DOCUMENT RESUME

ED 175 660 SE 028 648

AUTHOR Dubisch, Roy, Ed.
TITLE Supplementary and Enrichment Series: Functions. SP-1.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Scien Foundation, Washington, D.C.
PUB DATE 64
NOTE 47p.: For related documents, see SE 028 649-675; Contains occasional light and broken type

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS Curriculum; *Enrichment; *Graphs; *Instruction; Mathematics Education; Secondary Education; *Secondary School Mathematics; *Set Theory; Supplementary Reading Materials
IDENTIFIERS *Functions (Mathematics); *School Mathematics Study Group

ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include: (1) graphs; (2) constant, linear, and absolute-value functions; (3) composition and inversion; (4) one-to-one functions; and (5) ordered pairs. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-1

SUPPLEMENTARY and ENRICHMENT SERIES

FUNCTIONS

Edited by Roy Dubisch

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

FUNCTIONS

This pamphlet is essentially Chapter 1 and Section 9 of Chapter 4 of the SMSG text, Elementary Functions. A few minor changes have been made for clarity and to make the material self contained. Some background material on sets and a section on functions as sets of ordered pairs have been added.

It is intended for use as a unit in any course following a course in plane geometry and one-and-a-half or two years of algebra.

The material contained herein is basic to an understanding of the trigonometry of real numbers and the calculus as well as many other parts of mathematics.

FUNCTIONS

FUNCTIONS

1. Sets.

One of the most natural and familiar ideas of human experience is that of thinking about and identifying a collection of objects by means of a single word. Examples of such words are family, team, flock, herd, deck (of cards), collection, and so forth. We shall use the word set when talking about such a collection, and we shall restrict ourselves to sets that are clearly enough defined so that there is no possible ambiguity about their members. In other words, a set is a collection of objects, described in such a way that there is no doubt as to whether a particular object does or does not belong to the set.

As an illustration, think of the collection of books, pencils, tablets, etc., that is in your desk. You can easily tell whether or not a particular object belongs to this set: if an object is in your desk, then it is a member, or element, of this set; if an object is not in your desk, then it is not an element of this set. It is important to understand that it does not matter what objects are in your desk; to be an element of this particular set, the only requirement is that an object be in your desk and not somewhere else.

We have at our disposal two methods for describing a set: (1) the tabulation method, in which we list or tabulate every element of a set, and (2) the rule method, in which we describe the elements of a set by some verbal or symbolic statement without actually listing the elements. This latter method was used in the preceding paragraph when we defined a set by specifying that it contained all the objects in your desk. Other illustrations of the rule method in defining a set are the following: the set of all boys and girls who attend your school, the set of people who live in your home, the set of books in your school library, or the set of colors your mother is going to use in redecorating her kitchen.

Although the rule method for defining a set will be used predominantly, there are cases in which the only feasible way to define a set is by actually tabulating its elements. This may be because the elements of a set are not required to have anything in common except membership in the set. It is true that most, if not all, of the sets we shall be talking about will consist of things which are naturally assembled together, as, for example, the set of whole numbers. Nonetheless, a set may consist of things which have no obvious relation except that they happen to be grouped together, just as the set of objects which a nine-year-old boy calls his "treasure" may consist of a yo-yo, an Indian-head penny, a ball made of packed tinfoil, a collection of match books, a dried grasshopper, a pocket knife, and a pack of baseball cards. Perhaps such an example will help to clarify the idea that a set is a collection of things, not necessarily alike in any other respect, and that membership in the set is to be emphasized.

The notation which is customarily used when defining a set, whether by the tabulation method or the rule method, will be illustrated by another example. Consider the question: What is the set of all coins in your pocket at this moment? (The answer in this case might be the set with no elements--the empty set!) Suppose that you have three pennies, two nickels, a dime, and a quarter in your pocket, the pennies and nickels being distinguished by different dates. The set called for by the rule is the collection of these seven coins and no others. Using the tabulation method, we symbolize this by writing:

S = {1915 penny, 1937 penny, 1959 penny, 1942 nickel,
1950 nickel, dime, quarter}.

Capital S is the name for the set, and the names of the elements of the set are enclosed in the braces. The order in which the elements are listed within the braces does not matter. Alternatively, we may denote this same set by enclosing the rule in braces:

S = {*: * is a coin in your pocket}.

This is read, "S is the set of all * such that * is a coin in your pocket." The colon following the first * is a symbol for the phrase "such that", and the symbol * stands for any unspecified element of the set. We could just as well have used c, or x, or $\not{x}$, so that S = {c: c is a coin in your pocket} is still the set of coins in your pocket. The symbolism {*: * ...} is often called the "set-builder" notation.

In summary, we have illustrated two alternative ways for defining any particular set: (1) the tabulation method, and (2) the rule or set-builder method. As emphasized earlier, each of these methods has the essential characteristic that every object may be classified as either belonging to the set or not belonging to the set. In some cases either method can be used, as we did in describing the set of coins in your pocket. In other situations only one of the two methods may be practical.

To indicate membership in a set we use the Greek letter ϵ (epsilon). Thus, if a is an element of the set A, we write $a \epsilon A$. (This may be read, "a is an element of the set A," or "a is a member of the set A," or "a belongs to the set A," etc.) Likewise, we may wish to indicate that b is <u>not</u> an element of A. In this case we use epsilon with a diagonal line drawn through it, indicating negation, and write $b \notin A$.

At this point it may be helpful to review the ideas and symbolism of set thinking by means of examples of sets whose elements are numerical. Both the rule method and the tabulation method will be used in defining the sets.

$D = \{d: d \text{ is an integer and } 0 \leq d \leq 9\}$
$= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$

$E = \{e: e \text{ is an even integer and } e \in D\}$
$= \{0, 2, 4, 6, 8\}.$

$M = \{m: m \text{ is a positive integral multiple of } 3 \text{ and } m < 20\}$
$= \{3, 6, 9, 12, 15, 18\}.$

5 is an element of the set D: $5 \in D$.

5 is not an element of the set E: $5 \notin E$.

$P = \{x: x \text{ is a positive integer}\}$
$= \{1, 2, 3, 4, 5, \cdots\}$. The dots here signify that we do not stop at 5 but keep on going indefinitely. A set such as this with an unlimited number of elements is called an <u>infinite</u> set, whereas sets D, E, and M, above, are <u>finite</u> sets.

2 is an element of the set P: $2 \in P$.

$\frac{3}{4}$ is not an element of the set P: $\frac{3}{4} \notin P$.

Exercises 1

1. Use both the tabulation method and the rule method to specify the following sets:
   (a) the vowels;
   (b) the prime numbers less than 20;
   (c) the people who live in your house;
   (d) the odd multiples of three which are equal to or less than 21;
   (e) the two-digit numbers, the sum of whose digits is 8.

2. Represent the following sets by the rule method and tell why the tabulation method may be difficult or impossible:
   (a) the set of students in your school;
   (b) the integers greater than 7;
   (c) the people in your community who found a ten-dollar bill yesterday;
   (d) the books in your school library;
   (e) the rational numbers between 2 and 3.

3. Find a rule which will define the sets whose elements are tabulated in each of the following:
   (a) $A = \{2, 4, 6, 8, 10\}$;
   (b) $B = \{-3, -2, -1, 0, 1, 2, 3\}$;

(c) $C = \{1, 4, 9, 16, 25\}$;

(d) $D = \{2, 5, 8, 11, 14, 17\}$;

(e) $E = \{123, 132, 213, 231, 312, 321\}$.

---

2. Definition of Function.

One of the most useful and universal concepts in mathematics is that of a function, and this pamphlet, as its title indicates, will be devoted to the study of functions.

We frequently hear people say, "One function of the Police Department is to prevent crime," or "Several of my friends attended a social function last night," or "My automobile failed to function when I tried to use it." In mathematics we use the word "function" somewhat differently than we do in ordinary conversation; as you have probably learned in your previous study, we use it to denote a certain kind of association or correspondence between the members of two sets.

We find examples of such association on every side. For instance, we note such an association between the number of feet a moving object travels and the difference in clock readings at two separate points in its journey; between the length of a steel beam and its temperature; between the price of eggs and the cost of making a cake. Additional examples of such associations occur in geometry, where, for instance, we have the area or the circumference of a circle associated with the length of its radius.

In all of these examples, regardless of their nature, there seems to be the natural idea of a direct connection of the elements of one set to those of another; the set of distances to the set of times, the set of lengths to the set of thermometer readings, etc. It seems natural, therefore, to abstract from these various cases this idea of association or correspondence and examine it more closely.

Let us start with some very simple examples. Suppose we take the numbers 1, 2, 3, and 4, and with each of them associate the number twice as large: with 1 we associate 2, with 2 we associate 4, with 3 we associate 6, and with 4 we associate 8. An association such as this is called a function, and the set $\{1, 2, 3, 4\}$ with which we started is called the domain of the function. We can represent this association more briefly if we use arrows instead of words: $1 \to 2$, $2 \to 4$, $3 \to 6$, $4 \to 8$. There are, of course, many other functions with the same domain; for example, $1 \to 2$, $2 \to 1$, $3 \to 2$, $4 \to 5$.

It happens that these two examples deal with numbers, but there are many functions which do not. A map, for instance, associates each point on some bit of terrain with a point on a piece of paper; in this case, the domain of the function is a geographical region. We can, indeed, generalize this last example, and think of any function as a mapping; thus, our first two examples map numbers into numbers, and our third maps points into points.

What are the essential features of each of these examples? First, we are given a set, the domain. Second, we are given a rule of some kind which associates an object of some sort with each element of the domain, and, third, we are given some idea of where to find this associated object. Thus, in the first example above, we know that if we start with a set of real numbers, and double each, the place to look for the result is in the set of all real numbers. To take still another example, if the domain of a function is the set of all real numbers, and the rule is "take the square root", then the set in which we must look for the result is the set of complex numbers. We summarize this discussion in the following definition:

Definition 1. If with each element of a set A there is associated in some way exactly one element of a set B, then this association is called a function from A to B.

It is common practice to represent a function by the letter f (other letters such as g and h will also be used). If x is an element of the domain of a function f, then the object which f associates with x is denoted $f(x)$ (read "the value of f at x" or simply "f at x" or "f of x"); $f(x)$ is called the image of x. Using the arrow notation of our examples, we can represent this symbolically by

$$f: \quad x \rightarrow f(x)$$

(read "f takes x into $f(x)$"). This notation tells us nothing about the function f or the element x; it is merely a restatement of what $f(x)$ means.

The set A mentioned in Definition 1 is, as has been stated, the domain of the function. The set of all objects onto which the function maps the element of A is called the range of the function; in set notation,

$$\text{range of } f = \{f(x): \ x \in A\}.$$

The range may be the entire set B mentioned in the definition, or may be only a part thereof, but in either case it is included in B.

It is often helpful to illustrate a function as a mapping, showing the elements of the domain and the range as points and the function as a set of arrows from the points that represent elements of the domain to the points that represent elements of the range, as in Figure 1. Note that, as a consequence of Definition 1, to each element of the domain there corresponds one and only one element of the range.

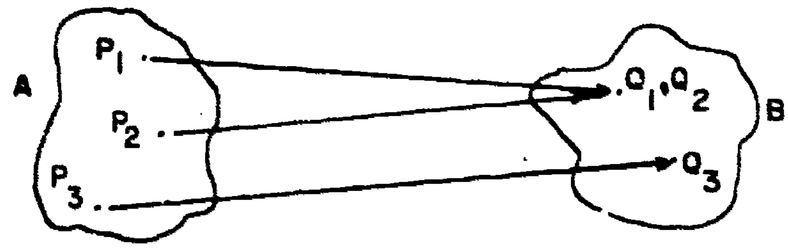


Figure 1. A function as a mapping.

If this condition is not met, as in Figure 2, then the mapping pictured is not a function. In terms of the pictures, a mapping is not a function if two arrows start from one point; whether two arrows go to the same point, as in Figure 1, is immaterial in the definition. This requirement, that each element of the domain be mapped into one and only one element of the range, may seem arbitrary, but it turns out, in practice, to be extremely convenient.

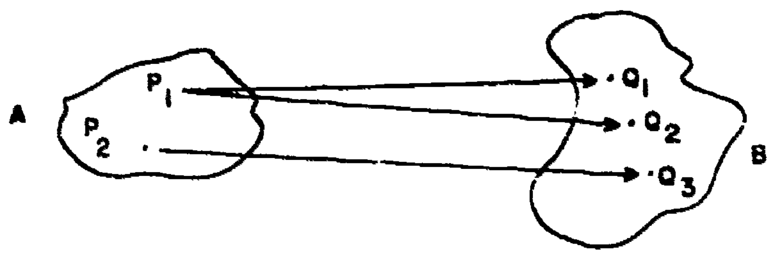


Figure 2. This mapping is not a function.

In this pamphlet, we are primarily concerned with functions whose domain and range are sets of real numbers, and we shall therefore assume, unless we make explicit exception, that all of our functions are of this nature. It is therefore convenient to represent the domain by a set of points on a number line and the range as a set of points on another number line, as in Figure 3.

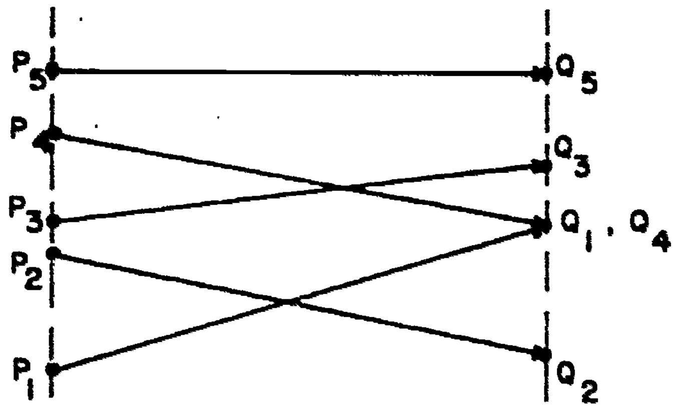


Figure 3. A function mapping
real numbers into real numbers.

More specifically, consider the function f, discussed earlier, which takes each element of the set {1, 2, 3, 4} into the number twice as great. The range of this function is {2, 4, 6, 8} and f maps its domain onto its range as shown in Figure 4. We note that, in this case, the image of the element x of the domain of f is the element 2x; hence we may write, in this instance, $f(x) = 2x$, and f is completely specified by the notation

$$f: x \rightarrow 2x, \quad x = 1, 2, 3, 4.$$

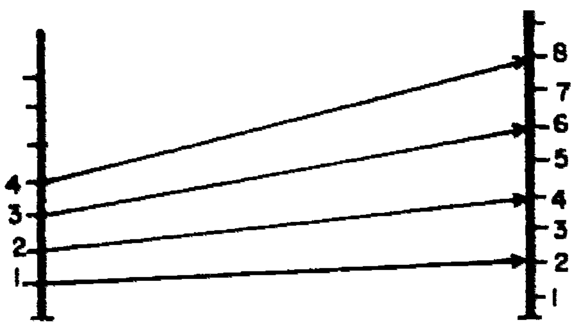


Figure 4. $f: x \rightarrow 2x, \quad x = 1, 2, 3, 4.$

In this case, the way in which f maps its domain onto its range is completely specified by the formula $f(x) = 2x$. Most of the functions which we shall consider can similarly be described by appropriate formulas. If, for example, f is the function that takes each number into its square, then it takes 2 into 4 (that is, $f(2) = 4$), it takes -3 into 9 (that is, $f(-3) = 9$), and, in general, it takes any real number x into x^2. Hence, for this function, $f(x) = x^2$, and we may write $f\colon\ x \to x^2$. The formula $f(x) = x^2$ defines this function f, and to find the image of any element of the domain, we can merely substitute in this formula; thus, if $a - 3$ is a real number, then $f(a - 3) = (a - 3)^2 = a^2 - 6a + 9$. Similarly, if we know that a function f has $f(x) = 2x - 3$ for all $x \in R$ (we use R to represent the set of real numbers), then we can represent f in our mapping notation as $f\colon\ x \to 2x - 3$, and to find the image of any real number we need only substitute it for x in the expression $2x - 3$; thus $f(5) = 2(5) - 3 = 7$, $f(\sqrt{2}) = 2\sqrt{2} - 3$, and if $k + 2$ is a real number, then

$$f(k + 2) = 2(k + 2) - 3 = 2k + 1.$$

Strictly speaking, a function is not completely described unless its domain is specified. In dealing with a formula, however, it is a common and convenient practice to assume, if no other information is given, that the domain includes all real numbers that yield real numbers when substituted in the formula. For example, if nothing further is said,.in the function $f\colon\ x \to \frac{1}{x}$, the domain is assumed to be the set of all real numbers except 0; this exception is made because $\frac{1}{0}$ is not a real number. Similarly, if f is a function such that $f(x) = \sqrt{1 - x^2}$, we assume, in the absence of any other information, that the domain is $\{x\colon\ -1 \leq x \leq 1\}$, that is, the set of all real numbers from -1 to $+1$ inclusive, since only these real numbers will give us real square roots in the expression for $f(x)$. When a function is used to describe a physical situation, the domain is understood to include only those numbers that are physically realistic. Thus, if we are describing the volume of a balloon in terms of the length of its radius, $f\colon\ r \to V$, the domain would include only positive numbers.

A humorist once defined mathematics as "a set of statements about the twenty-fourth letter of the alphabet". We may not agree about just how funny this statement is, but we must agree that it contains an element of the truth: we do make x work very hard. It is important to recognize that this arises out of custom, not necessity, and that any other letter or symbol would do just as well. The notations $f\colon\ x \to x^2$, $f\colon\ h \to h^2$, $f\colon\ t \to t^2$, and even $f\colon\ \# \to \#^2$ all describe exactly the same function, subject to our agreement that x, h, t, or $\#$ stands for any real number.

Another way of looking at a function, which may help you to understand this section, is to think of it as a machine that processes elements of its domain to produce elements of its range. The machine has an input and an output; if an element x of its domain is fed on a tape into the machine, the element $f(x)$ of the range will appear as the output, as indicated in Figure 5.

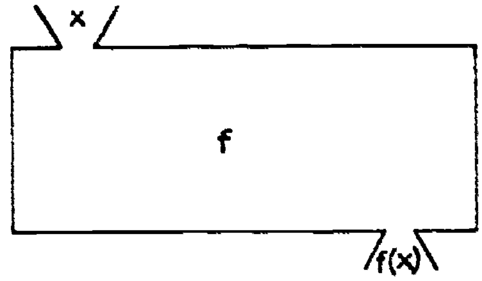


Figure 5. A representation of a function as a machine.

A machine can only be set to perform a predetermined task. It cannot exercise judgment, make decisions, or modify its instructions. A function machine f must be set so that any particular input x always results in the same output f(x); if the element x is not in the domain of f, the machine will jam or refuse to perform. Some machines--notably computing machines--actually do work in almost exactly this way.

Exercises 2

1. Which of the following do not describe functions, when $x, y \in R$?
   - (a) $f: x \rightarrow 3x - 4$
   - (b) $f: x \rightarrow x^3$
   - (c) $f: x \rightarrow y = x^2$
   - (d) $f: x \rightarrow$ all $y < x$
   - (e) $f: x \rightarrow 5x$
   - (f) $f: x \rightarrow 16 - x^2$

2. Depict the mapping of a few elements of the domain into elements of the range for each of the Exercises 1(a) and (c) above, as was done in Figure 4.

3. Specify the domain and range of the following functions, where $x, f(x) \in R$.
   - (a) $f: x \rightarrow x$
   - (b) $f: x \rightarrow x^2$
   - (c) $f: x \rightarrow \sqrt{x}$
   - (d) $f: x \rightarrow \frac{1}{x - 1}$
   - (e) $f: x \rightarrow \frac{3}{x^2 - 4}$

4. If $f: x \rightarrow 2x + 1$, find
   - (a) $f(0)$;
   - (b) $f(-1)$;
   - (c) $f(100)$;
   - (d) $f(\frac{3}{2})$.

5. Given the function $f: x \rightarrow x^2 - 2x + 3$, find

(a) $f(0)$; (c) $f(a)$;

(b) $f(-1)$; (d) $f(x - 1)$.

6. If $f(x) = \sqrt{x^2 - 16}$, find

(a) $f(4)$; (c) $f(5)$; (e) $f(a - 1)$;

(b) $f(-5)$; (d) $f(a)$; (f) $f(\pi)$.

7. If $f: x \rightarrow \frac{4}{3}x^3 - 12x^2 + \frac{98}{3}x - 20$ has the domain $\{1, 2, 3, 4\}$, (a) find the image of f, and (b) depict f as in Figure 4.

8. If $x \in R$, given the functions

$$f: x \rightarrow x$$

and

$$g: x \rightarrow \frac{x^2}{x}$$

are f and g the same function? Why or why not?

9. What number or numbers have the image 16 under the following functions?

(a) $f: x \rightarrow x^2$

(b) $f: x \rightarrow 2x$

(c) $f: x \rightarrow \sqrt{x^2 + 112}$

---

3. The Graph of a Function.

A graph is a set of points. If the set consists of all points whose coordinates (x,y) satisfy an equation in x and y, then the set is said to be the graph of that equation. If there is a function f such that, for each point (x,y) of the graph, and for no other points, we have $y = f(x)$, then we say that the graph is the graph of the function f. The graph is perhaps the most intuitively illuminating representation of a function; it conveys at a glance much important information about the function. The function $x \rightarrow x^2$, (when there is no danger of confusion, we sometimes omit the name of a function, as f in $f: x \rightarrow x^2$) has the parabolic graph shown in Figure 6. We can look at the parabola and get a clear intuitive idea of what the function is doing to the elements of its domain. We can, moreover, usually infer from the graph any limitations on the domain and the range. Thus, it is clear from Figure 6 that the range of the function there graphed includes only non-negative numbers, and in the function $f: x \rightarrow \sqrt{25 - x^2}$ graphed in Figure 7, the domain $\{x: -5 \leq x \leq 5\}$ and range $\{y: 0 \leq y \leq 5\}$ are easily determined, as shown by the heavy segments on the x-axis and y-axis, respectively.

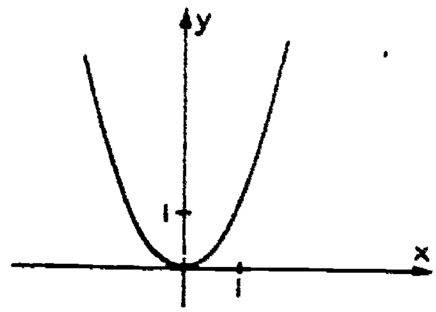


Figure 6. Graph of the function f: $x \to x^2$.

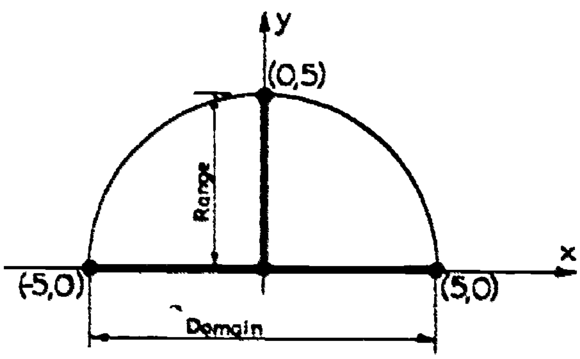


Figure 7. Graph of the function f: $x \to \sqrt{25 - x^2}$.

Another illustration: the function

$$f: x \to \frac{x}{2}, \quad 2 < x \leq 6$$

has domain $A = \{x: 2 < x \leq 6\}$ and range $B = \{f(x): 1 < f(x) \leq 3\}$. In this case we have used open dots at 2 on the x-axis and at 1 on the y-axis to indicate that these numbers are not elements of the domain and range, respectively. See Figure 8.

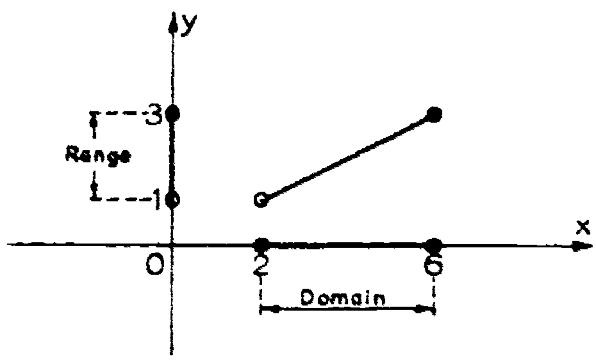


Figure 8. Graph of the function f: $x \to \frac{x}{2}$, $2 < x \leq 6$.

As might be expected, not 'ery possible graph is the graph of a function. In particular, Definition 1 requires that a function map each element of its domain into only one element of its range. In the language of graphs, this says that only one value of y can correspond to any value of x. If, for example, we look at the graph of the equation $x^2 + y^2 = 25$, shown in Figure 9, we can

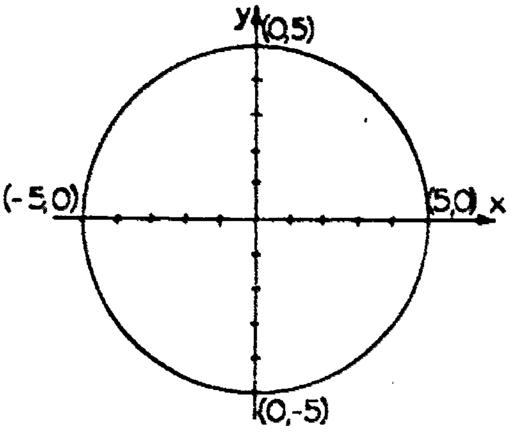


Figure 9. Graph of the set $S = \{(x,y): x^2 + y^2 = 25\}$.

see that there are many instances in which one value of x is associated with two values of y, contrary to the definition of function. To give a specific example, if $x = 3$, we have $y = 4$ or $y = -4$; each of the points $(3,4)$ and $(3,-4)$ is on the graph. Hence this is not the graph of a function. We can, however, break it into two pieces, the graph of $y = \sqrt{25 - x^2}$ and the graph of $y = -\sqrt{25 - x^2}$ (this makes the points $(-5,0)$ and $(5,0)$ do double duty), each of which is the graph of a function. See Figures 10 and 11.

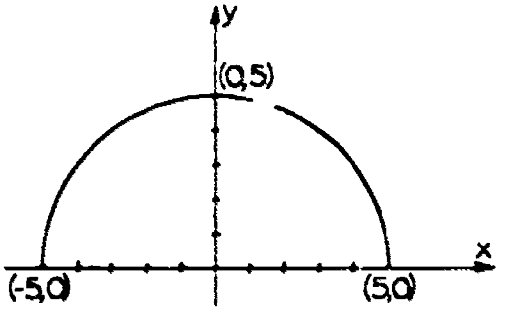


Figure 10.
Graph of $y = \sqrt{25 - x^2}$.

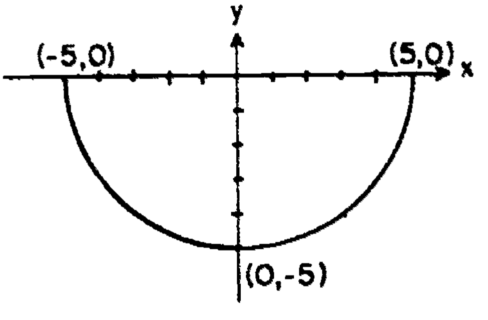
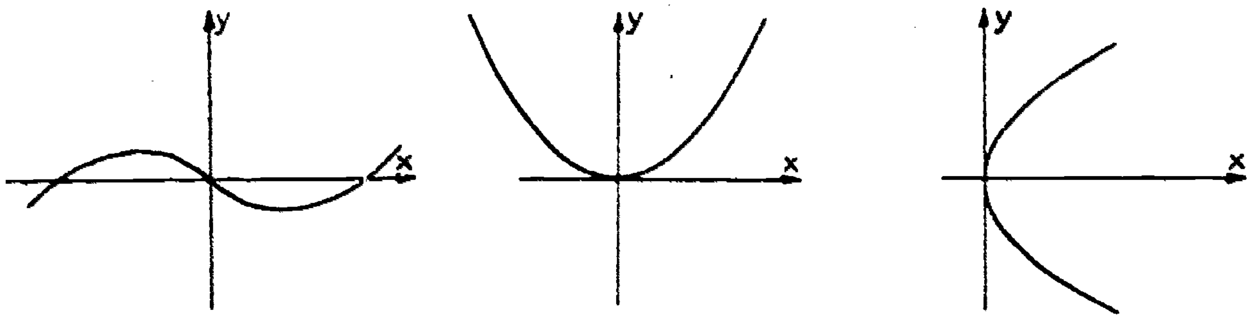


Figure 11.
Graph of $y = -\sqrt{25 - x^2}$.

18

If, in the xy-plane, we imagine all possible lines which are parallel to the y-axis, and if any of these lines cuts the graph in more than one point, then the graph defines a relation that is not a function. Thus, in Figure 12, (a) depicts a function, (b) depicts a function, but (c) does not depict a function.

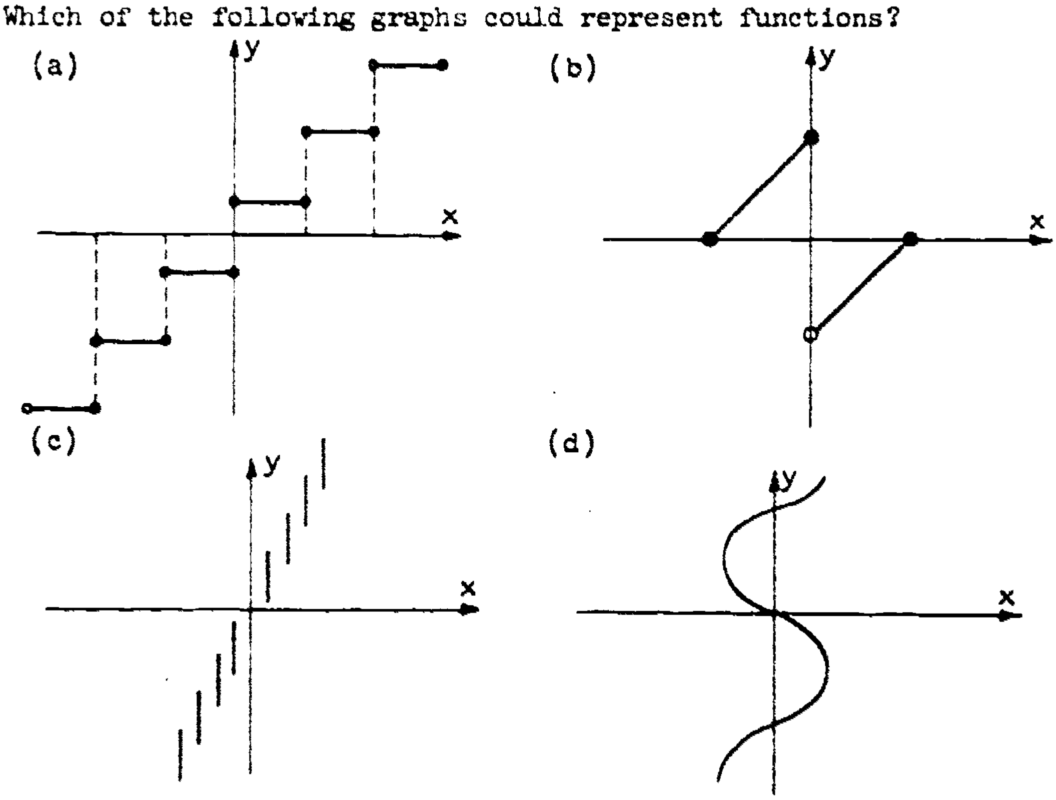


(a) (b) (c)

Figure 12. Function or not?

Exercises 3

1. Which of the following graphs could represent functions?



2. Suppose that in (a) above, $f: x \to f(x)$ is the function whose graph is depicted. Sketch

(a) $g: x \to -f(x)$; (b) $g: x \to f(-x)$.

3. Graph the following functions:

(a) f: $x \to 2x$;

(b) f: $x \to \frac{1}{x}$;

(c) f: $x \to y = 4 - x$ and x and y are positive integers;

(d) f: $x \to -\sqrt{4 - x^2}$.

4. Graph the following functions and indicate the domain and range of each by heavy lines on the x-axis and y-axis, respectively:

(a) f: $x \to y = x$ and $2 < y < 3$;

(b) f: $x \to \sqrt{9 - x^2}$;

(c) f: $x \to \sqrt{x}$ and $x < 4$.

---

4. Constant Functions and Linear Functions.

We have introduced the general idea of function, which is a particular kind of an association of elements of one set with elements of another. We have also interpreted this idea graphically for functions which map real numbers into real numbers. In Sections 2 and 3 our attention was concentrated on general ideas, and examples were introduced only for the purposes of illustration. In the present section we reverse this emphasis and study some particular functions that are important in their own right. We begin with the simplest of these, namely, the constant functions and the linear functions.

Let us think of a man walking north along a long, straight road at the uniform rate of 2 miles per hour. At some particular time, say time $t = 0$, this man passed the milepost located one mile north of Baseline Road. An hour before this, which we shall call time $t = -1$, he passed the milepost located one mile south of Baseline Road. An hour after time $t = 0$, at time $t = 1$, he passed the milepost located three miles north of Baseline Road. In order to form a convenient mathematical picture of the man's progress, let us consider miles north of Baseline Road as positive and miles south as negative. Thus the man passed milespost -1 at time $t = -1$, milepost 1 at time $t = 0$, and milepost 3 at time $t = 1$. Using an ordinary set of coordinate axes let us plot his position, as indicated by the mileposts, versus time in hours. This gives us the graph shown in Figure 13.

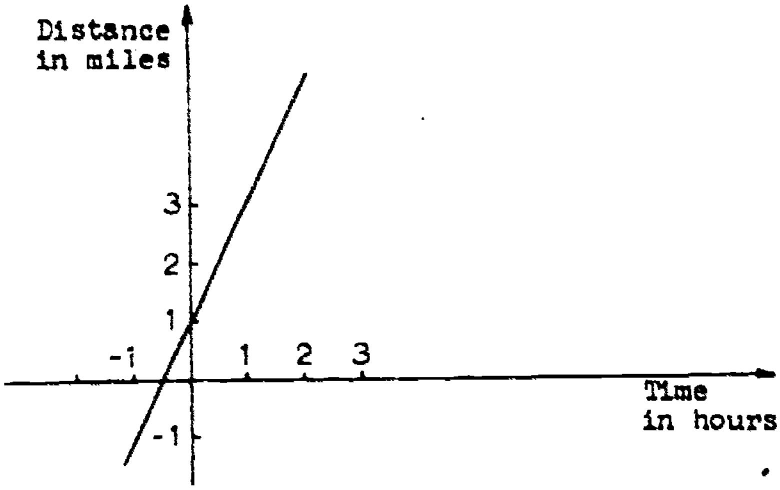


Figure 13. Graph of the function $f: t \to d = 2t + 1$.

In t hours the man travels $2t$ miles. Since he is already at milepost 1 at time $t = 0$, he must be at milepost $2t + 1$ at time t. This pairing of numbers is an example of a linear function.

Now let us plot the man's speed versus time. For all values of t during the time he is walking, his speed is 2 miles per hour. We have graphed this information in Figure 14.

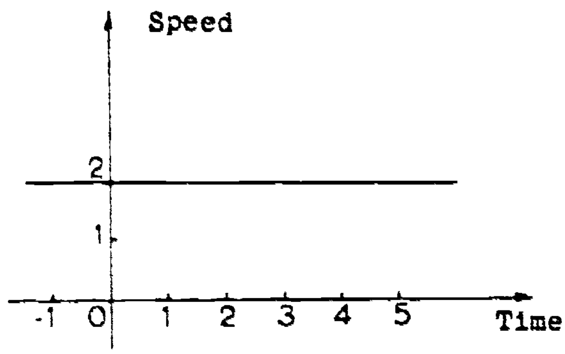


Figure 14. Graph of the function $g: t \to s = 2$.

When $t = -1$ his speed is 2; when $t = 0$ his speed is 2, etc.; with each number t we associate the number 2. This mapping, in which the range contains only the one number 2, is an example of a constant function.

Definition 2. If with each real number x we associate one fixed number c, then the resultant mapping,

$$f: x \to c,$$

is called a constant function.

The discussion of constant functions can be disposed of in a few lines. The function we just mentioned, for example, is the constant function $g: t \to 2$. The graph of any constant function is a line parallel to the horizontal x-axis. Constant functions are very simple, but they occur over and over again in mathematics and science and are really quite important. A well-known example from physics is the magnitude of the attraction of gravity, which is usually taken to be constant over the surface of the earth--though, in this age, we must recognize the fact that the attraction of gravity varies greatly throughout space.

The functions we examine next also occur over and over again in mathematics and science and are considerably more interesting than the constant functions. These are the linear functions. Since you have worked with these functions before, we can begin at once with a formal definition.

Definition 3. A function f defined on the set of all real numbers is called a linear function if there exist real numbers m and b, with $m \neq 0$, such that

$$f(x) = mx + b.$$

Example 1. The function $f: x \rightarrow 2x + 1$ is a linear function. Here $f(0) = 1$, $f(1) = 3$, $f(-1) = -1$. This function was described earlier in this section in terms of t, with $f(t) = 2t + 1$. Its graph can be found in Figure 13.

We note that the graph in Figure 13 appears to be a straight line. As a matter of fact, the graphs of all linear functions are straight lines (that is why we call them "linear" functions); you may be familiar with a proof of this theorem from an earlier study of graphs. In any case, we here assume it.

An important property of any straight line segment is its slope, defined as follows:

Definition 4. The slope of the line segment from the point $P(x_1,y_1)$ to the point $Q(x_2,y_2)$ is the number

$$\frac{y_2 - y_1}{x_2 - x_1},$$

provided $x_1 \neq x_2$. If $x_1 = x_2$, the slope is not defined.

Note that, by Definition 4, the slope of the line segment from the point $Q(x_2,y_2)$ to the point $P(x_1,y_1)$ is

$$\frac{y_1 - y_2}{x_1 - x_2}.$$

But

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1},$$

so that it is immaterial which of the two points P or Q we take first. Accordingly, we can speak of $\frac{y_2 - y_1}{x_2 - x_1}$ as the slope of the segment joining the two points, without specifying which comes first.

What about the geometric meaning of the slope of a segment? Suppose, for the sake of definiteness, we consider the segment joining $P(1,2)$ and $Q(3,8)$. By our definition, the slope of this segment is 3, since $\frac{8-2}{3-1} = 3$ (or $\frac{2-8}{1-3} = 3$). Note that this is the vertical distance from P to Q divided by the horizontal distance from P to Q, or, in more vivid language, the <u>rise</u> divided by the <u>run</u>.

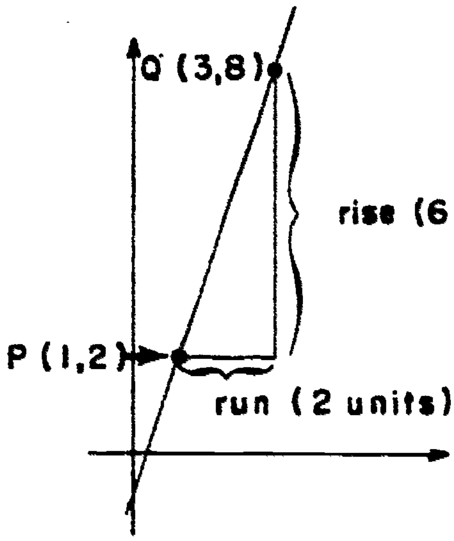


Figure 15.

Let us think of the segment $\overline{PQ}$ as running from left to right, so that the run is positive. If the segment rises, then the "rise" is positive and the slope, or ratio of rise to run, is positive; if, on the other hand, the segment falls, then the "rise" is negative, and the slope is therefore negative. The steeper the segment, the larger is the absolute value of its slope, and conversely; thus we can use the slope as a numerical measure of the "steepness" of a segment.

We have stated that slope is not defined if $x_1 = x_2$; in this case, the segment lies on a line parallel to the y-axis. It is important to distinguish this situation from the case $y_1 = y_2$ (and $x_1 \neq x_2$), in which a slope is defined and in fact has value 0; the segment is then on a line parallel to the x-axis.

If a line is the graph of a linear function $f: x \to mx + b$, then for any x_1 and x_2, $x_1 \neq x_2$, the slope of the segment joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is, by definition,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = m;$$

in other words, the slope m is independent of the choice of x_1 and x_2, and is therefore the same for every segment of the line. Hence we may consider the slope to be a property of the line as a whole, rather than of a particular segment. We shall also simplify our language a little and speak of the slope of the graph of a function as, simply, the slope of the function. We see, moreover, that we can read the slope of a linear function directly from the expression which defines the function: the slope of $f: x \to mx + b$ is simply m, the coefficient of x. Thus, the slope of the linear function $f: x \to 2x + 1$ is 2, the coefficient of x, and, similarly, the slope of $g: x \to -5x$ is -5.

Since the slope of a linear function $f: x \to mx + b$ is the number $m \neq 0$, it follows that the graph of a linear function is not parallel to the x-axis. Conversely, it can be proved that any line not parallel to either axis is the graph of some linear function. We assume that this, also, is known to you from previous work, and the proof is therefore omitted.

If the graphs of the functions $f_1: x \to m_1x + b_1$ and $f_2: x \to m_2x + b_2$ meet, there must be a value of x which satisfies the equation $f_1(x) = f_2(x)$, that is,

$$m_1x + b_1 = m_2x + b_2,$$

or

$$(m_1 - m_2)x = b_2 - b_1.$$

If $m_1 \neq m_2$, then the value $x = \dfrac{b_2 - b_1}{m_1 - m_2}$ satisfies this equation, and the lines do indeed meet. If $m_1 = m_2$ and $b_1 = b_2$, the functions f_1 and f_2 are the same, and there is only one line. If $m_1 = m_2$ and $b_1 \neq b_2$, the equation has no solution, and the lines do not meet. We conclude that lines with the same slope are parallel, and that two lines parallel to each other but not to the y-axis have equal slopes.

Note that lines having zero slope, that is, lines parallel to the x-axis, are graphs of constant functions. On the other hand, lines for which no slope is defined, that is, lines parallel to the y-axis, cannot be graphs of any functions because, with one value of x, the graph associates more than one value--in fact, all real values.

<u>Example 2</u>. Find the linear function g whose graph passes through the point with coordinates $(-2,1)$ and is parallel to the graph of the function $f: x \to 3x - 5$.

<u>Solution</u>. The graph of f is a line with slope 3. Hence the slope of g is the number 3, so that $g(x) = 3x + b$, for some as yet unknown b. Since $g(-2) = 1$, this implies that $1 = 3(-2) + b$, $b = 7$, and thus $g(x) = 3x + 7$ for all $x \in R$.

Exercises 4

1. Find the slope of the function f if, for all real numbers x,

   (a) $f(x) = 3x - 7$; (c) $2f(x) = 3 - x$;

   (b) $f(x) = 6 - 2x$; (d) $3f(x) = 4x - 2$.

2. Find a linear function f whose slope is -2 and such that

   (a) $f(1) = 4$; (c) $f(3) = 1$;

   (b) $f(0) = -7$; (d) $f(8) = -3$.

3. Find the slope of the linear function f if $f(1) = -3$ and

   (a) $f(0) = 4$; (c) $f(5) = 5$;

   (b) $f(2) = 3$; (d) $f(6) = -13$.

4. Find a function whose graph is the line joining the points

   (a) $P(1,1)$, $Q(2,4)$; (c) $P(1,3)$, $Q(1,8)$;

   (b) $P(-7,4)$, $Q(-5,0)$; (d) $P(1,4)$, $Q(-2,4)$.

5. Given $f: x \rightarrow -3x + 4$, find a function whose graph is parallel to the graph of f and passes through the point

   (a) $P(1,4)$; (c) $P(1,5)$;

   (b) $P(-2,3)$; (d) $P(-3,-4)$.

6. If f is a constant function, find $f(3)$ if

   (a) $f(1) = 5$; (b) $f(8) = -3$; (c) $f(0) = 4$.

7. Do the points $P(1,3)$, $Q(3,-1)$, and $S(7,-9)$ all lie on a single line? Prove your assertion.

8. The graph of a linear function f passes through the points $P(100,25)$ and $Q(101,39)$. Find

   (a) $f(100.1)$; (c) $f(101.7)$;

   (b) $f(100.3)$; (d) $f(99.7)$.

9. The graph of a linear function f passes through the points $P(53,25)$ and $Q(54,-19)$. Find

   (a) $f(53.3)$; (c) $f(54.4)$;

   (b) $f(53.8)$; (d) $f(52.6)$.

10. Find a linear function with graph parallel to the line with equation $x - 3y + 4 = 0$ and passing through the point of intersection of the lines with equations $2x + 7y + 1 = 0$ and $x - 2y + 8 = 0$.

11. Given the points $A(1,2)$, $B(5,3)$, $C(7,0)$, and $D(3,-1)$, prove that $ABCD$ is a parallelogram.

12. Find the coordinates of the vertex C of the parallelogram ABCD if AC is a diagonal and the other vertices are the points:
(a) $A(1,-1)$, $B(3,4)$, $D(2,3)$; (b) $A(0,5)$, $B(1,-7)$, $D(4,1)$.

13. If t is a real number, show that the point $P(t + 1, 2t + 1)$ is on the graph of $f: x \to 2x - 1$.

14. If you graph the set of all ordered pairs of the form $(t - 1, 3t + 1)$ for $t \in R$, you will obtain the graph of a linear function f. Find $f(0)$ and $f(8)$.

15. If you graph the set of all ordered pairs of the form $(t - 1, t^2 + 1)$ for $t \in R$, you will obtain the graph of a function f. Find $f(0)$ and $f(8)$.

16. If the slope of a linear function f is negative, prove that $f(x_1) > f(x_2)$ for $x_1 < x_2$.

---

5. The Absolute-Value Function.

A function of importance in many branches of mathematics is the absolute-value function, $f: x \to |x|$ for all $x \in R$. The absolute value of a number describes the size, or magnitude, of the number; thus, for example, $|2| = |-2| = 2$ (read $|2|$ as "the absolute value of 2"). A common definition of $|x|$ is the following:

Definition 5.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

A consequence of this definition is that no number has a negative absolute value ($-x$ is positive when x is negative); in fact, the range of the absolute-value function is the entire set of non-negative real numbers.

A very convenient alternative definition of absolute value is the following:

Definition 6. $|x| = \sqrt{x^2}$.

Since we shall make use of this definition in what follows, it is important that you understand it, and you must therefore be quite sure of the meaning of the square-root symbol, $\sqrt{}$. This never indicates a negative number. Thus, for example, $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3; $\sqrt{x}$ is nev· negative. It is true that every positive number has two real square roots, ·e of them positive and the other negative, but the symbol $\sqrt{}$ has been assigned the job of representing the positive root only, and if we wish to represent the negative root, we must use a negative sign before the radical. Thus, for example, the number 5 has two square roots, $\sqrt{5}$ and $-\sqrt{5}$.

The graph of the absolute-value function is shown in Figure 16.

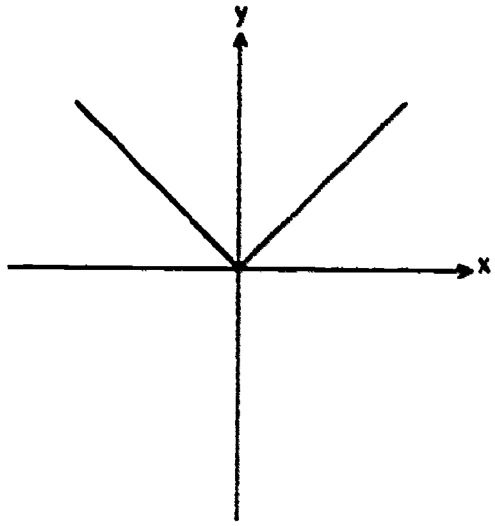


Figure 16. Graph of the function $f: x \to |x|$.

You should be able to see, from the first definition of this function given above, that this graph consists of the origin, the part of the line $y = x$ that lies in Quadrant I, and the part of the line $y = -x$ that lies in Quadrant II.

There are two important theorems about absolute values.

Theorem 1. For any two real numbers a and b, $|ab| = |a| \cdot |b|$.

Proof. $|a| \cdot |b| = \sqrt{a^2} \cdot \sqrt{b^2} = \sqrt{a^2b^2} = \sqrt{(ab)^2} = |ab|$.

Theorem 2. For any two real numbers a and b, $|a + b| \leq |a| + |b|$.

Proof. By Definition 6, Theorem 2 is equivalent to

$$\sqrt{(a + b)^2} \leq \sqrt{a^2} + \sqrt{b^2}. \qquad (1)$$

Now, if x and y are two non-negative numbers (i.e., positive or zero) such that $x \leq y$, then $x^2 \leq y^2$. For, if $x \leq y$, we know that there is a non-negative number, h, such that $x + h = y$. Then $x^2 + 2hx + h^2 = y^2$ where $2hx + h^2$ is a non-negative number. Hence $x^2 \leq y^2$. On the other hand, if $x^2 \leq y^2$, it follows that $x \leq y$. For, if $x^2 \leq y^2$, we have $0 \leq y^2 - x^2 = (y + x)(y - x)$ and, since x and y are non-negative, so is $y + x$. In fact, $y + x$ is positive unless $x = y = 0$ and, if $y + x$ is positive, $y - x$ cannot be negative since the product of a positive number and a negative number is a negative number. Thus either $x = y = 0$ or $y - x \geq 0$ so that $x \leq y$. In either case we have $x \leq y$.

From these remarks and the fact that $\sqrt{a^2 + b^2}$, $\sqrt{a^2}$, and $\sqrt{b^2}$ are all non-negative numbers, it follows that (1) holds if and only if

$$(a + b)^2 = a^2 + 2ab + b^2 \leq a^2 + 2\sqrt{a^2}\sqrt{b^2} + b^2. \qquad (2)$$

But (2) is certainly equivalent to

$$2ab \leq 2\sqrt{a^2}\sqrt{b^2}$$

so that we conclude that (1) holds if and only if

$$ab \leq \sqrt{a^2}\sqrt{b^2}. \qquad (3)$$

Now, inequality (3) is easy to prove. If one of a and b is negative and the other positive, then $ab < 0$ and $\sqrt{a^2}\sqrt{b^2} > 0$ so that (3) holds with the $<$ sign. Otherwise

$$ab = \sqrt{a^2}\sqrt{b^2}.$$

Hence, in any case (3) holds and therefore (1) holds.

Thus, for example, $|(-2)(3)| = |-6| = 6 = 2 \cdot 3 = |-2| \cdot |3|$, $|(-2) + (3)| = 1 < 5 = 2 + 3 = |-2| + |3|$, and $|(-2) + (-3)| = 5 = 2 + 3 = |-2| + |-3|$.

Exercises 5

1. (a) For what $x \in R$ is it true that $\sqrt{x^2} = x$?
   (b) For what $x \in R$ is it true that $\sqrt{x^2} = -x$?

2. (a) For what $x \in R$ is it true that $|x - 1| = x - 1$?
   (b) For what $x \in R$ is it true that $|x - 1| = -x + 1$
   (c) Sketch a graph of $f\colon x \rightarrow |x - 1|$.
   (d) Sketch a graph of $f\colon x \rightarrow |x| - 1$.

3. Solve:
   (a) $|x| = 14$; (b) $|x + 2| = 7$; (c) $|x - 3| = -1$.

4. For what values of x is it true that
   (a) $|x - 2| < 1$;
   (b) $|x - 5| > 2$;
   (c) $|x + 4| < 0.2$;
   (d) $|2x - 3| < 0.04$;
   (e) $|4x + 5| < 0.12$?

5. Show that $x^2 \geq x \cdot |x|$ for all $x \in R$.

6. Show that $|a - b| \leq |a| + |b|$.

7. Show that $\frac{1}{2}(a + b + |a - b|)$ is equal to the greater of a and b. Can you write a similar expression for the lesser of a and b?

8. Sketch: $y = |x| + |x - 2|$. (Hint: You must consider, separately, the three possibilities $x < 0$, $0 \leq x < 2$, and $x \geq 2$.)

9. If $0 < x < 1$, we can multiply both sides of the inequality $x < 1$ by the positive number x to obtain $x^2 < x$, and we can similarly show that $x^3 < x^2$, $x^4 < x^3$, and so on. Use this result to show that if $|x| < 1$, then $|x^2 + 2x| < 3|x|$.

10. Show that, if $0 < x < k$, then $x^2 < kx$. Hence, show that, if $|x| < 0.1$, then $|x^2 - 3x| < 3.1|x|$.

11. For what values of x is it true that $|x^2 + 2x| < 2.001|x|$?

---

6. Composition of Functions.

Our consideration of functions, to this point, has been concerned with individual functions, with their domains and ranges, and with their graphs. We now consider certain things that can be done with two or more functions somewhat as, when we start school, we first learn about numbers and then learn how to combine them in various ways. There is, as a matter of fact, a whole algebra of functions, just as there is an algebra of numbers. Functions can be added, subtracted, multiplied, and divided. The sum of two functions f and g, for example, is defined to be the function

$$f + g: \quad x \rightarrow f(x) + g(x)$$

which has for domain those elements that are both in the domain of f and the domain of g; there are similar definitions, which you can probably supply yourself, for the difference, product, and quotient of two functions. Because, for example, the number $(f + g)(x)$ can be found by adding the numbers $f(x)$ and $g(x)$, it follows that this part of the algebra of functions is so much like the familiar algebra of numbers that it would not pay us to examine it carefully. There is, however, one important operation in this algebra of functions that has no counterpart in the algebra of numbers: the operation of composition.

The basic idea of composition of two functions is that of a kind of "chain reaction" in which the functions occur one after the other. Thus, an automobile driver knows that the amount he depresses the accelerator pedal controls the amount of gasoline fed to the cylinders and this in turn affects the speed of the car. Again, the momentum of a rocket sled when it is near the end of its runway depends on the velocity of the sled, and this in turn depends on the thrust of the propelling rockets.

Let us look at a specific illustration. Suppose that f is the function $x \rightarrow 3x - 1$ (this might be a time-velocity function) and suppose that g is the function $x \rightarrow 2x^2$ (this might be a velocity-energy function). Let us follow what happens when we "apply" these two functions in succession--first f, then g--to a particular number, say the number 4. In brief, let us first calculate $f(4)$ and then calculate $g\big(f(4)\big)$. (Read this "g of f of 4".)

First calculate $f(4)$. Since f is the function $x \rightarrow 3x - 1$,

$f(4) = 3 \cdot 4 - 1 = 11$. Then calculate $g\big(f(4)\big)$, or $g(11)$. Since g is the function $x \to 2x^2$, $g(11) = 2 \cdot 11^2 = 242$. Thus $g\big(f(4)\big) = g(11) = 242$. In general, $g\big(f(x)\big)$ is the result we obtain when we first "apply" f to an element x and then "apply" g to the result. The function $x \to g\big(f(x)\big)$ is then called a <u>composite</u> of f and g, and denoted gf.

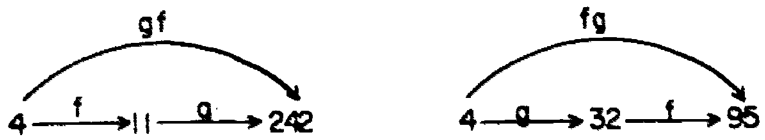
We say <u>a</u> composite rather than <u>the</u> composite because the order in which these functions occur is important. To see that this is the case, start with the number 4 again, but this time find $g(4)$ first, then $f\big(g(4)\big)$. The results are as follows:

$$g(4) = 2 \cdot 4^2 = 32 \quad \text{and} \quad f\big(g(4)\big) = f(32) = 3 \cdot 32 - 1 = 95.$$

Clearly $g\big(f(4)\big)$, which is 242, is not the same as $f\big(g(4)\big)$, which is 95.

<u>Warning</u>. When we write gf we mean that f is to be applied <u>before</u> g and then g is applied to $f(x)$. Since f is written <u>after</u> g is written, this can easily lead to confusion. You can avoid the confusion by thinking of the equation $(gf)(x) = g\big(f(x)\big)$.

It may be helpful to diagram the above process as follows: If gf is the function $x \to g\big(f(x)\big)$ and fg is the function $x \to f\big(g(x)\big)$, we have



Note particularly that fg is not the product of f and g mentioned earlier in this section. When we want to talk about this product, $f \cdot g$, we shall always use the dot as shown. Incidentally, for the above example, we have $(f \cdot g)(4) = f(4) \cdot g(4) = 11 \cdot 32 = 352 = 32 \cdot 11 = g(4) \cdot f(4) = (g \cdot f)(4)$.

To generalize this illustration, let us use x instead of 4 and find algebraic expressions for $(gf)(x)$ and $(fg)(x)$. We do this as follows:

$$(gf)(x) = g\big(f(x)\big) = g(3x - 1) = 2 \cdot (3x - 1)^2$$

and

$$(fg)(x) = f\big(g(x)\big) = f(2x^2) = 3(2x^2) - 1 = 6x^2 - 1 .$$

Again, note that $(gf)(x)$ and $(fg)(x)$ are not the same so the function gf is not the same as the function fg. In symbols, $gf \neq fg$. If, now, we substitute 4 for x we obtain

$$(gf)(4) = 2(3 \cdot 4 - 1)^2 = 242$$

and

$$(fg)(4) = 6 \cdot 4^2 - 1 = 95.$$

These results agree with the ones we obtained above.

We are now ready to define the general process that we have been illustrating.

Definition 7. Given two functions, f and g, the function $x \to g(f(x))$ is called a composite of f and g and denoted gf. The domain of gf is the set of all elements x in the domain of f for which $f(x)$ is in the domain of g. The operation of forming a composite of two functions is called composition.

Example 1. Given that $f\colon\ x \to 3x - 2$ and $g\colon\ x \to x^5$ for all $x \in R$, find

(a) $(gf)(x)$ (c) $f\big(g(x) + 3\big)$

(b) $(ff)(x)$ (d) $f\big(g(x) - f(x)\big)$

Solution.

(a) $(gf)(x) = g\big(f(x)\big) = g(3x - 2) = (3x - 2)^5$

(b) $(ff)(x) = f\big(f(x)\big) = f(3x - 2) = 3(3x - 2) - 2 = 9x - 8$

(c) $f\big(g(x) + 3\big) = f(x^5 + 3) = 3(x^5 + 3) - 2 = 3x^5 + 7$

(d) $f\big(g(x) - f(x)\big) = f(x^5 - 3x + 2) = 3(x^5 - 3x + 2) - 2 = 3x^5 - 9x + 4$

If we think of a function as a machine with an input and an output, as suggested in Section 2, we see that two such machines can be arranged in tandem, so that the output of the first machine feeds into the input of the second. This results in a "composite" process that is analogous to the operation of composition. It is illustrated in Figure 17. In this figure the machine for f and the machine for g have been housed in one cabinet. This compound machine is the machine for gf.

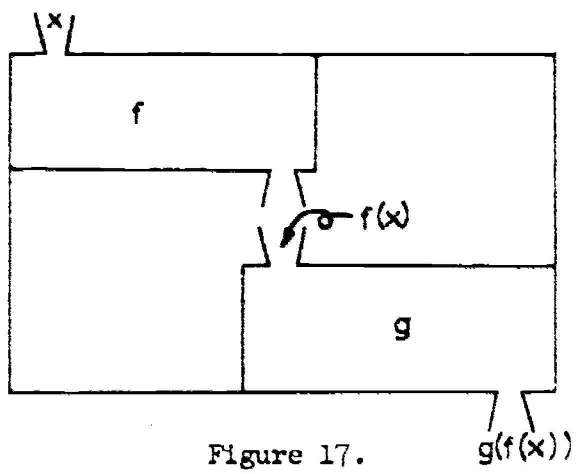


Figure 17.
Schematic representation of the composition of functions.

Note that the machine for gf will jam if either of two things happen:

(a) It will jam if a number not in the domain of f is fed into the machine.

(b) It will jam if the output $f(x)$ of f is not in the domain of g. Thus, once again we see that the domain of gf is the set of all elements x in the domain of f for which $f(x)$ is in the domain of g.

We have noted that the operation of composition is not commutative; that is, it is not always true that $fg = gf$. On the other hand, it is true that this operation is associative: for any three functions f, g, and h, it is always true that $(fg)h = f(gh)$. We shall not prove this theorem; we shall, however, illustrate its operation by an example.

Example 2. Given f: $x \to x^2 + x + 1$, g: $x \to x + 2$, and h: $x \to -2x - 3$, find

(a) fg;

(b) gh;

(c) (fg)h;

(d) f(gh).

Solution.

(a) $(fg)(x) = (x + 2)^2 + (x + 2) + 1 = x^2 + 5x + 7$, so
fg: $x \to x^2 + 5x + 7$

(b) $(gh)(x) = (-2x - 3) + 2 = -2x - 1$, so gh: $x \to -2x - 1$

(c) (fg)h: $x \to (-2x - 3)^2 + 5(-2x - 3) + 7$

(d) f(gh): $x \to (-2x - 1)^2 + (-2x - 1) + 1$

It is not altogether obvious from these expressions that (fg)h and f(gh) are the same function. But if you will simplify the expressions you will see that they are indeed the same.

Exercises 6

1. Given that f: $x \to x^2 - 1$ and g: $x \to x + 2$ for all $x \in R$, find
   (a) (fg)(-2);
   (b) (gf)(0);
   (c) (gg)(1);
   (d) (ffg)(1);
   (e) (fg)(x);
   (f) (gf)(x);
   (g) $\dfrac{(fg)(x) - (fg)(1)}{x - 1}$.

2. Let it be given that f: $x \to ax + b$ and g: $x \to cx + d$ for all $x \in R$.
   (a) Find (fg)(x).
   (b) Find (gf)(x).
   (c) Compare the slopes of fg and gf with the slopes of f and g.
   (d) Formulate a theorem concerning the slope of a composite of two linear functions.

3. Suppose that f: $x \to \frac{1}{x}$ for all real numbers x different from zero.
   (a) Find (ff)(1), (ff)(-3), and (ff)(8).
   (b) Describe ff completely.

4. Let it be given that j: $x \to x$ and f: $x \to x + 2$ for all $x \in R$.
   (a) Find fj and jf. (First find (fj)(x) for all $x \in R$.)
   (b) Find a function g such that fg = j. (That is, find g such that (fg)(x) = j(x) for all $x \in R$.)
   (c) Find a function h such that hf = j. Compare your result with that of (b).

5. (a) If $f: x \to x^2$ and $g: x \to x^3$, find expressions for $(fg)(x)$ and $(gf)(x)$.

   (b) If $f: x \to x^m$ and $g: x \to x^n$, find expressions for $(fg)(x)$ and $(gf)(x)$.

6. (a) If $f: x \to x^2$ and $g: x \to x^3$, find an expression for $(f \cdot g)(x)$, where $f \cdot g$ is the product of f and g; that is, $(f \cdot g)(x) = f(x)g(x)$. Compare with Exercise 5(a).

   (b) If $f: x \to x^m$ and $g: x \to x^n$ for all $x \in R$ (where m and n are positive integers), find an expression for $(f \cdot g)(x)$. Compare with Exercise 5(b).

7. Suppose that $f: x \to x + 2$, $g: x \to x - 3$, and $h: x \to x^2$ for all $x \in R$. Find expressions for

   (a) $(f \cdot g)(x)$;

   (b) $\big((f \cdot g)h\big)(x)$;

   (c) $(fh)(x)$;

   (d) $(gh)(x)$;

   (e) $\big((fh) \cdot (gh)\big)(x)$.

8. In Exercise 7, compare your results for (b) and (e). They should be the same. Do you think this result is true for any three functions f, g, and h that map real numbers into real numbers?

9. Would you say that $f(g \cdot h) = (fg) \cdot (fh)$ for any three functions f, g, and h that map real numbers into real numbers?

10. State which of the following will hold for all functions f, g, and h that map real numbers into real numbers:

$$(f + g)h = fh + gh;$$
$$f(g + h) = fg + fh.$$

11. Prove that the set of all linear functions is associative under composition; that is, for any three linear functions f, g, and h,

$$f(gh) = (fg)h.$$

---

7. Inversion.

Quite frequently in science and in everyday life we encounter quantities that bear a kind of reciprocal relationship to each other. With each value of the temperature of the air in an automobile tire, for example, there is associated one and only one value of the pressure of the air against the walls of the tire. Conversely, with each value of the pressure there is associated one and only one value of the temperature. Two more examples, numerical ones, will be found below.

Suppose that f is the function $x \to x + 3$ and g is the function $x \to x - 3$. Then the effect of f is to increase each number by 3, and the effect of g is to decrease each number by 3. Hence f and g are reciprocally related in the sense that each undoes the effect of the other. If we add 3 to a number and then subtract 3 from the result we get back to the original number. In symbols,

$$(gf)(x) = g\big(f(x)\big) = g(x + 3) = (x + 3) - 3 = x.$$

Similarly,

$$(fg)(x) = f\big(g(x)\big) = f(x - 3) = (x - 3) + 3 = x.$$

As a slightly more complicated example we may take

$$f:\ x \to 2x - 3 \quad \text{and} \quad g:\ x \to \frac{x + 3}{2}.$$

Here f says, "Take a number, double it, and then subtract 3." To reverse this, we must add three and then divide by 2. This is the effect of the function g. In symbols,

$$(gf)(x) = g\big(f(x)\big) = g(2x - 3) = \frac{(2x - 3) + 3}{2} = x.$$

Similarly,

$$(fg)(x) = f\big(g(x)\big) = f\left(\frac{x + 3}{2}\right) = 2\,\frac{x + 3}{2} - 3 = x.$$

In terms of our representation of a function as a machine, the g machine in each of these examples is equivalent to the f machine running backwards; each machine then undoes what the other does, and if we hook up the two machines in tandem, every element that gets through both will come out just the same as it originally went in.

We now generalize these two examples in the following definition of inverse functions.

<u>Definition 8</u>. If f and g are functions so related that $(fg)(x) = x$ for every element x in the domain of g and $(gf)(y) = y$ for every element y in the domain of f, then f and g are said to be <u>inverses</u> of each other. In this case both f and g are said to <u>have an inverse</u>, and each is said to be an <u>inverse</u> of the other.

As a further example of the concept of inverse functions let us examine the functions $f:\ x \to x^3$ and $g:\ x \to \sqrt[3]{x}$. In this case

$$(fg)(x) = f\big(g(x)\big) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

and

$$(gf)(x) = g\big(f(x)\big) = g(x^3) = \sqrt[3]{x^3} = x$$

for all $x \in R$.

If a function f takes x into y, that is, if $y = f(x)$, then an inverse g of f must take y right back into x, that is, $x = g(y)$. If we make a picture of a function as a mapping, with an arrow extending from each element of the domain to its image, as in Figure 18a, then to draw a picture of the inverse function we need merely reverse the arrows, as in Figure 18b.

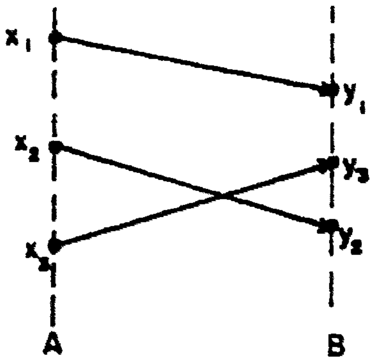


Figure 18a. A function.

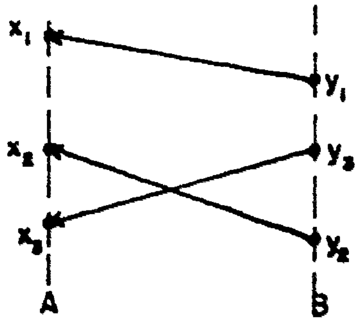


Figure 18b. Its inverse.

We can take any mapping, reverse the arrows in this way, and get a mapping. The important question for us, at this point, is this: If the original mapping is a function, will the reverse mapping necessarily be a function also? In other words, given a function, does there exist another function that precisely reverses the effect of the given function? We shall see that this is not always the case.

The definition of a function (Definition 1) requires that to each element of the domain there corresponds exactly one element of the range; it is perfectly all right for several elements of the domain to be mapped onto the same element of the range (the constant function, for example, maps every element of its domain into one element), but if even one element of the domain is mapped into more than one element of the range, then the mapping just isn't a function. In terms of a picture of a function as a mapping (such as Figures 1 and 3), this means that no two arrows may start from the same point, though any number of them may end at the same point. But if two or more arrows go to one point, as in Figure 19a, and if we then reverse the arrows, as in Figure 19b, we will have two or more arrows starting from that point (as in Figure 2), and the resulting mapping is not a function. Since the word "inverse" is used to describe only a mapping which is a function, we can conclude that not every function has an inverse. A specific example is furnished by the constant function $f\colon\ x \rightarrow 3$, since $f(0) = 3$ and $f(1) = 3$, and inverse of f would have to map 3 onto both 0 and 1. By definition, no function can do this.

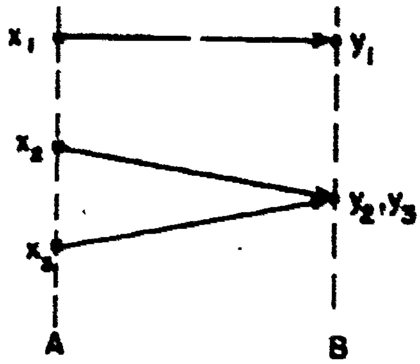


Figure 19a.

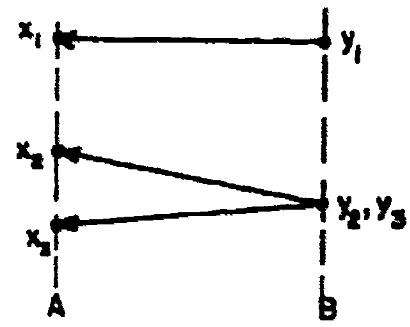


Figure 19b.

The preceding argument shows us just what kinds of functions do have inverses. By comparing the situation in Figures 18a and 18b with the situation in Figures 19a and 19b, we can see that a function has an inverse if and only if no two arrows go to the same point. In more precise language, a function f has an inverse if and only if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. A function of this sort is often called a "one-to-one" function. A formal proof of this theorem will be found in the next section.

Exercises 7

1. Find an inverse of each of the following functions:
   (a) $x \to x - 7$; (b) $x \to 5x + 9$; (c) $x \to \frac{1}{x}$.

2. Solve each of the following equations for x in terms of y and compare your answers with those of Exercise 1:
   (a) $y = x - 7$; (b) $y = 5x + 9$; (c) $y = \frac{1}{x}$.

3. Justify the following in terms of composite functions and inverse functions: Ask someone to choose a number, but not to tell you what it is. "Ask the person who has chosen the number to perform in succession the following operations. (i) To multiply the number by 5. (ii) To add 6 to the product. (iii) To multiply the sum by 4. (iv) To add 9 to the product. (v) To multiply the sum by 5. Ask to be told the result of the last operation. If from this product 165 is subtracted, and then the difference is divided by 100, the quotient will be the number thought of originally." (W. W. Rouse Ball)

---

8. One-to-One Functions.

Definition 8 leaves unanswered one important question: Can a function have more than one inverse? That is, if f and g are inverses of each other, does there exist a function $h \neq g$ such that f and h are also inverses of each other? As you might suspect, the answer is no, but we shall not prove it here. Consider, however, a picture of a function as a mapping, with arrows going (as in Figure 18a) from points representing elements of the domain to points representing elements of the range. To represent the inverse function, we merely reverse the direction of each arrow, as in Figure 18b. It seems intuitively clear that there is only one way to do this.

The fact that a function can have at most one inverse justifies our use of a distinctive notation for functions which are inverses of each other. If f and g are such functions, then we can say that g is <u>the</u> inverse of f and write $g = f^{-1}$. We read f^{-1} as "f inverse". Similarly we can write $f = g^{-1}$. Thus, $(f^{-1})^{-1} = f$.

<u>Warning</u>. Although the notation f^{-1} is strongly suggestive of "1 divided by f", it has nothing whatever to do with division. All it means is that

$$(ff^{-1})(x) = x \quad \text{and} \quad (f^{-1}f)(y) = y.$$

We now prove the basic theorems which relate to the existence of inverses.

<u>Theorem 3</u>. If a function f has an inverse, then $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 are two distinct elements of the domain of f.

<u>Proof</u>. We shall prove this theorem by assuming the contrary and then deriving a contradiction. Hence we assume that $f(x_1) = f(x_2)$. From this we see that $f^{-1}\big(f(x_1)\big) = f^{-1}\big(f(x_2)\big)$. Now, $f^{-1}f(x_1) = x_1$ and $f^{-1}f(x_2) = x_2$, so it follows that $x_1 = x_2$. But the elements x_1 and x_2 are supposed to be distinct (i.e., $x_1 \neq x_2$). This contradiction proves the theorem.

A vivid expression is used to describe functions f for which $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. This is the expression "<u>one-to-one</u>". If a function has an inverse, then by Theorem 3 it is one-to-one. Note that in this case the equation $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

We point out that the idea of a one-to-one function is fundamental to the process of counting a collection of objects. When we count a set of things we associate the number 1 with one of the things, the number 2 with another, and so on, until all the objects have been paired off with whole numbers. We do not give the same number to two different objects in the collection. In short, this "counting" function is one-to-one. As another example, suppose that there are 300 seats in a theater, and suppose that each seat is occupied by one and only one patron. Then, <u>without counting the people</u>, we can conclude that there must be 300 people sitting in these seats. These two examples deal with finite sets. On the other hand, the idea of a one-to-one function is fruitful even when the sets involved are not finite. Indeed, most of the applications deal with sets of this kind.

Now that we know that every function which has an inverse is one-to-one, it is natural to ask if the converse is true. Does every one-to-one function have an inverse? You might guess that the answer is yes. This is the content of Theorem 4.

Theorem 4. If f is a function which is one-to-one, then f has an inverse.

Proof. Using the hypothesis that f is one-to-one, we shall construct a function which will turn out to be f^{-1}. Given an element y of the range of f, then, since f is one-to-one, there exists one and only one element x in the domain of f such that $y = f(x)$. Now, associate the element x with the element y. This association defines a function $g: y \to x$. The domain of g is the range of f and the range of g is the domain of f. Finally, since

$$(fg)(y) = f(x) = y$$

and

$$(gf)(x) = g(y) = x,$$

we see that f and g are inverses of each other. Therefore f has an inverse and $f^{-1} = g$.

Definition 9. A function f is said to be strictly increasing if its graph is everywhere rising toward the right; if, that is, for any two elements x_1 and x_2 of the domain of f, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

An important corollary of Theorem 4 concerns strictly increasing functions.

Corollary. A function f which is strictly increasing has an inverse.

Proof. If x_1 and x_2 are any two elements of the domain of f, then either $x_1 < x_2$, in which case $f(x_1) < f(x_2)$ by hypothesis, or $x_2 < x_1$, in which case $f(x_2) < f(x_1)$. In either case, $f(x_1) \neq f(x_2)$. Hence f is one-to-one and therefore has an inverse by Theorem 4.

A similar result holds for strictly decreasing functions; see Exercise 5.

Theorems 3 and 4 provide an answer to our first question, which was: Under what circumstances does a function have an inverse? We summarize this answer in Theorem 5.

Theorem 5. A function has an inverse if and only if it is one-to-one.

As we might reasonably expect, there exists a rather simple relationship between the graph of a function f and the graph of its inverse f^{-1}. If, for example, r and s are real numbers such that $r = f(s)$, then $P(s,r)$ is, by definition, a point of the graph of f. But if $r = f(s)$, then $s = f^{-1}(r)$, and it follows, again by definition, that $Q(r,s)$ is a point of the graph of f^{-1}. Since this argument is quite general, we can conclude that, for each point $P(s,r)$ of the graph of f, there is a point $Q(r,s)$ of the graph of f^{-1}, and conversely; either graph can be changed into the other by merely interchanging the first and second coordinates of each point. To picture the relative positions of P and Q, we should plot a few points and contemplate the results. (See Figure 20, in which corresponding points of each pair $P(s,r)$, $Q(r,s)$ have been joined together.)

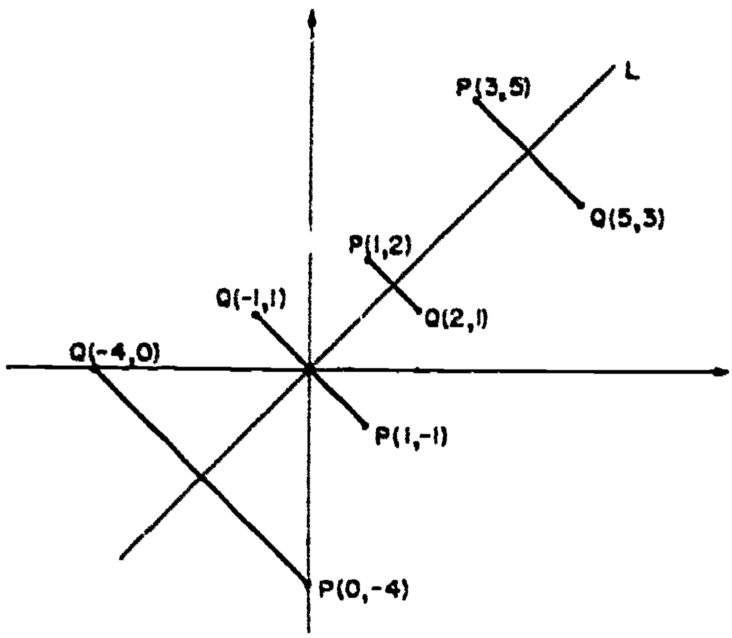


Figure 20.

The presence of the line $L = \{(x,y): y = x\}$ illustrates a striking fact: With respect to the line L, corresponding points are mirror images of each other! Thus we see that the graph of the inverse of a function f is the image of the graph of f in a mirror placed on its edge, perpendicular to the page, along the line L. This fact suggests the following (messy) way to obtain the graph of f^{-1} from that of f. Merely trace the graph of f in ink that dries very slowly, and then fold the paper carefully along the line L. The wet ink will then trace the graph of f^{-1} automatically. (See Figure 21.)

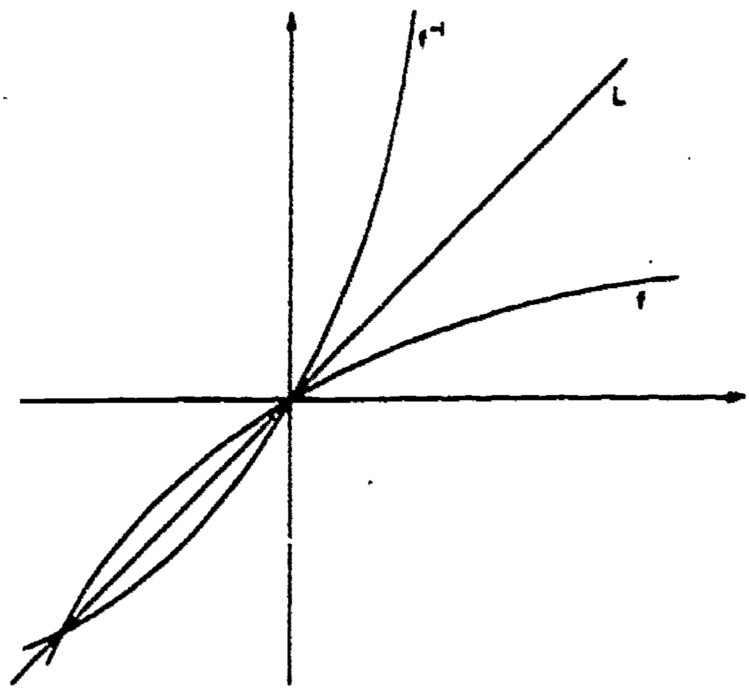


Figure 21.

Exercises 8

1. Find the inverse of each of the following functions:
(a) $x \to 4x - 5$; (b) $x \to \frac{3}{x} + 8$; (c) $x \to x^3 - 2$.

2. Solve each of the following equations for x in terms of y and compare your answers with those of Exercise 1:
(a) $y = 4x - 5$; (b) $y = \frac{3}{x} + 8$; (c) $y = x^3 - 2$.

3. Justify the following in terms of composite functions, inverse functions, and functions which associate integers with ordered pairs of digits. "A common conjuring trick is to ask a boy among the audience to throw two dice, or to select at random from a box a domino on each half of which is a number. The boy is then told to recollect the two numbers thus obtained, to choose either of them, to multiply it by 5, to add 7 to the result, to double this result, and lastly to add to this the other number. From the number thus obtained, the conjurer subtracts 14, and obtains a number of two digits which are the two numbers chosen originally." (W. W. Rouse Ball)

4. We know that each line parallel to the y-axis meets the graph of a function in at most one point. For what kind of function does each line parallel to the x-axis meet the graph in at most one point?

5. A function f is said to be strictly decreasing if, for any two elements x_1 and x_2 of its domain, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. Prove that every strictly decreasing function has an inverse.

6. (a) Sketch a graph of $f: x \to x^2$, $x \in R$. Show that f does not have an inverse.

   (b) Sketch graphs of $f_1: x \to x^2$, $x \geq 0$ and $f_2: x \to x^2$, $x < 0$, and determine the inverses of f_1 and f_2.

   (c) What relationship exists among the domains of f, f_1, and f_2? (f_1 is called the restriction of f to the domain $\{x: x \geq 0\}$, and f_2 is similarly the restriction of f to the domain $\{x: x < 0\}$.)

7. (a) Sketch a graph of $f: x \to \sqrt{4 - x^2}$ and show that f does not have an inverse.

   (b) Divide the domain of f into two parts such that the restriction of f to either part has an inverse.

8. Do Exercise 7 for $f: x \to x^2 - 4x$.

9. Divide the domain of $f: x \to x^3 - 3x$ into three parts such that the restriction of f to each has an inverse.

---

9. Functions as Sets of Ordered Pairs.

Our first example of a function was $f: x \to 2x$, $x = 1, 2, 3, 4$. We have $f(1) = 2$, $f(2) = 4$, $f(3) = 6$, and $f(4) = 8$. It is often useful to indicate this correspondence between the elements of the domain of a function (here, $\{1, 2, 3, 4\}$) and the elements of the range (here, $\{2, 4, 6, 8\}$) by writing down the pairs $\big(x, f(x)\big)$. Thus in our example we have the set of pairs

$$\{(1,2), (2,4), (3,6), (4,8)\}.$$

Clearly the order is important in these pairs; $(2,1)$ is not a proper pair for our function f although $(1,2)$ is. We call pairs of numbers (a,b) in which the order of the elements is to be considered, ordered pairs, and contrast them with sets where, for example, $\{a,b\} = \{b,a\}$ --order is not significant.

If our domain is not a finite set we cannot, of course, list all of the ordered pairs associated with the function but we can use our set-builder notation to indicate symbolically all such pairs. Thus if we have $f: x \to 2x$ where the domain of f is the set of all real numbers, we may write for the associated set of ordered pairs

$$\{(x,2x): x \in R\}.$$

Similarly, if we have $f: x \to x^2$ where the domain of f is the set of all integers, the associated set of ordered pairs is

$$\{(x,x^2): \ x \text{ an integer}\}.$$

We see that to every function is associated a set of ordered pairs. Is it true that, conversely, a function may be associated to every set of ordered pairs by defining f as f: first member of ordered pair, second member of ordered pair? The example

$$\{(1,2), (1,3)\}$$

shows that the answer is "no" since we would have $1 \to 2$ and also $1 \to 3$, contrary to Definition 1 where we required that exactly one element of the range of a function be associated with an element from the domain.

If, however, we consider only sets of ordered pairs in which any two pairs that have the same first element also have the same second element, it is clear that we can so define a function corresponding to this set of ordered pairs. Thus

$$\{(1,5), (2,3), (1,5)\}$$

describes the function f with domain $\{1,2\}$ and range $\{5,3\}$ where $1 \to 5$ and $2 \to 3$. (There is, of course, no need to list $(1,5)$ twice. We have $\{(1,5), (2,3), (1,5)\} = \{(1,5), (2,3)\}$.)

In fact, mathematicians sometimes define a function as a set of ordered pairs in which whenever two pairs have the same first element they also have the same second element. Thus, for example, instead of writing $f: x \to 2x$, $x \in R$ they write

$$f = \{(x,2x): \ x \in R\}.$$

We have indicated that it is easy to pass from looking at a function as a correspondence or mapping to considering it as a certain kind of set of ordered pairs and conversely. Which approach is used is simply a matter of convenience; we use whatever approach seems most useful for our purposes.

From our discussion of inverses of functions in terms of one-to-one functions it is easy to see when a function, as a set of ordered pairs, has an inverse; we simply require that whenever two ordered pairs of our function have the same second element, they also have the same first element. Thus the function

$$f = \{(1,3), (2,3)\}$$

has no inverse since $(1,3)$ and $(2,3)$ have the same second element but different first elements. By Theorem 5, the same conclusion would be reached if we regarded f as the mapping $1 \to 3$ and $2 \to 3$ since we would then have

$f(1) = 3$ and $f(2) = 3$ but $1 \neq 2$; i.e., the mapping is not one-to-one.

If a function does have an inverse it is easy to obtain it in the ordered-pair approach; we simply reverse the order in the pairs. Thus if

$$f = \{(1,5), (2,3)\}$$

we have

$$f^{-1} = \{(5,1), (3,2)\};$$

if

$$f = \{(x, \sqrt[3]{x}): \quad x \in R\}$$

we have

$$f^{-1} = \{(\sqrt[3]{x}, x): \quad x \in R\}$$

or, letting $y = \sqrt[3]{x}$ so that $y^3 = x$,

$$f^{-1} = \{(y, y^3): \quad y \in R\}.$$

Exercises 9

1. Write the following functions, defined as mappings, as sets of ordered pairs:

   (a) $f: \; x \rightarrow 3x - 1, \quad x \in \{0, 2, 5\}$; (c) $f: \; x \rightarrow 2, \quad x$ an integer;

   (b) $f: \; x \rightarrow x^3, \quad x \in R$; (d) $f: \; x \rightarrow x, \quad x \in R$.

2. Write the following functions, defined as sets of ordered pairs, as mappings:

   (a) $\{(0,1), (2,3), (4,5)\}$;

   (b) $\{(x, \sqrt{x}): \; x$ a positive real number$\}$;

   (c) $\{(x,-1): \; x \in R\}$;

   (d) $\{(0,-2), (-1,4), (5,15)\}$.

3. Which of the following sets of ordered pairs represent functions and which do not?

   (a) $\{(2,3), (5,1), (6,1), (3,2)\}$;

   (b) $\{(1,4), (2,3), (3,2), (2,5)\}$;

   (c) $\{(-1,1), (3,-2), (0,0)\}$;

   (d) $\{(-1,2)\}$.

4. Which of the functions of Problem 3 have inverses? For those that do, write the inverse as a set of ordered pairs.

5. Do as in Problem 4 for the functions of Problem 2.

10. Summary.

This chapter deals with functions in general and with the constant and linear functions in particular.

A function is an association between the objects of one set, called the domain, and those of another set, called the range, such that exactly one element of the range is associated with each element of the domain. A function can be represented as a mapping from its domain onto its range.

The graph of a function is often an aid to understanding the function. A graph is the graph of a function if and only if no line parallel to the y-axis meets it in more than one point.

A constant function is an association of the form $f: x \rightarrow c$, for some fixed real number c, with the set of all real numbers as its domain. The graph of a constant function is a straight line parallel to the x-axis.

A linear function is an association of the form $f: x \rightarrow mx + b$, $m \neq 0$. The domain and the range of a linear function are each the set of all real numbers. The graph of a linear function is a straight line not parallel to either axis, and, conversely, any such line is the graph of some linear function.

The slope of a line through $P(x_1,y_1)$ and $Q(x_2,y_2)$ is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

if $x_1 \neq x_2$. If $x_1 = x_2$, no slope is defined, and the line is parallel to the y-axis. Lines with the same slope are parallel, and parallel lines which have slopes have equal slopes. The slope of the graph of the linear function $f: x \rightarrow mx + b$ is the coefficient m; this number is also called the slope of the function.

The absolute-value function is conveniently defined as $f: x \rightarrow \sqrt{x^2}$. The domain of this function is the set of all real numbers and the range is the set of all non-negative real numbers.

If f and g are functions, then the composite function fg is $fg: x \rightarrow f\,g(x)$, with domain all x in the domain of g such that $g(x)$ is in the domain of f.

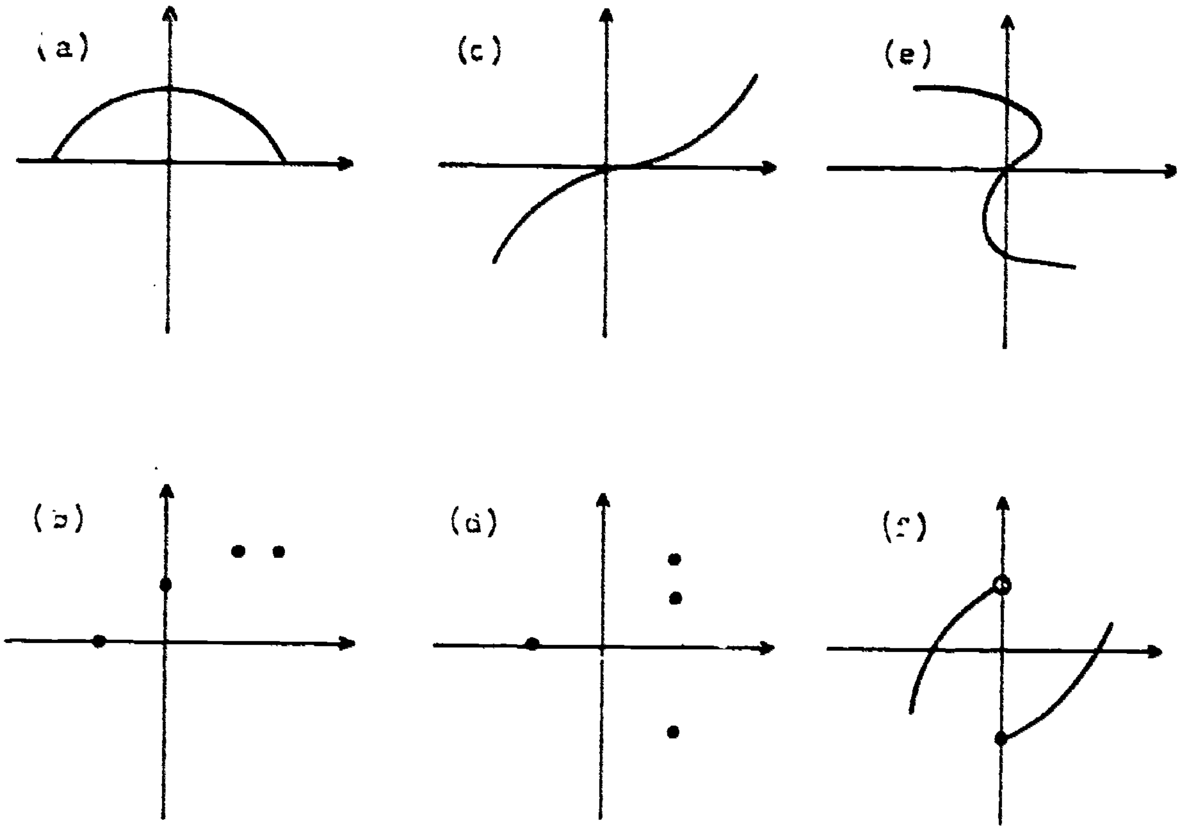
Given a function f, if there exists a function g such that $(gf)(y) = y$ for all y in the domain of f and $(fg)(x) = x$ for all x in the domain of g, then g is an inverse of f. Not all functions have inverses. A necessary and sufficient condition that a function have an inverse is that it be a one-to-one function; i.e., a function f such that $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$.

A function may also be considered as a set of ordered pairs in which if two pairs have the same first element they also have the same second element.

Miscellaneous Exercises

1. Describe how to obtain the composite, fg, of two functions f and g when the functions are considered as sets of ordered pairs.

2. Which graphs represent functions? Which of these functions have inverses?



3. What is the constant function whose graph passes through $(5,2)$?

4. For what values of a, b, and c will $f: x \rightarrow ax^2 + bx + c$ be a constant function?

5. What is the constant function whose graph passes through the intersection of L_1: $y = 3x - 2$ and L_2: $3y - 4x + 5 = 0$?

6. At what point do L_1: $y = ax + 4$ and L_2: $y = 5x + b$ intersect? Do they always intersect?

7. Write the linear functions f_1 and f_2 whose graphs intersect the x-axis at $P(-3,0)$ at angles of 45° and -45°, respectively.

8. If $10x + y - 7 = 0$, what is the decrease in y as x increases from 500 to 505? What is the increase in x as y decreases from -500 to -505?

9. Write the equation of the line through $(0,0)$ which is parallel to the line through $(2,3)$ and $(-1,1)$.

10. Write the equation of the line which passes through the intersection of L_1: $y = 6x + k$ and L_2: $y = 5x + k$ and has slope $\frac{5}{6}$.

11. Write the equation of the line which is the locus of points equidistant from L_1: $6x + 3y - 7 = 0$ and L_2: $y = -2x + 3$.

12. Write the equation of the line through $(8,2)$ which is perpendicular to (has a slope which is the negative reciprocal of the slope of) L_1: $2y = x + 3$.

13. In a manufacturing process, a certain machine requires 10 minutes to warm up and then produces y parts every t hours. If the machine has produced 20 parts after running $\frac{1}{2}$ hour and 95 parts after running $1\frac{3}{4}$ hours, find a function f such that $y = f(t)$, and give the domain of f.

14. If $ABCD$ is a parallelogram with vertices at $A(0,0)$, $B(8,0)$, $C(12,7)$, and $D(4,7)$, find
    (a) the equation of the diagonal AC;
    (b) the equation of the diagonal BD;
    (c) the point of intersection of the diagonals.

15. Repeat Problem 14, using parallelogram $ABCD$ with vertices at $A(0,0)$, $B(x_1,0)$, $C(x_2,y_2)$, and $D(x_2 - x_1, y_2)$.

16. Given the constant functions $f: x \to a$, $g: x \to b$, and $h: x \to c$, determine the compound functions $f(gh)$ and $f(hg)$. Does this result indicate that $gh = hg$?

17. Find an inverse of the linear function $f: x \to mx + b$.

18. Find a function f such that $ff = f$.

19. Sketch a graph of:
    (a) $f(x) = \frac{|x|}{x}$;
    (b) $|x| + |y| = 1$;
    (c) $y = |x - 1| - |x + 1|$.

20. If $f(x) = 2x - 5$ and $g(x) = 3x + k$, determine k so that $fg = gf$.

21. If $f(x) = x^2$ and $g(x) = \sqrt{16 - x^2}$, find the domains of fg and gf.

Suggestions for Further Reading

Allendoerfer, C. B., and Oakley, C. O.
Fundamentals of Mathematics. New York: McGraw-Hill Book Co., 1959. Chapter 9. Defines function in terms of ordered pairs and gives many examples of functions.

Brumfiel, C. F., Eicholz, R. E., and Shanks, M. E.
Algebra II. Reading, Mass.: Addison-Wesley Publishing Co., 1962. Chapter 6. Begins with a discussion of sets and relations. After defining function in terms of ordered pairs, the authors emphasize the mapping approach and consider the concept of a continuous function.

Evenson, A. B.
Modern Mathematics. Chicago: Scott, Foresman and Co., 1962. Chapters 5-6. A quite detailed discussion of functions. Chapter 5 discusses the more general concept of "relations"; Chapter 6 begins with the ordered pair approach and then moves to the mapping viewpoint.

Rose, J. H.
Algebra. New York: John Wiley and Sons, 1963. Chapters 1-2. Chapter 1 is on sets. Chapter 2 discusses functions as mappings.