DOCUMENT RESUME

ED 175 663 SE 028 651

AUTHOR	Kalman, Karl, Ed.
TITLE	Supplementary and Enrichment Series: The Complex Number System. SP-4.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Sc ce Foundation, Washington, D.C.
PUB DATE	64
NOTE	58p.: For related documents, see SE 028 648-675: Contains occasional light and broken type
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	*Algebra: Curriculum: *Enrichment: *Instruction: Mathematics Education: *Number Systems: Secondary Education: *Secondary School Mathematics: Supplementary Reading Materials
IDENTIFIERS	*Complex Numbers: *School Mathematics Study Group

ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is des-^ned to make material for the study of topics of special interest to . dents readily accessible in classroom quantity. Topics covered inclu e operations, standard form, equations, graphs, and conjugates. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-4

SUPPLEMENTARY and ENRICHMENT SERIES

THE COMPLEX NUMBER SYSTEM

Edited by Karl Kalman

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Printed in the United States of America

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

FOREWORD

This pamphlet is essentially a reprint of Chapter 5 of the text titled "Intermediate Mathematics" published by the School Mathematics Study Group.

The purpose of this publication is to make available to classes of students some new materials to be used in conjunction with standard programs. A class in second-year algebra using a standard textbook could, with some preparation, study the topic of "Complex Numbers" from this pamphlet. In order to do this, the students would need some experience with the properties of the "Real Numbers".

CONTENTS

COMPLEX NUMBER SYSTEMS

1. Introduction.

Consider equations of the form

$$ax^2 + bx + c = 0, \tag{1a}$$

where a, b, c are real numbers, $a \neq 0$. It is assumed that we understand the method for solving such equations and that the results depend in a very essential way on the value of the discriminant, $b^2 - 4ac$. If $b^2 - 4ac > 0$, the equation has two real solutions; if $b^2 - 4ac = 0$, the equation has one real solution; if $b^2 - 4ac < 0$, the equation has no real solution.

We ask whether we can extend our number system to include numbers of such a character that every quadratic equation with real coefficients has a solution regardless of the value of its discriminant. It is the task of this pamphlet to make such an extension of the system of real numbers. Actually we shall find that the system we derive for this purpose is a richer one than we bargain for: it gives us the solutions not only of all quadratic equations with real coefficients, but also of all polynomial equations of whatever degree with real coefficients. Even this does not quite describe the richness of the system we derive, but it is too soon to tell the whole story. Let it suffice to say that no further extensions will be necessary for the purposes of ordinary algebra.

The simplest example of a quadratic equation with a negative discriminant is the equation

$$x^2 + 1 = 0. \tag{1b}$$

If this equation is written in the form (1a) we have $a = 1$, $b = 0$, $c = 1$, and the discriminant is

$$b^2 - 4ac = -4,$$

so that we know that it has no real solutions. We can see this without evaluating the discriminant. Since the square of each real number is non-negative, we have $x^2 \geq 0$ for any real number x. Thus, if x is real, $x^2 + 1 \geq 0 + 1 = 1 > 0$, so that no real number is a solution of equation (1b).

To start we will look for a number system in which equation (1b) has a solution. It will turn out, in Section 5, that in this system every quadratic equation with real coefficients has a solution. Perhaps if you review the method of solving the quadratic equation you can see why this should be so.

Before undertaking our extension of the system of real numbers, it would be useful to look at the procedure followed in Chapter 1 of the SMSG text in Intermediate Mathematics each time the number system was extended. In this chapter the properties of the real number system were developed by starting with the natural number, followed by a consideration of the system of integers. After this came the rational numbers and finally the real numbers. In this development it was assumed that a new system could be constructed which would: (1) have as many as possible of the algebraic properties of the old system; (2) include all the numbers of the old system, in such a way that the new and the old algebraic operations, when applied to numbers of the old system, would be the same; (3) contain new numbers of the kind we need. We then discovered the rules for operating with the new numbers as logical consequences of the properties we assumed. For reference, a "List of Basic Properties of the Real Number System" is included in the Appendix.

Proceeding in the same way we now seek a new number system which contains the system of real numbers with all its familiar properties and also contains a number satisfying $x^2 + 1 = 0$, Equation (1b). We shall designate the system by the letter C and call it the system of complex numbers. Following are the specific properties we require of C:

Property C-1

(i) Two operations, addition (+) and multiplication (·) are defined in C. (It is to be understood that the result of an operation defined in a system is a number in the system, but when we wish to emphasize this fact we will say that the system is closed with respect to the operation.)

(ii) Addition is associative and commutative.

(iii) C possesses one and only one additive identity.

(iv) Each element of C has one and only one additive inverse.

(v) Multiplication is associative and commutative.

(vi) C possesses one and only one multiplicative identity.

(vii) Each element of C, other than the additive identity, has one and only one multiplicative inverse.

(viii) Multiplication is distributive with respect to addition.

Property C-2

(i) Every real number is a member of C.

(ii) The sum of two real numbers in C is the same as their sum in the real number system.

(iii) The product of two real numbers in C is the same as their product in the real number system.

(iv) The additive identity in C is the number 0 of the reals.

(v) The multiplicative identity in C is the number 1 of the reals.

Property C-3

The set C contains a special element i which has the property

$$i \cdot i = i^2 = -1.$$

We call the special element i the imaginary unit.

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2. Complex Numbers.

In Section 1 we stated a problem: to find a number system--that is, a set of elements and the operations of addition and multiplication defined for the set--having properties C-1, C-2 and C-3. Now we try to solve this problem. Let us first try to identify the set of elements.

Property C-3 implies that C contains at least one member not in the set of real numbers because the square of no real number is negative. By C-1, C is closed under the operations of addition and multiplication, so that if a and b are real numbers, the product bi is in C since b and i are, and it follows that $a + bi$ is in C since a and bi are. We see, then, that all numbers of the form

$$a + bi, \text{ where } a \text{ and } b \text{ are real,}$$

are included in C. The number i and every real number can be written in this form. We have $i = 0 + 1 \cdot i$. If a is any real number $a = a + 0 \cdot i$, since $0 \cdot i = 0$. (The statement that the product of 0 and any number is 0 can be proved for numbers in C exactly as it is done for integers.)

Now, however, if we add and multiply numbers of this form, take their additive and multiplicative inverses, add and multiply again, and so on, it would seem that we should encounter more and more numbers of the system not of this form. This is not so! The sum and product, additive and multiplicative inverses of numbers which can be written in the form $a + bi$, a and b real, can be written in the same form. We have not proved this, but after we complete our discussion of operations with these numbers you will see how such a proof can be constructed.

The results we have stated imply that if there is any system which solves our problem, then there is a simplest--that is, smallest possible--system which solves the problem. This is the system with the following property.

Property C-4

Each element of C can be written in the form $a + bi$, where a and b are real numbers.

We add C-4 to our list of basic properties, thus the system C which has Properties C-1, C-2, C-3 and C-4 is the system of complex numbers.

Historical Note. The adjectives "complex", "imaginary"--and, by contrast, "real"--which are standard terms sanctioned by years of use, serve to illustrate the "controversial" nature of our four fundamental properties. As recently as a hundred years ago many mathematicians believed that C-1, C-2, C-3 and C-4 contradicted one another, that is to say, that there could be no system with all these properties. The proof that this list of properties is just as respectable as that characterizing the "real" numbers was achieved through the work of the nineteenth century mathematicians Argand, Cauchy and Gauss. (Such a proof is outlined in Section 10.) Our continued use of the classical adjectives serves to remind us of the old controversy and of the work of the men who resolved it.

Exercises 2

1. For each of the following pairs of number systems state a property of the first which is not possessed by the second:

   (a) integers, natural numbers
   (b) rational numbers, integers
   (c) real numbers, rational numbers
   (d) complex numbers, real numbers.

2. The following equations have solutions in the system of real numbers if a, b, and c are real numbers. For each equation name the smallest number system in which the equation has a solution in the system if a, b, and c are in the system.

   (a) $a + x = b$
   (b) $ax = b \quad (a \neq 0)$
   (c) $ax + b = c \quad (a \neq 0)$

3. Write each of the following complex numbers in the form $a + bi$ where a and b are real numbers.

   (a) 1 (c) -1 (e) 3 (g) i^2.
   (b) 0 (d) i (f) $2i$

4. For each of the following pairs of number systems state a property of the first which is not possessed by the second:

   (a) natural numbers, integers
   (b) real numbers, complex numbers.

*5. Let S be the set of all real numbers which can be written in the form $a + b\sqrt{2}$, where a and b are rational numbers. Show that

(a) S is not the set of all real numbers.
(Hint: Show that $\sqrt{3}$ is not in S.)

(b) S is closed with respect to addition and multiplication of real numbers.

(c) the additive and multiplicative inverses of a number in S are also in S.

(d) S, with real addition and multiplication as operations, has all the properties listed in Property C-1.

(e) S is the smallest part of the real number system which has properties C-1, contains the rational numbers, and contains $\sqrt{2}$.

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3. Addition, Multiplication and Subtraction.

We now take up the task of deducing rules for calculating with the complex numbers. The remainder of this section is devoted to theorems which give formulas for the sum, product, and difference of two complex numbers. We postpone the discussion of division until Section 4.

Theorem 3a. $(a + bi) + (c + di) = (a + c) + (b + d)i$.

Proof: We suppose that $a + bi$ and $c + di$ are any two given complex numbers. Consider the expression

$$(a + bi) + (c + di).$$

Property C-1 assures us that addition in C is associative and commutative; therefore,

$$(a + bi) + (c + di) = (a + c) + (bi + di).$$

But Property C-1 also asserts that the distributive law holds, so $bi + di = (b + d)i$. Hence,

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

which we were required to prove.

Theorem 3b. $(a + bi)\cdot(c + di) = (ac - bd) + (ad + bc)i$.

Proof: Given complex numbers $a + bi$ and $c + di$, we consider the expression

$$(a + bi)(c + di).$$

Using the distributive law once, we obtain

$$(a + bi)(c + di) = a(c + di) + bi(c + di).$$

Applying the distributive law again, and using the commutative property of multiplication, we have

$$(a + bi)(c + di) = ac + adi + bci + bdi^2.$$

But $i^2 = -1$, so we can write

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

Using the commutative property of addition and once again making use of the distributive law, we obtain

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$

This complet.. the proof.

Example 3a. Express the sum of $2 + 3i$ and $5 + 2i$ in the form $a + bi$, where a and b are real numbers.

Solution: $(2 + 3i) + (5 + 2i) = (2 + 5) + (3 + 2)i = 7 + 5i$.

Example 3b. Express the product of $2 + 3i$ and $5 + 2i$ in the form $a + bi$, where a and b are real numbers.

Solution:
$$\begin{aligned}(2 + 3i)(5 + 2i) &= 2(5) - 3(2) + [2(2) + 3(5)]i \\ &= 10 - 6 + (4 + 15)i \\ &= 4 + 19i.\end{aligned}$$

Example 3c. Express the product of i, $2i$, and $1 - i$ in the form $a + bi$, a and b real.

Solution: $i \cdot 2i \cdot (1 - i) = -2(1 - i) = -2 + 2i$.

Now we consider subtraction. If z is a complex number, we denote the additive inverse of z by $-z$, so that by definition

(3a) $$z + (-z) = 0.$$

Also, just as with integers, we define $z_2 - z_1$ to be the solution z of the equation

(3b) $$z_1 + z = z_2,$$

where z_1, z_2 are given. (We leave as an exercise the proof that Equation (3b) cannot be satisfied by more than one complex number z.) It is easy to see that $z_2 + (-z_1)$ is a solution of Equation (3b).

$$z_1 + [z_2 + (-z_1)] = z_1 + [(-z_1) + z_2] = [z_1 + (-z_1)] + z_2$$
$$= 0 + z_2 = z_2.$$

We have therefore proved

(3c) $$z_2 - z_1 = z_2 + (-z_1).$$

Our problem now is to find $-z$ when $z = a + bi$ is given. Let $-z = x + yi$, where x and y are real. Since

$$z + (-z) = 0$$

we get

$$(a + bi) + (x + yi) = 0.$$

By the theorem on addition (Theorem 3a) this becomes

$$(a + x) + (b + y)i = 0 = 0 + 0 \cdot i$$

and this equation will be satisfied if

$$a + x = 0, \qquad b + y = 0,$$

that is, if $x = -a$ and $y = -b$. Thus, $(-a) + (-b)i$ is an additive inverse of $a + bi$, and since the inverse is unique we have proved

<u>Theorem 3c</u>. If $a + bi$ is a complex number (a and b real), then its additive inverse is

$$-(a + bi) = -a + (-b)i.$$

We can now summarize our discussion of subtraction in a theorem.

<u>Theorem 3d</u>. $(a + bi) - (c + di) = (a - c) + (b - d)i.$

<u>Proof</u>: Using Formula (3c), Theorem 3a and Theorem 3c we have

$$\begin{aligned}(a + bi) - (c + di) &= (a + bi) + [-(c + di)] \\ &= (a + bi) + [(-c) + (-d)i] \\ &= [a + (-c)] + [b + (-d)]i \\ &= (a - c) + (b - d)i.\end{aligned}$$

Exercises 3

1. Express each of the following sums in the form $a + bi$, where a and b are real numbers:

   (a) $(1 + 4i) + (3 + 5i)$
   (b) $(2 + 6i) + (2 - 6i)$
   (c) $(3 + 5i) + 2i$
   (d) $4 + (\pi + \pi i)$
   (e) $(\sqrt{2} + 3i) + (2i + 1)$
   (f) $(-1 + 5i) + 2i$
   (g) $8 + i$
   (h) $3 + (7i - 3)$
   (i) $(5 + 3i) + (7 + 2i) + (3 - 4i)$
   (j) $(3 + 2i) + (\sqrt{2} + 7i) + \sqrt{3}\, i$.

2. Add a complex number to each of the following to make the sum a real number. Can this be done in more than one way?

   (a) $2 - 5i$
   (b) $x - yi$ (x, y real numbers)
   (c) $\sqrt{2} - \sqrt{3}\, i$
   (d) $-5i$

3. Express each of the following products in the form $a + bi$, where a and b are real numbers:

   (a) $(2 + 3i)(4 + 7i)$
   (b) $(2 - 3i)(6 + 4i)$
   (c) $(3 - i)(1 - 2i)$
   (d) $i(3 + 5i)$
   (e) $2i(\sqrt{2} - i)$
   (f) $(8 + \sqrt{2}\, i)(1 + \sqrt{3}\, i)$
   (g) $(3 + 4i)(3 + 4i)$
   (h) $(1 + i)(1 - i)$
   (i) $6i \cdot 3i$
   (j) $7i(-2i)(1 - 6i)$
   (k) $(4 - 2i)(3 - 2i)(5i)$
   (l) $(4 - 3i)^2(2 - 5i)$
   (m) $(2 + 3i)(3 - 2i)(6 - 4i)$
   (n) $(c + di)(x + yi)$ (c, d, real numbers)
   (o) $(x - yi)(x + yi)$ (x, y, real numbers)

4. Find the additive inverses of each of the following complex numbers and express them in the form $a + bi$, where a and b are real numbers:

   (a) 3
   (b) i
   (c) $1 + i$
   (d) $2 + 3i$
   (e) $5 - 4i$
   (f) $-4 - 3i$
   (g) $a - bi$ (a, b real numbers)
   (h) $x + yi$ (x, y real numbers)

5. Express each of the following differences in the form $a + bi$, where a and b are real numbers:

(a) $(7 + 11i) - (2 + 3i)$
(b) $(5 - 6i) - (7 - 8i)$
(c) $(3 + 5i) - (3 - 5i)$
(d) $i - (1 + i)$
(e) $(\sqrt{3} + i) - (2 + \sqrt{2}\, i)$
(f) $\sqrt{4} - (1 - i)$
(g) $\pi - \pi i$
(h) $(2 + 3i) - (2 - 3i)$
(i) $(1 - i) - 2i$

6. Express the following powers of i in the form $a + bi$, where a and b are real numbers.

(a) i^3
(b) i^4
(c) i^9
(d) i^{15}
(e) i^{4n+1}, n is a natural number
(f) i^{79}

7. State a general rule for determining the n-th power of i where n is a natural number. Explain why the rule works.

8. Express each of the following quantities in the form $a + bi$, where a and b are real numbers.

(a) $i^3 + i^4$
(b) i^{4n+3}, n is a natural number
(c) $3i + 4i(5 - i)(5 + i)$
(d) $7i[(2 - 3i) + (4i + 10)]$
(e) $i[(3i + 6) - (2i + 7)]$
(f) $3(3 + 2i) + (6 + 8i) - 2(2 - 3i)$
(g) $(b + c - ai)(a + c - bi)(a + b - ci)$, where a, b, c are real numbers.
(h) $(\frac{1}{2} + \frac{\sqrt{3}}{2} i)^3$
(i) $i(i - 1)(i - 2)(i - 3)$

9. Show by substitution that $\frac{3}{4} + \frac{\sqrt{7}}{4} i$ is a solution of the equation $2z^2 - 3z + 2 = 0$.

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4. Standard Form of Complex Numbers.

Property C-4 asserts that each member of C can be expressed in the form $a + bi$, where a and b are real numbers. Our next theorem states that this representation is unique: given any complex number z, there is only one pair of real numbers a, b such that $z = a + bi$.

Theorem 4. If a, b, c, d are real numbers, then $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Proof: The "if" part of the statement "$a + bi = c + di$ if $a = c$ and $b = d$" is clear, since addition and multiplication have unique results. We have to prove the "only if" part:

$a + bi = c + di$ only if $a = c$ and $b = d$; that is, if $a + bi = c + di$ then $a = c$ and $b = d$.

Suppose, accordingly, that a, b, c, d are real numbers and that

$$a + bi = c + di.$$

Then by the theorem on subtraction (Theorem 3d),

$$(a - c) + (b - d)i = 0,$$

and

$$a - c = -(b - d)i.$$

We have to show that $a = c$ and $b = d$, or what is the same, that $a - c = 0$ and $b - d = 0$. Now if $b - d$ were not zero, we could write

$$\frac{a - c}{b - d} = -i,$$

or

$$-\left(\frac{a - c}{b - d}\right) = i.$$

But this would imply that i is a real number since a, b, c, d are real numbers and the difference and quotient of real numbers are real. Since we know that i is not a real number we conclude that $b - d = 0$. But if $b - d = 0$, then $-(b - d)i = 0$, and since $(a - c) = -(b - d)i$, it follows that $a - c = 0$. This completes the proof.

Example 4a. Find all pairs of complex numbers x, y for which $2x + 3yi = 6 + 3i$.

Solution: One solution of the problem is $x = 3$, $y = 1$. If the problem had required that x and y be real, then by the preceding theorem this would be the only solution. However, since we permit x and y to be complex, the preceding theorem is not directly applicable, and the equation may have other solutions; $x = 3 + 3i$, $y = -1$ is a solution, for example.

We can use Theorem 4 to find <u>all</u> complex solutions of this equation. Let $x = a + bi$, $y = c + di$ where a, b, c, d are real. Substituting in

$$2x + 3yi = 6 + 3i$$

we get

$$2(a + bi) + 3(c + di)i = 6 + 3i,$$

or

$$(2a - 3d) + (2b + 3c)i = 6 + 3i.$$

Since the expressions in parentheses in the last equation are real, it follows from the preceding theorem that the equation holds if and only if

$$2a - 3d = 6, \qquad 2b + 3c = 3;$$

or

$$c = \frac{3 - 2b}{3}, \qquad d = \frac{2a - 6}{3}.$$

Here a and b may be assigned values arbitrarily. Thus, all the solutions of the equation are given by

$$x = a + bi, \quad y = \frac{3 - 2b}{3} + \frac{2a - b}{3}i,$$

where a and b are any real numbers.

The representation of a complex number z as

$$z = a + bi,$$

where a and b are real numbers, is called the <u>standard form</u> of z. Note that z is real if and only if $b = 0$. (Why?) We therefore call a the <u>real part</u> of $a + bi$. The real number b is called the <u>imaginary part</u> of $a + bi$. Thus, we can say that a complex number is real if and only if its imaginary part is zero. A complex number $a + bi$ in which $a = 0$ is called a <u>pure imaginary number</u>. Thus, a complex number is a pure imaginary number if and only if its real part is zero. DO NOT CONFUSE the imaginary part of b of the complex number $a + bi$ with the pure imaginary number bi. Both the real and imaginary parts of $a + bi$ are real numbers: they are the real numbers a and b, respectively. Usually a complex number which is not real is called imaginary.

Examples 4b

	z	Real part of z	Imaginary part of z	Standard form of z
1.	0	0	0	$0 + 0i$
2.	$2 + i$	2	1	$2 + 1i$
3.	$1 - i$	1	-1	$1 + (-1)i$
4.	i	0	1	$0 + 1i$
5.	i^2	-1	0	$-1 + 0i$

In these examples only 0 and i^2 are real numbers; only 0 and i are pure imaginary numbers; $2 + i$, $1 - i$ and i are imaginary numbers.

Exercises 4

1. Find the real and imaginary parts of each of the following complex numbers:

(a) $(1 + i)^2$

(b) $1 + i^2$

(c) i^5

(d) $5 - i$

(e) $2x + 3i$

(f) $a - 2i$

(g) $(\sqrt{2} - i)^2$

(h) $(-1 + i\sqrt{3})^2$

(i) $(4 + i) - 7$

(j) $-2i^2$

(k) $3i$

(l) $2i + 1$

2. What real numbers must be added to each of the following complex numbers to make the sum a pure imaginary number? Can this be done in more than one way?

(a) $3 + 2i$

(b) $-4i$

(c) $5 - 2i$

(d) $5 - \sqrt{2}\, i$

3. Use Theorem 4a to find real values for x and y that satisfy the following equations:

(a) $x - yi = 3 + 6i$

(b) $2x + yi = 6$

(c) $x - 5yi = 20i$

(d) $8x + 3yi = 4 - 9i$

(e) $2x + 3yi - 4 = 5x - yi + 8i$

(f) $x - y + (x + y)i = 2 + 6i$

(g) $(1 + x) + i(2 - y) = 3 - 4i$

(h) $x + yi = 1 + i^2$

(i) $y^2 i^2 = i(1 - x^2)$

(j) $(x + i)^2 = y$

4. Express each of the following complex numbers in standard form:

(a) $3 + 2i + 5 + i$

(b) $(3 - 2i) - (5 - 2i)$

(c) $3i(4 - 2i)$

(d) $6 + 5i - (2 - 3i)$

(e) $(3 - 2i)(5 - 2i)$

(f) $(4 - i)(3 - 2i)$

(g) $(1 - i)(2 + 3i)(4 + 2i)$

(h) $(a + b - ci)(a + b + ci)$, where a, b, c are real numbers.

(i) $(x + yi)^3$, where x and y are real numbers.

5. Suppose $z = x + yi$, where x and y are real numbers, and $z^2 = 8 + 6i$. Solve for x and y.

*6. Suppose, for the sake of this exercise, that a and b are complex numbers. Show that $a + bi = 0$ and $a - bi = 0$ if and only if $a = 0$ and $b = 0$. Show also that the underlined word can be replaced by "or" only when we also assume that a and b are real numbers.

*7. Show that if z_1 is any non-real complex number, every complex number z can be expressed in one and only one way in the form $z = a + bz_1$, where a and b are real numbers.

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5. Division.

We have learned to add, multiply and subtract complex numbers. We now consider division. According to Property C-1 every complex number other than 0 has one and only one multiplicative inverse. We denote the multiplicative inverse of z by $\frac{1}{z}$, so that by definition

(5a) $$z \cdot \frac{1}{z} = 1.$$

Also, we define $\frac{z_2}{z_1}$ to be the solution of the equation

(5b) $$z_1 \cdot z = z_2$$

when this solution exists. (We leave as an exercise the proof that equation 5b cannot be satisfied by more than one complex number z.) It is easy to see that if $z_1 \neq 0$, Equation 5b has the solution $z_2(\frac{1}{z_1})$:

$$z_1 \cdot [z_2(\frac{1}{z_1})] = z_1[(\frac{1}{z_1})z_2] = [z_1(\frac{1}{z_1})]z_2 = 1 \cdot z_2 = z_2 .$$

We have therefore proved

(5c) $$\frac{z_2}{z_1} = z_2 \cdot \frac{1}{z_1} \qquad z_1 \neq 0.$$

Our task now is tofind the standard form of $\frac{1}{z}$ when $z = a + bi$ is given in standard form. Let us begin by considering a numerical example.

Example 5a. If $z = 2 + 3i$ find its multiplicative inverse $\frac{1}{z}$ in standard form.

Solution. We seek a number $x + yi$ (x and y real) satisfying

$$(2 + 3i)(x + yi) = 1.$$

If we multiply the factors on the left using the theorem on multiplication (Theorem 3b) we may write

$$(2x - 3y) + (3x + 2y)i = 1 + 0i.$$

Hence, from the theorem on standard form (Theorem 4),

$$2x - 3y = 1,$$
$$3x + 2y = 0.$$

Eliminating y, we have

$$(4 + 9)x = 2.$$

Hence,

$$x = \frac{2}{13}, \quad y = \frac{-3}{13};$$

and

$$x + yi = \frac{2}{13} + (-\frac{3}{13})i.$$

Now we can verify by substitution that

$$\frac{1}{2 + 3i} = \frac{2}{13} + (-\frac{3}{13})i.$$

We treat the general case in exactly the same way. Suppose $a + bi$, in standard form, is a non-zero complex number. Recall that this means that at least one of the two real numbers a, b is not 0. If there is a complex number $x + yi$, x and y being real numbers which satisfies the equation

(5d) $$(a + bi)(x + yi) = 1,$$

then by completing the multiplication in the left member we get

$$(ax - by) + (bx + ay)i = 1.$$

From the theorem on standard form (Theorem 4), this equation will be satisfied if and only if

(5e) $$ax - by = 1,$$
$$bx + ay = 0.$$

Thus, our problem is reduced to that of solving two linear equations with real coefficients for the real unknowns x and y. We solve these equations by elimination. To eliminate y, multiply the first equation by a, the second by b, and add. We get

$$(a^2 + b^2)x = a.$$

Our assumption that $a + bi \neq 0$; i.e., that at least one of the real numbers a, b is not zero, tells us that $a^2 + b^2 \neq 0$. Hence, we can write

$$x = \frac{a}{a^2 + b^2} .$$

In the same way, we eliminate x from Equations (5e). Multiplying the first equation by b, the second by a, and subtracting the first from the second, we get

$$(a^2 + b^2)y = -b.$$

As before, $a^2 + b^2 \neq 0$, so

$$y = \frac{-b}{a^2 + b^2} .$$

Now by substitution we can verify that

$$\frac{a}{a^2 + b^2} + \left(\frac{-b}{a^2 + b^2}\right)i$$

is a solution of Equation (5d) so that it is the unique multiplicative inverse of $a + bi$. We state our conclusion as a theorem.

Theorem 5. If $a + bi$ is a non-zero complex number in standard form, then its multiplicative inverse is

$$\frac{1}{a + bi} = \frac{a}{a^2 + b^2} + \left(\frac{-b}{a^2 + b^2}\right)i.$$

Now we can combine the results of this section to obtain a formula for the quotient of any two complex numbers when the denominator is not 0. We could state the result as a theorem, but the statement would be cumbersome. It is better to remember a procedure which we indicate by an example.

Example 5b. Find the quotient $\frac{8 + 5i}{2 + 3i}$ and express the answer in standard form.

Solution: By Formula 5c,

$$\frac{8 + 5i}{2 + 3i} = (8 + 5i)\left(\frac{1}{2 + 3i}\right).$$

By Theorem 5,

$$\frac{1}{2 + 3i} = \frac{2}{13} + \frac{-3}{13}i.$$

Combining these two equations and using the theorem on multiplication (Theorem 4b) we obtain

$$\frac{8 + 5i}{2 + 3i} = (8 + 5i)\left(\frac{2}{13} + \frac{-3}{13}i\right)$$
$$= \frac{31}{13} + \left(-\frac{14}{13}\right)i$$

as the quotient in standard form.

The following relations involving division of complex numbers can be proved on the basis of Property C-1 just as it is done for real numbers.

(5f) $z_1 z_2 = 0$ if and only if $z_1 = 0$ or $z_2 = 0$ (or both).

(5g) $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1 z_3}{z_2 z_4}$, if $z_2 \neq 0$, $z_4 \neq 0$.

(5h) $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1 z_4 + z_2 z_3}{z_2 z_4}$, if $z_2 \neq 0$, $z_4 \neq 0$.

We leave the proofs of these relations as exercises. (See Exercises 5, Problems 7-9.)

Exercises 5

1. Find the multiplicative inverses of each of the following complex numbers and write them in standard form:

(a) 1	(e) $1 + i$
(b) 5	(f) $2 + 3i$
(c) i	(g) $1 + i^2$
(d) $-i$	(h) $4 - 3i$

2. Does every complex number have a multiplicative inverse?

3. What complex numbers are their own multiplicative inverses?

4. What complex numbers are the additive inverses of their multiplicative inverses?

5. Express the following quotients in standard form:

(a) $\frac{1}{2 + i}$

(b) $\frac{3}{2i}$

(c) $\frac{1}{2i - 5}$

(d) $\frac{13 + 5i}{2i}$

(e) $\frac{1 + i}{2 - i}$

(f) $\frac{4 + 3i}{2 + 5i}$

(g) $\frac{7 + 6i}{3 - 4i}$

(h) $\frac{3i}{4 + 7i}$

(i) $\frac{-5i}{3 + 5i}$

(j) $\frac{1 + \sqrt{2}i}{1 - \sqrt{2}i}$

(k) $\frac{\sqrt{2} + \sqrt{3}i}{1 + \sqrt{2}i}$

(l) $\frac{a + bi}{a - bi}$; a,b real, $a - bi \neq 0$

(m) $\frac{a + 2bi}{2a - bi}$; a,b real, $2a - bi \neq 0$

(n) $\frac{-m - ni}{-m + ni}$; m,n real, $-m + ni \neq 0$

(o) $\frac{3x + 2yi}{x - yi}$; x,y real, $x - yi \neq 0$

6. Show that if $z_1 \neq 0$, the equation $z_1 \cdot z = z_2$ has no more than one solution.

7. Write in standard form all complex numbers z such that the real part of $\frac{1}{z}$ is $\frac{1}{2}$, and

(a) the imaginary part of z is zero.
(b) the imaginary part of z is $\frac{1}{2}$.
(c) the imaginary part of z is 1.

8. Prove that $z_1 z_2 = 0$ if and only if $z_1 = 0$, or $z_2 = 0$, or both are zero.

*9. Prove that $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1 z_3}{z_2 z_4}$, if $z_2 \neq 0$, $z_4 \neq 0$.

*10. Prove that $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1 z_4 + z_2 z_3}{z_2 z_4}$, if $z_2 \neq 0$, $z_4 \neq 0$.

11. Make use of the formulas in Problems 9 and 10 to obtain the following sums and products. Write the answers you obtain in standard form.

(a) $\frac{1 + i}{1 + 2i} + \frac{1 - i}{1 - 2i}$

(b) $\frac{1 + 2i}{3 + 4i} \cdot \frac{2 - i}{2i}$

(c) $\frac{2 + 36i}{6 + 8i} + \frac{7 - 26i}{3 - 4i}$

(d) $\frac{2 - 3i}{3 + 2i} + \frac{3 + 4i}{2 - 4i}$

(e) $\left(\frac{a + bi}{a - bi}\right)^2 + \left(\frac{a - bi}{a + bi}\right)^2$

$a + bi \neq 0$, $a - bi \neq 0$

*12. Show that the words "in standard form" may be omitted in Theorem 5 if we suppose merely that $a^2 + b^2 \neq 0$.

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6. Quadratic Equations.

We come now to a crucial test for the complex number system. Does it permit us to solve equations of the form

(6a) $$az^2 + bz + c = 0,$$

where a, b, c, are real numbers and

(6b) $$b^2 - 4ac < 0 \ ?$$

Let us first find the solutions of the quadratic equation on which we have so far focused our attention:

(6c) $$z^2 + 1 = 0.$$

If z is an arbitrary complex number, we have

$$z^2 + 1 = z^2 - (-1) = z^2 - i^2 = (z - i)(z + i).$$

This factorization of $z^2 + 1$ shows that if z is a complex number satisfying Equation (6c), then one of the factors $(z - i)$, $(z + i)$ must be zero, and z must be either i or -i. Conversely, we see that i and -i both satisfy Equation (6c). Therefore, we conclude that the solutions of Equation (6c) are i, -i.

Equation (6c) is a special case of the Equation

(6d) $$z^2 = r.$$

We know that if r is real and positive this equation has two real solutions. We have just seen that for a special negative value of r; namely, r = -1, this equation has two non-real complex solutions, i and -i. Let us next consider the general case in which r is negative.

If r is real and negative, then -r is real and positive, and $\sqrt{-r}$ is defined. We have

$$r = (-1)(-r) = (i)^2(\sqrt{-r})^2 = (i\sqrt{-r})^2;$$

and hence,

$$z^2 - r = z^2 - (i\sqrt{-r})^2 = (z - i\sqrt{-r})(z + i\sqrt{-r}).$$

Just as in the discussion of Equation (6c), we conclude that Equation (6d) has the two solutions $i\sqrt{-r}$, $-i\sqrt{-r}$, when r is real and negative.

For the case in which r is real and positive we introduced the notation $\sqrt{r}$ to describe the solution set of Equation (6d): one solution is $\sqrt{r}$ and the second $-\sqrt{r}$. It would be desirable to extend the definition of $\sqrt{r}$ for negative real r so that the description of the solution set of Equation (6d) would be the same for all r. The question is which of the two solutions $i\sqrt{-r}$, $-i\sqrt{-r}$ shall we take to be $\sqrt{r}$ if r is negative?

It should be clear that we have the problem of defining $\sqrt{r}$ unambiguously for positive r. The problem is resolved by defining $\sqrt{r}$ to be the non-negative solution of Equation (6d). The requirement that $\sqrt{r}$ be non-negative is simply an <u>agreement</u> adopted to make the meaning of $\sqrt{r}$ definite. However, this agreement makes no sense if the solutions of Equation (6d) are complex. We have not defined "positive" and "negative" for non-real complex numbers, and cannot define these terms for complex numbers in a way which is consistent with their usual meaning. We must make a new agreement for the case of negative r. Any agreement which definitely selects one of the solutions $i\sqrt{-r}$, $-i\sqrt{-r}$ of Equation (6d) will be satisfactory. We choose $\sqrt{r} = i\sqrt{-r}$, and accordingly make the following definition:

<u>Definition 6a</u>. Let r be any real number. We define $\sqrt{r}$ as follows:

(1) If $r \geq 0$, then $\sqrt{r}$ is the unique non-negative real number w such that $w^2 = r$.

(2) If $r < 0$, then $\sqrt{r} = i\sqrt{-r}$.

<u>Example 6a</u>.

$$\sqrt{-1} = i\sqrt{1} = i$$
$$\sqrt{-12} = i\sqrt{12} = 2\sqrt{3}\,i$$
$$\sqrt{(-2i)^2} = \sqrt{4i^2} = \sqrt{-4} = i\sqrt{4} = 2i.$$

<u>Example 6b</u>. Find the product $(\sqrt{-5})(\sqrt{-15})$.

<u>Solution</u>: We have

$$(\sqrt{-5})(\sqrt{-15}) = (i\sqrt{5})(i\sqrt{15}) = i^2\sqrt{5}\sqrt{15} = -\sqrt{75}.$$

Note that it is not correct to say

$$(\sqrt{-5})(\sqrt{-15}) = \sqrt{(-5)(-15)} = \sqrt{75}.$$

The statement $\sqrt{r}\sqrt{s} = \sqrt{rs}$ has been proved only for the case in which r and s are both positive. The statement is also true if r and s have opposite signs (Exercises 5, Problem 5), but as the foregoing example shows, it is false if both r and s are negative.

Example 6c. Find the product $(\sqrt{r})(\sqrt{r^3})$ if r is any real number.

Solution: We have to consider two cases. If $r \geq 0$ we have $r^3 \geq 0$, and

$$\sqrt{r}\sqrt{r^3} = \sqrt{r \cdot r^3} = \sqrt{r^4} = r^2.$$

If $r < 0$ we have $r^3 < 0$, and

$$\sqrt{r}\sqrt{r^3} = (i\sqrt{-r})(i\sqrt{-r^3}) = i^2\sqrt{(-r)(-r^3)} = -\sqrt{r^4} = -r^2 .$$

Now that we have given an unambiguous meaning to $\sqrt{r}$ for each real number r, we state as a theorem our previous conclusions about equations of the form $z^2 = r$, where r is any given real number.

Theorem 6a. If r is any given real number, the equation $z^2 = r$ has the roots $\sqrt{r}$ and $-\sqrt{r}$, and no others.

We now turn to the solution of the general quadratic equation

(6e) $\qquad az^2 + bz + c = 0, \quad a, b, c$ real and $a \neq 0$.

Recall that we were led to our study of complex numbers because we failed to find real solutions of Equation (6e) when its discriminant $b^2 - 4ac$ is negative.

Theorem 6b. The equation

$$az^2 + bz + c = 0, \quad a, b, c \text{ real and } a \neq 0,$$

has the solutions

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and no others.

There is nothing new if $b^2 - 4ac \geq 0$; this is the case of real solutions. We now prove that the formula holds if

$b^2 - 4ac < 0$, although in this case the solutions will not be real.

Divide by a and complete the square.

$$z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2},$$

(6f)

$$(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}.$$

We now have Theorem 6a which tells us that Equation (6f) has (complex) solutions whether $b^2 - 4ac$ is positive, negative, or zero.

Applying Theorem 6a, we obtain

$$z + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad z + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}};$$

so $$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad z = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The proof of Theorem 6b can be completed by showing that the numbers obtained actually satisfy the equation.

Example 6d. Find the solutions of $z^2 + z + 1 = 0$.

Solution: $a = b = c = 1$. By Theorem 6b the solutions are

$$\frac{-1 + \sqrt{-3}}{2} = \frac{-1 + i\sqrt{3}}{2}$$

and

$$\frac{-1 - \sqrt{-3}}{2} = \frac{-1 - i\sqrt{3}}{2}$$

Other statements about the relation between the solutions and coefficients of a quadratic equation can be established. In particular, if z_1 and z_2 are the complex solutions of the equation

$$az^2 + bz + c = 0,$$

then

(6g) $$z_1 + z_2 = \frac{-b}{a}, \quad z_1 \cdot z_2 = \frac{c}{a};$$

and

(6h) $$az^2 + bz + c = a(z - z_1)(z - z_2).$$

The proofs are left as exercises.

Exercises 6

1. Perform the indicated operations and write the answers you obtain in standard form.

   (a) $\sqrt{-25} + \sqrt{-4}$ (e) $\sqrt{-6} \cdot \sqrt{-8} \cdot \sqrt{-1}$

   (b) $2\sqrt{-5} - 6\sqrt{-20}$ (f) $\sqrt{-\frac{1}{4}}$

   (c) $\sqrt{-2} + 5\sqrt{-8} - \sqrt{-98}$ (g) $\dfrac{\sqrt{-81}}{3\sqrt{-8}}$

   (d) $\sqrt{-4} \cdot \sqrt{-5}$ (h) $\sqrt{\dfrac{-98}{-147}}$

2. Write each of the following complex numbers in standard form. Assume c is a real number.

   (a) $\sqrt{-(2)^2}$ (e) $\sqrt{(-c)^2}$

   (b) $\sqrt{(-2)^2}$ (f) $\sqrt{-c^2}$

   (c) $\sqrt{-(-2)^2}$ (g) $\sqrt{-(-c)^2}$

   (d) $\sqrt{c^2}$

3. Perform the indicated operations and write the answers you obtain in standard form. Assume a and b are positive real numbers.

   (a) $\sqrt{-a^2} + \sqrt{-b^2}$ (d) $\dfrac{5\sqrt{-a^2}}{3\sqrt{-a}}$

   (b) $\sqrt{-a^2} \cdot \sqrt{-4a^2b}$ (e) $\sqrt{-32a^2} - \sqrt{-50a^2}$

   (c) $\sqrt{-a}(\sqrt{-a} + \sqrt{-b})$ (f) $\sqrt{-a} - \sqrt{-a^3}$

   (g) $\sqrt{-a^2 - 2ab - b^2} + \sqrt{-(a + b)^2}$

*4. Examine the proof that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ if a and b are non-negative real numbers, and explain why the same argument cannot be used when a and b are negative.

5. Show that if $r < 0$ and $s > 0$, then $\sqrt{r}\sqrt{s} = \sqrt{rs}$.

In each of Problems 6-17 solve the given quadratic equation and express the solutions in standard form.

6. $z^2 + 1 = 0$

7. $z^2 + z - 1 = 0$

8. $z^2 + 2z + 2 = 0$

9. $z^2 - z + 1 = 0$

10. $3z^2 + 2z + 4 = 0$

11. $z^2 + 4z + 8 = 0$

12. $z^2 - 4z + 8 = 0$

13. $2z^2 + z + 1 = 0$

14. $z^2 - 4z - 8a = 0$ (a real)

15. $mz^2 + z + \frac{1}{m} = 0$ (m real, $m \neq 0$)

16. $z^2 - iz + 2 = 0$

17. $az^2 + c = 0$ (a,c real, $a \neq 0$)

18. The equation $z^3 - 8 = 0$ has the solution 2. Show that $z - 2$ is a factor of $z^3 - 8$, and use this fact to find two more solutions of the equation.

*19. Suppose z_1 and z_2 are the solutions of $az^2 + bz + c = 0$, where a, b, c are real and $a \neq 0$. Show that $z_1 + z_2 = -\frac{b}{a}$ and $z_1 z_2 = \frac{c}{a}$.

*20. If z_1 and z_2 are the solutions of the equation $az^2 + bz + c = 0$ show that the equation

$$az^2 + bz + c = a(z - z_1)(z - z_2)$$

holds for every element z of C. (This formula therefore provides a "factorization" of the expression $az^2 + bz + c$.)

21. Find the quadratic equations which have the following pairs of solutions:

(a) $z_1 = 1 - i$, $z_2 = 1 + i$

(b) $z_1 = i$, $z_2 = 2 + i$

(c) $z_1 = 0$, $z_2 = 0$

(d) $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$; a_1, b_1, a_2, b_2 being any four given real numbers.

*22. Solve the equation $z^2 = i$. [Hint: Writing z in standard form, $z = x + yi$, the given equation is equivalent to a pair of equations whose unknowns are real numbers: $x^2 - y^2 = 0$, $2xy = 1$.]

*23. Solve the equation $z^2 = -i$.

*24. Find an equation whose solutions are $1 + 2i$, $1 - i$, $1 + i$. Is there a quadratic equation having these numbers as solutions? If there is one, find it. If there is none, prove that there is none.

---

7. Graphical Representation: Absolute Value.

According to Property C-4 and Theorem 4a each complex number z may be written in one and only one way in the standard form $a + bi$, where a and b are real numbers. Thus, each complex number z determines, and is determined by, an ordered pair (a,b) of real numbers: a is the real part of z, b the imaginary part of z.

Recalling that ordered pairs of real numbers formed the starting point of coordinate geometry, we find that we are able to represent the complex numbers by points in the xy-plane. We agree to associate z with the point (a,b) if and only if $z = a + bi$, in standard form, and we set up a one-to-one correspondence between the elements of C and the points of the xy-plane.

It is customary to use the expression "Argand diagram" to describe the pictures obtained when the point (a,b) of the xy-plane is used to represent the complex number $a + bi$ given in standard form. Figure 7a is an example of an Argand diagram showing three points $(0,0)$, $(4,-5)$, $(-4,3)$ and the complex numbers they represent. Note that points on the x-axis correspond to realnumbers and points on the y-axis correspond to pure imaginary numbers. For the sake of brevity we shall often say "the point $z = x + yi$" instead of "the point (x,y) corresponding to the complex number $z = x + yi$."

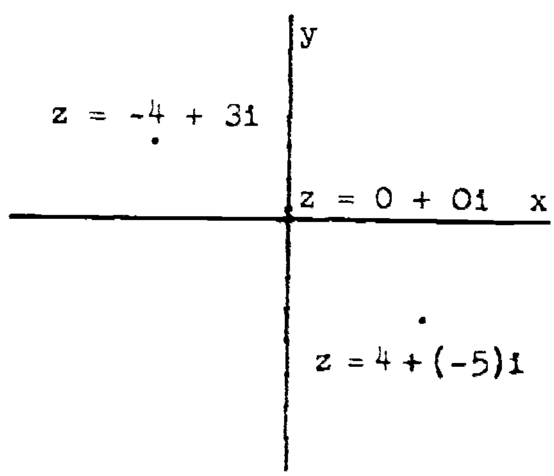

Figure 7a

The geometric representation of complex numbers by means of an Argand diagram serves a double purpose. It enables us to interpret statements about complex numbers geometrically and to express geometric statements in terms of complex numbers. As a first example, consider the formula for the coordinates of the midpoint of a line segment: The midpoint of the segment joining (x_1,y_1) and (x_2,y_2) is the point (x,y) given by the formulas

$$(7a) \qquad x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}.$$

In terms of complex numbers this may be stated: The midpoint of the segment joining the points $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is the point $z = x + yi$ given by

$$(7b) \qquad z = \frac{z_1 + z_2}{2}.$$

Note that we can express in one "complex" equation a statement which requires two "real" equations.

Now we can use Equation (7b) to establish a geometric interpretation of addition of complex numbers. Let z_1 and z_2 be two complex numbers and suppose that the points $0, z_1, z_2$ are not collinear. Let $z_3 = z_1 + z_2$ and consider the quadrilateral whose vertices are $0, z_1, z_2, z_3$ (Figure 7b). The midpoint of the diagonal from z_1 to z_2 is $\frac{z_1 + z_2}{2}$; that of the diagonal from 0 to z_3 is

$$\frac{0 + z_3}{2} = \frac{z_3}{2} = \frac{z_1 + z_2}{2}.$$

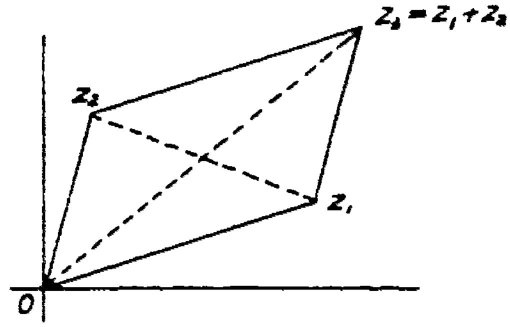

Figure 7b

Hence, the diagonals have a common midpoint. Since the diagonals of the quadrilateral bisect each other, the figure is a parallelogram. Thus, we have a geometrical construction for the sum of two complex numbers: If two complex numbers are plotted in an Argand diagram, their sum is the complex number corresponding to the fourth vertex of the parallelogram whose other three vertices are the origin and the two given points (and which has the segments joining z_1 and z_2 to the origin as sides.)

When the points $0, z_1, z_2$ are collinear the parallelogram collapses into a straight line and our construction fails. We shall discuss this case later.

Next we consider the geometric construction of the difference $z_2 - z_1$ of two complex numbers. Since $z_2 - z_1 = z_2 + (-z_1)$ we have only to find a geometric construction of the additive inverse $-z$ of the complex number z. By equation (7b) the midpoint of the segment joining z and $-z$ is

$$\frac{z + (-z)}{2} = \frac{0}{2} = 0,$$

that is, the midpoint is the origin. Thus, if a complex number is plotted in an Argand diagram, its additive inverse is the complex number corresponding to the point symmetric to the given point with respect to the origin (Figure 7c).

We could now describe geometric constructions for the product and quotient of complex numbers but these constructions are not very illuminating. After we have studied trigonometry and the relation between complex numbers and trigonometry we will be able to state simple and elegant geometric interpretations of these operations.

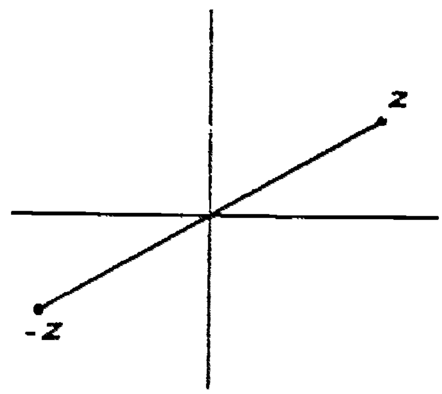

Figure 7c

Example 7a. Given $z_1 = 3 + i$, and $z_2 = 2 - 2i$, make use of an Argand diagram to find the difference $z_1 - z_2$.

Solution: Begin by plotting z_1 and z_2. Then locate the additive inverse of z_2, namely $-z_2$. This is easily done since we know that z_2 and $-z_2$ are symmetric with respect to the origin. The point $z_1 - z_2$ is the same as $z_1 + (-z_2)$. (See Figure 7d.)

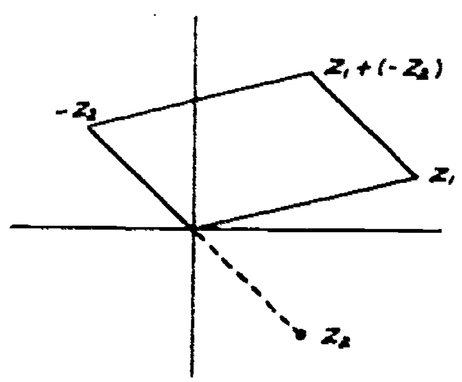

Figure 7d

The geometric representation of complex numbers suggests a definition of absolute value of a complex number. Recall that when real numbers are represented by points on a line, the absolute value of a real number is equal to its distance from the origin.

Accordingly, we define the absolute value $|z|$ of a complex number $z = a + bi$ to be the distance from the origin to the point (a,b). Using the distance formula our definition may be stated algebraically as follows:

<u>Definition 7a</u>. If $z = a + bi$, where a and b are real numbers, we write

$$|z| = \sqrt{a^2 + b^2},$$

and call $|z|$ the absolute value of z.

<u>Example 7b</u>. Show that the distance between the points z_1 and z_2 is $|z_2 - z_1|$.

<u>Solution</u>: If $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_1 i$ where x_1, y_1, x_2, y_2 are real numbers, then by the theorem on subtraction

$$z_2 - z_1 = (x_2 - x_1) + (y_2 - y_1)i.$$

By the definition of absolute value

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and this is the distance between the points (x_1, y_1) and (x_2, y_2).

When z_1 and z_2 are real numbers we know the following relations involving absolute value and the algebraic operations:

(7c) $$|z_1 \cdot z_2| = |z_1|\,|z_2|$$

(7d) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

(7e) $$|z_1 + z_2| \leq |z_1| + |z_2|$$

(7f) $$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|.$$

These relations continue to hold when z_1 and z_2 are complex numbers. Formulas (7c) and (7d) can be proved by calculation (Exercises 7, Problems 8-9), although we will present simpler proofs in the next section.

The algebraic proof of Formula (7e) is quite difficult but we can give an easy geometric proof. Consider the triangle whose vertices are 0, z_1, $z_1 + z_2$ in Figure 7b. The lengths of its sides are $|z_1|$, $|z_2|$, $|z_1 + z_2|$. Why? Since the length of a side of a triangle is less than the sum of the lengths of the other two sides, we have

$$|z_1 + z_2| < |z_1| + |z_2|.$$

We will show later that when the parallelogram collapses into a straight line we have either the inequality above or the equation

$$|z_1 + z_2| = |z_1| + |z_2|.$$

This will complete the proof of Formula (7e), which is often called the "triangle inequality". The discussion of (7f) is left as an exercise (Exercises 7, Problem 10).

For further discussion of the algebra and geometry of complex numbers it is convenient to introduce the notion of complex conjugate. We do this in the next section.

Exercises 7

1. Plot each of the following complex numbers in an Argand diagram. Label the points with the symbols z_1, z_2, etc.

   (a) $z_1 = 1$ (e) $z_5 = 2 + i$

   (b) $z_2 = i$ (f) $z_6 = -4 - 2i$

   (c) $z_3 = -1$ (g) $z_7 = \sqrt{2} - i$

   (d) $z_4 = -i$ (h) $z_8 = \pi - \sqrt{3}i$

2. Plot the additive inverse of each complex number in Problem 1. Label the point that corresponds to z_1 with the symbol $-z_1$, etc.

3. In each of the following problems find $z_1 + z_2$ and $z_1 - z_2$, and also construct them graphically.

   (a) $z_1 = 1 + i$, $z_2 = 2 + i$
   (b) $z_1 = 3 + 2i$, $z_2 = 2 + 3i$
   (c) $z_1 = -1 + 2i$, $z_2 = 2 - i$
   (d) $z_1 = -3 + 4i$, $z_2 = -1 - 3i$
   (e) $z_1 = -3 + i$, $z_2 = 1 + 4i$
   (f) $z_1 = -2i$, $z_2 = 2 - 4i$
   (g) $z_1 = 3$, $z_2 = -3 + 5i$
   (h) $z_1 = 4$, $z_2 = -4i$

4. Let $z_1, z_2, \ldots, z_8$ be the points given in Problem 1. Use Equation 7b to find the midpoints of the segments joining z_2 and z_5, z_3 and z_6, z_4 and z_7, and plot the points in an Argand diagram.

5. Find $|z|$ if:

   (a) $z = 3 - 4i$
   (b) $z = -2i$
   (c) $z = 1 + i^2$
   (d) $z = i^4 + i^7$
   (e) $z = \pi + \sqrt{2}i$

6. Show that if $z \neq 0$, $\left|\frac{z}{|z|}\right| = 1$.

7. Find the set of points described by each of the following equations

   (a) $z = 1$ (b) $z = |z|$ (c) $z = \frac{z}{|z|}$

8. Give an algebraic proof of the equality

$$|z_1 z_2| = |z_1| \cdot |z_2|,$$

if z_1 and z_2 are complex numbers.

9. Give an algebraic proof of the equality

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},$$

if z_1 and z_2 are complex numbers, and $z_2 \neq 0$.

10. Give a geometric proof of the inequality

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2| .$$

11. Suppose 0, $z_1 = a + bi$, $z_2 = c + di$ are collinear. If $z_3 = z_1 + z_2$ show that z_3 lies on the line through 0, z_1, and z_2.

12. Prove that the triangle with vertices $0, 1, z$ is similar to the triangle with vertices $0, z, z^2$ by showing that corresponding sides are proportional. (Hint: Note that the length of each side of the second triangle is equal to $|z|$ multiplied by the length of each side of the first triangle.) Use the result to describe a geometric construction for z^2.

---

8. Complex Conjugate.

Definition 8a. If $z = a + bi$, in standard form (a and b real), we call $a + (-b)i$ the complex conjugate, or simply the conjugate of z, and write

$$\bar{z} = \overline{a + bi} = a + (-b)i.$$

Since $a + (-b)i = a - bi$ we may also write

$$\bar{z} = \overline{a + bi} = a - bi.$$

Example 8a. $\overline{2 + 3i} = 2 - 3i$; $\overline{\left(\frac{1}{i}\right)} = -\bar{i} = i$.

It is easy to see that the conjugate of the conjugate of a complex number is the complex number itself. If $z = a + bi$ in standard form, we have

$$\overline{(\bar{z})} = \overline{(\overline{a + bi})} = \overline{a - bi} = a + bi$$

so that

(8a) $$\bar{\bar{z}} = z.$$

Thus, if the first of two numbers is the conjugate of the second, then the second is the conjugate of the first. We call such a pair of numbers conjugate.

Although we have not used the term "conjugate" before, conjugates of complex numbers have appeared in many of our statements about complex numbers. Thus, for example, the solutions of a quadratic equation with negative discriminant are conjugate. Also, the formula for the multiplicative inverse of $z = a + bi$ can be written

$$\frac{1}{a + bi} = \frac{a + (-b)i}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

or

(8b) $$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

From Equation (8b) we get immediately

(8c) $$z \cdot \bar{z} = |z|^2.$$

This last equation is important enough to deserve statement as a theorem and a new proof.

Theorem 8a. $z \cdot \bar{z} = |z|^2.$

Proof: If $z = a + bi$ in standard form, then

$$z \cdot \bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2i^2 = a^2-b^2(-1)$$
$$= a^2 + b^2 = (\sqrt{a^2 + b^2})^2 = |z|^2.$$

Now that we have proved Equation (8c) independently of Equation (8b) we can derive (8b) from (8c). In fact, it is convenient to use Theorem 8a directly in dividing complex numbers. The following example is illustrative.

Example 8b. Find the quotient $\frac{8 + 5i}{2 + 3i}$.

Solution: The conjugate of $2 + 3i$ is $2 - 3i$. Multiplying $\frac{8 + 5i}{2 + 3i}$ by $\frac{2 - 3i}{2 - 3i}$, and using Theorem 8a and Equation 5f, we get

$$\frac{8 + 5i}{2 + 3i} = \frac{2 - 3i}{2 - 3i} \cdot \frac{8 + 5i}{2 + 3i} = \frac{(2 - 3i)(8 + 5i)}{(2 - 3i)(2 + 3i)}$$
$$= \frac{(2)(8) - (5)(-3) + [8(-3) + 2(4)]i}{2^2 + 3^2}$$
$$= \frac{31}{13} + (\frac{-14}{13})i = \frac{31}{13} - \frac{14}{13}i .$$

If we plot z and $\bar{z}$ in an Argand diagram (Figure 8a),

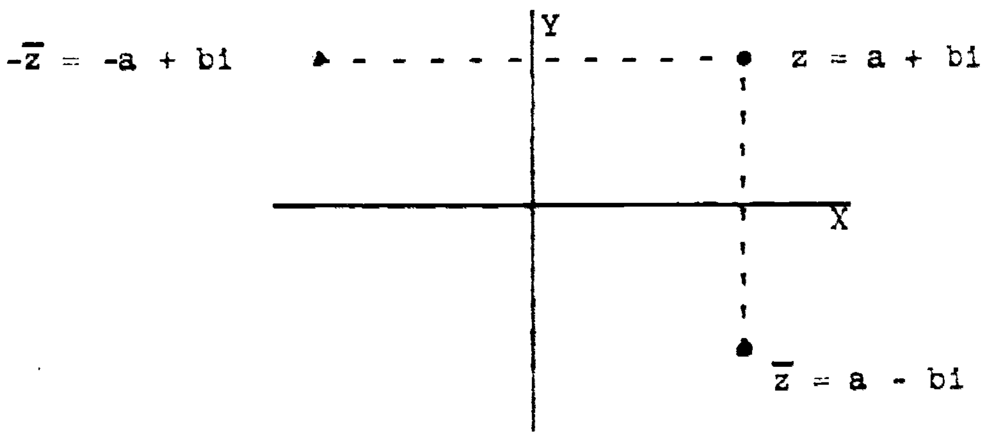

Figure 8a

we see that $\bar{z}$ is the reflection of z in the x-axis; that is, z and $\bar{z}$ are symmetric with respect to the x-axis. Similarly, $-\bar{z}$ is the reflection of z in the y-axis. From this diagram, or by direct calculation, we also see that $z + \bar{z} = 2a$ and $z - \bar{z} = 2bi$. With these equations we can express a and b in terms of z and $\bar{z}$. We thus obtain the following theorem:

Theorem 8b. If $z = a + bi$ in standard form, then

$$z + \bar{z} = 2a, \quad z - \bar{z} = 2bi;$$

or

$$a = \frac{1}{2}(\bar{z} + z), \quad b = \frac{1}{2}(\bar{z} - z).$$

Observe that since a complex number is real if and only if its imaginary part is 0 and pure imaginary if and only if its real part is 0, Theorem 8b has the following corollary.

Corollary. The complex number z is real if and only if $z = \bar{z}$ and pure imaginary if and only if $z = -\bar{z}$.

Theorem 8b permits us to state any relation between the real and imaginary parts of a complex number z as a relation between z and $\bar{z}$. In particular, many statements of analytic geometry can be expressed as a relation of this kind. Before considering examples we state the following theorem which simplifies the computation of conjugates.

Theorem 8c. If z_1 and z_2 are any complex numbers, then

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;

(b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$;

(c) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;

(d) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

The proofs are left as exercises (Exercises 8, Problem 5).

Example 8c. Show that, for any z, the reflection of the point $3iz + 2$ in the x-axis is the point $-3i\bar{z} + 2$.

Solution: The reflection of a point $3iz + 2$ in the x-axis is its conjugate, $\overline{3iz + 2}$. Using Theorem 8c twice we obtain

$$\overline{3iz + 2} = \overline{(3i)(z)} + \bar{2} = \overline{(3i)}(\bar{z}) + \bar{2}$$
$$= -3i\bar{z} + 2,$$

which was to be shown.

Example 8d. Show that the circle of radius 1 with center at the origin is the set of all points z which satisfy the equation

$$z \cdot \bar{z} = 1.$$

Solution: There are two possible approaches. We can start with the definition of this circle as the set of points whose distance from the origin is 1, and use the fact that the distance of the point $z = x + yi$ from the origin is $|z|$. Then z is on the circle if and only if

$$|z| = 1.$$

Squaring both sides of this equation and using Theorem 8a we get

$$z \cdot \overline{z} = |z|^2 = 1.$$

However, we can also start with the equation of the circle from analytic geometry:

$$x^2 + y^2 = 1.$$

If $z = x + yi$ then by Theorem 8b

$$x = \frac{1}{2}(z + \overline{z}), \quad y = \frac{1}{2}(z - \overline{z}).$$

Substituting for x and y, we obtain

$$[\tfrac{1}{2}(\overline{z} + z)]^2 + [\tfrac{1}{2}(\overline{z} - z)]^2 = 1,$$

or

$$\frac{1}{4}(\overline{z} + z)^2 - \frac{1}{4}(\overline{z} - z)^2 = 1.$$

Simplifying, we have

$$z \cdot \overline{z} = 1.$$

Example 8e. Show that the segments which join the points $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ to the origin are perpendicular if and only if the product $z_1 \cdot \overline{z}_2$ is pure imaginary.

Solution: Again, there are two approaches. We can either express the geometric conditions immediately in terms of z_1 and z_2, or state them first in terms of (x_1, y_1) and (x_2, y_2), and then use Theorem 8b. We will follow the first approach.

The segments joining z_1 and z_2 to the origin will be perpendicular if and only if the triangle with vertices $0, z_1, z_2$ is a right triangle. By the Pythagorean Theorem this will be true if and only if

$$|z_1|^2 + |z_2|^2 = |z_1 - z_2|^2.$$

Using Theorems 8a and 8c this equation may be written

$$z_1\overline{z_1} + z_2\overline{z_2} = (z_1 - z_2)(\overline{z_1 - z_2}) = (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$

$$z_1\overline{z_1} + z_2\overline{z_2} = z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2}$$

$$0 = -z_1\overline{z_2} - z_2\overline{z_1}$$

or, using Theorem 8c again and referring to Equation (7a),

$$z_1\overline{z_2} = -\overline{z_1}z_2 = -\overline{(z_1\overline{z_2})}.$$

By the Corollary to Theorem 8b this equation can hold if and only if the product $z_1\overline{z_2}$ is pure imaginary.

Finally, we can use Theorems 8a and 8c to establish Formulas 7c, 7d. We do the first as an example.

Example 8f. Show that $|z_1 \cdot z_2| = |z_1||z_2|$.

Solution: Since the numbers in the equation which is to be established are positive, it will suffice to prove $|z_1 \cdot z_2|^2 = |z_1|^2|z_2|^2$. (Why?) We have

$$|z_1 \cdot z_2|^2 = (z_1 \cdot z_2)\overline{(z_1 \cdot z_2)} = (z_1 \cdot z_2) \cdot (\overline{z_1} \cdot \overline{z_2})$$

$$= (z_1 \cdot \overline{z_1})(z_2 \cdot \overline{z_2}) = |z_1|^2|z_2|^2.$$

This completes the proof.

Exercises 8

1. Express the conjugate of each of the following complex numbers in standard form:

(a) $2 + 3i$	(d) -5	(g) πi^7
(b) $-3 + 2i$	(e) $-2i$	(h) $4 + i^6$
(c) $1 - i$	(f) $1 - i^5$	(i) $-3i + 3i^2$

2. Use conjugates to compute the following quotients. Write the answer in standard form.

(a) $\frac{2 + i}{1 + i}$

(b) $\frac{i}{i + 3}$

(c) $\frac{-i + 1}{2 + 5i}$

(d) $\frac{-4 + 3i}{2 + 3i}$

(e) $\frac{7 + 6i}{3 - 4i}$

(f) $\frac{3 + 2i}{4i}$

(g) $\frac{-5 + 6i}{-3 - 4i}$

(h) $\frac{3 - 6i}{2i}$

(i) $\frac{5}{3 + \sqrt{5}i}$

(j) $\frac{3 - \sqrt{-2}}{5 - \sqrt{-3}}$

(k) $\frac{\sqrt{3} - \sqrt{-7}}{\sqrt{3} - \sqrt{-5}}$

(l) $\frac{i^3 - 1}{i^2 - 1}$

(m) $\frac{a + bi}{2a + 3bi}$; a,b real, $2a + 3bi \neq 0$

(n) $\frac{x + yi}{2x - yi}$; x,y real, $2x - yi \neq 0$

(o) $\frac{(1 + i)(-1 + 2i) + (2 - i)}{2 - 3i}$

(p) $\frac{2i}{(i - 1)(i - 2)(i - 3)}$

3. For each of the following sketch in an Argand diagram the set of complex numbers z which satisfy the given equation.

(a) $z = \frac{1}{z}$ (b) $\bar{z} = \frac{1}{z}$

4. For each of the following sketch in an Argand diagram the set of points z that satisfies the given equation.

(a) $z + \bar{z} = 3$ (b) $z - \bar{z} = 2i$ (c) $z - \bar{z} = 3 + 2i$

5. Let $z_1 = x_1 + y_1 i$, $z_2 = y_2 i$ be any complex numbers, x_1, y_1, x_2, y_2 real. Prove each of the following.

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

(c) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

[Hint: Show that $\overline{(-z_2)} = -(\overline{z_2})$ and use (a).]

(d) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ [Hint: Show that $\overline{\left(\frac{1}{z_2}\right)} = \frac{1}{\overline{z_2}}$ and use (b).]

6. For any z, find the reflection of the point $z^3 - (3 + 2i)z^2 + 5iz - 7$ in the y-axis.

7. If $z^2 = (\overline{z})^2$, show that z is either real or pure imaginary.

8. Show that the product $z_1\overline{z_2}$ is pure imaginary if and only if $\frac{z_1}{z_2}$ is pure imaginary.

9. Prove that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

10. Suppose z_1 and z_2 are complex numbers and that

$$z_1 + z_2 \text{ and } z_1z_2$$

are real numbers. Show that either

$$z_1 \text{ and } z_2 \text{ are real,}$$

or

$$z_1 = \overline{z_2} .$$

11. Use the relation $z \cdot \overline{z} = |z|^2$ to show that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} .$$

12. Write the equation of the straight line $y = 3x + 2$ as an equation in z and $\overline{z}$.

13. Show that if $K \neq 0$ is any complex number and C is any real number, then $K\overline{z} + \overline{K}z = C$ is the equation of a straight line.

14. Show that the points z_1 and z_2 are symmetric with respect to the line $y = x$ if and only if

$$(1 - i)\overline{z_1} + (1 + i)z_2 = 0.$$

15. What is the relation between the line segments joining z_1 and z_2 to the origin if the product $z_1\overline{z_2}$ is real?

9. Polynomial Equations.

Linear and quadratic equations are special cases of polynomial equations. A polynomial is an expression of the form

$$P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-2} z^2 + a_{n-1} z + a_n \tag{9a}$$

where n is a non-negative integer and $a_1, a_1, a_2, \dots, a_{n-1}, a_n$ are any given complex numbers, $a_1 \neq 0$. The non-negative integer n is called the degree of the polynomial and the numbers $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are called its coefficients. A polynomial equation of degree n is an equation

$$P(z) = 0, \tag{9b}$$

where $P(z)$ is a polynomial of degree n. Linear equations are polynomial equations of degree 1; quadratic equations are polynomial equations of degree 2.

Examples 9a.

(a) $2z^3 - \frac{3}{5} z^2 + z - 2 = 0$ is a polynomial equation of degree 3 with rational coefficients.

(b) $z^5 - \sqrt{2} z^3 + 7z^2 - 3 = 0$ is a polynomial equation of degree 5 with real coefficients.

(c) $z^3 - 7\sqrt{z} + 3 = 0$ is not a polynomial equation.

(d) $5z^3 - (2 - i)z + (3 + 7i) = 0$ is a polynomial equation of degree 3 with complex coefficients.

(e) $z - 3 + \frac{1}{z^2} = 0$ is not a polynomial equation, but multiplying by z^2 we obtain the polynomial equation $z^3 - 3z^2 + 1 = 0$. Every solution of the first equation is a solution of the second, and every solution of the second equation is a solution of the first.

Ordinary algebra is mostly concerned with the solution of polynomial equations. Let us summarize some of the advantages that the complex number system C has over the real number system R in connection with polynomial equations.

There are certain quadratic equations whose coefficients are in R but which have no solutions in R; every such equation has solutions in C. This was proved in Section 6 for the case of real coefficients, but it is true even if the coefficients are complex numbers. For example, the equation

$$z^2 + (1 - 5i)z - (12 + 5i) = 0$$

has the two solutions $2 + 3i$ and $-3 + 2i$, a fact which may be checked by substitution. Methods for finding such solutions are beyond the scope of this pamphlet. The theorem that the solutions of any quadratic equation with complex coefficients are complex numbers is an unexpected and remarkable result. It shows us that we will not have to extend the complex number system in order to solve quadratic equations whose coefficients are in C. Recall that R does not have this property; indeed it was just for this reason that we extended R to C.

But the merits of C go far beyond this. <u>Every</u> polynomial equation with coefficients in C has solutions in C, and indeed all the solutions that could be expected are in C. This result, which is known as the Fundamental Theorem of Algebra, comes as an enormous bonus, when we recall that to solve the simple equation $x^2 = -1$ the new element i had to be invented. Conceivably, one might expect to need a new number j to solve $x^4 = -1$, for example. This is not the case! This equation has four and only four complex solutions, all of the form $a + bi$, where a and b are real numbers. (See Exercises 9.)

The first proof of the Fundamental Theorem was given by Gauss in 1799. Since then several other proofs have been developed and although some are quite simple, none is simple enough to be presented here. We shall, however, make a precise statement of the theorem in a form which is basic for the study of polynomials.

<u>Theorem 9</u>. Let

$$P(z) = a_1z^n + a_1z^{n-1} + \dots + a_{n-2}z^2 + a_{n-1}z + a_n$$

be a polynomial of degree n with complex coefficients. Then there exist n complex numbers $r_1, r_2, \dots, r_n$ (not necessarily distinct) such that

$$P(z) = a_o(z - r_1)(z - r_2)\dots(z - r_n).$$

If one of the factors in the factorization of $P(z)$ stated in Theorem 9 is $z - r$, r is called a <u>zero</u> of $P(z)$; if exactly m of these factors are $z - r$, r is called a <u>zero of multiplicity</u> m. A zero is called a <u>simple zero</u> if its multiplicity is one; otherwise, it is called a <u>multiple zero</u>. Since the total number of factors in Theorem 9 is n, the sum of the multiplicities of the zeros of a polynomial of degree n is n. This may also be stated: <u>The number of zeros, each counted with its multiplicity, of a polynomial of degree n is n.</u>

Since a product is 0 if and only if one of its factors is 0, it is clear that z is a solution of the polynomial equation

$$P(z) = 0$$

if and only if z equals one of the zeros of $P(z)$. According to Theorem 9 a polynomial of degree $n > 0$ has at least one zero (exactly one if $r_1 = r_2 = \ldots = r_n$) and may have as many as n zeros (exactly n if no two of the numbers $r_1, r_2, \ldots r_n$ are equal). It follows that every polynomial equation of degree $n > 0$ has at least one complex solution, and may have as many as n solutions, but has no more than n solutions.

<u>Example 9b</u>. Discuss the possible number of solutions of a polynomial equation of degree 3. Include examples.

<u>Solution</u>: The equation may have 1, 2, or 3 solutions. If it has one solution, this must be a triple zero (zero of multiplicity 3) of the polynomial. If it has two solutions, one must be a simple zero, the other a double zero (zero of multiplicity 2) of the polynomial. If it has three solutions, each must be a simple zero of the polynomial.

An example of the first case is given by the polynomial equation

$$z^3 - 3z^2 + 3x - 1 = (z - 1)^3 = 0.$$

The only solution of the equation is $z = 1$. 1 is a triple zero of the polynomial $z^3 - 3z^2 + 3z - 1$.

The equation

$$z^3 - z^2 - z + 1 = (z + 1)(z - 1)^2 = 0$$

has the solutions 1, -1. -1 is a simple zero and 1 a double zero.

The equation

$$z^3 + z = z(z - i)(z + i) = 0$$

has the solutions 0, i, $-i$. Each is a simple zero of $z^3 + z$.

Let $P(z)$ be a polynomial of degree n,

$$P(z) = a_o(z - r_1)(z - r_2)\ldots(z - r_n),$$

and define $Q(z)$ by

$$Q(z) = a_o(z - r_2)\ldots(z - r_n).$$

Then $Q(z)$ is a polynomial of degree $n - 1$ whose zeros are the zeros of $P(z)$, except possibly for r_1, and

$$P(z) = (z - r_1)Q(z).$$

Now suppose we have to determine the zeros of $P(z)$ and that we have found one zero, r_1. The remaining zeros will be the zeros of $Q(z)$ and to find $Q(z)$ we have only to divide $P(z)$ by $z - r_1$, since

$$\frac{P(z)}{z - r_1} = Q(z).$$

This fact enables us to reduce the solution of a polynomial equation of degree n to the solution of an equation of degree $n - 1$ once we have determined one solution of the original equation. The following example illustrates this.

<u>Example 9c</u>. Find all solutions of the equation $z^3 - 1 = 0$.

Solution: The solutions of the equation are the zeros of $z^3 - 1$. One zero is obviously 1. We divide $z^3 - 1$ by $z - 1$:

$$\begin{array}{r|llll}
 & z^2 & + z & + 1 & \\
\hline
z - 1 & z^3 & & & - 1 \\
 & \underline{z^3 - z^2} & & & \\
 & & z^2 & & \\
 & & \underline{z^2 - z} & & \\
 & & & z & \\
 & & & \underline{z - 1} & \\
 & & & & 0
\end{array}$$

The remaining solutions thus are the zeros of $z^2 + z + 1$, that is, the solutions of

$$z^2 + z + 1 = 0.$$

Solving this quadratic equation we get the roots $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Thus, the solutions of the given equations are 1, $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$, $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$.

In this example we observe that, as in the case of quadratic equations, the complex roots are conjugate. We can show that whenever the coefficients of a polynomial equation are <u>real</u> the complex solutions occur in conjugate pairs; that is, if z is a solution of such an equation, $\overline{z}$ is also a solution. Let z be a solution of

$$a_o z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0.$$

Then we have $\overline{a_o z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n} = \overline{0} = 0,$

and using Theorem 7c repeatedly we get

$$\overline{a_o}(\overline{z})^n + \overline{a_1}(\overline{z})^{n-1} + \ldots + \overline{a_{n-1}}(\overline{z}) + \overline{a_n} = 0.$$

Since the coefficients are real, $\overline{a_o} = a_o$, $\overline{a_1} = a_1$, ..., $\overline{a_{n-1}} = a_{n-1}$, $\overline{a_n} = a_n$ and we have

$$a_o(\overline{z})^n + a_1(\overline{z})^{n-1} + \ldots + a_{n-1}\overline{z} + a_n = 0,$$

so that $\overline{z}$ is also a solution of the equation.

Exercises 9

1. Determine the zeros and the multiplicity of each zero for the following polynomials.

   (a) $5(z - 1)(z + 2)^3$

   (b) $z^4(z + \frac{1}{2})^2(z - 3)$

   (c) $(z - 3 + 2i)^2(z + 1)^5$

2. Find the zeros of the following polynomials and state the multiplicity of each zero.

   (a) $z^5 + z^4 + 3z^3$

   (b) $z^4 + 2z^2 + 1$

   (c) $z^3 + 3z^2 + 3z + 1$

3. Write two polynomial equations whose only solutions are 1 and 2 such that:
   (a) the two equations have the same degree;
   (b) the two equations are of different degrees.

4. Discuss, with examples, the possible number of solutions of an equation of degree 4.

5. Find all solutions of $z^3 + 1 = 0$.

6. Find all solutions of the following equations, given one solution.
   (a) $3z^3 - 20z^2 + 36z - 16 = 0$ $\qquad z = 4$
   (b) $z^3 - 4z^2 + 6z - 4 = 0$ $\qquad z = 2$

7. Find all solutions of the following equations, given two solutions.
   (a) $z^4 + 2z^3 + z + 2 = 0$ $\qquad z = -1, -2$
   (b) $z^4 - 3z^3 - 3z^2 - 7z + 12 = 0$ $\qquad z = 4, 1$

8. Find the polynomial whose zeros include 1 and $-2i$ if:
   (a) the polynomial has the lowest possible degree.
   (b) the polynomial has real coefficients and has the lowest possible degree.
   (c) the polynomial has real coefficients, the lowest possible degree and $-2i$ is a double zero.

9. Given that $3 + \sqrt{2}\,i$ is a solution, find all solutions of the equation
$$z^4 - 6z^3 + 2z^2 + 54z - 99 = 0.$$

10. Given that $1 - \sqrt{5}\,i$ is a solution, find all solutions of the equation
$$z^4 - 2z^3 + 4z^2 + 4z - 12 = 0.$$

11. (a) Find a formula for the coefficients of the cubic polynomial whose zeros are r_1, r_2, r_3 if the coefficient of the highest power is 1.
   *(b) Do the same for the quadratic polynomial.
   *(c) Make a guess as to the form of a corresponding formula for a polynomial of degree 7.

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10. Miscellaneous Exercises.

1. If $z = 2 - 3i$, evaluate
   $-z$, $\overline{z}$, $|z|$, $|\overline{z}|$, $\frac{1}{z}$, $|z|^2$, $|z^2|$, and $\frac{4 + 5i}{z}$.

2. Write a quadratic equation having the solution $c + di$ and $c - di$, where c and d are real.

3. Is the set of numbers $\{1, -1, i, -i\}$ closed with respect to multiplication? Addition?

4. If $z = x + yi$, show that

   $$x \leq |z| \quad \text{and} \quad y \leq |z|.$$

5. Sketch the set of points z which satisfy each of the following conditions.

   (a) $|z - 2| = 3$

   (b) $|z + 2| > 3$

   (c) $|z - 2i| < 4$

   (d) $|z - z_0| \leq 5$

6. Write an equation in x and y which is equivalent to the equation $|z - (2 + 3i)| = 5$.
   Describe the set of points in an Argand diagram which satisfy the given equation.

7. Give a geometrical interpretation for the following relations.

   (a) $|z_1| < |z_2|$

   (b) $|z| = 5$

   (c) $z_1 + z_2 = 0$

   (d) $z_1 + \overline{z_2} = 0$

   (e) $z_1 - \overline{z_2} = 0$

8. Find all complex numbers z such that (Real part of z) = (Imaginary part of z), and $|z| = 1$.

9. Determine all quadratic equations with real coefficients which have $3 + 2i$ as a solution.

10. Plot the point corresponding to $3 + 5i$ in an Argand diagram, then multiply the given number successively by i, i^2, and i^3, and plot the three points which correspond to the resulting products. Finally, show that the three last named points together with the given point form the vertices of a square.

11. Show that if z_o is a solution of the equation $az^2 + bz + c = 0$, where a, b, c are real and $b^2 - 4ac < 0$, then $z_o\overline{z_o} = \frac{c}{a}$ and $z_o + z_o = -\frac{b}{a}$. Use the result to describe a geometric construction for z_o.

12. Find all quadratic equations with real coefficients having solutions z_1 and z_2 such that $z_1 + z_2 = 1$ and $z_1 z_2 = 4$.

13. Find all complex numbers z for which the real part of z^2 is 0. Show that if z belongs to this set, then $\frac{1}{z}$ also belongs to the set.

14. For what real values of r does the equation

$$rx^2 + (1 + r)x + 2 = 0$$

have non-real complex solutions? For what values of r does it have only one solution?

15. Show by an example that $a - bi$ need not be the complex conjugate of $a + bi$.

16. Find the equation of the perpendicular bisector of the line joining z_1 and z_2. [Hint: Use the fact that the perpendicular bisector of a line segment is the set of points equidistant from the endpoints.]

17. Let $z_o = x_o + y_o i$. Describe the set of points $z = x + yi$ which satisfy the inequality

$$\left|\frac{z - \overline{z}_o}{z - z_o}\right| < 1.$$

18. Let z_1 and z_2 be distinct non-zero complex numbers. Show that z_1 and z_2 represent points in an Argand diagram lying on a straight line through the origin if and only if $\frac{z_1}{z_2}$ is real.

19. Solve the equation $z^4 = -1$. (You may find it helpful to refer to Exercises 6, Problems 22 and 23.)

20. Show that it is impossible to satisfy all the order postulates of Chapter 1 in the complex number system. Consider the element i. Certainly $i \neq 0$, so either $i > 0$ or $i < 0$ if the "Trichotomy" property is to hold. Show that each of the assumptions $i > 0$, $i < 0$ leads to conclusions contradicting at least one of the order postulates.

21. Find all complex numbers x, y with the property that the conjugate of $x + yi$ is $x - yi$.

*22. If $z = x + yi$, show that

$$|x| + |y| \leq \sqrt{2}\,|z|.$$

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11. Construction of the Complex Number System.

In this chapter we have assumed that we have available a number system (which we called the complex number system) satisfying certain imposed requirements (the four fundamental properties C-1, C-2, C-3, C-4). In a sense we have stated what a complex number system ought to be. On the basis of the imposed requirements we have learned how to compute in such a system.

It is a fundamental (but sophisticated) question whether there actually exists a number system C fulfilling the requirements we set down in Section 1 and 2. We shall sketch the basic steps for constructing such a system. Many of the details will be left to the reader.

Let us return to our earlier developments. There we learned that the rule which associates with the complex number $a + bi$ the ordered pair of real numbers (a,b) sets up a one-to-one correspondence between the set of complex numbers and the set of ordered pairs of real numbers. This fact and the information which we have obtained on how we are compelled to add and multiply in C motivates the following proposal for constructing, on the basis of the real number system, a number system which meets the requirements we imposed on C.

Let K denote the set of ordered pairs of real numbers (a,b). These are the objects which we are to "add" and "multiply". Let us say: (a,b) = (c,d) if and only if a = c and b = d.

It is necessary to define operations of addition and multiplication for K. The facts we have deduced from the fundamental properties of the complex number system lead us to believe that the definitions which we shall put down are "reasonable" when we keep in mind our mission of constructing a complex number system with "real building blocks".

We define

<u>Addition</u>: (a,b) + (c,d) = (a + c, b + d).

<u>Multiplication</u>: (a,b)·(c,d) = (ac - bd, ad + bc).

Note that the operation of "addition" in K is <u>defined</u> in terms of the operation of addition in the real number system and that the operation of "multiplication" in K is <u>defined</u> in terms of addition, subtraction and multiplication in the real number system. Note that our definitions assure closure of the operations + and • of K: the "sum" of two ordered pairs of real numbers is an ordered pair of real numbers, the "product" of two ordered pairs of real numbers is an ordered pair of real numbers.

Two remarks are in order. First, we must distinguish "addition" and "multiplication" in K from addition and multiplication in the real number system. Two two kinds of addition and multiplication apply respectively to different kinds of objects. That is why we use the exaggerated plus sign + and the exaggerated times sign • for the operations of "addition" and "multiplication" in K.

Second, we emphasize that + and • are constructed from what we learned about addition and multiplication in C keeping in mind that our correspondence between a + bi and (a,b) identifies "real part" with "first component" and "imaginary part" with "second component". The spadework sets in at this stage. We verify first that K with the addition + and multiplication • satisfies the usual laws of algebra. This verification depends upon properties satisfied by the real number system. We easily verify that (0,0) is the <u>additive identity</u> for K, that (1,0) is the <u>multiplicative identity</u> for K, and that (-1,0) is the additive inverse of the multiplicative identity.

Explicitly, we have the following results:

$$(a,b) + (0,0) = (a,b), \qquad (a,b) \cdot (1,0) = (a,b),$$
$$(1,0) + (-1,0) = (0,0).$$

Verify these three statements.

Furthermore, $(0,1) \cdot (0,1) = (-1,0)$.

Hence, K possesses an element whose square is the additive inverse of the multiplicative identity. This sounds a bit heavy-handed but tells us that we have grounds for optimism as far as capturing something that will play the role of the all-important i. Let us go so far as to denote (0,1) by i. We may write

(11a) $$(a,b) = (a,0) + (0,b) = (a,0) + (b,0) \cdot (0,1)$$
$$= (a,0) + (b,0) \cdot i$$

Now if we restrict our attention to the special elements of K whose second components are zero, we see that they behave under + and $\cdot$ the same way that their first components do under the + and $\cdot$ of the real number system. That is,

(11b) $$(a,0) + (b,0) = (a + b,0),$$

(11c) $$(a,0) \cdot (b,0) = (ab,0).$$

Verify the statements (11b), (11c) and also the following two:

$$(a,0) + (-a,0) = (0,0);$$
$$(a,0) \cdot (\frac{1}{a}, 0) = (1,0), \quad a \neq 0.$$

We now define a notion of order among the special elements of the form (a,0). (Remark: We could not define a notion of order in K, even if we wanted to, which would yield the expected relation among the special elements (a,0). This remark applies to C also. If we had an order relation in C like that in R we could expect the square of each non-zero element to be positive. This would force i^2 into the unacceptable position of being both positive andnegative in the sense of the real number system.) We define

"<u>Less than</u>": $[(a,0) < (b,0)]$ means $(a < b)$.

It is now possible to show that the set of elements of the form (a,0) together with the operation of addition +, the operation of multiplication $\cdot$, and the relations of inequality $<$ satisfy the postulates for the real number system.

Verify this assertion.

We are thus justified in taking this set of awkward appearing elements $(a,0)$ with addition, multiplication and order so introduced as our real number system. With this understanding we verify that K has all the properties imposed on C. Note that $(-1,0)$ is the additive inverse of the multiplicative identity of our present real number system and that

(11d) $$i \bullet i = (-1,0).$$

Thanks to the fact that the elements $(a,0)$ may be taken as the real numbers, Property C-2 is satisfied. By Formula (11d), Property C-3 is satisfied. Further, Formula (11e) tells us that Property C-4 is satisfied. There remains to be verified only that $\boldsymbol{+}$ and $\bullet$ are commutative and associative, that the distributive law holds in K, and that each element has an additive inverse, in order to show that K has Property C-1.

The commutative and associative laws for $\boldsymbol{+}$ and $\bullet$ are readily verified as is the distributive law. As an illustration we consider the distributive law:

$$(a,b) \bullet [(c,d) + (e,f)]$$
$$= (a,b) \bullet (c + e,d + f)$$
$$= (a(c + e) - b(d + f), b(c + e) + a(d + f))$$

and $$[(a,b) \bullet (c,d)] \boldsymbol{+} [(a,b) \bullet (e,f)]$$
$$= (ac - bd, bc + ad) \boldsymbol{+} (ae - bf, af + be)$$
$$= ((ac - bd) + (ae - bf), (bc + ad) + (be + af))$$
$$= (a(c + e) - b(d + f), b(c + e) + a(d + f)).$$

We see that the distributive law holds.

Additive inverse? Since

$$(a,b) \boldsymbol{+} (-a,-b) = (0,0).$$

$(-a,-b)$ is the additive inverse of (a,b).

It is now simple to verify that a non-zero element (a,b) has a multiplicative inverse and hence that the equation

$$(a,b) \bullet (x,y) = (c,d), \qquad (a,b) \neq (0,0)$$

has a unique solution.

Given $(a,b) \neq (0,0)$, we verify that

$$(a,b) \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$$

$$= \left(a\left(\frac{a}{a^2 + b^2}\right) - \left(b\frac{-b}{a^2 + b^2}\right), a\left(\frac{-b}{a^2 + b^2}\right) + \left(b\frac{a}{a^2 + b^2}\right)\right)$$

$$= (1,0).$$

We now conclude that K together with + and • satisfies the conditions imposed on the complex number system.

At this stage it suffices to redesign our notation for the real numbers in K and to designate the real numbers by the letters a, b, c, ... to use the standard notations for the additive unit and the multiplicative unit, and to write + and • for + and • respectively. With these agreements each complex number is of the form

$$a + bi,$$

where a and b are real, and $i^2 = -1$.

APPENDIX

LIST OF BASIC PROPERTIES OF THE REAL NUMBER SYSTEM

Taken from Chapter 1 of "Intermediate Mathematics" (SMSG)

For arbitrary a, b, c in R:

$\underline{E}_1$ (Dichotomy) Either $a = b$ or $a \neq b$.

$\underline{E}_2$ (Reflexivity) $a = a$.

$\underline{E}_3$ (Symmetry) If $a = b$, then $b = a$.

$\underline{E}_4$ (Transitivity) If $a = b$ and $b = c$, then $a = c$.

$\underline{E}_5$ (Addition) If $a = b$, then $a + c = b + c$.

$\underline{E}_6$ (Multiplication) If $a = b$, then $ac = bc$.

$\underline{A}_1$ (Closure) $a + b$ is a real number.

$\underline{A}_2$ (Commutativity) $a + b = b + a$.

$\underline{A}_3$ (Associativity) $a + (b + c) = (a + b) + c$.

$\underline{A}_4$ (Additive Identity) $0 + a = a + 0 = a$.

$\underline{A}_5$ (Subtraction) For each pair a and b of real numbers, there is exactly one real number c such that $a + c = b$.

$\underline{M}_1$ (Closure) ab is a real number.

$\underline{M}_2$ (Commutativity) $ab = ba$.

$\underline{M}_3$ (Associativity) $a(bc) = (ab)c$.

$\underline{M}_4$ (Multiplicative Identity) $1 \cdot a = a \cdot 1 = a$.

$\underline{M}_5$ (Division) For each pair a,b of real numbers, $b \neq 0$, there is exactly one real number c such that $bc = a$.

$\underline{D}$ (Distributivity) $a(b + c) = ab + ac$.

O_1 (Trichotomy) If a and b are real numbers, exactly one of the following holds:

$$a = b, \qquad a < b, \qquad a > b.$$

O_2 (Transitivity) If $a < b$, and $b < c$, then $a < c$.

O_3 (Addition) If $a < b$, then $a + c < b + c$.

O_4 (Multiplication) If $a < b$ and $0 < c$, then $ac < bc$; but if $a < b$ and $c < 0$, then $bc < ac$.

O_5 (Archimedes) If a and b are positive real numbers and $a < b$, there is a positive integer n such that $na > b$.

O_6 (Density) If a and b are real numbers, $a \neq b$, then there is a real number c such that $a < c < b$ or $b < c < a$. Hence, between any pair of distinct real numbers there are infinitely many real numbers.

O_7 (R) If $\{a_0, a_1, a_2, \ldots, a_n \ldots\}$ and $\{b_0, b_1, b_2, \ldots, b_n \ldots\}$ are two sequences of real numbers with the properties

(i) $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$

(ii) $b_0 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots$

(iii) $a_n \leq b_n$, for every natural number n

(iv) $b_n - a_n < \frac{1}{10^n}$, for every natural number n

then there is one and only one real number c such that $a_n \leq c \leq b_n$, for every natural number n.