ and $y = c$ where a and c are real.

Hence, $\overline{x + yi} = x - yi$ if and only if x and y are real.

22. The proposition stated is true provided x and y are real. In this event we have

$|x| + |y| \leq \sqrt{2}|z|$ if and only if

$(|x| + |y|)^2 \leq 2|z|^2$.

Now, $|z|^2 = x^2 + y^2$ and we have

$|x|^2 + 2|x||y| + |y|^2 \leq 2|x|^2 + 2|y|^2$.

This reduces to $0 \leq |x|^2 - 2|x||y| + |y|^2$ or

$0 \leq (|x| - |y|)^2$ which is

true because the square of any real number is non-negative. Q.E.D.

The proposition is not true for all complex values of x and y as the following counter-example will show.

Let $x = 8 + 2i$ and $y = -1 + 4i$

then $|x| = \sqrt{68} = 2\sqrt{17}$ and $|y| = \sqrt{17}$.

$|x| + |y| = 3\sqrt{17}$

$z = x + iy = (8 + 2i) + i(-1 + 4i) = 4 + i$.

$|z| = \sqrt{17}$. It is false that $\sqrt{2}\sqrt{17} \geq 3\sqrt{17}$, hence, in this case $|x| + |y|$ is not equal to or less than $\sqrt{2}|z|$.

---

11. Construction of the Complex Number System.

Section 11 outlines Gauss's construction of the complex number system. As a source of historical information we suggest The Development of Mathematics, by E.T. Bell (McGraw-Hill, 1945, Second Edition): Wessell and Argand, p.177; Gauss, p.179; Cauchy, p.194.

---

12. Sample Test Questions for Chapter 5. (Answers on Page 55)

Note: In the questions included in this section a, b, c, d, x, y are real numbers and z is a complex number.

Part I: True-False.

Directions: If a statement is true, mark it T; if the statement is false, mark it F.

1. The imaginary part of $a + bi$ is bi.
2. The discriminant of the equation $x^2 + 2 = 0$ is -8.
3. Every complex number has an additive inverse.
4. A one-to-one correspondence can be established between points of the xy-plane and the elements of C.
5. The product of a complex number and its conjugate is a complex number.
6. The sum of a complex number and its conjugate is a pure imaginary number.

7. If the coefficients of a quadratic equation are real numbers, then the roots of the equation are real numbers.

8. $|z|$ is a non-negative real number.

9. The sum of z and $-\overline{z}$ is a real number.

10. If z is a complex number, z and $\overline{z}$ correspond to points in the xy-plane which are symmetric with respect to the y-axis.

11. The multiplicative inverse of $(x - yi)$ is $\frac{x - yi}{x^2 + y^2}$.

12. If $(a + bi)(x + yi) = 1$, then $ax - by = 1$.

13. $\overline{z_1 + z_2} = \overline{z}_1 - \overline{z}_2$.

14. $(a + bi)\overline{(a + bi)} = a^2 + b^2$.

15. If $|z| = 1$, then z is its own multiplicative inverse.

16. The set of numbers $\{1, -1, i, -i\}$ is closed under multiplication.

17. $|z_1| + |z_2| \leq |z_1 + z_2|$.

18. The reflection of $\overline{z}$ in the y-axis is $-z$.

Part II: Multiple Choice.

Directions: Select the response which best completes the statement or answers the question.

19. Which one of the following equations does not have a solution in the real number system?

(a) $x + 5 = 5$
(b) $(x + 5)^2 = 9$
(c) $x^2 + 5 = 0$
(d) $x^2 - 5 = 0$
(e) $\sqrt{x + 5} = 0$

20. What ordered pair of real numbers (x,y) satisfies the equation $x - 4yi = 20i$?

(a) (20,0)
(b) (0,-5)
(c) (0,20)
(d) (0,5)
(e) (0,0)

5

21. If $z = (5 - 6i) - (3 - 4i)$, then the standard form of z is

(a) $2 - (10)i$
(b) $2 + (2)i$
(c) $2 + (-10)i$
(d) $2 + (-2)i$
(e) $2 + (10)i$

22. The additive inverse of $c - di$ is

(a) $-c + di$
(b) $\frac{1}{c - di}$
(c) $c + di$
(d) 1
(e) 0

23. If the complex number $5 + 5i$ is represented by the point P in an Argand diagram, then the slope of the line segment joining P and the origin is

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 5 (d) 1 (e) 0

24. Which one of the following expressions does not represent a real number?

(a) $i^2 + \sqrt{2}$
(b) $3i^4$
(c) $\sqrt{(-3)^2}$
(d) $6 + 2i$
(e) $(2i)^6 - \sqrt{3}$

25. The multiplicative inverse of i is

(a) i (b) $-i$ (c) 1 (d) -1 (e) $-\frac{1}{i}$

26. Which one of the following equations has non-real solutions?

(a) $x - 4 = 6$
(b) $4x^2 - 3x + 6 = 0$
(c) $6x^2 + 5x - 8 = 0$
(d) $2x^2 - 14x + 3 = 0$
(e) $x^2 = \sqrt{14}$

27. The conjugate of -4 written in standard form is

(a) $4 + 0i$
(b) $-\frac{1}{4} - 0i$
(c) $\frac{-4}{16} - \frac{i}{16}$
(d) $-4 + 0i$
(e) None of these

28. Which one of the following is not equivalent to each of the other four?

(a) $\sqrt{(2)^2}$ (d) $\sqrt{-(2i)^2}$

(b) $\sqrt{(-2)^2}$ (e) $\sqrt{4}$

(c) $\sqrt{-(2)^2}$

29. The product of $(2 + 3i)$ and $(5 - 3i)$ is

(a) $19 + 9i$ (d) $1 - 21i$

(b) $19 + 21i$ (e) $10 - 9i$

(c) $1 + 9i$

30. When written in standard form the real part of $(2 - i)^2$ is

(a) 1 (b) -1 (c) 5 (d) -3 (e) 3

31. Given $z = -3i$, then $\bar{z}$ in standard form is

(a) $3i$ (b) $0 + 3i$ (c) $|3|i$ (d) $0 + (-3)i$ (e) $-3i$

32. The smallest set which contains the absolute value of every complex number is the set of

(a) natural numbers (d) rational numbers

(b) integers (e) complex numbers

(c) real numbers

33. The additive inverse of i is

(a) 1 (b) -1 (c) i (d) $-i$ (e) 0

34. Which one of the following pairs of complex numbers can be represented by points which are symmetric with respect to the origin in an Argand diagram?

(a) $3 + 2i$, $3 - 2i$ (d) $3 + 2i$, $-3 + 2i$

(b) $3 + 2i$, $2 + 3i$ (e) $3 + 2i$, $-2 - 3i$

(c) $3 + 2i$, $-3 - 2i$

35. In an Argand diagram the set of points defined by the equation $|z|^2 = 5$ is

(a) a point (d) a circle

(b) a straight line (e) two parallel lines

(c) two perpendicular lines

36. If z is a complex number such that $\frac{z}{\bar{z}} = -1$ and $z\bar{z} = 1$, then z is

(a) i
(b) $-i$
(c) i or $-i$
(d) 1 or -1
(e) i or $-i$ or -1.

37. Which of the following ordered pairs of real numbers (x,y) satisfies the equation $3x + 5yi - 8 = 5x - yi + 6i^2$?

(a) $(-4,1)$
(b) $(-1,0)$
(c) $(0,-1)$
(d) $(4,1)$
(e) $(-4,-4)$

38. Which of the following equations has the solutions $2 - i$ and $3i$?

(a) $z^2 - 4z + 5 = 0$

(b) $z^2 - (2 + 4i)z + (3 + 6i) = 0$

(c) $z^2 - (2 + 2i)z + (3 + 6i) = 0$

(d) $z^2 - (2 + 2i)z + (-3 + 6i) = 0$

(e) $z^2 - (2 - 2i)z + (6 - 3i) = 0$

39. The equation $z^3 - 2z^2 + z - 2 = 0$ has i as one of its solutions. The other solutions of the equation are

(a) $-1,2$ (b) $-i,-2$ (c) $-i,1$ (d) $-i,2$ (e) $0,-i$

40. Which one of the following complex numbers is the reflection of $2 - 3i$ in the y-axis?

(a) $-2 - 3i$
(b) $-2 + 3i$
(c) $2 + 3i$
(d) $-3 - 2i$
(e) $3 - 2i$

41. The solution set of the equation $z^2 + a^4 = 0$, where a is a real number, is

(a) $\{a^2, -a^2\}$
(b) $\{a, -a, ai, -ai\}$
(c) $\{a, -a, i, -i$
(d) $\{a^2i, -a^2i\}$
(e) $\{-a^2, a^2i, -a^2i\}$

42. The length of the line segment which joins the points representing $3 + 4i$ and $-4 + 5i$ is

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $5\sqrt{2}$ (d) 1 (e) 50

Part III: Matching.

Directions: In questions 43-49 choose the point on the Argand diagram which represents the given number. Write the letter which identifies the point of your choice on an answer sheet. Any choice may be used once, several times, or not at all.

43. $2 - 3i$

44. $3 - 0i$

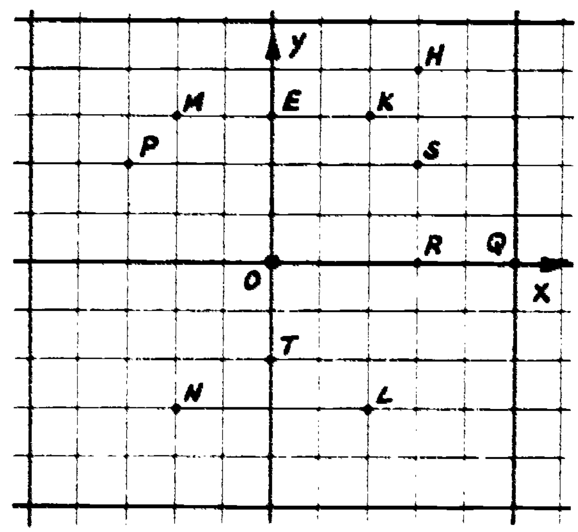
45. $\overline{-2 + 3i}$

46. $|3 + 4i|$

47. $(2 + 3i) + (1 - i)$

48. $(3 + 2i) - (5 + 5i)$

49. z_1 such that $|z_1| = 2$



Part IV: Problems.

50. Express the quotient $\frac{1 - i^3}{3 + i}$ in standard form.

51. If $z = 4 + 2i - i^6$, find the standard form of z.

52. Find the ordered pair of real numbers (x,y) that satisfies the equation $x - 15i = 5yi$.

53. Find the real values of x and y which satisfy the equation $x - y + (x + y)i = 2 + 6i$.

54. Solve the equation $(x + yi)(2 + i) + 3x - 11 = 0$ for real values of x and y.

55. For what real values of k does the equation $z^2 + kz + 1 = 0$ have solutions that are <u>not</u> real?

56. Write a quadratic equation with real coefficients which has $5 + i$ as one of its roots.

57. If $z_1 = -2 + i$ and $z_2 = 1 + 4i$, find $z_1 + z_2$ in standard form and exhibit the sum graphically.

58. Describe the set of points in the plane which satisfy the condition $|z|$ = the real part of z.

59. Solve each of the following equations and express the solutions in standard form:

(a) $3z^2 + z + 1 = 0$

(b) $z^2 + z + c = 0$, c is a positive integer

(c) $pz^2 + q = 0$, $p < 0$, $q > 0$, and p and q are real.

60. Given the following numbers: 2, -12, 41, $\frac{2}{3}$, $-\sqrt{16}$, 0, π, $\sqrt{-9}$, $\sqrt[3]{-27}$, $\sqrt{50}$, $\frac{1}{\sqrt{3}}$, $-\frac{3}{5}$, $\sqrt{5}$, $2\sqrt{16}$, $\sqrt{23}$, $\sqrt[3]{4}$, 1.74, $\sqrt{-3}$, $3.\overline{37}$, $-\frac{1}{\sqrt{7}}$, i^2, $2 + \sqrt{3}$, $2 - \sqrt{-5}$.

(a) Classify the given numbers into two lists; real numbers and imaginary numbers.

(b) Reclassify the real numbers into rational and irrational numbers.

Answers to Sample Test Questions.

Part I: True-False.

1.	F	7.	F	13.	F
2.	T	8.	T	14.	T
3.	T	9.	F	15.	F
4.	T	10.	F	16.	T
5.	T	11.	F	17.	F
6.	F	12.	T	18.	T

Part II: Multiple Choice.

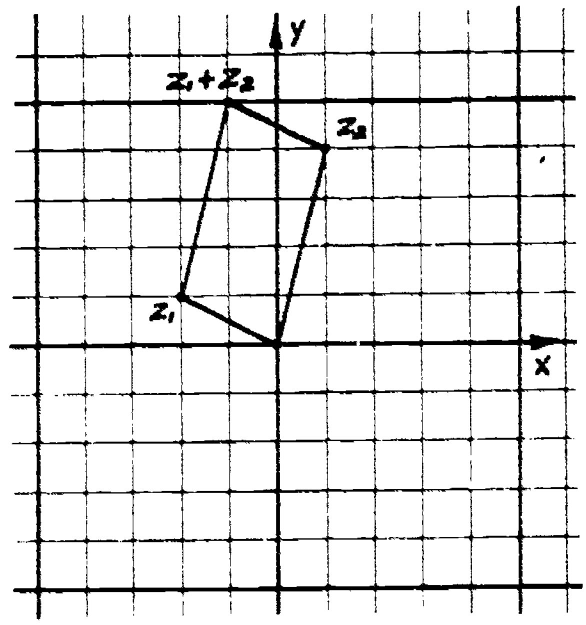
19.	(c)	27.	(d)	35.	(d)
20.	(b)	28.	(c)	36.	(c)
21.	(d)	29.	(a)	37.	(b)
22.	(a)	30.	(e)	38.	(c)
23.	(d)	31.	(b)	39.	(d)
24.	(d)	32.	(c)	40.	(a)
25.	(b)	33.	(d)	41.	(d)
26.	(b)	34.	(c)	42.	(c)

Part III: Matching

43. L
44. R
45. N
46. Q
47. S
48. N
49. T

Part IV: Problems

50. $\frac{2}{5} + \frac{1}{5}i$
51. $5 + 2i$
52. $(0, -3)$
53. $x = 4, \quad y = 2$
54. $x = 2, \quad y = -1$
55. $|k| < 2$
56. $z^2 - 10z + 26 = 0$
57. $z_1 + z_2 = -1 + 5i$



58. Non-negative part of x-axis.
59. (a) $z = -\frac{1}{6} + \frac{\sqrt{11}}{6}i, \quad -\frac{1}{6} + (-\frac{\sqrt{11}}{6})i$

    (b) $z = -\frac{1}{2} + \frac{\sqrt{4c - 1}}{2}i, \quad -\frac{1}{2} + (-\frac{\sqrt{4c - 1}}{2})i$

    (c) $z = \sqrt{-\frac{q}{p}} + 0i, \quad \sqrt{\frac{q}{p}} + 0i$

60. The table shows the answers to both parts (a) and (b).

REAL		IMAGINARY
RATIONAL	IRRATIONAL	
2		
-12		
		$4i$
$\frac{2}{3}$		
$-\sqrt{16}$		
0		
	π	
		$\sqrt{-9}$
$\sqrt{-27}$		
	$\sqrt{50}$	
		$i\sqrt{3}$
$-\frac{3}{5}$		
	$\sqrt{5}$	
$2\sqrt{16}$		
	$\sqrt{23}$	
	$\sqrt[3]{4}$	
1.74		
		$\sqrt{-3}$
$3.\overline{37}$		
		$-i\sqrt{7}$
i^2		
	$2 + \sqrt{3}$	
		$2 - \sqrt{-5}$