DOCUMENT RESUME

ED 156 455 SE 024 370

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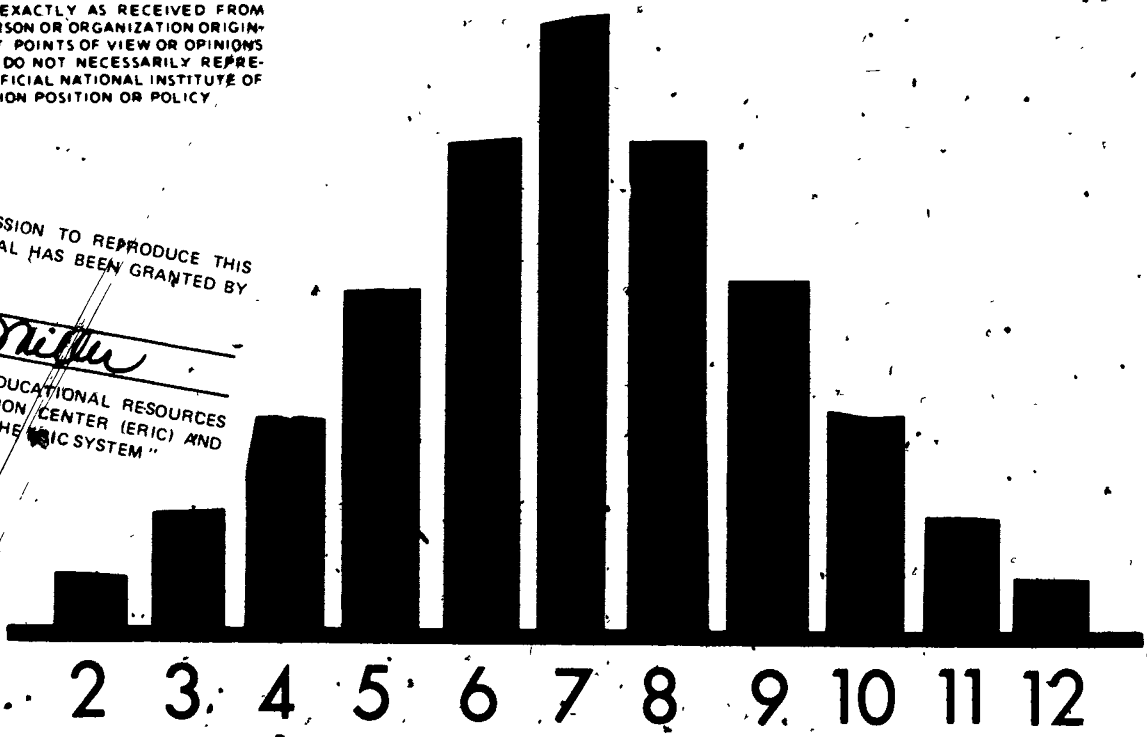
TITLE	Probability in Elementary Schools: A Guide for Teachers.
INSTITUTION	New York State Education Dept., Albany. Bureau of Mathematics Education.; State Univ. of New York, Albany.
SPONS AGENCY	National Science Foundation, Washington, D.C.; New York State Education Dept., Albany.
PUB DATE	Sep 77
NOTE	56p.
EDRS PRICE	MF-$0.83 HC-$3.50 Plus Postage.
DESCRIPTORS	Activity Units; Educational Games; Elementary Education; *Elementary School Mathematics; *Instructional Materials; Mathematical Concepts; Mathematics Education; *Probability; *Probability Theory; *Teaching Guides

ABSTRACT

This publication is a response to the need expressed by elementary teachers for a concise introduction to intuitive probability concepts and how they may be presented. Instruction is divided into five concept groups including: (1) basic understandings; (2) the probability of simple events; (3) the probability of compound events; (4) counting; and (5) sampling. Each concept group section contains a discussion of concepts, activities, and classroom games and projects. A bibliography is included. (MN)

PROBABILITY IN ELEMENTARY SCHOOLS

A guide for teachers



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FOREWORD

This manual is based on two publications of the School Mathematics Study Group:

Probability for Primary Grades, Teachers Commentary, Revised Edition, 1966.

Probability for Intermediate Grades, Teachers Commentary, Revised Edition, 1966.

Dr. George H. Ropes wrote this manual. The National Science Foundation and the New York State Education Department provided financial support.

(Reprint September 1977)

TABLE OF CONTENTS

	Page
INTRODUCTION	1
The Elementary School Probability Project	1
Probability Is Important	1
Probability in Elementary School	2
Activity-Based Learning	3
CONCEPT GROUP 1--Basic Understandings	6
1.1 Concepts	6
1.11 Terminology	6
1.12 Common Misunderstandings	6
1.13 Outcomes of Multiple Trials	6
1.14 Range of Results	6
1.2 Activities	6
1.3 Games	9
CONCEPT GROUP 2--The Probability of Simple Events	12
2.1 Concepts	12
2.11 Basic Statements about Probability	12
2.12 Probability by Comparing Outcomes	12
2.13 Probability by Counting Experimental Trials	12
2.14 Large Numbers of Trials	13
2.2 Activities	13
2.3 Games	20
CONCEPT GROUP 3--The Probability of Compound Events	22
3.1 Concepts	22
3.11 The Probability of This <u>Or</u> That	22
3.12 The Probability of This <u>And</u> That	22
3.13 Repeated Trials	23
3.14 Probability of This or Not This	23
3.2 Activities	24
3.3 Games and Problems	31
CONCEPT GROUP 4--Counting	34
4.1 Concepts	34
4.11 The Multiplication Principle	34
4.12 Permutations	34
4.13 Combinations	35
4.2 Activities	36
4.3 Games	41
CONCEPT GROUP 5--Sampling	43
5.1 Concepts	43
5.11 Random Samples	43
5.12 Predictions from Samples	43
5.2 Activities	43
5.3 Game and Class Project	47
BIBLIOGRAPHY	49

INTRODUCTION

The Elementary School Probability Project

During the 1974 school year, ten school districts in New York State[1] conducted classroom trials of probability lessons and materials in elementary grades. The project was under the direction of the Bureau of Mathematics Education of the New York State Education Department. Support was provided by the National Science Foundation. A coordinator was designated for each participating district.

Teachers in the project (12 or more per district) used School Mathematics Study Group texts as the primary reference source. They attended an in-service training course which met weekly.

Evaluation of the project led to the following conclusions:

1. Probability lessons are enthusiastically received by students at all elementary levels.
2. There is a place in elementary school mathematics for probability. It can provide meaningful applications and enrichment within the framework of the existing curriculum.
3. Probability offers opportunity for independent explorations.
4. Probability helps develop skills in data handling.
5. Lessons in probability are most successful after teachers have acquired some background knowledge of probability and have been exposed to ideas and activities related thereto.

This publication is a response to the need expressed by teachers for a concise introduction to intuitive probability concepts and how they may be presented.

Probability Is Important

Chance is involved in many games children play--spinning a spinner, drawing a card, or rolling a die are examples. If these were the only events to concern us, one would not assign the study of chance a high priority. However, life is filled with events which happen by chance or where chance appears to be a major factor. The daily choice of what to wear, meeting an acquaintance unexpectedly, a car accident--each is an event associated with chance. The sex of a newborn child, the occurrence of illness, choosing a career, death--these events too are heavily involved with chance.

[1]Buffalo, Cazenovia, Edgemont, Mamaroneck, New York City 12, New York City 31, Penn Yan, Plainedge, Valley Stream, and Whitesboro.

A knowledge of probability is becoming necessary in daily life. Decisions in industry, business, science, and agriculture often depend on probability. The theory of probability underlies all of mathematical statistics, a topic that is reaching into virtually every segment of human affairs. Biology, physics, chemistry, astronomy, marketing, distribution, and psychology are but a few of the fields where statistics plays an important role.

Probability in Elementary School

As is true of algebra and geometry, some of the fundamental concepts of probability are easily comprehended at the elementary school level. There are several considerations which, taken together, strongly imply that elementary school children should become acquainted with introductory ideas in probability.

1. Probability fits comfortably into the body of the elementary school mathematics curriculum. It offers an excellent application for fractions, a wide spectrum of material to be graphed, good experiences in counting and computing, and the challenge to think logically and clearly. In a short time, children begin to use the terminology of probability correctly. Work in probability also develops skills which are useful in science and social studies.

2. The materials children need to carry on probability experiments are simple and inexpensive. As they experiment, children have the opportunity to learn by doing.

3. Introducing probability into elementary schools induces positive student reaction. Working in situations where chance is a factor is fun and tends to improve student attitude toward mathematics.

This bulletin sets forth a number of elementary concepts in probability theory and suggests ways teachers can implement these concepts in their classrooms. Concepts are presented in related groups. For each such group there are recommended activities and games spanning a wide range of pupil ability. Some are intended for children in the lower grades; others are more appropriate for upper elementary levels. It is expected that teachers will select activities or games on the basis of the needs and capabilities of their students.

It is not mandatory that all concepts be presented. Time available, class interest, class ability, topics already covered, and other factors will influence the selection of concepts and activities for any one class. Although it is recommended that the general order of presentation follow the sequence outlined herein, some rearrangement can be made. For example, teachers may present Group V, Sampling, earlier in the sequence.

Students at the elementary level should depend primarily on experiment and observation as they explore questions involving probability. They should not be expected merely to memorize and apply formulas. It is, however, important for teachers to be able to answer probability questions correctly so students will not accept incorrect answers. We want children to discover concepts for themselves, but we also want to guard against their accepting false concepts.

Activity-Based Learning

One of the best ways to present concepts is to have students conduct experiments within a structure defined by the teacher.

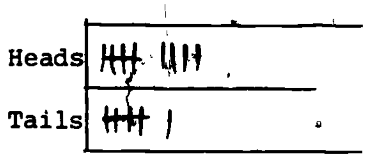
Initially, the teacher should make sure that each student knows what to do and what to look for. Conclusions should not be announced in advance. We should guard against the tendency of children to make results come out to fit pre-stated conclusions.

In suggesting an activity teachers may simply describe it in detail. Another frequently used method is to distribute a sheet of instructions, often called an "activity sheet," which can be gone over with the class as a whole or read individually by each student. Each of the two School Mathematics Study Group publications, Probability for Primary Grades and Probability for Intermediate Grades, contains many student activity pages. Other good sources of ideas for activities are the Nuffield Mathematics Project booklet, Probability and Statistics, and Experiments in Probability and Statistics by Donald Buckeye.

Once students have their materials and understand what to do, they proceed as their instructions indicate. It is essential for the teacher to circulate through the classroom to answer questions and help anyone who may not be on the right track. Children need not carry out each activity individually. Often it is preferable to set up groups of two or three children and assign an activity to each group. This helps children learn to work together.

How results are organized and recorded can have a major effect on the success of an activity. Learning to organize data is a paramount instructional objective. Children are from time to time asked to count and record outcomes. Some advance organization by the teacher will promote orderly and accurate work. A simple but effective procedure is to make a stroke tally, grouping by 5's, within a block arrangement:

Outcomes of Flipping a Coin



Such a table can be modified to accommodate the results of different experiments. The arrangement below (expanded to provide space for more trials) can be used when each trial has two outcomes.

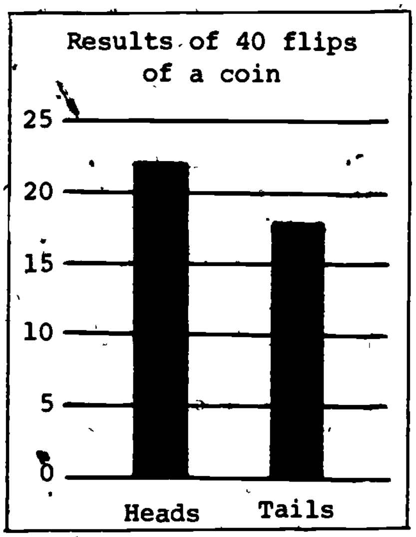
Trial No.	Outcomes 1st	Outcomes 2nd
1	H	T
2	H	H
3	T	H

Given adequate preliminary instruction, students usually develop considerable skill in making their own tables. It is helpful for teachers to compile all the results obtained in a given class. Where a trial consisted of flipping 5 coins, the compilation might look like this:

TOSSING FIVE COINS
(22 trials)

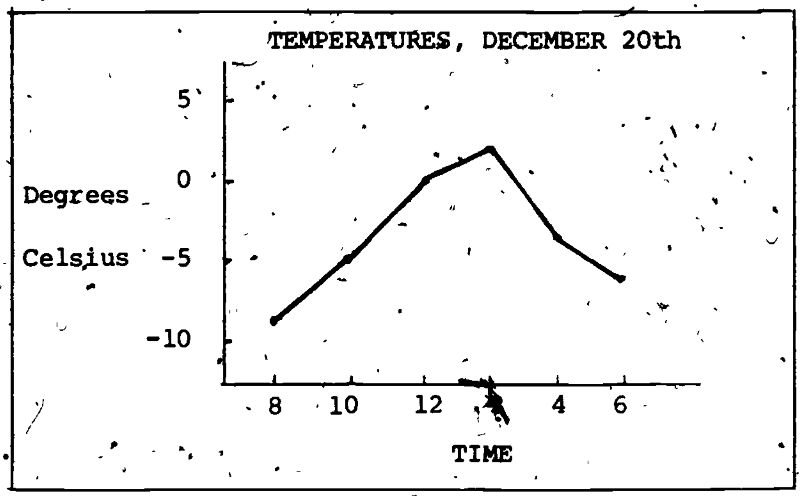
Heads	Tails	Number
5	0	||
4	1	||||
3	2	𝍸 ||
2	3	𝍸
1	4	|||
0	5	|

Children find it enjoyable to graph the results of their probability experiments. Bar graphs are simple and practical. Here is a bar graph showing the results of flipping a coin:



Results of 40 flips of a coin

A bar graph is equally effective for displaying the results of experiments in which there are more than two outcomes. Line graphs are useful for more complicated situations such as the relation between time of day and temperature.[2]



After students have obtained a set of outcomes from an experiment, the most important phase of the process begins--thinking about and discussing the results. Each child's perceptions and understanding are enlarged by seeing how his results compare with those of other children in the class. Students' questions, answers, and comments often add greatly to overall class understanding and provide clues to misunderstandings or wrong impressions children may have.

---

[2]For further information, see Improving Reading-Study Skills in Mathematics K-6, New York State Education Department, pages 20-22.

CONCEPT GROUP 1--Basic Understandings

Objective: To have children become familiar with the results of experiments where there are several equally likely outcomes. We are concerned only with instances where chance alone is at work. Any die used is presumed to be a fair die--there is equal likelihood of each face coming up. Coins are fair coins and are fairly thrown. Spinners have equal divisions and are spun so chance alone determines the outcome.

1.1 Concepts

1.11 Terminology: As they use devices whose outcomes are equally likely, children should become familiar with and understand the meaning of:

Likely	More likely	Certain	Possible
Unlikely	Less likely	Uncertain	Impossible

1.12 Common Misunderstandings. Nothing but chance affects the results of any one trial. Surprisingly, many children--particularly younger children --do not have this understanding and do not readily accept it. Some apparently believe that one color, number, or outcome is inherently more likely to occur. Other children believe that an effort of will can make a given outcome appear. Even more deep-seated among children--and some adults--is the impression that after a string of several appearances of one outcome--say eight successive heads--the chance of a different outcome on the next trial is substantially greater. Clearing up misunderstandings such as these is an important part of the study of probability.

1.13 Outcomes of Multiple Trials. As the number of trials in a two-outcome experiment increases, it becomes more and more likely that there will be the same number or nearly the same number of occurrences of each outcome. In 100 flips of a coin, a person might get 46 heads and 54 tails. If he makes more trials, he can expect that the number of heads will approach half the total number of trials and that the same will be true of the number of tails. This concept is not well recognized, perhaps because children do not often carry out experiments with an increasingly large number of trials. The effect of a large number of trials can be achieved by combining the results of all the trials made in a given class and by combining results from several classes.

1.14 Range of Results. The result of a series of trials can be any one of the possible outcomes. For example, if 4 coins are flipped, the result can be four heads or four tails or any combination of heads and tails totaling four.

1.2 Activities

1.21 Explaining the meaning of "more likely" and "less likely" is facilitated by setting up an experiment whose outcomes are not equally likely. Distribute blocks between two bags or containers so there is a much higher proportion of one color in one bag than in the other--for example, 6 red, 4 blue in the first and 2 red, 8 blue in the second.

Show the class what is in each bag. Ask which bag they would choose to draw from if they wanted a red block. If anyone doubts that a red block will be drawn more often from the first bag than from the second, have a number of drawings made (replacing the block drawn each time) and record the results. Point out that the way the blocks are distributed, a person is "more likely" to draw a red block from the first bag than from the second. Similarly, one is "less likely" to draw a red block from the second bag than from the first. Reinforce understanding by asking:

> Are we more likely to have snow in September than in January?
>
> Is it likely that the next person to enter our classroom will be a television star?

A different situation is involved when we ask if it is likely that the sun will set this evening. Since we know of no instance when the sun did not set, the setting of the sun on any one day is not only "likely" but "certain," or as certain as anything in this world can be. Give the class an opportunity to think of events that are certain to happen and some that are certain not to happen.

Suppose we ask if it is likely to rain in Moscow today. We do not know. It is "possible," but we are "uncertain." It is not "impossible." If we could get information or records, then we could say whether rain in Moscow at any given time of year is "likely" or "unlikely."

1.22 Have the class make a specified number of trials using a device with two outcomes, each equally likely. Possible activities are:

> Flip a coin and count heads and tails.
>
> Flip a checker and count the number of times each side appears face up.
>
> Spin a half-red, half-blue spinner and count reds and blues.
>
> Roll a die and count even faces (2, 4, or 6) and odd faces (1, 3, or 5).
>
> Draw a block from a bag containing one red block and one blue block, counting reds and blues drawn.

The optimum number of trials varies with grade level. First graders may find 5 trials appropriate; sixth graders can make 100 trials. Have students guess in advance what each outcome will be, record the guess, and then determine and record the actual result:

Guess	Actual Outcome
H	T
T	T
H	H
H	[illegible]

A simple game consists of scoring a point for the student when he guesses correctly and a point for "teacher" when he does not.

Questions to ask during and after the activity are:

What outcome do you think will appear next?

Do you always guess right?

Can you be sure in advance what the outcome of any one trial will be?

Would you be likely to get the same number of heads if you made the same number of trials again?

Suppose you got the same outcome 4 times in a row, say 4 heads, would you be more likely to get the opposite outcome (tails) on the next trial?

If you have compiled the results found by all class members or groups, additional questions are:

What outcome happened the largest number of times?

What outcome happened the fewest number of times?

Were there possible outcomes which did not occur for anyone in the class?

If we kept on long enough, would we be likely to get all possible outcomes?

It is possible, of course, that the overall results of the class experiment will in themselves be unlikely. The result "5 heads, one tail" may occur on six throws of a coin as often or more often than the result "3 heads, 3 tails." Such an occurrence provides an excellent opportunity to build children's understanding that unlikely events do happen. Discussion should bring out that over a large number of trials we expect a "more likely" event to happen more often than a "less likely" event; nevertheless, the reverse <u>can</u> happen. It is not too surprising then that in a class experiment where the number of trials is relatively small, an unlikely event occurs. Where possible in such a case, continue making more and more trials. As the number of trials increases, the number of occurrences of each event tends to approach the expected value.

1.23 After a class has had experience with a two-outcome device, have them make a number of trials with a device having three or more equally likely outcomes.

> Spin a spinner having one-third red, one-third blue, and one-third yellow. Count the number of times the pointer stops on each color.
>
> Spin a spinner with six equal sections numbered 1 through 6. Count how many of each number occur.
>
> Roll a tetrahedron having each of its four faces a different color. Keep a record of the color of the face on which the tetrahedron comes to rest.
>
> Draw a block from a bag containing three blocks, each a different color. Replace each block after recording the color drawn.

Follow-up questions should be similar to those suggested for two-way devices. It is important to accumulate results found by different individuals or groups and to discuss the characteristics of the composite results.

Have class members make a bar graph of the results of their own or their group's experiments. All graphs need not be arranged similarly. Bars can go from left to right as well as up and down. The scale used to determine the length of the bars can vary from one graph to another. Class members should practice saying what their own graph and those of others in the class indicate.

1.3 Games

1.31 I'M OK, YOU'RE OK. Two players alternate turns with a spinner, checker, coin, or other two-outcome device. The first player, A, selects one of the outcomes, say red; the second player, B, takes the other outcome, say blue. The game starts with both players' markers on the START space in the middle of the diagram shown below. On his turn, if a player's outcome appears, he moves his marker one space toward his Win Square. If it does not appear, his marker remains where it is. A does not move when it is B's turn, and vice versa. The game ends when one of the players reaches his Win Square.

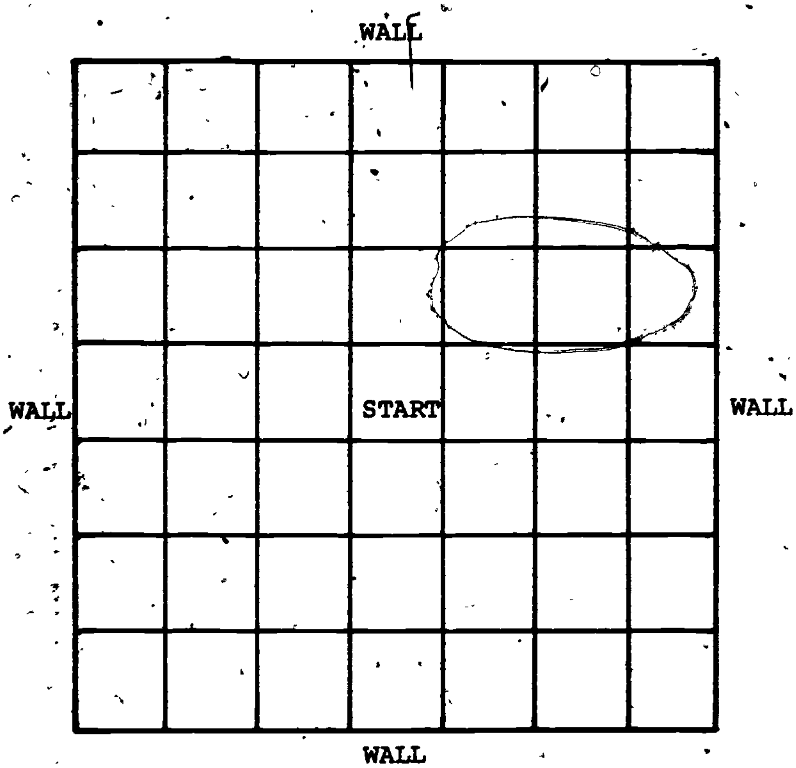
← RED BLUE →

RED WINS				START				BLUE WINS

1.32 HORSERACE. A game for 6 players, each of whom selects one of the numbers from 1 through 6, and puts a marker on the corresponding START position. The players take turns rolling a die. The player whose number comes up moves his marker (horse) one space forward on the diagram below. The winner is the first player to reach the WIN space.

START	1						1 Wins
	2						2 Wins
	3						3 Wins
	4						4 Wins
	5						5 Wins
	6						6 Wins

1.33 PRISONER's ESCAPE. For 2 to 4 players. Each player starts with his marker at START on the diagram shown below. Each player in turn rolls a die. If he rolls a 1, he moves Up; if he rolls a 2, he moves one space to the Right; if the roll is a 3, the move is one space Down; if a 4, one space Left. If the roll is a 5 or a 6, the player does not move. The first player to reach a square that touches a wall wins.



Variation: Roll a tetrahedron instead of a die. Mark the faces of the tetrahedron Up, Right, Left, and Down. Move according to which face of the tetrahedron is on the bottom after each roll. Also, North, East, South, and West can be substituted for Up, Right, Down, and Left.

Teacher questions:

> Does one player have a better chance of winning than another?
>
> Is it likely for one player to win three times in a row? What is the average number of rolls for a game of HORSERACE?

CONCEPT GROUP 2--The Probability of Simple Events

Objective: To have children learn how to determine the probability that a given event will occur, both by comparing the number of ways that are instances of the event with the total number of possible outcomes and by doing experiments.[3]

2.1 Concepts

2.11 Basic Statements about Probability. If an event cannot happen or never happens, the probability of the event is 0.

If an event is certain to happen, the probability of the event is 1.

In all other cases the probability of an event is a number between 0 and 1.

A short way of writing "the probability of an event" is "P(An event)." For example, P(Heads) means the probability that heads will appear.

2.12 Probability by Comparing Outcomes. In a situation where each outcome is equally likely to occur, the probability of a given event is the fraction:

$$P(\text{an event}) = \frac{\text{the number of outcomes that are instances of the event we are interested in}}{\text{the total number of possible outcomes of the experiment}}$$

An outcome that is an instance of the event we are interested in is commonly called a "favorable" outcome, hence the probability of an event can also be written as the fraction:

$$P(\text{an event}) = \frac{\text{the number of favorable outcomes}}{\text{the total number of possible outcomes}}$$

2.13 Probability by Counting Experimental Trials. Another way to determine the probability of one of the events or outcomes of an experiment is to make a large number of trials of the experiment under like conditions. The probability of a given event then is the fraction:

---

[3]The results of experiments or observations are called events or outcomes. Events may be single or multiple. If a die is tossed, 6 single events are possible--1, 2, 3, 4, 5, or 6. The event, "the die shows an odd number" is multiple. It happens if any one of three single events (1, 3, or 5) occurs.

$$P(\text{an event}) = \frac{\text{the number of times the event occurs}}{\text{the total number of trials made}}$$

This method of arriving at the probability of an event requires that we compile or obtain records of a large number of trials of an experiment and count the number of times the event occurs.

2.14 Large Numbers of Trials. With an increasing number of trials, the probability of an event determined experimentally will tend to get closer and closer to the probability of the same event determined by comparing favorable outcomes with total possible outcomes. What do we mean by "large" when we say make "a large number of trials"? Although one can never be absolutely certain that a given number of trials is enough, if in a given experiment we keep on making trials and calculating the value of the fraction:

$$\frac{\text{number of times an event occurs}}{\text{total number of trials made}}$$

we expect to find that at some point the probability stabilizes. It tends to reach a value which changes very little as more and more trials are made. At that point we can be confident that the value reached is close to the probability of the same event determined by comparing favorable outcomes with total possible outcomes.

In the following table of outcomes of tossing 20 coins, the probability of heads is approaching .50.

TOSSES OF 20 COINS

	1st	2nd	3rd	4th	5th	6th	7th
Number of Heads	6	10	11	9	13	10	12
Number of Tails	14	10	9	11	7	10	8
Cumulative Heads	6	16	27	36	49	59	71
Cumulative Tails	14	24	33	44	51	61	69
Total Coins Tossed	20	40	60	80	100	120	140
Probability--Heads	.30	.40	.45	.45	.49	.49	.51

If, after still more trials, the fluctuations above or below .50 become less and less, we would be justified in assuming that we had reached a close approximation to the desired probability.

2.2 Activities

2.21 As a means of introducing the meaning of probability, ask the class to think back to their experiments with coins, spinners, and dice and tell you:

How many tails they would expect for every two throws of a coin. [One]

How many reds they would expect for every two spins of a red-blue spinner. [One]

How many 2's they would expect for every 6 rolls of a die. [One]

Rephrase answers as shown below:

What we expect:	What we say:
1 tail in 2 throws	The probability of getting tails is 1 out of 2 or $\frac{1}{2}$
1 red in 2 spins	The probability of red is 1 out of 2 or $\frac{1}{2}$
1 two in 6 rolls	The probability of getting a two is 1 out of 6 or $\frac{1}{6}$.

Explain that to save time writing out the long word "probability," people use a short-cut:

For the probability of tails is $\frac{1}{2}$, we write $P(\text{Tails}) = \frac{1}{2}$

For the probability of a 2 is $\frac{1}{6}$, we write $P(2) = \frac{1}{6}$

Test understanding of this notation by asking:

If $P(\text{an event}) = \frac{2}{3}$, how often would you expect it to occur in 3 trials? [2 out of 3 times]

Sometimes the weatherman says that the probability of rain is .7. How would we write that in our short form? [$P(\text{rain}) = .7$]

How often do we expect a baseball player to get a hit in ten times at bat if his batting average is .300? [3 out of 10 times]

NOTE: For classes not familiar with decimals, treat .7 as another name for $\frac{7}{10}$ and .300 as another name for $\frac{300}{1000}$. Probabilities are sometimes expressed as percents--if P(An event) = 50%, one can say that there is a 50% chance that it will happen.

Introduce the idea of zero probability by asking:

How many times would you expect to get green when you spin a red-blue spinner? [0]

Explain that the probability of getting green on a red-blue spinner is 0 and that we write it, P(Green) = 0.

Here is another similar question:

What is the probability that when you toss a die once, an eight will appear? [0]

Point out that some events are certain to happen; the probability of such an event is 1.

What is the probability that the sun will set tonight? [1]

It follows that all events except those certain to happen and those certain not to happen have a probability between 0 and 1. If the probability is close to 0, it means that the event is not very likely to happen. If the probability is $\frac{1}{2}$ (or .5), it means that the chance of the event happening is equal to the chance of its not happening. If the probability is close to 1, it means that the event is very likely to happen.

Have the class suggest different events and estimate the probability of each.

2.22 Here is a way for classes to learn how to determine probabilities by comparing favorable outcomes with total possible outcomes:

We have been estimating or making guesses about the probability that the pointer will stop on blue when we spin a spinner with three equal sections--one red, one blue, and one yellow.

Are all the outcomes equally likely? [Yes]

How many of the possible outcomes are instances of the event "the pointer stops on blue"? [One]

How many possible outcomes are there? [Three]

Then the probability of getting blue is $\frac{1}{3}$. We can write it down like this:

$$P(\text{blue}) = \frac{1}{3}$$

In assigning a fraction as the probability of an event we identify the numerator and denominator as follows:

$$P(\text{an event}) = \frac{\text{the number of outcomes that are instances of the event we are interested in}}{\text{the total number of possible outcomes}}$$

In experiments where there are numerous outcomes--such as when two dice are thrown--a practical way to proceed is to make a table. We put the outcomes of rolling one die (1 through 6) on the bottom margin of the table and the outcomes of rolling the other die (also 1 through 6) along the left vertical margin. We draw horizontal and vertical lines dividing the table into squares. In each square formed, we put the sum of the outcome at the bottom and the outcome at the left:

Outcomes of Rolling Two Dice[4]

SECOND DIE						
6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

FIRST DIE

If we want to know the probability of getting a sum of 5 with two dice, we ask ourselves:

How many times does 5 appear as an outcome in the table above? [4]

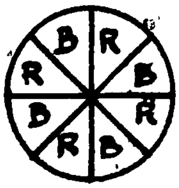
How many possible outcomes are there? [36]

Thus, $P(5) = \frac{4}{36}$ or $\frac{1}{9}$

The sheet on page 17 can be used to provide practice in determining probabilities.

[4]Strictly speaking, each entry is an ordered pair: (1,1), (1,2), (2,1), and so on. For simplicity, the sum of each ordered pair is shown in the table.

Practice Sheet on Probability

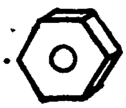
Find the probability of:	What is the number of favorable outcomes?	What is the total number of possible outcomes?	The P(event) is:
Drawing a black block from a bag containing 2 red blocks, 2 green blocks, and 1 black block	1	5	$\frac{1}{5}$
Having the red side down after you roll a regular tetrahedron with faces blue, red, white, and yellow	1	4	$\frac{1}{4}$
Drawing your own name from a hat in which there is one slip with your name and 7 slips with other names	1	8	$\frac{1}{8}$
Getting a "1" when you roll a die	1	6	$\frac{1}{6}$
Getting a number other than "1" when you roll a die	5	6	$\frac{5}{6}$
Getting a sum smaller than 6 with two dice?	10	36	$\frac{10}{36} = \frac{5}{18}$
Getting the same number on each die when you roll two dice	6	36	$\frac{6}{36} = \frac{1}{6}$
Drawing an "O" when you select one letter from the word "SCHOOL"	2	6	$\frac{2}{6} = \frac{1}{3}$
Landing on blue when you spin this spinner 	4	8	$\frac{4}{8} = \frac{1}{2}$

2.23 The experimental method of determining probability can be introduced as follows:

If we make a large number of trials of an experiment and determine the fraction of those times a particular event happens, that fraction approximates the probability of the particular event. Thus:

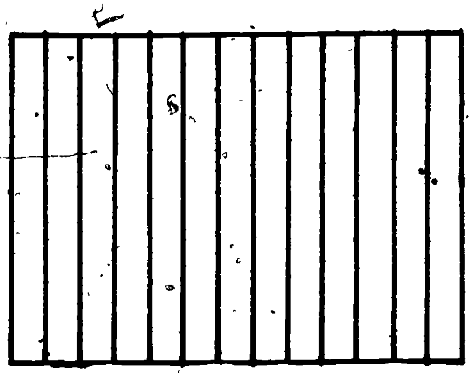
$$P(\text{an event}) = \frac{\text{the number of times the event happens}}{\text{the total number of trials we make}}$$

Activities which illustrate the process of determining probability experimentally include:

1. Shake thumbtacks in a cup and roll them onto a flat surface. What is the probability that a thumbtack comes to rest point upward? Rolling the thumbtacks into a shallow box helps keep them together. Ten is a good number to shake and roll at one time. Use thumbtacks with metal heads. Alternatively, hexagonal metal nuts may be used. Determine the probability that the nuts come to rest upright  rather than flat. 

2. Draw an M&M candy at random from an M&M candy package containing assorted colors. What is the probability of selecting a yellow candy? Give each student an M&M package and have him determine the number of times yellow is drawn compared to the total number of drawings. This fraction can be compared with the actual count of yellow candies out of total candies.

3. Drop toothpicks from a height of four feet or more onto a large piece of paper ruled vertically with lines as far apart as the length of the toothpicks used. What is the probability that a toothpick comes to rest across a vertical line?



Dropping 10 or more toothpicks at a time expedites the investigation. Do not count toothpicks which come to rest off the ruled paper. If one doubles the total number of toothpicks dropped and divides the product by the number coming to rest over a line, the answer should be close to π. An approximation of π is 3.14.

4. Drop a cylinder cut from a mailing tube and determine the probability that it lands on its side. The cylinder should be two inches or more in diameter and the drop should be three feet or more. By working with different sizes of cylinders, the change in the probability of coming to rest on a side can be investigated for shorter lengths and for longer lengths.

5. Plant a given kind of seed. Count seeds planted and seedlings that sprout after a given length of time. What is the probability of germination?

As they carry on any of these experiments, students should:

1. Make an estimate of the probability and record it. Students may want to revise their estimates as they begin to get data and should be allowed to do so. Each new estimate should also be recorded, together with the number of the trial at which the revised estimate was made.

2. Carry on the experiment carefully so all trials will be conducted under similar conditions.

3. Set up an adequate recording system.

4. Communicate results pictorially, using graphs where feasible.

5. Be prepared to discuss results obtained.

The total number of trials should be appropriately large. Calculations are simplified if trials are made in groups of 10 or 100. The probability of the event being investigated should be figured at regular points throughout the experiment, say every 100 trials. Have the students watch to see if the probability so calculated does tend to approach a certain value as the number of trials gets larger and larger.

If the probability of the event has been determined both experimentally and by comparing favorable outcomes with total possible outcomes, check to see how far apart the results are. The two probability figures may be reasonably close or there may be a discrepancy between them. Have the class discuss why a discrepancy might exist. Possible explanations are:

1. Happenstance.

2. Incorrect calculation in one or both of the methods.

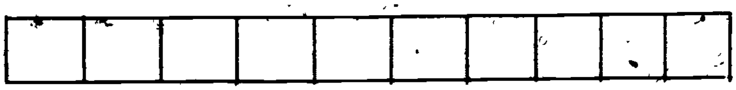
3. Insufficient care that the experimental trials were all carried out under similar conditions.

4. Use of an "unfair" device.

2.3 Games

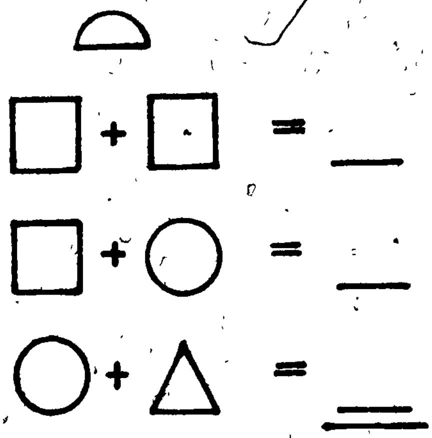
2.31 PICK-A-STATE. For 2 players. Each player thinks of a state of the United States and writes the name of that state on a piece of paper. The first player shows the state he picked. If the state written down by the second player is the same as that of the first player, he gets 2 points. If the second player's state borders on the first player's state, the second player gets 1 point. Then new states are picked. The two players alternate going first. The one with the most points after a series of games wins.

2.32 LARGEST NUMBER GAME. For 2 or more players and a leader. Each player draws the following diagram on a sheet of paper:



The leader has ten slips of paper or ten playing cards, each bearing a number from 0 through 9. Players take turns drawing a number, announcing it to the other players, and returning the slip or card to the leader. As each number is announced, each player decides in which empty square of his diagram he will put the number. He must put it in one square before the next number is drawn; once entered, no number can be changed or moved. At the end of ten draws, all squares will be filled. The player who has the largest 10-digit number wins.

2.33 THE HAT GAME. For 2 or more players and a leader. Each player draws the following diagram on a sheet of paper:



The leader puts in a hat 10 identical slips of paper or cards, each bearing a number from 0 through 9. Players take turns drawing a number, calling it out, and returning it to the hat. After a number is called, each player decides in what empty shape on his diagram he will put it. If he chooses squares, he puts the number in all 3 squares. If he chooses circles, it goes in both circles. If he chooses triangle or semicircle, it goes in the one triangle or semicircle. Numbers must always be entered before the next number is drawn; once entered, no number can be changed or moved. After four draws, all shapes will be filled. Each player then finds the sums of □ + □ , □ + ○

and ○ + △ , and combines these sums to get a grand total. The number in the semicircle does not count toward any total. The object of the game is to have the largest grand total.

The game is also interesting when the object is to arrive at the smallest grand total. To provide practice with multiplication, the plus signs in the diagram may be replaced with multiplication signs.

Questions:

If the first number drawn is a 5, which of the shapes should you put it in, if you're trying for the largest total?

How does knowing about probability help you in playing this game?

CONCEPT GROUP 3--The Probability of Compound Events

Objective: To have children learn how to determine the probability of compound events.

3.1 Concepts

3.11 The Probability of This Or That. If the probability of event A happening is P(A) and if the probability of event B happening is P(B), the probability of either A or B happening is:

$$P(A \text{ or } B) = P(A) + P(B)$$

provided the events are mutually exclusive.

Events are "mutually exclusive" when the happening of one of the events excludes the happening of any or all the others. If we are using a red-blue spinner, the events "red" and "blue" are mutually exclusive; when the spinner stops on red, blue is excluded as a possibility; when the spinner stops on blue, red is excluded as a possibility. If we are rolling a die, the events "even" and "2" are NOT mutually exclusive; when an even number is rolled, it could be a 2.[5]

3.12 The Probability of This And That. If the probability of event A happening is P(A) and if the probability of event B happening is P(B), the probability of A happening and then B happening is P(A) times P(B), that is:

$$P(A \text{ and } B) = P(A) \times P(B)$$

provided the events are independent.

Two events are independent when the probability of either of the events is not affected by the occurrence of the other event; in other words, neither event has an influence on the other. To check if event A and event B are independent:

1. Determine the probability of event A.
2. Assume that event B has already occurred. Determine what the probability of event A is, based on the assumption that event B has occurred.[6]

---

[5]The probability of This Or That for events that are not mutually exclusive and the probability of This And That for events that are not independent can be calculated using the basic formula, number of favorable outcomes divided by total possible outcomes, but determining the number of favorable outcomes may involve procedures beyond the scope of this publication.

[6]The probability of event A given that event B has occurred is often symbolized as $P(A|B)$.

If the two calculations of the probability of event A give the same result, then event A and event B are independent; if the results are different, the events are not independent.

As an example of independent events, consider event A to be getting red and event B to be getting blue on a red-blue spinner. The probability of event A, getting red, is $\frac{1}{2}$. Now assume we first get a blue, the probability of red stays exactly the same at $\frac{1}{2}$. The occurrence of Event B did not change the probability of event A, hence the two events are independent.

But suppose we are drawing from a bag containing a red block, a blue block, and a yellow block, and that we do not replace blocks after they are drawn. Event A is "drawing a red block" and event B is "drawing a blue block." At the outset, the probability of drawing red is $\frac{1}{3}$, but if we assume that a blue block has been drawn, the probability of drawing a red block is $\frac{1}{2}$. In this situation, therefore, event A and event B are not independent. Determining the probability of combined events that are not independent is discussed in the footnote on page 22.

3.13 Repeated Trials. The question, "What is the probability that event A happens twice in succession?" is the same as asking, "What is P(A and A)?" Just as we did with P(A and B), we multiply the probabilities and get:

$$P(\text{event A happens twice}) = P(A \text{ and } A) = P(A) \times P(A)$$

Similarly, the probability of event A happening three times in succession is P(A) x P(A) x P(A). The probability of event A happening n times in succession then is:

$$\underbrace{P(A) \times P(A) \times P(A) \times \ldots\ldots\ldots\ldots \times P(A)}_{n \text{ factors each } P(A)}$$

The foregoing assumes that all happenings of the event are independent, as they are in the cases we will be considering.

3.14 Probability of This or Not This. If event A is one of the possible events in a given trial, the event "Not A," also called the "complement" of event A, is all the possible outcomes other than event A. Event A and the complementary event Not A together include all possible outcomes; therefore it is certain that either event A happens or it does not happen. This being certain, the probability of event A happening plus the probability of event A not happening is 1. We can write:

$$P(A) + P(\text{Not } A) = 1$$

This statement is equivalent to

$$P(A) = 1 - P(\text{Not } A)$$
$$\text{and} \quad P(\text{Not } A) = 1 - P(A)$$

In some instances, the easiest way to determine the probability of an event is to determine the probability that it will not happen and then subtract that value from 1.

3.2 Activities

3.21 Probability of This or That. As an introductory step, children should have the opportunity of comparing the probability that This Event Or That Event will happen with the probability of each of the individual events.

The probability of rolling one of the numbers 1 through 6 with one die is $\frac{1}{6}$. Have class groups experiment to find out if the probability of rolling one of two different numbers (for example, a 3 or a 4) is greater than $\frac{1}{6}$ or less than $\frac{1}{6}$. Stroke tallies illustrate this:

The roll is a 3	卌 |
The roll is a 4	卌 |||
The roll is a 3 or a 4	卌 卌 ||||

It becomes clear that the number of times one of two events happens is greater than the number of times either of the events happens alone. Therefore, the probability of one of two events happening is greater than the probability of either event individually (except in cases where the probability of one of the two events is 0).

> NOTE: In any discussion of P(A or B), it is important to know whether or not the events are mutually exclusive.

Here is a procedure to explain how the probability of This or That can be calculated. Have children think of throwing one die and ask:

> What is the probability of your getting either a 3 or a 4 on one roll?

We can calculate this probability as we have done in the past by forming the fraction:

$$P(\text{An Event}) = \frac{\text{the number of favorable outcomes}}{\text{total number of possible outcomes}}$$

How many outcomes are either a 3 or a 4? [2]

How many possible outcomes are there? [6]

The probability therefore is:

$$P(3 \text{ or } 4) = \frac{2}{6} = \frac{1}{3}$$

Let's start again and look at the probability of rolling just a 3. It is $\frac{1}{6}$ and the probability of rolling a 4 is also $\frac{1}{6}$. If we add $\frac{1}{6}$ and $\frac{1}{6}$ we get $\frac{2}{6}$ or $\frac{1}{3}$, exactly the same result as obtained by comparing favorable and total possible outcomes. What we have demonstrated is another way to calculate the probability of This Or That, provided the events are mutually exclusive--simply add the probabilities of the events:

$$P(A \text{ or } B) = P(A) + P(B)$$

We must remember that the relation

$$P(A \text{ or } B) = P(A) + P(B)$$

applies only to events which are mutually exclusive. Suppose

Event (A) is throwing a prime number (2, 3, or 5) with one die
Event (B) is throwing an even number (2, 4, or 6) with one die

$$P(A) = \frac{1}{2}$$

$$P(B) = \frac{1}{2}$$

$$\text{but } P(A \text{ or } B) \neq 1$$

$$P(A \text{ or } B) = P(2, 3, 4, 5, \text{ or } 6) = \frac{5}{6}$$

3.22 The Probability of This And That. When we ask, "What is the probability of A and B happening?" or "What is the P(A and B)?" we mean "What is the probability that event A happens and then event B happens?"[7] As examples, one might ask, "What is the probability of getting a red on one spin of a red-blue spinner and a blue on the next spin?" or "What is the probability of getting a head on one flip of a coin and a tail on the next flip?"

A similar question is, "What is the probability of drawing a red block twice in succession from a bag containing one red block and one blue block if the block is replaced after each draw?" This is equivalent to asking, "What is P(A and A)," where A is the event "drawing a red block."

Begin by having students carry on experiments that answer the questions posed above and other similar experiments, to find out whether

[7] Or event A and event B happen together.

the probability of This And That is greater than or less than the probability of either of the events alone. The following table, enlarged to accommodate the recording of more trials, can be used in an experiment with a red-blue spinner:

Trial	1st Spin	2nd Spin	Red on 1st Spin and Blue on 2nd Spin
1	B	B	No
2	R	R	No
3	R	B	Yes
	No. Red___	No. Blue___	No. Red followed by Blue___

Twenty trials should serve to indicate that the number of times red appears on the first trial and blue on the second trial is less than the number of reds that occur and also less than the number of blues. Indeed, the probability of two events happening one after the other is less than the probability of either event individually (except in cases where the probability of one of the two events is 0).

> NOTE: In the discussion of P(A and B), it is important to make clear that in the cases considered, event A and event B are independent.

We now describe a procedure which may be used to explain how the probability of This And That may be calculated. Put one red block and one yellow block in a bag. State that any block drawn will be replaced before the next is drawn. Ask, "What is the probability of getting a red block on the first draw and a yellow on the second?" We can write this, "What is P(R and Y)?" Let us look at all possible outcomes. If we draw red first, the result of two draws can be either R and R or R and Y. If we draw a yellow first, the possible outcomes are Y and Y or Y and R; four possibilities in all:

First Draw	Second Draw	Possible Outcomes
R	R	R and R
	Y	R and Y
Y	R	Y and R
	Y	Y and Y

Are the outcomes equally likely? [Yes]

How many favorable outcomes are there? [One]

How many total possible outcomes? [Four]

The probability of red first and yellow second is therefore $\frac{1}{4}$.

Looking at the experiment in a different way, the probability of getting red on the first draw is $\frac{1}{2}$. The probability of getting yellow on the second draw after getting red on the first is $\frac{1}{2}$ of $\frac{1}{2}$, which is $\frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{4}$. This demonstrates another method of determining the probability of This And That (provided the events are independent)--multiply the probabilities of the individual events:

$$P(A \text{ and } B) = P(A) \times P(B)$$
$$\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$$

3.23 The Probability That An Event Will Not Happen. A suggested explanation and series of related questions to develop understanding of the relation between the probability of an event happening and the probability of its not happening follow.

Suppose there is a box of 12 gumdrops each of which is the same size. Five of them are yellow and the remainder are green. If you close your eyes, reach into the box, and take out a gumdrop, the probability you will take out a yellow one is $\frac{5}{12}$. It should be equally clear that the probability of your not getting a yellow gumdrop is $\frac{7}{12}$. It is certain that all the gumdrops are either yellow or not yellow and the probability of an event that is certain is 1, hence:

$$P(\text{gumdrop drawn is yellow}) + P(\text{gumdrop drawn is not yellow}) = 1$$
$$\frac{5}{12} + \frac{7}{12} = 1$$

In general:

$$P(\text{event A}) + P(\text{event Not A}) = 1$$

If we know the probability of an event happening, we can readily determine the probability that it will not happen. Likewise, if we know the probability that an event will not happen, we can determine the probability that it will happen.

Suppose we know that the probability of spinning a red is $\frac{1}{4}$. We can find the probability of not spinning a red as follows:

$$P(\text{Red}) + P(\text{Not spinning a red}) = 1$$
$$\frac{1}{4} + P(\text{Not spinning a red}) = 1$$
$$P(\text{Not spinning a red}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Similarly, if the probability of a person not going to school is $\frac{1}{10}$, the probability that the person will go to school is

$$P(\text{Going to school}) = 1 - \frac{1}{10} = \frac{9}{10}$$

Probabilities are sometimes expressed in terms of "odds." Where the probability that an event will happen is $\frac{2}{3}$, the probability that it will not happen (from the explanation just concluded) is $\frac{1}{3}$. This means that out of three possible outcomes, 2 will be favorable and 1 unfavorable and we say, "The odds are 2 to 1 in favor" or "The odds are 2 to 1 that the event will happen." Odds are stated as the ratio of favorable outcomes to unfavorable outcomes. The odds against an event happening are reversed--the ratio of unfavorable outcomes to favorable outcomes.

3.24 Practice Examples. Once the basic methods of calculating the probability of compound events are understood, it is desirable to provide students the opportunity to apply them to specific situations. What is the probability of:

Rolling no more than 4 (that is, rolling a 2 or a 3 or a 4) with two dice. $\boxed{\frac{6}{36} = \frac{1}{6}}$

Not getting a 5 with 2 dice. $\boxed{\frac{32}{36} = \frac{8}{9}}$

Rolling a 7 twice in a row with 2 dice. $\boxed{\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}}$

Getting a product of 10 when you roll two dice and multiply the numbers on each die. (Suggestion--first make a table of all possible outcomes.) $\boxed{\frac{2}{36} = \frac{1}{18}}$

Getting a head and a 6 when you flip a coin and roll a die. $\boxed{\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}}$

Getting a heart and then a spade in two draws with replacement from a deck of cards. $\boxed{\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}}$

Not getting either a heart or a spade in one draw from a deck of cards. $\boxed{\frac{1}{2}}$

Getting a picture card on one draw from a deck of cards. $\boxed{\frac{3}{13}}$

Getting a sum of 10 when you roll a tetrahedron with faces marked 1, 2, 3, 4 and an ordinary die marked 1 through 6. $\boxed{\frac{1}{24}}$

Drawing a red block from a bag containing 3 red blocks and 2 blue blocks and drawing a blue block from a bag containing 1 red block and 4 blue blocks.

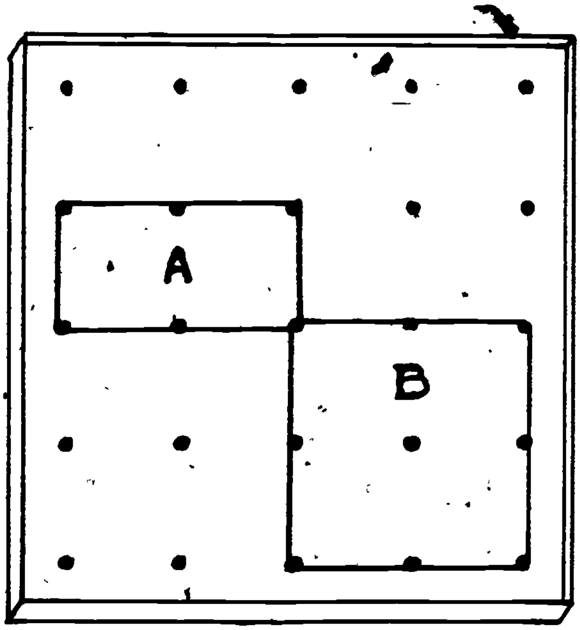
$$\boxed{\frac{3}{5} \times \frac{4}{5} = \frac{12}{25}}$$

3.25 The geoboard offers a variety of interesting ways to have children practice with and learn about probability. Consider a geoboard with five rows of five pegs to be a field. A parachutist jumps from an airplane and lands on the field. He has an equal probability of landing in any part of the field.

There are 16 squares where the parachutist might land; thus the probability of landing upon any one of them is $\frac{1}{16}$.

What is the probability that the parachutist lands in the top row of squares? $\boxed{\frac{1}{4}}$

Now we establish two different areas in the field by putting rubber bands on the geoboard--one we call Area A, the other Area B.



What is P(Landing in Area A)? $\boxed{\frac{1}{8}}$

What is P(Landing in Area B)? $\boxed{\frac{1}{4}}$

What is P(Landing in Area A or in Area B)?

$$\boxed{P(A \text{ or } B) = P(A) + P(B) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}}$$

We can check this by comparing outcomes.

A landing in how many squares gives a favorable outcome? [6]

How many total possible outcomes? [16]

What is P(Landing in Area A or in Area B)? $\left[\frac{6}{16} \text{ or } \frac{3}{8}\right]$

Suppose we consider Areas A and B as lakes.

What is the probability that on any one jump the parachutist does not land in a lake? $\left[1 - \frac{3}{8} = \frac{5}{8}\right]$

Now suppose the parachutist makes two jumps.

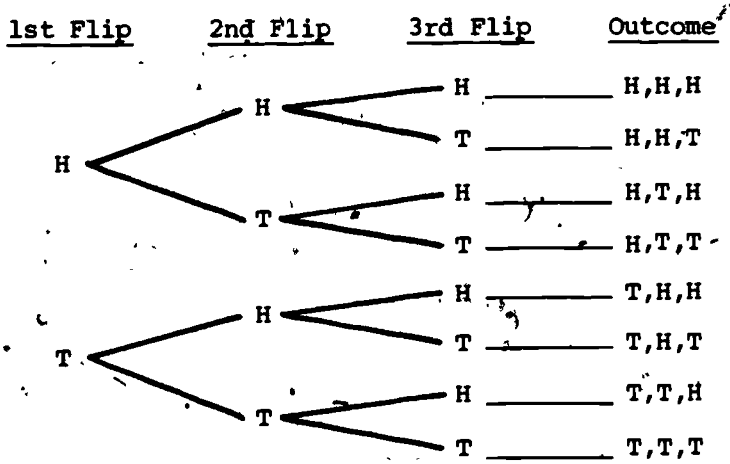
What is the probability that he lands in Area A on the first jump and in Area B the second jump?

$$\left[P(A \text{ and } B) = P(A) \times P(B) = \frac{1}{8} \times \frac{1}{4} = \frac{1}{32}\right]$$

What is the probability of landing in Area A twice in succession? $\left[\frac{1}{64}\right]$

For variety, change the sizes of Areas A and B or use additional areas. Also see Mathematics on the Geoboard by Niman and Postman for additional activities.

3.26 Tree diagrams are a helpful tool for calculating probabilities. If we are asked, "What is the probability of getting exactly two heads in three flips of a coin?" we can make a tree diagram with each branch representing one of the outcomes of each flip:



The probability of any one outcome is $\frac{1}{8}$. Of the 8 outcomes, there are three (H,H,T; H,T,H; T,H,H) where there are exactly two heads, hence the probability sought is $\frac{3}{8}$.

> What is the probability of getting more than one tail in three flips of a coin? [There are four outcomes where two or more tails occur, hence P(more than one tail) is $\frac{4}{8}$ or $\frac{1}{2}$.]

Here are additional problems where recourse to a tree diagram is useful:

> What is the probability that a family with three children will have at least one girl, assuming that the probability of having a boy is $\frac{1}{2}$? [$\frac{7}{8}$]
>
> What is the probability that the first two children in a family will be boys? [$\frac{1}{4}$]
>
> If one flips a penny, a nickel, and a dime, what is the probability of having the coins that land heads represent a value of 11 cents? [$\frac{1}{8}$]

3.3 Games and Problems

3.31 ODDS AND EVENS. This simple game has been played in Europe for centuries. Two players simultaneously bring their right hands from behind their backs using fingers to indicate a number from 0 through 5. If the sum of the two numbers is even, one player scores a point; if the sum is odd, the other player scores a point.

3.32 TELEPHONE NUMBERS. For a group of 3 to 10 students. Have each member of the group write his or her telephone number on a slip of paper. Ask how many think there are two persons in the group whose telephone numbers have the same last digit. Let children guess how large a group must be before the probability is $\frac{1}{2}$ that two group members have telephone numbers with the same last digit. After guesses have been recorded, compare telephone numbers within the group to determine if there is a match, that is, an instance of two telephone numbers with the same last digit. [With a group of 4 people, the probability is slightly less than $\frac{1}{2}$ that two in the group have the same last digit of their telephone numbers.]

An explanation of how the probability of approximately $\frac{1}{2}$ was determined is furnished as a matter of interest, and not with the expectation that it will be relayed to students.

We recall from Concept 3.14 that P(A) = 1 - P(Not A). In the terms of our problem, this means that the probability of two persons in a group having telephone numbers with the same last digit equals 1 minus the probability that no two people in the group have telephone numbers with the same last digit. We proceed as follows to calculate the latter probability. Suppose there is one person in a room. It is certain, that is, P = 1 or $\frac{10}{10}$, that no one else in the room has a telephone number with the same last digit.

A second person enters. His telephone number will have the same last digit in one case out of 10; therefore the probability that his number does not have the same last digit is $\frac{9}{10}$ and the probability that neither of the two persons in the room has a number with the same last digit is $\frac{10}{10} \times \frac{9}{10} = \frac{90}{100} = .9$.

A third person enters. The probability that he does not have a telephone number with the same last digit as the two already there is $\frac{8}{10}$ and the probability that no two persons in the room have numbers with the same last digit is $\frac{10}{10} \times \frac{9}{10} \times \frac{8}{10} = \frac{720}{1000} = .72$.

Following the same line of reasoning, after the fourth person enters the probability that no two have numbers with the same last digit becomes $\frac{10}{10} \times \frac{9}{10} \times \frac{8}{10} \times \frac{7}{10}$. The value of this fraction is $\frac{5040}{10000}$ or .504. Thus, we conclude that when there are four people in a group, the probability that two of them do have telephone numbers with the same last digit is 1 - .504, or just slightly less than $\frac{1}{2}$.

3.33 TELEPHONE NUMBERS IN A LARGER GROUP. In groups of 11 to 20 persons, ask for guesses as to the probability that two persons in the group have telephone numbers with the same last two digits in the same order. Record guesses and then compare telephone numbers. Using an extension of the procedure used in the preceding paragraph, it can be shown that in a group of 12 people, the probability that two of them have telephone numbers with the same last two digits in the same order is approximately $\frac{1}{2}$.

3.34 BIRTHDAYS. For an entire class. Have each class member write the month and day of his birthday on a slip of paper. Ask how many believe that there are two class members who share the same birthday. Ask for estimates of what the probability is that two members of the class have the same birthday. Again using the procedure of paragraph 3.32, it can be shown that with 23 persons in a group the probability is slightly more than $\frac{1}{2}$ that two persons have the same birthday (month and day).

3.35 QUACKY QUOTIENTS. This game, for two players, provides excellent practice in dividing whole numbers. Player A and Player B each select a number from 1 through 9 by spinning a spinner with 9 numbered sections or drawing a card from a set of cards numbered 1 through 9. Player A's number is divided by Player B's number. If the first digit of the quotient is a 1 or a 2 or a 3 (regardless of the decimal point), Player A wins a point. If the first digit of the quotient is 4, 5, 6, 7, 8, or 9, B wins a point. After several games have been played, ask if Quacky Quotients appears to be a fair game, that is, a game where each player has an equal chance of winning. The answer to this question lies in making a table showing all possible quotients for the digits 1 through 9. [50 of the 81 entries in the table have 1, 2, or 3 as a first digit, hence the probability is $\frac{50}{81}$ or a little more than .6 that Player A wins.]

Why can't we use 0 as one of the digits on the spinner or set of cards? [Division by zero is undefined.]

CONCEPT GROUP 4--Counting

Objective: To have children learn how to determine the number of different permutations and combinations of the members of a set.

4.1 Concepts

4.11 The Multiplication Principle. If one thing can be done in m ways and another in n ways, there are m x n ways to do the two things. Example: A girl has 4 blouses and 3 skirts. There are 4 x 3 or 12 different ways of choosing a blouse and a skirt.

4.12 Permutations. The different ways of arranging the members of a set, where each different order is considered a different arrangement, are called permutations.

For the letters a and b, there are two permutations, ab and ba. To find the number of permutations of a group of three different elements or objects, we ask these questions:

In how many ways can the first element be chosen? 3

After one element is placed first, in how many ways can the second place be filled? . . . 2

After one element is placed first and another second, in how many ways can third place be filled? 1

Any one of the three can be chosen for first and either of the two remaining chosen for second, so the number of permutations is: 3 x 2 x 1 = 6. For the group of letters a, b, and c, the six permutations are:

abc bac cab
acb bca cba

If there are four elements or objects in a group, the number of permutations is 4 x 3 x 2 x 1. This product is called "four factorial" and is written 4!. If there are seven objects in a group, the number of permutations is 7!. If there are n objects in a group, there are n! permutations of those objects. For any positive integer n,

$$n! = n(n - 1)(n - 2) \ldots (1)$$

Suppose we want to determine the number of permutations of any 3 objects out of a group of 5. In such a situation, we have 5 choices for the first object, 4 choices for the second, and 3 choices for the third, making 5 x 4 x 3 or 60 permutations in all. Note that the factors 5, 4, and 3 are the first three terms of 5!. In general, where r is not zero, we can find the number of permutations of r objects out of a group of n objects by taking the first r terms of n![8]

4.13 Combinations. Combinations are different selections of members from a set without regard for the order thereof.

There is just one combination of all the elements of a set. For the letters a and b, there is one combination, written either ab or ba. Similarly, there is only one committee of five that can be formed from a group of five persons.

How do we find the number of combinations of part of the elements of a group? Suppose we want the number of combinations of 2 out of three letters in the word "cat." By inspection, there are three, ca, ct, and at. Is there a general procedure to obtain the number of combinations of r out of n objects of a group? One way to do so is to start with the number of permutations of the n objects. For 2 out of the 3 letters of "cat," there are 6 permutations.

ca,ac   ct,tc   at,ta

Each of these 3 pairs has two permutations. To eliminate duplication of arrangements which are the same except for order, we divide 6 by 2 and arrive at the answer 3, the number of combinations of 2 out of the 3 letters of "cat." To review, we divided the number of permutations of 2 out of 3 letters by the number of permutations of 2 letters, thus:

$$\text{The combinations of 3 letters taken two at a time} = \frac{\text{the number of permutations of 2 out of 3 letters}}{\text{the number of permutations of 2 letters}}$$

From paragraph 4.12, the number of permutations of 2 out of 3 objects is the first 2 terms of 3! and the number of permutations of 2 objects is 2!, hence

$$\text{The combinations of 3 letters taken two at a time} = \frac{\text{the first 2 terms of 3!}}{2!}$$

More generally, the number of combinations of n objects taken r at a time is:[9]

---

[8] The number of permutations of r objects from a group of n objects is also given by the formula:

$$\frac{n!}{(n-r)!} \text{ (with 0! defined as 1)}$$

[9] The number of combinations of r objects from a group of n is also given by the formula:

$$\frac{n!}{(r!)(n-r)!} \text{ (where 0! is defined as 1)}$$

The number of combinations of n objects r at a time $= \frac{\text{the first r terms of } n!}{r!}$

Applying this to some examples, we have:

The number of combinations of 4 objects taken 3 at a time $= \frac{4 \times 3 \times 2}{3 \times 2 \times 1} = 4$

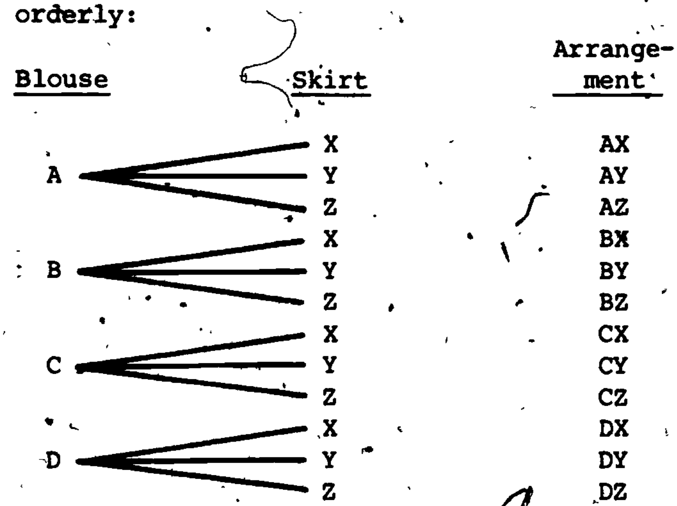
The combinations of 1 out of a group of 4 objects $= \frac{4}{1} = 4$

It is not accidental that the number of combinations of 3 objects out of a group of 4 turned out to be the same as the combinations of 1 out of 4 objects. In all cases, the number of combinations of r objects from a group of n objects is the same as the number of combinations of n - r objects out of n objects.

For the sake of consistency, we say that there is just one combination of none of the objects of a group.

4.2 Activities

4.21 The blouse and skirt example mentioned in Concept 4.11 is useful to illustrate and clarify the Multiplication Principle. Young children may enjoy working with cutout paper blouses and skirts. A tree diagram helps make counting orderly:



Four blouses and three skirts produce twelve arrangements.

How many arrangements are possible with 5 blouses and 4 skirts? [20]

Encourage children to look for the method of finding the number of arrangements without counting; i.e., to multiply the number of blouses by the number of skirts. Other problems to be worked on include:

What is the number of possible outcomes when a 3-way spinner is spun twice? [9] How many for a 12-way spinner? [144]

How many possible outcomes when a tetrahedron is rolled and a 6-way spinner is spun? [24]

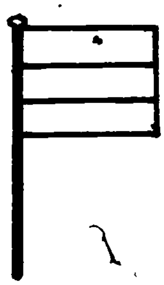
What is the number of ways a boy and a girl can be chosen class leaders if there are 6 boys and 8 girls to choose from? [48]

How many different double-dip cones can be made from 6 different flavors? [36] From 31 flavors? [961]

4.22 Introduce permutations by asking the students to make as many arrangements as they can of the letters of the word "PAN." Explain that each of the possible arrangements (PAN, PNA, ANP, APN, NPA, NAP) is called a permutation. Use the explanation in Concept 4.12 to show how the number of permutations of a given group of objects is determined. Introduce the notation $3! = 3 \times 2 \times 1$ and $4! = 4 \times 3 \times 2 \times 1$. Establish that the number of permutations of n different objects is n! and the number of permutations of r out of n objects is the first r terms of n!.

Among the many class activities and projects involving permutations and associated probability questions are:

Provide crayons and flag outlines such as that shown at the right. How many ways can a 3-stripe flag be made using all of three different colors? Extend to flags with more stripes. Let the class make a chart showing number of colors and number of ways flag can be colored. [6 ways for 3; 24 for 4; 120 for 5]



With flags having red, orange, and green stripes, what is the probability that the orange stripe and the green stripe will be next to each other? [4 out of 6 or $\frac{2}{3}$]

List and count the possible arrangements of 2 letters out of the word "TENS"; then 3 letters, then all the letters. [12; 24; 24]

Have a group of 2 children, one of 3 children, and one of 4 children line up in as many different ways as they can, keeping a record of the arrangements they make.

Count and record the number of different arrangements on a shelf that can be made with 2, 3, 4, or more books.

Find how many 3-letter arrangements are possible using the 26 letters of our alphabet. [26 x 25 x 24 where all three letters are different.]

How many different sets of initials are possible if each person has one surname and two given names? [26 x 26 x 26]

One "way" to answer a five-question true-false test is T-T-F-T-F. How many other "ways" are there? [31]

How many license plates can be made with one letter followed by 4 digits? [26 x 10 x 10 x 10 or, if we rule out 4-digit numbers starting with zero, the possibilities are 26 x 9 x 10 x 10.]

If a set of books is marked Volume I, Volume II Volume III, and Volume IV, what is the probability that the set will be put on a shelf in the right order by someone who is blindfolded? $\left[\frac{1}{24}\right]$

What is the probability that a permutation of the four letters A, R, S, and T spells a common English 4-letter word? $\left[4 \text{ out of } 24 \text{ or } \frac{1}{6}\right]$

A baseball team is the same regardless of how one lists the players. A batting order, however, differs depending upon order. How many batting orders are possible using 9 players? [9!]

The number of ways 10 boys can be arranged in a line is 10!. What is the probability that 10 boys will line up in descending order of height? $\left[\frac{1}{10!}\right]$

4.23 The procedure of Concept 4.13 can be used to explain how the number of combinations of a group of objects is calculated. Alternatively, children should be able to determine the number of combinations of a group of objects by listing them. A systematic approach to listing should be urged. Suggest starting with the letter or object at the left and work consistently from left to right. When listing the combinations of 2 letters from the word "PLAY," we would start with P, the left-hand letter, and take the next letter to the right to form PL. Next in order come PA and PY. Then go to L, the next letter not yet used as an initial letter, and list LA and LY. Finally, we go to A as the initial letter and make the combination AY.

The activities set forth below can provide experience either with listing or calculating combinations:

> If three coins are chosen from a group made up of a penny, a nickel, a dime, a quarter, and a half-dollar, how many different amounts of money are represented? [10]
>
> The manager of a baseball team had 6 outfielders on his squad. In how many ways could he pick three of them to play? [20]
>
> In how many ways could he assign the chosen three to the three outfield positions? [6]
>
> Determine the number of combinations of 3 council members that can be elected from a slate of 10 candidates. [120]
>
> Mary has 15 friends she would like to invite to a party but she can only invite 10 of them. In how many ways can she choose 10 out of the 15? [3003]
>
> A teacher has to pick 4 out of the 9 girls in her class for safety patrol. How many different groups of 4 girls can she choose? [126]
>
> How many different 5-card hands can be made from a standard 52-card deck of cards?
>
> $$\frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$$

The following problem can be solved by using one's knowledge of combinations or by using P(A and B):

> Two students are to be chosen at random from a class that consists of 8 boys and 13 girls. What is the probability that the two chosen are both boys? $\left[\frac{14}{105} \text{ or } \frac{2}{15}\right]$

4.24 One of the most fascinating relations in probability theory is named after a French mathematician who helped develop the theory. It is called the Pascal Triangle.

Distribute a sheet containing the table on page 40. For the information of teachers, the number of combinations appears in each circle but these figures should not be shown on the sheet distributed to students. The number of combinations in the circle at the left of each row is 1; there is only one combination of the objects of a group taken none at a time. This is equivalent to saying that there is only one way all the objects of a group can be left out. The number of combinations in the right-hand circle of each row is also 1; there is only one combination of all the objects of a group.

Have the class fill in the empty circles in the first five rows using procedures they are already familiar with to determine the various numbers of combinations. Much of the information will already have been calculated. After any row is completed, have students add the numbers of combinations in that row and put the row total in the column at the right, noting that each row total is twice that of the row above.

Number of Combinations

Number of Objects							Row Total
0	1 (0 of 0)						1
1	1 (0 of 1)	1 (1 of 1)					2
2	1 (0 of 2)	2 (1 of 2)	1 (2 of 2)				4
3	1 (0 of 3)	3 (1 of 3)	3 (2 of 3)	1 (3 of 3)			
4	1 (0 of 4)	4 (1 of 4)	6 (2 of 4)	4 (3 of 4)	1 (4 of 4)		
5	1 (0 of 5)	5 (1 of 5)	10 (2 of 5)	10 (3 of 5)	5 (4 of 5)	1 (5 of 5)	

Ask the class to try to find a way of obtaining the number of combinations in the 6th row of the triangle without calculating the number of combinations for each circle. The first and last entries are always 1; the second and next-to-last entries are always the same as the number of objects for the row. Some students may hit on the key; the number in any circle is the sum of the number above it and to the left and the number above it and to the right. Use this key to complete the 6th row and any subsequent rows desired.

The Pascal Triangle presents one of the best opportunities in all mathematics to explore number patterns and symmetrical relationships. The numbers making up the Pascal Triangle occur in several different mathematical situations, not only in the study of probability. Encourage students to look for patterns such as the progression of numbers on diagonal lines; common factors for all the numbers in a row other than the 1's at beginning and end; the symmetry of the rows. Much additional information about Pascal Triangle relationships is to be found in the March 1974 Arithmetic Teacher, pages 190-198.

It should be pointed out that the numbers of combinations shown in the Pascal Triangle are identical with the outcomes of flipping coins. If we are flipping two coins, for example, we can look at the row for two objects and find that there is 1 outcome with no heads (T,T); 2 outcomes with 1 head (T,H and H,T); one outcome with 2 heads (H,H); and a total of 4 possible outcomes.

Once children become familiar with the Pascal Triangle, they should be able to do the following problems:

> What is the probability of getting 4 heads and 1 tail when 5 coins are tossed? [In Row 5, there are 32 total outcomes. The 5th box in the row shows that there are 5 combinations of 4 objects (heads) out of a group of 5, hence the probability of 4 heads in 5 tosses is $\frac{5}{32}$.]
>
> In 3 tosses of a single coin, what is P(3 heads)? [$\frac{1}{8}$] [Tossing 1 coin 3 times is the same as tossing 3 coins once.]
>
> In 4 tosses of a single coin, what is P(2 tails)? [$\frac{3}{8}$]
>
> From 4 persons, how many committees of 3 can be chosen? [4]
>
> From 6 people, how many committees of 3 can be chosen? [20]
>
> If you are allowed to choose any 4 out of 6 problems on a test, how many different selections of 4 problems are possible? [15]

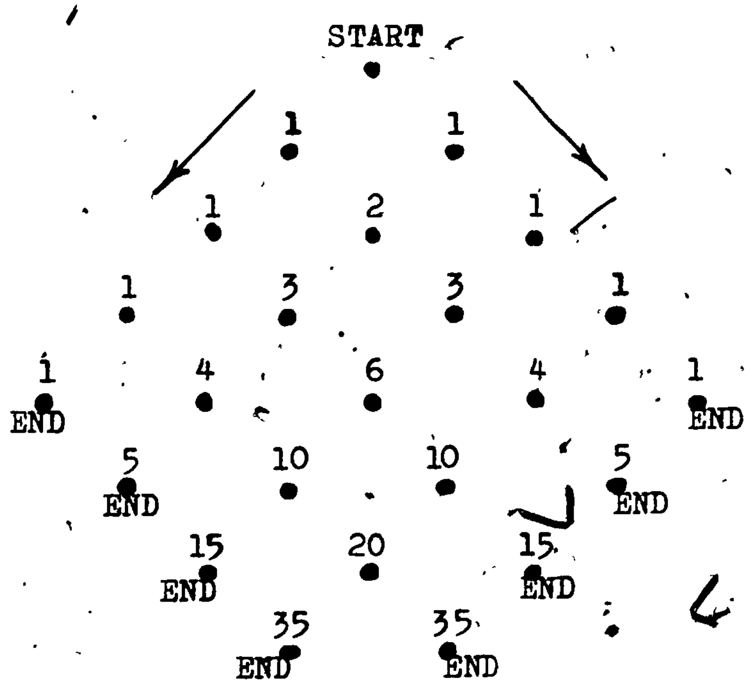
4.3 Games

4.31 BOX WORD. For a group of children. Make four identical boxes. In Box 1, put small pieces of paper on which are written the letters e, n, and t. In Box 2, put the letters t, r, and a. In Box 3, put a, b, and y; in Box 4, put a, g, and b. A child selects a box and draws the letters from it, one at a time. As each appears, the letter is written down in order. If the letters spell a word, the child scores a point. From time to time, the letters in the boxes can be replaced with others that spell different words.

Teacher Question:

What is the probability of scoring a point as each box is used? [Box 1 or Box 4, $\frac{1}{3}$; Box 3, $\frac{1}{6}$; Box 2, $\frac{1}{2}$]

4.32. GEOROLL. For 2 players. Use a 5 x 5 geoboard. Cut out a sheet of thin paper the size of the geoboard and press it down so the paper lies flat and the pegs stick through holes in the paper. Turn the geoboard so one corner points away from you. Mark the pegs so the geoboard looks like this:



Both players start by placing a marker on the top peg. Use a paperclip as a marker or a circular piece of paper that fits over the pegs readily. The players take turns flipping a coin. If a player flips a head, he moves one peg downward and to the left. If he flips a tail, he moves one peg downward and to the right. Each player's score is the sum of the numbers on the pegs he reaches as he goes from START to END. The player with the smaller sum wins.

Teacher Questions:

What is the lowest possible number of moves to win? [4]

What is the lowest winning score? [4]

What is the largest possible number of moves? [7]

What is the highest score? [77]

CONCEPT GROUP 5--Sampling

Objective: To give children experience with the process of sampling.

5.1 Concepts

5.11 Random Samples. A sample is a "random sample" if each member of the group from which it is taken has an equal probability of being chosen for the sample.

5.12 Predictions from Samples. It is likely that information from a relatively small random sample will give a reasonably accurate prediction about the make-up of the group from which the sample is drawn. The larger the sample, the greater the likelihood of accuracy in the prediction.

5.2 Activities

5.21 To introduce the idea of a random sample, propose the following situation to your class. The principal of a school wanted to find out how the students liked arithmetic. He didn't want to take time to question everyone, so he picked from each class a sample of three students who were good in arithmetic and asked their opinions. Do you think he got an accurate estimate of how the students in the school liked arithmetic? [Almost certainly not, because good students tend to like a subject more than poor students, and none of the poorer students had a chance to express opinions. Another problem revolves around the tendency of students to tell the principal what they think he would like to hear.]

What might the principal have done to get more accurate information, without asking everyone in school? Explain that results are more accurate and dependable when a sample is a "random sample;" that is, a sample where each person or thing in the group from which the sample is taken has an equal chance or probability of being chosen. Ask for comments on the following samples. Are they random samples?

Group from which sample is chosen	How sample was chosen
1. Automobile drivers	Taking every 3rd car from those on a suburban street between 11:00 A.M. and noon on a weekday morning.
2. The population of a city	Taking the first name on each page of the local telephone directory.
3. An elementary school class	Tossing a die to select one of the first six names and then every 6th name thereafter on an alphabetical listing of class members.

Sample No. 1 cannot be considered a random sample. A sample selected from cars on a suburban street in mid-morning would be likely to include far more women than men. For Sample No. 2, the answer is the same; families or individuals without a phone or with an unlisted phone are not included. For Sample No. 3, the selection method appears to give every class member a chance of being selected; hence it can be considered a random sample.

5.22 Random samples are sometimes chosen by referring to a table of random numbers; that is, a table of numbers lacking any pattern or regular arrangement.

Have a group of students use the local telephone directory as a table of random numbers to select a sample of 5 students from your class. A listing of all the students in the class should be available. Say there are 26 students on the list. Arbitrarily choose one number in the directory as a starting point. From that point, put a check on each subsequent number which has its last two digits in the sequence 01 through 26. Suppose the next 5 numbers so checked ended in 02, 19, 16, 22, and 11. Then the sample of 5 students consists of the 2nd, 19th, 16th, 22nd, and 11th names on the class listing.

5.23 As a demonstration of sampling and how it works in practice, put 10 blocks of the same shape and size in a bag, some blocks being of one color, the remainder another color. Tell the class only the total number of blocks in the bag. Have children, one after the other, draw a block from the bag, record its color, replace the block, and shake it up with all the others in the bag. After 5 blocks have been removed, ask the class for estimates of how many blocks of each color they think the bag contains. Continue removing more blocks until 10 have been withdrawn, recorded, and replaced. Allow estimates to be revised on the basis of the additional information available, if children wish to do so. Then open the bag and compare the contents with the estimates. The information developed by sampling will generally (but not necessarily) be close to the actual numbers in the bag.

This experiment can be varied by making a different distribution of the 10 blocks between the two colors, by using blocks of 3 or more colors, or by making more withdrawals before disclosing the bag's contents.

As the number of samples increases, predictions tend to approach actual quantities more closely. Children should appreciate that the larger the sample, the greater the likelihood that the information predicted is an accurate representation of the group from which the samples are taken.

5.24 In some situations, sampling is virtually the only technique to develop information about a large group. For example, suppose we want to find out how frequently each of the 26 letters of the alphabet is used in written English. It is clearly impossible to examine everything that is written in English; sampling is the only practical way to proceed. Ask for suggestions for appropriate samples from which we could determine the frequency with which we use A's, B's, C's, and so on in written English. Should we use passages from novels? A classic? A magazine? A newspaper? Textbooks? A comic book? Poetry? How long should samples be?

Agree on a selection of samples, say ten or more passages each of 150 to 200 words. The source or content is immaterial so long as there is variety. Have a count made of the number of times each letter is used throughout the samples. Accumulate the information and list the 26 letters of the alphabet according to frequency of use, the most frequent first. Compare your results with the order of frequency found in a survey made of a large number of passages:

E T A O I N S H R D L C U

The most frequently used letter in English is E; the next most frequent is T. Some sampling variation in the position of the letters after E and T is to be expected, though the probability is high that the order of frequency determined from the passages your class selected will not vary too much from the listing above.

*5.25 Sampling has a wide variety of practical uses. Television ratings are based on sampling, as are public opinion polls, marketing surveys, election forecasts, and many scientific investigations. For example, ecologists can use a sampling process to determine the approximate number of fish in a pond. They could proceed as follows:

a. Catch 100 of the fish in the pond, mark each of them with a harmless, indelible dye, and put them back in the pond.

b. Let some time elapse so the fish that are dyed can distribute themselves uniformly throughout the pond.

c. Now catch a sample of 50 fish and count the number marked with the dye. Suppose there are 4 dyed fish in the 50.

d. Set up a proportion based on the assumption that the ratio of marked fish to total fish is the same in the sample of 50 as it is throughout the whole pond. Then:

Ratio in sample of 50 = ratio in whole pond

$$\frac{4}{50} = \frac{100}{\text{Total}}$$

This method predicts that there are 1250 fish in the pond.

Teacher Questions:

> What circumstances could cause the estimated number of fish to be too high or too low? [Uneven distribution of fish after they are returned to the pond; death of fish caught and returned; fish caught by other fishermen.]
>
> Is 1250 likely to be a rough approximation of the actual number of fish in the pond? [Yes, if a reasonable margin of error is acceptable.]

The sampling technique just described can be paralleled in the classroom using bottlecaps. Have the children collect a large (but unknown) number of bottlecaps and store them in a container. Remove a double handful of bottlecaps, count them, mark them distinctively, and return them to the container. Shake them up thoroughly in the container, and then withdraw another double handful of bottlecaps. Count the number of marked bottlecaps and the total number of caps in the last sample withdrawn. An estimate of the unknown grand total number of bottlecaps is arrived at by equating the two ratios:

$$\frac{\text{No. of marked caps in last sample}}{\text{Total caps in last sample}} = \frac{\text{No. marked caps put back into container}}{\text{The unknown grand total number of caps}}$$

Left-handedness in school populations is a characteristic which can be studied by sampling techniques. First count the number of children in your class who write left-handed. Confining the investigation to those who write left-handed is more clear-cut than attempting to survey left-handedness in general. Next predict the number of left-handed writers in all classes at the same grade level by multiplying the total number of children in the grade by the fraction:

$$\frac{\text{left-handed writers in your class}}{\text{total number of children in your class}}$$

If there are 4 out of 26 in your class who write left-handed and 140 in the grade, the predicted number of left-handed writers in the grade is:

$$140 \times \frac{4}{26} = 22 \text{ (rounded to the nearer whole number)}$$

A count of the number of left-handed writers throughout the grade gives a check on the accuracy of the prediction. This investigation can be carried a step further by using the actual number of left-handed writers in the grade to predict the total number throughout the school. An alternative approach is to select say three students at random from each class in the school. Use the students so selected as your sample; determine the number of left-handed writers in the sample, and from that predict the total number of left-handed writers in the school. Which approach predicts more accurately?

5.3 Game and Class Project

5.31 SECRET MESSAGE or CRYPTOGRAM. A coded message in which each letter is replaced throughout by one other letter or symbol is called a "cryptogram."

Distribute copies of the following code message:

CRYPTOGRAM

XAG VGBL AJKJ JTHMGKPZC PZ
WZ GMN QPZJ. XSJB LWA W VGT.
WZN KWPLJN XSJ RGOJK. PZLPNJ
XSJKJ AJKJ HPJRJL GD KGRU
APXS VKPCSX LHJRUL PZ XSJQ.
"CGMN," XSJB LWPN. YZDGKXYZWXJMB,
PX XYKZJN GYX XG VJ W RGQQGZ
QWXJKPWM RWMMJN PKGZ HBKPXJL.

There are many avenues that will lead to a solution of this cryptogram. Give students some time to look for a solution on their own; for those who are unsuccessful, you can present the approach indicated in the next few paragraphs.

Can we use what we learned about the frequency of letters in English to help us find out what this message says? The letter that is used most frequently in the message is likely to stand for what letter of the alphabet? [E] What might the next most frequent letter stand for? [T]

Have the class count the number of times each letter of the alphabet is used in the secret message.

On the basis of frequency, what letter is most likely to stand for E? [J] Which one for T? [X] Have students put an E over each J in the message and a T over each X. What additional clues do we see? For the 15th word we have

T . E
X S J

This could be TIE, TOE, or THE. The 18th word helps us decide:

T . E . E
X S J K J

S certainly represents H and XSJKJ is either THESE or THERE. Now we look at the next word, AJKJ. It must end in either -ESE or -ERE, if our conclusions are correct up to this point. We know no words that fit the pattern -ESE, but -ERE could be HERE, MERE, or WERE. Best guess--XSJKJ AJKJ stands for THERE WERE!

We now have, or think we have, J as E, X as T, S as H, K as R, and A as W. From these we go on to find the words TO and WITH; and from those we gradually translate every word in the message. Solution:

> Two boys were exploring in an old mine. They saw a box and raised the cover. Inside there were pieces of rock with bright specks in them. "Gold," they said. Unfortunately, it turned out to be a common material called iron pyrites.

The whole decoding process was based on the information about frequency of letter use, developed through the technique of sampling. If additional code messages are made, they should be from 100 to 150 letters in length, preferably with E the most common letter.

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