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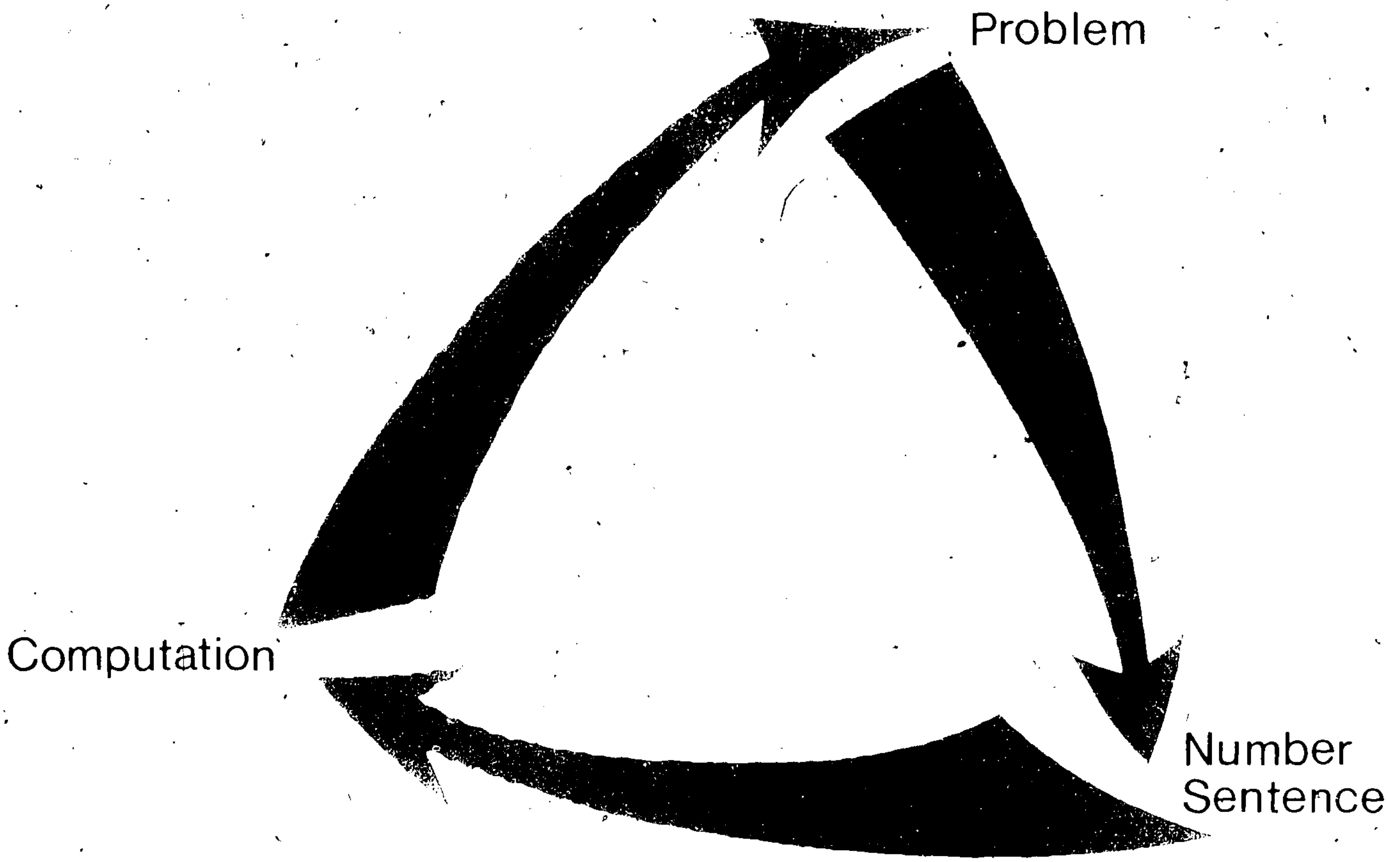
ABSTRACT

This guide is for the use of teachers in the primary grades and is organized around six concepts: sets, numbers and numeration; operations, their properties and number theory; relations and functions; geometry; measurement; and probability and statistics. Objectives and sample activities are presented for each concept. Separate sections deal with the processes of problem solving and computation. A section on updating curriculum includes discussion of continuing program improvement, evaluation of pupil progress, and utilization of media. (MP)

mathematics for georgia schools

VOLUME I PRIMARY GRADES



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FOREWORD

The Georgia Department of Education is constantly alert to the curricular changes which seem desirable as a result of studies and experiments in various fields. A committee was appointed in 1969 to rewrite the mathematics curriculum guides for the elementary schools incorporating findings from current curriculum studies in mathematics.

This committee was composed of rural and city public school teachers and supervisors, college teachers and one out-of-state consultant who is nationally known in mathematics education. They looked at the nation's best programs in mathematics education. They considered creative ideas of teaching which fit the age of space but which are as fundamental as adding two and two. They recognized that mathematics is an essential part of life itself and is a daily necessity for all people. The committee, taking the position that mathematics instruction is a process of initiating and nurturing understanding, felt that it would be necessary to discover techniques for accommodating the differing rates at which children develop.

LETTER TO PRIMARY GRADE TEACHER

This guide has been written to assist you in improving the teaching of mathematics in the primary grades. The committee and I have worked diligently for several years preparing this material and trust that the format is arranged so that it will be useful to you.

The guide has been organized around six central concepts called strands. They are entitled (1) Sets, Numbers and Numeration, (2) Operations, Their Properties and Number Theory, (3) Relations and Functions, (4) Geometry, (5) Measurement, (6) Probability and Statistics. These strands include the major mathematical concepts which undergird an updated mathematics program for children. The concepts are threads running through the curriculum and are expanded and enlarged in a spiral approach.

Each strand is introduced in terms of broad performance objectives which the teacher can make more specific by adapting them to the needs of particular children. There are one or more activities keyed to each objective. The list of objectives for each strand is placed at the end of the strand on a fold-out sheet. This allows the teacher to view the objectives as he selects activities to implement specific objectives. These activities are not sufficient to achieve the objectives. They are suggestions of kinds of experiences which will help reach the objectives.

The strands on Probability and Statistics and on Relations and Functions are included particularly because of new ideas in elementary school mathematics. It is hoped that teachers will accept the challenge of new topics, different approaches and experimental activities as a means of extending the spiral learning of mathematics for all pupils according to their potential.

There are separate sections in the guide which deal specifically with processes. Problem solving is considered a part of all mathematics and therefore is emphasized in a cross-strand approach. Computation, also, is viewed by the committee as permeating all strands, and the related section is intended to give detailed development for especially difficult procedures. Problem solving is *thinking through*, and computation is the manipulation of various symbols and terms used to express these thoughts.

Other sections are included to facilitate use of the guide by the teacher. While not prescriptive, the content and methods identified throughout the guide are of increasing importance in a contemporary mathematics program. The section on media lists instructional aids, and the use of aids is suggested in the activities of each strand. The correct use of the materials will help in the achievement of the objectives. Teachers should realize the importance of teaching children correct vocabulary and correct use of symbols. A glossary for the teacher is included to provide definitions which can be simplified into children's language. Words often used in daily communication, particularly some geometric terms, have a different meaning when considered mathematically. Symbols are to be understood as a means of stating problems and recording results after meaningful experiences with physical models of mathematical principles.

The teacher, guided by the objectives in each strand, should endeavor to determine those topics and activities most appropriate for realizing the objectives for the particular children being taught and should correlate these ideas with those in texts and other available materials. After a strand has been presented, the teacher should evaluate in terms of the objectives using instruments constructed for this purpose. Sample instruments are included in the Evaluating Pupil Progress section. In the bibliography are suggested materials designed to help implement achievement of the objectives. Early selection and purchase of materials for the library, a grade level or an individual class will insure access to books when needed.

Inservice programs for those who need help with the new ideas will result in a more competent faculty as well as increased knowledge on the part of the children. Assistance in improving local programs may be found in the section Continuing Program Improvement.

Gladys M. Thomason

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POINT OF VIEW

Curriculum planning is a continuous process of updating content, improving methods, analyzing objectives, measuring learning and appraising attitudes. The guide, *Mathematics for Georgia Schools*, is to help local curriculum committees and teachers of mathematics to identify the content, procedures and materials which will strengthen and enrich the mathematics educational program for the elementary school children of Georgia, and to measure the effectiveness of the program.

In the guide objectives are stated in behavioral terms. Local curriculum committees may find it helpful to state more specific objectives. The activities support the theory that learning is experiencing. The objectives and activities are organized into six strands written for primary grades and upper grades.

The ordering of the strands in the guide does not imply the ordering of presentation of subject matter; that is, one strand need not be completed (or even begun) before proceeding to another. The volume of material on different topics does not imply that one is more important than the other. Topics especially difficult to present and those not generally covered in currently available textbooks are developed in more detail. Individual teachers will need to make appropriate choices according to the needs of their pupils.

One strand which has emphasis is Relations and Functions, since most of mathematics involves relations between numbers and/or geometric figures. Since relations and functions are unifying concepts in mathematics, children should be encouraged to think in terms of them.

The guide does not restrict geometry to naming shapes and measuring them. Emphasis in geometry is placed on the relations between point sets such as *has the same shape, parallel to* and *congruent to*. The activities enable children to work with materials in order to learn these relations for themselves.

The emphasis on sets in this guide endorses the concept that the language of sets is a powerful tool in communicating mathematical ideas and can be used both to organize and describe.

Evaluation is a continuous and integral part of the successful elementary school program. The techniques of evaluation must include procedures for appraising interests and attitudes as well as skills and understanding.

Perhaps the one factor most essential to the success of the mathematics curriculum is understanding. To promote understanding a distinction is made between operations and computations. An operation is an assignment of a single number to an ordered pair of numbers. Computation is a process of manipulation of numerals by which one determines a name of the single number assigned to the ordered pair of numbers. The need to find a more efficient and enlightening method of instruction has led to the conclusion that clear understanding of essential mathematics concepts must precede, but certainly not supplant, the traditional point of emphasis, computation.

If mathematics instruction is viewed as a process of initiating and nurturing understanding, it will be necessary to discover techniques for accommodating the different development rates of children. Children develop concepts of mathematics from their experiences with physical objects. This guide is designed to help the teacher exercise professional judgment in adopting a mathematical program compatible with each child's ability.

CONCEPTS ACCORDING TO STRANDS

Sets, Numbers and Numeration

Operations, Their Properties and Number Theory

Relations and Functions

Geometry

Measurement

Probability and Statistics

SETS, NUMBERS AND NUMERATION

INTRODUCTION

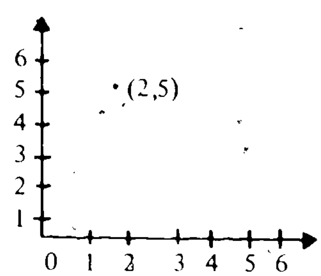
The concept of a set is a useful tool in the study of mathematics, and the language of sets enables one to communicate mathematical ideas with clarity and precision. In fact, the fundamental concept of whole numbers is based on the recognition of classes of (finite) sets. For example, there is a class of sets which can be placed in one-to-one correspondence with the set of fingers on one hand. The members of these sets may be big things or little things, animal, vegetable or mineral; yet they all have one property in common *five*. To understand what *five* means, then, is to recognize any set having this property.

Counting is a process of determining the whole number property, or cardinality, of finite sets. It is important to realize that counting techniques are developed slowly. It is not sufficient that a child be able to recite number words in order. He must also be able to establish a one-to-one correspondence between a set of number words and a set of objects and then realize that the last-named number in the correspondence is also the cardinal number assigned to the set.

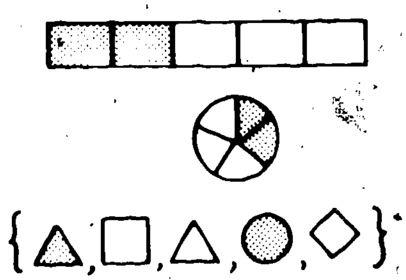
The place value scheme used in recording numbers greater than nine is an arbitrary convention and the basis for computational algorithms which children are expected to learn. Many of the difficulties which they have with computation is due to not understanding the convention, or coding scheme, called the decimal numeration system. For that reason, it is suggested that young children be provided many first-hand experiences with *bundling* concrete objects, such as popsicle sticks or tongue depressors, and then counting the bundles and singles. Such experiences should precede or, at least, accompany their learning to read and to record the count of tens (bundles) and ones (singles) — that is numbers greater than nine. Although early school experiences do not include computations with numerals of two or more digits, it is suggested that primary grade teachers examine the section on Difficulties in Computation in order to understand how children use place value in later grades.

Particular attention should be given in the primary grades to applications of whole numbers in which ordered pairs are used to quantify or tell the number story of certain kinds of experiences. For example, children use *ordered pairs of whole numbers* in all of the following contexts.

- A position context, in which the pair 2 and 5 are used in the ordinal sense; written "(2,5)" and read "2nd and 5th," the pair indicates a child's location in a seating chart – that is, the 2nd row, 5th seat.
- A rate context, in which the pair of numbers 2 and 5, written $\frac{2}{5}$ and read "2 for 5," tells a rate of exchange of balloons for pennies.
- A date context, in which the same pair used again in the ordinal sense, written "2/5" and read "February the 5th," identifies a particular day in the calendar sequence.
- A solution context, in which the pair 2 and 5, written "(2,5)" and read "2,5," is an element in the solution set of the open sentence $\square + 3 = \triangle$.
- A point-in-the-plane context, in which the same pair, written "(2,5)" and read "2,5," is associated with a point in the coordinate plane, as illustrated here.



- A ratio context, in which the pair, written "$\frac{2}{5}$" and read "2 to 5," expresses the ratio of the number of holidays to the number of school days in a particular week.
- A fraction context, in which the same pair, written "$\frac{2}{5}$" and read "2 out of 5," "2 over 5" or "2 fifths," is associated with a particular partitioning of units as illustrated below.



2 parts out of 5 parts in a unit strip.

2 parts out of 5 parts in a unit disc.

2 elements out of 5 elements in a unit set.

The use of ordered pairs in the latter context is introduced in this strand and is the basis for the development of the concept of rational numbers.

It is important to note that the meaning of the symbols "(2,5)" or "$\frac{2}{5}$" for the ordered pair of whole numbers 2 and 5 is dependent on the context in which the pair is used. All of the contexts indicated in the examples should be treated in the elementary school. Teachers should see other strands, in particular, the strand on Relations and Functions for activities involving other uses of ordered number pairs.

Whole numbers are also used in the context of direction from some arbitrarily selected point or position. For instance, young children use expressions such as "2 degrees below zero," "3 places to the right" or "5 steps up from the stair landing," etc. Primary grade teachers can capitalize on these verbal descriptions of experiences and provide readiness activities for the development of the concept of integers. Certain basic readiness experiences can begin even in the kindergarten and be expanded throughout the grades.

SETS, NUMBERS AND NUMERATION

Objectives Keyed to Activities | **ACTIVITIES**

obj. 2a — 1. Ask children to find and describe verbally collections of objects in the classroom such as the following.

set of dishes on the table
tray of milk cartons
bag of marbles
box of crayons
set of dolls in the doll house
sack of blocks
set of boys in the first row

Note that many different words are used to designate collections or sets. In developing a language of sets one needs to differentiate between the collection and the objects in the set. A box of crayons is different from the crayons. A bag of marbles is different from the marbles.

Obj. 2a — 2. Have in plastic bags a collection of objects such as an airplane, a boat, a car, a motorcycle. Have the child identify the members of the set. Have the plastic bag identified as the set holder. Remove the objects from the bag, and ask the children what is in the bag. When they reply "nothing," ask what one says a bag is when there is nothing in it. Some child will reply "empty." Say, "When we have the set holder empty, it represents the empty set." The empty set may be represented as { } or the set holder with no members.

obj. 2b — 3. Objects mentioned in activity 2 may easily be separated into subsets such as the subset of objects which travel in the air, the subset of objects which travel on water, the subsets which travel on land and other types of subsets according to the number of wheels, no wheels, wings, no wings, etc. Help the child see many subsets of the original set.

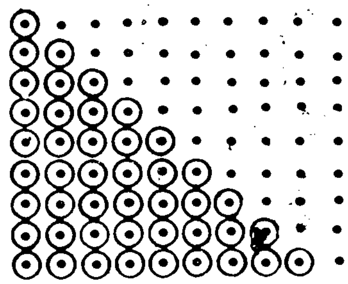
obj. 2a 2b — 4. Have triangular, circular and square discs of three or more colors. Have the child group this set by using yarns of various colors to show subsets of the original set such as the set of triangular shapes, the set of circular shapes, the set of square shapes. Other subsets of the original set may be made by selecting all discs of one color, then all of the second color and so forth.

obj. 1 — 5. Have the child's name written on two objects such as a flower and a flower pot. Have the flowers placed in some place where each child can match his name on the flower with his name on the flower pot. In matching these one-to-one the child not only learns to read his name but learns there is one flower for each flower pot. He can also learn to tell if there are pots with flowers missing for absent children. This process gives him experiences with the relations more than and less than as well as one-to-one correspondence. (Yellow jackets or bees and hives may be used in place of flowers. Butterflies and flowers, apples and leaves or seasonal objects may be used.)

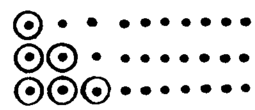
Many activities of matching one-to-one on felt board or using real objects need to be used. Supply a third object to be matched to the first two (as add to the flower and flower pot a bee for each flower) as readiness for the transitive property of one-to-one correspondence. If there is a flower for each flower pot and a bee for each flower, is there a bee for each flower pot?

obj. 3,4 — 6. The child learns the language of counting through counting rhymes, singing number songs and finger plays. Counting words are learned in order by rote.

obj. 1,2,3, 4,5 — 7. Early, the child copies peg, bead or other patterns so that he sees one as a group of one, and two as a group containing two, etc. Pegs may be copied from a full pattern as the following.



After the child becomes aware of patterns he completes patterns seeing only the beginning rows such as the following.



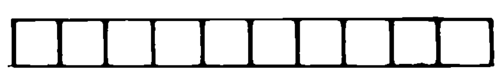
obj. 3,4, 5,6 — 8. The child learns to count by the comparison of sets. Given a set of three or four objects, the child constructs sets with one less or one more until he reaches the empty set on the lesser side. The child then orders the sets from the empty set to the largest set and orders the numbers associated with the sets. He assigns the names learned by rote to the sets having that number of members. By ordering the sets the student is able to answer questions such as the following.

Four is how many more than three?
Four is how many more than two?
Four is how many more than one?
One is how many less than four?
One is how many less than three? etc.

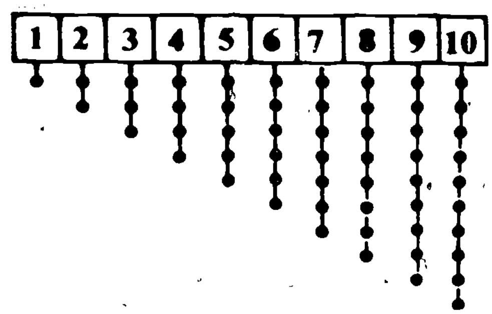
The concept of betweenness is developed by questions such as "What set is between the set of one and the set of three?"

obj. 1,2 — 9. As the child progresses toward the abstract he needs help with counting a fixed set of objects, such as a set of pictures on a page. To make this transition, the child places movable objects (as counters, chips, bottle caps) on the pictures on the page. Through a one-to-one correspondence of counters with pictures, the child recognizes the cardinal number of the set of pictures on the page to be the same as the cardinal number of the set of counters.

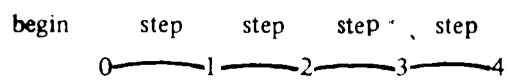
obj. 1,2,6, 7,9 — 10. The child develops the idea that the number that is one more than the preceding number is the next consecutive number in counting by ones. If a blank number line is presented to the child so that he can place a card with a picture on it in the first space and one in the second space, he can see that the set of pictures is two.



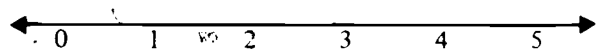
When three pictures are placed along this line in the first, second and third position, the last number named in counting is the cardinal number of the set. By placing a 3 in the third position the idea is conveyed that these three spaces make a set of three. Another form of presenting a beginning number line is to have a line of numerals with a set of objects associated with each number hanging from the block containing the numeral as illustrated here.



obj. 6,7 11. The child walks along a line putting a mark where he puts his foot as he steps. As the child begins he is at the zero point. By the point where he puts his foot for the first step, he places a one, where he steps the second time he places a two, etc. The result should be a line like this.



The child sees that the first step is from 0 to 1, the second 1 to 2 and third 2 to 3. These points become points on a number line.



The teacher should be careful to see that the unit from 0 to 1 be equal to the unit between each pair of the succeeding numbers. Repeat this experience using toy models of a rabbit, grasshopper, frog, jumping along the line, until the child recognizes that the points of landing form the set of numbered points.

obj. 5,8 12. Once the child has learned to order the numbers from one to nine, he is introduced to the numerals and number names for ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety.

Using bundles of ten, or a number frame, the child should realize that these names are used in counting by tens. One group of ten is ten, two groups of ten are twenty, three groups of ten are thirty. When the child reaches ten groups of tens, he should recognize that he has no symbol for 10 tens. Here the teacher supplies the name "one hundred."

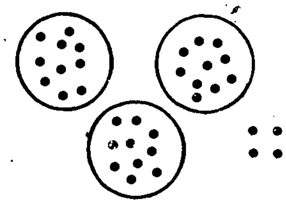
obj. 7,8 13. Have the child place ten objects (sticks, toothpicks, strips of cardboard or other objects) in bundles and count the number of bundles. The child should be shown that the coding system requires that the count of ones is recorded on the right of the numeral and the count of tens is recorded to the left of the ones. That is, ten is written "10" meaning one bundle (ten) and 0 singles (ones). Eleven, 11, then is one ten and one one; twelve, 12, is one ten and two ones, etc.

obj. 8 14. The child demonstrates that twenty-three is two groups (bundles) of ten and three singles. The child is required to place bundles and singles so that the tens are on the left and the singles on the right. This collection is called "two tens and three ones" and recorded, "23."

The above exercise should be repeated until the child has no difficulty in demonstrating two-place numerals in bundles of ten and singles.

obj. 8 15. The child counts random objects on a page by circling groups of ten and records the number of groups and the number of ones.

Example



three tens and four ones
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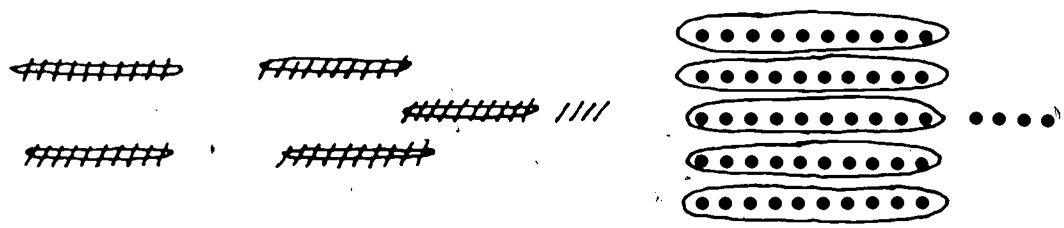
Provide many similar activities.

obj. 5,7,8 16. Have the child demonstrate the meaning of the symbol "54" as five tens and four singles by showing 5 bundles of ten and 4 singles. At a semi-concrete level he may draw bundles of ten and singles as shown.



obj. 9 17. The child may exhibit some awarenesses of ordinal uses of number which occur in out-of-school activities. For instance, in playing games children often use expressions such as, "I go first. You go second. He goes third." The teacher should build on pupils' experiences with ordinal ideas. In classrooms where desks are arranged in rows, the teacher may ask questions such as "Who is sitting in the second row, third seat?...the fifth row, first seat?" After class discussion of the ordering of rows in a classroom and the ordering of seats in each row, the teacher could sketch a seating chart on a large card, using only the ordinal number pairs in the cells of the chart as shown here.

1st,6th	2nd,6th	3rd,6th	4th,6th	5th,6th
1st,5th	2nd,5th	3rd,5th	4th,5th	5th,5th
1st,4th	2nd,4th	3rd,4th	4th,4th	5th,4th
1st,3rd	2nd,3rd	3rd,3rd	4th,3rd	5th,3rd
1st,2nd	2nd,2nd	3rd,2nd	4th,2nd	5th,2nd
1st,1st	2nd,1st	3rd,1st	4th,1st	5th,1st

The following day, the teacher should use the seating chart to call the roll. As he calls out the ordinal number pair "4th, 3rd," the child occupying that seat would stand to respond, "Here," or "Present." The teacher could vary the order in which he calls the roll. For instance, one day he might call the ordinal number pairs along a diagonal such as "(1st, 1st), (2nd, 2nd), (3rd, 3rd), (4th, 4th), (5th, 5th)." Another time he could call the pairs, "(1st, 6th), (2nd, 5th), (3rd, 4th)," etc. An alternative would be to call the child's name and have him respond with the ordinal number pair which identifies his location.

In all primary grade classrooms there are displays or charts which offer opportunities for children to learn ordinal uses of numbers. Page numbers, room numbers, house numbers, car tags and calendar dates are only a few of the many instances of the ordinal use of number which teachers may take advantage of to introduce and develop children's understanding and skills in the ordinal use of numbers.

obj. 10

18. Choose some physical model of a unit such as an apple, a circular cardboard or felt disc or a sheet of paper, and say to the class or a small group of children,

(for apples) "I am going to slice this apple like this."
(for disc) "I am going to cut across this disc like this."
(for paper) "I am going to fold this paper like this."

After partitioning the model in some way by slicing, or cutting, or folding so that there are two pieces of the same size, ask, "How many pieces do you see now? Count the pieces." After the children have counted and reported that there now are two pieces, write the numeral 2 on the board (or a large card so that all can see). Then hand one of the pieces of apple to a child, have a child shade one of the pieces of the disc or have a child color the paper region on one side of the fold line and ask,

(for apples) "How many pieces of the apple did I hand to John?"
(for disc) "How many pieces of the disc did Mary shade?"
(for paper) "How many parts of the paper did Susan color?"

After the children have responded, "One piece" or "One part," write the numeral "1" on the board (or card) so that the two numerals look like 1, 2 or $\frac{1}{2}$.

Read the numeral pair "one comma two" or "one over two" and say, "This pair of numbers tells the number story for what we have just done."

(for apple) "The number '2' is a count of the pieces of the apple we sliced, and the number '1' is a count of the pieces we gave away."

(for disc) "The number '2' is a count of the pieces of the disc we cut, and the number '1' is a count of the pieces we shaded."

(for paper) "The number '2' is a count of the parts of the paper we got by folding, and the number '1' is a count of the parts we colored."

The teacher, working slowly and carefully with the children, should repeat these kinds of activities many times. The physical acts of cutting, separating, folding, partitioning of the models should be done by teacher and children in the class. In first experiences with ordered pairs of counting numbers, it is suggested that the teacher not use precut or pre-partitioned models. The physical act of partitioning should be part of the learning activity.

Note that the word "divide" is not used to describe the act of partitioning a physical thing. Why? The word "divide" should be reserved for classroom use in talking about the operation of division on numbers.

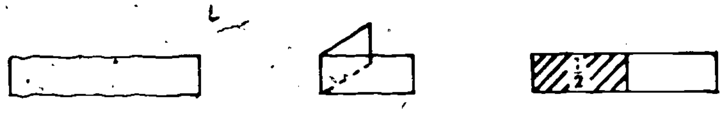
obj. 11

19. Repeat activity 18 with many different units in many different settings in which children assign ordered number pairs through (a) counting the number of pieces of parts or regions of the same size into which a unit has been separated and (b) counting the number of pieces or parts or regions which have been shaded or colored or given away or retained or whatever. As the children develop the concept of the counting numbers through 8, work with developing the concept of ordered pairs of counting numbers such as $\frac{1}{2}$, $\frac{2}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{5}$, $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, $\frac{6}{6}$, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$, $\frac{8}{8}$.

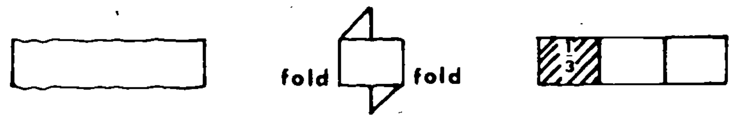
As children develop the concepts of larger numbers, introduce partitioning units into ten parts and twelve parts and naming the associated number pairs.

obj. 12

20. Provide children with fraction makers by cutting strips from sheets of paper. Give each child a strip and direct him to fold along a line so that the ends of the strip fit together. Then fold out and color one region. Write the pair of counting numbers which tells the number story that goes with that strip.

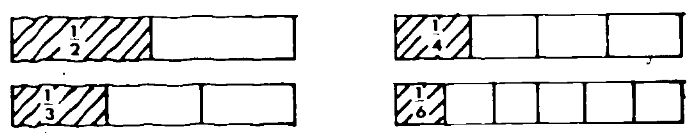


Give each child a second strip and direct him to fold the strip as in the illustration below. This, activity will involve some trial and error on the part of the child. The teacher will demonstrate with a strip; however, let each child work with folding until he has accomplished the fitting together of the parts of the strip. Then have him unfold and color one region.



Then have him write the number story that goes with that strip.

Continue having the children fold strips until on each child's desk there is a set of strips as pictured.



Then ask, "What do you notice about the colored parts of the strips?"

As the children respond, write on the board their statements. Suppose one child responds, "This is bigger than this." Write it down and ask another child if he can show that the first word "this" is bigger than the second word "this." The necessity for names of "this" will arise, and then you can write $\frac{1}{2}$ and $\frac{1}{3}$ in their respective places.

This is bigger than this.
$\frac{1}{2}$ is bigger than $\frac{1}{3}$

Also children may notice that one $\frac{1}{3}$ and two $\frac{1}{3}$'s make three $\frac{1}{3}$'s or $\frac{1}{3}$ and $\frac{2}{3}$ make $\frac{3}{3}$, etc. The teacher's continued encouragement to the children to talk about what they see can provide the first intuitive recognition of relationships among fractions. The encouragement to use number names for talking about pieces will help to develop the language of fractions. However, one must be careful not to insist on mastery of facts about fractions until children possess the concept of fraction.

obj. 11, 12

21. Repeat activity 20 except let the strips be of different length or width. The teacher needs to provide many experiences with partitioning different physical models of units and naming or writing corresponding number pairs. Thus, over a long period of time, the child abstracts the idea of ordered number pairs as fractions without regard to any particular model.

obj. 12, 13

22. To provide initial experiences and practice in working with ordered number pairs in the rate context, it may be advisable to begin with children's experiences in buying for themselves, as in buying bubble gum or ballons or pencils.

Have them make signs for a variety store or grocery store display. The teacher could provide a tray of balloons or pencils or wrapped candy mints for which the children must prepare the signs, such as $\frac{2}{5}$ or $\frac{3}{10}$ or $\frac{1}{2}$, which tells the corresponding rate stories.

Obj. 11, 12, 13

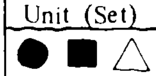
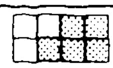
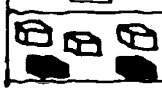
23. Early experiences in associating ordered number pairs with partitioning sets should develop slowly and carefully in order to avoid confusing rates and fractions. For instance say there are six children, four boys and two girls, in the first row. The ordered number pair, $\frac{2}{4}$ which tells the ratio of the number of girls in the set to the number of boys in the set is a rate. Whereas, the order number pair, $\frac{2}{6}$, which tells the ratio of the number of girls in the set to the number of children in the set is a fraction. In developing fractions associated with finite sets of objects, start with some physical model of a unit set. For instance, choose a set of 3 balls with 2 of them rubber balls and 1 of them a wooden ball, a set of 3 children with 2 of them boys and 1 of them a girl or a set of 3 crayons with 2 reds and 1 blue, and say, "How many members do you see in this set? Count the members."

It is important that the teacher's language be informal, but it must at the same time suggest that the class is considering the pieces of a unit set. The initial partitioning of the unit set into equivalent pieces is inherent in the tabulation of the members of the set, opposed to the folding or cutting of a cardboard unit of some kind in order to create pieces. After the children have counted and responded "3 members," write the numeral "3" on the chalkboard. Then ask, as appropriate, "How many balls in the set are rubber?" "How many children in the set are boys?" "How many crayons of the set are red?"

After the children have counted and responded "two," write the numeral "2" on the board so that the 2 numerals now look like (2, 3) or $\frac{2}{3}$. Read the numeral pair as "two comma three" or "two over three" and say, "This pair tells the number story of what we have just been talking about. The number '3' is a count of the members of the unit set, and the number '2' is a count of the members in a part of the unit set."

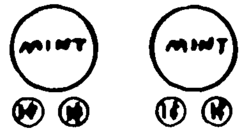
These kinds of activities should be repeated many times, with the emphasis on the unit set and a subset of the unit set. The corresponding number pair tells, in order, the count of members of pieces in the subset and the count of members or pieces in the unit set.

After experiences with sets of physical objects children may work with a chart as follows.

Partition of Unit (Set)	Number of Pieces in Unit Set	Number of Pieces Shaded	Ordered Number Pair (Fraction)
	3	2	$\frac{2}{3}$
	8	5	$\frac{5}{8}$
	4	1	$\frac{1}{4}$
	4	2	$\frac{2}{4}$
	2	1	$\frac{1}{2}$
	5	2	$\frac{2}{5}$

Consider the purchase of chocolate mints which sell for 1 for 2 cents.

The ordered number pair is $\frac{1}{2}$.



Pose the problem, "Suppose you buy 2 units. How many pennies will you need? What number pair tells that rate?"

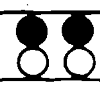
The ordered number pair is $\frac{2}{4}$.

Ask the question, "What can you say about $\frac{1}{2}$ and $\frac{2}{4}$?" "Do they tell the same rate?"

Other experiences with rate may involve comparisons such as "the rate of the number of boys to the number of girls on the first row" or "the rate of the number of dolls to the number of doll beds" or "the rate of the number of pens to the number of pencils in one's school bag."

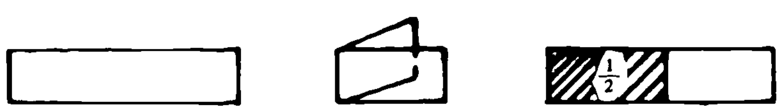
obj. 13

24. For the young child, ordered pairs of numbers such as $\frac{1}{2}$ and $\frac{2}{4}$ are closely related to physical models and in this context are associated with quite different partitions of units as depicted here.

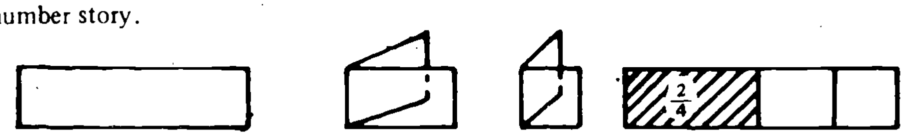
Partition of Unit Set	Number of Pieces in Unit Set	Number of Pieces Shaded	Fraction Name
	2	1	$\frac{1}{2}$
	4	2	$\frac{2}{4}$
	2	1	$\frac{1}{2}$
	4	2	2

Note that $\frac{1}{2}$ and $\frac{2}{4}$ in the first two examples in the chart tell the number story for quite different situations. It is suggested that no attempt be made to teach that $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent number pairs in this partition or similar ones of sets. In the last two examples the number pairs again tell the number story of different ways of partitioning physical units. Thus, $\frac{1}{2}$ and $\frac{2}{4}$ tell different number stories. In the latter case, however, the teacher can provide experiences from which children begin to abstract the notion that $\frac{1}{2}$ and $\frac{2}{4}$ are associated with equivalent amounts of a physical unit. Therefore, one agrees to say that $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent fraction names; note that one does not say $\frac{1}{2}$ is equal to $\frac{2}{4}$.

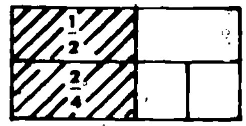
Provide many experiences in which children manipulate physical models as follows. Use paper strips as in activity 20. Each child has several paper strips, all of which are the same shape and size. Ask the children to fold a strip once so that edges meet, unfold and shade the piece of the strip on one side of the fold. Write the number story.



Then have them take another strip and fold so that edges meet. Then fold again so that those edges meet, unfold and shade the two pieces next to each other on the left as shown in the diagram. Write the number story.

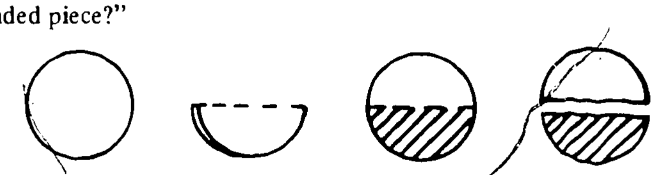


Say to the children, "Now match the finished strips. Put the strip with 2 pieces shaded over the strip with 1 piece shaded. What can you say about the shaded parts of both strips? If you cannot see both strips that way, try putting one just above the other like this.

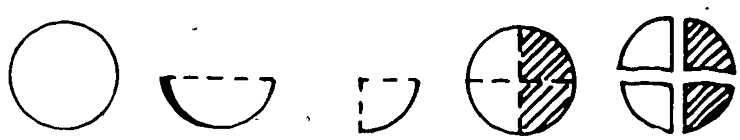


Now what can you say about the shaded parts of the two strips?" As they respond to the question, record their statements on the board, encouraging them to use the number pair names, as in the statement, "$\frac{1}{2}$ of that strip matches $\frac{2}{4}$ of this strip."

A second unit model might be a circular side of heavy paper which they can cut into pieces. Each child should have at least two discs of the same size. Say, "Fold the disc so that the edges meet. Now unfold. Shade the part on one side of the fold. Cut along the fold. What number story can we write for the shaded piece?"



Then, "Now take the other discs and fold as before. Then fold again so that the edges match. Unfold and shade two of the pieces next to each other. Cut along the folds. Now what number story can we write for the two shaded pieces?"



"Now see if you can match the shaded piece from both experiments. What can you say about the two pairs $\frac{1}{2}$ and $\frac{2}{4}$?" Perhaps the children will say, "We shaded the same amount of paper," or "$\frac{2}{4}$ matches $\frac{1}{2}$." Record their statements, and encourage them to search for and talk about how the two experiments were different and how they were alike. There should be many similar experiences with folding, cutting or partitioning units; shading, coloring or marking pieces; and matching the results in order that they eventually believe that $\frac{1}{2}$ of any unit in some way matches $\frac{2}{4}$ of the same or equivalent units.

In the primary grades perhaps children are likely to abstract the following equivalences among number pairs.

$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{6}{12}\right\}$ $\left\{\frac{1}{3}, \frac{2}{6}, \frac{4}{12}\right\}$ $\left\{\frac{2}{3}, \frac{4}{6}, \frac{8}{12}\right\}$

$\left\{\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{6}{6}, \frac{12}{12}\right\}$

obj. 14

25. The first experiences that the primary child should have with directed numbers should not require the assignment of $^{+}5$ or $^{-}5$ to a situation. He should use words like gain, loss, above, below, right and left to describe directions.

When the class is divided into two teams for any activity in science, social studies, spelling, etc., have the children refer to a correct answer as a gain and an incorrect answer as a loss. Instead of keeping score by adding a point each time a correct answer is given, let each team start with 50 points. The scorekeeper can then add 1 point for a gain and subtract 1 point for a loss. Encourage the child to use the words gain and loss.

obj. 14 26. In the study of weather, an activity that the children might enjoy would be to write on the board 15 or 20 dates from varying seasons. Write temperature readings for the dates. Include readings below zero, for example, September 8, 70°; October 15, 62°; December 5, 32°; January 30, 0°; February 5, -6°. Write this information on slips of paper, and have the child draw a date out of a box and record the temperatures. A class discussion could then follow the activity with the teacher asking questions like, "Who had the lowest temperature?" and, "Who had the highest temperature?" This would certainly give the teacher an opportunity to discuss temperatures represented by negative numbers.

obj. 14 27. Have the child play a game using an elevator in a building that has 4 basement floors and 10 stories above street level. Let the children take turns telling the elevator operator where they want to go by naming a positive or negative number. The class must first agree on which one they want to be represented by positive and which one they want to be represented by negative. Since they have had experiences with the thermometer, they will probably decide that above the street level should be positive.

obj. 14 28. Interesting games that will enable children to have experiences with directed numbers are described in books listed in the annotated references in the section on Utilization of Media.

SETS, NUMBERS AND NUMERATION

OBJECTIVES

The child should be able to do the following.

1. Place two sets in one to-one correspondence
2. a. Identify collections of objects as sets
   b. Select subsets of a given set
3. Assign the cardinal number to a set
4. Assign whole number names to sets of objects
5. Order the whole numbers
6. Put in one-to-one correspondence the ordered set of whole numbers and points on a line
7. Read and write the numerals 0, 1, 2, ... 9; read and write corresponding number words
8. Demonstrate the ability to use the place value code in writing two– and three-digit numerals
9. Demonstrate the ordinal use of numbers
10. Name the ordered pair of whole numbers associated with fractional parts of (a) units and (b) sets
11. Order several different fractional parts of equivalent units according to size, from smallest to largest, and name the corresponding fraction
12. Give an example of an ordered pair of whole numbers used in a rate context and an ordered pair of whole numbers used in a fraction context
13. Show that two or more different number pairs or fractions may be associated with equivalent fractional parts
14. Identify and describe everyday situations that require the use of directed whole numbers
15. Count by twos, threes and so on as well as by tens and hundreds, starting at different numbers

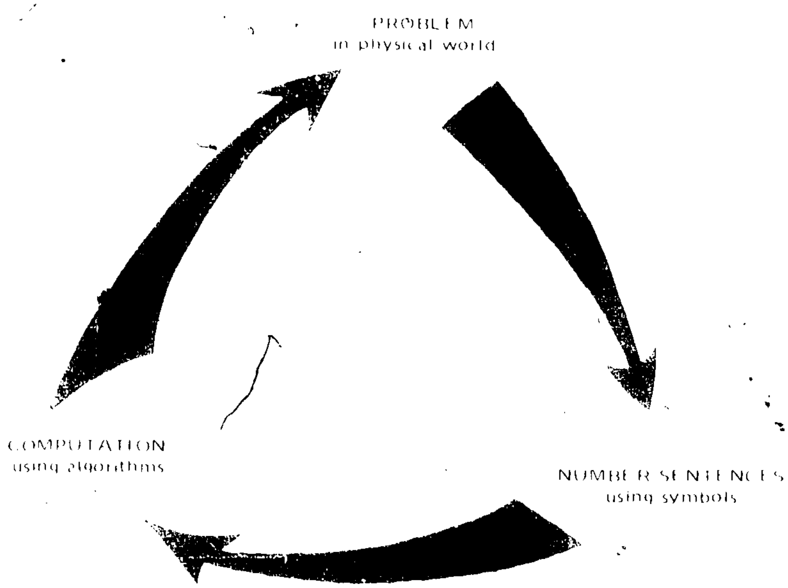
OPERATIONS, THEIR PROPERTIES AND NUMBER THEORY

INTRODUCTION

The purpose of this strand is two-fold. One purpose is to build the concepts of operations and their properties, and the other is to develop interest in number relationships through number theory.

In this guide, *operations and their properties* are separated from *computation* of numbers. The distinction between an operation and computation with numbers is an important one. An operation is a particular association of a given pair of members in a set with a particular member of the set. Computation, on the other hand, is the manipulation of numerals to determine the particular number (member) that results from the operation with the given pair of numbers (members).

The emphasis in studying operations is, initially, the association of a physical situation represented by mathematical symbols. In other words, the child learns to interpret a physical situation with mathematical symbols. The diagram that follows helps to explain how the concept of operations should be introduced to the child.



The child, for example, will learn that the physical action involved in joining is associated with the operation of addition. Likewise, separating a subset from a set is associated with subtraction. Again, physical situations can (and should) be used to introduce multiplication and division with whole numbers.

On the other hand, the child can also use the physical world to help interpret mathematical operations. He may find it helpful to think of 36−19=□ as resulting from a situation in which a child has 36 marbles and loses 19 of them.

In introducing mathematical operations, only operations with whole numbers will be initially be used. The activities for this part of the strand should emphasize a careful development and foundation for understanding the operations and their physical model interpretations. The children should manipulate objects, make up and act out stories. Teachers should be cognizant of the fact that the operational symbols are merely convenient modes of recording mathematical experiences, and that it can be disastrous to introduce the symbols before the children have had sufficient experiences with the manipulative materials or with concrete situations.

The teacher should wait until she feels that the children have understood an operation before she *tells* them the *name* of the operation, the *symbol* for it and how to record it. For example, after the children use many activities of *joining* disjoint sets and understand this process, they should readily accept the mathematical terminology. That is, the *joining* of two disjoint sets of objects is *associated with* the operation with numbers called "addition" and is shown by the symbol "+". The joining of two disjoint sets of 3 objects and 4 objects may be recorded using *goes with, associated with, matches* or *maps onto* as 4,3 *goes with* 7, or 4+3=7.

In using the operations, the child must know which is applicable to the situation in the problem at hand. For example, the number pair (6,2) can be associated with 4, with 8, with 3 or with 12. The child must select the appropriate operation for solution of his problem situation, and he must know which number is associated with the operation.

A special note should be made of the difficulty many children seem to have with the operations of subtraction and division as compared with addition and multiplication. Extra efforts must be made to help children understand and interpret these operations. As the child progresses in his understanding of the operations, he learns that addition and subtraction are inverse operations as are multiplication and division. He uses relationships in inverse operations to help him in his learning of basic number facts and in his computational proficiency.

The child should learn about properties of operations by manipulating objects and observing the number relationships on which the properties are based. In these experiences, the child through discovery may form generalizations and state them in his own words. In the early grades it is not as important for him to know the names of the properties as it is for him to apply them when appropriate. In the upper grades it is important for the child to identify the properties by name.

The second purpose of the strand is to develop interest in number theory. After acquiring a basic understanding of operations, the child may use this knowledge to explore number ideas through number theory. In working with operations, the learner begins with a pair of numbers to which a single number is assigned by a specific operation; in studying number theory the learner encounters such experiences as looking *inside* a single number. For example, one may look closely at the single number 49 to find the answer to questions as, "Is it a prime number? Is it an odd number? Is it a square number?" The child may also investigate number patterns in order to recognize numerous relationships of numbers.

In the early grades concrete objects such as beads of different colors and shapes, pegs on pegboards, strings of different lengths and colors may be organized into physical patterns to lead into the study of abstract patterns involving numbers. Investigating numbers and number patterns provides more challenging and appealing activities for the learner of mathematical concepts and basic facts than does the traditional drill activities or practice exercises.

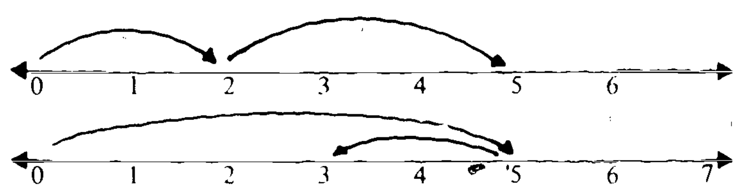
OPERATIONS, THEIR PROPERTIES AND NUMBER THEORY

Objectives Keyed to Activities	ACTIVITIES
	Addition and Subtraction
obj. 1a, 1b	1. *Joining of Sets* Place a rope, circle or hulahoop around John, Bill and Sam. Place another circle or hoop around Sue and Ann. Have these children come together inside a third circle to form a new set. The two sets have been *joined*. Discuss what has happened to obtain the new set and encourage the children to use such descriptions as "two sets are joined to make a third set."
obj. 1a	2a. *Associating Addition with the Joining of Sets* Use pieces of string or large rubber bands for enclosing sets of objects. Have the children bring some very small objects to be used on an overhead projector such as a life saver, a paper clip, a safety pin, a pencil, a toy airplane and a car. Place the objects on a table near the overhead projector. Ask a child to place a set of some of these objects on the overhead projector and place a fence around it using a string or rubber band. Discuss how many members are in the set. Write the numeral under the set. Ask another child to place another set of objects on the projector and enclose it. Discuss and write the numeral under the set. Now ask how many objects are in each of the two sets. Next, ask a child to join the two sets into one set by placing a fence around both sets. Then ask how many members are in the new set. Record this experience, for example, 3, 2 $\xrightarrow{\text{goes with}}$ 5. Discuss this association of a pair or numbers with a single number.
obj. 1a	2b. A variation of activity 2a can be done using clear plastic bags and small objects that can be easily handled and can be seen through the bags. The objects in each bag form a set. A larger bag can be used when joining sets. Discuss the association of a pair of numbers with a single number.
obj. 1a, 1b	3. *Using Numerals and Symbols to Represent Addition* Have the children make up simple stories to illustrate joining and separation of sets. Pictures or drawings help children to understand these actions with sets. Stories not only strengthen the understanding of joining and separation, but give the children an opportunity to learn to verbalize, for example, the teacher asks, "Who saw something yesterday that illustrates joining of sets?" Johnny says, "Mother found six eggs in a carton and then found two more on the egg tray of the refrigerator. So we have eight eggs in the refrigerator."
obj. 1a, 1b	4. *Separation of a Subset from a Set* In classroom settings similar to those described above, the children should have experience separating a subset for a set. For example, let the children act out a story similar to the following. Seven children were at the front of the room, and two children returned to their seats. How many children were left at the front of the room? Other experiences could involve physical objects such as sticks, beads and flannel cut outs. Pictures illustrating removal of subsets from sets can be used with the children discussing the actions depicted. Language for describing physical models and sets need to be developed before symbols are introduced.
obj. 1a, 1b, 2	5a. *Associate Subtraction with the Separation of a Subset from a Set* After the child has had numerous experiences with physical objects, the teacher can represent the separation of a subset

from a set with mathematical symbols. Thus, the story about the seven children in front of the room and two that return to their desks may be illustrated by 7 – 2 = □. By the same token, the children should have experience in interpreting a subtraction sentence as a story problem. For example, for the sentence, 6 – 4 = □ , encourage them to make up a story like the following. "There were six ducks. Four flew away. How many ducks were left?"

obj. 3 5b. Again ask four boys and three girls to walk to the front of the room and arrange themselves into two sets, one of four boys and the other of three girls. Ask other members of the class how many boys there are and how many girls there are. Ask the four boys and three girls to join into one set. Now ask, "How many children are there?" Ask a child to record on the chalkboard the number in each of the two sets and the number of the set after the boys and girls are joined, for example, 4, 3 → 7 or 4 + 3 = 7. Now there is a set of seven children. How can this set be separated and put back into the original sets? After the set of boys has been separated from the seven children, encourage discussion of what is being done. 7 – 4 = □. Ask the children how many girls must be in the other set. Now reassemble the four boys and three girls into one set of seven once more. This time, ask the three girls to separate from the set of seven. How many boys are left? 7 – 3 = □. Children may write the number sentences on the chalkboard to record the putting together and separation of the set of seven children.

obj. 1a, 1b 6. *Addition – Using a Number Line* The number line may also be used to illustrate addition and subtraction. When the children have had experience with the various types of joining and separating with sets, the number line can be introduced.



illustrates 2 + 3 = 5

illustrates 5 – 2 = 3

obj. 1a 7. *Match a Number Sentence with a Story* The children should have experience in matching or selecting from a choice of sentences a particular story problem. Initially, the matching should be addition only, then with subtraction only, and then finally mixed addition and subtraction.

John had three marbles. He got some more, then he had eight.
How many marbles did he get?

3 + 8 = □

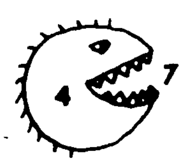
or

3 + □ = 8

or

□ – 3 = 8

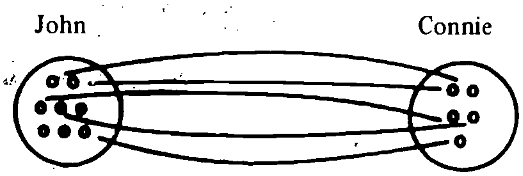
obj. 1a,1b,2 8a. *Take away and Comparative Subtraction* Take away subtraction is the removing of a subset from a set; this can be thought of as a number eater. For example, if a situation is reflected for 7 – 4 the child can think about this as a 4 eater, removing 4 from 7.



Not all subtraction can be thought of as take away. Some situations are comparative. For example, look at the following problem.

John has eight marbles. Connie has five marbles.
How many more marbles does John have than Connie?

In this problem the children should match the individual members of the set.



In this example, the child can see through matching that John has three more. The problem is represented by 8 − 5 = □. Comparative subtraction tends to present more difficulty for children than the take away subtraction. As a consequence, it should only be introduced after the children are confident of take away subtraction.

obj. 1b,2

8b. Place 5 animal cutouts on the flannel board. Ask the child to use all of the cutouts to make two sets. Discuss how many cutouts are in each set. Place that numeral under each set. Ask a child to join the two sets into a new set. How many cutouts are in the new set? Now place a numeral under this set telling how many members it has. Ask if anyone can tell a number sentence to illustrate what has been done. They may name "4 with 1" or "4+1;" "1 with 4" or "1+4;" "3 with 2" or "3+2;" "2 with 3" or "2+3;" "0 with 5" or "0+5;" "5 with 0" or "5+0." (Note: If an empty set is given as one of the two sets, discussion of the empty set and zero will probably be needed before continuing.)

obj. 1a,1b,4

8c. Give some child five pennies saying, "Here are some pennies; please put them on the overhead projector." Then, "How many pennies are there?" Then ask him to place the pennies into two sets. Ask, "How many are in each set?" Then ask, "How many are in the union of these two sets?" Emphasize that this means that 2+3 (or 1+4, etc., as the case may be) is another name for 5. Ask for some other names for 5. Explain that we use the symbol "=" to mean "is another name for" and can write any of the following:

2 + 3 = 5	5 = 1 + 4
5 = 2 + 3	2 + 3 = 1 + 4

Have the children write other sentences using the "=" sign based on this activity.

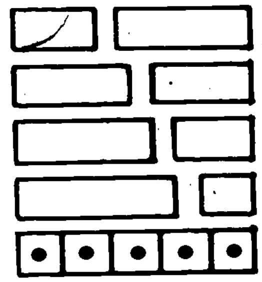
obj. 1a,2

8d. A variation of the activity can be used. Give several children each 5 animal cutouts. Ask each child to arrange his 5 objects into two sets, trying to make his two sets different from any other sets using the 5 objects. Sets may contain the following numbers of objects—3 and 2, 2 and 3, 1 and 4, 4 and 1, 0 and 5, 5 and 0. When this has been done numerals indicating the number of objects in each set can be placed under each set. Record on the chalkboard each child's sets in numeral form along with his name, such as (3, 2) Mary, (2, 3) John, (1, 4) Jack, (4, 1) Sue, (0, 5) Bill and (5, 0) Don. Then tell the children to join their two sets, and place the numeral indicating how many under the new set. Then discuss how this information can be recorded in a number sentence. Write the number sentences on the chalkboard, and discuss how all of the different combinations of sets and numbers of objects still give 5 in the *putting together* process.

obj. 1a,2

8e. Provide a set of cardboard or wooden rods (Cuisenaire rods) for each child, each set containing 10 one-rod lengths, several of each of the other lengths up through 10. Ask children to form all possible *trains*, each consisting of exactly two rods, and each pair being the same length as a given rod.

Example



This leads children to associate the number pairs (1,4), (2,3), (3,2), (4,1) with the length of 5. Attention can be called to the *is associated with* recording.

obj. 4 9a. *Associative Property of Addition* Lay out two collections, such as 3 red sticks, 4 blue sticks and 6 yellow sticks as follows.

Red	Blue	Yellow
|||	||||	||||||
|||	||||	||||||

Bring to the children's attention that in the collections there are the same number of red sticks, the same number of blue sticks and the same number of yellow sticks. Then say to the pupils, "Now watch what I do." In the top collection, move the blue sticks over with the red ones. Then say, "Now watch again." In the second collection, move the blue sticks over with the yellow ones. The final display will look like the following.

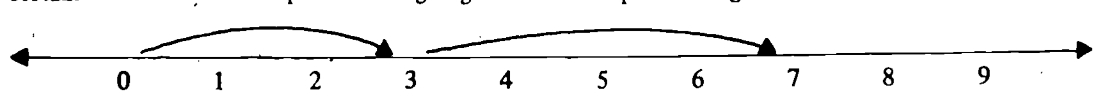
Red Blue	Yellow
|||||||	||||||
Red	**Blue Yellow**
|||	||||||||||

Ask if there are more sticks in the top collection or in the bottom one. Do not ask how many are in each collection, just if there are more or less. The children should match the two sets, either actually or mentally, rather than count, to determine that the collections are the same size. The blue sticks in the top collection could be moved with yellow, and the blue sticks in the bottom one could be moved with the red. Repeat the activity with different numbers of red, blue and yellow sticks. Ask, "What did we find always happened?" (The two sets matched.)

For emphasis, point out that you always began with the same numbers of red sticks in both collections, the same number of blue sticks and the same number of yellow sticks, and that whether the blue ones were put first with the red ones or the yellow ones, the two collections matched. Ask the children if this seems right to them.

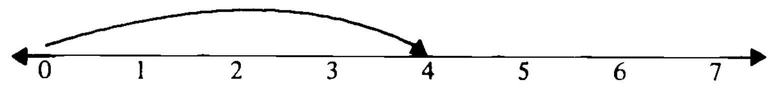
obj. 1a,2 9b. *Using the Number Line* Another type of joining action is illustrated with a number line since, for example, a step of size four is *joined* to a step of size three. In contrast to set union, in which the arrangement of the elements is irrelevant, with the number line the joining must be end to end with no skipping.

As an example draw a number line on the floor that children can walk on. The process of going a certain number of unit steps and then going more unit steps is *adding on* or addition.



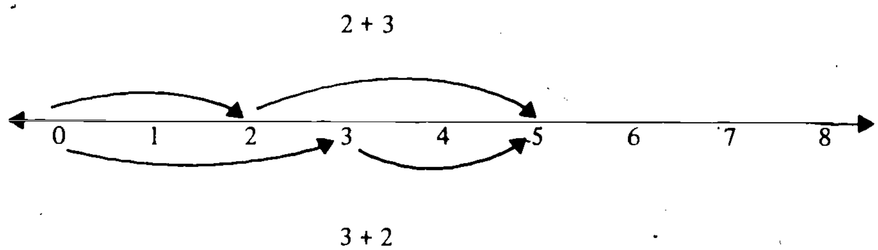
Move 3 steps. Move 4 more steps. The stopping point is 7, which is associated with the (3, 4) move, or (3, 4) associates with 7. This may also be written 3 + 4 associates with 7, with the symbol "+" indicating addition.

obj. 5 — 10. *Zero Property of Addition* Ask a child to walk 4 unit steps on the number line. Then walk no more steps. He should stop on 4. His steps can be recorded on a number line on the chalkboard as follows.



The number pair (4,0) can be recorded on the chalkboard to indicate steps the child was told to take. Discuss the significance of zero in its effect on 4. Now ask the children to walk off other pairs, such as, (1,0), (7,0), (4,0), (9,0), (11,0) and others. After this is done, children should have a better understanding of the zero property of addition.

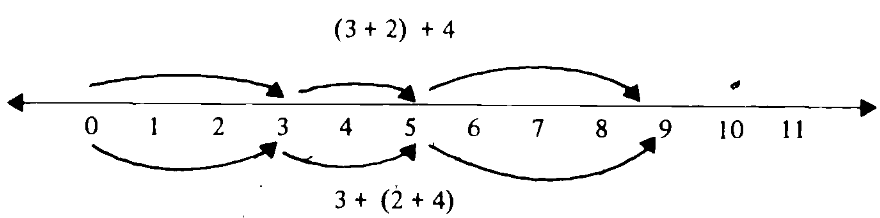
obj. 4 — 11. *Commutative Property of Addition* Commutativity of addition is shown on the number line in the following illustration.



Two children walk on opposite sides of the number line. One takes two unit steps, then three; the other takes three, then two. They notice they stop at the same point.
Emphasize the fact that they have stopped at the same point and that 2 + 3 = 3 + 2. Record this on the chalkboard.

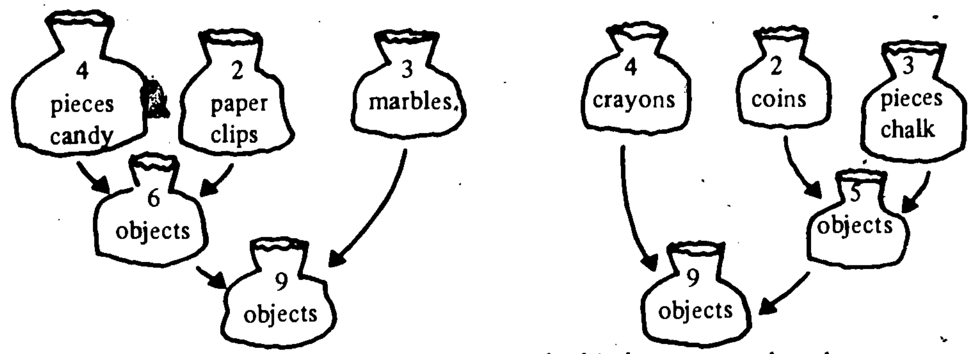
obj. 4 — 12a. *Associative Property of Addition* Associativity of addition can be shown in a similar manner. Ask two children to walk on opposite sides of the number line. Ask one child to walk 3 unit steps and 2 unit steps and mark this place. He then walks four more steps and stops.

Ask the other child to walk 3 unit steps and mark his place. Then ask him to take 2 unit steps and four unit steps and stop. They have both stopped at the same point.



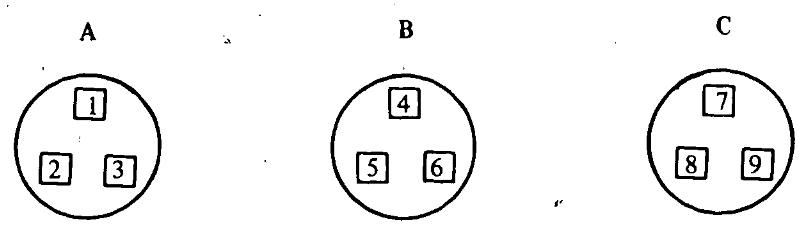
The record of what has been done is put on the chalkboard as (3 + 2) + 4 = 3 + (2 + 4).

obj. 4 — 12b. Associativity of addition can also be illustrated with the use of plastic bags and small objects as shown here.



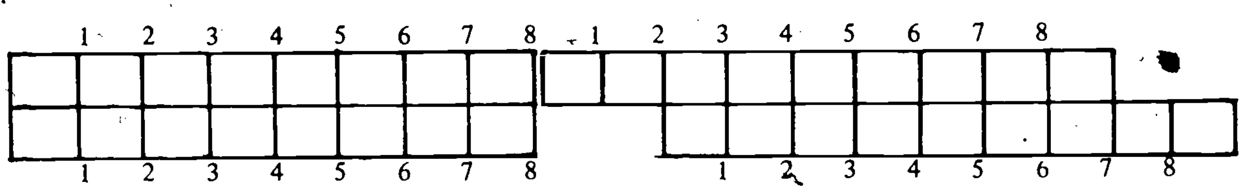
Children should match the objects one-to-one in the big bags to see that the same number of objects was obtained even though the objects were combined in a different order. The recording will read (4+2) + 3 = 4 + (2+3).

obj. 6 12c. Draw three sets on a transparency. Cut one inch squares of transparency paper and write the numerals 1, 2, 3, 4, 5, 6, 7, 8, 9 one numeral per square. Place these in the sets.



Ask a child to move one square from set C to another set so that the sum of the numbers in each set is the same.

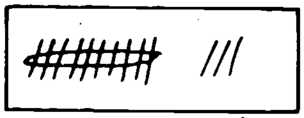
obj. 10 13. *Using Number Strips in Adding* Ask some of the more advanced students to make a slide rule of any length with strips of cardboard or two rulers.



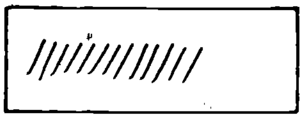
The two strips are calibrated on the same scale. The second strip, for example, can be moved two spaces to the right. Read where the 3 is on the second strip and look above it. Five appears directly above 3 on the upper strip. This shows that 2 + 3 = 5.

obj. 4 14. *Breaking Bundles in Separating Sets* When the children have had plenty of practice in separating sets, they can be given a real problem, as follows. Give each child a bundle containing ten sticks and three extras and say, "I am giving each of you a bundle of sticks and extras." Ask, "Can you take nine sticks away from the collection?" Ask the children who can do this to tell *where they got* the nine sticks to take away. Discuss the fact that to do this problem the bundle must be broken. It will be noted that this activity paves the way for later work with *borrowing*, however, at this first level, the emphasis should be on *finding* the requisite number of sticks to take away, not on how many are left. Having the collection in bundles and singles makes it a different problem from having the collection in a heap from which to count out; it is desirable to present this problem first within the context of where to find the nine sticks.

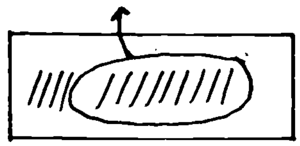
Children are given one bundle of ten sticks and three extras.



The bundle of ten is broken.



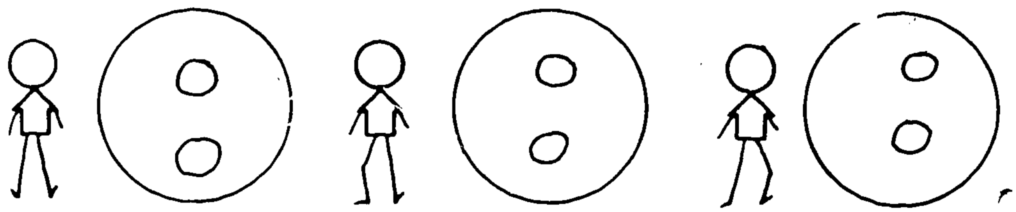
Nine are taken out.



Multiplication and Division

obj. la,lb

15. *Multiplication – Using Sets* Children have learned that (3,2) can be associated with 5 (joining action) or with 1 (separating or comparing). To show that (3,2) can be associated with 6, have children draw 3 stick figures to represent the boys with their plates at Tom's party. Tom wanted each boy to have 2 cookies on his plate. Let the children discover how many cookies Tom will need.

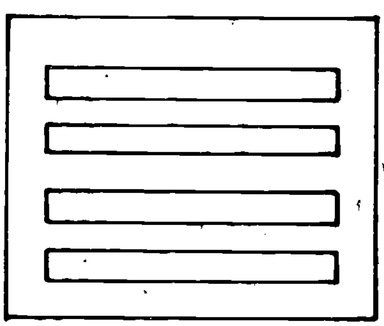


Point out that 3 names the number of boys, and 2 names the number of cookies for each boy, so the numbers pair (3, 2) describes the situation. The association is recorded (3, 2) → 6.

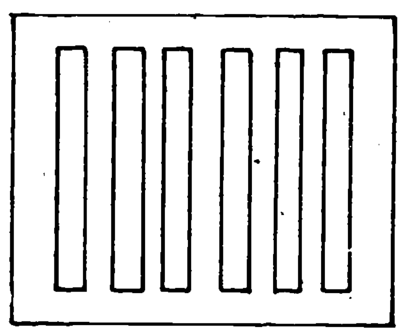
Repeat this activity using other numbers of children and other objects such as candy and straws. This could be worked into an art project.

obj. la,lb

16a. *Multiplication – Using Arrays* After experiences in which children make arrays of bottle caps, cubes, etc., and later actually arrange dots on paper to form arrays, much time can be saved by using the following idea to make an array. Cut slot cards using squares of light weight cardboard approximately 6" X 6". The following slot cards would be used to show a 4 by 6 array. (The slots are to be cut out leaving rectangular slot openings.)

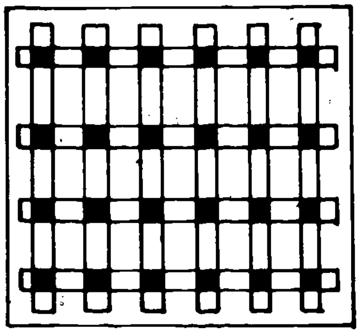


4 slots



6 slots

Slide the 4-slot card over the 6-slot card to give the effect shown.



The light will shine through the overlapping slots and form a very effective 4 by 6 array for pupils' individual use or for demonstrating 4 X 6 on the overhead projector. Turn the cards around and you have a model for 6 X 4.

obj. la,lb 16b. Give each child twelve counters, and ask if they can be arranged in 2 rows with the same number in each row. When the children are finished, ask how many are in each row. Then ask, "What pair of numbers describes your arrangement?" The children should see that the number pair (2, 6) describes the arrangement. Ask if the counters can be placed in 3 rows with the same number in each row, and give the number pair which describes the situation. Repeat for 4 rows, 5 rows, 6 rows, etc. Summarize that 12 can be associated with (2, 6), (3, 4), (4, 3), (6, 2). Repeat with a collection of nine counters, 10 counters, etc.

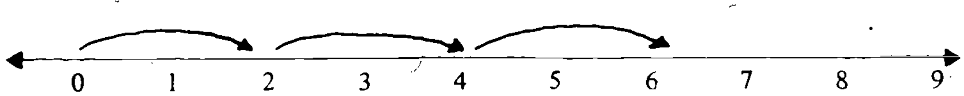
Multiplication and Division

obj. la,lb 17. *Multiplication – Using Cartesian Product* Multiplication may be shown as a Cartesian product by the following activities. Cut out shirts and pants from construction paper. Use 3 shirts (one yellow, one red, one blue) and 2 pairs of pants (one black, one white). Let the children by actual manipulation match each of the 3 shirts to each of the 2 pairs of pants to find out how many different outfits can be put together. Again, the pair (3,2) describes the situation, and the association is recorded (3, 2) → 6.

obj. la,lb 18. *Commutative Property of Multiplication* To introduce the commutative property of multiplication, have the children face the class and stand in 3 rows with 2 children in each row. When this is done, ask how many children are standing. (6). Record (3, 2) → 6.

Have the children turn their faces to the wall, and then pivot the columns so children again face the class. Now how many rows? (2). How many are in each row? (3). What number pair describes the situation? (2,3). Has the number of children changed? (no). Record (2, 3) → 6.

obj. lb 19. *Multiplication – Using a Number Line* Ask a child to start at zero and take 3 steps each of length 2. Ask what pair of numbers describe the situation. Emphasize 3 steps, each of length two so paired will be (3, 2). Now where is he? (6).



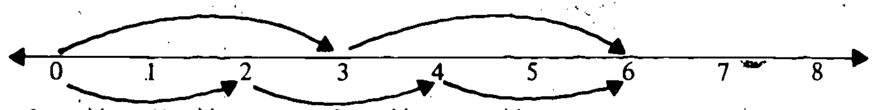
(3, 2) → 6

Have the children illustrate in number line each of the following (4,2), (1,3), (2,3), (5,2) etc. under agreement that the first number is the number of steps, and the second number is the size of the step.

obj. la,lb,2 20. *Recording Multiplication* When the preceding activities have been understood, introduce the symbol "X" in the activities, for example, 3 boys, 2 cookies each, 3 X 2. Then have the children make up stories showing 3 X 5, 5 X 4, 3 X 1, 1 X 3, etc.

obj. 1a,1b,2

21. *Commutative Property of Multiplication* Draw a number line on the floor. Place a child on each side of the line. Ask one of the children to show 2 X 3 (2 steps of length 3). Have the child on the opposite side of the line to show 3 X 2 (3 steps of length 2). What do they notice about where they finish?



Repeat for 3 X 4 and 4 X 3. Repeat for 3 X 1 and 1 X 3.

obj. 1a,1b

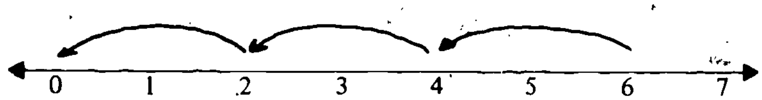
22. *Division – Using Arrays* Twelve children are asked to stand in a group in front of the room. Ask them to stand in rows with 3 in each row. Let the children arrange themselves if possible. What pair of numbers describe the situation? Twelve children, 3 in each row, (12,3) is the pair. Now ask how many rows were formed? (4). The association is recorded (12, 3) → 4.

obj. 1b

23. *Division – Using Sets* Have the children draw fifteen cookies on a piece of paper. Ask them to circle the cookies putting 3 cookies in each circle. What number pair describes the situation? (15,3). Now ask how many circles were drawn. (5). Record (15, 3) → 5.

obj. 1a,1b

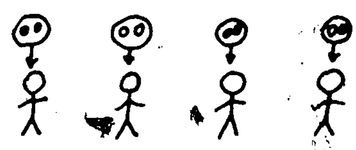
24. *Division – Using Number Line and Repeated Subtraction* Draw a number line or place a paper one on the floor. In order to show the number of 2 unit steps in a length of 6 units, have a child start on 6 and facing zero take steps of two unit lengths. What pair of numbers describes the situation? (6,2). How many steps were taken? (3). Record (6, 2) → 3.



This activity shows division as repeated subtraction and is a link between operations and computations.

obj. 2

25. *Recording Division and Using Sets* When the children have understood activities 22-24, introduce another way of recording these using the symbol " ÷ ". This activity illustrates division when one is to find the number of possible sets. Have the children make up stories to illustrate 8 ÷ 2, 8 ÷ 4, 6 ÷ 3, 12 ÷ 6, etc. For example, a child may say, "I have 8 marbles and I am going to give the boys 2 marbles each as long as they last." The teacher asks the child to draw a picture of this on the chalkboard.



Then the teacher asks how many boys received marbles. (4) As both symbols " ÷ " and " = " have been introduced the child should record what he has done as 8 ÷ 2 = 4.

obj. 1a,1b,2

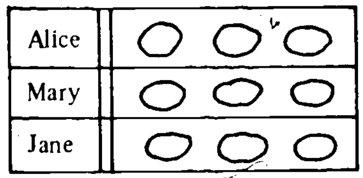
26. *Division – Using Sets* This activity illustrates division when one is to find the number of possible members in a given number of sets. Have the child draw a picture like the one shown.

9 cookies

3 girls

Alice	
Mary	
Jane	

Have the child note that he has a 9 cookie, 3 girl situation (9,3). Have the child place one of the 9 cookies by Alice's name, one by Mary's name and one by Jan's name. Repeat this procedure until all cookies have been placed on the chart.

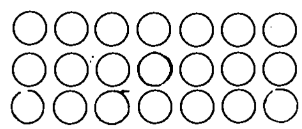


Ask how many cookies each girl received. (3). Then record 9 ÷ 3 = 3.

obj.
1a,1b,2,4

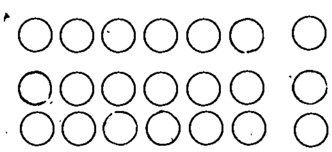
27. *Property of Multiplication over Addition* Give each child some dried beans for counters. For any given fact such as 3 X 7 have the child arrange 3 rows of beans with 7 beans in each row, then separate the rows of beans as shown.

3 X 7



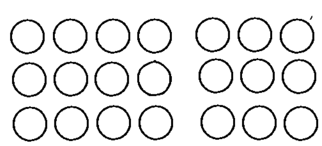
or

(3 X 6) + (3 X 1)



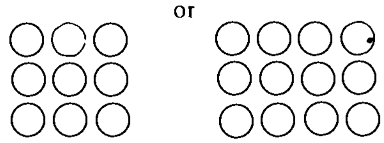
or

(3 X 4) + (3 X 3)



or

(3 X 3) + (3 X 4)

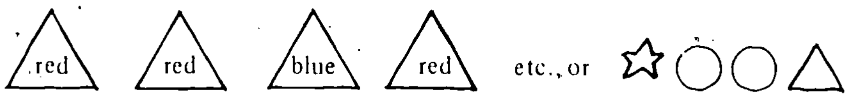


obj. 9

28a. *Continuing Patterns* Activities in which pupils learn to look for non-mathematical patterns have been suggested in other parts of the guide for pupils to begin to learn about numbers. They are suggested here to help pupils learn to look for likenesses and differences among members of sets.

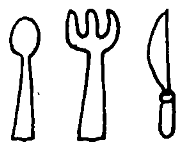
Give a child sets of two or three kinds of objects. You could use cardboard cutouts of different shapes, sizes and colors. Ask a child to look carefully at the order in which some of the shapes have been placed and continue the pattern. This can be done in the middle of the floor as a group of children observe the activity of the particular child.

Example



etc.

A set of plastic forks, spoons, and knives could be used. Ask a child to continue the pattern of arranging them in the order shown.



Logic blocks provide appropriate materials for numerous activities to help the pupil learn what is meant by a pattern.

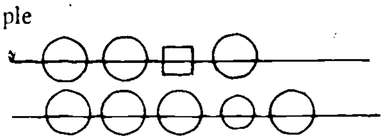
obj. 9

28b. Lessons using picture pages in textbooks as well as paper and pencil activities are much more abstract than work with concrete objects (as suggested in the activity just above) and should not be expected of a child who has not exhibited the ability to reproduce patterns using concrete objects.

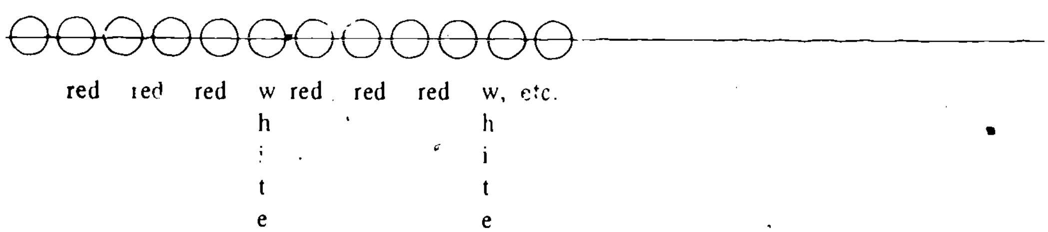
At this stage pages from work books or ditto sheets are appropriate for more development of patterns using less concrete media.

Give the child a ditto sheet of strings of beads, and ask pupils to continue the pattern already begun.

Example



Or make all beads the same size and shape and develop patterns by coloring the beads according to a pattern.



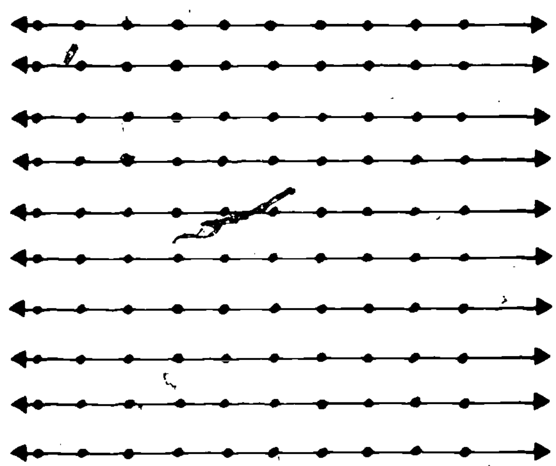
obj. 9

29a. *Skip Counting* Every kindergarten room and every first grade room should have a walk-on number line painted on the floor or made with masking tape on the floor along one wall of the room. After a pupil is able to count rationally and can step along the number line from 0 to 1, to 2, to 3, to 4, etc. and name the numbers as he steps, ask him to take steps two units long and to name the numbers as he steps from 0 to 2, to 4, to 6, to 8, to 10, etc. Each child can succeed at least to some degree in this activity though some pupils will be able to count much farther than others as they walk along the number line.

obj. 9

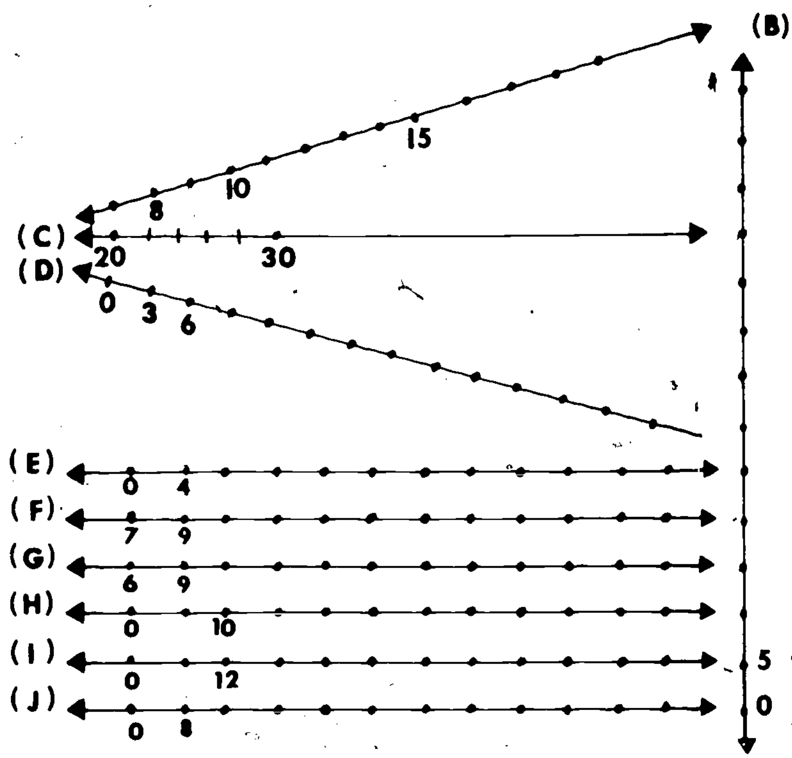
29b. Many variations of number lines will prove very useful for many mathematical activities. Transparencies for use in the overhead projector will prove invaluable. Spirit masters for making pupil work sheets will provide appropriate activities for individuals as they encounter difficulty with various concepts and skills throughout the year.

A few suggestions for transparencies and spirit masters are given here. Some uses are suggested but teachers will find many other effective ways to use them.



Label the points using appropriate numbers.

A ditto sheet similar to this is useful for skip counting. Pupils should be lead to realize that number lines may be horizontal, vertical or oblique. They need not always show zero as the first number.



obj. 11

29c Many counting situations can be learned from the chart below by counting by tens using any column. To count by fives, use both the first and sixth columns.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Count by twos. To get even numbers begin with zero and say one, skip one, say one, skip one, etc. For odd numbers begin with one and say one, skip one. To count by nines begin on the diagonal from the right. To count by elevens begin on the diagonal from the left.

obj. 6,9

30. *Discovering Patterns* The following sets vary greatly in difficulty and are not necessarily all recommended for use within one class. The teacher should use discretion and creativeness to provide sets appropriate for the levels of pupils with whom she is working.

Ask pupils to discover the missing numbers in sets similar to those below and write in the numerals.

Let pupils talk about the numbers in any or every set. The role of a teacher may well be considered that of a skillful interviewer. Try to design questions so that the child can answer them for himself, and from this he may venture to ask some questions which he is interested in because he created them.

A = { 1, ____ 3, ____ 5, ____ 7, ____ 9, ____ }

B = { 0, 2, 4, 6, ____ , ____ , 12, ____ , ____ , ____ }

C = { 0, 3, 6, 9, ____ , ____ , ____ , ____ , ____ , ____ , ____ }

D = { 1, 4, 7, 10, ____ , ____ , ____ , 21, ____ }

E = { 1, 3, 5, ____ , ____ , ____ , 13, ____ , ____ }

F = { 10, 20 ____ , ____ , ____ , ____ , 70, . . . }

Three dots after the last number listed in a set imply that the set continues in the same pattern and that it has no last member.

$$G = \{0, 1, 4, 9, \_\_\_, \_\_\_, \_\_\_, \_\_\_, \_\_\_\}$$

$$H = \{1, 3, 7, 15, \_\_\_, \_\_\_, \_\_\_\}$$

In this last set each succeeding term is one more than twice the previous one. Some children by the 4th grade will be able to discover this pattern, but children who are unable to discover the pattern should be given clues. Leave some small discovery for the child to make.

obj. 6 31a. *Odd and Even Numbers* If you have an overhead projector, ask a pupil to come to the projector and try to form an array having two rows or two columns using all of a set of objects which has been provided for him. Ask another child to volunteer to be secretary and keep a record on the blackboard of the information gained concerning the sets of objects to be studied.

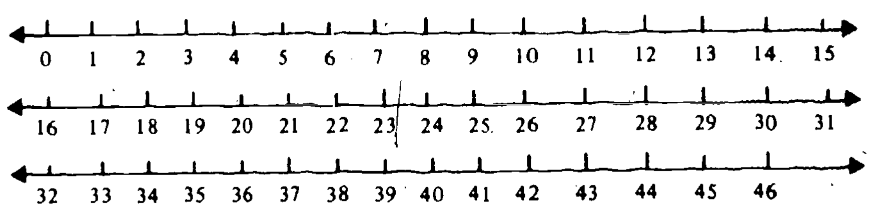
It will probably be better to begin with a set of at least four objects. As the pupil shows (or is unable to show) that a given number of objects can (or cannot) be arranged in pairs, the secretary should record the information on the board in some simple way such as "can," "can't" or "even," "not even."

Even	Not Even
4	5
6	3
2	7
8	9
10	11
.	.
.	.
.	.

Pupils can at this point be told that the number of a set which can be arranged in pairs is called an even number. The number of a set which cannot be arranged by two's is an odd number. From this discussion the child can conclude that every whole number is either even or odd.

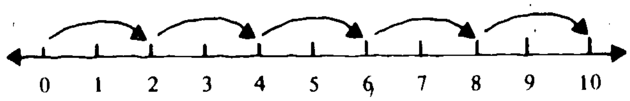
A very valuable concluding activity for any mathematical lesson is to encourage at least a few pupils to summarize *in their own words* what they have discovered or concluded from the day's mathematical experiences. These could be in poster form and displayed in the room. What he says is much more important than how he spells or punctuates.

obj. 6 31b. Transparencies of number lines may be used in many ways.



etc. to 100.

Most pupils are delighted to use the overhead projector and become involved in demonstrations and discussions of mathematical ideas. If an overhead projector is not available, use a chalkboard. A pupil starting at zero and using chalk may show jumps which land him on even numbers. Or starting at 1, he may show jumps which land him on the odd numbers.



obj. 6 31c. Using number line transparencies or other models, a child may think of a cricket which starts on 0 and skips over one point each time he jumps. He would first land on 2, then on 4, etc.

What points can he touch?
Could he land on 14? Why?
Could he land on 52? Why?
Could he land on 35? On 67?

obj. 6 31d. Make ditto sheets using examples similar to those below.

Which even number comes next?

8	______
32	______
46	______
60	______
98	______

Which even number comes before?

______	14
______	6
______	94
______	30
______	90

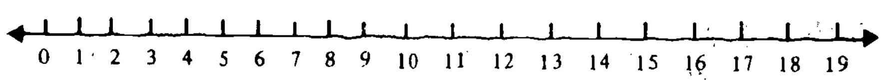
obj. 6 32a. *Sums of Odd and Even Numbers – Class Discussion Exercise* Below are sets of letters which are to be counted using words or groups of words. Count each group of letters and record the number in the column, "How Many In All." Then find out whether pairs can be made out of each group with no letters left unpaired. To show what is done draw a ring around the first two letters in the group, then the second pair, third pair and so on. Write "yes" if there are any letters left over and "no" if all the letters are circled. The first two are done as examples.

	How many in all?	Any letters left over?
(TO)P	3	Yes
(PL)(AY)	4	No
BEAUTIFUL	9	Yes
RUNNING		
TURTLE		
GO		
PROFESSIONAL		
NOVEMBER		
PHILOSOPHY		
CANDY		
CONVERSATIONAL		
TRANSPLANTS		
UNIMPEACHABILITY		
ACQUAINTANCES		
BLUEGREENYELLOW		

If a group can be separated into pairs with no left-overs, it came out even. Which groups came out even, the "yes" groups or the "no" groups?

32b. Make a list of the numbers of the sets that came out even when grouped in twos. These numbers are called even numbers.

32c. Draw a ring around the point just above each even number.



32d. You did not have a 25 letter group. Would a 25 letter group come out even?

Would a 28 letter group come out even?

A 21 letter group?

Can you tell by looking at the numeral that the number is even?
For all the even numbers the one's digit is either ____, ____, ____, ____, or ____.

Any whole number that is not even is called an odd number.
Write at least ten odd numbers.

____, ____, ____, ____, ____, ____, ____, ____, ____, ____.

For all odd numbers the one's digit is 1, 3, 5, 7, or 9.

32e. Put two of the even groups of letters together. See if the number of new group is even or odd.

T U R T L E G O

TURTLE has six letters.
GO has two letters.
Together they have
$6 + 2 = 8$ letters

The numbers 6 and 2 are even numbers. Is their sum even?
Try adding two other even numbers. Is their sum even?
Can you find two even numbers whose sum is not even?
Can you find two even numbers whose difference is not even?
Subtracting numbers, for example $12 - 4$, is the same as taking 4 letters out of a set of 12 letters.
PROFESSIONAL has 12 letters. Take off 4 letters (or cover 4 letters), and what kind of group is left?
If you cover an odd number of letters, what kind of group is left?
Put two of the odd groups together. Is the number of the new group odd or even?
Can you find two odd numbers whose sum is odd? Try!
Can you find two odd numbers whose difference is odd?

32f. Tell whether the sum is odd or even in each case.

2 + 3 ____	3 + 3 ____	11 + 13 ____
4 + 8 ____	9 + 9 ____	22 + 15 ____

126 + 103 ____

79 + 85 ____

32g. Three of the problems below have wrong answers. Without working the problems, put a *W* beside each wrong answer. Explain how you can tell.

126	226	79	82	391	838
+359	+634	87	−13	−183	−299
		73			
		+95			
485	860	335	68	208	538

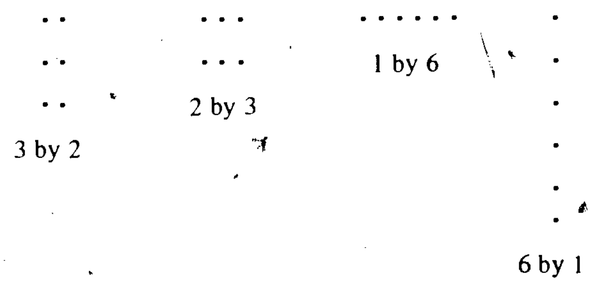
obj. 2

33a. *Prime Numbers* Before prime numbers can be understood, pupils must correctly use such terms as "divided," "is a factor of," "is divisible by."
Since $7 \times 3 = 21$, we say that 7 and 3 are factors of 21 and that 21 is divisible by 3 and 7. Since $1 \times 21 = 21$, 1 and 21 are factors of 21 also.

Of course, every number is divisible by 1, and 1 is a factor of every number. The set of factors of 21 is 21, 3, 7, 1.

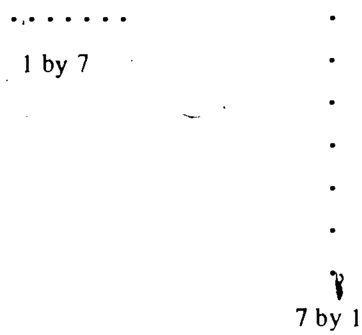
Ask pupils to use a set of 6 objects and see how many different arrays can be formed using all 6 objects.

Example



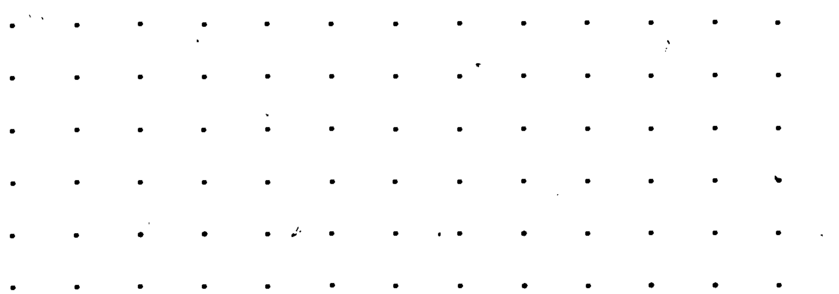
There are 4 possible arrays, hence there are 4 factors of 6, namely 1, 2, 3, 6. Have pupils count by 2's to show three 2's are 6, etc. Now try 7 objects. Let children form all possible arrays using all 7 objects.

Example



There are only 2 possible arrays, hence there are only two factors of 7, namely 1 and 7.

Dot paper is an excellent way to record the arrays as they are discovered.



As pupils study the number of factors they should observe that some numbers have exactly two factors. Others have more than two factors. Those which have exactly two factors are prime numbers; those which have more than two factors are composite numbers.

The Number	Number of Arrays	Number of Factors	Set of Factors	Prime or Composite
21	4	4	21, 7, 3, 1	Composite
7	2	2	1, 7	Prime

Ask the question about the number "1." Ask the pupils to read definitions of prime and composite numbers; the pupils should conclude that 1 is neither prime nor composite.

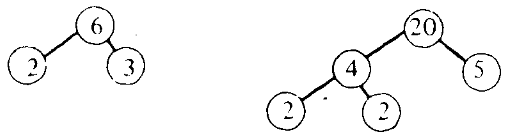
Make a poster listing all prime numbers less than 50, and display it in the room.

obj. 8

33b. Ask the pupils to construct a factor tree for each of the numbers of a set such as set A. Limit the numbers to be factored to those appropriate for the pupils. Even if a pupil has not mastered the basic multiplication facts he may be encouraged to use a multiplication table (preferably one which he constructed earlier.)

$A = \{4, 6, 7, 9, 12, 15, 18, 20, 24\}$

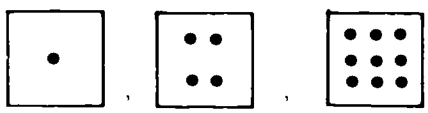
Sample

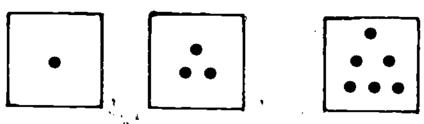


Notice that the factoring process is to be continued until each factor is a prime number. In the case of 7, or any other prime number, the only factors are 1 and the number itself. Since 1 is not prime, you cannot factor 7 into a product of primes.

obj. 9

34. *Continuing Patterns of Numbers* Ask the pupils to find at least the next 3 members in each series as seems appropriate at various levels.

a.  , ____, ____, ____

b.  , ____, ____, ____

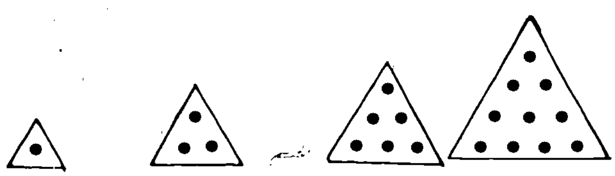
c. 0, 1, 4, 9, 16, ____, ____, ____

d. 1, 3, 5, 7, ____, ____, ____

e. 1, 1, 2, 3, 5, 8, ____, ____, ____

obj. 9

35a. *Triangular Numbers* Prepare triangular cardboard pieces as shown.



etc.

If each dot suggests one, ∴ suggests 3, etc. The numbers suggested by the triangles above are called triangular numbers. Ask pupils to name the first ten triangular numbers.

{1, 3, 6, 10, 15, , , , }

2 3 4 5 etc.

Find the difference between each two consecutive triangular numbers. Discuss the differences.

35b. Ask pupils to add any two consecutive triangular numbers. What kind of number do you get each time? Keep a record of the sums of any two consecutive triangular numbers.

Example

1 + 3 = ________

3 + 6 = ________

6 + 10 = ________

10 + 15 = ________

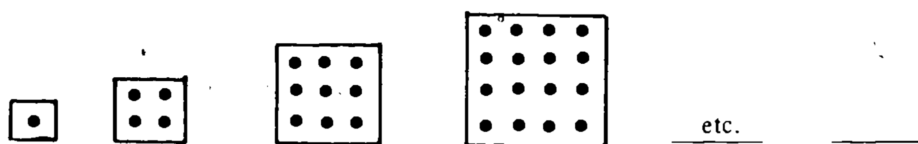
or

+	3	6	10	15	21
1	4				
3		9			
6			16		
10				25	
15					36
21					

35c. Describe the set of sums of two consecutive triangular numbers.

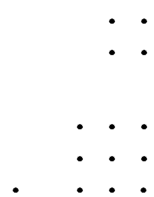
obj. 9

36a. *Square Numbers* Prepare card board pieces with dots arranged in rows and columns, with the same number of dots in rows as in columns.



1 4 9 16 25 36

How many numbers can be suggested by a pair of pieces placed side by side? Keep a record in a table as shown.



1 + 4 = 5

1 + 9 = 10

+	1	4	9	16	25	36
1			10	17	26	37
4				20		
9				25		
16						
25						61
36						

36b. Ask the pupils to find two consecutive square numbers whose sum is a square number. One such pair is 3 and 4 since $3^2 + 4^2 = 25$. Are there others?

obj. 9

37a. *Using Patterns* A discussion of patterns is appropriate for many levels of ability.

Circle the numerals which do not seem to belong.

$A = \{0, 2, 4, 6, 9, 10\}$

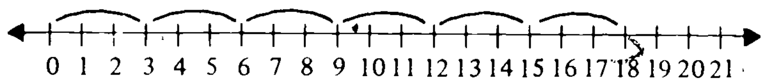
$B = \{1, 3, 5, 7, 9, 10\}$

$C = \{0, 3, 6, 9, 13, 15, 18\}$

$D = \{0, 4, 8, 10, 16, 20\}$

$E = \{0, 1, 4, 9, 12, 16, 25\}$

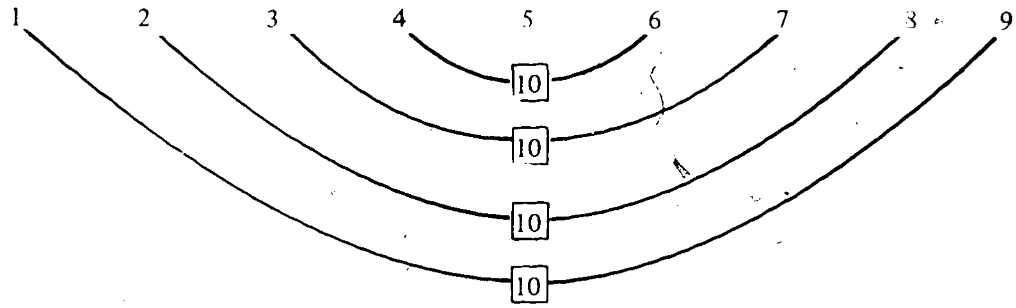
Some pupils may need to use the number line to help answer the question, for example, for *C* which numeral does not belong? Why?



If this set of numbers were continued according to the pattern, which of the following numbers would be in the set? {21, 22, 23, 24, 25, 26, 27, 28, 29, 30.}

obj. 10

37b. Use the following idea to add variety to meaningful practice with basic facts and to stress patterns in mathematics. Complete the pattern showing pairs of numbers whose sum is 10



Write two equations for each pair of different numbers whose sum is 10, for example, 1 + 9 = 10 and 9 + 1 = 10.

Use the same kind of pattern to show pairs of numbers whose sums are 5, 6, 7, 8, 9, 11 and 12.

obj. 4,10

37c. Parts of this activity could be used as pupils have sufficient experiences with open sentences, true sentences and false sentences.

The teacher should not expect the children to express their ideas formally. It is important that they understand the ideas of commutativity, associativity, distributivity, additive inverse, zero in multiplication, multiplicative identity, etc.

Ask pupils to complete each statement in such a way that each will be a true statement. (Let □, Δ and ○ represent whole numbers.)

a. 2 + 6 = 6 + ____

b. 3 X 4 = ____ X 3

c. □ + Δ = Δ + ____ where □ and Δ represent numbers

d. 3 X (2 + 4) = (3 X 2) + (3 X ____)

e. Δ X □ = ____ X Δ

f. Δ X (□ + ○) = (Δ X ____) + (Δ X ____)

g. 27 X 69 = 69 X ____

h. 38 X 0 = 52 X ____

i. If 27 X 1 = □, then □ is ____

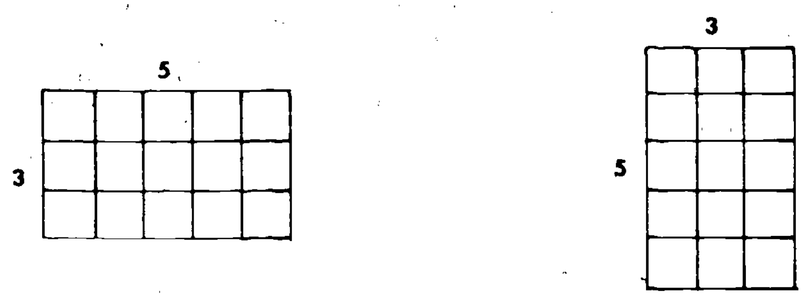
j. 7 - 7 = 8 - ____

Supplementary Activities for this Strand

obj. 4,10 38a. *Operations and Properties* Teachers can use a grid and a table of facts to develop and extend different ideas of operations as shown.

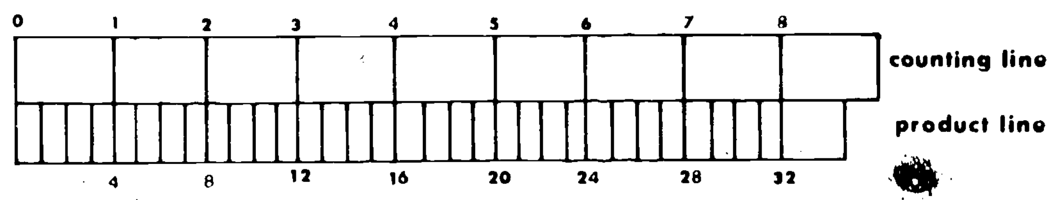
+	1	2	3
1	2	3	4
2			
3			

38b. Commutativity of multiplication using a grid is shown below.



38c. On the hundreds board or on grid paper, make a hundreds chart. Circle the multiples of 3, draw a triangle around the multiples of 4. Which numbers are both multiples of 3 and multiples of 4? Then identify multiples of 5, and ask which numbers are multiples of 3 and 5? 4 and 5? etc.

obj. 11 39. *Skip Counting*



The counting line will have to be adapted in order to count the number of 2's, 3's, 5's, etc. in a given length on the product line.

Two number strips, as above, can illustrate the operation of multiplication.

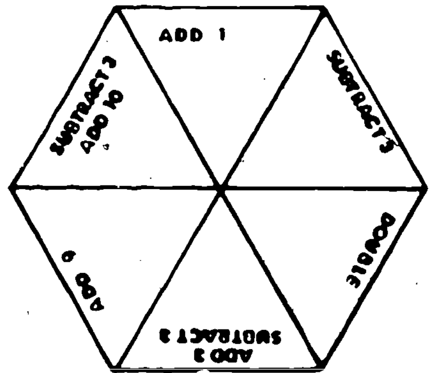
If 4 is the basic measure, the product line shows that 3 four's are 12.

$$3 \times 4 = 12$$

The reverse procedure would illustrate division. $12 \div 4 = 3$

obj. 10

40. *Recall or Basic Facts* Make a spinner as shown.



Let the children take turns spinning and following the direction on the side on which the spinner comes to rest. In case the direction tells him to subtract a number greater than he has accumulated, he loses his turn.

obj. 9

41a. *Closed Number Line* A clock is a very good example of a closed number line. The counting set of a clock uses the numbers one through twelve and then repeats them over and over. Thus, it can serve as a model for a modular number system.

Construct a number line on a 12-unit length of string, wire or some other pliable material which can be used as the rim of a circular clock face. Bring the end points together, lapping the 12 point over the 0 point thus making a circular number line. The numerals 12 and 0 then name the same point on the clock and could be designated either 0 or 12.

Therefore, the child should discover that to count to 3 on a clock, he must start at the point named 12 and move clockwise 3 spaces. When the device is used to show addition, any combination which results in 12 would terminate at the 12, 0 point.

Examples of addition on the circular number line.

2 + 3 = [5]

8 + 7 = [3]

This is similar to addition on a linear number line using whole numbers; to find the sum, move seven places clockwise from 8. The children should discover that the result is 3.

Examples of subtraction on the circular number line.

9 − 5 = [4]

This is similar to subtraction on a linear number line using whole numbers.

3 − 5 = [10]

The child will discover that he can move in a counter clockwise motion five spaces and the result is 10. This subtraction is not possible in the system of whole numbers he has been using. It is a good beginning toward development of the concept of a finite number system.

obj. 9

41b. After many experiences with addition on this circular number line, the children should be able to construct and use a table on this specific finite number system in both addition and subtraction.

Examples

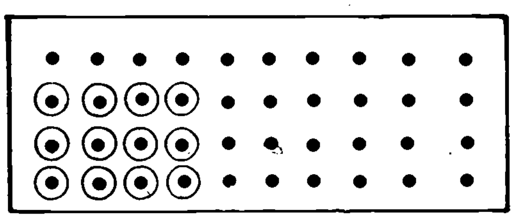
$11 + 3 = \boxed{2}$

$2 - 4 = \boxed{10}$

+	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												3
4										2		
5									2			
6												
7				1								
8								3		6		
9												
10				2								
11			2									
12												

obj. 1a,1b, 2,4

42. *Commutative Property of Multiplication* Using a geoboard, the children can form an array and determine the relationship $(3,4) \rightarrow 12$. The symbolism $3 \times 4 = 12$ can also be used.



$3 \times 4 = 12$

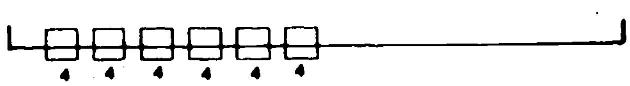
By arranging twelve objects on the geoboard the children can note that if there are four rows, each row will have three objects. The reverse is also true, three rows would have four objects each.

obj. 1a,1b, 2,4

43. *Distributive property of Multiplication over Addition*
from construction paper, clothes pins and string.

Tie string from one stationary point to another so clothes pins can be attached.

Attach 6 clothes pins to the string line.



Give 4 children 6 red circles each. Ask each child to clip one of his red circles to each clothes pin. How many red circles are on the line? $6 \times 4 = 24$

Now slide one pin (circles still attached) to the other end of the line.

Ask the child to write a number sentence showing the number of circles on the left side of the line.

$$5 \times 4 = 20$$

Now write a number sentence for the number of circles on the right side of the line.

$$1 \times 4 = 4$$

There are still 24 circles on the line as 20 + 4 = 24, therefore, 6 X 4 = (4X4) + (2X4).

Now slide another pin from the left to the right side of the line so that 2 pins are now on the right side and repeat the process to show that 6 X 4 = (4X4) + (2X4).

Slide another pin to the right so that 3 pins are now on the right side of the line. Show by the process explained above that 6 X 4 = (3X4) + (3X4).

(Note: A child having difficulty with 6X4 may not have difficulty with (5X4) + (1X4) or (4X4) + (2X4) or (3X4) + (3X4). By using any one of these the difficulty is overcome.)

OPERATIONS, THEIR PROPERTIES AND NUMBER THEORY

OBJECTIVES

The child should be able to do the following.

1. Match mathematical operations with physical representations
   (a) Select an appropriate mathematical operation for a given physical situation
   (b) Illustrate, with words or pictures, a given mathematical operation
2. Use the symbols $+, -, \times, \div, =, \neq, <, >$ correctly when writing number sentences
3. Write a subtraction sentence related to a given addition sentence. Write a division sentence related to a given multiplication sentence excluding division by zero
4. Use commutative, associative and distributive properties as mental or written computation is developed
5. Use the special properties of zero and one as mental or written computation is developed
6. Tell whether a number is even or odd and tell why
7. Write the set of prime numbers less than or equal to 50
8. Give the prime factorization of any whole number less than or equal to 24
9. Discover and extend some number patterns
10. Demonstrate immediate verbal recall of
    (a) any of the *basic* addition and subtraction facts
    (b) any of the *basic* multiplication and division facts through 50
11. Count by twos, threes,..., as well as by tens and hundreds, starting at different numbers

RELATIONS AND FUNCTIONS

INTRODUCTION

Relations, the idea of pairing or corresponding in a certain order, is basic to all mathematics. Beginning even at the pre-school level, the pupil, through experiences, can learn to recognize relations, to use them in formulating his own ideas and to show in communicating to others that he is intuitively developing a pattern of organized thinking in nonnumerical situations. By using relational thought patterns in his early experiences, he establishes readiness for extending these concepts in mathematical situations as he meets them in his development. Therefore, it behooves the teacher to see that from the beginning a foundation for correct concepts is laid so that unlearning will not be necessary later.

The objectives listed for this strand of the guide should not be seen as separate from each other, nor do they need to be mastered in the order given. They should be considered in three major categories--(1) objectives 1-3, making comparisons, (2) objectives 4-6, illustrating relations which lead to the basic principle in mathematics of trichotomy of numbers and (3) objectives 7-9, finding missing parts of given relations. Objective 10, recording relations, should be realized as a second part of each of the other objectives. All of the activities given here should be used and similar ones should be devised. The activities should not necessarily be used in the order printed but as the teacher feels they are needed in the development of the children's relational thinking.

Pupils encounter many nonnumerical relations; many can be found in stories for primary children. Some of these non-numerical relations such as *belongs to, is brother of* and *is in the same house as* should be used before numerical relations to illustrate the meaning of relations. These relations can also be used to lead into numerical relations since they can be examples of correspondences of *one-to-one, many-to-one, one-to-many* or *many-to-many*. Such correspondences are basic to the idea of number, to the relations *equal to, less than* and *more than* and to the operations with numbers.

There are several special kinds of relations with special names. One of these, called an equivalence relation, is associated with the process of classification. Classification is the process of partitioning a set of elements into different subsets in which no element can belong to more than one subset. This, too, can be introduced through nonnumerical situations. For example, a set of blocks can be separated into classes on the basis of color provided the colors are distinct. Or, a set of coins can be partitioned into subsets according to value. Such subsets, or classes, are called equivalence classes, and the relation exemplified by their membership, *same color as* or *same value as,* is called an equivalence relation. When school children are classified by grade in school, if no pupil can be in more than one grade, then the different grades represent equivalence classes, and the equivalence relation is *is in the same grade as.* Equivalence relations are very important in mathematics. The most familiar is *is equal to,* but many others are encountered as the pupil progresses through mathematics.

Another special kind of relation is that known as a function, or mapping. Although the concept of a function is one of the unifying themes of mathematics, it is unwise to introduce pupils to the concept by giving a formal definition. If the pupils have sufficient practice in pairing elements of one set with elements of another while studying relations in general, those having the special property required of functions will not be difficult to identify. It is for this reason that early activities on relations in the guide include the suggestion that pupils write out the ordered pairs associated with a relation. It is also suggested that pupils make graphs of relations. As the pupils observe many different kinds of graphs, those graphs characterizing functions will stand out in sharper focus.

Also important in mathematics are the special relations called order relations, such as *more than* and *less than.* These are used when such concepts as *heavier than, longer than, darker than* or *thinner than* are being considered. Measurements such as those of time, capacity and length consist of ordering the quality to be measured and then assigning numbers to correspond to that ordering. Thus the numerical order relation makes precise the intuitive one.

Activities in this strand include suggestions for introducing pupils to relations in general and to the special relations discussed above. As with other strands, the teacher will need to select those which are appropriate for his class, and supplement them as necessary. It should be re-emphasized, however, that familiarity with relations in general should precede formal work with special relations.

Mathematics can be viewed as an entity of systems each consisting of the following components – sets of elements, or basic units such as whole numbers, rational numbers, or points; relations or comparisons of these elements, such as *equal to, greater than* or *congruent to*; and operations such as multiplication or set union. Therefore, throughout each strand in elementary mathematics, recognizing and using relations constitutes a basic activity in which the pupil must engage in order to understand the concepts included in that strand.

RELATIONS AND FUNCTIONS

Objectives Keyed to Activities	ACTIVITIES
obj. 1	1. Before children can sort objects they must be able to recognize differences and similarities between them. To introduce this idea use the children themselves. Ask two boys to stand in front of the class. Ask, "As you look at these two boys, in what ways do you see that they are different?" Some examples of ways they might be alike are "They both have brown hair, blue shirts, brown shoes, two ears, two eyes and one nose." Some ways they might be different are "One has brown eyes and the other has blue eyes, one has solid colored pants and the other has plaid pants and one has longer hair than the other."
obj. 1	2. Hold up two objects and ask the children, "How are these alike and how are they different?" For example, hold up two toy cars, two dolls or two flowers. Repeat this a few times before using number 3.
obj. 1	3. Provide a set of logical blocks or a set of cardboard regions. These blocks should be in four shapes such as square, triangle, rectangle, circle or diamond. Each of these should be in two sizes and three colors such as red, yellow and blue. If blocks are used, it is possible to have two thicknesses so that in addition to the three attributes of size, shape and color, there will also be a fourth one, thickness. a. Hold up one of the blocks or regions and say, "Here is a block (or region). What kind of block is this?" If it is a block, it will have four attributes. If it is a region, as described above, it will have three attributes. Accept these attributes as they are named one at a time correctly. For example, the teacher may have held up a block that was a large, red, thick triangle and some children would have said, "A big block" or a "red triangle." b. Hold up two blocks and ask "How are these blocks alike?" and "How are these blocks different?" First use two blocks that are alike except for one attribute such as one large, red, thick triangle. After a few pairs of blocks with one different attribute, hold up two blocks that have two different attributes such as one large, red, thick triangle and one small, blue, thick triangle. Repeat this until the children are able to recognize the attributes of the blocks and the differences and similarities between two of the objects.
obj. 2a	4. Gather a set of objects, such as buttons, needles, thread, doll clothes, doll blankets, toy plates, toy cups, toy saucers, etc. Ask a child to sort the objects. At first the child might sort the objects in different sizes of buttons, kinds of doll clothes and so on. Later the child will probably sort all buttons together, all doll clothes together, and still at another time may place the doll clothes and doll blankets together and toy plates, cups and saucers together.
obj. 2a	5. Make three or four closed curves from lengths of rope or heavy cord to use as enclosures for sets of logical blocks as described in activity number 3. Ask some of the boys to place all of the triangular regions in the interior of one of the closed curves. Ask some of the girls to place all of the circular regions inside of another closed curve. Ask a different group to place all of the square regions inside of another closed curve. Ask a different group to place all of the square regions inside of another closed curve, etc.

obj. 1,2

6. Use the logical blocks or the cardboard regions as described in activity 3. Ask a child to begin sorting the set of blocks into two sets according to some relation which he had in mind. Ask other children to discover the relationship between the objects within each of the two sets and continue the sorting.

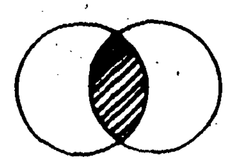
Example

All polygonal regions

All circular regions

obj. 2a

7. Use the logical blocks or cardboard regions described above. Use two overlapping closed curves. Ask one team of children to place all of the circular regions in one of the closed curves and another team to place all of the red regions in the other closed curve. Let them discuss among themselves where the red triangles should go until each team's task has been accomplished. The shaded area in the drawing should contain all of the red triangles and only the red triangles.

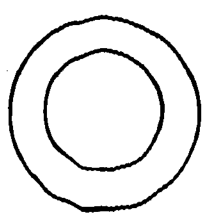


Set of all red regions

Set of all triangles

obj. 2a

8. To help children at a later stage develop the notion of the *inclusion* relation, e.g. the set of square regions is included in the set of rectangles, the following activity could be used. Ask a group to sort out all of the rectangles regions and then partition this set into two subsets (squares and non-squares). Use two closed curves as shown and ask which set, the squares or the non-squares should be placed in the inner most curve.



closed curves or rope or cord

obj. 1,2a

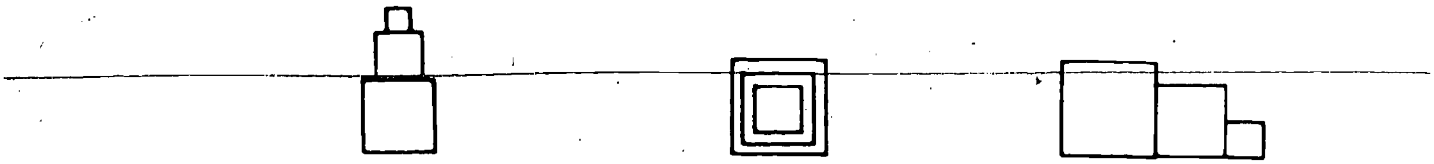
9. a. To introduce this activity hold up two logical blocks or two cardboard regions as described in activity 3 which have only one different attribute, for example, a large, red, thick circle and a large, blue, thick circle. Ask the pupils, "How are these blocks different from one another?" Repeat this a number of times, but be sure that one different attribute is not the same each time. The one difference in the example above was color, therefore, use shape, size or thickness for the one difference of the next pair of blocks.

b. Ask a child to pick out a block (from a set of blocks that is a mixture of shapes, colors and sizes) and place it on the floor. The next child is asked to pick out another block that is exactly one property different from the first block and place it next to the first block, etc. Ask children to continue placing blocks in a row in this manner until all of the blocks have been used. For example, if the first child picks a large, red triangle, the second child pick a large, blue triangle. The third child could choose a small, blue triangle and place it next to the large, blue triangle. Each child may tell the class the way in which his block is different from the previous one.

The game may be varied in many ways such as placing the blocks in a circle instead of a row or picking out a block different in two attributes. For other activities consult the reference list.

obj. 3,10

10. For activities to introduce relations of order, such as larger than, longer than, taller than, to the right of, weighs more than, etc., sets of objects in the classroom which have a range in size could be used. They could be arranged vertically or horizontally in order of size. If objects are not on hand, paper cut in square shapes or circular shapes can be used.



blocks　　　paper　　　blocks

obj. 4a,10

11. Ask each child to hold up one object each (book, ball, pencil, chalk, etc.). The relation *is holding* could be observed and recorded.
The symbol for *the relation of* in recording is →
The specific relation being used is written above the arrow.
This type of recording is called mapping.

is holding →

Tom → book
Jan → ball
Sue → pencil
John → chalk

obj. 4a,4c,10

12. Stories used in the primary grades provide many applications for relationships. The story, "The Three Little Pigs," can illustrate the relationship *built house of.*

built house of →

First Little Pig → straw
Second Little Pig → sticks
Third Little Pig → bricks

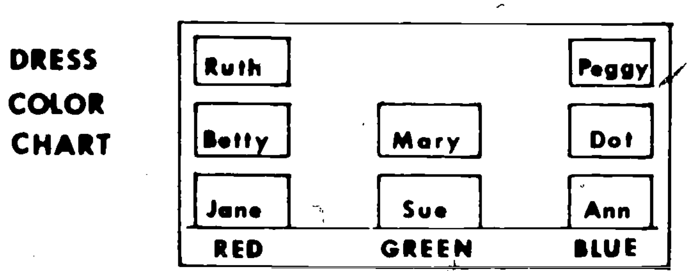
The story, "The Three Bears," can be used to illustrate the relationship *belongs to* and later the relationship *larger than.*

belongs to →

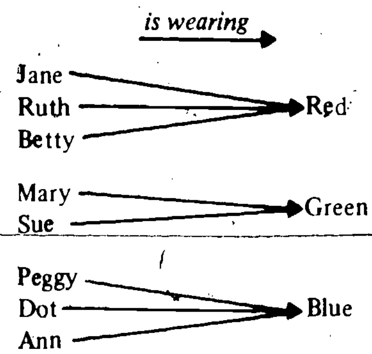
The little chair → Baby Bear
The little bowl → Baby Bear
The little bed → Baby Bear

obj. 4

13. Each girl wearing a red, green or blue dress can place a square piece of construction paper of the corresponding color on a chart, similar to the drawing below. The name of the girl can be written on the square before she places it on the chart.

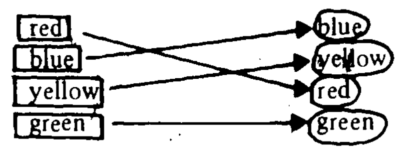


This relation could be recorded in other ways. In the recording below *is wearing* is the shorthand for the relation between some girls in the classroom and the color of their dresses.



obj. 4a,10

14. To illustrate one-to-one correspondence, one might pin rectangular pieces and circular pieces of corresponding colored construction paper on a display, and connect the corresponding colors with yarn or string.

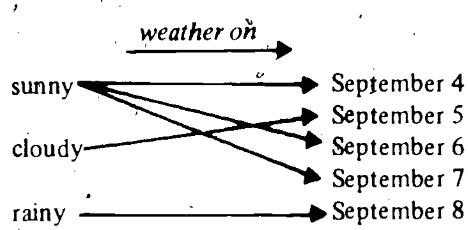


obj. 4a

15. Activities for matching or one-to-one correspondence might also arise from classroom activities as one child to desk, one cookie to each child or one piece of paper to each child. Many matching situations could arise from activities at home as each plate on the table to one person, one chair at the table for each person, one toothbrush for each person, one scoop of ice cream cone, one drinking straw per glass.

obj. 4b

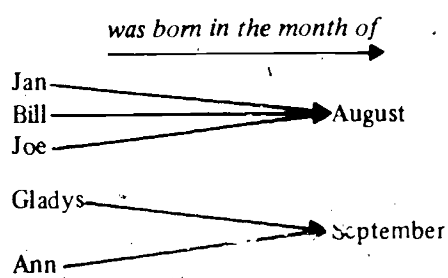
16. Ask the children to keep a weather chart for a week or a month. They may map the dates to the kinds of weather, such as, sunny, cloudy, rainy, etc.



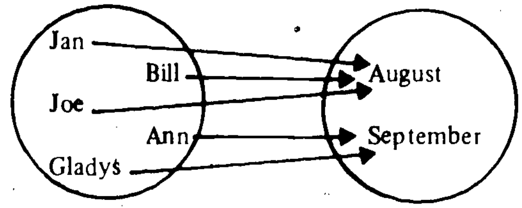
obj. 3a,10

17. For many-to-one relations, the names of the children could be illustrated by constructing a birthday chart or graph.

Example



Another way of mapping the relationship of the children to the months of their birthdate is shown here.

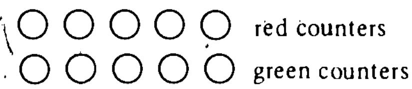


obj. 4c

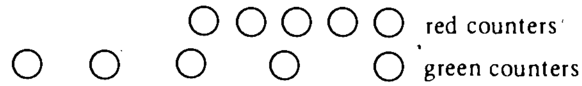
18. Examples such as two shoes to each child, two eyes to each child, two gloves to each child may be used to illustrate two-to-one correspondence.

obj. 4b,5

19. Arrange a set of 5 counters of one color and a set of 5 counters of another color as follows.



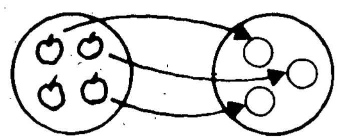
Quiz a child to see if he agrees the two sets have the same number. Then spread out the counters in one set as follows and see if the child still says they are the same in number.



If the spatial configuration obscures the number idea, the child does not have understanding of the relation *as many as* or *has the same number as.*

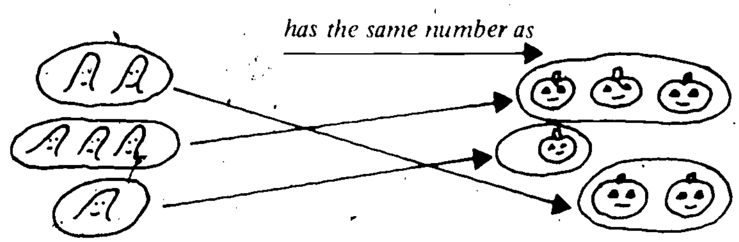
obj. 4a,5,10

20. Activities to illustrate the relations *more than* or *less than* should show that in matching two sets there may be one or more elements of one set which cannot be paired with elements of the other set. Activities may be performed and recorded. The children could pair themselves such as girls with boys to see whether there are (a) more boys than girls, (b) fewer boys than girls or (c) as many boys as there are girls. Or a set of apples and a set of plates may be corresponded and recorded as shown below.



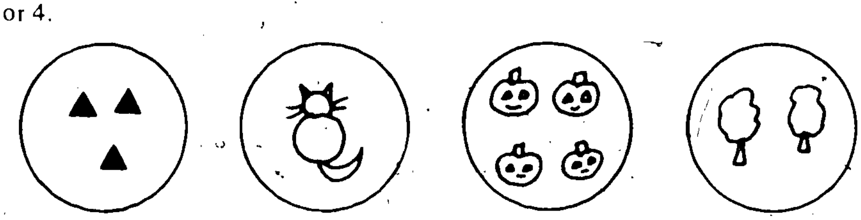
obj. 2b,5

21. On the bulletin board or on a flannel board match sets of objects in one-to-one correspondence. These sets may be varied to fit the season. For example, ghosts and jack-o-lanterns may be used for Halloween, Christmas trees and stars may be used for Christmas, etc.

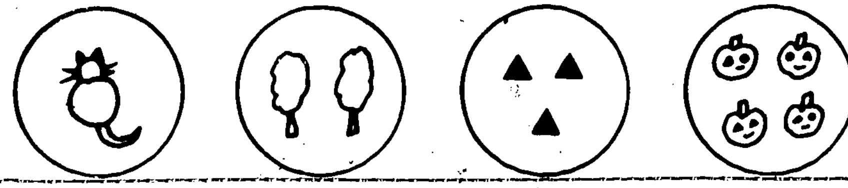


obj.

22. Arrange four sets of figures randomly on a flannel board, each showing the number property 1, 2, 3 or 4.



Discuss with the children the number property of each set, and then ask a child to place under the sets the numeral indicating the number property of each. Then ask the child to order the set of sets according to number property. Take time to discuss with the children why sets were ordered in this way.



obj. 1,2b

23. Make a collection of pictures and sets of objects which suggest the numbers 1-10. The collection should include many representations of each member. For example, two buttons tied together, two safety pins pinned together, a picture of two children, a pair of shoes, etc. Provide a shoe box and label each box one of the numbers 1-10. Ask the children to sort the items.

obj. 2b

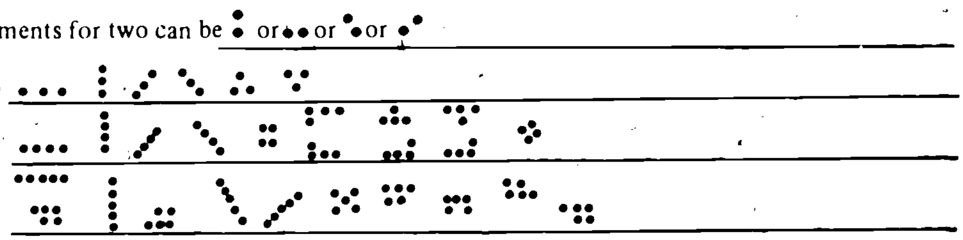
24. The child should make patterns of ways objects can be arranged for easy recognition of small numbers. Arrange pictures on the bulletin board in various patterns, small pictures in patterns on flash cards, and finally dots and small symbols as represented here. (At first it may seem more efficient to have one pattern for presenting a number such as three, but by working with all the patterns of three the child will accomplish this more effectively for later use.)

Arrangements for two can be ... or ... or ... or ...

Three as

Four as

Five as



(The eye span of the small child will probably not take in more than three dots in a straight line.)

obj. 3

25. Give the ordered arrangement of children seated in a row such as, John, Bill, Mary, Tom. Name or write a set of ordered pairs to show the relation *is to the left of* and the set of ordered pairs to show the relation *is to the right of.*

Example

Is to the left of {(John, Bill), (John, Mary), (John, Tom)}

Is to the right of {(Bill, John), (Mary, John), (Tom, Mary)}

obj. 5,6

26. After experiencing activities of non-numerical order relations, the children should have experiences in using numerical order relations, particularly those of ordinal numbers or counting. Further activities of numerical order relations should be given for the foundation of inequalities.

For example, given the set {1, 2, 3, 4, 5}, state the set of ordered pairs with the relation *is less than.*

Solution

(1,2)	(2,3)	(3,5)
(1,3)	(2,4)	(4,5)
(1,4)	(2,5)	
(1,5)		

These may also be recorded as $1 < 2$, $1 < 3$, etc.

obj. 5

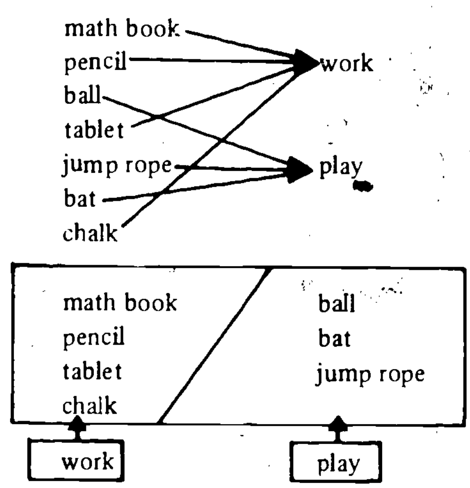
27. After activities such as 24, the children should have activities involving compound statements of inequalities, such as, $5 < 7 < 10$ or $8 > 6 > 3$. A number line should be used in these activities.

obj. 2a,4,10

28. The mapping of many-to-one leads to partitioning or classifying, which consists of separating a set into two or more subsets based on a given relation. Each subset is called an *equivalent class*, therefore, any two objects in the same class are said to be *in the same category as*. (Notice *equivalent* does not mean *equal*.) Children should first experience activities of partitioning a set into two subsets.

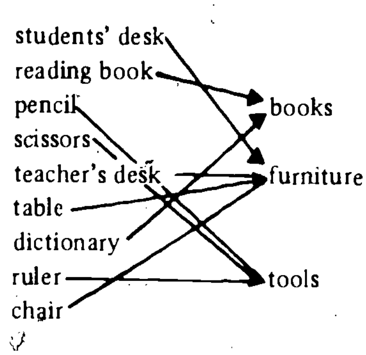
a. Provide a set of objects to be sorted by experimentation into *things that float* and *things that sink*. The set of objects should include some large objects which sink, some large objects which float, some small objects which sink and some small objects which float. Before each object is placed in the water the children may predict in which class the object belongs. Encourage each child to tell why he thinks the object will float or sink as he makes his prediction. Record the findings of the two classifications of the objects by listing *things that float* and *things that sink*.

b. Display a number of objects in the classroom used by the children for classwork or for play. Have the children to partition the set into two subsets, (1) for work and (2) for play. Then record their categories.

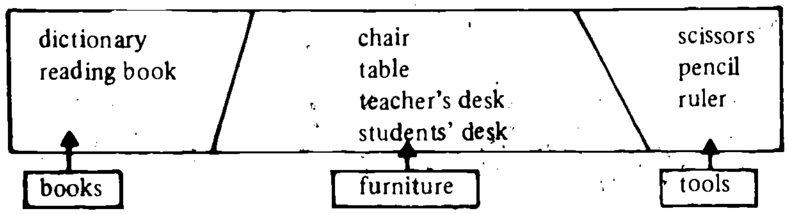


obj. 2,4,10

29. List the names of the objects and those categories as shown below. Have pupils choose correct category of each as indicated.



Then have the children write the names of the objects in a diagram similar to the one shown here.



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obj. 9

30. A list of members of a family is given. (mother, father, Ann, John, Sue)
List the members of the relation *is the brother of.*

Solution (John, Ann)
(John, Sue)

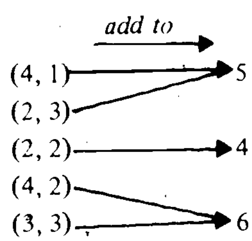
List the members of the relation *is the father of.*

Solution (father, John)
(father, Ann)
(father, Sue)

obj. 4c,10

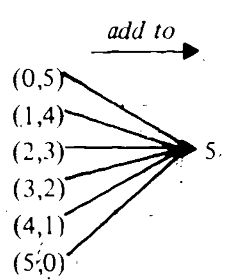
31. This activity provides a background of experiences in partitioning of numbers which establishes equivalence relations, the basis of many topics in elementary mathematics.

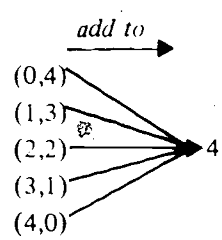
The operation of addition assigns a definite number called the sum to each ordered pair of whole numbers. Addition maps ordered pairs of elements of a set into single elements of the same set. A recording of such mappings might be as follows.

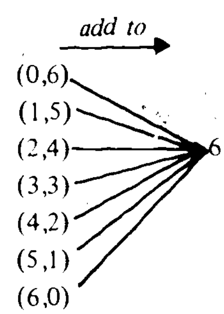


List some ordered pairs of whole numbers with the relation "*add to*," and ask the children to state the sums. The plus symbol (+) has not been used up to this point and should not be introduced until the children have had experiences of this type.

The children will probably suggest that there are other pairs of whole numbers which have the same sum. Lead them to find the other pairs of whole numbers whose sum is 5, those whose sum is 4 and those whose sum is 6. Record these.







This should be carried further to include other sums to lead children to generalize that in each set the ordered pairs *add to* a specific number or sum and forms a separate partition or class which they will later call equivalence classes.

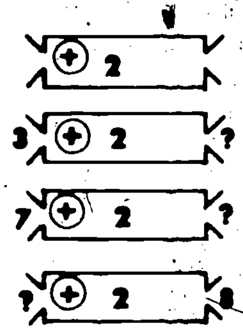
Recordings

+2

1 → 3
2 → 4
3 → 5
4 → 6
5 → 7
6 → 8

+2	
1	3
2	4
3	5
4	6
5	7
6	8

obj. 7,8, 9,10 32. Put the following on the board.



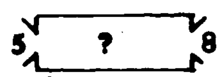
"Here is an *add 2* machine. Every number I put in has 2 added to it."

"Suppose I put a 3 into the machine. What number comes out? (5)"

"Suppose I put a 7 into the machine. What comes out? (9)"

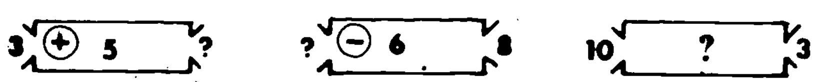
"Now suppose I got an 8 out of the *add 2* machine. What number went in?"

Supply as many examples as you wish. You should also use other machines such as *add 3, add 5, subtract 2*, etc. After the children are familiar with how the machine works, put the following on the board.

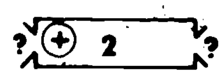


"What did this machine do to the 5? What kind of machine is it?"

Give several examples.
Then go to mixed missing facts machines.



Find some pairs for this example.



Then go on to the table form.

+2	
1	3
2	4

obj. 9 33. Given a relation expressed as an open sentence such as $\square + \triangle = 8$, tell the children to find as many pairs of this relation as he can.

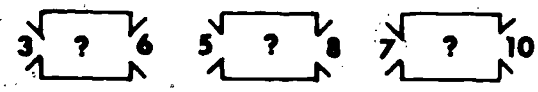
obj. 7 34. (a) Finding differences could be shown by using a subtraction machine, such as a *take three* machine.

(b) Subtraction could also be shown by giving the type of addition machine, such as *add four* and the output and asking, "What is the input?"

obj. 8

35. State the input and output of the machine and ask, "What does the machine do?" or "What is the operator?

Example



Recordings

?
→

3 → 6
4 → 7
5 → 8
6 → 9
7 → 10

?	
3	6
4	7
5	8
6	9
7	10

The operator would be "+3."

RELATIONS AND FUNCTIONS

OBJECTIVES

The child should be able to do the following.

1. Make comparisons by finding similarities and differences between two objects
2. Sort objects or symbols using the following
   (a) non-numerical relations and
   (b) numerical relations
3. Place objects in order by using a property such as length
4. Demonstrate correspondences such as the following
   (a) one-to-one
   (b) one-to-many
   (c) many-to-one
   (d) many-to-many
5. Illustrate the three basic numerical relations of *greater than, less than* or *as many as (equal to)*
6. Order two or more given numbers
7. Find a missing element of a pair when one member of the pair and the relation are given
8. Find the relation when a set of pairs is given
9. Find some pairs of elements when a relation is given
10. Picture or record relations, first using objects for representations, then by using diagrams or mappings and later by using tables and graphs

GEOMETRY

INTRODUCTION

The study of geometry should accompany the study of arithmetic from kindergarten through all the grades. The child's entire environment is composed of various geometrical figures, therefore, he needs to be able to recognize and understand spatial representations.

Geometry is the study of sets of points, that is, sets whose members are points. This does not mean that the concept of point should be developed first; on the contrary, the concept of point is an abstraction to be developed through experience. What it does mean is that care must be taken by the teacher to provide the proper contexts for this abstraction to take place. For example, in geometry the members of a set of points are seldom listed. Instead, a diagram is usually drawn to indicate the particular set to be considered. As with a listing, a diagram should indicate all the points and only those points to be included in the set. Thus if the points inside a square are under construction, the picture would look like ■ and not like □. Such distinctions become important when measure is involved. For example, area is appropriate for ■ (called a square disc), perimeter for □ (called a square). In particular, the diagrams for geometry are not like the cartoon figures used to illustrate a story; the child is expected to mentally visualize or fill in from a sketch such as  but not the diagram of a triangle (△). Although the orientation of a set of points is seldom important, recognition of sets of points under various orientations is important; a baseball diamond, for example, is a square.

For the primary level child, geometry is probably best thought of as exploration of space. Recognizing various orientations – the baseball diamond as a square, for example – exemplifies one kind of exploration of space. Recognizing relations represents another. Such exploration takes time and requires firsthand experience. *Telling* in order to save time can deprive children of this necessary foundation. The activities described in the following pages suggest ways in which children can become involved in learning geometry. The alert teacher will find many occasions for extending and reinforcing the ideas.

The Teaching of Vocabulary in Geometry

Just as the distinction between number and numeral is important in arithmetic, so precise use of terminology in geometry can help clarify the children's understanding and avoid misconceptions. However, caution must be used to avoid emphasis on word study at the expense of the more important *ideas.* The number and kind of activities suggested here reflect the intent of the committee that the bulk of the time devoted to geometry be spent in allowing the children firsthand experience with ideas and minimal time in word study.

The best test of whether a child understands the meaning of a word is whether he uses it correctly to express ideas. Thus for geometry, primary children should be able to use words such as *curve, between, unbounded,* etc., correctly when they ask or respond to questions.

The glossary at the end of the guide, or a mathematics dictionary are preferable to a general dictionary for determining technical distinctions. For example, in mathematics, *infinite* and *unbounded* are not synonomous, whereas *turn* is an acceptable substitute for the word *rotation* for a young child. For the primary child, word meaning will, in general, be learned in context, thus the responsibility for clear understanding and precise usage falls heavily on the teacher.

GEOMETRY

OBJECTIVES Keyed to Activities | **ACTIVITIES**

obj. 1a

1. It is easy for children to make the distinction between figures such as ዖ vs e. Have the children hold hands and form various shapes such as △, ○, □, ☆ called collection 1. This will illustrate the equivalence of these figures in what is sometimes called rubber sheet geometry. They should readily understand that it will be impossible for them to form ▲, ơ, or ➡ in this way, called collection 2. A mixed up collection of pictures drawn on the board for the children to try to make should lead them to sort out the can makes from the can't makes. They can further understand that by joining hands in a different way shapes such as ơ, ⌂ and ▽, called collection 3, can be made by moving in position. However, because they have to join hands in a different way, these figures are not equivalent to the figures in the first collection.

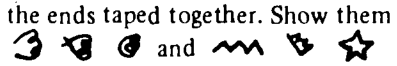
   Elastic thread or rubber bands could be used instead of children to illustrate the equivalence of the figures in the first collection, which are called simple closed curves. (Note: The word "alike" can be used with the children instead of the word "equivalence." The word "similar" should not be used in this connection because similar means having the same shape. See discussion under activity 19.)

obj. 1a

2. In playing the "Ring-Ring" Game use a long string, put a ring on the string, and then tie the two ends together. Place children around the string, and have them hold on to it. With eyes closed, the children pass the ring around the string until the teacher or selected child calls, "Stop! Who has the ring?" Children open their eyes and guess. However, if the string is tied like any one of the figures in collection 3 under activity 1, the child holding the end of the string cannot get the ring.

obj. 1a

3. An additional way of illustrating equivalence and non-equivalence is as follows. Divide the class into three groups. Give one group pieces of yarn which have not been tied, a second group pieces of yarn which have been tied but have the ends hanging free and the third group pieces of yarn with the ends taped together. Show them illustrations of shapes which are possible to make, such as

   Let them decide which shapes they can make with their yarn as it has been given them. They should discover the equivalence of the shapes which can be made within each group and the non-equivalence of the shapes between the groups.

obj. 1a

4. Use modeling clay for making discs, to show the equivalence of all discs, simple closed curves plus their interiors such as ⬢, ▲. As the children form and reform the clay, they will see that they cannot form △ or ✿ without breaking the clay or joining it in a way that it was not joined before.

obj. 1a

5. To show equivalence of space figures, a balloon is useful, since as it is blown up, it may assume various shapes all of which are equivalent. The geometric solids such as sphere, pyramids, cube, are, of course, hollow and are equivalent to the balloon. Boxes with tops, such as oatmeal boxes as well as rectangular boxes make better models than wooden blocks for showing equivalence to the balloon.

obj. 1a

6. For a further illustration of the kind of equivalence presented in activities 1-4, draw a figure on a broken balloon. By pulling the balloon varying amounts in different directions, the figure will be distorted into other shapes. As long as the sheet of rubber is not broken or folded, all the distortions will be equivalent. For other activities consult the annotated references listed in the section Utilization of Media.

obj. 1b

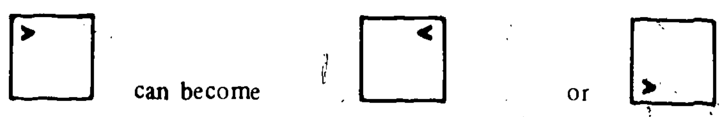
7. To illustrate rotations (turns), make a demonstration card with a design in one corner. Demonstrate that the design looks different when the card is given a quarter turn, half turn, etc., i.e., the design will be in a different corner, upside down, etc. The overhead projector could be used effectively for this illustration.

Next give the children the problem of picking out from a page of drawings those which could be a picture of the card rotated. Each child should have a small copy of the demonstration card to use to determine his answers. The page of drawings should include pictures which could not be made by rotation, as well as those which could, and the two kinds should be mixed so that each picture poses a problem. For example, if the demonstration card looks like [>] , then [<] or [<] cannot be made by rotations, but [<] or [v] can be made by rotation.

This activity should be repeated with different designs, and, as the children become more adept, different designs in two, three or four corners of the card should be used.

obj. 1b

8. If a transparency has been prepared for the exercise on rotation, it can be used to show reflection simply by turning the acetate sheet over.



The children should note that neither of these can be obtained from this card by rotating it as in the previous activity.

obj. 1c

9. Still another way to show reflection is to have some children use a felt-tipped pen and draw a design on some kind of absorbent paper such as newsprint. The ink will bleed through and show a reflection of the design on the reverse side of the sheet.

A worksheet similar to the one used in studying rotation should be prepared, again containing examples which could and could not be obtained by reflection.

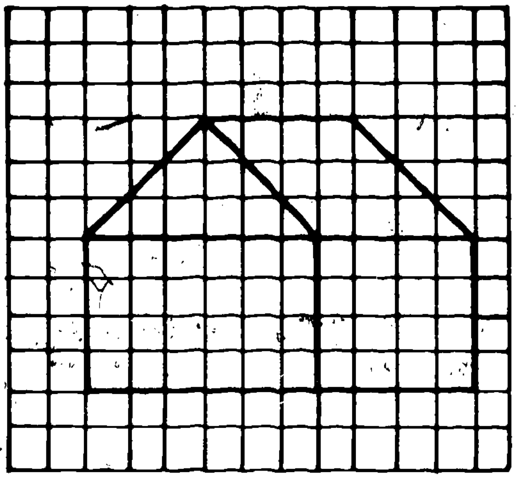
obj. 1c

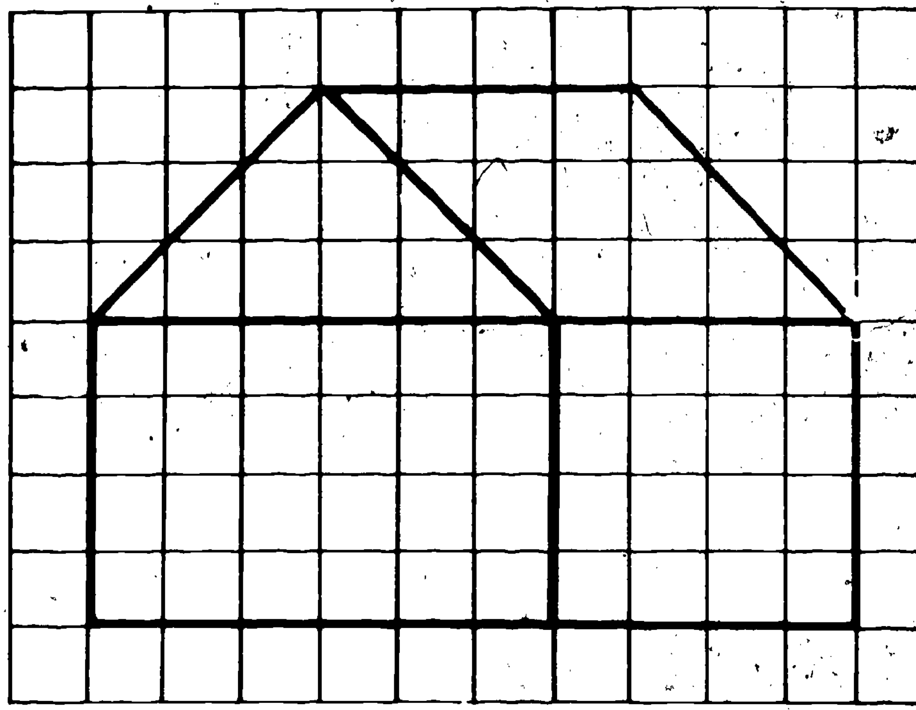
10. The mirror cards, listed in the section on Utilization of Media, are excellent for studying reflection. Also, ink blots and paper folding show reflections, as does cutting out designs by folding the paper and cutting through both thicknesses.

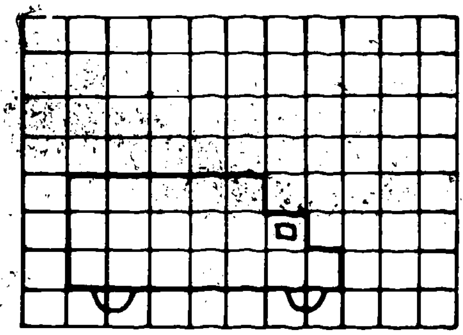
obj. 1e

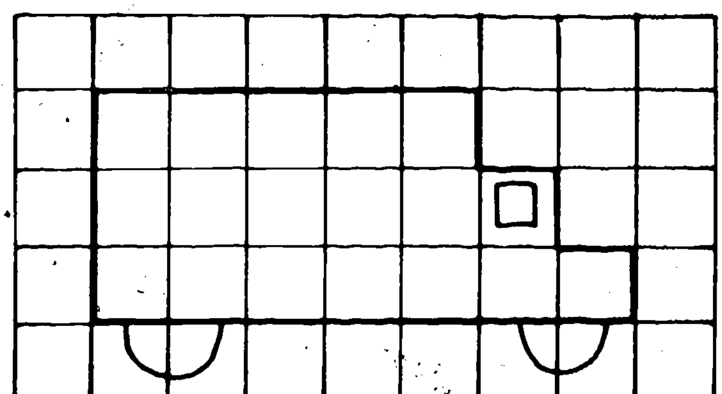
11. Copying designs on squared paper is a good way to illustrate uniform stretches and shrinks. The model as illustrated could be prepared on a transparency or the chalkboard and then copied by the children. The child's copy may be larger or smaller than the model.

When the children study maps they have another example of copies in miniature of some geographic subdivision. (Note: Recognizing figures which are enlargements or miniatures of others is probably easier for children than recognizing figures which are rotations or reflections of others.)



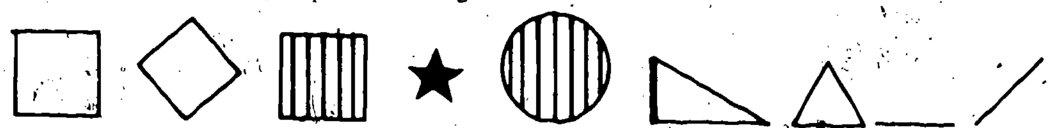




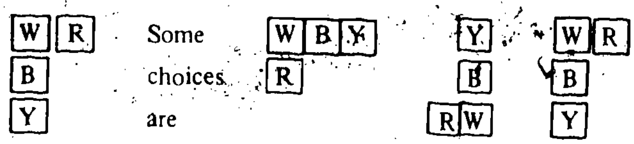


obj. 1

12. A work sheet would be helpful to check children's understanding of the various ways in which two point sets may be considered to be alike. On this sheet there should be examples of various kinds of equivalences explored thus far and a few should not be like any others on the page. Have the children tell which are alike. Some drawings should look as if they have been *turned around, flipped over*, etc. Some examples which might be included are shown here.



Block designs offer another means of checking the children's understanding of rotations and reflections. For example, ask "Which of the following are like this pattern?" (W=White, R=Red, etc.)



Use real blocks for younger children and pictures for older ones.

obj. 2a

13. The children are probably familiar with *inside the room, outside the house* and other similar terms which illustrate the fact that simple closed surfaces such as cubes, spheres, cylinders, have an *inside* and *outside*. The children need to understand how to precisely use the terms *inside* and *outside* with regard to geometry. For example, a block of wood does not have an inside but a wooden box does. The outside of the box or block does not include the object itself. To extend this notion to simple closed curves in the plane, draw a ring, or any simple closed curve on the floor and place toys inside and outside the closed curve. Ropes, hoops or children holding hands may be used for closed curves, and other children could place themselves inside or outside. A closed box such as a shoebox can be used to convey the idea of inside and outside a simple closed surface.

A good problem for older children to discuss is this. Suppose you have point Q identified inside a simple closed curve or on a surface. If the curve or surface is deformed into an equivalent figure, will Q still be on the inside? A diagram drawn on a piece of broken balloon can be a help in deciding the correct answer.

obj. 2b, 2c

14. In plane geometry, the words "parallel" and "perpendicular" are used with reference to *straight* lines in planes, but this is not true on the sphere. For example, the line down the middle of the road is parallel to the sides so long as the road is straight. When the road curves, we would say that the mid line and sided are concentric, not parallel. The child should have the concept of straightness as applied to lines. The everyday use of the word does not quite agree with the geometric idea, however. For example, a fold in a piece of paper will be straight even if the edges are not together. Also, walking catty-cornered across the street may still be walking a straight path.

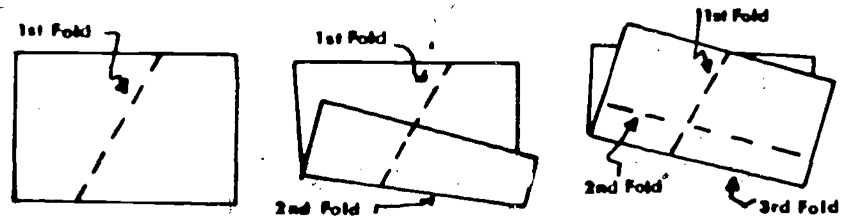
Some models of straight line segments are *lines* on writing paper, the edges of boxes and the intersection of walls and/or floors of the room. Since these tend to be either horizontal or vertical, some activity is needed to demonstrate that orientation or direction is irrelevent. For example, have two children hold the ends of a rope stretched taut. Have one child stand still and the other move around and raise or lower his end of the rope, moving toward and away from the class, always holding the rope taut. This example shows a few of the many lines through the one point, represented by the child who is standing still. To prevent any misconception, then, both children should move to a new position. In this way, the children will begin to develop an idea of the unlimited number of lines there are in all of space. A line, of course, is unbounded, as suggested by the arrows drawn on the number line, so that the rope actually shows only a line segment. This should probably be pointed out to the children, but not pursued in depth at this time.

obj. 2b

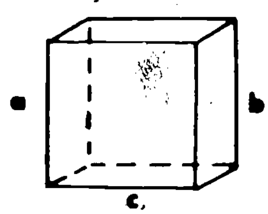
15. The children have seen parallel lines such as *lines* on the writing paper, but they need this brought to their particular attention. To do this, have them fold a piece of plain paper accordian style to make a fan. If the folding is typical, some folds will be parallel and others will not. To note the distinction, they will need to think about what would happen if the folds *kept going*.

obj. 2b

16. There is a way to fold paper so that, when properly done, parallel folds will ensure. The method, illustrated in fig. 1, is to first fold a sheet of paper, then place the folded edges together to make a second fold (note this makes a right angle). Again place the folded edges together to make a third fold. The second and third folds illustrate parallel line segments.



The children should be encouraged to look for further examples of parallel lines in the classroom and elsewhere. In doing this, some child is likely to notice that there are lines which do not meet but are not parallel either. These are called skew lines. To explain that two lines must be in the same plane to be called parallel but without using the word "plane," (since the child at this level should not be expected to know a definition of plane), lead the children to see that parallel lines are those which could be drawn side by side on a flat surface like the *lines* on their paper.



For example, a and b could be drawn side by side on a flat surface, hence are parallel; on the other hand, b and c could not be drawn side by side on a flat surface nor will they ever meet, so they are skew lines. A test for flatness, if this is necessary, would be to place a ruler edgewise on the surface to see if there are no gaps under it. This approach avoids the use of the term "distance" which may not be well understood even if it has been introduced earlier.

obj. 2c

17. The term "perpendicular" is used to describe two lines or segments which form a *square corner* (right angle). As mentioned above, paper may be folded to show a square corner, and there are many examples of perpendicular lines to be found in the classroom. The term "perpendicular" is also used to describe a relation between a line and a plane (also two planes). An example of this is the corner of the room and the floor. The concept of *plane* may be difficult for some young children to understand, therefore discretion is needed in the use of the word. In contrast, the idea of the *disc* – a bounded flat surface or portion of the plane – is very easy to understand. Care in the use of language will avoid misconception.

obj. 2b,2d

18. To illustrate various relations between line segments such as intersecting, parallel, perpendicular, have the children construct models of the skeletons of solids including pyramids, cubes and prisms with plastic soda straws and cleaners. Pipe cleaners make good joints since they can fit inside the straws and be bent at various angles. Let the children decide for themselves the number of straws needed and their arrangement by handling plastic or cardboard models of the solids.

obj. 1d,2d,2e

19. Recognition of the relations *same shape as* and *same size as* should be a natural by-product of the activities already described, but will need some expansion. Two point sets may be considered alike, for example, without being either the same size or the same shape (in rubber sheet geometry).

under uniform stretches and shrinkages two figures which are alike will have the same shape but will not be the same size; under turns, flips and slides, two figures which are alike will be the same size and the same shape.

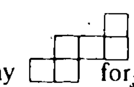
The concept of being the same size as is a subtle one and probably begins with the recognition that sometimes one thing fits exactly over another. This is called congruence. Thus the activities devoted to turns, flips and slides are important. As the child selects pictures which could or could not be a figure turned over, turned around or moved in some other way, he is learning to select those which can be made to fit exactly. A line segment which can be slid onto another, for example, provides a model for the child for the phrase "same length as." When he later learns to measure, it will make sense to him that he should get the same number if he measures each segment.

Accepting as the same size things which have the same measure, for example two 120 lb. individuals, one tall and one short, is quite another matter and requires more maturity on the part of the student. In the example just cited, by weighing the individuals we may not be measuring the attribute which the student thinks of as size. If measurement is to be made meaningful, there must be recognition of the attribute being measured. Thus the prescription here is for introducing the child at an early age to motions, with attendant practice in deciding whether things can be made to fit. In addition, recognition that no motion will make □ fit exactly on ■ alerts him to the various attributes which two figures may have.

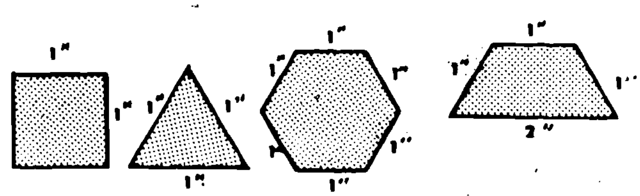
(NOTE: Activities 20-23 all provide first hand experience with arranging *units*. These units should be placed so as not to overlap, a basic step in measurement. Also involved is the selection of an appropriate unit such as tiles or discs for covering part of a plane, straws for paths and cubes for filling space. These remarks should not be discussed with the children, but the teacher should be alert for all counting opportunities as the children work.)

20. Give each child three blocks of the same color with instructions to see how many different patterns he can make. The patterns should not be alike by turn, flip or slide. The rules are that no gaps must be left between blocks, and they must be put together only at corners or whole sides. Discuss how many possible patterns can be made with three blocks. Ask if the children think all patterns cover the same amount of space.

    Repeat with four blocks, five blocks and more as the maturity of the children dictates.

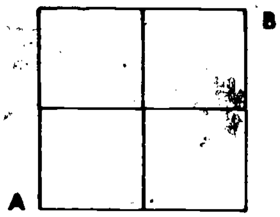
21. To build a basis for later study in measuring volume, after a child has built a pattern, say  for six blocks, ask how many blocks it would take to make the design two blocks high, three blocks high, etc. Care should be taken that in the beginning the children use irregular shapes such as the example given. Then when at a later time regular arrangements are selected for special attention, they will have had first-hand experience in counting blocks and can appreciate the computational shortcuts which regularity provides.

22. For young children the activity of building patterns is also called tiling. The children will need colored construction paper and cardboard or plastic *tiles* of various shapes. Large size parquetry blocks are also appropriate. Some suggested shapes are illustrated below.



Each child or group of children will need about 13 of a particular shape tile to work with at one time. Although all the children need not work with the same shape at the same time, by the time the activity is finished each child should have had the opportunity to work with each of the shapes available. The instructions to the children should be to try to cover a portion of the construction paper with a certain shape tile so that there are no gaps or overlapping edges. The problem they will be solving is to determine which of the given shapes will cover the portion of the paper. Thus at least one, preferably more than one, of the shapes which they have should not work. The construction paper should be of a contrasting color to the tile so that it will be easy to see when there are no gaps. Some children may be interested in using other shaped tiles (the diamond for example). At the end of the activity there should be class discussion leading to the selection of those shapes which will cover. Ask the children to predict others which could cover.

23. To make paths the children will need soda straws or sticks such as Tinker Toys, all of the same length, and a map drawn on tag board or newsprint. The map should look like this.



The map should be carefully made so that each side of each square is the same length as one of the straws. The first problem for the children is to determine the number of straws it would take to make the map. When they have determined this, possibly by laying out the straws on the map, the next problem is to determine the number of possible paths from A to B, staying on the lines of the map and only going up or to the right. Plan so that ample time can be allowed to explore various possibilities. The final problem is to determine if one of the five possible paths is shorter than the others. The children will probably have an opinion about this. They should try to prove if they are right by counting the straws in the path to discover the true relationship. With a faster group this activity could be pursued with a "3 X 3" map, etc.

24. In order that angle measurement not present difficulties in the upper grades, the idea of larger and smaller angles should be developed in the lower grades. To help develop this idea, the following activities are suggested.

    a. Use scissors; open the blades and ask the children to notice the size of the opening. Open the scissors a second time and ask whether the second opening is larger or smaller than the first.

    b. H[illegible] the scissors firmly with the opening remaining the same and rotate. Ask [illegible] the angle is greater, smaller or the same size.

    c. Hands of the clock may be used in the same manner.

GEOMETRY

OBJECTIVES

The child should be able to do the following.

1. Select from a collection of geometric figures those which are alike under the following
   (a) rubber sheet geometry
   (b) rotation (turn)
   (c) reflection (flip)
   (d) translation (slide)
   (e) uniform stretches and shrinks
2. Identify the following relations between point sets
   (a) inside and outside (for plane curves and space figures)
   (b) parallel (for lines)
   (c) perpendicular (for lines, planes)
   (d) has same size as
   (e) has same shape as
3. (a) Count the number of units in a given collection of units arranged in different configurations
   (b) Determine by sight which of two given angles is larger or smaller

MEASUREMENT

INTRODUCTION

Measurement is the process of relating a number to a property of an object or a set. Measure is a number which tells how many specified units have been determined.

Counting, the measure of "how many," yields the only exact measure. In this strand, counting is used to find the number of units of measure. Activities of this type should precede those in which the pupils find measurements from reading scales of measuring devices. In using a device, the *unit* of measure, which has the same property of the object to be measured, is compared to the object. The resulting measure is never the same as the true measure even with the most accurate instrument.

The different kinds of measurements presented in this strand are not necessarily to be studied in the order given or as separate studies from other strands. The development of measurement will depend upon the pupils' development in using relations, numbers, operations and properties.

Many of the difficulties in measuring that children have had in the past may have been due to the fact that applications were introduced too soon. The committee recommends some different approaches than have generally been used.

Time– Children are familiar with the phrases "time to go to school," "time for lunch," "now" and "not now," but they do not seem to realize time as an interval between events or that time is passing and continuous. Therefore, activities in measuring intervals of time as well as telling time are given. It is felt that by measuring intervals first, telling time or recording time should have more meaning. Note that in the approach described in the activities for telling time, the minute hand of the clock is used prior to the hour hand.

Weight– The weight of an object is the amount of force of gravity or the amount of pull of the earth on the object. Since this force is not visible, weight can be measured only by indirect means and, therefore, is difficult for children to comprehend. Also, the concept of weight is often confused with that of mass, which is actually the measure of the amount of matter. However, children are familiar with the idea of weightlessness of an astronaut and do realize that his mass does not decrease as he becomes weightless.

Capacity and Volume– Measurement of capacity and volume should begin only after the child realizes that two containers of different shape that hold the same amount of material have the same amount of space inside them.

Children use blocks, containers and other three-dimensional shapes in their preschool activities and begin to gain concepts of capacity and volume. Therefore, activities for capacity and volume will be considered before those of area and length in the guide. By filling containers and counting the units used, children should comprehend the ideas of capacity and volume.

In order to have more meaning, the measuring of areas of the faces and lengths of the edges should come after rather than prior to the study of volume.

Area– Activities for measuring area should begin with sorting the faces of three-dimensional shapes by size. These activities should develop from and also be a part of the study of geometry. The activities of tiling and counting units may seem time consuming in relation to the previous methods of teaching area measurement but should give real meaning later to measuring area indirectly by using formulas.

Length– The study of measuring length of line segments should develop similarly to that of measuring volume and area.

Temperature– The scales used in measuring temperature can be thought of as vertical number lines with special markings. Most classrooms will have a thermometer with the Fahrenheit scale, but pupils should also know about the Celsius or centigrade scale.

Money– Children should learn about money in planned situations in which each is an active participant. The study of money should be continuous. As new number concepts are learned, they should be used in problems involving money.

Measurement is a process. It should be studied in this manner at all grade levels and should develop through a number of stages– (1) making gross comparisons of objects or sets, (2) using units of measure devised by the children and (3) using units of measure called *standard* units.

In the first stage the children should have many experiences in making comparisons such as *greater than, less than* or *equal to.* These comparisons should be made between objects, sets of objects, dry materials and liquids which are familiar to the students. Children in kindergarten and the lower grades should experience filling and emptying containers, matching elements of sets, balancing materials, arranging shapes and similar activities to discover relationships between sizes and amounts of materials.

The second stage should develop from activities in the first stage. Units of measure will be needed to relate comparisons, and the first ones should be improvised or homemade. The children should make their own tables of measure so as to convert from one unit to another within the measurement of one property.

The third stage should be introduced only after the children have had experiences in the earlier ones and have seen a need for standard measure. Otherwise, their work will be manipulation of memorized symbols with no real comprehension of the process of measuring.

The students should make the measurements themselves using various measuring devices. The devices which are first used should have few markings. As the study progresses and operations with fractions are developed, additional markings can be added or different devices used. Activities should include those in which the children must choose the appropriate unit of measure as well as the appropriate instrument for measuring. Consideration must be given to both the property being measured and the size of unit applicable to what is being measured.

The students should have opportunities to measure and perform computations in both the English and metric systems. The emphasis should be placed on working within each system rather than on the conversion from one system to another. However, approximate relationships of the most often used units in the two systems should be discussed.

In the lower grades the measures of perimeters, areas, volumes and capacities should be found by counting the units of measure. This method should be continued in the upper elementary grades at which time activities are planned to lead them to discover the formulas.

The teacher should introduce the historical development of measurement to the pupils. It would be both interesting and helpful to them as they progress through the different stages of measurement.

MEASUREMENT

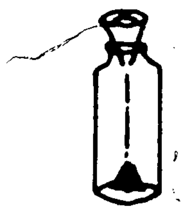
Objectives Keyed to Activities

ACTIVITIES

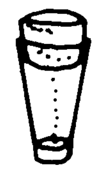
Time

obj. 1

1. Have the children measure short intervals of time using measuring devices such as sand clocks, water clocks or egg timers to see how long it takes for various activities. Examples of small group activities include the following. How long does it take for one member of the class to walk around the classroom, sharpen a pencil, to take up the mathematics papers, or to walk from the room to one end of the hall and return?



Sand Clock



Water Clock

obj. 1

2. Have the children measure longer intervals of time by using measuring devices such as a water clock which is larger than the one previously used, or a candle clock. A candle clock can be made by marking equally spaced divisions on a candle; each division would be an arbitrary unit for measuring time. Measure the time required for activities of the children such as length of the mathematics lesson, time required for the teacher to read a story and time required for one member of the class to walk around the school building.

obj. 1

3. Complete activities similar to those listed in Number 2 by using the minute hand of a clock. Choose activities no longer than 10 or 15 minutes in length. Repeat activities involving short intervals with the first setting of the minute hand between 12 and 6, then between 6 and 12. Next let the minute hand move so that it passes 12, such as from 9 to 2. The emphasis here should be on counting the number of minutes by using five-minute intervals and not on telling time. Bring to the student's attention that the clock dial is a circular number line.

obj. 1

4. Use curved strips cut from construction paper that will fit around the edge of the clock. Have strips to fit intervals of 5 minutes, 10 minutes, 15 minutes, 20 minutes, 25 minutes and 30 minutes. Ask the children to tell the number of minutes in the intervals as you place the strips against the edge of the clock.

obj. 1

5. Continue similar activities measuring intervals in minutes which are longer than 60 minutes and which begin on the hour, thus introducing the students to the use of the hour hand. Using drawings, play clocks or real clocks, have pupils determine the number of minutes in time intervals.

obj. 2

6. Set up play clocks or give drawings to provide practice in reading and writing clock time on the hour.

obj. 2

7. To introduce the readings of the quarter-hour and half-hour, use the appropriate strips from Number 4. Provide many activities for reading and writing clock time on the half-hour and quarter-hour. Include the use of the word "o'clock" meaning "of the clock."

obj. 2

8. Provide activities for reading the clock to the nearest minute. Have the children express clock readings in different ways. *First, teach "after the hour," and later "'til the hour."*

Examples--(1) 8:30 or half past eight, (2) 3:45, three forty-five or quarter 'til four.

obj. 2

9. Have children record time intervals using readings of a.m. and p.m., and call to their attention the 24 hours on the clock.

Examples--If school opens at 8:00 a.m. and closes at 3:00 p.m., how many hours is school in session?

If Johnny goes to sleep at 9:00 p.m. and sleeps 8 hours, at what time will he wake up?

obj. 3

10. Have an unmarked calendar chart for each month. Each day have the children record the day of the month in the appropriate square. Have a wall calendar nearby in order to relate today with the remainder of the month.

obj. 1

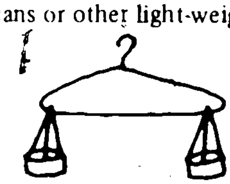
11. Provide activities in measuring intervals of time in days or weeks such as finding out for how many days a library book is checked out from the library, how many days from the time a seed is planted until it sprouts or how many days or weeks until some special time as Christmas or spring holidays.

Weight

obj. 1,5 of Rel.&Func.

12a. Have the children hold two objects of different weights in their hands and decide which is *heavier than the other* or whether one *is about as heavy as the other.* Repeat this activity with a number of different objects such as rocks, blocks or pieces of wood, or small packages of materials.

12b. Have the children compare the objects by using a simple balance. A balance may be made by using a coathanger with small cans or other light-weight containers hung for the weight pans.



Before the children use the balance the teacher should show that the weight pans are level when the pans are empty or when the objects that the children found are as heavy as each other. The teacher should show how the pans and the beam of the balance change positions as two objects of different weights are placed in the pans.

obj. 4

13a. Have the children use pairs of objects of different weights which were used in the previous activity, and find how much heavier one object is from the other in a given pair by using improvised units of weight. For improvised units use materials such as coins or play coins, nails of one size, paper clips, bags of tea, sugar cubes, macaroni shells, bottle caps or washers.

Have children place one object in each pan, then place improvised units in the pan with the lighter object until the pans are level. By counting the number of units used, the children will find how much heavier one object is than the other.

13b. Have the children find the approximate weight of given objects using the balance and improvised units. The children should place the object to be weighed in one pan and then place units of weight in the other pan until the pans balance. By counting the number of units used they will find the weight of the object.

13c. Have the children weigh an object first by balancing with one set of improvised units and then with a set of different type units, for example, paper clips and bottle caps. Since a paper clip is lighter than a bottle cap, the object will weigh a larger number of paper clips than bottle caps. Ask the children, "Which set of the units weighs more?" Since both sets balanced with the same object, they should weigh the same. Have the children compare the weights of the two sets of units by using the balance.

obj. 4

14a. Have the children make units of weight which will be *standard units* of the English system. Provide the children with one-pound weights of materials such as sugar, tea or sand. Have them use different materials to fill plastic bags so that each bag will weigh the same as the given one-pound weight. Close the bags tightly with rubber bands or tape.

14b. Have the children make other weights by filling some bags so that each will weigh the same as a given ½-pound weight and filling other bags to weigh the same as a given ¼-pound weight.

14c. Give the children some objects or bags of material for them to weigh by balancing the objects with the weights which they made in parts "a" and "b" of this activity. Be sure to include some objects for them to weigh which will require them to use combinations of their 1-pound, ½-pound and ¼-pound weights.

14d. Provide the children with bags which weigh one ounce each. These bags could contain materials such as tea, paper clips, sugar or sand. Have the children use these as weights to find the weights of some given objects.

14e. Ask the children to find relationships between the homemade weights such as the number of one-ounce weights which will balance one ½-pound weight.

obj. 4

15a. Have the children make *standard units* of weight of the metric system by filling bags with materials such as sand, sugar or rice so that each bag weighs the same as a given weight of 1 kilogram. If metric weights such as gram and kilogram weights are not available in science equipment for the elementary school, the teacher might borrow a set of these weights from the science department of the junior high or senior high school.

15b. Repeat this activity with the given weight of 1 gram or provide them with 1 gram weights. These weights may be made by linking paper clips or tying washers together instead of tying small bundles.

15c. Have them make weights of 5 grams, 10 grams, 50 grams and 100 grams.

15d. Have them balance different combinations of weights they have made. For example, they should see that two 5-gram weights balance with one 10-gram weight, but with the scales used and the materials used for making their weights there is room for error. The teacher might, then, balance some weights of a set of weights on the scales for them to see relationships.

15e. Have the children balance objects they used in activity Number 14 with their new set of gram and kilogram weights. They should realize that these units provide another way of naming the weight of objects. The purpose of this activity is for the children to see that there are two standard names for the weights of the materials. However, it is *not* the purpose at this level to have the children doing computation to convert from units of the English system to units of the metric system.

obj. 5

16. Have a number of objects available and ask the children to estimate the weights in English units. These materials could include a number of coins, bags of bottle caps, a brick, a pint of water and a book. Then, have them weigh the materials and see how close they guessed the weights. Repeat this process a number of times with other objects, including themselves. Try to use unusual objects.

Capacity and Volume

obj. 6

17. Since some children do not realize that if a liquid is poured from one container into another of a different shape, the amount of the liquid does not change, the following activity is suggested.

Have two cups of the same size and shape filled with water. Have the child pour the water from one of the cups into a bottle or jar. Ask him if the water in the bottle (or jar) is the same amount as that in the other cup.

For those children who have difficulty with this concept, provide more activities of this type.

obj. 6

18. Use six cubes and build a rectangular solid. Hand six cubes of the same size to a pupil and ask him to build a block like the one demonstrated. Ask him if the two blocks have the same number of cubes. One proof could be one-to-one correspondence of the cubes. Rearrange the demonstration block into a different rectangular shape. Now ask if his block takes up the same amount of space as the teacher's. *The understanding of this activity is basic to the understanding of volume.*

obj. 6
obj. 1,5 of Ref.&Func.

19. Have the children fit nesting boxes or jar lids together in order of size. Also have them place the items in rows in order of size.

Obj. 7

20a. Use two containers such as milk cartoons which are alike in all characteristics except height. Ask the child to show which holds more by pouring water or sand from one container to another. Ask him which holds less.

20b. Cut off the top part of a half-gallon milk carton so that the remaining part will hold the same amount as a guart carton. Ask the child to show which holds more by pouring water or sand from one to the other. He will probably be surprised to find that they hold the same amount.

20c. Use two containers which have different shapes and hold different amounts. Ask which holds more. Let the child find out by pouring from one container to another

obj. 7

21. Have the children find the number of cupfuls of water, sand or rice needed to fill a jar. At first it is advisable to fill a number of *units*, and as they are emptied count the number used. Later this will not be necessary; one unit may be filled and emptied again and again while the number of times it is being used is counted. Repeat this process a number of times with other containers.

obj. 7

22a. Have the children find the number of small boxes of sand or rice needed to fill larger boxes, such as shoe boxes.

22b. Have the children find the number of sugar cubes needed to fill some small boxes. Begin by finding the number of sugar cubes in their original box.

22c. Repeat the preceding activity, using marbles and then balls of cotton.

22d. Ask which of the materials used would be the best as a unit of measure, which would be the poorest and why.

Lead into a discussion of the need for *standard* units of measure, such as pints, quarts, gallons, cubic inches, which will be used in the upper grades.

obj. 8

23. Have a pupil fill a small metal can with water. Pour the water into another container. Reshape the original container by bending it. Have the pupil pour the water back into the container. He will probably be surprised to find that the capacity of the container has changed. Small metal cans such as fruit juice or tobacco cans are suitable for this activity.

Area

obj. 9

24. Provide rectangular shapes cut from sides of boxes. Ask the pupils to find the rectangular region which has the larger size and the one which has the smallest size. They should find this by counting the number of tiles used to cover the surfaces. If they have not seen that square regions are best for covering surfaces, let them use their own selection of tile shapes. The discussion should follow this activity to lead them to see that square tiles are best for this purpose. The strand on "Geometry" has similar activities.

obj. 9

25a. Ask the children to measure the area of a large rectangular region such as marked on the floor. Have newspapers for them to cut square tiles to use for units of measure. Quite a number might be needed to cover a large rectangle so as to measure the area.

25b. Repeat this activity a number of times, because the purpose of this activity is for the child to gain an understanding of surface measure (area) before the formula is introduced at the upper level. Some of the pupils may find the number of squares by multiplying the number of squares in length by the number of squares in width. It is best not to tell them this.

obj. 9

26. It will be agreed that some standard unit or units of measure are necessary to communicate with others. Introduce the square foot, and let the pupils cover large rectangles which have exact numbers of square feet. The unit square yard should be introduced in like manner if space permits. Merely count the number of square feet in a square yard if conversion is discussed.

obj. 9

27. Present rectangles which will not be exact numbers of units in area. The section which cannot be covered in square foot units should be covered with square inches and the measure determined. Do not require children to do computation, but have them count the units of square feet and square inches to cover the region.

obj. 9

28. Activity Numbers 26-28 should be repeated using square meters and square centimeters. The children need this hands on experience with metric measures.

obj. 9

29. Have the students use squared paper or squared acetate sheets to find sizes of 2-dimensional shapes. Provide grids which have one-inch squares and one centimeter squares.

Length

obj. 1,5 of Rel.&Func.

30. The activities for this objective should follow the same patterns as those for area. Use the length of edges of 3-dimensional solids, such as blocks or books used in earlier activities. The relations of *longer than, shorter than* or *about the same length as* should be made of such lengths.

obj. 10

31. Lengths should be measured using improvised unit lengths, such as new pencils, popsickle sticks, drinking straws, small lengths of cardboard. Then measurements of longer lengths should be made using longer units such as pointer sticks, long strips of cardboard or poles used for opening windows.

obj. 10

32. To introduce perimeter and circumference ask the children to find the distances around 2-dimensional shapes which are faces of 3-dimensional shapes such as cardboard boxes or the classroom. Pieces of strings or jump ropes may be used for units of measure. This measuring should be done in an informal manner; formulas will be studied in the upper grades.

obj. 11.

33. Standard units should also be introduced. Activities in measuring heights of children may be used to begin measure with inches and feet.

obj. 10

34. Lengths of different magnitudes should be estimated using standard units. Examples— width of a book, length of a pencil, length of chalk box, height of a student, width of a student, desk width of teacher's desk.

Money

obj. 13

35. Use pennies and dimes in the earlier activities and relate the numbers involved to our number base. If actual coins are not used, play money could be made by using construction paper and cutting the play coin the same size as the actual coin. Use brown for pennies, gray for nickels and white for dimes. On one side write "penny" and on the other "1¢". This type of homemade play money is preferred over some commercial products which bear little resemblance to the actual coins. Prepare enough for each child to be able to make change

Let pupils play store. Use items [illegible] of items familiar to pupils. At first, on the price tags use actual drawings of the coins needed. The coins can be drawn on a card and clipped to the item. Keep the prices less than or equal to 10¢ at first. Other coins should be added as children make progress and other items are used in the play store. Numerals should then be used instead of drawings to show the price of items.

obj. 13

36. One child may operate a bank in which children may exchange their coins for others of equal value.

obj. 13

37. Have pupils stack pennies so that the value of each stack is equal to that of a nickel or a dime. Have them use other coins to make stacks equal in value to a quarter, a half-dollar, a dollar.

obj. 13,14,15

38. To help children understand that the value of one dollar is equal to that of one hundred cents, use a hundreds board with round tags, or use a grid with round discs on the overhead projector. To show the value of the half dollar, divide the board or grid in half. To show the value of the quarters, divide the board or grid into four quadrants. For this activity the students should use ditto sheets containing the same information.

Record actual amounts of money needed for various situations such as the amount needed for lunch for each day of one week. The dollar sign and the decimal point should be used only after the children have had experience counting more than one hundred cents and know that 101¢ is equal in value to one dollar and one cent.

obj. 14

39. Let children play store as in Number 35, but provide situations involving making change. Reserve this activity for late in the second grade or for early third grade.

Temperature

obj. 15,16 of Sets, Nos. & Num.

40. Make a demonstration thermometer using cardboard and inserting [illegible], part of which has been colored red, through slots at the top and bottom. Mark the Fahrenheit scale on one side of the band and the centigrade (Celsius) scale on the other side. Adjust the elastic band up and down to indicate change in temperature. Discuss changes in outside temperature and temperatures in the classroom during the day. Relate temperature to the different seasons and to the kinds of clothing needed. The scale can be thought of and used as a vertical number line with special unit markings.

obj. 16

41. Keep a record of the high temperatures for a period of several days. Make a graph using these recordings. Use the high and low temperatures as exercises in addition and subtraction. Use only temperature readings of zero or above.

obj. 11 of Sets, Nos. & Num. 15

42. Some students may be ready for activities involving negative numbers. If so, temperatures below zero may be discussed to introduce negative numbers. The temperature scale may be considered as a vertical number line on which simple solutions to problems may be found by counting.

MEASUREMENT

OBJECTIVES

The children should be able to do the following.

1. State the number of units of time in a given interval between two specific events by using improvised time pieces, clocks and calendars
2. Tell time by using a clock
3. Identify the date by stating the month, day and year
4. Find the weight of an object using both improvised and standard units
5. Make a reasonable estimate of the weights of materials using specified units
6. Demonstrate that the capacity or volume of material does not change if its shape or position is changed
7. Determine the capacity or volume of a container by counting both the improvised and standard units needed to fill a container
8. Find by experimentation that changing the shape of a container changes its capacity
9. Determine the area of a region by covering the region and counting improvised and standard units
10. Determine lengths by using improvised and standard units
11. Give a reasonable estimate of lengths in specified units
12. Determine coins needed to obtain a given monetary value
13. Make change in coins by using the additive method
14. Record amounts of money using the cent symbol and the dollar symbol with a decimal
15. Read a temperature scale using the Fahrenheit and the centigrade (Celsius) scales

PROBABILITY AND STATISTICS

INTRODUCTION

Basic to statistics are the techniques of collecting, organizing, summarizing and analyzing data. Once the data are summarized and analyzed, the predictions that are made become the study of probability. Therefore, statistics and probability are studied together.

In the primary grades the study of statistics is limited to collecting and organizing data through techniques such as counting, tallying responses to the question, "How many?", searching for patterns and recording outcomes of game-like experiments.

The study of probability in the primary grades is limited to finding the chances of particular events occurring. For example, given a bag with three balls in it, all of which are a different color (red, green, blue), the primary grade child learns that the chance of drawing a blue ball from the bag is one out of three.

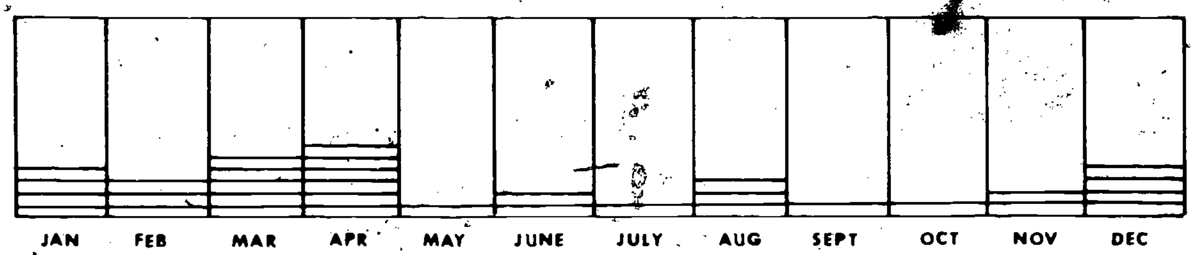
Not only are the activities in this strand readiness experiences for more formalized learning in statistics and probability later, but they also provide experiences in problem-solving and practice in counting and simple computations. Furthermore, children usually find them great fun and teachers find them excellent for motivational purposes.

PROBABILITY AND STATISTICS

OBJECTIVES Keyed to Activities | **ACTIVITIES**

obj. 1,2,3

1. List the birth date of each child. Have the children tally these under the 12 months. In addition to listing the birth dates and tallying them, they may construct a bar graph similar to the following.



Let each child place a rectangular disc in the appropriate column. After constructing the chart the children may then answer questions such as, "Which month has the most birthdays?" "Which month has least birthdays?" "In what month do most of the birthdays fall?" "Does it matter on which day of the month the birthday occurs?" The discussion of these questions would help them to sort out what is relevant.

obj. 1,3, 5,10

2. Have the children measure and record their heights on one large sheet of wrapping paper. Ask the following questions.

   a. Where is the mark for the shortest child? How tall is he?
   b. Where is the mark for the tallest child? How tall is he?
   c. Where are the marks for the other children? How tall are most of them?
   d. If we had a new student placed in our room today, how tall do you think he would be?
   e. If a visitor came into our room, how tall do you think he would be? What difference would it make if the visitor was a high school student? a grown-up?

Use these questions to promote discussion about this sample of measurements being representative of a restricted population, that is, the grade in which the measurements were made.

obj. 1,4,10

3. Have the children count and record the number of those who walked to school today, who rode their bikes to school, who came by bus or car. Ask questions such as, "Did most of you walk?" "Ride your bike?" "Come by family car?" "Come by bus?" "What do you think most of you will do about getting to school tomorrow?" "Is our discussion today likely to influence choices for getting to school tomorrow?" (Such discussions will begin the development of the idea that bias may enter the data.)

obj. 1,2,3, 4,10

4. Have the children tabulate the number of those who are wearing dresses or shirts of red, blue, green and brown. Only the specified colors are to be counted in this sample. It may be necessary to point out to the children that the purpose of this activity is to collect data for a restricted set of values, and that the presence or absence of other colors is irrelevant. Then ask, "Which of the specified colors is most popular?" "Which is least popular?" "Is there some reason for particular colors being worn today?" If, for instance, some boys are wearing uniforms because of a scheduled Cub Scout meeting, the data is biased. This sample is not a good one for determining the most popular of the four specified colors.

obj. 1,2,3,5

5. Have the child tally the number of cars that pass the school window, door or yard during a five minute period early in the day, around noon and after lunch. Stress that trucks and busses are not cars, and that make of the car is not to be considered. Ask, "In which period did the greatest number of cars pass?" "The fewest?" "Is there a reason for traffic being heavy near the school at this time of day?" "Would this be true at your home?"

obj. 1,3

6. Have the children construct bar graphs to show various distributions. For instance, identify the children according to the color of their eyes. Provide each child with a cube. Using the cubes to represent units the children can arrange them, one by one, in stacks in the chalk board tray. One stack would represent the number of brown-eyed children; a second stack would represent the number of blue-eyed children; and a third stack, the number of gray- or green-eyed children. Each stack of cubes corresponds to a bar on a bar graph. Ask questions such as, "Which stack is tallest?" and "What does that tell us?" Some children may record the data in a drawing of a bar graph.

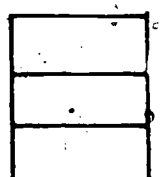
obj. 6

7. Provide children with objects such as beads or small blocks of two different colors. There should be the same number of blocks in each color. For instance, five red beads and five green beads may be given each child. Then ask the children to arrange the objects in a fixed pattern of alternating colors, by stringing the beads on a length of wire or arranging the blocks in a line on the desk top. They should discover that there are two different arrangements of colors which they can make. The difference depends on the choice of color they make for the first object in the arrangement. That is, the pattern *red, green, red, green, red, green* is different from the pattern *green, red, green, red, green, red*.

This same idea may be extended to more complex discrimination tasks involving three colors arranged in some fixed pattern.

obj. 1,3,6,7, 8,10,11

8. Have the child use crayons of three different colors to color a series of rectangular blocks.



Have him color the blocks in as many different arrangements as possible. He should find that the greatest number of different arrangements of colors is six. The different arrangements are called outcomes.

red	red	blue	blue	green	green
green	blue	red	green	red	blue
blue	green	green	red	blue	red

Similar activities would be helpful to the child. He could use three different books, three different blocks or three different children to be arranged in all possible ways. In using three children, if the teacher tells Mary, Jane and Jo to go to the front of the room, he could then ask, "What are the chances that they will walk up to the front in the order of Jane, Mary and Jo?" An acceptable response would be 1 out of 6. The teacher could then ask, "What are the chances that they will walk up in the order Mary, Jane and Jo?" The child will be able to see that each arrangement has an equal chance of happening. If John is not in the class, ask what chance does the arrangement Mary, John and Jo have of happening? The pupils should see that there is no chance of this arrangement occurring.

obj. 1, 3 8, 9	9. Give the child a spinner which has been colored in two distinct colors to form two regions of equal size. Ask the child to spin the spinner a number of times and to tally the times that the pointer stops on each color. Through this activity he will probably conclude that the pointer will stop on one color as often as on the other because the spinner was separated into two regions of equal size. Ask him a question which involves a color not on the spinner such as, "What is the chance of the spinner stopping on a purple?" Questions such as this one should lead him to discover when *an outcome has no chance of happening.*
obj. 1,3,7, 10,11	10. Play the game described in the previous activity with a spinner that is separated into four regions of equal size, one region colored yellow, one red and two blue. By tallying the stops of the pointer, the child will discover that the chance of getting blue on a stop is greater than the chance of getting any other color.
obj. 1,3,7, 10,11	11. Games played with two cubes with the six faces marked distinctly may be used in the classroom to show that the chance of getting some combinations is greater than others. Children's blocks or sugar cubes with simple markings such as +, X, ( ), | | | or colors may be used.

PROBABILITY AND STATISTICS

OBJECTIVES

The child should be able to do the following.

1. Collect data in different ways
2. Sort out relevant and irrelevant data
3. Use physical objects and pictures to record data
4. Distinguish between biased and unbiased data
5. Tell whether a sample of data represents a population
6. Arrange two or more objects in a number of ways, and collect the data
7. Count all the possible outcomes of an experiment which has a limited number of outcomes
8. Pick out events that have an equal chance of occurring
9. Select events that have no chance of happening
10. Show that in some instances one event has a better chance of occurring than another
11. Specify the chance of an event happening [illegible]

PROCESSES FOR SPECIAL CONSIDERATION

Problem Solving

Difficulties in Computation

PROBLEM SOLVING

Since problem solving is an integral part of mathematics, it is not possible to teach mathematics well and not be simultaneously solving problems. For this reason, many of the activities suggested in each of the strands in the guide are problem oriented. The teacher, however, has the major responsibility of selecting appropriate problems for his mathematics class.

WHAT IS A PROBLEM?

In the guide a problem situation is recognized as one in which habitual response are inadequate. Word problems may in fact be only verbal exercise for some pupils. On the other hand, open sentences or exercises such as 5 + 8 = □ are problems for those who do not know the associated fact. To say that habitual responses are indequate is to say that recognition and recall are insufficient cognitive processes for the activity of problem solving.

The verbal exercise which begins, "John had 4 marbles and his father gave him 5 more . . .," is not a problem for the pupil who *recognizes* that a joining action is associated with addition and then *recalls* the sum of 4 and 5. In contrast, the pupil who does not know the sum of 5 and 8 is faced with a problem. In that case, the teacher's task is to ask the right questions, "What do you think it is?" "Do you want to guess?" "Why do you say that?" "How could you find out for sure?" In response to the last question, the pupil's decision will depend on his prior experiences and the performance level at which he is confident of success. He may choose 5 counters and 8 counters; recreate the joining of sets of 5 and 8 and count the members in the superset. He may place a mark at 5 on the number line and count off 8 from there; or he may reason, using well-known facts, that 8 is the same as 5 + 3. Then 5 + 8 is the same as 5 + (5 + 3). *Observing* what happens as he carries out the experiment, he *verifies* and *records* his conclusions.

Consider another example.

> Mary has 2 skirts and 3 blouse. If all of them go together, how many different outfits can she assemble?

For pupils who have studied arrays as models for multiplication of whole numbers, this example is a drill exercise in recognition and recall. For pupils who have studied only repeated addition as the model for multiplication, it is a problem.

The selection of problems for mathematics classes becomes, then, a matter of selecting situations in which new relationships must be found among well-known facts or in which the context is so novel that pupils must explore, observe, make conjectures and test their conjectures.

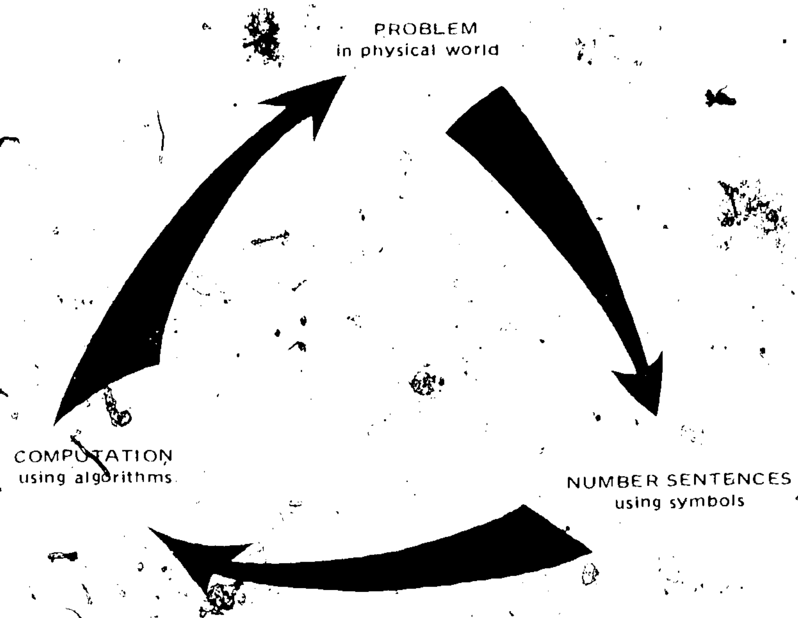
A Classroom Environment for Problem Solving

Given a problem situation, the role of the teacher is to provide an encouraging climate in which pupils feel free to make guesses, checking out their guesses at whatever performance level they are successful, talking about their observations and making decisions which are defensible in terms of the conditions and data given. Teachers can aid pupils in developing the right sort of guessing by asking questions such as the following. "What do you think? Which of these two cans holds more water?" "What do you think? Which is the greater number, $\frac{7}{8}$ or $\frac{7}{9}$?" Following the recognition or recording of pupils' conjectures, the teacher should ask, "How can we find out?" After pupils have carried out the experiment and observed what happens, the teacher should ask perhaps the most important question of all -- "Why did we think the answer was one thing when in fact it was something else?"

Collecting opinions from pupils at the outset may seem unusual in a mathematics class. However, it can give the teacher valuable insight into their thinking as well as focus attention on the nature of the problem at hand. The selection of problems, the questions asked by the teacher and reactions to pupils' guesses are all very important in developing and improving competencies in solving problems. Since some pupils are discouraged by being told that they are wrong, an important contribution from the teacher is his expectation that they *can* solve problems. His confidence can encourage easily frustrated pupils to try again and can be contagious. Persistence and confidence are characteristics of good problem solvers.

Other Factors Related to Success in Problem Solving

In addition to teaching guessing and encouraging flexible approaches to problem solving, the teacher needs also to provide for his pupils' acquisition of knowledge of fundamental concepts in mathematics, skills in computation and reading and other interpretive abilities.



Knowledge of fundamental concepts is acquired through a variety of planned experiences. Consider the word problem.

> Three boys were playing marbles and two other boys came to play with them. How many boys are now playing marbles?

When difficulties occur with exercises of this type, the difficulty is probably not one of reading nor one of computation, but one of not knowing the meaning of the operation called addition. Help for this difficulty involves the right kind of initial presentations of the operations concepts. It is for this reason that a distinction has been made in the guide between operation and computation. Teaching addition is different from teaching how to compute the sum; the former encompasses the *when* and *why* one adds, whereas the latter does not. If sufficient attention is given to the meaning and effects of operations, pupils should have less difficulty in recognizing verbal descriptions of situations for which the different operations are appropriate. On the other hand, if computation is given the major emphasis, it should not be surprising that pupils fail to recognize a situation which calls for a particular process. Teachers should see the strand on Operations, Their Properties and Number Theory.

Teachers also need to be alert to possible difficulties associated with the no-action situations. The joining of two sets of boys described in "Three boys were playing marbles and two other boys came to play. . ." is identified with the operation of addition. In the verbal description, the action of joining two sets is explicit. However, some pupils may have difficulty with verbal problems of the following type.

> Susan and Jane live on adjacent blocks. Susan counted 7 brick houses on the block where she lives, and Jane counted 5 brick houses on her block. How many brick houses are there in the two blocks?

The two sets of houses cannot be joined physically. Nevertheless, the reader must think about the two sets as *joined* in some superset and recognize that the situation described is an additive one.

Corresponding to the other operations are the same categories of *action* and *no action* problems. Quite often, it is helpful for pupils to recreate problem situations with manipulative or pictorial aids. Some will need physical experiences with counters or toys, while others will need only pictures or diagrams which they can draw. Problem difficulty has been assessed along the aids/action dimension as follows.

Aids	Problem Situation	
	Action	No Action
Physical	Easiest	
Pictorial		
None		Hardest

In light of the above discussion, the usefulness of cue words in verbal problems becomes highly questionable. The better strategy seems to be one of developing the concepts of operations and the concurrent development of language -- in both oral and written descriptions associated with the concepts. In the guide, it is suggested that pupils be given an equation such as $3 + 4 = \square$, or $3 + \square = 7$ and be asked to make up a word story with which the equation could be associated. By encouraging a diversity of responses, a teacher can aid pupils in learning many different settings and situations for which a particular operation is appropriate. At the same time, he is helping them develop the language skills one needs for reading and talking about mathematical ideas.

[illegible] the following word problems.

(1) John has [illegible] pennies in his pocket *and* 3 pennies in his lunchbox. If he *puts* them together, *how many* pennies does he have?

(2) John has 6 toy cowboys *and* 3 toy horses. If he *puts* one cowboy on each horse, *how many* cowboys will not have a horse to ride?

(3) Before school started in September, Mary's mother made her 6 new blouses *and* 3 new skirts. If she could *put* the blouses and skirts together in any combination, *how many* new school outfits would Mary have?

(4) Mary has 6 roses *and* 3 vases. *How many* roses could she *put* in each vase so that there are the same number of roses in each vase?

In each case, the word "and" serves its logical role as a conjunction and does not indicate the action involving the given sets. In each case, the word which suggests the action is "put(s)." In the first problem, two sets are put together to create a superset. In the second problem, two sets are put together for the purpose of comparing them. The putting together involves the action of matching objects from one set with objects of another set in a one-to-one correspondence until all the objects in one set are exhausted. In the third problem, the putting together involves the pairing of each object from one set with each object of the other set, that is, the creation of a third set in which the elements are outfits, or pairs (one skirt, one blouse). In the fourth problem, the putting together of the two sets involves the action of establishing a many-to-one correspondence of the given sets. In each case, the question is "How many . . . ?" There are no single, clear-cut word cues for corresponding mathematical operations.

Neither the ability to recall the number facts nor skills in reading as measured by standardized tests are sufficient cognitive factors for solving these problems. One must also possess an ability to recognize that certain kinds of action (real or implied) are associated with particular mathematical operations. The successful problem solvers, in this case, are those who associate addition with the action described in problem 1, subtraction with the action described in problem 2, multiplication with the action described in problem 3 and division with the action described in problem 4.

Teachers should provide a variety of experiences in which pupils can talk about their understanding of the operations. For instance, pupils may be given the four verbal problems above (or some similar set) and either of the following sets of corresponding expressions.

(a) (1) the sum of 6 and 3
(2) the difference of 6 and 3
(3) the product of 6 and 3
(4) the quotient of 6 and 3

(b) (1) $6 + 3$
(2) $6 - 3$
(3) 6×3
(4) $6 \div 3$

They should be asked to discuss (and demonstrate, if necessary) *why* the corresponding expressions give the correct solutions.

Other techniques teachers may use to help pupils improve their problem solving abilities through increased understanding of the fundamental operations are these.

- Use problems without numbers.
- Have pupils act out problem situations and solutions.
- Have pupils write only the corresponding number sentences for verbal problems (but not the answers).
- Have pupils make diagrams or drawings to illustrate their solutions.
- Use problem with insufficient data or irrelevant data.
- Have pupils make up problems for given pairs of numbers (e.g., formulate four different problems in which the numbers 24 and 8 are used, with each problem solution to involve a different operation on 24 and 8).

Skill in computation is, of course, a necessary condition for getting correct answers, and teachers should have pupils test the reasonableness of their answer as a check on their computations. Such tests may also provide checks on the appropriateness of processes selected. (For further discussion on skills in computation, teachers should see the section on Difficulties in Computation.) The tediousness of computation should not be allowed to obscure the principle inherent in a particular verbal exercise. For instance, if pupils are studying the principle related to finding the arithmetic mean of a sample of *n* measurements, the emphasis of classroom activities should be on the mathematical principle. In this case, once the pupils have identified the procedure to be used, it is helpful to use an adding machine to take care of the tedious task of summing all the measurements.

Reading and other interpretive skills should be developed as an integral part of the mathematics program, with special attention given these skills in class work with verbal problems. In addition to the techniques previously suggested for developing language and reading skills, the following suggestions should also prove beneficial.

Help pupils learn to use their textbooks. Take time to read with them certain narrative sections of the book, including the preface, if there is one. Take time to note the table of contents and to teach the use of the index. If the publisher has a glossary of mathematical symbols and word definitions, make it accessible to pupils. Encourage and help pupils find out who the authors are and secure biographical information about them.

Have pupils read library books and reference materials related to mathematical ideas. For instance, in the upper grades, when studying measurement, have a committee of pupils research a topic such as "The History of Linear Measures" and report their findings to the class. Identify science-related materials and problems (which involve mathematical principles and skills) for pupils to read and solve. Identify charts and graphs from the social studies program which require interpretive skills learned in mathematics. Help pupils to know that mathematics is a lively, purposeful and dynamic force in their culture.

Provide specific instruction in quantitive vocabulary. The study should include the role of prefixes and suffixes with word-roots which occur in mathematics, as in the words polygon, polynomial, triangle, trinomial, equalities, equidistant. The study of word endings should be included, as in the words polygonal, rectangular, radii. The study of prefixes, suffixes, roots and endings which have special meanings will facilitate the learning and spelling of new terms.

Have pupils formulate and write verbal problems, or have them jointly discuss how to explain a selected mathematical principle and write the verbal explanation on the board. Class discussions of the written material should lead to refinement and classification of the language used. Encourage pupils to avoid redundancies and ambiguous descriptions. An occasional mathematics period devoted to developing skills of conciseness and precision in language is well spent.

Have pupils read selected word problems for the purpose of finding specified information. What is the question? Describe the setting. Who is involved? What sets of physical things are involved? What is the order of events which occur over a period of time? What is the relationship or action described? Errors or inabilities in comprehending what they read may be due to the pupils' failure to read for the express purpose of noting details. Explanatory or descriptive materials which deal with mathematical ideas are marked by a conciseness which is not characteristic of reading materials in general. Know the reading abilities of pupils and select or adapt materials to their ability levels.

Present problems orally. Provide for the poor reader by using a tape recorder to record word problems. There are many opportunities for engaging all pupils in problem-solving activities through open discussion, through manipulation of physical models and through the use of pictures and diagrams. No pupil need be denied learning experiences in problem solving.

In addition to selecting and adapting instructional materials and procedures to the abilities of his pupils, the teacher himself can serve as a model in the use of language. For instance, the simple written expressions, "the product of 8 and 9" or "the sum of 8 and 9" are difficult for those pupils who have always read "8 X 9" as "8 times 9" or "8 + 9" as "8 plus 9." Teachers can easily provide experiences with such verbal language expressions through their own use of them in writing at the board and in reading symbolic expressions such as "8 X 9 = 72" as "The product of 8 and 9 is 72." Instead of asking pupils to "find the answer to...," the teacher could ask them to "find the sum of..." or "find the quotient of..." Consider the following ways of writing one simple fact in mathematics.

(1) $3 + 4 = 7$

(2) $7 = 3 + 4$

(3) $\begin{array}{r} 3 \\ +4 \\ \hline 7 \end{array}$

(4) Three plus 4 is equal to 7.

(5) Seven is the same as three plus four.

(6) The sum of 3 and 4 is 7.

(7) Seven is the sum of 3 and 4.

(8) Four more than 3 is 7.

(9) Seven is equal to 4 more than 3.

(10) Four added to 3 is 7.

(11) Seven is 4 added to 3.

If children hear and use the verbal variations in (6) through (11), they are more likely to be able to read such statements with understanding.

The ability to read any verbal problem with understanding and insight and find the solution is a complex ability which encompasses several cognitive factors and processes. There is no one step-by-step procedure which is best for every problem-solver. However, the following procedure is suggested as a guide for general classroom use.

- Tell a short story. That is, have the pupil read the verbal problem and revise the narrative so that it is in the form of a short story.
- Select the mathematical operation associated with the problem structure.
- Write the number story or open number sentence corresponding to the story.
- Solve the numerical problem.
- Answer the question asked in the verbal problem.
- Check to see if the solution is reasonable.

DIFFICULTIES IN COMPUTATION

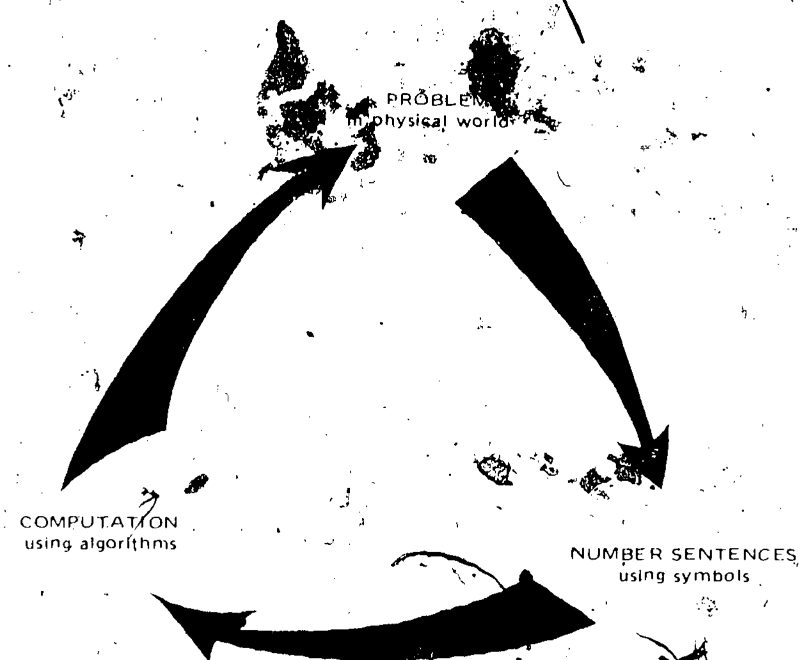
INTRODUCTION

In this section, the place of computation in the mathematics program is discussed. The committee has examined specific types of computations which students find difficult and has made some suggestions.

In the strand Operations, Their Properties and Number Theory a distinction is made between operations and computation. Computation is the processing of numerals. The process yields the name of that number which is assigned to a pair of numbers by the operation. For example, the operation of addition assigns a single number called the sum to the pair of numbers 17 and 18. By the process of computation, one determines that the name of this sum should be 35 in the decimal system of numeration.

In order for children to obtain skills in computing, they must first have an understanding of the operations. As described in the strand Operations, Their Properties and Number Theory, each operation should first be related to a physical situation, then illustrated through pictorial representations and number line activities and recorded with numerals and operational symbols.

The abstraction of the operations should be pulled from the physical world so that later in problem solving there will be no trouble when one also moves from the abstraction back to the physical world. The operations need to be clearly understood in order to organize and set up problems in symbols. The place of computation in solving problems may be seen in the following graphical illustration.



In the lower grades, teachers typically make a practice of offering children experiences with physical models associated with the operations and having them record the results using mathematical symbols. However, in order to develop any of the computational algorithms, the pupil needs to use physical models for the operations that emphasize bundling collections into [illegible] etc. For instance, the pupil sees that a collection of 9 objects joined to a collection of 6 objects yields a collection of (6 + 9) objects. He should then actively experience the bundling of this collection according to the ten's rule, or decimal coding scheme. The act of bundling and then recording the sum as one-five (1 bundle and 5 singles) should precede naming the sum "fifteen." Early experiences with small sums such as these are necessary for children to develop insight into computation as the processing of decimal numerals. It should be recognized that this bundling is the physical analogue of those maneuvers called *renaming* which occur again and again in the computational algorithms.

With larger numbers, one or more of the addends may need to be represented with bundles. For example, in determining the sum of 17 and 6, the 17 should be represented by 1 bundle of 10 and 7 singles. When the collection of 6 is joined to this, the resulting collection cannot be assigned a decimal name until more bundling is done. In the example in the second paragraph of this section, the sum 17 and 18 should be represented by 1 bundle of 10 and 7 singles joined with 1 bundle of 10 and 8 singles. Again the decimal name of the sum cannot be assigned until all possible bundling has been completed. The sum should then be recorded as 35. The pupils should work individually on many similar examples with physical objects before making the transition to using pencil and paper exclusively.

The stages in which children move from the basic concepts of the operations to the final computational skills is called "why (understanding)." And those skills which the children finally acquire to find the result with maximum speed are termed "how (skills)." The following examples show these stages.

	Why (Understanding)	How (Skill)
Problem 87 - 48 = ☐ Understanding the operation subtraction, decimal notation and regrouping, and knowing the number facts	87 = 80 + 7 = 70 + 17 48 = 40 + 8 = 40 + 8 30 + 9 = 39	87 -48 39
Problem 91 ÷ 7 = ☐ Understanding the meaning of division and the operation of subtraction, and knowing the number facts	91, -7, 77, -7, 70, . . . 14, -7, 7, -7 — counting 1, 2, . . . 12, 13 so 91 ÷ 7 = 13 Later 7) 91 — 70 (10), 21 — 21 (3), 0 so 91 ÷ 7 = 13	7) 91 = 13

The above activities allow children to discover their own patterns for computation. Since there is no single algorithm for any of the operations, the teacher should be willing to let children use any mathematically sound process they wish.

The development of computational skills continues to be one of the important goals of mathematics education. When pupils have difficulty in computation, the teacher will find it beneficial to diagnose the difficulties. Although diagnosis is time consuming, results will be worth the expenditure of time. Early diagnosis will prevent pupil frustration, failure and consequent lact of effort. If pupils do not understand the algorithms, additional problems of the same type will not prove beneficial. However, if they can see why they made mistakes and can learn the correct procedures, additional practice will then be helpful. Mistakes may result from lack of understanding basic facts rather than from failure to understand algorithm. Acquiring skill in basic facts requires practice, repetition and drill, but the teacher should use a wide variety of drill techniques.

An all-inclusive treatment of computational difficulties is beyond the scope of this guide. The examples given, however, will serve as models for diagnosing difficulties which the teacher will encounter in his own classroom.

Typical Computation Errors	Suggestions

I. Subtraction of whole numbers

A. Jim wants to buy a popsicle that costs 10¢. He has 7¢ in his pocket. How much money must he take from his piggy bank to be able to buy the popsicle?

Incorrect	Correct
17¢	3¢

The pupil does not realize that there are uses of subtraction other than take away. The example shown at the left is one example of how many more are needed.

Exemplify different models for subtraction. Some of these are (1) Take away— Use a set of 9 objects. Remove 3 objects. How many are left? (2) How many more— Use a set of three objects. Ask, "How many more are needed to make 9?" (3) Comparison— Compare a set of 9 objects with a set of 3 objects. Ask, "How many more are in the largest set?" (4) Inverse of Addition— Write 3 + □ = 9 or □ + 3 = 9. Ask, "What number would make this a true sentence?" See the strand entitled Operations, Their Properties and Number Theory.

Many errors in subtraction which pupils make throughout the elementary grades could be avoided with the use of proper vocabulary. On beginning the use of the symbol, "−," in subtraction, translate it "minus," not "take away." For example, do not read 9 − 3 as "nine take away three," but read it as "nine minus three." Also do not over-simplify in explaining by saying, "You always take the smaller number from the larger," since this leads to errors in computation later as illustrated in the next example.

B.

Incorrect	Correct
25 − 8 = 23	25 − 8 = 17
304 − 82 = 322	304 − 82 = 222

The pupil may understand the subtraction operation but not comprehend the subtraction algorithm.

This difficulty is due to a lack of understanding of place value or of different ways of naming a number. A pupil having such difficulty should have experience using place value devices with counters. These problems would require the use of counters bundled into tens and hundreds. From the use of manipulative materials the child should realize that in subtracting 8 from 25, he may represent 25 as 10 + 10 + 5 or 10 + 15. The teacher should lead the pupil to see that the logical choice here would be 10 + 15 for two reasons. He needs 8 or more in the ones column, and since the second column contains only multiples of 10, he must use one 10 from that column to combine with the 4 ones. Pupils should be encouraged to use methods other than the standard algorithm if they are better understood. Some alternate methods are as follows.

(1) Number line

Marking the 8 as an addend and 25 as the sum, and finding the missing addend.

(2) Counting on or counting back (similar to the complement method)

25 − 8 = 15 + (10 − 8)
which is 15 + 2
which is 17
so 25 − 8 = 17.

or

25 − 8 = 5 + 20 − 8
which is 5 + 12
which is 17
as 25 − 8 = 17

(3) Algorithm of integers (where appropriate)

24
−8
−4 (by 4 − 8 = − 4)
20 (by 2 tens − 0 tens = 2 tens or 20)
16 (by − 4 + 20 = 16.)

(4) Other methods which pupils may devise. They should be encouraged to analyze their methods to make sure that they are mathematically sound.

II. Division of Whole Numbers

For background for the division algorithm each pupil should progress through the following in sequence. Some pupils may move through this sequence more rapidly than others depending upon their understanding of each stage. If a student is having difficulty with the final algorithm, he needs to have exposure to each of these methods.

Division Algorithms

(1) Subtracting the divisor singly.

$32 \div 4$

$$
\begin{array}{rl}
32 & \\
\underline{-4} & 1 \\
28 & \\
\underline{-4} & 1 \\
24 & \\
\underline{-4} & 1 \\
20 & \\
\underline{-4} & 1 \\
16 & \\
\underline{-4} & 1 \\
12 & \\
\underline{-4} & 1 \\
8 & \\
\underline{-4} & 1 \\
4 & \\
\underline{-4} & 1 \\
0 &
\end{array}
$$

This illustrates that there are 8 fours in 32 or $8 \times 4 = 32$, therefore, $32 \div 4 = 8$

(2) For larger numbers it would be impractical to subtract the divisor singly. The process may be shortened by using multiples of the divisor.

$256 \div 4$

$$
\begin{array}{rrl}
4\,) & \overline{256} & \\
 & \underline{-40} & 10 \\
 & 216 & \\
 & \underline{-80} & 20 \\
 & 136 & \\
 & \underline{-80} & 20 \\
 & 56 & \\
 & \underline{-40} & 10 \\
 & 16 & \\
 & \underline{-16} & \underline{4} \\
 & 0 & 64
\end{array}
$$

This illustrates that there are 64 fours in 256 or $64 \times 4 = 256$; therefore, $256 \div 4 = 64$.

Another method of writing the preceding problem is

```
        4
       10
       20
       20
       10
  4 ) 256
      -40
      216
      -80
      136
      -80
       56
      -40
       16
      -16
        0
```

(3) Subtracting multiples of the divisor which are products of 4 and powers of 10.

```
4 ) 1296
    -400     100
     896
    -400     100
     496
    -400     100
      96
     -40      10
      56
     -40      10
      16
     -16       4
       0     324
```

As shown in the preceding example, there is an obvious advantage in subtracting multiples of the divisor which are large block multiples of tens, hundreds, etc.

A further refinement of the preceding method would be to choose the largest multiples of the divisor and powers of ten. To do so will give meaning to the traditional algorithm.

```
4 ) 1296
   -1200     300
      96
     -80      20
      16
     -16       4
       0     324
```

(4) The traditional algorithm

```
     Th H T U
         3 2 4
  4 ) 1  2 9 6
      1  2
           9
           8
           1 6
           1 6
             0
```

The letters representing thousands, hundreds, tens and ones placed above the quotient should help the pupil to realize that the 3 in the quotient means that there are 300 fours in 1296, the 2 means 20 fours in 96, etc. These letters should help him/to realize htat as he writes the 8 under the T that he is using a place value location allowing him to omit the 0.

Some pupils may prefer to write the zeros. The teacher should allow them to write the zeros if they find it helpful.

A. Incorrect

```
     86
7 ) 622
    56
     42
     42
      0
```

Correct

```
     88
7 ) 622
    56
     62
     56
      6
```

As an operation, division should be presented as the inverse of multiplication. See the strand entitled Operations, Properties and Number Theory. The algorithm generally used in processing division is called the long division algorithm. In example A., the error is actually one of computing differences. More work with processing subtraction problem should correct this kind of difficulty.

B. Incorrect

46 ÷ 7 = 6

Correct

46 ÷ 7 does not name a whole number. Instead, 46 = (7 X 6) + 4

Other difficulties may be related to not knowing subtraction or multiplication facts or to errors in precessing multiplication and can be remedied by reteaching and practice in related skills. However, the errors in examples B. through E. represent misunderstandings of a deeper nature, that is, misunderstandings of the questions about numbers which the process is designed to answer. To answer the question, "Does 7 divide 46," one asks the related question, "Is there a number which multiplies by 7 such that their product is 46?" Recall of the 7-facts or examinations of the multiplication chart reveals that there is no such number, and one concludes that 7 does *not* divide 46. That is, the mathematical expression "46 ÷ 7" does not name a whole number. However, in application to the real world one is often concerned not so much with the question, "Does one number divide another?," but with two questions– (1) How many subsets of a specified number of objects can be removed from a given set? and (2) How many objects remain? For example, (1) How many subsets of 7 objects each can be removed from a set of 46 objects and (2) How many objects are left over? The process employed in finding these answers is long division. In the process of answering the questions one must remember to find the *greatest* number of subsets which can be removed from the given set.

C. Incorrect / Correct

```
     218            38
4 )152         4 )152
    8              12
    7              32
    4              32
   32               0
   32
    0
```

D. Incorrect / Correct

```
      2             20
42 )849        42 )849
    84             84
     9              9
```

E. Incorrect / Correct

```
     23            203
9 )1827        9 )1827
   18             18
    27             27
    27             27
     0              0
```

In example C., 218 subsets of size 4 cannot be removed from a set of 152 objects. In example D., it is true that 2 subsets of size 42 can be removed from a set of 849 objects, but 2 is not the greatest number of such subsets. Also, in example E., 23 is not the greatest number of subsets of size 9 that can be removed from a set of 1827. In each of these three cases, if the pupil had used estimation or intelligent guessing in predicting an approximate number of subsets to be removed, he would have realized that the answer to the first question is not correct.

Note that the long division process can be used to determine whether or not one number divides another. For example, the expression, $5632 \div 176$, names a whole number if and only if there is a single whole number which multiplies by 176 such that their product is 5632, One can instead ask, (1) How many times can 176 be subtracted from 5632? and (2) How many remain? If the answer to question (2) is zero, then we can say that $5632 \div 176 = 32$. The related number sentence is $5632 = 176 \times 32$.

In summary, there are five types of difficulties associated with using the long division algorithm.

(1) Skill in estimation– The development of this skill should begin prior to the introduction of the division algorithm. Estimation skills can be developed in the context of questions such as, "About how many

times can a particular whole number be subtracted from a second whole number?" or "What is a multiple of the number called the 'divisor' which is close to but less than the number called the 'dividend?' Close to but greater than the number called the 'Dividend?' Do you think the answer to the first question is somewhere in between?"

(2) There are *two* answers in the long division process. That is, the process is designed to find answers to two questions.

(3) Division is impossible for many pairs of numbers. For instance, the expressions $622 \div 7$, $46 \div 7$, $849 \div 42$ do not name single whole numbers. In using the long division process, children should realize that if the process yields a non-zero remainder, one concludes that the dividend is not a multiple of the divisor. However, $152 \div 4$ and $1827 \div 9$ do name single whole numbers since the long division process yields zero remainders.

(4) The long division process is not a singular one. It is a combination of processes involving estimation, multiplication, subtraction and addition.

(5) Children, accustomed to memorizing multiplication and addition facts, find it more difficult to think in terms of division and what the operation of division means. See the strand on Operations, Their Properties and Number Theory.

III. Addition of Fractions

	Incorrect	Correct
A.	$\frac{2}{3} = \frac{4}{12}$	$\frac{2}{3} = \frac{8}{12}$
B.	$2\frac{5}{3} = 1\frac{2}{3}$	$2\frac{5}{3} = 3\frac{2}{3}$
C.	$\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

The causes of errors in addition of fractions may be due to errors in writing sets of equivalent fractions such as examples A. and B. Suggestions for teaching equivalent fractions may be found in the strands, Sets, Numbers and Numeration and Relations and Functions.

Suggestions for teaching addition of fractions may be found in Operations, Their Properties and Number Theory.

IV. Subtraction of Fractions

A.

Incorrect	Correct
$7\frac{1}{8} - 2\frac{7}{8}$	
$6\frac{11}{8}$	$7\frac{1}{8} = 6\frac{9}{8}$
$-2\frac{7}{8}$	$-2\frac{7}{8} = 2\frac{7}{8}$
$4\frac{4}{8}$	$4\frac{2}{8}$ $4\frac{1}{4}$

This error is a carry over from the subtraction of whole numbers. In such computation the pupil would write 1 beside the numeral in the ones place to represent the 1 ten which he had borrowed. Writing it in this manner was all right in computing with whole numbers since he was computing in base ten. However, in adding fractions this pupil did not understand that adding 1 to $\frac{1}{8}$ in this problem actually results in adding $\frac{8}{8}$ to $\frac{1}{8}$.

B.

Incorrect	Correct
$\begin{array}{r} 7 \\ -5\frac{1}{4} \\ \hline 2\frac{1}{4} \end{array}$	$\begin{array}{rcl} 7 & = & 6\frac{4}{4} \\ -5\frac{1}{4} & = & 5\frac{1}{4} \\ \hline & & 1\frac{3}{4} \end{array}$

Stress the important fact that 1 has many equivalents, and that the pupil should choose whatever form of 1 is needed to give a common denominator in the particular problem. For example, $1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5}$ . . . Notice in examples A. and B. the equivalent represent $\frac{8}{8}$ to $\frac{4}{4}$.

The pupil making error B. does not realize that the number "seven" should be represented by symbols other than 7.

C.

Incorrect	Correct
$\begin{array}{r} 3\frac{2}{5} \\ -1\frac{1}{3} \\ \hline 2\frac{1}{2} \end{array}$	$\begin{array}{rcl} 3\frac{2}{5} & = & 3\frac{6}{15} \\ 1\frac{1}{3} & = & 1\frac{5}{15} \\ \hline & & 2\frac{1}{15} \end{array}$

It is obvious that the pupil making error C. does not realize that he must have a common denomin when subtracting fractions. See the strand on Operations, Their Properties and Number Theory for a discussion on subtraction of fractions.

V. Multiplication of Fractions

A.

Incorrect	Correct
$\frac{2}{3} \times \frac{3}{4} = \frac{8}{9}$	$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ or $\frac{1}{2}$

This error is usually made after the student has learned to check for equivalent fractions or after he has studied proportion; in each case he is cross multiplying. Bring to his attention that in working with equivalent fractions the symbol "=" is used, whereas, if fractions are to be multiplied the " X " is used in such problems as A.

B.

Incorrect

$$3\frac{2}{5} \times 2\frac{2}{3} = 6\frac{4}{15}$$

Correct

(1)

$$3\frac{2}{5} \times 2\frac{2}{3} =$$

$$(3 + \frac{2}{5}) \times (2 + \frac{2}{3}) =$$

$$6 + \frac{4}{5} + \frac{6}{3} + \frac{4}{15} =$$

$$6 + \frac{12}{15} + 2 + \frac{4}{15} =$$

$$6 + \frac{12}{15} + 2 + \frac{4}{15} =$$

$$9\frac{1}{15}$$

The student may realize that $3\frac{2}{5}$ means $3 + \frac{2}{5}$ but he does not recognize the significance of this principle in this particular exercise. The student may use two different methods to solve this problem.

In (1) the sum of the partial products are given. It might be helpful for the student to compare this method to the partial products as used in multiplying whole numbers such as 23 X 45 where the sum of the partial products would be (5 X 3) + (5 X 20) + (40 X 20) or 15 + 60 + 120 + 800 or 995.

(2) $3\frac{2}{5} \times 2\frac{2}{3} =$

$\frac{17}{5} \times \frac{8}{3} =$

$\frac{136}{15} =$

$9\frac{1}{15}$

In (2) each mixed numeral is written and used as one fraction.

To record the numbers used in this method he should realize that $3\frac{2}{5}$ means $3 + \frac{2}{5}$. Therefore, $\frac{15}{5} + \frac{2}{5} = \frac{17}{5}$. The teacher should insist that the pupil use this procedure before using the short cut $3\frac{2}{5}$ is $\frac{5 \times 3 + 2}{5}$ or $\frac{17}{5}$.

If he uses the short cut with no understanding, he will in later grades make such mistakes as $3\frac{2}{5}$ is $\frac{2 \times 3 + 5}{5}$ or 11.

VI. Division of Fraction

Incorrect

$\frac{3}{4} \div \frac{2}{3} =$

$\frac{4}{3} \times \frac{2}{3} =$

$\frac{8}{9}$

Correct

$\frac{3}{4} \div \frac{2}{3} =$

$\frac{9}{12} \div \frac{8}{12} =$

$9 \div 8 =$

$1\frac{1}{8}$

$\frac{3}{4} \div \frac{2}{3} =$

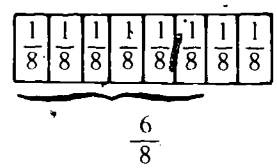
$(\frac{3}{4} \times \frac{3}{2}) \div (\frac{2}{3} \times \frac{3}{2}) =$

$\frac{9}{8} \div 1 =$

$1\frac{1}{8}$

See the Operations, Their Properties and Number Theory strand for a thorough discussion of the operation of division of rational numbers. The two major methods or algorithms for dividing with fractions are illustrated. The common denominator algorithm used in the first correct example can be justified by reading $\frac{9}{12} \div \frac{8}{12}$ as "9 twelfths divided by 8 twelfths," i.e. using the fractional parts, twelfths, as denominations.

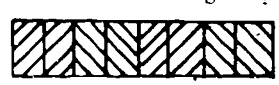
The reciprocal algorithm is used in the second correct example. By multiplying the dividend and the divisor each with the reciprocal of the divisor, the quotient remains unchanged and the divisor becomes 1. Of the two methods, the reciprocal method is more widely used; however, few pupils understand why it works and often invert the wrong fraction. The common denominator method may not be as well known, but it is more readily understood by children. The common denominator method may be illustrated by using strips as shown. (a) Find the number of eighths in six-eighths or $\frac{6}{8} \div \frac{1}{8} = \square$.



The pupil can easily see there are 6 eighths in six-eighths.

(b) Find the number of one-fourths in six-eighths or $\frac{6}{8} \div \frac{1}{4} = \square$.

The problem written as $\frac{6}{8} \div \frac{1}{4} = \frac{6}{8} \div \frac{2}{8}$ is more readily understood if a drawing is used.



The pupil can see that there are 3 two-eighths or 3 one-fourths in 6 eighths.

VII. Division of Decimals

Incorrect	Correct
$53.95 \div 8.3$	$83.\overline{)53.95}$ = 6.5
$8.3\overline{)53.95}$ = .65; 498; 41 5; 41 5	

Before beginning computation with decimals, it will be helpful to explain thoroughly that decimal fractions are an extension of base ten place value. The difficulty arises in the present example because the pupil does not understand the extension of place value. Division of decimals can be presented as the inverse of multiplication. In the example shown, 8.3 is a factor and 53.95 is a product. Ask the question, "Tenths × □ = hundredths or $\frac{1}{10} \times \square = \frac{1}{100}$?"

The pupil should see that 1/10 is a replacement for the □, and that the missing factor would be one tenth. One way of showing this method is as follows.

83 tenths) 5395 hundredths = 65 tenths or 6.5

A second method is to record the problem using common fractions so that the divisor and dividend have common denominators.

$$53.95 \div 8.3 = \frac{5395}{100} \div \frac{83}{10}$$

$$= \frac{5395}{100} \div \frac{830}{100}$$

A third method is to record the problem as one common fraction and multiply the numerator and denominator by a power of ten in order to make the denominator (divisor) a natural number.

$$\frac{53.95}{8.3} = \frac{53.95}{8.3} \times \frac{10}{10} = \frac{539.5}{83}$$

The third method is the one usually used with carets placed to show the placing of the decimals. If this method is used, care must be taken in recording to align the decimal of the quotient with the caret in the dividend.

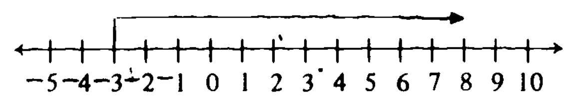
VIII. Subtraction of Integers

	Incorrect	Correct
A.	$^{+}8 - {}^{-}3 = 5$	$^{+}8 - {}^{-}3 = {}^{+}11$
B.	$^{+}8 - {}^{-}3 = {}^{-}11$	$^{+}8 - {}^{-}3 = {}^{+}11$

Errors in subtraction of integers are from two causes. (1) The pupil does not understand the operation with integers, or (2) he makes mistakes using combinations of whole numbers.

To alleviate errors due to the first cause consider subtraction of integers as the inverse of addition. In this problem, $^{+}8 - {}^{-}3$ may be stated as, "What number added to negative three will give a sum of eight?" (Note that $^{-}3$ is read "negative three" *not* "minus three." "Minus" is reserved for the subtraction symbol.

This problem of subtraction of integers, $^{-}3 + \square = {}^{+}8$ can be illustrated with a number line.



The points corresponding to the known addend and the sum are marked on the number line. The pupil should then count the number of units from the addend to the sum to find the missing addend. If he counts toward the right, the missing addend is a positive number; and if he counts to the left, the missing addend is negative. The pupil should have experiences using the number line to solve different types of subtraction problems with integers.

The pupil should be encouraged to observe many examples in order to discover the pattern that subtracting an integer gives the same result as adding its opposite. The teacher should provide many examples with correct answers.

They must be selected carefully in order to have the patterns necessary to guide discovery. One set of examples to use is as follows.

$^{+}8 - {}^{-}5 = \boxed{^{+}3}$	$^{-}8 + {}^{-}5 = \boxed{^{-}13}$
$^{+}8 + {}^{-}5 = \boxed{^{+}3}$	$^{+}4 - {}^{+}6 = \boxed{^{-}2}$
$^{-}8 - {}^{-}5 = \boxed{^{-}3}$	$^{+}4 + {}^{-}6 = \boxed{^{-}2}$
$^{-}8 + {}^{+}5 = \boxed{^{-}3}$	$^{-}4 - {}^{-}6 = \boxed{^{+}2}$
$^{-}8 - {}^{+}5 = \boxed{^{-}13}$	$^{-}4 + {}^{+}6 = \boxed{^{+}2}$

IX. Multiplication of Integers

Incorrect	Correct
$8 \times {}^{-}5 = {}^{-}40$	$^{-}8 \times {}^{-}5 = {}^{+}40$

Errors in multiplication of integers are from two causes. (1) The pupil does not know or understand the definition of the operation, or (2) he makes mistakes in multiplying the related whole numbers.

The definition of multiplication for integers is more difficult for pupils to understand because it is more difficult to relate the operations to objects or physical situations. Of course, a negative number times a negative number is most difficult. There are many methods of justifying the answer for this problem. The pupils should be exposed to several so that he may choose the one he believes. The physical situations may seem artificial to some students, but there are several that can be used. See those used in the strand on Operations, Their Properties and Number Theory.

A more mature student will enjoy a simple proof using several important properties for integers such as $a \times 0 = 0$, $a \times 1 = a$, $a \times {}^-a$. For example, if a student has a problem $^-6 \times {}^-9 = \square$ he may be able to understand a proof of multiplication of two specified integers as the one below although he may not be able to produce it.

Statement	Reason
$^+9 + {}^-9 = 0$	Inverse element for addition of integers
$^-6(^+9 + {}^-9) =$	$0\ a \times 0 = 0$
$^-6 \times {}^+9 + {}^-6 \times {}^-9 = 0$	distributive property
$^-54 + {}^-6 \times {}^-9 = 0$	multiplication facts
$^-54 + {}^+54 = 0$	additive inverse
Therefore	
$^-6 \times {}^-9 = {}^+54$	The additive inverse of a number must be unique.

An advanced student may want to do the proof for the general case $^-a \times {}^-b = {}^+ab$.

UPDATING CURRICULUM

Continuing Program Improvement
Evaluating Pupil Progress
Utilization of Media

CONTINUING PROGRAM IMPROVEMENT

Long range curriculum planning is necessary for the development of a good mathematics program. A curriculum committee should be in continuous operation to evaluate, to improve or to revise the program. The committee should schedule evaluations of the program for specified times; this schedule should be correlated with the schedule of evaluation of the pupils' progress.

The plan of the local program should be in writing and should be available to all of the elementary teachers. This guide is not a course of study for the local schools since it is not all inclusive. It should serve as a guideline in planning the mathematics program of the local elementary schools. The objectives in the local guide should be an expansion of those in the state guide. Ideally, the objectives at the local level should be written in specific, measurable, behavioral terms. The objectives should be determined before the textbooks and other materials are selected since the textbook is not the course of study but is one of the tools to aid instruction.

The mathematics program should

- be consistent with the total program of the local system so that there will be correlation with other curriculum areas;
- fulfill the objectives as presented in the local mathematics guide;
- be an expansion of the state mathematics guide;
- have consistent, accurate and precise vocabulary;
- include plans for best use of textbooks and other materials;
- include plans for best use of audio-visual aids and ETV;
- provide for continuous improvement;
- reflect research findings of recognized authorities in mathematics education.

Writing Local Guides

The local guides do not need to follow one pattern. They should be in accordance with local needs and should be in a form for best utilization. Some guides may be an enlargement on or extension of special units specifically selected by the local staff.

In order to achieve the goals of the mathematics curriculum it is essential to have well planned, regularly scheduled inservice programs. In these meetings, the staff of the local schools should utilize the state guide. The inservice program could center around one or more of the following.

- stating specific objectives, each one written in terms of the desired behavior, the situation involving such behavior and the criteria for success;
- writing additional activities in each of the strands or in specific strands;
- having indepth studies in one or more strands so as to develop them further;
- developing a mathematics laboratory and writing plans for its use considering effective staff utilization, organization of materials and facilitation of their use and instructions to students through such devices as activity cards;
- collecting and organizing various materials to use in teaching the strands;
- developing local guides to fit the local needs, such as to develop specific skills needed by local industries;
- developing charts for recording the evaluation of pupil progress.

Achievement of Goals

Pupil achievement should be in terms of growth, change and progress in the attainment of locally established objectives. Ability to use many processes in appropriate situations is an important facet of achievement. The mathematics curriculum should provide continuous opportunity for the pupil to progress in a program at his level of ability. The teacher has a sense of achievement if he has provided an opportunity for success in the program for his pupils.

The program should be flexible enough to encourage innovative practices such as trying out new materials and new methods. *Ideas for these may be obtained through participating in professional meetings, reading and studying professional journals and other publications and working with consultants and other resource people.*

The excellence of a program is perhaps best represented by the manner in which a suitable balance is achieved between challenge to the student on one hand and opportunity for success on the other.

Selection of Textbooks

Since there is multiple adoption of textbooks for the state public schools, the local systems need to establish criteria for selecting those which will be locally adopted. It is suggested that a local committee be composed of representatives from each school and from the different grade levels. This committee should begin its work one year prior to the selection of textbooks at the local level. It should study local needs considering both the experiences of the pupils and the background of the teachers.

When the committee has narrowed its selections to a few books, pilot programs should be instituted in representative cases where financially feasible and comparisons made of the results found before the committee makes the final selection. This practice would alleviate the dangers of having individual teachers making the choice of textbooks and of having no sequence throughout the school program.

The textbook committee should accept as its responsibility the selection of diversified textbooks for different levels to suit individual needs. Also, textbooks for supplemental uses could be chosen as well as those for experimental purposes.

If textbook money is limited, the committee should investigate sources of revenue for implementing the needs, such as the various federal funds. If sufficient funds are not available to purchase basic textbooks for all levels, a careful study should be made to determine the best usage of funds.

In selecting textbooks, the local committee should consider specific criteria. Among publications in which such listings may be found is the *Criteria for Selecting Textbooks* of the National Council of Teachers of Mathematics. Criteria such as the following should be taken into account.

- Throughout the series does the content include sufficient material to develop the concepts as stated in the objectives of the strands in the state mathematics guide?
- Is the spiral approach to learning emphasized in the content? Are topics explored in more depth at each level?
- Is the method of presentation conducive to logical thinking through pupil involvement and discovery of patterns and principles of mathematics?
- Does the presentation of material allow a child to move ahead at his own rate and interest level?
- Is the material appropriate for the child's vocabulary and reading level?
- Is the arrangement of material adaptable to all levels of ability?
- Is the review of previous level material distributed throughout the content areas rather than concentrated in the beginning of the book?
- Is the drill work designed to reinforce the basic concepts?
- In the series, is there multiple authorship which includes a mathematician and a teacher experienced at this level?
- Are there sufficient diagnostic and evaluative materials, including those for self-evaluation? (Such materials could be included in text, workbook or manual.)
- Do the teacher aids include a rationale, the pupil's material, suggestions for individualizing instruction, additional resources and evaluation procedures, and are these aids convenient to insure effective use?
- Are the textbooks durable, attractive, of convenient size, of good quality material, of suitable type and functionally arranged?

EVALUATING PUPIL PROGRESS

Evaluation is diagnostic, prescriptive and individualized, and should facilitate self-evaluation. The evaluation is developed by many methods and from many sources. Evaluation may be made in the context of projects, problem-solving situations, daily class participation, paper and pencil tests and standardized tests given for diagnostic purposes. For advice on selection of tests for specific purposes consult with the Guidance and Testing Unit of the Georgia Department of Education.

Projects should challenge the pupil without frustrating him, provide opportunity for applying mathematical concepts already learned, be appropriate to the interest and ability level of the individual and be shared after completion. Problems should arise naturally from classroom situations. A teacher may contrive problems to relate concepts learned to real situations and to challenge the pupil to reason and think beyond the level at which he has computational competency. Class participation should involve every child, build confidence in the individual, create an atmosphere conducive to building a positive self-image and should not be dominated by any one individual.

Daily performance of pupils including questions pupils ask and their response to teacher queations are excellent ways to evaluate the understanding of the pupils. Activities which allow pupils to demonstrate understanding of specific objectives are good instruments through which to measure understanding.

Tests should be given to determine pupil achievement of specific objectives, and can be varied by requiring descriptions of ways to solve specified problems rather than requiring numerical answers.

Teachers need to make tests that are short enough to give the pupil time to think and complete the test in the time alloted. Teachers need to be careful to give specific and clear instructions. Tests may be used as learning experiences when they are returned and discussed with pupils. Tests should be well balanced to cover the objectives which are to be tested. Tests should be checked and returned so that the pupil gets immediate feedback.

Evaluation procedures should provide the pupil with an understanding of his progress and difficulties. Evaluations should be made often enough to direct the development of the curriculum for the individual. Records should be kept in a form which clearly shows the progress of the pupil.

The charts given here are suggestions and only one strand has been used for illustrative purposes. The teacher may wish to sub-divide the larger topics into specific concepts which can be taught in short time periods. The process of developing his own charts will help the teacher to determine the direction in which he plans to move and assess what he has accomplished when he checks the progress of the pupils.

It is most important that the receiving teacher know at what level the pupil was performing at the end of the previous school year or at the time of transfer. For this reason, a cumulative record should be kept for each child. This chart may be checked by recording at the end of the presentation of a strand or at the end of the school year. To facilitate the record keeping, the teacher will probably prefer to keep a class record and then transfer the information to individual records. By seeing the class record as a whole, the teacher can see the strengths and weaknesses of the total class.

Records of individual pupil's progress should be kept in his cumulative folder and passed on to the next teacher. A chart should be made for each strand. (See Chart 1 - Pupil Progress.) Charts should be made on $8\frac{1}{2}''$ by $11''$ paper for easy filing.

There are three levels of progress indicated for each child. The first level indicates only that the teacher has presented this concept to the child. The second level indicates that the child uses the concept under directions. The third level indicates that the child uses the concept independently.

SAMPLE

Individual Pupil Progress Sheet

Sets, Numbers, Numerations

Primary Level

Name ____________________

Date ____________________

Grade ____________________

Objectives	Teacher has introduced	Pupil can accomplish with guidance	Pupil can accomplish independently	
1. Place two sets in one-to-one correspondence				
2. Select subsets of a given set				
3. Assign the cardinal number to a set				
4. Assign whole number names to sets of objects				
5. Order the whole numbers				
6. Put in one-to-one correspondence the ordered set of whole numbers and points on a line				
7. Read and write whole number numerals and number words				
8. Demonstrate the ability to use the place value code of writing numerals				
9. Demonstrate the use of another system of numeration				
10. Demonstrate the ordinal use of numbers				
11. Name the ordered pair of whole numbers associated with fractional parts of (a) units (b) sets				
12. Order fractional parts of equivalent units by size, and name the corresponding fraction				
13. Discriminate between an ordered pair of whole numbers used in a rate context and in a fraction context				
14. Show that two or more number pairs or fractions may be associated with equivalent fractional parts				
15. Identify and describe situations that require the use of directed whole numbers				
16. Count by twos Count by fives Count by tens Count by hundreds				
17. Identify odd and even numbers				
18. Identify given number patterns				
19. Complete given number patterns				

Chart 1

DIAGNOSTIC CLASS CHART*

Name of Pupils	Sets				Whole Numbers					Fractions						Number Theory			
	One-to-one correspondence	Selecting subsets	Cardinality of set	Number and name of set	Ordering	Matching with sets	Counting and recording numerals	Using place value	Ordinal use of number	Associate number pairs with fraction parts of sets	Associate number pairs with fraction parts of unit	Order parts by size and name fraction	Use in rate context	Use in fraction context	Recognize and name equivalent fractions	Count by twos, fives, tens, hundreds	Identify odd and even numbers	Identify given number patterns	Complete given number patterns
Adams, Sue																			
Brown, Wayne																			

Chart 2

*A diagnostic Class Chart with a sample of objectives from some of the strands is given. A diagnostic chart with objectives for the year will help the teacher see the strengths and weakness of the class before beginning a strand. Such a chart could also be used as the teacher completes his presentation of materials from each strand.

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TEACHER SELF EVALUATION CHART (Sample for First Grade)

Pupil's Name	Number names in order	Established relationship in counting	Copy number forms	Select-number that matches quantity in group	Draw number of objects a number symbol designates	Write number symbol that tells how many items	Selects specific number of objects from group	Place value— ones and tens	Counts, reads, writes numbers to fifty	Counts by tens to fifty	Place value to fifty	Counts by 2's to 20	Recognizes number words one to ten	Sums to six	Subtracts to six	Tells time	Calendar	Money
1. Allen, Mary																		
2. Battle, Kim																		
3.																		

Chart 3

A teacher may desire to make small learning step analyses of his pupils. This chart is a sample of small steps one teacher selected for her first grade.

UTILIZATION OF MEDIA

INTRODUCTION

The use of a variety of materials can improve learning and instruction, since the individual needs of children vary and individual children learn by different means. These materials help to stimulate new ideas and concepts enabling children to solve problems whose solutions are as yet unlearned responses. A variety of media should be readily accessible. In planning and organizing the housing of materials those often used and those occasionally used should be taken into consideration. This variety of media should include materials which are adaptable to each pupil's particular ability level.

Many valuable teaching aids may be made inexpensively, and pupil participation in their construction serve as a learning activity. A number of books are available which assist one in making aids. A number of these aids are listed in the references.

Many worthwhile, commercially prepared aids are now on the market. A special committee composed of those who determine the purpose and method of instruction in the local school should participate in the selection of instructional materials. These instructional materials should be examined or previewed to determine their probable effectiveness in the situation where they are to be used. Instructional materials in kits, packages and series should be thoroughly examined prior to approval of purchase.

Audio-visual equipment can be used in various ways, and teachers should be alert to innovations in the use of all media. For example, the overhead projector may be used in many ways other than for projecting transparencies. In the study of sets, collections of small objects may be placed on the overhead projector and their shadows can easily be seen by all pupils. The tape recorder may be used in work with individuals or small-groups; it is especially helpful with the slow reader.

The Georgia Department of Education television network has regularly scheduled programs in mathematics at all levels of elementary school. These programs supplement and enrich the classroom teacher's presentation. Schedules and guides giving information on each lesson are available from the director of programming.

Films and filmstrips should be previewed and carefully selected before use in the classroom. The Georgia Department of Education publishes an up-to-date list of films and filmstrips available through the Audio-visual Service.

Magazines such as *The Arithmetic Teacher* and *The Mathematics Teacher*, publications of the National Council of Teachers of Mathematics, contain excellent articles for teachers. Many valuable books are available through the Georgia Department of Education Readers' Services of the Public Library Unit and may be requested through the school or public librarian.

The list of materials which follows is not meant to be all-inclusive. These materials should be very useful, and teachers are encouraged to try many of them. The reference lists are limited and are annotated for the use of strands or subjects in the guide. The teacher may find these references useful in strands other than those for which they were annotated as well as references which are not listed.

The instructional aids for use in the activities of the strands are listed.

Teaching practices involving various instructional materials should be carefully evaluated in terms of effective results. The information relative to those materials whose use produces the most favorable results should be shared with teachers not only in the school system but with others. Provisions should be made for frequent revision of any locally approved list of instructional materials to allow for deletions and additions needed to update the curriculum.

Materials on media that are available through the Georgia Department of Education, State Office Building, Atlanta, Georgia 30334, and the offices from which they may be obtained are listed below.

Viewpoints; Instructional Materials – Selection at State and Local Levels, Suggestions for Use

Director of Elementary and Secondary Education

Catalog of Classroom Teaching Films for Georgia Schools

Audio Visual Unit

Selected List of Books for Teachers

Director, Readers' Services, Public Library Unit

Georgia Library List for Elementary and High Schools

Director, School Library Services Unit

In-school television schedules and guides for educational television

Director, Television Programming
Georgia Educational Television
Georgia Department of Education
1540 Stewart Avenue, S. W.
Atlanta, Georgia 30310

REFERENCES

Abbott, Janet S., *Learn to Fold – Fold to Learn.* Chicago: Lyons and Carnahan, 1968.
A pupil workbook about reflections; teacher's edition available also.

________________, *Mirror Magic. Chicago: Lyons and Carnahan, 1968.*
A pupil workbook about reflections and symmetry; teacher's edition available also.

Association of Teachers of Mathematics, *Notes on Mathematics in Primary Schools.* Cambridge: University Press, 1968.
Suggestions and lessons written by teachers for primary teachers use. This book is available from Cuisenaire Company of America, Inc., New Rochelle, New York.

Banks, J. Houston, *Learning and Teaching Arithmetic.* Boston: Allyn and Bacon, Inc., 1960.
Several chapters have suggestions related to difficulties with computation.

Bendick, Jeanne and Levin, Marcia, *Mathematics Illustrated Dictionary.* New York: McGraw-Hill Book Co., 1965.
This book is a student dictionary containing many of the terms as they are used in the guide. It also contains diagrams and pictures helpful to children.

Berger, Melvin, *For Good Measure: The Story of Modern Measurement.* New York: McGraw-Hill, 1969.
Interesting and little-known facts about the development of systems of measurement, the importance of precise measurement in science and industry and the many ways that measurement is used subconsciously.

Bloom, Benjamen, et. al., *Taxonomy of Educational Objectives. Handbook I: The Cognitive Domain.* New York: David McKay Co., 1956.
A help with making and interpreting tests.

Bowers, Henry and Joan, *Arithmetic Excursions: An Enrichment of Elementary Mathematics.* New York: Dover Publications, Inc., 1961.
Chapter 18 contains interesting information about figurate numbers, perfect numbers and amicable numbers.

Brumfiel, C. F., Eicholz, R. E., and Shanks, M. E., *Fundamental Concepts of Elementary Mathematics.* Reading, Massachusetts: Addison-Wesley Co., Inc., 1962.
A book in mathematics for teachers; provides background material for some concepts of geometry and other concepts of the guide.

Buros, O. K., *The Sixth Mental Measurements Yearbook.* Highland Park, New Jersey: Gryphon Press, 1965.
A help with making and interpreting tests.

Cambridge Conference, The, *Goals for the Correlation of Elementary Science and Mathematics.* Boston: Houghton Mifflin Co., 1969.
This book for teachers includes the development of relations and functions. The importance of application of the equivalence relation is emphasized.

Copeland, Richard W., *How Children Learn Mathematics.* New York: The Macmillan Co., 1970.
This book emphasizes how children learn mathematics, rather than techniques of teaching elementary school mathematics. The role of the teacher is suggested to be that of a skillful interviewer. Illustrations show the teacher how to use laboratory or manipulative materials which help the child learn mathematical concepts at a concrete operational level.

D'Augustine, Charles H., *Multiple Methods of Teaching Mathematics in the Elementary School.* New York: Harper and Row, 1968.

The book involves prospective teachers in the creation of exercises that can lead children to make discoveries of various number patterns and problems to lead to mathematical discovery. Many methods of presenting and using principles are given.

Davis, Robert B., *Explorations in Mathematics.* Palo Alto: Addison-Wesley Publishing Co., 1966.

Reference for the teacher. This book is especially helpful in providing activities on functions.

Dienes, Zolton P. and Golding, E. W., *Exploration of Space and Practical Measurement.* New York: Herder and Herder, 1966.

This book is a teacher's guide for developing geometry and measurement in the lower grades. Children are introduced to these topics by means of games of reflections, turning and measuring by using arbitrary units and later standard units.

__________, *Geometry of Congruence.* New York: Herder and Herder, 1967.

A booklet for teachers describing activities for pupils on reflections, rotations and translations.

__________, *Geometry of Distortion.* New York: Herder and Herder, 1967.

A book for teachers describing activities for pupils on topological equivalence and stretches and shrinkage.

__________, *Groups and Coordinates.* New York: Herder and Herder, 1967.

For teachers; describes activities for graphs.

__________, *Learning Logic; Logical Games.* New York: Herder and Herder, 1966.

This book gives the teacher, through narrative diagrams, directions for pupil activities with attribute blocks.

__________, *Sets, Numbers, and Powers.* New York: Herder and Herder, 1966.

A reference for the teacher; this booklet contains activities leading to an understanding of sets and numbers.

Duncan, Ernest G. and Quast, W. G., *Modern School Mathematics Workbook for Elementary Teachers.* Boston: Houghton Mifflin, 1968.

Provides a thorough treatise of mathematics needed by the elementary teacher through explanation, questioning and drill.

Ebel, R. L., *Measuring Educational Achievement.* Englewood Cliffs, New Jersey: Prentice-Hall, 1965.

A help with making and interpreting tests.

Elementary Science Study, Educational Development Center, *Attribute Games and Problems.* St. Louis: Webster Division, McGraw-Hill Co., 1968.

A variety of materials with teacher's guide for developing skills in problem solving, especially developing skills in classifying and setting up relationships between the classes.

Fitzgerald, William, et. al., *Laboratory Manual for Elementary Mathematics.* Boston: Prindle, Weber and Schmidt Inc., 1970.

An excellent reference for the teacher; this manual establishes a discovery approach for elementary teachers to find solutions to problems using many mathematical manipulative materials.

Ford Motor Company, *History of Measurement.* Dearborn, Mich.: Research Division.

This booklet gives historical development of measurement.

General Motors, *Precision – A Measure of Progress.* Detroit, Michigan: General Motors Corporation, 1952.

This Booklet gives historical development of precision of measurement.

Glennon, Vincent J., and Callahan, Leroy G., *Guide to Current Research, Elementary School Mathematics.* Washington: Association for Supervision and Curriculum Development, NEA, 1968.

Gives the teacher a ready reference to research pertinent to the field of mathematics as applied to elementary schools.

Grossnickle, Foster E., Brueckner, Leo J. and Reckzeh, John, *Discovering Meanings in Elementary School Mathematics.* Fifth edition. New York: Holt, Rinehart and Winston, Inc., 1968.

Illustrates how the basic principles of learning are applied in presenting a given topic. Great stress on structure as the dominant theme in elementary mathematics.

Hasford, Philip L., *Algebra for Elementary Teachers.* New York: Harcourt, Brace and World, Inc., 1968.

This book helps the elementary teacher understand operations in algebraic terminology.

Heinke, Clarence, *Fundamental Concepts of Elementary Mathematics.* Encino, Cal.: Dickinson Publishing Co., Inc., 1970.

The chapter entitled "Algorithms of Elementary Arithmetic" has helpful suggestions on computation.

Hoghen, Lancelot, *The Wonderful World of Mathematics.* Garden City, N. Y.: Doubleday, 1955.

The development of mathematics through the ages is described in story and pictures.

Hartung, Maurice L., and Walch, Ray, *Geometry for Elementary Teachers.* Glenview, Illinois: Scott, Foresman and Co., 1970.

A book for teachers which explains certain geometric relations; the basic constructions; and reflections, rotations, translations and stretches.

Horne, Sylvia, *Learning About Measurement.* Chicago: Lyons and Carnahan, 1968.

A book of student activities in measurement.

Huff, Darrell, *How to Lie with Statistics.* New York: W. W. Norton and Co., Inc., 1954.

This book could be used as interesting reference material for the teacher.

Johnson, Donovan A., Glenn, William H. and Norton, M. Scott, *Exploring Mathematics on Your Own.* St. Louis, Missouri: Webster Publishing Co., 1960.

These 18 booklets are readable sources for teachers. There are directions for numerous activities which illustrate specified relations. The booklets may be purchased separately. Some of the titles are *Topology The Rubber Sheet Geometry, The World of Measurement, Curves in Space, Pythagorean Theorem, Geometric Constructions, Probability and Chance, The World of Statistics.*

Johnson, Donovan A. and Rising, Gerald R., *Guidelines for Teaching Mathematics.* Belmont, Cal.: Wadsworth Publishing Co., Inc., 1967.

The chapter "Developing Computational Skills," has suggestions for overcoming difficulties in computation, and the chapter, "Evaluation of Achievement," has suggestions on various types of evaluation. Other chapters deal with basic techniques and materials. An excellent sourcebook for the teachers.

Kennedy, Leonard M., *Models for Mathematics in the Elementary School.* Belmont, Cal.: Wadsworth Publishing Co., Inc., 1967.

This book has descriptions of many aids to make and use in teaching different topics in elementary mathematics.

Linn, Charles F., *Puzzles, Patterns and Pastimes from the World of Mathematics.* Garden City, N. Y.: Doubleday, 1969.

Puzzles and mathematical games, both ancient and modern, to test the skill of the reader and to stimulate him to invent similar ones.

Mann, William et. al., *Measures.* Columbus, Ohio: Charles E. Merrill Publishing Co., 1968.

This booklet is one of the *Independent Learning Series.* It includes historical development in measurement and exercises for pupils.

Marks, John L., Purdy, Richard and Kinney, Lucien B., *Teaching Elementary School Mathematics for Understanding.* Second Edition. New York: McGraw-Hill Book Co., 1965.

Chapter six has some suggestions for techniques for fixing skills.

Merton, Elda L., and May, Lola June, *Mathematics Background for the Primary Teacher.* Wilmette, Ill.: John Colburn Associates, Inc., 1966.

A reference for teachers of K-3. The presentation is in the form of charts with explanations of eighteen topics taught at the primary level.

Mueller, Francis J., *Arithmetic, Its Structure and Concepts.* Second Edition. Englewood Cliffs, N. J.: Prentice Hall Inc., 1964.

Chapters two and three have extensive discussions of direct operations and inverse operations.

National Aeronautics and Space Administration and U. S. Office of Education, *What's Up There;* Teachers' Edition. Washington, D. C.: U. S. Government Printing Office, 1964.

A source book in space oriented mathematics for grades five through eight.

National Council of Teachers of Mathematics, 1201 Sixteenth Street, NW, Washington, D. C., 20036 –

Aids for Evaluators of Mathematics Textbooks, 1965.

A set of criteria to aid elementary and secondary teachers in selecting textbooks; pamphlet.

Boxes, Squares, and Other Things: A Teacher's Guide for a Unit in Informal Geometry.

Experiences for pupils in visualizing objects and in the concepts of transformations and symmetry.

Evaluation in Mathematics. Twenty-sixth Yearbook, 1961.

A discussion of and suggestions for evaluation of instruction.

Enrichment Mathematics. Twenty-seventh Yearbook, Second Edition, 1963.

Very brief discussion of topics pertinent to elementary school including rationale and appropriate activities.

Experiences in Mathematical Ideas, Vol. 1, 1970.

Designed to help teachers stimulate slow learners in grades 5-8. This project combines a text for teacher use with laboratory oriented package including loose-leaf materials to be duplicated for student worksheets.

Formulas, Graphs, and Patterns: Experiences in Mathematical Discovery, 1, 1966.

Describes activities for pupils; pamphlet.

Geometry: Experiences in Mathematical Discovery, 4, 1966.

Describes activities for pupils; pamphlet.

Growth of Mathematical Ideas, Grades K-12, Twenty-fourth Yearbook, 1959.

A general survey of mathematics curriculum including sections on "Ratio-like Numbers," "Fractions as Ordered Pairs of Numbers" and "Language and Symbols."

Instruction in Arithmetic, Twenty-fifth Yearbook, 1961.

Includes suggestions on computation. Provides background for teaching any modern elementary school mathematics program.

Mathematics Library – Elementary and Junior High School, 1968.

An annotated bibliography of enrichment books for grades K-9, classified by grade level.

More Topics in Mathematics for Elementary School Teachers. Thritieth Yearbook, 1969.

This book provides background information for teachers on the key principles of mathematics.

Paper Folding for the Mathematics Class, 1957.

Directions for paper folding activities which illustrate selected geometric relations.

Readings in Geometry from "The Arithmetic Teacher," 1970.

A booklet of articles containing suggestions for classroom activities.

Topics in Mathematics for Elementary School Teachers. Twenty-ninth Yearbook, 1964.

Gives the key principles for an understanding of the major topics. One chapter on the rational number system deals with various interpretations of rational numbers as well as the numeral forms -- fractional, decimal and mixed numerals -- used to represent rational numbers. A chapter on sets includes basic ideas on relations. One chapter gives a thorough discussion of number systems for whole numbers and rational numbers.

Nuffield Mathematics Project, *Teachers' Guides: Computation and Structure, Graphs Leading to Algebra* and *Shape and Size.* New York: John Wiley and Sons, Inc.

The teachers' guides are for elementary mathematics activities designed to encourage children to discover mathematical processes for themselves. The material is written in three main streams stated above. Each stream is written in a number of booklets in the stages of development of children.

Overman, James Robert, *The Teaching of Mathematics.* Chicago: Lyons and Carnahan, 1961.

Chapters 8-15 give suggestions on teaching addition, subtraction, fractions and denominate numbers.

Papy, Frederique, *Graphs and the Child.* Montreal: Algonquin Publishing Co., 1970.

A description of a series of ten lessons on graphs for six-year-olds helps the mathematical notion of relation to emerge.

A description of a series of ten lessons on graphs for six-year-olds helps the mathematical notion of relation to emerge.

Peterson, John A. and Hashisaki, Joseph, *Theory of Arithmetic. New York: John Wiley and Sons, 1967.*

Rational numbers are treated in terms of several interpretations, including both the fractions and rate-pair interpretations, which are appropriate to the elementary school curriculum.

Phillips, Jo M. and Zwoyer, R. E., *Motion Geometry, Book 1: Slides, Flips, and Turns.* New York: Harper and Row, 1969.

A pupil workbook about translations, reflections and rotations; teacher's edition, containing helpful notes, available. For students in the upper grades.

Riedesel, C. Alan, *Guiding Discovery in Elementary School Mathematics.* New York: Appleton-Century-Crofts, 1967.

This book provides prospective and inservice elementary school teachers with illustrative situations that make use of modern mathematical content and ideas to develop a guided discovery approach to teaching mathematics in the elementary school.

Sanders, N. M., *Classroom Questions: What Kinds?* New York: Harper and Row, 1966.

A help with making and interpreting tests.

School Mathematics Study Group, A. C. Vroman, Inc., 2085 E. Foothill Boulevard, Pasadena, California, 91109

Factors and Primes, 1965.

A book written for high school students; a good reference for the elementary school teacher. A teacher's commentary is available.

Mathematics for the Elementary School, 1962.

This book for elementary school mathematics includes student exercises in measurement.

Mathematics for Junior High School, Volume I, Parts 1-2; Volume II, Parts 1-2, 1961.

This book for junior high schools includes student exercises in measurement.

Introduction to Probability, Part 1, Basic Concepts, 1967.

This is excellent material that is appropriate for classroom use in grades 1-8.

Introduction to Probability, Part 2, Special Topics, 1967.

This is excellent material appropriate for use in the classroom in the upper elementary grades.

Probability for Primary Grades.

This booklet is for student use. A teacher's commentary and a set of spinners for the classroom are available.

Probability for Intermediate Grades.

This booklet is for student use. A teacher's commentary and a set of spinners for the classroom are available.

Secondary School Mathematics, 1970.

This material is nongraded for low achievers in mathematics in junior high school.

Starr, John U., *The Teaching of Mathematics in the Elementary Schools.* Scranton, Penn.: International Textbook Co., 1969.

This book emphasizes why and how various mathematical principles and concepts operate, and provides teachers with proven and class-tested techniques to help implement learning concepts.

Swain, Robert L., *Understanding Arithmetic.* Revised by Eugene D. Nichols. New York: Holt, Rinehart and Winston, Inc., 1965.

Two chapters give suggestions on computations with directed numbers and the complement method of subtraction. One chapter on number theory includes a discussion on divisibility.

Thorndike, R. L. and Hagen, Elizabeth, *Measurement and Evaluation in Psychology and Education.* New York: John Wiley (Second Edition), 1961.

A help with making and interpreting tests.

Torrance, E. Paul and Myers, R. E., *Creative Learning and Teaching.* Chapters 7, 8. New York: Dodd, Mead, 1970.

A help with making and interpreting tests.

Turner, Ethel M., *Teaching Aids for Elementary Mathematics.* New York: Holt, Rinehart and Winston, Inc., 1966.

This is a source book for teachers, including many activities for students. The activities using coordinates are useful.

Van Engen, Henry, et. al., *Foundations of Elementary School Arithmetic.* Chicago: Scott Foresman and Co., 1965.

An excellent reference with proper balance between a precise formulation of mathematical concepts and an informal presentation of the content. In the chapter on relations, the study of ordered number pairs is begun in the context of rate pairs. In a later chapter the study of ordered number pairs is continued in the context of fractions. A set of equivalent fractions is called a *rational number of arithmetic.* The various numeral forms are treated under computation involving rational numbers.

Wilhelms, Fred T., *Evaluation as Feedback and Guide.* Washington: Association for Supervision and Curriculum Development, NEA, 1967.

Stresses the idea that the feedback from evaluation controls the next steps and should be based on the objectives.

Wisner, Robert, *A Panorama of Number.* Glenview, Ill.: Scott Foresman and Co., 1965.

The author has employed a unique writing style which is interesting. Many interesting observations are made concerning prime numbers.

U. S. Department of Health, Education and Welfare. *Elementary Arithmetic and Learning Aids.* Washington, D. C.: U. S. Government Printing Office, 1965.

This book has descriptions of aids to make and use in elementary mathematics.

INSTRUCTIONAL AIDS

Bags, plastic-assorted sizes
Beads to string
Boxes, assorted sizes
Cards, assorted colors
Cards, decks of Old Maid, Rook, etc.
Chalk, assorted colors
Coins, real or play
Clocks, play
Clothes pins
Compasses
Construction paper, assorted colors
Counters (bottle caps, cardboard discs)
Crayons
Cubes, assorted colors
Dowel rods
Egg cartons
Elastic thread, assorted colors
Flannel board and flannel cutouts
Geoboard
Graph paper
  bulletin board size
  individual size
Logic blocks
Macaroni shells
Maps
Meter sticks
Mirror cards
Multibase blocks
Nails, one size
Number lines
  walk-on (first grade)
  chalkboard size
  individual size
  whole numbers
  fractions
  integers
Objects
  small, for sets
  different colored
Overhead projector
  transparency paper
Paper clips
Paper cups
Paper roll
Paper, squared
  assorted sizes
Pegboard and pegs
Place value charts
Polyhedra models, regular
  made of cardboard or plastic
Ribbon, assorted colors
Rubber bands, assorted sizes
Rulers, foot
  unmarked
  marked in inches
  marked in fractional parts of inches
Sampling box
Sampling ladle
Scales
Scissors
Spinners
  two-colored
  more than two colored
Sticks for bundling
String
Thermometers
  Fahrenheit
  Centigrade (Celsius)
Three-dimensional shapes with flat surfaces
Two-dimensional shapes for "tiling"
Spoons, plastic
Sugar cubes
Tape measures, including steel tapes
Tea
Washers or counting rings of one size
Yardsticks
Yarn, knitting, assorted colors.

GLOSSARY FOR TEACHERS

GLOSSARY FOR TEACHERS

The purpose of the glossary is for clarification of terms for the teacher and is not to be used for pupils. Each teacher should have a reputable student mathematics dictionary for use by the pupils.

ABSCISSA   If the ordered pair of numbers (a,b) are the coordinates of a point of a graph, the number a is the abscissa.

ABSOLUTE ERROR   One-half the smallest marked interval on the scale being used.

ABSOLUTE VALUE   The absolute value of the real number a is denoted by $|a|$. If $a > 0$ then $|a| = a$ and if $a < 0$, $|a| = {}^{-}a$. On the number line absolute value is the distance of a point from zero.

ACCURACY   The accuracy of a measurement depends upon the relative error. It is directly related to the number of significant digits in the measured quantity.

ADDITIVE IDENTITY   The number I in any set of numbers that has the following property: $I + a = a$ for all a in the set. The symbol for the identity is usually 0; in the complex numbers it is $0 + 0i$, and in some systems bears no resemblance to zero.

ADDITIVE INVERSE   For any given number a in a set of numbers the inverse, usually designated by $(-a)$ is that number which when added to a will give the additive identity.

ALGEBRAIC EXPRESSION   An algebraic expression may be a single numeral or a single variable; or it may consist of combinations of numbers and variables, together with symbols of operation and symbols of grouping.

ALGORITHM   (ALSO ALGORISM) Any pattern of computational procedure.

AMPLITUDE   The amplitude of a trigonometric function is the greatest absolute value of the second coordinates of that function. For a complex number represented by polar coordinates the amplitude is the angle which is the second member of the pair.

ANGLE   The set of all points on two rays which have the same end-point. The end-point is called the *vertex* of the angle, and the two rays are called the *sides* of the angle.

ANGULAR VELOCITY   The amount of rotation per unit of time.

APPROXIMATE MEASURE   Any measure not found by counting.

APPROXIMATION   The method of finding any desired decimal representation of a number by placing it within successively smaller intervals.

ARC   If A and B are two points of a circle with P as center, the arc $\widehat{AB}$ is the set of points in the interior of $\angle APB$ on the circle and on the angle.

AREA OF A SURFACE   Area measures the amount of surface.

ARGAND DIAGRAM   Two perpendicular axes, one which represents the real numbers, the other the imaginary numbers thus giving a frame of reference for graphing the complex numbers. These axes are called the real axis and the imaginary axis.

ARITHMETIC MEANS   The terms that should appear between two given terms so that all the terms will be an arithmetic sequence.

ARITHMETIC SEQUENCE   (ALSO PROGRESSION) A sequence of numbers in which there is a common difference between any two successive numbers.

ARITHMETIC SERIES   The indicated sum of an arithmetic sequence.

ARRAY   A rectangular arrangement of elements in rows and columns.

ASSOCIATIVE PROPERTY A basic mathematical concept that the order in which certain types of operations are performed does not affect the result. The laws of addition and multiplication are stated as $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

ASYMMETRIC Having no point, line or plane of symmetry.

AVERAGE A measure of central tendency. See mean, median and mode.

AXIS OF SYMMETRY A line is called an axis of symmetry for a curve if it separates the curve into two portions so that every point of one portion is a mirror image in the line of a corresponding point in the other portion.

BASE The first collection in a number series which is used as a special kind of one. It is used in combination with the smaller numbers to form the next number in the series. In the decimal system of numeration, eleven, which is one more than the base of ten, literally means ten and one. Twenty means two tens or two of the base. In an expression such as a^n, the quantity a is called the *base* and n the exponent.

BASE OF A NUMERATION SYSTEM The base of a numeration system is the number of units in a given digits place which must be taken to denote one in the next higher place.

BASE TEN A system of numeration or a place-value arithmetic using the value of ten as its base value.

BASIC FACTS The name given to any operational table in a base of place-value arithmetic, as, basic addition tables, subtraction tables, multiplication tables, division tables, power tables, logarithmic tables. Basic addition facts include all addition facts in which neither of the addends exceeds 9. Basic subtraction facts include all the subtractions facts which correspond to all basic addition facts. The products formed by ordered pairs composed of the numbers 0 through 9 are called basic multiplication facts. Basic division facts include all the division facts which correspond to the basic multiplication facts except for $a \times b = c$ where $b \neq 0$.

BETWEENNESS B is between A and C if A, B and C are distinct points on the same line and $AB + BC = AC$.

BIAS When the method of selecting samples does not satisfy the condition that every possible sample that can be drawn has an equal chance of being selected, the sampling process is said to be biased.

BINARY OPERATION An operation involving two numbers such as addition; similarly, a unary operation involves only one number as "the square of."

BINARY NUMERATION SYSTEM A system of notation with base two. It requires only two symbols, 0 and 1.

BOUNDED A point set S is bounded if and only if there is a circle (or sphere in 3-space) such that S lies entirely in the interior of that circle (sphere).

CARDINAL NUMBER If two sets can be put in one-to-one correspondence with each other, they are said to have the same cardinal number. A whole number which answers the question of how many in a given finite set is called the cardinal number of a set.

CARTESIAN COORDINATES In a plane, the elements of ordered pairs which are distances from two intersecting lines. The distances from one line is measured along a parallel to the other line.

CARTESIAN PRODUCT The Cartesian product of two sets A and B, written $A \times B$ and read "A cross B" is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$.

CELL The union of a simple closed surface and its interior.

CENTRAL TENDENCY A measure of the trend of occurrences of an event.

CHECK To verify the correctness of an answer or solution. It is not to be confused with *prove*.

CIRCLE The set of points in a given plane each of which is at a given distance from a given point of the plane. The given point is called the *center*, and the given distance is called the *radius*.

CIRCULAR FUNCTION A function which associates a unique point with each arc of a unit circle (as measured from a fixed point of the circle). The sine function associates with the measure of an arc the ordinate of its companion point and the cosine the abscissa of the point.

CIRCUMFERENCE The length of the closed curved line which is the circle.

CLOSED CURVE (SIMPLE) A path which starts at one point and comes back to this point without intersecting itself represents a *simple* closed curve.

CLOSURE, PROPERTY OF A set is said to have the property of closure for any given operation if the result of performing the operation on any two members of the set is a number which is also a member of the set.

COLLECTION Elements or objects united from the viewpoint of a certain common property; as collection of pictures, collection of stamps, numbers, lines, persons, ideas.

COMBINATION A combination of a set of objects is any subset of the given set. All possible combinations of the letters a, b and c are a, b, c, ab, ac, bc, abc.

COMMUTATIVE PROPERTY A basic mathematical concept that the order in which certain types of operations are performed does not affect the result. Addition is commutative, for example, $2 + 4 = 4 + 2$. Multiplication is commutative, for example, $2 \times 4 = 4 \times 2$.

COMPASS (OR COMPASSES) A tool used to construct arcs and circles.

COMPLEX FRACTION A fraction that has one or more fractions in its numerator or denominator.

COMPLEX NUMBER Any number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

COMPOSITE NUMBER A counting number which is divisible by a smaller counting number different from 1.

CONGRUENT Two configurations which are such that when superimposed any point of either one lies on the other. Two figures having the same size and shape.

CONJUGATE COMPLEX NUMBERS The conjugate of the complex number $a + bi$ is the complex number $a - bi$.

CONJUNCTION A statement consisting of two statements connected by the word *and*. An example is $x + y = 7$ *and* $x - y = 3$. The solution set for a conjunction is the intersection of the solution sets of the separate statements.

CONDITIONAL EQUATION An equation that is true for only certain values of the variable, for example, $x + 3 = 7$.

CONIC, CONIC SECTION The curves which can be obtained as plane sections of a right circular cone.

CONSISTENT SYSTEM A system whose solution set contains at least one member.

CONSTANT A particular member of a specified set.

COORDINATE PLANE A plane whose points are named by ordered pairs of numbers which measure the distances from two intersecting lines. Each distance is measured from one line along a parallel to the other line.

COTERMINAL ANGLES Two angles which have the same initial and terminal sides but whose measures in degrees differ by 360 or a multiple of 360.

COUNTABLE In set theory, an infinite set is countable if it can be put into one-to-one correspondence with the natural numbers.

COUNTING NUMBERS The counting numbers are 1, 2, 3, 4, . . .

CONVERGENT SEQUENCE A sequence that has a limit.

DECIMAL EXPANSION A digit for every decimal place.

DECIMAL FRACTION A decimal fraction is a fraction in which the denominator is a power of ten. The fractions $\frac{3}{10}$, $\frac{4}{100}$, $\frac{125}{1000}$, etc. are decimal fractions but $\frac{2}{3}$ is not.

DEDUCTIVE REASONING The process of using previously assumed or known statements to make an argument for new statements.

DEGREE In angular measure, a standard unit that is $\frac{1}{90}$ of the measure of a right angle. In arc measure, one of the 360 equal parts of a circle.

DEGREE OF A POLYNOMIAL The general polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x^1 + a^n$ is said to be of degree n if $a_0 \neq 0$.

DENOMINATOR The lower term in a fraction. It names the number of equal parts into which a number is to be divided.

DEPENDENT LINEAR EQUATIONS Equations that have the same solution set.

DEVIATION The difference between the particular number and the average of the set of numbers under consideration is the deviation.

DIFFERENCE The answer or result of a subtraction. Thus, $8 - 5$ is referred to as a difference, not as a remainder.

DIGIT Any one of the ten symbols used in our numeration system – 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (from the Latin, "digitus," or "finger").

DIHEDRAL ANGLE The set of all points of a line and two non-coplanar half-planes having the given line as a common edge. The line is called the *edge* of the dihedral angle. The *side* or *face* consists of the edge and either half-plane.

DIRECT VARIATION The number y varies directly as the number x if $y = kx$ where k is a constant.

DISC The union of a simple closed curve in a plane and its interior.

DISCRIMINANT The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is the number $b^2 - 4ac$.

DISJUNCTION A statement consisting of two statements connected by *or*, for example, $x + y = 7$ *or* $x - y = 3$. The solution set of disjunction is the union of the solution sets of the separate statements.

DISTRIBUTIVE PROPERTY Links addition and multiplication. Examples of distributive property are as follows.

$3 \times 14 = 3\,(10 + 4) = (3 \times 10) + (3 \times 4) = 30 + 12 = 42.$

$4 \times 3\frac{1}{2} = 4\,(3 + \frac{1}{2}) = (4 \times 3) + (4 \times \frac{1}{2}) = 12 + 2 = 14.$

$a\,(b + c) = ab + ac.$

DIVERGENT SEQUENCE A sequence that is not convergent.

DIVISIBLE An integer a is divisible by an integer b if and only if there is some integer c such that $b \times c = a$.

DIVISION The inverse of multiplication. The process of finding how many times one quantity or number is contained in another. For any real numbers a and b, $b \neq 0$, $a \div b$ means a multiplied by the reciprocal of b. Also, $a \div b = c$, if and only if $a = bc$.

DOMAIN OF A VARIABLE The set of all values of a variable is sometimes called its domain.

DUODECIMAL NUMERATION SYSTEM A system of notation with base twelve. It requires twelve symbols – 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E.

ELEMENTS In mathematics the individual objects included in a set.

EMPTY SET The set which has no elements. The symbol for this set is ϕ or $\{\,\}$.

END POINT The point on a line from which a ray extends is called the end point of the ray.

EQUALITY The relation *is equal to* denoted by the symbol "=."

EQUATION A sentence (usually expressed in symbols) in which the verb is "is equal to."

EQUIVALENT EQUATIONS Equations that have the same solution set.

EQUIVALENCE RELATION Any relation which is reflexive, symmetric and transitive, for example, reflexive: $a = a$; symmetric: if $a = b$ then $b = a$ and transitive: if $a = b$ and $b = c$, than $a = c$.

EQUIVALENT FRACTIONS Two fractions which represent the same number.

EQUILATERAL TRIANGLE A triangle whose sides have equal length.

ESTIMATE A quick and frequently mental operation to ascertain the approximate value of an involved operation.

EVEN NUMBER An integer that is divisible by 2. All even numbers can be written in the form $2n$, where n is an integer.

EXPANDED EXPONENTIAL FORM The expanded exponential form of a numeral is the form in which the additive, multiplicative and place value properties of a numeration system are explicitly indicated. The value of each place is written in exponential form, for example, $365 = 3(10^2) + 6(10^1) + 5(10^0)$

EXPONENT In the expressionon a^n the number n is called an exponent. If n is a positive integer it indicates how many times a is used as a factor.

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$$

Under other conditions exponents can include zero, negative integers, rational and irrational numbers.

EXPONENTIAL EQUATION An equation in which the independent variable appears as an exponent.

EXPONENTIAL FUNCTION A function defined by the exponential equation $y = a^x$ where $a > 0$.

EXTRANEOUS ROOTS Those roots in the solution set of a derived equation which are not members of the solution set of the original equation.

EXTRAPOLATING Estimating the value of a function greater than or less than the known values; making inferences from data beyond the point strictly justified by the data.

FACTOR The integer m is a factor of the integer n if $m \times q = n$ where q is an integer. The polynomial $R(x)$ is a factor of the polynomial $P(x)$ if $R(x) \times Q(x) = P(x)$ where $Q(x)$ is a polynomial. *Factorization* is the process of finding the factors.

FACTORIAL The expression "$n!$" is read n factorial. $n! = n(n-1)(n-2)\ldots 2 \times 1$.

FIGURATE NUMBERS Figurate numbers include the numbers more commonly referred to as square numbers, triangular numbers, etc.

FINITE SET In set theory, a set which is not infinite.

FRACTION A symbol $\frac{a}{b}$ where a and b are numbers, with b not zero.

FREQUENCY A collection of data is generally organized into several categories according to specified intervals or subcollections. A frequency is the number of scores or measures in a particular category.

FREQUENCY, CUMULATIVE The sum of frequencies preceding and including the frequency of measures in a particular category is the cumulative frequency.

FREQUENCY DISTRIBUTION A tabulation of the frequencies of scores or measures in each of the categories of data.

FUNCTION A relation in which no two of the ordered pairs have the same first member. Also, alternately, a function consists of (1) a set A called the domain, (2) a set B called the range, (3) a table, rule, formula or graph which associated each member of A with exactly one member of B.

FUNDAMENTAL THEOREM OF ARITHMETIC Any positive integer greater than one may be factored into primes in essentially one way. The order of the primes may differ but the same primes will be present. Alternately, any integer except zero can be expressed as a unit times a product of its positive primes.

FUNDAMENTAL THEOREM OF FRACTIONS If the numerator and denominator are both multiplied (or divided) by the same non-zero number, the result is another name for the fraction.

GEOMETRIC MEANS The terms that should appear between two given terms so that all of the terms will form a geometric sequence.

GEOMETRIC SEQUENCE A sequence in which the ratio of any term to its predecessor is the same for all terms.

GEOMETRIC SERIES The indicated sum of a geometric sequence.

GRAPH A pictorial representation of a set of points associated with a relation which involves one or more variables.

GREATEST INTEGER FUNCTION Is defined by the rule $f(x)$ is the greatest integer not greater than x. It is usually denoted by the equation $f(x) = [x]$.

GREATEST LOWER BOUND A lower bound a of a set S of real numbers is the greatest lower bound of S if no lower bound of S is greater than a.

HARMONIC MEAN A number whose reciprocal is the arithmetic mean between the reciprocals of two given numbers.

HEMISPHERE If a sphere is divided into two parts by a plane through its center, each half is called a hemisphere.

HISTOGRAM A bar graph representing a frequency distribution. The base of each of the contiguous rectangular bars is the range of measures within a particular category, and the height of each of the bars is the frequency of measures in the same category.

IDENTITY, IDENTICAL EQUATION A statement of equality, usually denoted by $\equiv$ which is true for all values of the variables. The values of the variable which have no meaning are excluded, for example, $(x + y)^2 \equiv x^2 + 2xy + y^2$.

INCONSISTENT SYSTEM OF EQUATIONS A system whose solution set is the empty set.

INDEPENDENT EVENTS Two events are said to be independent if the occurrence of one does not affect the probability of occurrence of the other.

INDEPENDENT SYSTEM OF EQUATIONS A system of equations that are not dependent.

INDEX The number used with a radical sign to indicate the root. ($\sqrt[3]{}$ In this example the index is three.) If no number is used, the index is two.

INDUCTIVE REASONING The process of drawing a conclusion by observing what happens in a number of particular cases.

INEQUALITY The relation in which the verb is one of the following – is not equal to, is greater than or is less than, denoted by the symbols $\neq$, $>$, $<$; respectively.

INFINITE DECIMAL (Also non-terminating) A decimal representation that has an unending string of digits to the right of the decimal point.

INFINITE REPEATING DECIMAL A decimal representation containing a finite block of digits which repeats endlessly.

INFINITE SET In set theory, a set which can be placed in one-to-one correspondence with a proper subset of itself.

INTEGER Any one of the set of numbers which consists of the natural numbers, their opposites and zero.

INTERCEPT If the points whose coordinates are $(a,0)$ and $(0,b)$ are points on the graph of an equation, they are called intercepts. The point whose coordinates are $(a,0)$ is the x-intercept, and the point whose coordinates are $(0,b)$ is the y-intercept.

INTERPOLATION The process of estimating a value of a function between two known values other than by the rule of the table of the function.

INTERSECTING LINES Two or more lines that pass through a single point in space.

INTERSECTION OF SETS If A and B are sets, the intersection of A and B, denoted by $A \cap B$, is the set of all elements which are members of both A and B.

INVERSE OF AN OPERATION That operation which, when performed after a given operation, annuls the given operation. Subtraction of a quantity is the inverse of addition of that quantity. Addition is likewise the inverse of subtraction.

INVERSE FUNCTION If f is a given function then its inverse is the function (provided f is one-to-one) formed by interchanging the range with domain. The symbol for inverse of f is f^{-1}.

INVERSE VARIATION The number y is said to vary inversely as the number x if $x \times y = k$ where k is a constant.

IRRATIONAL EQUATION An equation containing the variable or variables under radical signs or with fractional exponents.

IRRATIONAL NUMBER An irrational number is not a rational number. That is, it is a number that cannot be expressed in the form $\frac{a}{b}$ where a and b are integers. The union of the set of rationals and the set of irrationals is the set of real numbers.

JOINT VARIATION A quantity varies jointly as two other quantities if the first is equal to the product of a constant and the other two, for example, y varies jointly as x and w if $y = kxw$.

LATTICE POINTS An array of points named by ordered pairs.

LEAST COMMON MULTIPLE The least common multiple of two or more numbers is the common multiple which is a factor of all the other common multiples.

LEAST UPPER BOUND An upper bound b of a set S of real numbers is the least upper bound of S if no upper bound of S is less than b.

LINEAR EQUATION An equation in standard form in which the sum of the exponents of the variable in any term equals one.

LINEAR MEASURE A measure used to determine length.

LOGARITHM The exponent that satisfies the equation $b^x = n$ is called the logarithm of n to the base b for any given positive number n.

LOWER BOUND A number a is called a lower bound of set S of real numbers if $a \leqslant x$ for every $x \in S$.

MAGIC SQUARE A square of numbers possessing the particular property that the sums in each row, column and diagonal are the same.

MATRIX A rectangular array of numbers.

Example

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

MEAN In a frequency distribution, the sum of the n measures divided by n is called the mean.

MEASUREMENT A comparison of the capacity, length, etc., of a thing to be measured with the capacity, length, etc., of an agreed upon unit of measure. Non-standard units are used before standard units of measure are introduced.

MEDIAN In a frequency distribution, the measure that is in the middle of the range when elements are ranked from highest to lowest is called the median. In geometry, a median of a triangle is a line joining a vertex to the midpoint of the opposite side.

MODE In a frequency distribution, the interval in which the largest number of measures fall is called the mode. Alternately, in a frequency distribution, the measure which appears most frequently in the group is called the mode. There may be more than one mode in a set of measures.

MODULO ARITHMETIC For a given positive integer n, *modulo n* is obtained by using the integers $0 \ldots n - 1$ and defining addition and multiplication by letting the sum of $a + b$ and the product of $a\ b$ be the remainder after division by n of the ordinary sum and product of a and b. (This is often called clock arithmetic.)

MODULUS A statement of the type x is congruent to y modulus (or modulo) w, w is the modulus of the congruency. If 2 is congruent to 9, then the modulus is 7.

MULTIPLE If a and b are integers such that $a = b \times c$ where c is an integer, then a is said to be a multiple of b.

MULTIPLICATION A short method of adding like groups or addends of equal size. It may be illustrated on a number line by counting forward by equal groups.

MULTIPLICATIVE INVERSE The multiplicative inverse of a non-zero number a is the number b such that $a \times b = 1$. It is usually designated by $\frac{1}{a}$ or a^{-1}.

MUTUALLY DISJOINT SETS Two sets having no elements in common.

MUTUALLY EXCLUSIVE EVENTS Events which cannot occur simultaneously. Mutually exclusive subsets are subsets that are disjoint.

NATURAL NUMBERS Any of the set of counting numbers. The set of natural numbers is an infinite set; it has a smallest member (1) but no largest.

NULL SET A set containing no elements. It is sometimes called an empty set. The symbol for the null set is ϕ or $\{\}$.

NUMBER SYSTEM A number system consists of a set of numbers, two operations defined on the set, the properties belonging to the set and a definition for equivalence between any two members of the set.

NUMERATION SYSTEM A coding system for recording numerals. Modern systems of numeration are characterized by a set of symbols, or digits, a place value scheme and a base.

NUMERATOR The upper term in a fraction.

NUMERAL A written symbol for a number, for example, several numerals for the same number are 8, VIII, 7 + 1, 10 − 2, $\frac{16}{2}$

OBTUSE ANGLE If the degree measure of an angle is between 90 and 180, the angle is called an obtuse angle.

ODD NUMBER An odd number is an integer that is not divisible by 2; any number of the form $2n + 1$, where n is an integer.

ONE-TO-ONE CORRESPONDENCE A pairing of the members of a set A with members of a second set B such that each member of A is paired with exactly one member of B, and each member of B is paired with exactly one member of A.

OPEN SENTENCE An open sentence is a sentence involving one or more variables, and the question of whether it is true cannot be decided until definite values are given to the variables, for example, $x + 5 = 7$.

ORDERED N-TUPLE A linear array of numbers $(a_1, a_2, a_3, \ldots, a_n)$ such that a_1 is the first number, a_2 is the second number, a_3 is the third number, ... and a_n is the nth number.

ORDERED PAIR A pair of numbers (a,b) where a is the first member and b is the second member of the pair.

ORDINAL NUMBER A number that denotes order of the members in a set.

ORDINATE If an ordered pair of numbers (a,b) are coordinates of a point P, b is called the ordinate of P.

PARALLEL LINES Two straight lines in a plane that do not intersect however far extended.

PARALLELOGRAM A quadrilateral whose opposite sides are parallel.

PARAMETER An arbitrary constant or a variable in a mathematical expression, which distinguishes various specific cases.

PARTIAL PRODUCT Used in elementary arithmetic with regard to the written algorithm of multiplication. Each digit in the multiplier produces one partial product. The final product is then the sum of the partial products.

PARTIAL QUOTIENT In long division, any of the trial quotients that must be added to obtain the complete quotient.

PERIMETER The sum of the measures of the sides of a polygon. The measure of the outer boundary of a polygon.

PERIOD The number of digits set off by a comma in an integer or the integral part of a mixed decimal. In a repeating decimal the period is the sequence of digits that repeats.

PERIODIC FUNCTION A function from R to R, where R is the set of real numbers, is called periodic if, and only if, $f(x)$ is not the same for all x and there is a real number p such that $f(x + p) = f(x)$ for all x in the domain of f. The smallest positive number p for which this holds is called the period of the function.

PERMUTATION A permutation is an ordered arrangement of all or part of the members in a set. All possible permutations of the letters a, b and c are a, b, c, ab, ac, ba, bc, ca, cb, abc, acb, bac, bca, cab, cba.

PERPENDICULAR A line is perpendicular to a ray if and only if the end point of the ray is the only point of intersection of the two and the two angles formed are congruent.

PLACE VALUE The value of a numeral is dependent upon its position. In the number 324, for example, each digit has a place value 10 times that of the place value of the digit to its immediate right.

PLANE ANGLE Through any point on the edge of a dihedral angle pass a plane perpendicular to the edge intersecting each side in a ray. The angle formed by these rays is called the plane angle of the dihedral angle.

POLAR COORDINATES An ordered pair used to represent a complex number. The first member of the pair is the number of units in the radius vector, and the second member is the angle of rotation of the radius vector.

POLYGON A simple closed curve which is the union of line segments is called a polygon.

POLYHEDRON A solid bounded by plane polygons. The bounding polygons are the *faces*, the intersections of the faces are the *edges* and the points where three or more edges intersect are the *vertices*.

POLYNOMIAL An algebraic expression of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ sometimes designated by the symbol $P(x)$.

POLYNOMIAL EQUATION A statement that $P(x) = 0$.

POLYNOMIAL FUNCTION A function defined by a polynomial equation or $f : x \rightarrow P(x)$.

PRECISION The precision of a measurement is inversely related to the absolute error. Thus the smaller the absolute error, the greater the precision.

PRIME NUMBER A counting number other than one, which is divisible only by itself and one.

PRISM If a polyhedron has two faces parallel and its other faces in the form of parallelograms, it is called a prism.

PROBABILITY The numerical measure of the likelihood of an event is called the probability of the event. It is a rational number p such that $0 \leq p \leq 1$.

PROPER SUBSET A subset R is a proper subset of a set S if R is a subset of S and $R \neq S$.

PURE IMAGINARY A complex number $a + bi$ in which $a = 0$ and $b \neq 0$.

PYRAMID A polyhedron, one of whose faces is a polygon of any number of sides and whose other faces are triangles having a common vertex.

QUADRANTAL ANGLE If the terminal side of an angle with center at the origin coincides with a coordinate axis, the angle is called a quadrantal angle.

QUADRILATERAL A polygon formed by the union of 4 line segments.

QUINARY SYSTEM OF NUMERATION A system of notation with the base 5. It requires only five symbols or digits— 0, 1, 2, 3, 4.

RADIAN MEASURE Angular measure where the unit is an angle whose arc on a circle with center at vertex of angle is equal in length to the radius of the circle.

RADIUS Any line segment with endpoint at the center of a circle and the other endpoint on the circle is called a radius of the circle.

RADIUS VECTOR A line segment with one end fixed at the origin on the cartesian plane and rotating from an initial position along the positive x-axis so that its free end point generates a circle.

RANGE (STATISTICS) The range of the set of numbers is the difference between the largest and smallest numbers in a set.

RANGE (OF A FUNCTION) The set of all elements assigned to the elements of the domain by the rule of the function.

RATE PAIR An ordered pair of counting numbers which expressed a rate relation – e.g., a rate of exchange. In general, a rate pair $\frac{a}{b}$, where a and b are counting numbers, expresses a ratio of the number of elements in one set to the number of elements in a second set.

RATIONAL EXPRESSION A rational expression is a quotient of two polynomials or in symbols $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

RATIONAL NUMBER If a and b are whole numbers with b not zero, the number represented by the fraction $\frac{a}{b}$ is called a rational number.

RATIONAL NUMBERS OF ARITHMETIC In the elementary school, one generally defines a set of equivalent fractions to be a rational number. Alternatively, a rational number is an equivalence class of ordered pairs of integers a and b, $b \neq 0$.

RAY Let A and B be points on a line. Then ray $\overrightarrow{AB}$ is the set which is the union of the segment $\overline{AB}$ and the set of all points C for which it is true that B is between A and C. The point A is called the *end point* of $\overrightarrow{AB}$.

RECIPROCAL Multiplicative inverse.

RECIPROCAL FUNCTION Pairs of functions in the set of real numbers whose product is 1, for example, $(\sin \phi)(\csc \phi) = 1$.

RECTANGLE A parallelogram with right angles.

REFERENCE TRIANGLE For any angle on the Cartesian plane with vertex at the origin, the triangle formed by the radius vector, its projection on the x-axis and a line drawn from the end of the radius vector perpendicular to the x-axis is called the reference triangle.

REFLECTION IN A LINE A point P has a mirror image P' in the line $\overleftrightarrow{AB}$ if P, P' and $\overleftrightarrow{AB}$ all lie in the same plane with P and P' on opposite sides of $\overleftrightarrow{AB}$ and if the perpendicular distances PO and $P'O$ to the point O in $\overleftrightarrow{AB}$ are equal.

REFLEXIVE PROPERTY If a is any element of a set and if R is a relation on the set such that aRa for all a, then R is reflexive.

REGION The union of a simple closed curve and its interior.

RELATED ANGLE For any angle on the Cartesian plane, the related angle is the angle in the reference triangle formed by the radius vector and x-axis.

RELATION A relation from set A to set B (where A and B may represent the same set) is any set of ordered pairs (a,b) such that a is a member of A and b is a member of B.

RELATIVE ERROR Ratio of the absolute error to the measured value.

RELATIVE FREQUENCY The relative frequency is the frequency of a given category divided by the total number of measures in the category.

RELATIVELY PRIME Two integers are relatively prime if they have no common factors other than +1 or −1; two polynomials are relatively prime if they have no common factors except constants.

REPEATING DECIMAL A decimal numeral which never ends and which repeats a sequence of digits. It is indicated in this manner – 0.333 ... or $0.\overline{142857}$.

RESOLUTION OF VECTORS The process of finding the vertical and horizontal components.

RESTRICTED DOMAIN Domain of a function or relation from which certain numbers are excluded for reasons such as division by zero is not permitted and need for the inverse of a function to be a function.

RIGHT ANGLE Any of the four angles obtained at the point of intersection of two perpendicular lines. The angle made by two perpendicular rays. Its measure is 90 degrees.

RIGHT TRIANGLE A triangle with one right angle.

ROUNDING OFF Replacing digits with zero's to a certain designated place in a number with the last remaining digit being increased or decreased under certain specified conditions.

SAMPLE SPACE The set of all possible outcomes of an experiment.

SCALAR In physical science, a quantity having magnitude but no direction. In a study of mathematical vector, any real number.

SCALE A system of marks in a given order and at fixed intervals. Scales are used on rulers, thermometers and other measuring instruments and devices as an aid in measuring quantities.

SCIENTIFIC NOTATION A notation generally used for very large or very small numbers in which each numeral is changed to the form $a \times 10^k$ where a is a real number containing at most three significant digits such that $1 \leqslant a < 10$ and k is any integer.

Example

$6{,}708{,}345 = 6.71 \times 10^6$

$.000000052 = 5.2 \times 10^{-8}$

SEGMENT For any two points A and B, the set of points consisting of A and B and all points between A and B is the line segment determined by A and B. The segment is a geometrical figure while the distance is a number which tells how far A is from B.

SEQUENCE An ordered arrangement of numbers.

SERIES The indicated sum of a sequence.

SET A collection of particular things, as a set of numbers between 3 and 5, the set of points on the segment of a line or within a circle.

SET BUILDER NOTATION To describe the members of a very large or infinite set, it is often helpful to denote the set and its members as in this example— $\{x \mid x \in R \text{ and } 0 \leq x \leq 1\}$, read "The set of all x such that x is a member of the set R of rational numbers and x is greater than or equal to 0 and less than or equal to 1." The symbol device, $\{x \mid x \ldots\}$, read "the set of all x such that x..." is called set builder notation.

SIGNIFICANT FIGURE Any digit or any zero in a numeral not used for placement of the decimal point, for example, 703,000; .0056; 5.00.

SIMILAR Two geometric figures are similar if one can be made congruent to the other by using a transformation of similitude if one is a magnification or reduction of the other.

SKEW LINES Two lines which are not coplanar are said to be skew.

SLOPE The slope of a given segment (P_1P_2) is the number m such that $m = \frac{y_2 - y_1}{x_2 - x_1}$ where P_1 is the ordered pair (x_1, y_1) and P_2 is the ordered pair (x_2, y_2).

SOLID Any simple closed surface; the term is usually used with reference to polyhedra (rectangular solids, pyramids), cylinders, cones and spheres.

SOLUTION SET The truth set of an equation or a system of equations.

SPHERE The set of all points in space each of which is at a given distance from a given point. The given point is called the *center* of the sphere and the given distance is called the *radius*.

SQUARE A quadrilateral formed by four line segments of equal length which meet at right angles.

STANDARD DEVIATION The square root of the arithmetic mean of the squares of the deviations from the mean.

STATISTIC An estimate of a parameter obtained from a sample.

STATISTICS The concepts, measures and techniques related to methods of obtaining, organizing and analyzing data is included in statistics.

SUBSET A set contained within a set; a set whose members are members of another set. The fact that R is a subset of S is indicated by $R \subset S$.

SUBTRACTION To subtract the real number b from the real number a, add the opposite (additive inverse of b to a. $a - b = a + (-b)$. Also, $a - b = c$ if and only if $a = c + b$.

SUCCESSOR The successor of the integer a is the integer $a + 1$.

SUMMATION NOTATION The symbol $\sum_{k=n}^{n} a_k$. The symbol Σ, the Greek letter "sigma," corresponds to the first letter of the word "sum" and is used to indicate the summing process. The k and n represent the upper and lower indexes and indicate that the summing begins with the kth term and includes the nth term, for example,

$$\sum_{k=2}^{5} a_k = a_2 + a_3 + a_4 + a_5.$$

When the summation includes infinitely many terms is it written $\sum_{k}^{\infty} a_k$. In this case there is no last term a because ∞ is not a number. The symbol ∞ is used to indicate that the summation is infinite.

SYMMETRIC PROPERTY If a and b are any elements of a set and if R is a relation on the set such that aRb implies bRa, then the relation is said to have the symmetric property.

TERM In a phrase which has the form of an indicated sum, $A + B$, A and B are called *terms* of the phrase.

TERMINATING DECIMAL (Also finite decimal) A decimal representation that contains a finite number of digits.

TOPOLOGY A branch of mathematics which is the study of properties of point sets which are preserved.

TRANSITIVE PROPERTY If a, b and c are any elements of a set and if R is relation on the set such that aRb and bRc imply aRc, then the relation is said to have the transitive property.

TRAPEZOID A quadrilateral with at least two parallel sides.

TRIANGLE If A, B and C are three non-collinear points in a given plane, the set of all points in the segments having A, B, C as their end points is called a triangle.

UNBOUNDED Not bounded.

UNEQUAL Not equal, symbolized by $\neq$.

UNION OF SETS If A and B are two sets, the union of A and B is the set $A \cup B$ contains all the elements and only those elements that are in A or in B, for example, $A = \{2, 8, 3\}$, $B = \{5, 2, 7, 6\}$ then $A \cup B = \{2, 8, 3, 5, 7, 6\}$.

UNIQUE One and only one.

UPPER BOUND A number b is called an upper bound of a set S of real numbers if $b \geqslant x$ for every $x \in S$.

VARIABLE A letter used to denote any one of a given set of numbers. Another name for variable is placeholder in an equation, for example, $x + 5 = 7$.

VECTOR In physical science, a quantity having magnitude and direction. In mathematics a vector is a matrix of one row or one column as $(a_1\ b_1\ c_1)$ or

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

VERTEX The point of intersection of two rays.

VOLUME The amount of space occupied by a solid or enclosed within it.

WHOLE NUMBERS The whole numbers are 0, 1, 2, 3, 4, ...

SYMBOLS

$\neq$	is not equal to
$\approx$	is approximately equal to
$>$	is greater than
$\ngtr$	is not greater than
$<$	is less than
$\nless$	is not less than
$\geqslant$	is greater than or equal to
$\ngeqslant$	is not greater than or equal to
$\leqslant$	is less than or equal to
$\nleqslant$	is not less than or equal to
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\supset$	is a superset of
$\cong$	is congruent to
$\sim$	is similar to
ϵ	is an element of
$\notin$	is not an element of
U	universal set
S	solution set
S'	complement set
$A \times B$	Cartesian product set of sets A and B
a^{-n}	is interpreted as $\frac{1}{a^n}$ where $a \neq 0$
$\parallel$	is parallel to
$\perp$	is perpendicular to
$\overleftrightarrow{AB}$	straight line containing points A and B
$\overline{AB}$	straight line segment with end points A and B
$\overrightarrow{AB}$	ray from point A through point B
(a,b)	ordered pair a and b
$\{a\}$	set containing element a
$\square, \triangle$	frames, place holders or nonspecified elements
$\phi, \{\ \}$	the empty of null set
$\triangle ABC$	triangle with vertices A, B, and C applies to any polygon
$\angle ABC$	angle with point B as vertex
$\{\square \mid \square > 5\}$	the set of all $\square$ in the universal set such that $\square$ is greater than 5
$a{:}b$	ratio of a to b
$\cup$	union of two sets
$\cap$	intersection of two sets