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ABSTRACT

This is volume one of a three-volume set for teachers using SMSG junior high school text materials. Each unit contains a commentary on the text, answers to all the exercises, a copy of the questionnaire used for evaluating the material, and a summary of comments by the teachers using the text. Unit topics include: (1) numeration; (2) natural numbers and zero; (3) factoring and primes; (4) supplementary tests for divisibility and repeating decimals; (5) non-negative rational numbers; and (6) mathematical systems. (MP)

JUNIOR HIGH SCHOOL MATHEMATICS UNITS

VOLUME I, NUMBER SYSTEMS

Commentary for Teachers

SMSG

JUNIOR HIGH SCHOOL MATHEMATICS UNITS

VOLUME I, NUMBER SYSTEMS

Commentary for Teachers

Prepared by the SCHOOL MATHEMATICS STUDY GROUP
Under a grant from the NATIONAL SCIENCE FOUNDATION

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These units were prepared in the summer of 1958 at Yale University by a School Mathematics Study Group writing team. The units were taught in a number of classes during the academic year 1958-59. No major editing of them has been attempted, but typographical and other errors have been corrected.

When the units were used, the teachers were invited to submit comments concerning their experiences. A questionnaire was filled out by the teachers for each unit taught. A copy of this questionnaire and a summary of comments by the teachers is included for each unit.

In addition, in those cases where unit tests were prepared during the year, a collection of suggested test items appears at the end of the commentary on the unit.

This volume includes the units concerned with the structure of the number systems of arithmetic. These are units II, III, IV, IVa, V, XIV in the numbering system originally used.

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UNIT II

NUMERATION

For this unit there is little background needed except familiarity with the number symbols and the basic operations with numbers. The purpose of the unit is to deepen the student's understanding of the decimal notation for whole numbers and thus to delve a little deeper into the reasons for the mechanical operations for addition and multiplication, which he already knows. One of the best ways to accomplish this is to consider systems of number notation using other bases than ten. Since in using a new base the student is forced to look at the reasons for "carrying" and the other mechanical operations in a new light, he should gain deeper insight into the decimal system. A certain amount of computation in other systems is necessary to fix these ideas but such computation should not be regarded as an end in itself. However, some of the students may enjoy developing a certain proficiency in using new bases in computing.

Perhaps the most important reason for introducing ancient symbolisms for numbers is to contrast them with our system, in which not only the symbol, but its position, has significance. It should be pointed out that position has significance in other systems as they are presented here. The Roman system was a start in this direction in that XL represents a different number from LX, but the start was a very primitive one. The Babylonians also made use of position, but lacked a symbol for zero. The numeral zero is necessary to a positional system. In order for the pupils to appreciate the important characteristics of our system of numbers, the following facts should be brought out in comparing with other systems:

Egyptian	decimal	no place value	no zero
Babylonian	non-decimal	place value	zero (after 200 B.C.)
Roman	partly-decimal	no place value	no zero

The illustration of grouping in tens on page 6 suggests a method useful in mental calculation. For instance, to add 97 and 56, one can take 3 from 56, add 3 to 97 to make 100, and see that the sum is 153. On page 7, note also that parentheses are used to show that certain combinations are to be considered <u>as a single number</u>. In another unit a different use of parentheses will be discussed in greater detail.

Exponents are introduced on page 7, in a situation which shows clearly their usefulness for concise notation. Furthermore, their use serves to emphasize the role of the base and of position.

On page 9, question may arise about the notation for a number to base seven. We do not write "13_7" because the symbol "7" does not occur in a number system to this base. Replacing the numeral by the written word emphasizes this fact.

It is especially important to distinguish between a number and the symbols by which it is represented. Some of the properties usually connected with a number are really only properties of its notation. The facts that, in decimal notation, the numeral for a number divisible by 5 ends in 5 or 0, and that 1/3 has an unending decimal equivalent are illustrations. Most of the properties with which we deal are properties of the numbers themselves and are entirely independent of the notation in which they are represented. Examples of such properties of numbers are: $2 + 3 = 3 + 2$; the number eleven is a prime number; six is greater than five. The distinction between a number and the notation in which it is expressed should be emphasized whenever there is opportunity.

8

At several points numbers are represented by collections of x's. Exercises of this kind are important, because they show the role of the base in grouping the x's, as well as the significance of the digits in the numeral for the number.

On page 10, the symmetry of the table in Ex. 24 results from the fact that addition is commutative. Similar instances occur in later exercises, and present an opportunity to show that this is a property of the numbers, rather than of the notation in which they are represented.

On page 13, the exercises deal with tests for divisibility. The purpose is to compare evidences of divisibility in one notation with evidences in another notation; but the matter of tests of divisibility may be treated more fully if it seems desirable.

In Ex. 79-80, it is interesting to note that the sum of 11001 and 110 looks the same in the binary system and the decimal system, but the meaning is very different.

In Ex. 83, it can be noted that the peg board device is really a kind of abacus for the binary system.

The pupils will probably be interested to know that the binary system is used in many high-speed computing machines. They may therefore also be interested in the remainder method for changing a number from one base to another. Suppose that 25 in decimal notation is to be changed to binary numerals. The process rests on repeated division by 2, to identify the powers of 2 whose sum is 25. The division is shown below, followed by an interpretation of the results of the division at that stage. It will be noted that the remainders indicate digits in binary numerals. Recall that $2^0 = 1$.

$$
\begin{array}{lll}
2\underline{)25} & & 25 = 25 \times 2^0 \\
2\underline{)12}, & \text{remainder } 1 & 25 = 12 \times 2^1 + 1 \times 2^0 \\
2\underline{)\ 6}, & \text{remainder } 0 & 25 = 6 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2\underline{)\ 3}, & \text{remainder } 0 & 25 = 3 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2\underline{)\ 1}, & \text{remainder } 1 & 25 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2)\ 0, & \text{remainder } 1 & 25 = 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
 & & 25 = \quad 1 \quad 1 \quad 0 \quad 0 \quad 1_{two}
\end{array}
$$

The binary system is used for computers since there are only two digits, and an electric mechanism is either "on" or "off". Such an arrangement is called a flip-flop mechanism.

The base two has the disadvantage that, while only two different digits are used, many more digits are needed to express numbers in binary notation than in decimal, e.g.,

$$2000_{ten} = 11,111,010,000_{two}$$

People who work with computers very often use the base eight. To change from binary to octal and back is very simple by use of the table:

Binary	Octal
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7

For example, we have

$$2000_{ten} = 011,111,010,000_2 = 3720_8$$

Give pupils some exercises in changing from binary to octal and conversely, so that they can see the simple rule involved.

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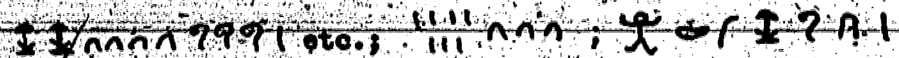
ANSWERS

1. 32 x 60 x 60 = 115200
   52 x 60 = 3120
   16 x 1 = 16
   118336

2. 

3. (a) LI (b) LXXV (c) CLX (d) DXII (e) MDCCCCLVIII

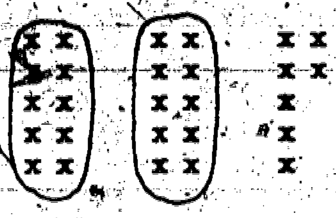
4. (a) 8 (b) 105 (c) 61 (d) 1250 (e) 503

5. Encyclopaedia Brittanica and World Book give explanations of these numerals.

6. He used Greek letters for 3 and 5 and other letters for 30 and 50. In our system, knowing 3 + 5 helps us to find 30 + 50.

7. See encyclopaedia.

8. From the Latin "digitus" or "finger."

10. 

11. Use 20, 30, 40, etc.

12. (a) 354 = (3 x 10 x 10) + (5 x 10) + (4 x 1)

    (b) 6421 = (6 x 10 x 10 x 10) + (4 x 10 x 10) + (2 x 10) + (1 x 1)

    (c) 709 = (7 x 10 x 10) + (0 x 10) + (9 x 1)

    (d) 320 = (3 x 10 x 10) + (2 x 10) + (0 x 1)

    (e) 2000 = (2 x 10 x 10 x 10) + (0 x 10 x 10) + (0 x 10) + (0 x 1)

13.

	I	V	X	L	C	D	M
I	I	V	X	L	C	D	M
V	V	XXV	L	CCL	D	MMD	$\overline{V}$
X	X	L	C	D	M	$\overline{V}$	$\overline{X}$
L	L	CCL	D	MMD	$\overline{V}$	$\overline{XXV}$	$\overline{L}$
C	C	D	M	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$
D	D	MMD	$\overline{V}$	$\overline{XXV}$	$\overline{L}$	$\overline{CCL}$	$\overline{D}$
M	M	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$	$\overline{D}$	$\overline{M}$

14. 3^4

15. 5^4 means $5 \times 5 \times 5 \times 5$

16. $4^3 = 4 \times 4 \times 4 = 64$

17. $4^3 = 64$; $3^4 = 81$

18. (a) $(4 \times 10^2) + (6 \times 10) + (8 \times 1)$

(b) $(5 \times 10^3) + (3 \times 10^2) + (2 \times 10) + (4 \times 1)$

(c) $(7 \times 10^3) + (0 \times 10^2) + (6 \times 10) + (2 \times 1)$

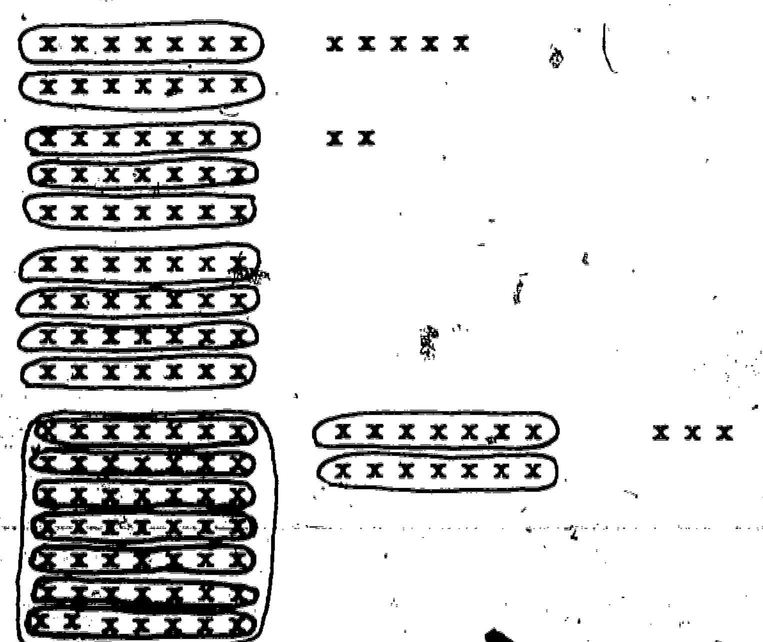
(d) $(5 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (2 \times 10) + (0 \times 1)$

19. eighty = quatre-vingts (or 4 twenties)

ninety = quatre-vingt-dix (or 4 twenties, ten)

score = twenty

20. (a) (b) (c) (d)



*21. $246_{seven} = (2 \times 49) + (4 \times 7) + (6 \times 1) = 98 + 28 + 6 = 132_{ten}$

(a) $56_{seven} = (5 \times 7) + (6 \times 1) = 35 + 6 = 41_{ten}$

(b) $241_{seven} = (2 \times 7 \times 7) + (4 \times 7) + (1 \times 1) = 98 + 28 + 1 = 127_{ten}$

(c) $500_{seven} = (5 \times 7 \times 7) + (0 \times 7) + (0 \times 1) = 245_{ten}$

(d) $4120_{seven} = (4 \times 7 \times 7 \times 7) + (1 \times 7 \times 7) + (2 \times 7) + (0 \times 1) =$

$1372 + 49 + 14 = 1435_{ten}$

23.

0	10	20	30	40	50	60	100	110
1	11	21	31	41	51	61	101	
2	12	22	32	42	52	62	102	
3	13	23	33	43	53	63	103	
4	14	24	34	44	54	64	104	
5	15	25	35	45	55	65	105	
6	16	26	36	46	56	66	106	

25. The two parts are "alike" in that they are symmetrical with respect to the diagonal indicated in Ex. 24.

26.

	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

27. Forty nine.

28. See table. Yes (see Ex. 25.)

29. 73_{ten}; 62_{seven}

30. Yes

31. ten

32. Seven

33. (a) 55_{seven} (b) 100_{seven} (c) 60_{seven} (d) 362_{seven}

(e) 1363_{seven}

34. 32_{seven} = 3 sevens + 2 ones
25_{seven} = <u>2 sevens + 5 ones</u>

5 sevens + seven ones = 6 sevens

* See explanation on page T-II-12, answer 69.

(a) $\begin{array}{r} 30 \\ \underline{10} \end{array}$ $40_{ten} = 55_{seven}$ (b) $\begin{array}{r} 47 \\ \underline{8} \end{array}$ $55_{ten} = 106_{seven}$ (c) $\begin{array}{r} 23 \\ \underline{19} \end{array}$ $42_{ten} = 60_{seven}$

(d) $\begin{array}{r} 137 \\ \underline{57} \end{array}$ $194_{ten} = 365_{seven}$ (e) $\begin{array}{r} 222 \\ \underline{313} \end{array}$ $535_{ten} = 1363_{seven}$

36. (a) $\begin{array}{r} 56 \\ \underline{14} \\ 42_{seven} \end{array}$ (b) $\begin{array}{r} 61 \\ \underline{35} \\ 23_{seven} \end{array}$ (c) $\begin{array}{r} 34 \\ \underline{25} \\ 6_{seven} \end{array}$ (d) $\begin{array}{r} 456 \\ \underline{263} \\ 163_{seven} \end{array}$

37. 34_{seven} = 3 sevens + 4 ones = 2 sevens + eleven ones

38.

	0	1	2	3	4	5	6
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Yes, two parts of this table are alike.

39. (a) 216 (b) 303 (c) 2314 (d) 252

*40. (a) $43_{seven} = (4 \times 7) + 3 = 28 + 3 = 31_{ten}$

(b) $526_{seven} = (5 \times 7 \times 7) + (2 \times 7) + (6 \times 1) = 245 + 14 + 6 = 265_{ten}$

(c) $304_{seven} = (3 \times 7 \times 7) + (0 \times 7) + (4 \times 1) = 147 + 4 = 151_{ten}$

(d) $260_{seven} = (2 \times 7 \times 7) + (6 \times 7) + (0 \times 1) = 98 + 42 = 140_{ten}$

41. (a) $37 \times 3 = 111$; $111_{ten} = 216_{seven}$

(b) $25 \times 6 = 150$; $150_{ten} = 303_{seven}$

(c) $211 \times 4 = 844$; $844_{ten} = 2314_{seven}$

(d) $15 \times 9 = 135$; $135_{ten} = 252_{seven}$

42. Division in base seven is done in a similar way to division in the decimal system. Refer to the multiplication table, base seven (Ex. 38) for each quotient.

(a) $\begin{array}{r} 5 \\ 6\overline{)42} \\ \underline{42} \end{array}$ (b) $\begin{array}{r} 50 \\ 6\overline{)420} \\ \underline{42} \\ 0 \end{array}$ (c) $\begin{array}{r} 52 \\ 6\overline{)435} \\ \underline{42} \\ 15 \\ \underline{15} \end{array}$ (d) $\begin{array}{r} 454 \\ 6\overline{)4053} \\ \underline{33} \\ 45 \\ \underline{42} \\ 33 \\ \underline{33} \end{array}$

43. $2_{seven}\overline{)501_{seven}}$ 234_{seven} R 1

44. $5_{seven}\overline{)652_{seven}}$ 123

45. The ones digit is 0, 2, 4, 6, or 8.
The ones digit is 5 or 0.
The ones digit is 0.

* See explanation on page T-II-12, answer 69.

46. 2, 4, 6; 11, 13, 15, 20, 22, 24, 26.
The sum of the digits is divisible by 2.

47. 10, 20, 30, 40, 50. The ones digit is zero.

48. (a) The sum of the digits is nine.
Rule: If the sum of the digits is divisible by nine, the number is divisible by nine.

(b) 108, 117, 126, 135. Yes.

(c) 101, 102, 105, 120.
If the sum of the digits is not nine, the number is not divisible by nine.

(d) Yes. See (a).

49. (a) The sum of the digits in each case is 6. If the sum of the digits is 6, the number is divisible by 6.

(b) Yes

(c) If the sum of the digits is not 6, the number is not divisible by 6.

(d) Yes. See (a).

50. Each number is one less than the base.

51. (a) If the sum of the digits is divisible by 3, the number is divisible by 3.

(b) If the sum of the digits is not divisible by 3, the number is not divisible by 3.

(c) See (a).

(d) Yes.

(e) 4, 13, 16 etc., or 10 ± 3n where n is any integer.

52. Yes.

53. Yes. Yes.

54. $50_{ten} = (1 \times 49) + (0 \times 7) + (1 \times 1) = 101_{seven}$

55. $145_{ten} = (2 \times 49) + (0 \times 7) + (5 \times 1) = 265_{seven}$

56. 1958 ÷ 10 = 195 R 8
195 ÷ 10 = 19 R 5
19 ÷ 10 = 1 R 9
1 ÷ 10 = 0 R 1

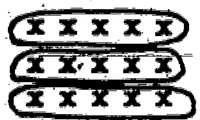
The successive remainders are the digits of the original number.

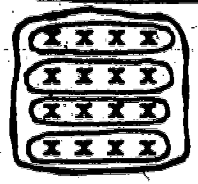
57. 524 ÷ 7 = 74 R 6
74 ÷ 7 = 10 R 4
10 ÷ 7 = 1 R 3
1 ÷ 7 = 0 R 1

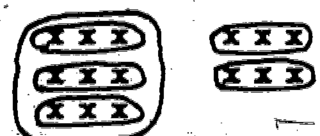
$1346_{seven} = (1 \times 343) + (3 \times 49) + (4 \times 7) + 6 = 524_{ten}$

58. Divide by seven as above until last quotient is zero. The digits of the number in base seven are the successive remainders. Note that the first remainder is the first digit, reading from right to left.

59. 5, 3, 4

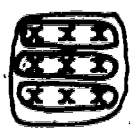
60.  x 31_{five}

61.  100_{four}

62.  x 121_{three}

63. (a)  x x x

(b)  x x x

(c)  x x

(d) Yes.

64. (a) Two (b) Two (c) Two

65. Two

66.

Base ten	Base six	Base four	Base three
1	1	1	1
2	2	2	2
3	3	3	10
4	4	10	11
5	5	11	12
6	10	12	20
7	11	13	21
8	12	20	22
9	13	21	100
10	14	22	101

*67. (a) $638_{nine} = (6 \times 9^2) + (3 \times 9) + (8 \times 1)$

(b) $245_{six} = (2 \times 6^2) + (4 \times 6) + (5 \times 1)$

(c) $1002_{three} = (1 \times 3^3) + (0 \times 3^2) + (0 \times 3) + (2 \times 1)$

*68. (a) $(6 \times 81) + (27) + 8 = 486 + 27 + 8 = 521$

(b) $(2 \times 36) + (24) + 5 = 72 + 24 + 5 = 101$

(c) $(27) + 2 = 29.$

*69. (a) $234_{five} = (2 \times 5^2) + (3 \times 5) + (4 \times 1)$

(b) $103_{five} = (1 \times 5^2) + (0 \times 5) + (3 \times 1)$

(c) $412_{five} = (4 \times 5^2) + (1 \times 5) + (2 \times 1)$

(d) 1¢, 5¢, 25¢, $1.25, etc.

*Exercise 69. In the expanded notation, we interpret the numeral 5 as expressed in the decimal system. The word "five" might preferably be used.

*70. (a) $146_{twelve} = (1 \times 12^2) + (4 \times 12) + (6 \times 1) = 144 + 48 + 6 = 198_{ten}$

(b) $3t2_{twelve} = (3 \times 12^2) + (10 \times 12) + (2 \times 1) = 432 + 120 + 2 = 554_{ten}$

(c) $47e_{twelve} = (4 \times 12^2) + (7 \times 12) + (11 \times 1) = 576 + 84 + 11 = 671_{ten}$

71. Many things are counted by twelves; our ordinary measure of length begins with 12 inches in one foot; twelve is divisible by 2, 3, 4, and 6.

72. (x x) x . 11_{two}

73.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111	10000	10001

74.

+	0	1
0	0	1
1	1	10

Four.

75.

x	0	1
0	0	0
1	0	1

Four. No.

*76. (a) $111_{two} = (1 \times 2^2) + (1 \times 2) + (1 \times 1)$

(b) $1000_{two} = (1 \times 2^3) + (0 \times 2^2) + (0 \times 2) + (0 \times 1)$

(c) $10101_{two} = (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2) + (1 \times 1)$

(d) $11000_{two} = (1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (0 \times 2) + (0 \times 1)$

77. $2^3 = 8$; $2^2 = 4$; $2^1 = 2$

$2^1 = 2$; $2^2 = 4$; $2^3 = 8$; $2^4 = 16$; $2^5 = 32$; $2^6 = 64$; $2^7 = 128$; $2^8 = 256$; $2^9 = 512$

78. (a) $4 + 2 + 1 = 7_{ten}$

(b) 8_{ten}

(c) $16 + 4 + 1 = 21_{ten}$

(d) $16 + 8 = 24_{ten}$

79. (a) 111_{two} (b) 1011_{two} (c) 110001_{two} (d) 110110_{two}

(a) $5 + 2 = 7_{ten}$; $111_{two} = 7_{ten}$

(b) $6 + 5 = 11_{ten}$; $1011_{two} = 11_{ten}$

(c) $22 + 27 = 49_{ten}$; $100001_{two} = 49_{ten}$

(d) $23 + 31 = 54_{ten}$; $100110_{two} = 54_{ten}$

80. (a) 10_{two} (b) 11_{two} (c) 111_{two} (d) 11_{two}

81. (a) $7 - 5 = 2_{ten}$; $10_{two} = 2_{ten}$

(b) $6 - 3 = 3_{ten}$; $11_{two} = 3_{ten}$

(c) $11 - 4 = 7_{ten}$; $111_{two} = 7_{ten}$

(d) $25 - 22 = 3_{ten}$; $11_{two} = 3_{ten}$

82. (a) $35_{ten} = 2^5 + 2 + 1 = 100011_{two}$

(b) $128_{ten} = 2^7 = 10,000,000_{two}$

(c) $12_{ten} = 2^3 + 2^2 = 1100_{two}$

(d) $100_{ten} = 2^6 + 2^5 + 2^2 = 1100100_{two}$

84. Base two: short addition and multiplication tables.

Base twelve: See (Ex. 71)

85. Four. The weights would be 1 lb., 2 lb., 4 lb., 8 lb.

86. The trick is based on the application of the binary numbers. Write the first fifteen binary numbers and use them as your guide. See Merrill reference pp. 26-28 and Swain p. 111.

1			
1	9	17	25
3	11	19	27
5	13	21	29
7	15	23	31

2			
2	10	18	26
3	11	19	27
6	14	22	30
7	15	23	31

4			
4	12	20	28
5	13	21	29
6	14	22	30
7	15	23	31

8			
8	12	24	28
9	13	25	29
10	14	26	30
11	15	27	31

16			
16	20	24	28
17	21	25	29
18	22	26	30
19	23	27	31

87. Yes. See Merrill reference pp. 23-26.

Sample Test Questions

PART I. TRUE - FALSE.

T 1. 12_{seven} and IX are numerals that refer to the same number.

T 2. Sticks, stones and notches are numerals if used to tell quantity.

T 3. The 3 in 357 stands for three hundreds.

F 4. The 4 in 428 is twice the value of the 2.

T 5. The 7 in 7654 is the same as 7×10^3.

F 6. The exponent 3 in 5^3 means to multiply 3 times 5.

T 7. Had man been created with eight fingers and eight toes, systems based on eight and sixteen would probably have been more common.

T 8. Without place value, we could not count beyond ten in our decimal system.

T 9. The cube of a number is gotten by multiplying the number together three times.

T 10. The base of a system and the number of digits in the system are numerically the same.

F 11. Doubling the number of digits in a system will give the number of basic addition facts in that system.

T 12. We can make a number symbol mean anything we want it to mean.

T 13. The number "8" (eight) has the same meaning in the twelve number system as in our common number system.

T 14. In performing computations there are more number combinations to remember in the ten number base than in the two base.

15. When we carry a figure in an addition or multiplication problem, the value of the figure we carry is the same in the base 12 as in the base 10.

F 16. When we borrow in the dozen system in a problem such as 154 - 2t, we actually borrow ten units.

T 17. In the dozen system, 7 × 9 = 53.

F 18. In the five system, 4 + 3 = 13.

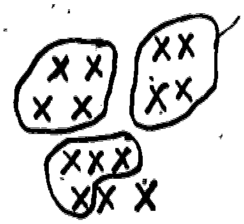
T 19. The larger the number base, the fewer the figures needed to represent large numbers.

F 20. In the number 842 base twelve, the 8 has a value that is twenty times the value of the 4 in the base twelve system.

T 21. In the number base five, the fraction $\frac{3}{4}$ is equal to $\frac{14}{22}$.

PART II. MULTIPLE CHOICE.

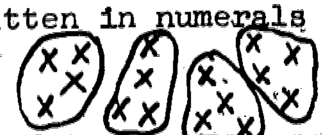
1. The x's at the right are grouped so that the number of x's can easily be written in a numeral to some base. What is the base?



   A. Base two.
   B. Base three.
   *C. Base four.
   D. None of these.

2. In which of the numerals below does the 1 stand for 1 eight?
   A. 21_{four}
   B. 21_{eight}
   C. 102_{three}
   *D. 1000_{two}

3. The number of x's at the right is written in numerals in four different bases. Which numerals



are correct?

a) 100_{four} b) 14_{twelve} c) 16_{ten} d) 31_{five}

A. a and c are correct.

B. b and c are correct.

C. a, b, and c are correct.

*D. All four are correct.

4. When you write numerals in base eight in order, what comes after 57_{eight}?

A. 58_{eight}

B. 61_{eight}

C. 100_{eight}

*D. 60_{eight}

5. Which of the numerals below represents the largest number?

A. 1011_{two}

B. 211_{three}

*C. 43_{five}

D. 21_{nine}

6. Which of the following numerals represent the same number?

a) 185_{twelve} b) 365_{eight} c) 1045_{six} d) 20002_{three}

*A. a, b, and c

B. a, c, and d

C. b, c, and d

D. a, b, c, and d

7. Which of the following are correct?

a) In the symbol 5^3, 5 is the base and 3 is the exponent.

b) In the symbol 5^3, 3 is the base and 5 is the exponent.

c) $5^3 = 5 \times 5 \times 5$

d) $5^3 = 3 \times 3 \times 3 \times 3 \times 3$

A. a and d are correct.

B. b and c are correct.

C. a and c are correct.

D. b and d are correct.

8. Which of the following shows the meaning of 423_{six}?

A. (4 × six × six × six) + (2 × six × six) + (3 × six)

B. 400 + 20 + 3

*C. (4 × six × six) + (2 × six) + 3

D. (4 × six) + (2 × six) + (3 × six)

9. Which of the following statements are true?

a) The Egyptians had no symbol for zero.

b) The Roman numerals CD and DC stand for the same number.

c) In the Roman numeral XX, the first X stands for a bigger number than the second X.

d) The Babylonians used the number sixty in writing numerals in somewhat the same way that we use the number ten.

A. a and b are correct.

B. c and d are correct.

*C. a and d are correct.

D. b and c are correct.

10. Which of the 2's in the base ten numeral below stands for 2 × one thousand?

a b c
21232

A. the one marked a

B. the one marked b

C. the one marked c

*D. None of them.

11. The reason people began to count by tens and to use ten as a base is probably that...

A. it is easy to multiply by ten.

*B. people have ten fingers.

C. ten is a "round" number.

D. decimal fractions are expressed in tenths.

12. If a zero is written at the right of a base nine numeral, the number the base nine numeral represents...

A. becomes twice as large.

B. becomes ten times as large.

C. stays the same.

*D. becomes nine times as large.

13. In what base are the numerals written if 2 × 2 = 10?

A. base two

B. base three

*C. base four

D. base five

14. If you have only the symbols 0, 1, 2, 3, 4, and 5 for a system of numerals, and can use as many of the symbols as you wish, a list of all possible bases is:

A. one, two, three, four, five, or six.

*B. two, three, four, five, or six.

C. two, three, four, five, six, or ten.

D. two, three, four, or five.

15. In this addition example, in what base are the numerals written?

A. base two

*B. base three

C. base four

$$\begin{array}{r} 120_{?} \\ +\ 210_{?} \\ \hline 1100_{?} \end{array}$$

25

D. You can't tell.

16. In this subtraction example, in what base are the numerals written?

A. base seven
B. base eight
*C. base nine
D. You can't tell.

```
  615?
- 364?
  241?
```

17. In this addition example, in what base are the numerals written

A. base two
B. base four
C. base ten
*D. You can't tell.

```
    101?
+ 10010?
  10111?
```

18. In this multiplication example, in what base are the numerals written?

A. base five
*B. base eight
C. base eleven
D. You can't tell

```
    34?
 x  23?
   124
   70
  1024?
```

19. In the division example below, the numerals are written in base eight. Which of the examples is correct?

a)
```
           175eight
3eight ) 567eight
         3
         26
         25
          17
          17
```

b)
```
           123eight
3eight ) 567eight
         3
         26
          6
         207
         207
```

c)
```
           175eight
3eight ) 567eight
         3
         26
         25
          17
          15
           2
```

d)
```
           189eight
3eight ) 567eight
         3
         26
         24
          27
          27
```

*A. Example a is correct.

B. Example b is correct.

*C. Example c is correct.

D. Example d is correct.

20. The numerals below are written in base ten. Which numerals represent numbers which are exactly divisible by nine?

a) 198 b) 373 c) 8901 d) 763

A. None of them is exactly divisible by nine.

B. a, b, and c are exactly divisible by nine.

*C. a and c are exactly divisible by nine.

D. All of them are exactly divisible by nine.

21. The numerals below are written in base seven. Which ones represent numbers which are exactly divisible by two?

a) 15_{seven} b) 32_{seven} c) 46_{seven} d) 101_{seven}

A. b and c are exactly divisible by two.

B. a and d are exactly divisible by two.

*C. a, c, and d are exactly divisible by two.

D. All of them are exactly divisible by two.

22. In which of the bases named below does the numeral 35 represent an even number?

A. in base ten

B. in base twelve

*C. in base nine

D. in none of these bases

23. In which of the bases named below does the numeral 32 represent an odd number?

A. in base four

B. in base ten

*C. in base five

D. in none of these bases

24. Below are four numerals written in expanded notation. Which of them are written correctly?

a) $(4 \times ten^2) + (9 \times ten^1) + (3 \times one) = 493_{ten}$

b) $(3 \times seven^3) + (6 \times seven^1) + (1 \times one) = 361_{seven}$

c) $(4 \times twelve^2) + (5 \times twelve^1) + (e \times one) = 45e_{twelve}$

d) $(2 \times five^2) + (2 \times five^1) + (4 \times one) = 224_{five}$

A. a and c are correct.

B. b and d are correct.

*C. a, c, and d are correct.

D. a and b are correct.

25. Which of the statements below are correct?

a) $75_{ten} = 1001011_{two}$

b) $31_{ten} = 111_{five}$

c) $907_{ten} = 637_{twelve}$

d) $184_{ten} = 270_{eight}$

A. Statements a and b are correct.

B. Statements a, b, and d are correct.

C. Statements a and d are correct.

*D. All the statements are correct.

PART III. PROBLEMS.

1. Write $3 \times 3 \times 3 \times 3 \times 3 \times 3$ using exponents. (3^6)

2. Which number is larger, 2^3 or 3^2? (3^2)

3. What number is represented by 9^2? (81)

4. Write the following in Arabic numerals: MDIV, MM. (1504) (2000)

Write the following base seven numbers in the base ten system:

5. 15_{seven}. (12_{ten})

6. 203_{seven}: (101_{ten})

Write 132 in the following systems:

7. Roman. (CXXXII)

8. Base Seven. (246_{seven})

9. Base Nine. (156_{nine})

10. A 2 in the third place of a base twelve number would represent what value? ($2 \times 144_{ten}$ or 288_{ten})

11. Write the base seven fraction $\frac{12}{34}$ in the base ten. ($\frac{9}{25}$)

PART IV. MATCHING.

Expressions in base other than ten.			Numbers in the ten base.	
(B)	1.	10101_{two}	A.	31
(B)	2.	41_{five}	B.	21
(A)	3.	27_{twelve}	C.	16
(C)	4.	$4_{five} \times 4_{five}$	D.	14
(C)	5.	$1001_{two} + 111_{two}$	E.	None of these
(E)	6.	$4_{six} - 4_{six}$		

Date ____________

UNIT: ____________

Name of Teacher: ____________

Name of School: ____________

City: ____________ State: ____________

Number of days given to the teaching (including testing) of this unit: ____________

Approximate dates: ____________

USE THE BACK OF THIS SHEET IF YOU NEED EXTRA SPACE TO ANSWER ANY OF THE QUESTIONS BELOW

1. Make a statement about the ability level of the pupils in the class and state whether your school uses some plan of homogeneous grouping.

2. What parts of the unit proved to be the most teachable?

3. What parts of the unit proved to be the most difficult to teach?

   Did you omit any part? ________

4. Did you use any supplementary developmental materials? ______

   If so, what were they, and at what points were they used?

5. Did you find it necessary to provide the pupils with additional material? ______

   If so, was it from textbooks or did you write your own?

6. Do you think that a unit on this topic should be included in regular textbooks for 7th and 8th grades? ________

7. Please make ANY additional comments about your teaching experience with this unit which you think would be helpful to the Panel responsible for preparing and experimenting with textbook materials for grades 7 and 8.

Summary of Teachers' Comments

Seventy-five teachers reported that they had taught this unit in 110 classes. Fifty of the classes were grouped according to ability (35 high, 9 medium, 6 low) and 60 were heterogeneously grouped. Most of the classes were in the seventh grade with the remainder in the eighth grade and a few in the sixth, ninth, or tenth grades. Opinion was close to unanimous that material in the spirit of this unit should be included in a seventh grade course.

The time spent on the unit varied from 10 to 27 days with an average of about 16 days. It is suggested that 15 - 18 days is a reasonable amount of time for the unit. There was some flagging of pupil interest when the unit was continued too long. Presumably, when teachers have become thoroughly familiar with the material and appropriate teaching techniques, a considerable reduction in time may be effected.

It is recommended that the section on divisibility be omitted or treated lightly except in the most capable classes. Teachers reported on the sections as follows:

Most teachable: (1) Other bases (2) Historical treatment (3) Operations

Most difficult: (1) Tests for divisibility (2) Base seven (3) Changing bases

A few representative comments of teachers follow:

> I have never seen interest as high toward mathematics. There was no need for motivation.
>
> I feel that more exercises should be included. The children need lots of practice before they can handle this material with understanding.
>
> I noticed the slower pupils expressed great interest at first but lost interest to some extent in the more difficult portions of the unit.
>
> I am sure now that the time it takes to teach it depends somewhat on how much experience one has in teaching this material.
>
> One parent commented that the advent of the new mathematics had made their dinner conversation very stimulating. She hoped the "new" math was going to continue next year.
>
> On the whole the unit was well received. I have a few pupils who were very poor producers, but did very fine work on this unit. On the other hand, a few pupils (2) did not make any effort to absorb this new material.

It was easier for me to teach this unit to the seventh graders than it had been previously to try to teach some of these same ideas to adults with mind sets of various kinds.

Base two was a novelty and caused much fun in the class.

UNIT III

NATURAL NUMBERS AND ZERO

Natural Numbers and Zero

Understandings	Skills and Abilities
1. Natural numbers are the numbers (which man has invented) for counting	
Number is an abstract concept -- an idea -- rather than something one can see, feel, etc.	
We become acquainted with number by counting sets of things, but eventually the idea is clear even without counting	
The idea of counting by matching (one-to-one correspondence) should be developed. The idea of "twice as many" is useful in its own right and in contrast to "just as many."	
2. Addition of natural numbers is a	2. Readiness to use the

Natural Numbers and Zero

Activities

1. Have a student count the windows in the room, another count the chairs, or other sets of objects.

   Ask - Do we use numbers in counting?

   Explain that man invented the first numbers for counting, and that this is an important idea, since we use it every day.

   Ask - We have already seen that numbers and numerals are different (review this); now let us see if we can get a better idea of what a number is.

   Can you remember how you learned the number 9? Have class discuss this. Bring out that a child counts sets of 9 things many times and gradually gets the idea of 9. So, although the idea of number starts with counting, eventually we get the idea so well that we can think of it without counting.

   Ask - Can you think of something else that is like this besides number? Suggest the color red, if a good one is not put forward. Have class discuss this. Bring out that the property of redness is again an idea (like threeness) -- not something that one can feel or see. One can see a red car but not the color red. One can see three apples but not the number 3.

   (other similar concepts that can be used are tallness, strongness, honesty, etc., but these are a little less precise) Examples of twice as many (eyes and people) should be given by the student. One might go on to four times as many, etc.

Exercises - 1

2. Provide each student with some object for counters, such as squares

commutative operation	commutativity property when appropriate
We can add two natural numbers a and b by finding a set with a members and an entirely different one with b members and counting. Since the counting steps are themselves immaterial, the commutativity property holds.	
	Ability to add from top to bottom and from bottom to top.
2. Multiplication of natural numbers is a commutative operation	2. Readiness to employ commutativity when appropriate
Not all operations are commutative	

of paper. Have them find 3 + 2 by making a pile of 3 on the left and 2 on the right and then counting. (It will be well to acknowledge that this kind of experience is familiar from elementary school.) Then have them write "3 + 2 = 5." Repeat this with the pile of 2 on the left, and have them write "2 + 3 = 5."

Have them write these sentences also using Roman numerals to emphasize that they are adding numbers -- not numerals.

Ask - We see that 3 + 2 = 2 + 3. Do you suppose that this holds for any two natural numbers? Does a + b = b + a for any two natural numbers?

(this way of writing is to mean that whenever we put a numeral for a and a numeral for b we always get a true sentence.)

Emphasize that this is the <u>commutative</u> <u>property</u> <u>of</u> <u>addition</u> for natural numbers.

Also point out that this is a property of natural numbers, and does not depend on what kind of numerals we use.

Have students do a few exercises in addition where they add from bottom to top and check by adding from top to bottom.

121796	83927	(It can be pointed out that this was an
83927	37869	
37869	121796	early way of writing additions.)

2. Have some diagrams on charts or board, of stars in a flag

```
x x x x    Ask - How many stars are there in the flag? Did you
x x x x
x x x x    count them all, one by one? If nobody says he counted
```

rows and columns and multiplied, point out that this would have been a quicker way to do it.

```
x x x      Now refer to this diagram and ask similar questions.
x x x
x x x      Bring out that this is the same diagram as before,
x x x
```

Use of symbol, <, for

"less than."

only rotated, so 3 rows of 4 each has to be the same as 4 rows of 3 each, so $3 \cdot 4 = 4 \cdot 3$.

Show a diagram of a 6 x 8 flag and ask how many stars. Rotate it 90° and then ask - Is $6 \cdot 8 = 8 \cdot 6$? Have students write "$6 \cdot 8 = 8 \cdot 6$."

Have students discuss whether multiplication of natural numbers is a commutative operation. Bring out that no matter what the numbers are, that number of objects could always be arranged in rows as above, so this holds for any natural numbers.

State and emphasize the <u>commutative</u> <u>property</u> <u>of</u> <u>multiplication</u> of natural numbers: $a \cdot b = b \cdot a$ becomes a true sentence whenever any numerals for natural numbers are put in place of 'a' and 'b'.

Present the terminology "factor" and "product."

A "factor" is one of the members of a product.

Do some exercises in multiplication to show that the commutative principle may be used for checking.

6937	5834
<u>5834</u>	<u>6937</u>

Point out that the individual steps are different in the two cases, and therefore it is not likely that the same mistake would be made in both cases.

Exercises - 2 and 2a

Since this may be the first experience of some students with the inequality symbols, <, >, and ≠, special attention should be given to their use in class discussion.

Columns A and B of the Exercises provide two sets of Exercises for the class. Those in Column B are considered more difficult. The teacher may wish to assign one set to a part of the class and the other to the remainder of the class, or he may prefer to use all Exercises with all pupils.

3. Addition of natural numbers is
an associative operation

Multiplication of natural numbers
is an associative operation

3. Provide students with counters and have them put down 3 and 2 and 4.

xxx xx xxxx Then have them push the first two piles together and count the total writing "(3 + 2) + 4 = 9." Then have them push the second two piles together and write "3 + (2 + 4) = 9."

Repeat for other numbers if necessary.

(A short review of use of parentheses may be necessary)

Point out that this is a simple but very important property of natural numbers. If we didn't have it we would need many, many parentheses. We would need parentheses in a numeral like 352, since this is 300 + 50 + 2.

State and emphasize the <u>associative property of addition</u> of natural numbers: $(a + b) + c = a + (b + c)$ becomes a true sentence whenever any numerals for natural numbers are put in place of 'a', 'b' and 'c'.

Ask - Is multiplication of natural numbers also an associative operation? Have students try a few examples, such as

$(3 \cdot 5) \cdot 2$ and $3 \cdot (5 \cdot 2)$*. (*some students may be able to see this using counters, but this may be confusing to some.)

Make sure they do not confuse the commutative property in these exercises.

State and emphasize the <u>associative property of multiplication</u> of natural numbers: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ becomes a true sentence whenever any numerals for natural numbers are put in place of 'a', 'b' and 'c'.

Ask - can you think of some operations that are not associative? Bring out that division and subtraction are not

$$(8 \div 2) \div 2 \neq 8 \div (2 \div 2)$$

$$(6 - 3) - 2 \neq 6 - (3 - 2)$$

4. Natural numbers have the distributive property for multiplication over addition

4. Readiness to use the distribute property

The student should be able to distinguish between the operation of the associative and distributive properties.

This property is very important in ordinary multiplying

Exercises - 3

Exercise 8 is given to emphasize the difference between the associative and the distributive properties.

4. Provide students with counters. Have them lay out 2 • 3 and 2 • 5 side by side

```
x x x      x x x x x
x x x      x x x x x
```

and ask - If we put these together, then we have 2 times what? (have them push the objects together and get 2 rows of 8) and have them write "(2 • 3) + (2 • 5) = 2 • 8."

Repeat for other numbers.

Have them lay out 3 • 5

```
x x x x x                        x x   x x x
x x x x x   and separate into    x x   x x x
x x x x x                        x x   x x x
```

Have them write "3 • 5 = (3 • 2) + (3 • 3)"

Repeat for other numbers of objects.

Ask - What property of natural numbers can you see from what we have been doing? Bring out the distributive property of multiplication over addition:

a • (b + c) = (a • b) + (a • c) becomes a true sentence whenever numerals for natural numbers are put in place of 'a', 'b', and 'c'

Do some exercises in replacing the letters by numerals, including Roman numerals, to emphasize again that this sentence is talking about numbers -- not numerals.

Ask - How do we multiply 356 by 7? Write out the computation on the board. Point out that when we do this we are really using the distributive property:

(7 • 300) + (7 • 50) + (7 • 6) is what we actually do and this is equal to (300 + 50 + 6) • 7.

Computational simplification is often possible by using the distributive property

The extension of the distributive property to the product of two binomials is important:
$(a+b)(c+d) = ad+bc+ac+bd$.

5. The set of natural numbers is closed with respect to the addition and multiplication operations.

Do some exercises in multiplying by first expressing a several-digit numeral in expanded form, to emphasize that the distributive property is being used.

Point out that the distributive principle can be used to make some computations easier if we watch for places to use it. For example, which is easier to do:

$$(343 \cdot 695) + (343 \cdot 214) \quad \text{or} \quad 343 \cdot (695 + 214)?$$

*Point out that in finding $357 \cdot 125$ we use the distributive property repeatedly:

$$(300 \cdot 125) + (50 \cdot 125) + (7 \cdot 125) =$$

$$[(300 \cdot 100) + (300 \cdot 20) + (300 \cdot 5)] + [(50 \cdot 100) + (50 \cdot 20) + (50 \cdot 5)]$$

$$+ [(7 \cdot 100) + (7 \cdot 20) + (7 \cdot 5)]$$

The student might get to the point of seeing through numerous numerical examples that the product of two binomials is the sum of all the products of members of the first by members of the second. See Exercise 6.

Exercises - 4

5. Ask - When we add two natural numbers, what do we get? Bring out that we always get a natural number, and that there is only one sum for any two natural numbers.

Point out that this is what we mean when we say that the set of natural numbers is closed with respect to the operation of addition. Do the similar thing for multiplication.

Ask - What about division? Bring out that when one natural number is divided by another we sometimes get a natural number and sometimes get some other kind of number. Hence the set of natural numbers is not closed with respect to the operation of division.

Do the similar thing for subtraction.

6. Subtraction is the inverse operation to addition

Division is the inverse operation to multiplication

Subtraction and division are not always possible for natural numbers

Addition and multiplication are the basic operations

6. Ability to use the concept of inverse operations in checking computations

7. The natural numbers have order

7. Readiness to employ the concept of betweenness

Exercises - 5

6. Ask - Suppose you think of 5 and then add 3. When you add 3 you get 8. What is the inverse of adding 3? (answer - subtracting 3)

How do we use this in checking addition and subtraction? Bring out that in addition of 2 numbers, subtraction may be used as a check. In subtraction, addition is the check, because subtraction is the inverse of addition.

Do a few exercises in checking.

Ask - What is the inverse operation to multiplication? Bring out the analogous ideas to those above.

Do some exercises in multiplication and division and checking.

Ask - Can we always subtract any natural number from another? Bring out that one cannot (if someone brings up zero or negatives for differences, point out that this is possible, but that the result is not a natural number.)

Ask - Can we always divide one natural number by another? Bring out that one cannot, without getting numbers which are not natural numbers.

Point out and discuss that addition and multiplication are basic operations, having the closure, commutative and associative properties, while their inverses do not.

Exercises - 6

7. Draw a row of dots in a horizontal line and ask students to label them using numerals. Write numerals beside the points

. . . .
1 2 3 4 5 6 7 8 9

at the far right of the board (when there is no more room to continue) make clear that the only reason to stop is the smallness of the board (and perhaps lack of time) -- that this naming of points

Added skill in use of "<".

8. Properties of the number 1 as the identity element for multiplication

8. No matter how many times you multiply by 1, the answer is the same.

9. Properties of the number 0.

9. To work confidently with the number zero. Especially to distinguish between 0/3 and 3/0.

can go on forever.

We can think of this kind of picture for natural numbers. They are ordered, or lined up. We can tell which come first and which are between two numbers.

Point out that we could make the row of dots vertical or on a slant if we wish.

Do some exercises using "<" and betweenness.

Exercises - 7

Have a student measure something, such as the length of a table. Make sure that the measurement has a fractional part.

Ask if in measuring we could get along very well with only the natural numbers. Have class discuss. Bring out that one soon needs fractions.

Make a list of activities (suggested by the students) in which the natural numbers are not sufficient.

8. Ask - What is the product of 1 and any number? What is the inverse of multiplying by 1? Do some exercises to bring this out.

Ask - What results from adding 1 to a number? How can this be used to develop the number system?

9. Ask - What happens when we multiply a number by zero? Why is this true?

Ask - What happens when we add or subtract 0? Why? Show what happens in numerous cases.

Ask - can we divide by zero? Can we divide a number into zero? There should be much discussion to fix the difference.

Exercises - 8

In the first exercise the student will notice that the answer is yes if the number involved is 1, 2, 3 but that the answer is no if it is 4 since the product of 2 and 2 is 4, where neither of them is 4. In fact, they may find without assistance that the answer is "yes" when the number is a prime number and "no" otherwise.

In exercise 5, various difficulties would appear. In the first place 0/0 would be equal to a/b since $0 \cdot a = b \cdot 0$. Another difficulty is that any integer multiplied by 0 is 0 and hence 0/0 would have an equal right to be any natural number. Here there are too many answers and for 3/0 there is no answer. Both are untenable.

ANSWERS

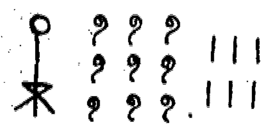
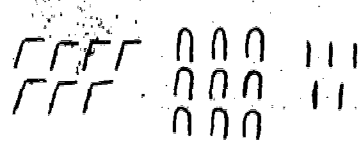
Exercises 1.

1. a, b, e, g, h, i

2. a) 1, 2, 3, 4, 5, 6
   b) 1+1, 2+1, 1+3, 0+5, 5+1, 1+6, 5+3
   c) IV, V, VI, VII, VIII, IX, X, XI

3. none; 4, 11, 15; 1; 13, 14, 22, 78, 86 (all of them).

4. a) 3 b) 6 c) 8 d) 7 e) 6 f) 9

5. a)  b) 

6. a)
   XXXXX
   XXXXX
   or
   XXX
   XXX
   XXX
   X
   b)
   XXXX
   XXXX
   XXXX
   c)
   XXXX
   XXXX
   XXXX
   XXXX
   d)
   XXXXX
   XXXXX
   XXXXX
   XXXXX
   XXXXX

7. a) 12 b) 16 c) 30

8. No, have them sit down. If chairs are needed, people will be left standing. <u>Only</u> if you knew how many chairs we had in the room and counted the people standing.

9. No, 148

10. Check grade book for students with no grade

11. a) 2 b) 3

12. XXXXX XXXXX
    OOOOO

13. No. That you don't count someone twice

14. Shoes and people, gloves and people, eyes and noses.

Exercises 2.

1.

A.

a) yes
b) yes
c) yes
d) yes
e) yes
f) no
g) yes
h) no
i) yes
j) no
k) no
l) yes
m) yes
n) no

B.

a) yes
b) no
c) yes
d) yes
e) yes
f) no
g) yes
h) =
i) < or ≠
j) > or ≠
k) > or ≠
l) =
m) =
n) =

2. 644; 110,596
155,752; $(130)_5$

7,554;
$(101100)_2$

1,413,758;
$(6109)_{12}$

3.

a) yes
b) yes
c) yes
d) yes
e) yes

a) yes
b) yes
c) yes
d) yes
e) no

4.

A

2,052; 25,620; 289,884
7,664,832; $(304)_8$

B

7,448; 952,767,671
$(14312)_5$; $(1,000,010)_2$; $(143,8et)_{12}$

5.

a) T
b) F
c) T
d) T
e) F
f) F
g) F

a) T
b) F
c) F
d) F
e) T
f) F
g) F

6. a) = a) =
   b) ≠ or > b) ≠ or >
   c) ≠ or > c) ≠ or >
   d) = d) =
   e) ≠ or < e) ≠ or <

Exercise 2 a

7. a, b, c are commutative

8. a, b, c, are commutative

9. -

10. 3V4 = 3+4+(3·4) = 7+12 = 19
    4V3 = 4+3+(4·3) = 7+12 = 19 } operation <u>is</u> commutative

Exercise 3

1.

A	B
a) (4+7)+2 = 4+(7+2)	a) 30 = 30
11+2 = 4+9	b) 80 = 80
13 = 13	c) 913 = 913
b) 17 = 17	d) 504 = 504
c) 217 = 217	e) 342,000 = 342,000
d) 270 = 270	
e) 17,280 = 17,280	

2. No - subtraction does not have the associate property.
3. No - division does not have the associative property.
   (8÷4)÷2 ≠ 8÷(4÷2)

4.

A	B
a) 6+(1+9) = 16	a) 72+(90+10) = 172
b) 5+(7+2) = 14 or (5+2)+7 = 14	b) (50+20)+36 = 106
c) 63+(75+25) = 163	or 50+(36+20) = 106
d) (26+4)+72 = 102	c) 28+(75+25) = 128
e) (340+60)+522 = 922	d) (83+17)+46 = 146
f) (45+15)+63 = 123	e) (3+7)+(5+15) = 30
g) 13+(36+4) = 53	f) (56+44)+(23+77) = 200
	g) (18+2)+(16+24) = 60

5. a) (13x10)x2 = 260 a) (2x5)x67 = 670
b) (5x2)x45 = 450 b) (25x4)x86 = 8600
c) 7x(25x4) = 700 c) 38x(50x2) = 3800
d) (50x2)x33 = 3300 d) (3x4)x11 = 132

6. yes - we use the associative and commutative principles
2x(5x6) = 6x(2x5)

7. ~~no - 2x(6x5) ≠ (2x6)x(2x5) you have introduced an additional factor of 2~~

8. yes - 2x(6+5) = (2x6)+(2x5). (Not yet; this is the distributive property of multiplication over addition.)

9. (2+3)·4 + 2 or 2+(3·4+2) other possibilities involve double parentheses.

Exercises 4.

	A	B
1. a)	33	52
b)	45	84
c)	45	30
d)	17	69
e)	38	74
f)	53	675
g)	72	46
h)	10	84
i)	105	208
j)	105	208
2. a)	48 = 48	108 = 108
b)	36 = 36	77 = 77
c)	25 = 25	132 = 132
d)	30 = 30	165 = 165
e)	119 = 119	115 = 115
f)	48 = 48	792 = 792
g)	42 = 42	3840 = 3840
h)	27 = 27	30,000 = 30,000
i)	90 = 90	112 = 112
j)	90 = 90	90 = 90

	A	B
3. a)	=	a) 6; 4
b)	+	b) 3; 2; 7
c)	·	c) 8; 8; 2+5
d)	=; ·	d) 6; 7; 4
e)	7	e) 2+3; 11; 11
f)	+ 4	f) +; 8·5; 8·6

	A	B
4. a)	(4·2)+(4·3)	a) 5(26+7)
b)	(7·4)+(7·6)	b) (8·14)+(8·17)
c)	9(8+2)	c) (7·213)+(7·787)
d)	(6·13) + (6·27)	d) 27(13+11)
e)	12(5+7)	e) 18(19+17)

	A	B
5. a)	(6+4) = (2·3) + (2·2) = 2(3+2)	a) (8+12) = (4·2) + (4·3)
b)	(12+9) = (3·4) + (3·3)	b) (14+21) = (7·2) + (7·3)
c)	(10+15) = (5·2)+(5·3)	c) (36+18) = (18·2)+(18·1)
d)	(24+18) = (6·4)+(6·3)	d) (40+16) = (8·5)+(8·2)
e)	(28+32) = (4·7)+(4·8)	e) (12+48) = (12·1)+(12·4)
f)	(21+14) = (7·3)+(7·2)	f) (56+42) = (14·4)+(14·3)
g)	(25+15) = (5·5)+(5·3)	g) (72+27) = (9·8)+(9·3)
h)	(3+6) = (3·1)+(3·2)	h) (7+63) = (7·1)+(7·9)

6. a) 78·45 = (70+8)·(40+5)
= 70(40+5)+8(40+5)
=(70·40)+(70·5)+(8·40)+(8·5)
= 2800 + 350 + 320 + 40
= 3510

a) 86·34 = (80+6)·(30+4)
= 80(30+4)+6(30+4)
=(80·30)+(80·4)+(6·30)+(6·4)
= 2400+320+180+24
= 2924

b) 13·76 = (10+3)·(70+6)
= 10(70+6) + 3(70+6)
=(10·70)+(10·6)+(3·70)+(3·6)
= 700+60+210+18
= 988

b) 53·19 = (50+3)·(10+9)
= 50(10+9)+3(10+9)
=(50·10)+(50·9)+(3·10)+(3·9)
= 500+450+30+27
= 1007

c) 567·84 = (500+60+7)·(80+4)
= 500(80+4)+60(80+4)+7(80+4)
=(500·80)+(500·4)+(60·80)+(60·4)+(7·80)+(7·4)
= 40,000 + 2000+4800+240+560+28
= 47,628

6) cont.

c) $623 \cdot 72 = (600+20+3)\cdot(70+2)$
$= 600(70+2)+20(70+2)+3(70+2)$
$= (600\cdot 70)+(600\cdot 2)+(20\cdot 70)+(20\cdot 2)+(3\cdot 70)+(3\cdot 2)$
$= 42000+1200+1400+40+210+6$
$= 44,856$

7) b, c, and d are true.

8) a, and c are true.

Exercises 5.

1) no, always an even number
no, it is not closed under addition

2) yes

3) yes

4) The sets of numbers in Ex. 1, 2, and 3 are all closed under multiplication

5) a) yes
b) no
c) no
d) yes

6) a) yes
b) no
c) no
d) yes

7) yes, due to the relationship of addition and multiplication

8) no

9) no

Exercises 6.

	A		B
1) a)	99	a)	1,438
b)	2437	b)	962
c)	113	c)	1,371
d)	1,003	d)	112,703
e)	1306	e)	110,000

	A	B
2) a)	62	111
b)	39	100
c)	89	2,002
d)	19	1,992
e)	89	19,219
3) a)	482	4,899
b)	6606	39,368
c)	16,820	24,432
d)	8,843	7,569
e)	33,222	165,821
4) a)	32	$54 + \frac{1}{13}$
b)	104	196
c)	14	508
d)	17	13
e)	13	12
5) a)	32	72
b)	21	1,000
c)	105	265
d)	27	29
e)	88	22,000
f)	1,462	50
g)	30,802	323
h)	0	5
i)	1,722	27
j)	18,226	665
6) a)	21	20
b)	84	104
c)	202	$13 + \frac{13}{14}$
d)	3	11
e)	46	11

Exercises 7

1) a) • 1 • 10 • 11 • 100 • 101 • 110 • 111 - - - - - -

b) 1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 20, 21, - - - -

c) 14

d) 8

e) 6

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Exercises 7, cont.

1) f) one dot
   g) one dot
   h) two dots; 15
   i) 16
   j) 33rd dot beyond right hand margin

2) a) 10 b) 10 c) 6 d) 23 e) 447 f) 14,692

3) 61

4) 15,643

5) a) yes b) '482' to the left of '516' c) 3 d) '33'

6) 6 people

7) 89; 29

8) no

9) no; no; if $a<b$ and $b<c$, then $a<c$ (transitive law)

10) 7; same; adding the same quantity to both sides of an inequality does not change the sense of the inequality. If $a<b$, then $a+c<b+c$

11) no; yes; b is between d and c.

Exercises 8

1) They are both 1; yes; no;

2) 0; 0

3) c must be equal to zero or $c = 0$

4) no, it is not defined, as division by zero is impossible (or not defined)

UNIT III

Sample Test Questions

PART I. TRUE - FALSE

T 1. The numbers which man invented for counting are called natural numbers.

T 2. The number of pupils in your class is a natural number.

F 3. Zero is a natural number.

T 4. $1 + (3 + 4) = (1 + 3) + 4$

T 5. Letters may be used as symbols to represent unknown numbers.

F 6. In the statement $(2 + 3) \times 6 = (2 \times 6) + (3 \times 6)$ we may interchange the "+" and the "×" and write $(2 \times 3) + 6 = (2 + 6) \times (3 + 6)$.

T 7. $(4 \times 2) \times 3 = 4 \times (2 \times 3)$.

F 8. The set of natural numbers is closed with respect to all arithmetic operations.

F 9. $(5 - 3) - 2 = 5 - (2 - 3)$.

F 10. The statement 73×25 may be written as $(70 + 5) \times (20 + 3)$.

T 11. The sum of <u>a</u> plus <u>b</u> plus <u>c</u> is equal to the sum of <u>b</u> plus <u>c</u> plus <u>a</u>.

F 12. $(20 \cdot 5) \div 4 = 20 \cdot (5 \div 4)$

F 13. If the sum of <u>a</u> and <u>n</u> is equal to <u>n</u> then <u>a</u> must be equal to <u>n</u>.

T 14. The set of numbers which are multiples of 3 is closed with respect to multiplication.

T 15. When we say the set of natural numbers is closed with respect to addition, we mean the sum of two natural numbers is always a natural number.

T 16. If c + 2d + 3 = 13 and d = 5 then c must equal zero.

F 17. Putting on shoes and putting on stockings is an example of the commutative property.

T 18. Parentheses do not change a problem when the operation is associative.

PART II. MULTIPLE CHOICE.

1. The sum of any two natural numbers...

   A. is not a natural number.

   B. is sometimes a natural number.

   *C. is always a natural number.

   D. is a natural number equal to one of the numbers being added.

   E. none of these.

2. If a and b are natural numbers then a + b = b + a is an example of

   *A. commutative property.

   B. associative property.

   C. distributive property.

   D. closure.

   E. none of these.

3. Choose the number from the following which is <u>not</u> a natural number.

   A. $\frac{2}{1}$

   B. 56

   C. 7

   D. 1

   *E. none of these.

4. "If a ticket to the movies cost 50 cents, how many could you buy for a quarter?" What is the best solution?

A. 50¢ divided by 25¢ = 2.

B. 25¢ divided by 50¢ = $\frac{1}{2}$.

C. 50¢ divided by zero = not possible.

D. Zero divided by 50¢ = infinity.

E. Zero tickets because half a ticket would have no value.

5. In which of the following numbers is zero not a placeholder?

A. 110.

B. 101.

C. 1.01

*D. 0.11

E. 10.1

6. Solve the following without doing the computation. "If the number named by (987 × 654) + (987 × 473) is divided by 987 the remainder would be..."

*A. zero.

B. 987.

C. a number between 473 and 654.

D. the sum of 654 and 473.

E. none of these.

7. The fact that a + (b + c) is exactly equal to a + (c + b) is an example of...

A. distributivity.

*B. commutativity.

C. closure.

D. associativity.

E. none of these.

8. The statement "the quotient obtained when zero is divided by a number is zero" is expressed

A. $\frac{a}{0} = 0$

*B. $\frac{0}{a} = 0$

C. $\frac{0}{0} = a$

D. $\frac{a}{a} = 0$

E. $a \times 0 = 0$

9. The sum of the natural numbers less than 5 and the even natural numbers less than 9 is

A. 12

B. 14

*C. 30

D. 46

E. none of these.

10. A simpler name for 336 × 42 is...

A. 378

B. 192

C. 42

D. 8

*E. none of these.

11. If a × b = 0 then

A. a must be zero.

B. b must be zero.

*C. either a or b must be zero.

D. neither a nor b must be zero.

E. all of the choices above are correct.

12. The product 356 × 7 is equal to...

A. (300 ÷ 50) × (6 + 7).

B. (3 × 7) + (5 × 7) + (6 × 7).

C. 300 × 50 × 6 × 7.

*D. (300 × 7) + (50 × 7) + (6 × 7).

E. none of these.

13. Four boys stand in a line. Tom stands between Dick and Harry. Harry is between Jack and Tom. Which of the following relationships is true?

A. Dick is between Tom and Jack.

*B. Tom is between Dick and Jack.

C. Jack is between Tom and Dick.

D. Harry is between Dick and Tom.

E. none of these statements is true.

14. If a, b, and c are natural numbers and b is between a and c, which of the following relations is true?

*A. b + 2 is between c + 2 and a + 2.

B. c + 2 is between a + 2 and b + 2.

C. a + 2 is between b + 2 and c + 2.

D. all of the above choices are true.

E. none of these.

15. A simpler name for 336 - 124 is

A. 336

B. 242

C. 124

D. 118

*E. none of these.

16. The operations which are associative are...

A. addition and subtraction.

*B. addition and multiplication.

C. subtraction and division.

*D. all of the above statements are correct.

E. none of the above statements are correct.

17. The inverse operation used to check multiplication is...

A. addition

B. subtraction.

C. multiplication.

*D. division.

E. none of these

18. If a V b = a(a + b) then 2 V 3 equals

A. 6

B. 7

*C. 10

D. 15

E. none of these

19. A simpler name for $25\overline{)675}$ is...

A. 25

*B. 27

C. 670

D. 700

E. none of these

20. The inverse operation for addition is...

A. addition.

*B. subtraction.

C. multiplication.

D. division.

E. none of these.

21. A simpler name for 85 + 32 is...

*A. 117

B. 85

C. 53

D. 32

E. none of these.

PART III. COMPLETION.

For problems 1 through 8 write one of the following symbols (<, =, >) in the blank space provided. Select the symbol that shows how these numbers compare. If the comparison is not known write "unknown."

1. The number of people in this room (=) the number of noses in this room
2. The number of states in the United States (>) the number of stripes in the flag.
3. The number of states in the United States (<) the number of United States senators.
4. 2^3 (<) 3^2.
5. 3 + a (<) 5 + a
6. 4 x (5 + 7) (=) (4 x 5) + (4 x 7)
7. a + b (unknown) b + c
8. (a + b) + c (=) (c + a) + b

Solve the problems from 9 through 12 and write the correct answers in the blanks.

9. 9 x (7 + 2) = (81).
10. 17 - (3 + 4) = (10).
11. (7 x 8) + (9 x 6) = (110).

12. $\frac{978 + 23}{7(5 + 8)}$ = (11).

13. If a<b and c is a natural number, then what is the relation between a x c and b x c? (a x c (<) b x c.)

14. What numbers belong both to the set of even numbers and to the set of natural numbers between 1 and 11? (2, 4, 6, 8, 10).

15. If the product of two natural numbers is 3 then one of the numbers must be (1) and the other (3).

16. How many natural numbers are there between 3 and 100? (96)

Date __________

UNIT: __________

Name of Teacher: __________

Name of School: __________

City: __________ State: __________

Number of days given to the teaching (including testing) of this unit: __________

Approximate dates: __________

USE THE BACK OF THIS SHEET IF YOU NEED EXTRA SPACE TO ANSWER ANY OF THE QUESTIONS BELOW

1. Make a statement about the ability level of the pupils in the class and state whether your school uses some plan of homogeneous grouping.

2. What parts of the unit proved to be the most teachable?

3. What parts of the unit proved to be the most difficult to teach?

   Did you omit any part?

4. Did you use any supplementary developmental materials: ______

   If so, what were they, and at what points were they used?

5. Did you find it necessary to provide the pupils with additional practice material: ______

   If so, was it from textbooks or did you write your own?

6. Do you think that a unit on this topic should be included in regular textbooks for 7th and 8th grades? ______

7. Please make ANY additional comments about your teaching experience with this unit which you think would be helpful to the Panel responsible for preparing and experimenting with textbook materials for grades 7 and 8.

UNIT III

Summary of Teachers' Comments

Reports from 76 teachers indicated that the time used on this unit varied from 5 to 30 days and that the pupils varied in ability. There seemed to be equal numbers of high, medium, and heterogeneous classes involved.

Tabulated results showing the topics regarded as easiest or most difficult to teach follow:

Topic	Easiest to Teach	Most difficult to Teach
Closure	2	43
Commutative and Associative	34	5
Distributive	0	32
Inverse	15	1
Betweenness	16	8
1	12	0
0	10	7
Division by zero	0	5
Inequalities	2	0
All	27	0
Counting	8	0
None	0	23

Two teachers reported that they had not taught all the topics. There is general agreement among the teachers who reported that Unit III is "---the most interesting and useful unit studied thus far"; "---one of the best units that I have ever taught". Most reports reflect the feeling that the closer examination of the familiar counting numbers and the equally familiar operations which they admit is not only important, but interesting to the student. Equally important is the rather gentle introduction to the use of letter symbols for numbers which it affords. A large number of classes omitted no topics and did not see the need for supplementary materials except in sections mentioned below. Most felt that this unit is desirable for the seventh grade; a small number would rather have it in the eighth grade.

More specific comments which occurred in significant numbers of reports follow:

1. An overwhelming majority of teachers reported that the section which introduces the concept of closure was the most difficult to teach. There was general feeling that classroom discussion and drill should be expanded beyond that indicated in the unit. In connection with this a little time might well be devoted to a discussion of the intuitive idea of "sets".

2. A sizable number of teachers indicated that students had difficulty with the distributive property. This section was frequently mentioned as one in which extra drill exercise might well be assigned.

3. While comparatively few teachers reported that the section on the number zero was especially difficult, a large number of them indicated that such things as division by zero and the conventional exclusion of zero from the set of counting numbers (even though zero "may be used to count" in a special sense) required additional emphasis and explanation.

Additional Materials Used

- Practice on principle
- Information on closure
- Cardboard squares and circles for grouping
- Number scale
- Simple interest problems
- Drill on inverse operations
- Formulas from geometry
- Abacus binomials
- Lateral binomials
- Trapezoid formula
- Brief proofs in algebra
- Operations Sets and Patterns

UNIT IV

FACTORING AND PRIMES

Vocabulary

common factor

common property

composite number

counting chart

even number

factor (verb)

greatest common factor (g.c.f.)

Goldbach conjecture

least common multiple (l.c.m.)

odd number

prime number

seive of Eratosthenes

unique factorization property of natural numbers

References

Banks, Elements of Mathematics, pp. 55 - 63

Courant and Robbins, What Is Mathematics?, pp. 25 - 32

Freund, A Modern Introduction to Mathematics, pp. 51 - 57

From The World of Mathematics

Bell, pp. 503 - 505

Teacher's Guide — Factoring and Primes

Understandings	Skills and Abilities
1. In a product of 2 (or more) numbers each of those numbers is a factor of the product.	1. To identify factors of a product.
2. Process of finding factors is called "factoring."	2. To identify factors of a natural number
3. Every natural number has the factor 1. Every number is a product of itself and 1.	
4. A natural number is called "even" if it has the factor 2. Otherwise it is called "odd".	4. To recognize quickly odd and even numbers by inspecting their decimal numerals.

Activities

1. Ask - What is 3 · 4? Write answer "12 = 3 · 4."

   Ask - What other numbers can be multiplied to get 12?

   Write all possibilities using two factors on board.

   Explain - In a product each of the numbers is called a "factor".

2. Repeat above process for some other natural numbers, as 24, 30, 36, 42, 48, 64.

   Explain - This process (of finding factors) is called "factoring."

   Point out that factors of some natural numbers may be found in several different ways.

   Do some class exercises in factoring.

3. Ask - Does 3 have the factor one? Does 7? 9? 11? etc.

   Bring out (by introduction on part of students) that every natural number has the factor one.

   Ask - If n is any natural number, what is n · 1?

   Bring out that the product of any natural number and one is that natural number.

   Use again the terminology: "n · 1 = n" becomes true whenever a numeral for any natural number is put in for n.

   Bring out also that this holds for any numbers -- not just natural numbers.

   Bring out that every natural number has itself as a factor, as well as one.

4. Ask - Think of some numbers that have the factor 2. Write on board: Do we know a name for the set of numbers that have the factor 2? Bring out that this set of natural numbers is called the set of even numbers. An even number is one which has the factor 2.

A product is even if one (or more) of its factors is even.	Skill in using induction and informal deduction.

Write numerals for some large natural numbers, some odd, some even.

Ask - Can you tell me which of these are even?

Bring out that those with even unit-digit are even and the others are odd.

Ask - Suppose we multiply a natural number by an even number. What do we get? Try a few and write on board (have students reach conclusions inductively) that such products are always even.

Ask - Can we be sure that this is always true? We tried it for only a few cases. Even if we tried a million cases that would still be only a few -- there would be so many left. Can we reason this out <u>for all</u> natural numbers?

Have students give their own arguments (They <u>must</u> be encouraged to say what <u>they</u> think, no matter how wild it may seem.) If no good argument is produced, bring out the associative law and suggest they try using it: Any even number has the factor 2, and so is $2 \cdot a$, where a represents any natural number. Then $(2 \cdot a) \cdot b = 2 \cdot (a \cdot b)$, for any natural number b. We see that this has the factor 2.

(Other arguments by students may be equally valid. This informal deduction should be encouraged by using any valid arguments proposed by students. It must be insisted that they argue in such a way that what they say is good <u>for all</u> natural numbers, and not just a finite number.)

The sum of two even numbers is even.

The sum of an odd and an even number is odd.

The sum of two odd numbers is even.

Point out that the above also shows that the set of even numbers is closed for multiplication.

Ask - Is 10 even? 10^2? 10^3? 10^n? Have students give their own arguments, but eventually point out that they can use the general principle just proved. (n is any natural number.)

Ask - Is $3 \cdot 10^5$ even? $5 \cdot 10^n$? Why? Again have students give their own arguments, but encourage them to use the two general principles just proved.

Ask - Suppose we add two even numbers. What do we get? Have students try a few and write on board. Get them to state the general principle they guess (induction).

Ask - How can we tell if this is true for all cases? Have students give their own arguments. If no good one is produced suggest that the distributive property can be used:

$$(2 \cdot a) + (2 \cdot b) = 2 \cdot (a + b)$$

Point out that the set of even numbers is closed for addition.

Ask - Suppose we add an odd number and an even number? Again have students do induction, by trying cases, guessing the general principle. Then insist that they produce informally deductive arguments to show that their guessed principle is indeed true. One possible argument: An odd number is an even number + 1. So odd + even is even + 1 + even. We know that even + even gives even, so odd + even is even + 1, which is odd.

Ask - The similar thing for odd plus odd, again using first induction, then informal deduction.

5. A natural number is called "prime" if it has exactly 2 different factors. (itself and one)

A natural number is called "composite" if it has more than 2 different factors (if it has factors other than itself and 1).

There are 3 kinds of natural numbers classified according to factorability: Primes, composites, and the number one.

One is not a prime. The only even prime is 2.

Point out that the set of odd numbers is not closed for addition.

Do some exercises with odds and evens, using non-decimal numeration. (It is possible that a bright student may use this as a basis for deductive arguments earlier.)

Ask - Now we said before that a natural number whose decimal numeral ends in an even digit is even. How can we tell if this is true for all natural numbers? Have students give their own arguments, as before. If no good argument is produced, have them look at some large number, such as 35787. Write it $3 \cdot 10^4 + 5 \cdot 10^3 + 7 \cdot 10^2 + 8 \cdot 10 + 7$. By previously proved principles, powers of ten are even. Even times anything is even, so the first four products are even. Sums of evens are even, so the sum of the first four products is even, so the number is even or odd, depending on whether the last digit is even or odd.

Exercises - 1

5. Ask - Think of some natural numbers that have exactly two different factors. Write on board.

Explain that these are called "prime".

Ask - How many of these are even? Have students substantiate statements deductively.

Explain that numbers with more than two different factors are called "composites".

Ask - If we put the set of all prime numbers with the set of all composite numbers, do we have the set of all natural numbers?

Bring out that the number one is not in either set.

6. Illustrative examples of a general principle never establish the truth of that principle (unless every single instance has been examined). If a generality has been guessed, a single instance in which it fails means that the generality is false.

6. Additional skill in induction and informal deduction.

7. The <u>unique factorization property</u> for natural numbers: Every composite number can be factored as a product of primes in only one way, except for order of the factors.

7. To factor a composite number as a product of primes.

Exercises — 2

6. Ask - What is the sum of the two primes 3 and 5? Ask for sums of several pairs of primes (always making sure not to include 2 as one of the primes) and have the students write the sums. Ask them what they notice about the numbers which are sums of two primes, and hope for the conclusion that they are all even (or at least composite). Then have them try to give a deductive argument.

Finally show an example like 2 + 3. Point out that the guessed principle was false, and that it takes only one example to show it false.

Explain that no matter how many examples we have (unless we have all possible examples) we do not have a proof.

Exercises - 3

When assignment on Goldbach's conjecture is handed in, ask if anybody has proved it. Explain that if anyone has, he will be famous by tomorrow, since mathematicians have tried for years to prove this but cannot.

7. Write "36 = 6 · 6" and ask how else 36 can be factored. Have students write the various factorizations.

Point out that there are composites for some of the factors, and show how to complete the factorization. 36 = 2 · 2 · 3 · 3 (One could be included, but is not necessary. Besides we just want the prime factors.) Have students complete the factorizations for the examples on their papers.

Repeat for several other numbers and encourage students to tell what they notice.

Bring out that the final factorizations are alike except for order (induction). Then

8. Given two (or more) numbers, any number which is a factor of each of them is called a "common factor" of them.

1 is a common factor of any set of numbers.

8. To identify factors common to several numbers.

For any set of natural numbers there is always a greatest common factor.

To find the g.c.f. of several numbers.

To simplify a fraction.

ask for a deductive argument that this is true for all natural numbers.

Exercises - 4

Explain - This is called the "unique factorization property of natural numbers" -- that every composite natural number can be factored into primes in only one way, except for order.

Point out that the commutative law makes it necessary to say "except for order."

8. Explain that if something is "common" to several things then (one meaning) it means that they are similar in some way. If two numbers have a certain number as a factor, we say the factor is common to both.

Do some exercises in seeing whether sets of numbers have a common factor, e.g. 12, 18, have 1, 2, 3 and 6 in common.

Bring out that 1 is a common factor of all members of any set of natural numbers.

Go back to exercises above and have students pick out the largest common factor. Explain this is called the "greatest common factor" and that for any set of numbers there is always a g. c. f.

Do some more exercises in finding common factors, using larger sets of numbers and more complicated examples, until difficulty arises, and then point out that we can use the unique factorization property: factor each number in the set, and pick out the g. c. f.

Explain - To find the simplest name of a fraction we can use this idea of g. c. f. Show some examples.

9. If one number is a factor of another than the second is called a "multiple" of the first.

9. To identify multiples of a number.

Even numbers are multiples of 2.

The product of two odd numbers is odd.

To recognize quickly multiples of 2, 3, 5, 9, and 10.

$$\frac{21}{42} = \frac{1 \cdot 3 \cdot 7}{2 \cdot 3 \cdot 7} = \frac{1}{2} \cdot \frac{3 \cdot 7}{3 \cdot 7} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Explain that we factor numerator and denominator and pick out the g.c.f. Then we separate the factors, the part which is g.c.f./g.c.f is 1, so we can leave it off.

(Do not let students "cancel" or make any marks reminiscent of that process. Do not let them divide numerator and denominator by the same thing. The essence of this procedure is the removal of a unit factor.)

Exercises - 5

9. Explain - If 3 is a factor of some natural number then that number is called a "multiple" of 3. Ask class to name several multiples of 3. Repeat for other numbers, such as 5, 6, 7, 9.

Ask - An even number is always a multiple of what?

Using binary numeration, how can we tell odds and evens?

Bring out that odds end in 1 -- evens in zero.

Ask - What is the product of two odd numbers? Bring out, using binary notation, that it must be odd, since any two odd numbers end in 1, so the last digit of the product will be 1.

Point out that the set of odd numbers is closed for multiplication.

Ask - In base seven numeration, if the units digit is zero, the number is a multiple of what?

Exercises - 6

10. Given two (or more) numbers, any number which is a multiple of each of them is called a "common multiple" of them.	10. To identify common multiples.
The smallest multiple common to two (or more) numbers is called their "least common multiple" (l.c.m.)	To find the least common multiple of several numbers.
	To apply the principle of l.c.m. to addition and subtraction of fractions.

Next day, after assignment is done, establish and emphasize that if the sum of digits is 9, then the number is a multiple of 9. Ask for deductive argument. A simple one is: In any number like 643, or 600 + 40 + 3, taking out all 9's from 600 leaves 6, all 9's from 40 leaves 4, so what is left is 6 + 4 + 3, the sum of the digits. Therefore if the sum of the digits is a multiple of 9, so is the number itself.

10. Ask - Do you recall what we mean by the word "common"?

What is the common multiple of two numbers?

Bring out that it means a multiple of both, and that each of the numbers is a factor of the common multiple.

Do some exercises in finding common multiples. Bring out that the product of two numbers is always a common multiple, but not the smallest.

Explain - Least common multiple of several numbers is the smallest number which is a multiple of all of them. Do some exercises in finding l.c.m. (least common multiple), in which it is brought out that for large numbers we can use the unique factorization property.

Exercises - 7

Point out that when we add fractions we find it convenient (but not necessary) to write their names with same denominators:

$$\frac{2}{3} + \frac{5}{9} = \frac{2}{3} \cdot 1 + \frac{5}{9} = \frac{2}{3} \cdot \frac{3}{3} + \frac{5}{9} = \frac{6}{9} + \frac{5}{9} = \frac{11}{9} \text{ (or } 1 + 2/9)$$

Explain that we multiply fractions by 1 in such a way as to make the denominators the least common multiple (do not say "least common denominator" here). It is not necessary to have the least common multiple, but if we have it, the denominators will be as small as possible. (Note also that 11/9 is just as acceptable as 1 + 2/9. Note also that 1 + 2/9 will always be written with the plus sign.)

Exercises - 8

ANSWERS TO EXERCISES

Unit--Factoring and Primes

Exercises 1. (Note that other answers are acceptable, as in 1) a).

		A		B
1)	a)	2×2×2 (Also: 2×4 or 1×8).	a)	2×2×2×2
	b)	2×13	b)	2×2×7
	c)	2×2×2×5	c)	2×2×2×2×3×3
	d)	2×3×3×3	d)	2×2×5×13
	e)	2×2×2×7	e)	2×2×5×5
	f)	2×2×2×3×3	f)	7×13
	g)	1×7	g)	1×13
2)	a)	3	a)	8
	b)	5	b)	$\frac{5}{6}$
	c)	$\frac{2}{3}$	c)	3
	d)	57 per cent	d)	72
	e)	7	e)	240
	f)	1	f)	$\frac{50}{51}$

3) a) The set of natural numbers between 1 and 30 inclusive.

b) The set of all even natural numbers from 2 to 30 inclusive

c) The set of all odd natural numbers from 1 to 29 inclusive

d) [4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30]

1×4	1×6	1×9	1×10	1×12	
2×2	2×3	3×3	2×5	2×6	etc.
				4×3	

e) [2, 3, 5, 7, 11, 13, 17, 19, 23, 29]

Exercises 2.

A

1. 56; 64; 102; 2568; 31,766; 570,000

B

1. 4683; 36,763; $(210)_3$; 500;

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Exercises 2. cont.

A		B	
a)	even	a)	even
b)	even	b)	odd
c)	even	c)	even
d)	even	d)	even
e)	odd	e)	odd
f)	odd	f)	odd
g)	even	g)	even
h)	even	h)	even
i)	even	i)	even
j)	even	j)	even
k)	odd	k)	even
l)	even	l)	even

3.

base 2		base 3	base 4	base 5
1		1	1	1
(10)	even numbers	2	2	2
11	end in zero	10	3	3
(100)	odd numbers	11	10	4
101	end in one	12	11	10
(110)		20	12	11
111		21	13	12
(1000)		22	20	13
1001		100	21	14
(1010)		101	22	20

Even numbers most easily recognized in base two. Any larger base which is an even number such as base 4, 6, 8 etc. would allow us to choose the even numbers by whether or not the last digit is zero or even.

Exercises 3.

1. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

2. g) Two can be factored in only one way and thus fits the definition of a prime number.
Yes, it's easier to exclude the even numbers (except 2) from the beginning.

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Exercises 3. cont.

2. h) No -
When you reach 11.

3. a) 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199
211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293

b) 17 c) 62 d) 25, 21, 16

e) group of 1 - 50
group 151-200 and 251-300

f) 3+5, 5+7, 11+13, 17+19, 29+31, 41+43, etc.

Exercises 4.

1. a) 50 b) 34 c) 246 d) 440 e) 432 f) 256 g) yes h) yes

2. They are all the sum of two prime numbers.
   a) -
   b) No
   c) No

Exercises 5.

1.

	A		B
a)	2×3	a)	2×5
b)	3×3	b)	5×5
c)	2×2×3	c)	2×3×3
d)	2×3×5	d)	2×7×11
e)	5×7	e)	3×3×5
f)	Prime	f)	Prime
g)	2×2×5×5	g)	2×2×5×5
h)	2×37	h)	3×3×5×7
i)	3×5×7	i)	3×7×11
j)	2×3×7	j)	2×2×3×3×3
k)	Prime	k)	7×13
l)	3×5×23	l)	2×2×2×2×2×2×2
m)	2×2×3×5×5	m)	3×3×3×3×3×3
n)	2×2×2×3×3	n)	2×2×2×5×5×5
o)	2×2×2×2×2×2	o)	2×2×2×2×2×3×5×11

Exercises 5. cont.

2.

A

b) 3^2

c) $2^2 \times 3$

g) $2^2 \times 5^2$

m) $2^2 \times 3 \times 5^2$

n) $2^3 \times 3^2$

o) 2^6

B

b) 5^2

c) 2×3^2

e) $3^2 \times 5$

g) $2^2 \times 5^2$

h) $3^2 \times 5 \times 7$

i) $3 \times 7 \times 11$

j) $2^2 \times 3^3$

l) 2^7

m) 3^6

n) $2^3 \times 5^3$

o) $2^5 \times 3 \times 5 \times 11$

3.

A

a) 6 = 2x3 = 1x6

b) 8 = 4x2 = 1x8

c) 24 = 8x3 = 6x4 = 12x2 = 24x1

d) 100 = 2x50 = 4x25 = 10x10 = 1x100 = 5x20

e) 150 = 2x75 = 3x50 = 5x30 = 6x25 = 10x15 = 1x150

B

a) 10 = 1x10 = 2x5

b) 16 = 1x16 = 2x8 = 4x4

c) 72 = 1x72 = 2x36 = 3x24 = 4x18 = 6x12 = 8x9

d) 81 = 1x81 = 3x27 = 9x9

e) 216 = 1x216 = 2x108 = 4x54 = 8x27 = 9x24 = 3x72 = 6x36 = 12x18

4.

30 = 2 x (3x5)
30 = (2x3) x 5

2x3x13 = 78
7x11 = 77
2x3x5x7 = 210
2x127 = 254

As can be seen above 78 has more unique factors than 77 as is the

Exercises 5. cont.

4. case in 210 and 254. Thus we can form a greater number of combinations.

5\. **A**

a) 2x5x7x11 = 770
10x77 = 770 5x154 = 770
35x22 = 770 7x110 = 770
55x14 = 770 11x70 = 770
2x385 = 770

b) 1) 2x3x7 = 42
7x[illegible] 42
2x21 = 42
3x14 = 42

2) 2x3x11 = 66
6x11 = 66
2x33 = 66
3x22 = 66

3) 2x3x13 = 78
6x13 = 78
2x39 = 78
3x26 = 78

4) 2x2x3 = 12
2x6 = 12
4x3 = 12

5) 2x3x3 = 18
2x9 = 18
3x6 = 18

6) 2x2x2x2x3 = 48
2 x 24 = 48
3 x 16 = 48
4 x 12 = 48
8 x 6 = 48

7) 7 x 7 = 49

8) 3x5x5 = 75
3x25 = 75
5x15 = 75

9) 2x2x2x2x2x2 = 64
2x32 = 64

B

a) Two ways (using 1 as a factor).
For example:
6 = 2x3
= 1x6

b) Four ways
For example:
30 = 1x30
= 2x15
= 3x10
= 5x6

9) 4x16 = 64
8x8 = 64

Exercises 5. cont.

A

c) Because one of the factors of 50 is repeated, resulting in a duplication of some of the possible pairs.

B

55	5x11	2	5x11, 1x55	2
130	done			
770	2x5x7x11	4	1x770, 2x385, 5x154, 7x110, 11x70, 10x77, 14x55, 22x35	
2310	2x3x5x7x11	5	1x2310, 2x1155, 3x770, 5x462, 7x330, 11x210, 6x385, 10x231, 14x165, 15x154, 21x110, 22x105, 33x70, 35x66, 55x42, 77x30	16
28,014	2x3x7x23 x29	5	1x28,014, 2x14,007 3x9, 338, 7x4,002, 23x1,218, 29x966 6x4,669, 14x2001, 46x609, 58x483, 21x1334, 69x406 87x322, 161x174 203x138, 667x42	16

c) Yes, if there are n unique prime factors each one used only once, then there are 2^{n-1} pairs of factors.

6. 16x7
8x14
4x28
2x56

6.
a) 2x500
4x250
5x200
8x125
40x25
20x50

b) 40 × 25

Excercise 6

I. a) 7 b) 3 c) 21 d) 6 e) 12 f) 13 g) 14

Exercise 6. cont.

II. a) $\frac{5}{7}$ b) $\frac{5}{7}$ c) $\frac{2}{3}$ d) $\frac{4}{5}$ e) $\frac{2}{11}$ f) $\frac{5}{13}$ g) $\frac{11}{5}$ or 2+1/5

h) $\frac{6}{7}$ i) $\frac{3}{7}$ j) $\frac{5}{11}$ k) $\frac{10}{49}$ l) $\frac{15}{26}$ m) $\frac{7}{2}$ or $3+\frac{1}{2}$

III. a) $\frac{2}{7}$ b) $\frac{7}{10}$ c) $\frac{7}{8}$ d) $\frac{50}{83}$ e) $\frac{4}{7}$ f) $\frac{3}{4}$ g) $\frac{25}{41}$

h) $\frac{112}{211}$ i) $\frac{11}{12}$ j) $\frac{2}{3}$ k) $\frac{7}{8}$ l) $\frac{3}{7}$

Exercise 7.

1) 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96
2) 14, 28, 42, 56, 70, 84, 98
3) 252, 261, 270, 279, 288, 297
4) 322, 345
5) 2, 4, 6, 8
6) 2, 4, 6, 8
7) 5, 10, 15
8) 1, 3, 7, 9, 11, 13, 17, 19
9) none
10) none
11) no
12) 10, 20, 30, 40, 50, 60, 70, -----------
13) 6, 10, 16, 20, 26, -----------
14) Any numeral which ends in zero.
15) nine gallons
16) three packages
17) 17; 25; 34; 34; 34; 34.

Exercise 8.

I.

A	B
a) 2: 2, 4, 6, 8, 10, ...98	a) 2: 2, 4, 6, 8, ...98
3: 3, 6, 9, 12 99	4: 4, 8, 12, 16 96
4: 4, 8, 12, 16 96	8: 8, 16, 24, 32 ...96

Exercise 8. cont.

I.

	A		B
b)	3: 3, 6, 9, 12 99	b)	6: 6, 12, 18, 24....96
	6: 6, 12, 18, 24 96		7: 7, 14, 21, 28....98
	9: 9, 18, 27, 36 99		8: 8, 16, 24, 32....96
c)	7: 7, 14, 21, 28 98	c)	11: 11, 22, 33, 44...99
	8: 8, 16, 24, 32 96		5: 5, 10, 15, 20 ...95
	9: 9, 18, 27, 36 99		
d)	13: 13, 26, 39, 52 91	d)	14: 14, 28, 42 98
	3: 3, 6, 9, 12, 15 99		12: 12, 24, 36, 48...96

II.

	A		B
a)	12, 24, 36, 48, 60, 72, 84, 96	a)	8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96.
b)	18, 36, 54, 72, 90	b)	No common multiple less than 100
c)	No common multiple less than 100	c)	55
d)	39, 78	d)	84

III.

	A		B
a)	12	a)	8
b)	18	b)	168
c)	504	c)	55
d)	39	d)	84

IV.

a)	210	a)	315
b)	660	b)	336
c)	210	c)	3315
d)	80	d)	200
e)	36	e)	156
f)	210	f)	450

V.

a)	24, 48, 72	a)	80
b)	24	b)	160
c)	48	c)	240
d)	96	d)	560
e)	120, + 144	e)	They are all multiples of the least common multiple
		f)	There is no greatest common multiple

Exercise 8. cont.

VI.

	A	B
	a) $1 + \frac{4}{9}$	a) $\frac{7}{16}$
	b) $\frac{87}{112}$	b) $\frac{19}{72}$
	c) $\frac{11}{40}$	c) $2 + \frac{13}{60}$
	d) $\frac{53}{80}$	d) $\frac{249}{280}$
	e) $\frac{15}{16}$	e) $\frac{13}{30}$
	f) $1 + \frac{59}{60}$	f) $\frac{25}{39}$
	g) $1 + \frac{13}{48}$	g) $\frac{1673}{3300}$
	h) 0	h) $\frac{102}{175}$
	i) $\frac{71}{80}$	i) $\frac{317}{330}$
	j) $\frac{53}{150}$	j) $1 + \frac{1}{96}$
	k) $\frac{13}{150}$	k) $\frac{2}{5}$

Unit IV

Sample Test Questions

1. Write each number as a product of prime factors:

   (a) 16 Ans. $2 \cdot 2 \cdot 2 \cdot 2$

   (b) 100 Ans. $2^2 \cdot 5^2$

2. Circle all numerals which are names of prime numbers:

   3   5 + 1   9 − 2   XIII $\frac{7}{2}$ IIII III

   $\frac{4}{2}$ 1 $1 \cdot 2$ 0   57 $(12)_5$

3. Find the greatest common factor of these numbers:

   (a) 15 and 25 Ans. 5

   (b) 18 and 27 Ans. 9

   (c) 60, 36, and 24 Ans. 12

4. Find the least common multiple of these numbers:

   (a) 6 and 8 Ans. 24

   (b) 7 and 9 Ans. 63

   (c) 16, 12, and 20 Ans. 240

5. Name the numbers in the set of natural numbers between 16 and 25 which have the factor 3. Ans. 18, 21, 24

6. Find the smallest number which has a factorization composed of three composites. Ans. 64

7. The greatest common factor of 48 and 60 is:

   (a) $2 \cdot 3$

   *(b) $2 \cdot 2 \cdot 3$

   (c) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$

   (d) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

   (e) None of the above

8. Which of the following is an odd number?

*(a) $(24501)_6$

(b) $(60654)_8$

(c) $(te\ 104)_{12}$

(d) $(110100)_2$

(e) None of the above

9. Every natural number has at least the following factors.

(a) Zero and one

(b) Zero and itself

*(c) One and itself

(d) Itself and two

(e) None of the above

10. In the complete factorization of a composite number

*(a) All the factors are prime.

(b) All the factors are composite.

(c) All the factors are composite except for the factor 1.

(d) All the factors are prime except for the factor 1.

(e) None of these

11. How many different prime factors does the number 72 have?

(a) 0

(b) 1

*(c) 2

(d) 2

(e) None of the above

12. The least common multiple of 8, 12, and 20 is:

(a) 2 . 2

(b) 2 . 3 . 5

*(c) 2 . 2 . 2 . 3 . 5

(d) 2 . 2 . 2 . 2 . 2 . 2 . 2 . 3 . 5

(e) None of the above

13. Which of the following is an even number?

(a) $(100)_3$

(b) $(100)_5$

(c) $(100)_7$

(d) $(100)_{11}$

*(e) None of the above

14. Which of the following numbers is odd?

(a) 17 . 18

(b) 18 . 11

(c) 11 . 20

*(d) 99 . 77

(e) None of the above

15. Which of the following is not a prime number?

(a) 271

(b) 277

(c) 281

*(d) 282

(e) 288

16. Which of the following is a list of all the factors of 12?

(a) 1, 2, 3, 4, 8, and 12

*(b) 1, 2, 3, 4, 6, and 12

(c) 1, 2, 3, 4, and 6

(d) 2, 3, 4, 6, and 12

(e) None of the above

17. How many factorizations, of two factors each, does 175 have?

(a) 2

*(b) 3

(c) 4

(d) 5

(e) None of the above

18. How many factors are there in the complete factorization of 182?

(a) 2

*(b) 3

(c) 4

(d) 5

(e) None of the above

19. Let a represent an odd number and b represent an even number. Then a + b must represent

(a) An even number

(b) A prime number

*(c) An odd number

(d) A composite number

(e) None of the above

20. Suppose p and q are natural numbers and q is a factor of p. Then

(a) q is a multiple of p

*(b) p is a multiple of q

(c) q must be a prime number

(d) The g.c.f. of p and q must be less than q

(e) None of the above

21. If N represents an odd number, the next larger odd number can be represented by:

(a) N + 1

*(b) $N + 2$

(c) $N + N$

(d) $2 \cdot N + 1$

(e) None of the above

True - False

T 22. Every composite number can be factored into primes in exactly one way, except for order.

T 23. Some odd numbers are not prime.

F 24. Every composite number has only two prime factors.

F 25. Some multiples of prime numbers are prime.

F 26. Goldbach proved that every even number greater than 2 is the sum of two primes.

F 27. The number 1 is prime.

F 28. All odd numbers have the factor 3.

F 29. No even number is prime.

T 30. The difference between any two different prime numbers greater than 100 is always an even number.

T 31. Every multiple of 18 is also a multiple of 6.

F 32. Every factor of 18 is also a factor of 6.

Date ____________

UNIT: ____________

Name of Teacher: ____________

Name of School: ____________

City: ____________ State: ____________

Number of days given to the teaching (including testing) of this unit: ____________

Approximate dates: ____________

USE THE BACK OF THIS SHEET IF YOU NEED EXTRA SPACE TO ANSWER ANY OF THE QUESTIONS BELOW

1. Make a statement about the ability level of the pupils in the class and state whether your school uses some plan of homogeneous grouping.

2. What parts of the unit proved to be the most teachable?

3. What parts of the unit proved to be the most difficult to teach?

   Did you omit any part? ________

4. Did you use any supplementary developmental materials? ______

   If so, what were they, and at what points were they used?

5. Did you find it necessary to provide the pupils with additional material? ________

   If so, was it from textbooks or did you write your own?

6. Do you think that a unit on this topic should be included in regular textbooks for 7th and 8th grades? ________

7. Please make ANY additional comments about your teaching experience with this unit which you think would be helpful to the Panel responsible for preparing and experimenting with textbook materials for grades 7 and 8.

UNIT IV

Summary of Teachers' Comments

There were fifty-seven teacher reports from centers which included classes of pupils of all levels of ability. The number of teaching days spent on the unit ranged from four to twenty-eight with a median of twelve days.

Two questions on the report form were: (1) What topics were the easiest to teach? And (2) What topics were the most difficult to teach? The topics listed by the teachers with the tabulated results follow:

Topic	Easy	Difficult
Primes	20	2
Factorization	24	1
G. C. F.	8	16
L. C. M.	9	23
Sieve of Eratosthenes	6	4
Deductive Proof	0	4
Vocabulary	0	2
Odds and Evens (other bases)	1	1
Composites	2	0
Multiplies	4	0
The entire unit	10	0

Four reports did not give an answer to either question.

In general, the reports indicated that this unit was teachable, easily understood and rather popular with the pupils. The unit was considered as a good motivating vehicle. Many questions were raised by the pupils. The pupils realized that the use of their imagination was important. Some pupils became interested in history and read biographies of famous mathematicians. Many enjoyed understanding processes previously done without understanding.

It was felt that this unit helped develop number consciousness, and also was an excellent preparation for algebra. The indications were that this unit definitely should be taught in Grade 7. The type of proofs used was new to the pupils and required much work in order to overcome the beginning difficulties. The vocabulary presented difficulties to the average pupil and the slow learner. The sections on G. C. F. and the L. C. M. caused the most trouble, with the L. C. M. taking the lead. It was felt that the L. C. M. followed too closely on the G. C. F. Some teachers felt that there should be more problems directed toward the talented group of pupils.

Listed below are direct quotations from some of the reports:

We enjoyed this unit and considered it to be another approach to our former knowledge of fractions. In adding and subtracting fractions we now use L. C. M. information and in multiplication and division the new knowledge of G. C. F. The value of these newer techniques cannot be stressed too strongly for their great insight into similar algebraic expressions of Grade 9. The pupils enjoyed using other bases and deciding on odd and evens. I was happy to see that they could recall the other bases quite efficiently.

I noticed that this unit developed a sort of a "number consciousness" that the pupils did not have before. They seemed to look at numbers a little differently and to associate them with such ideas as odd, even, prime, composite, even + even, even + one, odd + one with or without the factor two.

One of the most difficult parts of the teaching at first was to help pupils develop skill in using informal deduction to show that what they had suspected (by induction) was true. Often we had done some of this in class and I had illustrated some of the deductive arguments they could use, the pupils improved and would try to use some of the background they had built in studying Unit III.

The pupils' reaction to Unit IV was negative in the beginning. It was difficult to stimulate their interest and make them realize the value of this material. As we progressed, however, individuals began to see that what they had previously done mechanically, now had meaning which they understood.

The teacher who made the above statement gave some comments from the pupils as follows:

(a) We are learning this too late.
(b) It's a longer, but surer way to work.
(c) I was very glad I had studied Unit IV before I took the Iowa Reading Test.
(d) This kind of study gives us a fine background, so that we can talk more intelligently about numbers.

Some pupils brought in supplementary materials and all were extremely interested in the work of the unit. We had reports based on an article from the June, 1958

Fortune and an article from the December, 1958 Scientific American. In retrospect, I believe that I should have given more work involving computations with fractions, applying the concepts of G. C. F. and L. C. M. There was some tendency to confuse the two ideas.

The section of this unit which produced the most interesting discussion was the process of reasoning inductively - then deductively about the even-odd relationships. Some of the deductive proofs presented by the pupils were not conclusive, but the germ of the method was present. Several pupils were so interested I gave them some "number theory" problems to prove.

SUPPLEMENTARY TESTS FOR DIVISIBILITY AND REPEATING DECIMALS

1. Introduction. This unit is for unusually good students. One of the purposes of this monograph is to encourage the student to be self-reliant. The teacher should only give him as much help as is necessary to keep him going and refuse help when it becomes apparent that the student wants the teacher to save him mental effort. The teacher might even want to make application of these principles himself and see how much he can do without the help of this commentary.

2. Here the student should see the advantage of multiplying the numbers "algebraically" using the distributive property and commutative property. Probably one of the hardest things to see in this section is that if 9b + r is divided by 9, the remainder is the same as when r is divided by 9. Fundamentally the explanation is this: Every number can be divided by 9 yielding a quotient, q, and a remainder r. Thus if N is the number, it can be written 9q + r where r is the remainder, that is, is zero or a natural number less than 9. Conversely, if a number is written in the form 9q + r where r is a natural number less than 9, then r is the remainder when the number 9q + r is divided by 9. Thus what the expansion of 156,782 in powers of ten shows, is that it can be written in the form 9q + (1+5+6+7+8+2). Then the sum in parentheses can be written in the form 9t + 2, where 2 is the remainder after division by 9. Hence 156,782 is equal to 9(q+t) + 2 which means that 2 must be the remainder when the sum of the digits is divided by 9. This is also dealt with in the students' material. The student can probably see this better in terms of numbers, although it may be easier for the teacher to see in letters. While the student should eventually get to the use of letters, the important thing is for

him to see what is happening. This can probably best be done by working out a number of numerical problems along the lines given in the text.

Thus, in exercise 2, the student might write $69 + 79 = 7 \times 9 + 6 + 8 \times 9 + 7 = 7 \times 9 + 8 \times 9 + 6 + 7 = (7 + 8) \times 9 + 6 + 7$. He should be required to name what properties of numbers (commutative, associative and distributive properties) he is using. This is a good opportunity to impress these properties on him, but it should not be done to the point of boredom. When he gets to the point that it is clear that he knows what is going on, he should not be required to name the property each time. For a sum of three he could show from first principles or, by combining terms, reduce it to summing two things twice just as $a+b+c = (a+b)+c$.

In exercise 3, the student would of course write 69 as $9 \times 7 + 6$ and 79 as $11 \times 7 + 2$. In exercise 4, one would proceed as above except that one would multiply instead of add. The same results would hold for 23 since one could write 79 as $3 \times 23 + 10$ and 69 as $3 \times 23 + 0$. They would hold equally well for any number.

The general principle should be formulated as soon as the student can do so himself. The remainder when the sum (or product) of any two numbers is divided by a given number, is the same as when the sum (or product) of the remainders is divided by a given number.

In exercise 5 the answer is 1^{20}, which is 1 whether the divisor is 9 or 3. If the divisor is 99, we can write 10^{20} as 100^{10} and see that then the remainder will be 1^{10} which is also 1. The bright student might want to know what would happen if the divisor were 999. Then one would write 10^{20} as $1000^{6} \times 100$ and so the remainder would be 1×100 which is 100. By a similar argument the remainder when 7^{20} is divided by 6 is 1.

The student might want to carry this farther and consider the remainder when 10^{20} is divided by 7. This could be worked out as follows: 10^{20} could be replaced by 3^{20} which is equal to 9^{10}. That remainder would be the same as that for 2^{10} which is $(2^3)^3 \times 2$ or $8^3 \times 2$ which has the remainder $1^3 \times 2$ or 2. A student bright enough to be interested in developing this should have no trouble with manipulating the exponents. He might want to explore what would be the last digit in the huge number 3^{40}. This would be just the remainder when 3^{40} is divided by 10. Since 3^4 has a remainder of 1 when divided by 10, the answer would be 1^{10} or 1. This exercise might also be done by looking at the pattern of last digits in the powers of 3: 3, 9, 27, 81. The last digits form the pattern 3, 9, 7, 1, 3, 9, 7, 1 and so forth. These things could be explored still further.

For exercise 7 the simplest test for divisibility by 4 is to test whether the number consisting of the last two digits is divisible by 4. For instance, 178524 is divisible by 4 since 24 is. This is because 178524 is equal to 178500 + 24 and any multiple of 100 is divisible by 4. Similarly to test for divisibility by 8 one uses the last three digits. Another test for divisibility by 4 would be to see that if the last digit is 0, 4 or 8 the number is divisible by 4 if the next to the last digit is even. If the last digit is 2 or 6, the number is divisible by 4 if the next to the last digit is odd. The reasons should of course be found.

For exercise 8, the number written to the base twelve would be a multiple of twelve plus the last digit. Thus the number would be divisible by any divisor of 12 if the last digit is divisible by this divisor. That is, if the number is divisible by 6, its last digit must be divisible by 6. Other numbers in place of 6 would be 2, 3, 4, 12.

If the base were 7, since 7 has no factors but itself and 1, the only number that could correctly go in the blank would be 7.

Exercise 9 is rather fundamental but requires some insight on the part of the student. Just as a decimal terminates when the divisor is a divisor of a power of 10, so a "decimal" to be base 7, will terminate when the divisor is a divisor of a power of 7. That is, 1/7 and 1/49 will have terminating "decimals" to the base 7. The decimal expansion of a fraction depends on the number base. But whether or not it is rational is independent of the base. A number that is a prime number when expressed in the decimal notation is a prime number when expressed to any other base, since the property of being prime is a property of the number and not of the way in which it happens to be written. You may not want to tackle this point at this stage with your students, but you perhaps should be prepared to meet it if it occurs.

The answer to number 10 would be: the remainder when 2 + 2 + 2 + 2 + 2, or 10, is divided by 7. Hence the remainder is 3. Number 11 is easy and the answers to numbers 12 and 13 are developed in the text after the exercises.

In what follows it is probably wise to deal with the discussion with letters since here the advantage is quite clear and the manipulation is not very complex.

Probably the students will be interested in doing quite a little casting out of the nines in numerical examples. There is an amusing little book by E. T. Bell on "Numerology" which the students might like to read.

In the next set of exercises, the justification of the casting out of nines is that the remainder when the sum (or product) of two numbers is divided by 9 is the same as the remainder when the sum of the digits is divided by 9, as well as the properties described in exercises 2 and 4 of the previous sets. For instance, suppose a product ab is to be checked. The remainder when ab is divided by 9 is the same as the remainder when the product of their remainders is divided by 9 which is the same as the remainder when the product of the sums of their digits is divided by 9. So whichever way one does it, he is computing the remainder after division by 9 -- it does not matter anywhere whether a number or the sum of its digits, is divided by 9 as far as the remainder goes. The check of course only checks the remainder after division by 9. The digits for instance could be scrambled without altering the value of their sum and the answer could be completely wrong. But of course this is not very likely to happen.

For exercise 4, the remainders would be 3 for division by 7, and, since the remainder when 7 is divided by 6 or 3 is 1, the remainder when the number is divided by 6 or 3 would be the same as when the sum: 5+3+2+1+4+3 is divided by 6 or 3. The sum is equal to 18 which is divisible by both 6 and 3. Hence the given number is divisible by 6 and 3, that is, has a remainder 0 when divided by 6 or 3.

The short-cuts asked for in exercise 5 are described in the text that follows. As exercise 4 shows, one would cast out threes, or sixes or twos in the number system to the base 7, since 6, 2 and 3 are divisors of 7 - 1 just as 3 and 9 are divisors of 10 - 1.

For exercise 7, it may be seen that scrambling the digits does not alter the value of their sum nor the remainder when the numbers are divided by 9. Hence each of the numbers is of the form 9n + r and 9t + r, with the remainders the same. Then if we subtract one from the other, we get 9n + r - 9t - r which is equal to 9(n - t), that is a multiple of 9. Thus the sum of its digits is a multiple of 9. When the sum of the digits given is a multiple of 9, one cannot be sure whether the missing digit is 9 or 0. Otherwise the trick can be worked by adding the sum given and seeing what number added to this sum will give a multiple of 9. For instance in the example given, one must add 4 to 14 to get a multiple of 9 and hence 4 is the missing digit.

3. Divisibility by 11. The exercises are solved in the text which follows them. You may have a little trouble with the product of -1 and -1 and you may want to avoid this; this can be done in the example worked out by seeing that 10^3 is $10^2 \times 10$ which would have a remainder of $1 \times (-1)$ or -1. If you wish to avoid negative numbers completely, you could confine yourself to the first test. Or you could show the test for two-digit numbers by noticing that if the second digit is larger than the first one, the remainder after division by 11 is the second digit less the first one. (e.g. $79 = 77 + 2$) If the second digit were smaller than the first one, one could add 11 to it and subtract the first digit, giving a correct remainder (e.g. for 73, subtract 7 from 3 + 11 and see that the remainder is 7). This is somewhat laborious, however.

It is important in testing for the remainder after division by 11 by the second method to start at the right hand end of the number. If you are merely testing for divisibility it does not matter at which end you start.

For exercise 2, the test for divisibility by 8 would be analogous to that for 11 in the decimal system.

For exercise 3, the number 157,892 is equal to 157(1000) + 892 or $157 \times 999 + (157 + 892)$. Hence if d is any divisor of 999, the remainder when 157,892 is divided by d is the same as when 157 + 892 is divided by d. The prime divisors of 999 are 3, 37. In fact $999 = 3^3 \times 37$.

This test would work for any divisor of 999, that is, for 3, 9, 27, 37, 3x37, 9x37 as well as, of course, for 999 itself.

The students might also be interested in writing 157,892 as equal to 157(1001) + 892 - 157. Then if d is any divisor of 1001, the remainder when 157,892 is divided by d is the same as when 892 - 157 is divided by d. The prime divisors of 1001 are 7, 11 and 13. This would give a test for divisibility by 7 and 13 as well as 11. This has some connection with the fact that the decimal equivalents of 1/7 and 1/13 have six digits in the repeating portion. This can be developed further.

For exercise 4, consider a number $534{,}623_{seven}$. Then in the numeral system to base seven, 534,623 is equal to $534_{seven} \times 666_{seven} + 534_{seven} + 623_{seven}$. Hence we would have to find the divisors of 666_{seven}. Now $666_{seven} = 6_{seven} \times 111_{seven}$. To see if 111_{seven} has factors it is probably easiest to convert to the base ten. So $111_{seven} = 49 + 7 + 1 = 57$ which has 3 and 19 as its factors. Actually we could have tested 111_{seven} for divisibility by 3 by adding the digits since this works for 3 in the system to the base seven as well as to the base ten. So in the number system to the base seven, this kind of test would work for 19, that is 25_{seven} as well as for 2, 3, 6 which are written the same way in both systems.

Similarly for the number system to the base twelve, we would seek the divisors of eee_{twelve} which is $e_{twelve} \times 111_{twelve}$ and e stands for "eleven." We cannot say that 111_{twelve} is divisible by 3 since 3 is not a divisor of one less than the base. But we have to convert it to the decimal notation. It is 144 + 12 + 1 = 157, which is a prime number. Hence a number system to the base twelve, this kind of test would work for 3 and 157 in the decimal system, that is 3_{twelve} and 111_{twelve}.

The connection between repeating decimals and tests for divisibility should begin to emerge for the student here. Suppose m is some number which is divisible by neither 2 nor 5. Then some power of 10 is 1 more than a multiple of m. This can be seen as follows. We know that the remainders when the powers of 10 are divided by m are the numbers from 1 to m-1 inclusive. Since there are infinitely many powers of 10, two of the powers must have the same remainder, that is $10^a - 10^b$ will be divisible by m for some natural numbers a and b. Suppose b is smaller than a, then the difference may be written

$$10^b(10^{a-b} - 1).$$

Since m divides this product and has no factors except 1 in common with 10^b it must divide $10^{a-b} - 1$, which is what we wanted to show. Thus we have shown that some power of 10 has a remainder of 1 when divided by m. Call k the smallest such power greater than zero.

From this we can conclude two things. First, the remainders when we compute the decimal expansion of 1/m will be just the remainders when the powers of 10 are divided by m. As soon as we get a remainder 1, the decimal begins to repeat and not before. So our number k is the number of digits in the repeating portion of the decimal.

Second, we may write any number in the form

$$a + b \times 10^k + c \times 10^{2k} + d \times 10^{3k} + \ldots.$$

Since the remainder when 10^k is divided by m is 1, the remainder when our number is divided by m is the same as the remainder when

$$a + b + c + d + \ldots$$

is divided by m. We must notice, of course, that a, b, c, d are not digits in general but they are natural numbers less than 10^k.

How far the student can progress here, remains to be seen. Certainly he should not be pushed.

For casting out the elevens, instead of adding the digits, we would take the units digit, minus the tens digit, plus the hundreds digit and so forth. An example would be this:

$$\begin{array}{r} 637 \\ \times\ \underline{152} \end{array} \qquad \begin{array}{l} 7 - 3 + 6 = 10 \\ 2 - 5 + 1 = \underline{-2} \end{array}$$

product: 96,824   product: -20

$$4 - 2 + 8 - 6 + 9 = 13 \qquad 0 + 2 = 2$$
$$3 - 1 = 2$$

We come out with 2 in each case. This test is a little less likely to check when the answer is wrong but it is much more difficult to apply.

For the corresponding trick mentioned in exercise 7, one cannot scramble the digits indiscriminately. The simplest direction to give would be to reverse the order of the digits and add the two numbers if the number of digits is even but subtract the two numbers when the number of digits is odd. This can be seen as follows: Suppose there are three digits a, b, c. Then the remainder for the given number would be that for a - b + c and for its reverse c - b + a. Then the difference would have zero remainder. If the number has four digits a, b, c, d, the sum for the given number would be - a + b - c + d and for its reversal would be - d + c - b + a. Then the sum would have a zero remainder. Then it would not be sufficient to know the sum of all but one of the digits in the answer. You would have to ask for all the digits but the last, or something like that. This trick is much more complicated. Its only advantage is that it does not have the case of failure that that for nine has.

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Answers to Exercises

pages 3 - 5

1. (a) No answer required. Examples follow:

Sum of digits 7247 is 20 Not divisible by 9.

$\frac{20}{9}$ = 2 remainder 2

Sum of digits 427986 is 36 Divisible by 9.

$\frac{36}{9}$ = 4 remainder 0

Sum of digits 9499794 is 51 Not divisible by 9.

$\frac{51}{9}$ = 5 remainder 6

Sum of digits 187434 is 27 Divisible by 9.

$\frac{27}{9}$ = 3 remainder 0

(b) No answer required. Example follows:

$\frac{7247}{9}$ = 805 remainder 2

Sum of digits is 20; $\frac{20}{9}$ = 2 remainder 2

(c) The remainder when a number is divided by 9 is the same as the remainder when the sum of the digits of that number is divided by 9.

(d) No answer required. Examples follow:

$\frac{407}{9}$ = 45 remainder 2; $\frac{11}{9}$ = 1 remainder 2

$\frac{877}{9}$ = 97 remainder 4; $\frac{22}{9}$ = 2 remainder 4

2. (a) The uniqueness property of addition. (refer to page 2 of the commentary and to page 5 of the students' text)

(b) Yes

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3. (a) Yes

(b) Same reason as number 2

4. (a) Yes

(b) Yes

(c) Yes

(d) Yes

5. (a) $\frac{(9+1)^{20}}{9}$ has a remainder of 1^{20} or 1

(b) $\frac{(3 \times 3 + 1)^{20}}{3}$ has a remainder of 1^{20} or 1

(c) $10^{20} = 100^{10} = (99 + 1)^{10}$ has a remainder of 1^{10} or 1

6. $\frac{(6+1)^{20}}{6}$ has a remainder of 1

7. (a) divisibility by 4: If the number formed by the last two digits is divisible by 4, then the number is divisible by 4.

(b) divisibility by 8: If the number formed by the last three digits is divisible by 8, then the number is divisible by 8.

(c) divisibility by 25: If the number formed by the last two digits is divisible by 25 (e.g., 00, 25, 50, 75), then the number is divisible by 25.

8. 2, 3, 4, 6, 12 (0 is divisible by 12)

9. (a) multiples of powers of 7. (This includes negative powers of 7, e.g., 13/49 in the decimal system is .16 in the system to the base 7.)

(b) $(.12541\ldots)_7$

10. remainder 3

11. (a) $\frac{(9+1)^{20} - 1}{9}$ $1 - 1 = 0$ (refer to exercise 5)

(b) $\frac{(6+1)^{108} - 1}{6}$ $1 - 1 = 0$ (as above)

12. Cast 9's from the sum as you go along. (see page 7 in students' text)

page 9

1.
```
   927   sum of digits: 18   sum of digits:  9   Sum of digits: 9
   865   sum of digits: 19   sum of digits: 10   Sum of digits: 1
  4635                                           Product:       9
 5562
7416
801855
```

Sum of digits: 27
Sum of digits: 9

2. Answered in the text

3. 810855

4. 3, 0, 0

5. Answered in the text

6. Casting out the sixes.

7.
```
7543
5437
2106      sum of digits = 9
```

$2 + 1 + 0 = 3$, other digit is 6,
or $1 + 0 + 6 = 7$, other digit is 2,
or $2 + 1 + 6 = 9$, other digit is 0 or 9.

page 12

1. (1) a. $758 = 7 \times 10^2 + 58$ which has the remainder $7 + 3 = 10$ when divided by 11 since 100 has the remainder 1.

b. $758 = 7 \times 10^2 + 5 \times 10 + 8$ which has the remainder $8 - 5 + 7 = 10$

(2) a. $7246 = 72 \times 10^2 + 46$ which has the same remainder as has $72 + 46$, that is, $6 + 2$ or 8.

b. $7246 = 7 \times 10^3 + 2 \times 10^2 + 4 \times 10 + 6$ which has the same remainder as has $6 - 4 + 2 - 7 = -3$, that is, 8.

(3) a. $81675 = 8 \times 10^4 + 16 \times 10^2 + 75$ which has the same remainder as has $8 + 5 + 9 = 22$. Hence the remainder is 0.

b. $81675 = 8 \times 10^4 + 1 \times 10^3 + 6 \times 10^2 + 7 \times 10 + 5$ which has the same remainder as has $5 - 7 + 6 - 1 + 8 = 11$. Hence the remainder is zero.

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2. Since 8 is 1 more than 7, a number to the base 7 can be tested for divisibility by 8 in the same way that we can test for divisibility by 11 in the decimal system. For instance, consider $(5326)_7$.

Using the first method we have $(5326)_7 = (53)_7 \times (10^2)_7 + (26)_7$. Since 8 is $(11)_7$, the remainder when $(10^2)_7$ is divided by 8 is 1. Thus the remainder when the given number is divided by 8 is the same as when $(53)_7 + (26)_7$ is divided by $(11)_7$. But $(53)_7 - (44)_7 = 6$ and $(26)_7 - (22)_7 = 4$ and hence the remainder is the same as when $6 + 4$ is divided by 8, that is, 2.

Using the second method we have $(5326) = 5 \times (10^3)_7 + 3 \times (10^2)_7 + 2 \times (10)_7 + 6$. When this is divided by 8 the remainder is the same as that for $6 - 2 + 3 - 5 = 2$.

3. (see page 6 of the teachers' commentary).

3, 9, 27, 37, 3 x 37, 9 x 37, 999.

4. (See page 6 of the teachers' commentary).

In the numeral system to the base seven, we can test for division by grouping in triples all the divisors of $7^3 - 1 = 342$. These divisors are:

2, 3, 6, 9, 18, 19, 38, 57, 114, 171, 342.

In the number system to the base twelve, we would have the divisors of $12^3 - 1 = 1727$. These factors are 11, 157, 1727.

5. (a) Yes. The remainders when the powers of 10 are divided by 11 are -1, 1, -1, 1, ... which has a period of 2 since 11 is a divisor of $10^2 - 1$.

(b) The divisors 3 and 9 listed in the answers to number 3 are divisors of $10^1 - 1$ as well as of $10^3 - 1$. All the others have three digits in the decimal equivalent of their reciprocals and for these, grouping the digits in threes gives a divisibility test.

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See also the teachers' commentary.

page 14

	3	7	9	11	13	17	19	21	37	101	41
1	1	1	1	1	1	1	1	1	1	1	1
10^1	1	3	1	10	10	10	10	10	10	10	10
10^2	1	2	1	1	9	15	5	16	26	100	18
10^3	1	6	1	10	12	14	12	13	1	91	16
10^4	1	4	1	1	3	4	6	4	10	1	37
10^5	1	5	1	10	4	6	3	19	26	10	1
10^6	1	1	1	1	1	9	11	1	1	100	10
10^7	1	3	1	10	10	5	15	10	10	91	18
10^8	1	2	1	1	9	16	17	16	26	1	16
10^9	1	6	1	10	12	7	18	13	1	10	37
10^{10}	1	4	1	1	3	2	9	4	10	100	1
10^{11}	1	5	1	10	4	3	14	19	26	91	10
10^{12}	1	1	1	1	1	13	7	1	1	1	18
10^{13}	1	3	1	10	10	11	13	10	10	10	16
10^{14}	1	2	1	1	9	8	16	16	26	100	37
10^{15}	1	6	1	10	12	12	8	13	1	91	1
10^{16}	1	4	1	1	3	1	4	4	10	1	10

The number of digits in the cycle in the repeating decimal is the same as the number of digits in the cycle of remainders. Since, for example, there are five remainders in the column headed by 41, one can test for divisibility by 41 by grouping the digits in fives.

For the questions on page 15 look on page 8 in the teachers' manual.

NON-NEGATIVE RATIONAL NUMBERS

The treatment in this Unit is somewhat more sophisticated than the treatment in the Unit on Natural Numbers and Zero. It is assumed that the student has some skill with fractions, and in converting fractions to decimals. The Exercises in many instances are very challenging. Some teachers may wish to omit sections 11 and 12 or to teach these sections to the more gifted pupils only. Probably this unit is more appropriate for Grade 8 than Grade 7.

Other previous knowledge assumed is: the natural numbers and zero, called the whole numbers, and some familiarity with the commutative, associative and distributive properties of these numbers. The notation for 6/2 when the denominator is a factor of the numerator as well as simple fractions like $6/4 = 1\frac{1}{2}$, are assumed. The fact that 1 is the identity element for multiplication, that zero is the identity element for addition and that zero times any number is zero, as well as the property that if the product of two whole numbers is zero, one or both must be zero, is assumed though the student may not be too sure of some of these properties. The inequality relationships of whole numbers are not assumed, though if they were known some condensation would be possible.

Much of this chapter can be taught experimentally and the definitions of equality, sum and product should be on the basis of experimental work on the part of the student.

Section 3. Here it should be made clear that we <u>define</u> the product of two rational numbers as we do because of our idea of what they should be. The students themselves should get to the point where they see that to get the product of two rational numbers, they compute the fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators of the given fractions.

In exercise 1 the explanation might be that 7/12 is dividing one pie into 12 parts and taking seven of them. But the idea is that the number which 12 is multiplied by to get 7 is more important.

Exercises 3 and 4 are leading up to the next section. Exercise 5 should bring out the idea that the natural numbers are included in the set of rational numbers. For exercise 9, one can use intuitive ideas like dividing pies into parts. This is probably sufficient for most students. But it is better eventually to do it like this: Suppose a/b = c/b. Then multiply both sides of the equation by b and get a = c. (We are here using the property that "equals multiplied by equals are equals".)

Section 4. Here the students should gradually develop the idea of equality of two fractions. The "cross-multiplying" is inherent in the definition of equality of two fractions and is not in itself bad if it comes after due experience, and on the basis of formulation of the students. Most of the students should probably test for equality by the long ways many times before they get to the point of divising a short-cut. The short-cut should not be devised for them.

Exercises. In exercise 1, the two ways referred to are reducing to lowest terms and converting in other ways to two fractions with the same denominator. Of course cross multiplying is another way when the students are ready for it. In exercise 2, it is interesting that doing it this way yields the same definition of equality. We have ac/bc = ca/da and hence bc = ad. The purpose of exercise 8 is to point out the difference between the distributive property and the associative property. That is, to divide 2+6 or 6+12 by 3, one must divide each number by 3, getting 2/3+2 or 2+4 respectively; but to divide 2x6 or 6x12 by 3 one can divide either factor of the product by 3 but not both. In exercise 9, the statement is true for all the rational numbers we have so far and hence the answer is "yes". However if we include also the rational number zero which appears later, the reciprocal of zero does not exist and, since the reciprocal of a number which does not exist cannot itself exist, there is no reciprocal of the reciprocal of zero.

5. Division by zero. In this section and the exercises, the student should explore the difference between 3/0 and 0/3.

Probably the easiest way to distinguish is to perform the indicated multiplication: 3/0 is that number which gives 3 when you multiply it by 0, but every number multiplied by 0 is 0 and cannot be 3. Hence there is no such number. On the other hand 0 times 3 is 0 and hence 0/3 should be the same as 0. From this point on, 0 should be included among the rational numbers of the chapter. There are quite different difficulties with 0/0. One respect in which it would cause trouble is in the definition of equality. By the definition, 0/0 would be equal to all other fractions. Also 0/0 could be any number since zero multiplied by any number is zero.

Section 6. The phrase "invert and multiply" may appear here after due experience. A little better description would be that dividing by a rational number gives the same result as multiplying by its reciprocal. But again this should grow out of experience.

One way to approach the answer to exercise 1 would be this: How many halves are there in 3 - answer 6. It would be possible by this means to show that $a/(1/c) = ac$ but it would be more difficult to deal with the general case along these lines. Exercise 3 shows that the answer to exercise 4 is "no".

Section 7. Probably many more exercises should be given here. This would be a good place to review greatest common factor and least common multiple. A few literal fractions might be brought in with caution. Exercise 7 is an illustration of the distributive property. Though it is mentioned in the summary in the next section, it does not seem necessary to stress it here.

Section 8. It is probably true that at no point in the student's career should he memorize the properties listed in this section. But as he uses them more and more he should become familiar with them, not to satisfy the teacher, but to assist him in spots where he is not just sure whether what he wants to do is "allowed" or not.

Section 9. Here we run into a little difficulty because we have been calling "rational numbers" a subset of the complete set of rational numbers. There was no special harm up to this point since the statements made were all true of the larger set. But when we have inequalities we need to make a distinction between positive and negative rational numbers.

Exercise 2. This could of course be solved using the results of the next exercise but one is not supposed to do this. It could be done intuitively: if you divide a certain number of objects into a smaller number of parts, each part is larger. It can also be done by converting the fractions to those with the same denominators: $a/b = ad/bd$ and $a/d = ab/bd$. Thus if $a/b > a/d$, $ad > ab$ and hence $d > b$. Exercise 5 can be done this way.

In exercise 3 it should be apparent with a little cogitation that 5/6, 8/9 and 18/19 are in increasing order since they differ from 1 by smaller amounts. The others will have to be put in order in other ways.

Exercise 4 is an example of what is called Farey Series and is sometimes designated as F_7. The answer is

0, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 2/5, 3/7, 1/2, 4/7, 3/5,
2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 1

References can be found to this in many books on theory of numbers.* Two of the most interesting properties are these:

I. If a/b and c/d are any two successive fractions, bc - ad = 1. For example, if the fractions are 3/7 and 1/2, then (7 x 1) - (3 x 2) = 1.

II. If a/b, c/d, e/f are three successive fractions, then c/d = (a + e)/(b + f). For instance, consider 4/5, 5/6, 6/7. Then 5/6 = (4 + 6)/(5 + 7).

Similar results would occur with 7 replaced by any other natural number.

Exercise 8 should not be too difficult numerically but it is harder algebraically - in fact probably beyond the capabilities of most of the students. Solutions follow:

a. $(r + s)/2 = r/2 + s/2$. Now $r = r/2 + r/2$ and if $s > r$, then $s/2 > r/2$ which shows that $r < (r + s)/2$. Similarly one would show the other inequality.

*For example see: Hardy, "Introduction to theory of numbers", 2nd ed., p.23 ff., Clarendon Press, Oxford, 1945.

b. The inequality is the same as $1/s < (r + s)/2rs < 1/r$. We can multiply through by rs (this would have to be established or assumed) and this becomes the inequality of part a.

c. This is just the reciprocal of part b.

d. Here it is convenient to use exercise 2, and see that $a/b < (a+c)/(b+d)$ if and only if $a(b+d) < b(a+c)$ if and only if $ad < bc$.

Section 10. In this section there is much room for exploration on the part of the student. It should soon develop that .333333... is not exact for 1/3 no matter how far the decimal is carried unless one puts a 1/3 at the end. Some decimals in the list have a string of zeroes, which can be omitted and others have a repeating pattern. In exercise 6, the students should come to a clear statement that a decimal terminates if and only if the rational number can be expressed as the quotient of two natural numbers in which the denominator is a power of 10. When this stage is reached they are ready to go on to the next exercises.

Exercise 7. The students should have as much experience as is necessary along the lines of this exercise so that they will formulate for themselves the conclusion that is stated below. The same can be said about exercise 7. Throughout this section the students should be discouraged from reading ahead. The zest of discovery should be theirs.

Section 11. Exercise 1: In .33 1/3 the 1/3 is of course one third of .01, in .333 1/3 it is one third of .001, etc.

Section 12. Exercise 4: The number of digits in the repeating part of the decimal equivalent of 1/p where p is a prime number, is always a divisor of p - 1. A proof can be found in books on the theory of numbers. The idea of the proof is this: First, one shows that the remainder when 10^{p-1} is divided by p, is 1. That shows that the decimal has the remainder 1 after p-1 divisions and hence repeats beginning with the p-th division. If the repeating portion is smaller than p-1, it would repeat again and again until it comes to the p-th division and since it has to come out exactly the number of digits in the repeating portion must be a divisor of p-1. The first statement made is the hardest to prove. It is done this way: the remainders when 10, 2•10, 3•10, •••, (p-1)10 are divided by p are all different and there are p-1 of them. Hence they are in some order 1, 2, 3, 4, 5, •••, (p-1). Hence $10 \cdot 2 \cdot 10 \cdot 3 \cdot 10 \cdots (p-1)10 - 2 \cdot 3 \cdot 4 \cdots (p-1)$ is divisible by p. Since p does not divide the product $2 \cdot 3 \cdot 4 \cdots (p-1)$ we can divide by the product and see that $10^{p-1} - 1$ is divisible by p.

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- Answers -

Excercises A.

1) a) 1, 2, 3, 4 ......

b) 0, 1, 2, 3, 4, ...

c) 0, 1, 1/2, 4, $\frac{13}{3}$, (Note these are non-negative numbers, Non-negative integers would be the same as b))

d) 7/8, 2, 5, 7 1/3

2) a) 8 = 2×4 b) 21 = 7×3 c) 150 = 15×10 d) 29 = $5 \times 5\frac{4}{5}$

3) 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144
The above natural numbers are all factors of 144, (including 144 if we think of 144 as factorable into 1×144)

4) 7, 14, 21, 28, 35, 42, 49, 56, 63, 70

5) 13, 26, 39, 52, 65

6) 1/8 of 3 or 3 1/8's

7) a) 4/5 is the number such that if it is multiplied by 5, 4 is obtained.
b) 7/3 " " " " " " " " " " 3, 7 " "
c) 1/8 " " " " " " " " " " 8, 1 " "
d) 6/11 " " " " " " " " " " 11, 6 " "
e) $\frac{100}{9}$ " " " " " " " " " " 9, 100 "

8) a) 1, 2, 3, 5, 6, 10, 15, 17, 30, 34, 51, 85, 102, 170, 255, 510
b) all the numbers above
c) all the numbers above

9) 6, 10, 15, 34, and 51

10) d is a multiple of a, and b, and c
a is a factor of d, b is a factor of d, c is a factor of d
d is a multiple of a×b, a×c or b×c
Similarly a×b, a×c, b×c, are all factors of d
a × b × c is a factor of d, d is a multiple of a × b × c

Exercises B.

1) 7/12: means 7 × 1/12 or 1/12 × 7. In words, seven twelfths or one-twelfth of seven.

5/3 : means 5 × 1/3 or 1/3 × 5 or five thirds or one-third of five

10/6 : means 10 × 1/6 or 1/6 × 10 or ten sixths or one-sixth of ten

14/24: means 14 × 1/24 or 1/24 × 14 or fourteen twentyfourths or one twentyfourth of fourteen

Excercises B. cont.

2) 10; 7; if b = 2 $\frac{3b}{b} \times b = \frac{6}{2} \times 2 = 6$

if b = 7 $\frac{3b}{b} \times b = \frac{21}{7} \times 7 = 21$

Thus $\frac{3b}{b} \times b = 3b$

3) $\frac{3}{5}$; $\frac{7}{10}$; $\frac{11}{8}$

4) $\frac{18}{30}$; $\frac{140}{200}$; $\frac{77}{56}$

5) yes - it is of the form $\frac{a}{b}$ where a and b are natural numbers

6) a) $(\frac{1}{2}) \times (\frac{3}{5}) = 1 \times (\frac{1}{2}) \times 3 \times (\frac{1}{5})$ def. of rational number and the associate property

$= 1 \times 3 \times (\frac{1}{2}) \times (\frac{1}{5})$ commutative property

$= 3 \times \frac{1}{10} = \frac{3}{10}$ using value of product of two rational numbers which are in each numerator

b) $(\frac{2}{3}) \times (\frac{3}{4}) = 2 \times (\frac{1}{3}) \times 3 \times (\frac{1}{4})$ same reason as above

$= 2 \times 3 \times (\frac{1}{3}) \times (\frac{1}{4})$

$= 6 \times \frac{1}{12} = \frac{6}{12}$

c) $(\frac{5}{6}) \times (\frac{8}{9}) = 5 \times (\frac{1}{6}) \times 8 \times (\frac{1}{9})$ same reason as above

$= 5 \times 8 \times (\frac{1}{6}) \times (\frac{1}{9})$

$= 40 \times \frac{1}{54} = \frac{40}{54}$

d) $(\frac{1}{4})\times(\frac{2}{3})\times(\frac{7}{8}) = 1 \times (\frac{1}{4})\times 2\times(\frac{1}{3})\times 7\times(\frac{1}{8})$ same reason as above

$= 1 \times 2 \times 7 \times (\frac{1}{4}) \times (\frac{1}{3}) \times (\frac{1}{8})$

$= 14 \times (\frac{1}{96}) = \frac{14}{96}$

7) yes

8) a) $\frac{18}{11}$ b) $\frac{8}{9}$ c) $\frac{1}{60}$ d) $\frac{14}{24}$ e) $\frac{35}{35} = 1$ f) $\frac{216}{96}$

9) Their numerators are equal also. The denominators would have to be equal.

10) To find the product of two rational numbers, find the product of the numerators and the product of the denominators.

$\frac{3}{6 \times 12} \neq \frac{1}{(2 \times 4)}$ These fractions are unequal - The division was distributed which is incorrect as 6 and 12 are factors, not addends.

9) This is true if $a \neq 0$, $b \neq 0$ - reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, reciprocal of $\frac{b}{a}$ is $\frac{a}{b}$.

Exercises D.

1) yes, $\frac{0}{3} = 0$ as $\frac{0}{3}$ would have to be defined so that $\frac{0}{3} \times 3 = 0$ and the only way you can arrive at a product of zero is for at least one of the factors to be zero. As $3 \neq 0$, then $\frac{0}{3} = 0$

2) If $\frac{4}{0}$ is 4, then by definition $\frac{4}{0} \times 0 = 4$ so that $4 \times 0 = 4$ which is contradictory as any number multiplied by zero is equal to zero.

3) a) 2 b) 3 c) 10 d) 100 e) 1000 f) 1,000,000

4) No, for to give it any meaning would mean we could divide by zero

5) No, for to find a value for $\frac{0}{0}$, we would have to find a number so that $0 = 0 \cdot$ <u>some number</u> - We see here that <u>any</u> number would work, this means $\frac{0}{0}$ has no definite value and is therefore meaningless.

Exercises E.

1) $\frac{3}{\frac{1}{2}} = \frac{3}{1} \times \frac{2}{1} = 6$ $\qquad \frac{\frac{3}{1} \times 2}{\frac{1}{2} \times 2} = \frac{6}{1} = 6$

2) a) $\frac{2}{3}$ b) $\frac{27}{14}$ c) $\frac{9}{7}$ d) $\frac{5}{12}$ e) 4 f) $\frac{25}{11}$

3) a) $\frac{7}{9}$ b) $\frac{4}{7}$

4) No

5) $\frac{ad}{bc}$

6) The quotient of two rational numbers is equal to the product of the (dividend / numerator) and the reciprocal of the (divisor / denominator)

Exercises F.

1) a) $\frac{10}{8}$ or $\frac{5}{4}$ b) $\frac{9}{5}$ c) $\frac{17}{16}$ d) $\frac{59}{40}$ e) $\frac{219}{168}$ or $\frac{73}{56}$ f) $\frac{57}{40}$ g) $\frac{25}{24}$

2) a) $\frac{15}{8}$ b) 15

3) No, there are various ways of explaining this. One might be the numerator tells "how many" (numerates) while the denominator tells us "what kind or size" (denomination)

4) $\frac{30}{91}$

5) First change the rational numbers so they have equal denominators, then subtract one numerator from the other, and place over the common denominator.

6) a) $\frac{5}{8}$ b) can't do c) $\frac{1}{8}$ d) can't do e) $\frac{1}{16}$

7) $\frac{111}{32}$

8) a) $\frac{7}{8}$ b) $\frac{b}{c}$ c) 0 d) 0

9) $\frac{ad+bc}{bd}$

10) We use it in finding like denominators (or, if preferred, a common denominator) so we can change the fractions, if necessary, to fractions of equal value, having this common denominator.

11) Yes, as it is of the form $\frac{a}{b}$ where a is any whole number and b is the natural number one.

12) If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers such that $\frac{a}{b} \times \frac{c}{d} = 0$, then either $\frac{a}{b} = 0$, $\frac{c}{d} = 0$, or both.

Proof: 1) By definition, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

2) ac and bd are whole numbers - whole numbers are closed under multiplication

3) If $\frac{ac}{bd} = 0$, then by definition of division, $ac = 0 \cdot bd$,

4) Thus $ac = 0$

5) Thus $a = 0$ or $c = 0$ or both, from the known property for whole numbers.

Exercises G.

1) 1) closure 2) identity element for add. and mult. 3) Associativity for add. and mult. 4) Commutativity for add. and mult. 5) Distributive property. 6) No. 7) product = 0, one or both are zero. 8) zero multiplied by any whole number is zero.

Exercises H.

1) $\frac{7}{10} < \frac{9}{10}$; $\frac{7}{10} > \frac{1}{10}$; $\frac{7}{10} > \frac{1}{2}$; $\frac{7}{10} > \frac{3}{5}$; $\frac{7}{10} < \frac{3}{4}$; $\frac{7}{10} > \frac{3}{7}$

$\frac{7}{10} > 0$; $\frac{7}{10} = \frac{21}{30}$; $\frac{7}{10} < \frac{7}{5}$; $\frac{7}{10} < \frac{7}{6}$; $\frac{7}{10} < \frac{7}{8}$; $\frac{7}{10} < \frac{7}{9}$; $\frac{7}{10} > \frac{7}{11}$; $\frac{7}{10} > \frac{7}{12}$

2). Suppose we have two rational numbers $\frac{a}{b}$ and $\frac{a}{c}$ where $\frac{a}{b} < \frac{a}{c}$.

Changing to common denominator we have $\frac{ac}{bc}$ and $\frac{ab}{bc}$

now - $\frac{ac}{bc} = \frac{a}{b}$ and $\frac{ab}{bc} = \frac{a}{c}$

As $\frac{a}{b} < \frac{a}{c}$, $ac < ab$, thus $c < b$

3) 3/4, $\frac{5}{6}$, $\frac{8}{9}$, $\frac{25}{27}$, $\frac{18}{19}$

4) $\frac{1}{7}$, $\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{2}{7}$, ($\frac{1}{3}$ or $\frac{2}{6}$), $\frac{2}{5}$, $\frac{3}{7}$, ($\frac{3}{6}$ or $\frac{2}{4}$ or $\frac{1}{2}$), $\frac{4}{7}$, $\frac{3}{5}$, ($\frac{2}{3}$ or $\frac{4}{6}$), $\frac{5}{7}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$

Leaving out the halves and the thirds we would have something like this

$\frac{1}{7}$, $\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{2}{7}$, $\frac{2}{6}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{3}{6}$, $\frac{4}{7}$, $\frac{3}{5}$, $\frac{4}{6}$, $\frac{5}{7}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$

(See Teacher's Manual for two properties on this Farey series)

5) See No. 2 above. The demonstration of these properites consists simply of changing each rational number to one having a common denominator and comparing the numerators.

6) $\frac{22}{7} > \frac{355}{113}$; $\frac{355}{113}$ is closer to π

7) Given any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ - (where $b \neq 0$ and $d \neq 0$)

Changing to common denominators

we have $\frac{ad}{bd}$ and $\frac{cb}{bd}$ - with the same denominators. We now compare the numerators which due to closure are whole numbers. Then $ad > cb$, $ad = cb$, or $ad < cb$ according to the property previously stated for whole numbers

Exercises H. cont.

8) a) let $r = \frac{1}{3}$ and $s = 2/5$ $\quad \frac{1}{3} < [\frac{(\frac{1}{3} + \frac{2}{5})}{2}] < \frac{2}{5}$

$$\frac{1}{3} < [\frac{\frac{11}{15}}{2}] < \frac{2}{5}$$

$$\frac{1}{3} < \frac{11}{30} < \frac{2}{5}$$

$$\frac{10}{30} < \frac{11}{30} < \frac{12}{30}$$

Let $r = \frac{1}{2}$ and $s = \frac{2}{3}$

$$\frac{1}{2} < [\frac{(\frac{1}{2} + \frac{2}{3})}{2}] < \frac{2}{3}$$

$$= \quad \frac{1}{2} < [\frac{\frac{7}{6}}{2}] < \frac{2}{3}$$

$$= \quad \frac{1}{2} < \frac{7}{12} < \frac{2}{3}$$

$$= \quad \frac{6}{12} < \frac{7}{12} < \frac{8}{12}$$

b) Let $r = \frac{1}{3}$ and $s = 2/5$

$$\frac{1}{\frac{2}{5}} < \frac{\frac{1}{\frac{1}{3}} + \frac{1}{\frac{2}{5}}}{2} < \frac{1}{\frac{1}{3}}$$

$$= \quad \frac{5}{2} < \frac{11}{4} < 3$$

$$= \quad \frac{10}{4} < \frac{11}{4} < \frac{12}{4}$$

Let $r = \frac{1}{2}$ and $s = 2/3$

$$\frac{1}{\frac{2}{3}} < \left[\frac{\frac{1}{\frac{1}{2}} + \frac{1}{\frac{2}{3}}}{2}\right] < \frac{1}{\frac{1}{2}}$$

$$= \quad \frac{3}{2} < \frac{7}{4} < 2$$

$$= \quad \frac{6}{4} < \frac{7}{4} < \frac{8}{4}$$

Exercises H. cont.

c) Let $r = \frac{1}{3}$ and $s = 2/5$

$$\frac{1}{3} < \frac{2}{\frac{1}{\frac{1}{3}} + \frac{1}{\frac{2}{5}}} < \frac{2}{5}$$

$$= \frac{1}{3} < \frac{4}{11} < \frac{2}{5}$$

$$= \frac{55}{165} < \frac{60}{165} < \frac{66}{165}$$

Let $r = \frac{1}{2}$ and $s = 2/3$

$$\frac{1}{2} < \frac{2}{\frac{1}{\frac{1}{2}} + \frac{1}{\frac{2}{3}}} < \frac{2}{3}$$

$$= \frac{1}{2} < \frac{4}{7} < \frac{2}{3}$$

$$= \frac{21}{42} < \frac{24}{42} < \frac{28}{42}$$

d) Let $r = \frac{1}{3}$ and $s = \frac{2}{5}$

$$\frac{1}{3} < \frac{1 + 2}{3 + 5} < \frac{2}{5}$$

$$= \frac{1}{3} < \frac{3}{8} < \frac{2}{5}$$

$$= \frac{40}{120} < \frac{45}{120} < \frac{48}{120}$$

Let $r = \frac{1}{2}$ and $s = \frac{2}{3}$

$$\frac{1}{2} < \frac{1 + 2}{2 + 3} < \frac{2}{3}$$

$$= \frac{1}{2} < \frac{3}{5} < \frac{2}{3}$$

$$= \frac{15}{30} < \frac{18}{30} < \frac{20}{30}$$

9) They all show this

10) For all parts of problem 8, let $r = \frac{a}{b}$, $s = \frac{c}{d}$ be any two positive rational numbers where $\frac{a}{b} < \frac{c}{d}$ or $ad < cb$

Exercises H. cont.

10) a) $\frac{a}{b} < \frac{\frac{a}{b} + \frac{c}{d}}{2} < \frac{c}{d}$

$$\frac{a}{b} < \frac{ad + bc}{2bd} < \frac{c}{d}$$

$$\frac{2ad}{2bd} < \frac{ad + bc}{2bd} < \frac{2bc}{2bd} \quad \text{or} \quad 2ad < ad + bc < 2bc$$

but $2ad = ad + ad$ and $ad + ad < ad + bc$ (as $ad < bc$)

Similarly, $ad + bc < bc + bc$ (as $ad < bc$)

b) $\frac{\frac{1}{\frac{c}{d}}} < \frac{\frac{1}{\frac{a}{b}} + \frac{1}{\frac{c}{d}}}{2} < \frac{1}{\frac{a}{b}}$

$$\frac{d}{c} < \frac{bc+ad}{2ac} < \frac{b}{a}$$

$$\frac{2ad}{2ac} < \frac{bc+ad}{2ac} < \frac{2bc}{2ac} \quad \text{or} \quad 2ad < bc + ad < 2bc$$

Same reasons as in a) above.

c) $\frac{a}{b} < \frac{2}{\frac{1}{\frac{a}{b}} + \frac{1}{\frac{c}{d}}} < \frac{c}{d}$

$\frac{a}{b} < \frac{2ac}{bc+ad} < \frac{c}{d}$ We now can make the numerators equal and can compare denominators. From previous work we know that when numerators are equal, the smaller denominator means the larger fractional value. Thus we see that the required inequalities are equivalent to $2bc > bc+ad$ and $bc + ad > 2ad$.

d) $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

$$\frac{ad(b+d)}{bd(b+d)} < \frac{bd(a+c)}{bd(b+d)} < \frac{cb(b+d)}{bd(b+d)}$$

or $ad(b+d) < bd(a+c) < cb(b+d)$

$abd + add < abd + bcd < cbb + bcd$

As $ad < bc$, the left member $(abd + add)$ is less than the middle member. Similarly, $abd + bcd < cbb + bcd$ - as $ad < bc$

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Exercises I.

1) $\frac{1}{2} = .5\underline{0}$; $\frac{1}{3} = .\underline{3}$; $\frac{1}{4} = .25\underline{0}$; $\frac{1}{5} = .2\underline{0}$; $\frac{1}{6} = .1\underline{6}$; $\frac{1}{7} = .\underline{142857}$;

$\frac{1}{8} = .125\underline{0}$; $\frac{1}{9} = .\underline{1}$; $\frac{1}{10} = .1\underline{0}$; $\frac{1}{11} = .\underline{09}$

2) Some of the decimal equivalents were repeating decimals, such as $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{8}$, and $\frac{1}{10}$ were exact.

3) $\frac{1578}{100}$; $\frac{17,893}{10,000}$; $\frac{12}{10,000}$

4) Yes, the denominator will always have the same number of zeros as the number of decimal places.

5) .156; .0057; 7.89; 358.9
Yes, due to the nature of the division process with powers of ten, there will be no remainder, if division is carried far enough.

6) Every terminating decimal can be expressed in the form $\frac{n}{10^k}$ where n and k are natural numbers. Likewise, any rational number of the form $\frac{n}{10^k}$ can be expressed as a terminating decimal.

7) If the denominator is a factor of 10 or a factor of a power of ten, it will be a terminating decimal

8) They will be only 2 or 5 (or powers of 2 or 5 or both)

9) Yes; 32 divides 100,000
40 divides 10,000
16 divides 10,000
64 divides 1,000,000

Exercises J.

1) 1/3 of 1/100; No; Yes; 1/3 of 1/1000

2) .000857 $\underline{142857}$....or 6/7,000; .000057$\underline{142857}$....or 4/70,000

3) .9, .99, .999, .9999
.1 .01 .001 .0001
If we multiplied $.33^1/_3$ by 3, we would get exactly 1

4) a) .375 c) .$\underline{405}$... e) .$\underline{123}$... g) .$\underline{076923}$...
b) .113$\underline{36}$... d) .056 f) .$\underline{12345}$... h) .$\underline{0588235294117647}$...

Exercises J. cont.

5). In $\frac{1}{17}$ there are 16 digits, in $\frac{1}{7}$, 6 digits (at most the number of digits will be one less than the denominator).

6) It will repeat, as the denominator is made up of prime factors other than 2 and 5. As far as 412 places, possibly.

7) 726; zero

Exercises K.

1) $\frac{4}{33}$; $\frac{121}{999}$

2) $\frac{34}{99}$; $\frac{133}{99}$; $\frac{1330}{99}$; $\frac{21}{37}$; $\frac{12,340}{9999}$; $\frac{1,918,490}{333}$

3) Count the number of digits in the repeating part of the decimal. If the number of digits is k, multiply n by 10^k

4) The number of digits in the repeating part of the decimal equivalent of $\frac{1}{p}$ where p is a prime number, is always a divisor of p-1

Sample Test Questions

Part 1. True - False

F 1. Division is the inverse of subtraction.

T 2. 2/9 can mean either 1/9 of 2 or 2 × 1/9.

T 3. In adding rational numbers, if the denominators are equal, the numerator of the sum is the sum of the numerators and the denominator is the common denominator.

T 4. The following numbers are all examples of rational numbers: 3/4, 5, 8/3, and 1

F 5. Division of rational numbers is associative.

T 6. The fractions 0/a and 0/b represent the same rational number if neither a nor b is zero.

F 7. The set of non-negative rational numbers is not closed with respect to multiplication.

F 8. If z is different from zero, then 3z/z equals 2z.

T 9. The expression a + b = c may be written a = c - b.

F 10. 7 is a multiple of 28.

F 11. The decimal equivalent of any rational number will always terminate.

T 12. The symbol 24/8 stands for both a natural number and a non-negative rational number.

F 13. Zero is not a rational number.

F 14. The sum of 2 rational numbers having equal denominators is a rational number whose numerator is the sum of the numerators and whose denominator is the sum of the common denominator.

T 15. Every repeating decimal may be represented as a rational number.

T 16. The product of zero and any rational number is zero.

F 17. If one rational number has a larger numerator than a second rational number, the first is always the larger.

T 18. Even if $a = 0$, $a/7$ is a rational number.

F 19. If two rational numbers have the same denominator, they are always equal.

T 20. The number $7/16 < 8/17$.

F 21. The fractions 3/4 and 16/20 represent the same rational number.

T 22. Another way of saying that 14 is divisible by 7 is to say that 14 is a multiple of 7.

T 23. $a - b$ is a natural number only if $a > b$.

T 24. In the division problem 1/2 divided by 1/3, we are looking for a number which when multiplied by 1/3 gives 1/2.

T 25. The reciprocal of 1/2 is 2.

F 26. The sum of $a/c + b/c = \frac{a + b}{2c}$

F 27. The reciprocal of the reciprocal of 3 is 1/3.

F 28. Even if b equals 0, a/b is a rational number.

T 29. The fractions r/s and t/u are equal if and only if $r \times u = s \times t$.

PART II. MULTIPLE-CHOICE

1. In adding rat*onal numbers with unlike denominators, which of the following may never be used in finding the least common multiple of the two denominators?

   A. multiplying the two denominators.

   B. doubling the largest denominator.

   *C. adding the two denominators.

D. dividing the product of the two denominators by 2.

E. All of the above are correct.

2. Which of the following pairs of numbers are both divisible by some number greater than one?

A. 7, 3

B. 8, 9

*C. 7, 28

D. 5, 23

E. None of the above.

3. You can find the product of two rational numbers by...

A. multiplying the two numerators.

B. multiplying the two denominators.

*C. dividing the result of A by the result of B.

D. multiplying the denominators and dividing the numerator.

E. multiplying the numerators and dividing the denominators.

4. Which one of the following fractions will give a repeating decimal?

A. 1/2

B. 3/4

C. 5/8

*D. 6/11

E. 7/20

5. The quotient of two natural numbers is always...

A. zero.

B. a rational number.

*C. equal to one.

D. a natural number.

E. none of the above is correct.

6. Which of the following is true?

A. (x/z)(t/k) equals x plus z plus t plus k

B. (x/z)(t/k) equals (xk)/(zt)

C. (x/z)(t/k) equals (x/t)/(zk)

*D. (x/z)(t/k) equals (xt)(zk)

E. None of the above.

7. Terminating decimals...

A. never stop.

*B. are exact.

C. never end in a sequence of zeros.

D. are not exact.

E. None of the above.

8. If a/x = y/d and a = 6 and d = 12, then x . y equals...

A. 1/2.

B. 2.

C. 24.

D. cannot be determined from the information given.

E. none of these.

9. The decimal number, 2.71828..., is:

A. a repeating decimal.

B. a rational number.

C. equal to the quotient of two integers.

*D. can't be determined from the given information.

E. none of the above.

10. We can change the <u>denominator</u> of the fraction (2/3) / (4/5) to the number "1" without changing the value of the fraction by...

A. adding 5/4 to the numerator and denominator.

B. subtracting 5/4 from the numerator and denominator.

*C. multiplying both the numerator and denominator by 5/4.

D. dividing both the numerator and denominator by 5/4.

E. none of the above.

11. In the following list of fractions, the fraction which does <u>not</u> terminate is

A. 1/2

B. 1/4

C. 1/5

*D. 1/6

E. 1/8

12. The fact that $1/4 \times 12$ is equal to $12 \times 1/4$ is due to which of the following principles?

A. associative principle.

*B. commutative principle.

C. distributive principle.

D. closure property.

E. none of the above.

13. A repeating decimal...

A. always starts repeating immediately following the decimal point.

B. never repeats.

*C. continues to repeat, no matter how far the division is carried out.

D. are not periodic decimals.

E. none of the above.

14. The number of digits in the repeating parts of 57.173456173... is...

A. 3

*B. 6

C. 8

D. 9

E. none of the above.

15. Which of the following is not a correct way of writing the quotient 2/3 ÷ 4/9

A. (2/3)/(4/9)

B. $\frac{(2/3) \times (9/4)}{(4/9) \times (9/4)}$

C. (2/3) × (9/4)

*D. (2/3) / (9/4)

E. all of the above are correct.

PART III. MATCHING.

(B)	1. (3/4) × (4/9) = 12/36	A.	(a.k)/(b.k) = a/b
(C)	2. (2/3) ÷ (4/9) = 18/12	B.	(a/b) × (c/d) = (a.c)/(b.d)
(A)	3. 1/3 = 12/36	C.	(a/b) ÷ (c/d) = (a.d)/(b.c)
(A)	4. 2/6 = 1/3	D.	(a/b) ÷ (c/d) = (a.c)/(b.d)
(F)	5. (3/4) × 4 = 3	E.	(a/b) × (c/d) = (a.d)/(b.c)
(A)	6. 12/18 = 2/3	F.	(a/b) × b = a

PART IV. COMPLETION.

Find the missing numbers for each of the following:

(9) 1. 3/4 = ? /12

(18) 2. 6/7 = ? /21

(64) 3. 5/8 = 40/?

(4/7) 4. 28/49 = ? /?

(3/7) 5. 15/35 = ? /?

(3/5) 6. 102/170 = ? /?

7. Arrange these numbers in order of size, the smallest first and largest last.

21/28 16/20 18/21 Ans.: 21/28, 16/20, 18/21.

(7/33) 8. The rational number which is equivalent to .212121 is ?

(22) 9. In division by 23 the number of the possible remainders is ?

Find the exact decimal equivalents for these rational numbers. Carry out the division until decimals terminate.

(.25) 10. 1/4

(.625) 11. 5/8

(.35) 12. 7/20

(.0625) 13. 1/16

(.53125) 14. 17/32

(b - 1) 15. It was shown in the unit that in the decimal equivalent of a/b the number of digits in the repeating part cannot be greater than ____.

PART V. MULTIPLE CHOICE.

1. Which of the following properties for rational numbers does not fit our everyday experiences?

   A. One is the identity for multiplication.

   B. Division is the opposite of multiplication.

   C. There is an associative property for addition.

   D. Subtraction is the opposite of addition.

   *E. None of the above.

2. John had 3/4 of a dollar and spent 1/5 of it. George had 1/5 of a dollar and spent 3/4 of it. They both spent 15 cents. What property for numbers does this life situation illus-

trate?

A. Distributive property over addition.

B. Associative property for multiplication.

C. Identity property for multiplication.

*D. Commutative property for multiplication.

E. None of the above.

3. If $\frac{m}{n} \cdot \frac{r}{s} = 1$ where $\frac{m}{n}$ and $\frac{r}{s}$ are rational numbers, then we know that

A. m = 1 or r = 1 or both

*B. $\frac{m}{n} = \frac{s}{r}$

C. r = 0 or m = 0

D. m = 1 or n = 1 or both

E. None of the above

4. Which of the following would be a correct way to give an explanation of "3 . 1/5"?

A. Divide 5 objects into 3 parts and take one of them.

B. Divide 1 object into 3 parts and take 5 of them.

*C. Divide 1 object into 5 parts and take 3 of them.

D. Divide 3 objects into 3 parts and take 1 of them.

E. None of the above.

5. Here are 3 numbers: $\frac{m}{n}$, $\frac{m+1}{n}$, $\frac{m}{n+1}$. If we arrange them in order, from the smallest to the largest, we would have

A. $\frac{m}{n}$, $m + \frac{1}{n}$, $\frac{m}{n} + 1$

B. $\frac{m}{n} + 1$, $\frac{m}{n}$, $m + \frac{1}{n}$

C. $m + \frac{1}{n}$, $\frac{m}{n} + 1$, $\frac{m}{n}$

D. $\frac{m+1}{n}, \frac{m}{n}, \frac{m+1}{n}$

*E. None of the above

6. Another name for the inverse for multiplication of a rational number is the

*A. Reciprocal.

B. Opposite.

C. Reverse.

D. Zero.

E. None of the above.

7. Jim and Jerry work together doing odd jobs. Each gets half the money they earn. One week they had two jobs which paid $5 and $3. When they split the money, Jim thought they could divide the $5 in half, and then divide the $3 in half and then add these two shares together. Jerry thought they should add the $5 and $3 together and then divide the total earnings. When they tried both methods, they got the same amount each way. What property for arithmetic numbers does this life situation illustrate?

A. Commutative property for addition.

*B. Distributive property.

C. Associative property for multiplication.

D. Commutative property for multiplication.

E. None of the above.

8. If $\frac{r}{s} + \frac{t}{u} = 0$ where r and t are non-negative rational numbers then we know that

*A. $\frac{r}{s} = 0$ and $\frac{t}{u} = 0$

B. $r = 0$, $s = 0$, $t = 0$ and $u = 0$

C. Either $\frac{r}{s} = 1$ or $\frac{t}{u} = 1$

D. $\frac{t}{u}$ is the reciprocal of $\frac{r}{s}$

E. None of the above

9. Which one of the following properties does <u>not</u> hold for rational numbers?

A. The set of rational numbers is closed with respect to multiplication.

B. Multiplication is commutative.

C. Multiplication is associative.

D. There is an identity for multiplication.

*E. Every rational number has an inverse for multiplication.

10. The decimal for the number $\frac{3}{23}$ will

A. End.

B. Repeat in cycles of 3 digits.

*C. Repeat in cycles of less than 23 digits.

D. Will not repeat or end as 23 is prime.

E. None of the above.

11. If "$\frac{m}{n}$" represents the simplest form of a fraction, then the decimal will end if

A. n is any even number.

B. n has no prime factors.

C. n is any odd number.

*D. n has no prime factors other than 2 or 5 or both.

E. None of the above.

12. The rational number halfway between 3/4 and 13/16 is

A. 7/8

B. 11/16

C. 24/32

D. 32/24

*E. None of the above.

13. If $\frac{a}{b} \cdot \frac{c}{d} = 0$, where $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then we know that

*A. a = 0 or c = 0 or both

B. a = 0 or b = 0 or both

C. b = 0 or d = 0 or both

D. $\frac{a}{b}$ is the reciprocal of $\frac{c}{d}$

E. None of the above.

14. Indicate the relative size of the following pairs of fractions by using the symbols <. >. =.

(=) A. 20/32 . 125/200

(>) B. 15/18 . 36/45

(=) C. 21/49 . 15/35

(>) D. 14/35 . 7/21

15. Calculate the following quotients:

(4/7) A. 3/4 ÷ (7/8 ÷ 2/3)

(9/7) B. (3/4 ÷ 7/8) ÷ 2/3

(Associative) C. This shows that the operation of division of rational numbers does not have the ________________ property.

16. Place the rational numbers in the following list in the answer spaces at the left.

(0/3)	(2/7)	0/3	0.0
(3.0)	(6)	3/0	0.3
(0.0)	____	0/0	2/7
(0.3)	____	3.0	6

17. A. How many different remainders can appear when one natural number is divided by another natural number if the decimal equivalent does not terminate?

(One less than the divisor.)

B. What remainder must appear if the decimal equivalent terminates?

(Zero.)

C. Arrange the following decimal equivalents of rational numbers in increasing order from left to right:

.6, .57, .601, .561, 6, 5, .06, .05

(.05, .06, .561, .57, .6, .601, 5, 6)

D. Arrange the following rational numbers and decimal equivalents in increasing order from left to right:

1/3, 1/4, .31, 1/5, .21, .91, 2/7, .37

(1/5, .21, 1/4, 2/7, .31, 1/3, .37, .91)

18. What rational number would you add to .3 to make it exactly equal to 1/3? (1/30 or .03$\underline{3}$...)

19. Match the descriptions on the left with the answers on the right.

(2) A. A check to determine if two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, are equal — 1. $\frac{ac}{bd}$

2. ad = bc

3. ab = cd

(6) B. The sum of $\frac{a}{b}$ and $\frac{c}{b}$ — 4. $\frac{ad}{bc}$

(7) C. The sum of $\frac{a}{b}$ and $\frac{c}{d}$ 5. $\frac{ca + ab}{ba}$

(4) D. The answer for $\frac{a}{b} \div \frac{c}{d}$ 6. $\frac{a + c}{b}$

7. $\frac{ad + cb}{bd}$

(1) E. The answer for $\frac{a}{b} \cdot \frac{c}{d}$ 8. $\frac{a + c}{b + d}$

20. Which symbol (<, >, or =) should be inserted between each pair of rational numbers to make it a true statement?

(>) A. 6/7 5/7

(<) B. 6/7 6/5

(=) C. 5/9 20/36

(<) D. 6/7 7/8

(<) E. 243/37 243/36

(=) F. 7/13 539/1001

21. Carry out the following computations, reducing answers to simplest form

(0) A. (0/5) × (1/6) × (2/7) =

(27/64) B. (3/4) × (1/2) × (9/8) =

(3/2) C. (1/3) × (4) × (4/8) × (9/4) =

(3/4) D. (3/2) - (3/4) =

22. Find the product 1/5 . 1/4. Explain your answer in terms of a dollar. (1/20)

23. Define: (See text)

A. Rational number

B. Terminating decimals

C. Decimal equivalent of rational numbers

D. Ordering of rational numbers

E. Repeating decimals

F. Equality of rational numbers

24. A ______ is a symbol we use for writing a rational number on the board.

25. List two meanings of 3/8. (See text)

26. ~~List <u>two</u> ways to prove 2 rational numbers are equal in value.~~ (See text)

Date ______________________

UNIT: ____

Name of Teacher: ____________________________________

Name of School: _____________________________________

City: ________________ State: ________________

Number of days given to the teaching (including testing) of this unit: ________________

Approximate dates: __________________________________

USE THE BACK OF THIS SHEET IF YOU NEED EXTRA SPACE TO ANSWER ANY OF THE QUESTIONS BELOW

1. Make a statement about the ability level of the pupils in the class and state whether your school uses some plan of homogeneous grouping.

2. What parts of the unit proved to be the most teachable?

3. What parts of the unit proved to be the most difficult to teach?

   Did you omit any part? ________

4. Did you use any supplementary developmental materials? ______

   If so, what were they, and at what points were they used?

5. Did you find it necessary to provide the pupils with additional material? ________

   If so, was it from textbooks or did you write your own?

6. Do you think that a unit on this topic should be included in regular textbooks for 7th and 8th grades? ________

7. Please make ANY additional comments about your teaching experience with this unit which you think would be helpful to the Panel responsible for preparing and experimenting with textbook materials for grades 7 and 8.

UNIT V

Summary of Teachers' Comments

The number of days given to the unit varied from 7 to 43 with an average of 18. The classes varied in ability to this extent: 17 high level, 5 medium level, and 14 heterogeneously grouped classes. Many classes made use of supplementary developmental and practice materials.

The ease or difficulty of the sections in the opinion of the responding teachers is indicated in the following list:

Section	Easiest to Teach	Most Difficult to Teach
Whole numbers and divisibility	5	0
Fractional notation	5	0
Multiplication of rationals	8	2
Equality of rationals	5	1
Division by zero	4	2
Division of rationals	7	5
Addition of rationals	6	5
Summary of properties	2	2
Order of rationals	1	9
Decimal equivalents	6	2
Repeating decimals	6	5
Fractional equivalents of repeating decimals	3	8

Teachers of this unit generally agreed on the importance of most of the material in this unit, but found it slightly harder to maintain high interest than on some of the other units. However, 22 out of 24 felt the material should be introduced.

A majority felt it necessary to use supplementary material, particularly for added practice material. This was the most commonly voiced need. Added materials used by different teachers included the following:

Work on geometric progressions including the "Achilles and the tortoise" paradoxes.
More work on percent from regular texts
Introduction of negative numbers (because of title of unit)
Ratio and proportion problems
Practice exercises from corresponding Maryland unit - also use of this unit for developmental work
Language of sets and simple diagrams to clarify meaning of operations on rationals
Drill problems on manipulation with rationals
Formula for changing repeating decimal to equivalent fraction

No clear conclusions appear possible from the data on ease or difficulty of sections except that the first few sections in general went easily, but that the work on the order relations of the rationals and the work on repeating decimals in general were hard. Several teachers noted, however, that the material on the repeating decimals, though not easy, was particularly stimulating and exciting for the students.

Typical quotations of statements made by teachers follow:

> I feel that this was most valuable in strengthening the pupils' understanding of the various properties. As we work with these units, I find an increasing desire on the part of my students to know reasons for operations. They also show a growing ability to generalize.

> The level of verbalization is very high for 7th and 8th grades in all of these units. Poor readers encounter major difficulties. A number of my better students have complained that the statements of the exercises were hard to understand.

> We had more fun with this unit than any of the others so far. It was also much tougher for most of the students. We spent considerable time making various "proofs". I believe this was most valuable and will carry over into future mathematics work. I am sure the path into algebra will be much smoother for these students than for those in "regular" classes. There is much here to tie arithmetic to algebra and vice versa. I believe that the material in this unit could most effectively be taught at the 8th grade level. With some revision and simplification, I plan to continue to use it with the accelerated 7th grade groups.

UNIT XIV

MATHEMATICAL SYSTEMS

It is particularly important that teachers using this unit have clearly in mind both its objectives and the method of approach suggested to be used with it.

The main objective is to lead the students to achieve some appreciation of the nature of mathematical systems. It is not intended nor desirable that the children should memorize the various tables or drill for mastery of the operations introduced here.

The teacher should read this unit through before beginning to plan his presentation and give considerable thought both to how to present some introductory motivation and even more how to lead the students to "discover" the various relationships and properties of the unit for themselves and in advance of the reading of the text. The text itself attempts to suggest problems and processes for doing this as does this teacher's guide. However, these can only be effective if carefully planned for by the teachers. The process of discovering, of perceiving for one's self is a vital step in achieving our major objective: an appreciation of the nature of some types of mathematical systems. This is close to an appreciation of the nature of modern mathematics and of the work of mathematicians.

One of the most important activities of modern mathematicians is the search for common elements or properties often found in apparently diverse situations or systems. Sometimes these common elements are deliberately built into new systems which are constructed as generalizations or abstractions of old systems as when the number system is extended from the system of natural numbers to the whole numbers to the (positive and negative) integers to the rational numbers, etc., etc.

Sometimes these common elements are observed in systems less clearly related at first glance as when the rotations of a rectangle into itself are conceived of as forming an algebraic system with a "multiplication" table which is discussed in the unit.

Frequently, the systems developed out of the intellectual curiosity of mathematicians and their search for patterns in diverse abstract situations have been exactly the tools needed and seized upon by scientists in their attack on the problems of our physical world. The theory of groups, which actually has as its logical beginnings the properties discussed in this unit, had its chronological beginnings in the early 19th century in problems relating to the solution of equations. Matrices, some of which form groups and give further examples of the principles of this unit, were invented largely by the Englishman Cayley a little later. Within our generation the German physicist Werner Heisenberg has used matrices in the formulation of the quantum mechanics so important in modern physics. Analogous stories relate the development of radio by Marconi back to the differential equations of Maxwell, and point out that all the outgrowths of Einstein's relativity theory owe much to his use of the tensor calculus developed by the Italian geometers Ricci and Levi-Civita. All of these stories have the same theme that both mathematicians and scientists are always seeking unifying principles or patterns. Frequently mathematics developed solely for the intrinsic interest of its properties and structure is later found to fit the needs of science, but for both science and mathematics we need to develop students who can see and understand patterns and structure.

In this unit we are studying mathematical systems involving sets of elements and binary operations. Such systems which have certain simple additional properties are called groups and their study is a major branch of so-called "modern algebra". We shall not use all of these technical terms. However, other substantial objectives incidental to the major concern of this unit and appropriate for secondary school students are

1. Increased understanding of the nature and occurence of the commutative, associative, and distributive properties, as well as the concepts of closure, identity element, inverse element.
2. Increased understanding of the inverse operations of division and subtraction and their relationship to inverse and identity (zero and one) elements.

Additional discussions of these ideas and problem materials may be found in

Carl B. Allendoerfer and Cletus O. Oakley, Principles of Mathematics. New York: McGraw-Hill Book Company, Inc., 1955

W. W. Sawyer, Prelude to Mathematics. Baltimore, Md.: Penguin Books, 1955

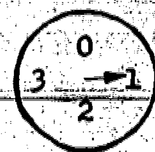
B. W. Jones, Elementary Concepts of Mathematics. New York: The Macmillan Company, 1941

MATHEMATICAL SYSTEMS

Understandings	Skills and Abilities
1. There are other kinds of addition besides the familiar kind.	1. To add in modular situations.
In modular arithmetics, we can check all of the properties by studying tables.	To appreciate that the properties of modular addition are similar to those of ordinary addition.
Modular addition is commutative and associative.	
Zero is an identity for addition.	
Subtraction is the inverse of addition.	To subtract in modular situations.

Activities

1. Use a 4-minute clock, such as might be used to time rounds and intermissions in a boxing match. (This may be drawn on the board or a model can be constructed.)



Ask - On this clock, what is 1 + 2?, 2 + 3?, etc.

Point out that this kind of addition is different from ordinary addition.

Have students construct an addition table for this kind of addition.

Explain - This is a new kind of arithmetic. Let us see in what ways it is like ordinary arithmetic and in what ways it is different.

Bring out that all the facts are before us (in the table), so we can answer all our questions about properties by looking at the table.

Bring out that addition is commutative (the table is symmetric), associative (for this many examples will have to be checked).

There is an identity for addition.

Subtraction (the opposite of addition) is always possible.

e.g., 1 - 3 = 2 because 2 + 3 = 1

Explain - This kind of arithmetic is called "modular arithmetic" and the number 4 is called the "modulus". We do not actually need a clock for it (we just add in the ordinary way and take away multiples of the modulus.). We can make addition tables of modular arithmetic for any modulus.

Have students make addition tables mod 5 and mod 8.

2. A binary operation is a process which associates with two given elements in a set, a third element. This third element need not be different from the first two and it may not even be in the original set, but it must be unique.

2. To recognize operations.

3. A set is closed under an operation if for any two members of the set, the process associates with them a member of the same set.

3. To decide whether a set is closed to an operation.

Exercises - 1

2. Explain - We have talked about operations before. Now we have seen a different kind of operation. Let us think more about what an operation is.

   In discussion bring out that whenever we have a way to associate with two things a 3rd one (and only one 3rd one), we call that process an operation.

   Give some examples of processes which are operations and some which are not;

   e.g., for natural numbers a and b, let a * b mean 2 • a + 3 • b or let it mean 2 • a - 3 • b. These are operations because for every two natural numbers, a, b, there may be a unique natural number. However, this system is not closed because if b is equal to or greater than $\frac{2a}{3}$ there is no natural number equal to 2a - 3b. But if a * b means to take twice one of them and then add the other we do not always get the same natural number (there are two possibilities) so this is not an operation.

Exercises - 2

3. Explain - We say a set is closed to an operation if we can always do that operation on <u>any</u> two members of the set and when we do, we get a member of the set. (the "two" members of the set may be the same one, like 2 + 2)

   Give some examples of sets closed to certain operations, such as,

   The set of even nos. is closed to +, since for any two

4. An algebraic system consists of a set of things, one or more operations and certain properties concerning the operations or members of the set

A member of a system is an identity for a given operation if the result of applying the operation to it and any member of the system is that member.

To identify an identity.

even nos. we can add and we always get an even no.

The same set is not closed to -, because the operation 4 - 12 cannot be performed, for example.

4. and 5. Tell the students that they are to choose the entries in the table; remind them of the symmetry test for commutativity.

Exercises - 3

4. Explain - We have seen that there are other mathematics than ordinary arithmetic. Modular arithmetics are examples of <u>algebraic</u> <u>systems</u>.

An algebraic system consists of a set of things (not necessarily numbers), one or more operations, and some rules, or properties, concerning the operations and the things in the set.

Point out that for the modular systems we have all these things, so they are mathematical systems.

Explain - Zero is the identity for addition in ordinary arithmetic. This means that when we add it to any number we get that number.

In any mathematical system, if we find a member and an operation so that when we apply that operation to this member and another we get the other, then we call that special member an "identity for that operation."

Ask - What is the identity for multiplication in ordinary arithmetic?

Have students make a 5-clock, using "5" rather than "0" and point out that 5 is the identity for addition mod 5. (It was labeled "0" before. The point here is that it need not be 0.)

5. If there is an identity for an operation in a mathematical system, and if the result of applying the operation to two members of the system is the identity, then those members are called inverses of each other.

5. To decide whether a member of a system has an inverse for a given operation. If so, to identify it.

6. The members of a mathematical system may not be numbers.

In some mathematical systems the members are motions of some kind.

In systems involving motions:
To recognize the members,
To apply the operations,
To decide the properties of the operations,
To identify and use an identity,
To identify and use inverses.

Exercises - 4

5. Explain - For the operation of addition, in mod 5 we have an identity (named "0" or "5", as we choose).

Ask - Take 3 in this mathematical system. What must we add to it to get the identity? Repeat question for 1, 4, 5. Point out that for every member of the system there is something that can be added to produce the identity. We "prove" this by noting that the identity is to be found somewhere on each line of the table.

Explain - If we add two things and get the identity, then we call them "inverses of each other."

2 is the inverse of 3, 3 is the inverse of 2.

Exercises - 5

6. Review the definition of an algebraic system.

Explain - There are algebraic systems without numbers in them. Just by writing tables we could make up lots of them. But let's make up an algebraic system of <u>changes</u> of a rectangle.

Provide students with rectangles for rotating. These can be cardboard, light books, etc.

Explain - The things in this system will not be numbers, but will be changes in the position of a rectangle.

We will mark the corners, and will change the rectangle so that it has the same appearance as when we started, except that the marks on the corners can be moved around.

Let us see what changes will be in our system.

Show the changes which would result from rotations about horizontal and vertical axes and by rotating in the plane of the rectangle. Call these 'H', 'V', and 'R', respectively.

Explain - Another change is the one which leaves it exactly as it is (either leave it alone, or change it in such a way that it ends up exactly as it is).

Call this change 'I'.

These 4 things will be the members of our system.

Explain - Now let's define an operation.

Our operation will be to do first one change and then another.

Have class make up a name for the operation, such as "multiplication" or "changification".

Have them also make up an operation symbol, such as "H*V" or "HoV".

Start class filling in a table for this operation and let them finish it individually.

Ask - What are the properties of this operation?

Have students find

That it is in fact an operation.

That it is commutative.

That it is associative (this will take a lot of checking)

There is an identity.

There is an inverse for every member.

Exercises - 6

Note that the commutativity of * and ~ is shown by the symmetric arrangement of the elements in the table.

Some introduction to important aspects of <u>deductive</u> reasoning is possible here -- for example in 4, <u>every</u> combination would have to be tried to prove the associative principle to hold, <u>but</u> only <u>one</u> "counter example" would be needed to prove that it did not hold.

8. Modular arithmetics may be thought of as algebraic systems with two operations.

8. To calculate in modular arithmetics.

0 times any member is 0.

Exercises - 7

7. Ask - Do the natural numbers make up a mathematical system?

Discuss this, having students give properties.

Two operations, + and • for which there is closure, commutativity, associativity, and distributivity for multiplication over addition.

Two operations, subtraction and division for which there is not closure, and which are not commutative or associative.

There is an identity for •, none for +.

No number except 1 has an inverse for multiplication.

Since there is no identity for +, there is no question to ask about inverses.

Explain - The whole numbers make up a mathematical system, too. Describe its set as the set of natural numbers with zero put in.

Repeat the discussion as for the system of natural nos.

Exercises - 8

8. Explain - We have seen modular arithmetic for addition. If we put in multiplication we will get different mathematical systems. With both operations they would be more like ordinary arithmetic.

Have students make multiplication tables for mod 5 and mod 8 and list the properties:

Commutativity, associativity, closure.

Identity

Inverses for mod 5; in mod 8 not every element has an inverse.

0 times anything is 0.

A product of non-zero factors may be zero in some systems.

Modular arithmetics have the distributive property of multiplication over addition.

To identify and review the properties of modular arithmetics.

To recognize whether for two operations the distributive property holds, for one over the other.

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If a product is zero, at least one factor is zero, in mod 5. This is not true in mod 8 arithmetic.

The inverse of multiplication (division) is always possible in mod 5 as long as the divisor is not zero. Division is not always possible in mod 8 arithmetic, even with a non-zero divisor.

Now consider mod 5 and mod 8 arithmetic with both operations. Have students check whether the distributive property for multiplication over addition holds.

(Some deduction is possible here, since it can be reasoned that we do calculations in whole numbers and then take away multiples of the modulus. Therefore, since the distributive property holds for whole numbers, it will hold for modular arithmetics.)

Have students check to see whether, in mod 5 and mod 8 addition is distributive over multiplication.

(whether for any a, b, c, $a + (b \cdot c) = (a+b) \cdot (a+c)$

Do some arithmetic exercises in modular arithmetics, including finding square roots. (Explain that to find a square root of a number, we find the number(s) which when multiplied by itself give that original number.)

ANSWERS

Exercise 1

1. Mod 3

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Mod 5

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Mod 6

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Mod 8

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

2. $1 + 2 = 3$

$3 - 1 = 2$

$0 - 3 = 2$

$4 + 1 = 0$

$3 + 2 = 0$

$3 + 4 = 2$

$1 - 2 = 4$

$3 - 2 = 1$

$2 - 4 = 3$

$4 + 0 = 4$

$(3 + 2) + 4 = 4$

$3 + (2 + 4) = 4$

$(4 - 2) + 3 = 0$

$(3 - 1) + (2 + 4) = 3$

$4 - (2 - 3) = 0$

$(3 + 4) - 3 = 4$

$(2 - 3) - 3 = 1$

$(1 - 4) + 4 = 1$

$3 + (3 - 1) = 0$

$(3 + 2) + (2 - 1) - (4 - 3)$

$= (0) + (1) - (1)$

$= 0$

3. 4 times; at 3; in the same position

Divide the time by 5 and the remainder gives you the position of the clock hand.

Exercise 2

1. A

(a) 1

B

(a) 12

(b) 6 (b) 4

(c) 8 (c) 2

(d) 2 (d) 5

(e) 3 (e) 8 (not assoc.)

(f) 11 (f) 1

2. a, b, c, and e are commutative

a, c, and e are associative; in b, $3 + (5 + 7)$ for example, is not defined.

Look for symmetry of numbers to diagonal from upper left to lower right.

3. (a) No, No

(b) Yes, Yes

(c) Yes, Yes

(d) Yes, Yes

(e) Yes, Yes

(f) Yes, Yes

(g) No, No

(h) Yes, Yes

(i) Yes, Yes

4.

	3	5	7	9
3	12	16	20	24
5	16	20	24	28
7	20	24	28	32
9	24	28	32	36

5.

	1	2	3	4
1	4	5	6	7
2	7	8	9	10
3	10	11	12	13
4	13	14	15	16

Exercise 3

1. a, c and e are closed

b, d are not closed, because other numbers arise that are not in either column or row on the outside.

2. (a) closed
   (b) closed
   (c) closed
   (d) not closed
   (e) closed
   (f) closed (this is assuming $10 - 15 = -5$ has meaning here)
   (g) closed
   (h) not closed
   (i) closed
   (j) closed
   (k) not closed
   (l) closed
   (m) not closed

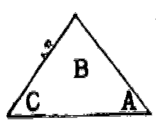
Exercise 4

1. (a) table a - 5, table c - 12, table d - 0, table e - 2
   (b) table a - 1 + 4, 2 + 3, 5 + 5; Yes
   table c - 1 + 11, 2 + 10, 3 + 9, 4 + 8, 5 + 7, 6 + 6, 12 + 12; Yes
   table d - 0 + 0; No
   table e- 2+ 2, 3 + 1; Yes

Exercise 5

1. (a) Yes (b) Yes (c) Yes (d) Yes

2. I^oM means first do nothing and then flip the triangle about the vertical axes
   M^oI will result in



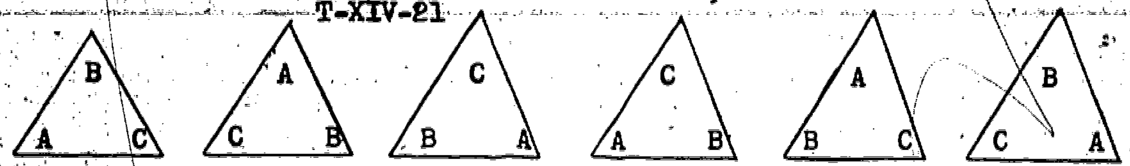
   or $M^oI = M$

   (a)

o	I	M
I	I	M
M	M	I

   (b) Yes
   (c) The operation is both commutative and associative.
   (d) Yes, I
   (e) ~~Yes, each~~ member is its own inverse

3. The Set:



I R$_1$ R$_2$ S$_1$ S$_2$ S$_3$

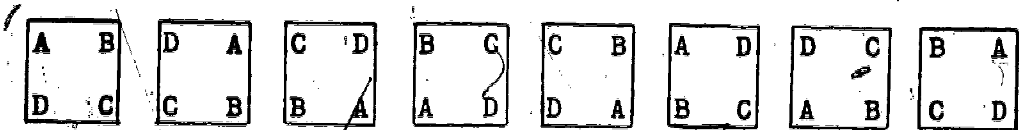
*	I	R_1	R_2	S_1	S_2	S_3
I	I	R_1	R_2	S_1	S_2	S_3
R_1	R_1	R_2	I	S_3	S_1	S_2
R_2	R_2	I	R_1	S_2	S_3	S_1
S_1	S_1	S_2	S_3	I	R_1	R_2
S_2	S_2	S_3	S_1	R_2	I	R_1
S_3	S_3	S_1	S_2	R_1	R_2	I

It is not commutative

There is an identity change, I

Each change has an inverse

4.



I R_1 R_2 R_3 S_1 S_2 H V

	I	R_1	R_2	R_3	S_1	S_2	H	V
I	I	R_1	R_2	R_3	S_1	S_2	H	V
R_1	R_1	R_2	R_3	I	V	H	S_1	S_2
R_2	R_2	R_3	I	R_1	S_2	S_1	V	H
R_3	R_3	I	R_1	R_2	H	V	S_2	S_1
S_1	S_1	H	S_2	V	I	R_2	R_1	R_3
S_2	S_2	V	S_1	H	R_2	I	R_3	R_1
H	H	S_2	V	S_1	R_3	R_1	I	R_2
V	V	S_1	H	S_2	R_1	R_3	R_2	I

The operation is NOT commutative.

There is an identity change, I

5.

+	0	1
0	0	1
1	1	0

same as table to left

6. (a) the successive application of two tire switches.

(b) The table would be the same as the table in 3 of this exercise.

(c) Yes

(d) No

(e) No

7. |A| |B| |C| I

|A| |C| |B| S_1

|B| |C| |A| S_2

|B| |A| |C| S_3

|C| |A| |B| S_4

|C| |B| |A| S_5

X	I	S_1	S_2	S_3	S_4	S_5
I	I	S_1	S_2	S_3	S_4	S_5
S_1	S_1	I	S_5	S_4	S_3	S_2
S_2	S_2	S_3	S_4	S_5	I	S_1
S_3	S_3	S_2	S_1	I	S_5	S_4
S_4	S_4	S_5	I	S_1	S_2	S_3
S_5	S_5	S_4	S_3	S_2	S_1	I

The operation is NOT commutative

There is an identity change and each change has an inverse.

Exercise 6

1. (a) R (b) X (c) X (d) W (e) P_1 (f) P_3 (g) P_0

(h) V (i) △ (j) W (k) R (l) P_2

2. * - P_2 and ∼ - △

3. * is commutative -- a proof here would be to demonstrate that commutativity holds for all pairs of elements in the set. The truth of this can be seen from the table page 14. Note symmetry of table with respect to diagonal from top left to lower right.

∼ is commutative -- proof by demonstration, as in table c, page 14.

4. (a) $P_0 * (P_1 * P_2) = P_0 * P_1 = P_3$

(b) P_3

(c) P_2

(d) P_2

(e) P_1

(f) P_1

(g) X

(h) X

(i) Δ

(j) Δ

Yes, the way in which the three elements are operated on makes no difference -- the same answer appears.

You could prove this by demonstrating that associativity held for <u>all</u> combinations of triple elements in the set.

To prove you wrong, someone else would need only to find <u>one</u> case that was not associative.

Exercise 7

1. (a) Properties of closure, commutativity, associativity, identity element for multiplication. No inverse

(b) Same as (a) above plus the identity element for addition.

(c) Properties of closure, commutativity, associativity, identity element for multiplication. No inverse

(d) Same as (c) above, <u>minus</u> the identity element.

(e) Properties of closure, commutativity, associativity, identity element for addition. No inverse

(f) Same as (e) above.

(g) Same as (e) above, <u>minus</u> the identity element.

(h) Not closed; does have properties of commutativity and associativity; no identity element. No inverse

2. We could not make complete tables for above operations, as the set of elements is infinite.

Exercise 8

1\. A

(a) 6
(b) 7
(c) 0
(d) 1
(e) 6
(f) 5
(g) 2
(h) 1
(i) 0
(j) 1
(k) 1
(l) 0
(m) 9
(n) 6
(o) 3

B

(a) 5
(b) 3
(c) 6
(d) 2
(e) 0
(f) 1
(g) 2
(h) 5
(i) 0
(j) 11
(k) 0
(l) 1
(m) 0
(n) 5
(o) 6

2\. A

(a) 7
(b) 6
(c) 2
(d) 1
(e) 0
(f) 5
(g) 4
(h) 1
(i) 4
(j) 0
(k) 4

B

(a) 6
(b) 7
(c) 6
(d) 5
(e) 6
(f) 4
(g) 4
(h) 6
(i) 1
(j) 0
(k) 8

3. (a) 6
   (b) No answer - impossible
   (c) 3 and 7
   (d) 2
   (e) 0
   (f) 0 and 4

   (a) 6
   (b) 3 and 7
   (c) 0, 2, 4, 6
   (d) 0
   (e) 4
   (f) 9

4. (a) 4
   (b) 4
   (c) 7
   (d) 3
   (e) 4
   (f) 7
   (g) 6
   (h) 6
   (i) 6
   (j) 5
   (k) Yes
   (l) Yes

   (a) 2
   (b) 6
   (c) 7
   (d) 4
   (e) 3
   (f) 7
   (g) 8
   (h) 8
   (i) 8
   (j) 7
   (k) Mod 7
   (l) Mod 14

5. (a) 3 is the square root of 3; 2 is the square root of 4; 4 is the square root of 4; 1 is the square root of 1; and 0 is the square root of 0.
   (b) Yes, 1
   (c) No
   (d) Yes, 3 and 0
   (e) Yes, 2 and 5
   (f)

Number	Square Roots of the Number
0	0
1	1 and 5
2	none
3	3
4	2 and 4
5	none

6. (a) 4, 9, 16, 25, etc.
   (b) Yes
   (c) No, 2 does not have a square root in the system of natural numbers.
   (d) No
   (e)

Number	1	4	9	16	25	36	49	64	81	100
Sq. Roots of the Number	1	2	3	4	5	6	7	8	9	10

UNIT XIV

Sample Test Questions

PART I. TRUE - FALSE

T 1. Operations can be defined by tables.

T 2. A symbol can be made to mean anything providing we define it.

F 3. The identity for multiplication in ordinary arithmetic is zero.

F 4. The identity for addition in ordinary arithmetic is one.

T 5. The identity for addition with a 4-minute clock (modular 4) is 4.

T 6. The additive inverse of 2 in the mod 4 system is 2.

T 7. In ordinary arithmetic, the inverse of division by 4 is multiplication by 4.

F 8. All algebraic systems are sets of numbers.

T 9. In mod 6 arithmetic, $\frac{0}{3} = 2$.

T 10. Negative numbers are not needed in modular systems.

PART II. COMPUTATION.

Find the sums:

(11) 1. (9 + 2) mod 12

(0) 2. (5 + 4 + 3) mod 6

Find the differences:

(3) 3. (5 - 2) mod 6

(5) 4. (3 - 5) mod 7

Find the products:

(6) 5. [(3 + 7) × 6] mod 9

(1) 6. 3^2 mod 8

Find the quotients:

(4) 7. $\frac{2}{3}$ mod 5

(0) 8. $\frac{0}{7}$ mod 11

PART III. MULTIPLE CHOICE.

The table below describes a mathematical system. It is to be used in answering questions 1, 2, and 3 below.

⊙	A	B	C	D
A	C	D	[illegible]	B
B	D	A	B	C
C	A	B	C	D
D	B	C	D	A

1. Which one of the following statements is true?

   A. The set (A, B, C, D) is <u>not</u> closed with respect to the operation. ⊙

   *B. The operation ⊙ is commutative.

   C. The operation ⊙ is commutative for <u>only</u> elements A and B.

   D. The operation ⊙ is <u>not</u> associative.

   E. None of the above.

2. The identity for the operation ⊙ is:

   A. D.

   B. B.

   *C. C.

   D. both A and B

   E. none of the above.

3. In the mathematical system:

A. only B has an inverse.

B. only D has an inverse.

C. only A and C have inverses.

D. none of the elements has an inverse.

*E. all the elements have inverses.

4. For what modulus is 2 - 5 = 4 true?

A. mod 9

B. mod 6

C. mod 8

*D. mod 7

E. none of the above.

5. For the system consisting of the set of odd numbers and the operation of multiplication:

A. the system is not closed.

B. the system is not commutative.

C. the system has no identity element.

*D. none of the above is correct.

E. all of the above are correct.

6. For the system consisting of the set of even whole numbers and the operation of addition:

A. the system is not closed.

*B. the system has an identity element.

C. the system has an inverse for addition for each element.

D. all of the above are correct.

E. none of the above is correct.

7. A mathematical system consists of several things. Which of the following is always necessary in a mathematical system?

A. numbers

B. an identity element

C. the commutative property

*D. one or more operations

E. none of the above

8. When a product has two identical factors, we call one of them:

*A. the square root of the product.

B. the square of the product.

C. the inverse element of the product.

D. the identity element of the product.

E. none of the above.

9. Modular arithmetics are:

A. not commutative.

B. not associative.

C. not distributive.

D. not closed with respect to addition.

*E. none of the above.

Use the mathematical system as described below in answering questions 10, 11, and 12 below. The set of elements in our system is the set of changes of a rectangle.

The elements are:

I means leave alone — H means flip on the horizontal axis — V means flip on the vertical axis — R means turn halfway around its center

The following is an illustration of our operation *

V * H means do change V and then do change H.

Thus V * H = R

10. H * H is:

A. H

*B. I

C. R

D. V

E. none of the above.

11. I * R is:

*A. R

B. V

C. I

D. H

E. R * H

12. (H * V) * V is:

A. I

B. V * V

*C. H * I

D. V

E. none of the above.

Date ____________

UNIT: ____________

Name of Teacher: ____________

Name of School: ____________

City: ____________ State: ____________

Number of days given to the teaching (including testing) of this unit: ____________

Approximate dates: ____________

USE THE BACK OF THIS SHEET IF YOU NEED EXTRA SPACE TO ANSWER ANY OF THE QUESTIONS BELOW

1. Make a statement about the ability level of the pupils in the class and state whether your school uses some plan of homogeneous grouping.

2. What parts of the unit proved to be the most teachable?

3. What parts of the unit proved to be the most difficult to teach?

   Did you omit any part? ________

4. Did you use any supplementary developmental materials? ______

   If so, what were they, and at what points were they used?

5. Did you find it necessary to provide the pupils with additional material? ________

   If so, was it from textbooks or did you write your own?

6. Do you think that a unit on this topic should be included in regular textbooks for 7th and 8th grades? ________

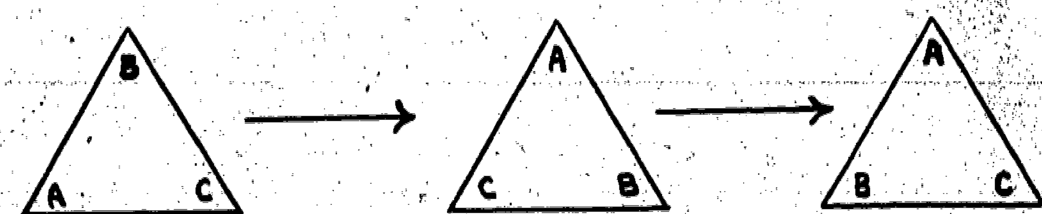
7. Please make ANY additional comments about your teaching experience with this unit which you think would be helpful to the Panel responsible for preparing and experimenting with textbook materials for grades 7 and 8.

Summary of Teachers' Comments

Only two teachers reported on unit XIV; both had pupils of average ability; both thought the material should be included in the seventh grade curriculum; no material was omitted; one teacher spent 8 days, and the other spent 10 days on this unit.

One teacher found the introductory material to Exercises 4 easiest to teach. The other found rectangle changes easiest. The following points were mentioned as being difficult to teach:

1. Algebraic systems without numbers.
2. Motivation for the concepts of closure and inverses.
3. Problem 3 of Exercises 5 needs careful explanation. The following additional information was considered useful: The axes will remain stationary; e.g. the flip about the axis through B in the figure will always be a flip about a vertical axis. This is important when such a flip follows the turning of the triangle about its center. Example: First (turn the triangle 120° about its center) then (flip about the vertical axis) would be represented by the following sketches:



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