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ABSTRACT

This is part two of a two-part manual for teachers using SMSG text materials for grade 9 students whose mathematical talents are underdeveloped. The overall purpose for each of the chapters is described and the mathematical development detailed. Background information for key concepts, answers for all exercises in each chapter, and suggested test items are provided. Chapter topics include: (1) factors and exponents; (2) radicals; (3) polynomials; (4) rational expressions; (5) truth sets of open sentences; (6) truth sets and graphs of sentences in two variables; (7) systems of open sentences; (8) quadratic polynomials; and (9) functions. (MN)

School Mathematics Study Group

Introduction to Algebra

Teacher's Commentary, Part II

Unit 46

REVISED EDITION

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Chapter 11

FACTORS AND EXPONENTS

One of the most important ideas in the structure of numbers and algebraic expressions is that of "factor". It is necessary for the student to realize that "factor" has little meaning unless restrictions are placed upon our discussion. Basic to our development is the definition of a proper factor. An integer is said to be <u>factorable</u> if it has a <u>proper</u> factor. It is <u>prime</u> if it does not have a proper factor.

It is very important to keep in mind that these ideas of factors and factoring depend on the set over which we do the factoring. Over the set of positive integers, 4 is a factor of 12. If we permitted all integers, -4 and 4 would both be factors of 12. If we factor over the rational numbers, then $\frac{2}{3}$ is a factor of 12, as well. If we factor over the real numbers, any number except 0 is a factor of 12, and the idea has become meaningless. When we speak of factoring a positive integer, we shall always mean over the set of positive integers, unless a different set is specified.

It is of more than passing interest that the positive integers can be generated by using the element 1 and the operation of addition, and that they can also be generated by the set of prime numbers and the operation of multiplication. These ideas will be helpful when we study polynomials, whose structure closely resembles that of the integers.

Divisibility as a subject is not a significant part of this course. We use it to develop some of the ideas we need about factors and prime numbers. It is a subject relatively easy to understand and usually interesting to students. It is doubtful that a great amount of time should be devoted to the extensive lists of exercises provided, or to the divisibility by 4, 7, 8, or 11. Though divisibility is used to introduce the topic of factoring, we soon move to the definition of a proper factor in terms of multiplication.

11-1. Factors and Divisibility.

The point of the story of the farmer with eleven cows is that the number 12 has a property which 11 does not have, namely: 12 has factors other than one and itself. Specifically, 12 can be divided by 2, 4, and 6 with remainder zero. Eleven cannot be divided by these numbers with remainder zero. The reason the stranger got his cow back was that $\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$ equals $\frac{11}{12}$, instead of 1.

The fact that any positive integer has itself and 1 as factors follows from the property of 1: $a \cdot 1 = a$. But some positive integers have factors other than 1 and itself. We call this kind of a factor a proper factor. Observe the similarity of the factor-proper factor relation to the subset-proper subset relation. "Proper" is a word people use to indicate the really interesting cases of a particular concept.

After the definition of proper factor, the question could be asked, "Does it follow from this definition that m also can equal neither 1 nor n?" If m is a proper factor of n we mean that there is a positive integer q ($q \neq 1$, $q \neq n$) such that $mq = n$. Then another name for q is $\frac{n}{m}$. If $m = 1$, then $q = \frac{n}{m} = \frac{n}{1} = n$. This is a contradiction. If $m = n$, then $q = \frac{n}{m} = \frac{n}{n} = 1$. This is a contradiction.

Answers to Oral Exercises 11-1a: page 458:

1. (a) Yes, since $8 \times 3 = 24$
   (b) No. There is no integer q such that $5q = 24$.
   (c) No. There is no integer q such that $9q = 24$.
   (d) Yes, since $24 \times 1 = 24$
   (e) No
   (f) Yes, since $1 \times 5 = 5$
2. {6, 12, 18, 24, ...}
3. {12, 24, 36, 48, ...}
4. {1, 2, 3, 4, 6, 12}
5. {2, 3, 4, 6}

6. Yes Yes
7. Yes
8. Yes. There is a positive integer q such that $p = mq$. Hence $\frac{p}{m} = q$, which means that $\frac{p}{m}$ is an integer and $m(\frac{p}{m}) = p$. If m is a proper factor of p, then so is $\frac{p}{m}$.
9. Find the quotient of the number and the known factor.

Answers to Problem Set 11-1a: pages 458-459:

Problems 6 through 12 are meant to reinforce the idea that if m is a factor of n, then $\frac{n}{m}$ is a factor of n.

1. Yes, $13 \times 7 = 91$
2. Yes, $30 \times 17 = 510$
3. Yes, $12 \times 17 = 204$
4. Yes, $10 \times 10,000 = 100,000$
5. No. (151,821 is not a multiple of 6.)
6. $\frac{51}{3}$
7. $\frac{57}{3}$
8. $\frac{65}{5}$
9. $\frac{161}{7}$
10. $\frac{87}{29}$
11. $\frac{437}{23}$
*12. $\frac{q}{c}$
13. {1,2,3,6,9,18}
14. {2,3,6,9}
15. {18,36,54,···}
16. (a) $85 = 5 \times 17$
(b) $51 = 3 \times 17$
(c) $52 = 2 \times 26$ or 4×13 or $2 \times 2 \times 13$
(d) 29: no proper factors
(e) $93 = 3 \times 31$
(f) $92 = 2 \times 46$ or 4×23 or $2 \times 2 \times 23$
(g) 37: no proper factors
(h) $94 = 2 \times 47$
(i) $55 = 5 \times 11$
(j) 61: no proper factors
(k) 23: no proper factors
(l) $123 = 3 \times 41$
(m) $57 = 3 \times 19$
(n) $65 = 5 \times 13$
(o) $122 = 2 \times 61$
(p) $68 = 2 \times 34$ or 4×17 or $2 \times 2 \times 17$
(q) $95 = 5 \times 19$
(r) $129 = 3 \times 43$
(s) $141 = 3 \times 47$
(t) 101: no proper factors

We would expect that students know the rules of divisibility for 2, 5, and 10, but the teacher should make certain that every student does. The following discussion is for the teacher's information. We hope that the teacher will not just lay out the rules but will aid the student to discover as much as possible for himself.

Any integer can be represented in the form $10t + u$, where t is a non-negative integer and u is an integer such that $0 \leq u \leq 9$. For example,

$$36 = 3(10) + 6,$$
$$178 = 17(10) + 8,$$
$$156237 = 15623(10) + 7.$$

The advantage of this form is that we can learn what rules of divisibility can be based on the last digit of the numeral. For example, $178 = 17(10) + 8$. Two is a factor of $17(10)$ because 2 is a factor of 10. In other words, the 17 doesn't matter since 2 is a factor of 10. Thus, whether or not 178 is divisible by 2 depends only on the last digit, 8. Since 2 is a factor of 8, 2 is a factor of 178.

Stating the previous argument more generally, since 2 is a factor of 10, 2 is a factor of $10t$. Thus 2 is a factor of $10t + u$, if and only if 2 is a factor of u, the last digit.

In an analogous manner, since 5 and 10 are factors of $10t$, each is a factor of $10t + u$ if and only if each is a factor of u. On the other hand, 3, 4, 7, 11, and 13 are not factors of $10t$ for every t. Thus rules for divisibility by these numbers cannot be based on only the last digit of the numeral.

Let us examine the last two digits of any numeral. In general we could write any number $(100)h + d$ where h is a non-negative integer and d is an integer such that $0 \leq d \leq 99$. Any number which is a factor of 100 will be a factor of $100h$. Therefore, any number which is a factor of 100 will be a factor of $(100)h + d$ if and only if it is a factor of d, the last two digits. Since the prime factorization of 100 is $2^2 \cdot 5^2$, any number made up of at most two 2's and two 5's will divide 100. Such numbers are 2, 4, 10, 20, 25, 50, and 100. But 2, 10, 20, and 50 are more easily checked by a single or double application of the last digit rules, so this leaves 4 and 25. Thus, a number is divisible by 4 or 25 if and only if the number denoted by the last two digits in its decimal notation is divisible by 4 or 25.

Another interesting test is based on the sum of the digits of a numeral. If the digits of a four digit decimal numeral are a, b, c, d, the number is

$$\begin{aligned} 1000a + 100b + 10c + d &= 999a + a + 99b + b + 9c + c + d \\ &= (999a + 99b + 9c) + (a + b + c + d) \\ &= (111a + 11b + c)9 + (a + b + c + d) \end{aligned}$$

A decimal numeral with any number of digits may be treated similarly.

Since 3 is a factor of 9, 3 is a factor of $(111a + 11b + c)9$. Hence, if 3 is a factor of $(a + b + c + d)$ it is a factor of the original number. Furthermore, if 3 is not a factor of $(a + b + c + d)$, then 3 is <u>not</u> a factor of the original number.

We conclude, then, that divisibility by 3 can be tested by determining whether 3 is a factor of the sum of the digits of the decimal numeral.

We notice, incidentally, that 9 has a similar test for divisibility. A number is divisible by 9 if and only if the sum of the digits of its decimal numeral is divisible by 9.

When finding the prime factorization of numbers in later sections, one finds the prime factors in the order 2, 3, 5, 7, ... The student should, therefore, know divisibility rules for 2, 3, and 5, at least. The rules for 5, 9, and 25 are also helpful in other situations. The tests for divisibility by 7 and 11 are left as starred exercises in the student's text. They are not very important except that students might find them interesting.

One last bit of "ammunition" for the teacher. Tell the student to write any three-digit number, say 456. Tell him to make it a six-digit number by appending the same three digits. The six-digit number in our example is then 456,456. Then tell him to divide successively by 7, 11, and 13. Lo and behold, he will end up with 456.

$$7\overline{)456456}\ \ 65208 \qquad 11\overline{)65208}\ \ 5928 \qquad 13\overline{)5928}\ \ 456$$

The reason is that appending 456 to 456 is the same as multiplying 456 by 1001. That is, (456)(1001) = 456,456. The number 1001 = 7 × 11 × 13. Thus, 1001 is divisible by 7, 11, and 13. This "pick-a-number" problem serves also to emphasize the meaning of variable. If n is the number chosen at the start of the problem, it is evident that, in accordance with the definition of variable, it represents an unspecified number from a set of numbers (in this case the phrasing of the problem indicates the domain: $100 \leq n \leq 999$.) Following out the directions of this problem, we have in terms of n

$$n$$

$$1001\ n$$

$$\frac{1001\ n}{7}$$

$$\frac{1001\ n}{7 \times 11}$$

$$\frac{1001\ n}{7 \times 11 \times 13},$$

$$\text{then}\quad \left(\frac{7 \times 11 \times 13}{7 \times 11 \times 13}\right) n = 1n = n.$$

Answers to Problem Set 11-1b; pages 461-464:

1. (a) divisible by 2
   (b) divisible by 2 and 3
   (c) divisible by 2 and 3
   (d) divisible by 2
   (e) divisible by 2, 5, and 10
   (f) divisible by 2 and 3

(g) divisible by 3 and 5
(h) divisible by 2, 5, and 10
(i) divisible by 2
(j) divisible by 2
(k) divisible by 2 and 3
(l) divisible by 2, 3, 5, and 10

2. Yes, by the associative property $2 \cdot 3 \cdot n = 6 \cdot n$. Thus, 6 is a factor of $2 \cdot 3 \cdot n$.

3. The numbers in (a), (c), (e), (f), (g), (i), and (j) have 6 as a factor; the others do not.

4. If a number is divisible by 3 and by 5, it is divisible by 15.

5. The numbers in parts (a), (d), (g), and (h) are divisible by 15; the others are not.

*6. A number is divisible by 9 if the sum of the digits is divisible by 9.

*7. All the numbers are divisible by 4 except those in the first two columns. A number is divisible by 4 if the number named by the last two digits is divisible by 4.

*8. A number is divisible by 12 if it is divisible by 4 and by 3. The number in part (k) of problem 1 is divisible by 4 and by 3. The numbers in parts (b), (c), and (e) are not divisible by 12.

*9. A number is divisible by 8 if the number named by the last 3 digits is divisible by 8.

*10. If the difference of the sums of sets of alternate digits in the common name of a number is a multiple of 11, then the number itself is a multiple of 11.

*11. If the difference between <u>twice</u> the number represented by the last digit and the number represented by the <u>remaining</u> digits is a multiple of seven, then the number itself is a multiple of seven.

The proof of this rule for divisibility by 7 may be of interest to some students. It can be established as follows: Let N be the number to be tested. If c is the right hand digit, our process gives us $\frac{N - c}{10} - 2c$. If we assume this is divisible by 7, then

$$\frac{N - c}{10} - 2c = 7k \quad \text{for some integer } k.$$

$$N - c = 20c + 70k$$

$$N = 21c + 70k$$

$$N = 7(3c + 10k) \quad \text{i.e. } 7 \text{ divides } N.$$

As an answer to the last question a reply of "no" is acceptable on the grounds that application of the rule is more time consuming than actual division. The proof, however, if shared with interested students may serve a purpose of its own.

11-2. <u>Prime Numbers and Prime Factorization</u>.

We have a long range objective in the discussion of this section. Students often get the impression that $x^2 - 2$ cannot be factored when in fact a pair of its factors are $(x - \sqrt{2})(x + \sqrt{2})$. So we wish to emphasize the set of numbers over which numbers are factored. If we want the factors of $x^2 - 2$ to be polynomials over the integers, then $x^2 - 2$ cannot be factored. But if we allow the factors to be polynomials over the real numbers then $x^2 - 2$ can be factored. For this reason we wish to spell out what <u>kind</u> of factors we want, not only in this chapter but also in Chapter 13.

In this chapter we want the factors of positive integers to be positive integers. We do not gain anything if we admit negative integers as factors of positive integers. If negative factors were admitted then 12 could be factored in each of the following ways: $2^2 \cdot 3$, $(-2)^2 \cdot 3$, $2(-2)(-3)$.

The idea of what is needed to "generate" a set of numbers is interesting and very basic. It also prepares the student to understand the famed Sieve of Eratosthenes which, we hope, will be constructed by every student as Problem 4 of Problem Set 11-2a. As an optional exercise the student might be asked to use the prime numbers 2, 3, 5, and 7 and generate by multiplication as many integers from 2 to 100 as possible. For example:

$4 = 2 \times 2$	$10 = 2 \times 5$	$16 = 2 \times 2 \times 2 \times 2$
$6 = 2 \times 3$	$12 = 2 \times 2 \times 3$	$18 = 2 \times 3 \times 3$
$8 = 2 \times 2 \times 2$	$14 = 2 \times 7$	$20 = 2 \times 2 \times 5$
$9 = 3 \times 3$	$15 = 3 \times 5$	$21 = 3 \times 7$

The set of numbers generated includes all the integers from 2 to 100 that are not prime.

Answers to Problem Set 11-2a; page 465:

1. 3 and addition generate the set {3, 6, 9, 12, ...}, the set of all multiples of 3 greater than 0. Each element of the set except the first has 3 as a proper factor.
2. The set of all positive integers with proper factor 5 could be generated by the number 5 and addition. The set would be {10, 15, 20, ...}.
3. 7 and addition will generate the set of all positive integers with factor 7. The set is {7, 14, 21, 28, ...}.
4. Every student should complete this sieve and have it for reference in his notebook.

1	2	3	~~4~~$_2$	5	~~6~~$_2$	7	~~8~~$_2$	~~9~~$_3$	~~10~~$_2$
11	~~12~~$_2$	13	~~14~~$_2$	~~15~~$_3$	~~16~~$_2$	17	~~18~~$_2$	19	~~20~~$_2$
~~21~~$_3$	~~22~~$_2$	23	~~24~~$_2$	~~25~~$_5$	~~26~~$_2$	~~27~~$_3$	~~28~~$_2$	29	~~30~~$_2$
31	~~32~~$_2$	~~33~~$_3$	~~34~~$_2$	~~35~~$_5$	~~36~~$_2$	37	~~38~~$_2$	~~39~~$_3$	~~40~~$_2$
41	~~42~~$_2$	43	~~44~~$_2$	~~45~~$_3$	~~46~~$_2$	47	~~48~~$_2$	~~49~~$_7$	~~50~~$_2$
~~51~~$_3$	~~52~~$_2$	53	~~54~~$_2$	~~55~~$_5$	~~56~~$_2$	~~57~~$_3$	~~58~~$_2$	59	~~60~~$_2$
61	~~62~~$_2$	~~63~~$_3$	~~64~~$_2$	~~65~~$_5$	~~66~~$_2$	67	~~68~~$_2$	~~69~~$_3$	~~70~~$_2$
71	~~72~~$_2$	73	~~74~~$_2$	~~75~~$_3$	~~76~~$_2$	~~77~~$_7$	~~78~~$_2$	79	~~80~~$_2$
~~81~~$_3$	~~82~~$_2$	83	~~84~~$_2$	~~85~~$_5$	~~86~~$_2$	~~87~~$_3$	~~88~~$_2$	89	~~90~~$_2$
~~91~~$_7$	~~92~~$_2$	~~93~~$_3$	~~94~~$_2$	~~95~~$_5$	~~96~~$_2$	97	~~98~~$_2$	~~99~~$_3$	~~100~~$_2$

The student may be interested to notice that we were only able to carry out the procedure for 2, 3, 5, 7 since in this process (4 steps) all the multiples of the next prime 11 which are less than 100 have already been crossed out. We also notice that the first number we actually cross out when crossing out all numbers having n as a common factor is n^2. Take 5 for example. The multiples of 5 are $2 \cdot 5$, $3 \cdot 5$, $4 \cdot 5$, $5 \cdot 5$, ...

Now $2 \cdot 5$ and $4 \cdot 5$ are already crossed out because each has 2 as a proper factor. The multiple $3 \cdot 5$ is already crossed out because it has 3 as a proper factor. Thus the first multiple crossed out when crossing out multiples of 5 is $5 \cdot 5$ or 5^2.

To further explain why there are no numbers less than or equal to 100 to cross out when crossing out 11's, list the multiples of 11 as:

$2 \cdot 11$, $3 \cdot 11$, $4 \cdot 11$, $5 \cdot 11$, $6 \cdot 11$, $7 \cdot 11$, $8 \cdot 11$, $9 \cdot 11$, $10 \cdot 11$, $11 \cdot 11$

Then go through the crossing-out process again to show that the first number crossed out would have to be $11 \cdot 11$.

We are getting at the prime factorization of integers in this section. It is essential that the student learn to find the prime factors of integers because we use this idea for reducing fractions, finding the lowest common denominator of fractions, and simplifying radicals. The Sieve of Eratosthenes provides a very natural way to obtain prime factors.

Problem Set 11-2b provides practice in finding the prime decompositions of integers. If the student can write these without using the method developed here, so much the better. There is no particular reason why he <u>must</u> begin with the smallest prime and then use successively larger primes, but this procedure is systematic and most students need encouragement along these lines. The advantages of the method should be emphasized, but the teacher should not <u>insist</u> on its use. If students use exponents to express their answers, so much the better: but if the students do not demand exponents the teacher should avoid them at this point. The long expressions obtained become motivation for exponents in a later section.

Assuming that practically all students will use the systematic approach to prime factorization, be sure of the following:

1. The student should use rules of divisibility for at least 2, 3, and 5 (11 if taught).
2. The student should know where he can stop.

Example: $2\overline{)202}$ → 101

$1 + 0 + 1$ is not divisible by 3

101 is not divisible by 5

$\frac{101}{7} = 14\frac{3}{7}$

After trying to divide by 7, you are through, since 101 is less than 121, which is the first number crossed out by 11.

One other way to determine the "stopping point" is to observe that if in factoring n we have tried all the prime numbers up to $\sqrt{n}$ then any further attempted divisions will result in quotients that are numbers we have already tried so that we have "tried all possibilities". This is especially valuable when we are trying to decide if a number is prime.

The students may be interested in this but don't expect many of them to learn it, nor should much time, if any, be spent in class on the subject.

The proof of this "unique factorization" theorem is far beyond anything which the students can understand at the present time. An aid in convincing the student, perhaps, would be to begin with a number like 72, and factor it in various ways: 24 × 3, 9 × 8, 2 × 2 × 3 × 2 × 3, 6 × 12, and then to reduce 24 × 3 and 9 × 8 and 6 × 12 further until only prime factors remain. You will get three 2's and two 3's no matter how you do it (not always in the same order, of course, but the associative and commutative laws let you put them in order). The fact is that 72 is "made up" of three 2's and two 3's and this result is obtained no matter how you begin and carry out your factoring of 72. If you are interested, a formal proof of the unique factorization property of the positive integers may be found in Courant and Robbins, <u>What is Mathematics</u>, Oxford, 1941, page 23.

<u>Answers to Oral Exercises 11-2b</u>: pages 468-469:

1. (a) 4 = 2 × 2
   (b) 6 = 2 × 3
   (c) 15 = 3 × 5
   (d) 18 = 2 × 3 × 3
   (e) 35 = 7 × 5
   (f) 28 = 2 × 2 × 7
   (g) 27 = 3 × 3 × 3
   (h) 99 = 3 × 3 × 11
   (i) 46 = 2 × 23
   (j) 51 = 3 × 17
   (k) 24 = 2 × 2 × 2 × 3
   (l) 91 = 7 × 13
   (m) 48 = 2 × 2 × 2 × 2 × 3
   (n) 26 = 2 × 13
   (o) 20 = 2 × 2 × 5
   (p) 75 = 3 × 5 × 5
   (q) 66 = 2 × 3 × 11
   (r) 108 = 2 × 2 × 3 × 3 × 3
   (s) 144 = 2 × 2 × 2 × 2 × 3 × 3
   (t) 225 = 3 × 3 × 5 × 5

2. 101

3. (a) 5
   (b) 35
   (c) 21
   (d) 98
   (e) 54
   (f) 539
   (g) 385
   (h) 45
   (i) 105
   (j) 247 [13(20 - 1) = 260 - 13]

Answers to Problem Set 11-2b; page 469:

1. (a) 49 = 7 × 7 not prime
   (b) 53 prime
   (c) 37 prime
   (d) 105 = 3 × 5 × 7 not prime
   (e) 111 = 3 × 37 not prime
   (f) 121 = 11 × 11 not prime
   (g) 97 prime
   (h) 10101 = 3 × 3367 not prime
   (i) 9999 = 3 × 3 × 1111 not prime

2. 2 is the <u>only</u> even prime.

*3. There is no largest odd prime number; that is, there are infinitely many odd primes. The proof credited to Euclid is a simple one and the teacher should look it up if he is not familiar with it.

4. (a) 5 × 13 = 65
   (b) 3 × 17 = 51
   (c) 7 × 13 = 91
   (d) 2 × 3 × 13 = 78
   (e) 2 × 3 × 17 = 102
   (f) 132 = 2 × 2 × 3 × 11
   (g) 5 × 29 = 145
   (h) 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 = 256
   (i) 3 × 3 × 3 × 3 × 3 = 243
   (j) 5 × 5 × 5 × 5 = 625
   (k) 2 × 2 × 2 × 2 × 3 × 23 = 1104
   (l) 2 × 2 × 3 × 61 = 732
   (m) 3 × 7 × 13 × 37 = 10101
   (n) 999 = 3 × 3 × 3 × 37

5. (a) 121
   (b) 143
   (c) 209
   (d) 133
   (e) 161
   (f) 231
   (g) 741
   (h) 81
   (i) 2945

Note: Rather than make these routine multiplication problems, discuss and encourage the use of properties of addition and multiplication by which some of these can be done mentally.

For example, (i) can be done as

$$5 \times 31(20 - 1) = 5 \times (620 - 31) = 5(600 - 11)$$
$$= 3000 - 55 = 2945$$

The relationship between prime factorization and divisibility is of particular importance in relation to the process of finding the least common multiple. There is often a bit of difficulty in verbalizing the requirement that all prime factors of the divisor "must be contained in the dividend at least as many times as each appears in the divisor". It is hoped that exponents can offer a partial assist here.

Answers to Oral Exercises 11-2c; page 473:

1. (a) 25 (b) 36 (c) 8 (d) 27 (e) 49 (f) 1000 (g) 144 (h) 125 (i) 16 (j) 243 (k) 32 (l) 625

2. (a) 5 (b) 3 (c) 3 (d) 2 (e) 7 (f) 11 (g) 3 (h) 5 (i) 7 (j) 2 (k) 7 (l) 3

3. (a) $20 = 2^2 \cdot 5$
(b) $16 = 2^4$
(c) $49 = 7^2$
(d) $100 = 2^2 \cdot 5^2$
(e) $75 = 3 \cdot 5^2$
(f) $18 = 2 \cdot 3^2$
(g) $144 = 2^4 \cdot 3^2$
(h) $68 = 2^2 \cdot 17$
(i) $50 = 2 \cdot 5^2$
(j) $27 = 3^3$

4. (a) True (b) False (c) False (d) True (e) False (f) False

Answers to Problem Set 11-2c; pages 473-475:

1. (a) 2^3 (b) 2^4 (c) 7^2 (d) 5^3 (e) 3^4 (f) 2^5 (g) 3^3 (h) 3^5 (i) 11^2 (j) 13^2 (k) 5^4 (l) 7^3

2. (a) $2 \cdot 7^2$ (b) $2^4 \cdot 3^3$ (c) $2 \cdot 3 \cdot 43$ (d) $2^2 \cdot 3^2 \cdot 5$ (e) 2^{10} (f) $2 \cdot 3^3 \cdot 7$ (g) 3^6 (h) $3 \cdot 5^2 \cdot 11$ (i) $2^6 \cdot 3^2$ (j) $2 \cdot 3^2 \cdot 61$ (k) $2 \cdot 3^5$ (l) $3^3 \cdot 5^3$

3. (a) True (b) False (c) True (d) False (e) False (f) True

4. (a) $2^2 \cdot 3^2$ divides $2^2 \cdot 3^2 \cdot 11$
   (b) $2 \cdot 3 \cdot 5$ does not divide $5^3 \cdot 7$
   (c) 2^4 divides $2^6 \cdot 3$
   (d) 3^3a^2b does not divide 3^4ab^2 unless $a = b$ or $a = 3$.
   (e) $2^2 \cdot 3^2 \cdot 5a^3b$ divides $2^2 3^2 5 \cdot 7a^4b^2$.
   (f) $5^2 \cdot 7$ does not divide $2^2 \cdot 3 \cdot 7 \cdot 1933c$

5.

(a)	32	(i)	225	(q)	135
(b)	16	(j)	225	(r)	25
(c)	343	(k)	12	(s)	13
(d)	27	(l)	9	(t)	25
(e)	128	(m)	48	(u)	-5
(f)	243	(n)	1296	(v)	-5
(g)	125	(o)	3375	(w)	1
(h)	32	(p)	3375	(x)	1

pages 475-478: 11-2 and 11-3

6. 25 square inches

7. 125 cubic inches

8. 5^2 suggests the area of a square - hence "5 squared".

   5^3 suggests the volume of a cube - hence "5 cubed".

11-3. Factors and Sums.

Another application of prime factorization of integers is presented in this section. Find two factors of 72 with the property that their sum is 22. This might seem like a game to the student, but we have a serious purpose. This is the kind of thinking which is done in factoring the quadratic polynomial $x^2 + 22x + 72$.

Perhaps classwork with this material should begin with very simple examples for which the students might give the answers orally. Here are some easy examples:

Product	Sum	Factors
3	4	3 and 1
8	6	4 and 2
4	4	2 and 2
9	10	9 and 1

The first and fourth examples point up the need for the factor 1 in this particular context.

Answers to Problem Set 11-3; pages 478-479:

1. In addition to the pairs of factors listed in the example of this section, the following are possibilities:

   (a) $2 \cdot 2 \cdot 3^2$ and 2

   (b) 2^3 and 3^2

   (c) $2^3 \cdot 3$ and 3. The sum of these numbers is 27.

2. (a) $2 \cdot 3^2$: 3 and 6 (g) 20 and 10
   (b) $2 \cdot 3^2$; 9 and 2 (h) 15 and 15
   (c) 2^5: 8 and 4 (i) Not possible
   (d) $1 \cdot 7^2$: 49 and 1 (j) 18 and 12
   (e) 9 and 6 *(k) 243 and 4
   (f) $2 \cdot 7$: 14 and 1 *(l) 15 and -12

3. (a) $48 = 2^4 \cdot 3$: 2^3 and $2 \cdot 3$
   (b) $2^3 \cdot 3$ and 2
   (c) Not possible
   (d) 2^4 and 3
   (e) $150 = 2 \cdot 3 \cdot 5^2$: Not possible
   (f) $2 \cdot 5$ and $3 \cdot 5$
   (g) 5^2 and $2 \cdot 3$
   (h) Not possible
   (i) 150 and 1
   (j) $288 = 2^5 \cdot 3^2$: 2^4 and $2 \cdot 3^2$
   (k) $330 = 2 \cdot 3 \cdot 5 \cdot 11$: $2 \cdot 11$ and $3 \cdot 5$

4. If ℓ and w represent the measures of the length and the width, $\ell \cdot w = 225$ and $\ell + w = 34$. $225 = 3^2 \cdot 5^2$
   The length and the width are 25 feet and 9 feet.

5. $\ell \cdot w = 54$ and $\ell + w = 29$. $54 = 2 \cdot 3^3$
   The length and the width are 27 feet and 2 feet.

6. If b and h represent the measures of the base and the height, $b \cdot h = 36$ and $b + h = 15$. $36 = 2^2 \cdot 3^2$
   The base and the height are 12 inches and 3 inches.

7. $b \cdot a = 600$ and $b + a = 62$. $600 = 2^3 \cdot 3 \cdot 5^2$

8. (a) 24 and 2 (c) There are none. $60 = 2^2 \cdot 3 \cdot 5$
   (b) 15 and 4 (d) 28 and 25 $700 = 2^2 \cdot 5^2 \cdot 7$

11-4. Laws of Exponents.

The first part of the section involving multiplication should not present any difficulties to the student.

In the definition

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ factors}},$$

be careful to say, "a used as a factor n times," not "a multiplied by itself n times". Consider the meaning of each of these phrases: "Two used as a factor 3 times means $2 \cdot 2 \cdot 2$, which is eight. Two multiplied by itself 3 times means

$2 \cdot 2 = 4$, 2 multiplied by 2 once,

$4 \cdot 2 = 8$, multiplied by 2 twice,

$8 \cdot 2 = 16$, multiplied by 2 three times, or 2^4.

Clearly, saying "multiplied by itself" can lead to confusion.

In the second part we want to show that rules for simplifying fractions containing powers can be generalized, but you need not emphasize these rules. The student should be encouraged to think on the basis of the definition. For example:

$$\frac{xy^3}{x^2y^2}$$

The student should reason, "There is one x factor in the numerator and two in the denominator: this is the same as just one x factor in the denominator because $\frac{x}{x} = 1$. There are three y factors in the numerator and two y factors in the denominator: this is the same as one y factor in the numerator, since $\frac{y^2}{y^2} = 1$. Avoid the use of the word "cancel," but don't treat the words as a "dirty word" should it come from a student. Just require the student to explain his use of the word in terms of the theorem, $\frac{a}{a} = 1$, $(a \neq 0)$.

In applying the so-called laws of exponents, students frequently encounter more difficulties in working with numerical expressions than in dealing with variables. For example, a student may have no trouble recognizing the fact that $a^2 \cdot a^3 = a^5$ or that $a^2 \cdot b^2 = (ab)^2$. On the other hand he might be strongly tempted to write $2^2 \cdot 2^3 = 4^5$ or $2^2 \cdot 3^2 = 6^4$. It has also been observed that a student will accept the fact that $a^2 + b^2$ permits of no further simplification and at the same time will be inclined to write $3^2 + 4^2 = 7^2$. Consequently great care must be taken to ensure a student's understanding of the significance of the <u>base</u> in all manipulations with exponents.

<u>Answers to Oral Exercises 11-4a</u>: page 481:

1. (a) 2^7 (f) a^8 (k) x^{a+b}
   (b) 3^5 (g) m^5 (l) 2^{m+n}
   (c) a^{10} (h) c^4 (m) a^{x+y}
   (d) 10^7 (i) x^3 (n) x^{2+3a}
   (e) 4^3 (j) y^9 (o) $2^3 \cdot a^3 \cdot b^3$

2. (a) False (e) False (i) True
   (b) False (f) False (j) False
   (c) True (g) False (k) False
   (d) False (h) True (l) True

3. (a) not true for all values
   (b) not true for all values
   (c) true for all values
   (d) true for all values

   It would seem unwise to generalize at this stage from $(a^2)^3 = a^6$ to $(a^m)^n = a^{mn}$. Get the student to see that $(a^2)^3 = a^2 \cdot a^2 \cdot a^2$. From this he will easily get a^6. On the other hand $a^2 \cdot a^3 = a^5$.

<u>Answers to Problem Set 11-4a</u>: page 482:

1. (a) $2^8 = 256$
   (b) m^9
   (c) $3^6 = 729$
   (d) a^6
   (e) m^{14}
   (f) x^{12}
   (g) $10^8 = 100,000,000$
   (h) $9^3 = 729$
   (i) x^4
   (j) a^{41}

2. (a) 2^4x^3
   (b) 3^7a^4
   (c) 3^7a^4 or $2187\ a^4$
   (d) x^{3a}
   (e) 3^6 or 729
   (f) 3^42^3 or 648
   (g) $3^3a^3b^5$ or $27\ a^3b^5$
   (h) $(3kmt)^2$ or $3^2k^2m^2t^2$

3. (a) 2^9a^6
   (b) $3^5 \cdot 7^2ab^2$
   (c) $3^5x^3y^4$
   (d) $2 \cdot 17^2ab^2$
   (e) $2^4 \cdot 3^3 \cdot m^4n^2p$
   (f) $2^7 \cdot 3^2a^6b^3c$
   (g) 3^6x^4
   (h) $3^3a^3b^3$
   (i) $2^2a^2b^2$

<u>Answers to Oral Exercises 11-4b</u>; page 486:

1. (a) m^3
   (b) $\frac{3x}{2}$
   (c) a^9
   (d) $3ax^4$
   (e) $\frac{1}{a^9}$
   (f) $\frac{a}{y^2}$
   (g) $\frac{3a}{4m}$
   (h) 2^2a
   (i) 3^3y
   (j) $\frac{5}{2am^2}$
   (k) $\frac{3x^2}{y^2}$
   (l) $\frac{7^3}{2 \cdot 3 \cdot 5^2}$
   (m) $\frac{5m^2}{2^3x}$

2. (a) False
   (b) False
   (c) True
   (d) False
   (e) True

3. Only (c) is true for all values of the variables.

4. Division by zero has no meaning, since 0 has no reciprocal.

Answers to Problem Set 11-4b; pages 487-488:

1. m^8
2. $\frac{x^2}{2^2} = \frac{x^2}{4}$
3. $\frac{5^2}{x^3} = \frac{25}{x^3}$
4. a^5b^4
5. $a^8b^4c^5$
6. $\frac{3^2b^2}{2a^3} = \frac{9b^2}{2a^3}$
7. $\frac{2 \cdot 3}{x^4y^3} = \frac{6}{x^4y^3}$
8. $\frac{2m^8}{3t^8}$
9. $\frac{81}{16a^3x^7}$
10. $\frac{11b^2c^2}{8y^2}$
11. $\frac{3c^2}{2hby}$
12. $\frac{7a^2c}{2b}$
13. $-\frac{10cd^2}{y^2}$
14. $\frac{y^3z}{2x}$
15. $\frac{3a}{2b}$
16. $\frac{8}{9ab}$
17. $-b^2$
18. $\frac{64m}{97n}$
19. $\frac{-1}{32m^2}$
20. $\frac{1}{32m^2}$
21. x^a
22. $\frac{7a^2}{5}$
23. mn^5
24. $\frac{3^2 \cdot 5}{2^3} = \frac{45}{8}$

*25. 1

*26. $\frac{135}{32a^2b}$

*27. $\frac{2z}{25x}$

*28. $\frac{a + 2b}{ab}$

29. (a) True $(\frac{2}{3})^2 = \frac{2}{3} \cdot \frac{2}{3}$ by the definition of an exponent

$$= \frac{2^2}{3^2}$$

(b) Not true, since $\frac{2^2}{3^2} = \frac{2}{3} \cdot \frac{2}{3}$ while $\frac{2}{3} = \frac{2}{3} \cdot 1$

(c) Not true, 152 is not a multiple of 27. (Notice that 3^3 is a factor of 3^3 but not of 5^3.)

(d) True. $6^2 + 9^2 = 2^2 \cdot 3^2 + 3^2 \cdot 3^2$

$= 3^2(2^2 + 3^2)$ by the distributive property

(e) True. $2x^2 + 4y^2 = 2(x^2 + 2y^2)$

$2(x^2 + 2y^2)$ is an even number, since

$x^2 + 2y^2$ is an integer

__Answers to Problem Set 11-4c; pages 490-491:__

1. a^{10}
2. d^9
3. a^4b^8
4. x^6y^3
5. $4a^2m^6$
6. $27c^9d^6$
7. $25a^4$
8. $27x^{10}$
9. r^4s^8
10. $(a^2)^3 = a^6$
11. $\frac{6}{m}$
12. $\frac{1}{15}$
13. $\frac{3a}{4b^2}$
14. a^2
15. $-\frac{a^2}{3}$
16. $-2x^2yz$
17. a^4c
18. $-\frac{a^2}{k}$
19. $\frac{b}{a}$
20. $\frac{3b^3}{2}$

Be sure to have students simplify fractions before substituting numerical values, in problems 21-28.

21. $-2(2)^2(-2)^2(3)^2 = -288$
22. $((-2)(2)(-2)(3))^2 = 576$

23. $\frac{-2ad}{3b^2}$; $\frac{(-2)(2)(-3)}{3(-2)^2} = 1$

24. $\frac{bc^2}{4a}$; $\frac{(-2)(3)^2}{4(2)} = -\frac{9}{4}$

25. $\frac{1}{(ab)^3}$; $\frac{1}{((2)(-2))^3} = -\frac{1}{64}$

*26. $\frac{(2 + (-2))^3}{2^3 + (-2)^3} = \frac{0^3}{0}$ Since division by zero is not permitted, the expression does not name a number when a is 2 and b is -2.

*27. $\frac{-3a^2}{b^2c^2}$; $-\frac{1}{3}$

*28. $\frac{9}{17}$

11-5. Adding and Subtracting Fractions.

We wish to apply the prime factorization of integers to the problem of finding the least common multiple of the denominators. We do not want blind adherence to the method developed, however, but we do want to give the student a systematic way of approaching the problem. For example, if the student were asked to add the fractions $\frac{1}{2} + \frac{1}{6}$, the lowest common denominator can be quickly determined by inspection, and the student should do it this way. But if he is asked to add $\frac{1}{57} + \frac{1}{95}$, it may not be easy to determine the lowest common denominator by inspection. But by prime factorization

$$57 = 3 \cdot 19,$$

$$95 = 5 \cdot 19,$$

and the lowest common denominator is $5 \cdot 3 \cdot 19$.

It is good technique, both here and in later work on factoring, to leave expressions in factored form as long as possible, for these factors indicate structure of the expression, which may otherwise be obscurred In the above example, after the fractions had a common denominator, the numerators were, of course, multiplied out and combined; but, as you saw, it was to our advantage to leave the denominator in factored form until the very end. Then we know that the fraction cannot be reduced unless the numerator has as a factor one of the factors of the L.C.D.

Answers to Oral Exercises 11-5a; page 493:

1. Multiples of 4: 4, 8, 12, 16, 20
   Multiples of 5: 5, 10, 15, 20, 25
   Least common multiple of 4 and 5 is 20.

2. Multiples of 2: 2, 4, 6, 8, 10
   of 10: 10, 20, 30, 40, 50
   Least common multiple of 2 and 10 is 10.

3. Multiples of 9: 9, 18, 27, 36, 45
   of 12: 12, 24, 36, 48, 60
   Least common multiple of 9 and 12 is 36.

4. Multiples of 3: 3, 6, 9, 12, 15
   of 5: 5, 10, 15, 20, 25
   of 6: 6, 12, 18, 24, 30
   Least common multiple of 3, 5, and 6 is not named.

5. Multiples of 2: 2, 4, 6, 8, 10
   of 7: 7, 14, 21, 28, 35
   of 12: 12, 24, 36, 48, 60
   Least common multiple of 2, 7, and 12 is not named.

6. Multiples of 2: 2, 4, 6, 8, 10
   of 6: 6, 12, 18, 24, 30
   of 11: 11, 22, 33, 44, 55
   Least common multiple of 2, 6, and 11 is not named.

7. Multiples of 5: 5, 10, 15, 20, 25
   of 10: 10, 20, 30, 40, 50
   of 15: 15, 30, 45, 60, 75
   Least common multiple of 5, 10, and 15 is not named.

8. Multiples of a: $a, 2a, 3a, 4a, 5a$
   of $3a$: $3a, 6a, 9a, 12a, 15a$
   Least common multiple of a and $3a$ is $3a$.

9. Multiples of $2a$: $2a, 4a, 6a, 8a, 10a$
   of $3a$: $3a, 6a, 9a, 12a, 15a$
   Least common multiple of $2a$ and $3a$ is $6a$.

10. Multiples of x^2: $x^2, 2x^2, 3x^2, 4x^2, 5x^2$
    of $2x^2$: $2x^2, 4x^2, 6x^2, 8x^2, 10x^2$
    Least common multiple of x^2 and $2x^2$ is $2x^2$.

Answers to Problem Set 11-5a; page 493:

1. Positive multiples of 3: {3, 6, 9, 12, 15,···}
   of 5: {5, 10, 15,···}
   Least common multiple of 3 and 5 is 15.

2. Positive multiples of 4: {4, 8, 12, 16, 20, 24,···}
   6: {6, 12, 18, 24,...}
   8: {8, 16, 24,...}.
   Least common multiple of 4, 6, and 8 is 24.

3. Positive multiples of 12: {12, 24, 36, 48, 60,···}
   of 15: {15, 30, 45, 60,···}
   Least common multiple of 12 and 15 is 60.

4. Positive multiples of 10: {10, 20, 30, 40 60,..., 150,...}
   of 15: {15, 30, 45, 60, , 150,...}
   of 25: {25, 50, 75, 100,..., 150,...}
   Least common multiple of 10, 15, and 25 is 150, but several more multiples of each of the numbers must be written before we arrive at a common multiple by this method. Experience with exercises of this type should help make the student receptive to the suggestion of a more efficient method for finding the least common multiple of several numbers.

5. Positive multiples of 2: {2, 4, 6, 8, 10, ···, 100, ···}
of 10: {10, 20, 30, 40, 50, ···, 100, ···}
of 25: {25, 50, 75, 100, ···}
Least common multiple of 2, 10, and 25 is **50**.

6. Positive multiples of 4: {4, 8, 12, 16, 20, ··· , 180, ···}
of 18: {18, 36, 54, ··· , 180, ···}
of 20: {20, 40, 60, 80, ··· , 180, ···}
Least common multiple of 4, 18, and 20 is 180.
The comment for question 4 applies here also.

7. ~~Positive multiples of~~ 30: {30, 60, 90, 120, 150, 180, ···}
of 36: {36, 72, 108, 144, 180, ···}
Least common multiple of 30 and 36 is 180.

8. Multiples of 2a: {2a, 4a, 6a, 8a, ···}
of 8a: {8a, 16a, 24a, ···}
Least common multiple of 2a and 8a is 8a.

9. Multiples of 2x: {2x, 4x, 6x, ··· , 30x, ···}
of 5x: {5x, 10x, 15x, 20x, 25x, 30x, ···}
of 6x: {6x, 12x, 18x, 24x, 30x, ···}
Least common multiple of 2x, 5x, and 6x is 30x.

10. Multiples of 2x: {2x, 4x, 6x, ···}
of 3y: {3y, 6y, 9y, ···}
The least common multiple in this case depends on what numbers x and y represent.

Answers to Oral Exercises 11-5b; page 496:

1. $2 \cdot 3 \cdot 5$
2. $2^2 \cdot 3^2 \cdot 5^2 \cdot 7$
3. $2^2 \cdot 3^2 \cdot 7^2$
4. $2^2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13$
5. $2^2 \cdot 3$
6. $2^2 \cdot 3$
7. $2^2 \cdot 3^2$
8. $3 \cdot 5^2$
9. $2^2 \cdot 7^2$
10. $2 \cdot 3 \cdot 5^2 \cdot 7^2$
11. $2^3 \cdot 3^2 \cdot 5^2$

Answers to Problem Set 11-5b; page 496:

1. $2^2 \cdot 5^2$
2. $2^4 \cdot 3$
3. $2 \cdot 3^2 \cdot 7$
4. $2^3 \cdot 3 \cdot 5$
5. $2^4 \cdot 5$
6. $2^2 \cdot 3 \cdot 5^2$
7. $2^4 \cdot 3^3$
8. $2^3 \cdot 3 \cdot 11 \cdot 17$
9. $2^2 \cdot 5 \cdot 13$
10. $2 \cdot 3 \cdot x^2$

Answers to Oral Exercises 11-5c; pages 499-500:

1. (a) 36 (b) 50 (c) 52 (d) 88 (e) 120

2. (a) 30 (b) 132 (c) 28 (d) 24 (e) 120

3. (a) $\frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 5}$; new numerator $2 \cdot 3^2 \cdot 5$

   (b) $\frac{11}{11}$; new numerator 11

   (c) $\frac{2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 5}$; $2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$

   (d) $\frac{5 \cdot 7}{5 \cdot 7}$; $5 \cdot 7$

   (e) $\frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 5}$; $2 \cdot 3 \cdot 3 \cdot 5$

   (f) $\frac{2 \cdot 2}{2 \cdot 2}$; $a \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

Answers to Problem Set 11-5c; page 500:

1. $\frac{29}{18}$
2. $\frac{5}{32}$
3. $\frac{77}{180}$
4. $\frac{68}{65}$

5. It is better to reduce $\frac{3}{51}$ first.

$$\frac{1}{85} + \frac{1}{17} = \frac{1}{85} + \frac{5}{85}$$
$$= \frac{6}{85}.$$

6. $\frac{47}{225}$

7. $\frac{49}{180}$

*8. $-\frac{61}{114}$

*9. $\frac{8}{39}$

*10. $\frac{25}{2 \cdot 5 \cdot 17} + \frac{22}{2 \cdot 5 \cdot 17} - \frac{119}{2 \cdot 5 \cdot 17} = -\frac{72}{2 \cdot 5 \cdot 17}$
$$= -\frac{36}{85}$$

Answers to Problem Set 11-5d; pages 501-502:

1. (a) $\frac{x}{20} + \frac{5}{24} = \frac{6x + 25}{2^3 \cdot 3 \cdot 5}$
$$= \frac{6x + 25}{120}$$

(b) $\frac{37x}{72}$

(c) $-\frac{55a}{216}$

(d) $\frac{22m + 75}{100}$

(e) $\frac{157c}{225}$

(f) $\frac{196n^2 + 7n - 6}{196}$

(g) $\frac{35 + 28d + 50d}{2^2 \cdot 5 \cdot 7} = \frac{35 + 78d}{140}$

(h) $\frac{99k}{280}$

(i) $\frac{41x}{72}$

*(j) $\frac{6x + 121a}{132}$

*(k) $\frac{967a}{1575}$

*(l) $\frac{70x + 175x - 44 + 42}{2^3 \cdot 5 \cdot 7}$
$$= \frac{245x - 2}{280}$$

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2. (a) True $\frac{8}{15} = \frac{64}{2^3 \cdot 3 \cdot 5}$, $\frac{13}{24} = \frac{65}{2^3 \cdot 3 \cdot 5}$

(b) False

(c) False

(d) True

(e) True $\frac{26}{32} = \frac{429}{2^4 \cdot 3 \cdot 11}$, $\frac{28}{33} = \frac{448}{2^4 \cdot 3 \cdot 11}$

3. (a) $(\frac{1}{2} - \frac{1}{3}) > \frac{1}{7}$ since $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(b) $\frac{4}{15} = \frac{36}{3^3 \cdot 5}$

$\frac{7}{27} = \frac{35}{3^3 \cdot 5}$, so $\frac{4}{15} > \frac{7}{27}$.

(c) $\frac{5}{12} > \frac{5}{13}$ In fact, $\frac{a}{b} > \frac{a}{b+1}$ for every positive a and b.

(d) Both fractions name the same number.

(e) Neither is greater since $(-\frac{3}{24})$ and $(-\frac{5}{40})$ are both names for $(-\frac{1}{8})$.

(f) $\frac{10}{16} > (-\frac{5}{8})$. A positive number is always greater than a negative number.

(g) $.499 > .49$

(h) $(-.0009) > (-.009)$

(i) $.900 > .09000$

(j) $\frac{15}{30} > .49$ since $\frac{15}{30} = \frac{50}{100}$ and $.49 = \frac{49}{100}$

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4. (a) $\frac{6}{27} > \frac{5}{27}$, $\frac{5}{27} > \frac{5}{28}$;

thus $\frac{6}{27} > \frac{5}{28}$ (transitive property of order)

(b) $\frac{2}{3} < \frac{5}{7}$, since $\frac{2}{3} = \frac{14}{3 \cdot 7}$ and $\frac{5}{7} = \frac{15}{3 \cdot 7}$

(c) $\frac{6}{16} = \frac{9}{24}$

(d) $\frac{1}{2} + \frac{1}{3} = \frac{130}{12 \cdot 13}$

$\frac{11}{12} - \frac{1}{13} = \frac{131}{12 \cdot 13}$

Thus $\frac{1}{2} + \frac{1}{3} < \frac{11}{12} - \frac{1}{13}$

5. $200 + \frac{1}{12}(600) > \frac{1}{3}(600)$

$200 + \frac{1}{12}(700) > \frac{1}{12}(700)$

In each of the above cases \$200 plus $\frac{1}{12}$ of sales is better.

$200 + \frac{1}{12}(1000) < \frac{1}{3}(1000)$

Here $\frac{1}{3}$ of sales is better.

Let s be the sales in dollars for which the offers are equal. Then s satisfies $200 + \frac{s}{12} = \frac{s}{3}$. This sentence is equivalent to the sentence

$$s = 800.$$

The offers are equal if his sales amount to \$800.

*6. There is no algebraic approach. What we want is a number which has the largest possible proper factor. By inspection we decide $2 \times 47 = 94$ will do, because the next larger prime is 53, and 2×53 is too large. Thus, John and Bob could ask for $94 + 47 = 141$ cents.

Answers to Review Problem Set; pages 504-507:

1. (a) 2, 3, 4, 8, 9, 12, 18, 24, etc.
   (b) 2, 3, 4, 6, 12
   (c) 3, 9
   (d) 2, 5, 10, 20, 40, 3, 15, etc.
   (e) 3
   (f) 5, 25, 3

   These are some of the more likely responses.

2. (a) $2^2 \cdot 3^2 \cdot 5$
   (b) $2 \cdot 3^3 \cdot 7$
   (c) $3 \cdot 5^2 \cdot 41$
   (d) $2^3 \cdot 3^2 \cdot 11$
   (e) $2^4 \cdot 3 \cdot 7 \cdot 19$

3. (a) Yes, since 9 and 4 are factors
   (b) Yes, since 75 is divisible by 25
   (c) Yes, since 3 and 5 are factors
   (d) Yes, since 27 is a factor of 81000 and of 729

4. (a) b^6
   (b) x^4m^3
   (c) n^2
   (d) $\frac{1}{y}$
   (e) $3^4a^7b^3x$
   (f) $\frac{3a^4}{x^2}$
   (g) $\frac{1}{7mn}$
   (h) $x^2 + 4$
   (i) $\frac{(x + y)^2}{x - y}$
   (j) $\frac{3x(a + b)^3}{5a^4}$
   (k) $(-2)^5a^5$ or $-32a^5$

5. (a) $\frac{77}{120}$
   (b) $\frac{x}{9}$
   (c) $\frac{35ax + 14bx + 15cx}{210}$
   (d) $\frac{84m - 35n - 8c}{140}$
   (e) $\frac{849y + x}{120}$

6. (a) True
   (b) False
   (c) True
   (d) False
   (e) True

7. The sentence in part (b) is the only one which is true for all values of the variables.

8. (a) $(x + y)(x - y)$, if the numbers are x and y
   (b) $\frac{x}{y} + 2xy$, or $\frac{y}{x} + 2xy$, if the numbers are x and y
   (c) $n + 7 = 3(2n)$, where n is the given number.
   (d) If the first speed is n miles per hour, then $2n = 3(n - 5)$.
   (e) If ℓ is the number of inches in length of the first piece, then $\ell + (2\ell + 5) = 76$.
   (f) If n is the number of pounds of solution and s is the number of pounds of salt, then $.10n = s$.
   (g) If the regular price was r dollars, then $r - .10r = 95$.
   (h) If n is the number of lbs. of candy at \$2 per lb., then $2(n) + 1(n + 5) = 23$.
   (i) If t is the number of hours walked, then $2t + 3t = 14$.

9. If the first even integer is n, then the next two are $(n + 2)$ and $(n + 4)$.

$$n + (n + 2) + (n + 4) = 5920$$
$$3n + 6 = 5920$$
$$3n = 5914$$
$$n = 1971\tfrac{1}{3}$$

There are no three consecutive even integers whose sum is 5920. The assumption that there were such integers led to a contradiction.

10. If n is the length in feet

$$n + n + 7\frac{1}{4} + 7\frac{1}{4} = 47,$$
$$2n + 14\frac{1}{2} = 47,$$
$$2n = 32\frac{1}{2},$$
$$n = 16\frac{1}{4}.$$

The length is $16\frac{1}{4}$ feet.

11. If x is the number of inches in the first side, then $2x$ is the number of inches in the second side and $\frac{3}{2}(2x)$ is the number of inches in the third side.

$$x + 2x + \frac{3}{2}(2x) = 87$$
$$x + 2x + 3x = 87$$
$$6x = 87$$
$$x = 14\frac{1}{2}$$

The lengths of the sides are $14\frac{1}{2}$ inches, 29 inches, and $43\frac{1}{2}$ inches.

12. (a) $\{1\}$
(b) All numbers greater than 12
(c) $\{3\}$
(d) $\{-\frac{10}{39}\}$
(e) All numbers greater than -3
(f) $\{4, -4\}$
(g) $\emptyset$
(h) $x - 2 = 2$ or $x - 2 = -2$
$x = 4$ or $x = 0$
$\{4, 0\}$
(i) $\{-5, 2\}$
(j) All real numbers

13. (a) 32 and 3
(b) 24 and 4
(c) 60 and 10
(d) 75 and 8
(e) 600 and 1
(f) Not possible

14. If ℓ is the number of feet in the length and w is the number of feet in the width, then

$$\ell w = 800 \quad \text{and} \quad \ell + w = 60.$$

The length and width must be 40 feet and 20 feet, respectively.

15. If b is the number of units in the base and a is the number of units in the altitude, then

$$ba = 104 \quad \text{and} \quad b + a = 21.$$

The base and altitude are 8 and 13 units.

Suggested Test Items

1. Which of the following are prime numbers: 2, 5, 9, 13, 39, 47?
2. Write the two smallest primes which are greater than 23.
3. Find the prime factorization of each of these numbers. Use exponents in the factored form.

   (a) 120 (d) 630
   (b) 171 (e) 3375
   (c) 64

4. The prime factorizations of numbers A, B, C, and D are given below.

   $A = 2^2 \cdot 3 \cdot 5$; $B = 2 \cdot 3^3 \cdot 5^2$; $C = 2^3 \cdot 3^2 \cdot 11$; $D = 3 \cdot 5$

   Write in factored form the least common multiple of

   (a) numbers A and B
   (b) numbers A and C
   (c) numbers A, B, and C
   (d) numbers B and D

5. Find a common name for each of these numbers.

(a) $2^3 \cdot 5^2$

(b) 3^5

(c) $2^2 \cdot 3^3$

(d) 10^3

(e) $(m^3)(m^4)$

(f) $(8ab^2c^3)(4b^3c)$

(g) $(2b^2)^3$

(h) $(5r)^3(r^2)^2$

6. For what values of a, b, and c are each of the following true?

(a) $(3c^2)(2c) = 6c^3$

(b) $(c^2)^3 = c^5$

(c) $\frac{a^5}{a^2} = a^3$

7. Write a simpler expression for each of the following. If the domain of any variable must be restricted, indicate this.

			Restriction?
(a)	$\frac{5a^2b^3}{2a^4b} =$	______	______
(b)	$\left(\frac{7m}{2n^2}\right)^2$	______	______
(c)	$\frac{2^4xy^5}{2x^3}$	______	______
(d)	$\frac{2x^3y}{8y^3}$	______	______
(e)	$\frac{a^2b^3c}{5a^3b^3c^2}$	______	______
(f)	$\frac{(2x)^3y}{4}$	______	______
(g)	$\frac{(2xy)^3}{8x^3y^4}$	______	______

8. Is it true that:

(a) $2^3 \cdot 3^3 = 6^3$? Explain why or why not?

(b) $2 \cdot 3^3 + 2^3 \cdot 3 = 2 \cdot 3(3^2 + 2^2)$? Explain.

9. Find two numbers which have the product p and the sum s for each of the following:

	p	s
(a)	36	13
(b)	96	20
(c)	72	27

10. Find a common name for each of the following expressions

(a) $\frac{w}{12} + \frac{5w}{8}$

(b) $\frac{3}{16} + \frac{4c}{15} - \frac{c}{12}$

(c) $\frac{7d}{25} - \frac{d}{10} + \frac{3d}{20}$

Answers to Suggested Test Items

1. 2, 5, 13, 47

2. 29, 31

3. (a) $2^3 \cdot 3 \cdot 5$

   (b) $3^2 \cdot 19$

   (c) 2^6

   (d) $2 \cdot 3^2 \cdot 5 \cdot 7$

   (e) $3^3 \cdot 5^3$

40

4. (a) $2^2 \cdot 3^3 \cdot 5^2$
   (b) $2^3 \cdot 3^2 \cdot 5 \cdot 11$
   (c) $2^3 \cdot 3^3 \cdot 5^2 \cdot 11$
   (d) $2 \cdot 3^3 \cdot 5^2$

5. (a) 200 (e) m^7
   (b) 243 (f) $32ab^5c^4$
   (c) 108 (g) $8b^6$
   (d) 1000 (h) $125r^7$

6. (a) all values
   (b) 0, 1
   (c) $a \neq 0$

7. (a) $\frac{5b^2}{2a^2}$, $a \neq 0$, $b \neq 0$
   (b) $\frac{49m^2}{4n^4}$, $n \neq 0$
   (c) $\frac{2^3y^5}{x^2}$, $x \neq 0$
   (d) $\frac{x^3}{4y^2}$, $y \neq 0$
   (e) $\frac{1}{5ac}$, $a \neq 0$, $b \neq 0$, $c \neq 0$
   (f) $2x^3y$, no restriction
   (g) $\frac{1}{y}$, $x \neq 0$, $y \neq 0$

8. (a) True. $2^3 \cdot 3^3 = 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3$

$$= 6 \cdot 6 \cdot 6$$

$$= 6^3$$

(b) True, by the distributive property

$$2 \cdot 3^3 + 2^3 \cdot 3 = 2 \cdot 3 \cdot 3^2 \cdot 2 \cdot 3 \cdot 2^2$$

$$= 2 \cdot 3(3^2 + 2^2)$$

9. (a) 9 and 4

(b) 12 and 8

(c) 24 and 3

10. (a) $\frac{17w}{24}$

(b) $\frac{45 + 44c}{2^4 \cdot 3 \cdot 5}$

(c) $\frac{33d}{2^2 \cdot 5^2}$

Chapter 12

RADICALS

Having prepared the way by factoring integers and studying exponents, we proceed to a study of radicals. Prime factorization and its uniqueness will play an important role in the development. Work on the simplification of radicals is based on two fundamental properties, for which simple proofs are given in the text.

The major thrust in the chapter is the evolvement of a theorem which enables us to recognize that there is a very large class of positive integers having square roots which are not rational numbers. We make an assumption that these are coordinates of points on the number line and therefore deduce the existence of real numbers which are irrational. The concept is extended informally to include cube roots and other n^{th} roots. The chapter closes with a reference to an even larger class of irrational real numbers besides those which are "roots". Even though the student may have encountered only one of these, perhaps π, this may give him a sense of the "complete" real number system.

The method of finding approximate square roots will be the iteration method, sometimes called the "divide and average" method. This consists of an initial estimate in the form of a positive integer and its subsequent improvement. It is possible to <u>prove</u>, although we do not do it in the text, that this method converges in the sense that each new approximation is better than the previous one.

12-1. Square Roots.

We approach the notion of finding square roots as the inverse of the operation of squaring. This is accomplished initially by recognition of known squares or by prime factorization. We note that if a non-zero number n has a square root a, then it also has a square root $-a$. To see that 10 and -10 are the only roots we need only to recall the following: If $0 < a < b$, then $a^2 < b^2$, and if $a < b < 0$ then $b^2 < a^2$. This same property may be reapplied in connection with cubes and n^{th} powers in Section 12-8.

Answers to Oral Exercises 12-1; pages 511-512:

1. (a) 9
(b) 16
(c) 25
(d) $\frac{1}{4}$
(e) $\frac{4}{9}$
(f) $\frac{1}{9}$
(g) 4
(h) 49
(i) 144

2. (a) 4, -4
(b) 7, -7
(c) 10, -10
(d) $\frac{2}{3}, -\frac{2}{3}$
(e) $\frac{1}{5}, -\frac{1}{5}$
(f) $\frac{3}{4}, -\frac{3}{4}$
(g) 8, -8
(h) 12, -12
(i) 11, -11

Answers to Problem Set 12-1; pages 512-513:

1. (a) 81
(b) 169
(c) $\frac{1}{9}$
(d) 196
(e) 625
(f) $\frac{9}{25}$

2. (a) $\frac{3}{5}, -\frac{3}{5}$
(b) 6, -6
(c) $\frac{9}{4}, -\frac{9}{4}$
(d) 13, -13
(e) 15, -15
(f) $\frac{1}{17}, -\frac{1}{17}$

3. (a) $\{25, -25\}$
(b) $\{18, -18\}$
(c) $\{\frac{7}{3}, -\frac{7}{3}\}$
(d) $\{\frac{1}{2}, -\frac{1}{2}\}$
(e) $\{11, -11\}$
(f) $\{\frac{4}{9}, -\frac{4}{9}\}$

4. (a) $2 \times 3 \times 5$
$2 \times 2 \times 3 \times 3 \times 5 \times 5$
(b) $2 \times 2 \times 3$
$2 \times 2 \times 2 \times 2 \times 3 \times 3$
(c) 7 is prime
7×7
(d) $2 \times 3 \times 3$
$2 \times 2 \times 3 \times 3 \times 3 \times 3$
(e) 2×11
$2 \times 2 \times 11 \times 11$
(f) 3×13
$3 \times 3 \times 13 \times 13$

5. (a) $3 \times 3 \times 7 \times 7$
3×7

(b) $2 \times 2 \times 11 \times 11$
2×11

(c) $2 \times 2 \times 3 \times 3 \times 7 \times 7$
$2 \times 3 \times 7$

(d) $2 \times 2 \times 13 \times 13$
2×13

(e) $2 \times 2 \times 2 \times 2 \times 7 \times 7$
$2 \times 2 \times 7$

(f) $2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 7 \times 7$
$2 \times 3 \times 3 \times 7$

6. If x represents the positive number, then

$$x^2 - 3 = 166$$

$$\{13\}$$

The positive number is 13.

12-2. Radicals.

Although the radical symbol has appeared in previous chapters, we give it a more formal introduction here. Emphasis should be placed on the fact that the radical indicates specifically the positive square root, in every case except the square root of zero, and is therefore completely unambiguous. The student is asked to deduce the fact that an expression such as $\sqrt{-9}$ has no meaning.

Answers to Oral Exercises 12-2; page 514:

1. The symbols in (a), (c), (d), and (f) name real numbers; the others do not.

2. (a) 11

(b) 3

(c) $\sqrt{-16}$ does not name a real number.

(d) $\frac{12}{5}$

(e) 16

(f) $\sqrt{-49}$ does not name a real number.

Answers to Problem Set 12-2; pages 514-5:

1. (a) 8

(b) $\frac{5}{3}$

(c) 5

(d) does not name a real number

(e) $\frac{7}{13}$

(f) 0

pages 515-517: 12-2 and 12-3

2. (a) $\{4, -4\}$

(b) $\{4\}$

(c) $\{\sqrt{14}, -\sqrt{14}\}$

(d) $\{17, -17\}$

(e) $\{\frac{25}{3}, -\frac{25}{3}\}$

(f) $\{\frac{25}{3}\}$

3. (a) $\sqrt{256} = \sqrt{2^8}$
$= \sqrt{2^4 \cdot 2^4}$
$= 2^4$
$= 16$

(b) $\sqrt{1089} = \sqrt{3^2 \cdot 11^2}$
$= 3 \cdot 11$
$= 33$

(c) $\sqrt{2304} = \sqrt{2^8 \cdot 3^2}$
$= 2^4 \cdot 3$
$= 48$

(d) $\sqrt{1024} = \sqrt{2^{10}}$
$= \sqrt{2^5 \cdot 2^5}$
$= 2^5$
$= 32$

(e) $\sqrt{576} = \sqrt{2^6 \cdot 3^2}$
$= 2^3 \cdot 3$
$= 24$

(f) $\sqrt{1936} = \sqrt{2^4 \cdot 11^2}$
$= 2^2 \cdot 11$
$= 44$

4. (a) 1 and 3
(b) 1 and 2
(c) 1 and 5
1 and 5
(d) 1
1
(e) 1

12-3. Irrational Numbers.

The development in this section is one of major importance. We begin with the realization by the student that certain integers do not have square roots which are themselves integers. This can be done effectively by an exploitation of the uniqueness of prime factorization. We exhibit a necessary as well as a sufficient condition for a positive integer to have an integer as its square root. The condition is merely that the proper prime factors come in pairs.

This is followed by the principal theorem which states that if an integer is to have a square root which is to be a rational number, then this rational number must be an integer. We thereby produce a device for the recognition of a large class of numerals, such as $\sqrt{10}$, representing numbers which are not rational. With the assumption that such numbers are associated with points on the number line, we obtain a set of irrationals, i.e. real numbers which are not rational.

The teacher may wish to use construction methods with the diagonal of a square and the Pythagorean Theorem to bolster the assumption that such numbers as $\sqrt{2}$ can be "located" on the number line. The extent to which this can be done will depend, of course, on the background of the students.

A word about the proof. Should the student's difficulty in walking through the steps be insurmountable, the teacher may wish to skip the formal argument. However, in this case an effort should be made to insure that the student understands the full significance of the conclusion.

We base the argument on prime factorization. Starting with an assumption that $\frac{a}{b}$ is a square root of an integer n where a and b are integers with no factor in common except 1, we ultimately show that the denominator b must be 1. We do this first by showing that b must be a factor of $(a \times a)$. It follows, then, that either b is 1 or else it has at least one prime factor in common with $(a \times a)$. The next step will probably be easier for the student to understand if the teacher produces some numerical examples. We say, in effect, that any prime factor in $(a \times a)$ will also be a factor of a. Thus, if b has a prime factor in common with $(a \times a)$, then b will have a prime factor in common with a. Since this contradicts the assumption that the rational number $\frac{a}{b}$ is in lowest terms, we deduce that $b = 1$. The initial hypothesis that $\frac{a}{b}$ is in lowest terms will also dispose of the special case where $b = a$.

The section ends with a formal introduction to the word irrational. We also recognize the symbol $-\sqrt{n}$ as representing the negative square root.

Answers to Oral Exercises 12-3a; pages 519-520:

1. If the integer 3 has a rational square root, it must be an integer. But 1 is too small since $1^2 = 1$, and 2 is

too large since $2^2 = 4$. There are no integers between 1 and 2; hence, 3 has no rational square root.

2. 2

3. 5 does not have a rational square root. (The reason is the same as in the first problem above.)

4. If 6 has a rational square root, it must be an integer. Since $2^2 < 6$ and $3^2 > 6$, 6 has no rational square root.

5. 3

6. 4

7. 13 has no rational square root. If it did, the square root would have to be an integer and thus a factor. But 13 is prime and has no proper factors.

8. 5

9. 17 has no rational square root, since it is a prime.

10. $18 = 2^2 \cdot 3$ Since the prime factors do not come in pairs, 18 does not have any integral square root and hence it does not have any rational square root.

Answers to Problem Set 12-3a; pages 520-521:

1. (a) If 11 has a rational square root, it must be an integer. Since 11 is a prime, its prime factors do not occur in pairs. Therefore, it does not have an integral square root and so it does not have a rational square root.

   (b) 6

   (c) 7

   (d) no rational square root (same reason as in 1(a))

   (e) $35 = 5 \cdot 7$ Prime factors do not occur in pairs. Hence, 35 has no integral square root and therefore no rational square root.

   (f) 25

2. (a) 2×2, which is 4

(b) $3 \times 3 \times 5$ does not have a rational square root. A rational square root would have to be an integer. The prime factors, however, do not occur in pairs.

(c) $3 \cdot 7$, which is 21

(d) $2 \cdot 3 \cdot 7 \cdot 11$, which is 462

(e) no rational square root

(f) $2^2 \cdot 13$, which is 52

3. (a) 12

(b) $5 \cdot 7 \cdot 23 = 805$

(c) none

(d) $2448 = 2^4 \cdot 3^2 \cdot 17$—no rational square root, since there is only one factor 17

(e) $592 = 2^4 \times 37$ - no rational square root

(f) 941 is a prime. Hence, it does not have a rational square root.

4. (a) $1025 = 5^2 \cdot 41$. It does not have a rational square root.

(b) $252 = 2^2 \cdot 3^2 \cdot 7$ - no rational square root

(c) no rational square root

(d) 37

(e) $675 = 3^3 \cdot 5^2$ - no rational square root

(f) 30

Answers to Oral Exercises 12-3b; page 522:

1. (a) irrational (d) irrational (g) irrational
(b) irrational (e) irrational (h) rational
(c) rational (f) irrational (i) irrational

pages 522-525: 12-3 and 12-4

2. (a) $\{\sqrt{2}, -\sqrt{2}\}$ (d) $\{\sqrt{5}, -\sqrt{5}\}$ (g) $\{\sqrt{8}, -\sqrt{8}\}$
(b) $\{\sqrt{3}, -\sqrt{3}\}$ (e) $\{\sqrt{6}, -\sqrt{6}\}$ (h) $\{3, -3\}$
(c) $\{2, -2\}$ (f) $\{\sqrt{7}, -\sqrt{7}\}$ (i) $\{\sqrt{10}, -\sqrt{10}\}$

Answers to Problem Set 12-3b; pages 522-523:

1. (a) rational (d) rational (f) rational
(b) irrational (e) irrational (g) rational
(c) rational

2. (a) $\{\sqrt{5}, -\sqrt{5}\}$ (d) $\emptyset$
(b) $\{\sqrt{7}, -\sqrt{7}\}$ (e) $\{6\}$
(c) $\{9, -9\}$ (f) $\{\sqrt{11}\}$

3. (a) $\{\sqrt{5}, -\sqrt{5}\}$ (d) $\{\sqrt{3}, -\sqrt{3}\}$
(b) $\{5, -5\}$ (e) $\{-\sqrt{11}\}$
(c) $\{\sqrt{2}, -\sqrt{2}\}$ (f) $\{1, -1\}$

4. (a) $\{\sqrt{2}, -\sqrt{2}\}$ (d) $\{-\sqrt{3}, \sqrt{3}\}$
(b) $\{-2, 2\}$ (e) $\{\sqrt{3}, -\sqrt{3}\}$
(c) $\{7\}$ (f) $\{\sqrt{3}, -\sqrt{3}\}$

5. m is 3, k is 2.

12-4. Simplification of Radicals.

As a key to the simplification of radicals we develop the property that for any two non-negative real numbers a and b, $(\sqrt{a})(\sqrt{b}) = \sqrt{ab}$. It might be well to present the student with a few more examples to begin with so that he may more confidently anticipate the result. The demonstration rests on a basic property of exponents and the definition of square root. Attention should be called to the fact that the distributive property, since it holds for all reals, also applies to irrationals.

Answers to Oral Exercises 12-4a; page 526:

1. 2
2. $\sqrt{6}$
3. 4
4. $\sqrt{18}$
5. $2 + \sqrt{6}$
6. 6
7. 6
8. $\sqrt{18} + 6$
9. $\sqrt{35}$
10. $\sqrt{18}$ or $(\sqrt{2})(3)$

Answers to Problem Set 12-4a; page 526-527:

1. (a) 6
   (b) $\sqrt{30}$
   (c) 3
   (d) 9
   (e) $\sqrt{60}$
   (f) 15

2. (a) $\sqrt{6} + 4$
   (b) $5 + \sqrt{30}$
   (c) 9
   (d) 0
   (e) $\sqrt{105}$
   (f) $\sqrt{12}$ or $2(\sqrt{3})$

3. (a) $\sqrt{2x}$, (domain: $x \geq 0$)
   (b) $\sqrt{16y}$, (domain: $y \geq 0$)
   (c) $\sqrt{36a}$, (domain: $a \geq 0$)
   (d) $\sqrt{2}(\sqrt{2x} + \sqrt{8}) = \sqrt{4x} + \sqrt{16}$
   $= \sqrt{4x} + 4$ (domain: $x \geq 0$)
   (e) $6m$, (domain: all real numbers)
   (f) $\sqrt{y^2}$ or y, (domain: $y \geq 0$)

4. (a) $\{4\}$
   (b) $\{6\}$
   (c) $\{\sqrt{6}\}$
   (d) $\{12\}$
   (e) $\{9\}$
   (f) $\{\sqrt{2}, -\sqrt{2}\}$

5. (a) 1

(b) -1

(c) $3 + 2\sqrt{2}$

(d) $\sqrt{x^2} + x$, (domain: $x \geq 0$) $x + \sqrt{x}$ is also correct

(e) $\sqrt{y^2} - 1$, (domain: $y \geq 0$) $y - 1$ is also correct

(f) $\sqrt{a^2} + 2\sqrt{ab} + \sqrt{b^2}$, (domain: $a \geq 0$, $b \geq 0$)

$a + b + 2\sqrt{ab}$ is also correct

Here we convey the notion that the simplest form of a radical is one in which the smallest integer, and the lowest power of the variable remain under the radical sign. (Once again factorization provides the tool.) Particular notice should be made of the fact that $\sqrt{x^2} = x$ is not a true sentence for $x < 0$. The student should develop the habit of writing $\sqrt{x^2} = |x|$. The example in the text $\sqrt{5x^2y^3} = \sqrt{x^2}\cdot\sqrt{y^2}\sqrt{5y} = |x|\cdot|y|\cdot\sqrt{5y}$ shows the absolute value of y in the simplified version. Again this is intended to strengthen a habit pattern for writing $\sqrt{y^2} = |y|$. On the other hand, an alert student may observe that the expression $\sqrt{y^3}$ is defined only for non-negative values of the variable. Hence the sentence $\sqrt{5x^2y^3} = |x|y\cdot\sqrt{5y}$ is true for all values of y for which the sentence has meaning. Thus the student should be encouraged in his suspicion that the second absolute value symbol is in this case superfluous. A review of the meaning and significance of absolute value is recommended at this point.

<u>Answers to Oral Exercises 12-4b</u>; pages 529-530:

1. 7
2. $5\sqrt{3}$
3. $3^2\sqrt{2}$, which is $9\sqrt{2}$
4. 3^3, which is 27
5. 2^5, which is 32
6. $11\sqrt{11}$

pages 530-531: 12-4

7. $11\sqrt{11}$
8. $5\sqrt{5}$
9. $5\sqrt{2}$
10. $3\sqrt{2}$
11. $3\sqrt{3}$
12. $|a|$
13. $|m|$
14. $3|b|$
15. $4|t|$
16. $t\sqrt{t}, \quad t \geq 0$

Answers to Problem Set 12-4b; pages 530-531:

1. (a) $2\sqrt{5}$ (d) $5\sqrt{6}$
   (b) $4\sqrt{2}$ (e) $4\sqrt{3}$
   (c) $2\sqrt{6}$ (f) $3\sqrt{5}$

2. (a) $2\sqrt{12} = 2\cdot 2\sqrt{3}$
   $= 4\sqrt{3}$
   (d) $6\sqrt{3}$
   (b) $3\sqrt{36} = 3\cdot 6$
   $= 18$
   (e) $6\sqrt{2}$
   (c) $4\sqrt{6}$ (f) 33

3. (a) is in simplest form (d) $6\sqrt{7}$
   (b) $\sqrt{12}\sqrt{6} = \sqrt{12\cdot 6}$
   $= \sqrt{2\cdot 6\cdot 6}$
   $= 6\sqrt{2}$
   (e) $2\sqrt{7}$
   (c) $\sqrt{2}\sqrt{10} = \sqrt{2\cdot 2\cdot 5}$
   $= 2\sqrt{5}$
   (f) 10

4. (a) $2|x|\sqrt{6}$ (d) $4\sqrt{2a}, \quad a \geq 0$
   (b) $2x\sqrt{6x}, \quad x \geq 0$ (e) $4|a|\sqrt{2}$
   (c) $2x^2\sqrt{6}$ (f) $4a\sqrt{2a}, \quad a \geq 0$

5. (a) $\{2\sqrt{14}, \quad -2\sqrt{14}\}$ (d) $\{4\sqrt{3}, \quad -4\sqrt{3}\}$
   (b) $\{9\sqrt{2}, \quad -9\sqrt{2}\}$ (e) $\{7\sqrt{11}, \quad -7\sqrt{11}\}$
   (c) $\{6\sqrt{5}, \quad -6\sqrt{5}\}$ (f) $\{\sqrt{6}\}$

6. (a) $5x\sqrt{x}$, $x \geq 0$ (d) $25|y|$

(b) $\sqrt{3x}\sqrt{6x} = \sqrt{3 \cdot 3 \cdot 2x^2}$ (e) $14\sqrt{m}$, $m \geq 0$

$= 3x\sqrt{2}$, $x \geq 0$

(c) $x^3\sqrt{5x}$, $x \geq 0$ (f) $4x\sqrt{3}$, $x \geq 0$

12-5. Simplification of Radicals Involving Fractions.

A second property of radicals, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, $a \geq 0$, $b > 0$, provides the basis for this simplification. The argument follows the same line of reasoning as in the previous section.

Answers to Oral Exercises 12-5a; page 533:

1. $\frac{2}{3}$
2. $\frac{4}{5}\sqrt{2}$
3. $\frac{12}{7}$
4. $\frac{5}{11}\sqrt{3}$
5. $\sqrt{\frac{8}{2}} = \sqrt{4}$ $= 2$
6. 5
7. $2\sqrt{2}$
8. 3
9. $\sqrt{3}$
10. $\frac{35}{13}$

Answers to Problem Set 12-5a; pages 534-535:

1. (a) $\frac{4}{5}$ (d) $\frac{4}{3}|x|$

(b) $\frac{7}{8}$ (e) $\frac{2}{3|x|}\sqrt{2}$, $x \neq 0$

(c) $\frac{3}{4}\sqrt{3}$ (f) $\frac{45}{7}$

2. (a) $\sqrt{3}$ (d) $\sqrt{3}|x|$

(b) $\sqrt{7}$ (e) $\sqrt{5}$, $m > 0$

(c) $\sqrt{6}$ (f) $\frac{\sqrt{7^3 \cdot 11^2}}{\sqrt{7}} = \sqrt{7^2 \cdot 11^2} = 77$

pages 534-537: 12-5

3. (a) $\frac{2}{5}\sqrt{3}$

(b) $\frac{7}{|a|}$, $a \neq 0$

(c) $\frac{21}{\sqrt{3}}$, $7\sqrt{3}$ is also correct

(d) $\frac{x\sqrt{x}}{|y|}$, $x \geq 0$, $y \neq 0$

(e) $\frac{\sqrt{x^3}}{\sqrt{x^2}} = \sqrt{x}$, $x > 0$

(f) 14a, $a > 0$

4. (a) 2

(b) $\sqrt{5}$

(c) $\frac{\sqrt{2}}{3|a|}$, $a \neq 0$

(d) 1, $y > 0$

(e) $\frac{1}{2}\cdot\sqrt{7}$

(f) $\frac{5a}{9|y|}\cdot\sqrt{3a}$, $a \geq 0$, $y \neq 0$

5. (a) $\{3\}$

(b) $\{2\sqrt{2}\}$

(c) $\{\sqrt{7}\}$

(d) $\{2\}$

(e) $\frac{\sqrt{2}}{\sqrt{72}} = \frac{1}{x}$ is equivalent to $\sqrt{2}\,x = \sqrt{72}$, if $x \neq 0$.
Truth set: $\{6\}$

(f) $\{3 + \sqrt{3}\}$

Page 535. The technique of rationalizing the denominator, or rationalizing the numerator is introduced here. We make use of the multiplication property of 1. We do not consider rationalizing the denominator as a "must", but rather let the choice rest on the context of the problem. For example, in some problems involving the distributive property it may be preferable to rationalize the numerator.

Answers to Oral Exercises 12-5b; page 537:

1. (a) $\frac{\sqrt{3}}{3}$

(b) $\frac{\sqrt{2}}{2}$

(c) $\frac{\sqrt{5}}{5}$

(d) $\frac{\sqrt{21}}{7}$

(e) $\frac{\sqrt{15}}{5}$

(f) $\frac{\sqrt{30}}{3}$

pages 537-538: 12-5

2. (a) $\frac{2}{\sqrt{6}}$ (d) $\frac{1}{\sqrt{3}}$

(b) $\frac{5}{\sqrt{15}}$ (e) $\frac{2}{\sqrt{14}}$

(c) $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2}{2\sqrt{2}}$ (f) $\frac{7}{\sqrt{35}}$

$= \frac{1}{\sqrt{2}}$

Answers to Problem Set 12-5b; pages 538-539:

1. (a) $\sqrt{2}$ (d) $\frac{1}{7}\sqrt{14}$

(b) $\frac{\sqrt{11}}{11}$ (can also be written $\frac{1}{11}\sqrt{11}$) (e) $\frac{\sqrt{4}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

(c) $\sqrt{3}$ (f) $\frac{1}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{4}\sqrt{2}$

2. (a) $\frac{6}{\sqrt{42}}$ (d) $\frac{1}{\sqrt{2}}$

(b) $\frac{5}{\sqrt{55}}$ (e) $\frac{\sqrt{8}}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4}{\sqrt{6}}$

(c) $\frac{1}{\sqrt{3}}$ (f) $\sqrt{\frac{12}{5}} = \frac{2\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6}{\sqrt{15}}$

3. (a) $\sqrt{\frac{9}{50}} = \frac{3}{5\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ (d) $\frac{1}{5}\sqrt{15b}$, $b \geq 0$

$= \frac{3\sqrt{2}}{10}$

(e) [illegible] $x > 0$

(b) $\frac{\sqrt{7}}{2}$

(f) $\sqrt{\frac{2}{7b}} = \frac{\sqrt{2}}{\sqrt{7b}} \cdot \frac{\sqrt{7b}}{\sqrt{7b}} = \frac{\sqrt{14b}}{7b}$, $b > 0$

(c) $\frac{5\sqrt{21}}{21}$

pages 538-540: 12-5 and 12-6

4. (a) $\frac{1}{\sqrt{5}}$ (d) $\frac{m}{\sqrt{3m}}$, $m > 0$

(b) $\frac{8}{3\sqrt{2}}$ (e) $\frac{t^2}{\sqrt{2t}}$, $t > 0$

(c) $\frac{5}{\sqrt{21}}$ (f) $\frac{2|y|}{\sqrt{6}}$

5. (a) $\frac{\sqrt{6}}{36}$ (d) $\frac{\sqrt{5}}{30}$

(b) $\frac{3\sqrt{70}}{50}$ (e) $\frac{5}{16}$

(c) $\frac{6\sqrt{385}}{539}$ (f) $\frac{16}{27}\sqrt{6}$

12-6. Sums and Differences of Radicals.

There may be some necessity for convincing the student that and expression such as $\sqrt{3} + \sqrt{2}$ cannot be further simplified. He should at all costs avoid the temptation to infer that $\sqrt{3} + \sqrt{2} = \sqrt{3 + 2} = \sqrt{5}$. It is conceivable that such an example as $\sqrt{9} + \sqrt{4} \neq \sqrt{13}$ will help the argument. On the other hand, if no temptation is present, it may be just as well to avoid the issue entirely.

Answers to Oral Exercises 12-6; page 540:

1. $5\sqrt{2}$
2. $3\sqrt{3}$
3. $8\sqrt{5}$
4. $6\sqrt{5}$
5. $4\sqrt{2}$
6. $\sqrt{2}$
7. 0
8. $3\sqrt{a}$, $a \geq 0$
9. $\sqrt{a}$, $a \geq 0$
10. $4 + \sqrt{2}$

Answers to Problem Set 12-6; pages 541-542:

1. (a) $7\sqrt{5}$
   (b) $3\sqrt{5}$
   (c) $10\sqrt{a}$, $a \geq 0$
   (d) $6\sqrt{a}$, $a \geq 0$
   (e) $4\sqrt{2}$
   (f) $2\sqrt{2}$

2. (a) $3\sqrt{2}$
   (b) $-\sqrt{2}$
   (c) $3\sqrt{3} + 3\sqrt{2}$
   (d) $3\sqrt{3} - 3\sqrt{2}$
   (e) $4\sqrt{3} + 3\sqrt{2}$
   (f) $5\sqrt{3} - 3\sqrt{2}$

3. (a) $5\sqrt{3} + 4\sqrt{2}$
   (b) $3\sqrt{2}$
   (c) $(2\sqrt{2} + 2)\sqrt{a}$, $a \geq 0$
   (d) $12\sqrt{2}$
   (e) $\frac{5}{2}\sqrt{2}$
   (f) $\frac{1}{6}\sqrt{6}$

4. (a) $5\sqrt{2}$
   (b) $3\sqrt{2}$
   (c) $6\sqrt{2} + 2 - 2\sqrt{5}$
   (d) $\frac{14}{15}\sqrt{5}$, $\frac{14}{3\sqrt{5}}$ is also correct
   (e) $\frac{20}{3}\sqrt{15}$, $\frac{100}{\sqrt{15}}$ is also correct
   (f) $2\sqrt{2}$

5. (a) $|a|\sqrt{16a} + 2\sqrt{a^3} = a \cdot 4\sqrt{a} + 2a\sqrt{a}$, $a \geq 0$
   $= 6a\sqrt{a}$, $a \geq 0$
   Because of the restriction on a we do not need the absolute value sign. If $a \geq 0$, $|a| = a$.
   (b) $4|x|\sqrt{2}$
   (c) $2|t|\sqrt{3}$
   (d) $\frac{29}{6}\sqrt{5b}$, $b \geq 0$

12-7. Approximate Square Roots in Decimals.

The acceptance of the fact that a numeral such as $\sqrt{10}$ does represent a real number is not sufficient to provide the student with a satisfactory sense of the relative size of the number or a feeling of where this number might be located on the number line with any degree of accuracy. We therefore wish to provide the student with a basis for forming reasonable approximations in terms of decimals. At the same time we want to avoid computational tricks or gimmicks which tend to obscure the understanding of what is actually taking place.

Consequently, we have not included the conventional algorithm for finding square root. We use instead a method of iterated approximations. This, we feel, has the advantage of keeping the student aware of the process involved, as well as giving him a continuing association with the number line. It is important for the student to become acquainted with processes of iteration, since they lend themselves very naturally to computation on electronic computers. There is, of course, no reason why the teacher should not also acquaint his students with the conventional algorithm.

We begin by locating the given square root on an interval between two integers. This is easy for the student to understand since he merely finds the integer whose square immediately precedes and the integer whose square immediately follows the square of the number he is looking for.

At the outset we follow this by a trial and error approach in locating the interval between the nearest tenths. Computation is more involved, but the idea is very clear. Thus, a student through experimental multiplications can arrive at a fair approximation in terms of one place decimals.

We also introduce the interval notation in terms of two inequality symbols. This is convenient, but not necessary, if difficulties seem to be arising.

Answers to Oral Exercises 12-7a; page 543:

1. 2 and 3	3. 5 and 6	5. 10 and 11
2. 3 and 4	4. 8 and 9	6. 7 and 8

pages 543-544: 12-7

7. 9 and 10
8. 9 and 10
9. 11 and 12
10. 5 and 6

Answers to Problem Set 12-7a; page 544:

1. (5.0, 5.1) $(5.0)^2 = 25.00$, $(5.1)^2 = 26.01$
2. (8.0, 8.1) $(8.0)^2 = 64.00$, $(8.1)^2 = 65.61$
3. (11.0, 11.1) $(11.0)^2 = 121.00$, $(11.1)^2 = 123.21$
4. (9.2, 9.3)
5. (3.6, 3.7)
6. (4.5, 4.6)
7. (7.4, 7.5)
8. (9.4, 9.5)
9. (7.9, 8.0)
10. (4.3, 4.4)

Noting that the same trial and error approach in terms of hundredths would produce tedious computations we now introduce an alternative technique. However, students who wish to carry out more extensive computations to arrive at a better approximation by the primitive method might be encouraged to do so. This certainly in no way obscures the nature of the search.

Our final method is based on a rather simple procedure. It may, however, be necessary to lead the student through several examples before the idea becomes completely clear. As stated before, the proof that each successive approximation will be an improvement over the previous one will not be given in the text. It is on a level of sophistication beyond the scope of the ninth grade and will be presented in the teachers' commentary only. However, it should be quite possible for the student to perceive the plausibility of the argument, which will lead to its acceptance on grounds more valid than mer dience.

The selection of the two nearest integers is carried out as before. This time, however, we single out the one which appears to be closer to our desired square root and call this our <u>first approximation</u>.* In this connection it should be noted that the "closer" integer may have to be selected on intuitive grounds, since we do not present a formal device for determining this. In most cases, however, the choice will be fairly obvious. Where there is some margin of doubt, it would be well to select the larger number. It is possible to show that a second approximation will inevitably improve on the first in this case.

Our integer having been selected, we form the product of our integer and an unknown factor n such that this product is equal to the square of the number we are seeking. In the given example, we noted that $3 \cdot n = 10$. On the assumption that $p^2 = 10$, we assert that if $3 < p$, then $n > p$, and vice versa. For students who request a proof of this assertion, and we hope there are some, the following should be comprehensible.

If $a \cdot b = 10$ and $a < \sqrt{10}$,

then $a \cdot b < b \cdot \sqrt{10}$.

Thus $10 < b\sqrt{10}$,

and $\sqrt{10} < b$.

The students should be able to supply the arguments for the individual steps.

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*The choice of an integer as a first approximation may seem inefficient and unduly restrictive to many teachers and to the more alert students. The aim has been to keep the computation simple and the arguments uncluttered. It also seemed a logical starting point since our development has been based on the prime factorization of integers. However, there is no objection whatever to a student's experimenting with a "first approximation" of his own preference which appears to be more accurate than the "closest integer" as long as this does not cause any confusion.

We now utilize the fact that $3 < p$ and $n > p$, where $n = \frac{10}{3}$ to bring home the fact that the desired square root lies between our two factors, i.e. $3 < \sqrt{10} < \frac{10}{3}$. It seems natural, then, to take the point halfway between these two factors as our second approximation.

It may be necessary at this juncture to secure the notion in the mind of the student that a point halfway between two other points is determined by computing the average, or arithmetic mean of the two numbers.

If we go through the process again using our second approximation as the new first factor, we will arrive at a better approximation. And so on into the night. The teacher, however, should use discretion in deciding whether or not very much should be made of this. Perhaps some of the better students can give it a try.

To simplify matters we have confined our approach to the finding of square roots of positive integers, and of positive integers which are reasonably small. Perhaps the chase should end here. On the other hand for the better student it is possible to work with the so-called standard form of a number, thus reducing a larger problem to one involving a smaller number and powers of ten. The following example illustrates the approach.

To estimate $\sqrt{354000}$ we note that this can be written as $\sqrt{35.4}\cdot\sqrt{10^4}$. A first estimate to $\sqrt{35.4}$ is 6. The first estimate to $\sqrt{354000} = 6\cdot 10^2 = 600$. If a second estimate to $\sqrt{35.4}$ turns out to be n, then the desired second estimate is $n\cdot 10^2$.

The following is a proof of the assertion that our method guarantees that a second approximation is better than the first.

Let m be the positive integer whose square root we want to approximate. Let p be the positive integer such that

$$p < \sqrt{m} < p + 1$$

Then $$0 < \sqrt{m} - p < 1.$$

Let us regard p as the first approximation of $\sqrt{m}$. We form a second approximation

$$\frac{p + \frac{m}{p}}{2},$$

which equals $\frac{p^2 + m}{2p}$.

This number is greater than $\sqrt{m}$, since

$$\frac{p^2 + m}{2p} - \sqrt{m} = \frac{p^2 - 2p\sqrt{m} + m}{2p}$$

$$= \frac{(\sqrt{m} - p)^2}{2p} > 0.$$

Now, what we want to show is that the difference $\frac{p^2 + m}{2p} - m$ is less than the difference $\sqrt{m} - p$.

In other words, we want to show that

$$\sqrt{m} - p - \left(\frac{p^2 + m}{2p} - \sqrt{m}\right) > 0.$$

The left side of this inequality can be written

$$\sqrt{m} - p - \frac{(\sqrt{m} - p)^2}{2p} = (\sqrt{m} - p)\left(1 - \frac{\sqrt{m} - p}{2p}\right) .$$

We need to show that $(\sqrt{m} - p)\left(1 - \frac{\sqrt{m} - p}{2p}\right)$ is positive.

We know that

$$0 < \sqrt{m} - p < 1 ,$$

and since $2p > 1$

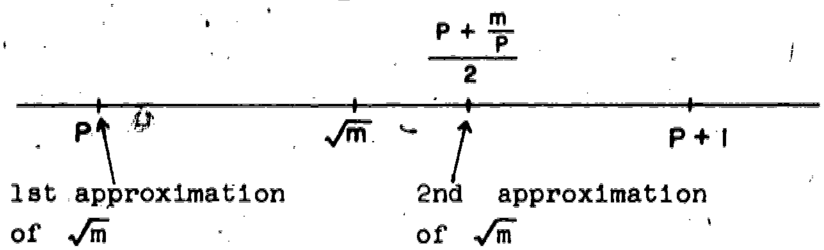
$$0 < \frac{\sqrt{m} - p}{2p} < 1 .$$

Hence

$$1 - \frac{\sqrt{m} - p}{2p} \quad \text{is positive,}$$

and the product $(\sqrt{m} - p)\left(1 - \frac{\sqrt{m} - p}{2p}\right)$ is positive. This completes our proof.

The diagram below shows the relative positions of the numbers p, $\sqrt{m}$, and $\frac{p + \frac{m}{p}}{2}$.



It is noteworthy that the proof does not depend on an assumption that p is the <u>closer</u> of the two integers. An argument entirely analogous to the above can be made for the case in which the larger integer, $p + 1$, is chosen as the first approximation.

<u>Answers to Oral Exercises 12-7b</u>; page 547:

1. (a) 3 (f) 11
   (b) 5 (g) 10
   (c) 9 (h) 9
   (d) 8 (i) 6
   (e) 3 (j) 2

2. (a) 4.5 (d) 7.5
   (b) 5.5 (e) 2.25
   (c) 8.5 (f) $\frac{163}{18}$ or 9.1

<u>Answers to Problem Set 12-7b</u>; pages 547-548:

1. (a) 4 (d) 11
   (b) 6 (e) 10
   (c) 9 (f) 3

page 547: 1-7.

2. (a) First approximation 4

$4n = 17, \quad n = \frac{17}{4}$

$\sqrt{17} \approx \frac{4 + \frac{17}{4}}{2} = 4.1$

(b) First approximation 6

$\frac{6 + \frac{41}{6}}{2} = \frac{77}{12} \qquad \sqrt{41} \approx 6.4$

(c) $\frac{9 + \frac{83}{9}}{2} = \frac{82}{9} \qquad \sqrt{83} \approx 9.1$

(d) $\frac{11 + \frac{130}{11}}{2} = \frac{251}{22} \qquad \sqrt{130} \approx 11.4$

(e) $\frac{10 + 10.7}{2} = \frac{20.7}{2} \qquad \sqrt{107} \approx 10.35$

(f) $\frac{3 + \frac{11}{3}}{2} = \frac{10}{3} \qquad \sqrt{11} \approx 3.3$

3. (a) $\frac{5 + \frac{23}{5}}{2} = \frac{24}{5} \qquad \sqrt{23} \approx 4.8$

(b) $\frac{4 + \frac{19}{4}}{2} = \frac{35}{8} \qquad \sqrt{19} \approx 4.4$

(c) $\frac{6 + \frac{37}{6}}{2} = \frac{73}{12} \qquad \sqrt{37} \approx 6.1$

(d) $\frac{9 + \frac{89}{9}}{2} = \frac{85}{9} \qquad \sqrt{89} \approx 9.4$

(e) $\frac{10 + 11.3}{2} = \frac{21.3}{2} \qquad \sqrt{113} \approx 10.6$

(f) $\frac{7 + \frac{48}{7}}{2} = \frac{97}{14} \qquad \sqrt{48} \approx 6.9$

4. (a) $\sqrt{72} = 6\sqrt{2}$
$\sqrt{72} \approx 6(1.414)$
$\sqrt{72} \approx 8.484$

(b) $\sqrt{18} = 3\sqrt{2}$
$\sqrt{18} \approx 4.242$

(c) $6\sqrt{3} \approx 6(1.732)$
≈ 10.392
$\sqrt{108} \approx 10.392$

(d) $\sqrt{48} = 4\sqrt{3}$
$\sqrt{48} \approx 6.928$

(e) $\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$
$\approx .577$
$\frac{1}{\sqrt{3}} \approx .577$

(f) $\sqrt{2450} = 35\sqrt{2}$
$\sqrt{2450} \approx 49.490$

5. (a) $\sqrt{41}$ 2nd approximation is 6.4

(b) $\sqrt{34}$ $\frac{6 + \frac{34}{6}}{2} = \frac{35}{6}$
$\sqrt{34} \approx 5.8$

(c) $\sqrt{74}$ $\frac{9 + \frac{74}{9}}{2} = \frac{155}{18}$
$\sqrt{74} \approx 8.6$

12-8. Cube Roots and n^{th} Roots.

This section is purposefully unpretentious. No work in cube roots goes beyond the recognition stage and no thought is given to a possible algorithm for finding an approximati ı. A point is raised about there being only one real cube root in contrast to the square root situation. Attention should be called to the fact that $\sqrt[3]{a}$ is defined for every real number a.

"n^{th}" roots are introduced primarily for purposes of identification.

The existence of the most copious of all the subsets of real numbers, namely the set of irrationals which are not "roots", usually referred to as transcendental numbers, should be revealed to the student. His previous contact will undoubtedly be limited. If his curiosity is unsatisfied, mention might be made of trigonometric functions, logarithms, etc.

pages 550-552: 12-8

Answers to Oral Exercises 12-8a; page 550:

1. 3
2. -2
3. 4
4. $\frac{1}{2}$
5. $-\frac{1}{3}$
6. a, no restriction on the domain
7. 6
8. $\frac{1}{15}$
9. 6a, no restriction on the domain
10. -(-2) or 2

Answers to Problem Set 12-8a; pages 550-551:

1. (a) -3
   (b) -2
   (c) $\frac{1}{2}$
   (d) $-\frac{4}{3}$
   (e) 5
   (f) 21

2. (a) 3
   (b) -a
   (c) -b
   (d) 6x
   (e) 4xy
   (f) $\frac{m}{5}$

3. (a) {-3}
   (b) {2}
   (c) $\frac{1}{2}t^3 - 12 = 20$

   $\frac{1}{2}t^3 = 32$

   $t^3 = 64$

   $t = 4$

   {4}
   (d) $\{\frac{1}{8}\}$
   (e) {3}
   (f) $m = (81)^3$

   $\{(81)^3\}$

Answers to Problem Set 12-8b; page 552:

1. $\sqrt{-16}$ and $\sqrt[4]{(-2)^5}$ do not represent real numbers; the rest of the symbols do.

2. $\sqrt{4}$, integer; $\sqrt{-16}$, none; $\sqrt[3]{8}$ integer; $\sqrt[3]{-8}$ integer

pages 552-558

$\sqrt{2}$, irrational

$-\sqrt{3}$, irrational

$-\sqrt{9}$, integer

$-\sqrt[3]{-64}$, integer

$\sqrt[4]{2^4}$, integer

$\sqrt[4]{(-2)^5}$, none

$\sqrt{2\frac{1}{4}}$, rational

Summary of the Fundamental Properties of Real Numbers.

This section is designed to serve a dual purpose. It should constitute a convenient review. It should also provide the student with some insights as to the nature of a mathematical structure. It is significant that we do not in a literal sense describe the numbers themselves other than to say that they are elements in a set. But we do completely characterize them by means of operations, relations, and properties thereof. Thus the real number system is defined in an abstract way, a way which reveals the essential mathematics involved.

A word, as to the completeness property. Since we have been talking throughout the course about the real numbers, it was felt that this property must be included in the fundamental list. As indicated in the text, the list without this property is a characterization of the rational numbers. Because a description of the property requires an understanding of the concept of "least upper bound" it seemed advisable to omit discussion from the summary. However, should the teacher wish to satisfy the curiosity of some of the interested students, it might be helpful to refer to Studies in Mathematics, Volume III, Structure of Elementary Algebra, Chapter 5, for a clear exposition of this topic.

There is always an opportunity for some dispute as to what constitute so-called fundamental properties and what ones belong on the list of derived properties. The selection here is not made because of strict mathematical reasons but is to a large extent a matter of convenience and common agreement.

It is hoped that some of the better students will be able to prove the additional properties by means of the first set, and in this way be led to a fuller realization of the deductive point of view.

Answers to Review Problem Set; pages 558-560:

1. $-\sqrt{2}$ real, irrational

   $\sqrt{9}$ real, integer

   $\sqrt{-7}$ not a real number

   $(\sqrt{3})^2$ real, integer

   $\frac{1}{\sqrt{2}}$ real, irrational

   $\frac{\sqrt{2}}{3-3}$ not a real number

   $3 + 2$ real, integer

   $\frac{2}{\sqrt{-2}}$ not a real number

   $-\sqrt{36}$ real, integer

   $\frac{5}{4}$ real, rational

   $\sqrt{\frac{2}{3}}$ real, irrational

   $\frac{6}{0}$ does not name a number

   $\frac{63}{7}$ real, integer

   $\frac{0}{6}$ real, integer

2. (a) $2\sqrt{3}$
   (b) $\sqrt{3}$
   (c) 8
   (d) $\frac{3}{7}\sqrt{7}$
   (e) 32
   (f) $7\sqrt{22}$

3. (a) $\emptyset$
   (b) $\{\sqrt{5}, -\sqrt{5}\}$
   (c) $\{\sqrt{5}\}$
   (d) $\{6\}$
   (e) $\{\sqrt{2}, -\sqrt{2}\}$
   (f) all non-negative real numbers

4. (a) True
   (b) False
   (c) True
   (d) True
   (e) False
   (f) True

5. (a) 2a
   (b) $3|x|\sqrt{2}$
   (c) $4a, \quad a \geq 0$
   (d) 5b
   (e) $-4x|y|\sqrt{2x}, \quad x \geq 0$
   (f) $2, \quad a > 0$

6. (a) $\frac{1}{6}\sqrt{30}$
   (b) 1
   (c) $-\frac{1}{4}\sqrt{[illegible]}$
   (d) $\sqrt{3}$
   (e) $-\frac{4\sqrt{3}}{3}$
   (f) 0

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7. (a) $x^2 - x - 6$ (d) $2n^2 + n - 1$

(b) $y^2 + 4y + 3$ (e) $a^2 - b^2$

(c) $m^2 - m - 30$ (f) $4x^2 - 1$

8. (a) $\frac{1}{2}$ (d) $-\frac{7m}{q}$, $m \neq 0$, $q \neq 0$

(b) $\frac{3}{5}$ (e) $\frac{5}{6}$, $a \neq 0$

(c) $\frac{9}{16}$ (f) $b + 1$, $b \neq 0$

9. Let x be the number of nickels.
Then $2x$ is the number of dimes
and $\frac{2x}{5}$ is the number of quarters.

$$5x + 10(2x) + 25\left(\frac{2x}{5}\right) = 175$$
$$35x = 175$$
$$x = 5$$

He has 5 nickels, 10 dimes, and 2 quarters.

10. Let x be the number of miles the plane can fly away from its base. Then the open sentence for the problem is

$$\frac{x}{100} + \frac{x}{120} = 6$$
$$6x + 5x = 3600$$
$$x = \frac{3600}{11}$$
$$= 327\frac{3}{11}$$

The plane can fly $327\frac{3}{11}$ miles before it must turn back.

Suggested Test Items

1. Simplify.

(a) $\sqrt{27}$ (c) $\sqrt[3]{8}$ (e) $\frac{\sqrt{18}}{\sqrt{3}}$

(b) $\sqrt{\frac{4}{3}}$ (d) $\sqrt{24} - \frac{1}{3}\sqrt{54}$ (f) $(-\sqrt{2})(-\sqrt{18})$

2. Simplify.

(a) $\sqrt{15}\sqrt{20}$

(b) $\frac{\sqrt{15}}{\sqrt{20}}$

(c) $\sqrt{45} + \sqrt{20}$

(d) $\sqrt{336}$

(e) $\sqrt{\frac{2}{5}}\sqrt{\frac{3}{5}}\sqrt{\frac{1}{6}}$

(f) $\sqrt{3}(\sqrt{12} + \sqrt{2})$

3. Simplify assuming that variables represent positive numbers.

(a) $\sqrt{\frac{1}{2x}}$

(b) $\sqrt{\frac{1}{ab}}\sqrt{a^3b}$

(c) $\frac{\sqrt{\frac{x}{y^2}}}{\sqrt{\frac{1}{x^2}}}$

(d) $\sqrt{\frac{2a}{3}}\sqrt{\frac{3a}{4}}\sqrt{\frac{4a}{5}}\sqrt{\frac{5a}{6}}$

(e) $\sqrt{4a^3} + \sqrt{\frac{4}{9a}}$

(f) $\sqrt{2}(\sqrt{2} - \sqrt{18})x$

4. Simplify indicating the restrictions on the domain of the variables.

(a) $\sqrt{a^3}$

(b) $\sqrt{x^2}$

(c) $\sqrt{\frac{1}{y^2}}$

(d) $\sqrt{\frac{1}{a^3}}$

(e) $\sqrt{a^2b^3}$

(f) $\sqrt{y^5}$

5. Find the truth sets of the following.

(a) $\sqrt{18}\ x = \sqrt{2}$

(b) $7m^2 - 5 = 16$

(c) $t^2 + \frac{1}{2} = \frac{3}{4}$

(d) $\frac{y}{\sqrt{2}} = \sqrt{450}$

(e) $4k^2 - 40 = 32 - 2k^2$

(f) $\sqrt{2}\ y + \sqrt{2} = \sqrt{18}$

6. Find the truth sets of the following.

(a) $\sqrt{y} = -2$

(b) $x^2 = -7$

(c) $\sqrt{m^2} = m$

(d) $\sqrt[3]{y} = -2$

(e) $m^2 = 144$

(f) $k^2 = 3$

7. Rationalize the denominator of each and indicate the restrictions on the domains of the variables.

(a) $\frac{2}{\sqrt{2}}$

(b) $\frac{1}{\sqrt{24}}$

(c) $\frac{\sqrt{4}}{\sqrt{12}}$

(d) $\frac{1}{\sqrt{x}}$

(e) $\frac{1}{\sqrt{a^3b}}$

(f) $\sqrt{\frac{2}{7b}}$

8. Find a second approximation for each.

(a) $\sqrt{43}$

(b) $\sqrt{83}$

(c) $\sqrt{23}$

9. Given $\sqrt{5} \approx 2.2361$ find an approximation for each.

(a) $\sqrt{20}$

(b) $\frac{1}{\sqrt{5}}$

(c) $\sqrt{125}$

Answers to Suggested Test Items

1. (a) $3\sqrt{3}$

(b) $\frac{2\sqrt{3}}{3}$ or $\frac{2}{\sqrt{3}}$

(c) 2

(d) $\sqrt{6}$

(e) $\sqrt{6}$

(f) 6

2. (a) $10\sqrt{3}$

(b) $\frac{1}{2}\sqrt{3}$

(c) $5\sqrt{5}$

(d) $4\sqrt{21}$

(e) $\frac{1}{5}$

(f) $6 + \sqrt{6}$

3. (a) $\frac{1}{2x}\sqrt{2x}$

(b) a

(c) $\frac{x}{y}\sqrt{x}$

(d) $\frac{a^2}{3}\sqrt{3}$

(e) $(\frac{6a^2 + 2}{3a})\sqrt{a}$

(f) $-4x$

4. (a) $a\sqrt{a}$, domain: $a \geq 0$

(b) $|x|$, domain: all real numbers

(c) $\frac{1}{|y|}$, domain: $y \neq 0$

(d) $\frac{1}{a^2}\sqrt{a}$, domain: $a > 0$

(e) $|a|b\sqrt{b}$, domains: a, all real numbers; b, non-negative reals

(f) $y^2\sqrt{y}$, domain: $y \geq 0$

5. (a) $\{\frac{1}{3}\}$

(b) $\{-\sqrt{3}, \sqrt{3}\}$

(c) $\{-\frac{1}{2}, \frac{1}{2}\}$

(d) $\{30\}$

(e) $\{-2\sqrt{3}, 2\sqrt{3}\}$

(f) $\{a\}$

6. (a) $\emptyset$

(b) $\emptyset$

(c) The non-negative real numbers

(d) $\{-8\}$

(e) $\{12, -12\}$

(f) $\{\sqrt{3}, -\sqrt{3}\}$

7. (a) $\sqrt{2}$

(b) $\frac{1}{12}\sqrt{6}$

(c) $\frac{\sqrt{3}}{3}$

(d) $\frac{\sqrt{x}}{x}$, $x > 0$

(e) $\frac{\sqrt{ab}}{a^2b}$, $a > 0$ and $b > 0$

(f) $\frac{1}{7b}\sqrt{14b}$, $b > 0$

8. (a) 6.6

   (b) 9.1

   (c) 4.8

9. (a) 4.4722

   (b) .4472

   (c) 11.1805

Chapter 13

POLYNOMIALS

The underlying concept in this chapter is the similarity between the behavior of polynomials and that of the integers themselves. Factoring of polynomials is featured in a manner similar to that of the factoring of integers in Chapter 11.

The definition of a polynomial is developed gradually. The student will begin with the set of integers and one or more variables. A polynomial over the integers is defined as any expression which may be formed from variables and elements of this set by indicated operations of addition, subtraction, multiplication, or taking opposites. The student must be reminded that "or" is inclusive in mathematical usage so that our definition of polynomial means that any finite combination of indicated addition, subtraction, multiplication, and taking opposites which involves variables and elements of our set will produce a polynomial. The point here is that we shall study operations on expressions just as we studied operations on numbers earlier.

The question of the need in the definition for "taking opposites" may be raised on the ground that multiplication by (-1) produces the same result. It is important here to be reminded that the term polynomial has to do with form. Thus $(-1)(x)$ and $(-x)$, though names for the same number for any real value of x, are, strictly speaking, different polynomials. The first involves multiplication of x by -1, the other involves the taking of the opposite of x. This notion of the special significance of the form in which an expression is written may be a new concept to the student. It should be developed with care.

The chapter is devoted mainly to factoring "polynomials over the integers" and we use the single word "polynomial" to refer to these expressions. We would say, "polynomials over the rational numbers," or "polynomials over the real numbers" if we need to refer to these classes of expressions.

These references to p   nomials "over the rationals," "over the reals," etc. are mad   hint to the student that there is considerably more to investigate than that which is covered in this text.

We say that a polynomial is factorable over the integers if it can be written as the product of two or more polynomials over the integers, excluding 0 and ± 1 as factors.

13-1. Polynomials.

The reason polynomials are developed so carefully is that we want the student to realize that when we wish to write a certain phrase in factored form, we want the factors themselves to be expressions of the same kind as the phrase we started with. If we are factoring a polynomial over the integers, we want our factors to be polynomials over the integers. If we were to factor a polynomial over the rational numbers, we would find factors which were polynomials over the rational numbers, etc. Thus, we say that the factored form of a polynomial is an indicated product of polynomials of the same type. Notice that now we are dealing with the algebra of polynomials, and we speak of "multiplying polynomials," meaning of course, that the resulting polynomial names the same number as the indicated product.

It is important for teacher and student alike to realize that we have placed no restrictions on the domain of the variable when we talk about polynomials over the integers. The variable could be any real number, or in fact, by a slight extension of the definition of variable could be any "indeterminate" such as, for example, a vector or matrix neither of which is a number. Later when the student solves an equation involving a polynomial over the integers he will realize that it is permissible for the truth set to contain elements that are not integers. Thus the sentence $2x + 7 = 0$, which involves the polynomial over the integers, $2x + 7$, has $\{-\frac{7}{2}\}$ as its truth set.

Answers to Oral Exercises 13-1a; page 562:

1. (a) $m(m + 2)$ (f) $2(b^2 + 2)$
   (b) $3(n^2 + 2)$ (g) $2x(x + 3)$
   (c) $3y(y + 2)$ (h) $2(2z^2 - 5)$
   (d) $a(a - 1)$ (i) $2r(2r - 5)$
   (e) $b(b + 1)$

   The commutative property permits us, for example, to give the answer in part (a) as $(m + 2)m$.

Answers to Oral Exercises 13-1b; page 566:

1. The expressions in parts (a), (b), (d), (e), (h), and (i) are polynomials. The others are not.

2. (a) polynomial in one variable
   (b) polynomial with no variable
   (c) not a polynomial
   (d) polynomial in two variables
   (e) polynomial in one variable
   (f) not a polynomial
   (g) not a polynomial
   (h) polynomial in one variable
   (i) polynomial in two variables
   (j) not a polynomial
   (k) not a polynomial

Answers to Problem Set 13-1b; page 567:

The purpose of problems in this set is to give the students another way to recognize a polynomial.

1. A few suggested examples of a polynomial in one variable are: $3a$, $2a + 1$, $4x^2 + 8x$, $6m^2 + 5m + 1$. These are not expected to be the required answers; only suggestions.

2. Examples of polynomials in two variables are $x^2 - m^2$, $a^2 + 2ab$, $m^2 + 4my + 4y^2$, $x^3 - x^2 - k - 1$.

3. Examples of polynomials in three variables are: amn, $3bxy + 2z$, $2b^2cd + cd^2$, $4a^2 + 8b^2 - 2c^2 + 1$.

4. (b) $y + 3$
   (c) $2(y + 3)$
   (d) $2(y + 3) - 9$
   (e) The expression is a polynomial in one variable.

5. If the variable chosen were n, then we would have:
   (a) $-n$
   (b) $4n$
   (c) $4n + 6$
   (d) $3(4n + 6)$
   (e) $\frac{3(4n + 6)}{-n}$
   (f) By definition, a polynomial involves only the indicated operations of subtraction, multiplication, addition, and taking opposites. Since we indicate division by the opposite of the variable, our example does not meet the requirements of our definition of a polynomial. $\frac{3(4n + 6)}{-n}$ is not a polynomial.

6. (a) $2x^2 - 4x$, yes (d) $\frac{1}{2}t + \frac{1}{2}$, no
   (b) $3y^2 + 5y$, yes (e) $u^2 + 4u - 12$, yes
   (c) $m^2 + 5m + 6$, yes (f) $x^2y^3 + x^3y^2$, yes

7. (a) $3(2x + 1)$ (d) $4z(z - 2)$
   (b) $5(x + y)$ (e) $x(3x - 2)$
   (c) $h(2h + 1)$ (f) $ax(x + 1)$

13-2. Factoring.

This section defines what we mean by factoring a polynomial (over the integers). The student learns what we mean by prime polynomials and the term "prime factorization". It is shown that factors which are integers are left in their simplest form.

For example, $6(x + 5)$ is not written as $2 \cdot 3(x + 5)$.

Answers to Oral Exercises 13-2; page 570:

1. $x + 3x^2$ can be factored as $x(1 + 3x)$, since x and $1 + 3x$ are polynomials over the integers as is $x + 3x^2$. We would not factor $x + 3x^2$ as $\frac{1}{3}x(3 + 9x)$ since $\frac{1}{3}x$ is not a polynomial over the integers.

2. No. The factor $3x + 6$ is not a prime polynomial.

3. $6x(x + 5)$

4. $2ar^2(2 + 9r)$

Answers to Problem Set 13-2; pages 570-571:

1. The sentences in (a), (b), (c), (d), (g), (i), and (j) are true for all values of the variables. The others are not.

2. The expressions in parts (a), (c), and (d) are polynomials. The others are not.

3. The expresssions in parts (a), (c), (e), and (h) are prime polynomials. The others are not.

4. (a) $8(2r + 3)$ (d) $9q(q - 3)$
   (b) $6a(a^2 + 2a + 3)$ (e) $4a(r + 6x + 8z)$
   (c) $5(b - 7c)$ (f) $11a^2x^2(2 - 3x)$

13-3. Common Monomial Factoring.

Answers to Oral Exercises 13-3a; page 573:

1. (a) $3(a + b)$ (c) $3x(x^2 + 2x + 3)$
   (b) $2x(x^2 + 2x + 4)$ (d) $3y(y - 2)$

2. (a) $7(1 + 4a)$, 7 (c) $5mn(1 + 3mn)$, $5mn$
   (b) $4x(x - 2)$, $4x$ (d) $7ax(r - 2x)$, $7ax$

pages 573-574: 13-3 and 13-4

3. The expressions in parts (a), (c), and (e) are monomials. The others are not.

Answers to Problem Set 13-3a; page 574:

1. In parts (a), (b), (d), (e), and (g)

2.

(a)	$3(x + 2)$	(k)	$5(x^2 + 2x + 4)$
(b)	$8(c + 3)$	(l)	$2a(1 + 9a)$
(c)	$2(a - 9)$	(m)	$3m(2m - 3)$
(d)	prime	(n)	$2a(c + 3d + 5bd)$
(e)	$3(2x + 3)$	(o)	prime
(f)	prime	(p)	$3x(x - 4x^2 + 2)$
(g)	$a(a + b)$	(q)	$b(x^2 + y^2)$
(h)	$c^2(c + d)$	(r)	$2b(b^2 - 3b - 2)$
(i)	$a(a^2 + a + 2)$	(s)	$3a(x + 5y - 3a)$
(j)	$cn^2(cn^2 - 1)$	(t)	$7bx(ab - 9c^2)$

13-4. Quadratic Polynomials.

An attempt is made in this and the next few sections to present a unified approach to the process of factoring. Rather than exhibit a compendium of different polynomial types each with its specialized technique, the discussion centers on the quadratic polynomial

$$Ax^2 + Bx + C.$$

In substance the process of factoring involves the identification of two integers a and b in a so-called second form

$$Ax^2 + ax + bx + C,$$

which can then be factored by a direct application of the distributive property. This in effect, exactly reverses the multiplication process developed in earlier chapters.

The integers a and b are determined by the conditions

$$a + b = B \quad \text{and} \quad ab = AC.$$

If $A = 1$, the latter, of course, reduces to $ab = C$. In this way the form $x^2 + Bx + C$ with leading coefficient 1, may be considered as a special case of the general type above. For simplicity, however, the study of this special case precedes the more general treatment.

Two important special forms, the <u>difference of two squares</u> and <u>perfect squares</u>, are treated in sections 13-5 and 13-6, respectively, as special cases of the general quadratic. As such they are analyzed by means of the method applied in the previous section. The student, however, will quickly learn, (and should definitely be encouraged to do so), that the factored form in these cases can readily be determined by inspection thus eliminating the need for intermediate steps.

Since the "perfect square" concept is of major importance in the study of quadratic equations, the recognition test in section 13-6 should be given special attention. The process of "completing the square" is given an informal introduction at this point. This will help the student prepare for the work in Chapter 18.

The term "perfect square" as applied here is restricted to polynomials with integral coefficients. It may be necessary later on to extend this notion to include such polynomials as

$$x^2 + x + \frac{1}{4} \qquad \text{or even} \qquad x^2 + \sqrt{5}x + \frac{5}{4}.$$

However, such an extension should probably be postponed until the need arises.

There are essentially two motives behind the approach to factoring as used in this chapter aside from the mathematical virtues inherent in any unifying theory. They are:

(1) The factoring process is exhibited directly as a systematic reversal of the multiplication process.

(2) The student is not obliged to confront a wide variety of different types with the sometimes "painful" necessity of choosing the most appropriate method for

dealing with each separate type.

The factoring of quadratic polynomials plays a prominent role in the study of open sentences. Thus it has greater relevance for the student of introductory algebra than do the other types.

<u>Answers to Problem Set 13-4a</u>; pages 577-578:

1. (a) $(b^2 + b) + (b + 1)$
   (b) $(m^2 + 2m) - (3m + 6)$
   (c) $(a^2 - 12a) + (2a - 24)$
   (d) $(b^2 - 6b) - (4b - 24)$

2. (a) $(c + 3)c + (c + 3)5$
   (b) $(3c + 4)3c + (3c + 4)(-4)$
   (c) $(5m - 4)7m + (5m - 4)3$
   (d) $(2x + 1)x + (2x + 1)7$

3. (a) $(m + 2)(m + 3)$
   (b) $(y + 3)(y + 4)$
   (c) $(z - 2)(z + 3)$
   (d) $(x + 5)(x - 4)$

4. (a) $(3x + 2)(x + 3)$
   (b) $(2r - 3)(5r + 2)$
   (c) $(3q - 5)(2q + 3)$
   (d) $(2y - 5)(4y - 3)$
   (e) $(2m + 1)(2m + 1)$
   (f) $(3a + 4)(4a + 1)$

5. (a) $(x - 4)(x + b)$
   (b) $(c + d)(5 + a)$
   (c) $(3m + 2b)(3a + 4)$
   (d) $(a - 2b)(m^2 + n)$
   (e) $(x - y)(5 + a)$
   (f) $(2s + 3)(3s + 4)$

<u>Answers to Oral Exercises 13-4b</u>; page 582:

1. (a) 2, 5 (b) -2, -5 (c) 2, 4 (d) -2, -4 (e) -4, 2 (f) 4, -2 (g) 8, 1 (h) -8, -1 (i) 1, 1 (j) -1, -1 (k) -2, 1 (l) 2, -1

2. (a) $x^2 + 4x + 2x + 8$
   (b) $x^2 + 8x + x + 8$
   (c) $x^2 - 4x - 2x + 8$
   (d) $x^2 + x - 8x - 8$
   (e) $x^2 + 4x - 2x - 8$
   (f) $x^2 - 4x + 2x - 8$
   (g) $x^2 + 4x + 3x + 12$
   (h) $x^2 - 6x - 2x + 12$
   (i) $x^2 + x + 12x + 12$
   (j) $x^2 + 3x - 4x - 12$
   (k) $x^2 + 6x - 2x - 12$
   (l) $x^2 - 12x + x - 12$

<u>Answers to Problem Set 13-4b</u>; page 583:

1. (a) $x^2 + 5x + 3x + 15$
   (b) $r^2 - 5r + 3r - 15$
   (c) $s^2 + 9s - s - 9$
   (d) $t^2 + 3t + 2t + 6$
   (e) $w^2 - 11w + w - 11$
   (f) $m^2 - 6m + 2m - 12$
   (g) $a^2 - 4a - 3a + 12$
   (h) $y^2 + 6y + 4y + 24$
   (i) $z^2 - 6z - 5z + 30$
   (j) $c^2 + 4c - 6c - 24$

2. (a) $t^2 + 12t + 35 = t^2 + 7t + 5t + 35$
   $= (t + 7)t + (t + 7)5$
   $= (t + 7)(t + 5)$

   (b) $w^2 - 7w + 10 = w^2 - 2w - 5w + 10$
   $= (w - 2)w + (w - 2)(-5)$
   $= (w - 2)(w - 5)$

(c) $(r - 11)(r - 2)$
(d) $(a - 7)(a - 11)$
(e) $(m + 6)(m - 3)$
(f) $(z + 7)(z - 3)$
(g) $(a + 11)(a - 5)$
(h) $(y - 4)(y - 3)$
(i) $(x + 1)(x - 4)$
(j) $(b - 8)(b - 1)$

3. (a) $(y - 9)(y + 2)$
(b) $(a + 5)(a - 1)$
(c) $(x + 3)(x - 10)$
(d) $(w - 4)(w - 4)$
(e) $(w - 7)(w + 1)$
(f) $(k + 7)(k - 1)$
(g) $(x - 16)(x + 1)$
(h) $(x - 2)(x - 9)$
(i) $(m - 4)(m + 4)$
(j) $(t + 1)(t + 1)$

Answers to Oral Exercises 13-4c; page 589:

1. (a) $3x - 7$
$5x$
$5 - 2x$

(b) $2x^2 - 3x$
$x^2 + 5$
$x^2 - 2x + 5$
$2x^2$
$3 + 2x + 5x^2$
$9 - x^2$

(c) $2x^2 - 3x$
$x^2 + 5$
$x^2 - 2x + 5$
$2x^2$
$3 + 2x + 5x^2$
$9 - x^2$
$x^3 + 3x^2$

(d) $3x - 7$
$5x$
$5 - 2x$
$x^3 + 3x^2$

(e) $2x^2 - 3x$
$x^2 - 2x + 5$
$3 + 2x + 5x^2$

(f) $x^2 + 5$
$x^2 - 2x + 5$
$3x - 7$
$3 + 2x + 5x^2$
$5 - 2x$
$9 - x^2$

(g) $2x^2 - 3x$, -3
$x^2 - 2x + 5$, -2
$5x$, 5
$3 + 2x + 5x^2$, 2
$5 - 2x$, -2

(h) $2x^2 - 3x$, 2
$x^2 + 5$, 1
$x^2 - 2x + 5$, 1
$2x^2$, 2
$3 + 2x + 5x^2$, 5
$9 - x^2$, -1

2. (a) $(a)(b) = 24$
$a + b = -11$
$a = -8, \quad b = -3$

(b) $(a)(b) = 24$
$a + b = 10$
$a = 6, \quad b = 4$

(c) $(a)(b) = -24$
$a + b = -23$
$a = -24, \quad b = 1$

(d) $(a)(b) = -24$
$a + b = 10$
$a = -2, \quad b = 12$

(e) $a = 8, \quad b = -3$

(f) $a = -6, \quad b = 4$

(g) $a = 24, \quad b = 1$

(h) $a = -12, \quad b = -2$

Answers to Problem Set 13-4c; pages 590-591:

1. (a) $2y^2 + 7y + 3 = 2y^2 + 6y + y + 3$
$= (y + 3)(2y) + (y + 3)(1)$
$= (y + 3)(2y + 1)$

(b) $3y^2 + 7y + 2 = 3y^2 + y + 6y + 2$
$= (3y + 1)y + (3y + 1)2$
$= (3y + 1)(y + 2)$

(c) $(2a + 1)(2a + 1)$
(d) $(n - 1)(3n - 1)$
(e) $(3x - 2)(x - 1)$
(f) $(4x - 1)(2x - 3)$
(g) $(h - 12)(h + 2)$
(h) $(x - 1)(7x + 13)$
(i) $(3x - 1)(x + 7)$
(j) $(x + 1)(5x - 7)$
(k) $(2h + 1)(h - 5)$
(l) $(2u - 3)(2u - 3)$
(m) $(v + 9)(v + 4)$
(n) $(3x + 7)(2x + 3)$
(o) $(7t + 2)(2t + 5)$
(p) $(5y - 1)(7y - 3)$

2. (a) $(8x + 1)(x + 15)$ (i) $(4x + 1)(2x - 15)$
   (b) $(4x - 1)(2x - 15)$ (j) $(x - 1)(8x + 15)$
   (c) $(2x - 1)(4x + 15)$ (k) $(2x + 3)(4x - 5)$
   (d) $(x + 5)(8x - 3)$ (l) $(x + 3)(8x + 5)$
   (e) $(x - 5)(8x - 3)$ (m) $(8x + 1)(x - 15)$
   (f) $(4x + 5)(2x + 3)$ (n) $(2x + 1)(4x + 15)$
   (g) $(2x - 5)(4x + 3)$ (o) $(x - 1)(8x - 15)$
   (h) $(x - 3)(8x + 5)$ (p) $(2x + 5)(4x + 3)$

3. (a) $(2x + 3)(2x - 3)$ (f) $(2y - 6)(2y + 6)$
   (b) $(2h + 3)(2h + 3)$ (g) $(2y + 6)(2y + 6)$
   (c) $(z - 5)(z + 5)$ (h) $(x + 6)(x + 5)$
   (d) $(v + 1)(v + 1)$ (i) $(x + 4)(5x - 18)$
   (e) $(2y - 9)(y + 4)$ (j) $(x - 12)(5x - 6)$

13-5. <u>Differences of Squares</u>.

$a^2 - b^2 = (a + b)(a - b)$ may be stated in words: "the difference of the squares of two numbers is equal to the product of the sum of the numbers and the difference of the numbers." Some students may wonder about factoring expressions like $x^2 - 5$ when they see how to factor $x^2 - 4$. The former cannot be factored over the integers but, as we shall see in section 13-7, it can be factored over the reals as follows:

$$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5}).$$

For this reason the teacher should avoid statements like, "It is not factorable," but instead say, "It is not factorable over the integers."

Similarly, $x^2 + 9$, cannot be factored over the integers or over the reals, but if we were to extend our number system to include what we call the <u>complex numbers</u>, we could factor this expression

$$x^2 + 9 = x^2 - (-9) = (x - 3i)(x + 3i).$$

pages 594-595: 13-5

We see that both of the examples fit in with the form

$$a^2 - b^2 = (a + b)(a - b).$$

Answers to Oral Exercises 13-5; pages 594-595:

1. (a) $(n - 5)(n + 5)$ (f) $(2x - 1)(2x + 1)$
   (b) $(t + 6)(t - 6)$ (g) $(ab - 2)(ab + 2)$
   (c) $(7 - a)(7 + a)$ (h) $(3z + 5)(3z - 5)$
   (d) $(2x - 5)(2x + 5)$ (i) $(4a - 2b)(4a + 2b)$
   (e) $(m - n)(m + n)$ (j) $(1 - 2t)(1 + 2t)$

2. (a) $t^2 - 1$ (f) $9m^2 - 4$
   (b) $x^2 - 36$ (g) $u^2 - v^2$
   (c) $s^2 - 100$ (h) $s^2t^2 - u^2$
   (d) $x^2 - 49$ (i) $1 - 4k^2$
   (e) $4x^2 - 1$ (j) $w^2 - 225$

Answers to Problem Set 13-5; pages 595-596:

1. (a) $(n + 4)(n - 4)$ (f) $(w + 1)(w - 1)$
   (b) $(x - 5)(x + 5)$ (g) $(rs + 2)(rs - 2)$
   (c) $(t - 9)(t + 9)$ (h) $(3rs - 2)(3rs + 2)$
   (d) $(9 - t)(9 + t)$ (i) $(5n - 1)(5n + 1)$
   (e) $(8 + s)(8 - s)$ (j) $(4 - 7k)(4 + 7k)$

2. (a) $n^2 - 16$ (f) $w^2 - 225$
   (b) $t^2 - 1$ (g) $x^2 - 400$
   (c) $x^2 - 36$ (h) $m^2n^2 - 1$
   (d) $s^2 - 100$ (i) $4m^2n^2 - x^2$
   (e) $x^2 - 49$ (j) $4x^2 - 9$

3. (a) $(x + y)(x - y)$
   (b) $(x - 5)(x + 3)$
   (c) $(5y - 2)(y - 3)$
   (d) $5a(a + 1)$
   (e) $(m - 1)(m - 1)$
   (f) $(3ab - 5)(3ab + 5)$
   (g) $6z(1 - 3z)$
   (h) $(t - 1)(2t + 5)$
   (i) $(1 - 12t)(1 + 12t)$
   (j) $9(x^2 + 9)$
   (k) $(u + 17)(u + 1)$
   (l) $ab(a + b)$

4. (a) 396
   (b) 896
   (c) 899
   (d) 2496
   (e) 1591
   (f) 391x
   Hint: $(17x)(23) = x(20 + 3)(20 - 3)$.
   (g) 1596x
   (h) 2499xy
   (i) 9999
   (j) $(6)(6)(4)(11) = (36)(44)$
   $= 1584$

*5. (a) $(n + 1)(n - 1)$
   (b) $(x + 2)x$
   (c) $(x + a + 1)(x - a + 1)$
   (d) $(x + a)(x - a + 2)$
   (e) 4mn
   (f) $(x - y)(x + y - 1)$

*6. (a) $899 = 30^2 - 1$
   $= (30 - 1)(30 + 1)$, not prime.
   (b) $1591 = 40^2 - 3^2$
   $= (40 - 3)(40 + 3)$, not prime

pages 597-602: 13-6

Answers to Oral Exercises 13-6; page 601:

1. The expressions in parts (a), (e), (f), and (h) are perfect squares.

2. (a) 9
   (b) 25
   (c) 4

3. (a) $(x + 4)^2$ (d) $(2y + 4)^2$ or $4(y + 2)^2$
   (b) $(n - 3)^2$ (e) $(4m + 1)^2$
   (c) $(x + y)^2$ (f) $(2a + b)^2$

Answers to Problem Set 13-6; pages 602-604:

1. (a) yes (e) yes (i) yes
   (b) no (f) no (j) no
   (c) no (g) yes (k) yes
   (d) no (h) no (l) yes

2. (a) $x^2 - 8x + (16)$
   (b) $x^2 + 8x + (16)$
   (c) $n^2 + 2n + (1)$
   (d) $t^2 + 10t + (25)$
   (e) $y^2 - 16y + (64)$
   (f) $x^2 + (8x) + 16$ or $x^2 + (-8x) + 16$
   (g) $y^2 + (24y) + 144$ or $y^2 + (-24y) + 144$
   (h) $9s^2 + 6s + (1)$
   (i) $u^2 - (10u) + 25$ or $u^2 + (-10u) + 25$
   (j) $a^2 + 12a + (36)$
   (k) $4s^2 + 4st + (t^2)$
   (l) $(x^2) + 6xy + 9y^2$
   (m) $4s^2 + (12s) + 9$ or $4s^2 + (-12s) + 9$
   (n) $(16v^2) + 40v + 25$

(o) $49x^2 - (56xy) + 16y^2$ or $49x^2 - (-56xy) + 16y^2$

(p) $(v + 1)^2 + 4(v + 1) + (4)$

3. (a) $(x + 6)^2$
   (b) $(2s - 3)^2$
   (c) $2(m^2 + 6m + 36)$
   (d) $3(u + 1)^2$
   (e) $(v - t)^2$
   (f) $(6k - 1)^2$
   (g) $3(4t^2 + 12t + 9)$
   (h) $a(a - b)^2$
   (i) $2(2t + 1)^2$
   (j) $(x - 10)^2$

4. (a) $x^2 + 4x + 4$
   (b) $a^2 - 6a + 9$
   (c) $y^2 + 20y + 100$
   (d) $x^2 - 14x + 49$
   (e) $x^2 - 2xy + y^2$
   (f) $r^2 + 12r + 36$

5. (a) $(21)^2 = (20 + 1)^2$
   $= 400 + 40 + 1$
   $= 441$

   (b) $(41)(41) = (40 + 1)^2$
   $= 1600 + 80 + 1$
   $= 1681$

   (c) $(19)^2 = (20 - 1)^2$
   $= 400 - 40 + 1$
   $= 361$

   (d) $(39)^2 = (40 - 1)^2$
   $= 1600 - 80 + 1$
   $= 1521$

   (e) $(29)(29) = (30 - 1)^2$
   $= 900 - 60 + 1$
   $= 841$

(f) $(51)^2 = (50 + 1)^2$

$= 2500 + 100 + 1$

$= 2601$

(g) $(49)^2 = (50 - 1)^2$

$= 2500 - 100 + 1$

$= 2401$

(h) $(99)^2 = (100 - 1)^2$

$= 10,000 - 200 + 1$

$= 9,801$

(i) $(101)(101) = (100 + 1)^2$

$= 10,000 + 200 + 1$

$= 10,201$

6. (a) $(y - 6)(y - 2)$
(b) $(x + 3)^2$
(c) $a(m^2 + n^2)$
(d) $(t - 3)(3t + 2)$
(e) $(3s + 4)(3s - 4)$
(f) $(u + v)^2$
(g) $(x + 1)(6x - 7)$
(h) $3(y + 1)(y - 1)$
(i) $(u + 7)(u - 2)$
(j) $(y - 5)^2$
(k) $(1 - 3t)(1 + 3t)$
(l) $(s + 2)(s + 1)$
(m) $8(y^2 + 1)$
(n) $(2x + 1)(x + 2)$
(o) $8(m + 1)(m - 1)$
(p) $(2y - 1)^2$
(q) $a(x + 5)^2$
(r) $3(r + 4)(r - 4)$
(s) $4x(x - 5)$
(t) $5(m + 2)(m - 2)$
(u) $3a(x - 3)(x - 2)$
(v) $(t + 1)(3t + 5)$
(w) $x(a - b)(a + b)$
(x) $2(k - 3)(k + 2)$
(y) $(t - 1)(t + 1)(t^2 + 1)$
(z) $(y + 6)(5y - 12)$

pages 604-608: 13-7 and 13-8

Sections 13-7 and 13-8 on polynomials over the reals and over the rationals are intended to give the student a broader look at the subject of polynomials in general and the factoring of polynomials in particular. These sections may be omitted without serious loss of continuity if the teacher feels that the material may tend to confuse the issue. Such an omission might be seriously recommended for classes of particularly slow students.

The question of whether or not a given polynomial can be factored is now seen to depend on the particular set of coefficients under consideration. For example the polynomial $x^2 - 7$ is readily seen to be unfactorable over the integers. However, if this same expression is regarded as a polynomial over the real numbers, then the factored form is $(x + \sqrt{7})(x - \sqrt{7})$.

Once again the point is emphasized that factors must be of the same type as the expression being factored. In this sense an instruction which merely directs that a student factor $x^2 - 7$ without an accompanying statement relative to the coefficient set is ambiguous.

Polynomials over the rationals are shown in section 13-8 to be expressible as the product of a rational number and a polynomial over the integers. This essentially reduces the problem of factoring over the rationals to one of factoring over the integers. For simplicity sake this is the procedure recommended for this chapter.

In subsequent work it will probably be helpful to offer two options to the student in factoring an expression such as

$$x^2 + x + \frac{1}{4}.$$

He may wish to write

$$\frac{1}{4}(2x + 1)(2x + 1) \qquad \text{or} \qquad (x + \frac{1}{2})(x + \frac{1}{2}).$$

For the present it is perhaps wise to stick to the former approach.

pages 604-609: 13-7 and 13-8

In order to avoid uncertainties in the student's mind it is perhaps wise to establish a pattern of priorities in the general approach to factoring over the real numbers. The first attempt should always be to factor over the integers. If the polynomial contains fractional coefficients which are rational, then a common monomial factor consisting of the reciprocal of the L.C.D. of the fractions involved should be extracted. Factors containing irrational coefficients might be thought of as a kind of "last resort". It should be stressed, however, that the latter procedure is legitimate only for polynomials over the reals, and cannot be applied if the initial problem is stated in terms of rational or integral polynomials.

Answers to Oral Exercises 13-7; page 606:

1. $(x + \sqrt{7})(x - \sqrt{7})$
2. $(y - \sqrt{11})(y + \sqrt{11})$
3. $(3m - \sqrt{5})(3m + \sqrt{5})$
4. $(2z + \sqrt{3})(2z - \sqrt{3})$
5. $(k\sqrt{7} - n)(k\sqrt{7} + n)$
6. $(3x - y\sqrt{5})(3x + y\sqrt{5})$
7. $(ax - \sqrt{3})(ax + \sqrt{3})$
8. $(5ay - b\sqrt{7})(5ay + b\sqrt{7})$
9. $(a\sqrt{2} - \sqrt{3})(a\sqrt{2} + \sqrt{3})$
10. $(a - \sqrt{c})(a + \sqrt{c})$

Answers to Problem Set 13-7; page 606:

1. $(m + \sqrt{5})(m - \sqrt{5})$
2. $(t - 4)(t + 4)$
3. $(2x - \sqrt{7})(2x + \sqrt{7})$
4. $(2y - 3)(2y + 3)$
5. $(ab - \sqrt{3})(ab + \sqrt{3})$
6. $(z - 3a)(z + 3a)$
7. $(k - b\sqrt{7})(k + b\sqrt{7})$
8. $(6u + v\sqrt{5})(6u - v\sqrt{5})$
9. $(x\sqrt{3} - \sqrt{2})(x\sqrt{3} + \sqrt{2})$
10. $(r - \sqrt{c})(r + \sqrt{c})$

Answers to Oral Exercises 13-8; page 609:

1. $\frac{1}{2}(x^2 - 9)$
2. $\frac{1}{3}(t^2 + 2t + 1)$
3. $\frac{1}{5}(x^2 - 4)$
4. $\frac{2}{3}(y^2 + 2)$
5. $\frac{1}{4}(m^2 + 2)$
6. $\frac{1}{6}(9s^2 - 1)$
7. $\frac{1}{2}(y^2 - 4)$
8. $\frac{1}{6}(m^2 + 3m + 6)$
9. $\frac{1}{12}(4y^2 - 3)$
10. $\frac{1}{3}(9 - 2x^2)$

Answers to Problem Set 13-8; page 609:

1. $\frac{1}{2}(t + 3)(t - 3)$
2. $\frac{1}{3}(y + 2)^2$
3. $\frac{1}{5}(z^2 + 4)$
4. $\frac{1}{12}(x - 2)(x + 2)$
5. $\frac{1}{8}(x - 3)(x + 2)$
6. $\frac{1}{3}x(x + 2)$
7. $\frac{1}{2}(m + 2)(m - 2)$
8. $\frac{1}{6}(a + b)^2$
9. $\frac{1}{4}(x - 4)(x + 4)$
10. $\frac{1}{5}(5 - t)(5 + t)$

13-9. Truth Sets of Polynomial Equations.

When an equation is put in a form in which one side is a polynomial and the other side is 0, we call it a polynomial equation in standard form. The truth set of a polynomial equation can be found if we can factor the polynomial and then make use of the property that for any real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Answers to Oral Exercises 13-9; page 612:

(a) {-3,-2}
(b) {2,-4}
(c) {3,-3}
(d) {5, 2}
(e) {9,-1}
(f) {0, 1}

Answers to Problem Set 13-9; pages 612-613:

1. (a) {-1,-4}
   (b) {2, 3}
   (c) {-2,-1}
   (d) {-1, 9}
   (e) {-2, 7}
   (f) {-5, 3}
   (g) {-9,-8}
   (h) {1}

2. (a) $t^2 + 2t - 15 = 0$, {-5, 3}
   (b) $x^2 + 11y + 18 = 0$, {-9,-2}
   (c) $y^2 - 4y + 3 = 0$, {1, 3}

(d) $b^2 - 6b + 5 = 0$, $\{1, 5\}$

(e) $m^2 - 5m - 66 = 0$, $\{11, -6\}$

(f) $a^2 - 13a - 30 = 0$, $\{15, -2\}$

(g) $x^2 - 5x + 6 = 0$, $\{2, 3\}$

*(h) $18 + 7x - x^2 = 0$, $\{-2, 9\}$

*(i) $x^2 - 9x + 20 = 0$, $\{4, 5\}$

*(j) $y^2 - 49 = 0$, $\{7, -7\}$

*(k) $x^2 + 8x + 36 = 0$, This cannot be factored over the integers; we cannot find the truth set by present methods.

3. (a) If a is the number, then a^2 is the square of the number and the open sentence is

$$a^2 = 7 + 6a.$$

$$a^2 + (-(7 + 6a)) = 7 + 6a + (-(7 + 6a))$$

$$a^2 + (-7) + (-6a) = 7 + 6a + (-7) + (-6a)$$

$$a^2 - 7 - 6a = (7 + (-7)) + (6a + (-6a))$$

$$a^2 - 6a - 7 = 0 + 0$$

$$(a - 7)(a + 1) = 0$$

$$a = 7 \text{ or } a = (-1).$$

In the open sentence, if $a = 7$, then

$a^2 = 7^2 = 49$ and $7 + 6a = 7 + 42 = 49$

If $a = (-1)$, then

$a^2 = (-1)^2 = 1$ and $7 + 6a = 7 - 6 = 1$

The truth set of the original sentence is $\{-1, 7\}$.

(b) If w is the number of inches in the width of the rectangle, then $(w + 5)$ is the number of inches in the length of the rectangle and the open sentence is

$$w(w + 5) = 84.$$

The numbers -12 and 7 satisfy this sentence. However, the teacher should emphasize the fact that the domain of the variable for a measurement of length is the set of positive real numbers and (-12) is not within the domain. Consequently the truth set for the problem is $\{7\}$.

(c) The number can be 1 or 9.

(d) If m is the larger number, then the smaller one is $m - 8$. Then

$$m(m - 8) = 84$$
$$m^2 - 8m - 84 = 0$$
$$(m - 14)(m + 6) = 0$$
$$m = 14 \text{ or } m = -6.$$

The larger number can be either 14 or -6. The smaller number $(m - 8)$ is then 6 or -14. So two pairs satisfy the conditions of the problem

$$(14, 6) \quad \text{and} \quad (-6, -14).$$

(e) (5, 7) The pair $(-3, -1)$ is not acceptable if the odd numbers are considered a subset of the counting numbers, and thus positive.

*(f) $\ell(14 - \ell) = 24$

The truth set of this sentence is $\{12, 2\}$. If the length is to be longer than the width, then we use only

$$\ell = 12$$

and $\quad 14 - \ell = 2.$

So the dimensions are 12 ft. and 2 ft.

*(g) $(a + 4)(a - 3) = a^2$, where a is the number of feet in the length of the side of the square.

(1) 12 ft. (2) 16 ft. and 9 ft.

<u>Answers to Review Problem Set</u>; pages 616-620:

1. (a) 19 (e) -11
   (b) 16 (f) -26
   (c) 22 (g) 0
   (d) -14 (h) 11

2. (a) 11a (g) 2d
   (b) 5b (h) -5a
   (c) 3a (i) 2c - 2b
   (d) 0 (j) 5n - 4m
   (e) -3t (k) 2x - 3y + 4z
   (f) -12a

3. (a) 20 (h) 0
   (b) -27 (i) -8a
   (c) -32 (j) -18m
   (d) 30 (k) 18m
   (e) 30 (l) -3.75a
   (f) 14 (m) $-\frac{21}{8}b$
   (g) -24 (n) 0

4. (a) ab + ac (f) 9ac + 15bc
   (b) ab - ac (g) -a - b - c
   (c) -am - bm (h) -4a + 3b - 2m + 7
   (d) 2mn + 6mp (i) 0
   (e) $-6m^2 + 12mn$ (j) $a^2 + 5a + 6$

(k) $m^2 - 7m + 12$

(l) $a^2 + 2a - 24$

(m) $6a^2 + 17a + 12$

(n) $8b^2 - 2b - 1$

(o) $15c^2 - 28c + 12$

(p) $4m^2 - 23mn + 15n^2$

(q) $16m^2 - n^2$

(r) $ac + ad + bc + bd$

5. (a) 4

(b) 5

(c) -3

(d) -5

(e) 4

(f) 11

6. (a) $7x - 5$

(b) $5y - 73$

(c) $4x^2 + 5x - 2$

(d) $-3n + 3$

(e) $a + 2b$

(f) $9x - 5y$

(g) $6c - 4$

(h) $2m + 9n - 14$

(i) $8x - 7y - 4z$

(j) $7r - 10s + 8t$

7. (a) $6(a + 2b)$

(b) $4(c)$

(c) $m(m + 2b + 1)$

(d) $(-3)(a - 2b + 4c)$

(e) $2a(b + 4x - 5y)$

(f) $(x + 3)(x + 7)$

(g) $(2r + 5s)(3m + 7)$

(h) $(a + b)(c - d)$

(i) $(3m - 4n)(5m + 7r)$

(j) $(5a - b)(3c - 4d)$

(k) $(b + 5)(b + 6)$

(l) $(a - 10)(a + 7)$

(m) $(m + 8)(m - 1)$

(n) $(c - 2d)(3c - 5d)$

(o) $a(2b - c)(b - 2c)$

(p) $ab\,(m - 8)(m + 8)$

(q) $3t(2c - 5d)(2c + 5d)$

(r) $(x - 8)^2$

(s) $(3x + 10)^2$

(t) $(2x - 5a)^2$

(u) This is a prime polynomial over the rationals. However, over real numbers, $5 - 4x^2$ can be factored: $(\sqrt{5} - 2s)(\sqrt{5} + 2s)$.

8. (a) $\frac{11}{30}$

(b) $\frac{2a + b}{4}$

(c) $\frac{5m + 6n + 27}{90}$

(d) $-\frac{29}{10a}$

(e) $-\frac{1}{6}$

(f) $\frac{m^2}{36}$

(g) $\frac{a^5b^5}{m^2n}$

(h) $\frac{b^3y^2z^2}{ax^4}$

(i) $-m^4$

(j) $\frac{-25x^4}{9m^2}$

(k) $\frac{21}{16}$

(l) $\frac{5ab}{m}$

(m) $\frac{7am^3n^3}{x}$

(n) -1

9. (a) $\{15\}$

(b) $\{\frac{10}{3}\}$

(c) $\{19\}$

(d) $\{-2\}$

(e) $\{-9, 9\}$

(f) $\{-\frac{1}{3}, 1\}$

(g) $\{\frac{1}{2}, 4\}$

(h) $\{0, 5\}$

(i) all numbers greater than -7.

(j) all numbers greater than -1, but less than 7.

(k) $\emptyset$

10. (a) 24 feet, 16 feet

(b) Ann is 36 years old, Jerry is 12 years old.

(c) 8 girls, 10 boys

(d) 11, 4

(e) 7 inches

Suggested Test Items

1. Which of the following expressions are:

   polynomials ____________
   prime polynomials ______________
   perfect squares ___________
   differences of squares _______________

   Assume in each case that we are considering polynomials over the integers.

   (a) $16m^2 - 1$ (f) $6x^2 + y^2$
   (b) $6x^2 + 15y$ (g) $3a - b$
   (c) $x^2 - 10x + 25$ (h) $\frac{4x^2}{9}$
   (d) $\frac{x^2 - a^2}{7}$ (i) $x^2 - 5x + 4$
   (e) $x^2 + 7x - 3x - 21$ (j) $a^2 + 2ab + b^2$

2. Find a prime factorization of each of the following:

   (a) $m^2 - 4m + 3$ (f) $ax + 7x + 3a + 21$
   (b) $2ax^2 + 2ay^2$ (g) $a^2 + 16a + 64$
   (c) $c^2 - 100$ (h) $2x^2 - 6x - 8$
   (d) $ax^2 + 7ax + 12a$ (i) $2y^2 + 3y - 5$
   (e) $9y^2 - c^2$ (j) $3x^2 - 301x + 100$

3. Find the truth sets of each of the following equations:

   (a) $x^2 + 11x = 0$ (c) $4c^2 - 49 = 0$
   (b) $x^2 + 8x = 20$ (d) $2x^2 + 20x + 50 = 0$

4. Write each of the following indicated products as an indicated sum.

   (a) $(5x - 4y)(5x + 4y)$ (d) $(m - 4)^2$
   (b) $(3a + b)^2$ *(e) $(x + \sqrt{2})^2$
   (c) $(w + 5)(w + 2)$ (f) $(c + 8)(c - 3)$

5. Complete each of the following so that the resulting polynomial is a perfect square.

(a) $c^2 + 12c + (\quad)$ (d) $y^2 + (\quad) + 49$

(b) $m^2 - 10m + (\quad)$ (e) $a^2 + (\quad) + 81b^2$

(c) $4a^2 - 4a + (\quad)$ (f) $25x^2 - (\quad) + 9y^2$

Answers to Suggested Test Items

1. polynomials: (a), (b), (c), (e), (f), (g), (i), (j).

   prime polynomials: (f), (g).

   perfect squares: (c), (j).

   difference of squares: (a).

2. (a) $(m - 3)(m - 1)$ (f) $(x + 3)(a + 7)$

   (b) $2a(x^2 + y^2)$ (g) $(a + 8)^2$

   (c) $(c + 10)(c - 10)$ (h) $2(x - 4)(x + 1)$

   (d) $a(x + 4)(x + 3)$ (i) $(2y + 5)(y - 1)$

   (e) $(3y + c)(3y - c)$ (j) $(3x - 1)(x - 100)$

3. (a) $\{0, -11\}$ (c) $\{\frac{7}{2}, -\frac{7}{2}\}$

   (b) $\{2, -10\}$ (d) $\{-5\}$

4. (a) $25x^2 - 16y^2$ (d) $m^2 - 8m + 16$

   (b) $9a^2 + 6ab + b^2$ *(e) $x^2 + 2\sqrt{2}x + 2$

   (c) $w^2 + 7w + 10$ (f) $c^2 + 5c - 24$

5. (a) $c^2 + 12c + 36$ (d) $y^2 + 14y + 49$

   (b) $m^2 - 10m + 25$ (e) $a^2 + 18ab + 81b^2$

   (c) $4a^2 - 4a + 1$ (f) $25x^2 - 30xy + 9y^2$

Chapter 14

RATIONAL EXPRESSIONS

Although this chapter deals with the definition of, and operations with, rational expressions, the term "rational expression" itself is not introduced until section 14-5. Earlier in the chapter, indicated quotients of polynomials are introduced; and multiplication and addition of such indicated quotients are considered in sections 14-3 and 14-4, respectively. It is in section 14-5 that a formal definition of rational expression is introduced. It is an extension of the definition of polynomial in Chapter 13, in that the operation of division is permitted, and the polynomials are seen to constitute a subset of the set of rational expressions.

It is also pointed out in section 14-5 that any rational expression, though it may not be the indicated quotient of two polynomials, can be "simplified" to such a form. Therefore, since the skills of adding and multiplying any two indicated quotients of polynomials have already been developed, any two rational expressions can be added or multiplied by first changing them to this form. Subtraction presents no special problem, since, by definition, an expression indicating subtraction may be restated in terms of addition; this is done in section 14-4. Division, however, does present a special problem, and section 14-6 is devoted to the long division process regularly presented in elementary algebra. In this section, the process of dividing two polynomials is compared to that of changing an "improper fraction" to a "mixed number" in arithmetic; this is a sound pedagogical device, in that new understandings and skills are derived from old ones.

In fact, this chapter is motivated in part by an analogy between the integers and the polynomials (over the integers) and between rational numbers and rational expressions. As one example of the fruits of this analogy (besides the one mentioned above), every rational number can be expressed as the indicated quotient of two integers, and every rational expression can be expressed as the indicated quotient of two polynomials.

The analogy can be established in only a casual way at this time, however, since the student in this course has had no experience with the idea of an indeterminate, which gives expressions a life of their own, independent of numbers. So far as the student is concerned, expressions are names for numbers. Both polynomials and rational expressions are subsets of the set of all numerals. Thus, the implication in this chapter is that we have an algebra of <u>numerals</u> with the same structure as the algebra of <u>numbers</u>.

Even though the student's appreciation of the analogy between numbers and expressions is necessarily limited, there are certain understandings the teacher should have in teaching this chapter. The following discussion is concerned with such understandings.

Consider, for example, the distributive property:

$$a(b + c) = ab + ac.$$

We have always understood a, b, and c, to be real numbers, so that we are dealing with an assertion about real numbers. The assertion involves two phrases "$a(b + c)$" and "$ab + ac$" and enables us to replace either phrase by the other, in any statement about real numbers, without altering the validity of the statement. However, suppose we forget, for the moment, that we are talking about real numbers (as was commonly done at one time in elementary algebra). Then the distributive property (or "law") becomes a "rule" for transforming algebraic expressions, that is, a rule in the "game" of "symbol pushing". From this point of view, the various fundamental properties, with which we have been working, constitute the complete set of rules of the game. Attention is thus shifted from the system of real numbers to the <u>language</u> used to talk about the real numbers. Although blind symbol pushing is highly undesirable, it is a fact that we do work with expressions from this point of view. This is what we are doing whenever we discuss the <u>form</u> of an expression. The difference is that symbol-pushing at this level is not mechanical but is with reference to an algebraic system. We shall now describe more carefully this system.

In the first place, we "add" and "multiply" expressions by use of what we have called "indicated" sums and products. Thus, if A and B are expressions then $A + B$ and $A \cdot B$ are also expressions. We also write $A = B$ provided that for each permitted value of each variable involved in A and B, the numerals "A" and "B" name the same number. This is actually a <u>definition of equality</u> for expressions. The numerals "0" and "1" are themselves expressions and serve as the additive and multiplicative identities. With these agreements, the following <u>basic properties</u> could be found for expressions and have, in fact, been used many times in the course:

1. If A and B are expressions, then $A + B$ is an expression.
2. If A and B are expressions, then $A + B = B + A$.
3. If A, B, and C are expressions, then $(A + B) + C = A + (B + C)$.
4. There is an expression 0 such that $A + 0 = A$ for every A.
5. For each expression A, there is an expression $-A$ such that $A + (-A) = 0$.
6. If A and B are expressions, $A \cdot B$ is an expression.
7. If A and B are expressions, $AB = BA$.
8. If A, B, and C are expressions, then $(AB)C = A(BC)$.
9. There is an expression 1 such that $A \cdot 1 = A$ for every A.
10. For each expression A different from 0, there is an expression $\frac{1}{A}$ such that $A \cdot \frac{1}{A} = 1$.
11. If A, B, and C are expressions, then $A(B + C) = AB + AC$.

Thus we see that the class of rational expressions satisfies the axioms for a <u>field</u>. The class of all polynomials (or all polynomials in one variable over the integers) is a sub-system of the class of rational expressions and has all of these properties except Number 10. Notice also that the rational numbers satisfy all of these properties, and the integers satisfy all except Number 10--hence, the parallel between rational expressions and polynomials, on the one hand, with rational numbers and integers on the other.

Once these general properties are established, we can study rational expressions and polynomials as algebraic systems in their own right independently of their connection with real numbers. Our work with factoring, simplification of rational expressions and division of polynomials forms a small fragment of the study of these general systems, although we have not presented it explicitly as such. This way of looking at the language of algebra, which is implicit in much of what has gone before, will turn up frequently in later courses in algebra. A good student automatically shifts to this point of view about algebra as he matures. However, if this occurs before he understands, at least intuitively, that an algebraic system is involved, only confusion will result. This is why it is important to go back to the real numbers whenever students show signs of mechanical manipulation of symbols. For further discussion see Studies in Mathematics, Volume III, Sections 6-1 - 6-6.

14-1. Polynomials and Integers.

In this opening section, an analogy is made between the integers and the polynomials over the integers. Although the analogy is necessarily flimsy at this time, it is worth making because of the use that will be made of it in sections 14-5 and 14-6.

The analogy rests on the fact that polynomials (over the integers), considered as numerals, have the same closure properties as the integers.

> To take addition for example, the integers are closed under addition. That is, the sum of any two integers is an integer. (The whole idea of closure may have to be reviewed at this time.) Now, the indicated sum of two polynomials is also a polynomial, since addition is one of the permissible operations in generating a polynomial. Therefore, the indicated sum of two polynomial numerals is another polynomial numeral. In this way, it makes sense to the student to say that the polynomials are closed under addition. (Again, as

teachers, we must not overlook the fact that when polynomials are considered as abstract expressions, not as numerals, closure of polynomials under addition is more substantive.)

In a similar way, it is established that the polynomials, like the integers, are closed under subtraction and multiplication, but not under division.

Answers to Oral Exercises 14-1; page 622:

1. A polynomial (over a set of numbers) is an expression which indicates addition, subtraction, multiplication or taking opposites of elements of the set of numbers and a set of variables. Any numeral for a number of the set or any variable is also a polynomial.

2. Polynomials and integers share the following closure properties:
   (a) closed under multiplication, addition and subtraction.
   (b) not closed under division.

Answers to Problem Set 14-1; pages 622-623:

1. In every instance the resulting expression is a polynomial.
   (a) $5x - 2$
   (b) $5x - 11$
   (c) $12a + 4$
   (d) $a^2 - 1$
   (e) $3a^2 + 3a + 1$
   (f) $3a^2 - 4a + 1$
   (g) $2ax - 2x + 7a - 7$
   (h) $x^2 + 10x + 25$
   (i) 0
   (j) $3x^3 - 3x^2 + 5x - 7$

2. (a) $\{-\frac{1}{5}\}$
   (b) $\emptyset$
   (c) $\{-\frac{2}{7}\}$

3. If the height of the smaller triangle is h inches, then the area of the triangle is $\frac{1}{2}(4)h$ square inches. The area of the second triangle is $\frac{1}{2}(4)(\frac{3}{2}h)$ square inches. The open sentence is

$$\frac{1}{2}(4)h + \frac{1}{2}(4)(\frac{3}{2}h) = 15.$$

$$2h + 3h = 15$$
$$5h = 15$$
$$h = 3$$

The height of the smaller triangle is 3 inches; the height of the larger, $\frac{9}{2}$ inches.

14-2. Quotients of Polynomials.

Indicated quotients of polynomials are singled out for discussion here. As mentioned earlier, they will be identified as rational expressions in a later section.

The principal purpose of this section is to provide the student with more experience in restricting the domain of a variable--or variables--so that the expression in which it appears represents a real number. In this case, of course, the emphasis is on indicated quotients of polynomials, and the restrictions are aimed at avoiding zero denominators.

It is true that the student has had such experiences before. However, this is a more concentrated attack on the problem. In the next sections, where indicated quotients of polynomials are multiplied and added, restrictions on domains must be kept constantly in mind. It is hoped that this section will help the student do this.

Answers to Oral Exercises 14-2; pages 625-626:

1. Values for which the denominator of the expression is zero

2. $\frac{3}{x + 1}$ names a real number for all x except $x = -1$.

3. The domain of x for the expression $\frac{1}{x} + 1$ does not include 0 because division by zero is not defined; that is, $\frac{1}{0} + 1$ is not a real number.

4. (a) no restrictions on x
   (b) Exclude 0.
   (c) no restrictions on m or x
   (d) x cannot be negative.
   (e) Exclude 0.
   (f) Exclude $-\frac{1}{2}$.
   (g) no restriction on a
   (h) no restriction on x
   (i) Exclude -2
   (j) Exclude 1 and -2.
   (k) no restrictions on a or on b
   (l) Exclude $\frac{5}{3}$.
   (m) Exclude all numbers less than -1.
   (n) All values of x must be excluded because for all values of x the denominator is zero and the indicated quotient does not name a real number.
   (o) no restrictions on x

<u>Answers to Problem Set 14-2</u>; page 626:

1. no restriction
2. Exclude 1.
3. Exclude -3.
4. Exclude -5 from the domain of b.
5. Exclude $\frac{5}{2}$.
6. no restriction on either domain
7. Exclude 0 from the domains of x and y
8. no restriction on either domain
9. Exclude $-\frac{4}{7}$
10. Exclude all values. This expression never names a real number since its denominator is always 0.
11. Exclude -2 and -3.
12. no restriction
13. Exclude 0 and -1.
14. Exclude -7 and 3.
15. Exclude $-\sqrt{2}$ and $\sqrt{2}$.

<u>Answers to Oral Exercises 14-3a</u>; pages 629-630:

1. "$\frac{2x - 1}{2x - 1} = 1$" is true for all values of x except $\frac{1}{2}$.

2. (a) $\frac{2}{3x^2}$, $x \neq 0$

   (b) 1, $k \neq 0$

   (c) $\frac{1}{3}$, $a \neq 0$, $a \neq -2$

   (d) $\frac{6}{m}$, $m \neq 0$, $x \neq 0$

   (e) $\frac{m^2 + 2m}{6}$, no restrictions on m

   (f) $\frac{1}{2}$, $x \neq 0$, $x \neq -2$

   (g) $\frac{x^2 - 4}{4x}$, $x \neq 0$

   (h) 1, $k \neq -3$

<u>Answers to Problem Set 14-3a</u>; page 630:

1. $\frac{22}{x^2}$, $x \neq 0$

2. $\frac{a^2}{c^2}$, $b \neq 0$, $c \neq 0$

3. $\frac{15x + 60}{x + 2}$, $x \neq -2$, $x \neq 1$

4. $\frac{3b}{2}$, $b \neq 0$, $a \neq -12$

5. 2, $m \neq 0$, $m \neq -\frac{1}{2}$

6. $\frac{2}{3}$, $a \neq -b$, $a \neq b$

7. $\frac{a - b}{3} \cdot \frac{2}{b - a} = \frac{a - b}{3} \cdot \frac{2}{(-1)(a - b)}$

   $= -\frac{2}{3}$, $a \neq b$

8. $\frac{a^2b^3}{4}$, $a \neq 0$, $c \neq 0$

9. ab, $b \neq 0$

10. 1, $k \neq 1$, $k \neq -3$

11. $\frac{5x^2 + 30x + 45}{3x^3 - 18x^2 + 27x}$, $x \neq 0$, $x \neq 3$

12. $\frac{x + 1}{3}$, $x \neq -1$, $x \neq 1$

13. $\left(\frac{(x + 1)x + (x + 1)2}{x + 2}\right) \cdot \frac{5}{x + 1}$

    $= \frac{(x + 1)(x + 2)5}{(x + 2)(x + 1)}$

    $= 5$, $x \neq -2$, $x \neq -1$

Answers to Problem Set 14-3b; pages 632-633:

1. $\frac{x - 2}{x}$, $x \neq 0$, $x \neq -2$

2. $\frac{m^2 + 10m + 25}{3m}$, $m \neq 0$, $m \neq 5$.

3. $\frac{4y}{y + 5}$, $y \neq 0$, $y \neq 5$, $y \neq -5$

4. $x + 5$, $x \neq -2$, $x \neq 3$

5. $\frac{x + 1}{x}$, $x \neq 0$, $x \neq 1$, $x \neq 2$

6. $a^2 - 36$, $a \neq 6$, $a \neq -6$

7. $\frac{b}{1 + b}$, $b \neq 1$, $b \neq -1$

8. $\frac{3m^2 - m - 2}{m}$, $m \neq 0$, $m \neq -1$

9. $x + 2$, $x \neq 0$, $x \neq 2$, $x \neq -7$

10. $\frac{38m}{5}$, $m \neq 0$, $m \neq 1$, $m \neq -1$

11. $\frac{a + 6}{x}$, $x \neq 0$, $a \neq 6$, $a \neq -6$

12. $\frac{35x^2 + 28x}{8x + 10}$, $x \neq 0$, $x \neq -\frac{5}{4}$

13. $\frac{5a^2b + 25ab + 30b}{3a^3 + 27a^2 + 60a}$, $a \neq 0$, $a \neq -4$, $a \neq -5$

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14. The denominator $5y - 2(2y) - y$ is zero for all values of y. It is impossible to perform the indicated operation.

15. Let x be the larger part (number).
Then $80 - x$ is the smaller part.
Then we have

$$\frac{1}{2}x + 5(80 - x) = 76$$

Solution set: {72}

The parts are 72 and 8.

16. Let x be the tens' digit. Then $12 - x$ is the units digit and the number is

$$10x + (12 - x).$$

$$x = 3(12 - x)$$

Solution set: {9}

The number is $10(9) + 3$ or 93.

Answers to Oral Exercises 14-4a; page 636:

1. $6a^3$
2. $(x + 2)(x + 3)$
3. a^2b^2
4. a^2c^2
5. $(x - 2)^2$
6. $48m^3n$
7. $24a^2$
8. $45a^2b^2$
9. $8(m + 3)^2$
10. $72a^2b^3$
11. $60a^2b$
12. $30(a - b)^2$

Answers to Problem Set 14-4a; pages 636-637:

1. $\frac{2}{b} + \frac{3}{7b} = \frac{2}{b} \cdot \frac{7}{7} + \frac{3}{7b}$

$= \frac{14}{7b} + \frac{3}{7b}$

$= \frac{17}{7b}$, $b \neq 0$

2. $\frac{11}{3a}$, $a \neq 0$

3. $\frac{4c + 1}{12c^2}$, $c \neq 0$

4. $\frac{25x + 6}{60x^2}$, $x \neq 0$

5. $\frac{25ay - 4}{30y^2}$, $y \neq 0$

6. $\frac{2n + 3m}{6m^2n}$, $m \neq 0$, $n \neq 0$

7. $\frac{14}{3(x - 1)}$, $x \neq 1$

8. $\frac{3 - x}{2(x + 3)^2}$, $x \neq -3$

9. $\frac{9a + 4b}{5(a^2 - b^2)}$, $a \neq b$, $a \neq -b$

10. $\frac{2a}{(a - b)^2} - \frac{3}{a - b} = \frac{2a}{(a - b)^2} + \frac{-3}{a - b}\left(\frac{a - b}{a - b}\right)$

$$= \frac{2a}{(a - b)^2} + \frac{-3a + 3b}{(a - b)^2}$$

$$= \frac{-a + 3b}{(a - b)^2}, \quad a \neq b$$

11. $\frac{75 - 9x}{25x^2}$, $x \neq 0$

12. $\frac{bc + ac + ab}{abc}$, $a \neq 0$, $b \neq 0$, $c \neq 0$

13. $\frac{2x + 5}{x(x - 1)}$, $x \neq 1$, $x \neq 0$

14. $\frac{12x - 21}{(x + 3)(x - 3)}$, $x \neq 3$, $x \neq -3$

15. $\frac{5m - 8}{(m - 1)(m - 2)}$, $m \neq 1$, $m \neq 2$

pages 637-640: 14-4

16. $\frac{-8x}{(x + 5)(x - 3)}$, $x \neq 3$, $x \neq -5$.

17. $\frac{x^2 - 2xy - y^2}{(x + y)(x - y)}$, $x \neq y$, $x \neq -y$

18. $\frac{5}{a - b}$, $a \neq b$

19. $\frac{y^2 - 3y + 8}{2y^2}$, $y \neq 0$

20. $\frac{x^2 + 8x + 1}{x^2 - 1}$, $x \neq 1$, $x \neq -1$

Answers to Problem Set 14-4b; pages 639-640:

1. $\frac{16x - 15}{3x(x - 1)}$, $x \neq 0$, $x \neq 1$

2. $\frac{11x + 26}{x^2 + 2x}$, $x \neq 0$, $x \neq -2$

3. $\frac{a^2 + 2a - 4}{a^2(a - 4)}$, $a \neq 0$, $a \neq 4$

4. $\frac{-c^2 + 2c + 6}{c^2(c + 2)}$, $c \neq 0$, $c \neq -2$

5. $\frac{3(b - 6)}{b(b - 5)}$, $b \neq 0$, $b \neq 5$

6. $\frac{16 + 5b}{7(b - 1)^2}$, $b \neq 1$

7. $\frac{-y^2 + 8y - 18}{y(y - 3)(y + 3)}$, $y \neq 0$, $y \neq 3$, $y \neq -3$

8. $\frac{1}{x - 1}$, $x \neq 1$, $x \neq -1$

9. $\frac{8y}{(y + 4)(y - 2)}$, $y \neq -4$, $y \neq 2$

10. $\frac{9 - 5x}{3x(x + 2)}$, $x \neq 0$, $x \neq -2$

11. $\frac{9m + 5}{(m + 5)(m - 4)(m - 3)}$, $m \neq -5$, $m \neq 4$, $m \neq 3$

12. $\frac{7a - 7b + 6}{(a - b)^2}$, $a \neq b$

13. $\frac{2(x + 4)}{(x - 1)(x + 1)^2}$, $x \neq 1$, $x \neq -1$

14. $\frac{2x^2 - 7x + 24}{(x + 8)(x - 1)(x - 5)}$, $x \neq -8$, $x \neq 1$, $x \neq 5$

15. $\frac{v^2 - 6v - 4}{(v + 4)(v - 3)(v - 5)}$, $v \neq -4$, $v \neq 3$, $v \neq 5$

16. $\frac{2y(2y - 5)}{(y - 4)(y + 2)(y - 2)}$, $y \neq 4$, $y \neq -2$, $y \neq 2$

17. $\frac{-x + 2}{(x + 1)(2x - 1)}$, $x \neq -1$, $x \neq \frac{1}{2}$

18. $\frac{19x + 12}{2(x - 2)(3x + 4)}$, $x \neq 2$, $x \neq -\frac{4}{3}$

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19. $\frac{7a + 4}{(a + 1)(a - 1)}$, $a \neq -1$, $a \neq 1$

20. $\frac{11z + 65}{6(z - 5)(z + 5)}$, $z \neq 5$, $z \neq -5$

Note: The answers above have numerator and denominator in factored form. This is not a requirement for a simplified form. For example $\frac{8y}{y^2 + 2y - 8}$ would be a perfectly acceptable form for the answer to problem 9 above.

14-5. Rational Expressions and Rational Numbers.

In section 14-1, it was pointed out that the integers and the polynomials (over the integers) are both closed under addition, subtraction, and multiplication, but not under division. Thus, an analogy was established at that time.

In this section, the analogy is extended. If the operation of division is introduced in the set of integers, the set of rational numbers is generated. As the student has seen earlier, every rational number can be expressed as the indicated quotient of two integers.

Since something has been made of the analogy between the polynomials and the integers, and since the quotient of two integers represents a rational number, it seems natural to use the word "rational" in connection with the indicated quotient of two polynomials. This is indeed what is done. The indicated quotients of polynomials which have been the subject of the past three sections are called rational expressions.

In fact, this can be given as the definition of a rational expression. However, the following definition is given in the student's text:

pages 642-645: 14-5

A rational expression is one which indicates at most the operations of addition, subtraction, multiplication, division, and taking opposites.

This definition has at least two advantages:

(1) It parallels the definition given for polynomial in Chapter 13.

(2) It makes it easy to identify rational expressions which are not in the form of the indicated quotient of two polynomials. e.g.,

$$\frac{2 - \frac{3}{x - 5}}{x - 2 + \frac{1}{x}} .$$

Although a rational expression does not have to appear as the indicated quotient of two polynomials over the integers, this is usually considered a preferred form of a rational expression. Simplification of a rational expression to a quotient of two polynomials constitutes a major part of the problem set for this section.

Answers to Oral Exercises 14-5; page 645:

1. Since the set of polynomials is a subset of the set of rational expressions any polynomial is a rational expression. An example is $x^2 + 2x + 1$.

2. Any expression which involves the operations with radicals, absolute values or division by the zero expression is not a rational expression. An example is $\sqrt{x^2 + y^2}$.

3. Previous comments apply. An example is $\frac{x^2 + 2y^2 + m^3}{3xy}$.

4. (a) polynomial and rational expression
   (b) rational expression only
   (c) polynomial and rational expression
   (d) Neither - it is a sentence.

(e) Neither - the denominator is the zero expression which excludes the expression from the set of rational expressions.

(f) rational expression only

(g) Neither - it involves an indicated square root.

(h) Neither - although it is another name for a number which can be expressed by a polynomial. Here we are talking not about numbers but about <u>forms</u> of expression. This expression does not have the form of a polynomial.

(i) Neither - the operation of taking the absolute value is not among those prescribed for polynomials or rational expressions.

(j) Neither - it is a sentence.

(k) rational expression only

(l) polynomial and rational expression

(m) Neither - even though it is another name for 0 and "0" is a polynomial, this <u>form</u> is not that of a polynomial.

(n) rational expression only

(o) rational expression only - Even though simplification would result in a polynomial, the form would be different, and so would be the domain of the variable.

<u>Answers to Problem Set 14-5</u>; pages 645-646:

1. (a) polynomial and rational expression in 2 variables
   (b) polynomial and rational expression in 1 variable
   (c) polynomial and rational expression in 1 variable
   (d) rational expression in 2 variables
   (e) rational expression in 1 variable
   (f) polynomial and rational expression
   (g) polynomial and rational expression in 1 variable
   (h) none of these
   (i) none of these - sentence in one variable
   (j) polynomial and rational expression
   (k) polynomial and rational expression in 3 variables
   (l) polynomial and rational expression in 3 variables

(m) polynomial and rational expression

(n) rational expression in 3 variables

(o) none of these

(p) none of these - sentence in one variable

(q) rational expression in 1 variable

2. Many answers are possible

(a) an example is $(x + 5)^3$

(b) an example is $\frac{x^2}{3x}$

(c) an example is $|x|$

3. (a) $\dfrac{\frac{1}{2}m + \frac{1}{3}}{\frac{1}{3}m} \cdot \dfrac{6}{6} = \dfrac{(\frac{1}{2}m + \frac{1}{3})(6)}{(\frac{1}{3}m)(6)}$

$= \dfrac{3m + 2}{2m}, \quad m \neq 0$

(b) $\dfrac{3 + \frac{1}{a}}{\frac{1}{a}} \cdot \dfrac{a}{a} = \dfrac{(3 + \frac{1}{a}) \cdot a}{\frac{1}{a} \cdot a}$

$= 3a + 1, \quad a \neq 0$

(c) $b - 1, \quad b \neq 0$

(d) $\dfrac{3 + 5x + x^2}{x^2}, \quad x \neq 0$

(e) $\dfrac{y^2 + y}{3}, \quad y \neq 0$

(f) $\dfrac{x + 1}{2(x - 1)}, \quad x \neq 1,$

$x \neq -1$

(g) $\dfrac{3x + 7}{x + 2}, \quad x \neq -2$

(h) $y(y^2 + 2)$ or $y^3 + 2y,$

$y \neq 0$

(i) $\dfrac{3}{u^2v^2}, \quad u \neq 0, \quad v \neq 0$

(j) $\dfrac{3y + 1}{3y - 1}, \quad y \neq 0, \quad y \neq \frac{1}{3}$

pages 647-649: 14-6

14-6. Dividing Polynomials.

With the establishment of a link between the integers and the polynomials and between rational numbers and rational expressions, an expression such

$$\frac{2x^2 + x - 5}{x - 3}$$

can be compared to an expression such as

$$\frac{168}{11}.$$

The second case represents what many students may call an "improper fraction"--one in which the numerator is not smaller than the denominator.

In the course of this section, the student will come to see that the first case above is "improper" in the sense that the degree of the numerator is not less than the degree of the denominator.

Thus, both kinds of expressions are simplified by the same process--long division.

The section begins with an explanation of long division as repeated subtraction, since this process is one often poorly understood by students. Then the process is extended to apply to the division of one polynomial by another.

The fundamental idea of the section may be represented by the following property of the system of polynomials:

> Let N and D be polynomials with D different from the zero polynomial. Then <u>there exist</u> polynomials Q and R with R of lower degree than D such that $N = QD + R$.

This property is analogous to the following property of the system of integers:

> Let n and d be positive integers with d different from 0. Then <u>there exist</u> non-negative integers q and r with r less than d such that $n = qd + r$.

Answers to Oral Exercises 14-6a; page 650:

1. Several ways are possible, such as $1 + \frac{11}{4}$, $2 + \frac{7}{4}$, etc.
   In the form $\frac{n}{d} = q + \frac{r}{d}$, $0 \leq r < d$, $\frac{15}{4} = 3 + \frac{3}{4}$.

2. $\frac{x}{5} = 2 + \frac{2}{5}$ $\quad x = 5(2) + 2$ $\quad$ The dividend is 12.

3. $\frac{16}{19} = 0 + \frac{16}{19}$

4. (a) $d = 5$
   (b) $q = 2$
   (c) $r = 4$
   (d) $n = 50$
   (e) $d = 44$
   (f) $q = 2$ $\quad d = 5$
   (g) $d = 7$
   (h) $r = 4$ $\quad d = 11$
   (i) $r = 2$ $\quad d = 5$
   (j) $n = 57$ $\quad d = 5$

Answers to Problem Set 14-6a; page 650-651:

1. $229 = 17(13) + 8$
   $\frac{229}{17} = 13 + \frac{8}{17}$

2. $486 = 23(21) + 3$
   $\frac{486}{23} = 21 + \frac{3}{23}$

3. $192 = 131(1) + 61$
   $\frac{192}{131} = 1 + \frac{61}{131}$

4. $768 = 47(16) + 16$
   $\frac{768}{47} = 16 + \frac{16}{47}$

5. $18 = 2(9) + 0$
   $\frac{18}{9} = 2 + \frac{0}{9}$

6. $41 = 4(10) + 1$
   $\frac{41}{10} = 4 + \frac{1}{10}$

7. $22 = 4(5) + 2$
   $\frac{22}{5} = 4 + \frac{2}{5}$

8. $45 = 3(15) + 0$
   $\frac{45}{15} = 3 + \frac{0}{15}$

9. If h is the number of hours he walked, then $4h$ was the number of miles he walked. His rate was $\frac{4h}{h}$ or 4 miles per hour.

10. If x is the number of quarters in each pile, then

$$29.75 = 3(.25)x + 2(.25)$$
$$.75x = 29.25$$
$$x = 29.25(\tfrac{4}{3})$$
$$x = 39$$

The truth set is $\{39\}$. There were 39 quarters in each pile.

Answers to Oral Exercises 14-6b; pages 654-655:

1. (a) 2 (b) 3 (c) 1 (d) 0 (e) 2 (f) 1

2. (a) $8x$ (b) $5x$ (c) $2a$ (d) $3a$ (e) -7 (f) $3m^2$

3. (a) $-6x^2$
   (b) $-y^2 + 4y + 2$
   (c) $4x + 2$
   (d) $14x - 2$
   (e) $7x + 5$
   (f) $-2x^3 - 3x^2$
   (g) $y^3 - 3y^2 - y$
   (h) $3x + 7$
   (i) 0
   (j) $-\frac{7}{6}x^2 + x - \frac{3}{8}$

Answers to Problem Set 14-6b; pages 655-656:

1. (a) $5x^2 - 7x - 10 = (5x + 3)(x - 2) + (-4)$
   dividend quotient divisor remainder

   (b) $x^3 + 2x^2 - x + 10 = (x^2 - x + 2)(x + 3) + 4$
   dividend quotient divisor remainder

   (c) $x^3 + 2x^2 + 2x - 1 = (x^2 + x + 1)(x + 1) + (-2)$
   dividend quotient divisor remainder

pages 655-658: 14-6

2. (a) $\frac{x^3 - 2x^2 + 2x + 1}{x - 1} = x^2 - x + 1 + \frac{2}{x - 1}$

dividend: $x^3 - 2x^2 + 2x + 1$ quotient: $x^2 - x + 1$
divisor: $x - 1$ remainder: 2

(b) $\frac{x^3 + 4x^2 + 3x - 5}{x + 2} = x^2 + 2x - 1 + \frac{-3}{x + 2}$

dividend: $x^3 + 4x^2 + 3x - 5$ quotient: $x^2 + 2x - 1$
divisor: $x + 2$ remainder: -3

(c) $\frac{3x^2 - 10x - 3}{x - 4} = 3x + 2 + \frac{5}{x - 4}$

dividend: $3x^2 - 10x - 3$ quotient: $3x + 2$
divisor: $x - 4$ remainder: 5

3. (a) x^2 ; x^2
(b) $4x^3$; $-15x^3$
(c) $-5x$; $-7x + 4$
(d) $2x^2$; 0
(e) $3x^3$; $-6x^4 - 3x^3 - 2x^2$
(f) x ; $x + 1$
(g) $3x$; $14x - 7$
(h) $4x^2$; $-3x^2 + 2x + 4$
(i) $3x$; $9x^2 - 10x + 7$
(j) $2x^2$; $-7x^4 + 8x^3 - 3x^2 - x + 2$

Answers to Oral Exercises 14-6c; page 658:

1. d is a factor of n if r is zero in the sentence

$$\frac{n}{d} = q + \frac{r}{d}.$$

2. (a) $x - 3$ is a factor since the remainder is zero.
(b) $x + 3$ is not a factor since the remainder is one, not zero.
(c) $x - 1$ is a factor since the remainder is zero.
(d) $14x - 2$ is a factor since the remainder is zero.
(e) $2x - 3$ is not a factor. The remainder is 2, not zero.
(f) $x^2 - 2x - 1$ is a factor since the remainder is zero.

Answers to Problem Set 14-6c; pages 658-659:

1. $\frac{x^2 + 5x + 6}{x + 3} = x + 2$ since $(x + 2)(x + 3) = x^2 + 5x + 6$

2. $\frac{4x^2 - 4x + 1}{2x - 1} = 2x - 1$ since $4x^2 - 4x + 1 = (2x - 1)(2x - 1)$

3. $\frac{2x^2 - 4x + 3}{x - 2} = 2x + \frac{3}{x - 2}$ since $2x^2 - 4x + 3 = 2x(x - 2) + 3$

4. $\frac{5x^3 + 4x^2 - 3x + 7}{x + 1} = 5x^2 - x - 2 + \frac{9}{x + 1}$

5. $\frac{x^3 - 3x^2 + 7x - 1}{x - 3} = x^2 + 7 + \frac{20}{x - 3}$

6. $\frac{x^4 - 9x^2 - 1}{x + 3} = x^3 - 3x^2 - \frac{1}{x + 3}$

7. $\frac{5x^3 - 11x + 7}{x + 2} = 5x^2 - 10x + 9 - \frac{11}{x + 2}$

8. $\frac{x^3 + 1}{x + 1} = x^2 - x + 1$

9. $\frac{x^4 - 1}{x - 1} = x^3 + x^2 + x + 1$

10. $\frac{2x^3 - 2x^2 + 5}{x - 6} = 2x^2 + 10x + 60 + \frac{365}{x - 6}$

11. $\frac{2x^5 + x^3 - 5x^2 + 2}{x - 1} = 2x^4 + 2x^3 + 3x^2 - 2x - 2$

12. $\frac{x^5 + 1}{x + 1} = x^4 - x^3 + x^2 - x + 1$

13. $\frac{9x^2 + 12x + 4}{3x + 2} = 3x + 2$

*14. $\frac{6x^3 - x^2 - 5x + 4}{3x - 2} = 2x^2 + x - 1 + \frac{2}{3x - 2}$

15. $3x + 7$

16. $2x^2 + x - 4$

17. $3x^2 - 1$

*18. (a) $x + 2$
(b) $3x + 5$
(c) $5x - 1$
(d) $2x + 4$
(e) $x - 3$

Answers to Review Problem Set; pages 660-664:

1. All are rational except in (i), (j), (k), and (r).
   All are polynomials except in (h), (i), (j), (k), (l), (m), (n), (o), and (p).
   The expressions in (b), (q), and (r) are polynomials in one variable.
   The expressions in (a), (b), (c), (d), and (q) are polynomials over the integers.
   The expressions in (a), (b), (c), (d), ~~(e), (f), (g),~~ and (q) are polynomials over the rationals.
   The expressions in (a), (b), (c), (d), (e), (f), (g), (q), and (r) are polynomials over the reals.

2. (a) $3\sqrt{2}$
   (b) $5\sqrt{2}$
   (c) $\frac{\sqrt{3}}{2}$
   (d) $2\sqrt{3}$
   (e) $|a|$
   (f) $6\sqrt{2} + 6\sqrt{3}$
   (g) $\frac{\sqrt{2} - 4\sqrt{3}}{2}$
   (h) $\sqrt{3}\ a^2$
   (i) $|x + y|$
   (j) $\frac{|x|}{2}\sqrt{2}$

3. (a) $3\sqrt{2} + 9$
   (b) $2\sqrt{3} + 1$
   (c) $4\sqrt{3} - 6\sqrt{2}$
   (d) $5 + 2\sqrt{6}$
   (e) -3

4. (a) $3a(1 + 2b)$
   (b) $(x + y)(4)$
   (c) $3x(m - 4 + 4y)$
   (d) $(x - 6)(x - 5)$
   (e) $(x - 24)(x + 2)$
   (f) prime
   (g) $(a - b)(a + b - 4)$
   (h) $(x + y)(x - y - 4)$
   (i) $3a^2b^3(ab^2 - 2 + 4a^2b)$
   *(j) $(3a - 2)(2a - 5)$
   *(k) $(3a - 2)(2a + 5)$
   (l) $4(4x - 5a)(4x + 5a)$
   (m) $2(4a - 9)(4a + 9)$
   *(n) $(m + n - a)(m + n + a)$

5. (a) $\frac{46}{231}$
   (b) $\frac{3b - 2a}{ab}$
   (c) $\frac{15a + 2b}{3x}$
   (d) $\frac{15b + 91a}{175a^2b}$
   (e) $\frac{8x^2 - 3x + 6}{4x(x - 2)}$
   (f) $\frac{4a - 5}{2(a - 2)}$
   (g) $\frac{-2x^2 + 8x}{(x-3)(x+3)(x-1)}$
   (h) $\frac{2m^2 + m + 12}{m^2 - 4}$

pages 662-664

6. (a) $3x + 5$

(b) $2x + 5 + \frac{10}{x - 1}$

(c) $x^2 - x - 2$

(d) $x^4 + x^3 + x^2 + x + 1$

(e) $2x + 3$

7. (a) $\{\frac{7}{12}\}$

(b) the set of all numbers greater than $(-\frac{9}{2})$

(c) $\{\frac{5}{7}\}$

(d) $\{\frac{5}{12}\}$

(e) $\{7, -\frac{5}{4}\}$

(f) $\{5, 8\}$

(g) $\{-\frac{3}{2}\}$

(h) $\{25\}$

(i) $\{-\frac{5}{4}\}$

(j) $\{-9, 9\}$

(k) $\{5, 1\}$

(l) $\{-7, 2\}$

*(m) $\{0, \frac{2}{3}, -\frac{2}{3}\}$

*(n) $\{3, -3\}$

*(o) the set of all numbers equal to or greater than 8; or equal to or less than 2

8.
$$\begin{array}{r}
2x + 7 \\
x-3 \overline{)\, 2x^2 + x - 20} \\
\underline{2x^2 - 6x} \\
7x - 20 \\
\underline{7x - 21} \\
1
\end{array}$$

Since the remainder polynomial is not zero, $x - 3$ is not a factor of $2x^2 + x - 20$.

9. (a) $\sqrt[3]{\pi^3}$ is not rational

(b) $\sqrt{.04} = .2$, is rational

(c) $\sqrt[3]{-1} = -1$, is rational

(d) $\sqrt{\frac{81}{100}} = \frac{9}{10}$, is rational

(e) $\sqrt{11}$ is not rational

10. The "average" of $\frac{x + 3}{x}$ and $\frac{x - 3}{x}$ is $\frac{1}{2}(\frac{x + 3}{x} + \frac{x - 3}{x})$ which is 1.

11. If the number is x, then the sentence is $x^2 = 91 + 6x$.

$$x^2 - 6x - 91 = 0$$

$$(x + 7)(x - 13) = 0$$

The truth set is $\{-7, 13\}$. The number can be -7 or it can be 13.

12. If the smaller integer is x, then the next successive integer is $x + 1$. Their reciprocals are $\frac{1}{x}$ and $\frac{1}{x + 1}$, respectively. The sentence is

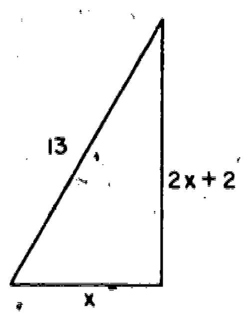
$$\frac{1}{x} + \frac{1}{x + 1} = \frac{15}{56}.$$

$$56(x + 1) + 56(x) = 15(x)(x + 1)$$

$$15x^2 - 97x - 56 = 0$$

$$(15x + 8)(x - 7) = 0$$

The truth set is $\{7\}$ since the domain permits only integers. The first integer is 7 and the second is 8.

13. 

If the length of the shorter leg is x feet, then the longer leg has length $2x + 2$ feet. The Pythagorean relation applies, resulting in the sentence

$$x^2 + (2x + 2)^2 = 13^2.$$

$$5x^2 + 8x - 165 = 0$$

$$(5x + 33)(x - 5) = 0$$

The truth set is $\{5\}$ since negative values must be excluded from the domain. The legs are 5 feet and 12 feet long.

14. If the passenger train travels at r miles per hour, then the jet travels at $10r$ miles per hour. The sentence is

$$10r = 120 + 8r.$$

$$r = 60$$

Since the truth set is $\{60\}$, the rate of the passenger train is 60 miles per hour and the rate of the jet is 600 miles per hour.

15. If n is the number of pounds with nut centers, then $40 - n$ is the number of pounds with cream centers. The value of the nut centers is $n(1.40)$ dollars and of the creams $(40 - n)(1.00)$ dollars. The value of the mixture is $40(1.10)$ dollars, so our sentence is

$$n(1.40) + (40 - n)(1.00) = 40(1.10).$$
$$.40n = 4$$
$$n = 10$$

The truth set is $\{10\}$. Thus there should be 10 pounds of nut centers and 30 pounds of creams.

16. If the faster train travels at the rate of x miles per hour, then the slower train travels at the rate of $\frac{2}{3}x$ miles per hour. The distances traveled by the trains are $x(3\frac{1}{5})$ miles and $(\frac{2}{3}x)(3\frac{1}{5})$ miles, respectively. The sentence is

$$\frac{16}{5}x + \frac{32}{15}x = 160.$$
$$x = 160(\frac{15}{80})$$
$$x = 30.$$

The truth set is $\{30\}$. The faster train is going at an average rate of 30 miles per hour, the slower at an average rate of 20 miles per hour.

Suggested Test Items

1. Classify the following numbers by writing the letter which identifies the number in the spaces provided. Some numbers may fall into more than one classification.

   (a) $\sqrt{\frac{32}{2}} - 12 \cdot 2$

   (b) $\sqrt{11} - \sqrt{2}$

   (c) $\dfrac{\frac{4}{5}}{\frac{5}{4}}$

   (d) $\dfrac{1 + \frac{1}{3}}{1 - \frac{1}{3}}$

   (e) $-6 + \frac{3}{2} - \frac{7}{14}$

   (f) $\frac{5}{6} \cdot \frac{2}{3} - 5 \cdot \frac{1}{9}$

   (g) $\sqrt{\frac{4}{5}} \cdot \sqrt{\frac{5}{4}}$

   (h) $\sqrt{11 - 2}$

   (i) $\sqrt{2 - 11}$

   positive integers ____________________
   integers ____________________
   rational numbers ____________________
   real numbers ____________________
   none of the above ____________________

2. If a rational expression is not a polynomial how does it differ from a polynomial?

3. Classify the following expressions by writing the letter which identifies the expression in the spaces provided. Some expressions may fall into more than one classification.

   (a) $(x - 3)(x + 2)$

   (b) $s + \frac{1}{3}t$

   (c) $\sqrt{x + 1}$

   (d) $\frac{3}{x^2} + \frac{1}{x}$

   (e) $s^2 \sqrt{3} + 1$

   (f) $\dfrac{(x - 1)(x - 2)}{(x - 2)}$

   (g) $|x + 1|$

   polynomials over the integers ____________________
   polynomials over the rational numbers ____________________
   polynomials over the real numbers ____________________
   rational expressions ____________________
   none of the above ____________________

4. State the conditions under which the expressions

$\frac{(x + 1)(x + 3)}{x + 3}$ and $x + 1$ name the same number.

5. Explain what is meant by "simplifying" a rational expression.

6. Simplify. (In each case indicate the domain of the variable.)

(a) $5 - \frac{3}{x + 2}$

(b) $\frac{3a}{a - 2} \cdot \frac{a^2 - 4}{6a^2}$

(c) $\frac{z - 3}{2z} - \frac{z + 5}{z^2}$

(d) $\frac{3}{x^2 - 4x - 5} + \frac{4}{x^2 + x}$

(e) $\frac{a^2 + 11a - 26}{a^2 - 5a + 6}$

(f) $\frac{x + 1}{x^2 + x} - \frac{1}{x - 1}$

(g) $\frac{a^2 - a}{a^2 - 1} \cdot \frac{a^2 - 3a - 4}{a^2 - 5a + 4}$

(h) $\frac{2b}{b - 1} \cdot \frac{1 - b}{4b^3}$

(i) $\frac{1 + \frac{1}{x}}{2 + \frac{2}{x}}$

7. Determine polynomials q and r (with r of lower degree than $x + 1$) such that

$$2x^2 - 4x + 1 = (x + 1)q + r.$$

8. Determine polynomials q and r (with r of lower degree than the divisor) such that

(a) $\frac{3x^3 + x - 4}{x - 2} = q + \frac{r}{x - 2}$

(b) $\frac{x^4 - 1}{x - 1} = q + \frac{r}{x - 1}$

(c) $\frac{x + 7}{x^2 + 6} = q + \frac{r}{x^2 + 6}$

9. Is $x + 2$ a factor of $x^5 + 2x^4 - x^2 + 5x + 14$? (That is, can a polynomial q be found so that

$\frac{x^5 + 2x^4 - x^2 + 5x + 14}{x + 2} = q$, with $r = 0$?)

10. Find the truth set of each of the following sentences.

(a) $(5x - 2)(2x + 7) = 0$

(b) $3x^2 + 11x + 6 = 0$

(c) $\frac{5}{x} + \frac{7}{2x} = 3$

(d) $\frac{2x}{x - 1} + \frac{3x}{x^2 - 1} = 2$

(e) $\frac{x}{x + 1} + \frac{2}{x + 1} = 1$

11. Solve each of the following problems.

(a) Two trains 400 feet apart are approaching each other on the same track. One is traveling 70 feet per second and the other is going 90 feet per second. After how much time will the trains collide?

(b) A school dance committee has an assortment of candy bars to be sold at 10¢ each and twice as many cups of soft drink to be sold at 15¢ each. The total income from the affair, including $177 for tickets, was $313 after selling out all refreshments. How many drinks and bars were sold?

(c) If the committee in part (b) made a profit of 60% of cost, what was their profit for the evening and how much did the dance cost?

Answers to Suggested Test Items

1. positive integers (c), (d), (g), (h)
   integers (a), (c), (d), (e), (f), (g), (h)
   rational numbers (a), (c), (d), (e), (f), (g), (h)
   real numbers (a), (b), (c), (d), (e), (f), (g), (h)
   none of the above (i)

2. If a rational expression is not a polynomial expression, then it involves an indicated division.

3. polynomials over the integers (a)
   polynomials over the rational numbers (a), (b)
   polynomials over the real numbers (a), (b), (e)
   rational expressions (a), (b), (d), (e), (f)
   none of the above (c), (g)

4. $\frac{(x+1)(x+3)}{x+3}$ and $x + 1$ name the same number if x is different from -3.

5. "Simplifying" a rational expression means writing the rational expression as an indicated quotient of two polynomials which do not have common factors.

6. (a) $\frac{5x+7}{x+2}$, (domain: $x \neq -2$)

   (b) $\frac{a+2}{2a}$, (domain: $a \neq 2$, $a \neq 0$)

   (c) $\frac{z^2 - 5z + 10}{2z^2}$, (domain: $z \neq 0$)

   (d) $\frac{7x - 20}{x(x-5)(x+1)}$, (domain: $x \neq 0$, $x \neq -1$, $x \neq 5$)

   (e) $\frac{a+13}{a-3}$, (domain: $a \neq 3$, $a \neq 2$)

   (f) $\frac{-1}{x^2 - x}$, (domain: $x \neq 1$, $x \neq 0$)

   (g) $\frac{a}{a-1}$, (domain: $a \neq 1$, $a \neq -1$, $a \neq 4$)

   (h) $\frac{-1}{2b^2}$, (domain: $b \neq 1$, $b \neq 0$)

   (i) $\frac{1}{2}$, (domain: $x \neq 0$, $x \neq -1$)

7. $2x^2 - 4x + 1 = (x+1)(2x-6) + 7$

8. (a) $\frac{3x^3 + x - 4}{x - 2} = 3x^2 + 6x + 13 + \frac{22}{x-2}$

   (b) $\frac{x^4 - 1}{x - 1} = x^3 + x^2 + x + 1 + \frac{0}{x-1}$

   (c) $\frac{x+7}{x^2+6} = 0 + \frac{x+7}{x^2+6}$

9. $\frac{x^5 + 2x^4 - x^2 + 5x + 14}{x + 2} = x^4 - x + 7 + \frac{0}{x + 2}$

$r = 0$; therefore, $x + 2$ is a factor of $x^5 + 2x^4 - x^2 + 5x + 14$

10. (a) $\{\frac{2}{5}, -\frac{7}{2}\}$

(b) $\{-\frac{2}{3}, -3\}$

(c) $\{\frac{17}{6}\}$

(d) $\{-\frac{2}{5}\}$

(e) $\emptyset$

11. (a) Each train will travel t seconds before the collision. The sentence is $70t + 90t = 400$.

$$160t = 400$$

$$t = \frac{5}{2}$$

The truth set is $\{\frac{5}{2}\}$. The trains will collide after $2\frac{1}{2}$ seconds.

(b) If n is the number of candy bars sold and $2n$ is the number of cups of soft drink sold, the sale of the candy bars amounted to $.10n$ dollars and the sale of the cups of soft drink amounted to $.15(2n)$ dollars. The income from the refreshments was $136. So the sentence is

$$.10n + .30n = 136.$$

$$.4n = 136$$

$$n = 340$$

The truth set is $\{340\}$. Therefore, 340 candy bars were sold and 680 cups of soft drink were sold.

(c) The dance cost n dollars. The committee made a profit of $.60n$. Total income from the affair was $313. The sentence is $n + .6n = 313$

$$1.6n = 313$$

$$n = 195.625$$

The truth set is $\{195.625\}$. Therefore, the profit was $117.33 and the cost was $195.62.

Chapter 15

TRUTH SETS OF OPEN SENTENCES

In this chapter we take a more careful look at the process of finding the truth set of a sentence. By developing a theory of equivalent equations and equivalent inequalities, we are able to determine when a new sentence has the same truth set as the original sentence without having to check in the original sentence.

Special attention is focused on the domain of the variable as determined by the form of the original sentence under consideration. We reestablish the convention that unless otherwise indicated the domain of the variable for each sentence will be the set of all real numbers for which that sentence has meaning.

Alertness should be developed for the possibility of a zero denominator for certain real values of the variable.

Background material on open sentences and equivalent sentences will be found in Studies in Mathematics, Volume III, Sections 6.8 - 6.16.

15-1. Equivalent Open Sentences.

The important concept being emphasized here is an understanding of why sentences are equivalent. If your students have any trouble with the technique of deciding what to do to a sentence to obtain a simpler sentence, you may want to point out how an indicated addition can be "undone" by adding the opposite (as in adding $(-y - 7)$ in the second example) and an indicated multiplication can be "undone" by multiplying by the reciprocal (as in multiplying by $\frac{1}{3}$ in the first example).

The number zero plays a prominent role throughout the chapter. At the outset the student should be fully aware that zero has no reciprocal, hence multiplication by zero is not a reversible step.

Answers to Oral Exercises 15-1a; page 668:

1. (a) adding -6
   (b) adding 10
   (c) multiplying by $\frac{1}{5}$
   (d) adding $-\frac{2}{3}$
   (e) adding $(-y - 7)$
   (f) multiplying by $\frac{4}{3}$
   (g) adding $(x + 6)$
   (h) multiplying by 10
   (i) multiplying by $-\frac{1}{6}$
   (j) adding $-w$

2. (a) $\{4\}$
   (b) $\{1\}$
   (c) $\{2\}$
   (d) $\{3\}$
   (e) $\{4\}$
   (f) $\{\frac{1}{3}\}$
   (g) $\{0\}$
   (h) $\{\frac{3}{4}\}$

Answers to Problem Set 15-1a; pages 668-670:

1. (a) $\{2\}$; $\{2\}$ Yes, they are equivalent.
   (b) $\{-2\}$; $\{38\}$ No, they are not equivalent.
   (c) $\{6\}$; $\{6\}$ Yes, they are equivalent.
   (d) $\{3\}$; $\{4\}$ No, they are not equivalent.

2. (a) $3x - 8 = 12$ add 8 to each side to obtain $3x = 20$

   $3x = 20$ add (-8) to each side to obtain $3x - 8 = 12$

   (b) $4y = 12$ multiply both sides by $\frac{1}{4}$ to obtain $y = 3$

   $y = 3$ multiply both sides by 4 to obtain $4y = 12$

   (c) $9t + 7 = 2t$ add $(-2t)$ to each side to obtain $7t + 7 = 0$

   $7t + 7 = 0$ add $2t$ to each side to obtain $9t + 7 = 2t$

(d) $7x - 2 = 3x + 5$	add $(-3x)$ to each side to obtain $4x - 2 = 5$
$4x - 2 = 5$	add $3x$ to each side to obtain $7x - 2 = 3x + 5$
(e) $3 + 5 = 3t + 5$	add (-5) to each side and then multiply by $\frac{2}{3}$ to obtain $2 = 2t$
$2 = 2t$	multiply each side by $\frac{3}{2}$ and then add 5 to each side to obtain $3 + 5 = 3t + 5$
(f) $\frac{3}{4}h - 1 = h$	multiply each side by 4 to obtain $3h - 4 = 4h$
$3h - 4 = 4h$	multiply each side by $\frac{1}{4}$ to obtain $\frac{3}{4}h - 1 = h$
(g) $\frac{k}{4} + \frac{1}{2} = \frac{k}{2} - \frac{1}{4}$	multiply each side by 4 to obtain $k + 2 = 2k - 1$
$k + 2 = 2k - 1$	multiply each side by $\frac{1}{4}$ to obtain $\frac{k}{4} + \frac{1}{2} = \frac{k}{2} - \frac{1}{4}$
(h) $3x^2 = 27$	multiply each side by $\frac{1}{3}$ to obtain $x^2 = 9$
$x^2 = 9$	multiply each side by 3 to obtain $3x^2 = 27$
(i) $14 + n = 25 + 3n$	add $(-13 - n)$ to each side to obtain $1 = 12 + 2n$
$1 = 12 + 2n$	add $(13 + n)$ to each side to obtain $14 + n = 25 + 3n$

3. Where the sentences are equivalent, only one of the given solutions is necessary. Both are included since one could start with either sentence.

(a)

$$2x = 12$$
$$(\tfrac{1}{2})2x = \tfrac{1}{2}(12)$$
$$x = 6$$

$$x = 6$$
$$(2)x = (2)6$$
$$2x = 12$$

They are equivalent.

(b)

$$12 + 3n = 5n$$
$$12 + 3n + (-3n) = 5n + (-3n)$$
$$12 = 2n$$

$$12 = 2n$$
$$12 + (3n) = 2n + (3n)$$
$$12 + 3n = 5n$$

They are equivalent.

(c)

$$5y - 4 = 3y + 8$$
$$5y - 4 + (-3y + 4) = 3y + 8 + (-3y + 4)$$
$$2y = 12$$
$$(\tfrac{1}{2})2y = \tfrac{1}{2}(12)$$
$$y = 6$$

$$y = 6$$
$$(2)y = (2)6$$
$$2y = 12$$
$$2y + (3y - 4) = 12 + (3y - 4)$$
$$5y - 4 = 3y + 8$$

They are equivalent.

(d)

$$7s - 5s = 12$$
$$2s = 12$$
$$(\tfrac{1}{2})2s = \tfrac{1}{2}(12)$$
$$s = 6$$

$$s = 6$$
$$(2)s = (2)6$$
$$2s = 12$$

They are equivalent.

(e)

$$3x + 9 - 2x = 7x - 12$$
$$3x + 9 - 2x + (12 - x) = 7x - 12 + (12 - x)$$
$$21 = 6x$$
$$(\tfrac{1}{6})21 = (\tfrac{1}{6})6x$$
$$\tfrac{7}{2} = x$$

$$\tfrac{7}{3} = x$$

They are not equivalent since the truth set of the first sentence is $\{\frac{7}{2}\}$ while the truth set of the second sentence is $\{\frac{7}{3}\}$.

(f)

$$2x^2 + 4 = 10$$
$$2x^2 + 4 + (-4) = 10 + (-4)$$
$$2x^2 = 6$$
$$(\tfrac{1}{2})2x^2 = (\tfrac{1}{2})6$$
$$x^2 = 3$$

$$x^2 = 4$$

They are not equivalent. 2 is a truth number of the sentence $x^2 = 4$, but it is not a truth number of the sentence $2x^2 + 4 = 10$.

4. (a) equivalent (e) equivalent
(b) equivalent (f) equivalent
(c) equivalent (g) equivalent
(d) equivalent (h) equivalent

Note: (g) may be questioned by the student since carry-out operations on the second sentence does not appear to produce the first sentence. Remind him that $(-1)0 = (-1)(2x - x^2)$ does produce $0 = x^2 - 2x$.

5. (a) {2} (g) ∅
(b) {2} (h) {15}
(c) {-1} (i) {4}
(d) {1} (j) {9}
(e) {0} (k) {-3}
(f) {6} (l) {80}

Page 670. Here we warn the student of the hazards of multiplication by an expression containing the variable. In the first example the multiplier has a zero denominator when $x = 3$, and in the second the multiplier becomes zero for $w = 0$. In both cases the truth set is altered.

The general purpose of this section is to caution the student against the type of manipulation which fails to preserve equivalence. The first example illustrates a more serious danger involving the "loss" of a truth value. In most situations the student can redeem the situation by checking, but in this case the missing element may be gone forever. A "safe" method for solving such problems is given later.

Answers to Oral Exercises 15-1b; page 673:

1. They are equivalent, because the second sentence can be obtained from the first by multiplying each side by 3. Since "multiplying each side by 3" is a reversible step, it guarantees that the new sentence is equivalent to the first. One could also answer "yes" on the basis that the truth sets of the two sentences are the same. Both reasons are important but in this chapter we are emphasizing the first.

2. They are not equivalent. The truth set for $x = 3$ is $\{3\}$ while the truth set for $x^2 = 3x$ is $\{0, 3\}$. The second sentence can be obtained from the first by multiplying both sides of the first by x, but this step does not guarantee equivalence unless x is a non-zero real number (that is, zero excluded from the domain). Since x can be zero in this case, equivalence is not guaranteed. Thus we look at the truth sets.

3. They are equivalent, since the second can be obtained from the first by multiplying both sides by a real number $(\frac{1}{7})$.

4. They are equivalent.

5. They are not equivalent. The second sentence can be obtained from the first by multiplying both sides by $\frac{1}{x}$ but $\frac{1}{x}$ is not a real number for every x, thus equivalence is not guaranteed. The truth sets are $\{0, 1\}$ and $\{1\}$.

6. They are not equivalent; the truth set for $t = 1$ is $\{1\}$ while the truth set for $t^2 = t$ is $\{0, 1\}$.

7. They are not equivalent; the truth set for $m^2 = m$ is $\{0, 1\}$ while the truth set for $m = 1$ is $\{1\}$.

8. They are not equivalent; the truth set for $5(y - 1) = y(y - 1)$ is $\{5, 1\}$ but the truth set for $5 = y$ is $\{5\}$.

9. They are not equivalent; the truth set for $6 = m$ is $\{6\}$ but the truth set for $6(m - 1) = m(m - 1)$ is $\{6, 1\}$.

10. They are not equivalent; the truth set for $x(x + 1) = 0$ is $\{0, -1\}$ but the truth set for $x + 1 = 0(\frac{1}{x})$ is $\{-1\}$.

<u>Answers to Problem Set 15-1b</u>; pages 673-675:

1.

	(i) The phrase is zero,	(ii) The phrase is not a real number,
(a)	if x is 3	never
(b)	if y is (-4)	never
(c)	never	if t is 2
(d)	never	if h is 0
(e)	if x is 0	never
(f)	if t is 2 or 3	never
(g)	never	if t is (-1) or 0
(h)	never	if x is 1 or (-1)

2.

	(i) (zero)	(ii) (not a real number)	(iii) (a non-zero real number)
(a)	if y is -5	no	no
(b)	if x is 0	no	no
(c)	no	no	yes
(d)	if x is 0	no	no
(e)	no	if y is 6	no
(f)	no	no	yes
(g)	if x is 1, 2, or 3	no	no

3. (a) $2x + 3 = x + 5$ — The sentences are equivalent since $x + 3$ is a real number for every x.

   (b) $12 = 4x$ — The sentences are equivalent since 4 is a real number.

   (c) $2y - 2 = y + 2$ — The sentences are equivalent since $y - 2$ is a real number for every y.

   (d) $x^2 = x$ — The sentences are not equivalent since the truth set of $x = 1$ is $\{1\}$ while the truth set of $x^2 = x$ is $\{0, 1\}$. This "happened" because we multiplied each side by an expression that is not a non-zero real number for every value of the variable.

   (e) $w^2 - 3w = 0$ — The sentences are not equivalent since the truth set of $w = 0$ is $\{0\}$ and the truth set of $w^2 - 3w = 0$ is $\{3, 0\}$. We multiplied by $(w - 3)$ which is not a non-zero real number for every value of the variable.

   (f) $\frac{x^2}{5} + 3x = 5x$ — The sentences are not equivalent since the truth set of $x = 10$ is $\{10\}$ but the truth set of $\frac{x^2}{5} + 3x = 5x$ is $\{0, 10\}$. We multiplied by an expression which is not a non-zero real number for every value of x.

4. (a) This might not yield an equivalent sentence. $x - 1$ is not a non-zero real number for all values of x. We must multiply by a non-zero real number to guarantee equivalent sentences.

   (b) This always yields an equivalent sentence. Adding any real number produces equivalent sentences.

   (c) This might not yield an equivalent sentence. x is not a non-zero number for all values of x.

(d) This might not yield an equivalent sentence. $\frac{1}{x-1}$ is not a real number for all values of x.

(e) This might not yield an equivalent sentence. $\frac{1}{x}$ is not a real number for all values of x.

(f) This might not yield an equivalent sentence. $\frac{1}{x}$ is not a real number for all values of x.

5. (a) $\{3\}$
   (b) $\{0, 2\}$
   (c) $\{0, 3\}$
   (d) $\{0, 4\}$
   (e) $\{5\}$
   (f) $\{14\}$
   (g) $\emptyset$
   (h) $\{\frac{39}{2}\}$
   (i) $\{\frac{9}{2}\}$

Page 675.

We now give more careful consideration to the question of the domain of the variable. The student may need some reinforcement at this point on the concept of certain values of the variable being excluded by the form of the sentence. For example, any value of x for which a denominator becomes zero must be excluded. If we restrict the domain throughout a problem, and multiply only by quantities which are defined and are non-zero for every value of the variable in this restricted domain, we can be sure of obtaining an equivalent sentence. It must always be borne in mind that in this process any new sentence must be thought of as having the same domain for the variables as the original sentence. For this reason the student is urged to accompany his final statement with a statement about the domain.

In the example given, the accompanying statement about the domain might have seemed superfluous in the first two cases, since the truth sets of the final sentence did not contain excluded values of the variable.

However in the last case, we saw that the explicit definition of domain was of major importance. It was only through this that we could conclude that our solution was the empty set.

Notice: the truth set of $x = k$ is $\{k\}$, but the truth set of the sentence "$x = k$ (Domain: $x \neq k$)" is $\emptyset$.

The teacher may want to point out that a statement about the domain, whether explicit or implied is to be construed as essential to the full meaning of the sentence. Thus, truth sets are determined in part by domains.

<u>Answers to Oral Exercises 15-1c</u>; page 679:

1. (a) $x \neq 2$
   (b) $m \neq 0$
   (c) all real numbers
   (d) $q \neq 0$, $q \neq -1$, $q \neq -2$
   (e) all real numbers
   (f) $x \neq 1$, $x \neq 2$

2. (a) Multiplying each side by x. Since zero is excluded from the domain, we have multiplied by a non-zero real number and hence equivalence is preserved.
   (b) Multiplying each side by $\frac{1}{3}$. Equivalence is preserved for we have multiplied by a non-zero real number.
   (c) Multiplying each side by $(x - 2)$. Equivalence is preserved by multiplying by a non-zero real number.
   (d) Adding $(-2x - 5)$ to each side. Equivalence is preserved by adding a real number.

<u>Answers to Problem Set 15-1c</u>; pages 679-680:

1. (a) $x = 2$
   (b) $x = -1$
   (c) $x = 0$
   (d) $x = 3$
   (e) $x = 1$ or $x = -1$
   (f) $\frac{1}{x^2 + 1}$ is a real number for all values of x.

2. (a) all real numbers
   (b) $t \neq 0$
   (c) $t \neq 0$, $t \neq 5$
   (d) $y \neq 3$
   (e) $k \neq 0$, $k \neq 1$
   (f) $m \neq 6$, $m \neq -5$

3. (a) Multiply each side by $(5 + m)$ which is a non-zero real number for all values of the variable in the domain.

   (b) Multiply each side by $(y - 1)(y - 2)$ which is a non-zero real number since $y \neq 1$, $y \neq 2$.

   (c) Multiply each side by $|x|$ which is a non-zero real number since $x \neq 0$.

   (d) Using the multiplication property of one:

$$\frac{x - 2}{(x - 2)(x - 3)} = 1 \quad (\text{domain } x \neq 2, \ x \neq 3)$$

$$\frac{x - 2}{x - 2} \cdot \frac{1}{x - 3} = 1$$

$$\frac{1}{x - 3} = 1$$

   The second sentence, thus, is equivalent to the first since $x \neq 2$, $x \neq 3$.

4. (a) $\{\frac{1}{3}\}$ (d) $\{\frac{12}{5}\}$

   (b) $\{3\}$ (e) $\{\frac{1}{2}\}$

   (c) $\{-\frac{5}{3}\}$ (f) $\{12\}$

5. (a) $\{4\}$ (d) $\{5\}$

   (b) $\emptyset$ (e) $\{8\}$

   (c) $\{-4, 4\}$ (f) $\emptyset$

<u>Page 681</u>.

Attention here is called to the fact that some rational expressions such as $\frac{1}{x^2 + 1}$ are defined for all values of x even though the variable appears in the denominator. In the second example the way is prepared for use of an L.C.D. in solving equations involving fractions.

Usually in order to simplify a fractional equation we multiply by an expression that is a product of factors of the denominators in the equation. This expression may not be a non-zero real number and we have been warned that this may not give an equivalent equation. We find, however, that we can avoid trouble if we are careful to exclude from the domain of the variable the values which make the multiplier zero; so we must be careful to exclude values of the variable which make any one of the denominators zero. Thus in $\frac{1}{x} = \frac{1}{1 - x}$ we require that $x \neq 0$ and $x \neq 1$.

Answers to Oral Exercises 15-1d; page 683:

1.	Domain	$x \neq 0$	x
2.	Domain	$t \neq 0$	$2t$
3.	Domain	$m \neq 1$	$3(m - 1)$
4.	Domain	$a \neq 0$	$2a$
5.	Domain	$t \neq 0$, $t \neq 1$	$t(t - 1)$
6.	Domain	$y \neq 0$, $y \neq -1$	$y(y + 1)$
7.	Domain	$x \neq 0$	$6x$
8.	Domain	$t \neq 0$	t
9.	Domain	$m \neq 0$	$3m$
10.	Domain	$x \neq 0$, $x = -1$	$x(x + 1)$

Answers to Problem Set 15-1d; pages 683-684:

1. (a) 0, $\{-\frac{1}{10}\}$
   (b) no restrictions, $\{60\}$
   (c) 0, $\{9\}$
   (d) 0, $\{3\}$
   (e) 0, $\{3, -3\}$
   (f) 0, $\{\frac{1}{14}\}$

2. (a) 0, $\{6\}$
   (b) 0, $\{\frac{1}{2}\}$
   (c) 0, $\{-2\}$
   (d) 0 and 3, $\{5\}$
   (e) 0 and 1, $\emptyset$
   (f) no restrictions, $\{-\frac{1}{3}\}$
   (g) all real numbers except 0

3. (a) no restrictions, $\{3\}$
   (b) 1 and 3, $\{\frac{7}{2}\}$
   (c) 3, $\{1\}$
   (d) no restrictions, $\{2\}$
   (e) no restrictions, $\{0, 3\}$
   (f) 0 and 1, $\{3\}$
   (g) no restrictions, $\{9, -9\}$

4. (a) -1 and 0, $\{-7\}$ (e) -1, $\emptyset$
   (b) -2 and 1, $\{-1\}$ (f) 5, $\emptyset$
   (c) no restrictions, $\{-\frac{1}{2}\}$ (g) no restrictions, $\emptyset$
   (d) no restrictions, $\{3\}$

15-2. Equations Involving Factored Expressions.

The keys to this section are the two statements relative to the number zero. These establish a kind of reversibility in the reasoning process. We can say

1) If $x - 3 = 0$ or $x + 2 = 0$, then $(x - 3)(x + 2) = 0$.
2) If $(x - 3)(x + 2) = 0$, then $x - 3 = 0$ or $x + 2 = 0$.

This enables us to conclude that the sentence in factored form, $(x - 3)(x + 2) = 0$, is equivalent to the compound sentence $x - 3 = 0$ or $x + 2 = 0$. A review of truth sets of compound sentences is in order here.

As in previous situations, the associative law gives us the right to extend a binary principle to an expression involving more than two factors. The teacher, however, may not wish to make too much of this, since most students will probably make the extension intuitively.

Answers to Oral Exercises 15-2a; pages 685-686:

1. The first and third are true when $x = -3$.
   The second and third are true when $x = 4$.

2. For $(x + 3)(x - 4) = 0$ to be true it is required that $x + 3 = 0$ or $x - 4 = 0$. If x is 5, this condition is not satisfied, so $(x + 3)(x - 4) = 0$ must be false.

3. $\{-3\}$; $\{4\}$

4. $\{-3, 4\}$

Answers to Problem Set 15-2a; pages 686-687:

1. (a) {-2, 5} (d) {1, -1}
   (b) {3, -1} (e) {0, -3}
   (c) {6} (f) {1}

2. (a) $\{\frac{1}{2}, -1\}$ (d) $\{-\frac{3}{2}, -\frac{2}{3}\}$
   (b) $\{\frac{3}{4}\}$ (e) $\{\frac{1}{2}, -1, 1\}$
   (c) {0, 1} (f) $\{\frac{1}{2}, 1\}$

3. (a) {1, 2, 3} (d) {1, -1, 0}
   (b) {0, -1, 1} (e) $\{1, -1, \frac{3}{2}\}$
   (c) $\{0, \frac{1}{2}, -\frac{2}{3}\}$ (f) ∅

4. (a) {2, -1} (d) {4, -3}
   (b) {11, -11} (e) {-2, -3}
   (c) {0, 5, -5} (f) $\{\sqrt{2}, -\sqrt{2}\}$

5. (a) {0, 1} (d) ∅
   (b) {-8, 2} (e) {6, 1}
   (c) $\{-\frac{1}{2}, 3\}$ (f) $\{\frac{3}{2}, -9\}$

6. (a) Domain: $x \neq 0$, $\{\frac{1}{3}, 3\}$
   (b) Domain: $x \neq 0$, {2, -1}
   (c) Domain: $x \neq 3$, {2}
   (d) Domain: $m \neq 0$, $m \neq 3$, $\{\frac{1}{2}, 5\}$
   (e) Domain: $y \neq 3$, {-3, 2}
   (f) Domain: $z \neq 0$, $z \neq 4$, {2}

7. Let x represent the integer we are seeking.
The open sentence for this problem is

$\frac{1}{x} = \frac{x}{4}$ (domain: all integers except zero)

$4 = x^2$

$0 = x^2 - 4$

$0 = (x - 2)(x + 2)$

$\{2, -2\}$

The integers are 2 and -2.

8. If n represents the integer, the open sentence we write is

$n + \frac{1}{n} = \frac{10}{3}$ (domain: all integers except zero)

$3n^2 + 3 = 10n$

$3n^2 - 10n + 3 = 0$

$(3n - 1)(n - 3) = 0$ (domain: all integers except zero)

Truth set: $\{3\}$ Note: $\frac{1}{3}$ is not in the domain.

The integer is 3.

9. (a) If q represents the rate of the current in miles per hour, $15 + q$ is the rate in miles per hour of the boat going downstream and $15 - q$ is the rate in miles per hour of the boat going upstream.
The open sentence is

$$\frac{120}{15 + q} = \frac{60}{15 - q} .$$

(b) The domain is the set of all non-negative real numbers except 15.

(c) $\{5\}$

The rate of the current is 5 miles per hour.

15.

10. (a) If b represents the circumference in feet of the back wheel, $b - 3$ is the circumference in feet of the front wheel, then the open sentence is

$$\frac{60}{b - 3} = \frac{90}{b}.$$

(b) The domain is the set of all positive real numbers greater than 3.

(c) {9}

The circumference of the back wheel is 9 feet and the circumference of the front wheel is 6 feet.

Page 688.

There are times when we are tempted to perform operations on sentences which will not necessarily yield equivalent sentences. A common temptation is to eliminate a factor which we see in every term by multiplying by its reciprocal. As pointed out before, if the reciprocal is not a real number—if its denominator is zero for some x, — this may cause trouble. We repeat an earlier example showing this, and follow with an alternative procedure involving addition, which does preserve equivalence. This is a good time to point up again the fact that addition of a quantity which may become zero for some value of the variable is perfectly safe, while multiplication is not. Once again the equivalence of a sentence in factored form with a compound sentence involving the connective "or" is illustrated.

Answers to Oral Exercises 15-2b; page 689:

1. $x^2 - x = 0$
2. $x(x - 1) = 0$
3. $2m - m^2 = 0$
4. $m(2 - m) = 0$
5. $0 = 4y^2$

6. $z(z - 1) - 2(z - 1) = 0$,
$(z - 2)(z - 1) = 0$
are each equivalent to the given sentence.

7. $(t - 3)(t + 1) = 0$

8. $(t + 2 - 2)(t + 2) = 0$,
$t(t + 2) = 0$
are each equivalent to the given sentence.

9. $(x - 3)(1 - x) = 0$

10. $(1 - y)^2 - 5(1 - y) = 0$,
$(1 - y - 5)(1 - y) = 0$,
$(-y - 4)(1 - y) = 0$
are each equivalent to the given sentence.

Answers to Problem Set 15-2b; pages 689-690:

1. (a) $\{0, 3\}$ (d) $\{0, -9\}$
   (b) $\{0, -5\}$ (e) $\{2, 1\}$
   (c) $\{0, 4\}$ (f) $\{5, -1\}$

2. (a) $\{0, 4\}$ (d) $\{5, 1\}$
   (b) $\{0, 5\}$ (e) $\{0, \frac{3}{5}\}$
   (c) $\{4, \frac{1}{2}\}$ (f) $\{0, \frac{1}{2}\}$

3. (a) $\{0, 5\}$ (d) $\{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\}$ or $\{\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\}$
   (b) $\{0, \frac{1}{2}\}$ (e) $\{2\}$
   (c) $\{0, \frac{9}{4}\}$ (f) $\{3, 2\}$

4. (a) $\{\frac{1}{2}, 18\}$ (d) $\{\frac{3}{2}, -4\}$
   (b) $\{-2, -1\}$ (e) $\{2\}$
   (c) $\{0, 3\}$ (f) $\{0, -3, \frac{1}{2}\}$

5. If d represents the number, then the open sentence is $d^2 = 4d$.

$$d^2 - 4d = 0$$

$$d(d - 4) = 0$$

$$\{0, 4\}$$

The numbers are 0 and 4.

6. If c represents the integer, c + 1 is its successor and 6(c + 1) is six times its successor, then the open sentence is

$$c(c + 1) = 6(c + 1).$$

$$c(c + 1) = 6(c + 1) \text{(domain, all positive integers)}$$

$$c^2 - 5c - 6 = 0$$

$$(c - 6)(c + 1) = 0$$

$\{6, \}$ **Note: -1 is not in the domain of the variable.**

The integer is 6.

The successor is 7.

15-3. Squaring.

Basic to the disucssion of solving sentences by squaring is the notion that implication in this case is a "one way street." We can confidently assert that if $a = b$, then it must follow that $a^2 = b^2$. The reverse, however, is definitely not true. We cannot say that if $a^2 = b^2$, then a must be equal to b. It is easy to furnish a counter example such as

$$(-3)^2 = (3)^2 \text{ but } -3 \neq 3.$$

It is apparent that squaring both sides of an equation usually does not yield an equivalent equation. However, in solving certain equations involving square roots or absolute values it may be necessary or desirable to square both sides. We do so, then bear carefully in mind that we may expect to find a larger truth set for the new equation. We must therefore test the members of this truth set to find which ones really make the original equation true.

pages 693-694: 15-3

The point is developed that the new truth set will always contain the elements of the truth set of the original sentence. That is, nothing will be lost in the process, but so-called extra elements may have been added as a result of the squaring process. These are sometimes referred to as "extraneous roots."

Answers to Oral Exercises 15-3; page 694:

1.	{9}	6.	{49}
2.	{1}	7.	{6}
3.	{5}	8.	$\{\frac{1}{4}\}$
4.	{10}	9.	$\{\frac{1}{2}\}$
5.	{50}	10.	$\{\frac{1}{8}\}$

Answers to Problem Set 15-3; pages 694-695:

1.	(a) {9}		(d) {4}	
	(b) {8}		(e) {2}	
	(c) {1}		(f) ∅	
2.	(a) {4}		(d) ∅	
	(b) {5}		(e) {1}	
	(c) $\{\frac{1}{12}\}$		(f) {83}	
3.	(a) {9}		(d) {1}	
	(b) {0,-1}		(e) {-1,5}	
	(c) {0}		(f) $\{1,-\frac{1}{3}\}$	
4.	(a) $\{\frac{3}{4}\}$		(d) ∅	
	(b) {3, -3}		(e)	
	(c) {1}		(f) ∅	

With regard to this set of problems it should be pointed out to the student that checking his answers is absolutely necessary. It is more than just a safeguard against arithmetical error.

page 695: 15-3 and 15-4

5. If x represents the number, then the open sentence is

$\sqrt{x} = x - 12$

Square both sides.

$x = x^2 - 24x + 144$

$0 = x^2 - 25x + 144$

$0 = (x - 16)(x - 9)$. Truth set of new sentence: $\{16, 9\}$

Check: $\sqrt{16} = 16 - 12$; $\sqrt{9} \neq 9 - 12$

The truth set of the original sentence: $\{16\}$

The number is 16.

6. If m represents the number, then the open sentence is

$m + |m| = 8.$

$|m| = 8 - m$

$m^2 = 64 - 16m + m^2$

$16m = 64$

$m = 4$ Truth set of this sentence: $\{4\}$

Check in the original sentence: $4 + |4| = 8.$

Hence the truth set is $\{4\}$.

The number is 4.

7. If d represents a number, then the open sentence is

$d + |d| = 0.$

$|d| = -d$

all non-positive real numbers

The number is any negative number or zero.

15-4. Equivalent Inequalities.

This section highlights the similarity between the notion of equivalence with respect to equations and equivalence with respect to inequalities. In this connection a review of the properties of order in Chapter 9 is expedient. Special attention should be focused on the fact that multiplication by a negative real number reverses the order of the inequality.

Students may experience some difficulty at first in dealing with the symbols $<$, $>$, $\leq$. It should be carefully explained that there are two ways to indicate a reversal of order. For example, if we start with

$$x < 3$$

and we want to multiply both sides by -2, we can write the new sentence as

$$(-2)(3) < -2x$$

or as

$$-2x > (-2)(3)$$

In the first way, the two members of the sentence are interchanged with the order symbol remaining fixed. In the other way, the members were left fixed and the symbol changed. It may be well to let the student select whichever method comes most naturally to him.

Answers to Oral Exercises 15-4; page 699:

1. (a) All real numbers greater than 5
   (b) All real numbers less than 4
   (c) All real numbers greater than or equal to 10
   (d) All real numbers less than or equal to -6
   (e) All real numbers greater than 6
   (f) All real numbers less than or equal to $\frac{7}{4}$
   (g) All real numbers less than 3
   (h) All real numbers greater than 4
   (i) All real numbers greater than or equal to 4
   (j) All real numbers greater than -7

Answers to Problem Set 15-4; pages 699-700:

1. (a) $x < 27$
   all real numbers less than 27
   (b) $t > 31$
   all real numbers greater than 31

(c) $y < -9$
all real numbers less than -9

(d) $11 < x$
all real numbers greater than 11

(e) $x > 2$
all real numbers greater than 2

(f) $x > 3$
all real numbers greater than 3

2. (a) all real numbers greater than 2
(b) all real numbers less than 2
(c) all real numbers less than (-2)
(d) all real numbers greater than (-7)
(e) all real numbers greater than $\frac{2}{3}$
(f) all real numbers less than (-9)

3. (a) all real numbers less than 12
(b) $0 < m < 9$
(c) all numbers less than (-6)
(d) $\emptyset$
(e) all real numbers greater than -1
(f) $0 < x < 3$

4. (a) all real numbers greater than $\frac{4}{5}$
(b) all real numbers greater than 2
(c) all real numbers less than 8
(d) all real numbers greater than $-\frac{5}{2}$
(e) all negative real numbers greater than -2
(f) $\emptyset$

Answers to Review Problem Set; pages 701-703:

1. {-1}
2. {7}
3. $\{-\frac{1}{2}\}$
4. {3}
5. $\{-\frac{1}{3}, 3\}$
6. {4,0}
7. {6,-6}
8. {-48}
9. {0,1}
10. {3,-2}
11. All real numbers greater than 2
12. $\{\frac{1}{5}\}$
13. All real numbers greater than $\frac{1}{5}$
14. {-3}
15. {35}
16. ∅
17. {-3}
18. {-1}
19. all real numbers less than $\frac{7}{2}$
20. {-2}
21. {1,4}
22. $\{-\frac{1}{3}\}$
23. all real numbers
24. all real numbers greater than -2
25. {-8,-3}
26. {8}
27. {0,2}
28. all real numbers greater than 12
29. {3, -3}
30. {-4}
31. ∅
32. all non-positive real numbers
33. all real numbers except (-1)
34. {1}

35. If x represents the denominator, then $\frac{x-7}{x} = \frac{2}{3}$.

    {21}

    The numerator is 14. The denominator is 21.

36. If q represents the average rate in miles per hour on the first trip, then $\frac{150}{q} = \frac{200}{q+10}$.

    {30}

    The average rate was 30 miles per hour.

37. If r represents the number,
then $\frac{r}{3} = \frac{2}{5}(r + 3)$.
$\{-18\}$
The number is -18.

38. If c represents the integer,
then $\frac{4}{c-5} = \frac{5}{c-4}$.
$\{9\}$
Yes, the integer is 9.

39. If x is the number of hours each boy works,
then $\frac{x}{2} + \frac{x}{5} = 1$.
$\{\frac{10}{7}\}$
It will take the boys $\frac{10}{7}$ hours to do the job.

40. If s represents the number,
then $\frac{3}{s+2} = \frac{7}{s}$.
$\{-\frac{7}{2}\}$
The number is $(-\frac{7}{2})$.

41. If y represents the number,
then $\frac{7}{y} + \frac{12}{y^2} = -1$.
$\{-3, -4\}$
The number is (-3) or (-4).

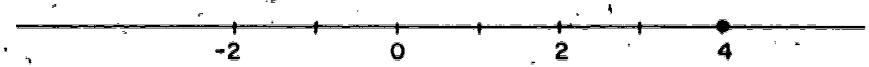
42. If t represents the number of students in each row,
then $t(t - 9) = 90$.
$\{15\}$ (Since $t > 0$)
There are 6 rows. (Since $(t - 9)$ represents the number of rows.)

43. If d represents the number,
then $d + |d| > 6$.
$d > 3$
The number is any real number greater than 3.

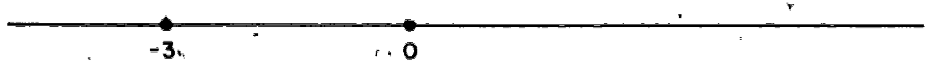
44. If h represents the number,
then $\frac{1}{h} = \frac{1}{36}$.

{6, -6}

The number is 6 or -6.

45. 

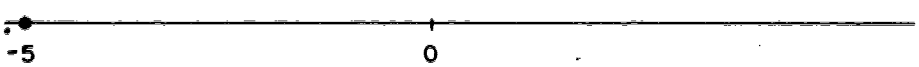
46. 

47. 

48. 

49. 

50. 

51. 

Suggested Test Items

1. When solving equations, which of the following steps will always preserve equivalence?

   (a) add 6 to each side; domain, all real numbers
   (b) multiply each side by x; domain, all real numbers
   (c) multiply each side by $\frac{1}{x - 1}$; domain, $x \neq 1$
   (d) add $\frac{1}{x}$ to each side; domain, $x \neq 0$
   (e) add $-2x + 5$ to each side; domain, $x \neq 0$
   (f) square both sides; domain, all real numbers
   (g) multiply each side by $x - 3$; domain, all real numbers
   (h) multiply each side by $\frac{x}{x - 2}$; domain, $x \neq 2$

2. For each of the expressions, give the restrictions on the domain such that each expression will be a <u>real number</u> for every value of x in the domain.

   (a) $\frac{1}{x - 1}$ (d) $\frac{1}{7}(x - 3)$
   (b) $\frac{1}{x(x - 1)}$ (e) $\frac{x - 3}{(x - 1)(x - 2)}$
   (c) $x - 3$ (f) $(x - 2)(x + 1)$

3. Give the restrictions on the domain such that each expression will be a <u>non-zero real number</u> for every value of y in the domain.

   (a) $\frac{1}{y - 1}$ (d) $\frac{1}{7}(y - 3)$
   (b) $\frac{1}{y(y - 1)}$ (e) $\frac{y - 3}{(y - 1)(y - 2)}$
   (c) $y - 3$ (f) $(y - 2)(y + 1)$

4. For each pair of sentences, determine whether the two sentences are equivalent and give a reason for your conclusion.

(a) $3x + 6 = 8, \quad x = \frac{2}{3}$

(b) $\frac{x - 2}{x + 2} = 0, \quad x = 2$

(c) $|x| + 1 = 4, \quad x^2 - 9 = 0$

(d) $\frac{x}{x - 2} = \frac{2}{x - 2}, \quad x = 2$

(e) $\frac{x}{x - 2} = \frac{3}{x - 2}, \quad x = 3$

(f) $\frac{1}{x} = 5, \quad 1 = 5x$

(g) $x(x - 3) + 3(x - 3) = 0, \quad x - 3 = 0$

(h) $x(x - 3) - 3(x - 3) = 0, \quad x - 3 = 0$

(i) $x^2 = 9, \quad x = 3$

5. Solve

(a) $3x - 7 = 7x - 11$

(b) $2x + 3 > x - 2$

(c) $(x - 1)(x + 1) = 0$

(d) $x^2 - 2x - 35 = 0$

(e) $(x + 3)(x - 2) = 6$

(f) $\sqrt{x^2 - 9} = 4$

(g) $\frac{3}{x} < 5$ and $x > 0$

(h) $\frac{5}{|x|} > 7$

6. Solve

(a) $\frac{1}{y} = 5$

(b) $\frac{1}{m - 2} = 4$

(c) $\frac{x}{x - 1} = \frac{13}{11}$

(d) $\frac{y + 1}{2} = \frac{y}{5}$

(e) $1 + \frac{2}{x - 1} = \frac{x}{x - 1}$

(f) $\sqrt{4x} = x - 3$

7. Solve

(a) $4 = \sqrt{x - 9}$

(b) $\frac{6}{x^2 + 1} < 1$

(c) $\frac{x + 1}{x} + \frac{1}{x(x - 1)} = \frac{3}{x - 1}$

(d) $\frac{2(x - 1)}{x} + \frac{2}{x(x + 1)} = \frac{1}{x + 1}$

(e) $x + \frac{1}{x} = - \frac{17}{4}$

(f) $1 - \frac{7}{x} = \frac{18}{x^2}$

Answers to Suggested Test Items

1. (a) preserves equivalence
   (b) does not always preserve equivalence since x could be 0
   (c) preserves equivalence since $\frac{1}{x - 1}$ is a real number for every x in the domain
   (d) preserves equivalence
   (e) preserves equivalence even if the domain is unrestricted
   (f) does not always preserve equivalence
   (g) does not always preserve equivalence
   (h) does not always preserve equivalence since $\frac{x}{x - 2}$ is zero for $x = 0$

2. (a) $x \neq 1$
   (b) $x \neq 0$ and $x \neq 1$
   (c) no restrictions
   (d) no restrictions
   (e) $x \neq 1$ and $x \neq 2$
   (f) no restrictions

3. (a) $y \neq 1$
   (b) $y \neq 0$ and $y \neq 1$
   (c) $y \neq 3$
   (d) $y \neq 3$
   (e) $y \neq 1$ and $y \neq 2$ and $y \neq 3$
   (f) $y \neq 2$ and $y \neq -1$

4. Giving a reason could consist of deriving one equation from the other using only reversible steps or showing the truth sets are the same.
   (a) Equivalent because the second can be obtained from the first by adding -6 to both sides, then multiplying both sides by $\frac{1}{3}$.

(b) Equivalent because the truth set of $\frac{x - 2}{x + 2} = 0$ is $\{2\}$ and the truth set of $x = 2$ is $\{2\}$.

(c) Equivalent because the truth set for each is $\{-3, 3\}$.

(d) Not equivalent since the truth set of $\frac{x}{x - 2} = \frac{2}{x - 2}$ is $\emptyset$ and the truth set of $x = 2$ is $\{2\}$.

(e) Equivalent since the second can be obtained from the first by multiplying both sides by $x - 2$ which is a non-zero real number for every x in the domain.

(f) Equivalent. The second can be obtained from the first by multiplying each side by x which is a non-zero real number (domain, $x \neq 0$).

(g) Not equivalent. The truth set of the first sentence is $\{-3, 3\}$, the second $\{3\}$.

(h) Equivalent. The truth set of each is $\{3\}$.

(i) Not equivalent since the truth set of the first is $\{3, -3\}$ and of the second is $\{3\}$.

5. (a) $\{1\}$
(b) all real numbers greater than -5
(c) $\{-1, 1\}$
(d) $\{-[illegible], 7\}$
(e) $\{-4, 3\}$
(f) $\{-5, 5\}$
(g) all real numbers greater than $\frac{3}{5}$
(h) all real numbers between $-\frac{2}{7}$ and $\frac{5}{7}$ except 0

6. (a) $\{\frac{1}{5}\}$
(b) $\{\frac{9}{4}\}$
(c) $\{\frac{13}{2}\}$
(d) $\{-\frac{5}{3}\}$
(e) $\emptyset$
(f) $\{0\}$

7. (a) $\{25\}$
(b) all numbers greater than $\sqrt{}$
(c) $\{3\}$
(d) $\{\frac{1}{2}\}$
(e) $\{-\frac{1}{4}, -4\}$
(f) $\{9, -2\}$

Answers to Challenge Problems

1. If the number of steers is s and the number of cows is c, then,

$$250s + 260c = 10{,}000$$

$$250s = 10{,}000 - 260c$$

$$s = \frac{10000 - 260c}{250}$$

$$s = 40 - \frac{26c}{25}$$

If s and c are positive integers, then 26c must be divisible by 25. This is true when c = 25, 50, 75, ⋯, a multiple of 25, because 26 and 25 are relatively prime to each other.

If $c = 25$, $\frac{26c}{25} = 26$ and $s = 40 - 26 = 14$.

If $c = 50$, $\frac{26c}{25} = 52$ and $s = 40 - 52 = -12$.

If $c = 75$, $\frac{26c}{25} = 78$ and $s = 40 - 78 = -38$.

It is thus apparent that if $c \geq 50$, s is a negative number. Hence, c can only be 25

and $s = 40 - 26 = 14$

So he can buy 25 cows and 14 steers.

If we were to solve the original equation instead for c,

$$c = \frac{1000 - 25s}{26}$$

s would have to be chosen so as to make 1000 - 25s divisible by 26. Though this can be done, it is plainly more difficult than the other approach.

2\. (a) While the small hand travels over a number of minute markings, x, the large hand travels over 12x of these units. Since the hour hand is at 3 o'clock position, it has a 15-unit "head start" over the minute hand at the time 3:00. Thus

$$12x = x + 15,$$
$$11x = 15,$$
$$x = \frac{15}{11}.$$

Thus, the truth set of the equation is $\{\frac{15}{11}\}$ and the time when the hands are together is $16\frac{4}{11}$ minutes after 3 o'clock.

(b) In part (a) both hands came to the same minute division; in part (b) the minute hand is to come to a reading 30 units ahead of the hour hand. Hence, an equation for part (b) is

$$12x = (x + 15) + 30.$$

This equation is satisfied if $x = 4\frac{1}{11}$.

And the hands will be opposite each other at $49\frac{1}{11}$ minutes after 3 o'clock.

3\. If the width of the strip is w feet, then the number of feet in the length of the rug is $20 - 2w$, and the number of feet in the width of the rug is $14 - 2w$. Hence, two names for the area of the rug are available, and appear as sides of the equation:

$$(20 - 2w)(14 - 2w) = (24)(9),\ 0 < w < 7.$$
$$280 - 68w + 4w^2 = 216$$
$$4w^2 - 68w + 64 = 0$$
$$w^2 - 17w + 16 = 0$$
$$(w - 16)(w - 1) = 0$$

1 is the only truth number, since 16 is not in the domain of the variable.
The width of the strip is 1 foot.

4. If x is the number of units in the length of the smaller leg, then $2x + 2$ is the number of units in the longer leg. Hence, by the Pythagorean relationship,

$$x^2 + (2x + 2)^2 = 13^2, \; 0 < x < 13.$$
$$x^2 + 4x^2 + 8x + 4 = 169$$
$$5x^2 + 8x - 165 = 0$$
$$(5x + 33)(x - 5) = 0$$
$$x = 5 \qquad \left(-\frac{33}{5} \text{ is not in the domain of the variable}\right)$$

5. The first three are irrational. The fourth number is rational since $\sqrt[3]{-1} \cdot \sqrt{.16} = (-1)(.4) = -.4$.

6. If the two-digit number is $10t + u$, the sum of its digits is $t + u$, and

$$\frac{10t + u}{t + u} = 4 + \frac{3}{t + u}.$$

Multiplying both sides by $t + u$ (which is certainly not zero)

$$10t + u = 4t + 4u + 3$$
$$6t - 3u = 3$$
$$2t - u = 1$$

$u = 2t - 1$ and u is a positive integer ≤ 9.

If $t = 1$, then $u = 1$, and the number is 11;
if $t = 2$, then $u = 3$, and the number is 23;
if $t = 3$, then $u = 5$, and the number is 35;
if $t = 4$, then $u = 7$, and the number is 47;
if $t = 5$, then $u = 9$, and the number is 59.

Hence, the solutions are 11, 23, 35, 45, 59.

7. $\frac{8}{5}$

8. (a) The possible integral factorizations would be

$$(x + 1)(x + 12)$$
$$(x + 2)(x + 6)$$
$$(x + 3)(x + 4)$$

so the possible values of k are 13, 8, 7 .

(b) Since k must be the product of two numbers whose sum is 6, the possibilities are 1×5, 2×4, 3×3.

Thus, k may be 5, 8, or 9.

(c) In order for $x^2 - 6\sqrt{3}\,x + k$ to be a perfect square we must have

$$\left(\frac{6\sqrt{3}}{2}\right)^2 = k .$$

Hence k must be 27.

9. $(n + 3)^2 - n^2 = \big((n + 3) - n\big)\big((n + 3) + n\big)$

$$= 3(2n + 3)$$

so $(n + 3)^2 - n^2$ is divisible by 3 for all integers n.

10.

$$\begin{array}{r|l} & x^3 - 2x^2 \\ \hline x - 3 & x^4 - 5x^3 + 6x^2 - 3 \\ & \underline{x^4 - 3x^3} \\ & \quad - 2x^3 + 6x^2 - 3 \\ & \quad \underline{- 2x^3 + 6x^2} \\ & \qquad\qquad - 3 \end{array}$$

Therefore,

$$x^4 - 5x^3 + 6x^2 - 3 = (x^3 - 2x^2)(x - 3) - 3$$

and $x - 3$ is <u>not</u> a factor.

11. (a) The degree of r is less than 3.

(b) The degree of q is 97.

12. Any value of x could be used. For example, if $x = 0$, we obtain $1 = 2 \cdot (-1) + r$ and hence, $r = 3$. A better value is $x = 1$ since in this case

$$2 \cdot 1^4 + 1 = 2(1^3 + 1^2 + 1^1 + 1)(1 - 1) = r$$

$$3 = 2 \cdot 4 \cdot 0 + r$$

$$3 = r$$

The idea is that with this value of x the first term on the right hand side of the equation is automatically zero regardless of what the number $2(x^3 + x^2 + x + 1)$ is. (See the next problem.)

13. In this problem we do not know q and it would be a great deal of trouble to find it. However, the choice of 1 for the value of x gives

$$5 \cdot 1^{100} + 3 \cdot 1^{17} - 1 = q(1 - 1) + r$$

$$7 = q \cdot 0 + r$$

$$7 = r$$

14. If d is the distance in miles one way $(d > 0)$ and the rate is r miles per hour, the time one way is $\frac{d}{r}$ hours. On the return, if the rate is q miles per hour, the time is $\frac{d}{q}$ hours. The total distance, $2d$ miles, divided by the total time, $\frac{d}{r} + \frac{d}{q}$ hours, gives the average speed for the entire trip.

$$\frac{2d}{\frac{d}{r} + \frac{d}{q}} = \frac{2d}{\frac{d}{r} + \frac{d}{q}} \cdot \frac{rq}{rq} \qquad \text{Note: } d \neq 0,\ r \neq 0,\ q \neq 0$$

$$= \frac{2drq}{d(q + r)}$$

$$= \frac{2rq}{q + r}$$

The average speed is $\frac{2rq}{q + r}$ miles per hour.

15. <u>Theorem</u>. If a and b are distinct positive real numbers, then

$$\frac{a+b}{2} > \sqrt{ab}.$$

<u>Proof</u>: If $a + b - 2\sqrt{ab} > 0$,

then $a + b - 2\sqrt{ab} + 2\sqrt{ab} > 0 + 2\sqrt{ab}$
(addition property of order)

or $a + b > 2\sqrt{ab}$.

Hence, $\frac{a+b}{2} > \sqrt{ab}$ (multiplication property of order)

Therefore, we have only to prove that,

$$a + b - 2\sqrt{ab} > 0.$$

Observe that

$$a + b - 2\sqrt{ab} = a - 2\sqrt{a}\sqrt{b} + b$$
$$= (\sqrt{a} - \sqrt{b})^2.$$

Since $a \neq b$, $\sqrt{a} \neq \sqrt{b}$ and thus, $\sqrt{a} - \sqrt{b} \neq 0$.

Since the square of any non-zero real number is positive, it follows that $a + b - 2\sqrt{ab} > 0$.

16. $x^4 = 1$

$x^4 - 1 = 0$

$(x^2)^2 - 1^2 = 0$

$(x^2 - 1)(x^2 + 1) = 0$

This sentence is equivalent to the compound sentence

$$x^2 - 1 = 0 \quad \text{or} \quad x^2 + 1 = 0$$

The truth set of $x^2 - 1 = 0$ is $\{-1, 1\}$.

The truth set of $x^2 + 1 = 0$ is empty.

Hence the truth set of $x^4 = 1$ is $\{-1, 1\}$.

17. $n^2 - n + 41$ will produce a prime number for every n in the set $\{0, 1, 2, \cdots 40\}$. It fails to give a prime for $n = 41$ since then the sum of the last two terms is zero, leaving 41^2 which can be factored 41×41.

Of course 1, 5, or 400 examples do not make a proof and we cannot assume the truth of the next or some other example. This is a common error in reasoning and is frequently the basis of common superstitions.

18. Proof: Given: $a > b$, $a > 0$, and $b > 0$

Either $\sqrt{a} < \sqrt{b}$, $\sqrt{a} = \sqrt{b}$, or $\sqrt{a} > \sqrt{b}$.

Assume $\sqrt{a} < \sqrt{b}$.

Then $a < b$. If $x < y$ then $x^2 < y^2$

This contradicts the assumption that $a > b$.

Assume $\sqrt{a} = \sqrt{b}$.

Then $a = b$.

But this also contradicts the assumption that

$a > b$.

Thus $\sqrt{a} > \sqrt{b}$.

19. If the rat weighs x grams at the beginning of the experiment, it will weigh $\frac{5}{4}x$ grams after the rich diet and $\frac{3}{4}(\frac{5}{4}x)$ at the end of the experiment. Thus, the difference is $\frac{15}{16}x - x = -\frac{1}{16}x$ grams. The result is that the rat loses $\frac{1}{16}x$ grams of his weight.

20. If x is the number of quarts of white paint, then $3x$ is the number of quarts of grey paint. We write the following open sentence.

$$x + 3x = 7 \cdot 4$$

$$4x = 4 \cdot 7$$

$$x = 7 \qquad 3x = 21$$

Thus the man bought 1 gallon and 3 quarts of white and 5 gallons and 1 quart of grey.

21. $\frac{x}{2x - 5} = \frac{4}{6}$.

Multiply both sides by $6(2x - 5)$.

$$6x = 4(2x - 5)$$
$$6x = 8x - 20$$
$$-2x = -20$$
$$x = 10$$

The sides are 10 units, and 15 units.

22. If the length of the bin is x feet, then the width is $12-x$ feet and $70 = x(12-x)2$.
The length of the bin is 7 feet and the width is 5 feet.

23. The square is 6 feet on a side and the rectangle is 12 feet long and 3 feet wide.

24. Assume that $(x + m)(x + n) = x^2 + px + q$.
Then, $mn = q$ and $m + n = p$.
Also, $(x - m)(x - n) = x^2 -(m + n)x + mn = x^2 - px + q$.

25. $x^2 + px + 36$ is a perfect square for $p = 12$ or $p = -12$.
12 is the smallest value $|p|$ can have for $x^2 + px + 36$ to be factorable. Values of p for which $x^2 + px + 64$ is factorable are obtained as follows:
Note that $p = m + n$ where $mn = 2^6$.

$m \cdot n$	$m + n$
$2^6 \cdot 1$	65
$2^5 \cdot 2$	34
$2^4 \cdot 2^2$	20
$2^3 \cdot 2^3$	16

Positive values of p are 16, 20, 34, 65 and negative values are -16, -20, -34, -65. The perfect squares are given by $p = 16$ or $p = -16$. Note that 16 is the smallest value $|p|$ can have.

The student should be able to generalize the results in the two examples and guess that $2n$ is the smallest positive value of p for which $x^2 + px + n^2$ is factorable and that this gives the perfect square, $(x + n)^2$.

The largest value of p for which $x^2 + px + n^2$ is factorable is $n^2 + 1$, in which case

$$x^2 + px + n^2 = (x + 1)(x + n^2).$$

26. If k quarts of weed killer are used, then $40-k$ quarts of water are used.

$$\frac{k}{40 - k} = \frac{3}{17}$$

There should be 6 quarts of weed-killer.

27. (a) $(t + 1)(t^2 - t + 1)$

(b) $(s + 2)(s^2 - 2s + 4)$

(c) $(3x + 1)(9x^2 - 3x + 1)$

28. (a) $(t - 1)(t^2 + t + 1)$

(b) $(s - 2)(s^2 + 2s + 4)$

(c) $(2x - 1)(4x^2 + 2x + 1)$

Chapter 16

TRUTH SETS AND GRAPHS OF SENTENCES IN TWO VARIABLES

In this chapter a systematic study is made of open sentences of the first degree in two variables. The truth sets of such sentences are exhibited as ordered pairs of real numbers, attention being called to the fact that these sets have infinitely many elements. To aid in the understanding of the relationships involved, the graphs of the truth sets are introduced, based on the establishment of a one-to-one correspondence between pairs of real numbers and points in the plane.

The fact that the graph of a sentence of this type forms a line is developed intuitively by means of examples. These examples make plausible the subsequent assertion that if an open sentence is of the form $Ax + By + C = 0$, where A, B, and C are real numbers with A and B not both zero, then its graph is a line. The assertion is also made that every line in a plane is the graph of a sentence of this type.

16-1. Open Sentences in Two Variables.

Emphasis is placed on the ordered pair of real numbers as the characteristic element of the truth set of the first degree equation in two variables. It should be noted that we have avoided use of the traditional term, "linear equation", so as to avoid conflict with the more generalized notion of linear function, or linear transformation, which the student is apt to encounter later on. For a space of two dimensions these latter appear in the form

$x' = ax + by, \quad y' = cx + dy$ without the constant term. Students working in this area for the first time may have difficulty grasping the concept of an infinite set of truth values for a single equation. Practice in determining number pairs will help adjust them to this new idea. Construction of tables is valuable practice.

pages 713-714: 16-1

Answers to Oral Exercises 16-1a; pages 713-714:

1. (1, 2), $(2, \frac{3}{2})$, (3, 1), $(0, \frac{5}{2})$, $(4, \frac{1}{2})$ all make the sentence $x + 2y = 5$ true.

   (8, 3), (1, 3), (0, 5), (2, 4), (7, 1) all make the sentence $x + 2y = 5$ false.

2. "(5, 1)" is an abbreviation for "x is 5 and y is 1."
   "(-2, 7)" is an abbreviation for "x is -2 and y is 7."

3. "(a, b)" is an abbreviation for "x is a and y is b."
   Two numerals are required to name one element of this set.
   Here a and b are any real numbers.

4. x

5. The truth set of a sentence in two variables contains infinitely many elements.

6. One

7. Two

8. One has no trouble deciding which is the value of x or the value of y. The numeral on the left always represents the value of x, and the numeral on the right always represents the value of y.

9. (-3, 8), (-1, 6), (0, 5), (1, 4), (6, -1), (7, -2)

10. Infinitely many

Answers to Problem Set 16-1a; pages 714-715:

1. The ordered pairs in (a), (b), (d), (f), (h), and (i) satisfy the sentence; the others do not.

2. (a) (5, 1)
   (b) (6, 0)
   (c) (0, 6)
   (d) (-1, 7)
   (e) (-2, 8)
   (f) (3, 3)
   (g) (20, -14)
   (h) (-9, 15)
   (i) (-6, 12)
   (j) (-7, 13)

3. (a) True
   (b) True
   (c) False
   (d) True
   (e) True
   (f) False
   (g) True
   (h) True
   (i) True
   (j) False
   (k) True

4. (a) (1, 2), (0, 3), (4, -1)
   (b) (0, -2), (3, -5), (-1, -1)
   (c) (8, 1), (6, -1), (3, -4)
   (d) (2, 0), (1, -2), (3, 2)
   (e) (1, 2), (2, -5), (-1, 16)
   (f) (2, 5), (4, 4), (0, 6)
   (g) $(1, -\frac{9}{4})$, $(\frac{1}{3}, -\frac{1}{4})$, $(\frac{1}{5}, \frac{1}{4})$
   (h) $(4, -\frac{1}{4})$, $(8, \frac{7}{4})$, $(1, -\frac{7}{4})$
   (i) $(0, -\frac{1}{9})$, $(1, \frac{8}{9})$, $(\frac{1}{3}, \frac{2}{9})$
   (j) (1, 1), $(3, -\frac{15}{7})$, $(4, -\frac{26}{7})$

   Many other answers are possible.

5. (a) x + y = 11
   (b) The domain of x is the set of integers from 1 to 9 and the domain of y the set of integers from 0 to 9, where x represents the tens' digit and y the ones' digit.
   (c) {(2, 9), (9, 2), (3, 8), (8, 3), (4, 7), (7, 4), (5, 6), (6, 5)}

Page 715. The y-form is presented here with two motives. It is a convenient form for the calculation of truth elements, and it points out certain relationships between the variables, leading to the concept of slope and y-intercept. Care should be taken in discussing the form y = ax + b to guarantee that the student comprehends the distinction between the variables x and y, and the "specified" numbers a and b. We have avoided in this instance the somewhat ambiguous term "constant" on the ground that it raises questions as to the nature of a variable. Though the y-form is used predominantly in the chapter, it must be kept in mind that the form $Ax + By + C = 0$ with A and B not both zero, is more general since it includes the type $x = k$, which cannot be converted to the y-form.

Answers to Problem Set 16-1b; pages 717-718:

1. (a) $y = -x + 7$
   $(2, 5), (0, 7), (-1, 8)$
   (b) $y = -3x + 12$
   $(2, 6), (0, 12), (-1, 15)$
   (c) $y = 3x - 2$
   $(2, 4), (0, -2), (-1, -5)$
   (d) $y = -x + \frac{1}{2}$
   $(2, -\frac{3}{2}), (0, \frac{1}{2}), (-1, \frac{3}{2})$
   (e) $y = -\frac{2}{5}x + 8$
   $(2, \frac{36}{5}), (0, 8), (-1, \frac{42}{5})$
   (f) $y = -6x - 4$
   $(2, -16), (0, -4), (-1, 2)$
   (g) $y = -\frac{2}{3}x + \frac{1}{2}$
   $(2, -\frac{5}{6}), (0, \frac{1}{2}), (-1, \frac{7}{6})$
   (h) $y = -2x + \frac{4}{3}$
   $(2, -\frac{8}{3}), (0, \frac{4}{3}), (-1, \frac{10}{3})$
   (i) $y = 4x - \frac{5}{8}$
   $(2, \frac{59}{8}), (0, -\frac{5}{8}), (-1, -\frac{37}{8})$
   (j) $y = -\frac{11}{12}x + \frac{7}{12}$
   $(2, -\frac{15}{12}), (0, \frac{7}{12}), (-1, \frac{3}{2})$

2. (a) $y = -x + 4$
$(1, 3), (2, 2), (3, 1)$

(b) $y = x - 7$
$(3, -4), (5, -2), (7, 0)$

(c) $y = -3x + 2$
$(0, 2), (2, -4), (-1, 5)$

(d) $y = 11x + 5$
$(1, 16), (\frac{1}{11}, 6), (2, 27)$

(e) $y = -\frac{2}{3}x + \frac{4}{3}$
$(0, \frac{4}{3}), (1, \frac{2}{3}), (3, -\frac{2}{3})$

(f) $y = 2x - \frac{9}{2}$
$(1, -\frac{5}{2}), (0, -\frac{9}{2}), (\frac{1}{2}, -\frac{7}{2})$

(g) $y = \frac{1}{4}x - 4$
$(8, -2), (16, 0), (4, -3)$

(h) $y = -\frac{2}{3}x + 2$
$(3, 0), (6, -2), (9, -4)$

(i) $y = \frac{7}{3}x - \frac{10}{3}$
$(0, -\frac{10}{3}), (3, \frac{11}{3}), (2, \frac{4}{3})$

(j) $y = -\frac{20}{7}x + \frac{1}{14}$
$(\frac{1}{2}, -\frac{19}{14}), (-\frac{1}{2}, \frac{3}{2}), (0, \frac{1}{14})$

(k) $y = 3x + \frac{3}{2}$
$(\frac{1}{2}, 3), (1, \frac{9}{2}), (2, \frac{15}{2})$

(l) $y = -\frac{1}{12}x + \frac{7}{12}$
$(12, -\frac{5}{12}), (0, \frac{7}{12}), (2, \frac{5}{12})$

(m) $y = -\frac{3}{2}x + \frac{1}{2}$
$(1, -1), (2, -\frac{5}{2}), (-1, 2)$

(n) $y = \frac{2}{3}x - \frac{1}{3}$

$(0, -\frac{1}{3}), (3, \frac{5}{3}), (6, \frac{11}{3})$

(o) $y = -\frac{1}{4}x + \frac{6}{5}$

$(4, \frac{1}{5}), (8, -\frac{4}{5}), (0, \frac{6}{5})$

(p) $y = -\frac{7}{2}x + \frac{3}{2}$

$(2, -\frac{11}{2}), (1, -2), (-1, 5)$

(q) $y = \frac{2}{15}x - \frac{7}{15}$

$(1, -\frac{1}{3}), (0, -\frac{7}{15}), (-1, -\frac{3}{5})$

(r) $y = \frac{9}{2}x - 6$

$(2, 3), (4, 12), (6, 21)$

(s) $y = -\frac{3}{7}x + \frac{2}{7}$

$(1, -\frac{1}{7}), (2, -\frac{4}{7}), (-2, \frac{8}{7})$

(t) $y = 5x - 21$

$(0, -21), (1, -16), (4, -1)$

(Many other answers are possible for the ordered pairs.)

3. $x + y = 15$

$\{(6, 9), (9, 6), (8, 7), (7, 8)\}$

69, 78, 87, 96 are the numbers.

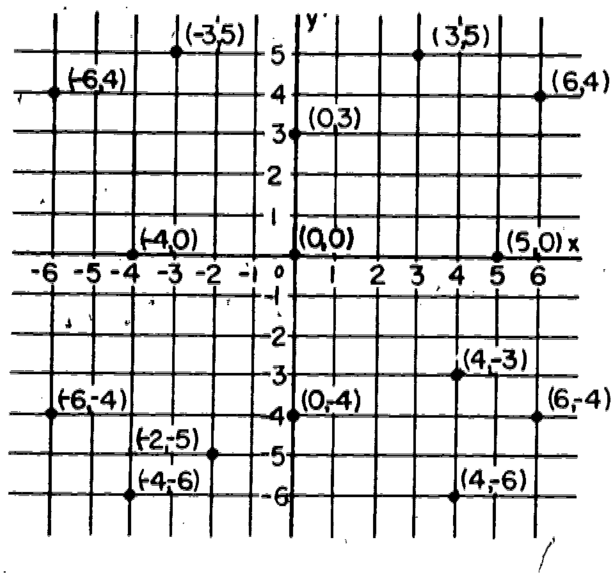
16-2. <u>Graphs of Ordered Pairs of Real Numbers</u>.

Correspondence between pairs of real numbers and points in the plane, the foundation stone of analytic geometry, is presented in a natural way without elaboration. It is felt that a student readily grasps this notion intuitively, and with a modest amount of practice can learn to plot points with confidence. The word coordinate has already been established in the student's vocabulary. It is hoped that "abscissa" and "ordinate" will cause little additional trouble.

pages 718-724: 16-2

Answers to Problem Set 16-2a; page 722:

1. and 2.



Answers to Problem Set 16-2b; pages 723-724:

1. E: (5, 2) H: (-5, -5)

   F: (-4, 4) P: (4, -5)

   G: (-7, 2) Q: (7, -3)

2. (a) (3, 1) (c) (1, -6)
   (b) (-4, -2) (d) (-3, 5)

3. (a) (5, -2) (d) (-1, -3)
   (b) (0, 3) (e) (5, 0)
   (c) (3, 6) (f) (0, -12)

The fact that points in quadrants II and III have negative abscissas and that points in quadrants III and IV have negative ordinates should be established firmly and early. This will help to avoid considerable difficulty in later work.

Answers to Oral Exercises 16-2c; pages 725-726:

1. negative; positive
2. negative; negative
3. (-1, -6); (2, -3)
4. negative; positive
5. negative; negative
6. II, III; III, IV
7. I; III
8. II
9. IV

Answers to Problem Set 16-2c; page 726:

1. (1, 4), (6, 2), (7, 3)
2. (-2, 3), (-5, 7)
3. (-2, -1), (-3, -7), (-7, -3)
4. (4, -1), (4, -9), (3, -7)

The ordered pair (0, 0) is in no quadrant.

16-3. The Graph of a Sentence in Two Variables.

Through examples we infer that there is a close connection between the set of points whose coordinates satisfy an equation of the first degree in two variables and a line in the plane. This inference is based on an observation of the way in which points, though plotted more or less at random, appear to "line up" in a definite pattern.

An intuitive notion of the meaning of the word "line" is relied on here. If the student has had little or no contact with geometry, we should not venture far beyond the concept of something which can be represented by a pencil mark traced along the edge of a ruler. However, an effort should be made to strengthen the notion of a line as an infinite set of points.

The basic assumption that the graph of an equation of the type under discussion is in fact a line enables the student to construct a graph by means of a straightedge and two predetermined points. Auxiliary points are recommended as a "check".

To support this assumption the teacher may wish to point out the existence of two analogous concepts. The first, a geometric one, asserts that for any two given distinct points in the plane there is one and only one line which contains both of them. The companion assertion in algebra states that for any two different pairs of real numbers there is one and only one equation of the first degree in two variables which has both of these elements in its truth set. In this connection we consider two equivalent first degree equations as being essentially the same equation.

One might go a step farther and avoid the possible confusion generated by the concept of equivalence. The assertion could be phrased as follows: "For any two different pairs of real numbers there is one and only one equation of the form $y = ax + b$, or $x = k$, which has these elements in its truth set." This anchors the coefficients. Whereas the form $Ax + By + C = 0$ allows them to be any three numbers proportional to the original A, B, and C.

The proof of the above assertion is given below.

Let (x_1, y_1) and (x_2, y_2) by any two different pairs of real numbers.

If $x_1 = x_2$, both pairs of numbers satisfy the sentence $x = x_1$, which is of the form $x = k$. Assume now that (x_1, y_1) and (x_2, y_2) also satisfy a sentence of the form $y = ax + b$. Then

$$y_1 = ax_1 + b \quad \text{and} \quad y_2 = ax_1 + b, \text{ since } x_1 = x_2.$$

But this would mean that $y_1 = y_2$, which is contrary to our hypothesis that the two number pairs are different. Hence in the case for which $x_1 = x_2$, the sentence $x = x_1$ is the only sentence of first degree satisfied by both number pairs.

Now consider the case where $x_1 \neq x_2$. Certainly no sentence of the form $x = k$ will be satisfied by both number pairs. Assume that there is a sentence of the form $y = ax + b$ which is satisfied by (x_1, y_1) and by (x_2, y_2). Then the following two sentences are true.

$$y_1 = ax_1 + b$$

$$y_2 = ax_2 + b$$

From this it follows that

$$y_1 - y_2 = a(x_1 - x_2)$$

and that
$$a = \frac{y_1 - y_2}{x_1 - x_2}.$$

Consequently
$$b = y_1 - \frac{y_1 - y_2}{x_1 - x_2} x_1 .$$

This means that if there is a sentence in y-form satisfied by both number pairs, then this sentence is

$$y = \frac{y_1 - y_2}{x_1 - x_2} x + y_1 - \frac{y_1 - y_2}{x_1 - x_2} x_1$$

The only question that remains is: do both number pairs really satisfy this equation? It can be easily verified that they do.

Answers to Oral Exercises 16-3a; pages 732-733:

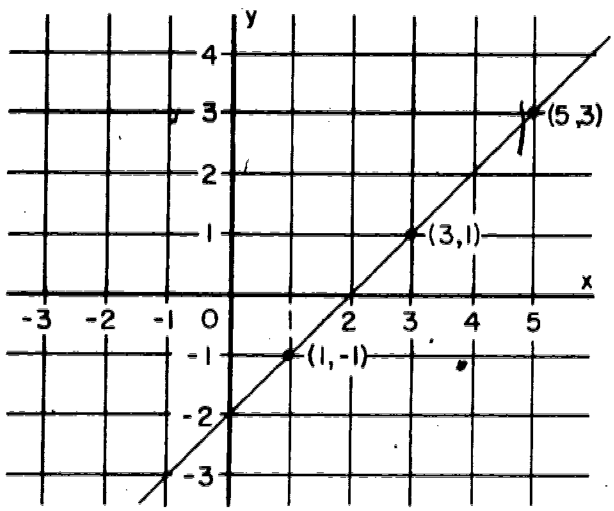
1. (a) 4; 0; -2
   (b) -7; -1
   (c) 0; -4; 3
   (d) 4; 8; 10
   (e) -1; 0; -2

2. (a) yes
   (b) no
   (c) yes
   (d) yes
   (e) no
   (f) yes

Answers to Problem Set 16-3a; page 733:

1. (a) for example (3, 1), (1, -1)

(b)

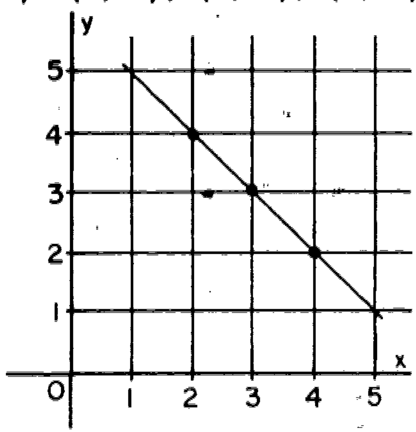


(c) yes
(d) yes
(e) yes
(f) yes

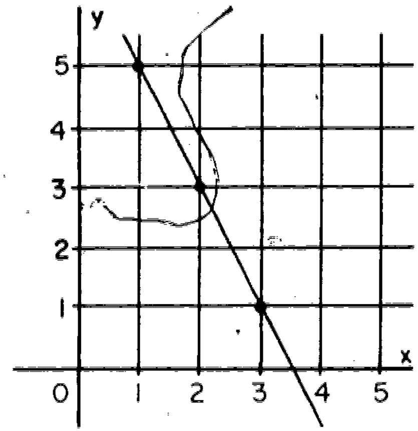
The graph of a second degree equation is taken up in detail in a later chapter. The topic is touched on only superficially here to point up the fact that the line graph is characteristic of a first degree equation only. In this connection it should be noted that even a first degree equation in two variables will have a line as its graph if and only if the domain of both variables is the set of all real numbers. If, on the other hand, the domain of the variables of such an equation is restricted to the integers, then the graph will be a set of discrete points. There are, however, no graphs of this latter type in the present chapter.

Answers to Problem Set 16-3b; page 735:

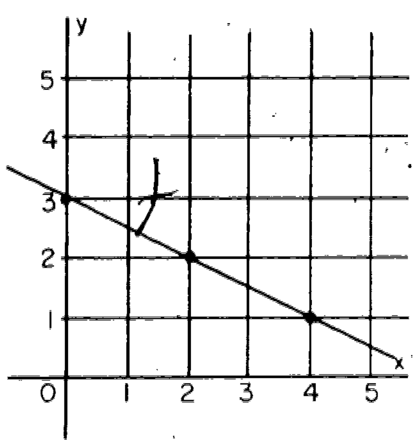
(a) (3, 3), (4, 2), (2, 4)



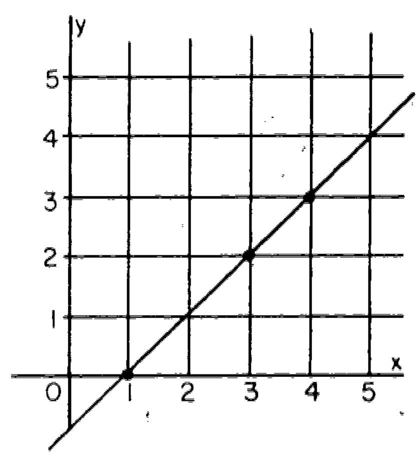
(b) (1, 5), (2, 3), (3, 1)



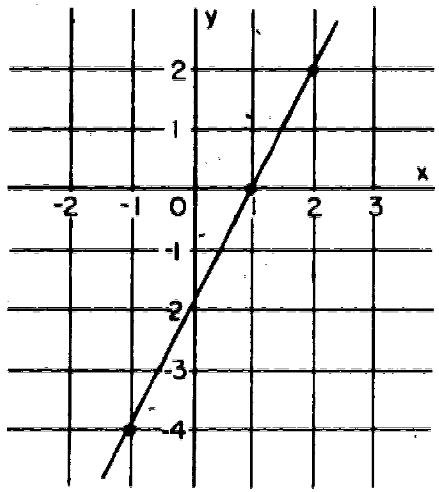
(c) (0, 3), (2, 2), (4, 1)



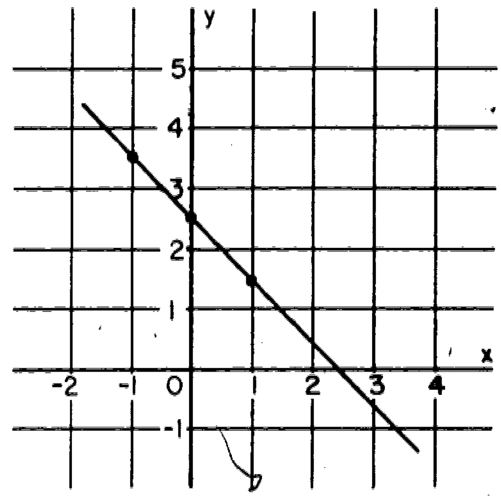
(d) (3, 2), (4, 3), (1, 0)



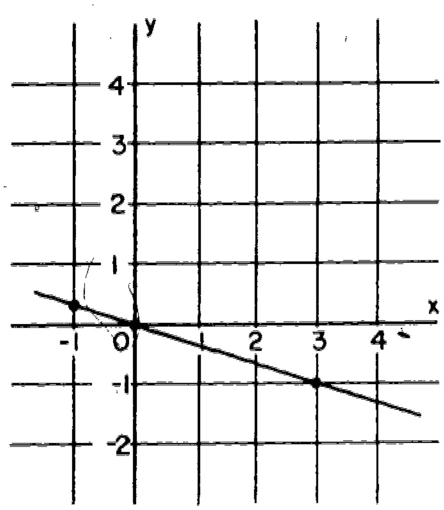
(e) (1, 0), (-1, -4), (2, 2)



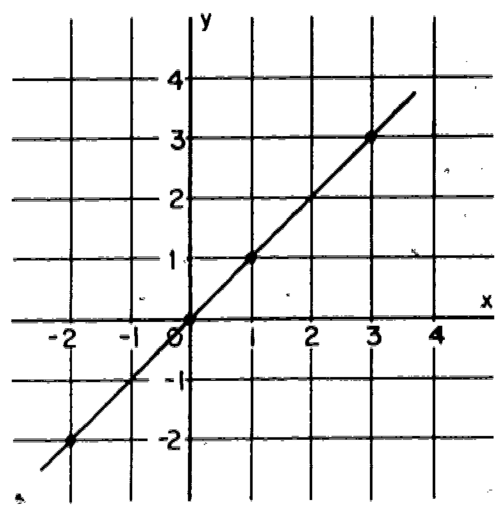
(f) $(1, \frac{3}{2})$, $(0, \frac{5}{2})$, $(-1, \frac{7}{2})$



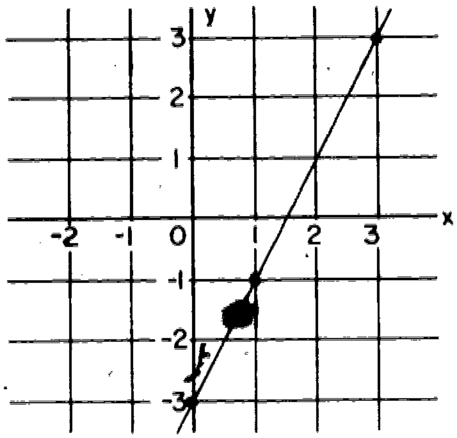
(g) (0, 0), $(-1, \frac{1}{3})$, (3, -1)



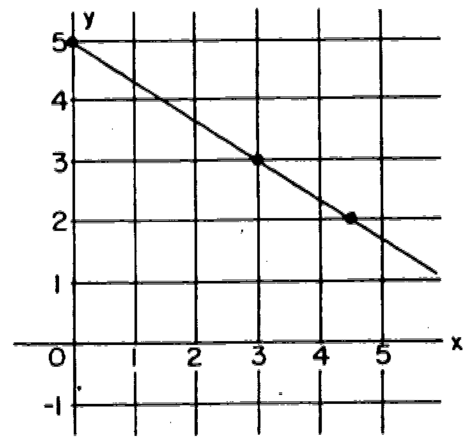
(h) (3, 3), (1, 1), (-2, -2)



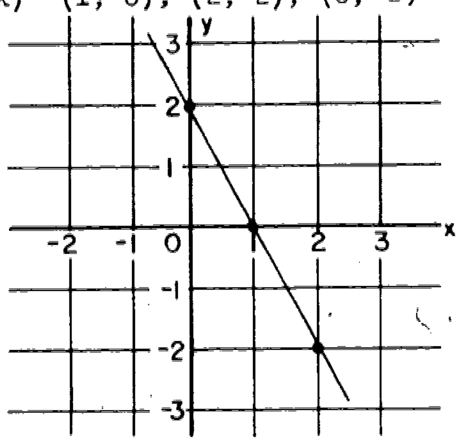
(i) (3, 3), (1, -1), (0, -3)



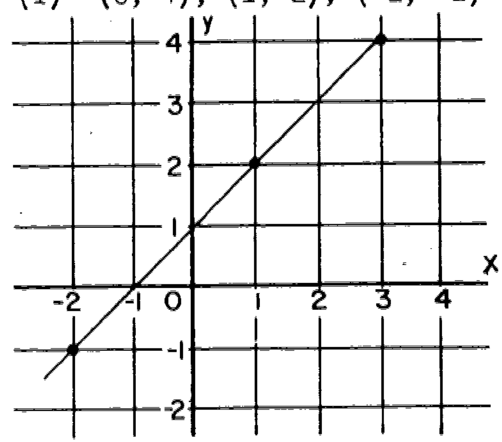
(j) (3, 3), (0, 5), ($\frac{9}{2}$, 2)



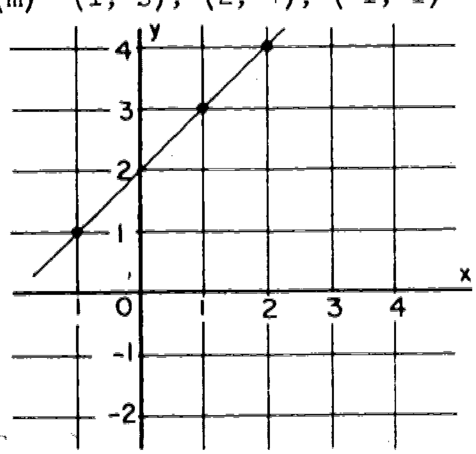
(k) (1, 0), (2, -2), (0, 2)



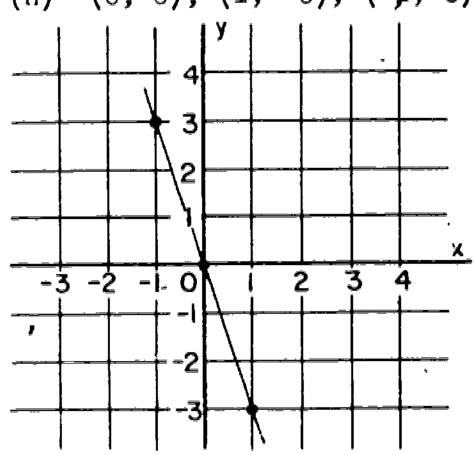
(l) (3, 4), (1, 2), (-2, -1)



(m) (1, 3), (2, 4), (-1, 1)



(n) (0, 0), (1, -3), (-1, 3)



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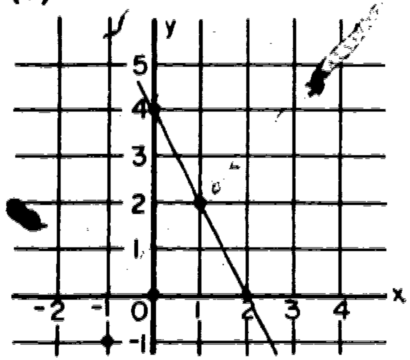
2. (a) (1, 2), (0, 4), (2, 0)

(b)

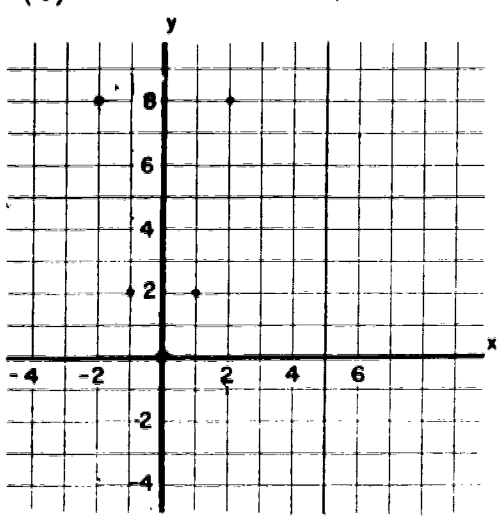


(c) (0, 0), (-1, -1)

(e) no

3. (a) (1, 2), (2, 8), (0, 0), (-1, 2), (-2, 8)

(b)

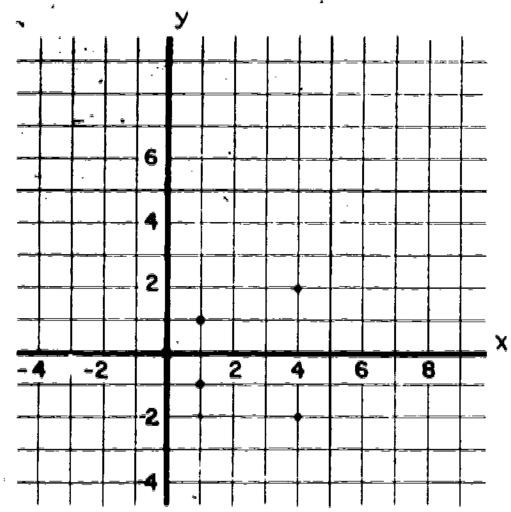


The points do not lie on a line.

(c) Yes.

4. (a) (0, 0) (1, 1) (1, -1) (4, 2), (4, -2)

(b)



The points do not lie on a line.

(c) Yes.

Page 736.

Here the student is introduced to what may seem at first to be a contradiction, namely that equations such as $x = 5$, or equations such as $y = 2$, can be interpreted as equations in two variables. This idea, which is fortified by the use of a zero coefficient, will be of considerable importance in the study of systems of equations in Chapter 17.

Answers to Oral Exercises 16-3c; page 739:

1. (a) the set of points whose abscissas are 3, that is, the line parallel to the y-axis and 3 units to the right of the y-axis
   (b) the line parallel to the y-axis and four units to the left of the y-axis
   (c) the line parallel to the x-axis and five units above the x-axis

(d) the line parallel to the x-axis and two units below the x-axis

(e) the line parallel to the y-axis and three units to the left of the y-axis

(f) the line parallel to the y-axis and seven units to the right of the y-axis

(g) the line parallel to the x-axis and three units above the x-axis

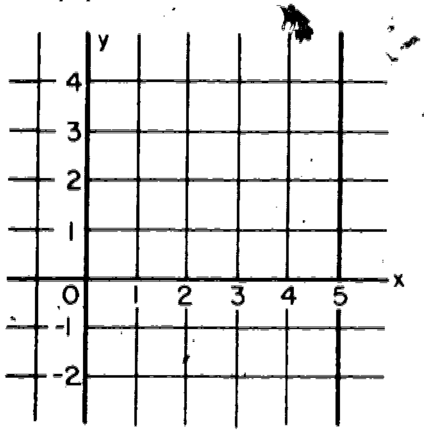
(h) the line parallel to the x-axis and three units below the x-axis

(i) the y-axis

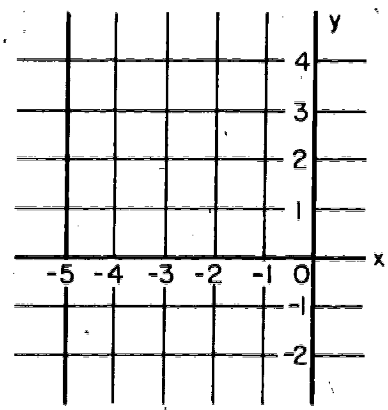
(j) the x-axis

Answers to Problem Set 16-3c; pages 739-740:

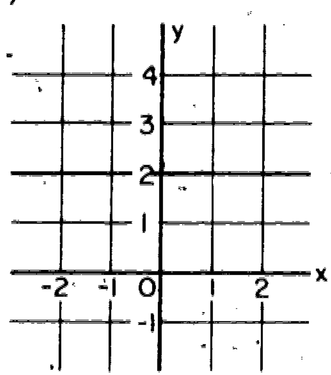
1. (a)



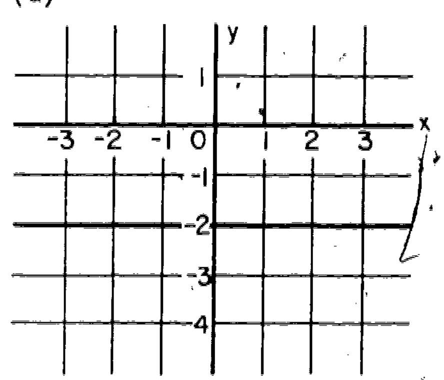
(b)



(c)



(d)



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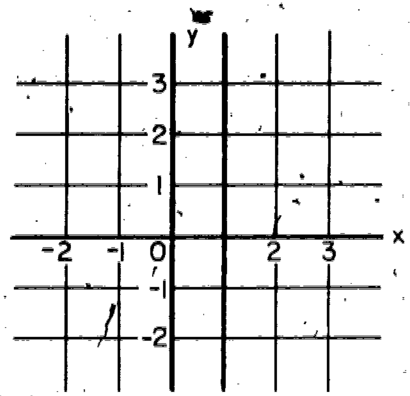
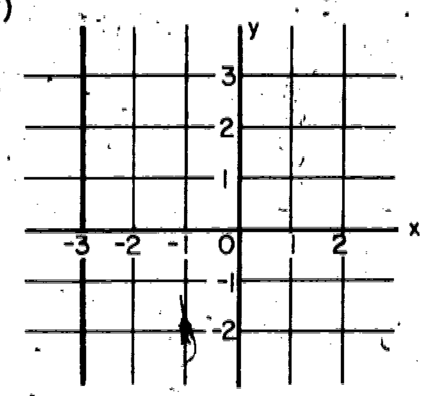
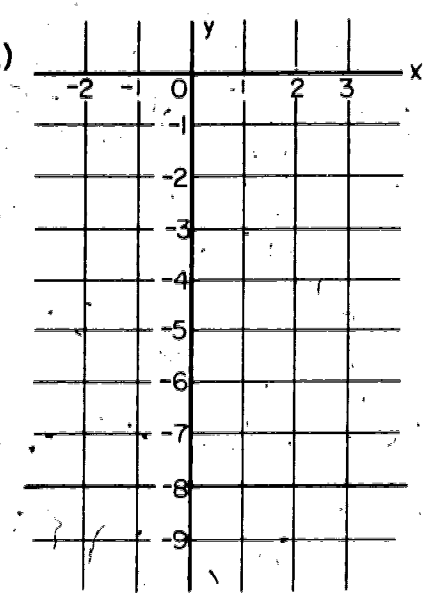
(e)



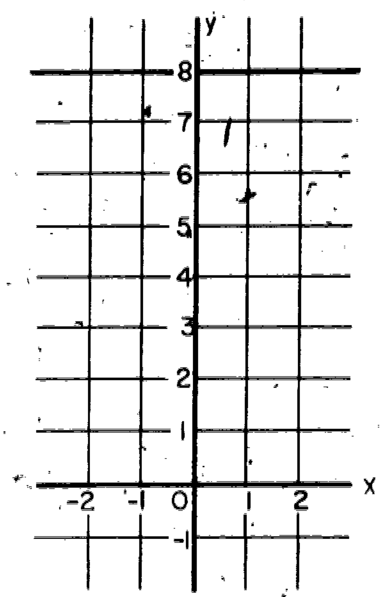
(f)



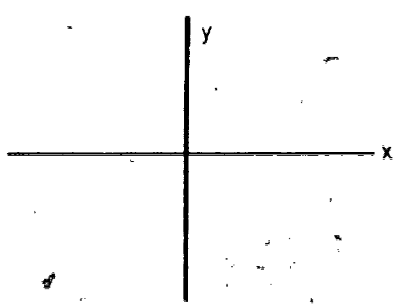
(g)



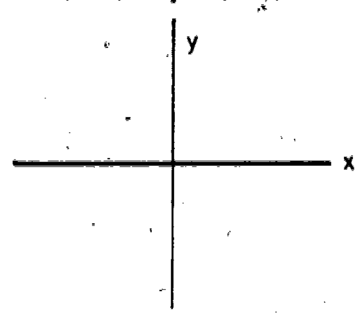
(h)



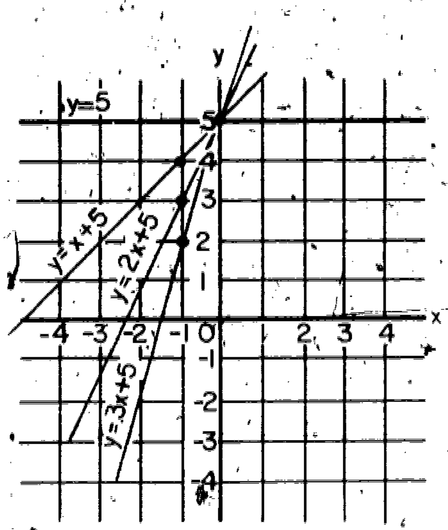
(i)



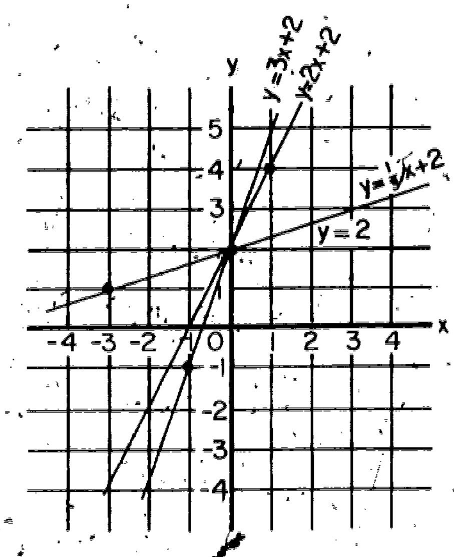
(j)



2.

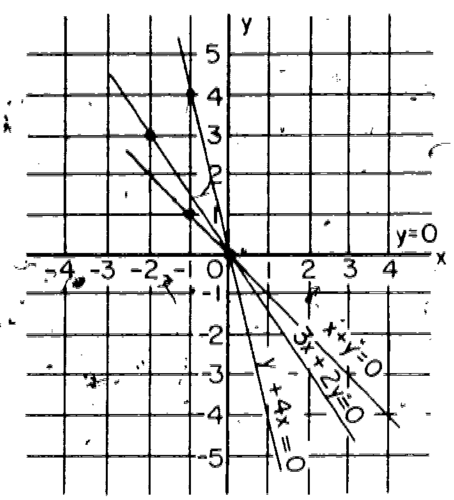


4.



All the graphs pass through the point (0, 2).
All the equations have a constant term 2.

3.



All the graphs pass through the origin. None of the equations contain a constant term.

pages 740-742: 16-4

16-4. Intercepts and Slopes.

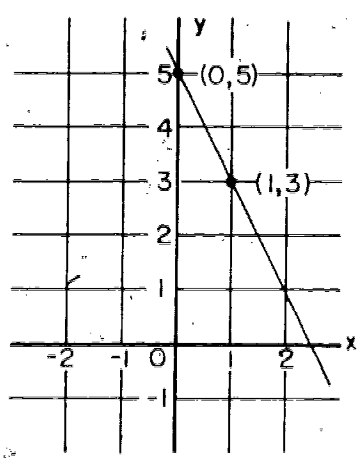
Once again a heavy emphasis is placed on the y-form of an equation. It is very important that the student realize that the definition of both slope and y-intercept are based on this particular form. Thus the equation $x = k$ has as its graph a line for which the slope is not defined as a real number, nor does the graph of such a line have a y-intercept.

Answers to Oral Exercises 16-4a; page 742:

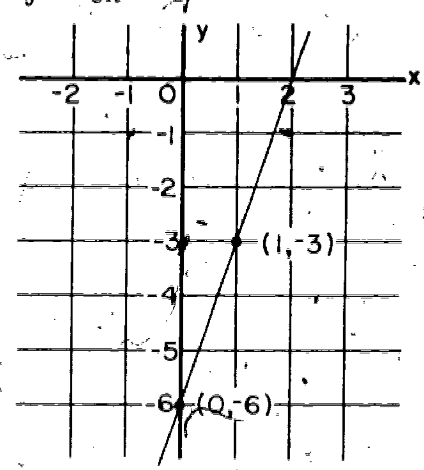
1. (0, 2)
2. (0, -5)
3. (0, 12)
4. $(0, -\frac{3}{4})$
5. $y = -x + 2$ (0,2)
6. $y = -2x + 7$ (0,7)
7. $y = -\frac{3}{4}x + \frac{9}{10}$ $(0, \frac{9}{10})$
8. $y = -.8x + .16$ (0, .16)
9. $y = x - 12$ (0, -12)
10. $y = 2x - \frac{1}{3}$ $(0, -\frac{1}{3})$
11. $y = -x + \frac{5}{2}$ $(0, \frac{5}{2})$
12. $y = -\frac{1}{3}x + 2$ (0, 2)
13. $y = \frac{1}{2}x - 2$ (0, -2)
14. $y = -\frac{3}{4}x + 2$ (0, 2)
15. $y = -\frac{2}{3}x + \frac{5}{3}$ $(0, \frac{5}{3})$

Answers to Problem Set 16-4a; pages 742-743:

1. $y = -2x + 5$

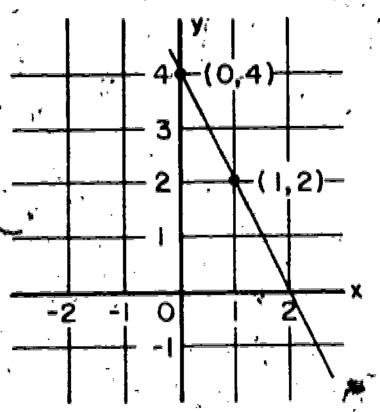


2. $y = 3x - 6$

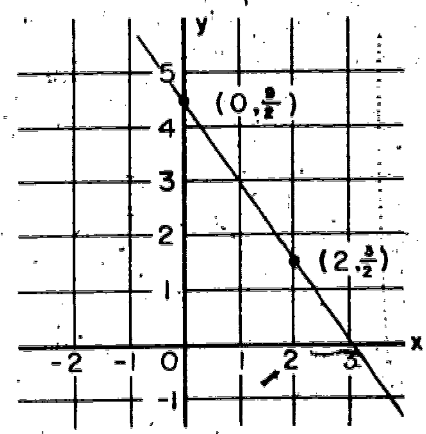


pages 742-743: 16-4

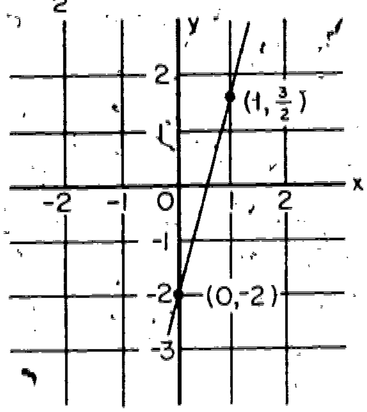
3. $y = -2x + 4$



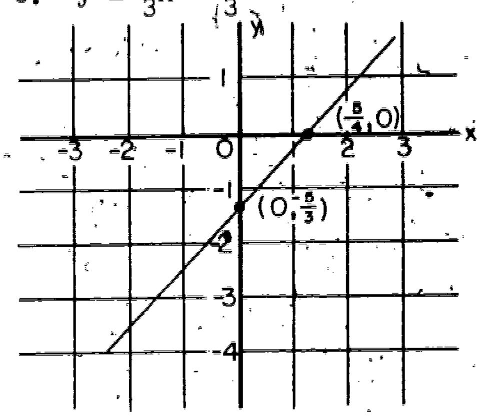
4. $y = -\frac{3}{2}x + \frac{9}{2}$



5. $y = \frac{7}{2}x - 2$



6. $y = \frac{4}{3}x - \frac{5}{3}$



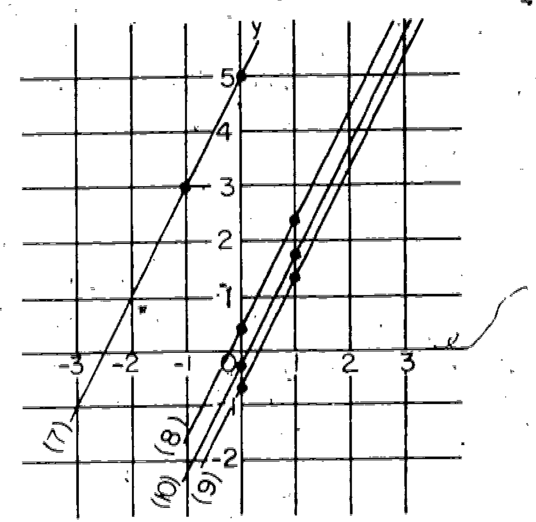
7. $y = 2x + 5$

8. $y = 2x + \frac{1}{3}$

9. $y = 2x - \frac{3}{5}$

10. $y = 2x - \frac{1}{4}$

These graphs seem to be parallel lines. In the y-forms the coefficients of x are each 2 .



<u>Answers to Oral Exercises 16-4b</u>, page 744:

1. (a) (0, 2)
   (0, -5)
   (0, 2)

   (b) $y = 3x + 2$
   and
   $y = 3x - 5$

   (c) $y = 2x + 2$

   (d) none

2. (a) (0, -2)
   (0, 2)
   (0, -2)

   (b) all have graphs which are parallel to each other

   (c) none

   (d) $y = \frac{1}{2}x - 2$
   and
   $2y = x - 4$

3. (a) (0, 7)
   (0, -2)
   (0, -2)

   (b) all have graphs which are parallel to each other

   (c) none

   (d) $y = -3x - 2$
   and
   $-\frac{1}{3}y = x + \frac{2}{3}$

4. (a) $(0, \frac{3}{4})$
   $(0, -\frac{3}{4})$
   $(0, -\frac{3}{4})$

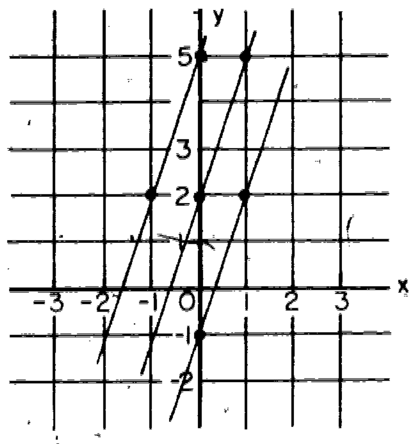
   (b) none have graphs which are parallel to each other

   (c) no two graphs are parallel to each other

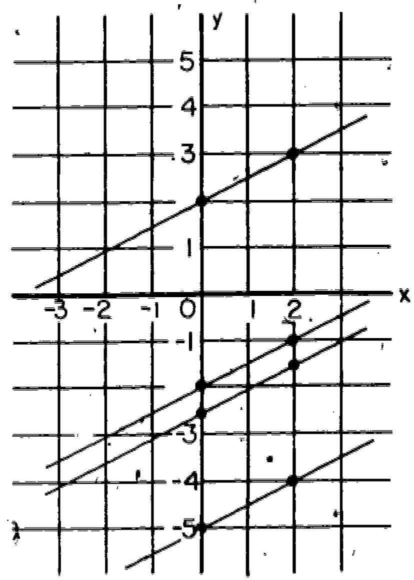
   (d) none

Answers to Problem Set 16-4b; page 745:

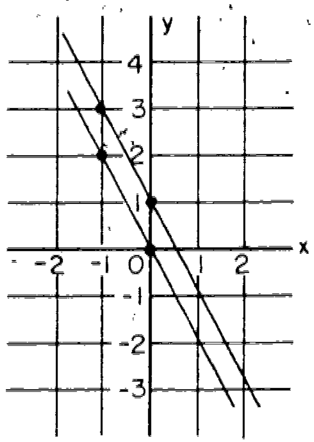
1. $y = 3x + 2$
   $y = 3x + 5$
   $y = 3x - 1$



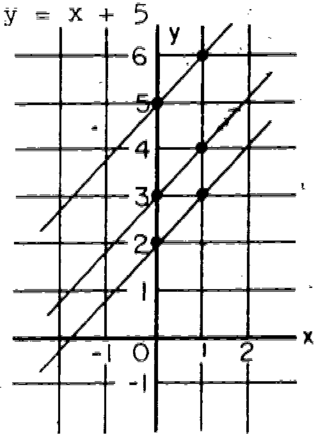
2. $y = \frac{1}{2}x - 5$
   $2y - x = -5$
   $y = \frac{1}{2}x + 2$
   $x - 2y = 4$



3. $y = -2x + 1$
   $y + 2x = 0$



4. $y = x + 3$
   $y - x = 2$
   $y = x + 5$



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We have essentially a choice between two possible definitions of the slope of a line: the coefficient of x in the y-form of the equation; the ratio of the vertical change to the horizontal change from one point to another on the (non-vertical) line. In a course in analytic geometry in which a line is given a geometric meaning, the second of these would be taken as a definition and the first proved as a theorem. Here we have considered a line more in the light of the truth set of a given equation. Hence it is natural to take the first as a definition and develop its connection with the second. In this way we avoid some of the complications which arise under the ratio definition with respect to the distinction between a line and a line segment. The theorem which establishes the connection between these two concepts of slope is stated but not proved, since the proof involves the solution of a system of two equations in two variables. However, the teacher may wish at this time to discuss the general nature of the proof in informal terms. It can be developed fairly simply as follows.

The theorem states that if (c, d) and (e, f) are the coordinates of any two points on a line with $c \neq e$, then the ratio $\frac{d - f}{c - e}$ is equal to the coefficient a in the y-form of the corresponding equation.

Proof:

Since the number pairs (c, d) and (e, f) satisfy the sentence $y = ax + b$, it follows that

$$d = ac + b$$

and $$f = ae + b\ .$$

Hence $d - f = ac - ae = a(c - e)$, from which the conclusion

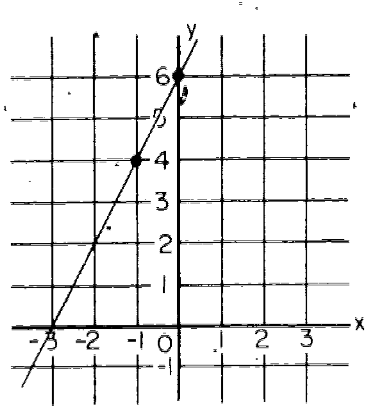
$\frac{d - f}{c - e} = a$ follows. In presenting this proof to the student it should be emphasized that the operations were legitimate only because of the hypotheses (1) that the points were elements in the truth set of the equation, and (2) that $c - e \neq 0$. The student should be cautioned against an assumption that one may "subtract" open sentences whether true or false. In the above proof, the sides of each of the respective equations have already been established as names for the same number before the subtraction process. The coefficient definition also encompasses in an uncomplicated way the notion of a zero slope or a negative slope, as well as the notion that slope is not defined for vertical lines.

Answers to Oral Exercises 16-4c; page 749:

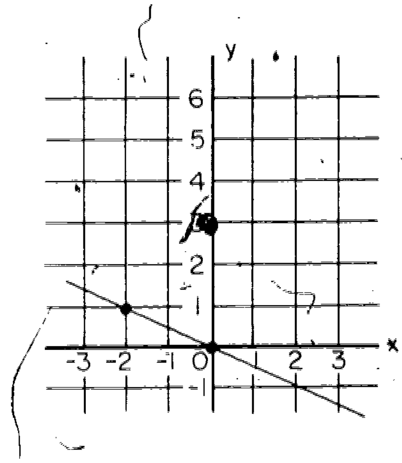
1. (a) slope: $\frac{3}{4}$
   y-intercept: (0, 1)
   (b) slope: -3
   y-intercept: (0, 3)
   (c) slope: 17
   y-intercept: (0, -6)
   (d) slope: $-\frac{2}{3}$
   y-intercept: (0, 0)
   (e) slope: 0
   y-intercept: (0, 5)
   (f) slope: not defined
   y-intercept: none
2. (a), (d), (e)
   (b), (c)

Answers to Problem Set 16-4c; pages 749-750:

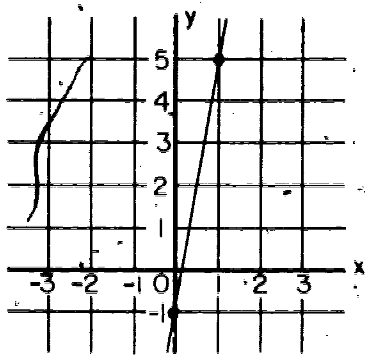
1. (a) slope: 2
   y-intercept: (0, 1)
   (b) slope: 2
   y-intercept: (0, 1)
   (c) slope: $-\frac{3}{2}$
   y-intercept: (0, 3)
   (d) slope: $-\frac{1}{3}$
   y-intercept: $(0, \frac{1}{3})$
   (e) slope: $-\frac{2}{3}$
   y-intercept: (0, 2)
   (f) slope: not defined
   y-intercept: none
2. (a) $y = 2x + 6$



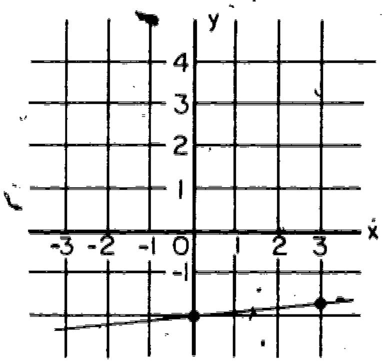
   (b) $y = -\frac{1}{2}x$



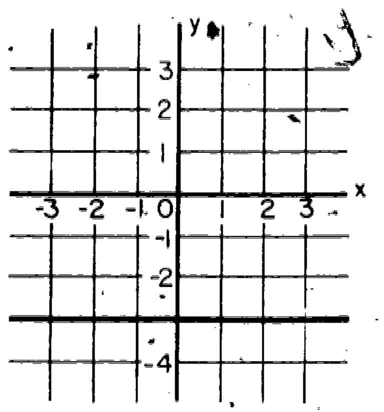
(c) $y = 6x - 1$



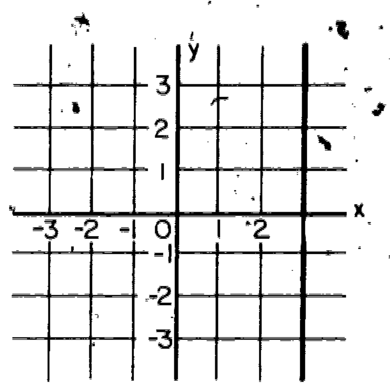
(d) $y = \frac{1}{12}x - 2$



(e) $y = -3$



(f) $x = 3$



3. $y = 3x + 1$
4. $y = -2x + 3$
5. $a_1 = a_2$

<u>Answers to Oral Exercises 16-4d</u>; page 755:

1. (a) 3
   (b) 1
   (c) 1
   (d) $\frac{5}{13}$
   (e) 0
   (f) $\frac{7}{12}$

2. (a) increases
   (b) decreases
   (c) increases
   (d) decreases
   (e) decreases
   (f) increases

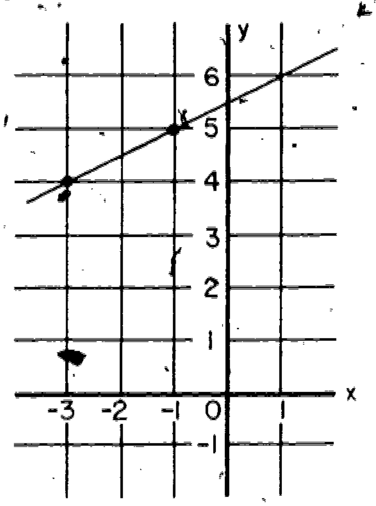
Answers to Problem Set 16-4d; pages 755-756:

1. (a) -2 (d) $\frac{1}{3}$

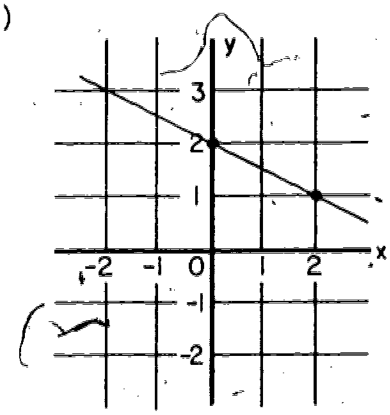
   (b) $\frac{13}{5}$ (e) 0

   (c) $\frac{5}{4}$ (f) not defined

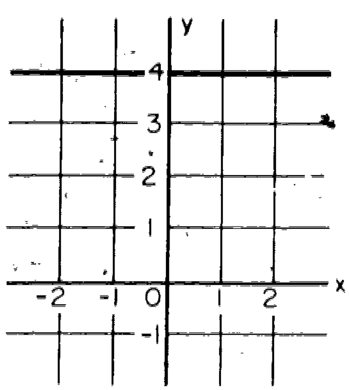
2. (a)



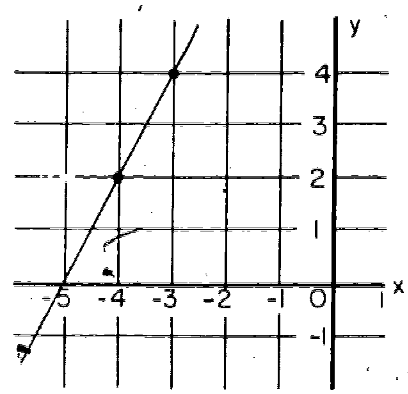
(b)



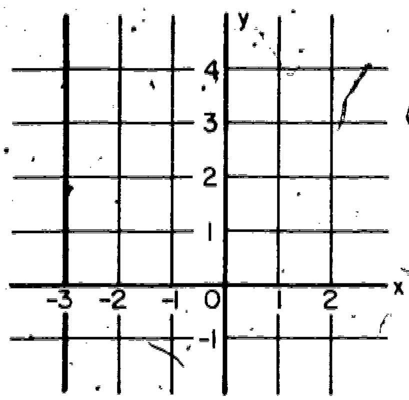
(c)



(d)



(e)



3. (a) $y = 3x + 3$
   (b) $y = 3x + 3$
   (c) $y = 2x + 3$
   (d) $y = 3x$
   (e) $y = -2x + \frac{4}{3}$
   (f) $y = -\frac{3}{2}x + 1$

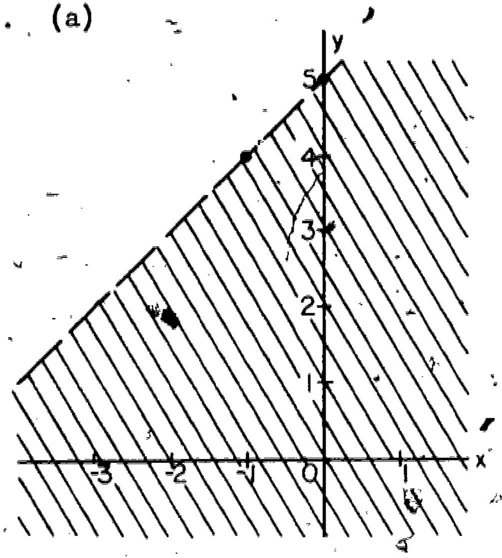
16-5. Graphs of Sentences Involving an Order Relation.

The approach in this section is based on an assumption, made without formal proof, that a line divides the plane into two half planes, thereby creating three mutually exclusive and all inclusive subsets of the set of all points in the plane, namely the set of points on one side of the line, the set of points on the other side, and the set of points on the line itself. The major assertion, made plausible by examples, is that the graph of an inequality of the first degree in two variables consists of all the points on one side of a line whose equation is formed by changing the order symbol to "=". The teacher may wish to support this assertion with more examples than are given in the text. It may also help to have the student practice selecting two points, one on one side of the line, and one on the other. Then show that one of these satisfies the inequality, while the other does not.

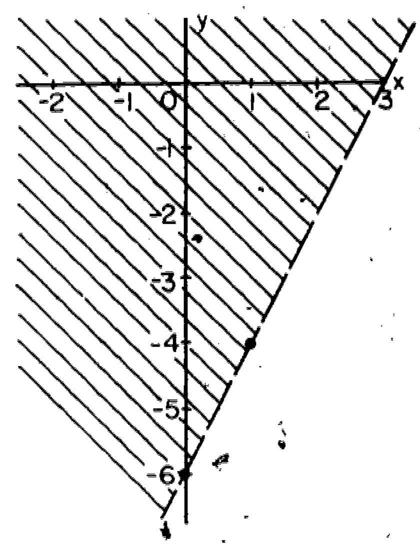
The machinery for constructing such a graph consists essentially in determining on which side of the line an element of the truth set lies, and then indicating this particular half plane as the graph. It is important that the student recognize the distinction between the relation "<" and "≤", the latter requiring that the line be included in the graph. The student should avoid an inference that the symbol ">" is associated with points "above" the line while the symbol "<" is associated with points "below" the line.

Answers to Problem Set 16-5; pages 761-762:

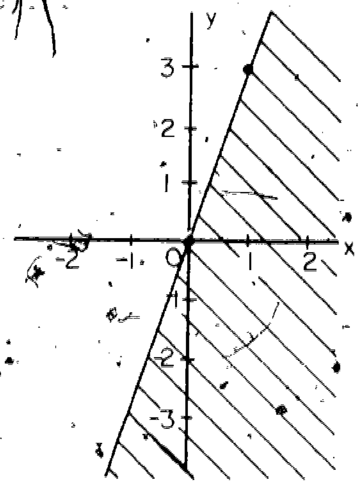
1. (a)



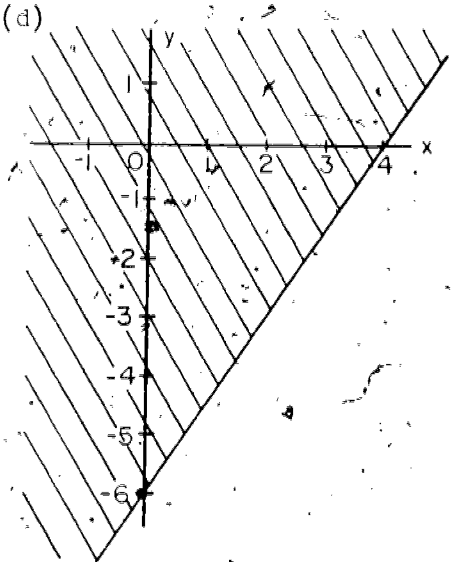
(b)



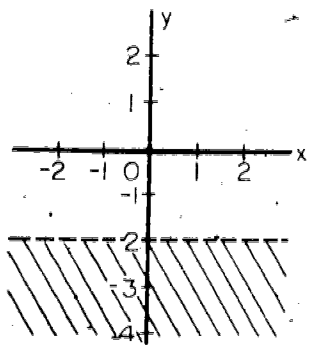
(c)


y
x
3
2
1
-1
-2
-3
-2
-1
0
1
2

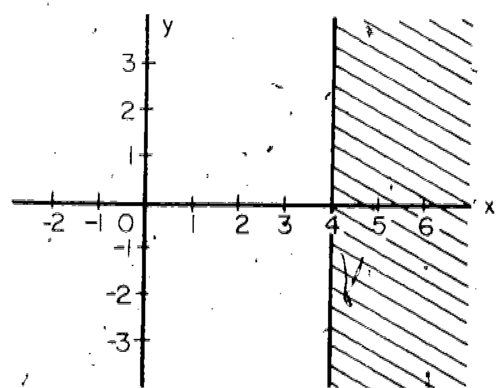
(d)


y
x
1
-1
0
1
2
3
4
-1
-2
-3
-4
-5
-6

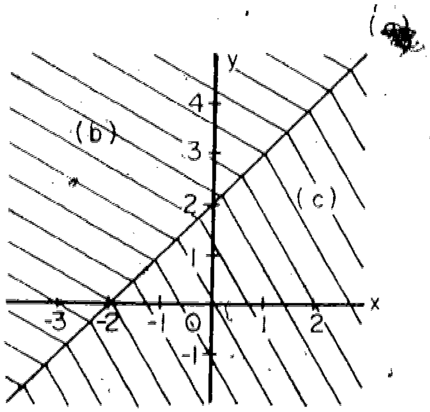
(e)


y
x
2
1
-2
-1
0
1
2
-1
-2
-3
-4

(f)


y
x
3
2
1
-2
-1
0
1
2
3
4
5
6
-1
-2
-3

2.


(a)
y
x
(b)
(c)
4
3
2
1
-3
-2
-1
0
1
2
-1

pages 762-764

3. Yes

$y = x + 2$

<u>Answers to Review Problem Set</u>; pages 763-766:

1. (a) -8
   (b) -11
   (c) 11
   (d) $\frac{1}{2}$
   (e) $20a$
   (f) $-2a + 6b + c$
   (g) 6
   (h) $\frac{-4a}{9\ cm}$
   (i) $\frac{8a^2y^3z^2}{21m^3}$
   (j) $-a - 3b$
   (k) $2m + 2n - 10$
   (l) $5x + 2y - 9xz + 9yz + 18z^2$

2. (a) $a(m + n)$
   (b) $6(m + n)$
   (c) $14a(a + 3)$
   (d) $6c(3 + 2d)$
   (e) $3(3 - 2a)$
   (f) $ab(6a + b)$
   (g) $3m - 2n$
   (h) $3(x - 3)$
   (i) $m(20)$
   (j) $a(1 + a + a^2)$
   (k) $b(y^2 + y - 1)$
   (l) $2\pi(r - t)$
   (m) $5(4y - 3x - 2)$
   (n) $4y + 5x + 7z$
   (o) $2a(c - 2b - 1)$
   (p) $(a + 1)(x + y)$
   (q) $cy - cx + ay + ax$
   (r) $(mn + 1)(m + n)$
   (s) $(-4b + 3a)(2c - d)$
   (t) $(x + 5)(x + 4)$
   (u) $(x - 2)(x - 6)$
   (v) $(b - 4)(b + 3)$
   (w) $4(m - 5)(m - 2)$
   (x) $b(c - 6)(c + 1)$
   (y) $x(1 - x)(1 + x)(1 + x^2)$

3. (a) $\{\frac{17}{3}\}$
   (b) $\{4\}$
   (c) $\{2\}$
   (d) $\{-20\}$
   (e) $\{-1\}$
   (f) $\{-14\}$
   (g) $\{3\}$
   (h) $\{18\}$
   (i) $\{28\}$
   (j) set of real numbers greater than 2
   (k) set of real numbers greater than 6
   (l) set of real numbers less than -6
   (m) set of real numbers less than $\frac{7}{3}$
   (n) set of real numbers greater than 3
   (o) $\{1,3\}$
   (p) $\{3,7\}$

(q) {4,-3}

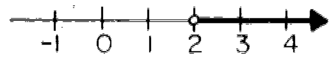
(r) {-6,4}

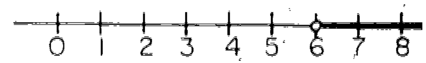
(s) {6}

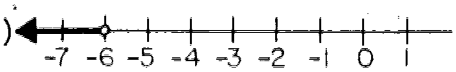
(t) {0,3}

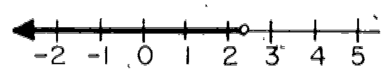
(u) {4,-4}

(v) {3,-3}

4. (j) 

(k) 

(l) 

(m) 

5. (a) Let n be the first integer. The next consecutive integer is $n + 1$.

$$n + (n + 1) = 63$$
$$2n = 62$$
$$n = 31$$

Hence, the truth set is {31}.
The integers are 31 and 32.

(b) Let n be a number. Another number is $5n$.

$$n + 5n = 90$$
$$6n = 90$$
$$n = 15$$

Hence, the truth set is {15}.
The two numbers are 15 and 75.

(c) If George is a years old, then James is $a + 5$ years old.

$$a + (a + 5) = 37$$
$$2a = 32$$
$$a = 16$$

Hence, the truth set is {16}.
George is 16 and James is 21.

(d) Let w be the length in feet of one of the sides of the lot. Then, $120 - w$ is the length in feet of an adjacent side.

$$w(120 - w) = 2700$$

Domain: $w < 60$
By convention the width is less than the length so $w < 120 - w$.

$$120w - w^2 = 2700$$

$$w^2 - 120w + 2700 = 0$$

$$(w - 90)(w - 30) = 0$$

$$w - 90 = 0 \quad \text{or} \quad w - 30 = 0$$

$$w = 90 \quad \text{or} \quad w = 30 \qquad \text{Domain: } w < 60$$

Hence, the truth set is $\{30\}$.
The width is 30 ft.; the length is $120 - 30$, or 90 ft.

(e) Let a be the altitude in inches of the triangle.

$$\frac{1}{2}a(16) = 60$$

$$8a = 60$$

$$a = \frac{15}{2}$$

Hence, the truth set is $\{\frac{15}{2}\}$.
The altitude is $\frac{15}{2}$ inches.

(f) Let n be the first number. Then, the next two consecutive numbers are $n + 1$ and $n + 2$.

$$n + (n + 1) + (n + 2) = 108$$

$$3n + 3 = 108$$

$$3n = 105$$

$$n = 35$$

Hence, the truth set is $\{35\}$.

The numbers are 35, 36, 37.

*(g) Let x be an odd number. Then, the next consecutive odd number is $x + 2$.

$$x + (x + 2) < 110 \qquad x + (x + 2) > 99$$
$$2x + 2 < 110 \qquad 2x + 2 > 99$$
$$2x < 108 \qquad 2x > 97$$
$$x < 54 \qquad x > 48\frac{1}{2}$$

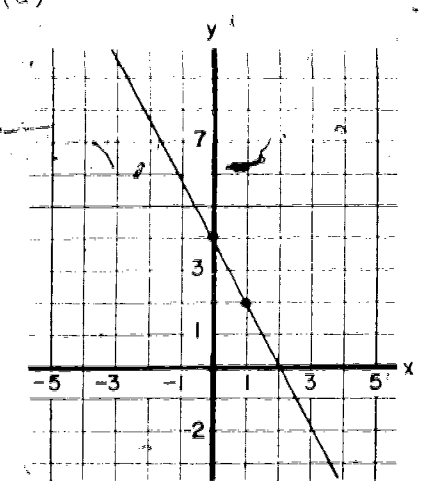
Hence, the truth set is the set of all integers which are less than 54 and greater than $48\frac{1}{2}$.
The numbers could be 53 and 55, 51 and 53, or 49 and 51.

*(h) Let s be an even integer. Then, $s + 2$ is the next consecutive even integer.

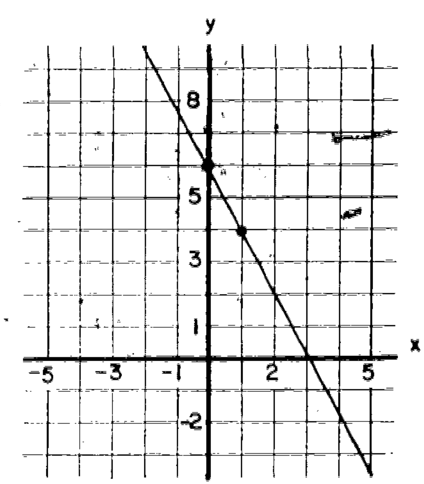
$$s + (s + 2) = 35$$
$$2s = 33$$
$$s = \frac{33}{2}$$

Hence, the truth set is $\{\frac{33}{2}\}$.
But $\frac{33}{2}$ is not an even integer.
Therefore, there is no even integer s such that $s + (s + 2) = 35$.

6. (a)



(b)



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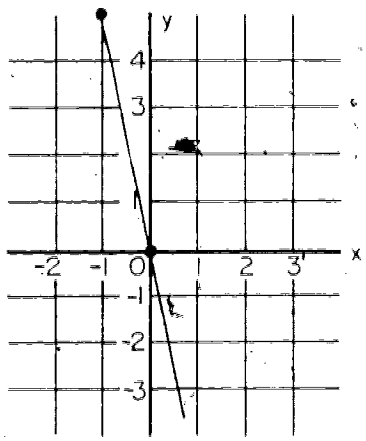
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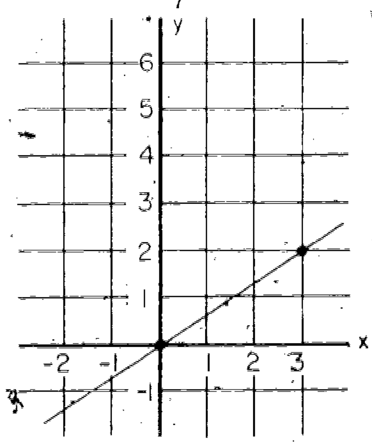
4
3
1
x
-2
-1
0
1
2
3
-1
-2
-3

(d)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6
5
4
3
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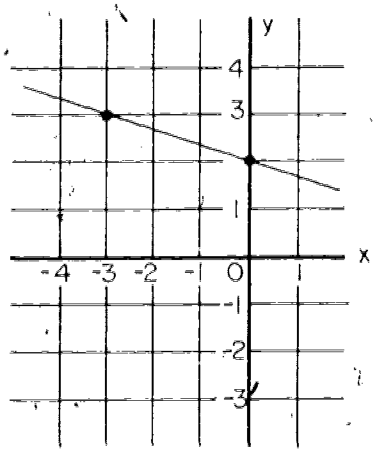
x
-2
-1
1
2
3
-1

(e)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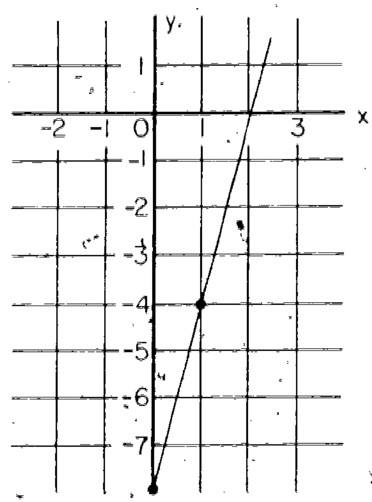
x
-4
-3
-2
-1
0
1
-1
-2
-3

(f)


y
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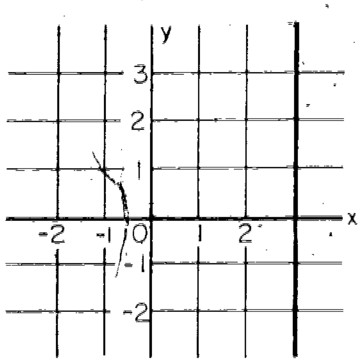
-2
-3
-4
-5
-6
-7

(g)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3
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1
x
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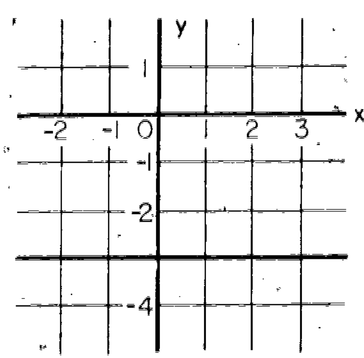
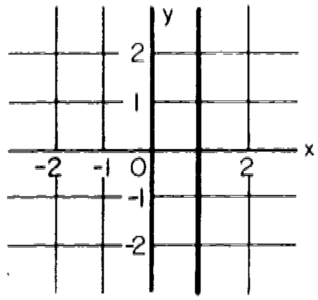
-1
0
1
2
-1
-2

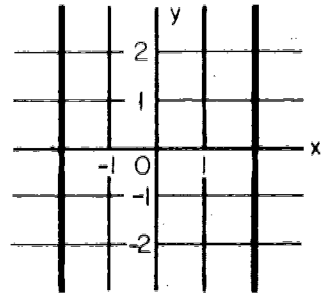
(h)


y
1
x
-2
-1
0
1
2
3
-1
-2
-4

(i)



(j)



7. (a) $y = -3x + 2$
   (b) $y = \frac{2}{3}x + \frac{3}{2}$
   (c) $y = -\frac{5}{8}x$
   (d) $y = 4x + 4$
   (e) $y = -8x - 3$

8. (a) slope: 3
   y-intercept: (0, -5)
   (b) slope: -2
   y-intercept: (0, 1)
   (c) slope: -7
   y-intercept: (0, 5)
   (d) slope: $\frac{5}{3}$
   y-intercept: $(0, \frac{2}{3})$
   (e) slope: 2
   y-intercept: $(0, \frac{7}{2})$
   (f) slope: 0
   y-intercept: (0, 5)

Suggested Test Items

1. Consider the equation $2x - 5y - 10 = 0$
   (a) Write the equation in the y-form.
   (b) What is the slope of the line which is the graph of this equation?
   (c) What is the y-intercept of this line?
   (d) Draw the graph of the equation.
   (e) What is an equation of the line parallel to the given line and with the y-intercept (0, 1)?
2. What is the value of a such that the line with equation $3x + 2y - 6 = 0$ contains the point (a,3)?
3. What is the value of b such that the point (2,-3) is on the line with equation $2x - by = 3$?

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4. Determine the slope of the line whose equation is:

   (a) $y - 3 = 0$

   (b) $x = 2y - 2$

   (c) $3y + x - 4 = 0$

5. Given the equation $x^2 - 1 - y = 0$ and the ordered pairs $(0,2)$, $(-3,8)$, $(1,1)$, $(-2,4)$, $(0,-1)$, $(0,1)$, $(-2,3)$, which of the given ordered pairs are elements of the truth set of the given equation?

6. Give a reason why or why not the equation in Problem 5 has a graph which is a line.

7. Draw the graph of each of the following with reference to a different set of axes.

   (a) $2x + y + 2 = 0$

   (b) $y = \frac{3}{2}x - 2$

   (c) $2x - 1 = 0$

   (d) $3x + 2y < 6$

   (e) $2y + 3 = 0$

   (f) $y - x \geq 0$

   (g) $x - 2y > 2$

   (h) $x + 2 = y$ or $x = y$

   (i) $x = 2y + 3$

   (j) $x + 2 = y$ and $x = y$

8. Draw a line such that the ordinate of each point on the line is twice the opposite of the abscissa. What is the equation of this line?

9. From one point to another on a line the horizontal change is -3 units and the vertical change is 6 units. What is the slope of the line?

10. If the line described in Problem 9 contains the point $(0, -\frac{1}{2})$, what is an equation of the line?

11. Is the point $(3,10)$ on the line containing the points $(2,7)$ and $(0,3)$? Give a reason for your answer.

12. If the ordered pair $(a,2)$ is in the truth set of the sentence $2x + y > 4$, is $(a + 1,2)$ also in the truth set? Give reasons for your answer.

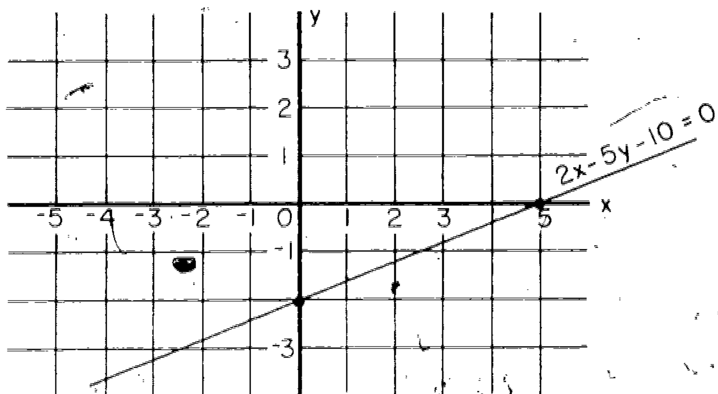
Answers to Suggested Test Items

1. (a) $y = \frac{2}{5}x - 2$

   (b) The slope is $\frac{2}{5}$

   (c) The y-intercept is $(0,-2)$

   (d)



   (e) $y = \frac{2}{5}x + 1$

2. $3(a) + 2(3) - 6 = 0$

   $a = 0$

3. $2(2) - b(-3) = 3$

   $b = -\frac{1}{3}$

4. (a) 0

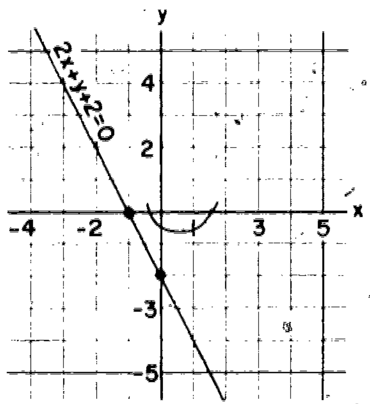
   (b) $\frac{1}{2}$

   (c) $-\frac{1}{3}$

5. $(-3,8)$, $(0,-1)$, $(-2,3)$

6. The graph of $x^2 - 1 - y = 0$ is not a line since only a first degree equation in two variables has a graph which is a line.

7. (a)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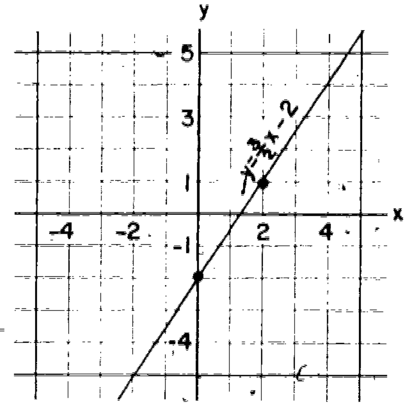
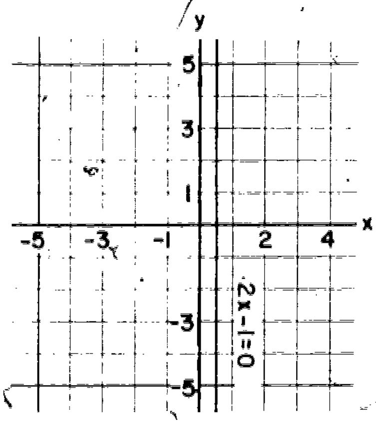
2x+y+2=0
4
2
-4
-2
3
5
x
-3
-5

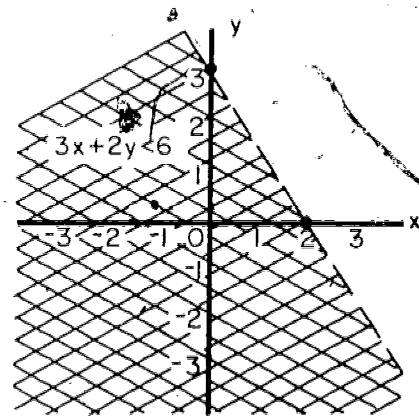
(b)


y
5
3
y=3/2x-2
1
-4
-2
2
4
x
-1
-4

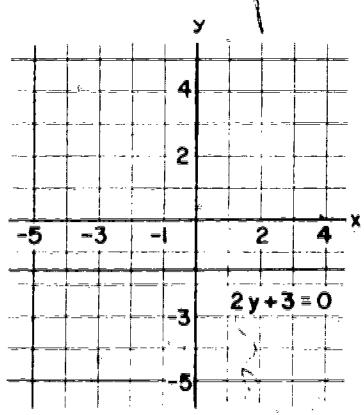
(c)


y
5
3
1
-5
-3
-1
2
4
x
2x-1=0
-3
-5

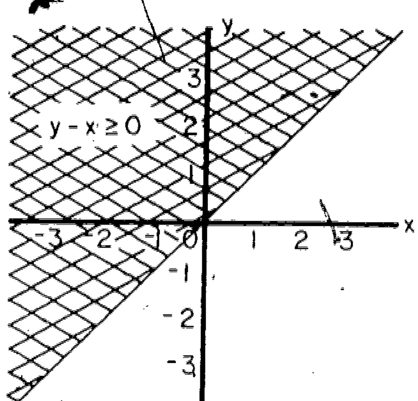
(d)


y
3
2
3x+2y<6
1
-3
-2
-1
0
1
2
3
x
-1
-2
-3

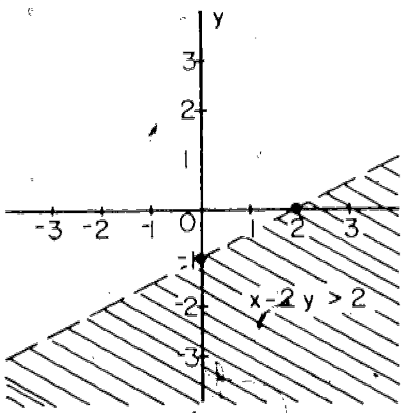
(e)


y
4
2
-5
-3
-1
2
4
x
2y+3=0
-3
-5

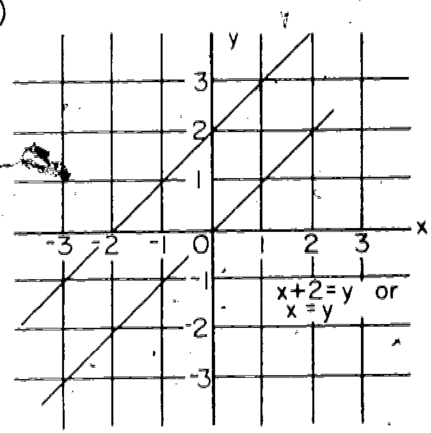
(f)


y
3
y-x≥0
2
1
-3
-2
-1
0
1
2
3
x
-1
-2
-3

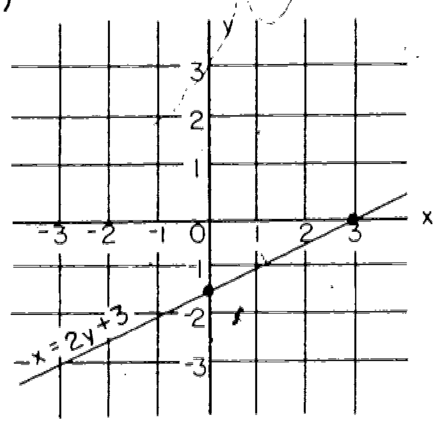
(g)



(h)

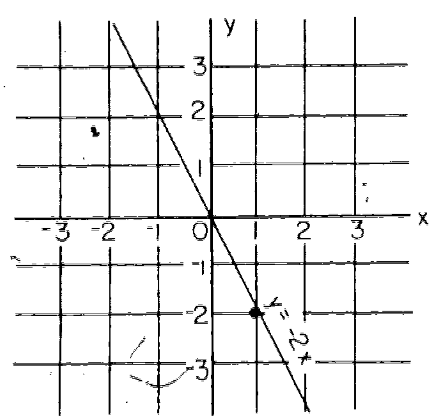


(i)



(j) Φ

8.



$y = -2x$

9. slope = $\frac{\text{vertical change}}{\text{horizontal change}} = -2$

10. $y = -2x - \frac{1}{2}$

11. The slope of the line containing the points $(2,7)$ and $(0,3)$ is 2. The y-intercept is $(0,3)$. Therefore the equation of the line is

$$y = 2x + 3 .$$

The point $(3,10)$ is not on the line since

$$10 \neq 2(3) + 3 .$$

12. Since $(a,2)$ is in the truth set we know that

$$2a + 2 > 4 .$$

Is $2(a + 1) + 2$ greater than 4?

$2(a + 1) + 2 = 2a + 2 + 2$, by the distributive property.

$2a + 2 + 2 > 2a + 2$, by the addition property of order, since $2 > 0$.

Therefore $2a + 2 + 2 > 4$, by the transitive property of order.

$(a + 1,2)$ is in the truth set of $2x + y > 4$.

Chapter 17

SYSTEMS OF OPEN SENTENCES

In this chapter the students extend their ideas about sentences, truth sets and graphs to study compound first degree sentences in two variables. The most common sentence of this type is the conjunction

$$Ax + By + C = 0 \text{ and } Dx + Ey + F = 0,$$

which arises in many contexts where two variables have two conditions placed on them simultaneously. Such a conjunction is called a system of equations and is usually written

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0. \end{cases}$$

Because the truth set of a conjunction is the set of elements for which both sentences are true simultaneously, such a system is often called a "simultaneous system".

The problem of determining the truth set of a compound sentence in two variables is again one of finding an equivalent compound sentence whose truth set is obvious. Here we are aided by intuitive geometry of lines. Two distinct lines either intersect in exactly one point or they are parallel. If the lines given by the equations of the system intersect, then that point of intersection has coordinates satisfying the equations of the system. This ordered pair is the one and only element in the truth set of the system, and this point is the graph of the system. Thus, in the case of one solution the problem of finding equivalent systems is one of finding pairs of equations of lines through this point of intersection. The most simple equations of two such lines are those whose graphs are a horizontal and a vertical line through the point of intersection. All methods of solving systems of first degree equations are actually procedures for finding these two lines.

In the first two sections of this chapter we consider a system of equations with exactly one solution and show how to generate from these equations any other first degree equation also having this solution. Thus, if the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

has exactly one solution, say (r,s), then for any $a \neq 0$, $b \neq 0$, the equation

$$a(Ax + By + C) + b(Dx + Ey + F) = 0$$

is also satisfied by (r,s). Through examples and discussion, the student should learn to select the multipliers a and b so that either the coefficient of x or of y is 0 in the new equation. Thus, with two proper choices of a and b, we obtain an equivalent system of the form

$$\begin{cases} x = h \\ y = k \end{cases},$$

whose truth set is $\{(h,k)\}$.

It should be emphasized that the above analysis depended on the fact that the system has <u>exactly one</u> solution, thus guaranteeing that the corresponding lines intersect in exactly one point. In the cases for which the lines are parallel (including the case of coinciding lines), it is necessary to give an algebraic development of equivalent sentences, one which does not depend on intuitive geometric ideas. The fact is established in this section that any system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

is equivalent to

$$\begin{cases} a(Ax + By + C) + b(Dx + Ey + F) = 0 \\ Dx + Ey + F = 0 \end{cases}$$

and to

$$\begin{cases} Ax + By + C = 0 \\ a(Ax + By + C) + b(Dx + Ey + F) = 0 \end{cases}$$

for any $a \neq 0$, $b \neq 0$. This fact is "proved" for a special system. Since the proof for the general case would be so similar in every detail it is not included. The above "theorem" is used in Sections 3, 4, and 5 to solve systems of equations.

A student may need help in seeing that if there is a number $k \neq 0$ such that

$$D = kA, \quad E = kB, \quad F = kC,$$

then the multipliers a and b may be so chosen that the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

is equivalent to

$$\begin{cases} Ax + By + C = 0 \\ 0 = 0 \end{cases} \text{ and to } \begin{cases} 0 = 0 \\ Dx + Ey + F = 0, \end{cases}$$

which have a common truth set whose graph is a line. In this case, "$Ax + By + C = 0$" and "$Dx + Ey + F = 0$" are equivalent and their graphs coincide.

If there is a number $k \neq 0$ such that

$$D = kA, \quad E = kB, \quad \text{but} \quad F \neq kC,$$

then the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

has no solution. The graphs of "$Ax + By + C = 0$" and "$Dx + Ey + F = 0$" are parallel lines.

Throughout this development it is well to insert an occasional compound sentence with the connective "or". Such examples should remind students of the fundamental difference between conjunctions and disjunctions and emphasize the importance of correct connectives. The graph of "$2x - y + 3 = 0$ and $x + 2y + 1 = 0$" is one point, whereas the graph of "$2x - y + 3 = 0$ or $x + 2y + 1 = 0$" is the set of all the points on the two lines. In this connection, emphasize the difference between the statement "the graphs of the equations of the system", which refers to a pair of graphs, and the statement "the graph of the system", which refers to either one point, one line, or the null set.

17-1. Systems of Equations.

The word system is used as a synonym for conjunction, that is, to indicate a compound sentence with connective and. This is a good time to review the conditions for a true compound sentence.

> "A or B" is false if A, B are both false sentences; otherwise it is true.
> "A and B" is true if A, B are both true sentences; otherwise it is false.

Thus, the truth set of a system of sentences is the set of elements each of which makes every sentence true.

Answers to Problem Set 17-1; pages 769-770:

1. The given ordered pair satisfies the system in each part except (a), (d), and (e).
2. (9,1) is an element of the truth set of the compound sentence in every part except (a).
3. Answers will vary.
4. Answers will vary.
5. Answers will vary.

17-2. Graphs of Systems of Equations.

It should not be difficult to convinc students that although the graphs of the equations of a system allow us to make an estimate of the truth set of the system, we must not accept this estimate as the truth element until it is shown to satisfy both equations. This sort of discussion should then motivate the development of the following sections, in which procedures are established for solving systems by finding equivalent systems.

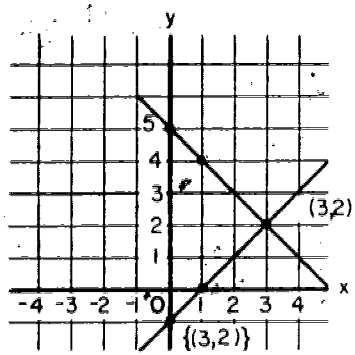
The situation is analogous to that of the beginning of the course when truth numbers of sentences were guessed and then checked; later, the guesswork was eliminated by considering the concept of equivalent sentences.

There is a sharp difference between the "graphs of the sentences in a system" and "the graph of the system". The latter is the graph of a conjunction, and in the case of a system of first degree equations with one solution, it is a single point.

Answers to Problem Set 17-2; pages 773-775:

1. (a) {(5,6)}
   (b) {(-3,4)}
   (c) $\{(-2, -\frac{7}{2})\}$
   (d) {(0,0)}

2. (a)



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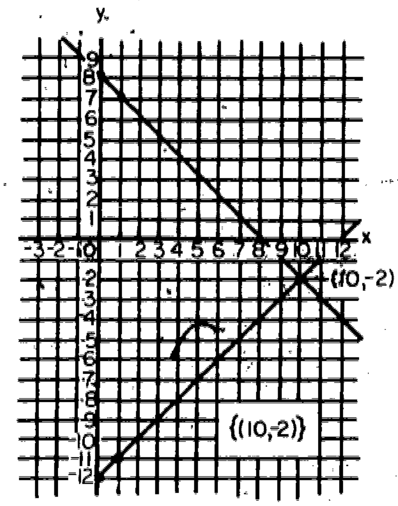

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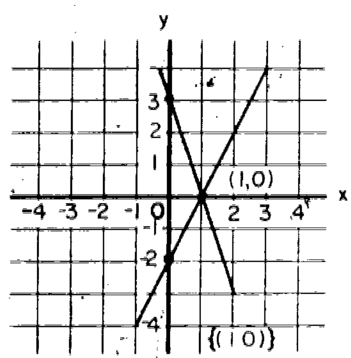
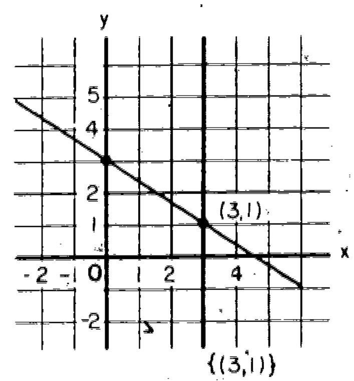


(d)



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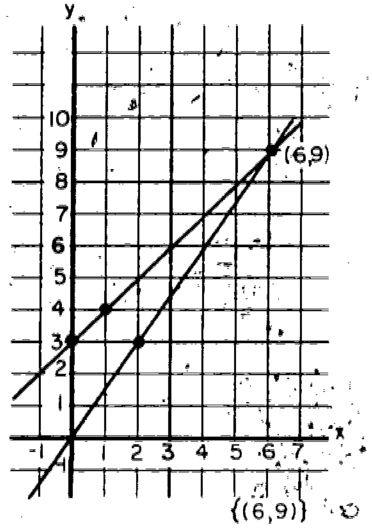

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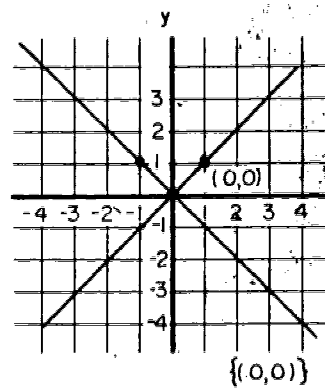

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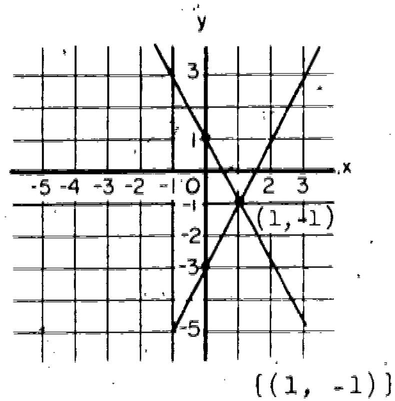
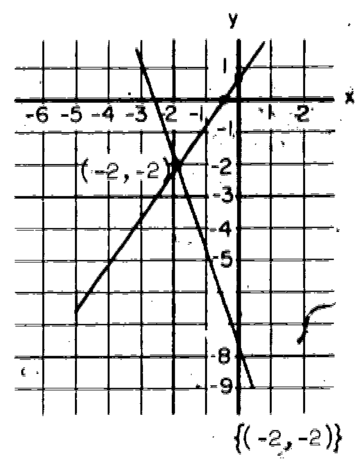
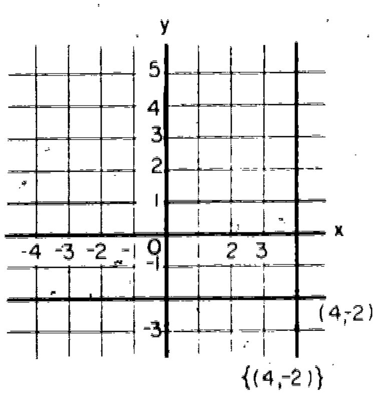


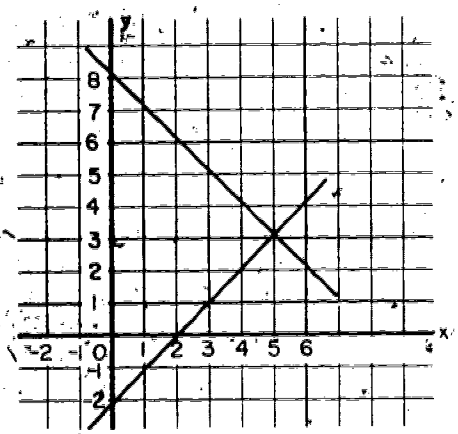
(h)



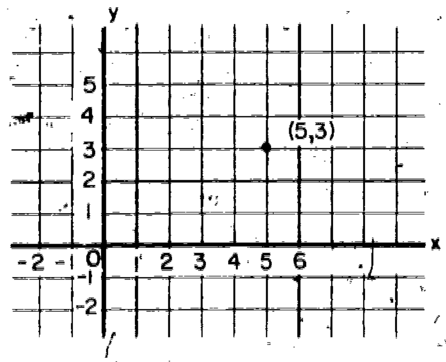
(i)



3. (a)



(b)



4. a point

17-3. Solving Systems of Equations.

In this section we develop the idea of equivalent systems. There is one hurdle here for the students; if this hurdle is passed, the succeeding development should run smoothly. It is this: given any two first degree equations, $Ax + By + C = 0$ and $Dx + Ey + F = 0$, which represent intersecting lines, we may choose any two non-zero real numbers a and b and be certain that the generated equation

$$a(Ax + By + C) + b(Dx + Ey + F) = 0$$

represents a line and, in fact, a line containing the point of intersection of the original lines. Furthermore, every line through this point of intersection (other than the original lines) can be obtained by certain choices of a and b. The verification of this fact follows in part from the observation that a number pair (s,t) satisfies an equation if and only if the point (s,t) is on the graph of the equation.

<u>Answers to Problem Set 17-3a</u>; pages 778-779:

1. (a) yes   (e) yes
   (b) yes   (f) yes
   (c) yes   (g) yes
   (d) yes   (h) yes

2. Answers will vary

3. $$\begin{aligned}(2x - y + 5) + (x + y - 2) &= 0\\ 3x + 3 &= 0\\ 3x &= -3\\ x &= -1\end{aligned}$$

$$\begin{aligned}(2x - y + 5) + (-2)(x + y - 2) &= 0\\ (2x - y + 5) + (-2x - 2y + 4) &= 0\\ -3y + 9 &= 0\\ -3y &= -9\\ y &= 3\end{aligned}$$

<u>Answers to Oral Exercises 17-3b</u>; page 783:

1. $a = -3,\ b = 1$ (a = -6, b = 2; a = -9, b = 3; etc., are
   $a = 1,\ b = -2$ also correct.)

2. $a = 1,\ b = 1$
   $a = 1,\ b = -1$

3. $a = 1,\ b =$ [illegible]
   $a = 1,\ b = -2$

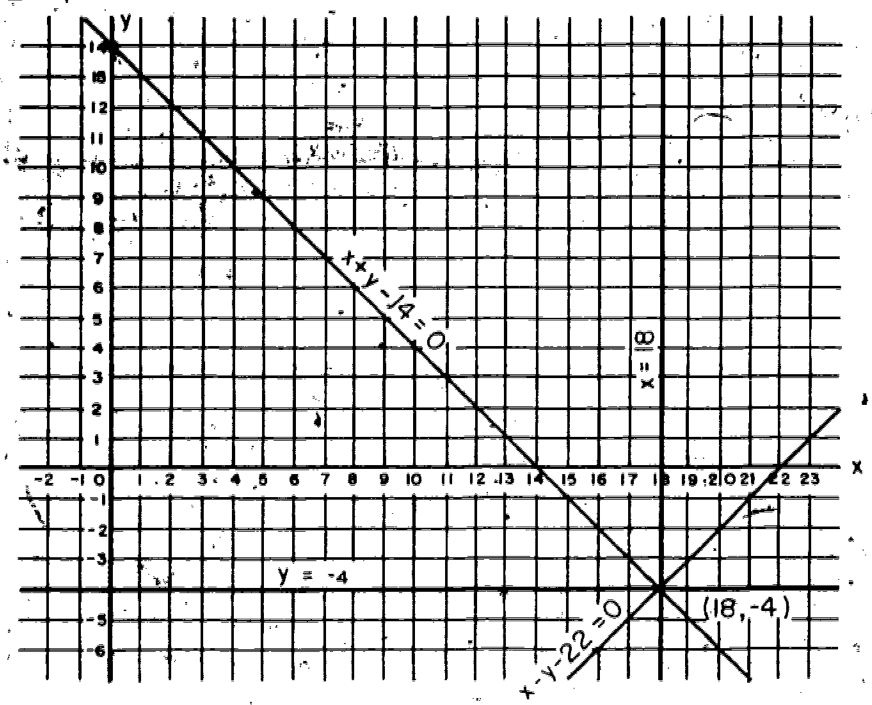
4. $a = 1,\ b = 2$
   $a = -2,\ b = 1$

5. $a = 3,\ b = 1$
   $a = -2,\ b = 1$

6. $a = 2,\ b = 1$
   $a = 1,\ b = -2$

Answers to Problem Set 17-3b; page 784:

1. $\begin{cases} x = 18 \\ y = -4 \end{cases}$ Truth set of the system: $\{(18,-4)\}$



The graph of the system is the point (18,-4)

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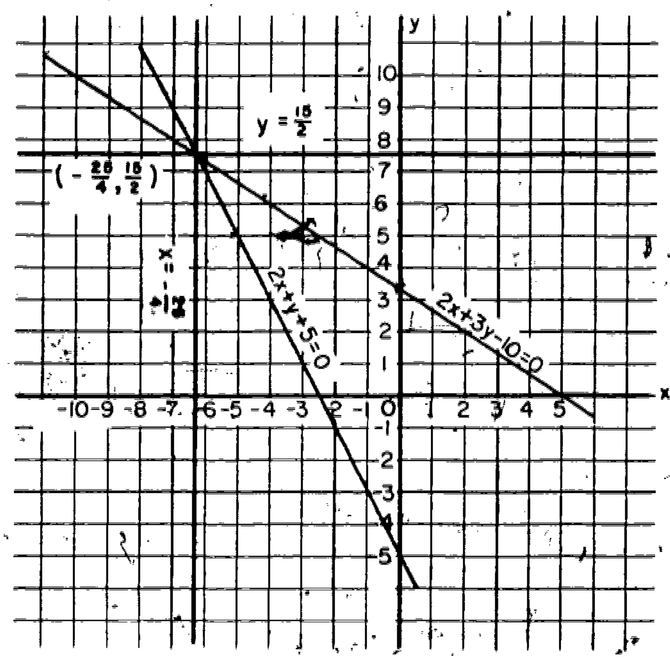
page 784: 17-3

2. $\begin{cases} x = -\frac{25}{4} \\ y = \frac{15}{2} \end{cases}$ $\{(-\frac{25}{4}, \frac{15}{2})\}$



The graph of the system is the point $(-\frac{25}{4}, \frac{15}{2})$.

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3. $\begin{cases} x = -9 \\ y = 5 \end{cases}$ $\{(-9,5)\}$

4. $\begin{cases} x = 8 \\ y = -2 \end{cases}$ $\{(8,-2)\}$

5. $\begin{cases} x = -1 \\ y = -6 \end{cases}$ $\{(-1,-6)\}$

6. $\begin{cases} x = 7 \\ y = 9 \end{cases}$ $\{(7,9)\}$

7. $\begin{cases} x = -5 \\ y = -4 \end{cases}$ $\{(-5,-4)\}$

8. $\begin{cases} x = \frac{1}{2} \\ y = 4 \end{cases}$ $\{(\frac{1}{2}, 4)\}$

9. $\begin{cases} x = -\frac{5}{2} \\ y = 2 \end{cases}$ $\{(-\frac{5}{2},2)\}$

10. $\begin{cases} x = -2 \\ y = 9 \end{cases}$ $\{(-2,9)\}$

Answers to Problem Set 17-3c; page 787:

1. (a) $\begin{cases} 2x - 3y - 4 = 0 \\ 7y - 14 = 0 \end{cases}$

   (b) $\begin{cases} 2x - 3y - 4 = 0 \\ 7x - 35 = 0 \end{cases}$

   (c) $\begin{cases} 2x - 3y - 4 = 0 \\ -14x + 42y - 14 = 0 \end{cases}$

2. (a) $\begin{cases} -7y + 14 = 0 \\ x + 2y - 9 = 0 \end{cases}$

   (b) $\begin{cases} 14x - 70 = 0 \\ x + 2y - 9 = 0 \end{cases}$

   (c) $\begin{cases} 5x - 39y + 53 = 0 \\ x + 2y - 9 = 0 \end{cases}$

3. Answers will vary.

Answers to Problem Set 17-3d; page 790:

1. $\begin{cases} x - y - 1 = 0 \\ x + y - 5 = 0 \end{cases}$

$\begin{cases} (x - y - 1) + (x + y - 5) = 0 \\ x + y - 5 = 0 \end{cases}$

$\begin{cases} 2x - 6 = 0 \\ x + y - 5 = 0 \end{cases}$

$\begin{cases} x = 3 \\ x + y - 5 = 0 \end{cases}$

If there is a solution, it must be of the form (3,b), and b must satisfy the following equivalent sentences:

$$3 + b - 5 = 0$$
$$b - 2 = 0$$
$$b = 2$$

(3,2) is the solution of the system.

The above form is an idealization of the complete set of steps in solving the system. Of course, it is hoped and desired that students will quickly learn to omit some of the steps by choosing the multipliers and forming new sentences in one step, as well as making the "substitution" to obtain b in one step. No matter how short the process, the idea of equivalent systems should remain.

2. {(-4,-3)}
3. {(1,-1)}
4. {(2,-7)}
5. {(-5,2)}
6. {(-2,-3)}
7. {(3,-4)}
8. {(2,5)}
9. {(-2,-1)}
10. {(6,7)}
11. (a) one
    (b) infinitely many

17-4. Systems of Equations with Many Solutions.

The students should be reminded that a compound sentence with connective and (that is, a system of sentences) is true when both sentences are true; otherwise it is false. Thus, in the system

$$\begin{cases} Ax + By + C = 0 \\ 0 = 0, \end{cases}$$

"0 = 0" is true, and, therefore the system is true for all number pairs making "Ax + By + C = 0" true.

This situation will occur when the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

is one for which we can find a non-zero number k such that A = kD, B = kE, and C = kF, and the system is equivalent to

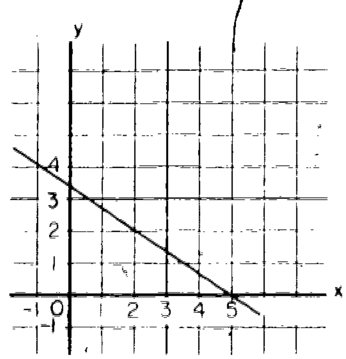
$$\begin{cases} Ax + By + C = 0 \\ (Ax + By + C) - k(Dx + Ey + F) = 0 \end{cases}$$

$$\begin{cases} Ax + By + C = 0 \\ 0 = 0 \end{cases}$$

In other words, in this case the individual sentences in the system are equivalent (one having been obtained from the other by multiplying by a non-zero real number). Thus, the graph of such a system is the one line represented by the two equations.

Answers to Problem Set 17-4; pages 792-794:

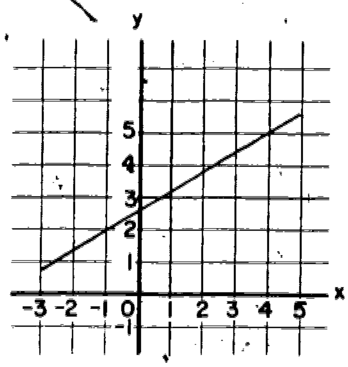
1. (a)



(b) yes
"4x + 6y - 20 = 0" can be obtained by multiplying each side of "2x + 3y - 10 = 0" by 2, a non-zero real number.

(c) The truth sets and graphs of the truth sets of equivalent sentences are the same.

2. (a)

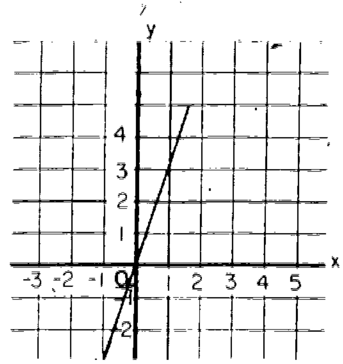


(b) yes
"3x - 6y + 15 = 0" can be obtained by multiplying each side of "x - 2y + 5 = 0" by 3, a non-zero real number.

(c) $\begin{cases} x - 2y + 5 = 0 \\ 0 = 0 \end{cases}$

(d) infinitely many

3. (a)



(b) yes

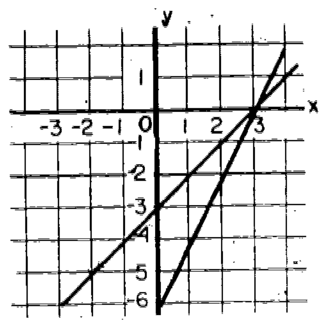
yes

"$12x - 4y = 0$" can be obtained by multiplying both sides of "$3x - y = 0$" by 4, a non-zero real number.

(c) yes

(d) infinitely many

4. (a)



(b) no

no

(c) no

(d) one

5. (a) Please see the first paragraph on page 555.

(b) They are the same graph.

6. All systems, except the one in part (c), have infinitely many solutions.

7. (a) $\begin{cases} 3x - 5y + 4 = 0 \\ 0 = 0 \end{cases}$

(b) $\begin{cases} x - 3y - 1 = 0 \\ 0 = 0 \end{cases}$

(c) It is not possible.

(d) $2x + y - 6 = 0$ and $0 = 0$

17-5. Systems with No Solution.

Again, the definition of a conjunction of sentences tells us that the system

$$\begin{cases} Ax + By + C = 0 \\ -5 = 0 \end{cases}$$

is false for every number pair, because "$-5 = 0$" is false.

Such a situation occurs when the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

is one for which there is a non-zero real number k such that $A = kD$, $B = kE$, and $C \neq kF$, and the system is equivalent to

$$\begin{cases} Ax + By + C = 0 \\ (Ax + By + C) - k(Dx + Ey + F) = 0 \end{cases}$$

$$\begin{cases} Ax + By + C = 0 \\ C = kF \end{cases}$$

Clearly, the second sentence is false and the system has an empty truth set. For cases in which $B \neq 0$ and $E \neq 0$ it may be instructive for the class to put the sentences of such a system in y-form

$$\begin{cases} y = -\frac{A}{B}x - \frac{C}{B} \\ y = -\frac{D}{E}x - \frac{F}{E} \end{cases}$$

and note that $-\frac{A}{B} = -\frac{D}{E}$ and $-\frac{C}{B} \neq -\frac{F}{E}$. This implies that the lines have the same slope (are parallel) and different y-intercepts (do not coincide).

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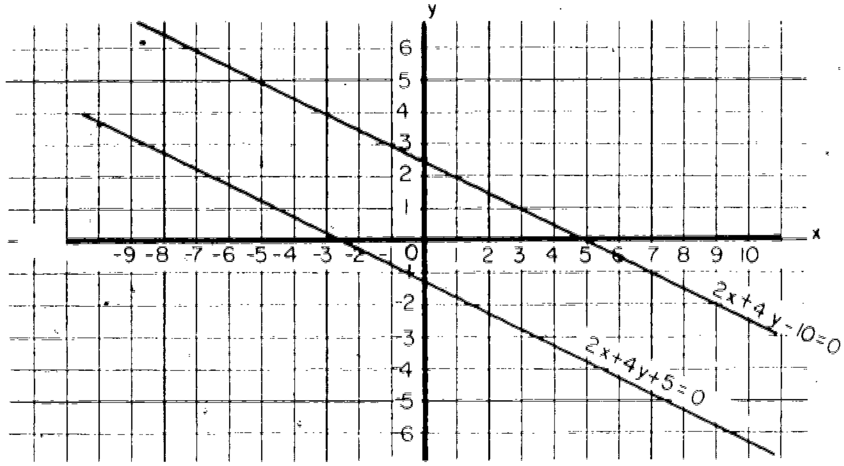
Answers to Problem Set 17-5; pages 795-797:

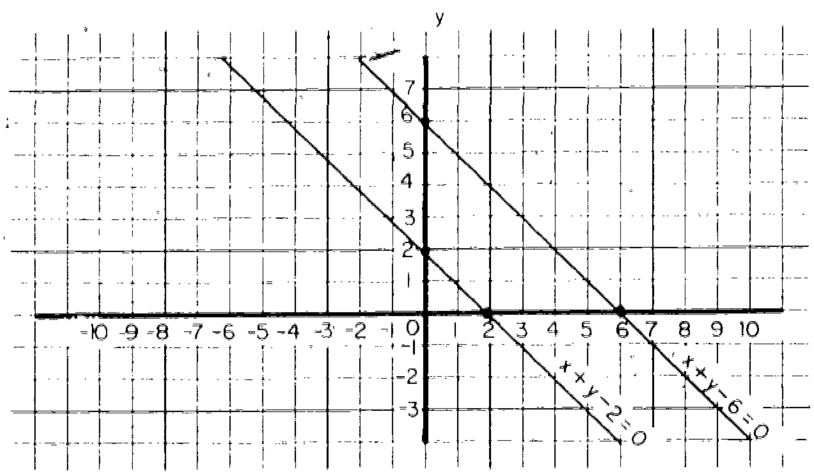
1. (a)



(b) The graphs are parallel lines.

(c) The sentences are not equivalent.
The coefficients of x and the coefficients of y are the same in the two equations, but the constant terms are different.

2. (a)



(b) The sentences are not equivalent.
The coefficients of x and the coefficients of y in the two equations are the same, but the constant terms are different.

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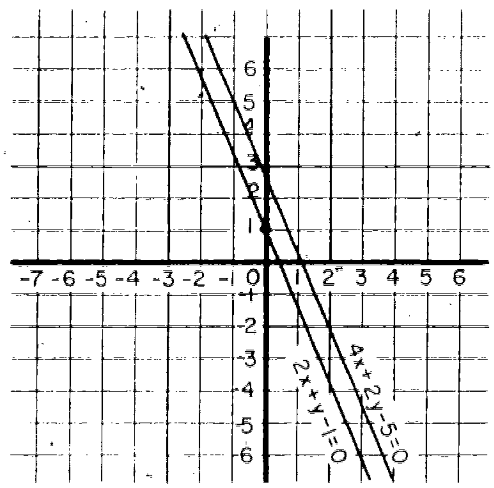
(c) $\begin{cases} x + y - 6 = 0 \\ (-1)(x + y - 6) + (1)(x + y - 2) = 0 \end{cases}$

$\begin{cases} x + y - 6 = 0 \\ -x - y + 6 + x + y - 2 = 0 \end{cases}$

$\begin{cases} x + y - 6 = 0 \\ 4 = 0 \end{cases}$

(d) The truth set is the empty set.

3. (a)



(b) The sentences are not equivalent. The coefficients of of x and y in the second sentence are each twice the corresponding coefficient in the second sentence. The constant terms are not related in this way.

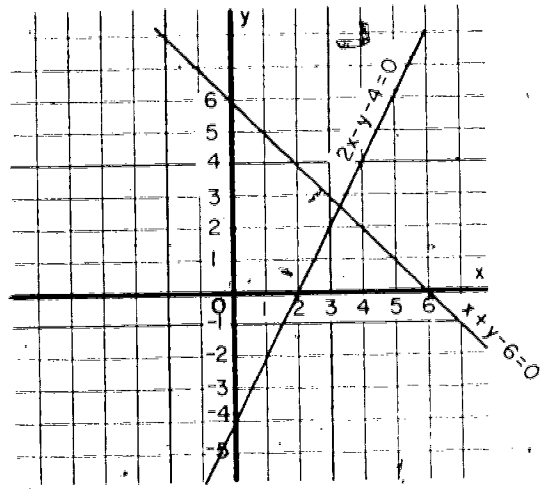
(c) $\begin{cases} 2x + y - 1 = 0 \\ (-2)(2x + y - 1) + (1)(4x + 2y - 5) = 0 \end{cases}$

$\begin{cases} 2x + y - 1 = 0 \\ -4x - 2y + 2 + 4x + 2y - 5 = 0 \end{cases}$

$\begin{cases} 2x + y - 1 = 0 \\ -3 = 0 \end{cases}$

(d) The truth set is the empty set.

4. (a)



(b) The sentences are not equivalent. The coefficients of x and y are not related in any special way.

(c) This cannot be done. No choice of a and b in the sentence "a(2x - y - 4) + b(x + y - 6) = 0" will result in a sentence in which the coefficients of x and y are both zero.

(d) There is one element.

5. There is a number $k \neq 0$ such that if you multiply the coefficients of x and y in one sentence by k, you get the coefficient of x and y in the other sentence. The constant terms are not so related.
The graphs of the individual sentences are parallel lines.

6. (a) infinitely many
(b) none
(c) one
(d) none
(e) none
(f) none
(g) infinitely many
(h) none

7. (a) $\begin{cases} x + 2y - 5 = 0 \\ (-2)(x + 2y - 5) + (1)(2x + 4y + 3) = 0 \end{cases}$

$\begin{cases} x + 2y - 5 = 0 \\ -2x - 4y + 10 + 2x + 4y + 3 = 0 \end{cases}$

$\begin{cases} x + 2y - 5 = 0 \\ 13 = 0 \end{cases}$

(b) No such system can be determined.

(c) $\begin{cases} 2x + y - 3 = 0 \\ (-3)(2x + y - 3) + (1)(6x + 3y - 5) = 0 \end{cases}$

$\begin{cases} 2x + y - 3 = 0 \\ -6x - 3y + 9 + 6x + 3y - 5 = 0 \end{cases}$

$\begin{cases} 2x + y - 3 = 0 \\ 4 = 0 \end{cases}$

17-6. Another Method for Solving Systems.

This section differs from the chapter as a whole in that all other sections are treated from the point of view of equivalent systems. The equivalence approach is desirable for a number of reasons. It is easily extended to linear systems consisting of any number of sentences. It is the basis for the "addition-subtraction" method, long a standard technique in elementary algebra texts. It is strengthened by, and in turn strengthens, the concept of equivalent sentences, developed earlier in this text.

Nevertheless, the "substitution" method, also a familiar feature of most texts, is very efficient in many cases and is useful enough in later work to warrant some attention; and this section is devoted to an explanation of it. If, for any reason, it should seem desirable to omit this section, no loss of continuity would result.

The text is careful to work with numbers in developing the substitution method. Thus, in Example 2, for instance, the problem-solving process begins by stating that if a and b are numbers such that (a,b) is a solution of the system

$$\begin{cases} 2x = y - 7 \\ x + 2y - 4 = 0, \end{cases}$$

then "$2a = b - 7$" and "$a + 2b - 4 = 0$" must be true sentences From the first of these, the name "$2a + 7$" is derived for the number b, and this name is in turn substituted for "b" in the second sentence.

This differs from presentations that simply substitute "2x + 7" for "y", without the introduction of the "(a,b)" notation. There is of course nothing wrong with this, but it needs to be accompanied by an awareness that, in making such a substitution, one is working with a <u>particular</u> (x,y). It is not true that, for <u>any</u> (x,y) satisfying the second sentence, it is true that y = 2x + 7. To the beginning student, however, this distinction in the use of x and y is likely to be unclear. The "(a,b)" notation emphasizes that we are working with a particular pair of numbers.

In the examples of the text, the process in each case begins with the statement "<u>If</u> (a,b) is a solution..." . Thus, as a final step, it is necessary to check in each sentence of the system, since there may be no solution at all. Otherwise, we have not shown that there is a solution; we have simply shown that if there is a solution, we know what it is. However, if in the study of sections 17-4 and 17-5, the class has established a way of telling beforehand that a system has exactly one solution, then the checking is no longer a logical necessity. The statements "There is exactly one solution" and "If there is a solution, it is (-1,1)", for instance, certainly imply that (-1,1) is the solution. Even so, checking may be desirable as a guard against arithmetic errors.

<u>Answers to Problem Set 17-6</u>; pages 801-802:

1. {(1,1)}
2. {(1,3)}
3. {(1,-1)}
4. {(6,-4)}
5. {(5,6)}
6. {(-2,-5)}
7. {(10,6)}
8. {(10,15)}
9. {(2,3)}
10. {(0,1)}

<u>Answers to Problem Set 17-7</u>; pages 804-805:

1. The larger number is -8 and the smaller number is -15.
2. The larger number is 23 and the smaller number is 7.
3. The boy has 6 dimes and 4 quarters.

4. The number of five dollar bills is 40.

5. The number is 30.

6. The man bought 10 pounds of 35-cent nuts and 20 pounds of 50-cent nuts.

7. Each man is paid $10 per day and each boy is paid $6 per day.

8. 50 pounds of seed worth $1.05 per pound and 150 pounds of seed worth 85 cents per pound

9. He invested $5,200 at 5% and $4,800 at 6%.

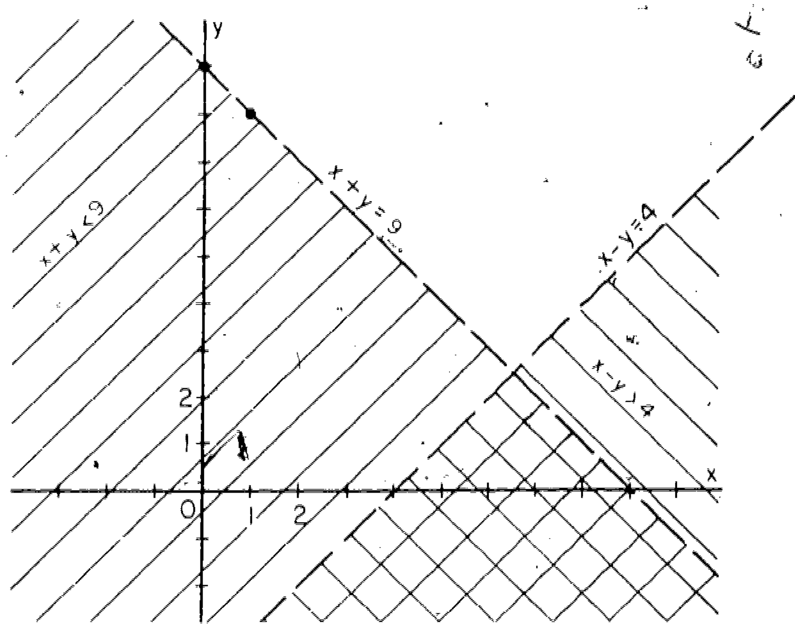
10. At the point whose coordinates are (3,1).

17-8. Systems of Inequalities.

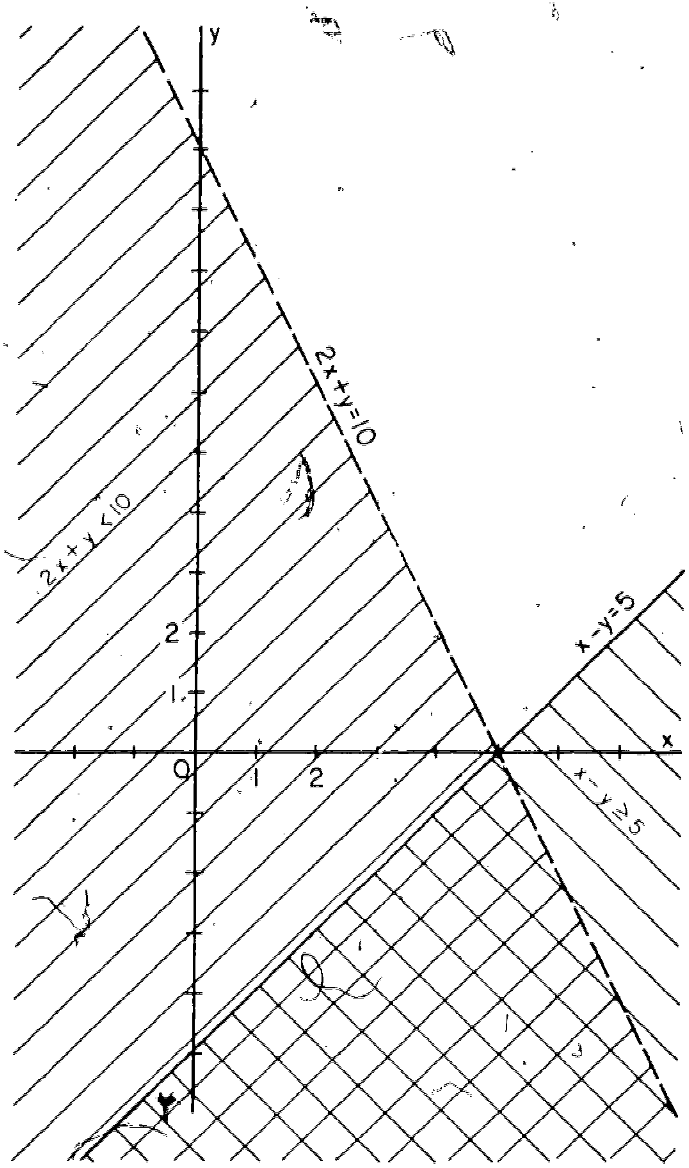
As before, the definition of a conjunction will be the key idea here. A number pair satisfies a system of sentences if and only if it satisfies both sentences. Thus, a point is in the graph of a system of inequalities if and only if its coordinates satisfy both inequalities.

Answers to Problem Set 17-8; page 810:

1.



2.



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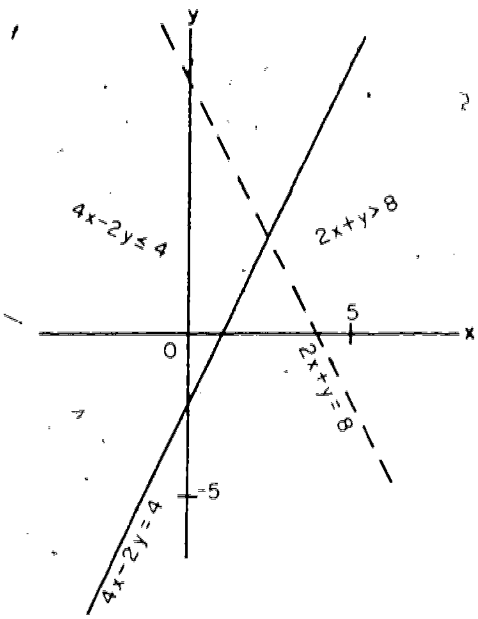
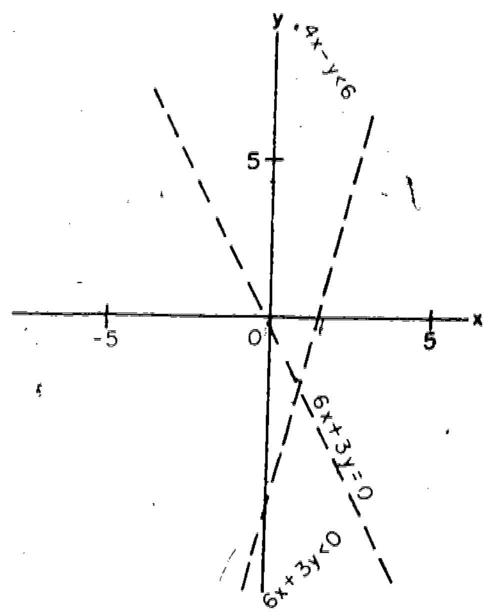
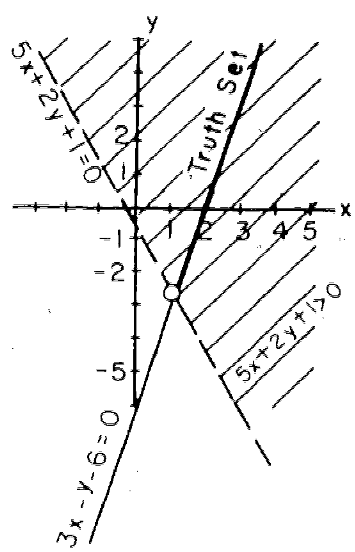
3.



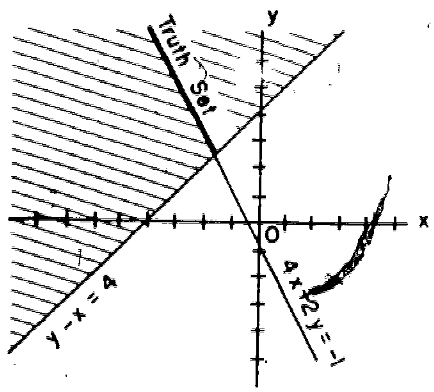
4.



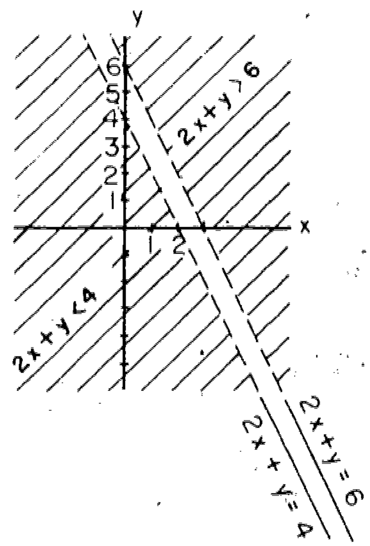
5.



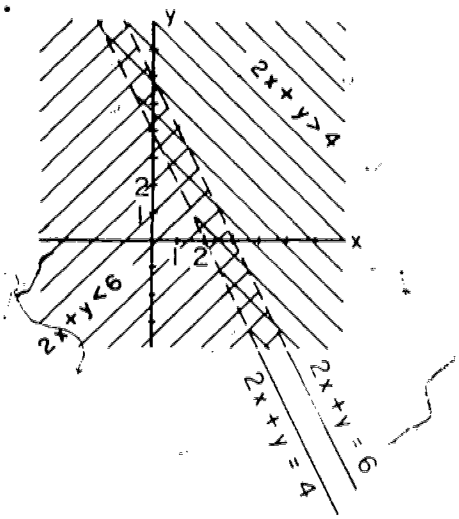
6.



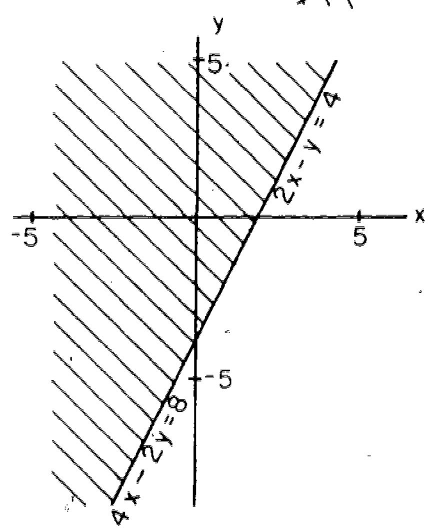
7.



8.

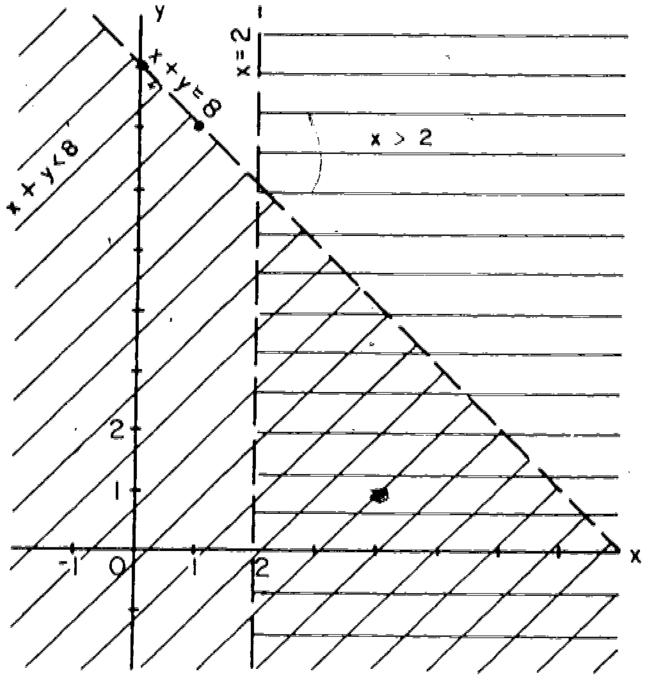


9.

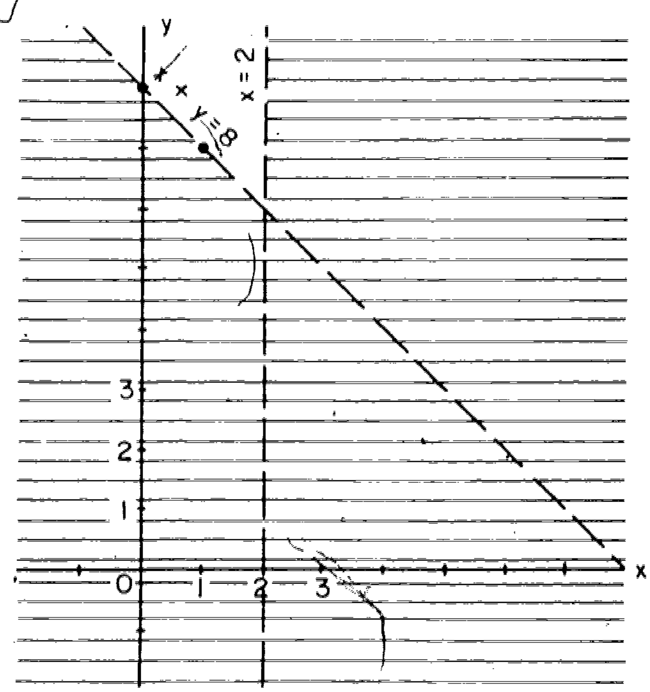


The graph of the solution set consists of the shaded region and the line.

10.



11.



Answers to Review Problem Set; pages 811-814:

1. (a) (d) (e)

   $(x - 2y + 3)(2x + y - 8) = 0$ means

   $x - 2y + 3 = 0$ or $2x + y - 8 = 0$.

   $3 < x < 7$ means $3 < x$ and $x < 7$.

2. (a) $3(2) - (3) = 6 - 3 = 3$

   $2 - 3(3) + 7 = 2 - 9 + 7 = 0$

   Since the two clauses of the compound sentence are not equivalent, there is at most one number pair that is the solution of the system. Hence the truth set is $\{(2,3)\}$.

   (b) $(2,3)$

   (c) $a = 1$, $b = -3$. There are other possibilities such as $a = 2$, $b = -6$.

   (d) $a = -3$, $b = 1$. There are other possibilities.

   (e) $-3a + 7b = 0$ is true for $a = 7$ and $b = 3$. There are other possibilities.

3. (a) $\{4\}$

   (b) all numbers greater than -36.

   (c) $\{\frac{109}{132}\}$

   (d) $\{-2, \frac{3}{4}, 7\}$

   (e) $\{-2, -3\}$

   (f) the set of all number pairs whose ordinate is at least 2 more than the opposite of the abscissa.

   (g) $\{(0, 0)\}$

   (h) $\{(1, -\frac{1}{2})\}$

   (i) $\emptyset$

   (j) the set of all number pairs where the ordinate is 2 more than $-\frac{5}{2}$ times the abscissa.

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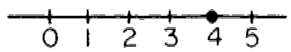
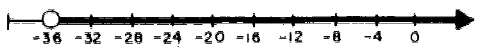
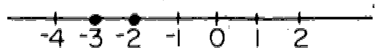
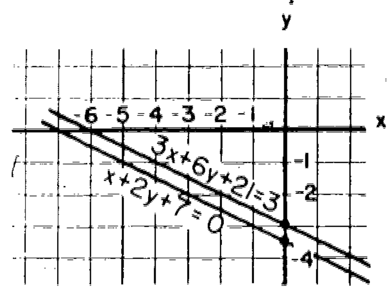
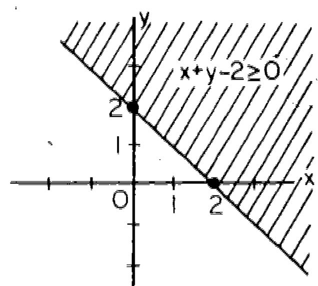
4. (a)


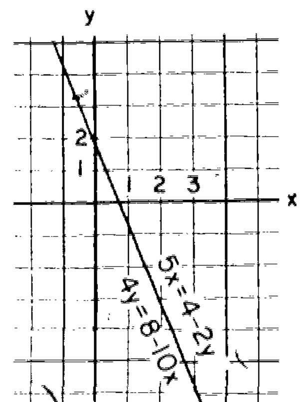
(b)


(e)


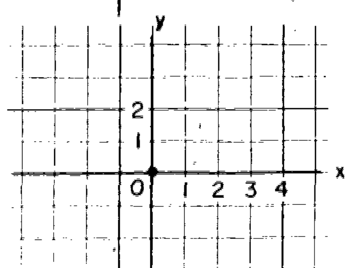
(i)


There is no common point.

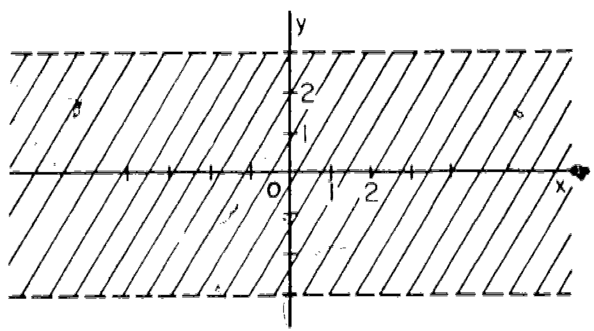
(f)


(j)


The two lines coincide.

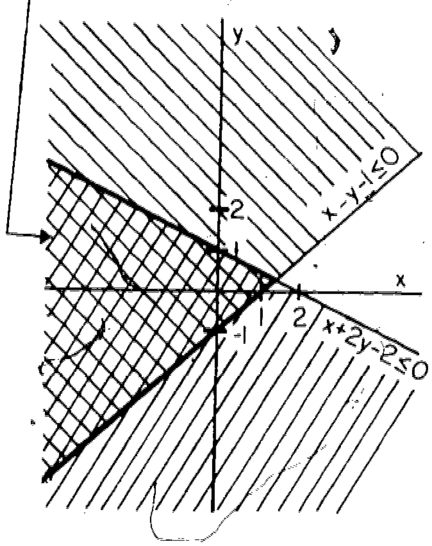
(g)


5. (a)

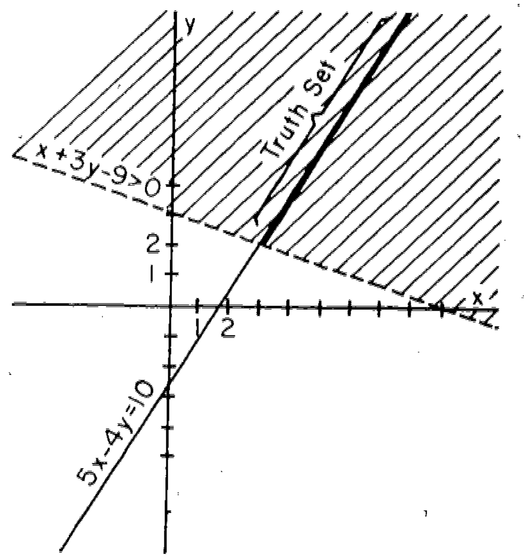

(b)


6. To every number pair there corresponds exactly <u>one point</u> in the plane and to every point in the plane there corresponds exactly one number pair for a given set of axes in the plane.

7. (a) Truth set area between lines including solid lines.



(b)



8. (a) $-2x^2$
   (b) third degree
   (c) 1
   (d) 7
   (e) 4
   (f) 1
   (g) 0
   (h) yes
   (i) yes

9. (a) -3
   (b) -1
   (c) third quadrant
   (d) 3
   (e) second quadrant

10. (a) $2^2 \cdot 7^2 \cdot x^4$ or $196x^4$
(b) $-7a + 18b$
(c) $\frac{a^2}{2}$
(d) $2ab$
(e) $\frac{7x}{30}$
(f) 81
(g) $27a$
(h) $| x + 5 |$
(i) $\frac{6a^3 + 9a + 7}{3a^2}$
(j) $\frac{4c - 5a}{6abc}$

11. Let x represent the number of days of slow growth and y represent the number of days of fast growth. Then

$$\begin{cases} x + y = 365 \\ .10x + .25y = 50 \end{cases}$$

$$(x + y - 365) + (-10)(.10x + .25y - 50) = 0$$

$$y - 2.5y = 365 - 500$$

$$-1.5y = -135$$

$$y = \frac{2}{3}(135) = 90$$

There were 90 days of fast growth and 275 days of slow growth.

12. There were 160 snails from population B and 40 snails from population A if the migrants were a representative sample from each population.

13. If f is the number of grams in the fresh weight and d is the number of grams of dry weight, then

$$\begin{cases} f - d = 100 \\ d = \frac{2}{27} f \end{cases}$$

$$(f - d - 100) + (d - \frac{2}{27}f) = 0$$

$$\frac{25}{27}f = 100$$

$$f = 108$$

The fresh weight was 108 grams.

14. If f is the average weight of the females in micrograms and m is the average weight of the males in micrograms then

$$\begin{cases} 8m + 2f = 10(720) \\ 2f = 940 + 980 \end{cases}$$

$$(8m + 2f - 7200) - (2f - 1920) = 0$$
$$8m = 7200 - 1920 = 5280$$
$$m = 660$$

The average weight of the 8 males is 660 micrograms each.

Suggested Test Items

1. Is each of the following compound sentences a system of sentences?
   (a) $x + y = -2$ and $2x - y = 8$
   (b) $2x + 6y + 3 = 0$ or $x - y = 8$
   (c) $x = 3$ and $y = -5$
   (d) $3x + y \neq 4$ and $x < 4$
   (e) $4x + y = 8$ and $2y = -8x + 16$
   (f) $3x + 2y + 1 = 0$ and $6x + 4y - 9 = 0$

2. For which of the six sentences in problem 1 is $(3,-5)$ an element of the truth set?

3. (a) Draw the graphs of the individual sentences of the system

$$\begin{cases} 3x + y + 3 = 0 \\ x - 2y + 1 = 0 \end{cases}$$

   (b) Draw the graph of the truth set of the system in 3(a).
   (c) Explain why the graphs in (a) and (b) differ.

4. Draw the graphs of the individual sentences of each of the six compound sentences in problem 1.

5. Draw the graph of the truth set of each of the compound sentences in problem 1.

6. Is the truth set of the system $\begin{cases} x - 2y = 3 \\ 3x - 2y - 7 = 0 \end{cases}$ contained in the truth set of any of the following sentences? If so, in which?

(a) $4(x - 2y + 3) + 7(3x - 2y - 7) = 0$

(b) $-3(x - 2y - 3) + 1(3x - 2y - 7) = 0$

(c) $a(x - 2y - 3) + b(3x - 2y - 7) = 0$, where a and b are non-zero real numbers.

7. Find the truth set of each of the following by determining an equivalent system in which one of the sentences contains only the variable x, and the other sentence only the variable y; or, if this is not possible, find the truth set by determining an equivalent system in which one of the sentences is $0 = 0$ or a false sentence, such as $4 = 0$.

(a) $\begin{cases} x + y - 7 = 0 \\ x - y - 5 = 0 \end{cases}$

(b) $\begin{cases} x - y = 5 \\ 2x + 3y = 10 \end{cases}$

(c) $\begin{cases} 2x - 3 = 7y \\ 10x - 35y = 15 \end{cases}$

(d) $\begin{cases} y = 2x - 3 \\ 10x - y = 11 \end{cases}$

(e) $\begin{cases} \frac{1}{2}x - \frac{2}{3}y - 2 = 0 \\ -\frac{1}{4}x + \frac{1}{3}y + 3 = 0 \end{cases}$

8. Can two integers be found whose difference is 13 and the sum of whose successors is 28? Explain.

9. How many pounds each of 95-cent and 90-cent coffee must be mixed to obtain a mixture of 90 pounds to be sold for 92 cents per pound?

10. A man made two investments, the first at 4% and the second at 6%. He received a yearly income from them of \$400. If the total investment was \$8,000, how much did he invest at each rate?

11. At a service station Jack bought ten gallons of gas and a quart of oil, spending a total of \$3.75. Jim bought eight gallons of gas and two quarts of oil, and he spent \$3.54. What was the cost per gallon of gasoline and the cost per quart of oil?

12. The digits of an integer between 0 and 100 have the sum 12, and the tens digit is 3 more than twice the units digit. What is the integer?

13. Draw the graph of each of the following systems.

(a) $x + y = 5$ and $x < 2$

(b) $\begin{cases} y < x + 1 \\ x + y < 5 \end{cases}$

(c) $\begin{cases} x - y < 3 \\ y = x - 4 \end{cases}$

14. Consider the sentence "$2x - y > 3$". The number pair (2,3) does not satisfy the sentence, and (3, -1) does satisfy the sentence. Which of the following is true, and why?

(a) (2,3) and (3,-1) are on the same side of the line "$2x - y = 3$".

(b) The point (3,-1) is on the line "$2x - y = 3$".

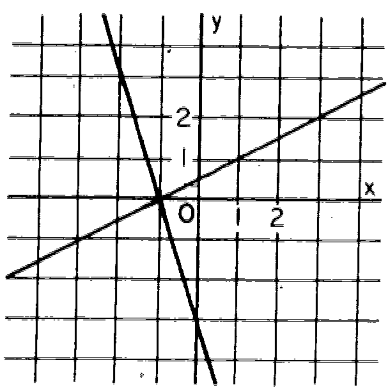
(c) (2,3) and (3,-1) are on opposite sides of the line "$2x - y = 3$".

Answers to Suggested Test Items

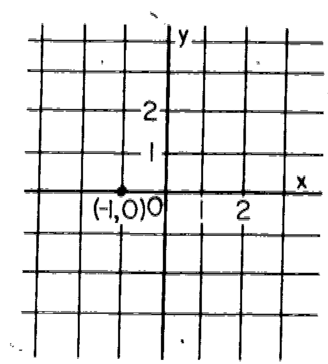
1. (b) is not a system of sentences, since it is a compound sentence formed with "or". Each of the others is a system of sentences.

2. (3,-5) is in the truth set of each of (b), (c), and (d).

3. (a)

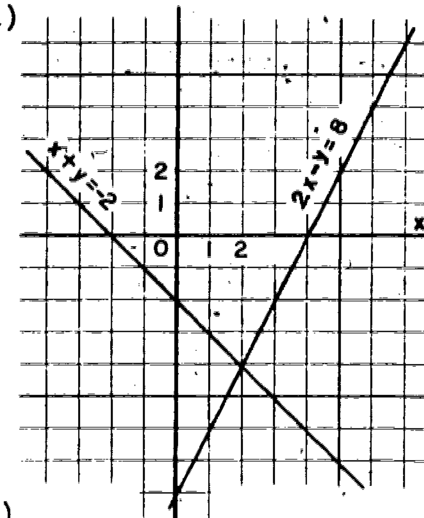


(b)

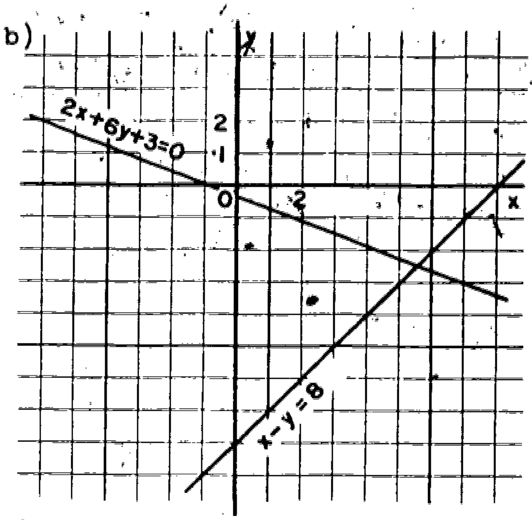


(c) These graphs differ because in (a) the required graph is made up of points whose coordinates satisfy one or both of the sentences comprising the compound sentence, whereas in (b) only number pairs satisfying both sentences are in the truth set, so the graph is the point (-1,0).

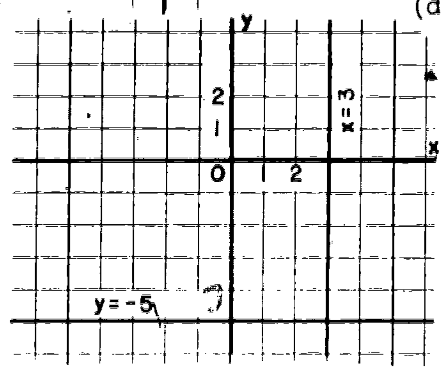
4. (a)


x+y=-2
2x-y=8
x
0 1 2
2
1

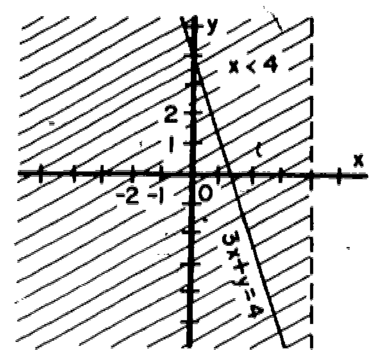
(b)


y
2x+6y+3=0
x-y=8
x
0 2
2
1

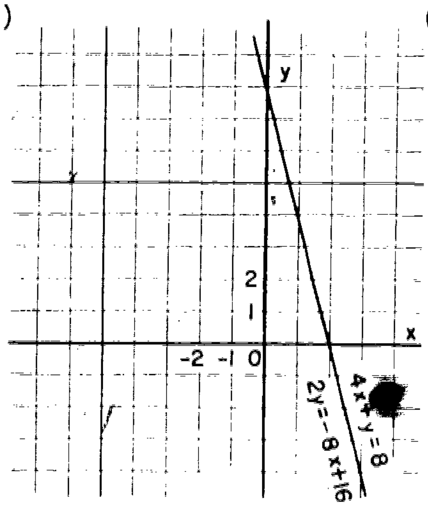
(c)


y
x=3
y=-5
x
0 1 2
2
1

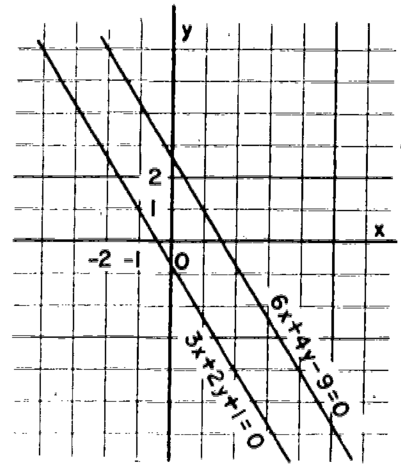
(d)


y
x < 4
3x+y=4
x
-2 -1 0
2
1

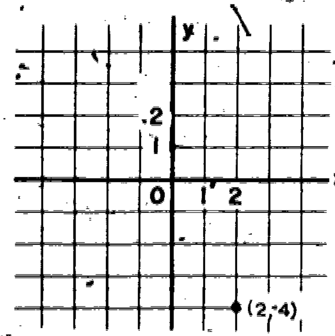
(e)


y
x
-2 -1 0
2
1
4x+y=8
2y=-8x+16

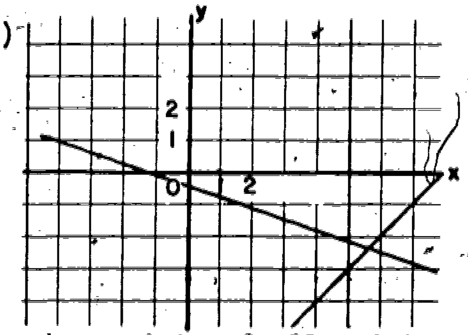
(f)


y
x
-2 -1 0
2
1
3x+2y+1=0
6x+4y-9=0

5. (a)

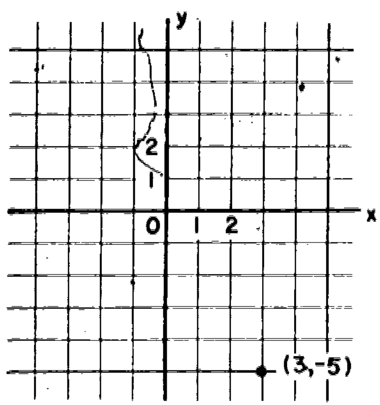


(b)

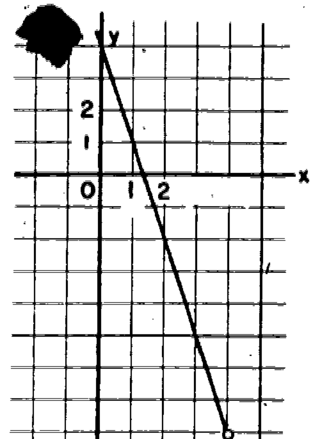


The graph consists of all points whose coordinates satisfy at least one of the sentences of the compound sentence.

(c)

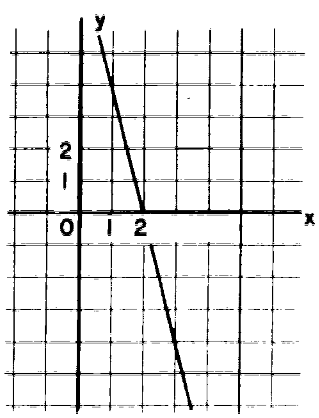


(d)

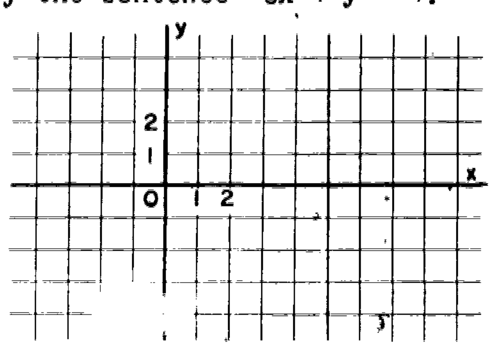


The graph consists of all points with abscissa less than 4 which satisfy the sentence $3x + y = 4$.

(e)



(f)



The truth set is the empty set.

6. Yes. The truth set of the system is contained in sentences (b) and (c).

7. (a) $\begin{cases} x + y - 7 = 0 \\ x - y - 5 = 0 \end{cases}$

$$1(x + y - 7) + 1(x - y - 5) = 0$$
$$x + y - 7 + x - y - 5 = 0$$
$$2x - 12 = 0$$
$$x = 6$$
$$1(x + y - 7) + (-1)(x - y - 5) = 0$$
$$x + y - 7 - x + y + 5 = 0$$
$$2y - 2 = 0$$
$$y = 1$$
$$\{(6,1)\}$$

(b) $\begin{cases} x - y = 5 \\ 2x + 3y = 10 \end{cases}$

$$3(x - y - 5) + 1(2x + 3y - 10) = 0$$
$$3x - 3y - 15 + 2x + 3y - 10 = 0$$
$$5x - 25 = 0$$
$$x = 5$$
$$-2(x - y - 5) + 1(2x + 3y - 10) = 0$$
$$-2x + 2y + 10 + 2x + 3y - 10 = 0$$
$$5y = 0$$
$$y = 0$$
$$\{(5,0)\}$$

(c) $\begin{cases} 2x - 3 = 7y \\ 10x - 35y = 15 \end{cases}$

$$-5(2x - 3 - 7y) + 1(10x - 35y - 15) = 0$$
$$-10x + 15 + 35y + 10x - 35y - 15 = 0$$
$$0 = 0$$

Therefore the two sentences of the system are equivalent, and any number pair satisfying one of the sentences satisfies the other. The solution set is infinite.

(d) $\begin{cases} y = 2x - 3 \\ 10x - y = 11 \end{cases}$

$$1(y - 2x + 3) + 1(10x - y - 11) = 0$$
$$y - 2x + 3 + 10x - y - 11 = 0$$
$$8x - 8 = 0$$
$$x = 1$$
$$5(y - 2x + 3) + 1(10x - y - 11) = 0$$
$$5y - 10x + 15 + 10x - y - 11 = 0$$
$$4y + 4 = 0$$
$$y = -1$$
$$\{(1, -1)\}$$

(e) $\begin{cases} \frac{1}{2}x - \frac{2}{3}y - 2 = 0 \\ -\frac{1}{4}x + \frac{1}{3}y + 3 = 0 \end{cases}$

$$1(\tfrac{1}{2}x - \tfrac{2}{3}y - 2) + 2(-\tfrac{1}{4}x + \tfrac{1}{3}y + 3) = 0$$
$$\tfrac{1}{2}x - \tfrac{2}{3}y - 2 + \left(-\tfrac{1}{2}x\right) + \tfrac{2}{3}y + 6 = 0$$
$$4 = 0$$

Therefore the truth set of the system is $\emptyset$.

8. If the greater integer is x and the lesser integer y, then

$$\begin{cases} x - y = 13 \\ (x + 1) + (y + 1) = 28 \end{cases}$$
$$\begin{cases} x - y = 13 \\ x + y = 26 \end{cases}$$
$$1(x - y - 13) + 1(x + y - 26) = 0$$
$$x - y - 13 + x + y - 26 = 0$$
$$2x - 39 = 0$$
$$x = 19\tfrac{1}{2}$$

But the domain of x is the set of integers.

Thus, there is no integer x which meets the conditions of the problem.

9. If it is to contain x pounds of 95-cent coffee and y pounds of 90-cent coffee, then

$$\begin{cases} x + y = 90 \\ 95x + 90y = (92)(90) \end{cases}$$
$$\{(36, 54)\}$$

Therefore 36 pounds of 95-cent coffee and 54 pounds of 90-cent coffee should be used.

10. If m dollars was invested at 4% and n dollars at 6%,

$$\begin{cases} m + n = 8,000 \\ .04m + .06n = 400 \end{cases}$$
$\{(4000, 4000)\}$.

Thus, \$4,000 was invested at each rate.

11. If gasoline costs g cents per gallon and oil costs q cents per quart,

$$\begin{cases} 10g + q = 375 \\ 8g + 2q = 354 \end{cases}$$
$\{(33, 45)\}$

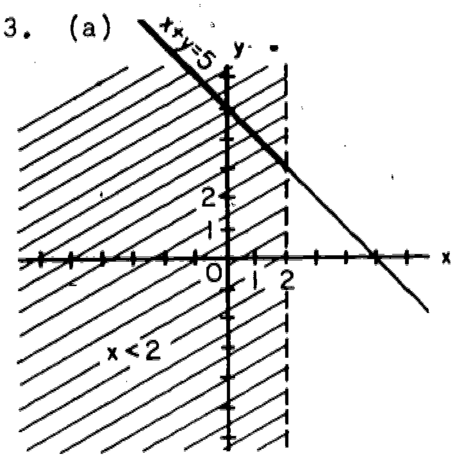
Gasoline costs 33 cents a gallon, and oil costs 45 cents a quart.

12. If u is the units digit and t is the tens digit of the integer, then

$$\begin{cases} t + u = 12 \\ t = 2u + 3. \end{cases}$$
$\{(9,3)\}$

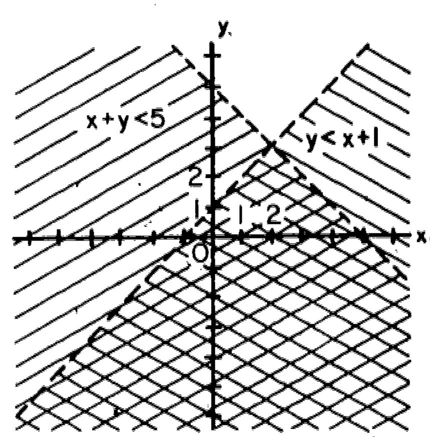
The integer is 93.

13. (a)



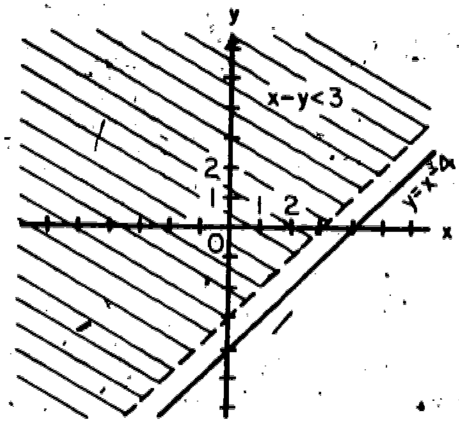
The truth set is the set of number pairs which are in the truth set of $x + y = 5$ and which have abscissa less than 2.

(b)



The truth set is comprised of the number pairs corresponding to points in the doubly shaded region.

(d)



The truth set is $\emptyset$.

14. The graph of a first degree inequality in two variables is the set of all points on one side of the line of the corre sponding equality. Thus, (c) is the only correct response.

Chapter 18

QUADRATIC POLYNOMIALS

Quadratic polynomials were first studied in Chapter 13. Now that coordinate systems have been introduced (Chapter 16), it is possible to get a deeper insight into the quadratic polynomials in x,

$$Ax^2 + Bx + C, \quad A \neq 0,$$

by considering the graphs of the corresponding sentences,

$$y = Ax^2 + Bx + C, \quad A \neq 0.$$

In this chapter we use the graphs of quadratic polynomials to motivate the development of the <u>standard form</u> of a quadratic polynomial in x. To do this we first study graphs of "ax^2" (where a is any non-zero real number). Then we note how the graph of "$a(x - h)^2$" is obtained from the graph of "ax^2" by horizontal translation or "movement" in the plane. Finally, the graph of "$a(x - h)^2 + k$" is obtained from the graph of "$a(x - h)^2$" by a vertical movement. Thus, since any quadratic polynomial "$Ax^2 + Bx + C$", $A \neq 0$, can be written in the standard form "$a(x - h)^2 + k$", $a \neq 0$, and since the graph of "$a(x - h)^2 + k$" can be obtained by movements of the graph of "ax^2", we conclude that the shape of the graph is determined by the number a and its position (axis, vertex) by the numbers h, k.

The graph of "$Ax^2 + Bx + C$" intersects the x-axis at points whose abscissas satisfy the <u>quadratic equation</u>

$$Ax^2 + Bx + C = 0.$$

Here the interesting by-play between quadratic polynomial and corresponding quadratic equation should be pointed out. Because of the shape of the graph (parabola) of the quadratic polynomial, it is suspected that there are none, one, or at most two points at which its graph intersects the x-axis.

The corresponding quadratic equation, whose solutions are the abscissas of the points on the x-axis, can be solved in complete generality by using the standard form of the quadratic polynomial to show that the two sentences

$$Ax^2 + Bx + C = 0, \quad A \neq 0$$

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad \text{or} \quad x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \quad A \neq 0$$

are equivalent. Thus, there are no solutions if $B^2 - 4AC$ is negative, one solution if $B^2 - 4AC = 0$, and two solutions if $B^2 - 4AC$ is positive. In no case are there more than two solutions. As a consequence we can prove that our presumption about the graph of the quadratic polynomial was correct: it intersects the x-axis in, at most, two points.

The second of the two equivalent sentences above is often called the quadratic "formula". In our terminology it is a sentence equivalent to the quadratic equation from which the truth numbers are easily read. The quadratic formula is not developed in the text because as a mechanical device for grinding out solutions it leaves much to be desired. It is much more important that the student should learn to write "$Ax^2 + Bx + C$" in the standard form "$a(x - h)^2 + k$" and solve the resulting equation by factoring. In this way he is constantly aware of the shape and position of the graph and at the same time aware of the possibility of factoring over the reals (if and only if $\frac{k}{a}$ is negative or zero).

18-1. Graphs of Quadratic Polynomials.

A quadratic polynomial in x is defined as a polynomial which can be simplified to the form $Ax^2 + Bx + C$, where A, B, and C are any real numbers, with the important restriction that A be different from zero. Oral Exercises 18-1a are designed to give the student a chance to test his ability to recognize a quadratic polynomial. The following points are brought out in these exercises:

> Although A cannot be zero, either B or C or both may be zero.

pages 815-818: 18-1

The coefficients A, B, and C may be any real numbers ($A \neq 0$); they are not restricted to integers.

An expression such as $(x + 1)^2 + 5$ (Problem 13) is a quadratic polynomial since it can be expressed in the form $Ax^2 + Bx + C$. (This will come up again later when standard form is studied.)

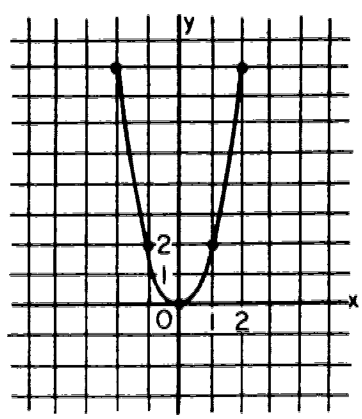
Answers to Oral Exercises 18-1a; page 816:

1. yes; 2, 4, 5
2. yes; 1, 6, 8
3. yes; $\frac{1}{2}$, -6, -8
4. yes; -3, 5, $-\frac{1}{4}$
5. yes; -1, 6, $\sqrt{2}$
6. yes; 48, 8, 0
7. yes; -1, 0, 14
8. no
9. yes; -1, 1, 8
10. no
11. no
12. yes; 1, 0, 0
13. yes; 1, 2, 6
14. yes; 1, t, k

Answers to Problem Set 18-1b; page 818:

1. Graph of polynomial $2x^2$

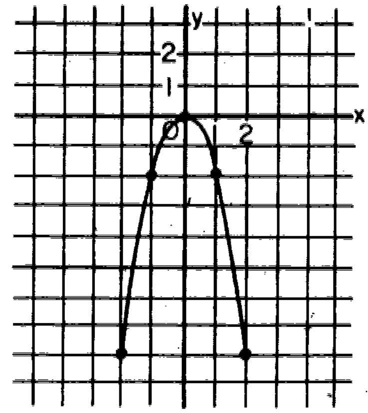
(x, y):
(-2,8)
(-1, 2)
(0, 0)
(1, 2)
(2, 8)



page 818: 18-1

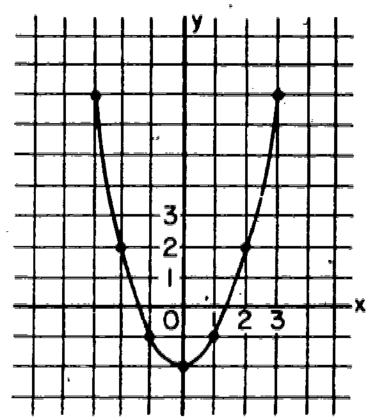
2. Graph of polynomial $-2x^2$

(x, y):
(-2, -8)
(-1, -2)
(0,)
(1, -2)
(2, -8)



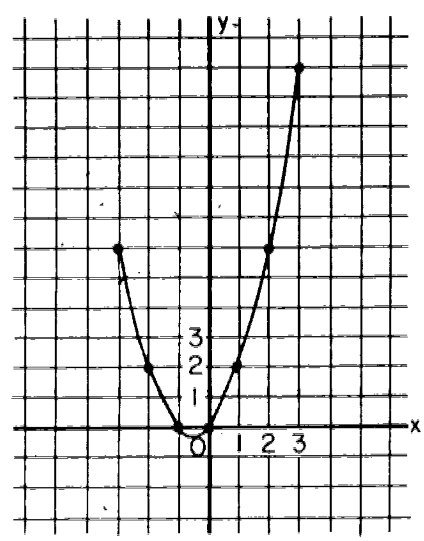
3. Graph of polynomial $x^2 - 2$

(x, y):
(-3, 7)
(-2, 2)
(-1, -1)
(0, -2)
(1, -1)
(2, 2)
(3, 7)



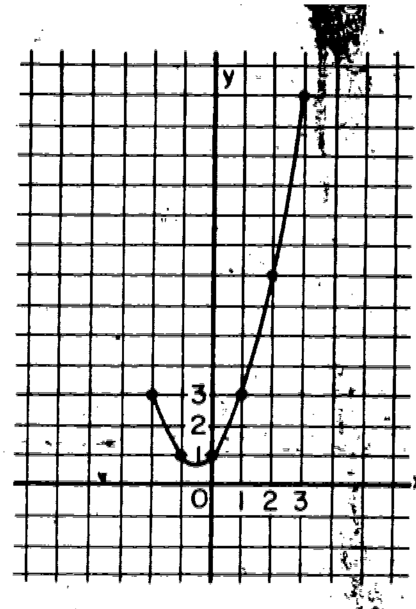
4. Graph of polynomial $x^2 + x$

(x, y):
(-3, 6)
(-2, 2)
(-1, 0)
(0, 0)
(1, 2)
(2, 6)
(3, 12)



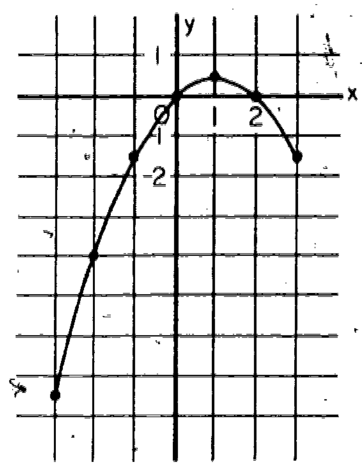
5. Graph of polynomial $x^2 + x + 1$

(x, y):
(-2, 3)
(-1, 1)
(0, 1)
(1, 3)
(2, 7)
(3, 1[illegible])



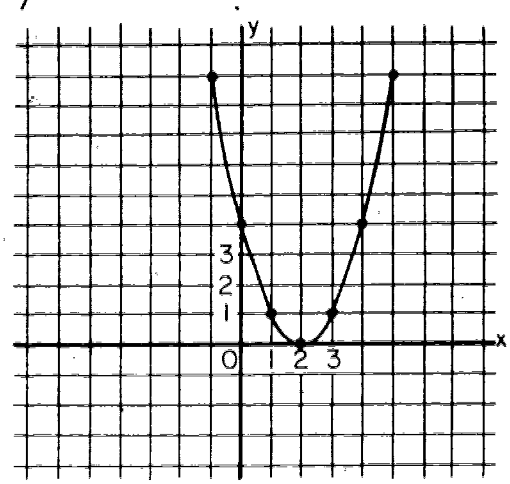
6. Graph of polynomial $-\frac{1}{2}x^2 + x$

(x, y):
$(-3, -\frac{15}{2})$
(-2, -4)
$(-1, -\frac{3}{2})$
(0, 0)
$(1, \frac{1}{2})$
(2, 0)
$(3, -\frac{3}{2})$



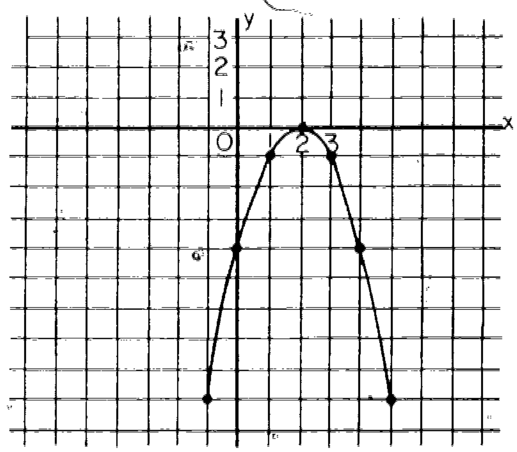
7. Graph of polynomial $x^2 - 4x + 4$

(x, y):
(-1, 9)
(0, 4)
(1, 1)
(2, 0)
(3, 1)
(4, 4)
(5, 9)



8. Graph of polynomial $-x^2 + 4x - 4$

(x, y):
(-1, -9)
(0, -4)
(1, -1)
(2, 0)
(3, -1)
(4, -4)
(5, -9)



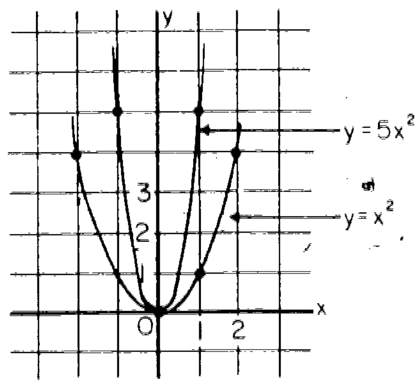
The graphs of polynomials of the form ax^2, $(a \neq 0)$ receive emphasis here because, in the next two sections, the graph of any quadratic polynomial will be drawn by reference to the graph of a polynomial of this form. In describing the effect of the number a on the shape of the graph, students may come up with such expressions as "Making the value of a smaller spreads the graph out more."

Note that the graph of any polynomial ax^2 contains the origin. The word "vertex" will be introduced in a subsequent section.

Answers to Problem Set 18-1c; page 820:

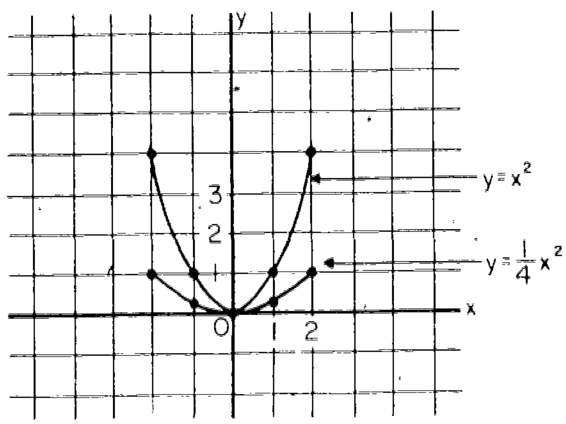
1. Each ordinate of the graph of $2x^2$ is twice the value of the corresponding ordinate of the graph of x^2.

2.



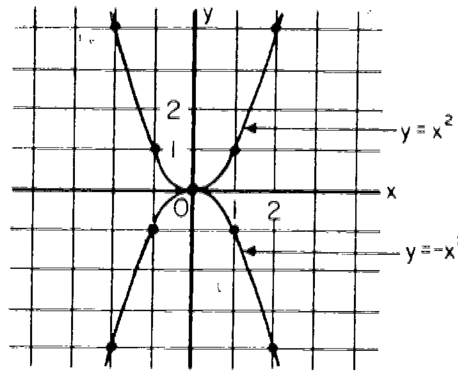
3. Each value of $\frac{1}{2}x^2$ is half the corresponding value of x^2. Therefore, each ordinate of the graph of $\frac{1}{2}x^2$ is half the corresponding ordinate of the graph of x^2.

4.



5. Each value of $-\frac{1}{2}x^2$ is the opposite of the corresponding value of $\frac{1}{2}x^2$. Therefore, each ordinate of the graph of $-\frac{1}{2}x^2$ is the opposite of the corresponding ordinate of the graph of $\frac{1}{2}x^2$.

6.



7. Each value of $-ax^2$ is the opposite of the corresponding value of ax^2. Therefore, each ordinate of the graph of $-ax^2$ is the opposite of the corresponding ordinate of the graph of ax^2.

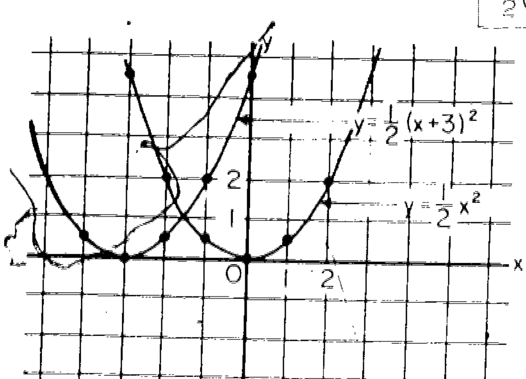
8. no

If a is 10, the graph of ax^2 is less spread out than the graph of x^2. If a is .1, the graph of ax^2 is spread out more than the graph of x^2.

Answers to Problem Set 18-1d; page 822:

1.

x	-4	-3	-2	-1	0	1	2
$\frac{1}{2}x^2$	8	$\frac{9}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2
$\frac{1}{2}(x+3)^2$	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{9}{2}$	8	$\frac{25}{2}$



The graph of "$y = \frac{1}{2}(x+3)^2$" can be obtained by "moving" the graph of "$y = \frac{1}{2}x^2$" three units to the left.

2\. (a) The graph of "$y = 3(x + 4)^2$" can be obtained by moving the graph of "$y = 3x^2$" 4 units to the left.

(b) The graph of "$y = 3(x - 4)^2$" can be obtained by moving the graph of "$y = 3x^2$" 4 units to the right.

(c) The graph of "$y = 2(x - 2)^2$" can be obtained by moving the graph of "$y = 2x^2$" 2 units to the right.

(d) The graph of "$y = 2(x + 2)^2$" can be obtained by moving the graph of "$y = 2x^2$" 2 units to the left.

(e) The graph of "$y = -2(x - 3)^2$" can be obtained by moving the graph of "$y = -2x^2$" 3 units to the right.

(f) The graph of "$y = -\frac{1}{2}(x + 1)^2$" can be obtained by moving the graph of "$y = -\frac{1}{2}x^2$" 1 unit to the left.

(g) The graph of "$y = \frac{1}{3}(x + \frac{1}{2})^2$" can be obtained by moving the graph of "$y = \frac{1}{3}x^2$" $\frac{1}{2}$ unit to the left.

*(h) The graph of "$y = 5(x + 7)^2$" can be obtained by moving the graph of "$y = 5(x - 7)^2$" 14 units to the left.

3\. If $h > 0$, the graph of "$y = a(x - h)^2$" can be obtained by moving the graph of "$y = ax^2$" h units to the right.

If $h = 0$, the graph of "$y = a(x - h)^2$" is the same as the graph of "$y = ax^2$".

If $h < 0$, the graph of "$y = a(x - h)^2$" can be obtained by moving the graph of "$y = ax^2$" $|h|$ units to the left.

Notice why $|h|$ and $|k|$ (instead of merely h and k) are used in the summary in this section. If $k > 0$, no trouble arises in saying "move k units upward". However, if $k < 0$, confusion results. For example, if k is -6, one would not want to say "move -6 units downward". The use of absolute value obviates this difficulty. A similar argument applies to h.

<u>Answers to Oral Exercises 18-1e</u>; pages 825-826:

1. The graphs in parts (a), (c), (d), and (e)

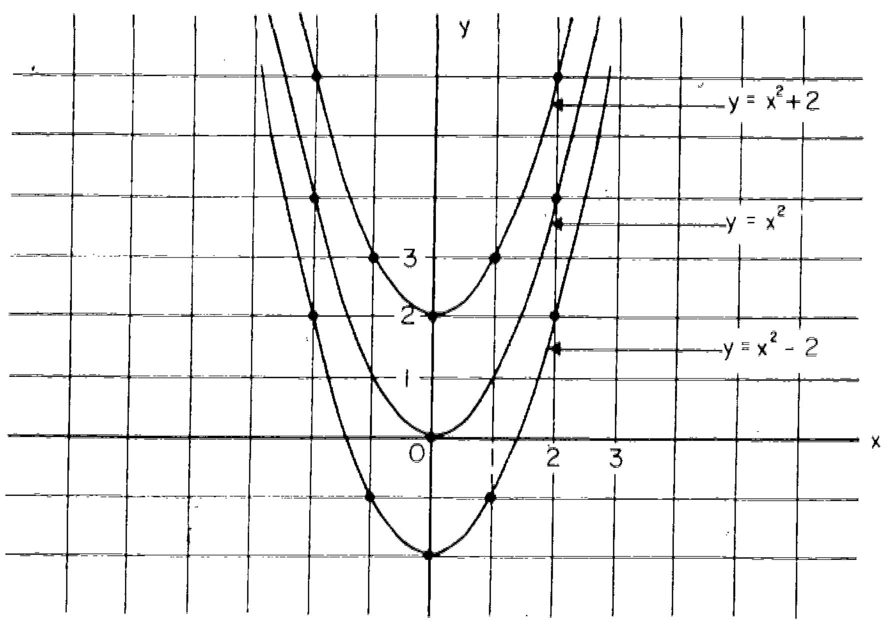
2. The number a

3. (a) down
   (b) up
   (c) left
   (d) right
   (e) left and down
   (f) left and up
   (g) right and down
   (h) right and up
   (i) no movement

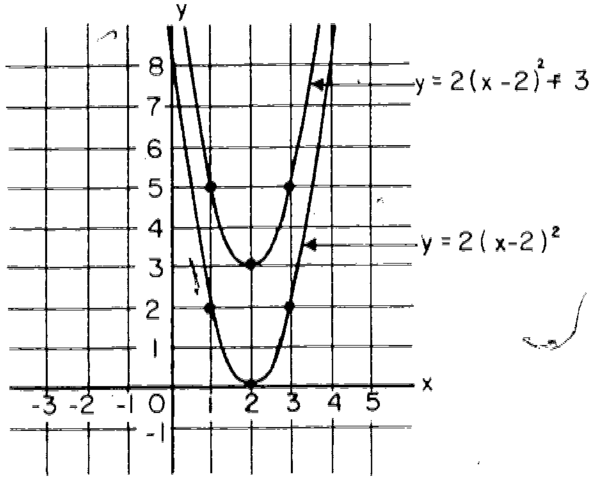
<u>Answers to Problem Set 18-1e</u>; pages 826-827:

1. The graph of "$y = x^2 - 2$" can be obtained by moving the graph of "$y = x^2$" 2 units downward.

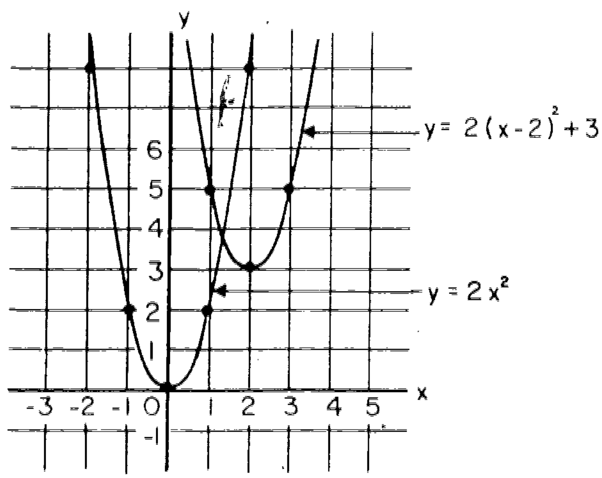
   The graph of "$y = x^2 + 2$" can be obtained by moving the graph of "$y = x^2$" 2 units upward.



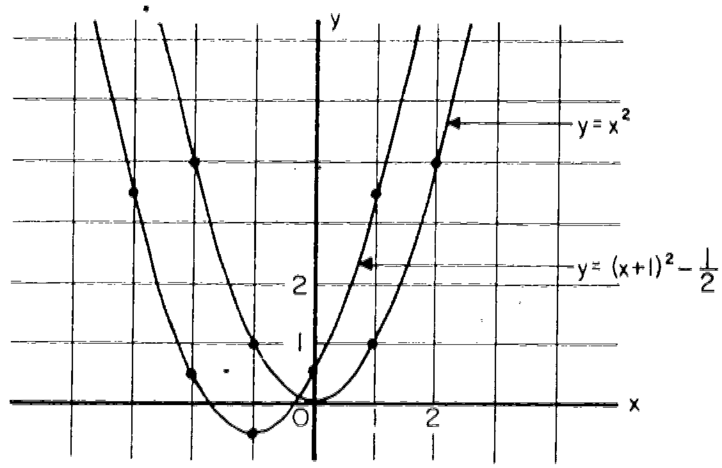
2. The graph of "$y = 2(x - 2)^2 + 3$" can be obtained by moving the graph of "$y = 2(x - 2)^2$" 3 units upward.



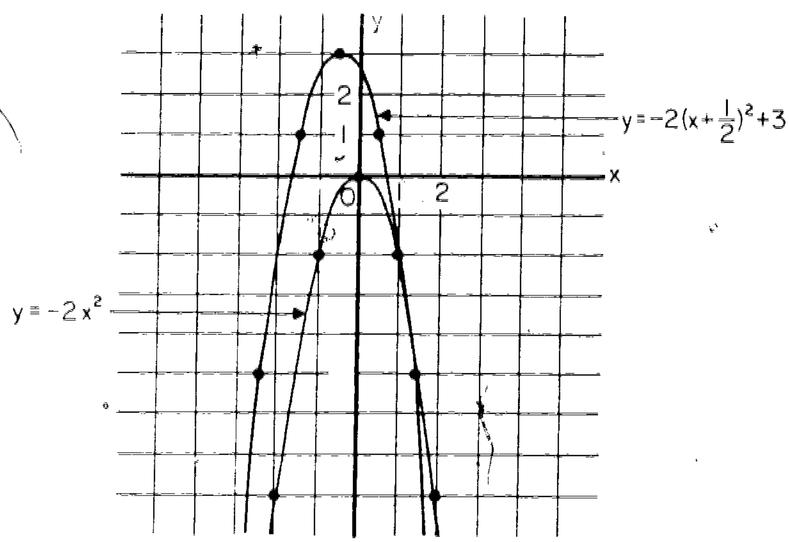
3. The graph of "$y = 2(x - 2)^2 + 3$" can be obtained by moving the graph of "$y = 2x^2$" 2 units to the right and 3 units upward.



4. The graph of "$y = (x + 1)^2 - \frac{1}{2}$" can be obtained by moving the graph of "$y = x^2$" 1 unit to the left and $\frac{1}{2}$ unit downward.



5. The graph of "$y = -2(x + \frac{1}{2})^2 + 3$" can be obtained by moving the graph of "$y = -2x^2$" $\frac{1}{2}$ unit to the left and 3 units upward.



6. (a) The graph of "$y = 3(x - 7)^2 + \frac{1}{2}$" can be obtained by moving the graph of "$y = 3x^2$" 7 units to the right and $\frac{1}{2}$ unit upward.

(b) The graph of "$y = 3(x - \frac{1}{2})^2 + 7$" can be obtained by moving the graph of "$y = 3x^2$" $\frac{1}{2}$ unit to the right and 7 units upward.

(c) The graph of "$y = 2x^2 + \frac{5}{2}$" can be obtained by moving the graph of "$y = 2x^2$" $\frac{5}{2}$ units upward.

(d) The graph of "$y = 2(x + \frac{5}{2})^2$" can be obtained by moving the graph of "$y = 2x^2$" $\frac{5}{2}$ units to the left.

(e) The graph of "$y = -(x + 3)^2 - 4$" can be obtained by moving the graph of "$y = -x^2$" 3 units to the left and 4 units downward.

(f) The graph of "$y = x^2 + 14$" can be obtained by moving the graph of "$y = x^2$" 14 units upward.

(g) The graph of "$y = 5x^2 + 14$" can be obtained by moving the graph of "$y = 5x^2$" 14 units upward.

(h) The graph of "$y = 5(x - 2)^2 + 14$" can be obtained by moving the graph of "$y = 5x^2$" 2 units to the right and 14 units upward.

(i) The graph of "$y = -8(x - 8)^2 - 8$" can be obtained by moving the graph of "$y = -8x^2$" 8 units to the right and 8 units downward.

*(j) The graph of "$y = 4(3 - x)^2 - 6$" can be obtained by moving the graph of "$y = 4x^2$" 3 units to the right and 6 units downward. "$(3 - x)^2 = (x - 3)^2$" is true for all values of x.

7. (a) $y = (x - 5)^2$

(b) $y = x^2 - 5$

(c) $y = x^2 + 5$

(d) $y = (x + 5)^2$

(e) $y = (x - 5)^2 - 5$

(f) $y = -(x - \frac{1}{2})^2 + 7$

(g) $y = 2(x + 3)^2 - 6$

(h) $y = \frac{1}{3}(x - \frac{1}{2})^2 - 1$

*(i) $y = \frac{1}{2}x^2$

Page 828. In general, the vertex of the graph of "$y = a(x - h)^2 + k$" is the point (h, k) and the axis is the line whose equation is "$x = h$".

Answers to Oral Exercises 18-1f; page 828:

1. Coordinates of vertex: $(0, 0)$
   Equation of axis: $x = 0$
2. Coordinates of vertex: $(0, 0)$
   Equation of axis: $x = 0$
3. Coordinates of vertex: $(0, 0)$
   Equation of axis: $x = 0$
4. Coordinates of vertex: $(2, 0)$
   Equation of axis: $x = 2$
5. Coordinates of vertex: $(2, 3)$
   Equation of axis: $x = 2$
6. Coordinates of vertex: $(2, -3)$
   Equation of axis: $x = 2$
7. Coordinates of vertex: $(-2, 0)$
   Equation of axis: $x = -2$
8. Coordinates of vertex: $(-2, \frac{1}{2})$
   Equation of axis: $x = -2$
9. Coordinates of vertex: $(-2, -\frac{1}{2})$
   Equation of axis: $x = -2$
*10. Coordinates of vertex: (h, k)
   Equation of axis: $x = h$

18-2. Standard Forms.

The standard form of a quadratic polynomial, $a(x - h)^2 + k$ is discussed carefully for two reasons. First, once the standard form is determined, the graph of the polynomial is easily obtained from the graph of ax^2. Secondly, the standard form leads to a procedure for solving any quadratic equation, coming up in Section 18-3.

Answers to Oral Exercises 18-2a; page 831:

1. $(3x - 2)^2 + 5$ is not in standard form. In the standard form, $a(x - h)^2 + k$, the coefficient of x^2 within the parenthetical phrase itself is one.

2. The distributive property must be applied in order to get a coefficient of one for x^2. Then the square must be completed.

Answers to Problem Set 18-2a; pages 831-832:

1. (a) $x^2 - 6x + 9$
   (b) $2x^2 - 12x + 18$
   (c) $2x^2 - 12x + 24$
   (d) $-5x^2 - 5x + \frac{39}{4}$
   (e) $-x^2 - 6x - 11$
   (f) $2x^2 + 5$ is in the form $Ax^2 + Bx + C$

2. (a) $(x + 5)^2 - 27$
   (b) $(x + 3)^2 + 1$
   (c) $(x - 8)^2 - 68$
   (d) $(x - 8)^2 - 64$
   (e) $(x - 10)^2 - 95$
   (f) $(x + \frac{1}{2})^2 + \frac{3}{4}$
   (g) $(x - \frac{3}{2})^2 - \frac{17}{4}$
   (h) $(x - \frac{3}{2})^2 - \frac{1}{4}$
   (i) $2x^2 + 5$ is in standard form
   *(j) $5(x - 1)^2 - 10$
   *(k) $2(x + 3)^2 - 25$
   *(l) $\frac{1}{2}(x - 3)^2 - \frac{5}{2}$

3. (a) $(x - 1)^2 + 4$
   (b) $(x - \frac{1}{2})^2 + \frac{7}{4}$
   (c) $(x + \frac{3}{2})^2 - \frac{5}{4}$
   (d) $x^2 - 25$
   *(e) $3(x - \frac{1}{3})^2 - \frac{1}{3}$
   *(f) $6(x - \frac{1}{12})^2 - \frac{361}{24}$

4. (a) The graph of "$y = x^2 - 2x + 5$" can be obtained by moving the graph of "$y = x^2$" 1 unit to the right and 4 units upward.
   (b) The graph of "$y = x^2 - x + 2$" can be obtained by moving the graph of "$y = x^2$" $\frac{1}{2}$ unit to the right and $\frac{7}{4}$ units upward.
   (c) The graph of "$y = x^2 + 3x + 1$" can be obtained by moving the graph of "$y = x^2$" $\frac{3}{2}$ units to the left and $\frac{5}{4}$ units downward.

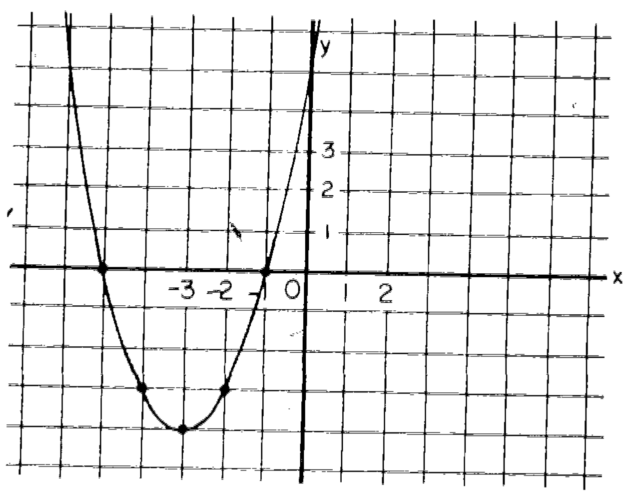
(d) The graph of "$y = (x + 5)(x - 5)$" can be obtained by moving the graph of "$y = x^2$" 25 units downward.

(e) The graph of "$y = 3x^2 - 2x$" can be obtained by moving the graph of "$y = 3x^2$" $\frac{1}{3}$ unit to the right and $\frac{1}{3}$ unit downward.

(f) The graph of "$y = 6x^2 - x - 15$" can be obtained by moving the graph of "$y = 6x^2$" $\frac{1}{12}$ unit to the right and $\frac{361}{24}$ units downward.

5. (a) "$y = x^2 + 6x + 5$" in standard form:

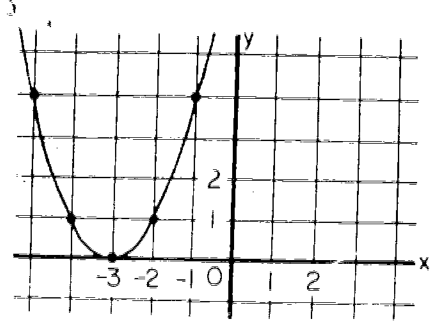
$$y = (x + 3)^2 - 4$$



The graph intersects the x-axis at the points $(-1, 0)$ and $(-5, 0)$.

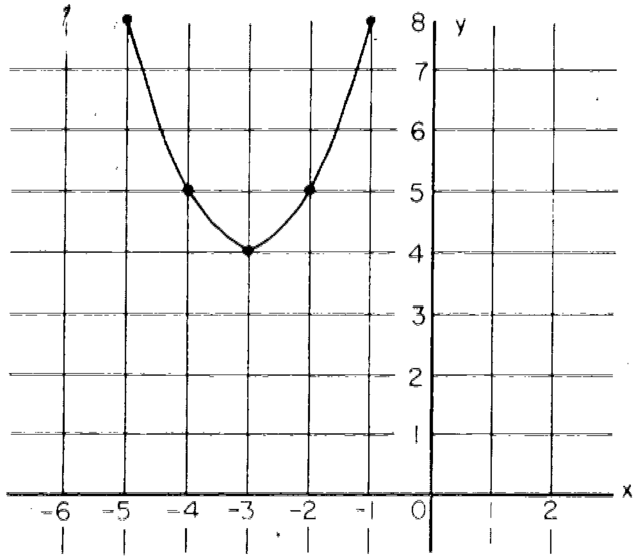
(b) "$y = x^2 + 6x + 9$" in standard form:

$$y = (x + 3)^2$$



The graph intersects the x-axis at the point $(-3, 0)$.

(c) "$y = x^2 + 6x + 13$" in standard form:
$y = (x + 3)^2 + 4$



The graph does not intersect the x-axis.

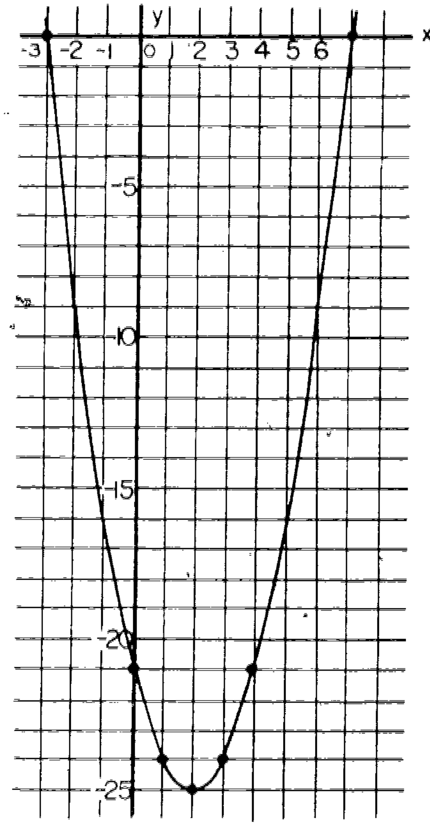
The approximate nature of determining truth sets from a graph may not be appreciated by students unless some time is given to cases in which the truth numbers are not integral and hence are not likely to be read exactly from a graph.

Therefore, some class time might be given to graphing an open sentence such as "$y = 3x^2 - 2x - 1$", which intersects the x-axis at points whose abscissas are 1 and $-\frac{1}{3}$.

<u>Answers to Problem Set 18-2b</u>;

1. "$y = x^2 - 4x - 21$" in standard form:

$$y = (x - 2)^2 - 25$$

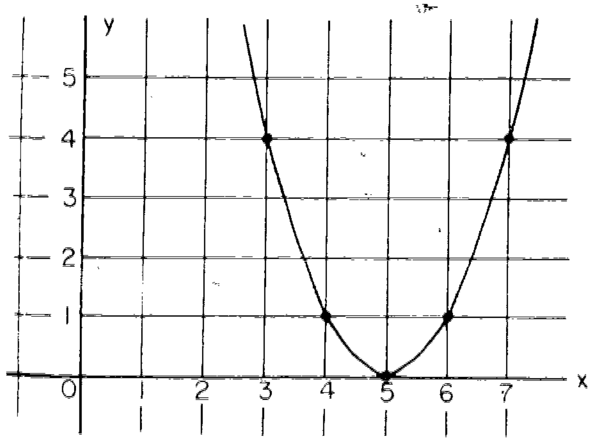


The abscissas of the points at which the graph intersects the x-axis are -3 and 7.

The truth set of "$x^2 - 4x - 21 = 0$" is $\{-3, 7\}$.

2. "$y = x^2 - 10x + 25$" in standard form:

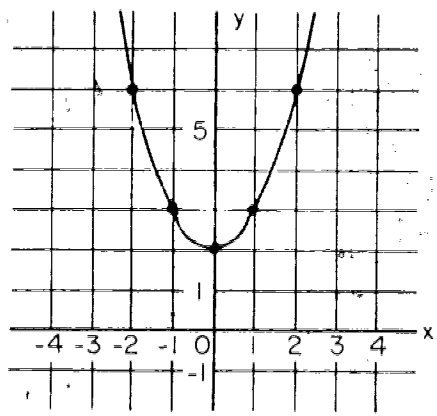
$$y = (x - 5)^2$$



The abscissa of the point at which the graph intersects the x-axis is 5.

The truth set of "$x^2 - 10x + 25 = 0$" is $\{5\}$.

3. "$y = x^2 + 2$" is in standard form.



The graph does not intersect the x-axis.
The truth set of "$x^2 + 2 = 0$" is the empty set.

4. The truth numbers of "$Ax^2 + Bx + C = 0$" are abscissas of the points where the graph of "$y = Ax^2 + Bx + C$" intersects the x-axis. If "$Ax^2 + Bx + C = 0$" has more than two truth numbers then the graph of "$y = Ax^2 + Bx + C$" must intersect the x-axis at more than two points. But our notion about the shape of a parabola tells us that there are at most two points of intersection with the x-axis. Hence, it is reasonable to expect that there are at most two truth numbers for the equation.

5. The graph of "$y = Ax^2 + Bx + C$" intersects the x-axis at exactly one point. We sometimes say, in this case, that the graph touches (is tangent to) the x-axis at this point.

6. The graph of "$y = Ax^2 + Bx + C$" does not intersect the x-axis at any point. The graph is entirely above or entirely below the x-axis.

18-3. Quadratic Equations.

In Example 1 of Section 18-3, the student may need some help in making the step from the sentence

$$(x - 3)^2 - (5)^2 = 0$$

to the sentence

$$(x - 3 - 5)(x - 3 + 5) = 0.$$

From his work in Chapter 13, he knows that the sentence

$$(n)^2 - (5)^2 = (n - 5)(n + 5)$$

is true for any real number n. If "x - 3" is used as a name for the number n, the sentence

$$(x - 3)^2 - (5)^2 = (x - 3 - 5)(x - 3 + 5)$$

results, and this is what is involved in the step mentioned in Example 1.

Similar steps occur in the other examples. And in the problems following this section, the student is given an opportunity to write factors of expressions such as $(x - 3)^2 - (5)^2$.

Answers to Problem Set 18-3a; pages 837-838:

1. (a) $(x - 1)^2 - 4 = (x - 1 + 2)(x - 1 - 2)$
   $= (x + 1)(x - 3)$

   (b) $(x + 2)^2 - 9 = (x + 2 + 3)(x + 2 - 3)$
   $= (x + 5)(x - 1)$

   (c) $(x - 2)^2 - 1 = (x - 2 + 1)(x - 2 - 1)$
   $= (x - 1)(x - 3)$

   (d) $(x + \frac{3}{2})^2 - 1 = (x + \frac{3}{2} + 1)(x + \frac{3}{2} - 1)$
   $= (x + \frac{5}{2})(x + \frac{1}{2})$

   (e) $(x - 1)(x - 6)$

   (f) $(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})$

   (g) $(x + 3 + \sqrt{5})(x + 3 - \sqrt{5})$

   (h) $(x - \frac{1}{2} + \sqrt{7})(x - \frac{1}{2} - \sqrt{7})$

   (i) $(x - \frac{3}{2} + \frac{\sqrt{3}}{2})(x - \frac{3}{2} - \frac{\sqrt{3}}{2})$

   (j) $(x + \frac{5}{6} + \frac{\sqrt{5}}{6})(x + \frac{5}{6} - \frac{\sqrt{5}}{6})$

pages 837-838: 18-3

2. (a) $\{-3, 1\}$

(b) $\{-(2+\sqrt{2}), -(2-\sqrt{2})\}$

(c) $\{-(3+\sqrt{6}), -(3-\sqrt{6})\}$

(d) $\{-(3+2\sqrt{3}), -(3-2\sqrt{3})\}$

(e) $\{-(5+\sqrt{19}), -(5-\sqrt{19})\}$

(f) $\{-(3+\sqrt{5}), -(3-\sqrt{5})\}$

(g) $\emptyset$

(h) $\{\sqrt{6}, -\sqrt{6}\}$

(i) $\emptyset$

(j) $\emptyset$

(k) $\{-\frac{3}{2}, 4\}$

(l) $\{3, \frac{1}{2}\}$

3. Let w be the width of the rectangle in feet. Then, the length is $w + 6$ feet. The area of the rectangle is 36 square feet.

$w(w + 6) = 36$; Domain: $w > 0$

$w^2 + 6w - 36 = 0$

$(w + 3)^2 - 45 = 0$

$(w + 3 - 3\sqrt{5})(w + 3 + 3\sqrt{5}) = 0$

$w + 3 - 3\sqrt{5} = 0$ or $w + 3 + 3\sqrt{5} = 0$

$w = -3 + 3\sqrt{5}$ or $w = -3 - 3\sqrt{5}$

are all equivalent.

Hence, the truth set is

$\{-3 + 3\sqrt{5}\}$.

$(-3 - 3\sqrt{5})$ is a negative number and is not in the domain of w. The positive number leads to the following solution:

width: $-3 + 3\sqrt{5}$ ft.

length: $3 + 3\sqrt{5}$ ft.

4. Let n be the larger number. The other number is $n - 5$. Their product is 100.

$$n(n - 5) = 100$$
$$n^2 - 5n - 100 = 0$$
$$(n^2 - 5n + \frac{25}{4}) - 100 - \frac{25}{4} = 0$$
$$(n - \frac{5}{2})^2 - \frac{425}{4} = 0$$
$$(n - \frac{5}{2} - \frac{5\sqrt{17}}{2})(n - \frac{5}{2} + \frac{5\sqrt{17}}{2}) = 0$$
$$n - \frac{5}{2} - \frac{5\sqrt{17}}{2} = 0 \quad \text{or} \quad n - \frac{5}{2} + \frac{5\sqrt{17}}{2} = 0$$
$$n = \frac{5 + 5\sqrt{17}}{2} \quad \text{or} \quad n = \frac{5 - 5\sqrt{17}}{2}.$$

are all equivalent.

Hence, the truth set is $\left\{\frac{5 - 5\sqrt{17}}{2}, \frac{5 + 5\sqrt{17}}{2}\right\}$.

One number is $\frac{5 + 5\sqrt{17}}{2}$ and the other is $\frac{-5 + 5\sqrt{17}}{2}$

or

one number is $\frac{5 - 5\sqrt{17}}{2}$ and the other is $\frac{-5 - 5\sqrt{17}}{2}$.

5. Let x be the number.

$$x + \frac{1}{x} = 4 \qquad \text{Domain: } x \neq 0$$
$$x^2 + 1 = 4x$$
$$x^2 - 4x + 1 = 0$$
$$(x^2 - 4x + 4) + 1 - 4 = 0$$
$$(x - 2)^2 - 3 = 0$$
$$(x - 2 - \sqrt{3})(x - 2 + \sqrt{3}) = 0$$
$$x - 2 - \sqrt{3} = 0 \quad \text{or} \quad x - 2 + \sqrt{3} = 0$$
$$x = 2 + \sqrt{3} \quad \text{or} \quad x = 2 - \sqrt{3}$$

are all equivalent.

Hence, the truth set is $\{2 + \sqrt{3}, 2 - \sqrt{3}\}$.
The number is $2 + \sqrt{3}$ or $2 - \sqrt{3}$.

page 838: 18-3

6. Let x be the number.

$x + \frac{1}{x} = -4$ Domain: $x \neq 0$

$x^2 + 1 = -4x$

$x^2 + 4x + 1 = 0$

$(x^2 + 4x + 4) + 1 - 4 = 0$

$(x + 2)^2 - 3 = 0$

$(x + 2 - \sqrt{3})(x + 2 + \sqrt{3}) = 0$

$x + 2 - \sqrt{3} = 0$ or $x + 2 + \sqrt{3} = 0$

$x = -2 + \sqrt{3}$ or $x = -2 - \sqrt{3}$

are all equivalent.

Hence, the truth set is $\{-2 + \sqrt{3}, -2 - \sqrt{3}\}$.
The number is $-2 + \sqrt{3}$ or $-2 - \sqrt{3}$.

7. Let n be the number.

$14n + n^2 = 11$

$n^2 + 14n - 11 = 0$

$(n^2 + 14n + 49) - 11 - 49 = 0$

$(n + 7)^2 - 60 = 0$

$(n + 7 - 2\sqrt{15})(n + 7 + 2\sqrt{15}) = 0$

$n + 7 - 2\sqrt{15} = 0$ or $n + 7 + 2\sqrt{15} = 0$

$n = -7 + 2\sqrt{15}$ or $n = -7 - 2\sqrt{15}$

are all equivalent.

Hence, the truth set is $\{-7 + 2\sqrt{15}, -7 - 2\sqrt{15}\}$.
The number is $-7 + 2\sqrt{15}$ or $-7 - 2\sqrt{15}$.

8. Let ℓ be the length in feet and $\ell - 3$ be the width in feet.

$\ell(\ell - 3) = \frac{187}{4}$ Domain: $\ell > 3$

$\ell^2 - 3\ell - \frac{187}{4} = 0$

$(\ell^2 - 3\ell + \frac{9}{4}) - \frac{187}{4} - \frac{9}{4} = 0$

$(\ell - \frac{3}{2})^2 - \frac{196}{4} = 0$

$$(\ell - \frac{3}{2} - \frac{14}{2})(\ell - \frac{3}{2} + \frac{14}{2}) = 0$$

$\ell - \frac{17}{2} = 0$ or $\ell + \frac{11}{2} = 0$

$\ell = \frac{17}{2}$ or $\ell = -\frac{11}{2}$ Domain: $\ell > 3$

are all equivalent.

Hence, the truth set is $\{\frac{17}{2}\}$.

The length is $\frac{17}{2}$ feet.

*9. Let n be the number. Then, $\frac{1}{n}$ is its reciprocal.

$n + \frac{1}{n} = 0$ Domain: $n \neq 0$

$n^2 + 1 = 0$

are all equivalent. However, $n^2 + 1$ cannot be factored over the real numbers, and the truth set of the equation is $\emptyset$. Therefore, there is no number n such that $n + \frac{1}{n} = 0$.

*10. Let n be the width of the original sheet in inches. Then, the length of the original sheet is $n + 8$ inches. The length of the base of the box is $(n + 8) - 4$ inches, the width of the base is $n - 4$ inches, and the height of the box is 2 inches.

$\big((n + 8) - 4\big)(n - 4)(2) = 256$ Domain: $n > 4$

$(n + 4)(n - 4)(2) = 256$

$2n^2 - 32 = 256$

$n^2 - 144 = 0$

$(n + 12)(n - 12) = 0$

$n + 12 = 0$ or $n - 12 = 0$

$n = -12$ or $n = 12$ Domain: $n > 4$

are all equivalent.

Hence, the truth set is $\{12\}$.

The dimensions of the original sheet are 12" by 20".

pages 838-839: 18-3

11. Let the height of the window be x feet. Then the length of the rope is $x + 8$ feet.

$$x^2 + (28)^2 = (x + 8)^2$$
$$x^2 + 784 = x^2 + 16x + 64$$
$$16x = 720$$
$$x = 45$$

are all equivalent.

Hence, the truth set is $\{45\}$.
The window is 45 feet above the ground.

*12. Let r be John's average speed to Chicago. Then, his average speed on the return trip was $r - 6$.

$$\frac{336}{r} = \frac{336}{r-6} + 1 \qquad \text{Domain: } r > 6$$
$$\frac{336}{r} = \frac{336 + r - 6}{r - 6}$$
$$r^2 - 6r - 2016 = 0$$
$$(r - 48)(r + 42) = 0$$
$$r - 48 = 0 \quad \text{or} \quad r + 42 = 0$$
$$r = 48 \quad \text{or} \quad r = -42 \qquad \text{Domain: } r > 6$$

are all equivalent.

Hence, the truth set is $\{48\}$.
John's average speed to Chicago was 48 mph.
His average speed on the return trip was 42 mph.

Answers to Review Problem Set; pages 839-841:

1. (a) By moving the graph of "$y = \frac{1}{3}x^2$" 2 units downward.
   (b) By moving the graph of "$y = \frac{1}{3}x^2$" 2 units to the right.
   (c) By moving the graph of "$y = \frac{1}{3}x^2$" 7 units to the left.
   (d) By moving the graph of "$y = \frac{1}{3}x^2$" $\frac{1}{2}$ unit upward.

(e) By moving the graph of "$y = \frac{1}{3}x^2$" 6 units to the right and 8 units upward.

(f) By moving the graph of "$y = \frac{1}{3}x^2$" 5 units to the left and 10 units upward.

(g) By moving the graph of "$y = \frac{1}{3}x^2$" $\frac{1}{4}$ unit to the left and 2.5 units downward.

(h) By moving the graph of "$y = \frac{1}{3}x^2$" n units to the right and t units upward.

2. (a) $(x + 3)^2 - 7$; $a = 1,$ $h = -3,$ $k = -7$

(b) $(x - 5)^2 - 32$; $a = 1,$ $h = 5,$ $k = -32$

(c) $(x - 2)^2$; $a = 1,$ $h = 2,$ $k = 0$

(d) $(x + 6)^2 - 16$; $a = 1,$ $h = -6,$ $k = -16$

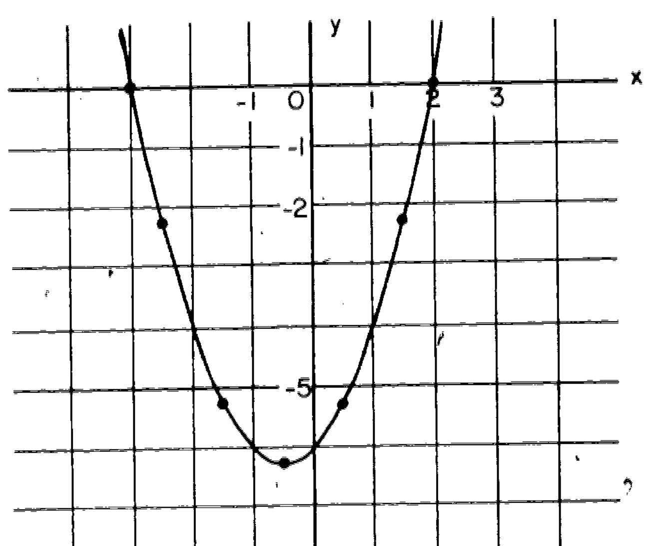
(e) $(x - \frac{7}{2})^2 - \frac{57}{4}$; $a = 1,$ $h = \frac{7}{2},$ $k = -\frac{57}{4}$

(f) $(x + \frac{5}{2})^2 + \frac{135}{4}$; $a = 1,$ $h = -\frac{5}{2},$ $k = \frac{135}{4}$

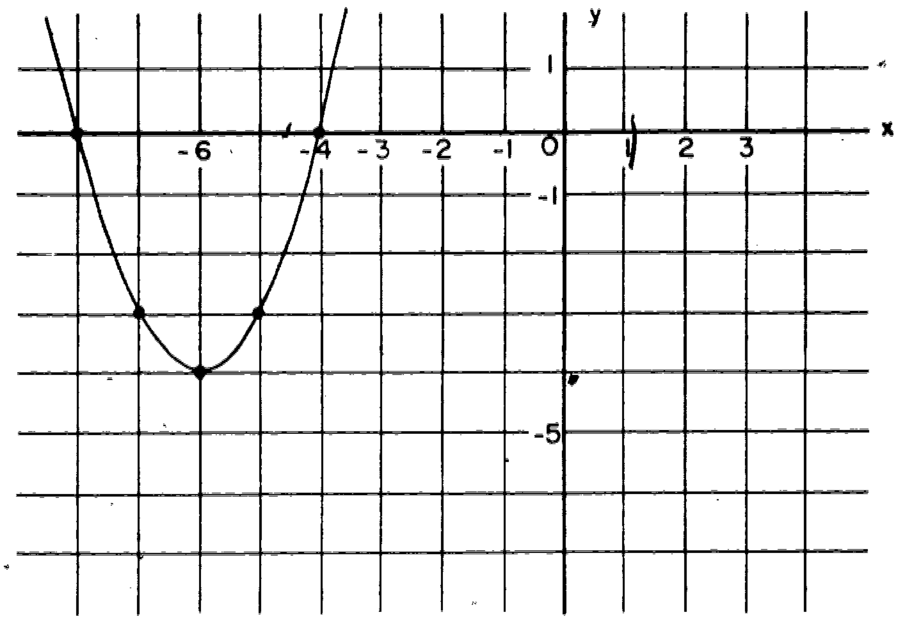
(g) $2(x + 2)^2 - 1$; $a = 2,$ $h = -2,$ $k = -1$

(h) $2(x - \frac{9}{4})^2 + \frac{15}{8}$; $a = 2,$ $h = \frac{9}{4},$ $k = \frac{15}{8}$

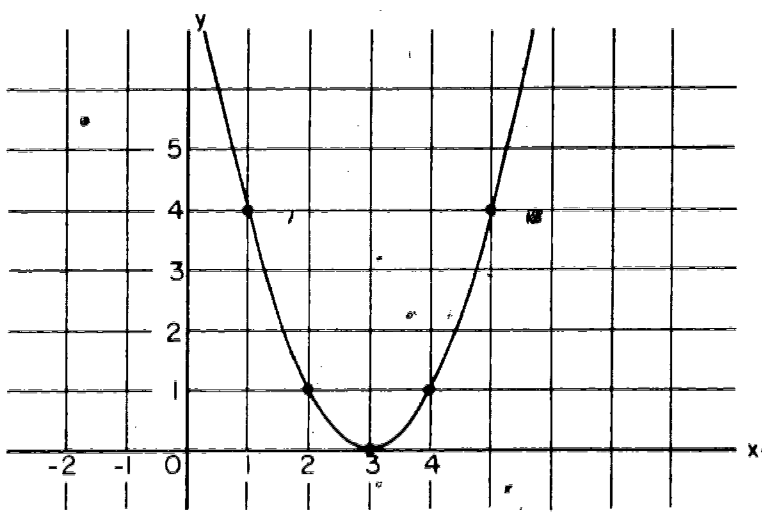
3. (a) "$y = x^2 + x - 6$" in standard form: $y = (x + \frac{1}{2})^2 - \frac{25}{4}$



(b) "$y = x^2 + 12x + 32$" in standard form:
$y = (x + 6)^2 - 4$

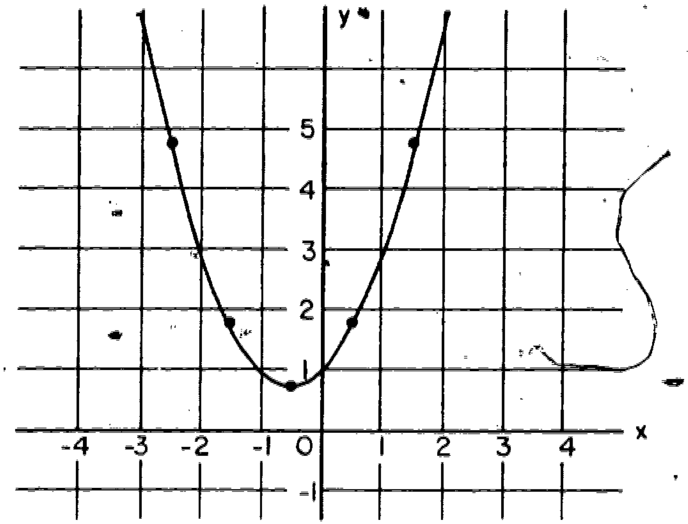


(c) "$y = x^2 - 6x + 9$" in standard form:
$y = (x - 3)^2$



(d) "$y = x^2 + x + 1$" in standard form:

$$y = (x + \frac{1}{2})^2 + \frac{3}{4}$$



4. (a) $\{-3, 2\}$ (c) $\{3\}$

(b) $\{-4, -8\}$ (d) $\emptyset$

5. (a) $\{2, 4\}$ (e) $\{2 - \sqrt{3}, \ 2 + \sqrt{3}\}$

(b) $\{3, 7\}$ (f) $\{-1, -6\}$

(c) $\{1 - \sqrt{2}, \ 1 + \sqrt{2}\}$ (g) $\emptyset$

(d) $\{-\frac{1}{2}, \frac{1}{3}\}$ (h) the set of all real numbers

6. (a) Coordinates of vertex: $(0, 2)$
Equation of axis: $x = 0$

(b) Coordinates of vertex: $(0, -7)$
Equation of axis: $x = 0$

(c) Coordinates of vertex: $(4, 0)$
Equation of axis: $x = 4$

(d) Coordinates of vertex: $(-7, 0)$
Equation of axis: $x = -7$

(e) Coordinates of vertex: $(7, -\frac{1}{2})$
Equation of axis: $x = 7$

(f) Coordinates of vertex: $(-2, -8)$
Equation of axis: $x = -2$

(g) Coordinates of vertex: (2, -4)
Equation of axis: $x = 2$

(h) Coordinates of vertex: (-5, 10)
Equation of axis: $x = -5$

7. (a) a line
(b) a parabola
(c) a line
(d) a parabola
(e) a pair of lines
The sentences
$y^2 - x^2 = 0$
$(y - x)(y + x) = 0$
$y = x$ <u>or</u> $y = -x$
are equivalent.
(f) a parabola
(g) a line
(h) a parabola
(i) a line
(j) a line
(k) a line
(l) a point
(m) a pair of lines

8. Let n be the smaller number. The other number is $n + 7$.

$(n + 7)^2 = 2(n)^2$

$n^2 + 14n + 49 = 2n^2$

$n^2 - 14n - 49 = 0$

$(n^2 - 14n + 49) - 49 - 49 = 0$

$(n - 7)^2 - 98 = 0$

$(n - 7 + 7\sqrt{2})(n - 7 - 7\sqrt{2}) = 0$

$n - 7 + 7\sqrt{2} = 0$ or $n - 7 - 7\sqrt{2} = 0$

$n = 7 - 7\sqrt{2}$ or $n = 7 + 7\sqrt{2}$

are all equivalent.

Hence, the truth set is $\{7 - 7\sqrt{2},\ 7 + 7\sqrt{2}\}$.

One number is $7 - 7\sqrt{2}$ and the other is $14 - 7\sqrt{2}$

or

one number is $7 + 7\sqrt{2}$ and the other is $14 + 7\sqrt{2}$.

9. Let s be the smaller number. The other number is $s + 7$.

$s^2 = 2(s + 7)^2$

$s^2 = 2s^2 + 28s + 98$

$0 = s^2 + 28s + 98$

$$(s^2 + 28s + 196) + 98 - 196 = 0$$
$$(s + 14)^2 - 98 = 0$$
$$(s + 14 - 7\sqrt{2})(s + 14 + 7\sqrt{2}) = 0$$
$$s + 14 - 7\sqrt{2} = 0 \quad \text{or} \quad s + 14 + 7\sqrt{2} = 0$$
$$s = -14 + 7\sqrt{2} \quad \text{or} \quad s = -14 - 7\sqrt{2}$$
are all equivalent.

Hence, the truth set is $\{-14 + 7\sqrt{2}, \ -14 - 7\sqrt{2}\}$.
One number is $-14 + 7\sqrt{2}$ and the other is $-7 + 7\sqrt{2}$

or

one number is $-14 - 7\sqrt{2}$ and the other is $-7 - 7\sqrt{2}$.

10. Let w be the width of the rectangle in feet. Then the length is $47 - w$ feet.

$$w(47 - w) = 496$$
$$47w - w^2 = 496$$
$$w^2 - 47w + 496 = 0$$

Domain: $w < \frac{47}{2}$. If the length is greater than the width, $w < 47 - w$.

$$(w^2 - 47w + \frac{2209}{4}) + 496 - \frac{2209}{4} = 0$$
$$(w - \frac{47}{2})^2 - \frac{225}{4} = 0$$
$$(w - \frac{47}{2} - \frac{15}{2})(w - \frac{47}{2} + \frac{15}{2}) = 0$$
$$(w - \frac{62}{2}) = 0 \quad \text{or} \quad (w - \frac{32}{2}) = 0$$
$$w = 31 \quad \text{or} \quad w = 16 \qquad \text{Domain: } w < \frac{47}{2}$$
are all equivalent.

Hence, the truth set is $\{16\}$.
The width is 16 feet and the length is $47 - 16$ or 31 feet.

*11. Let one number be x and the other $9 - x$. Their product will be the ordinate of the vertex of the graph of $x(9 - x)$ since the vertex is the highest point on this graph. "$y = 9x - x^2$" in standard form is "$y = -(x - \frac{9}{2})^2 + \frac{81}{4}$". The vertex of the graph of "$y = -(x - \frac{9}{2})^2 + \frac{81}{4}$" has coordinates $(\frac{9}{2}, \frac{81}{4})$. The product $x(9 - x)$ has the largest value, $\frac{81}{4}$, when x is $\frac{9}{2}$. Each of the two numbers is $\frac{9}{2}$.

*12. The vertex of the graph of "$c = n^2 - 12n + 175$" is the minimum point on the graph. Therefore, the ordinate of the vertex is the smallest value of the cost per boat. "$c = n^2 - 12n + 175$" in standard form is "$c = (n - 6)^2 + 139$". The coordinates of the vertex are (6, 139). The cost per boat will be the smallest when he manufactures 6 boats each day.

Suggested Test Items

1. Draw the graph of $y = x^2$.
   (a) Explain how the graph of "$y = -x^2$" can be obtained from the graph of "$y = x^2$".
   (b) Explain how the graph of "$y = x^2 - 4$" can be obtained from the graph of "$y = x^2$".
   (c) Explain how the graph of "$y = (x - 2)^2$" can be obtained from the graph of "$y = x^2$".

2. Consider the polynomial $2(x - 3)^2 + 4$.
   (a) Does the graph of the polynomial have a highest point? Does it have a lowest point? Write the coordinates of this point.
   (b) In how many points does the graph intersect the x-axis?
   (c) Write the equation of the axis of the graph (parabola).

3. Which of the following polynomials are in standard form?

(a) $x^2 + 2x + 1$	(d) $(2x - 1)^2 + 3$
(b) $3x^2 + 5$	(e) $-(x - 5)^2$
(c) $2(x - 1)^2 + 3$	(f) $(x - 3)(x - 2)$

4. Obtain the standard form of the polynomial $2x^2 + 4x + 8$. Show each step of the process and indicate what properties you have used.

5. Solve each of the following equations.

(a) $25(x + 3)^2 - 4 = 0$

(b) $3x^2 + 6x - 2 = 0$

6. Write in standard form the polynomial whose graph can be obtained from the graph of $y = 4x^2$ by the following movements.

(a) 2 units upward

(b) 1 unit to the left

(c) 3 units to the right and 4 units downward

7. How must the graph of $y = -2x^2$ be moved to obtain the graph of each of the following?

(a) $y = -2(x - 3)^2$

(b) $y = -2(x + 1)^2 + 3$

8. Find the value of c if the graph of $y = cx^2$ contains the point $(-2, -12)$.

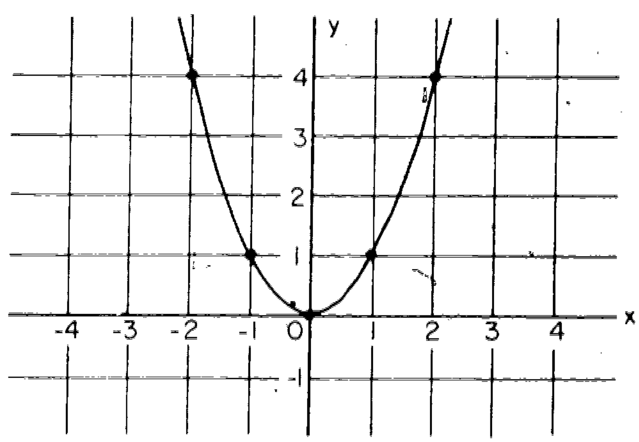
9. Find the value (or values) of k such that the graph of $y = 3(x + 1)^2 + k$ satisfies the following conditions.

(a) intersects the x-axis in exactly one point

(b) does not intersect the x-axis at all

Answers to Suggested Test Items

1.



(a) Each ordinate of the graph of "$y = -x^2$" is the opposite of the corresponding ordinate of the graph of "$y = x^2$".

(b) The graph of "$y = x^2 - 4$" can be obtained by moving the graph of "$y = x^2$" 4 units downward.

(c) The graph of "$y = (x - 2)^2$" can be obtained by moving the graph of "$y = x^2$" 2 units to the right.

2. (a) The graph of the polynomial $2(x - 3)^2 + 4$ has no highest point. It does have a lowest point. The coordinates of this point are $(3, 4)$.

   (b) The graph does not intersect the x-axis.

   (c) $x = 3$.

3. The polynomials in (b), (c), and (e).

4. $2x^2 + 4x + 8 = 2(x^2 + 2x) + 8$ — The distributive property is used.

   $= 2(x^2 + 2x + 1) + 8 - 2$ — The addition property of zero is used in completing the square.

   $= 2(x + 1)^2 + 6$

5. (a) $\{-\frac{17}{5}, -\frac{13}{5}\}$

   (b) $\{\frac{-3 - \sqrt{15}}{3}, \frac{-3 + \sqrt{15}}{3}\}$

6. (a) $y = 4x^2 + 2$

   (b) $y = 4(x + 1)^2$

   (c) $y = 4(x - 3)^2 - 4$

7. (a) The graph of "$y = -2x^2$" must be moved 3 units to the right to obtain the graph of "$y = -2(x - 3)^2$".

   (b) The graph of "$y = -2x^2$" must be moved 1 unit to the left and 3 units upward to obtain the graph of "$y = -2(x + 1)^2 + 3$".

8. $c = -3$

9. (a) $k = 0$

   (b) $k > 0$

Chapter 19

FUNCTIONS

19-1. The Function Idea.

This chapter deals with one of the most important and most basic ideas in mathematics--the idea of a function. The teacher can find additional discussion of this concept in Studies in Mathematics, Volume III, pages 6.17-6.25.

In the past it has been customary to postpone a careful study of functions to a much more advanced mathematical setting. Because of this, the subject is surrounded by an aura of difficulty which is completely undeserved. The idea is simple and, as will become evident as we proceed, is involved implicitly in our most elementary considerations. In this respect the function concept is in the same category as the set concept.

The point of departure is an initial set, which we have called D in anticipation of the notion of domain. From the beginning we wish to establish in the student's mind that a function embraces three aspects, that is, two sets and a rule of association. In effect, however, the second set, or range, is completely predetermined by the rule and the domain.

Throughout, the student should be made aware of the great variety of ways in which a function may be described. He should resist the impulse to infer that function is just another name for equation.

Answers to Oral Exercises 19-1a; pages 845-846:

1. {7, 8, 9}
2. {3, 6, 9}
3. {1, 4, 9}
4. {2, 5, 10}
5. {4, 7, 12}
6. {3, 5, 7}
7. {1, 0, 9}
8. {7}
9. {2, 3, 4}
10. {1, 8, 27}

Answers to Problem Set 19-1a; pages 846-848:

1. (a) {-1, -2, -3}
   (b) {2, [illegible], 6}
   (c) {1, [illegible], 3}
   (d) {6, 5, 4}
   (e) $\{1, \frac{1}{2}, \frac{1}{3}\}$
   (f) {-1, 2, 5}

2. (a) {1, 2, 3, 4, ...}
   (b) {1, 2, 3, 4, ...}
   (c) {1, 2, 3, 4, ...}

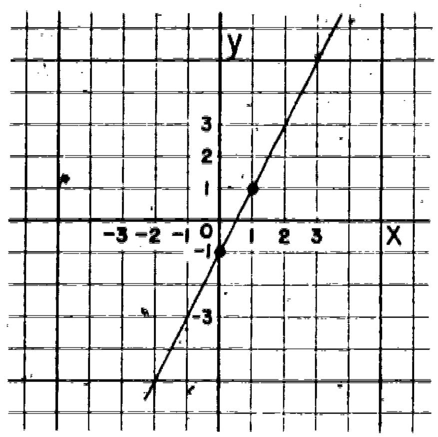
3. (a) {7}
   (b) {1, -1}
   (c) {-1, 0, 1}

4. (a) Domain: the set of counting numbers
   Range: the set of positive odd integers
   Missing numbers: 11, 13, 15, 17, 19, ...
   25 is associated with 13.
   1999 is associated with 1000.
   The rule is "with each counting number associate one less than twice the number".

   (b) Domain: the set of positive real numbers
   Range: the set of real numbers which are greater than -1
   33
   It does not associate any numbers with 0 and -1.

   (c) Domain: the set of real numbers
   Range: the set of real numbers
   -27; 45; 1999

(d) Domain: the set of real numbers
Range: the set of real numbers
-3; -2; 9; -11; 25



(e) Domain: the set of real numbers which are greater than -1 and less than 1
Range: the set of real numbers which are greater than -3 and less than 1
0; -2; $\frac{1}{3}$; $-\frac{7}{3}$; no number, 2 is not in the domain

(f) Domain: the set of negative real numbers
Range: the set of real numbers which are less than -1
-17; -27; no number, 0 is not in the domain

Pages 849-850.

The definition of function which we have given is, strictly speaking, a definition of real function since we have restricted the domain and range to real numbers. In later courses, the student will meet more general types of functions in which the domain and range can be sets other than sets of real numbers. Such a function might, for example, have sets of points in the plane as its domain of definition. As an illustration, associate with each point (x, y) of the plane the abscissa x of the point. In this case the domain is the set of all points of the plane and the range is the set of all real numbers. If we associate with each point (x, y) of the plane the point (-x, y) the result is a function with both domain and range equal to the set of all points in the plane.

In the discussion about functions, it is important to emphasize at every opportunity the following points:

(1) To each number in the domain of definition, the function assigns one and only one number. In other words we do not have "multiple-valued" functions. However, the same number can be assigned to many different elements of the domain.

(2) The essential idea of function is found in the actual association from numbers in the domain to numbers in the range and not in the particular way in which the association happens to be described.

(3) Always speak of the association as being from the domain to the range. This helps fix the correct idea that we are dealing with an ordered pairing of numbers in which the number from the domain is mentioned first and the assigned number from the range is mentioned second.

(4) Not all functions can be represented by algebraic expressions.

Although the above points are not absolutely vital as far as elementary work with functions is concerned, they become of central importance later. Also, many of the difficulties which students have with the idea of function can be traced to confusion on these matters. Therefore, it becomes important to make certain that the student understands these points from his very first contact with the function concept.

The definition of a function as a set of ordered pairs appears frequently in the literature of mathematics, and is regarded with favor by many mathematicians. In this text the inclusion of the rule of association as a part of the function concept has seemed preferable for many reasons. However, the discussion at the end of Section 19-1b should give the student some preparation for the alternative definitions he may encounter in subsequent study.

Answers to Oral Exercises 19-1b; page 851:

1. Domain: {1, 2, 3}
   Range: {5, 10, 15}

2. {2, 3, 5}; {4, 9, 25}

3. {4, 9, 16, 25}; {2, 3, 4, 5}

4. {8, 9, 10, 11, 12}; {62, 65, 70, 71}

5. Domain: the set of all real numbers
   Range: the set of all real numbers

*6. {1, 2, 3, 4}; {2, 4, 6, 8}

7. {0, 1, 2}; {0, 3, 6}

8. {2, $\sqrt{2}$, $\sqrt[3]{2}$}; {7, $4\sqrt{2} - 1$, $4\sqrt[3]{2} - 1$}

9. Domain: the set of real numbers
   Range: the set of real numbers

10. {10, 11, 12, 13, 14, ... 98, 99}
    {1, 2, 3, 4, ... 18}

Answers to Problem Set 19-1b; pages 852-853:

1. (a) {1, 2, 3, 4, 5, ...}; {0, 1, 2, 3, 4}

   (b) Domain: the set of positive real numbers
   Range: the set of real numbers which are greater than $\frac{2}{3}$
   The rule of this function is described by $\frac{1}{3}(n + 2)$.

   (c) {1, 2, 3, ..., 12}
   {28, 30, 31}

   (d) Domain: the set of real numbers
   Range: the set of real numbers
   Every real number is twice some real number.

(e) $\{\frac{1}{4}, \frac{1}{8}, \frac{3}{4}, \frac{1}{16}, \frac{1234}{10000}\}$
{2, 3, 4}

(f) {1,2,3,4,5,6,7,8,9,10}
{2,3,5,7,11,13,17,19,23,29}

(g) {1, 2, 3, ... 12}
{0, 1}

2. (a) (i)

x	1	2	3	4	5	6	...
3x	3	6	9	12	15	18	...

(ii) 3x

Domain: the counting numbers
Range: the positive multiples of 3

(b) (i)

x	1	2	3	4	5	6	7	8	9
x + 5	6	7	8	9	10	11	12	13	14

(ii) x + 5

Domain: the first nine counting numbers
Range: the set of integers which are greater than 5 and less than 15

(c) (i)

x	1	2	3	4	5	6	7
3x + 2	5	8	11	14	17	20	23

(ii) 3x + 2

Domain: the first seven counting numbers
Range: {5, 8, 11, 14, 17, 20, 23}

(d) (i)

x	1	2	3	4	...
x + 1	2	3	4	5	...

(ii) x + 1

Domain: the positive integers
Range: the set of positive integers which are greater than 1

(e) (i)

x	1	2	3	4	...
-x	-1	-2	-3	-4	...

(ii) -x

Domain: the positive integers
Range: the negative integers

(f) (i)

x	1	4	9	16	25	36	...
$\sqrt{x}$	1	2	3	4	5	6	...

(ii) $\sqrt{x}$

Domain: the set of integers which are squares of positive integers
Range: the positive integers

3.

2	3	4	5	6	7	8	9	10	11
2	3	2	5	2	7	2	3	2	11

2, 3, 5, 7, 11
Note that these are prime numbers.

4. 1, 2, 2, 2, 3
Yes

5. (a) $x \neq 3$
(b) $x \geq 1$
(c) $x \neq 0$
(d) all x
(e) the set of all real numbers whose absolute value is greater than or equal to 1
(f) $x \neq 2$ and $x \neq -2$

6. The sets in parts (a), (b), (d), and (f) define functions.

29?

19-2. The Function Notation.

The function notation must be handled with great care. In the beginning one should do a great many examples and exercises of the type, "What is the value of f at 2?" or "What number is represented by f(2)?" Check the students on this at every opportunity. It is essential for everyone to understand that the symbol "f", when used for a function, stands for the complete function and not just for the rule or some special way of representing the function. Notice that we have avoided using the misleading expression "the function f(x)" as a substitute for the correct expression "the function f". In other words, f(x) is a number which is the value of f at x, not the function itself.

If two variables x and y are related by the sentence y = f(x), where f is a given function, then x is sometimes called the independent variable and y the dependent variable in the relation. This terminology, which is used by many, has been avoided here since it is so easily abused. It leads to expressions such as "y is a function of x" which tend to obscure the function concept.

Students may experience some difficulty at first with functions defined in several parts. Attention should be called to the fact that the rule may differ depending on which part of the domain the value of the variable is chosen from, but that the function is to be thought of as a single entity.

It is well to accustom students to the shortened form of description, as this is in very general use.

Again we adopt a convention similar to that in the study of open sentences. If an open phrase is used to describe a function and no mention is made of a particular domain, then we assume the implied domain to be the set of all real numbers for which the phrase, or expression, has meaning.

pages 856-858: 19-2

When a restricted domain is specified, the student should be alert to the fact that a function will have no meaning at a point outside the domain, even through it may be possible to compute a value. For example, if the function f is defined for positive integers only then $f(.5)$ and $f(\frac{3}{4})$ and $f(-6)$ have no meaning.

Answers to Oral Exercises 19-2; page 857:

1. 6, -4, 0, 1, 2a, 4a
2. The answer each time is 3.
3. 0, 3, $\frac{4}{5}$, 0
4. No, they are not the same function since their domains are different. Two functions are the same if and only if they have the same rule and the same domain.
5. Yes, f and g are the same function. The choice of letter for the number in the domain is arbitrary.

Answers to Problem Set 19-2; pages 858-859:

1. 2, $\frac{3}{2}$, $-\frac{1}{2}$, 1, a + 1

   The domain is the set of all real numbers.

2. (a) 3
   (b) -1
   (c) $2\frac{1}{4}$
   (d) $\frac{1}{2}$
   (e) 2
   (f) 5
   (g) -1
   (h) $2 - \frac{a}{2}$
   (i) $2 - \frac{a}{4}$
   (j) $2 - a$
   (k) $2 - \frac{1}{2a}$

3. (a) the set of real numbers
   (b) the set of non-negative real numbers
   (c) 3
   (d) 3
   (e) 0
   (f) 0
   (g) 0.
   (h) 1
   (i) 25

4. (a) $x + 1$
   (b) $\{1, 2, 3, 4, \ldots\}$
   (c) -1 and $\frac{1}{2}$ are not elements of the domain of g and hence $g(-1)$ and $g(\frac{1}{2})$ are meaningless.
   (d) No, since the domains are different

5. It is the same function.

6. (a) $2 - \frac{x}{2} = -1$
   Sol'n set: $\{6\}$
   (b) all real numbers greater than 4
   (c) $\{5\}$
   (d) $\{\frac{4}{3}\}$
   (e) $x < 0$
   (f) $x \geq 2$

7. (a) $D = \{1, 2, 3, 4, 5\}$
   (b) $R = \{2, 3, 4, 5, 6\}$
   (c) $f(1) = 2$, $f(4) = 5$, $f(6)$ is not defined, $-f(2) = -3$, $f(3) + 1 = 5$
   (d) $f(x) = x + 1$, where x is an integer such that $1 \leq x \leq 5$

19-3. Graphs of Functions.

The graph of a function gives us a quick way to picture certain properties of the function. In most cases we are primarily interested in the "shape" of the graph rather than in the precise location of individual points, although there may be certain key points which need to be located carefully and doubt about the shape of a portion of the graph can frequently be resolved by locating a few judiciously chosen points. Other kinds of information can also be helpful in determining the general shape of a graph. For example, without locating any points whatsoever, we know that the graph of $y = 3x^2 + 1$ must be above the line $y = 1$ since $3x^2 \geq 0$ for all x. Also, since $0 < a < b$ implies $0 < a^2 < b^2$, we know that the graph of $y = 3x^2 + 1$ rises to the right. It must be impressed on the student that the objective in drawing a graph is not to locate a lot of points but rather to discover the shape of the graph by any methods which can be applied. The location of certain carefully chosen points of the graph is one of the methods.

Example 1. The graphs of the function f defined by:

$$f(x) = 2x - 1, \quad 0 \leq x < 2$$

and the function F defined by:

$$F(x) = 2x - 1, \quad -2 < x < 2$$

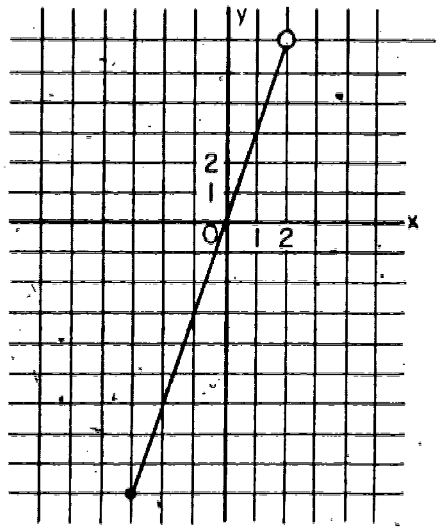
are different since the first graph is the line segment joining the points $(0, -1)$ and $(2, 3)$ with the first point included and the second excluded, and the second graph is the line-segment joining the points $(-2, -5)$ and $(2, 3)$ with both end-points excluded.

As indicated in the text this may well be the first time that the students have encountered a graph of a line in the plane with end points determined by a specified domain. In previous graphs of sentences in two variables emphasis has been placed on the infinite extent of such graphs. Hence attention must be called to this new idea. Open circles and blackened dots will be familiar from graphs on the real number line.

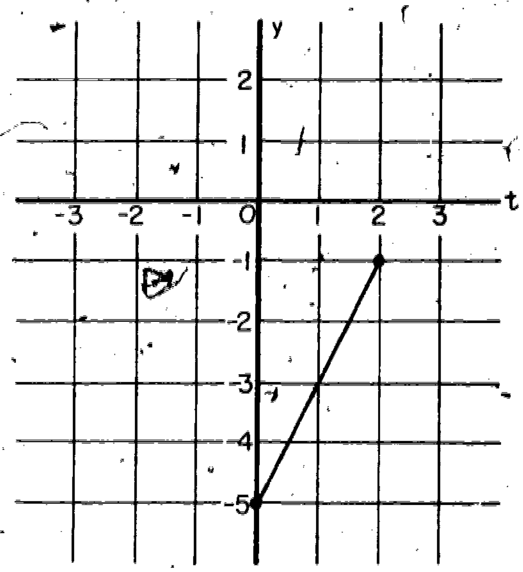
page 861: 19-3

Answers to Problem Set 19-3a; pages 861-862:

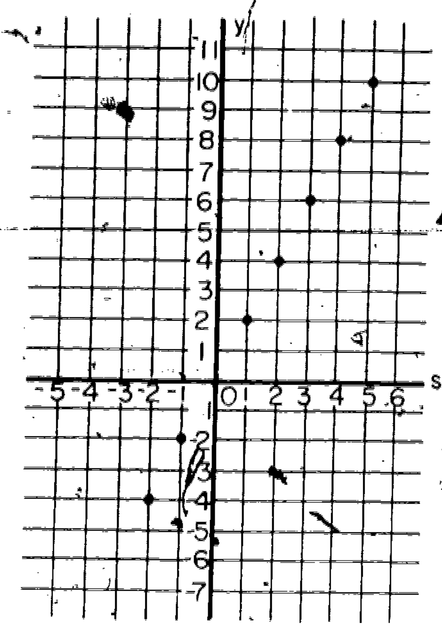
1. (a)



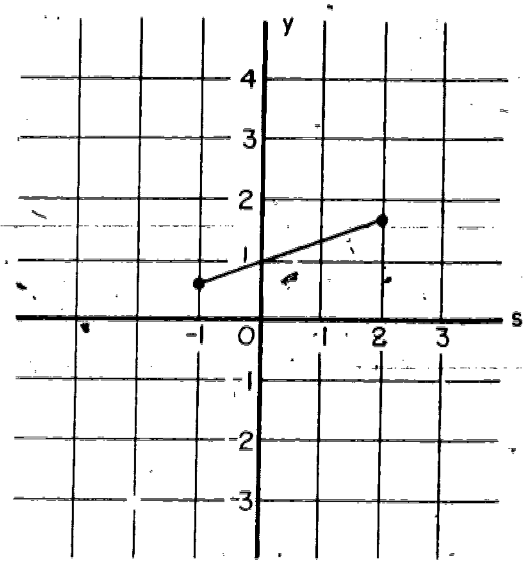
(b)



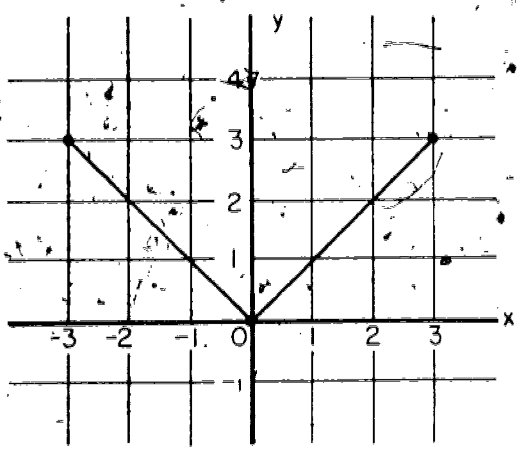
(c)



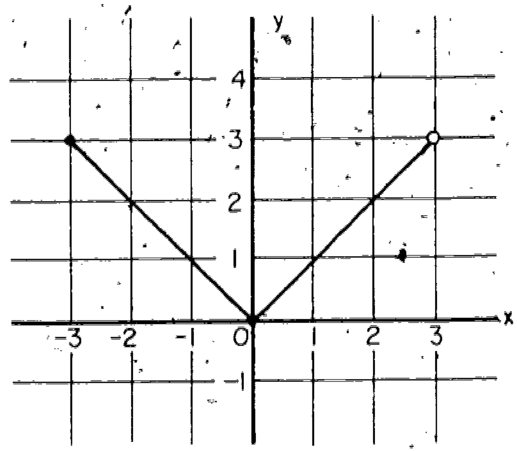
(d)



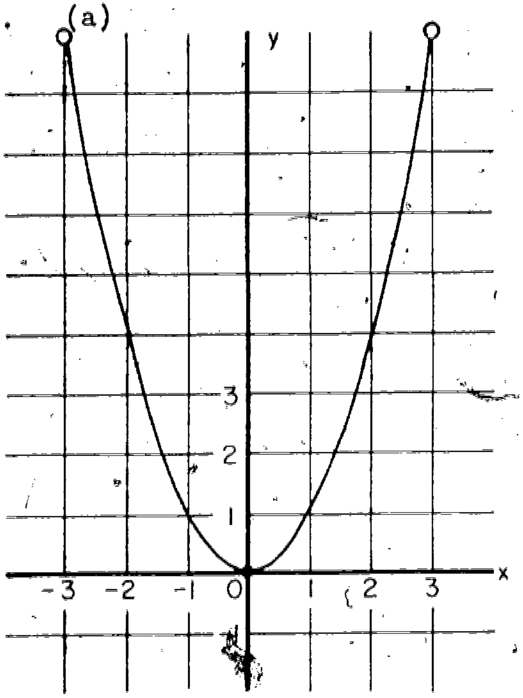
(e)


y
x
4
3
2
1
-1
-3 -2 -1 0 1 2 3

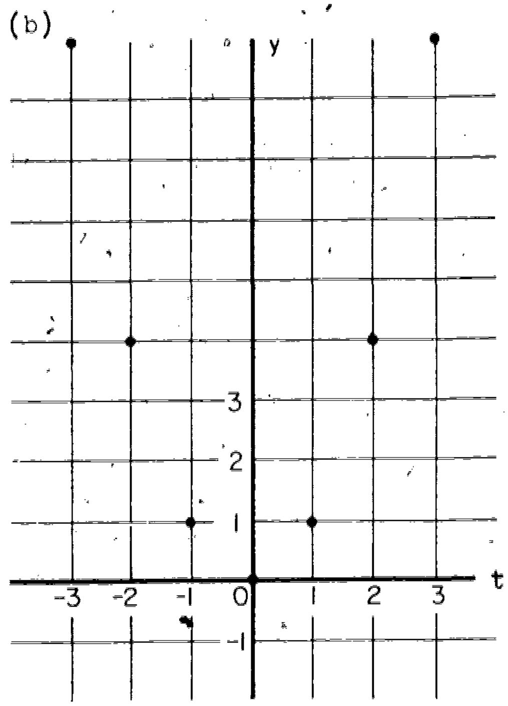
(f)


y
x
4
3
2
1
-1
-3 -2 -1 0 1 2 3

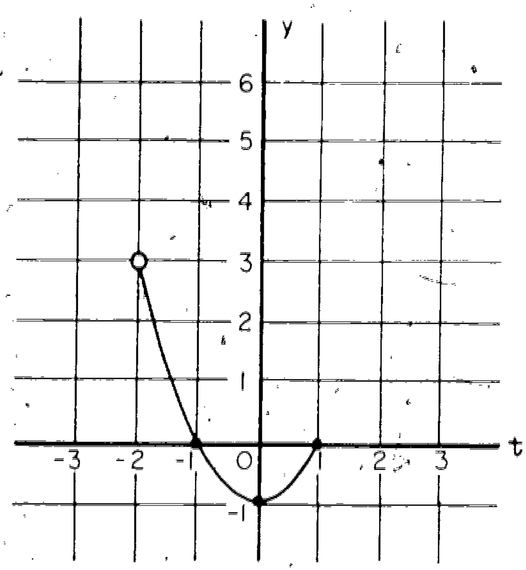
2. (a)


y
x
3
2
1
-3 -2 -1 0 1 2 3

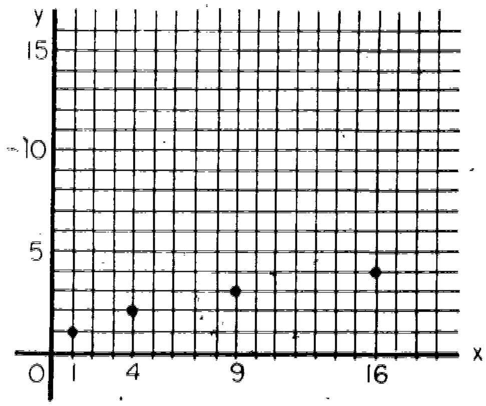
(b)


y
t
3
2
1
-1
-3 -2 -1 0 1 2 3

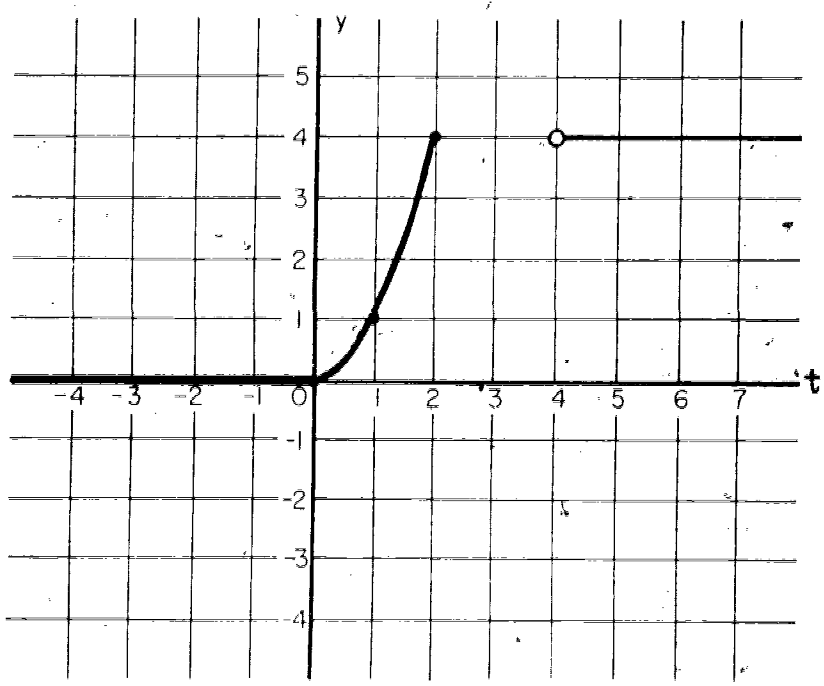
(c)


y
6
5
4
3
2
1
-3 -2 -1 0 1 2 3
-1
t

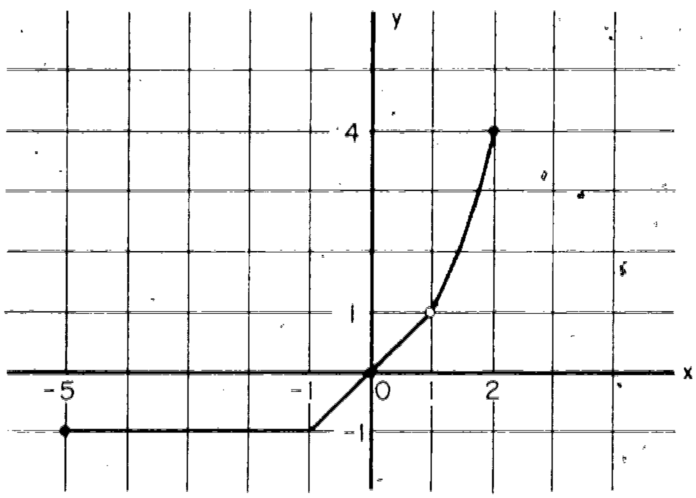
(d)


y
15
10
5
0 1 4 9 16
x

(e)


y
5
4
3
2
1
-4 -3 -2 -1 0 1 2 3 4 5 6 7
t
-1
-2
-3
-4

(f)



3. (a) Domain: the set of real numbers equal to or greater than -3 and less than 2

Range: the set of real numbers greater than or equal to -9 and less than 6

(b) Domain: the set of real numbers greater than or equal to 0 and less than or equal to 2

Range: the set of real numbers greater than or equal to -5 and less than or equal to -1

(c) Domain: the set of integers greater than -3 and less than or equal to 5

Range: the set of integers greater than -6 and equal to or less than 10

(d) Domain: the set of real numbers greater than or equal to -1 and equal to or less than 2

Range: the set of real numbers greater than or equal to $\frac{2}{3}$ and less than or equal to $1\frac{2}{3}$

(e) Domain: the set of real numbers greater than or equal to -3 and less than or equal to 3
Range: the set of real numbers greater than or equal to zero and less than or equal to 3

(f) Domain: the set of real numbers greater than or equal to -3 and less than 3
Range: the set of real numbers greater than or equal to zero and less than or equal to 3

4. (a) Domain: the set of real numbers greater than -3 and less than 3
Range: the set of real numbers greater than or equal to 0 and less than 9

(b) Domain: {-3, -2, -1, 0, 1, 2, 3}
Range: {0, 1, 4, 9}

(c) Domain: the set of real numbers greater than -2 and less than or equal to 1
Range: the set of real numbers greater than or equal to -1 and less than 3

(d) Domain: {1, 4, 9, 16}
Range: {1, 2, 3, 4}

(e) Domain: the set of all real numbers less than or equal to 2 or greater than 4
Range: the set of all real numbers equal to or greater than 0 and equal to or less than 4

(f) Domain: the set of all real numbers greater than or equal to -5 and less than 1, greater than 1 and less than or equal to 2
Range: the set of all real numbers greater than or equal to -1 and less than 1, greater than 1 and less than or equal to 4

Pages 862-863.

In defining a function stress was placed on the assignment to each element in the domain of exactly one number. Thus, as previously stated, we exclude double, and multiple valued functions which at one time were included in the general category of functions. The graph provides a very convenient means of determining whether or not a set of points exhibits more than one ordinate for each abscissa. We can apply the test of intersection by any vertical line in more than one point as a way of screening out associations which are not functions.

Answers to Problem Set 19-3b; pages 864-865:

1. The figures in (a), (e), (h), and (i) are the only ones which are graphs of functions.

    (b) Each element in the domain has been assigned two different numbers.

    (c) The element 1 in the domain has been assigned more than one number.

    (d) Some elements in the domain have been assigned two different numbers.

    (f) Some elements in the domain have been assigned two different numbers.

    (g) Some elements in the domain have been assigned more than one number.

    (j) Some elements in the domain have been assigned two different numbers.

    (k) Some elements in the domain have been assigned more than one number.

    (l) Some elements in the domain have been assigned two different numbers.

pages 865-867: 19-3

2. (a) $\frac{7}{4}$, 0, -2

   (b) the set of real numbers greater than or equal to -4 and less than or equal to 5

   (c) the set of real numbers greater than or equal to -3 and less than or equal to 4

3. (a) Find the ordinate that corresponds to each abscissa x in the domain.

   (b) Each element in the domain has a corresponding point on the graph of G.

   (c) G is the graph of the function g. If a = c, then b ≠ d, since (a, b) and (c, d) are two distinct points. But this would mean that the functions assigns two different numbers, b and d, to the number a (or c) in the domain. This cannot be. Therefore a ≠ c.

Answers to Review Problem Set; pages 866-877:

1. The sets in (b) and (e) cannot define functions because some elements in their domains have been assigned more than one number.

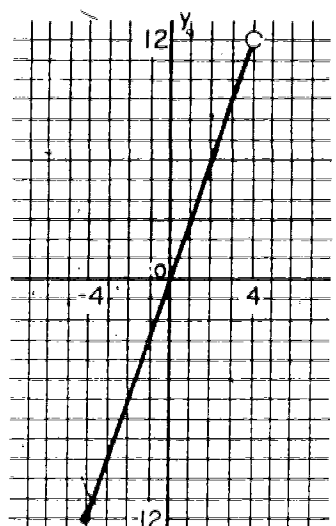
2. (a) (0, 0)

   (b) 3

   (c) -3, 0, $\frac{2}{2}$, 12

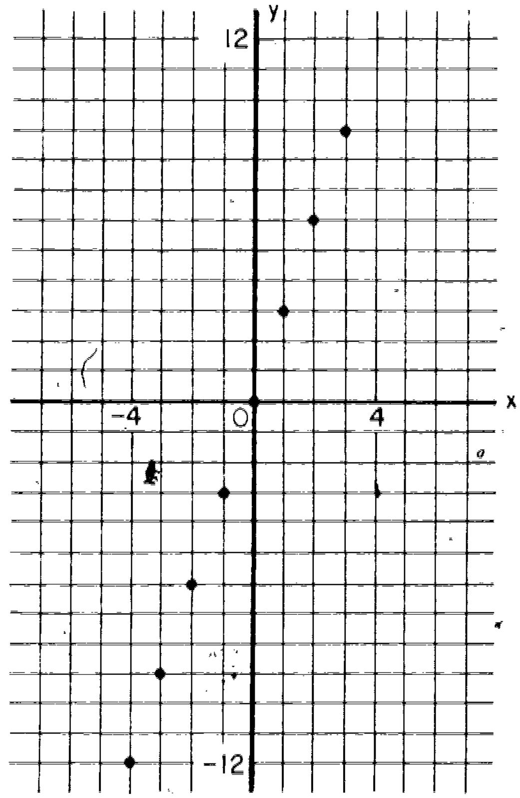
   (d) The graph of "y = 3x + 2" is obtained by moving the graph of "y = 3x" up 2 units.

   (e) The graph of "3y = 9x" is the same as the graph of "y = 3x".



(f)

(f)



g is not the same function as f.

3. (a) the set of real numbers

(b) the set of real numbers

(c) (-3), 3, 0, (-1.75), $(-\sqrt{2})$, (-x)

(d) (1, -1), (2, -2), (3, -3), (4, -4), (5, -5)

(e) (2, -2)

(f) (-a, -b)

4. (a) Each ordinate of the graph of "$y = -x^2$" is the opposite of the corresponding ordinate of the graph of "$y = x^2$".

(b) The graph of "$y = x^2 + 2$" can be obtained by moving the graph of "$y = x^2$" 2 units upward.

(c) The graph of "$y = (x + 1)^2$" can be obtained by moving the graph of "$y = x^2$" 1 unit to the left.

(d) Exactly one number is assigned to each number in the domain.

(e) The set of real numbers greater than or equal to 0 and less than or equal to 9

(f) 0, 2, $\frac{9}{4}$. Since 4 is not in the domain, h(4) is undefined.

5. (a) {(3, -2)}
   (b) {(-1, -1)}
   (c) {(5, -1)}

6. (a) Since the graph of "x - y + 2 = 0" is not the same as the graph of "x + y - 4 = 0", the graphs intersect in only one point.

   (b) Since (1, 3) satisfies x - y + 2 = 0 and also x + y - 4 = 0, it satisfies (x - y + 2) + (x + y - 4) = 0.

   (c) Yes, the graph of each is the same point.

7. (a) $x \neq 1$ and $x \neq 0$

   (b) 20x(x - 1)

   (c) Both sides of the given open sentence have been multiplied by 20x(x - 1) which is a non-zero number for every x in the domain.

   (d) {5, $\frac{12}{17}$}

8. (a) The domain of definition is the set of all positive real numbers.

   (i)

positive real number a	$\frac{1}{2}$	1	$\sqrt{2}$	$\frac{2}{3}$	2	3	4	. . .
$\frac{1}{3}(a + 2)$	$\frac{5}{6}$	1	$\frac{1}{3}(\sqrt{2} + 2)$	$\frac{8}{9}$	$\frac{4}{3}$	$\frac{5}{3}$	2	. . .

   (ii) To each positive real number a associate the number $\frac{1}{3}(a + 2)$.

(b) The domain of definition is the set of all positive real numbers.

(i)

positive integer n	1	2	3	4	5	6	7	8	9	10	...
n^{th} prime	2	3	5	7	11	13	17	19	23	29	...

(ii) No function is known whose domain of definition is the positive integers, whose range is a set of primes, and whose rule is an algebraic expression.

(c) Domain is the set of all positive integers less than or equal to the number of dollars which you possess.

(i)

number of dollars invested	1	2	3	...	20	...	100	...
interest at 6%	0.06	0.12	0.18	...	1.20	...	6.00	...

(ii) If x is the number of dollars invested, associate $0.06x$.

(d) Domain is the set of all positive real numbers.

(i)

diameter in inches	$\frac{1}{4}$	$\frac{1}{\pi}$	$\frac{1}{2}$	$\frac{2}{\pi}$	$\sqrt{2}$	2
circumference in inches	$\frac{\pi}{4}$	1	$\frac{\pi}{2}$	2	$\sqrt{2}\pi$	2π

(ii) If x is the diameter, the circumference is πx.

9. If his average speed going was g miles per hour, then his average speed returning was $g - 6$ miles per hour.

$$\frac{336}{g - 6} = \frac{336}{g} + 1, \quad g > 6$$

Truth set: $\{48\}$

The average speed was 48 miles per hour going and 42 miles per hour returning.

10. If x is the number,

$$x + \frac{1}{x} = 4 \text{ and } x \neq 0.$$

Truth set: $\{2 + \sqrt{3},\ 2 - \sqrt{3}\}$

The number is either $2 + \sqrt{3}$ or $2 - \sqrt{3}$.

11. Notice that if the perimeter of a rectangle is equal to 10 ft. then the length of a side must be less than 5 ft.

(a) The domain of definition here is the set of all positive integers between 0 and the number of dollars which you possess. (Some might want to think of borrowing as negative investment, in which case the domain would include negative integers depending on your credit.)

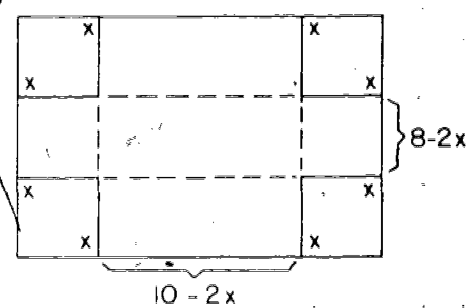
(b) Since the area of a triangle is equal to one-half the base x times the altitude a, we must have $\frac{1}{2}xa = 12$ or $a = \frac{24}{x}$. In this case the base can have any length whatsoever except 0. Therefore the domain is the set of all positive real numbers.

(c) The bottom of the box has dimensions $8 - 2x$ by $10 - 2x$. Hence the volume is given by

$$(8 - 2x)(10 - 2x)x$$

or

$$4x(4 - x)(5 - x).$$



Obviously the square which is removed must have its side less than 4, so that $0 < x < 4$. In other words the domain of definition is the set of all real numbers between 0 and 4.

12. If $x < 0$, then $|x| = -x$, so that $\frac{x}{|x|} = -1$.
If $x > 0$, then $|x| = x$, so that $\frac{x}{|x|} = 1$.
Therefore

$$k(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

and hence, k and g are the same functions.

13. (a) $H(2) = 3$ (b) $H(\frac{1}{3}) = -\frac{8}{9}$ (c) $H(-\frac{1}{3}) = -\frac{8}{9}$

(d) $-H(-2) = -3$ (e) $H(-1) + 1 = 1$ (f) $H(3)$ is not defined.

(g) $H(a) = a^2 - 1$, for $-3 < a < 3$.

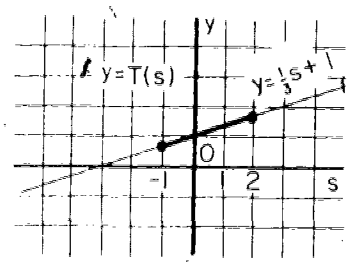
(h) $H(t - 1) = t^2 - 2t$ for $-2 < t < 4$. Notice that,

if $-2 < t < 4$, then $-3 < t - 1 < 3$,

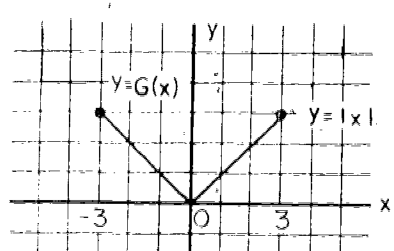
so that $H(t - 1)$ is defined (i.e. $t - 1$ is in the domain of definition of H.)

(i) $H(t) - 1 = t^2 - 2$, for $-3 < t < 3$.

14. (a)

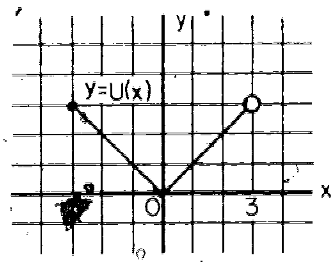


(b)

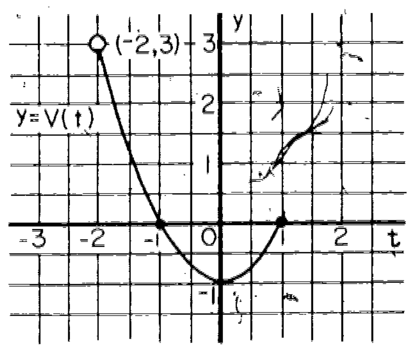


31

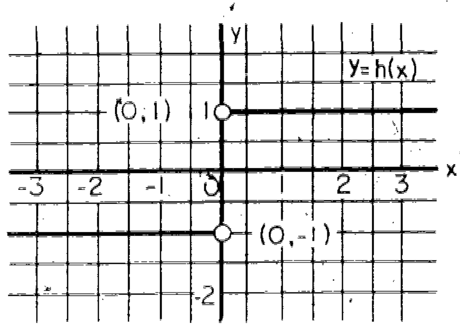
(c)



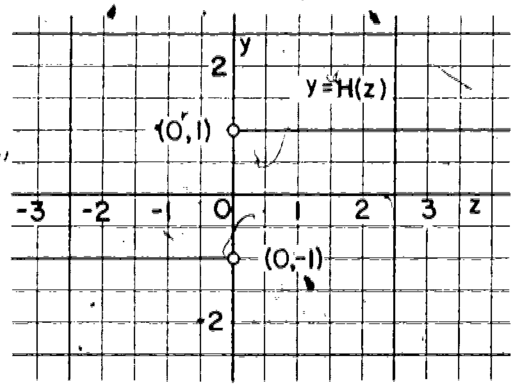
(d)



(e)



(f)



15. (a) Domain: all s such that $-1 \leq s \leq 2$
Range: all y such that $\frac{2}{3} \leq y \leq \frac{5}{3}$

(b) Domain: all x such that $-3 \leq x \leq 3$
Range: all y such that $0 \leq y \leq 3$

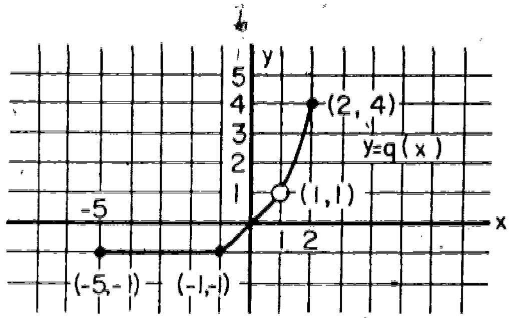
(c) Domain: all x such that $-3 \leq x < 3$
Range: all y such that $0 \leq y \leq 3$

(d) Domain: all t such that $-2 < t \leq 1$
Range: all y such that $-1 \leq y < 3$

(e) Domain: all non-zero real numbers
Range: $\{-1, 1\}$

(f) Same function as in (e)

16.

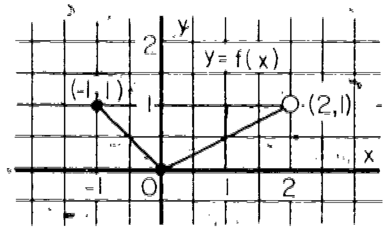


17. To each number x such that $-2 \leq x \leq 4$ assign the number $-\frac{1}{2}x + 1$.

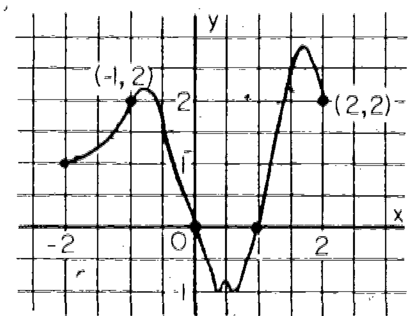
18. Domain: all x such that $-1 \le x < 2$

Range: all y such that $0 \le y \le 1$

$$f(x) = \begin{cases} -x, & -1 \le x \le 0 \\ \frac{1}{2}x, & 0 < x < 2 \end{cases}$$



19. There are many functions which satisfy all of these conditions. One example is:



20.

$$f(x) = g(x)$$

$$x^2 + x = 2x + 6$$

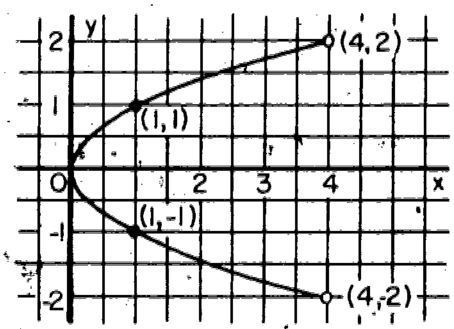
$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$\{3, -2\}$$

Thus, if x is 3 or -2, $f(x) = g(x)$.

21. This is not the graph of a function.



22. (a) The graph is a horizontal line, $y = B$.

(b) The graph is the x-axis.

(c) Since the points $(-3, 0)$ and $(1, 2)$ are on the graph, we must have

$$\begin{cases} 0 = A\cdot(-3) + B \\ 2 = A\cdot 1 + B \end{cases}$$

This is a system of equations in the unknowns A and B. The solution is $A = \frac{1}{2}$, and $B = \frac{3}{2}$. The problem can also be solved by obtaining the equation of the line determined by the points $(-3, 0)$, $(1, 2)$ and then writing it in the y-form.

(d) The domain of definition is the set of all real numbers x such that $-3 \le x \le 1$.

(e) $$\begin{cases} 1 = A\cdot(-1) + B \\ 3 = A\cdot 3 + B \end{cases}$$

Solving for A and B, we obtain $A = \frac{1}{2}$ and $B = \frac{3}{2}$. Again, we could obtain A and B by writing the equation of the line determined by the points $(-1, 1)$, $(3, 3)$ in the y-form. Note that the function in (e) is different from that in (c).

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(f) The slope is $\frac{1}{2}$ and the y-intercept is $(0, \frac{3}{2})$.

(g) The domain of definition is the set of all real x such that $-1 < x < 3$.

23. The equation of the line L is $x + 2y + 1 = 0$. When $y = 2$, $x = -5$, and when $y = -2$, $x = 3$. Therefore

$$h(x) = -\frac{1}{2}x - \frac{1}{2}, \quad -5 < x < 3.$$

24. The expressions in (a) and (e) are the only ones that define a function whose graph is a line.

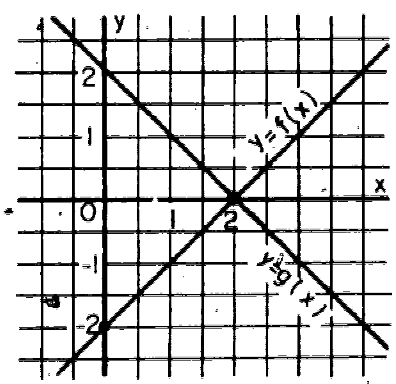
25. (a) $g(x) = -f(x)$ (d) $g(x) = f(|x|)$

(b) $g(x) = |f(x)|$ (e) $g(x) = f(-x)$

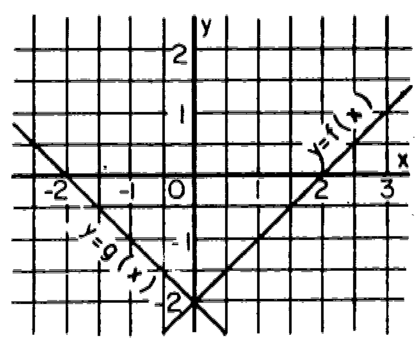
(c) $g(x) = \frac{1}{f(x)}$ (f) $g(x) = f(x^2)$

26. (a) The graph of g is obtained by rotating the graph of f one-half revolution about the x-axis.

(a)



(e)

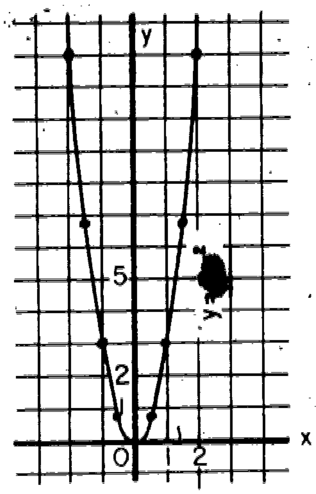


(e) The graph of g is obtained by rotating the graph of f one-half revolution about the y-axis.

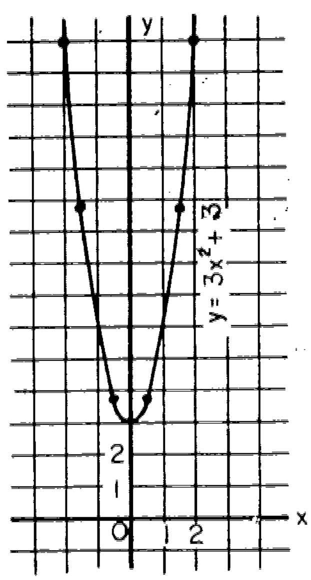
27. The graph of the equation $(y - F(x))(y - G(x)) = 0$ is the set of all points which are on either the graph of F or the graph of G.

28. (a)

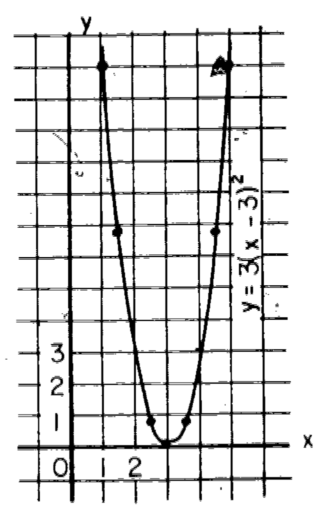
x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	12	$\frac{27}{4}$	3	$\frac{3}{4}$	0	$\frac{3}{4}$	3	$\frac{27}{4}$	12



(b)



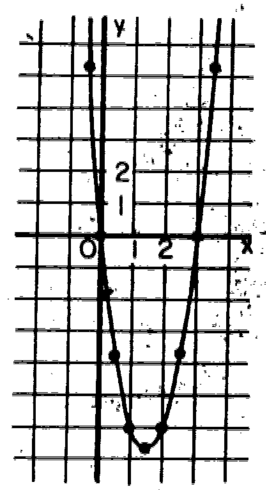
(c)



(d) $3x(x - 3) = 3x^2 - 9x$

$= 3(x^2 - 3x + \frac{9}{4}) - \frac{27}{4}$

$= 3(x - \frac{3}{2})^2 - \frac{27}{4}$



(e) The graph of (d) can be obtained by moving the graph of (a) $\frac{3}{2}$ units to the right and $\frac{27}{4}$ units down.

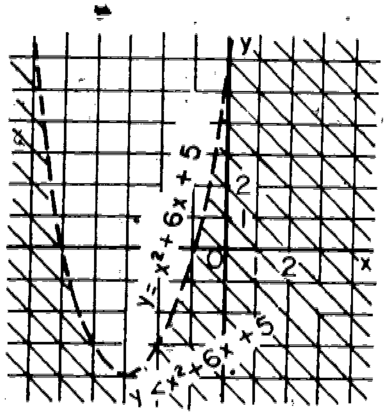
29. (a) $y = -x^2$

(b) $y = x^2 + 3$

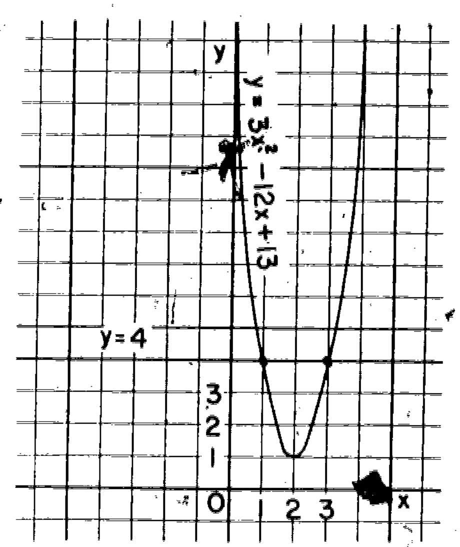
(c) $y = (x + 2)^2$

(d) $y = (x - 1)^2 - 2$

30. (a)



(b)



The graph is the two points (1, 4) and (3, 4).

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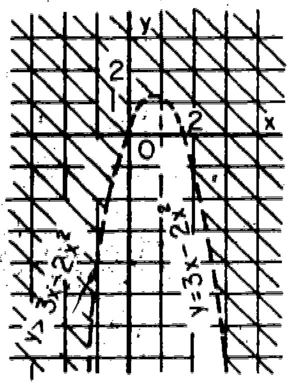
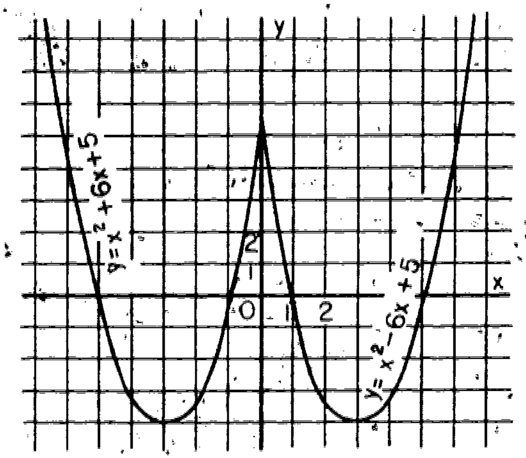
(c)



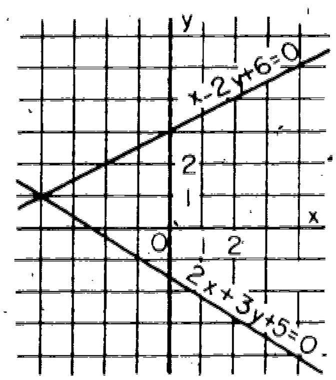
(d)



(d) We recall here that $y = x^2 - 6|x| + 5$ implies:

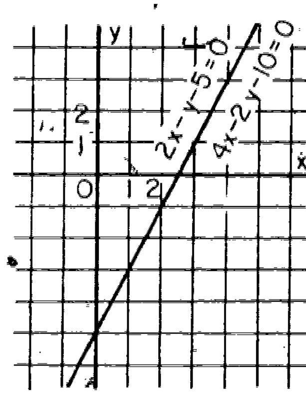
$$y = \begin{cases} x^2 - 6x + 5, & x \geq 0 \\ x^2 + 6x + 5, & x < 0. \end{cases}$$

31. (a)



Truth set is $\{(-4, 1)\}$

(b)



Truth set consists of all (x, y) which satisfy the open sentence $2x - y - 5 = 0$.

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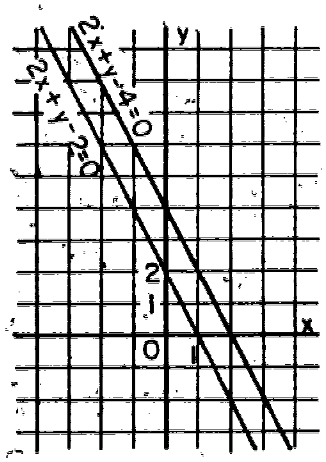
(c)



Truth set is $\emptyset$.
The lines are parallel.

32. If the graph of $Ax + By + C = 0$ is to contain the origin, then C must be equal to zero. Since $a(5x - 7y - 3) + b(3x - 6y + 5) = 0$ represents any line passing through the intersection of the lines whose equations are $5x - 7y - 3 = 0$ and $3x - 6y + 5 = 0$, we must choose a and b, not both 0, so that $-3a + 5b = 0$. One obvious choice is $a = 5$ and $b = 3$. Therefore, our compound sentence now becomes

$$5(5x - 7y - 3) + 3(3x - 6y + 5) = 0.$$

Therefore, $34x - 53y = 0$ is the required equation.

33. (a) $\{(7, -2)\}$
(b) $\{(\frac{1}{2}, 1)\}$
(c) $\{(\frac{7}{10}, \frac{4}{5})\}$
(d) $\emptyset$
(e) $\{(3, -4)\}$
(f) $\{(0, 0)\}$
(g) $\{(\frac{1}{2}, \frac{1}{3})\}$
(h) $\{(1, \frac{1}{2})\}$
(i) $\{(6, 1)\}$
(j) $\emptyset$

34. If the numbers are x and y, then

$$\begin{cases} x + y = 56 \\ x - y = 18 \end{cases}$$

Truth set: $\{(37, 19)\}$

The numbers are 37 and 19.

This problem can also be solved by using only one variable as follows:

If x is the larger number, then the smaller number is $56 - x$, and

$$x - (56 - x) = 18$$

The truth set of this sentence is $\{37\}$.

Hence the numbers are 37 and 19.

35. If Sally is x years old and Joe is y years old, then

$$\begin{cases} x + y = 30 \\ x - y = 4 \end{cases} \quad \text{OR} \quad \begin{cases} x + y = 30 \\ y - x = 4 \end{cases}$$

Notice that we do not know which is the older, so the problem can be answered in two ways. Notice also that the information "In five years" is irrelevant since the difference in their ages is the same now as it is at any time.

Truth set: $\{(17, 13)\}$ or $\{(13, 17)\}$

Sally is 17 years old and Joe is 13 or Joe is 17 years old and Sally is 13.

This problem can be set up in four different ways using one variable:

$\begin{cases} \text{Sally } n \text{ years old} \\ \text{Joe } 30-n \text{ years old} \end{cases}$ $\begin{cases} \text{Sally } n \text{ years old} \\ \text{Joe } (n+4) \text{ or } (n-4) \text{ years old} \end{cases}$

$\begin{cases} \text{Joe } n \text{ years old} \\ \text{Sally } 30-n \text{ years old} \end{cases}$ $\begin{cases} \text{Joe } n \text{ years old} \\ \text{Sally } (n+4) \text{ or } (n-4) \text{ years old} \end{cases}$

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36. If he uses a pounds of almonds and c pounds of cashews, then

$$\begin{cases} a + c = 200 \\ 1.50a + 1.20c = 1.32(200) \end{cases}$$

Truth set: $\{(80, 120)\}$

He should use 80 pounds of almonds and 120 pounds of cashews.

37. If the tens' digit is t and the units' digit is u, then

$$\begin{cases} u = 2t + 1 \\ (10t + u) + u = 3t + 35 \end{cases}$$

Truth set: $\{(3, 7)\}$

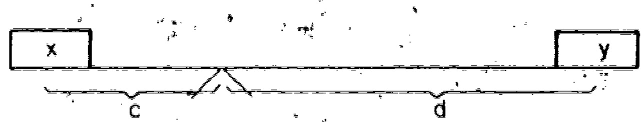
The number is 37.

Using one variable, the problem can be worked as follows:

If t is the tens' digit, then
$2t + 1$ is the units' digit, and

$$(10t + 2t + 1) + 2t + 1 = 3t + 35$$

38. From a principle in physics we know that the lever shown below



balances if $cx = dy$ where c and d measure the distances from the fulcrum, or point of balance, and x and y measure the weights of the objects on the lever.

If Hugh is h feet from the point of balance and Fred is f feet from the point of balance, then

$$\begin{cases} f + h = 9 \\ 100f = 80h \end{cases}$$

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The truth set: $\{(4, 5)\}$

Hugh is 5 feet, Fred is 4 feet from the point of balance.

39. If one boy weighs a pounds and the other boy weighs b pounds, then

$$\begin{cases} a + b = 209 \\ 5a = 6b \end{cases}$$

Truth set: $\{(114, 95)\}$

One boy weighs 114 pounds, the other 95 pounds.

40. If the speed of the current is c m.p.h. and the speed of the boat in still water is b m.p.h., then

$$\begin{cases} \frac{3}{2}(b + c) = 12 \\ 6(b - c) = 12 \end{cases}$$

Truth set: $\{(5, 3)\}$

The speed of the current is 3 m.p.h. and the speed of the boat in still water is 5 m.p.h.

This problem is not easily done with one variable.

41. If apples cost a cents per pound, and bananas cost b cents per pound, then

$$\begin{cases} 3a + 4b = 108 \\ 4a + 3b = 102 \end{cases}$$

Truth set: $\{(12, 18)\}$

Apples are 12 cents per pound and bananas are 18 cents per pound.

42. If A walks at a miles per hour, and B walks at b miles per hour, then

in 60 hours, A walks $60a$ miles and B walks $60b$ miles;

in 5 hours, A walks $5a$ miles and B walks $5b$ miles.

$$\begin{cases} 60a = 60b + 30 \\ 5a + 5b = 30 \end{cases}$$

The truth set: $\{(\frac{13}{4}, \frac{11}{4})\}$

A walks at $3\frac{1}{4}$ miles per hour.

B walks at $2\frac{3}{4}$ miles per hour.

43. If there are x quarts of the 90% solution and y quarts of the 75% solution, then there are $.90x$ quarts of alcohol in the 90% solution and $.75y$ quarts of alcohol in the 75% solution. Furthermore, there are $.78(20)$ quarts of alcohol in the mixed solution.

$$\begin{cases} .90x + .75y = .78(20) \\ x + y = 20 \end{cases}$$

Truth set: $\{(4,16)\}$

He should use 4 quarts of the 90% solution.

44. If the average speed of car A is a miles per hour and the average speed of car B is b miles per hour, then

$$\begin{cases} \frac{300}{a} = \frac{275}{b} - \frac{1}{2} \\ \frac{300}{a} = \frac{240}{b} + \frac{1}{5} \end{cases}$$

Truth set: $\{(60,50)\}$

A's average speed was 60 m.p.h. and

B's average speed was 50 m.p.h.

45. Suppose there were x tickets sold at 25 cents, and y tickets sold at 75 cents.

Open sentence: $x + y = 311$ and $25x + 75y = 10875$.

Truth set: $\{(249,62)\}$ Hence, there were 249 pupils and 62 adult tickets sold.

46. If there are x girls, then Elsie has $x - 1$ sisters. Hence, there must be $x - 1$ boys. Similarly if there are y boys, then Jimmie has $y - 1$ brothers and $2(y - 1)$ sisters.

Open sentence: $x - 1 = y$ and $2(y - 1) = x$.

Truth set: $\{(4, 3)\}$
Hence, there are 4 girls and 3 boys in the family.

47. Suppose there were x three cent stamps purchased, and y four cent stamps.

Open sentence: $x + y = 352$ and $3x + 4y = 1267$.

Truth set: $\{(141, 211)\}$
Hence, there were 141 three cent and 211 four cent stamps purchased.

48. Suppose there were x one dollar bills and y five dollar bills.

Open sentence: $x + y = 154$ and $x + 5y = 465$.

Truth set: $\{(76\frac{1}{4}, 77\frac{3}{4})\}$
He has not counted correctly, since the number of bills must be an integer.

49. If the number of feet in the width is x and the number of feet in the perimeter is 94, then the number of feet in the length is $47 - x$. The number of square feet in the area is $x(47 - x)$.

$$x(47 - x) = 496$$

$$x^2 - 47x + 496 = 0$$

$$(x^2 - 47x + \frac{2209}{4}) + 496 - \frac{2209}{4} = 0$$

$$(x - \frac{47}{2})^2 - \frac{225}{4} = 0$$

$$(x - \frac{47}{2} + \frac{15}{2})(x - \frac{47}{2} - \frac{15}{2}) = 0$$

$$(x - 16)(x - 31) = 0$$

$$x - 16 = 0 \quad \text{or} \quad x - 31 = 0$$

$$x = 16 \quad \text{or} \quad x = 31$$

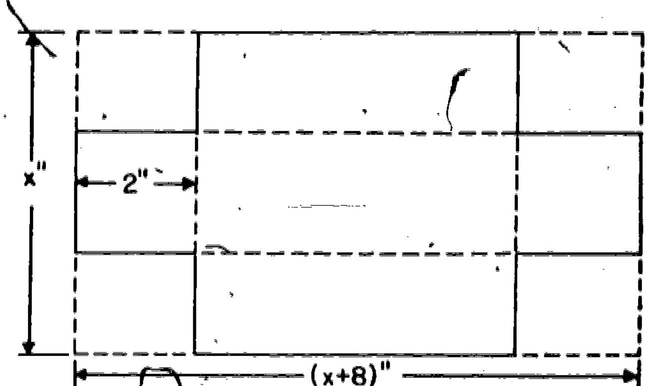
If we require that the width x is less than the length, then we get

$$x = 16, \qquad 47 - x = 31.$$

The dimensions are 16 feet and 31 feet.

50. If the sheet of metal is x inches wide, it is $x + 8$ inches long.

The box is $x - 4$ inches wide, $x + 8 - 4$ inches long, and 2 inches deep.



$$2(x - 4)(x + 4) = 256; \qquad \text{Domain: } x > 0$$

$$2(x^2 - 16) = 256$$

$$x^2 - 16 = 128$$

$$x^2 - 144 = 0$$

$$(x + 12)(x - 12) = 0$$

$$x + 12 = 0 \text{ or } x - 12 = 0$$

$$x = -12 \text{ or } x = 12$$

are all equivalent. Hence the truth set of the sentence is $\{12\}$. -12 is not in the domain of x.

The sheet of metal is 12 inches wide and 20 inches long.

51. If one leg is y feet long, the other leg is $y - 1$ feet long, and the hypotenuse is $y + 8$ feet long.

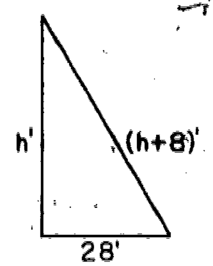
$$y^2 + (y - 1)^2 = (y + 8)^2 \qquad \text{Domain: } y > 0$$

Truth set: $\{21\}$

The lengths of the sides of the triangle are 20 feet, 21 feet, and 29 feet.

52. If the window is h feet above the ground, the rope is $h + 8$ feet long.

$h^2 + 28^2 = (h + 8)^2$ Domain: $h > 0$.



Truth set: $\{45\}$

The window is 45 feet above the ground.

53. If a leg of the triangle is x feet long.

$$x^2 + x^2 = 3^2 \quad \text{Domain: } x > 0$$

Truth set: $\{\frac{3\sqrt{2}}{2}\}$

Each leg is $\frac{3\sqrt{2}}{2}$ units long.

54. If the diagonal is d inches long, a side is $d - 2$ inches long.

$$(d - 2)^2 + (d - 2)^2 = d^2 \quad \text{Domain: } d > 0$$

Truth set: $\{4 + 2\sqrt{2}\}$

The diagonal is $4 + 2\sqrt{2}$ inches long.

55. If the sheet is t feet long, the width is $t - 3$ feet.

$$t(t - 3) = 46\tfrac{3}{4} \quad \text{Domain: } t > 0$$

Truth set: $\{\frac{17}{2}\}$

The sheet is $8\frac{1}{2}$ feet long.

56. If one of the numbers is n, the other is $9 - n$, and their product is $n(9 - n)$.

$$\begin{aligned} n(9 - n) &= 9n - n^2 \\ &= -(n^2 - 9n + \tfrac{81}{4}) + \tfrac{81}{4} \\ &= -(n - \tfrac{9}{2})^2 + \tfrac{81}{4} \end{aligned}$$

The vertex of the parabola is $(\frac{9}{2}, \frac{81}{4})$ and it opens downward. Hence, the value of n which makes $n(9 - n)$ largest is $\frac{9}{2}$.

The numbers are $\frac{9}{2}$ and $\frac{9}{2}$.

57.
$$\begin{aligned} c &= n^2 - 10n + 175 \\ &= (n^2 - 10n + 25) + 150 \\ &= (n - 5)^2 + 150 \end{aligned}$$

The vertex of the parabola is at $(5, 150)$.
He should manufacture 5 boats a day; this will result in a minimum cost of $150 per boat.

58. If n is one of the numbers, $9 - n$ is the other number.

$$n^2 - (9 - n)^2 = 25$$

Truth set: $\{\frac{53}{9}\}$

The numbers are $\frac{53}{9}$ and $\frac{28}{9}$.

59. If n is the number

$$14n + n^2 = 11.$$

Truth set: $\{-7 + 2\sqrt{15}, \; -7 - 2\sqrt{15}\}$

The number is $(-7 + 2\sqrt{15})$ or $(-7 - 2\sqrt{15})$.

Suggested Test Items

1. The function f assigns to each positive integer x, the number $2x$.

   (a) What is the domain of f?

   (b) What is the range of f?

   (c) What numbers are represented by the following symbols:

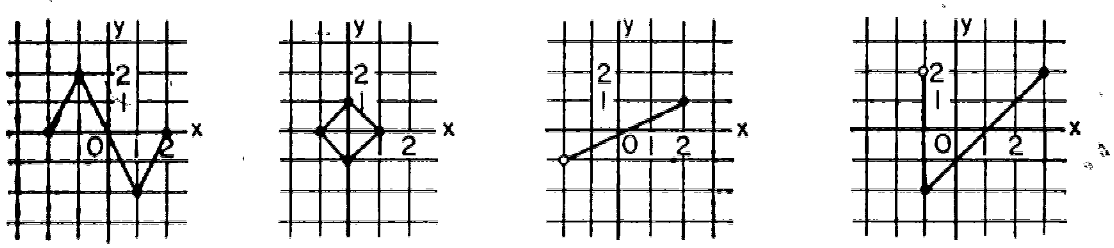
   $f(3)$, $f(100)$, $f(-1)$, $f(x)$, $-f(2)$, $3 \cdot f(4)$, $f(1) + 1$

   (d) Draw the graph of the function f for $x < 6$.

(e) For what x is it true that $4 < f(x) < 8$?

(f) If g is a function defined by $g(t) = 2t$ for some domain, under what conditions is g the <u>same</u> function as f?

2. Examine the following graphs:



A B C D

(a) Which of the above figures is <u>not</u> the graph of a function?

(b) Tell how you decide whether a graph is <u>not</u> the graph of a function.

3. (a) Draw a graph of the function h defined by

$$h(x) = \begin{cases} -x & -3 < x < 0 \\ 2x & 0 \leq x \leq 3 \end{cases}$$

(b) What is the domain of h?

(c) What is the range of h?

4. State the domain of definition and the range of each of the functions described by:

(a) $F(x) = |x|$

(b) $h(t) = \sqrt{t}$

(c) $H(x) = \frac{1}{x}$

5. F is a function which associates with each positive integer the remainder after dividing the integer by 3.

(a) What is the domain of F?

(b) What is the range of F?

(c) What is F(5), F(3), F(1), F(0)?

(d) Graph the function for the positive integers less than 7.

(e) Describe the set of positive integers a such that f(a) = 0.

(f) Describe the set of positive integers b such that f(3b + 1) = f(1).

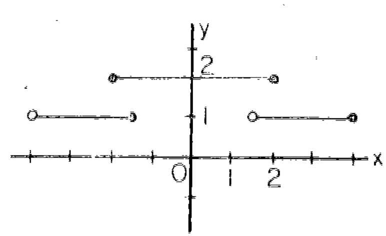
6. c is a number in the domain of a function H. What symbol represents the number H assigns to c?

7. Which of the following defines a function and which does not? Give a reason for your answer.

(a) The set {(3, 1), (2, 2), (1, 1), (3, 2)}

(b) The sentence: corresponding to each real number x there is a real number y such that $y^2 = x$.

(c) The graph



(d) The set {(1, 1), (2, 1), (3, 1), (4, 1)}

(e) The sentence: $y - x^2 = 0$ for all real numbers, x.

(f) The table which assigns to each score a grade

Score	100 - 95	94 - 89	88 - 80	79 - 70
Grade	5	4	3	2

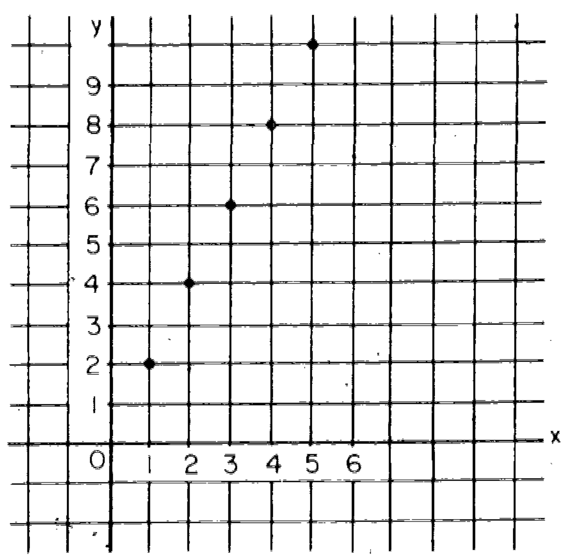
Answers to Suggested Test Items

1. (a) the set of positive integers

(b) the set of positive even integers

(c) $f(3) = 6$, $f(100) = 200$, $f(-1)$ is meaningless since -1 is not in the domain of f, $f(x) = 2x$, $-f(2) = -4$, $3 \cdot f(4) = 24$, $f(1) + 1 = 3$

(d)



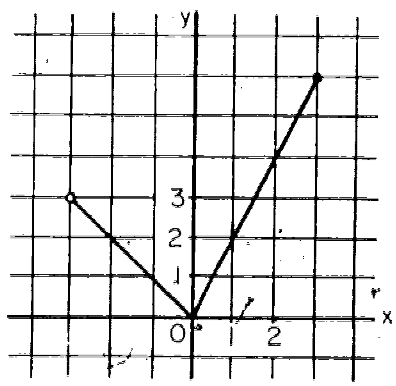
(e) $x = 3$

(f) The domain of g must be the same as the domain of f, that is, the set of positive integers.

2. (a) B and D

(b) If the intersection of the graph and any vertical line is more than one point, then the graph is not the graph of a function.

3. (a)



(b) the set of real numbers x such that $-3 < x \leq 3$

(c) the set of real numbers y such that $0 \leq y \leq 6$

4. (a) Domain: the set of real numbers
Range: the set of non-negative reals

(b) Domain: the set of non-negative real numbers
Range: the non-negative reals

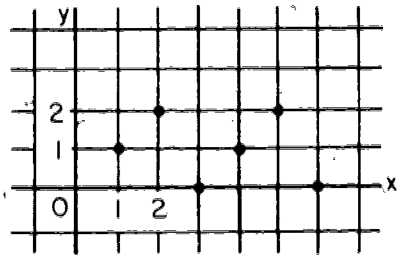
(c) Domain: the set of real numbers except 0
Range: the reals except zero

5. (a) the set of positive integers

(b) the integers 0, 1, 2

(c) $F(5) = 2$, $F(3) = 0$, $F(1) = 1$, $F(0)$ is meaningless since 0 is not in the domain of F

(d)



(e) The set of positive multiples of 3

(f) The set of positive integers

6. H(c)

7. (a) This set does not define a function since it assigns two numbers to 3.

(b) This sentence does not define a function since it assigns two numbers to each number in the domain (except zero).

(c) This graph does not represent a function since there is a vertical line which will intersect the graph in more than one point, for example, the vertical line $x = 2$.

(d) This set of ordered pairs defines a function since to each number in the domain the number 1 is assigned.

(e) The sentence defines a function, since one number is assigned to each number in the domain.

(f) The sentence defines a function, since one number is assigned to each number in the domain.

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Answers to Challenge Problems; pages 878-885:

1. Each point is moved to a point with the same ordinate, but whose abscissa is the opposite of the abscissa of the original point.

(a) (2, 1) goes to (-2, 1)
(2,-1) goes to (-2,-1)
($-\frac{1}{2}$, 2) goes to ($\frac{1}{2}$, 2)
(-1,-1) goes to (1,-1)
(3, 0) goes to (-3, 0)
(-5, 0) goes to (5, 0)
(0, 2) goes to (0, 2)
(0,-2) goes to (0,-2)

(b) (-2, 1) goes to (2, 1)
(-2,-1) goes to (2,-1)
($\frac{1}{2}$, 2) goes to ($-\frac{1}{2}$, 2)
(1,-1) goes to (-1,-1)
(-3, 0) goes to (3, 0)
(5, 0) goes to (-5, 0)
(0, 2) goes to (0, 2)
(0,-2) goes to (0,-2)

(c) (c,-d) goes to (-c,-d)

(d) (-c, d) goes to (c, d)

(e) (-c, d) goes to (c, d)

(f) The points on the y-axis go to themselves.

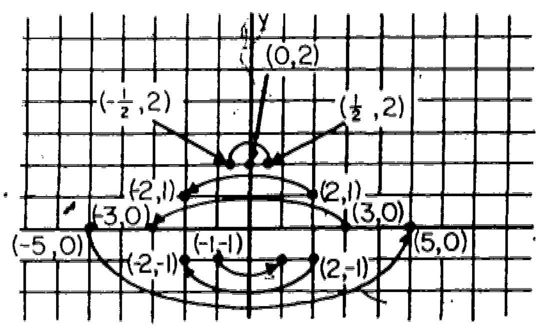


Figure for Problem 1.

2. (a) (1, 1) goes to (3, 1)
(-1, 1) goes to (1, 1)
(-2, 2) goes to (0, 2)
(0,-3) goes to (2,-3)
(3, 0) goes to (5, 0)

(b) (-1, 1) goes to (1, 1)
(-3, 1) goes to (-1, 1)
(-4, 2) goes to (-2, 2)
(-2,-3) goes to (0,-3)
(1, 0) goes to (3, 0)

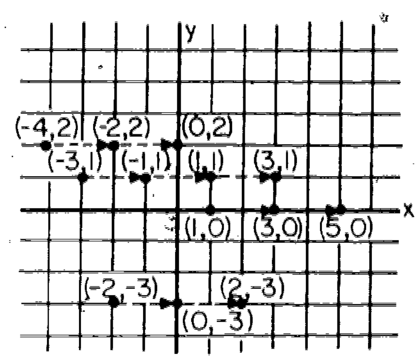


Figure for Problem 2.

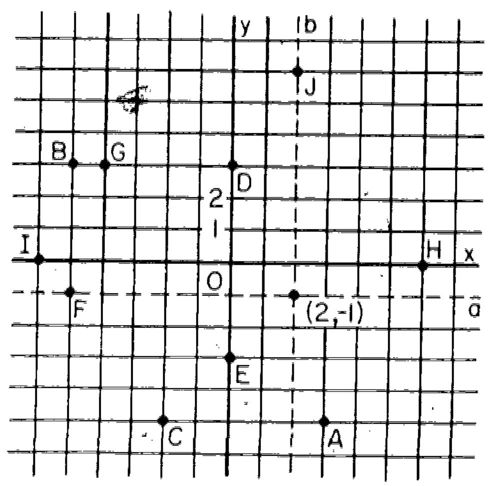
(c) $(c - 2, d)$ goes to (c, d)

(d) $(-c - 2, d)$ goes to $(-c, d)$

(e) No points go to themselves.

3.

	(a, b)
A	$(1, -4)$
B	$(-7, 4)$
C	$(-4, -4)$
D	$(-2, 4)$
E	$(-2, -2)$
F	$(-7, 0)$
G	$(-6, 4)$
H	$(4, 1)$
I	$(-8, 1)$
J	$(0, 7)$



4. L_1: $y = -\frac{3}{2}x - 7$; $b = -\frac{3}{2}a$

L_2: $y = 2x - 5$; $b = 2a - 5$

L_3: $y = -\frac{1}{2}x$; $b = -\frac{1}{2}a + 5$

L_4: $y = -9$; $b = -5$

L_5: $|x - 5| = 1$; $|a - 7| = 1$

5. Each point of the plane is moved to a point having the same abscissa as before, and the ordinate of the new point is the opposite of the ordinate of the original point. This amounts to rotating the points of the plane one-half revolution about the x-axis.

(a) (2, 1) goes to (2,-1)
(2,-1) goes to (2, 1)
$(-\frac{1}{2}, 2)$ goes to $(-\frac{1}{2},-2)$
(-2,-3) goes to (-2, 3)
(3, 0) goes to (3, 0)
(-5, 0) goes to (-5, 0)
(0, 5) goes to (0,-5)
(0,-5) goes to (0, 5)

(b) (2,-1) goes to (2, 1)
(2, 1) goes to (2,-1)
$(-\frac{1}{2},-2)$ goes to $(-\frac{1}{2}, 2)$
(-2, 3) goes to (-2,-3)
(3, 0) goes to (3, 0)
(-5, 0) goes to (-5, 0)
(0,-5) goes to (0, 5)
(0, 5) goes to (0,-5)

(c) (a,-b) goes to (a, b)

(d) (-a, b) goes to (-a,-b)

(e) (a,-b) goes to (a, b)

(f) All points on the x-axis go to themselves.

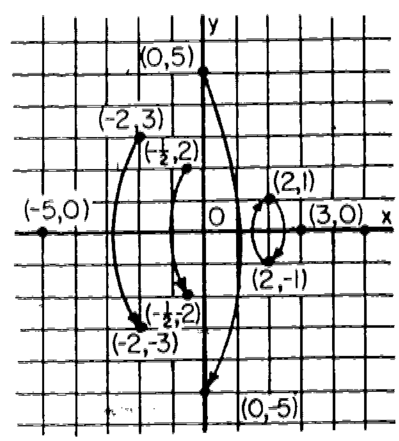


Figure for Problem 5.

6. The points of the plane move up two units and to the left three units.

(a) (1, 1) goes to (-2, 3)
(-1,-1) goes to (-4, 1)
(-2, 2) goes to (-5, 4)
(0,-3) goes to (-3,-1)
(3, 0) goes to (0, 2)

(b) (4,-1) goes to (1, 1)
(2,-3) goes to (-1,-1)
(1, 0) goes to (-2, 2)
(3,-5) goes to (0,-3)
(6,-2) goes to (3, 0)

(c) (a, b - 2) goes to (a - 3, b)

(d) (-a + 3, -b - 2) goes to (-a, -b)

(e) No point goes to itself

(f) Moving (a, b) to (a, b - 2) has the effect of sliding the points of the plane down two units.

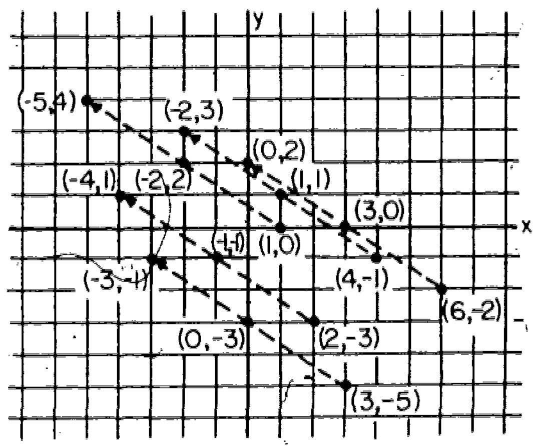


Figure for Problem 6

7. If $y = 0$, $|x| = 5$, which means that x can be 5 or -5. Similarly, (0,5) and (0,-5) satisfy the sentence.

x	-5	-3	-3	-1	-1	0	0	1	1	3	3	5
|x|	5	3	3	1	1	0	0	1	1	3	3	5
|y|	0	2	2	4	4	5	5	4	4	2	2	0
y	0	2	-2	4	-4	5	-5	4	-4	2	-2	0

The graph shown is the graph of $|x| + |y| = 5$, as well as the combined graphs of the four open sentences:

$x + y = 5$, and $0 \le x \le 5$
$x - y = 5$, and $0 \le x \le 5$
$-x + y = 5$, and $-5 \le x \le 0$
$-x - y = 5$, and $-5 < x < 0$

It was necessary in these to limit the values of x so that only the indicated segments of the lines would be included.

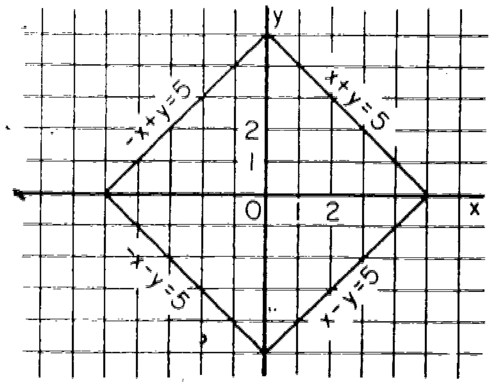


Figure for Problem 7

8. (a) The given sentence is equivalent to the following four-part compound sentence:
$x + y > 5$,
or $x - y > 5$,
or $-x + y > 5$,
or $-x - y > 5$.

The graphs of the corresponding equations: "$x + y = 5$ or $x - y = 5$ or etc." are then drawn with dotted lines. Now note that:

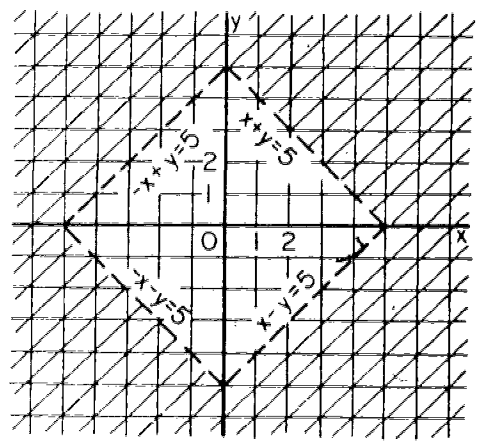


Figure for Problem 8(a)

$x + y > 5$ becomes $y > -x + 5$
$-x + y > 5$ becomes $y > x + 5$

So the area above each of the lines where "$x + y = 5$" and "$-x + y = 5$" is shaded.

Also: $x - y > 5$ becomes $y < x - 5$
$-x - y > 5$ becomes $y < -x - 5$

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So the area below each of the lines where "$x - y = 5$" and "$-x - y = 5$" is shaded.

Therefore the graph of $|x| + |y| > 5$ is all of the plane outside the graph of $|x| + |y| = 5$.

(b) In the same line of reasoning

$|x| + |y| < 5$ implies that:

$x + y < 5$,

and $x - y < 5$,

and $-x + y < 5$,

and $-x - y < 5$.

Hence, the graph is the area inside the graph of

$$|x| + |y| = 5.$$

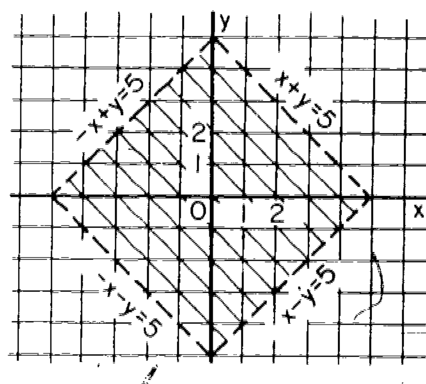


Figure for Problem 8(b).

Verify on the number line that "$|y| < k$" is equivalent to "$y < k$ and $-y < k$", whereas "$|y| > k$" is equivalent to "$y > k$ or $-y > k$".

(c) The graph is the same as that for (b), except that the lines are solid to indicate that the graph of

$$|x| + |y| = 5$$

is included, as well as the graph of $|x| + |y| < 5$.

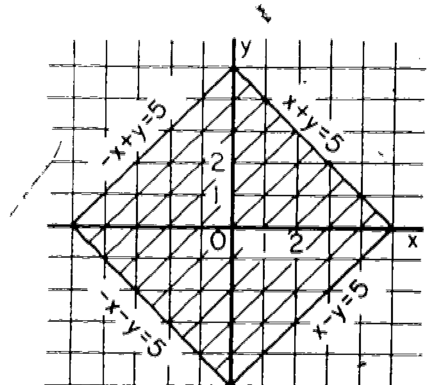


Figure for Problem 8(c)

9.

x	-7	-7	-6	-6	-4	-4	-3	-2	1	3	4	4	7	7
$\|x\|$	7	7	6	6	4	4	3	2	1	3	4	4	7	7
$\|y\|$	4	4	3	3	1	1	0	-1	-2	0	1	1	4	4
y	4	-4	3	-3	1	-1	0	impossible		0	1	-1	4	-4

The four open sentences whose graphs form the same figure are:

$x - y = 3$, and $x \geq 3$
$x + y = 3$, and $x \geq 3$
$-x + y = 3$, and $x \leq -3$
$-x - y = 3$, and $x \leq -3$

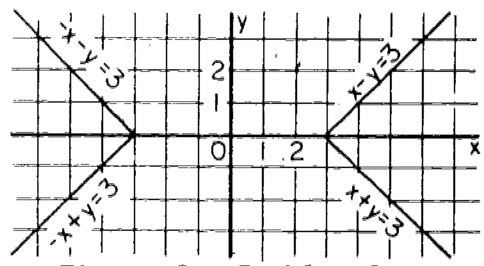


Figure for Problem 9.

10. (a) $y = -2|x|$
(b) $y = 2|x - 3|$
(c) $y = 2|x + 2|$
(d) $y = 2|x| + 5$
(e) $y = 2|x - 2| - 4$

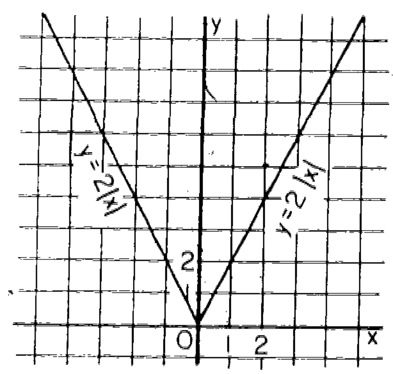


Figure for Problem 10.

11. The slope is $\frac{(-4) - 2}{3 - (-3)}$ or -1. The slope of the line containing $(-3, 2)$ and (x, y) is $\frac{y - 2}{x - (-3)}$. The slope of the line containing $(3, -4)$ and (x, y) is $\frac{y - (-4)}{x - 3}$. Since -1 and $\frac{y - 2}{x - (-3)}$ are names for the same number,

$$\frac{y - 2}{x - (-3)} = -1, \text{ provided } x \neq -3.$$

Then $y - 2 = (-1)(x + 3)$, by multiplying both sides by "$(x + 3)$", with the restriction that $x \neq -3$. Then

$$y = -x - 1.$$

Since -1 and $\frac{y - (-4)}{x - 3}$ are names for the same number

$$\frac{y - (-4)}{x - 3} = -1, \text{ provided } x \neq 3.$$

Then $y + 4 = (-1)(x - 3)$,

$y = -x - 1$.

12. (a) The slope is $\frac{2 - 3}{(-5) - 0}$ or $\frac{1}{5}$, and the y-intercept is $(0, 3)$, so the equation is "$y = \frac{1}{5}x + 3$".

(b) The slope is $\frac{(-4) - 8}{0 - 5}$ or $\frac{12}{5}$, and the y-intercept is $(0, -4)$, so the equation is "$y = \frac{12}{5}x - 4$".

(c) The slope is $\frac{-7 - (-2)}{-3 - 0}$ or $\frac{5}{3}$, and the y-intercept is $(0, -2)$, so the equation is "$y = \frac{5}{3}x - 2$".

(d) The slope is $\frac{6 - (-2)}{0 - 5}$ or $-\frac{8}{5}$, and the y-intercept is $(0, 6)$, so the equation is "$y = -\frac{8}{5}x + 6$".

(e) Two expressions for the slope are

$\frac{y - 3}{x - (-3)}$ and $\frac{2 - 3}{0 - (-3)} = -\frac{1}{3}$, if $x \neq -3$.

Then $\frac{y - 3}{x + 3} = -\frac{1}{3}$ and $y - 3 = -\frac{1}{3}(x + 3)$.

(f) Two expressions for the slope are

$\frac{y - 3}{x - (-3)}$ and $\frac{3 - 3}{-5 - (-3)} = 0$, if $x \neq -3$.

Then $\frac{y - 3}{x + 3} = 0$ and $y - 3 = 0$.

(g) The slope is $\frac{5 - 3}{-3 - (-3)}$ or $\frac{2}{0}$. But $\frac{2}{0}$ is not a number. Hence, the line is vertical. The vertical line through the point $(-3, 3)$ has the equation "$x + 3 = 0$".

(h) Two expressions for the slope are

$\frac{y-2}{x-4}$ and $\frac{2-1}{4-(-3)} = \frac{1}{7}$, if $x \neq 4$.

Then,

$$\frac{y-2}{x-4} = \frac{1}{7},$$

$$y - 2 = \frac{1}{7}(x - 4).$$

13. (a)

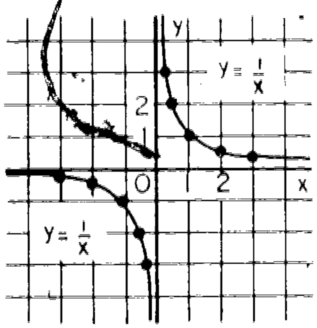


Figure for Problem 13(a)

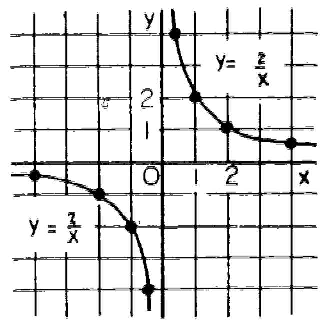


Figure for Problem 13(a)

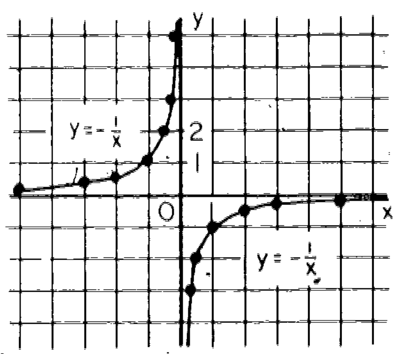


Figure for Problem 13(a)

(b)

If the variable x is given increasing positive values, then the values of $\frac{k}{x}$ decrease if $k > 0$ but they increase if $k < 0$.

14. (a) $\frac{25}{w}$, where $w > 0$

(b) This is inverse variation, and the constant of variation is 25.

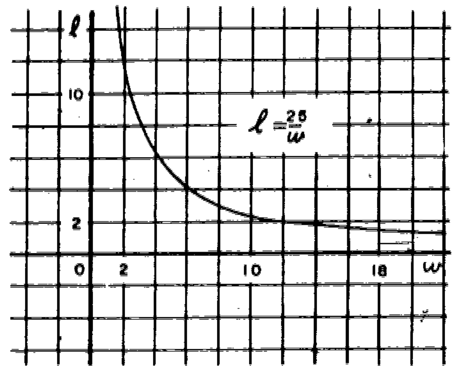
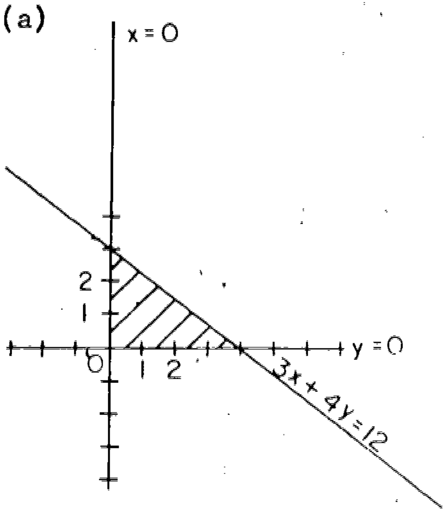


Figure for Problem 14

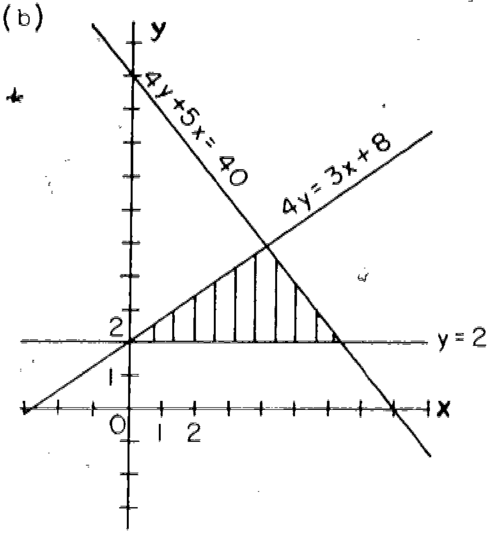
Points used include:

w	$1\frac{1}{4}$	2	4	5	$6\frac{1}{4}$	$12\frac{1}{2}$	20
ℓ	20	$12\frac{1}{2}$	$6\frac{1}{4}$	5	4	2	$1\frac{1}{4}$

15. (a)



(b)



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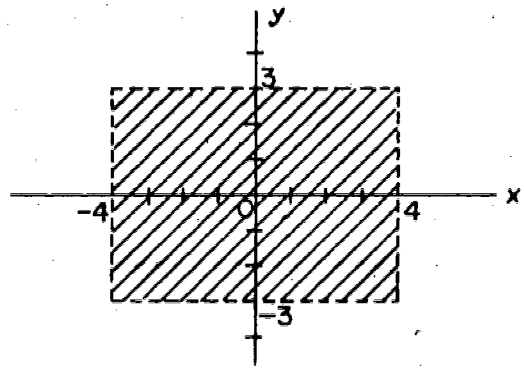
(c)



16. We know that the product of two positive or negative real numbers is positive. Since $x^2 = x \cdot x$, it follows that x^2 is a product of two positive numbers if $x > 0$, or of two negative numbers if $x < 0$; that is, $x^2 > 0$ for any real number $x \neq 0$. We know further that if $ab = 0$, where a and b are real numbers, then at least one of them is zero. Since $x^2 = x \cdot x = 0$, it follows that $x = 0$. Conversely, if $x = 0$, then $x^2 = x \cdot x = 0 \cdot 0 = 0$. Since points with positive ordinates are above the x-axis, it follows that the graph of $y = x^2$ has positive ordinates for all $x \neq 0$ and a single point $(0, 0)$, for $x = 0$, lies on the x-axis.

17. If (a, b) is a point on the graph, then $b = a^2$ is true. Since $b = (-a)^2 = a^2$, it follows that the open sentence is also true for the ordered pair $(-a, b)$; in other words, $(-a, b)$ is also on the graph.

18. If x is positive, multiplication of the members of "$x < 1$" by x yields $x^2 < x$. If, for the same value of x, the ordinate of $y = x^2$ is denoted by y_1 and the ordinate of $y = x$ by y_2, then for $0 < x < 1$ we have $y_1 < y_2$. In other words, the graph of $y = x^2$ lies below the graph of $y = x$.

19. Here, we obtain $x < x^2$ and $y_2 < y_1$. Hence, for $x > 1$, the graph of $y = x^2$ lies above the line $y = x$.

20. Multiplication of the members of "$a < b$" by a and b yields $a^2 < ab$ and $ab < b^2$, respectively. Hence, by the transitive property of order we obtain $a^2 < b^2$. If we denote $y_a = a^2$ and $y_b = b^2$, then by the above property for $b > a$ we obtain $y_b > y_a$ for all $b > a > 0$. Hence it follows that the graph of $y = x^2$ continues to rise as we move from 0 to the right.

21. The horizontal line $y = a$ where $a > 0$ (since the graph of $y = x^2$ is above the x-axis, a cannot be negative) and the graph of $y = x^2$ have equal ordinates at the points of intersection. Therefore, $x^2 = a$.

    Since $x^2 - a = (x - \sqrt{a})(x + \sqrt{a})$, $x^2 - a = 0$ has the truth set $\{\sqrt{a}, -\sqrt{a}\}$ for $a \neq 0$ and the truth set $\{0\}$ for $a = 0$. It follows that there can be at most two points of intersection.

22. Since the slope of a line containing the points (a, b) and (c, d) is $\frac{d - b}{c - a}$, $(c \neq a)$, we obtain easily for the points $(0, 0)$ and (a, a^2) the slope $\frac{a^2 - 0}{a - 0} = a$. Hence, we conclude that the slope of the line containing $(0, 0)$ and (a, a^2) approaches 0 as a approaches 0. But a line passing through $(0, 0)$ with a slope close to zero apparently approaches the x-axis, which touches the parabola at $(0, 0)$. If we note that the segment of the line between $(0, 0)$ and (a, a^2), for a close to 0, nearly coincides with the arc of the parabola, it is plausible that the graph must be flat near the origin.

23. $Ax^2 + Bx + C = A(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}) + C - \frac{B^2}{4A}$, $A \neq 0$

$$= A(x + \frac{B}{2A})^2 + \frac{4AC - B^2}{4A}$$

$$= a(x - b)^2 + k$$

where $a = A$, $h = -\frac{B}{2A}$, $k = \frac{4AC - B^2}{4A}$

24. (a) $a(x - h)^2 + k = ax^2 - 2ahx + (ah^2 + k) = 3x^2 - 7x + 5$

If $a = 3$, $-2ah = -7$, $ah^2 + k = 5$, then

$a = 3$ implies that $-2(3)h = -7$; i.e. $h = \frac{7}{6}$;

$a = 3$ and $h = \frac{7}{6}$ implies that $(3)(\frac{7}{6})^2 + k = 5$, i.e.

$k = \frac{11}{12}$.

Hence, $3x^2 - 7x + 5 = 3(x - \frac{7}{6})^2 + \frac{11}{12}$.

(b) $ax^2 - 2ahx + (ah^2 + k) = 5x^2 - 3x + \frac{13}{20}$

If $a = 5$, $-2ah = -3$, $ah^2 + k = \frac{13}{20}$, then

$a = 5$, $h = \frac{3}{10}$, $k = \frac{1}{5}$.

Hence, $5x^2 - 3x + \frac{13}{20} = 5(x - \frac{3}{10})^2 + \frac{1}{5}$

(c) $ax^2 - 2ahx + (ah^2 + k) = Ax^2 + Bx + C$, for every real number x. This is possible if

$a = A$, $-2ah = B$, and $ah^2 + k = C$.

If $a = A$, then the sentence "$-2ah = B$" is equivalent to "$-2Ah = B$," that is, $h = -\frac{B}{2A}$. Also, if $a = A$ and $h = -\frac{B}{2A}$, then "$ah^2 + k = C$" is equivalent to "$A \cdot \frac{B^2}{4A^2} + k = C$," that is, $k = C - \frac{B^2}{4A} = \frac{4AC - B^2}{4A}$.

25. (a) From the fact that

$a(x - h)^2 + k = a\left((x - h)^2 + \frac{k}{a}\right), \quad a \neq 0$

it follows that it is factorable if $\frac{k}{a} \leq 0$.

(b) If $\frac{k}{a} = -p^2$ where $p = \frac{m}{n}$, m and n are integers, then $a(x - h)^2 + k = a(x - h)^2 - p^2 = a(x - h - p)(x - h + p)$ $= \frac{a}{n^2}\left(n(x - h) - m\right)\left(n(x - h) + m\right)$ where m, n, h are integers with a possible exception of the constant factor $\frac{a}{n^2}$. If $n = 1$, that is, p an integer, then $\frac{a}{n^2}$ is also an integer. Hence, if $-\frac{k}{a}$ is a perfect square of an integer, the above polynomial is factorable over the integers.

(c) If $\frac{k}{a} < 0$, $\frac{k}{a} = 0$, or $\frac{k}{a} > 0$ the truth set of $a(x - h)^2 + k$ contains two, one, or no real numbers, respectively.

26. (a) The domain of definition of Q is the set of all real numbers x such that $-1 \leq x < 0$ or $0 < x \leq 2$, i.e. all numbers between -1 and 2, except 0 and including -1 and 2.

(b) The range of Q consists of the number -1 along with all x such that $0 < x \leq 2$.

(c) $Q(-1) = -1$, $Q(-\frac{1}{2}) = -1$, $Q(0)$ is not defined, $Q(\frac{1}{2}) = \frac{1}{2}$, $Q(\frac{3}{2}) = \frac{3}{2}$, $Q(\pi)$ is not defined.

(d) R is the same function as Q.

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The preliminary edition of this volume was prepared at a writing session held at the Stanford University during the summer of 1960. Revisions were prepared at Yale University in the summer of 1961, taking into account the classroom experience with the preliminary edition during the academic year 1960-61. This edition was prepared at Stanford University in the summer of 1962, again taking into account the classroom experience with the Yale edition during the academic year 1961-62.

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