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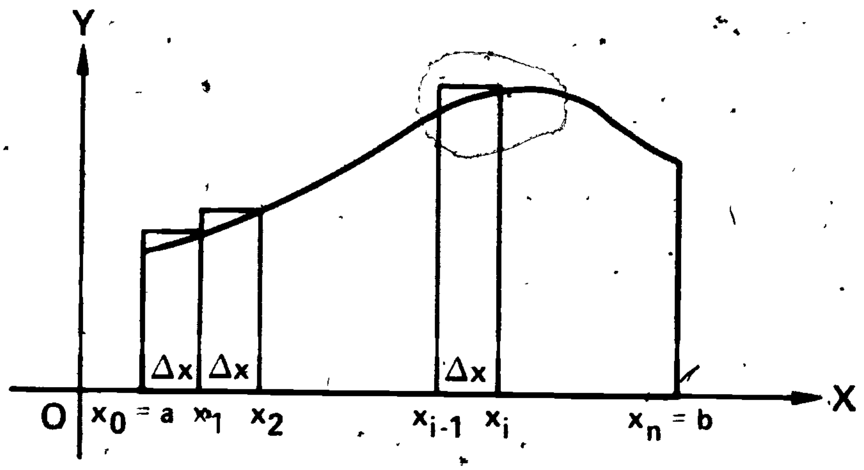
ABSTRACT

The purpose of this instructional guide is to assist teachers of calculus in the organization and presentation of the course content to best meet the needs of the student. The behaviors expected of the student have been organized into eleven units. These units include the topics recommended for those students preparing for the CEEB advanced placement BC test administered in May of each year. Within each unit is an introduction, a list of the instructional objectives for that unit, at least two sample performance objectives for most instructional objectives, sample assessment measures and answers for the assessment measures. Special features of this guide are a Performance Objective Index which keys each objective to currently approved text materials, and a list of suggested assignments. (MN)

Montgomery County Public Schools

CALCULUS

Instructional Guide



Donald L. Hymes

Instructional Guide

for

CALCULUS

Secondary Mathematics

Montgomery County Public Schools
Rockville, Maryland

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POINT OF VIEW FOR MATHEMATICS EDUCATION IN MONTGOMERY COUNTY

A provocative activity which teachers often use with pupils at various levels is that of trying to imagine a world without numbers -- what would it be like? Such a world is quite difficult to imagine. The idea of number continues to play an important role in virtually all aspects of our world; and mathematics, therefore, constitutes a program of considerable importance in the schools.

As a discipline, mathematics is truly the art and science of abstraction. Characteristics of the physical world are converted into abstract ideas and symbols; these are then manipulated through mathematical operations to produce information and theorems about less easily observed aspects of the world. Recent evidence supports the contention that children's experiences with concrete materials are vital to ... later conceptual development. The school program thus proceeds from the concrete to the abstract.

The concepts of mathematics acquire greater meaning when they can be applied to the world in which we live. Because the variety and extent of mathematical applications have grown so rapidly in recently years, it is impossible for any one person to be conversant with the entire field. The school program must therefore be developed so that mathematical applications are selected and presented as efficiently as possible and with the intent of challenging pupils at all levels to see mathematics as an independent discipline as well as a tool for the advancement of other disciplines.

THE MONTGOMERY COUNTY MATHEMATICS PROGRAM

The Montgomery County mathematics program is designed and implemented to take into account the logical and relatively sequential nature of mathematics. Equally important is the realization that the rate at which individual students learn mathematics varies significantly. The mathematics program is structured to encourage various approaches which allow students to progress at their individual rates.

In general terms, the instructional program in mathematics should help each student to:

- Develop basic skills in using the vocabulary and symbols of mathematics
- Develop skills in recognizing common geometric shapes
- Develop basic skills in computing
- Develop basic skills in working with geometric shapes
- Develop basic skills in measuring
- Develop understanding of the vocabulary and symbols of mathematics
- Develop concepts related to common geometric shapes
- Develop understanding of computation
- Develop understandings in measurement
- Develop an understanding of basic principles related to the structure of mathematics
- Develop understanding and basic skills in problem solving
- Apply the principles of mathematical reasoning to the solution of problems
- Appreciate the significance of mathematics in daily living and its contribution to our cultural heritage
- Use mathematics as needed in daily living

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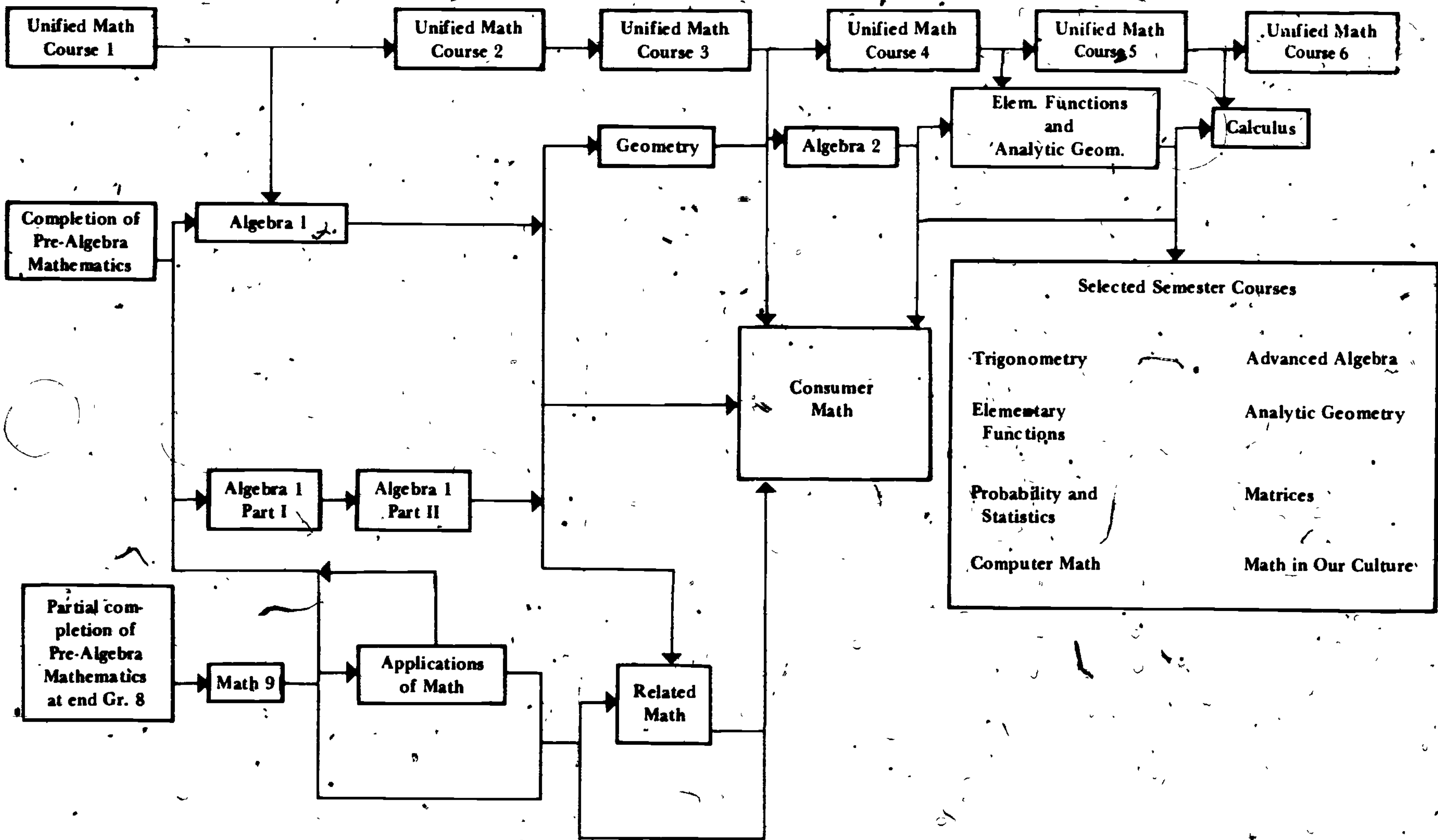

MATHEMATICS PROGRAM PATTERNS CHART

INTRODUCTION

I. PURPOSE

The purpose of this instructional guide is to assist teachers of Calculus in the organization and presentation of the course content to best meet the needs of the students.

The Goals of Education, adopted by the Board of Education of Montgomery County in 1973, emphasize that the school program should be "...related to the needs, interests, and concerns of each student." Calculus serves to meet the needs of students accelerated in mathematics to the point of enrolling in this college-level subject while still in high school. Many students seek advanced placement in college mathematics through enrollment in this course; since advanced placement is partially based on May examinations, timing and order of presentation of topics are emphasized in this guide.

The Goals of Education further state that:

> In addition to acquiring academic skills, each individual should develop his /her intellectual capabilities to the fullest extent possible. Therefore, the school will encourage each pupil to think creatively, to reason logically, to apply knowledge usefully, to deal with abstract concepts, to solve problems.

The study of Calculus involves the use of the highest levels of these skills.

The Point of View for Mathematics Education in Montgomery County describes mathematics as the "...art and science of abstraction," yet states: "The concepts of mathematics acquire greater meaning when they can be applied to the world in which we live." The study of Calculus is unique in that the use of abstract symbolism becomes an instrument of application to make clear the solution of problems otherwise difficult or impossible to solve.

The place of Calculus in the sequence of mathematics courses in Montgomery County is shown on page vii of this guide. The instructional objectives of the course are described in the MCPS Program of Studies.

By the end of Calculus, the student should be able to:

- apply the rules for limits of sums, of products, of quotients, of polynomials, and of a function between two functions which have the same limit when the limit is taken as the variable approaches a constant, and as it approaches infinity

- apply the rules for determining the derivatives of a monomial function and for finding the sum, product, and quotient of differentiable functions

- apply the method of implicit differentiation to specific examples of the general function $y = u^{p/q}$
- apply the chain rule for derivatives to specific examples involving composite functions
- construct the equations of the tangent line and the normal line at selected points of an algebraic function
- construct the graph of an algebraic function, listing the critical points
- identify and construct the solution of problems that call for minimizing or maximizing a function
- apply Rolle's Theorem, the Mean Value Theorem, and l'Hopital's Rule where appropriate
- apply the Fundamental Theorem of Calculus to determine the area of a given region
- identify and compute the sum of Riemann integrals
- apply the definite integral to solve problems involving area, velocity, and acceleration; volume of a solid; area of a surface of revolution; total amount of mass; and work as the product of a given force moved through a given distance
- apply formulas for differentiation and integration of trigonometric, inverse trigonometric, logarithmic, and exponential functions
- select and apply suitable methods for the evaluation of a set of assorted integrals
- apply the rules of hyperbolic functions and inverse hyperbolic functions in problems of equilibrium, tension, area, volume, and center of gravity
- apply the theorems on sequences to construct the limits of sequences
- apply the comparison, ratio, and integral tests for convergence to a given series
- compute the value of given trigonometric or logarithmic functions using Maclaurin's or Taylor's series expansion

II. ACKNOWLEDGMENTS

Under the direction of William J. Clark, Director of Division of Academic Skills, and Ellen L. Hocking, Coordinator of Secondary Mathematics, this

instructional guide was prepared by the following professionals during the period 1971-77:

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III. OVERVIEW

The course of study for Calculus has been constructed to assist the teacher in organizing the content of Calculus to meet the needs of the students.

Each of the eleven units contains an introduction, a list of instructional objectives, a list of performance objectives, sample assessment measures for each objective, and answers for the assessment measures.

Special features of this guide are a Performance Objective Index, which keys each objective to currently approved text materials, and a list of Suggested Assignments, which may assist the teacher in pacing progress through the content material.

IV. STUDENT AUDIENCE

Calculus is an elective, one-credit course, offered to students who have successfully completed the objectives for Elementary Functions and Analytic Geometry. Many students enroll in Calculus in preparation for the Advanced Placement Examination or the College Level Examination Program (CLEP) of the College Entrance Examination Board. A combination of either score, the student's mathematics record, and other test scores may lead to advanced placement in the mathematics program of the college chosen.

V. ORGANIZATION AND SUGGESTED USES OF GUIDE

The behaviors expected of the students in any content area need to be clearly understood by both teacher and student. In this instructional guide, the behaviors have been organized into eleven units. These units include the topics recommended for those students preparing for the CEEB Advanced Placement BC Test administered in May of each year. Within each unit are the instructional objectives for that unit, at least two sample performance objectives for most instructional objectives, sample assessment measures, and suggested assignments. The Performance Objective Index keys the objectives to the textbook.

The fact that the advanced placement tests are scheduled for the middle of May requires that, for the BC test, Units I through VI be completed during the first semester. Units X and XI may be studied after the placement tests, provided materials from Unit XI on velocity and acceleration vectors have been included in Unit III. For the AB Placement Tests, Units I through V should be completed in the first semester, and a review of analytic geometry should be done instead of Units VIII and IX.

BIBLIOGRAPHY

BASIC TEXTBOOKS

Thomas, Jr., George B. Elements of Calculus and Analytic Geometry, 2nd ed. Reading, MA: Addison-Wesley Publishing, 1976.

SUPPLEMENTARY MATERIALS

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The Calculus Problem Solver, New York: R.E.A., 1974

Hackett, Shirley O. Barron's How To Prepare for Advanced Placement Examinations Mathematics. Woodbury, N. Y.: Barrons Educational Series, 1971.

McAloon, Kenneth, and Tromba, Anthony. Calculus of One Variable. Vol. 1 BC. New York: Harcourt Brace Jovanovich, Inc., 1972.

TV Calculus - Work Manual, 2 Vols., Owings Mills, Md.: Maryland State Department of Education, 1973.

SUGGESTED ASSIGNMENTS

A list of suggested assignments in the basic text has been compiled as an aid in pacing class instruction in Calculus.

This material, referenced to the text, has been compiled to provide the teacher with a key relating the instructional objectives to the topics of Calculus, to the reading on each topic, and to suggested assignments of problems. For most of the units, it has been presented in the order that it appears in the text. The 96 lessons should not be interpreted as being all of the same length or importance.

UNIT I

LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENTS	INSTRUCTIONAL OBJECTIVE
1	Definition of Limit of a Function	pp. 44-46	pp. 46-47; 1, 3, 4, 6, 9	1
2	Theorems About Limits	pp. 47-50	pp. 50; 1, 3, 5, 7, 9, 10	2
3			pp. 50; 13, 14, 18, 19	
4	More Theorems About Limits	pp. 50-55	pp. 55; 1-21 (odd)	2
5			pp. 56; 25-31	
6	Infinity	pp. 56-59	pp. 59-60; 2, 4, 5, 7, 9, 10, 11, 17, 19	3
7	Continuity	pp. 92-97	pp. 97; 1, 2, 3, 6, 9	4

UNIT II

LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENTS	INSTRUCTIONAL OBJECTIVE
8	Polynomial Functions and Their Derivatives	pp. 62-66	pp. 66-67; 2, 3, 5, 9, 13, 16, 17, 18, 20	1, 2, 3, 4
9	Rational Functions and Their Derivatives	pp. 67-70	p. 70; 1, 2, 3, 5, 8, 11, 14, 16, 17, 18	5, 6, 7, 8
10	Implicit Relations and Their Derivatives	pp. 71-76	p. 77; 3, 6, 9, 13, 17, 22, 24	7, 9
11			p. 77; 27, 31, 33	
12	Increment of a Function	pp. 78-80	pp. 80-81; 2, 5, 6, 9	11
13	Chain Rule for Derivatives	pp. 84-87	p. 87; 2, 4, 7, 9	10
14	Differentials dx and dy Formulas for Differentials in Notation of Differentials	pp. 88-91	p. 91; 3, 4, 5, 9, 12, 14, 17, 19, 20	11

UNIT III

LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENTS	INSTRUCTIONAL OBJECTIVE
15	Tangents and Normals	pp. 294-296	pp. 297-298; 1, 3, 5, 8, 9, 11, 15	1
16	Increasing or Decreasing Functions	pp. 102-105	p. 105; 3, 7, 10	2
17	Related Rates	pp. 105-108	p. 108; 1, 3, 7, 8, 9	3
18			p. 108; 2, 5, 6, 12, 14, 15	
19	Significance of the Sign of the 2nd Derivative	pp. 109-113	pp. 113-114; 3, 5, 7, 15	2
20			pp. 113-114; 8, 17, 18	
21	Maxima and Minima	pp. 118-123	pp. 123-125; 3, 4, 9, 18, 13, 23	3
22			pp. 123-125; 5, 7, 15, 24, 14, 28	
23	Rolle's Theorem	pp. 125-127	p. 127; 1-4	4
24	Mean Value Theorem	pp. 127-128	p. 130; 2, 3, 5, 6, 7, 8	5
25	l'Hôpital's Rule	p. 253	p. 253; 91, 92	6

UNIT IV

LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENTS	INSTRUCTIONAL OBJECTIVE
26	The Indefinite Integral	pp. 139-143	p. 143; 2, 3, 4, 9, 12, 16, 19, 22-25	1
27	Applications of Indefinite Integral	pp. 143-144	p. 145; 1, 3, 5, 8, 9, 14, 16	2
28	Limits of Trigonometric Expressions	p. 151	p. 155; 1, 5, 8, 11, 13, 20	I-2
29	Differentiation of Sines and Cosines	pp. 152-154	p. 155; 25, 29, 35, 36, 38	VI-1
30	Integration of Sines and Cosines	p. 154	p. 155; 40, 46, 50, 52, 54, 58, 60	VI-2

UNIT IV (cont.)

LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENT	INSTRUCTIONAL OBJECTIVE
31	Area Under a Curve	pp. 156-159	pp. 159-160; 1-5	2
32	Computation of Areas as Limits	pp. 161-163	pp. 163-164; 2, 4, 5	2
33	Areas by Calculus	pp. 164-166	p. 167; 1, 4, 6, 8, 12, 15	3
34	Trapezoid Rule for Approximating an Integral	pp. 174-176	p. 176; 1, 3, 4, 5	4
35	Simpson's Rule for Approximating an Integral	pp. 282-284	p. 284; 1, 2, 5, 6	4

UNIT V

LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENT	INSTRUCTIONAL OBJECTIVE
36	Area Between Two Curves	pp. 183-184	pp. 184-185; 1, 2b, 2d, 2f, 4	1
37	Distance	pp. 185-187	p. 187; 1, 3, 4, 11, 12, 13, 15	2
38	Volumes-Disk	pp. 188-194	pp. 194-195; 1, 3, 9a, 11a, b	3
39	Shell		p. 195; 5, 8, 9b, 11c, d, e, 12	
40	Other		p. 195; 13, 14, 15	
41	Length of Curve	pp. 199-202	pp. 202-203; 1, 2 5, 6	4
42			pp. 202-203; 3, 4, 7, 8, 9	
43	Area of Surface of Revolution	pp. 203-206	pp. 206-207; 1, 2, 4, 5, 7	5
44	Average Value of a Function	pp. 206-208	p. 208; 1, 3, 5. 6, 7	6
45	Moments and Center of Mass	pp. 208-212	p. 212; 2, 5, 6, 7, 11 p. 214; 9, 10	7
46	Work	pp. 214-217	pp. 216-217; 1, 3, 4, 6, 9	8

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LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENTS	INSTRUCTIONAL OBJECTIVE
47	Derivatives and Integration of Trigonometric Functions	pp. 221-222	pp. 222-223; 13, 15, 17, 19, 22, 38, 41, 42, 44, 47	1, 2
48	Derivatives and Integration of Inverse Trig Functions	pp. 228-230	p. 230; 5, 7, 9, 11, 13, 16, 18, 20	3
49	Natural Logarithm	pp. 230-231	p. 231; 2, 3, 6, 8	4
50	Derivative of ln x	pp. 232-233	p. 233; 2, 4, 8, 10, 16, 21, 24, 28, 30	5
51	Properties of ln x Graph of ln x	pp. 233-236	p. 234; 4, 10; p. 236; 1, 3, 4	4
52	Exponential Function	pp. 236-241	p. 242; 1a, h, 1, m, 2, 5, 7, 15, 17, 19	6, 7
53			p. 242; 3, 4, 8, 13, 16, 22, 25, 30, 31	
54	The Function e^u	pp. 243-245	p. 245; 3, 6, 7, 13, 14, 15	8
55	The Function $\log_a u$	p. 246	p. 247; 1, 6b, 8a, 9b, d	9
*	Differential Equations	pp. 247-249	p. 249; 1, 2, 5, 7, 9	IX-9

* This topic can be done either here or with unit IX.

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LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENTS	INSTRUCTIONAL OBJECTIVE
56	Powers of Trigonometric Functions	pp. 257-260	p. 260; 1, 5, 6, 7a, 8, 9, 10, 15, 24	1
57	Even Powers of Sines and Cosines	p. 261	pp. 261-262; 2, 5, 6, 7, 10	
58	Integrals with Terms $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-a^2}$, a^2+u^2, a^2-u^2	pp. 262-266	pp. 265-266; 1, 3, 6, 10, 12 pp. 265-266; 2, 8, 11, 13, 15	1

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LESSON	ASSIGNMENT TOPIC	READING	SUGGESTED ASSIGNMENTS	INSTRUCTIONAL OBJECTIVE
60	Integrals with $ax^2 + bx + c$	pp. 266-267	p. 267; 1, 3, 5, 6, 9	1
61 62	Integration by Partial Fractions	pp. 268-270	p. 270; 1, 4, 6, 11	1
63 64	Integration by Parts	pp. 271-274	pp. 273-274; 2, 3, 4, 8, 13 pp. 273-274; 10, 16, 19, 20	2
65	Integration of Rational Functions of sin x and cos x, and Other Trigonometric Integrals	pp. 274-276	pp. 275-276; 1, 4, 7	1
66	Further Substitutions	pp. 276-278	p. 278; 1, 3, 16, 17	1
67	Improper Integrals	pp. 278-282	p. 288; 1, 5, 8, 10, 12, 17, 18	3

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70 71 72	Tests for Convergence of a Series of Constants	pp. 403-409	p. 409; 3, 5, 6, 7, 9 p. 409; 4, 8, 10, 12, 13 p. 409; 11, 14, 15, 16, 17, 19	2, 3, 4, 5
73 74	Power Series Expansions of Functions	pp. 409-412	p. 412; 1-4	6
75 76	Taylor's Theorem	pp. 412-419	pp. 419-520; 1, 2, 3, 7, 8 pp. 419-420; 4, 5, 6, 9	7
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80	Absolute and Conditional		p. 434; 1, 2, 4, 5, 7, 8, 10	

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81	1st Order Equations with Variables Separable	pp. 439-441	p. 441; 1, 2, 3, 6, 7, 10	1, 2
82	First-order Homogeneous Equations	pp. 441-442	p. 442; 1, 3, 6	3
83	First-order Linear	pp. 442-444	pp. 443-444; 1, 3	4
84			pp. 443-444; 2, 6, 9, 10	
85	First-order Exact	p. 444	p. 444; 1, 3, 8, 10	5
86 87	Homogeneous Linear Second-order Differential Equations with Constant Coefficients	pp. 446-448	p. 448; 1, 4, 6	6
88	Non-homogeneous Linear Second-order Differential Equations with Constant Coefficients	pp. 448-449	p. 449; 1-9(odd), 2	7
89	Applicants of Solutions of Differential Equations	pp. 247-249 pp. 366-367 pp. 450-453	p. 249; 1, 2, 3 p. 367; 7 p. 452-453; 2, 3, 4	8

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91	Derivatives and Integrals of Hyperbolic Functions	pp. 356-359	p. 359; 2, 3, 5, 6, 7, 8, 9; 10, 13, 14, 16, 17	2, 3
92	Derivatives of Inverse Hyperbolic Functions	pp. 361-364	p. 365; 6-10	4
93	Integrals of Inverse Hyperbolic Formulas	p. 365	p. 365; 11-15	5
*	The Hanging Cable	p. 367	p. 367; 7	IX-8

*This topic can be done in Unit IX if results are expressed in exponential function form.

UNIT IX

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96	Normal Vector	pp. 385-387	p. 388; 3, 4, 5 ($dT/d\phi$ only), 11	6

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*The only approved text is G. B. Thomas, Jr., Elements of Calculus and Analytic Geometry, Addison-Wesley, 1976.

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SUGGESTED ASSIGNMENTS

Elements of Calculus and Analytic Geometry, by G. B. Thomas, Jr., (Addison Wesley, 1976) is, at the time of publication, the only text approved for Calculus.

This material, referenced to the text, has been compiled to provide the teacher with a key relating the instructional objectives to the topics of Calculus, to the reading on each topic and to suggested assignments of problems. It has been presented in the order that it appears in the text for most of the units. The 96 lessons should not be interpreted to be all of the same length of importance.

UNIT I: LIMITS AND CONTINUITY

INTRODUCTION

The theory of limits is a difficult but necessary beginning for the study of calculus. An aim of this unit is to strike a balance between mathematical intuition and the rigor of formal proof.

Unit I begins by introducing the concept of neighborhoods as they apply to the definition of a limit. Emphasis should be placed on the definition of the limit of a function, giving time to geometric interpretation as well as the determination of particular epsilon and delta neighborhoods for given functions. Comprehension of the definition is necessary for understanding the limit theorems and their proofs. Applications of these theorems will enable the students to begin developing skills in determining limits.

It should be emphasized that although this may be the student's first formal study of limits, it certainly will not be his last. The background established in this unit will serve as a foundation for future units in the course.

INSTRUCTIONAL OBJECTIVES

1. Determine the limit of a function and construct a proof that it is the limit by applying the definition of a limit (epsilon-delta).

2. Apply the limit theorems to determine the limit of a function as the variable approaches a constant.

3. Apply the limit theorems to determine the limit of a function as the variable approaches infinity.

4. Apply the definition of continuity of a function.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) State the definition of the limit of a function.

   b) Determine the limit of a given algebraic function.

   c) Construct an epsilon-delta proof to verify the choice of a limit for a given function.

2. a) Determine the limit of a given linear function as $x \to a$

   b) Determine the limit of the sum of two functions as $x \to a$

c) Determine the limit of the product of two functions as $x \to a$.

d) Determine the limit of the quotient of two functions as $x \to a$.

e) Apply $\lim_{x \to 0} \frac{\sin x}{x} = 1$ to determine the limit of expressions containing trigonometric functions.

3. a) Determine the limit of the sum of two functions as $x \to \infty$.

b) Determine the limit of the product of two functions as $x \to \infty$.

c) Determine the limit of the quotient of two functions as $x \to \infty$.

4. a) State the definition of continuity of a function at a given point.

b) Determine the continuity of a function at a given point by using the definition of continuity.

c) Determine the continuity of a function over a given interval.

SAMPLE ASSESSMENT MEASURES: LIMITS AND CONTINUITY

PERFORMANCE OBJECTIVE I - 1a: State the definition of the limit of a function.

1. State the limit of f(x) as $x \to a$.

2. Complete the following definition.

$$L = \lim_{x \to a} \leftrightarrow \forall \epsilon > 0 \, \exists \delta > 0 \ni \_\_\_\_ .$$

OR

$$L = \lim_{x \to a} \leftrightarrow \forall \epsilon > 0 \, \exists \text{ a deleted neighborhood N of a } \ni \_\_\_\_ .$$

PERFORMANCE OBJECTIVE I - 1b: Determine the limit of a given algebraic function.

1. $\lim_{x \to 0^-} x\sqrt{2x + \frac{9}{x^2}}$

a) 3 b) -3 c) 0 d) 9 e) ∞

2. $\lim_{x \to 5} \frac{2/x - 2/5}{x - 5}$ is ________ ?

PERFORMANCE OBJECTIVE I - 1c: Construct an epsilon-delta proof to verify the the choice of a limit for a given function.

1. Prove $\lim_{x \to 2} (5 - 3x) = -1$.

2. Prove $\lim_{x \to -1} (3x^2 - 2x + 4) = 9$.

3. Prove $\lim_{x \to 2} (1 + 1/x) = 3/2$.

PERFORMANCE OBJECTIVE I - 2a: Determine the limit of a given linear function as $x \to a$.

1. Determine $\lim_{x \to 4} (8 - 3x)$.

2. Determine $\lim_{x \to -1/2} (3x/5 + 2/3)$.

PERFORMANCE OBJECTIVE I - 2b: Determine the limit of the sum of two functions as $x \to a$.

1. Determine $\lim_{x \to 2^+} ([2-x] + |x-3|)$.

2. Determine $\lim_{x \to 3} (x^3/5 - 4x^2/3 + 7) - \lim_{x \to 3} ((3x^3 - 20x^2)/15 + 13)$.

PERFORMANCE OBJECTIVE I - 2c: Determine the limit of the product of two functions as $x \to a$.

1. Determine $\lim_{x \to 1} \left(\sqrt{5x-x^2}\,\mathrm{sgn}(x-3)\right)$.

2. Determine $\lim_{x \to a} \sqrt{\sqrt{x} - 5} \cdot \lim_{x \to a} \sqrt{\sqrt{x} + 5}$.

PERFORMANCE OBJECTIVE I - 2d: Determine the limit of the quotient of two functions as $x \to a$.

1. Determine $\lim_{x \to 3} \frac{x^3-27}{x^2-9}$.

2. Determine $\lim_{x \to 3} \frac{(6 - \sqrt{x^2+27})}{(x-3)}$.

3. Determine $\lim_{x \to -1} \frac{x^5+1}{x+1}$.

PERFORMANCE OBJECTIVE I - 2e: Apply $\lim_{x \to 0} \frac{\sin x}{x} = 1$ to determine the limit of expressions containing trigonometric functions.

1. $\lim_{x \to 0} \frac{\sin 3x}{x}$ is

a) 1 b) 3 c) 1/3 d) 0 e) ∞

2. Determine $\lim_{x \to \infty} x^2 \sin(1/x^2)$.

PERFORMANCE OBJECTIVE I - 3a: Determine the limit of the sum of two functions as $x \to \infty$.

1. $\lim_{x \to \infty} (\text{sgn } x + \sin x)$

a) is nonexistent b) is infinity c) oscillates between 0 and 2

d) oscillates between -1 and 1 e) is 1

2. Determine $\lim_{x \to \infty} \left[\frac{\sin x}{x} + \sin(x-4) \right]$.

PERFORMANCE OBJECTIVE I - 3b: Determine the limit of the product of two functions as $x \to \infty$.

1. Determine $\lim_{x \to \infty} \left[\frac{x^4 - b^4}{3x^4} \right] \left[\frac{2x + 3}{x - 6} \right]$.

2. Determine $\lim_{n \to \infty} (100 + 1/n)^2 \left[1 + \frac{n-1}{n^2} \right]^{100}$.

a) 100 b) not defined c) ∞ d) 10,000 e) 1

PERFORMANCE OBJECTIVE I - 3c: Determine the limit of the quotient of two functions as $x \to \infty$.

1. $\lim_{n \to \infty} \frac{2n^2 - 2n - 4}{2n - n^2}$ is

a) 2 b) -2 c) 0 d) ∞ e) 1

2. Determine $\lim_{y \to 0} \frac{\sqrt{1/y}\,\sqrt{(1/y) - 5^6}}{1/y + 5^6}$.

PERFORMANCE OBJECTIVE I - 4a: State the definition of continuity of a function at a given point.

1. Define what is meant by the statement: $f(x)$ is continuous at $x=b$.

2. Choose the combination of conditions listed below that are necessary to define continuity of a function at a point.

a) $f(x)$ has a minimum value m and a maximum value M on the interval $[a,b]$.

b) $f(x)$ has a definite finite value $f(c)$ at c.

c) $f(x)$ is defined on the interval $[a,b]$.

d) as $x \to c$ then $f(x) \to f(c)$ as a limit.

e) f is defined for some neighborhood of c.

a) a, c, d b) b, d c) a, b, d d) b, c, e e) b, d, e

PERFORMANCE OBJECTIVE I - 4b: Determine the continuity of a function at a given point, using the definition of continuity.

1. $f(x) = \frac{x^2-1}{x-1}$ has a point of discontinuity. Redefine $f(x)$ at the point of discontinuity so that f is continuous. Justify your solution.

2. Determine whether $\frac{x^3-8}{x^2-4}$ is continuous at $x = 2$. Justify your conclusion.

PERFORMANCE OBJECTIVE I - 4c: Determine the continuity of a function over a given interval.

1. Determine whether f(x) is continuous over [-4,4]. Justify your conclusion.

$$f(x) = \begin{cases} x^2-2, & x>2 \\ -2, & x=2 \\ x^2+7x-14, & x<2 \end{cases}$$

2. Determine whether f(x) is continuous over (-5,5). Justify your conclusion.

$$f(x) = 1 - x + [x] - [1-x]$$

ANSWERS TO ASSESSMENT MEASURES

1. a) (1) $L = \lim_{x \to a} f(x) \Leftrightarrow \forall \epsilon > 0 \; \exists \; \delta > 0 \ni$

$|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$

or

$L = \lim_{x \to a} f(x) \Leftrightarrow \forall \epsilon > 0, \exists$ deleted neighborhood N of a $\ni$

$L - \epsilon < f(x) < L + \epsilon$ whenever $X \in N$

(2) $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$

or

$L - \epsilon < f(x) < L + \epsilon$ whenever $X \in N$

b) (1) b

(2) $\frac{-2}{25}$

c) (1) $\delta \leq \frac{\epsilon}{3}$

(2) $\delta = \min \left\{1, \frac{\epsilon}{11}\right\}$

(3) $\delta = \min \{1, 2\epsilon\}$

2. a). (1) -4

(2) $\frac{11}{10}$

b) (1) 0

(2) -6

c) (1) -2

(2) $\sqrt{a - 25}$

d) (1) $\frac{9}{2}$

(2) $-\frac{1}{2}$

(3) 5

e) (1) b

(2) 1

ANSWERS TO ASSESSMENT MEASURES: (continued)

3. a) (1) a. is nonexistent.

(2) 1

b) (1) $\frac{2}{3}$

(2) d. 10,000

c) (1) b. -2

(2) 1

4. a) (1) See definition of continuity of a function at a point.

(2) e. b, d, e

b) (1) $f(x) = \begin{cases} \frac{x^2 - 1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$ Show $\lim_{x \to 1} f(x) = f(1)$

(2) Function is discontinuous at 2 = 2 because f(2) is undefined.

c) (1) Discontinuous at x = 2. Use definition to justify.

(2) Discontinuous at integral values

UNIT II: DERIVATIVES OF ALGEBRAIC FUNCTIONS

INTRODUCTION

In this unit, derivatives of algebraic functions should be determined initially by direct application of the definition. It will soon become apparent to the student that this method, while effective, is tedious. The limit theorems developed in Unit I can then be applied to establish rules for determining the derivatives of various algebraic functions. The application of these rules provides the student with an efficient method for differentiating algebraic functions.

Particular emphasis should be placed upon the chain rule, as its role will be continued throughout the development of derivatives of transcendental functions. In the latter part of this unit, the method of implicit differentiation will be applied to algebraic equations. The unit concludes with a discussion of the role of differentials.

INSTRUCTIONAL OBJECTIVES

1. Determine the derivative of a function using the definition.
2. Apply the rule for determining the derivative of x^n.
3. Apply the rule for determining the derivative of cu, where c is a constant and u is a differentiable function of x.
4. Apply the rule for determining the derivative of the sum of a finite number of differentiable functions.
5. Apply the rule for determining the derivative of the product of two differentiable functions.
6. Apply the rule for determining the derivative of the quotient of two differentiable functions where the divisor is non-zero.
7. Apply the rule for determining the derivative of u^n where u is a differentiable function of x and n is rational.
8. Determine higher order derivatives.
9. Apply the method of implicit differentiation to algebraic equations.
10. Apply the chain rule for derivatives.
11. Apply the rules for differentiation to construct differentials.

UNIT II. DERIVATIVES OF ALGEBRAIC FUNCTIONS

PERFORMANCE OBJECTIVES

By the end of this unit the student should have mastered the objectives listed below.

1. a) State the definition of the derivative of a function.

   b) Apply the definition to a function of the form u^n where n is a positive integer.

   c) Apply the definition to a function of the form u^n where n is negative.

   d) Apply the definition to a function of the form u^n where n is rational.

2. a) State the rule for determining the derivative of x^n, where n is a rational number.

   b) Apply the rule for determining the derivative of x^n where n is an integer.

   c) Apply the rule for determining the derivative of a given expression x^n where n is a rational number.

3. a) State the rule for determining the derivative of a given expression cu where c is a constant and u is a differentiable function of x.

   b) Apply the definition of the derivative to derive the rule for determining the derivative of cu where c is a constant and u is a differentiable function of x.

   c) Apply the rule for determining the derivative of a given expression cu where c is a constant and u is a differentiable of x.

4. a) State the rule for determining the derivative of the sum of a finite number of differentiable functions.

   b) Apply the definition of the derivative to derive the rule for determining the derivative of the sum of a finite number of differentiable functions.

   c) Apply the rule for determining the derivative of the sum of a finite number of given differentiable functions.

5. a) State the rule for determining the derivative of the product of two differentiable functions.

   b) Apply the definition of the derivative to derive the rule for determining the derivative of the product of two differentiable functions.

   c) Apply the rule for determining the derivative of the product of two given differentiable functions.

6. a) State the rule for determining the derivative of the quotient of two differentiable functions where the divisor is non-zero.

b) Apply the definition of the derivative to derive the rule for determining the derivative of the quotient of two differentiable functions where the divisor is non-zero.

c) Apply the rule for determining the derivative of the quotient of two given differentiable functions where the divisor is non-zero.

7. a) State the rule for determining the derivative of u^n where u is a differentiable function of x, and n is a rational number.

b) Apply the definition of the derivative to derive the rule for determining the derivative of u^n where u is a differentiable function of x and n is a positive integer.

c) Apply the rule for determining the derivative of a given expression u^n where n is a rational number.

8. a) Determine the second derivative of a given algebraic expression.

b) Determine the third derivative of a given algebraic expression.

c) Determine the least value of n for which the nth derivative of a given polynomial is zero.

9. a) Apply the method of implicit differentiation to determine $\frac{dy}{dx}$ for a given algebraic equation.

b) Apply the method of implicit differentiation to determine $\frac{d^2y}{dx^2}$ for a given algebraic equation in terms of x and y.

10. a) State the chain rule for derivatives.

b) Apply the chain rule for derivatives to determine $\frac{dy}{dx}$ where y is a given composite function of x.

c) Apply the chain rule for derivatives to determine $\frac{dy}{dx}$ where x and y are given functions of the parameter t.

d) Apply the chair rule for derivatives to determine $\frac{d^2y}{dx^2}$ where x and y are given functions of the parameter t.

11. a) Demonstrate a geometric interpretation of dx and dy.

b) Determine the differential of y for a given algebraic expression.

SAMPLE ASSESSMENT MEASURES: DERIVATIVES OF ALGEBRAIC FUNCTIONS

PERFORMANCE OBJECTIVE II - 1a: State the definition of the derivative of a function.

1. State the algebraic definition of the derivative of a function.

PERFORMANCE OBJECTIVE II - 1b: Apply the definition to a function of the form u^n where n is a positive integer.

1. Apply the definition of a derivative to find dy/dx, given $y=x^3$. Show all work.

2. Apply the definition of the derivative of a function to determine dy/dx, given $y = (3x-5)^2$. Show all work.

PERFORMANCE OBJECTIVE II - 1c: Apply the definition of a derivative to a function of the form u^n where n is negative.

1. Apply the definition of the derivative of a function to find dy/dx, given $y = 1/(3x+4)$. Show all work.

2. Apply the definition of the derivative of a function to find dy/dx, given $y = 1/(x^2-5)$. Show all work.

PERFORMANCE OBJECTIVE II - 1d: Apply the definition of the derivative of a function to a function of the form u^n where n is rational.

1. Apply the definition of the derivative of a function to find dy/dx, given $y = \sqrt{x^2+5}$. Show all work.

2. Apply the definition of the derivative of a function to find dy/dx, given $y = 2/\sqrt[3]{x}$. Show all work.

PERFORMANCE OBJECTIVE II - 2a: State the rule for determining the derivative of x^n where n is a rational number.

1. State the rule for determining dy/dx, given $y = x^n$.

PERFORMANCE OBJECTIVE II - 2b: Apply the rule for determining the derivative of x^n where n is an integer.

1. Determine dy/dx, given $y = x^7$.

2. Determine dy/dx, given $y = 1/x^4$.

PERFORMANCE OBJECTIVE II - 2c: Apply the rule for determining the derivative of a given expression x^n where n is a rational number.

1. Determine dy/dx, given $y = x^{3/4}$.

2. Determine dy/dx, given $y = 1/\sqrt{x}$.

PERFORMANCE OBJECTIVE II - 3a: State the rule for determining the derivative given expression cu where c is a constant and u is a differentiable function of x.

1. State the rule for finding dy/dx, given y = cu, c is a constant, and u = f(x).

PERFORMANCE OBJECTIVE II - 3b: Apply the definition of the derivative to derive the rule for determining the derivative of cu where c is a constant and u is a differentiable function of x.

1. Apply the definition of derivative to show that for y = cu, where c is constant and u is a differentiable function of x, dy/dx = c(du/dx).

PERFORMANCE OBJECTIVE II - 3c: Apply the rule for determining the derivative of a given expression cu, where c is a constant and u is a differentiable function of x.

1. Determine dy/dx, given $y - 5x^3$.

2. Determine dy/dx, given $y = 7/\sqrt{x}$.

PERFORMANCE OBJECTIVE II - 4a: State the rule for determining the derivative of the sum of a finite number of differentiable functions.

1. State the rule for finding dy/dx, where $y = u + v$; u and v are both differentiable functions of x.

PERFORMANCE OBJECTIVE II - 4b: Apply the definition of the derivative to derive the rule for determining the derivative of the sum of a finite number of differentiable functions.

1. Apply the definition of derivative to show that $dy/dx = du/dx + dv/dx$ for $y = u + v$, where u and v are differentiable functions of x.

PERFORMANCE OBJECTIVE II - 4c: Apply the rule for determining the derivative of the sum of a finite number of the sum of a finite number of given differentiable functions.

1. Determine dy/dx, given $y = 3x^4 - 5x^3 + 7x$.

2. Determine dy/dx, given $y - x^5/3 - x^3/7 + 2/x + 8$.

PERFORMANCE OBJECTIVE II - 5a: State the rule for determining the derivative of the product of two differentiable functions.

1. State the rule for determining the derivative of $y = uv$, where u and v are both differentiable functions of x.

PERFORMANCE OBJECTIVE II - 5b: Apply the definition of the derivative to derive the rule for determining the derivative of the product of two differentiable functions.

1. Use the definition of derivative to show that for $y = uv$, u and v differentiable functions of x, $dy/dx = u(dv/dx) + v(du/dx)$.

PERFORMANCE OBJECTIVE II - 5c: Apply the rule for determining the derivative of the product of two given differentiable functions.

1. Determine dy/dx, given $y = (2x-1)\ (x^2+x+1)$.

2. Determine dy/dx, given $y = (\sqrt{x} + 7x^3)\ (4x^2-7x+5)$.

3. Determine dy/dx, given $y - (1/x^3 = 4/x^5 + 2/x^7)\ (\sqrt{x} + 2\sqrt[3]{x} + 3\sqrt[4]{x})$.

PERFORMANCE OBJECTIVE II - 6a: State the rule for determining the derivative of the quotient of two differentiable functions where the divisor is non-zero.

1. State the rule for determining dy/dx when $y = u/v$, u and v are differentiable functions of x, and $v \neq 0$.

PERFORMANCE OBJECTIVE II - 6b: Apply the definition of the derivative to derive the rule for determining the derivative of the quotient of two differentiable functions where the divisor is non-zero.

1. Use the definition of the derivative to show that for $y = u/v$, u and v differentiable functions of x, and $v \neq 0$, dy/dx = $\dfrac{v(du/dx) - u(dv/dx)}{v^2}$.

PERFORMANCE OBJECTIVE II - 6c: Apply the rule for determining the derivative of the quotient of two given differentiable functions where the divisor is non-zero.

1. Determine dy/dx, given $y = \dfrac{x^3 - 5x^2 + x}{x - 2}$.

2. Determine dy/dx, given $y = \dfrac{x^3}{6x^2 - 4x + 5}$.

PERFORMANCE OBJECTIVE II - 7a: State the rule for determining the derivative of u^n where u is a differentiable function of x and n is a rational number.

1. State the rule for determining dy/dx when $y = u^n$ where u is a differentiable function of x.

PERFORMANCE OBJECTIVE II - 7b: Apply the definition of the derivative to derive the rule for determining the derivative of u^n where u is a differentiable function of x and n is a positive integer.

1. Use the definition of the derivative to show that for $y = u^n$, where u is a differentiable function of x, $dy/dx = nu^{n-1}(du/dx)$.

PERFORMANCE OBJECTIVE II - 7c: Apply the rule for determining the derivative of a given expression u^n where n is a rational number

1. Determine dy/dx, given $y = (5x^3 - 2x)^{12}$

2. Determine dy/dx, given $y = \sqrt{3x^4 + 2x^2 - 7x}$.

PERFORMANCE OBJECTIVE II - 8a: Determine the second derivative of a given algebraic expression.

1. Determine d^2y/dx^2, given that $y = 5x^3 - 6x^2 + 7x - 8$.

2. Determine d^2y/dx^2, given $y = (5x^3 - 7x + 4)^2$.

3. Determine d^2y/dx^2, given $y = \dfrac{3x^2}{x + 1}$.

PERFORMANCE OBJECTIVE II - 8b: Determine the third derivative of a given algebraic expression

1. Determine d^3y/dx^3, given $y = 4x^5 - 3x^3 + 7x$.

2. Determine d^3y/dx^3, given $y = 3x^2 + 9x + 8$.

3. Determine d^3y/dx^3, given $y = (3x-1)^4$.

PERFORMANCE OBJECTIVE II - 8c: Determine the least value of n for which the nth derivative of a polynomial is zero.

1. If $f(x) = x^n$, integer $n>0$, the first derivative of $f(x)$ which is identically zero is

a) the first b) the (n-1)st c) the (n+1)st d) the (n+2)nd

e) the nth.

2. What is the least value of n for which the nth derivative of $3x^7 - 5x^4 + 2x^3$ is equal to zero?

PERFORMANCE OBJECTIVE II - 9a: Apply the method of implicit differentiation to determine dy/dx for a given algebraic equation.

1. Determine dy/dx, given $y^2 = x$.

2. Determine dy/dx, given $3x^2 - 4xy + 7y^2 = 8$.

3. Determine dy/dx, given $y = \sqrt{\frac{3x + y}{x - 2y}}$.

PERFORMANCE OBJECTIVE II - 9b: Apply the method of implicit differentiation to determine d^2y/dx^2 for a given algebraic equation in terms of x and y.

1. Determine d^2y/dx^2, given $x^3 + y^3 = 5$

2. Determine d^2y/dx^2, given $\sqrt{xy} = y$

PERFORMANCE OBJECTIVE II - 10a: State the chain rule for derivatives.

1. State the chain rule for derivatives.

PERFORMANCE OBJECTIVE II - 10b: Apply the chain rule for derivatives to determine dy/dx where y is a given composite function of x.

1. Determine dy/dx, given $y = u^2 + 1$, $u = x^2 + x$.

2. Determine dy/dx, given $y = \sqrt{x^3 + 4x^2 - 7}$.

3. Determine dy/dx, given $y = \left(\frac{x^2 - 5}{x^3 + 2}\right)^4$.

PERFORMANCE OBJECTIVE II - 10c: Apply the chain rule for derivatives to determine dy/dx where x and y are given functions of the parameter t.

1. Determine dy/dx, given $x = t^2 - 1$ and $y = 6t - t^3$

2. Determine dy/dx, given $x = \sqrt{t^2 + 2t}$ and $y = \frac{t^3 + 3t^2 + 3t}{3}$.

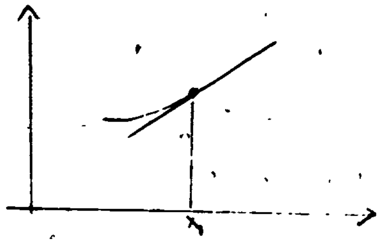
PERFORMANCE OBJECTIVE II - 10d: Apply the chain rule for derivatives to determine d^2y/dx^2 where x and y are given functions of the parameter t.

1. Determine d^2y/dx^2, given $x = t^4$ and $y = t^2 - 1$.

2. Determine d^2y/dx^2, given $x = t^2 + 2t$ and $y = t^3 - 3t$.

PERFORMANCE OBJECTIVE II - 11a: Demonstrate a geometric interpretation of dx and dy.

1.



Complete the diagram with appropriate labeling to show the relationship between dy and dx.

PERFORMANCE OBJECTIVE II - 11b: Determine the differential of y for a given algebraic expression.

1. Determine dy, given $y = (3x+1)^3$.

2. Determine dy, given $3y^3 = (4x-2)^{1/2}$.

3. Determine dy, given $x^2y^3 - 3x^3y^2 = 4x$.

ANSWERS TO ASSESSMENT MEASURES

1 a)

(1) $\lim\limits_{x \to 0} \frac{\Delta y}{\Delta x}$, $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$, OR $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$.

b)

(1) $$dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \{3x^2 + 3x(\Delta x) + (\Delta x)^2\}}{\Delta x}$$

$$= 3x^2$$

(2) $$dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\{3(x+\Delta x) - 5\}^2 - (3x-5)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{9(x^2 + 2x(\Delta x) + (\Delta x)^2) - 30(x+\Delta x) + 25 - (9x^2 - 30x + 25)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\{18x + 9\Delta x - 30\}}{\Delta x}$$

$$= 18x - 30$$

c)

(1) $$dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{1}{3(x+\Delta x) + 4} - \frac{1}{3x + 4}\right] \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{3x + 4 - \{3(x+\Delta x) + 4\}}{(3x+4)[3(x+\Delta x) + 4]}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\Delta x(-3)}{(3x+4)\{3(x+\Delta x) + 4\}}$$

$$= \frac{-3}{(3x+4)^2}$$

1 c)

(2) $dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{1}{(x+\Delta x)^2 - 5} - \frac{1}{x^2 - 5}\right]$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{x^2 - 5 - \{(x+\Delta x)^2 - 5\}}{\{(x+\Delta x)^2 - 5\}(x^2-5)}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{-2x(\Delta x) - (\Delta x)^2}{\{(x+\Delta x)^2 - 5\}(x^2-5)}\right]$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{\{(x+\Delta x)^2 - 5\}(x^2-5)}$$

$$= \frac{-2x}{(x^2-5)^2}$$

1 d)

(1) $dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left\{\frac{\sqrt{(x+\Delta x)^2+5} - \sqrt{x^2+5}}{\Delta x}\right\} \cdot \left\{\frac{\sqrt{(x+\Delta x)^2+5} + \sqrt{x^2+5}}{\sqrt{(x+\Delta x)^2+5} + \sqrt{x^2+5}}\right\}$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2+5 - (x^2+5)}{\Delta x(\sqrt{(x+\Delta x)^2+5} + \sqrt{x^2+5})}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x(\sqrt{(x+\Delta x)^2+5} + \sqrt{x^2+5})}$$

$$= \frac{2x}{\sqrt{x^2+5} + \sqrt{x^2+5}} = \frac{x}{\sqrt{x^2+5}}$$

(2) $dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to 0} \frac{1}{\Delta x}\left[\frac{2}{\sqrt[3]{x+\Delta x}} - \frac{2}{\sqrt[3]{x}}\right]$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left\{\frac{2\sqrt[3]{x} - 2\sqrt[3]{x+\Delta x}}{\sqrt[3]{x+\Delta x}\,\sqrt[3]{x}}\right\} \cdot \left[\frac{(2\sqrt[3]{x})^2 + 4\sqrt[3]{x}\,\sqrt[3]{x+\Delta x} + (2\sqrt[3]{x+\Delta x})^2}{(2\sqrt[3]{x})^2 + 4\sqrt[3]{x}\,\sqrt[3]{x+\Delta x} + (2\sqrt[3]{x+\Delta x})^2}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left\{\frac{8x - 8(x+\Delta x)}{\sqrt[3]{x(x+\Delta x)}\,(4\sqrt[3]{x^2} + 4\sqrt[3]{x(x+\Delta x)} + 4\sqrt[3]{(x+\Delta x)^2}}\right\}$$

$$= \frac{-8}{\sqrt[3]{x^2}(12\sqrt[3]{x^2})} = \frac{-2}{3x\sqrt[3]{x}}$$

2 a)

(1) $dy/dx = nx^{n-1}$

b)

(1) $7x^6$

(2) $-4x^{-5}$

c)

(1) $(3/4)x^{-1/4}$

(2) $(-1/2)x^{-3/2}$

3 a)

(1) $dy/dx = c(du/dx)$

b)

(1) $$dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{c(u+\Delta u) - cu}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c\Delta u}{\Delta x} = c \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = c(du/dx)$$

c)

(1) $dy/dx = 15x^2$

(2) $dy/dx = (-7/2)x^{-3/2}$

4 a)

(1) $dy/dx = du/dx + dv/dx$

b)

(1) $$dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(u+\Delta u) + (v+\Delta v) - (u+v)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta u + \Delta v}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

$$= du/dx + dv/dx$$

4 c)

(1) $dy/dx = 12x^3 - 15x^2 + 7$

(2) $dy/dx = (5/3)x^4 - (3/7)x^2 - 2/x^2$

5 a)

(1) $dy/dx = u(dv/dx) + v(du/dx)$

b)

(1) $$dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(u+\Delta u)(v+\Delta v) - uv}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{uv + u\Delta v + v\Delta u + \Delta u\Delta v - uv}{\Delta x}$$

$$= \lim_{\Delta x \to 0} u\frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} v\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\Delta v$$

$$= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \Delta v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= u(dv/dx) + v(du/dx) + \Delta v\,(du/dx) \qquad \text{As } \Delta x \to 0, \Delta v \to 0;$$

$$= u(dv/dx) + v(du/dx) \qquad \text{thus } \Delta v(du/dx) \to 0.$$

c)

(1) $dy/dx = 6x^2 + 2x + 1$

(2) $dy/dx = (1/(2\sqrt{x}) + 21x^2)(4x^2 - 7x + 5) + (\sqrt{x} + 7x^3)(8x - 7)$

(3) $dy/dx = (-3/x^4 + 20/x^6 - 14/x^8)(\sqrt{x} + 2\sqrt[3]{x} + 3\sqrt[4]{x}) +$
$(1/x^3 - 4/x^5 + 2/x^7)(1/(2\sqrt{x}) + (2/(3\sqrt[3]{x^2})) + (3/(4\sqrt[4]{x^3})))$

6 a)

(1) $$dy/dx = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

6 b) (1) $dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}\right]$

$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{v(u+\Delta u) - u(v+\Delta v)}{(v+\Delta v)v}\right]$

$= \lim_{\Delta x \to 0} \frac{1}{(v+\Delta v)v}\left[v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}\right]$

$= \lim_{\Delta x \to 0} \frac{1}{(v+\Delta v)v}\left[v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right]$

$= \frac{1}{v^2}[\, v(du/dx) - u(dv/dx)\,]$ since $\lim_{\Delta x \to 0} (v+\Delta v) = v$ as $\lim_{\Delta x \to 0} \Delta v = 0$

c) (1) $dy/dx = \frac{2x^3 - 11x^2 + 20x - 2}{(x-2)^2}$

(2) $dy/dx = \frac{6x^4 - 8x^3 + 15x^2}{(6x^2-4x+5)^2}$

7 a) (1) $dy/dx = nu^{n-1}(du/dx)$

b) (1) $dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(u+\Delta u)^n - u^n}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{nu^{n-1}\Delta u + n(n-1)u^{n-2}(\Delta u)^2 + n(n-1)(n-2)u^{n-3}(\Delta u)^3 + \ldots + (\Delta u)^n}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}[\, nu^{n-1} + n(n-1)u^{n-2}\Delta u + n(n-1)(n-2)u^{n-3}(\Delta u)^2 + \ldots + (\Delta u)^{n-1}\,]$

$= nu^{n-1}(du/dx)$ since $\Delta u \to 0$ as $\Delta x \to 0$.

7 c)

(1) $dy/dx = 12(5x^3 - 2x)^{11}(15x^2 - 2)$

(2) $dy/dx = \dfrac{12x^3 + 4x - 7}{2\sqrt{3x^4+2x^2-7x}}$

8 a)

(1) $d^2y/dx^2 = \dfrac{d(dy/dx)}{dx} = \dfrac{d(15x^2 - 12x+7)}{dx} = 30x - 12$

(2) $d^2y/dx^2 = 2(375x^4 - 420x^2+120x+49)$

(3) $d^2y/dx^2 = \dfrac{6}{(x+1)^3}$

b)

(1) $y''' = 240x^2 - 18$

(2) $d^3y/dx^3 = 0$

(3) $y''' = 648(3x-1)$

c)

(1) c)

(2) 8

9 a)

(1) $dy/dx = 1/(2y)$

(2) $dy/dx = \dfrac{2y - 3x}{7y - 2x}$

(3) $dy/dx = \dfrac{3 - y^2}{2xy - 6y^2 - 1}$

9 b)

(1) $d^2y/dx^2 = \dfrac{-2xy^3 - 2x^4}{y^5}$

(2) $d^2y/dx^2 = \dfrac{2xy - 2y^2}{(x-2y)^3}$

10 a)

(1) $dy/dx = (dy/du)\,(du/dx)$ OR $y' = f'(g(x))\,g'(x)$

b)

(1) $dy/dx = 2(x^2+x)\,(2x+1)$

(2) $dy/dx = \dfrac{3x^2+8x}{2\sqrt{x^3+4x^2-7}}$

(3) $dy/dx = \dfrac{4(x^2-5)^3\,(-x^4+15x^2+4x)}{(x^3+2)^5}$

c)

(1) $dy/dx = \dfrac{3(2-t^2)}{2t}$

(2) $dy/dx = (t+1)\sqrt{t^2+2t}$

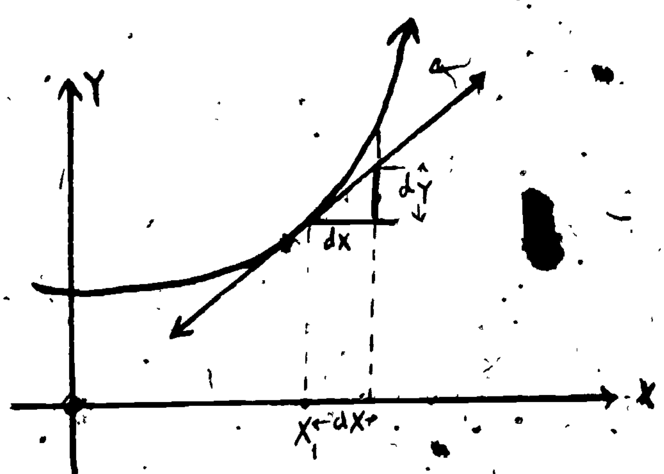
d)

(1) $d^2y/dx^2 = -\dfrac{1}{4t^6}$

(2) $d^2y/dx^2 = \dfrac{3}{4(t+1)}$

11 a)

(1)



II a)

(1) $dy = 9(3x+1)^2\,dx$

(2) $dy = \dfrac{2x}{27y^5}\,dx$

(3) $dy = \dfrac{4 + 9x^2y^2 - 2xy^3}{3x^2y^2 - 6x^3y}\,dx$

UNIT III: APPLICATIONS OF THE DERIVATIVE

INTRODUCTION

The purpose of this unit is to enable the student to apply the theory and methods of differentiation developed in Unit II. Emphasis should be placed on this unit as it is the student's introduction to calculus as a tool for solving problems. The role of differential calculus acquires new importance when applied to problems involving rates of change, maxima and minima, and velocity and acceleration. Use of the first and second derivatives refines and extends techniques of graphing.

In addition to reinforcing methods of differentiation, Unit III develops three important new theorems: Rolle's Theorem, Mean Value Theorem, and l'Hôpital's Rule. Note: Teachers whose students are preparing for the advanced placement examination may wish to include at this time the applications of the derivative involving velocity and acceleration vectors and speed rather than waiting until the unit on vectors at the end of the course.

INSTRUCTIONAL OBJECTIVES

1. Determine the equations of the tangent and the normal lines of a curve.
2. Apply the first and second derivatives to determine the critical points of a function.
3. Determine the solution of word problems involving the use of first or second derivatives.
4. Apply Rolle's Theorem for a function.
5. Describe the Mean Value Theorem for a function.
6. Apply l'Hôpital's rule for determining limits.
7. Apply differentials to determine the linear approximation of a function.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) Construct an equation of a line tangent to a given curve through a point on the curve.

   b) Construct an equation of a line normal to a given curve through a point on the curve.

   c) Construct an equation of a line tangent to a given curve through a point not on the curve.

d) Construct an equation of a line normal to a given curve through a point not on the curve.

2. a) Apply the first derivative to locate the critical points of a function.

   b) Apply the second derivative to identify the critical points of a function.

   c) Construct the graph of a function indicating the coordinates of critical points and points of change in concavity.

3. a) Construct the velocity and the acceleration functions from a distance function.

   b) Construct solutions of problems involving velocity and acceleration from a given distance function.

   c) Construct solutions of problems involving related rates.

   d) Construct solutions of problems involving minimizing or maximizing a function.

4. a) State Rolle's Theorem.

   b) Apply Rolle's Theorem to isolate the roots between a given pair of numbers for $f(x) = 0$.

5. a) State the Mean Value Theorem.

   b) Demonstrate a geometric interpretation of the Mean Value Theorem.

6. a) State l'Hôpital's rule.

   b) Identify problems for which it is appropriate to use l'Hôpital's rule.

   c) Apply l'Hôpital's rule to problems of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

7. a) Determine the change in the range for a specified change in the domain using differentials.

   b) Describe a geometric interpretation of a differential.

SAMPLE ASSESSMENT MEASURES: APPLICATIONS OF THE DERIVATIVE

PERFORMANCE OBJECTIVE III - 1a: Construct an equation of a line tangent to a given curve through a point on the curve.

1. The equation of the line tangent to the curve $y = 3x^2 - 9x - 6$ at the point $(1, -12)$ is ______.

2. If $x = (t + 7)^{\frac{1}{2}}$ and $y = 4t^3 - 1$, then the equation of the tangent line at $t = 2$ is _____.

3. Determine the equation of the line tangent to $f(x) = 4 - x^2$ at the point where its slope is -2.

PERFORMANCE OBJECTIVE III - 1b: Construct an equation of a line normal to a given curve through a point on the curve.

1. The equation of the line normal to the curve $y = x^3 - 7x + 4$ at the point $(0, 4)$ is _____.

2. If $x = t - t^2$ and $y = t - t^3$, the equation of the line normal to the curve at $t = 1$ is _____.

3. Determine the equation of the line normal to $f(x) = 3x^2 - (\frac{9}{2})x + 1$ where the slope of the tangent is 3/2.

PERFORMANCE OBJECTIVE III - 1c: Construct an equation of a line tangent to a given curve through a point not on the curve.

1. The equation(s) of the tangent to $h(x) = 4x - 3x^2$ passing through the point (0, 3) is/are _____.

2. Determine the equation(s) of the tangent, through the point (2, -2), to the hyperbola $x^2 - y^2 = 16$.

PERFORMANCE OBJECTIVE III - 1d: Construct an equation of a line normal to a given curve through a point not on the curve.

1. The equation of the line passing through the point (2, 1) and normal to the curve $y^2 = 4x$ is _____.

2. Determine the equation(s) of the normal line to $y^2 = (x - 1)^3$ through (29, 0).

PERFORMANCE OBJECTIVE III - 2a: Apply the first derivative to locate the critical points of a function.

1. Locate all critical points of the function $y = x^4 - x^3$.

2. Locate all critical points of the function $y = x^2(x-1)^2$.

PERFORMANCE OBJECTIVE III - 2b: Apply the second derivative to identify the critical points of a function.

1. Using first and second derivatives of $y = 2x^3 - 9x^2 + 12x + 12$, locate all relative maxima and minima points.

2. Locate all relative maxima and minima points of $y = x^4 - x^3$, using the second derivative for verification.

3. Locate all relative maxima and minima points of $y = \frac{x}{2 + x^2}$, using the second derivative for verification.

PERFORMANCE OBJECTIVE III - 2c:	Construct the graph of a function, indicating the coordinates of critical points and points of change in concavity.

1. Construct the graph of $y = \frac{1}{4}(x^4 - 6x^2 + 6)$, labeling the coordinates of all relative maxima and minima with M and m, respectively, and all points of inflection with I.

2. Construct the graph $x^2y - 4y = x^2$, labeling the coordinates of all relative maxima and minima with M and m, respectively, and all points of inflection with I.

3. Construct the graph of $x^2y = (x - 2)^2$, labeling the coordinates of all relative maxima and minima with M and m, respectively, and all points of inflection with I.

PERFORMANCE OBJECTIVE III - 3a: Construct the velocity and the acceleration functions from a distance function.

1. The position s of an object is given by $s = \frac{1}{6}t^3 - t^2 + 1$. Write the acceleration and velocity functions from the distance function.

2. The position s of an object is given by $s = 3t - \frac{1}{t}$. Write the acceleration and velocity functions from the distance function.

PERFORMANCE OBJECTIVE III - 3b: Construct solutions of problems involving velocity and acceleration from a given distance function.

1. The position s of an airplane t seconds after touching the ground is given by $s = 180t - 10t^2$. Find how far the plane will travel before it stops.

2. The position x of a moving particle at time t is given by the formula $x = \frac{t}{3 - t}$ for $0 \le t \le 2$. Write the velocity and the acceleration functions from the distance function and find the particle's acceleration when $t = 2$.

PERFORMANCE OBJECTIVE III - 3c: Construct solutions of problems involving related rates.

1. A spherical snowball is melting. Its volume is decreasing at a rate of 200 cc per minute. At what rate (with respect to time) is the radius r changing when r = 10? ($V = \frac{4}{3}\pi r^3$).

2. The radius r of a right circular cone is decreasing at the rate of 2 inches per minute and the altitude h is increasing at the rate of 3 inches per minute. When the radius is 18 inches and the altitude is 20 inches, find the rate at which the volume is changing ($V = \frac{1}{3}\pi r^2 h$).

3. A weight is attached to one end of a rope which passes over a pulley 20 feet above the ground. A man who keeps his hand four feet off the ground grasps the other end of the rope and walks away at the rate of two feet per second. When the man is 12 feet from the point directly under the pulley, the weight is off the ground and rising. At what rate is it rising?

PERFORMANCE OBJECTIVE III - 3d: Construct solutions of problems involving minimizing or maximizing a function.

1. Express the number 12 as the sum of two positive real numbers in such a way that the product of one of them by the square of the other is as large as possible.

2. A man has a stone wall alongside a field. He has 1200 feet of fencing material, and he wishes to make a rectangular pen, using the wall as one side. What should the dimension of the pen be in order to enclose the largest possible area?

3. In a certain locality and type of soil, it is found that if 20 orange trees are planted per acre, the yield will be 500 oranges per tree, and that the yield per tree is reduced by 15 for each additional tree per acre. What is the best number of trees to plant per acre?

4. What is the area of the largest rectangle with lower base on the x-axis and upper vertices on the curve $y = 12 - x^2$?

PERFORMANCE OBJECTIVE III - 4a: State Rolle's Theorem.

1. State Rolle's Theorem (in symbols, where possible).

PERFORMANCE OBJECTIVE III - 4b: Apply Rolle's Theorem to isolate the roots between a given pair of numbers for $f(x) = 0$.

1. Determine whether the following function has exactly one real root in the interval $-1 < x < 1$. $f(x) = 2x^3 - 3x^2 - 12x - 6$.

2. Show that between any two zeros of a polynomial, there is at least one zero of the derivative of the polynomial.

PERFORMANCE OBJECTIVE III - 5a: State the Mean Value Theorem.

1. State the Mean Value Theorem (in symbols where possible).

PERFORMANCE OBJECTIVE III - 5b: Demonstrate a geometric interpretation of the Mean Value Theorem.

1. Demonstrate geometrically the relationships of the Mean Value Theorem.

2. Read each of the following statements very carefully and decide whether it is True or False. If you answer False, you must supply a counter-example to the statement to justify your answer. No justification need be given for an answer of True.

   a) If f is continuous on the closed interval $[a, b]$, f' exists on the open interval (a, b) and $f(a) < f(b)$, then f is increasing throughout $[a, b]$.

   b) If f' exists throughout $[a, b]$, then there is at least one point w in (a, b) such that $f'(w) = [f(b) - f(a)] / (b-a)$.

   c) Since Rolle's Theorem is a special case of the Mean Value Theorem, then if f satisfies the hypothesis of the Mean Value Theorem, it must also satisfy the hypothesis of Rolle's Theorem.

PERFORMANCE OBJECTIVE III - 6a: State l'Hôpital's rule.

1. State l'Hôpital's rule (in symbols, where possible).

PERFORMANCE OBJECTIVE III - 6b: Identify problems for which it is appropriate to use l'Hôpital's rule.

1. For which of the following is it appropriate to use l'Hôpital's rule?

a) $\lim_{x \to 1} \frac{2x^2 - x - 3}{x + 1}$

b) $\lim_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3} - 2}$

c) $\lim_{x \to \infty} \frac{\sqrt{x^2 - 1}}{2x + 1}$

d) $\lim_{x \to \infty} \frac{x^2 + 10}{6x^2 + 2}$

e) $\lim_{x \to 0} \frac{1}{\sqrt{x + 1}} - \frac{1}{\sqrt{x}}$

PERFORMANCE OBJECTIVE III - 6c: Apply l'Hôpital's rule to problems of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

1. Evaluate $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$.

2. Evaluate $\lim_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3} - 2}$.

3. Determine $\lim_{x \to \infty} \frac{x^2 - 1}{4x^2 + x}$.

PERFORMANCE OBJECTIVE III - 7a: Determine the change in the range for a specified change in the domain using differentials.

1. If $y = x^2$, approximate the change in y when x changes from 2 to 2.1.

2. By means of differentials, find an approximate value for $(32.04)^{\frac{4}{5}}$.

PERFORMANCE OBJECTIVE III - 7b: Describe a geometric interpretation of a differential.

1. Describe a geometric interpretation of a differential.

ANSWERS TO ASSESSMENT MEASURES

1. a)

(1) $3x + y = -9$

(2) $288x - y = 833$

(3) $2x + y = 5$

b)

(1) $x - 7y + 28 = 0$

(2) $x + 2y = 0$

(3) $4x + 6y = 1$

c)

(1) Equations of tangent line are $2x + y = 3$ and $10x - y = -3$

(2) $5x - 3y = 16$

d)

(1) Equation of normal line is $x + y = 3$.

(2) $x + 3y = 29$, $x - 3y = 29$

2. a)

(1) $(0, 0)$ and $(\frac{3}{4}, -\frac{27}{256})$

(2) $(0, 0)$ $(\frac{1}{2}, \frac{1}{16})$, $(1, 0)$

b)

(1) Relative maximum at $(1, 17)$ and relative minimum at $(2, 16)$

(2) Relative minimum at $(\frac{3}{4}, \frac{-27}{256})$, no relative maximum

(3) Relative minimum at $(-\sqrt{2}, \frac{-\sqrt{2}}{4})$

Relative maximum at $(\sqrt{2}, \frac{\sqrt{2}}{4})$

c)

(1) Points to be labeled should be $m(-\sqrt{3}, -\frac{3}{4})$; $I(-1, \frac{1}{4})$

$M(0, \frac{3}{2})$, $I(1, \frac{1}{4})$, and $m(\sqrt{3}, -\frac{3}{4})$.

NOTE: Graph is symmetric to y - axis.

(2) M(0, 0)

NOTE: Graph is symmetric to y - axis.

$x = -2$, $x = 2$, and $y = 1$ are asymptotic lines

(3) Note that $y = 1 - 4x^{-1} + 4x^{-2}$

m (2, 0) and I $(3, \frac{1}{9})$

$x = 0$ and $y = 1$ are asymptotic lines

3. a)

(1) $v = \frac{1}{2}t^2 - 2t$, $a = t - 2$

(2) $v = 3 + t^{-2}$, $a = -2t^{-3}$

b)

(1) $v = 180 - 20t$

Plane travels 810 units before it stops.

(2) $v = 3(t - 3)^{-2}$, $a = -6(t-3)^{-3}$, $a = 6$ when $t = 2$

c)

(1) $\frac{dr}{dt} = -\frac{1}{2\pi}$ cm per minute

(2) $\frac{dv}{dt} = -156\pi$ cubic inches per minute

(3) 6/5 ft./sec.

d)

(1) Non-squared number is 4 and squared number is 8.

(2) 300 feet by 600 feet

(3) $y = (20 + x)(500 - 15x)$, $y' = 200 - 30x$

$y' = 0$ when $x = 6\frac{2}{3}$

$y(6) = 10660$, $y(7) = 10665$

Plant 27 trees per acre.

(4) $A = 2x(12 - x^2)$, $A' = 24 - 6x^2$

$A' = 0$ when $x = 2$ and $A'' = -24$ when $x = 2$

Maximum area is 32 square units.

4. a)

(1) If $f(x)$ is continuous $\forall x \in [a, b]$ and if $f(a) = f(b) = 0$ and if $f'(x)$ exists $\forall x \in (a, b)$, then $\exists c \in (a, b) \ni f'(c) = 0$.

b)

(1) $f'(x) = 6x^2 - 6x - 12$

$f'(x) = 6(x^2 - x - 2)$, $f'(x) = 6(x-2)(x+1)$

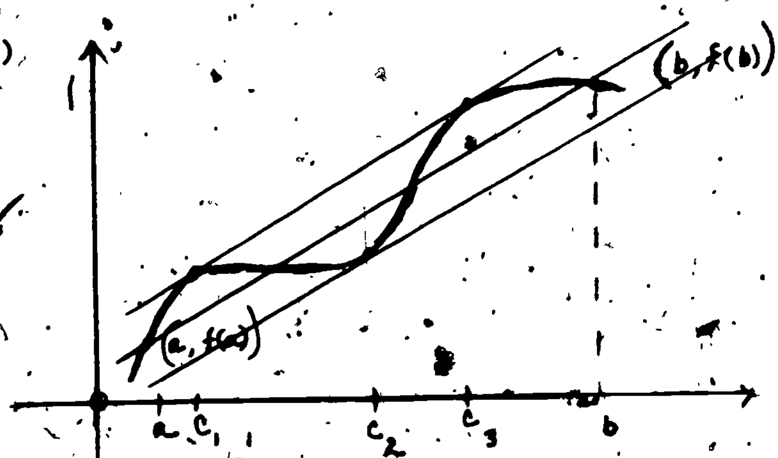
Therefore there is no c on (-1, 1) such that f'(c) = 0. Thus the function does not change direction on (-1, 1). There is at most one real root on (-1, 1). f(-1) = 1, f(1) = -19. Therefore there is exactly one real root.

(2) See page 109, Thomas <u>Calculus</u>, 1972.

5. a)

(1) If $f(x)$ is continuous $\forall x \in [a, b]$ and if $f'(x)$ exists $\forall x \in (a, b)$, then $\exists c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$.

b) (1)



Slope of tangent line = $f'(c)$ = slope of chord = $\frac{f(b) - f(a)}{b - a}$

(2) a) F, $f(x) = (x - 1)^2,\ 0 \le x \le 3$

b) T

c) F; $f(x) = (x - 1)^2,\ 0 \le x \le 3$, $f(0) = 1$ and $f(3) = 4$

6. a) If $f(a) = g(a) = 0$ or if $f(a) = g(a) = \infty$ and if $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists,

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

b) (b) is in the form $\frac{0}{0}$;

(c) and (d) are in the form $\frac{\infty}{\infty}$

c)

(1) 8

(2) 2

(3) $\frac{1}{4}$

7. a)

(1) 0.4

(2) Let $y = x^{\frac{4}{5}}$ then $y' = \frac{4}{5}x^{-\frac{1}{5}}$

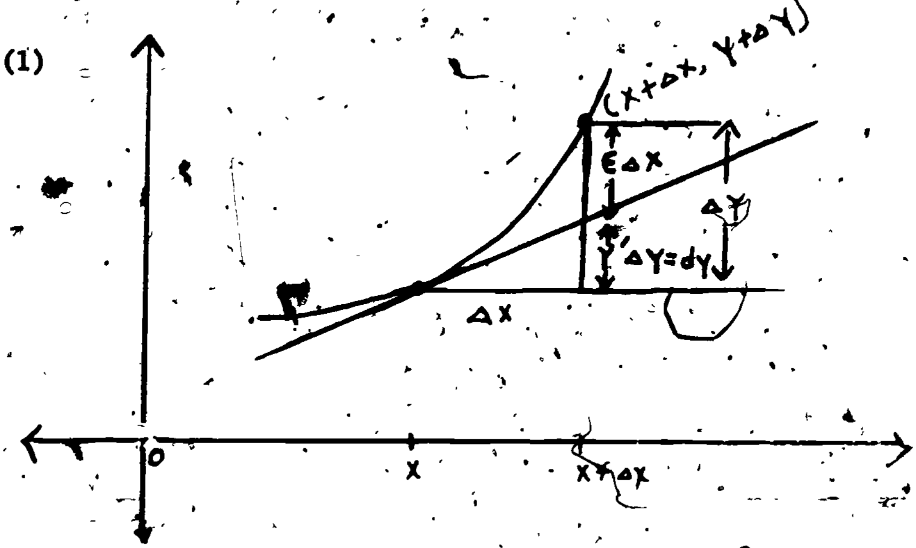
$dy = \frac{4}{5} x^{-\frac{1}{5}} dx$

$dy = \frac{4}{5}(32)^{-\frac{1}{5}} \left(\frac{4}{100}\right)$

$dy = \frac{4}{5} \cdot \frac{1}{2} \cdot \frac{4}{100} = \frac{16}{1000} = .016$

Thus: $32.04^{\frac{4}{5}} \doteq (32)^{\frac{4}{5}} + 0.16 = 16.016$

b) (1)



Differential of y which approximates the change in y for the curve y = f(x) is the slope of the tangent line at x times the change in x.

UNIT IV: THE INTEGRAL

INTRODUCTION

Previous units have presented materials related to differential calculus. In Unit IV, the student is introduced to the other branch of calculus, integral calculus. The first treatment of the integral presented in this unit is that of the indefinite integral, or the antiderivative. This concept is developed through the solution of simple differential equations.

The development of the definite integral involves the concept of area under a curve as a limit of sums of areas of inscribed rectangles. This idea, which is first pursued intuitively through geometric representation, is formally presented in the statement and proof of the Fundamental Theorem of Integral Calculus.

Methods for numerical approximation of the definite integral are introduced through the trapezoidal and Simpson rules. The unit concludes with statements of the definition and theorems involving the Riemann integral.

INSTRUCTIONAL OBJECTIVES

1. Apply the rules for $\int du$, $\int a\, du$ (a=constant), $\int (du+dv)$, and $\int u^n du$ to the solution of differential equations.

2. Determine the approximate area under a curve.

3. Describe the derivation of the Fundamental Theorem of Integral Calculus (F T I C).

4. Apply the rules for approximating the value of a definite integral.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) State the rules for $\int du$, $\int a\, du$, $\int (du + dv)$ and $\int u^n\, du$.

   b) Integrate a given expression of the form $\int du$, $\int a\, du$, $\int (du + dv)$, or $\int u^n\, du$.

2. Determine the approximate area under a given curve over a given interval, using a finite number of inscribed (or circumscribed) rectangles.

3. a) State a definition of area under a curve as a limit of sums.

   b) State the Mean Value Theorem for Areas (MVT).

c) Interpret MVT for areas geometrically.

d) State the two F.T.I.C.

e) Apply the two F.T.I.C.

4. a) Apply the trapezoidal rule to compute the approximate values of a given definite integral.

b) Apply Simpson's rule to compute the approximate value of a given definite integral.

SAMPLE ASSESSMENT MEASURES: THE INTEGRAL

PERFORMANCE OBJECTIVE IV - 1a: State the rules for $\int du$, $\int a\,du$, $\int (du + dv)$ and and $\int u^n\,du$.

State the rule for the following:

1. $\int du$

2. $\int a\,du$ (a is constant)

3. $\int (du + dv)$

4. $\int u^n\,du$

PERFORMANCE OBJECTIVE IV - 1b: Integrate a given expression of the form $\int du$, $\int a\,du$, $\int (du + dv)$, on $\int u^n\,du$.

1. Evaluate the following integrals:

   (a) $\int dt$

   (b) $\int 4\,dx$

   (c) $\int (4x + 5)dx$

   (d) $\int (3x - 2)^4 dx$

2. Evaluate the following integrals:

   (a) $\int (x^3 - \frac{1}{x^2})dx$

   (b) $\int \frac{5x\,dx}{\sqrt{3 - 2x^2}}$

   (c) $\int (3 + 4x)^{\frac{3}{2}}\,dx$

PERFORMANCE OBJECTIVE IV - 2: Determine the approximate area under a given curve over a given interval, using a finite number of inscribed (or circumscribed) rectangles.

1. Approximate the area under the curve $y = x^2$ from $x = 0$ to $x = 1$ using 8 subdivisions in the interval $0 \leq x \leq 1$ (inscribed, circumscribed, or both acceptable).

2. Approximate the area under the curve $y = x^3$ from $x = 0$ to $x = 3$ using 24 subdivisions with interval $0 \leq x \leq 3$ (inscribed, circumscribed, or both acceptable).

PERFORMANCE OBJECTIVE IV - 3a: State a definition of area under a curve as a limit of sums.

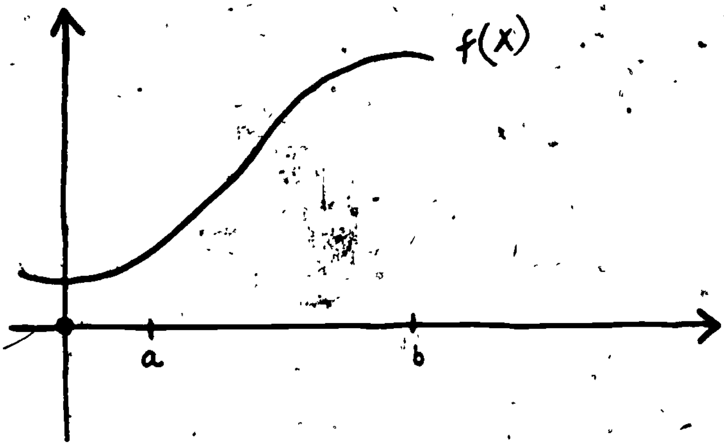
1. State a definition of area under a curve as a limit of sums.

PERFORMANCE OBJECTIVE IV - 3b: State the Mean Value Theorem for Areas (MVT).

1. State the Mean Value Theorem for Areas.

PERFORMANCE OBJECTIVE IV - 3c: Interpret the MVT for area geometrically.

1. Complete the diagram with appropriate labeling to show the geometric interpretation of the Mean Value Theorem for Areas.



PERFORMANCE OBJECTIVE IV - 3d: State the two Fundamental Theorems of Integral Calculus.

1. State (in symbols) the First Fundamental Theorem of Integral Calculus.

2. State (in symbols) the Second Fundamental Theorem of Integral Calculus.

PERFORMANCE OBJECTIVE IV - 3e: Apply the two FTIC.

1. Evaluate: $\int_1^5 (3x-2)\, dx$

2. Find $F'(x)$ for $F(x) = \int_0^x \frac{dt}{1+t^2}$.

3. $\int_0^1 x(1-\sqrt{x})^2\, dx$

4. Find $F'(x)$ for $F(x) = \int_x^1 \frac{dt}{\sqrt{1-t^2}}$.

PERFORMANCE OBJECTIVE IV - 4a: Apply the trapezoidal rule to compute the approximate value of a given definite integral.

1. Use the trapezoidal rule with n = 6 to estimate $\int_1^3 x^3\,dx$ and compare this approximation with the exact value of the integral.

2. Use the trapezoidal rule with n = 6 to estimate $\int_0^{\pi} \sin(x)\,dx$ and compare this approximation with the exact value of the integral.

PERFORMANCE OBJECTIVE IV - 4b: Apply Simpson's rule to compute the approximate value of a given definite integral.

1. Use Simpson's rule with n = 6 to estimate $\int_{-1}^{3} x^3\,dx$ and compare this approximation with the exact value of the integral.

2. Use Simpson's rule with n = 6 to estimate $\int_0^{\pi} \sin(x)\,dx$ and compare this approximation with the exact value of integral which is 2.

ANSWERS TO ASSESSMENT MEASURES

1. a) (1) $u + c$

(2) $au + c$ Note: 1. - 4. c is constant

(3) $u + v + c$

(4) $\frac{u^n}{n + 1} + c \quad (n \neq -1)$

b) (1)

(a) $t + c$

(b) $4x + c$

(c) $2x^2 + 5x + c$

(d) $\frac{1}{15}(3x - 2)^5 + c$

(2)

(a) $\frac{x^4}{4} + \frac{1}{x} + c$

(b) $-\frac{5}{2}(3 - 2x^2)^{\frac{1}{2}}$

(c) $\frac{1}{10}(3 + 4x)^{\frac{5}{2}} + c$

2. (1) $\frac{35}{128} < A < \frac{51}{128}$

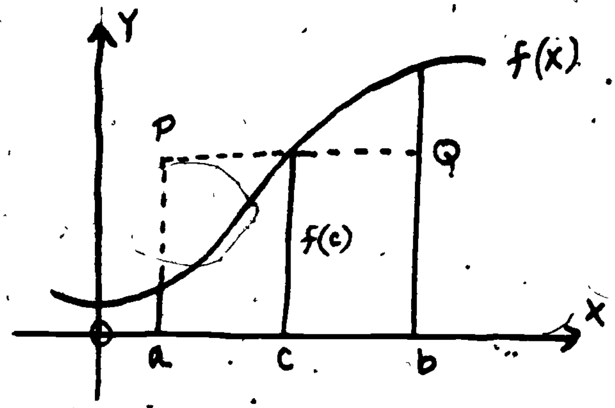
(2) $\frac{4761}{256} < A < \frac{5625}{256}$

3. a) (1) $A_b^a = \lim_{n\to\infty} \sum_{K=1}^{n} f(C_K)\,\Delta x$

b) Let f be a non-negative continuous function over the domain $[a, b]$. Let A_a^b denote the area under the graph of f over the domain. Then there is at least one number c between a and b such that

$$A_c^b = f(c) \cdot (b - a).$$

c)



$A_a^b = f(c) \cdot (b - a)$

Area under curve is equal to area of rectangle aPQb.

d) (1) $A_a^b = \lim_{n \to \infty} \sum_{K=1}^{n} f(C_K)\Delta x = \int_a^b f(x)\,dx = F(b) - F(a)$, where f is continuous and integrable on [a, b]

(2) $F(x) = \int_a^x f(t)dt$ for $x \in [a, b]$ implies $F'(x) = f(x)$ for $a < x < b$.

e) (1) 28

(2) $F'(x) = \frac{1}{1 + x^2}$

(3) 1/30

(4) $-\frac{1}{\sqrt{1 - x^2}}$

4. a)

(1) $\int_1^3 x^3 dx = 20$: trap - $20\frac{2}{9}$

(2) Trap. $= \frac{\pi}{6}\left[2 + \sqrt{3}\right] \doteq 1.95$

b)

(1) $\int_1^3 x^3 dx = 20$: Simpson's = 20

(2) Simpson $= \frac{\pi}{9}\left[4 + \sqrt{3}\right] \doteq 2.00$ (3 significant digits)

UNIT V: APPLICATIONS OF THE DEFINITE INTEGRAL

INTRODUCTION

The statement of the Fundamental Theorem of Integral Calculus established conditions for which the limit of sums could be evaluated by the definite integral. In Unit V, these conditions are applied to various physical situations. Problems involving distance, area, volume, center of mass, length of a curve, surface area, and work are solved by direct application of the Fundamental Theorem of Integral Calculus.

Note: Ample class time should be provided for students to practice with a large variety of problems. This unit offers an excellent opportunity for the teacher to work with small groups and individual students.

INSTRUCTIONAL OBJECTIVES

1. Apply the Fundamental Theorem of Integral Calculus (FTIC) to compute areas.
2. Apply the FTIC to problems involving motion.
3. Apply the FTIC to compute volumes.
4. Apply the FTIC to compute the length of a position of a plane curve.
5. Apply the FTIC to compute surface areas.
6. Apply the rule for the average value of a function over an integral.
7. Apply the FTIC to compute the center of mass.
8. Apply the FTIC to compute work.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) Determine the area of the region bounded by the horizontal axis, two given vertical lines, and a given function.

   b) Determine the area of the region bounded by two given functions.

   c) Determine the area of the region defined by a polar equation.

2. a) Construction the distance function from a given velocity function.

   b) Determine the distance traveled in a specified period of time from a given velocity function.

c) Construct the velocity function from a given acceleration function.

d) Determine the distance in a specified period of time for a given acceleration function.

3. a) Determine the volume of a solid of revolution, using the slicing method.

b) Determine the volume of a solid of revolution, using the cylindrical shell method.

c) Determine the volume of a solid having a known cross section.

4. a) Determine the length of a portion of a curve defined in rectangular coordinators.

b) Determine the length of a portion of a curve defined in polar coordinates.

5. a) Determine the surface area generated by the rotation of a portion of a curve defined in rectangular coordinates.

b) Determine the surface area generated by the rotation of a portion of a curve defined in polar coordinates.

6. a) State the rule for the average value of a function.

b) Determine the average value of a given function over a given interval.

7. a) Determine the center of mass for a portion of a given curve.

b) Determine the center of mass for a given region of a plane.

c) Determine the center of mass for a given solid.

8. a) State the definition of work.

b) Determine the work done by a varying force acting over a directed distance.

SAMPLE ASSESSMENT MEASURES: APPLICATIONS OF THE DEFINITE INTEGRAL

PERFORMANCE OBJECTIVE V - 1a:	Determine the area of the region bounded by the horizontal axis, two given vertical lines and a given function.

1. Determine the area of the region bounded by the curves $y = 3x^2$, $x = 1$, $x = 3$, and the x-axis.

2. Determine the area of the region below the parabola $y = x^2 - 4$, above the x-axis and between the lines $x = -3$, $x = 3$.

PERFORMANCE OBJECTIVE V. - 1b:	Determine the area of the region bounded by two given functions.

1. Determine the area bounded by the curves $y^2 = x$ and $y = x^3$.

2. The total area bounded by the cubic $y = x^3 - x$ and the line $y = 3x$ is equal to:

   a) 4 b) $\frac{16}{3}$ c) 8 d) $\frac{32}{8}$ e) 16

PERFORMANCE OBJECTIVE V. -	1c: Determine the area of the region defined by a polar equation.

1. Determine the area bounded by the curve $r = 2 + \cos\theta$ and the lines $\theta = 0$, $\theta = \pi$.

2.
   a) Sketch the curve whose equation is $r = 2 + \sin\theta$.

   b) Determine an integral for the area of the region inside the above curve but outside the circle $r = 2$. Do not evaluate.

PERFORMANCE OBJECTIVE V - 2a: Construct the distance function from a given velocity function.

1. The velocity of a particle is given by $v = t^2 - t - 2$.
The formula for the distance traveled is ________________.

2. A body has velocity $v = 3t^2 - 12t + 9$. At $t = 0$, $s = 4$, what is the distance $\underline{s}$ in terms of $\underline{t}$?

PERFORMANCE OBJECTIVE V. - 2b: Determine the distance traveled in a specified period of time from a given velocity function.

1. If $v = t^2 - t + 3$ represents the velocity of a moving body as a function of time, find the total distance traveled by a body moving according to this law from $t = 0$ to $t = 3$. (Assume $\underline{v}$ is meters per second, $\underline{t}$ is seconds.)

2. A body moves along a straight line so that its velocity $\underline{v}$ at time $\underline{t}$ is given by $v = 4t^3 + 3t^2 + 3$. The distance it covers from $\underline{t} = 0$ to $\underline{t} = 2$ equals:

a) 41 b) 30 c) 60 d) 55 e) None of these

3. A body moves along a straight line so that its velocity $\underline{v}$ at time $\underline{t}$ is given by $v = t^2 - 3t + 2$. Determine the total distance it covers from $t = 0$ to $t = 4$.

PERFORMANCE OBJECTIVE V - 2c: Construct the velocity function from a given acceleration function.

1. Determine the velocity function from the acceleration function $a = 6t^2$. The initial velocity is zero.

2. A particle moving in a straight line starts from rest at $t = 0$ with an acceleration $a = 10 - 5t$. Determine the velocity function.

3. Determine the velocity function from the acceleration function $a = \frac{t}{\sqrt{t^2 - 4}}$

PERFORMANCE OBJECTIVE V - 2d: Determine the distance in a specified period of time for a given acceleration function.

1. A particle moves in a line with acceleration $a = 12t$ and when $t = 0$, $v = 8$. The total distance traveled between $t = 0$ and $t = 3$ equals:

a) 30 b) 78 c) 30 d) 46 e) None of these

2. The acceleration of a body is $(4t + 1)^{-\frac{1}{2}}$ with the initial velocity $v_0 = 1$. What distance does it travel between $t = 1$ and $t = 3$?

PERFORMANCE OBJECTIVE V - 3a: Determine the volume of a solid of revolution using the slicing method.

1. Determine the volume generated when the area enclosed by $y = 3x - x^2$ and $y = 0$ is rotated about the x-axis.

   a) 27π b) $\frac{81\pi}{10}$ c) $\frac{648\pi}{15}$ d) $\frac{27\pi}{4}$ e) $\frac{27\pi}{2}$

2. Determine the volume generated when the area enclosed by $y = x^3$, $x = 1$, $y = 0$ is rotated about the y-axis.

PERFORMANCE OBJECTIVE V - 3b: Determine the volume of a solid of revolution using the cylindrical shell method.

1. Determine the volume generated when the area enclosed by $y = x^3$, $x = 1$, $y = 0$ is rotated about the y-axis. (Use the method of cylindrical shells.)

2. Determine the volume generated when the area enclosed by $y = 4 - x^2$ and $y = 0$ is rotated about the line $x = 3$. (Use the method of cylindrical shells.)

PERFORMANCE OBJECTIVE V - 3c: Determine the volume of a solid having a known cross-section.

1. Determine the volume of a solid if its base is the circle $x^2 + y^2 = 9$ and each plane section perpendicular to the x-axis is a square.

2. Determine the volume of the solid whose base is a region bounded by the line $y = 4$ and the parabola $x^2 = 8y$ and whose plane section perpendicular to the y-axis is an equilateral triangle.

a) $32\sqrt{3}$ b) $\frac{64\sqrt{3}}{3}$ c) $64\sqrt{3}$ d) 32 e) None of these

PERFORMANCE OBJECTIVE V - 4a: Determine the length of a portion of a curve defined in rectangular coordinates.

1. Find the length of the arc of $y = 6x^{\frac{3}{2}}$, $0 \le x \le 1$.

2. The length of the arc of $y = 4x^2$ cut off by the line $y = 4$ is given by the integral:

a) $\int_0^1 (1 + 64x^2)^{\frac{1}{2}} dx$ b) $\int_{-1}^1 (1 + 8x)^{\frac{1}{2}} dx$

c) $\int_{-1}^1 (1 + 64x^2) dx$ d) $2\int_0^1 (1 + 64x^2)^{\frac{1}{2}} dx$

e) $2\int_{-1}^0 (1 + 8x)^{\frac{1}{2}} dx$

3. Determine the length of the arc of the curve $x = t^3$, $y = t^2$ from $t = 0$ to $t = 1$.

PERFORMANCE OBJECTIVE V - 4b: Determine the length of a portion of a curve defined in polar coordinates.

1 Determine the length of the curve $r = 2a \sin \theta$.

2 Determine the length of the curve $r = 4 \sin^2 \frac{\theta}{2}$.

PERFORMANCE OBJECTIVE V - 5a: Determine the surface area generated by the rotation of a portion of a curve defined in rectangular coordinates.

1. Determine the surface area of a frustum of a cone generated by revolving the line $3x + 4y = 24$ about the x-axis, $0 \leq x \leq 6$.

2. Determine the surface area generated when $y = x^3$ is revolved about the x-axis, $0 \leq x \leq 2$.

PERFORMANCE OBJECTIVE V - 5b: Determine the surface area generated by the rotation of a portion of a curve defined in polar coordinates.

1. Determine the area generated when $r^2 = \cos 2\theta$ is rotated about the polar axis.

2. The cardioid $r = 2(1 + \cos\theta)$ is rotated about the polar axis. Determine the surface area generated.

PERFORMANCE OBJECTIVE V - 6a: State the rule for the average value of a function.

1. State the rule for the average value of a function.

PERFORMANCE OBJECTIVE V - 6b: Determine the average value of a given function over a given interval.

1. Determine the average value of $y = 3x^2 - 4x + 7$ from $x = 2$ to $x = 3$.

2. Determine the average value of $y = \sqrt{3x - 2}$ for $6 \leq x \leq 22$.

PERFORMANCE OBJECTIVE V - 7a: Determine the center of mass for a portion of a given curve.

1. Find the center of mass of the [illegible] of a circle $x^2 + y^2 = a^2$.

2. Find the center of mass of a rod of length ℓ if its density varies as the square of the distance from one end.

PERFORMANCE OBJECTIVE V - 7b: Determine the center of mass for a given region of a plane.

1. Find the coordinates of the center of mass of the area bounded by $y = x^2$, $x = 3$, and $y = 0$.

2. Determine the center of mass of the region bounded by $y = x^3$ and $y^2 = x$.

PERFORMANCE OBJECTIVE V - 7c: Determine the center of mass for a given solid.

1. Determine the coordinates of the center of mass of the solid of revolution produced when the region bounded by $x = 0$, $y = 0$, and $y = 4 - x^2$ is rotated about the y-axis.

2. The region bounded by $x = 0$, $y = 0$, and $y = 4 - x^2$ is rotated about the x-axis. The density varies inversely as the distance from the x-axis. Determine the coordinates of the center of mass of the solid of revolution.

3. The region bounded by $x = 0$, $y = 0$, and $y = 4 - x^2$ is rotated about the x-axis. The density varies directly as the distance from the y-axis. Determine the coordinates of the center of mass of the solid of revolution.

PERFORMANCE OBJECTIVE V - 8a: State the definition of work.

1. State the definition of work.

PERFORMANCE OBJECTIVE V - 8b: Determine the work done by a varying force acting over a directed distance.

1. We are given a pail that weighs 3 kg and contains 30 kg of sand at the start. The pail is lifted slowly a distance of 5 meters, and as it is lifted sand leaks out of a hole at the uniform rate of 3 kg sand per meter lifted. Find the work done in lifting the pail.

2. A conical vessel is 12 feet across the top and 15 feet deep. If it contains a liquid weighing 3 pounds per cubic foot to a depth of 10 feet, find the work done in pumping the liquid to a height of 3 feet above the top of the vessel.

ANSWERS TO ASSESSMENT MEASURES

1. a)

(1) $A = \int_1^3 3x^2 dx$

$= x^3 \Big]_1^3$

$= 26$ sq.

(2) $A = 2\int_2^3 (x^2 - 4)dx$

$= 2\left[\frac{1}{3}x^3 - 4x\right]_2^3$

$= \frac{14}{3}$ sq.

b)

(1) $\frac{5}{12}$

(2) C

c)

(1) $A = \frac{1}{2}\int_0^\pi \left[2 + \cos\theta\right]^2 d\theta$

$= \frac{1}{2}\int_0^\pi (4 + 4\cos\theta + \cos^2\theta)d\theta$

$= 2\theta + 2\sin\theta\Big]_0^\pi + \frac{1}{2}\int_0^\pi \cos^2\theta\, d\theta$

$= 2\pi + \frac{1}{4}\int_0^\pi (1 + \cos 2\theta)d\theta$

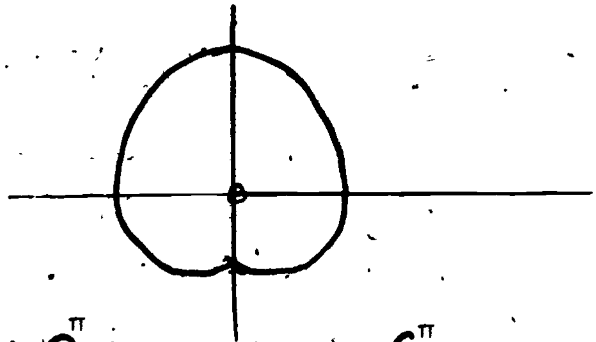
$= 2\pi + \frac{1}{4}(\theta + \frac{1}{2}\sin 2\theta)\Big]_0^\pi$

$= 2\pi + \frac{1}{4}\pi$

$= \frac{9}{4}\pi$

(2)

(a)



(b) $\frac{1}{2}\int_0^{\pi} (2 + \sin \theta)^2 d\theta - \frac{1}{2}\int_0^{\pi} 2^2 d\theta$

2. a)

(1) $s = \frac{1}{3} t^3 - \frac{1}{2} t^2 - 2t + c$

(2) $s = t^3 - 6t^2 + 9t + 4$

b) (1) 13.5 m

(2) b

(3) $\frac{17}{3}$

c)

(1) $v = 2t^3$

(2) $v = 10t - \frac{5}{2} t^2$

(3) $v = (t^2 - 4)^{\frac{1}{2}} + c$

d)

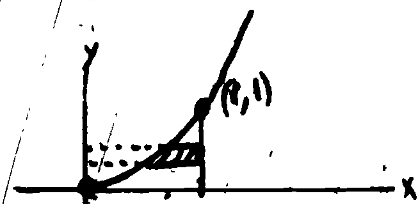
(1) b

(2) $\frac{13\sqrt{13} - 5\sqrt{5} + 12}{12}$

3. a)

(1) b

(2)



$$\int_0^1 \left[\pi(1)^2 - \pi x^2\right] dy$$

$$\pi\int_0^1 dy - \pi\int_0^1 y^{\frac{2}{3}} dy$$

$$\pi\left[y - \frac{3}{5}y^{\frac{5}{3}}\right]_0^1$$

$$\pi\left[1 - \frac{3}{5}\right]$$

$$\frac{2\pi}{5}$$

b)

(1) $V = \int_0^1 2\pi xy\,dx$

$$= 2\pi\int_0^1 x^4\,dx$$

$$= 2\pi\left[\frac{1}{5}x^5\right]_0^1$$

$$= \frac{2\pi}{5}$$

(2) $V = \int_{-2}^{2} 2\pi(3 - x)\, y\, dx$

$= 64\pi$

c)

(1) 144 cubic units

(2) $64\sqrt{3}$

4. a)

(1) $\frac{2}{243}(82^{\frac{3}{2}} - 1)$ $\qquad s = \int_{0}^{1} (1 + 81x)^{\frac{1}{2}}\, dx$

(2) D.

(3) $\ell = \int_{0}^{1} \sqrt{9t^4 + 4t^2}\, dt = \int_{0}^{1} t\sqrt{9t^2 + 4}\, dt$

$= \frac{1}{27}(13^{\frac{3}{2}} - 8)$

b)

(1) $2\pi a$

$ds = \sqrt{r^2 d\theta^2 + dr^2}$

$L = \int_{0}^{\pi} \sqrt{(2a \sin \theta)^2\, d\theta^2 + (2a \cos \theta\, d\theta)^2}$

$= \int_{0}^{\pi} 2a\, d\theta$

(2) $L = \int_0^{2\pi} \sqrt{(4 \sin^2 \frac{\theta}{2})^2 d\,\theta^2 + (4 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\,\theta)^2}$

$= 4\int_0^{2\pi} \sin \frac{\theta}{2}\, d\,\theta$

$= 16$

5. a)

(1) $S = \int_0^6 2\pi y \left(\frac{5}{4}\, dx\right) = \frac{225\pi}{4}$

(2) $S = \frac{\pi}{27}\left[145\sqrt{145} \quad -1\right]$

b)

(1) $S = \int 2\pi y\, ds$

$= 2\int_0^{\frac{\pi}{4}} 2\pi(r \sin\theta) \left(\frac{1}{\cos 2\theta}\right)^{\frac{1}{2}} d\theta$

$= 4\pi\int_0^{\frac{\pi}{4}} \sin\theta\, d\theta = 2\pi(2-\sqrt{2})$

(2) $S = \int 2\pi y\, ds$

$= \int_0^{\pi} 2\pi r \sin\theta\, (2\sqrt{2})\, (1 + \cos\theta)^{\frac{1}{2}}\, d\,\theta$

$= \frac{128\pi}{5}$

6. a) $y \text{ ave.} = \frac{1}{b-a}\int_a^b f(x)dx$

b)

(1) 16

(2) $\frac{56}{9}$

7. a)

(1) $\bar{y} = \frac{\int_{-a}^{a} y\sqrt{1 + (-\frac{x}{y})^2}\,dx}{\pi a} = \frac{\int_{-a}^{a} a\,dx}{\pi a} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$

$\bar{x} = 0,\ \bar{y} = \frac{2a}{\pi}$

(2) $\bar{x} = \frac{\int_0^{\ell} x(kx^2)\,dx}{\int_0^{\ell}(kx^2)\,dx} = \frac{3}{4}\ell$

Center of mass is $\frac{3}{4}$ of way from referenced end of rod.

7. b)

(1) $M = \int_0^3 x^2\,dx = 9$

$\bar{x} = \frac{1}{9}\int_0^3 x\,(x^2dx), \quad \bar{y} = \frac{1}{9}\int_0^3 \frac{x^2}{2}\,(x^2dx)$

$\bar{x} = \frac{9}{4}, \quad \bar{y} = \frac{27}{10}$

(2) $M = \int_0^1 (x^{\frac{1}{2}} - x^3)\,dx = \frac{5}{12}$

$\bar{x} = \frac{12}{5}\int_0^1 x(x^{\frac{1}{2}} - x^3)\,dx, \quad \bar{y} = \frac{12}{5}\int_0^1 \left(\frac{x^{\frac{1}{2}} + x^3}{2}\right)\left(x^{\frac{1}{2}} - x^3\right)dx$

$\bar{x} = \frac{12}{25}, \quad \bar{y} = \frac{3}{7}$

c)

(1) $M = \int_0^2 2\pi\,xy\,dx = 8\pi$ (Shell Method Used)

$\bar{y} = \frac{1}{8\pi}\int_0^2 \frac{y}{2} \cdot 2\pi\,xy\,dx$

$\bar{x} = 0, \; \bar{y} = \frac{4}{3}, \bar{z} = 0$

(2) $M = \int_0^2 \pi\,y^2dx\,\left(\frac{k}{y}\right) = \frac{16\pi}{3}$ (Slicing Method Used)

$\bar{x} = \frac{3}{16\pi}\int_0^2 x\pi\,y^2dx\left(\frac{k}{y}\right)$

$\bar{x} = \frac{3}{4}, \; \bar{y} = 0, \; \bar{z} = 0$

(3)

$\bar{x} = \int \rho\,\tilde{x}\,\pi\,y^2\,dx$ $\qquad \bar{x} = \frac{32}{35} \quad \bar{y} = 0, \quad \bar{z} = 0$

$= \dfrac{\int_0^2 (k\,x)(x)(4 - x^2)^2dx}{\int_0^2 kx(4 - x^2)^2dx}$

8. a) (1) Work equals force through a distance. $W = \int_a^b F \, ds$

b) (1) $w = \int_0^5 (33 - 3x) \, dx$

$w = 33x - \frac{3x^2}{2} \Big]_0^5$

$w = 127.5$ kg-m

(2) $w = \int_0^{10} 3\pi r^2 (18 - h) \, dh$

$\frac{r}{h} = \frac{6}{15}$, so $r = \frac{2}{5} h$

$w = \int_0^{10} \frac{12\pi}{25} (18h^2 - h^3) dh$

$w = \frac{12\pi}{25} \left[6h^3 - \frac{h^4}{4} \right]_0^{10}$

$w = \frac{12\pi}{25} \left[6000 - 2500 \right] = 1680\pi$ ft-lbs

UNIT VI: TRANSCENDENTAL FUNCTIONS

INTRODUCTION

The methods of differentiation and integration of algebraic functions that were developed in the previous units are extended in Unit VI to include the transcendental functions. The more familiar inverse trigonometric functions are developed first. Logarithmic and exponential functions are introduced with the definition of the natural logarithm. This is followed by a treatment of the exponential function as the inverse of the logarithm function.

A particular application of these functions is the solution of problems involving rate of growth or decay.

INSTRUCTIONAL OBJECTIVES

1. Apply the rules for differentiating the trigonometric functions.
2. Apply the rules for integrating the trigonometric functions.
3. Apply the rules for differentiating the inverse trigonometric functions.
4. Apply the rules for integrating expressions of the form $\int \frac{du}{1-u^2}$, $\int \frac{du}{1+u^2}$, and $\int \frac{du}{u\sqrt{u^2-1}}$.
5. Apply the formula for the derivative of the natural logarithm function.
6. Apply the rule for $\int \frac{du}{u}$.
7. Apply the rules for the derivatives of e^u and a^u.
8. Apply the rules for $\int e^u \, du$ and $\int a^u \, du$.
9. Apply the method of logarithmic differentiation.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) State the rules for differentiating the trigonometric functions.

   b) Construct the derivatives of expressions containing the trigonometric functions.

   c) Construct the derivatives of expressions containing powers of the trigonometric functions.

d) Construct the derivatives of expressions containing products and quotients of trigonometric functions.

2. a) State the rules for integrating the trigonometric functions.

b) Integrate expressions containing the trigonometric functions.

c) Integrate trigonometric expressions of the form $u^n\,du$.

3. a) State the rules for differentiating the inverse trigonometric functions.

b) Construct the derivative for a given inverse trigonometric function.

4. a) State the rules for integrating expressions of the form $\int \frac{du}{1-u^2}$, $\int \frac{du}{1+u^2}$, and $\int \frac{du}{u\sqrt{u^2-1}}$.

b) Integrate expressions of the form $\int \frac{du}{\sqrt{1-u^2}}$, $\int \frac{du}{1+u^2}$, and $\int \frac{du}{u\sqrt{u^2-1}}$.

5. a) State the rule for the derivative of the natural logarithm function.

b) Construct the derivative for an expression containing the natural logarithm function.

6. a) State the rule for $\int \frac{du}{u}$.

b) Integrate an expression of the form $\int \frac{du}{u}$.

7. a) State the rule for the derivative of e^u.

b) State the rule for the derivative of a^u.

c) Construct the derivative of an expression containing e^u.

d) Construct the derivative of an expression containing a^u.

8. a) State the rule for $\int e^u\,du$.

b) State the rule for $\int a^u\,du$.

c) Integrate an expression of the form $\int e^u\,du$.

d) Integrate an expression of the form $\int a^u\,du$.

9. a) Construct an equivalent equation in logarithmic form from a given equation.

b) Construct the derivative of a given equation by first writing the equation in logarithmic form using the base _e_.

SAMPLE ASSESSMENT MEASURES: TRANSCENDENTAL FUNCTIONS

PERFORMANCE OBJECTIVE VI - 1a:	State the rules for differentiating the trigonometric functions.

State the derivative of each of the following:

1. $\frac{d(\sin u)}{dx}$

2. $\frac{d(\cos u)}{dx}$

3. $\frac{d(\tan u)}{dx}$

4. $\frac{d(\cot u)}{dx}$

5. $\frac{d(\sec u)}{dx}$

6. $\frac{d(\csc u)}{dx}$

PERFORMANCE OBJECTIVE VI - 1b:	Construct the derivatives of expressions containing the trigonometric functions.

Determine $\frac{dy}{dx}$ for:

1. $y = \sin(2x + 5)$
2. $y = \cos(5x - 2)$
3. $y = 2 \tan(4x^2 - 5)$
4. $y = \frac{1}{3} \cot(6x - 3)$
5. $y = 7 \sec(2x - 3)$
6. $y = \sqrt{2} \csc(x^2 - 3)$

PERFORMANCE OBJECTIVE VI - 1c: Construct the derivatives of expressions containing powers of the trigonometric functions.

Determine $\frac{dy}{dx}$ for:

1. $y = \sin^2 2x$
2. $y = \cos^4 (x^2 + 2x)$
3. $y = \tan^{-3} x$
4. $y = -\cot^{-2} (3x^2 - 7)$
5. $y = 2 \sec^5 \sqrt{x}$
6. $y = 3 \csc^3 5x$

PERFORMANCE OBJECTIVE VI - 1d: Construct the derivatives of expressions containing products and quotients of trigonometric functions.

1. Determine $\frac{dy}{dx}$ for:

   a) $y = \sin 3x \cos 7x$

   b) $y = \tan 5x \sec 2x$

2. Determine $\frac{dy}{dx}$ for:

   a) $\frac{4 \sin^2 x}{3 \cos^3 x}$

   b) $y = \frac{\sec^2 2x}{\csc^3 x^2}$

PERFORMANCE OBJECTIVE VI - 2a: State the rules for integrating the trigonometric functions.

Determine the following:

1. $\int \sin u \, du$
2. $\int \cos u \, du$
3. $\int \tan u \, du$
4. $\int \cot u \, du$
5. $\int \sec u \, du$
6. $\int \csc u \, du$
7. $\int \sec^2 u \, du$
8. $\int \sec u \tan u \, du$
9. $\int \csc^2 u \, du$
10. $\int \csc u \cot u \, du$

PERFORMANCE OBJECTIVE VI - 2b: Integrate expressions containing the trigonometric functions.

1. $\int \cos 2x \, dx$
2. Evaluate: $\int_0^{\pi/3} \sec^2 x \, dx$
3. $\int \sec (7x - 5) \tan (7x - 5) \, dx$

PERFORMANCE OBJECTIVE VI - 2c: Integrate trigonometric expressions of the form $u^n \, du$.

1. $\int \tan 2x \sec^2 2x \, dx$
2. $\int \cot^4 x \csc^2 x \, dx$

PERFORMANCE OBJECTIVE VI - 3a: State the rules for differentiating the inverse trigonometric functions.

State the derivative of the following:

1. $\frac{d\,(\text{arc sin } u)}{dx}$

2. $\frac{d\,(\text{arc cos } u)}{dx}$

3. $\frac{d\,(\text{arc tan } u)}{dx}$

4. $\frac{d\,(\text{arc cot } u)}{dx}$

5. $\frac{d\,(\text{arc sec } u)}{dx}$

6. $\frac{d\,(\text{arc csc } u)}{dx}$

PERFORMANCE OBJECTIVE VI - 3b: Construct the derivative for a given inverse trigonometric function

1. Determine $\frac{dy}{dx}$ for $y = \text{arc sin } 2x$.

2. Determine $\frac{dy}{dx}$ for $y = \text{arc cot } x^3$.

3. Determine $\frac{dy}{dx}$ for $y = \text{arc sec } 3x^2$.

PERFORMANCE OBJECTIVE VI - 4a: State the rules for integrating expressions of the form $\int \frac{du}{\sqrt{1-u^2}}$, $\int \frac{du}{1+u^2}$, and $\int \frac{du}{u\sqrt{u^2-1}}$.

1. $\int \frac{du}{\sqrt{1-u^2}}$

2. $\int \frac{du}{1+u^2}$

3. $\int \frac{du}{u\sqrt{u^2-1}}$

PERFORMANCE OBJECTIVE VI - 4b: Integrate expressions of the form

$$\int \frac{du}{\sqrt{1-u^2}}, \int \frac{du}{1+u^2}, \text{ and } \int \frac{du}{u\sqrt{u^2-1}}.$$

1. $\int \frac{dx}{\sqrt{1-49x^2}}$

2. $\int \frac{dx}{1+7x^2}$

3. $\int \frac{dx}{x\sqrt{4x^2-1}}$

PERFORMANCE OBJECTIVE VI - 5a: State the rule for the derivative of the natural logarithm function.

1. State $\frac{d(\ln u)}{dx}$

PERFORMANCE OBJECTIVE VI - 5b: Construct the derivative for an expression containing the natural logarithm function.

1. Construct $\frac{dy}{dx}$ for $y = \ln(x^2 + 3x + 4)$.

2. Construct $\frac{dy}{dx}$ for $y = \ln(\tan^2 x)$.

3. Construct $\frac{dy}{dx}$ for $y = \ln(2x + 7)^3$.

PERFORMANCE OBJECTIVE VI - 6a: State the rule for $\int \frac{du}{u}$.

1. $\int \frac{du}{u}$

PERFORMANCE OBJECTIVE VI- 6b: Integrate an expression of the form $\int \frac{du}{u}$.

1. $\int \frac{dx}{3x + 7}$

2. $\int_1^3 \frac{(x + 1)dx}{x^2 + 2x + 5}$

3. $\int \frac{\cos x \, dx}{\sin x}$

PERFORMANCE OBJECTIVE VI - 7a: State the rule for the derivative of e^u.

1. State the rule for $\frac{d(e^u)}{dx}$.

PERFORMANCE OBJECTIVE VI - 7b: State the rule for the derivative of a^u.

1. State the rule for $\frac{d(a^u)}{dx}$

PERFORMANCE OBJECTIVE VI - 7c: Construct the derivative of an expression containing e^u.

1. Construct: $\frac{d(e^{-x^3})}{dx}$

2. Construct: $\frac{d(e^{\text{arc cos } x})}{dx}$

3. Construct the derivative for $\sin y = e^{2x} + \ln x$.

PERFORMANCE OBJECTIVE VI - 7d: Construct the derivative of an expression containing a^u.

1. Construct $\frac{dy}{dx}$ for $y = 2^{\tan x}$.

2. Construct $\frac{dy}{dx}$ for $y = 5^{\sec 2x}$.

PERFORMANCE OBJECTIVE VI - 8a: State the rule for $\int e^u\,du$.

1. State the rule for $\int e^u\,du$.

PERFORMANCE OBJECTIVE VI - 8b: State the rule for $\int a^u\,du$.

1. $\int a^u\,du =$

a. $a^u + c$ b. $a^u \ln a + c$ c. $\frac{a^u}{\ln a} + c$

d. $\frac{a^{u+1}}{u+1} + c$ e. None of these

PERFORMANCE OBJECTIVE VI - 8c: Integrate an expression of the form $\int e^u\, du$.

1. $\int \frac{5dx}{e^{x}}$

2. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$

3. $\int_e^{e^3} \frac{dx}{x \ln x}$

PERFORMANCE OBJECTIVE VI - 8d: Integrate an expression of the form $\int a^u\, du$.

1. Evaluate: $\int_0^{1.3} 2^x\, dx$.

2. Evaluate: $\int_0^{1} 7^{3t-1}\, dt$.

3. Evaluate: $\int_0^{\frac{\pi}{3}} \sin\theta \cdot 2^{\cos\theta}\, d\theta$.

PERFORMANCE OBJECTIVE VI - 9a: Construct an equivalent equation in logarithmic form from a given equation.

1. Write $y^2 = x(x - 1)$ in logarithmic form, base e.

2. Write $y^{\frac{3}{2}} = \sqrt[4]{\dfrac{(x^2 + 1)\,(3x - 7)^{\frac{3}{2}}}{(2x - 3)(x^2 - 9)}}$ in logarithmic form, base e.

PERFORMANCE OBJECTIVE VI - 9b: Construct the derivative of a given equation by first writing the equation in logarithmic form using base e.

1. Given: $y = \tan x^{\sin x}$, determine the derivative $\frac{dy}{dx}$.

2. Given $y = \dfrac{x\sqrt[3]{x^2 - 1}}{(x-1)^{\frac{5}{3}}}$, $x > 1$; determine the derivative $\frac{dy}{dx}$.

ANSWERS TO ASSESSMENT MEASURES

1. a)

(1) $\cos u \frac{du}{dx}$

(2) $-\sin u \frac{du}{dx}$

(3) $\sec^2 u \frac{du}{dx}$

(4) $-\csc^2 u \frac{du}{dx}$

(5) $\sec u \tan u \frac{du}{dx}$

(6) $-\csc u \cot u \frac{du}{dx}$

b)

(1) $2 \cos (2x + 5)$

(2) $-5 \sin (5x - 2)$

(3) $16x \sec^2 (4x^2 - 5)$

(4) $-2 \csc^2 (6x - 3)$

(5) $14 \sec (2x - 3) \tan (2x - 3)$

(6) $-2\sqrt{2}\, x \csc (x^2 - 3) \cot (x^2 - 3)$

c)

(1) $4 \sin 2x \cos 2x$ or $2 \sin 4^x$

(2) $-8 (x + 1) \cos^3 (x^2 + 2x) \sin (x^2 + 2x)$

(3) $-3 \tan^{-4} x \sec^2 x$

(4) $-12x \cot^{-3} (3x^2 - 7) \csc^2 (3x^2 - 7)$

(5) $5x^{-\frac{1}{2}} \sec^5 \sqrt{x} \tan \sqrt{x}$

(6) $-45 \csc^3 5x \cot 5x$

d)

(1)

(a) $-7 \sin 3x \sin 7x + 3 \cos 3x \cos 7x$

(b) $2 \tan 5x \sec 2x \tan 2x + 5 \sec^2 5x \sec 2x$

(2)

(a) $\dfrac{8 \cos^2 x \sin x + 12 \sin^3 x}{3 \cos^4 x}$

(b) $\dfrac{4 \sec^2 2x \tan 2x + 6x \sec^2 2x \cot x^2}{\csc^3 x^2}$

2. a)

(1) $-\cos u + c$

(2) $\sin u + c$

(3) $-\ln|\cos u| + c = \ln|\sec u| + c$

(4) $\ln|\sin u| + c$

(5) $\ln|\sec u + \tan u| + c$

(6) $-\ln|\csc u + \cot u| + c$

(7) $\tan u + c$

(8) $\sec u + c$

(9) $-\cot u + c$

(10) $-\csc u + c$

2. (continued)

b)

(1) $\frac{1}{2}\sin 2x + c$

(2) $\sqrt{3}$

(3) $\frac{1}{7}\sec(7x - 5) + c$

c)

(1) $\frac{1}{4}\tan^2 2x + c$

(2) $\frac{-\cot^5 x}{5} + c$

3.a)

(1) $\dfrac{\left(\frac{du}{dx}\right)}{\sqrt{1 - u^2}}$

(b) $\dfrac{-\left(\frac{du}{dx}\right)}{\sqrt{1 - u^2}}$

(3) $\dfrac{\left(\frac{du}{dx}\right)}{1 + u^2}$

(4) $\dfrac{-\left(\frac{du}{dx}\right)}{1 + u^2}$

(5) $\dfrac{\left(\frac{du}{dx}\right)}{|u|\sqrt{u^2 - 1}}$

(6) $\dfrac{-\left(\frac{du}{dx}\right)}{|u|\sqrt{u^2 - 1}}$

b)

(1) $\frac{dy}{dx} = \frac{2}{\sqrt{1 - 4x^2}}$

(2) $\frac{dy}{dx} = \frac{-3x^2}{1 + x^6}$

(3) $\frac{dy}{dx} = \frac{2}{|x|\sqrt{9x^4 - 1}}$

4. a)

(1) arc sin u + c or -arc cos u + c

(2) arc tan u + c or -arc cot u + c

(3) arc sec u + c or -arc csc u + c

b)

(1) $\frac{1}{7}$ arc sin 7x + c

(2) $\frac{1}{\sqrt{7}}$ arc tan $\sqrt{7}$ x + c

(3) arc sec 2x + c

5. a)

(1) $\frac{1}{u} \cdot \frac{du}{dx}$

b) (1) $\frac{2x + 3}{x^2 + 3x + 4}$

(2) $\frac{2 \sec^2 x}{\tan x}$ or $\frac{2}{\sin x \cos x}$

(3) $\frac{6}{2x + 7}$

6. a)

(1) $\ln|u| + c$

b)

(1) $\frac{1}{3} \ln |3x + 7| + c$

(2) $\frac{1}{2} \ln \frac{5}{2}$

(3) $-\ln |\sin x| + c$

7. a)

(1) $e^u \frac{du}{dx}$

b)

(1) $a^u \ln a \frac{du}{dx}$

c)

(1) $-3x^2 e^{-x^3}$

(2) $-\frac{e^{\text{arc cos } x}}{\sqrt{1 - x^2}}$

(3) $\frac{2xe^{2x} + 1}{x \cos y}$

d)

(1) $\frac{dy}{dx} = 2^{\tan x} \ln 2 \sec^2 x$

(2) $\frac{dy}{dx} = 2(5^{\sec 2x} \ln 5 \sec 2x \tan 2x)$

8. a)

(1) $e^u + c$

b)

(1) C

c)

(1) $-\frac{5}{4} e^{-4x} + c$

(2) $\ln | e^x + e^{-x} | + c$

(3) ln 3

d)

(1) $\frac{2^{1.3} - 1}{\ln 2}$

(2) $\frac{1}{21 \ln 7} (7^3 - 1)$ or $\frac{114}{7 \ln 7}$

(3) $\frac{2 - 2^{\frac{1}{2}}}{\ln 2}$

9. a)

(1) $2 \ln y = \ln x + \ln(x - 1)$

(2) $\frac{3}{2} \ln y = \frac{1}{4} \ln(x^2 + 1) + \frac{1}{8} \ln(3x - 7) - \ln(2x - 3) - \ln(x^2 - 9)$

b)

(1) $y (\sec x + \cos x \ln(\tan x))$

(2) $y \left(\frac{-5x - 3}{3x (x^2 - 1)}\right)$

UNIT VII: METHODS OF INTEGRATION

INTRODUCTION

The development of integral calculus thus far has been restricted to polynomials or special functions which are clearly the derivatives of other functions. It should be rather obvious to the student that he/she may need to integrate functions which are not so convenient and, moreover, that more than a "trial and error" approach is needed. The purpose of Unit VII is to develop the techniques needed to integrate various standard types of expressions.

Emphasis in this unit is upon the acquisition of skills involving the various techniques of integration such as substitution, parts, and partial functions. The student's success depends upon his/her ability to recognize the type of integral involved so that the appropriate technique can be applied. Practice is the key to this unit, and adequate time must be allotted.

INSTRUCTIONAL OBJECTIVES

1. Integrate expressions, using the method of substitution.
2. Integrate expressions, using the method of integration by parts.
3. Apply the rules for evaluating improper integrals.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) Integrate given expressions, using substitution of trigonometric identities.

   b) Integrate given expressions of the form $\sqrt{a^2 - u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2 - a^2}$, a^2+u^2, and $a^2 - u^2$, using trigonometric substitution.

   c) Integrate, using the method of partial fractions.

   d) Integrate given expressions, using a change of variables. (e.g., $f(x) = z^n$, $z = \tan(x/2)$, etc.)

2. a) State the rule for integration by parts.

   b) Derive the rule for integration by parts.

   c) Apply the method of integration by parts to given expressions.

3. a) State the conditions for which an integral is classified as an improper integral.

   b) Identify given integrals which satisfy the conditions for an improper integral.

   c) Evaluate a given improper integral.

SAMPLE ASSESSMENT MEASURES; METHODS OF INTEGRATION

PERFORMANCE OBJECTIVE VII - 1a:	Integrate given expressions, using substitution of trigonometric identities.

1. $\int \sec^4 2x \, dx$

2. $\int \cos^2 3x \, dx$

PERFORMANCE OBJECTIVE VII - 1b:	Integrate given expressions of the form $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-a^2}$, a^2+u^2 and a^2-u^2, using trigonometric substitution.

1. $\int_0^3 \frac{dx}{9+x^2}$

2. $\int \frac{dx}{x\sqrt{4-x^2}}$

3. $\int \sqrt{9-x^2} \, dx$

PERFORMANCE OBJECTIVE VII - 2a: State the rule for integration by parts.

1. State the rule for integration by parts.

PERFORMANCE OBJECTIVE VII - 2b: Derive the rule for integration by parts.

1. Derive the rule for integration by parts.

PERFORMANCE OBJECTIVE VII - 2c: Apply the method of integration by parts to given expressions.

1. $\int x^2 \ln 3x \, dx$

2. $\int e^{2x} \sin 2x \, dx$

3. $\int x^2 e^{3x} \, dx$

PERFORMANCE OBJECTIVE VII - 1c: Integrate using the method of partial fractions.

1. $\int \frac{x\,dx}{x^2 - 5x + 6}$

2. $\int \frac{(3x^2 + 1)\,dx}{(x-1)(x^2 + 1)}$

3. $\int \frac{4x^2 - 5x + 2}{x^3 - 2x^2 + x}\,dx$

PERFORMANCE OBJECTIVE VII - 1d: Integrate given expressions using a change of variables (e.g., $f(x) = z^m$, $z = \tan \frac{x}{2}$, etc.).

1. $\int \frac{\sqrt{x + 1}}{x + 2}\,dx$

2. $\int y^5 \sqrt{2 + y^3}\,dy$

3. $\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x}$

PERFORMANCE OBJECTIVE VII - 3a: State the conditions for which an integral is classified as an improper integral.

1. State the conditions for which an integral is classified as an improper integral.

PERFORMANCE OBJECTIVE VII - 3b: Identify given integrals which satisfy the conditions for an improper integral.

1. Determine which of these are improper integrals:

a) $\int_0^{10} \frac{dx}{x^2 + 1}$

b) $\int_{-1}^{1} \frac{dx}{\sqrt{x}}$

c) $\int_1^{\infty} x^{-2}\, dx$

d) $\int_0^1 \sqrt{1 - x^2}\, dx$

e) $\int_0^1 \frac{dx}{x^{-\frac{1}{2}}}$

PERFORMANCE OBJECTIVE VII - 3c: Evaluate a given improper integral.

1. $\int_0^3 \frac{dx}{\sqrt{3 - x}}$

2. $\int_0^\infty e^{-x}dx$

3. $\int_{-1}^1 \frac{dx}{x^3}$

ANSWERS TO ASSESSMENT MEASURES

1. a)

(1) $\frac{1}{6}\tan^3 2x + \frac{1}{2}\tan 2x + c$

(2) $\frac{1x}{2} + \frac{1}{12}\sin 6x + c$

b)

(1) $\frac{\pi}{12}$

(2) $\frac{1}{2}\ln\left|\frac{x}{2+\sqrt{4-x^2}}\right| + c$ or $\frac{1}{2}\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + c$

(3) $\frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + c$

c)

(1) $\ln\left|\frac{(x-3)^3}{(x-2)^2}\right| + c$

(2) $\ln (x-1)^2\sqrt{x^2+1} + \tan^{-1}x + c$

(3) $\ln\left|x^2(x-1)^2\right| - \frac{1}{x-1} + c$

d)

(1) $2\sqrt{x+1} - 2\tan^{-1}\sqrt{x+1} + c$

(2) $\frac{2}{15}(2+y^3)^{\frac{5}{2}} - \frac{4}{9}(2+y^3)^{\frac{3}{2}} + c$

(3) $\frac{\pi}{3\sqrt{3}}$ or $\frac{\sqrt{3}}{9}\pi$

2. a) $u\,dv = uv - \quad v \quad du + c$

b) Since $d(uv) = u\,d(v) = v\,d(u)$, then $\int (u\,dv + v\,du) = uv + c \leftrightarrow$
$\leftrightarrow \int u dv + \int v\,du = uv + c \quad \leftrightarrow \int u\,dv = uv - \int v\,du + c$

c) (1) $\frac{x^3}{3} \ln|3x| - \frac{x^3}{9} + c$

(2) $\frac{e^{2x}}{4}(\sin 2x - \cos 2x) + c$

(3) $\frac{e^{3x}}{27}(9x^2 - 6x + 2) + c$

3. a)

(1) The function becomes infinite at a value of x in the interval of integration $[a, b]$.

(2) One or both of the limits of integration are infinite, such as

$$\int_a^{\infty} f(x)\,dx \quad \text{or} \int_{-\infty}^{b} f(x)\,dx\,.$$

b)

(1) (b), (c)

c)

(1) $2\sqrt{3}$

(2) 1

(3) Limit does not exist (integrand meaningless).

UNIT VIII: SEQUENCES AND SERIES

INTRODUCTION

The study of sequences and series provides a challenge to the student's mathematical ingenuity. Although rules for testing convergence and determining limits are developed, much of the student's success depends upon his/her cleverness in manipulating the given expressions to fit the various rules.

The practical application of series can be demonstrated by the representation of familiar transcendental functions as power series or Taylor series. The evaluation of these series can then be computed to approximate the function for a particular value.

INSTRUCTIONAL OBJECTIVES

1. Determine the limit of a sequence and construct a proof that is correct by applying the definition of limit of a sequence.

2. Identify series that satisfy the necessary condition for convergence.

3. Apply the comparison test for convergence of a series.

4. Apply the ratio test for convergence of a series.

5. Apply the integral test for convergence of a series.

6. Express transcendental functions as power series.

7. Apply Taylor's theorem with remainder to transcendental functions.

8. Apply the definition of absolute convergence to series.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) State the definition of the limit of a sequence.

   b) Determine the limit of a given sequence.

   c) Determine the limit of a given sequence and construct the proof that it is correct by applying the definition.

2. a) State the rule for the necessary condition for convergence of an infinite series.

b) Apply the rule to a given series to determine whether it satisfies the necessary condition for convergence.

3. a) State the rule for the comparison test for the convergence of a series.

   b) Determine whether a given series converges, by the comparison test.

4. a) State the rule for the ratio test for the convergence of a series.

   b) Determine whether a given series converges, by the ratio test.

5. a) State the integral test for the convergence of a series.

   b) Determine whether a given series converges, by the integral test.

6. a) Construct a power series expansion for a given transcendental function.

   b) Determine the approximate value of a given transcendental function for a particular value of the variable.

   c) Determine the interval of convergence for a given power series.

7. a) State Taylor's theorem.

   b) Construct a Maclaurin series expansion for a given transcendental function.

   c) Construct a Taylor series expansion for a given transcendental function.

   d) Apply the Taylor series expansion to approximate the value for a given transcendental function for a particular value of the variable.

   e) Apply the Lagrange remainder to determine the accuracy of the value of a transcendental function obtained by the use of the Taylor series expansion.

8. a) State the definition of absolute convergence.

   b) Apply the definition of absolute convergence to a given alternating series.

PERFORMANCE OBJECTIVE VIII - 1a: State the definition of the limit of a sequence.

1. State the definition of the limit of a sequence.

PERFORMANCE OBJECTIVE VIII - 1b: Determine the limit of a given sequence.

1. Evaluate: $\lim_{n\to\infty} \left(3 - \frac{(-1)^n}{n}\right)$.

2. Evaluate: $\lim_{n\to\infty} \frac{3n^3}{5n^4 + n^3}$.

3. Evaluate: $\lim_{n\to\infty} \frac{5n^2 - 3n + 2}{6n^2 + 7n}$

4. Evaluate: $\lim_{n\to\infty} \frac{\ln 2n}{n}$.

5. Evaluate: $\lim_{n\to\infty} \frac{e^n}{n^3}$.

PERFORMANCE OBJECTIVE VIII - 1c: Determine the limit of a given sequence and construct the proof that it is correct by applying the definition.

1. Given: $S_n = \frac{2n + 3}{3n + 1}$, determine L; and using $\epsilon = \frac{1}{100}$, calculate the smallest value of n which will prove that $\lim_{n\to\infty} S_n = L$.

2. Given: $S_n = \frac{n}{n - 2}$, $n > 2$, determine L; and find n as a function of ϵ which will prove that $\lim_{n\to\infty} S_n = L$ according to the definition.

3. Given: $S_n = \frac{n}{n^2 - 1}$, $n > 1$, determine L, and find n as a function of ϵ which will prove that $\lim_{n\to\infty} S_n = L$ according to the definition.

PERFORMANCE OBJECTIVE VIII - 2a: State the rule for the necessary condition for convergence of an infinite series.

1. State the necessary condition for convergence of an infinite series.

PERFORMANCE OBJECTIVE VIII - 2b: Apply the rule to a given series to determine whether it satisfies the necessary condition for convergence.

1. Determine which of the following series satisfy the necessary condition for convergence.

a) $\sum_{n=1}^{\infty} \frac{5n - 2}{2 + 3n}$ b) $\sum_{n=1}^{\infty} \frac{n - 1}{3n^2 + 2}$ c) $\sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n}$

d) $\sum_{n=1}^{\infty} \frac{1}{n!}$ e) $\sum_{n=1}^{\infty} \frac{e^n}{5n}$ f) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

PERFORMANCE OBJECTIVE VIII - 3a: State the rule for the comparison test for the convergence of a series.

1. State the rule for the comparison test for convergence of a series.

PERFORMANCE OBJECTIVE VIII - 3b: Determine whether a given series converges by the convergence test.

1. Determine whether $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 1}$ converges or diverges using the comparison test.

2. Determine whether $\sum_{n=3}^{\infty} \frac{1}{\sqrt{2n}}$ converges or diverges using the comparison test.

PERFORMANCE OBJECTIVE VIII - 4a: State the rule for the ratio test for the convergence of a series.

1. State the rule for the ratio test for the convergence of a series.

PERFORMANCE OBJECTIVE VIII - 4b: Determine whether a given series converges by the ratio test.

1. Show that the series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges by the ratio test.

2. Determine whether $\sum_{n=1}^{\infty} \frac{3^n}{n^3 + 1}$ converges or diverges according to the ratio test. Show the value of ρ (rho).

3. Determine whether the following series converges or diverges according to the ratio test. Indicate the value of ρ (rho).

$$\sum_{n=1} \frac{n(n+1)}{n!}$$

4. Determine rho using the ratio test on $\sum_{m=1}^{\infty} \frac{m+1}{m^2+1}$.
Does the series converge or diverge?

PERFORMANCE OBJECTIVE VIII - 5a: State the integral test for the convergence of a series.

1. State the integral test for the convergence of a series.

PERFORMANCE OBJECTIVE VIII - 5b: Determine whether a given series converges by the integral test.

1. Determine whether $\sum_{n=1}^{\infty} 2n/(n^2+1)$ converges or diverges using the integral test.

2. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ell n\, n)^2}$ converges or diverges using the integral test. Indicate the function of n after integrating.

PERFORMANCE OBJECTIVE VIII - 6a: Construct a power series expansion for a given transcendental function.

1. Construct a power series <u>in x</u> for cos x including the nth term.

2. Construct a power series <u>in x</u> for e^{-x} including the nth term.

3. Construct a power series <u>in x - a</u>, where a = 2, for ln x including the nth term.

PERFORMANCE OBJECTIVE VIII - 6b: Determine the approximate value of a given transcendental function for a particular value of the variable

1. Construct a power series <u>in x</u> for sin x and determine the approximate value of the power series for x = ½ using the first three terms.

2. Construct a power series <u>in x</u> for ln(1 + x) and determine the approximate value for x = 1 using the first four terms.

PERFORMANCE OBJECTIVE VIII - 6c: Determine the interval of convergence for a given power series.

1. Determine the interval of convergence for the power series listed below:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!} + \dots$$

2. Determine the interval of convergence for the power series listed below:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{-1} \frac{x^n}{n} + \dots$$

PERFORMANCE OBJECTIVE VIII - 7a: State Taylor's theorem.

1. State Taylor's Theorem

PERFORMANCE OBJECTIVE VIII - 7b: Construct a Maclaurin series expansion for a given transcendental function.

1. Construct a Maclaurin series expansion for arc tan x. Indicate the first four terms and the nth term.

2. Construct a Maclaurin series expansion for $\frac{\sin x}{x}$. Indicate the first four terms and the nth term.

3. Construct a Maclaurin series expansion for $\cos^2 x$. Indicate the first four terms and the nth term.

PERFORMANCE OBJECTIVE VIII - 7c: Construct a Taylor series expansion for a given transcendental function.

1. Construct a Taylor series expansion for $e^{\frac{x}{2}}$. Indicate the first three terms and the nth term.

2. Construct a Taylor series expansion for $\cos(x + \frac{\pi}{4})$ with $a = \frac{\pi}{4}$. Indicate the first three terms and the nth term.

3. Construct a Taylor series expansion for $\cos^2 x$. Indicate the first five terms and the nth term.

PERFORMANCE OBJECTIVE VIII - 7d: Apply the Taylor series expansion to approximate the value for a given transcendental function for a particular value of the variable.

1. Approximate the value for $\sqrt{e}$ using only the first four terms in the Taylor series expansion for e^x. Let $a = 0$.

2. Approximate the value for $\sin 31^\circ$ using only the first three terms in the Taylor series expansion for $\sin x$. Let $a = 0$.

3. Approximate the value for $\ln 1.2$ using only the first two terms in the Taylor series expansion for $\ln \left| \frac{1 + x}{1 - x} \right|$. Hint: $1.2 = \frac{6}{5} = \frac{1 + x}{1 - x}$

PERFORMANCE OBJECTIVE VIII - 7e: Apply the Lagrange remainder to determine the accuracy of the value of a transcendental function obtained by the use of the Taylor series expansion.

1. Determine the error according to the Lagrange remainder for $e^{.1}$ evaluated by the terms $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$.

2. Determine the error, according to the Lagrange remainder, for cos .1 evaluated by the first three terms in the Taylor series, $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

PERFORMANCE OBJECTIVE VIII - 8a: State the definition of absolute convergence.

1. State the definition of absolute convergence.

2. A series $\sum_{K=1}^{\infty} u_K = u_1 + u_2 + \ldots$ is absolutely convergent if ...

PERFORMANCE OBJECTIVE VIII - 8b: Apply the definition of absolute convergence to a given alternating series.

1. Determine the convergence of $1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \ldots$

2. Determine the convergence of $\frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} - \frac{1}{4 \cdot 3^4} + \ldots$

3. Determine the convergence of $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + \ldots + (-1)^n \frac{n+1}{n} + \ldots$

ANSWERS TO ASSESSMENT MEASURES

1. a)

(1) If for every positive number ϵ there exists a real number n_o such that

$$| S_n - L | < \epsilon \text{ for } n > n_o$$

then the sequence is said to converge to the limit L.

b)

(1) 3

(2) 0

(3) $\frac{5}{6}$

(4) 0

(5) Limit does not exist.

c)

(1) $L = \frac{2}{3}$, n = 78

(2) $L = 1$, $n = \frac{2 + 2\epsilon}{\epsilon} + 1$ (not a unique solution)

(3) $L = 0$, $n = \frac{1 + \sqrt{1 + 4\epsilon^2}}{2\epsilon} + 1$ (not a unique solution)

2. a)

(1) A necessary condition for the congruence of an infinite series

$$a_1 + a_2 + \cdots + a_n + \cdots$$

is that $\lim_{n\to\infty} a_n = 0$.

b)

(1) b, c, d, f

3. a)

(1) If each term of a positive series $\sum_{K=1}^{\infty} u_K$ is less than the corresponding term of a known convergent series, then $\sum u_K$ converges. But if each term of $\sum u_K$ is greater than the corresponding term of a known divergent positive series, then $\sum u_K$ diverges.

b)

(1) Converges by comparison with $\frac{1}{n^2}$

(2) Diverges by comparison with $\frac{1}{n}$

4. a) In a positive series $u_1 + u_2 + \ldots + u_m$, where $\rho = \lim_{n \to \infty} \frac{u_{n+1}}{u_n}$

(1)

(a) The series converges if $\rho < 1$.

(b) The series diverges if $\rho > 1$.

(c) The series may converge or diverge if $\rho = 1$.

b)

(1) $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right|$

$$\lim_{n \to \infty} \frac{\frac{2}{3}^{n+1}}{\frac{2}{3}^{n}} = \frac{2}{3} < 1$$

$\therefore$ converges

(2) $\rho = 3 > 1 \therefore$ diverges

(3) rho = 0; therefore series converges

(4) rho = 1; therefore series may either converge or diverge

5. a)

(1) If f(x) is a continuous, positive, decreasing function for which f(n) is the Nth term u_n of the series, then $\sum_{n=1}^{\infty} u_n$ converges if and only if the integral $\int_1^{\infty} f(x)\,dx$ converges.

b)

(1) Diverges

(2) $\frac{-1}{\ln n}$; converges

6. a)

(1) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} + \dots$

(2) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + (-1)^{n-1} \frac{x^{n-1}}{(n-1)!} + \dots$

(3) $\ln 2 + \frac{1}{2}(x-2) - \frac{1}{2^2}\frac{(x-2)^2}{2} + \frac{1}{2^3}\frac{(x-2)^3}{3} - \dots + \frac{(-1)^{n+1}}{2n}\frac{(x-2)^n}{n} + \dots$

b)

(1) (a) $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ (b) .4794

(2) (a) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ (b) $.58\overline{3}$ or $\frac{7}{12}$

c)

(1) Convergent for all values of x.

(2) Convergent on the interval $-1 < x \le 1$

7. a)

(1) A power series in the form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

b)

(1) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$

(2) $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + (-1)^{n-1}\frac{x^{2(n-1)}}{(2n-1)!} + \cdots$

(3) $1 - \frac{2x^2}{2!} + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \cdots + \frac{(-1)^n\, 2^{2n-1}}{(2n)!} + \cdots$

c)

(1) $e^{\frac{x}{2}} = e^{\frac{a}{2}}\left[1 + \frac{1}{2}(x-a) + \frac{1}{2^2}\frac{(x-a)^2}{2!} + \cdots + \frac{1}{2^n}\frac{(x-a)^{n-1}}{(n-1)!} + \cdots\right]$

(2) $\cos(x + \frac{\pi}{4}) = -(x - \frac{\pi}{4}) + \frac{(x - \frac{\pi}{4})^3}{3!} - \frac{(x - \frac{\pi}{4})^5}{5!}$

$$+ \cdots (-1)^n\frac{(x - \frac{\pi}{4})^{2n-1}}{(2n-1)!} + \cdots$$

(3) $\cos^2 a - \sin 2a\,(x-a) - 2\cos 2a\,\frac{(x-a)^2}{2!} + 2^2\sin 2a\,\frac{(x-a)^3}{3!}$

$$+ 2^3\cos 2a\frac{(x-a)^4}{4!} + \cdots + \frac{2^{4m-4}\sin 2a(x-a)^{4m-3}}{(4m-3)!} - \frac{2^{4m-3}\cos 2a(x-a)^{4m-2}}{(4m-2)!}$$

$$+ \frac{2^{4m-2}\sin 2a(x-a)^{4m-1}}{(4m-1)!} + \frac{2^{4m-1}\cos 2a\,(x-a)^{4m}}{(4m)!} + \cdots$$

Block m (m = 1, 2, 3, ...)

d)

(1) 1.646

(2) .5150

(3) .1823

e)

(1) 4.2×10^{-9}

Teacher Note: $R_n(x, 0) = f^{n+1}(c) \dfrac{x^{n+1}}{(n+1)!}$; $0 \le c \le .1$

$$R_5(.1, 0) = \frac{e^c (.1)^6}{6!}$$

$$\le \frac{3 (.1)^6}{6!}$$

$$< 3 \frac{(10^{-6})}{720}$$

$$< 4.2 \times 10^{-9}$$

(2) 1.4×10^{-9}

Teacher Note: $\left| R_{2n+1}(x, 0) \right| \le \left| \dfrac{x^{2n+2}}{(2n+2)!} \right|$

$$|R_5| \le \left| \frac{x^6}{6!} \right| = \left| \frac{(.1)^6}{6!} \right|$$

$$|R_5| \le 1.4 \times 10^{-9}$$

8. a)

(1). A series with mixed signs is said to converge absolutely if the series constructed by taking the absolute value of the terms converges,

(2) ... if $\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots$ converges.

b)

(1) Converges conditionally

Teacher Note: p - series with $p < 1$

Theorem II - Thomas

(2) Absolutely convergent (comparison test)

(3) Diverges $\lim_{n \to \infty} \left| \frac{n+1}{n} \right| \neq 0$

UNIT IX: DIFFERENTIAL EQUATIONS

INTRODUCTION

An elementary treatment of ordinary differential equations is presented in Unit IX. First-order equations are developed by types: separable, homogeneous, linear, and exact. Applications involving rate of growth and decay should be examined.

The study of second-order equations is limited to linear equations with constant coefficients. Solution techniques developed for homogeneous equations are then extended to special types of nonhomogeneous equations.

As in Unit VII, practice is necessary for the student to recognize the type equation in order to apply the appropriate technique.

INSTRUCTIONAL OBJECTIVES

1. Identify the order and degree of ordinary differential equations.
2. Determine the solution of separate first-order differential equations.
3. Determine the solution of homogeneous first-order differential equations.
4. Determine the solution of linear first-order differential equations.
5. Determine the solution of exact first-order differential equations.
6. Determine the solution of homogeneous linear second-order differential equations with constant coefficients.
7. Determine the solution of nonhomogeneous linear second-order differential equations with constant coefficients.
8. Determine the solution of differential equations, given initial conditions.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) Identify the order of given differential equations.

   b) Identify the degree of given differential equations.

2. a) Identify the separable equations from given first-order differential equations.

   b) Determine the solution of a given separable first-order differential equation.

3. a) Identify the homogeneous equations from given first-order differential equations.

   b) Construct a given first-order homogeneous differential equation in the form $dy/dx = F(v)$ where $v = y/x$.

   c) Determine the solution of a given first-order homogeneous differential equation.

4. a) Identify the linear equations from given first-order differential equations.

   b) Construct the standard form of a given first-order linear differential equation.

   c) Determine the integrating factor for a given first-order linear differential equation.

   d) Determine the solution of a given first-order linear differential equation.

5. a) Identify the exact equations from given first-order differential equations.

   b) Determine the solution of a given first-order exact differential equation.

6. a) Identify the homogeneous equations from given linear second-order differential equations.

   b) Construct an equation using differential operator notation from a given homogeneous linear second-order differential equation with constant coefficients.

   c) Determine the characteristic equation of a given homogeneous linear second-order differential equation with constant coefficients.

   d) Determine the solution of a given homogeneous linear second-order differential equation whose characteristic equation has two real unequal roots.

   e) Determine the solution of a given homogeneous linear second-order differential equation whose characteristic equation has two real equal roots.

   f) Determine the solution of a given homogeneous linear second-order differential equation whose characteristic equation has imaginary roots.

7. a) Identify the nonhomogeneous equations from given linear second-order differential equations.

b) Determine the solution of a given nonhomogenous linear second-order differential equation $d^2y/dx^2 + a_1\ dy/dx + a_2y = F(x)$ where $F(x)$ is a polynomial.

c) Determine the solution of a given nonhomogeneous linear second-order differential equation $d^2y/dx^2 + a_1\ dy/dx + a_2y = F(x)$ where $F(x)$ is of the form e^{ax}.

d) Determine the solution of a given nonhomogeneous linear second-order differential equation $d^2y/dx^2 + a_1\ dy/dx + a_2y = F(x)$ where $F(x)$ is of the form sin ax or cos ax.

e) Determine the solution of a given nonhomogeneous linear second-order differential equation $d^2y/dx^2 + a_1\ dy/dx + a_2y = F(x)$ where $F(x)$ is a linear combination of polynomials and transcendental functions.

8. a) Construct the solution to problems requiring the use of first-order differential equations (e.g., growth and decay, modified Ohm's law, etc.).

b) Construct the solution to problems requiring the use of second-order differential equations (e.g., hanging cable, motion on a straight line, oscillary motion, etc.).

PERFORMANCE OBJECTIVE IX - 1a: Identify the order of given differential equations.

1. State the order of the differential equation $y'' + 3y\,y' + 2y = 0$.

2. State the order of the differential equation $y^4 \cdot y^{(3)} + 3y^2\,y^{(1)} + y = 0$.

PERFORMANCE OBJECTIVE IX - 1b: Identify the degree of given differential equations.

1. State the degree of the differential equation $xy'' + x(y')^2 = e^x$.

2. State the degree of the differential equation $\left(\frac{dy}{dx}\right)^5 + \left(\frac{d^3y}{dx^3}\right)^2 + x^6 - 3x = 0$.

PERFORMANCE OBJECTIVE IX - 2a: Identify the separable equations from given first-order differential equations.

1. Determine which of these first-order differential equations are separable!

(a) $\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$

(b) $xy^2 (1 + x^2) dy + (1 + y^3) dx = 0$

(c) $2xydy = (x^2 - y^2)dx$

(d) $\frac{dx}{dt} = 3 - .03x$

(e) $(x - 2y) dy + (y + 3x)dx = 0$

(f) $x^3 y' = y^2 (x - 4)$

(g) $y' = xy + x$

PERFORMANCE OBJECTIVE IX - 2b: Determine the solution of a given separable first-order differential equation.

1. Solve the differential equation $y' = e^{2x + y}$.

2. Solve the differential equation $y' = xy^2 + x$.

3. Solve the differential equation $(2x + 5) dy + (3y + 2) dx = 0$.

PERFORMANCE OBJECTIVE IX - 3a: Identify the homogeneous equations from given first-order differential equations.

1. Determine which of these first-order differential equations are homogeneous.

   a) $2xy\,dy = (x^2 - y^2)\,dx$

   b) $x^2 y' - 3xy + y^2 = 0$

   c) $y' + 5y = 2 \cos x$

   d) $(3x + 2y)\,dx + (y + x)\,dy = 0$

   e) $(2e^y + e^x)dx + e^x (1 - xe^{-x})\,dy = 0$

   f) $(x^3 + y^3)\,dx = 3xy^2\,dy$

   g) $dy + y\,dx + \sqrt{x^2 - y^2}\,dx = 0$

PERFORMANCE OBJECTIVE IX - 3b: Construct a given first-order homogeneous differential equation in the form $\frac{dy}{dx} = F(v)$ where $v = \frac{y}{x}$

1. Construct $(2x - 3y)dx + (x+y)dy = 0$ in the form $\frac{dy}{dx} = F(v)$, where $v = \frac{y}{x}$

2. Construct $-xdy + ydx + \sqrt{x^2 - y^2}dx = 0$ in the form $\frac{dy}{dx} = F(v)$, where $v = \frac{y}{x}$

PERFORMANCE OBJECTIVE IX - 3c: Determine the solution of a given first-order homogeneous differential equation.

1. Solve the differential equation $(2x - y)\,dx + (x + y)dy = 0$.

2. Solve the differential equation $xdy - ydx = \sqrt{x^2 + y^2}dx$.

PERFORMANCE OBJECTIVE IX - 4a: Identify the linear equations from given first-order differential equations.

1. Determine which of these differential equations are linear:

(a) $\ln y \frac{dy}{dx} = \frac{y}{x} + x$

(b) $\frac{1}{x} \frac{dx}{dy} = 3$

(c) $x^3 \frac{dy}{dx} + y = e^x$

(d) $y \frac{dy}{dx} + x = y$

(e) $\frac{dy}{dx} + \frac{x}{y} = x$

(f) $\frac{dy}{dx} + \frac{\sin y}{x} = {}^-1$

(g) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = {}^-0$

PERFORMANCE OBJECTIVE IX - 4b: Construct the standard form of a given first-order linear differential equation.

1. Construct, in standard form, the differential equation

$x^2 \frac{dy}{dx} + 5\,xy = \frac{\cos x}{x^3}$.

2. Construct, in standard form, the differential equation

$y \ln y \, dx + (x - \ln y)\, dy = 0$.

PERFORMANCE OBJECTIVE IX - 4c: Determine the integrating factor for a given first-order linear differential equation.

1. Determine the integrating factor for $y' - y = x^3$.

2. Determine the integrating factor for $xdy - 2ydx = (x - 2)e^x dx$.

3. Determine the integrating factor for $(2yx^3 + x)\,dy + ydx = 0$.

PERFORMANCE OBJECTIVE IX - 4d: Determine the solution of a given first-order linear differential equation.

1. Solve the differential equation $dy + 2xy\,dx + x\,dx = 0$

2. Solve the differential equation $x\,dy - 2y\,dx = x \sin x\,dx$.

PERFORMANCE OBJECTIVE IX - 5a: Identify the exact equations from given first-order differential equations.

1. Which of these differential equations possess exact differentials?

a) $(1 + e^{3x})\,dy + 3ye^{3x}\,dx = 0$

b) $x \sin 3y\,dy + y \sin 3x\,dx = 0$

c) $2xy\,dx + (1 + x^2)\,dy = 0$

d) $\sin x \cos^2 y\,dx + \cos^2 x \sin y\,dy = 0$

e) $y(x-2y)dx - x^2dy = 0$

f) $(x^2 - y^2)dx - 2xy\,dy = 0$

g) $e^{xy}\,dx + xe^x dy = 0$

h) $\tan^{-1}(xy)\,dy + \frac{1}{2xy} \ln(1 + x^2 y^2)dx = 0$

i) $\left(\frac{\ln y}{x} + \frac{1}{y}\right) - \left(\frac{x}{y^2} - \frac{\ln x}{y}\right) y' = 0$

PERFORMANCE OBJECTIVE IX - 5b: Determine the solution of a given first-order exact differential equation.

1. Solve the differential equation $(x^2 - y^2)\,dx - 2xy\,dy = 0$.

2. Solve the differential equation $(1 + e^{3x})dy + 3ye^{3x}\,dx = 0$.

3. Solve the differential equation

$(\cos y + y \cos x)\,dx + (\sin x - x \sin y)\,dy = 0$, PERFORMANCE OBJECTIVE IX - 6a: Identify the homogeneous equations from given linear second-order differential equations.

1.

a) $y'' + y' - 2y = 5e^{-x}$

b) $y'' + y = e^{x}$

c) $y'' = y + (x^2 + 1)$

d) $y'' = 6x^2 + 3x - 2$

e) $y'' - y = y'$

PERFORMANCE OBJECTIVE IX - 6b: Construct an equation using differential operator notation from a given homogeneous linear second-order differential equation with constant coefficients.

1. Construct an equation using differential operator notation from the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

2. Construct an equation using differential operator notation from the equation $\frac{15\,d^2y}{dx^2} = 14\frac{dy}{dx} + y$.

PERFORMANCE OBJECTIVE IX - 6c: Determine the characteristic equation of a given homogeneous linear second-order differential equation with constant coefficients.

1. Write the characteristic equation for the differential equation $y'' = 2y' - y$.

2. Write the characteristic equation for the differential equation $-\frac{1}{4}y'' = y$.

PERFORMANCE OBJECTIVE IX - 6d: Determine the solution of a given homogeneous linear second-order differential equation whose characteristic equation has two real unequal roots.

1. Solve the differential equation $y'' - 3y' + 2y = 0$.

2. Solve the differential equation $2y'' = 9y - 3y'$.

PERFORMANCE OBJECTIVE IX - 6e: Determine the solution of a given homogeneous linear second-order differential equation whose characteristic equation has two real equal roots.

1. Solve the differential equation $y'' - 10y' + 25y = 0$.

2. Solve the differential equation $-y'' = \frac{4}{3}y' + \frac{4}{9}y$.

PERFORMANCE OBJECTIVE IX - 6f: Determine the solution of a given homogeneous linear second-order differential equation whose characteristic equation has imaginary roots.

1. Solve the differential equation $y'' + 16y = 0$.

2. Solve the differential equation $y'' = 6y' - 13y$.

PERFORMANCE OBJECTIVE IX - 7a: Identify the nonhomogeneous equations from given linear second-order differential equations.

1. Identify the non-homogeneous equations from the following linear second-order differential equations:

a) $y'' + y' - 2y = 5e^{-x}$

b) $y'' = y - y'$

c) $y'' = y + (x^2 + 1)$

d) $y'' = x^3 - 3x + 2$

e) $y'' - y' = y + \sin(e^x)$

f) $6y = y'' - 2y'$

PERFORMANCE OBJECTIVE IX - 7b: Determine the solution of a given non-homogeneous linear second-order differential equation $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = F(x)$ where $F(x)$ is a polynomial.

1. Solve $y'' + 2y' - 8y = x + 1$.

2. Solve $y'' + y' - 2y = x^2 - x - 2$

PERFORMANCE OBJECTIVE IX - 7c: Determine the solution of a given non-homogeneous linear second-order differential equation $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = F(x)$ is of the form e^{ax}.

1. Solve $y'' = 2y' + 5y = e^{2x}$.

2. Solve $2y'' + 3y' + y = e^{-x}$.

3. Solve $y'' - 4y' + 4y = e^{2x}$.

PERFORMANCE OBJECTIVE IX - 7d: Determine the solution of a given non-homogeneous linear second-order differential equation $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = F(x)$ where $F(x)$ is of the form $\sin ax$ or $\cos ax$.

1. Solve $y'' + y' - 6y = -\sin x$.

2. Solve $y'' - 2y' + 2y = \cos x$.

3. Solve $y'' + y = \cot^2 x$.

PERFORMANCE OBJECTIVE IX - 7e: Determine the solution of a given non-homogeneous linear second-order differential equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = F(x)$$ where F(x) is a linear combination of polynomials and transcendental functions.

1. Solve $y'' - y' - 2y = x + e^x$.

2. Solve $y'' - 2y' + 5y = e^x + \sin 2x$.

3. Solve $\frac{1}{4}y'' + y = \sin x + \cos x + \sin 2x$.

PERFORMANCE OBJECTIVE IX - 8a: Construct the solution to problems requiring the use of first order differential equations (e.g., growth and decay, modified Ohm's law, etc.)

1. The population of a state increases continuously at a rate proportional, at any time, to the population at that time. The population doubles in 40 years. After 60 years the ratio of the population P to the initial population P_o is what number?

2. In some chemical reactions, the rate of conversion of a substance at any instant t is found to be proportional to the quantity of the substance still untransformed at that instant. If V is the quantity of the substance that has been converted at time t and A is the original quantity of the substance, find V as a function of t.

PERFORMANCE OBJECTIVE IX - 8b: Construct the solution to problems requiring the use of second order differential equations. (e.g., hanging cable, motion on a straight line, oscillary motion, etc.)

1. A particle starts from rest at (3, 0) and moves along the x-axis, its acceleration being always toward the origin and equal to four times the distance from the origin. Determine its position and velocity at any time t.

2. A mass weighing 16 pounds is attached to a spring, for which the spring constant K = 18 pounds/foot, and is brought to rest. Determine the displacement of the mass at time t if a force equal to sin 2t is applied to it. Hint: Resultant force = applied force - spring force.

UNIT IX: DIFFERENTIAL EQUATIONS

ANSWERS TO ASSESSMENT MEASURES

1. a)

    (1) 2

    (2) 3

    b)

    (1) 1 (y is the dependent variable)

    (2) 2

2. a)

    (1) a, b, d, f, g

    b)

    (1) $\frac{1}{e^y} = -\frac{1}{2}e^{2x} + c$

    (2) Arc tan $y = \frac{x^2}{2} + c$

    (3) $|(3y + 2)^2 (2x + 5)^3| = c$

3. a)

    (1) a, b, d, f

    b)

    (1) $\frac{dy}{dx} = \frac{3v - 2}{v + 1}$

    (2) $\frac{dy}{dx} = v + \sqrt{1 - v^2}$

    c)

    (1) $\ln(2x^2 + y^2)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \tan^{-1} \frac{y}{\sqrt{2}\, x} = c$

(2) $\frac{\sqrt{x^2 + y^2} + y}{x^2} = e$

4. a)

(1) b, c.

b)

(1) $\frac{dy}{dx} + \frac{5y}{x} = \frac{\cos x}{x^5}$

(2) $\frac{dx}{dy} + \left(\frac{1}{y \ln y}\right) x = \frac{1}{y}$

c)

(1) e^{-x}

(2) $\frac{1}{x^2}$

(3) y

d)

(1) $y = ce^{-x^2} - \frac{1}{2}$

(2) $y = \frac{2 \sin x}{x} + \frac{2 \cos x}{x^2} - \cos x + \frac{c}{x^2}$

5. a)

(1) a, c, f, h, i

b)

(1) $\frac{x^3}{3} - xy = c$

(2) $ye^{3x} + y = c$

(3) $x \cos y + y \sin x = c$

6. a)

(1)

(a) $y'' + y' - 2y = 0$

(b) $y'' + y = 0$

(c) $y'' - y = 0$

(d) $y'' = 0$

(e) $y'' - y' - y = 0$

b)

(1) $(D^2 - 4D + 4)y = 0$

(2) $(15D^2 - 14D - 1)y = 0$

c)

(1) $r^2 - 2r + 1 = 0$

(2) $r^2 + 4 = 0$

d)

(1) $y = c_1 e^x + c_2 e^{2x}$

(2) $y = c_1 e^{2x} + c_2 e^{-3x}$

e)

(1) $y = (c_1 x + c_2)e^{5x}$

(2) $y = (c_1 x + c_2)e^{-\frac{2}{3}x}$

f)

(1) $y = c_1 \cos 4x + c_2 \sin 4x$

(2) $y = e^{3x} (c_1 \cos 2x + c_2 \sin 2x)$

7. a)

(1) a, c, d, e

b)

(1) $y = c_1 e^{-4x} + c_2 e^{2x} - \frac{1}{8}x - \frac{5}{32}$

(2) $y = c_1 e^{-2x} + c_2 e^{x} - \frac{1}{2}x^2 + \frac{1}{2}$

c)

(1) $y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{5}e^{2x}$

(2) $y = c_1 e^{-\frac{x}{2}} + c_2 e^{-x} - xe^{-x}$

Teacher note: $y = Axe^{-x}$ if using method of undetermined coefficients.

(3) $y = (c_1 + c_2 x + \frac{1}{2}x^2)e^{2x}$

d)

(1) $y = c_1 e^{-3x} + c_2 e^{2x} + \frac{7}{50}\sin x + \frac{1}{50}\cos x$

(2) $y = e^{x}(c_1 \cos x + c_2 \sin x) + \frac{1}{5}(\cos x - 2 \sin x)$

(3) $y = c_1 \cos x + c_2 \sin x + \cos x \ln |\csc x + \cot x| - 2$

(e)

(1) $y = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4}(1 - 2x - 2e^{x})$

(2) $y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{17}(4 \cos 2x + \sin 2x) + \frac{1}{4}e^{x}$

(3) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{4}{3}(\sin x + \cos x) - x \cos 2x$

8. a)

(1) $\frac{P_{60}}{P_0} = 2\sqrt{2}$

(2) $V = A(1-e^{-kt})$

b)

(1)

(a) $x = 3 \cos 2t$

(b) $V = -6 \sin 2t$

(2) Displacement $= \frac{-1}{48} \sin 6t + \frac{1}{16} \sin 2t$

Teacher note: $m = \frac{w}{g}$ where $g = 32$ ft/sec²,

UNIT X: HYPERBOLIC FUNCTIONS

INTRODUCTION

The study of hyperbolic functions introduces the student to a new system based upon the unit hyperbola. Analogies to the trigonometric functions and the unit circle should be emphasized.

A major benefit of this unit is the reinforcement of the exponential and logarithmic functions. Unit X provides an excellent opportunity for the students to explore new mathematical relations on their own.

INSTRUCTIONAL OBJECTIVES

1. Apply the definitions of the hyperbolic functions to verify identities.
2. Apply the rules for the derivatives of the hyperbolic functions.
3. Apply the rules for determining integrals of expressions containing hyperbolic functions.
4. Apply the rules for differentiating the inverse hyperbolic functions.
5. Apply the rules for integrating differentials of inverse hyperbolic functions.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) State the definitions of the hyperbolic functions.

   b) Construct the proof of given identities involving hyperbolic functions.

2. a) State the rules for differentiating the hyperbolic functions.

   b) Construct the derivatives for given expressions containing the hyperbolic functions.

3. a) State the rules for determining the integrals of the form

   $\int \sinh u\,du$, $\int \cosh u\,du$, $\int \text{csch}^2 u\,du$, $\int \text{sech}\,u \tanh u\,du$, $\int \text{csch}\,u$ $\int \coth u\,du$, $\int \text{sech}^2 u\,du$.

   b) Integrate given expressions containing hyperbolic functions.

4. a) State the rules for differentiating inverse hyperbolic functions.

   b) Construct the derivative for given expressions containing the inverse hyperbolic functions.

5. a) State the rules for integrating $\int \frac{du}{\sqrt{1+u^2}}$, $\int \frac{du}{\sqrt{u^2-1}}$, $\int \frac{du}{1-u^2}$,

   $\int \frac{du}{u\sqrt{1-u^2}}$ and $\int \frac{du}{u\sqrt{1+u^2}}$.

   b) Integrate given expressions of the form $\int \frac{du}{\sqrt{1+u^2}}$, $\int \frac{du}{\sqrt{u^2-1}}$,

   $\int \frac{du}{\sqrt{1-u^2}}$, $\int \frac{du}{u\sqrt{1-u^2}}$ and $\int \frac{du}{u\sqrt{1+u^2}}$.

SAMPLE ASSESSMENT MEASURES: HYPERBOLIC FUNCTIONS

PERFORMANCE OBJECTIVE X - 1a: State the definitions of the hyperbolic functions.

1. State the definitions of the six hyperbolic functions.

PERFORMANCE OBJECTIVE X - 1b: Construct the proof of given identities involving hyperbolic functions.

1. Prove: $\cosh(2x) = \cosh^2 x + \sinh^2 x$.

2. Prove: $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.

PERFORMANCE OBJECTIVE X - 2a: State the rules for differentiating the hyperbolic functions.

1. State the rules for differentiating the six hyperbolic functions.

PERFORMANCE OBJECTIVE X - 2b: Construct the derivatives for given expressions containing the hyperbolic functions.

1. Determine y' if $y = \sinh(x^2)$.

2. Determine y' if $y = \operatorname{sech}^2 3x$.

3. Determine y' if $y = \tanh(\tan x)$

4. Determine y' if $\sinh(xy) = \cosh(\ln y)$.

5. Determine y' if $y = \ln(\tanh 2x)$.

PERFORMANCE OBJECTIVE X - 3a: State the rules for determining the integrals of the form: $\int \sinh u\,du$, $\int \cosh u\,du$, $\int \operatorname{csch}^2 u\,du$, $\int \operatorname{sech} u \cdot \tanh u\,du$, $\int \operatorname{csch} u \coth u\,du$, and $\int \operatorname{sech}^2 u\,du$

1. State the rule for integrating each of the following:

   a) $\int \sinh u\,du =$

   b) $\int \cosh u\,du =$

   c) $\int \operatorname{sech}^2 u\,du =$

   d) $\int \operatorname{sech} u \tanh u\,du =$

   e) $\int \operatorname{csch}^2 u\,du =$

   f) $\int \operatorname{csch} u \coth u\,du =$

PERFORMANCE OBJECTIVE X - 3b: Integrate given expressions containing hyperbolic functions.

1. $\int x \sinh 3x^2\,dx =$

2. $\int \tanh^3 2x \operatorname{sech}^2 2x\,dx =$

3. $\int \sinh^2 2x\,dx =$

4. $\int \tanh^3 x\,dx =$

5. $\int e^x \cosh x\,dx =$

PERFORMANCE OBJECTIVE X - 4a: State the rules for differentiating inverse hyperbolic functions.

1. State the rules for differentiating the six inverse hyperbolic functions.

PERFORMANCE OBJECTIVE X - 4b: Construct the derivative for given expressions containing the inverse hyperbolic functions.

1. Determine y' if $y = \cosh^{-1} e^{x}$.

2. Determine y' if $y = \tanh^{-1} \frac{1}{2x}$.

3. Determine y' if $y = \operatorname{sech}^{-1} (\cos x)$.

4. Determine y' if $x = a \operatorname{sech}^{-1}(\frac{y}{a}) - \sqrt{a^2 - y^2}$

PERFORMANCE OBJECTIVE X - 5a: State the rules for integrating $\int \frac{du}{\sqrt{1+u^2}}$, $\int \frac{du}{\sqrt{u^2-1}}$, $\int \frac{du}{1-u^2}$, $\int \frac{du}{u\sqrt{1-u^2}}$, $\int \frac{du}{u\sqrt{1+u^2}}$

1. State the rule for integrating each of the following:

(a) $\int \frac{du}{\sqrt{1+u^2}}$

(b) $\int \frac{du}{\sqrt{u^2-1}}$

(c) $\int \frac{du}{1-u^2}$

(d) $\int \frac{du}{u\sqrt{1-u^2}}$

(e) $\int \frac{du}{u\sqrt{1+u^2}}$

PERFORMANCE OBJECTIVE X - 5b: Integrate given expressions of the form

$$\int \frac{du}{\sqrt{1+u^2}}, \int \frac{du}{\sqrt{u^2-1}}, \int \frac{du}{1-u^2}, \int \frac{du}{u\sqrt{1-u^2}}, \text{ and } \int \frac{du}{u\sqrt{1+u^2}}.$$

1. $\int \frac{dx}{\sqrt{x^2 + 25}}$

2. $\int \frac{\sin \theta \, d\theta}{\sqrt{\cos^2 \theta - 25}}$

3. $\int \frac{dx}{2x^2 + 12x + 16}$

4. $\int \frac{dx}{x\sqrt{1-x}}$

5. $\int \frac{\cot \theta}{\sqrt{1+\sin^2 \theta}} d\theta$

ANSWERS TO ASSESSMENT MEASURES

1. a)

(1) $\cosh(u) = \frac{1}{2}(e^{u} + e^{-u})$

(2) $\sinh(u) = \frac{1}{2}(e^{u} - e^{-u})$

(3) $\tanh(u) = \dfrac{e^{u} - e^{-u}}{e^{u} + e^{-u}}$

(4) $\coth(u) = \dfrac{e^{u} + e^{-u}}{e^{u} - e^{-u}}$

(5) $\operatorname{sech}(u) = \dfrac{2}{e^{u} + e^{-u}}$

(6) $\operatorname{csch}(u) = \dfrac{2}{e^{u} - e^{-u}}$

b)

(1) --

(2) --

2. a)

(1) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(2) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

(3) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^{2} u \dfrac{du}{dx}$

(4) $\frac{d}{dx}(\coth u) = -\text{csch}^2 u \frac{d}{dx}$

(5) $\frac{d}{dx}(\text{sech } u) = -\text{sech } u \tanh u \frac{d}{dx}$

(6) $\frac{d}{dx}(\text{csch } u) = -\text{csch } u \coth u \frac{du}{dx}$

b)

(1) $y' = 2x \cosh(x^2)$

(2) $y' = -6 \text{ sech}^2 3x \tanh 3x$

(3) $y' = \text{sech}^2(\tan x) \sec^2 x$

(4) $y' = \dfrac{y^2 \cosh(xy)}{\sinh(\ln y) - xy \cosh(xy)}$

(5) $y' = 4 \text{ csch } 4x$

3. a)

(1)

(a) $\cosh u + C$

(b) $\sinh u + C$

(c) $\tanh u + C$

(d) $-\text{sech } u + C$

(e) $-\coth u + C$

(f) $-\text{csch } u + C$

b)

(1) $\frac{1}{6} \cosh 3x^2 + C$

(2) $\frac{1}{8} \tanh^4 2x + C$

(3) $\frac{1}{8} \sinh 4x - \frac{x}{2} + C$

(4) $\ln(\cosh x) - \frac{\tanh^2 x}{2} + C$

(5) $\frac{e^{2x}}{4} + \frac{x}{2} + C$

4. a)

(1) $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1 + u^2}} \cdot \frac{du}{dx}$

$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2} \cdot \frac{du}{dx}, \ |u| < 1$

$\frac{d}{dx}(\text{sech}^{-1} u) = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}$

$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}$

$\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2} \frac{du}{dx}, \ |u| > 1$

$\frac{d}{dx}(\text{csch}^{-1} u) = \frac{-1}{|u|\sqrt{1+u^2}} \cdot \frac{du}{dx}$

b)

(1) $y' = \frac{e^x}{\sqrt{e^{2x} - 1}}$

(2) $y' = \frac{2}{1 - 4x^2}$

(3) $y' = \sec x$

(4) $y' = \frac{-y}{\sqrt{a^3 - y^2}}$

5. a)

(1)

(a) $\sinh^{-1} u + C$

(b) $\cosh^{-1} u + C$

(c) $\tanh^{-1} u + C$ if $|u| < |$

$\coth^{-1} u + C$ if $|u| > |$

(d) $-\text{sech}^{-1} |u| + C$

(e) $-\text{csch}^{-1} | u | + C$

b)

(1) $\sinh^{-1} (\frac{x}{5}) + C$

(2) $-\cosh^{-1} (\frac{\cos \theta}{5}) + C$

(3) $-\frac{1}{2} \tanh^{-1} (x + 3) + C$ if $(x + 3)^2 < 1$

$-\frac{1}{2} \coth^{-1} (x + 3) + C$ if $(x + 3)^2 > 1$

(4) $-2 \text{ sech}^{-1} (\sqrt{x}) + C$

(5) $-\text{csch}^{-1} |\sin \theta | + C$

UNIT XI: VECTORS

INTRODUCTION

In analytic geometry the student was introduced to vectors. Unit XI applies calculus to vectors. Particular emphasis should be given to the velocity and acceleration vectors for a distance function, including the determination of speed. A discussion of tangential and normal vectors concludes the unit.

INSTRUCTIONAL OBJECTIVES

1. Determine the position vector for a curve defined by a pair of parametric equations.
2. Determine the velocity vector from a position vector.
3. Determine the speed at which a particle moves along the curve.
4. Determine the acceleration vector from a velocity vector.
5. Apply the definition to determine the tangential vector for a given curve.
6. Apply the definition to determine the normal vector for a given curve.

PERFORMANCE OBJECTIVES

By the end of this unit, the student should have mastered the objectives listed below.

1. a) State the definition of the position vector.

   b) Construct the position vector for a given curve defined by a pair of parametric equations.

2. a) State the definition of a velocity vector.

   b) Construct the velocity vector from a given position vector.

3. a) State the definition of speed.

   b) Determine the speed of a particle at a given time.

4. a) State the definition of an acceleration vector.

   b) Construct the acceleration vector from a given velocity vector.

5. a) State the definition of the tangential vector.

   b) Construct the tangential vector to a given curve.

6. a) State the definition of the unit normal vector.

   b) Construct the unit normal vector to a given curve.

SAMPLE ASSESSMENT MEASURES: VECTORS

PERFORMANCE OBJECTIVE XI - 1a: State the definition of the position vector.

1. State the definition of a position vector.

PERFORMANCE OBJECTIVE XI - 1b: Construct the position vector for a given curve defined by a pair of parametric equations.

1. Construct the position vector for the curve defined by $x = t + 1$ and $y = t^2 - 1$ where $t>0$.

2. Construct the position vector for the curve defined by $x = r \cos \omega t$ and $y = r \sin \omega t$ where r and ω are positive constants.

PERFORMANCE OBJECTIVE XI - 2a: State the definition of a velocity vector.

1. State the definition of a velocity vector.

PERFORMANCE OBJECTIVE XI - 2b: Construct the velocity vector from a given position vector.

1. Construct the velocity vector given the position vector $\vec{R} = t^2\vec{i} + \ln(t+1)\vec{j}$.

2. Construct the velocity vector given the position vector $\vec{R} = e^{-2t}\vec{i} + e^{t}\vec{j}$.

3. Construct the velocity vector given the position vector $\vec{R} = (3\sin t)\vec{i} + (2\cos t)\vec{j}$.

PERFORMANCE OBJECTIVE XI - 3a: State the definition of speed.

1. State the definition of speed.

PERFORMANCE OBJECTIVE XI - 3b: Determine the speed of a particle at a given time.

1. Determine the speed at $t = 0$ when $\vec{R} = (2\sin t)\vec{i} + (\cos 2t)\vec{j}$.

2. Determine the speed at $t = \pi/6$ when $\vec{R} = (\tan t)\vec{i} + (\sec t)\vec{j}$.

PERFORMANCE OBJECTIVE XI - 4a: State the definition of an acceleration vector.

1. State the definition of an acceleration vector.

PERFORMANCE OBJECTIVE XI - 4b: Construct the acceleration vector from a given velocity vector.

1. Construct the acceleration vector given the velocity vector

   $\vec{v} = (2\cos t)\vec{i} + (3\sin t)\vec{j}$.

2. Construct the acceleration vector given the velocity vector

   $\vec{v} = e^{t}\ \vec{i} + e^{-2t}\ \vec{j}$.

PERFORMANCE OBJECTIVE XI - 5a: State the definition of the tangential vector.

1. State the definition of a tangential vector.

PERFORMANCE OBJECTIVE XI - 5b: Construct the tangential vector to a given curve.

1. Construct the tangential vector to the curve defined by

$\vec{R} = (2\sin t)\vec{i} + (2\cos t)\vec{j}$

2. Construct the tangential vector to the curve defined by

$\vec{R} = (\sin^3 t)\vec{i} + (\cos^3 t)\vec{j}$

PERFORMANCE OBJECTIVE XI - 6a: State the definition of the unit normal vector.

1. State the definition of the unit normal vector.

PERFORMANCE OBJECTIVE XI - 6b: Construct the unit normal vector to a given curve.

1. Construct the unit normal vector when $\vec{T} = (\cos t)\vec{i} + (-\sin t)\vec{j}$.

2. Construct the unit normal vector when $\vec{R} = (2\cos t)\vec{i} + (\cos 2t)\vec{j}$.

ANSWERS TO ASSESSMENT MEASURES

1 a) (1) $\vec{R} = x\,\vec{i} + y\,\vec{j}$

b) (1) $\vec{R} = (t+1)\,\vec{i} + (t^2-1)\,\vec{j}$

(2) $\vec{R} = r \cos \omega t\,\vec{i} + r \sin \omega t\,\vec{j}$

2 a) (1) $\vec{v} = d\vec{R}/dt = (dx/dt)\,\vec{i} + (dy/dt)\,\vec{j}$

b) (1) $\vec{v} = (2t)\vec{i} + (1/(t+1))\,\vec{j}$

(2) $\vec{v} = (-2e^{-t})\vec{i} + (e^{t})\vec{j}$

(3) $\vec{v} = (3\cos t)\vec{i} - (2\sin t)\vec{j}$

3 a) (1) Speed is the magnitude of the velocity vector, i.e.,
$|ds/dt| = \sqrt{(dx/dt)^2 + (dy/dt)^2}$

b) (1) 2

(2) $2\sqrt{5}/3$

4 a) (1) $\vec{a} = d\vec{v}/dt = (d^2x/dt^2)\vec{i} + (d^2y/dt^2)\vec{j}$

b) (1) $\vec{a} = (-2\sin t)\vec{i} + (3\cos t)\vec{j}$

(2) $\vec{a} = (e^{t})\vec{i} + (-2e^{-2t})\,\vec{j}$

5 a)

(1) $\vec{T}$ is the unit vector which is tangent to a curve at any point P.

$\vec{T} = d\vec{R}/ds = (dx/ds)\,\vec{i} + (dy/ds)\,\vec{j}$

b)

(1) $\vec{T} = \cos t\,\vec{i} - \sin t\,\vec{j}$

(2) $\vec{T} = \sin t\,\vec{i} - \cos t\,\vec{j}$

6 a)

(1) A unit vector perpendicular to the tangential vector at any point P on the curve.

$\vec{N} = d\vec{T}/d\phi$, where ϕ is the slope angle of inclination of the tangential vector.

Teacher Note: $\vec{T} = (dx/ds)\,\vec{i} + (dy/ds)\,\vec{j}$

$\vec{T} = \cos\phi\,\vec{i} + \sin\phi\,\vec{j}$

$\vec{N} = -\sin\phi\,\vec{i} + \cos\phi\,\vec{j}$

or $\vec{N} = -(dy/ds)\,\vec{i} + (dx/ds)\,\vec{j}$

b)

(1) $\vec{N} = \sin t\,\vec{i} + \cos t\,\vec{j}$

(2) $\vec{N} = \dfrac{2\cos t}{\sqrt{1 + 4\cos^2 t}}\,\vec{i} - \dfrac{1}{\sqrt{1 + 4\cos^2 t}}\,\vec{j}$

Teacher Note: $\vec{T} = \dfrac{-1}{\sqrt{1 + 4\cos^2 t}}\,\vec{i} - \dfrac{2\cos t}{\sqrt{1 + 4\cos^2 t}}\,\vec{j}$