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ABSTRACT

This book contains the first four chapters of the second course of a pilot mathematics sequence for the seventh and eighth grades. The content of the sequence is to serve as a vehicle for the development of relevant computational skills, mathematical reasoning, and geometric perception in three dimensions and is to reflect the application of mathematics to the social and natural sciences. The material is divided into five types of sections: (1) activities; (2) short reading sections; (3) questions; (4) sections for the student with a weaker background; and (5) sections for the strongly motivated student. The material in the first four chapters of the second course includes: the cube, volume, powers of ten, and signed numbers. (MN)

MATHEMATICS

FOR JUNIOR HIGH SCHOOL

pilot edition *chapters 1-4*

second course

Boston University Mathematics Project

PREFACE

This book starts the second course of a mathematics sequence for the seventh and eighth grades. The content for the sequence is being selected to serve as a vehicle for the development of relevant computational skills, mathematical reasoning, and geometric perception in three dimensions. The application of mathematics to the social and natural sciences is also an important factor in the selection of material.

The style of the sequence encourages individual as well as group work, thus developing the communication skills in the context of mathematics. Strong emphasis is placed on student activities, many of which are manipulative.

To serve a broad spectrum of students in heterogeneous classes, the material is divided into five types of sections. Three types constitute the main core:

 Activities by the whole class, small groups or individuals;

 Short reading sections, to be assigned and discussed or to be read in class; and

 Questions to be worked out at home or in class.

Sections indicated by  are intended to help the student who has a weaker background, and sections indicated by  provide extra challenge and pleasure for the strongly motivated student.

The development of this project is supported by a grant from the National Science Foundation.

Uri Haber-Schaim
Project Director

CONTENTS

1. THE CUBE

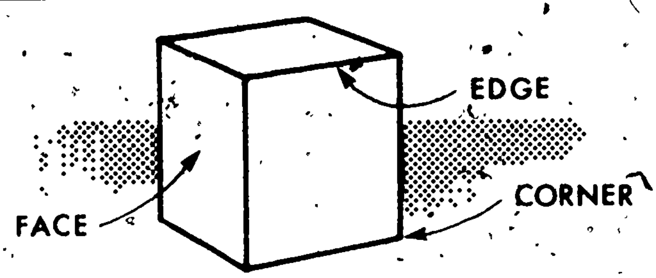
SECTION 1 VISUALIZING THE CUBE

When we describe a table we are likely to speak of its height, length, and width. Describing a swimming pool, we speak of length, width, and depth. We can put a pair of shoes, a typewriter, or any other object in a rectangular box. The box has a length, a width, and a depth. They are called the dimensions of the box. All objects in our world have three dimensions. Yet we do much of our communicating on two-dimensional surfaces such as a sheet of paper or a television screen.

Because of this fact we have two kinds of problems: how to visualize a three-dimensional object from a two-dimensional drawing, and how to draw a three-dimensional object on a two-dimensional surface.

We shall begin this course with work on the first of these problems. As an introduction we raise some questions about cubes. A cube (Figure 1) has six square <u>faces</u>, twelve <u>edges</u> of the same length, and eight <u>corners</u>.

Figure 1



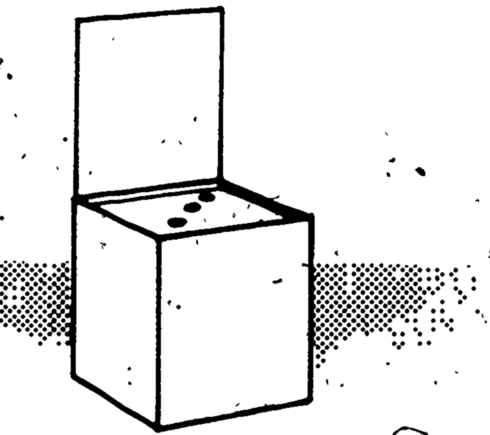
[?]

1. Suppose you were to paint each face of a cube. What is the smallest number of colors you would need so that no two faces having a common edge are painted the same color?

2. Suppose a large die fits snugly into a shipping box.

(a) In how many different ways can the die be placed in the box so that the face with three dots will be showing when the lid of the box is opened? (Figure 2)

Figure 2



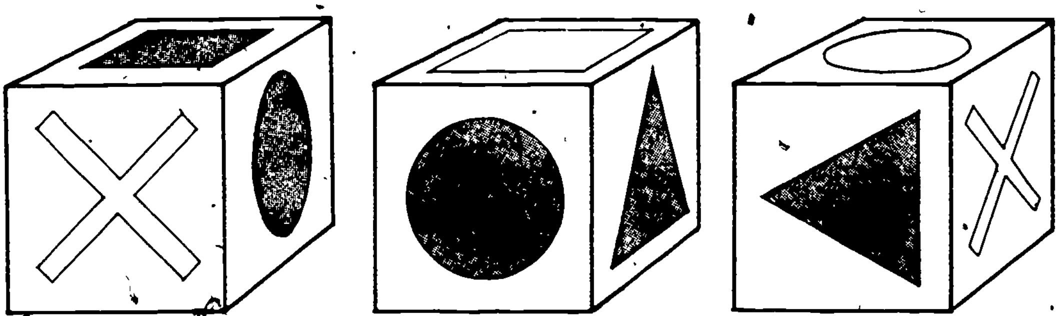
(b) In how many ways can the die be placed in the box so that the face with the three dots will not be showing when the lid is opened?

(c) All together, in how many ways can the die be placed in the box?

3. Consider two lines drawn on a sheet of paper. If both lines are perpendicular to a third line, then they must be parallel to each other. If two lines are perpendicular to a third line on a cube, must they also be parallel to each other?

★

Painted on each face of a cube is one of these shapes: ■, ●, ✕, □, ▲, or ○. Three views of this cube are shown in Figure 3.



(a) (b) (c)

Figure 3

(a) Use a piece of chalk to draw these shapes on the faces of a wooden cube so that three views of it will correspond to the three views shown in Figure 3.

(b) On your cube which shape is opposite ● ?

(c) Which shape is on the bottom of Figure 3(a)?

4. There are calendars that show the day of the month by using two cubes as in Figure 4. Using both cubes you can form all the numbers 01, 02, ... , 31. How are the numbers painted on the faces of the two cubes to make such a calendar?

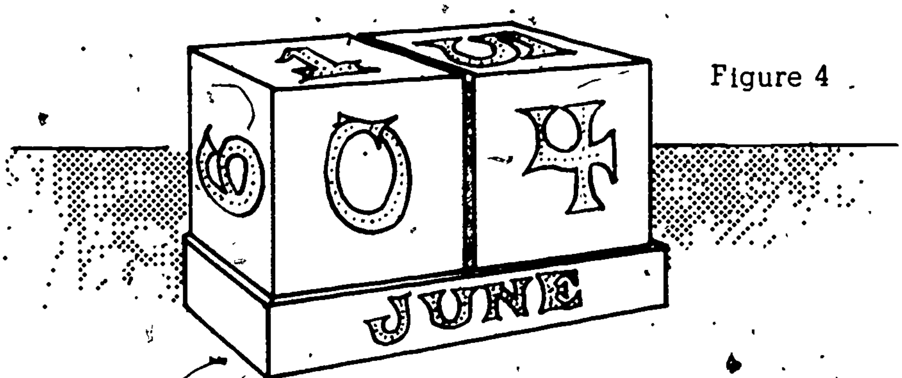
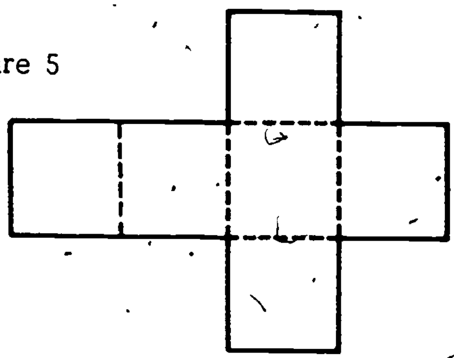


Figure 4

SECTION 2 PATTERNS FOR CUBES

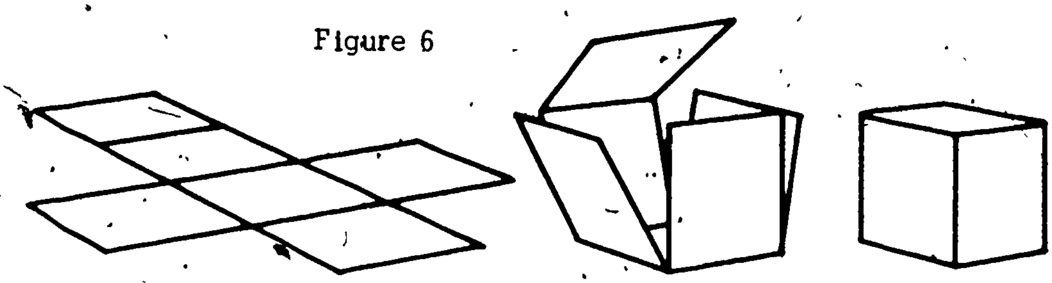
A solid cube and its faces are very different things. A cube may be heavy or light; it makes no sense to say that a cube has heavy faces. On the other hand, the faces of a cube may be rough or smooth, but the cube itself is neither. There are ways to help you visualize how the faces of a cube fit together to form a cube. One good way is to make a cube by cutting out a pattern like the one in Figure 5 and making folds on the dashed lines.

Figure 5



The steps for making a cube from this pattern are shown in Figure 6.

Figure 6



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5. Which patterns in Figure 7 can be folded into a cube?

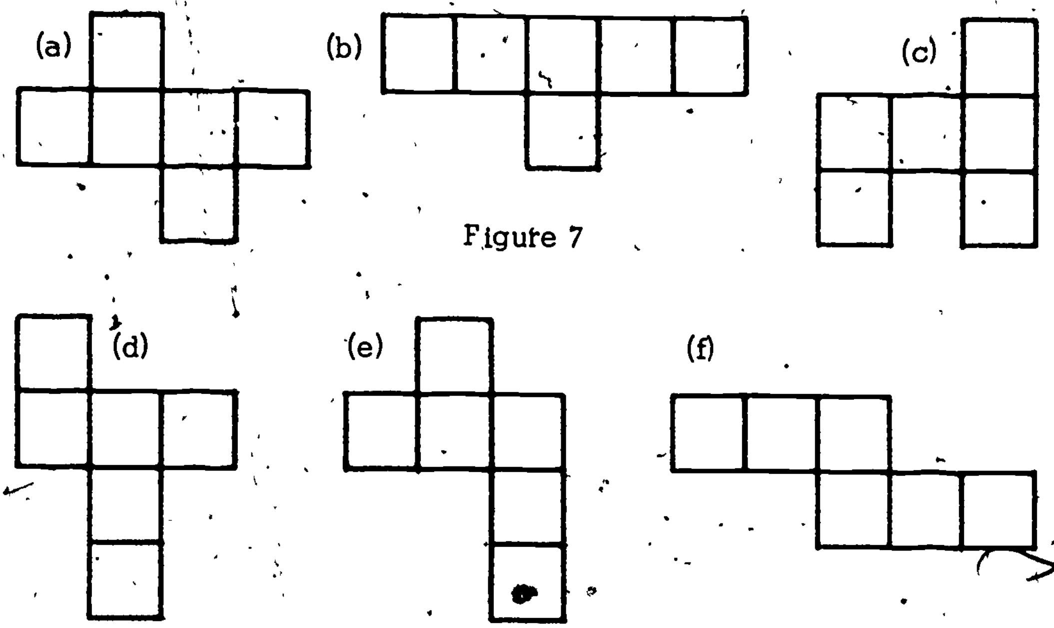


Figure 7

6. Each pattern in Figure 8 will make a cube. If the face labeled B in each pattern becomes the bottom when folded, which face becomes the top? Label each of these faces with the letter T.

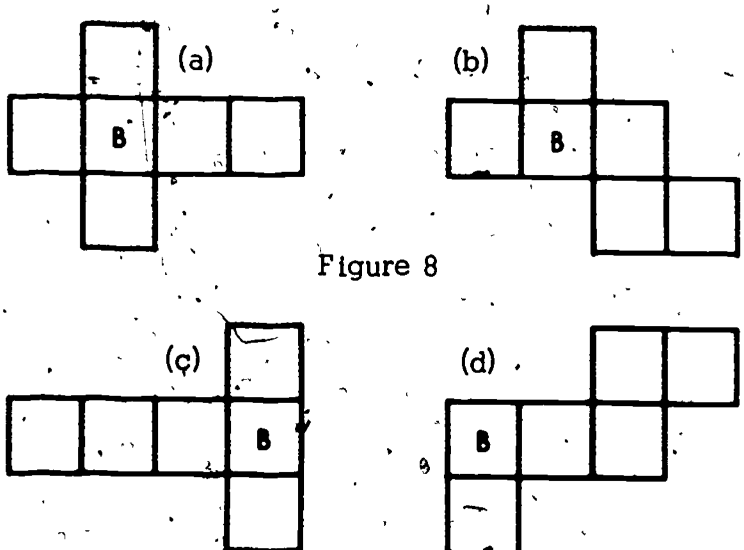
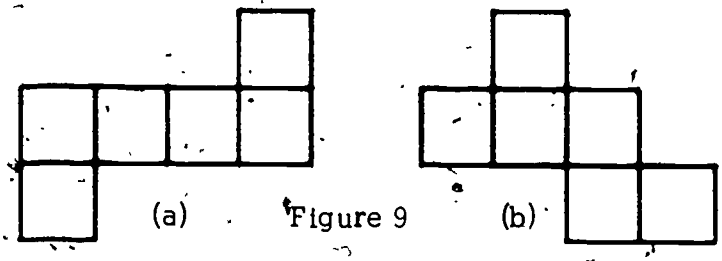


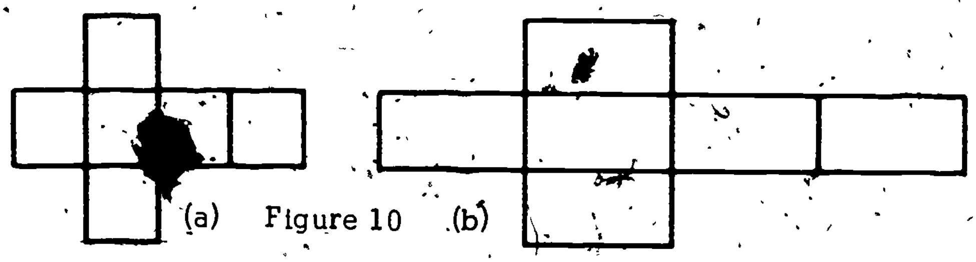
Figure 8

7. On a die, when you add the numbers on opposite faces you always get seven. Number the faces on the patterns in Figure 9 so that the cubes are numbered the same way that dice are.



(a) Figure 9 (b)

8. The pattern in Figure 10(a) can be used to make a box that will hold one die.

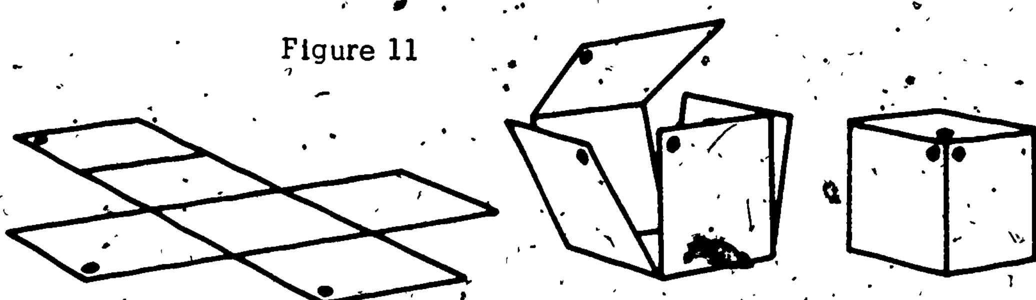


(a) Figure 10 (b)

Can the pattern in Figure 10(b) be used to make a box that will hold two dice?

★ Figure 11 shows how the three corners marked ● on the pattern come together when the pattern is folded to make a cube.

Figure 11



Your teacher will give you four patterns that make closed three-dimensional objects. Color code the corners of each pattern so that the corners that come together are colored the same and the corners that do not come together are colored differently. Then cut out the patterns to check your color code.

SECTION 3 UNFOLDING THE CUBE

We saw how some flat patterns could be folded into hollow cubes. We can also do the reverse. That is, by starting with a hollow cube, we can flatten it out by cutting along some of the edges and unfolding it.

Imagine a hollow cube, as in Figure 12, with a stripe running over four of the faces. Decide how you would unfold the cube so that the stripe is broken in

(a) four places
(b) two places
(c) one place.

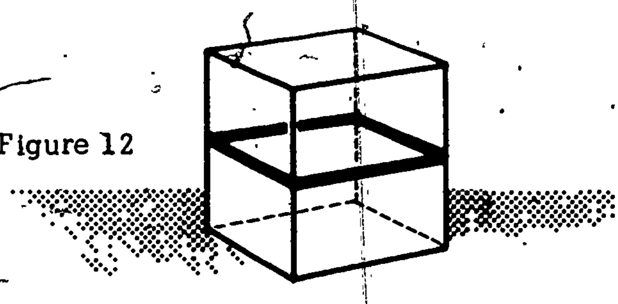


Figure 12

Draw a flat pattern for each. Be sure to mark the stripes on each pattern. Then compare your patterns with your classmates'. Check your results by folding the patterns into cubes.

9. Two adjacent faces of a cube (that is, two faces sharing an edge) are painted red.

(a) Show how you can unfold the cube so that the red faces remain adjacent on the unfolded pattern.

(b) Show how you can unfold the cube so that the red faces are no longer adjacent on the unfolded pattern.

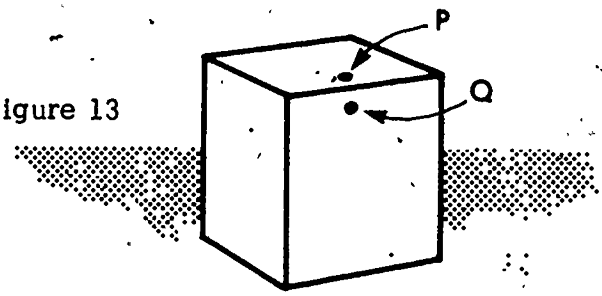
10. Three faces of a cube that meet at the same corner are painted blue.

(a) Can you unfold the cube in such a way that the three faces still meet at a corner?

(b) On the cube, every two of the three blue faces are adjacent. Is this also true when you unfold the cube?

11. Figure 13 shows two points on the faces of a cube. Point P is on one face near an edge, and point Q is on an adjacent face near the same edge.

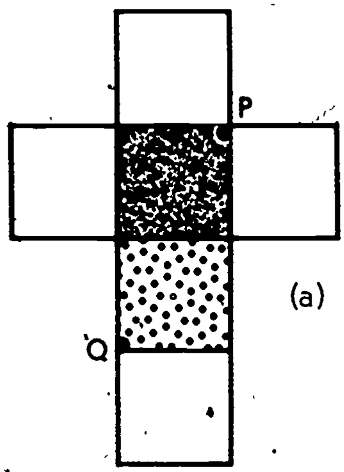
Figure 13



(a) How would you unfold the cube so that the two points are as close as possible on the unfolded pattern?

(b) How would you unfold the cube so that the two points are as far apart as possible on the unfolded pattern?

(a) Mark points P and Q on a pattern as in Figure 14(a). Cut it out so that it fits over your wooden cube. Wrap the pattern around your cube and tape it together. Now hold the cube so that you can see points P and Q as in Figure 14(b).



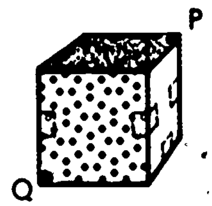
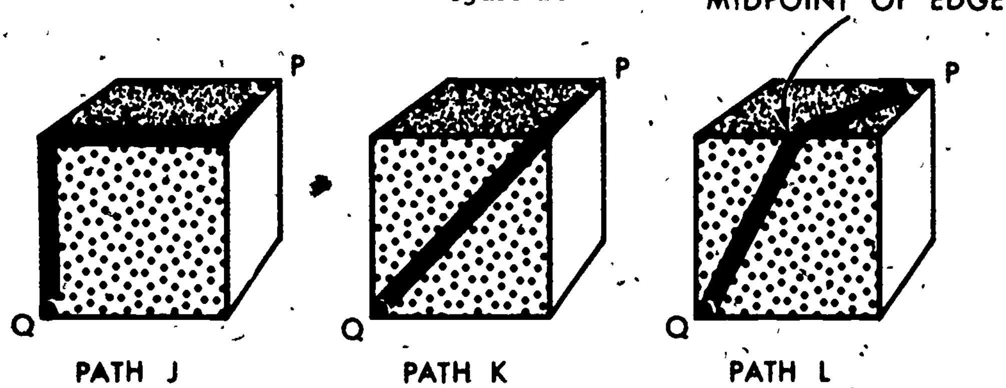


Figure 14

Figure 15 shows three paths, paths J, K, and L, that connect points P and Q. Use different colors to copy these paths onto the pattern covering your cube. Measure each path to find out which is the shortest.

Figure 15



What do you think the three paths will look like when the pattern is unfolded? Unfold the pattern. What do you notice about the shortest path when you look at the unfolded pattern?

(b) Figure 16 shows midpoints marked on two faces of a cube. What is the shortest path between them? Draw an unfolded pattern for this cube with the two points and the shortest path marked on it. Is the path unbroken?

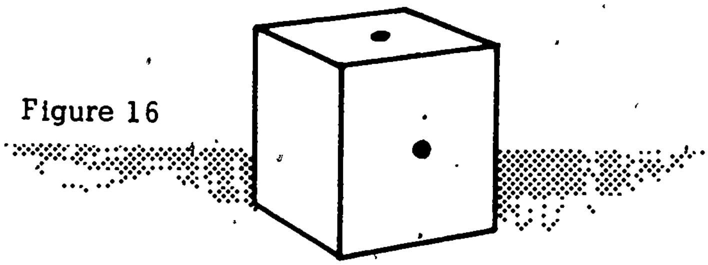


Figure 16

(c) Is the path shown in Figure 17 the shortest path over the faces connecting points P and Q? Use a pattern to help you find out.

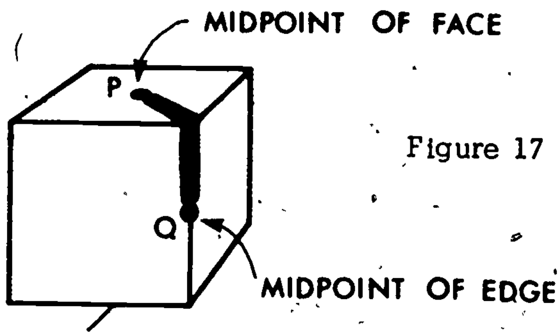


Figure 17

(d) Choose two different points on adjacent faces of your cube. Between these points find the shortest path crossing the edge common to both faces.

12. In Figure 18, which is the shortest path over the faces connecting P and Q? Mark points P and Q on your cube and unfold the cube so that the path in Figure 18(a) is not cut. Again mark points P and Q on your cube and unfold the cube so that the path in Figure 18(b) is not cut. What do you notice about the two paths? Which path is shorter? Can you find the shortest path between P and Q?

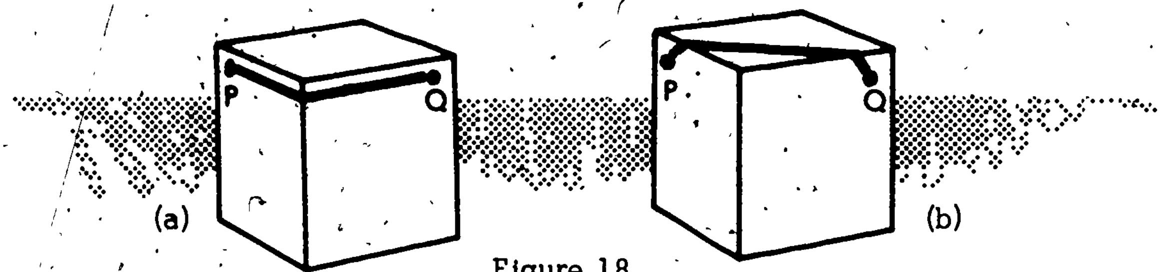


Figure 18

13. Imagine a planet in the shape of a cube. A map of this planet can be made by building a paper model and unfolding it.

    Figure 19 shows one possible shape for such a map. The points P, Q, R, and S stand for cities. Use your ruler to find the distance on the map between cities

    (a) P and Q
    (b) R and S
    (c) P and S.

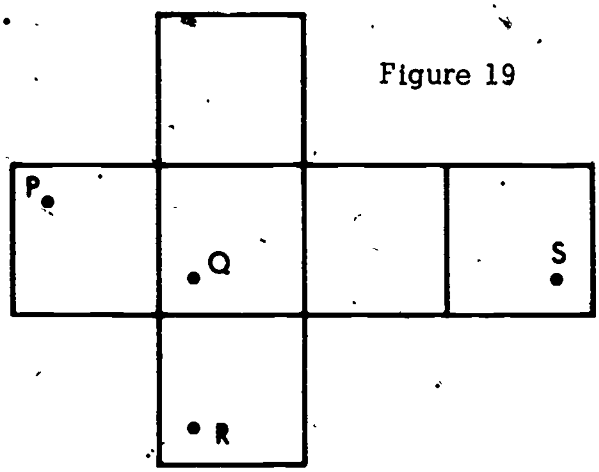


Figure 19

SECTION 4 CROSS SECTIONS OF A CUBE

If we take a three-dimensional object and slice it with a saw or a knife we see a two-dimensional face that is called a <u>cross section</u>. You have seen cross sections before — for example, a cross section of an apple, of a loaf of bread, or of the trunk of a tree (Figure 20).

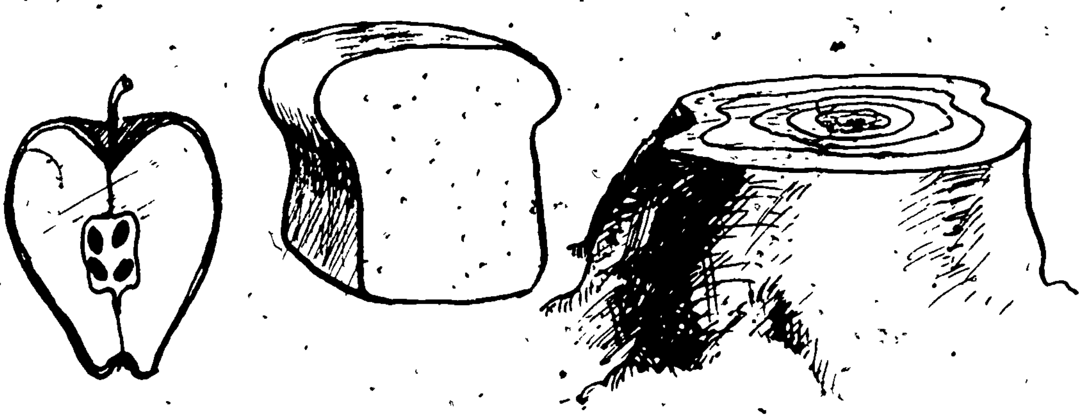


Figure 20

What kinds of cross sections does a cube have?

Get together in small working groups, and practice using the clay cutter. With the cutter make a number of clay cubes about 3 cm on a side. Without cutting the cubes, try to answer the following questions. Then use the cutter and the cubes to check your answers.

?

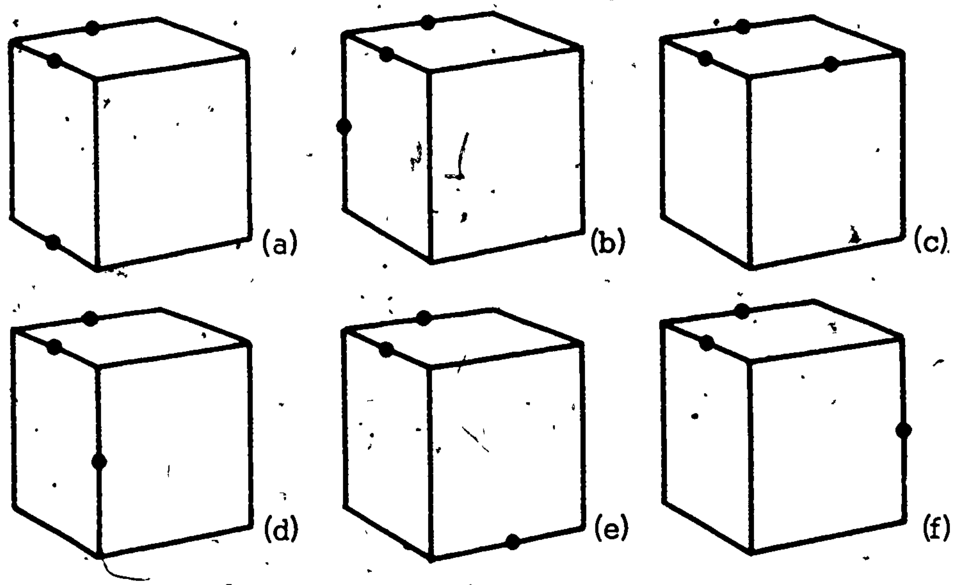
14. When you slice a cube parallel to one of its faces, what will the shape of the cross section be?

15. Show that it is possible to get a square cross section when a cube is not cut parallel to one of its faces.

16. Show how you can slice a cube and get a rectangle that is
    (a) taller than it is long
    (b) longer than it is tall.

17. Each of the cubes in Figure 21 is marked with three points. All of these points are midpoints of edges. Suppose you cut each of the cubes along a plane passing through the marked points. What will the cross section look like in each case?

Figure 21



18. Any cross section that you can cut from a cube has the shape of a polygon. Make a list of the different-shaped cross sections you get in Question 17. List them by the number of sides that the polygons have.

19. Choose one of the polygon cross sections and answer the following questions about it.

(a) Is each side of the polygon on one of the faces of the cube? Why?

(b) Were two sides of the polygon cut from the same face?

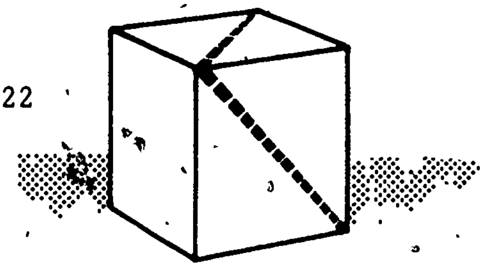
(c) Compare your answers to parts (a) and (b) with the answers of classmates who chose polygon cross sections that were different from yours.

20. Explain the fact that no cross section of the cube can have more than six sides.

21. Suppose you have a block of wood that has rectangular rather than square faces. Will it be possible to cut a cross section that has more than six sides? Why?

22. In Figure 22, two dashed lines are shown. Imagine that you have cut the cube along these lines.

Figure 22



(a) What would the shape of the cross section be?

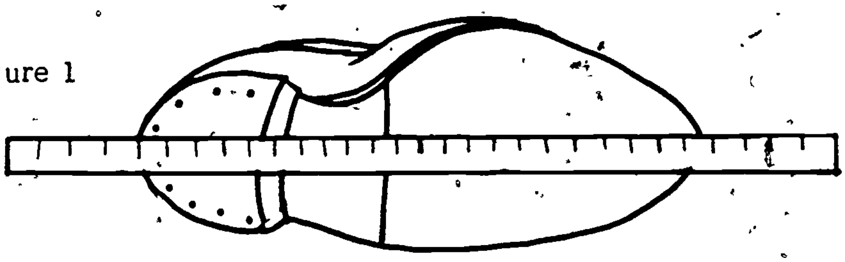
(b) On the cross section how big is the angle between the dashed lines?

2. VOLUME

SECTION 1 LENGTH AND AREA

Suppose you want to measure the length of a shoe. You have a ruler that has centimeter marks on it but no printed numbers. What would you do? You would line up a mark on the ruler with one end of the shoe and count the number of centimeter marks there are from this end to the other end of the shoe (Figure 1).

Figure 1



Most commercial rulers have their centimeter marks numbered, and so the counting has been done for you. However, if you want to read tenths of a centimeter, you still have to do your own counting. In any case, measuring length involves counting units of length.

1. What is the length of the shoe in Figure 1?

2. You know that an inch is a larger unit of length than a centimeter. Would the length of a shoe given in inches be a larger or a smaller number than the same length given in centimeters?

3. One meter equals 100 centimeters (cm). Express the following lengths in cm.

   (a) 2.50 m

   (b) 3.6 m

   (c) 0.42 m

   (d) 0.668 m

4. Express the following lengths in meters.

   (a) 300 cm

   (b) 106 cm

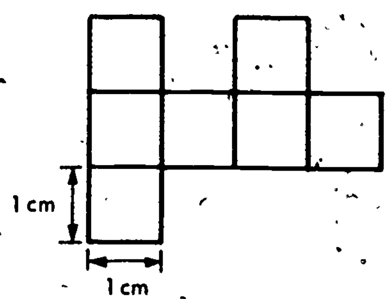
   (c) 25 cm

   (d) 5 cm

   (e) 4.7 cm

Measuring an area means <u>counting</u> units of area. We use a square that is 1 cm by 1 cm as the unit of area. This unit is called a square centimeter and is written as 1 cm^2. For example, in Figure 2 the area of the figure is 7 cm^2.

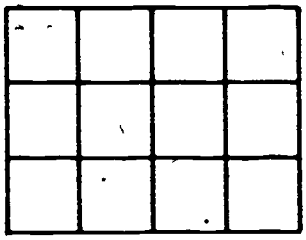
Figure 2



When we measure the area of rectangles, we can use a short-cut for the counting process. We can imagine a rectangle to be broken up into rows 1 cm wide and each row broken up into squares of

1 cm by 1 cm. This is illustrated in Figure 3. There are three rows and each row contains four squares. Each of these squares has an area of 1 cm^2. Therefore, the area of the rectangle is

$$3 \times 4 \times 1 \text{ cm}^2 = 12 \text{ cm}^2$$

Figure 3 

The product length times width gives an area in cm^2 only if both length and width are expressed in cm. To remind us, we write the units next to the numbers that express the length and the width. In our example,

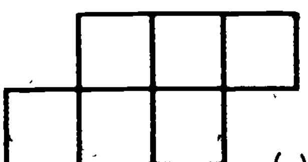
$$3 \text{ cm} \times 4 \text{ cm} = 12 \text{ cm}^2$$

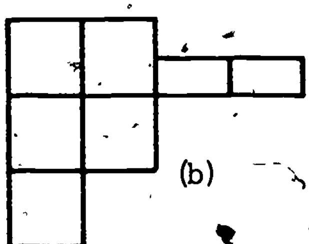
In general,

$$\text{Area of rectangle (in cm}^2) = \text{length (in cm)} \times \text{width (in cm)}$$

5. By counting, find the area in cm^2 of each figure in Figure 4.

Figure 4


(a)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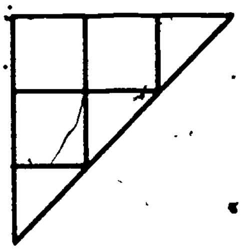

(c)

6. Calculate the areas in cm^2 of the rectangles whose length and width are given below.

   (a) 15 cm, 6 cm

   (b) 150 cm, 0.3 cm

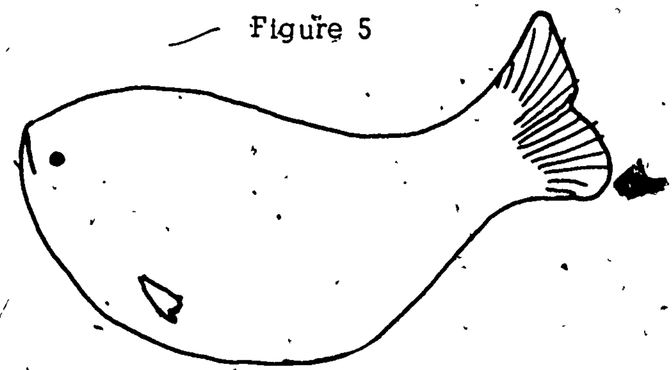
   (c) 2.8 m, 5.5 cm

   (d) 25 cm, 1.6 m

7. A square of 1 m by 1 m has an area of one square meter ($1\ m^2$). How many cm^2 are there in $1\ m^2$? How many m^2 are there in $1\ cm^2$?

8. Trace the figure shown in Figure 5. Draw two rectangles in such a way that their areas will bracket the area of the figure. What can you say about the area of the figure?

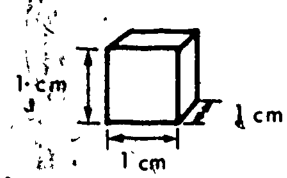
Figure 5



SECTION 2 UNITS OF VOLUME

Just as measuring area involves counting units of area, measuring volume involves counting units of volume. A square of 1 cm on each side is a common unit of area. Similarly, a common unit of volume is a cube whose edges are 1 cm long (Figure 6). This unit of volume is called a <u>cubic centimeter</u>, which is abbreviated as cm^3.

Figure 6



Counting the unit cubes in the solid shown in Figure 7 tells us that its volume is 5 cm^3.

Figure 7

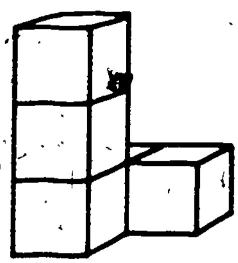
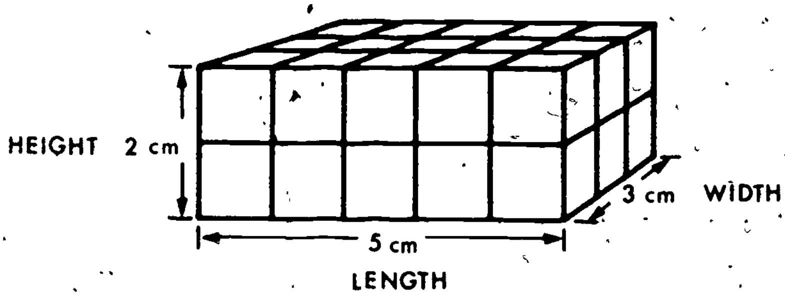


Figure 8 shows us an example of a rectangular solid — a solid whose faces are rectangles. There is a shortcut for finding the volume of a rectangular solid, and it is similar to the shortcut for finding the area of a rectangle.

Figure 8



First we find the number of cubic centimeters in a layer that is 1 cm high. Figure 9 shows one layer of the solid in Figure 8, and

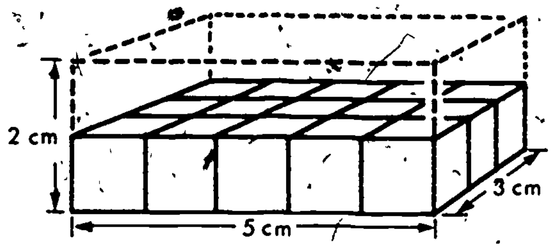


Figure 9

it is 5 cm long and 3 cm wide and 1 cm high. This layer contains 5×3 cubes. The edges of each cube are 1 cm long. The volume of this layer, therefore, is

$$5 \times 3 \times 1 \text{ cm}^3 = 15 \text{ cm}^3$$

Since the solid in Figure 8 is 2 cm high and each layer is 1 cm high, we have two layers. Therefore, the volume of the whole solid is

$$5 \times 3 \times 2 \times 1 \text{ cm}^3 = 30 \text{ cm}^3$$

That is, the number of cubic centimeters in a rectangular solid is equal to the product of the length, the width, and the height when each dimension is given in centimeters.

To remind ourselves of the need to have each dimension given in the same unit, we write the unit of length, width, and height after the respective numbers:

$$5 \text{ cm} \times 3 \text{ cm} \times 2 \text{ cm} = 30 \text{ cm}^3$$

The reasoning we used to calculate the volume of the rectangular solid in Figure 8 can be used for any rectangular solid. Thus the volume of any rectangular solid is given by

$$\text{Volume} = \text{length} \times \text{width} \times \text{height}$$

Of course, all three dimensions must be in cm if the volume is to be in cm^3.

9. What are the volumes of the solids in Figure 10?

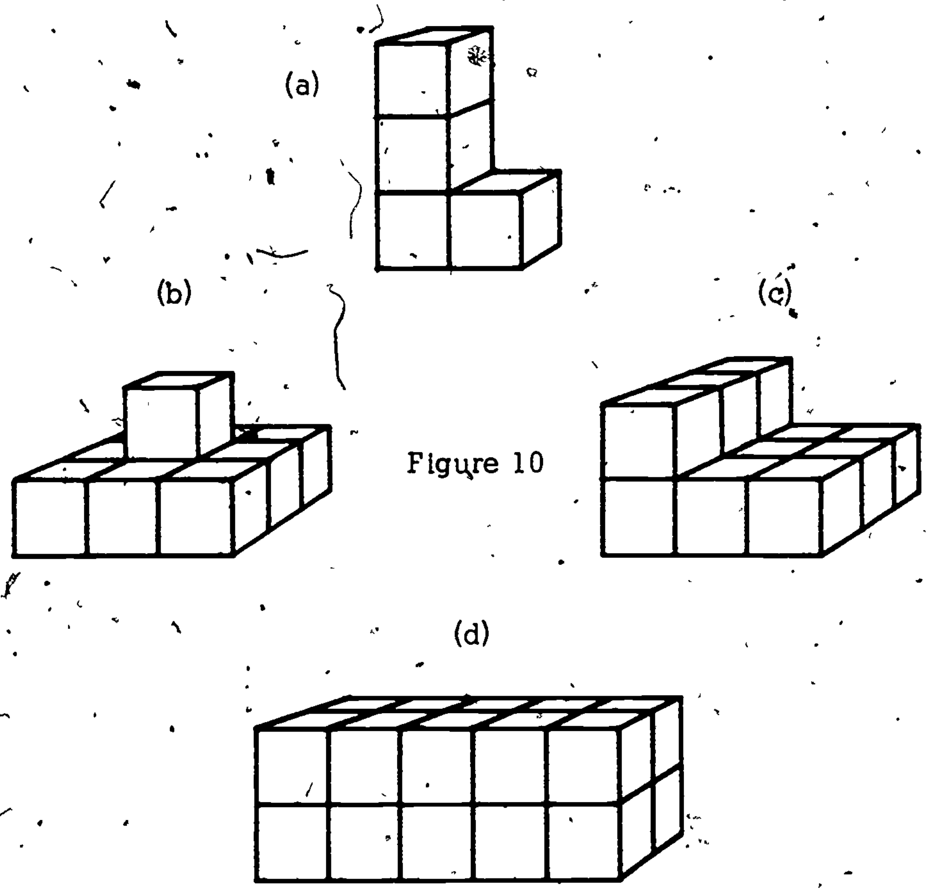
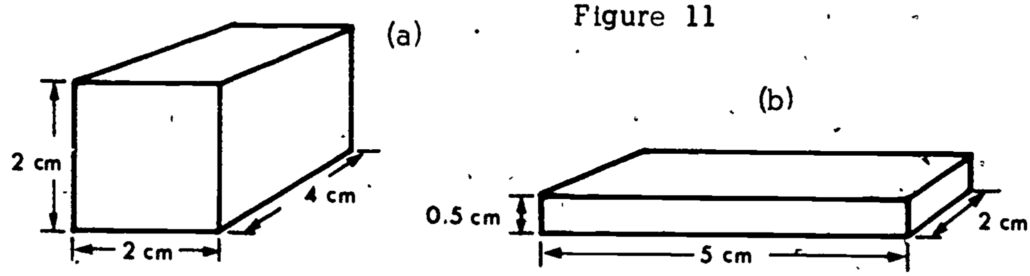


Figure 10

10. Count or use the shortcut to find the volumes of the rectangular solids in Figure 11.

Figure 11



11. A rectangular box is 1 m long, 2 m wide, and 60 cm high. What is its volume in cubic centimeters?

12. Find the volumes of the solids in Figure 12.

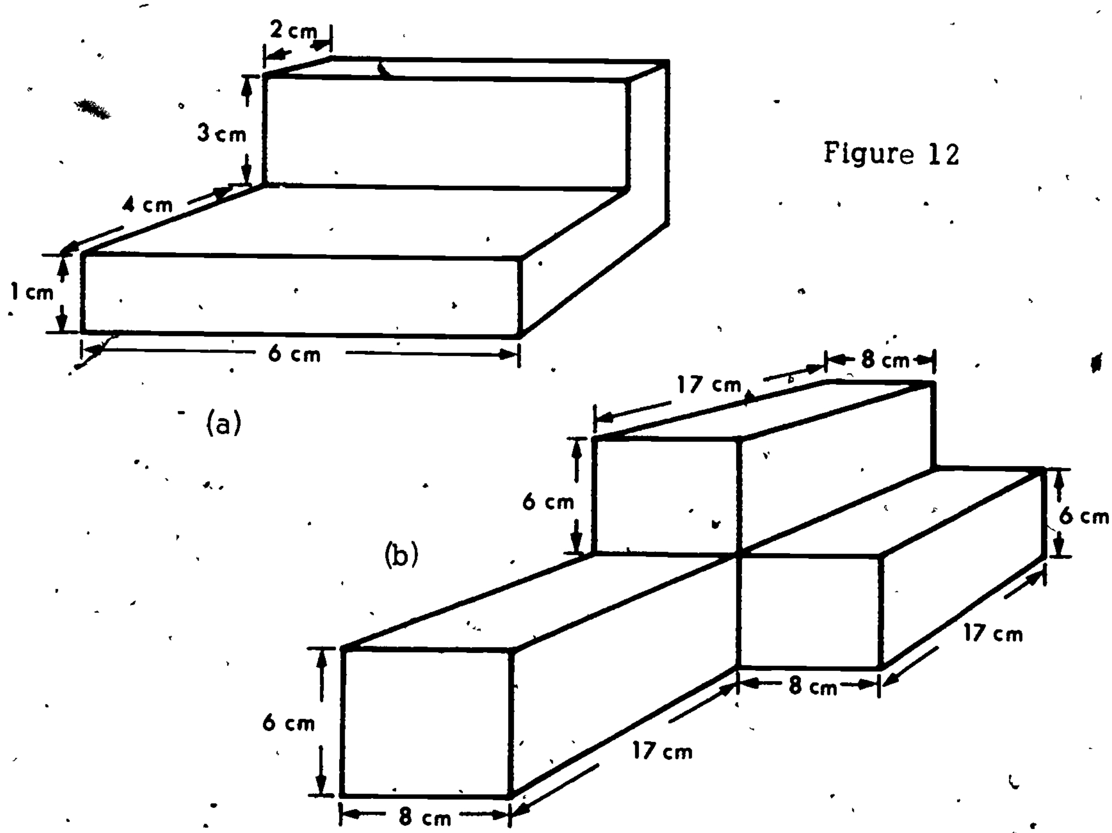


Figure 12

13. What could be the dimensions (length, width, height) of the rectangular solids whose volumes are given below?

(a) 12 cm^3

(b) 30 cm^3

(c) 500 cm^3

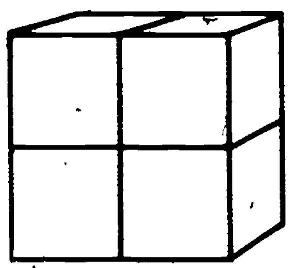
14. A cube has a volume of 27 cm^3. How long is its edge?

15. The rectangular solid in Figure 13 consists of four equal cubes. The volume of the solid is 500 cm^3.

(a) How long is an edge of one of the cubes?

(b) Find the dimensions of the solid.

Figure 13



SECTION 3 VOLUME AND SURFACE AREA

The <u>surface area</u> of a solid is the sum of the areas of all its faces. A rectangular solid has six rectangular faces. Its surface area is the sum of the areas of these six faces.

Consider the rectangular solid illustrated in Figure 14. Its volume is

$$8 \text{ cm} \times 6 \text{ cm} \times 4 \text{ cm} = 192 \text{ cm}^3$$

To find the surface area of the solid, we note that the front face has an area of 8 cm × 4 cm. Another face of equal area is hidden in the back. Added together, the area of the two faces is

$$2 \times (8 \text{ cm} \times 4 \text{ cm})$$

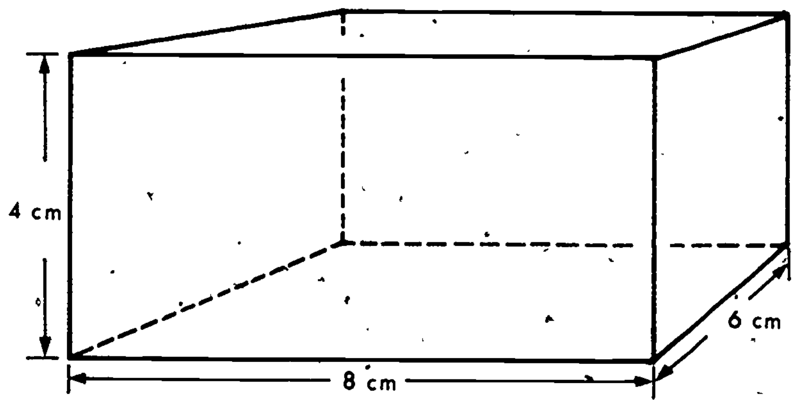


Figure 14

Similarly, the top and the bottom faces together have an area of

$$2 \times (8 \text{ cm} \times 6 \text{ cm})$$

and the two end faces together have an area of

$$2 \times (6 \text{ cm} \times 4 \text{ cm})$$

Added together, the total surface area is

$$2 \times (8 \text{ cm} \times 4 \text{ cm}) + 2 \times (8 \text{ cm} \times 6 \text{ cm}) + 2 \times (6 \text{ cm} \times 4 \text{ cm})$$

$$= 64 \text{ cm}^2 + 96 \text{ cm}^2 + 48 \text{ cm}^2$$

$$= 208 \text{ cm}^2$$

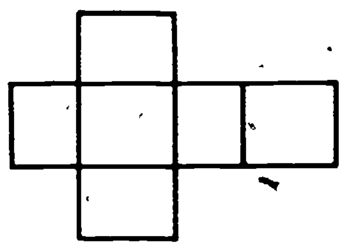
16. Explain why only two of the factors in any of the preceding products have units.

17. A rectangular solid measures 4 cm by 5 cm by 10 cm. What is its volume? What is its surface area?

18. A box has a length of 8 cm, a width of 7 cm, and a height of 6 cm.

(a) If the pattern shown in Figure 15 can be used to build this box, what must be the measurement of each of the line segments on the pattern?

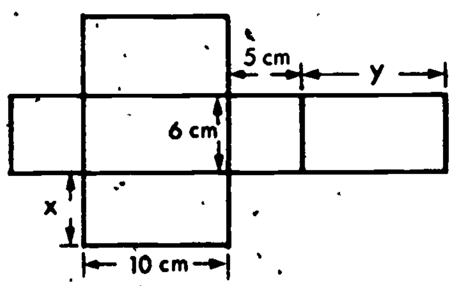
Figure 15



(b) How does the surface area of the box compare with the area of the pattern that is used to make the box?

19. A pattern for a rectangular box is shown in Figure 16.

Figure 16



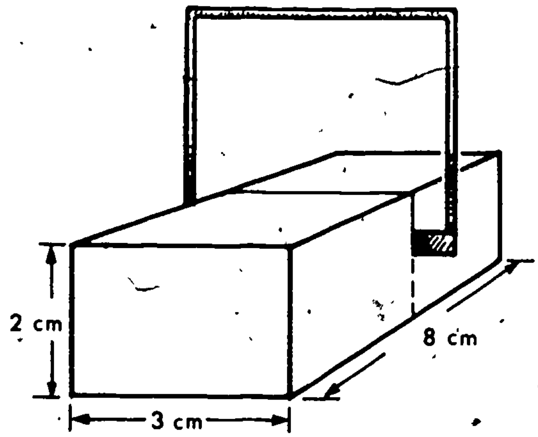
(a) What are the lengths of the edges marked x and y?

(b) What is the area of the surface of the box?

(c) What is its volume?

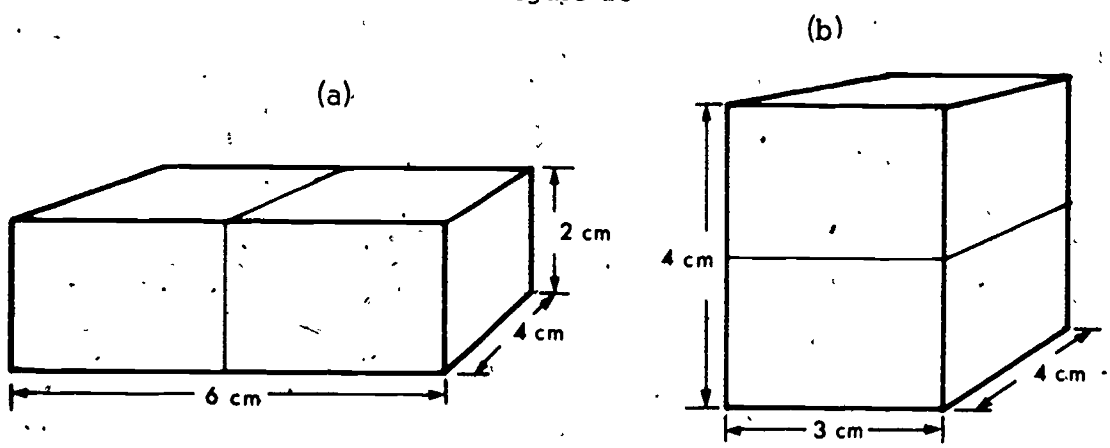
Figure 17 shows a rectangular solid. Suppose that you cut the solid along the plane indicated and rearrange the two halves as

Figure 17



shown in either Figure 18(a) or 18(b). Do you think the volume of the new solid is different from the volume of the old one? Do you think the new surface area is different from the old one?

Figure 18



Use your clay cutter to make a rectangular solid with the same dimensions as those in Figure 17. How many ways are there to cut this solid into halves that are each rectangular solids?

Make at least two such cuts in the solid that you made, and then rearrange all the pieces to make different rectangular solids. How do the volume and the surface area of each of these solids compare with those of the original?

Which of the rectangular solids has the smallest surface area? Which has the largest? What do you notice about their dimensions?

Can you cut the original solid in other ways to make the surface area as small as you wish—for example, 10 cm^2? As large as you wish—for example, 1000 cm^2? Explain your answer.

SECTION 4 VOLUMES OF RIGHT PRISMS AND RIGHT CYLINDERS

Let us look again at a rectangular solid (Figure 19) and at the formula for its volume:

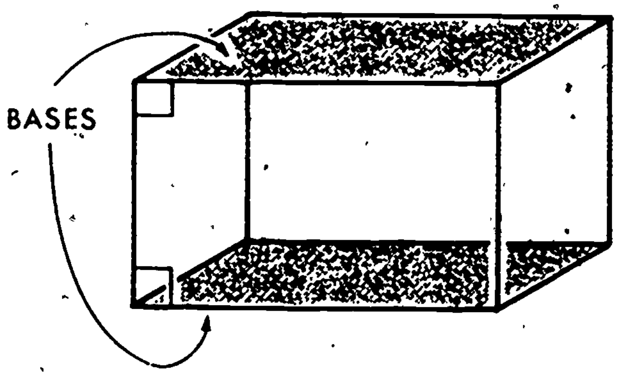
$$\text{Volume} = \text{length} \times \text{width} \times \text{height}$$

The area of one of a pair of opposite faces is the product

$$\text{length} \times \text{width}$$

Each of these faces is perpendicular to the height of the solid; that means each face meets the height at right angles. It is customary to call either of these faces the base of the solid.

Figure 19



The volume of the rectangular solid can be expressed as

Volume = area of base × height

This formula is actually a shortcut for adding up the volumes of individual layers that are each of unit height (see Figure 9).

?

20. The length and the width of a rectangular solid are 4 cm and 3 cm respectively. Its height is 7 cm.

   (a) What is the area of the base?

   (b) What is the volume of the solid?

21. The dimensions of a rectangular solid are 12 cm, 5 cm, and 20 cm.

(a) If you choose the face with the edges of 5 cm and 20 cm as the base, what will be the height of the solid? ___

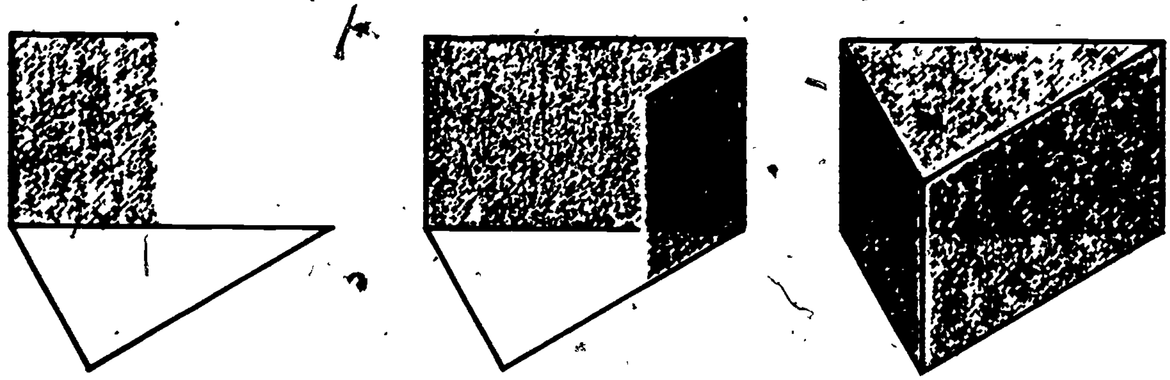
(b) Calculate the area of the base and the volume of the solid.

(c) Choose another face as the base. Calculate the area of this base and the volume. Compare your results with part (b).

We can use the same shortcut for other solids that we can think of as being made up of layers identical to each other.

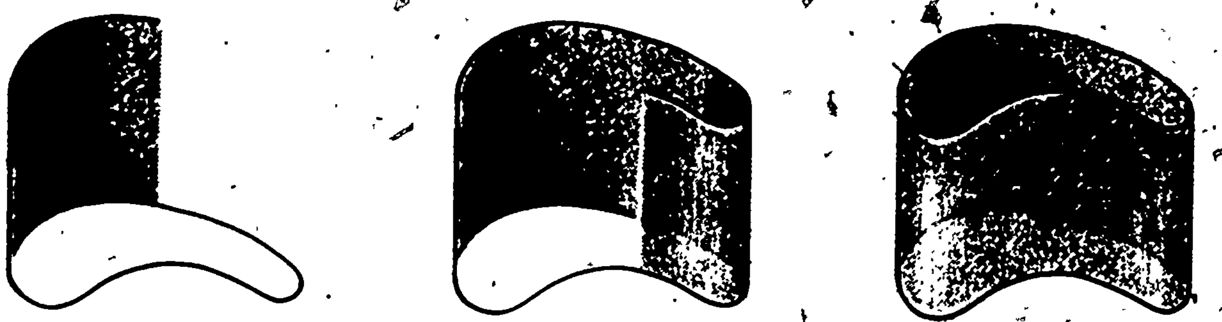
What do some of these solids look like? Here is a way to visualize the construction of such a solid. Start with a polygon region as a base, and then imagine a line segment perpendicular to the plane of the polygon (see Figure 20). Suppose that you can move the line segment along the polygon, keeping it perpendicular to the base. The other end of the segment would then trace out another polygon identical to the base. Such a solid is called a right prism because the moving segment, which is the height of the prism, meets the base at right angles.

Figure 20



We can also visualize a solid by starting with a closed curve in a plane. This is illustrated in Figure 21. The resulting solid is called a <u>right cylinder</u>.

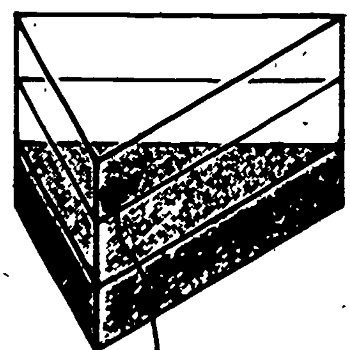
Figure 21



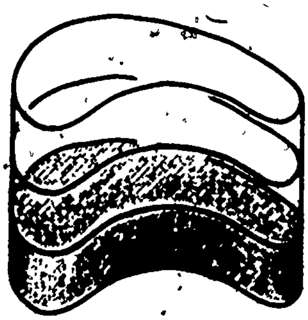
By looking at Figure 22 (a) and (b) we can see that these solids can be thought of as being made up of identical layers.

Figure 22

(a)



(b)



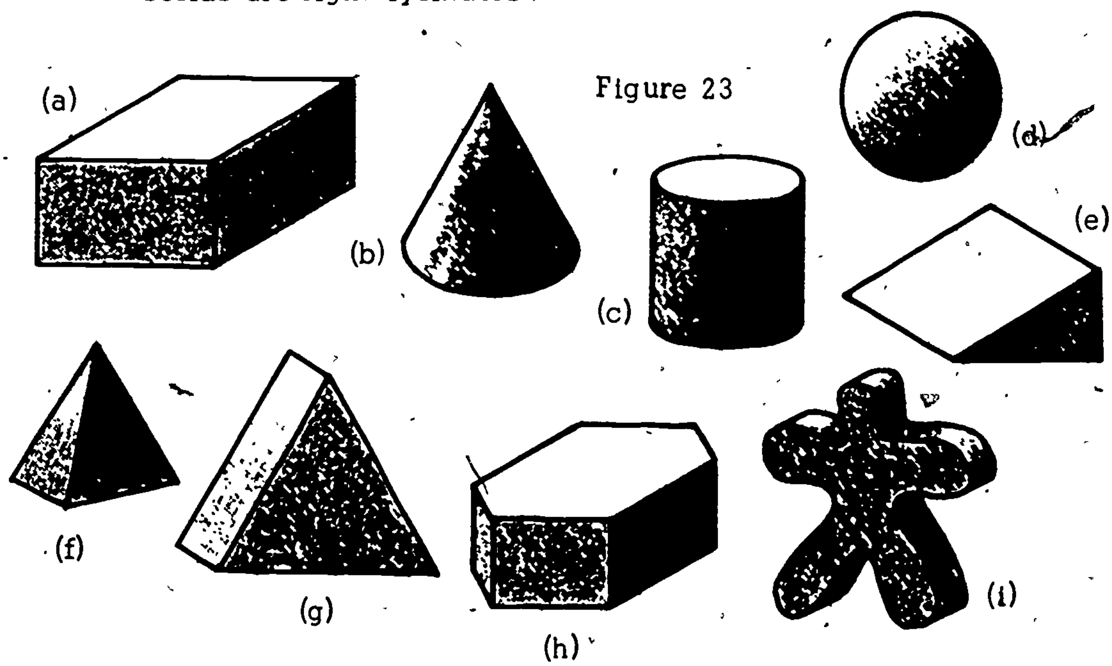
Therefore, we can find the volume of a right prism or a right cylinder by adding up the volumes of all the layers. Since the layers are identical, we can use the shortcut

Volume = area of base × height

22. Draw a triangle on a piece of paper. Using your pencil to represent the perpendicular line segment, show how you would construct a right prism that has this triangle as its base.

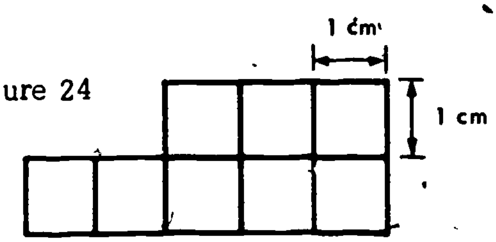
23. Which of the solids in Figure 23 are right prisms? Which solids are right cylinders?

Figure 23



24. Each square making up the polygon shown in Figure 24 is 1 cm^2.

Figure 24



(a) What is the area of the polygon?

(b) If you cover the polygon with one layer of cubes 1 cm on the edge, how many cubes will you need?

(c) What is the volume of this layer?

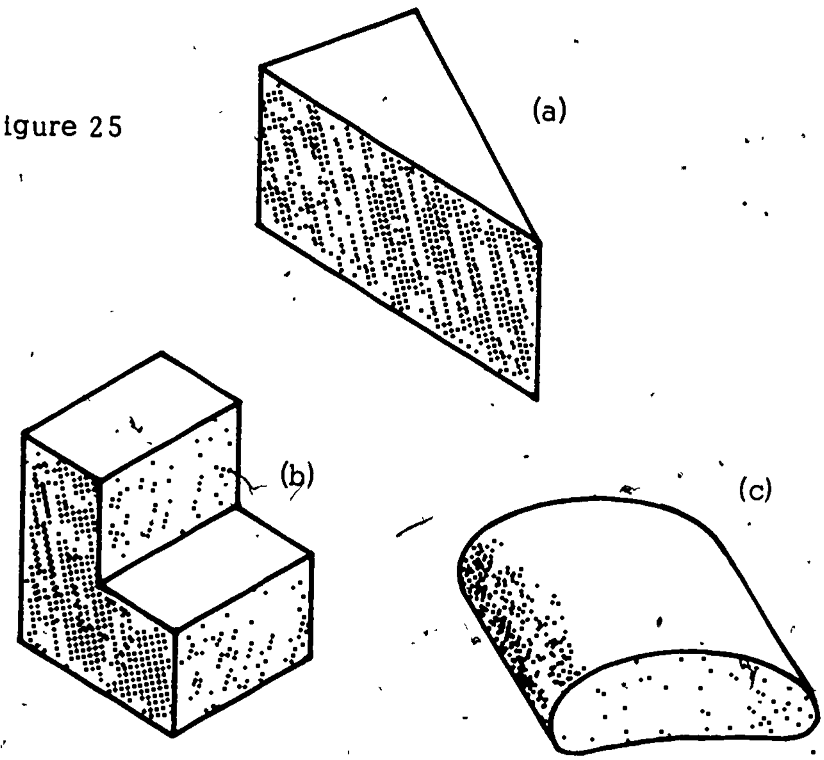
(d) How does the number of cubic centimeters compare with the number of square centimeters?

(e) How would you find the volume of a block having the same base but with a height of 3 cm?

(f) How would you find the volume if the height is 4.5 cm?

25. The wedge in Figure 25(a) has a base area of 48 cm^2 and a height of 7 cm. The solid in Figure 25(b) has a base area of 67 cm^2 and is 19 cm thick. The solid in Figure 25(c) has a base area of 36 cm^2 and a height of 9.5 cm.

Figure 25



(a) Identify the solids as right prisms or right cylinders.

(b) For each, name the base and find the volume.

26. What is the volume of each solid in Figure 26?

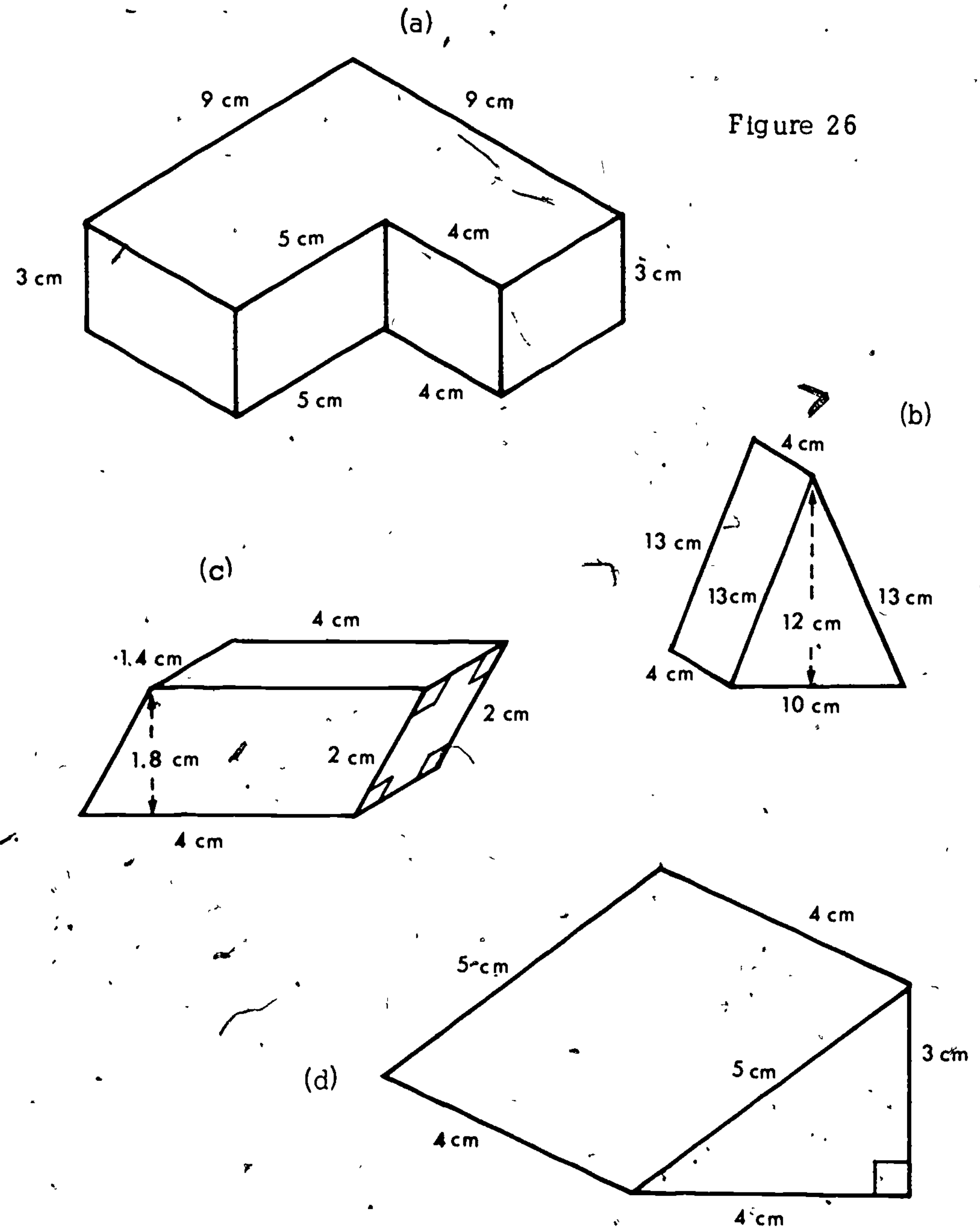
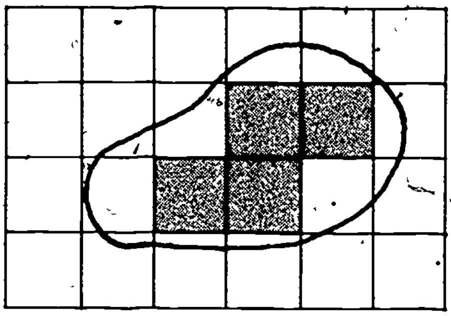


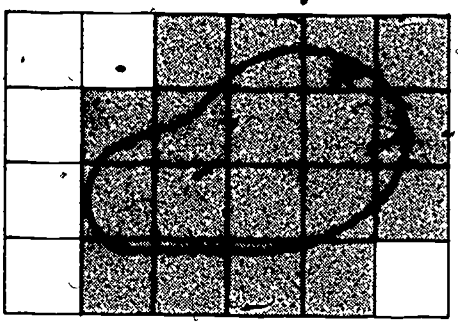
Figure 26

Finding the volume of a right cylinder is easy if we know the area of the base. When the base has an irregular shape, we can only bracket the area. Figure 27 shows the base of a cylinder

Figure 27



(a)



(b)

placed on a grid of unit squares. Counting the squares entirely within the base, Figure 27(a), and the squares needed to completely cover the base, Figure 27(b), we can bracket the area of the base:

$$4 \text{ cm}^2 < \text{area of base} < 18 \text{ cm}^2$$

The average of the smaller and larger bracketing values is usually a good approximation of the area of a region. In this case

$$\frac{4 \text{ cm}^2 + 18 \text{ cm}^2}{2} = 11 \text{ cm}^2$$

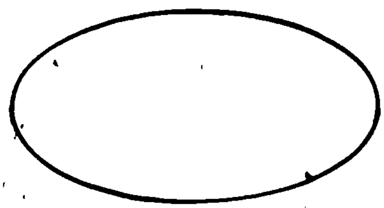
Multiplying the area of the base (11 cm^2) by the height (10 cm) gives us the approximate volume of the cylinder,

$$11 \text{ cm}^2 \times 10 \text{ cm} = 110 \text{ cm}^3$$

To remind yourself of how to improve the bracketing, refer to Sections 3 and 4 of Chapter 7 of the first year course.

27. Find the approximate volume of a right cylinder that is 5 cm high and has the base shown in Figure 28.

Figure 28



SECTION 5 VOLUMES OF IRREGULAR SOLIDS

In the previous section we learned that we can use layers to help us visualize the volumes of right prisms and right cylinders.

Suppose that you want to find the volume of an irregular solid such as the lump of clay illustrated in Figure 29. Again, you

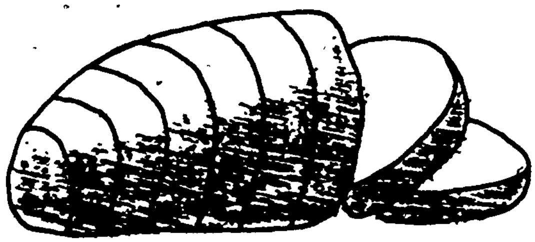
Figure 29

can use layers to help you visualize its volume. However, there is no formula that gives a shortcut for finding that volume because the layers are not all identical.

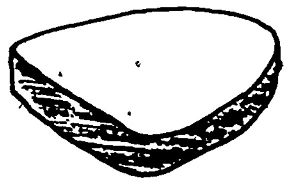
By using the clay cutter you can slice the lump into layers 1 cm thick, as shown in Figure 30. You can see from Figure 30(b) that each slice has the approximate shape of a right cylinder.

Figure 30

(a)



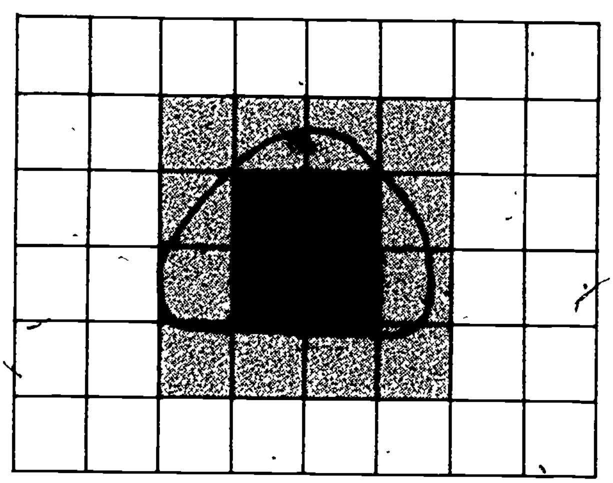
(b)



Note that you can use a centimeter grid to bracket the area of the base, just as you did with a right cylinder. Figure 31 shows the outline of the base of a slice. Since $4\text{ cm}^2 <$ area of base $< 16\text{ cm}^2$, the area of the base of the slice is about

$$\frac{4\text{ cm}^2 + 16\text{ cm}^2}{2} = 10\text{ cm}^2$$

Figure 31



And so, because the slice is 1 cm thick, the approximate volume of the slice is

$$10\ \text{cm}^2 \times 1\ \text{cm} = 10\ \text{cm}^3$$

To find the volume of the solid we must find the volume of each remaining slice and then add all the volumes. You can use this method to calculate the approximate volume of any solid so long as you can find the area of each of its cross sections.

28. Bernie cut a lump of clay into 5 layers. With his centimeter grid he bracketed the area of each base and found:

Layer	Smaller Bracket	Larger Bracket	Thickness
Layer 1	14 cm^2	33 cm^2	1 cm
Layer 2	12 cm^2	29 cm^2	1 cm
Layer 3	9 cm^2	22 cm^2	1 cm
Layer 4	7 cm^2	21 cm^2	1 cm
Layer 5	6 cm^2	19 cm^2	1 cm

What is a good estimate of the volume of this lump of clay?

Make an irregular solid out of clay. By slicing it into 1 cm slices, estimate its volume.

SECTION 6 A USEFUL PROPERTY OF LIQUIDS

Water and other liquids have a very useful property. The volume of a liquid stays the same regardless of the shape of the container holding the liquid. This means that if we want to measure the volume of a liquid we can transfer the liquid into any container we choose and then measure its volume. Furthermore, this allows us to measure the volume of any container — by filling it with water and then measuring the volume of the water.

Usually we measure liquids in a measuring cup or graduated cylinder (see Figure 32). Each marking along the side of the cup or the cylinder indicates the level of a certain volume of liquid.

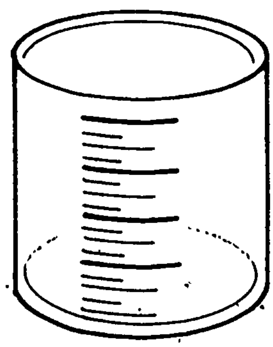
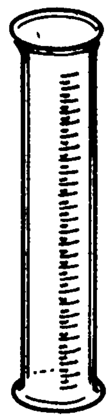


Figure 32



Volume Estimation Game

Figure 33 illustrates a method for estimating the volume that a container can hold. First, imagine that it is filled with clay. Next, in your mind, reshape the clay to form a rectangular solid and estimate the dimensions of this solid. Finally, find the approximate volume by multiplying the dimensions together.

To check an estimate, fill the container with water and measure the volume of the water with a graduated cylinder or a metric measuring cup.

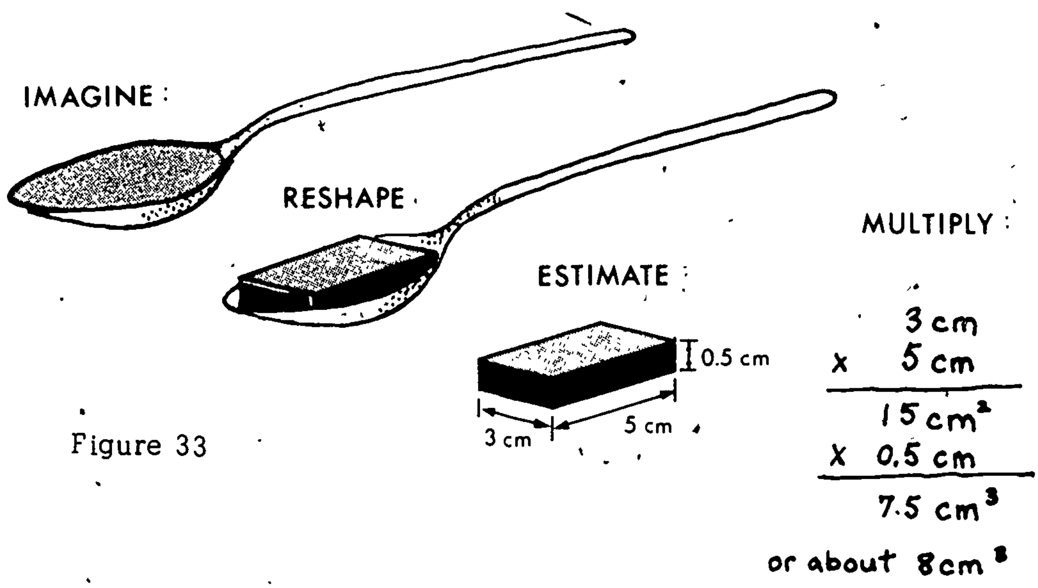


Figure 33

Get together in small groups.

Your teacher will show you a number of common household containers that can hold water – for example, a teaspoon, a teacup, an ice-cream scoop, a coffee can, a soup can, a paper cup, a cake pan, and so on.

Use the method shown in Figure 33 to estimate the volume of water that each container can hold. Then for each container, average the estimates in your group.

Next, use water to measure the volume that each container holds. Use a commercially available graduated cylinder or metric cup measure to measure the actual volume. (If you don't have either measuring device, you can make one by marking off levels corresponding to known amounts of water like 25 cm^3, 50 cm^3, 75 cm^3, and so on.)

The group with the smallest total error is the winning group.

?

29. If you look closely at the markings on a graduated cylinder, you will notice that they are equally spaced. Since it gets wider near the top, the measuring cup shown in Figure 34 is not cylinder shaped. Must the markings be closer together or farther apart near the top?

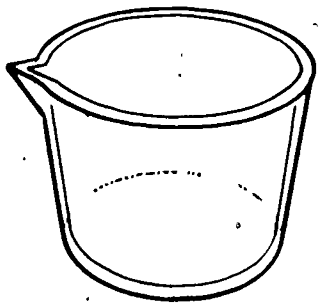


Figure 34

30. A certain measuring container holds a maximum of 250 cm^3 of water. A larger container is full of water. This water fills the measuring container exactly $2\frac{1}{2}$ times. What is the volume of water that the larger container holds?

31. A measuring cup has some water in it, and the level reads 175 cm^3. A marble is dropped into the water and the level rises to the 181 cm^3 mark. What is the volume of the marble?

32. A 1-pound coffee can holds 1000 cm^3 of water. Suppose we take a certain-sized paper cup and discover that eight paper cups full of water exactly fill the coffee can. How much water does the paper cup hold?

33. The water from 18 identical paper cups fills 4 identical soup cans.

(a) How many paper cups hold just as much water as one soup can holds?

(b) What is the ratio of the volume of one soup can to the volume of one paper cup?

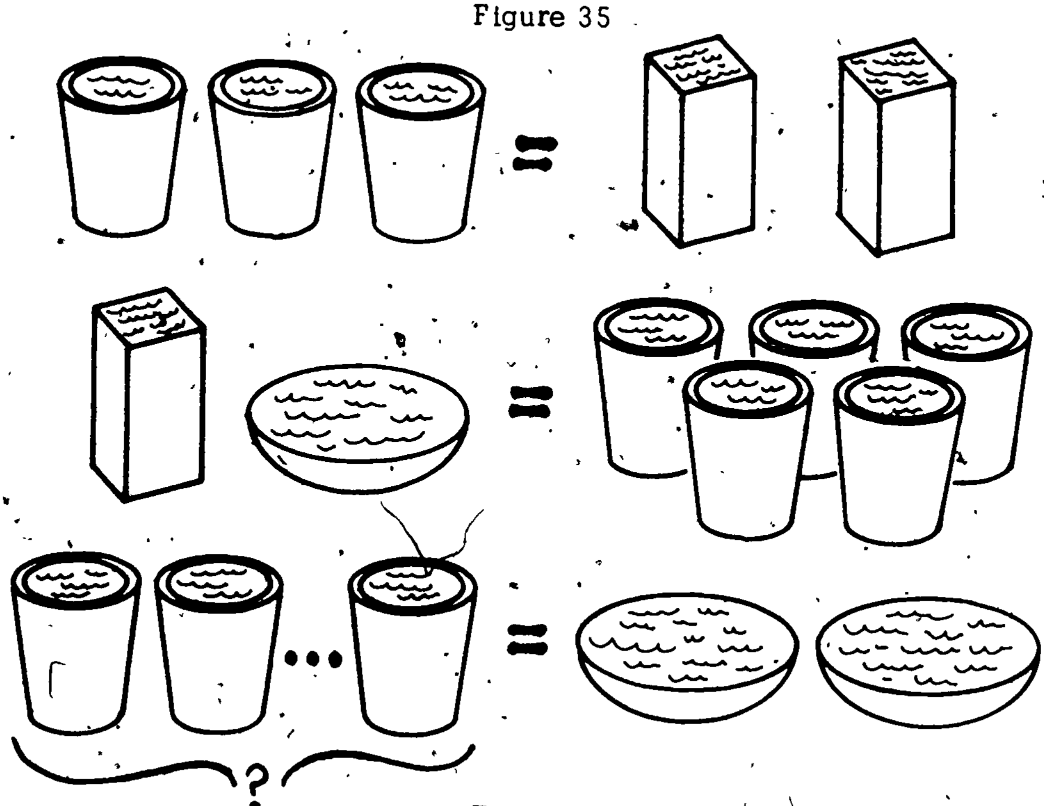
34. (a) What is the ratio of the volume of one tablespoon to the volume of one teaspoon?

(b) If 48 teaspoons of water fill a cup, how many tablespoons will fill a cup?

35. Four soup cans hold the same amount of water as one large glass. Three of those glasses hold as much water as one mixing bowl. How many soup cans full of water would fill the mixing bowl?

36. How many glasses hold the same volume of water as two bowls? (See Figure 35.)

Figure 35



SECTION 7 LITERS AND CUBIC METERS

Why do people use different units of length such as centimeters, meters, and kilometers rather than just one unit? The reason is that we all prefer to use numbers larger than one and smaller than, say, one thousand. This is why we find it convenient to describe the length of a pencil in centimeters, the length of a room in meters, and the distance between two cities in kilometers.

When you are concerned with describing areas, it is convenient to use square centimeters for the area of a postcard, square meters for the area of a room, and square kilometers for the area of a city.

37. (a) How many meters are in 1 km?
    (b) How many centimeters are in 1 km?
    (c) 2.5 km equals how many meters? How many cm?
    (d) 1.8 cm equals how many meters? How many km?

38. (a) How many square centimeters are in 1 m^2?
    (b) What is the area of your classroom in m^2? In cm^2?

39. (a) A chalkboard is 2.40 m long and 1.20 m high. What is its area in m^2? In cm^2?
    (b) In which unit do you prefer to give the area of the chalkboard?

As you already know, 1 m^2 equals 10,000 cm^2. Areas of a few thousand cm^2 can be conveniently expressed in either cm^2 or

m^2. And so there is no need for a unit between cm^2 and m^2. However, the situation is different when we work with volumes.

A cube that is 1 m on the edge is 1 cubic meter (1 m^3) and has a volume of 100 cm × 100 cm × 100 cm = 1,000,000 cm^3. A cubic meter is too big a unit to express the volume of many household containers such as pots and buckets. On the other hand, a cubic centimeter is too small a unit. Therefore, we use a unit of volume between the two: namely, the volume of a cube 10 cm on the edge. This unit is called a <u>liter</u>, and it is abbreviated as ℓ.

$$1\ \ell = 10\text{ cm} \times 10\text{ cm} \times 10\text{ cm} = 1,000\text{ cm}^3$$

A liter is a little larger than a quart. If you know how large a quart is, you can visualize how large a liter is.

The liter is so widely used, particularly for liquid, that 1 cm^3 is often referred to as 0.001 liter and is written as 1 ml (1 milliliter).

40. A box has dimensions of 75 cm, 80 cm, and 50 cm.

    (a) Find the volume of the box in cm^3, ℓ, and m^3.

    (b) Which description of the volume is the most convenient?

41. Suppose that you cut 1 m^3 up into cubes 1 cm on a side. Next, you stack the cubes into a tower that has a square base of 10 cm on a side.

    (a) How tall would the tower be?

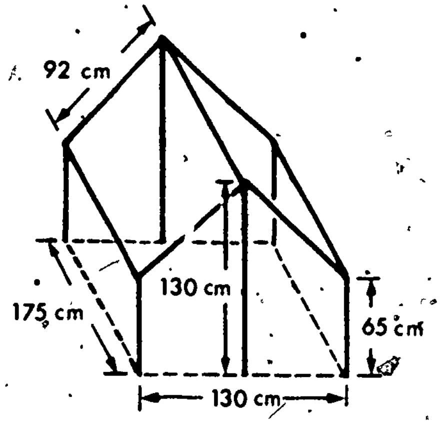
    (b) Would this tower fit into your classroom?

(c) Would this tower be shorter or taller than your school?

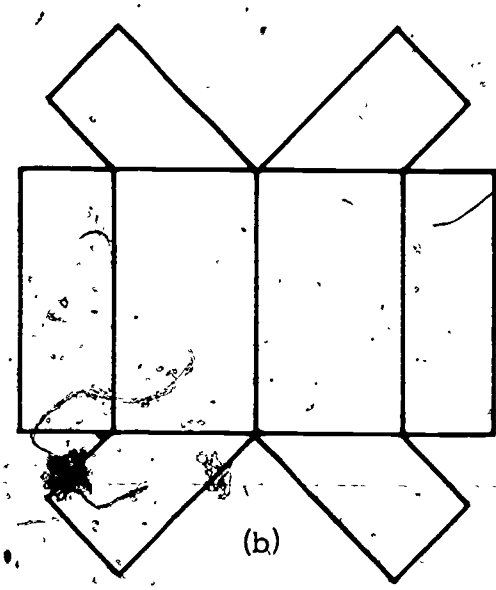
(d) Could you rearrange this tower so that it would fit into your classroom?

42. A certain kind of tent comes with a set of metal poles that fit together to form a skeleton, illustrated in Figure 36(a), and with a large piece of canvas to go over the skeleton. When the canvas is laid flat, it looks like the pattern shown in Figure 36(b). The tent, when assembled, measures 175 cm long, 130 cm wide, 130 cm tall to the highest point, 65 cm tall at the sides, and 92 cm along the slant of the roof. See Figure 36(a).

Figure 36



(a)



(b)

(a) What is the total length of all the poles?

(b) What are the dimensions of the canvas pattern?

(c) What is the total surface area of the canvas?

(d) How big is the volume of the tent when it is assembled?

(e) Suppose that you were unable to purchase a tent that has these dimensions and volume. Explain why it would be useful to know the information obtained from parts (a) through (d).

43. A paint company sells paint in 4-liter (1-gallon) cans. The company claims that this paint will cover 37 square meters of wall (400 square feet). How thick (on the average) will the paint be if it covers as much surface as advertised?

44. The New England Aquarium publicizes that it has aquariums totaling 2 million liters of water.

(a) How many cubic meters of water is that?

(b) Almost all of this water is in one cylinder-shaped tank that has a base area of 125 m^2. How tall is that tank?

45. The largest building in the world is the Boeing Jumbo Jet assembly building in Everett, Washington. It has a volume of 5,820,000 m^3. How many liters of air does the building hold?

46. The Great Wall of China is 2400 km long and averages 7.6 m in height and 6.0 m in width. Estimate its volume.

3. POWERS OF TEN

SECTION 1 LARGE NUMBERS

We saw in Chapter 2 that calculating volumes often involves large numbers. Large numbers also occur when we work with areas of states, populations of countries, budgets of cities, and many other topics. For example, the area of California is approximately 400,000 km^2, the population of India is about 600 million persons, and the amount of money New York City spent in 1973 was $10 billion.

Note that when we write such large numbers we often use terms like "approximately" or "about." We do this for one of two reasons. In many cases an approximate value is all we are interested in, and in other cases the exact value is not known.

When we have a number in the billions, we may be interested only in the billions. Therefore, in writing the number, we prefer to use the label "billion" rather than to write out all nine zeros as place holders. In tables we often see "in thousands of km^2" or "in millions of dollars" printed at the head of a column. This is also done so that we do not have to write out zeros or other digits as place holders.

[?]

1. On a highway there is a sign that reads "Entering Mathville, pop. 13,871."

   (a) How accurate do you think this number can be?

   (b) How would you express this number in a way that would indicate that you know it only approximately?

2. Do you think anybody knows the present population of India to the nearest million? Nearest thousand?

3. Colorado is a mountainous state with an area of 269,000 km^2.

   (a) How accurate do you think this number is?

   (b) How would you write this number so that its lack of accuracy is recognized?

4. The population of the United States is about 200,000,000.

   (a) Write this number using the label "million."

   (b) Write it using the label "thousand."

5. Using the following labels, rename the number 3,600,000.

   (a) thousand

   (b) hundred thousand

   (c) million

6. About 2,300,000 copies of the magazine <u>Sports Illustrated</u> are sold each week. How many millions of copies is that? How many thousands? Do you think that the same exact number of copies is sold each week?

7. When the Hoover Dam was constructed, a lake was made that contains 36,700,000,000 m^3 of water.

   (a) How accurate can this number be?

   (b) How many km^3 of water are contained in the lake?

8. Use Table 1 to answer the following questions.

(a) How many chickens were raised in Rhode Island in 1972? How many eggs were produced that year?

(b) Which is larger, the number of eggs produced in Rhode Island in 1972 or the number of chickens raised in Maine during the same year?

(c) The number of chickens raised in Pennsylvania in 1972 was 13.3 million. Compare this number to the total number of chickens raised in New England during that year.

(d) Approximately how many more eggs were produced in New Hampshire in 1972 than in 1971?

TABLE 1: Eggs and Chicken Production in New England States, 1971 and 1972

State	Eggs produced (in millions)		Chickens raised (in thousands)	
	1971	1972	1971	1972
Connecticut	830	924	3,620	3,320
Maine	1,368	1,402	5,423	5,886
Massachusetts	513	535	1,959	2,200
New Hampshire	312	313	1,230	1,255
Rhode Island	70	57	245	160
Vermont	96	114	469	479
Total	3,189	3,345	12,946	13,300

Source: Adapted from a table that appeared in U.S. Department of Commerce, Statistical Abstract of the United States: 1973. (94th Edition) Washington, D.C., 1973.

9. Table 2 shows the approximate areas of several states in the U.S.A.

TABLE 2

State	Area (in thousands of km^2)
Iowa	145
Alaska	1450
Maryland	26
Connecticut	13
Michigan	147
Arizona	294
Rhode Island	3
Colorado	269

(a) What is the area of Rhode Island in km^2?

(b) What is the difference between the area of Arizona and the area of Michigan?

(c) By how much is Alaska larger than Iowa?

SECTION 2 VISUALIZING LARGE NUMBERS

Suppose that you could earn ten dollars an hour and that you worked for 10 hours a day, 300 days a year. How long would it take you to earn one million dollars? Working 300 days a year for 10 hours a day you would work

3 hundred × ten = 3 thousand hours

Your annual earnings would be

3 thousand × ten = 30 thousand dollars

Therefore, it would take more than 30 years to earn one million dollars.

How long would it take to earn one billion dollars? Since

$$1 \text{ billion} = 1 \text{ thousand} \times 1 \text{ million}$$

it would take one thousand times as long, or over 30 thousand years to earn one billion dollars!

Large numbers such as 1 billion or even 1 million are difficult to imagine. A good way to help us visualize such large numbers is to think of them as products of smaller numbers with which we are more familiar.

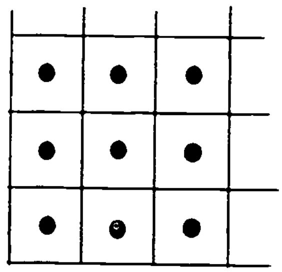
You can use Table 3 to help you calculate some products that lead to large numbers.

TABLE 3

×	Ones	Tens	Hundreds	Thousands	Ten Thousands
Ones	ones	tens	hundreds	thousands	ten thousands
Tens	tens	hundreds	thousands	ten thousands	hundred thousands
Hundreds	hundreds	thousands	ten thousands	hundred thousands	millions
Thousands	thousands	ten thousands	hundred thousands	millions	ten millions
Ten Thousands	ten thousands	hundred thousands	millions	ten millions	hundred millions

10. Each square in the grid shown in Figure 1 contains one dot and is 1 cm on a side. Suppose that this grid extended 10 m across and 10 m vertically. What would be the total number of dots in the grid?

Figure 1



11. Suppose that you walked 5 km in one hour for 10 hours a day, 300 days a year for 10 years.

(a) What distance would you have covered?

(b) How many times around the equator is this? (The equator is about 40,000 km long.)

12. The distance from the earth to the moon is about four hundred thousand km. To visualize this distance you might think of it as the following product:

Distance earth to moon=
(distance New York to Detroit) × (number of trips)

The distance from New York to Detroit is about one thousand km. How many trips would you have to make between New York and Detroit to cover a distance equal to the distance between the earth and the moon?

13. A paper manufacturer produces 45 billion sheets of paper a year. The sheets are packed in cartons that contain 5 thousand sheets each. The cartons are shipped on 300 working days.

(a) How many cartons does he ship a year?

(b) How many cartons make up an average day's shipment?

(c) Write the number of sheets produced in a year as the product of the number of sheets in one carton, the number of cartons in an average shipment, and the number of working days in one year.

14. If $100 billion were evenly distributed among 200 million people, how much would each person receive?

15. Suppose that you have enough cubic-centimeter blocks to build one large cube whose volume is one cubic meter. To put each block in place takes you only one second.

(a) If you work nonstop for ten hours, how many blocks will you have put in place?

(b) Do you think that by working around-the-clock (without stopping to eat or sleep) you could finish building the cubic-meter block in five days? About how many days do you think it would actually take?

How good are you at guessing how big a number is? Make a chart like the following and put a checkmark in the column that you think gets you "in the right ballpark" for each row. Where possible, think of the number that you are estimating as a product of more familiar quantities. You may use almanacs, encyclopedias, or other sources to resolve disagreements.

LARGE QUANTITIES \ LABELS	Tens	Hundreds	Thousands	Ten Thousands	Hundred Thousands	Millions	Ten Millions	Hundred Millions
Distance in km across U.S. from Atlantic to Pacific								
Number of cars manufactured in the U.S. last year								
Population of the U.S.S.R.								
Number of leaves on an average-sized tree in summer								
Number of stories in the world's tallest building								
Distance in km from the from the earth to the sun								
Number of high schools in the U.S.								
Number of stars in the sky visible to the unaided eye								
Population of New York City								

SECTION 3 APPROXIMATING PRODUCTS OF WHOLE NUMBERS

H You should be able to calculate simple products such as 30 × 200 without using paper and pencil. A good way to do this is to do the multiplication in four steps.

Step 1. Rename the numbers by using word labels so that each number is one digit times a label. The product 30 × 200 becomes 3 tens × 2 hundreds.

Step 2. Multiply the two digits, 3 × 2 = 6.

Step 3. Multiply the values represented by the word labels, tens × hundreds = thousands.

Step 4. Put everything together:
3 tens × 2 hundreds = 6 thousands.

The reason for using word labels in renaming numbers is to make it easier to do the multiplication mentally. It is easier to remember that tens × hundreds = thousands than to count zeros in your head.

You can use Table 3 to find products such as tens × hundreds. However, the better you are at remembering the table, the more quickly you will be able to do such problems in your head.

16. Calculate the following products. (Try to do it mentally.)

(a) 2 tens × 3 tens

(b) 2 tens × 3 hundreds

(c) 4 ones × 2 hundreds

(d) 4 tens × 5 thousands

(e) 6 hundreds × 5 hundreds

17. Calculate the following products without using paper and pencil.

(a) 40×20

(b) 20×50

(c) 200×40

(d) 6000×100

(e) 3000×3000

H Calculators are very handy for finding multidigit products. However, one wrong punch of the key can produce a completely unreasonable answer. To see if the answer is reasonable you do not need to redo the calculation. You can simply approximate the product and compare the calculator's answer with the approximation.

Suppose that you multiply 238 by 34 on a calculator (or by longhand on paper) and you read off 1292 as the answer. Can this be correct? To check your answer you round off 238 to 200 and 34 to 30. Multiplying 200×30 mentally gives you

$$200 \times 30 = 2 \text{ hundreds} \times 3 \text{ tens} = 6 \text{ thousands}$$

Clearly 238×34 cannot possibly equal 1292. An error must have been made in entering the numbers in the calculator.

Of course, small errors cannot be detected by this method. But as we said at the beginning of this chapter, approximate answers are often sufficient when dealing with large numbers.

18. Round off each of the following numbers to one nonzero digit and the correct place value.

(a) 8,324

(b) 3

(c) 58

(d) 256,000

(e) 76,120

(f) 985

19. Approximate the following products. (Try to do it in your head.)

(a) 46×22

(b) 225×67

(c) $6,400 \times 25$

(d) $2,300 \times 975$

(e) $52,000 \times 3,600$

20. Without doing the calculations, determine which of the following results are definitely wrong. Why?

(a) $52 \times 37 = 6,426$

(b) $14 \times 78 = 192$

(c) $6,400 \times 860 = 5,504,000$

(d) $450 \times 320 = 14,400$

(e) $75,000 \times 3,300 = 247,500,000$

SECTION 4 EXPONENTIAL NOTATION

Look back at Table 3. Note that in the "tens" column the values follow a distinct pattern: The value of a given entry is ten times the value of the entry above it. Writing digits instead of words, we have

$$100 = 10 \times 10$$

$$1{,}000 = 10 \times 100 = 10 \times 10 \times 10$$

$$10{,}000 = 10 \times 1{,}000 = 10 \times 10 \times 10 \times 10$$

Any number appearing as part of a product is called a <u>factor</u>. In the product 10×10 the number 10 is a factor twice. For products such as 10×10 where a factor is repeated, we often use a shorthand notation. We write 10×10 as 10^2. The raised 2 to the right of the 10 is called an <u>exponent</u> or a <u>power</u>, and it tells us how many times the 10 is to be used as a factor. When a product is written using exponents we say that it is written in <u>exponential notation</u>.

The numbers in the sequence

10 100 1,000 10,000 100,000 1,000,000

can be expressed as products with 10 as the repeated factor and can, therefore, be written in exponential notation as

10 10^2 10^3 10^4 10^5 10^6

(We can write 10 as 10^1 because we can think of 10 as a product, 10×1, where 10 appears as a factor once; but, we have no need to write 10 this way.)

The written symbol 10^4 is read as "ten to the fourth." This is really short for "ten to the fourth power." Similarly, 10^9 is read as "ten to the ninth." The number 10^2 is read either as "ten to the second" or as "ten squared."

21. In exponential notation write the number of cubic centimeters in
    (a) one cubic meter
    (b) one liter

22. The area of Canada is about 10^7 km^2. Write this number in words.

23. In 1972 there were about one hundred million automobiles registered in the United States.
    (a) Write this number by using zeros.
    (b) Write it in exponential form.

24. Which is larger?
    (a) ten thousand or 10^5?
    (b) 1,000,000 or 10^4?
    (c) 10^{10} or 1 billion?

25. How many thousands are in 10^4?

We saw how by using exponents we can write powers of ten in a compact way. We can also extend this shorthand to numbers that are not powers of ten. For example, the distance from the

1.

earth to the sun is 150,000,000 km. Since

$$150{,}000{,}000 = 1.5 \times 100{,}000{,}000$$

we can write 150,000,000 km as 1.5×10^8.

Any large number can be broken up into a product of a smaller number and a power of ten. The smaller number is called the <u>coefficient</u> of the power of ten. In 1.5×10^8 the coefficient is 1.5 and 10^8 is the power of ten.

There is a general procedure for finding the correct power of ten for any given number and coefficient. Suppose that you wish to write 3,250 as 3.25 times a power of ten, or

$$3{,}250 = 3.25 \times 10^{\square}$$

What exponent should be placed in the box? The exponent tells us how many times we multiply 3.25 by 10. The result of multiplying by 10 once is that the decimal point is moved to the right one place:

$$3.25 \times 10 = 32.5$$

Two multiplications by a factor of 10 move the decimal point two places to the right:

$$3.25 \times 10^2 = 325$$

Here the decimal point is not written but is implied. Multiplying by another factor of 10 requires the use of zero as a place holder. Again, the decimal point is after the zero:

$$3.25 \times 10^3 = 3{,}250$$

This is the answer we are looking for. Therefore, the correct exponent is 3.

We can summarize this procedure as follows. When you wish to write a number in exponential notation with a given coefficient, count the number of times you must move the decimal point to the right to change the coefficient into the number that you started with. The exponent is the number of times the decimal moves to the right.

Here are two additional examples.

$$360000 = 3.6 \times 10^{\square}$$

Here we have to move the decimal point 5 times. Therefore, $\square = 5$.

$$250000 = 250 \times 10^{\square}$$

In this example we move the decimal point 3 times. Therefore, $\square = 3$.

There are many ways to write a large number in exponential notation:

$$150{,}000{,}000 = 1.5 \times 10^{8}$$

or

$$150{,}000{,}000 = 15 \times 10^{7}$$

or

$$150{,}000{,}000 = 150 \times 10^{6}$$

and so on. You can always choose the coefficient when writing a number in exponential notation and then find the correct power of ten that goes with it.

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26. Fill in the box with the correct exponent.

(a) $250{,}000 = 25 \times 10^{\square}$

(b) $8{,}200{,}000 = 82 \times 10^{\square}$

(c) $12{,}500{,}000 = 125 \times 10^{\square}$

27. Fill in the box with the correct exponent.

(a) $3,600 = 3.6 \times 10^{\square}$

(b) $420,000 = 4.2 \times 10^{\square}$

(c) $18,000,000 = 1.8 \times 10^{\square}$

(d) $890,000,000,000 = 8.9 \times 10^{\square}$

28. Fill in the box with the correct exponent.

(a) $1,256 = 1.256 \times 10^{\square}$

(b) $80,000 = 0.8 \times 10^{\square}$

(c) $250,000,000 = 25 \times 10^{\square}$

(d) $250,000,000 = 2.5 \times 10^{\square}$

29. Write these numbers by using zeros.

(a) 5×10^6

(b) 16×10^2

(c) 5.6×10^4

(d) 4.55×10^5

(e) 8.6×10^9

SECTION 5 SIGNIFICANT DIGITS AND STANDARD NOTATION

You know from your work with decimals that 2 = 2.00, 53.6 = 53.60, 62.1 = 62.1000, and so on. In short, you can write as many zeros as you wish after the last digit on the right-hand side of the decimal point without changing the value of the number

Since these zeros have no purpose when we deal with abstract numbers, we usually do not write them.

The situation is different with numbers that communicate results of measurements. A length reported as 87 cm tells us that the reporter knows the length only to the nearest centimeter. Had she reported it as 87.0, it would mean that she knew the length to the nearest tenth of a centimeter. Similarly, if you make a rough estimate of the capacity of a bottle to the nearest tenth of a liter, you might report it as, say, 1.2 ℓ. A careful measurement might show that the capacity is 1.205 ℓ. However, if the more careful measurement shows that the capacity is 1.2 ℓ to the nearest thousandth of a liter, then the correct way of reporting it is 1.200 ℓ.

If the measurement is sufficiently precise, it is generally an accepted convention to write a zero even after the last digit beyond the decimal point. This tells the reader that the digit at that place is known to be zero.

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30. Explain the difference between 28.2 cm and 28.20 cm.

31. Suppose you measured the width of a card to the nearest tenth of a centimeter. Which of the following would be a correct statement of your results?

(a) 6 cm

(b) 6.0 cm

(c) 6.00 cm

32. Which of the following statements report measurements to the nearest tenth of a meter?

(a) 87.40 m

(b) 62.0 m

(c) 757.2 m

(d) 430 m

(e) 2.0 m

Within the convention we just described, a zero in a number such as 3.80 m tells us that we know the number to the nearest hundredth of a meter.

The situation is quite different in a number like 120 cm^2. Here the zero may mean that the number is known to the nearest square centimeter: that it is indeed 120 cm^2 and not, say, 122 cm^2. On the other hand, the zero may only indicate the place of the decimal point. In other words, its only purpose is to show that the number is known only to the nearest ten square centimeters.

In the first case, we say that the zero is <u>significant</u>; it means what it says just as any other digit does. Nonzero digits are always significant; they always mean what they say.

The situation is more ambiguous for a number such as 35,000. Are all these zeros significant? Only the first two, or perhaps none? It is clearly necessary to have a notation that removes this ambiguity. Such a notation exists and is called <u>standard notation</u>.

Standard notation is a special case of exponential notation in which the coefficient has one nonzero digit to the left of the decimal point. For example, the number 380,000 can be written in standard notation as 3.8×10^5, or 3.80×10^5, or 3.800×10^5 – depending on how many digits are significant. Note that 380,000 can also be written as 380×10^3. However, this form of exponential notation does not remove the ambiguity.

33. In each of the following numbers none of the zeros is significant. Write each number in standard notation.

(a) 5,000,000

(b) 700,000

(c) 8,500,000

(d) 3,650,000

(e) How many significant digits are there in each of the numbers in parts (a)-(d)?

34. How would you write one billion in standard notation if only one digit is significant?

35. In each of the following numbers only one zero is significant. Write each number in standard notation.

(a) 610,000

(b) 47,000

(c) 150,000

(d) 7,500

(e) 20

36. Each of the following numbers has three significant digits. Write each number in standard notation.

(a) 200,000,000

(b) 51,000,000

(c) 30,600

(d) 507,000

(e) 81,000,000

37. Write each of the following numbers in standard notation to show how many significant digits each of them has.

(a) 3,600 km (measured to the nearest km)

(b) 14,000 m (measured to the nearest hundred m)

(c) 130,000,000 km (reported to the nearest million km)

(d) $11,000,000,000 (reported to the nearest billion dollars)

38. The distance between two cities is given as 2.5×10^3 km. Express this distance in meters. Has the number of significant digits changed?

39. In the number 1.40×10^6, the "1" is in the millions place, the "4" is in the hundred thousands place, and the "0" is in the ten thousands place. Identify the place value of every digit in each of the following numbers.

(a) 3.50×10^3

(b) 4.06×10^8

(c) 9.005×10^6

40. If the population of a state is correctly written as 1.40×10^6, you know that it is precise to the nearest ten thousand persons. How precise are the following data?

(a) The population of the People's Republic of China in 1970 was 7.60×10^8.

(b) 9.66×10^6 passenger cars were produced in the United States in 1973.

(c) In 1974 there were 1.625×10^7 head of cattle in the State of Texas.

(d) In the 1972 presidential election, 7.8×10^7 people voted.

(e) In 1973 the Los Angeles Times printed 1.0×10^6 newspapers daily.

41. Turn back to Question 3 of this chapter. To how many significant digits do you think the area of Colorado is given? Use standard notation to express the area of Colorado.

42. Referring to Question 7 of this chapter, write in standard notation the number of m^3 and the number of km^3 in the lake made by the Hoover Dam. The number given has three significant digits.

43. According to the 1974 edition of the Guinness Book of World Records, the largest recorded number of consecutive sit-ups done on a hard surface without feet held down is 25,222. These were done by an eight-year-old boy in Idaho Falls in 1972.

(a) How many significant digits are indicated in this record number of sit-ups?

(b) Is there any advantage to writing this number in standard notation?

SECTION 6 MULTIPLYING AND DIVIDING NUMBERS IN EXPONENTIAL NOTATION

Light travels 3×10^8 m in one second. How far does it travel in two seconds? To answer this question we multiply 3×10^8 by 2. The product $2 \times (3 \times 10^8)$ m actually consists of two multiplications.

Since it does not matter which of the two multiplications we do first we have

$$2 \times (3 \times 10^8)\ \mathrm{m} = (2 \times 3) \times 10^8\ \mathrm{m} = 6 \times 10^8\ \mathrm{m}$$

How far does light travel in one year or in about 3×10^7 seconds? Here we multiply 3×10^8 by 3×10^7. Again, we can rearrange the factors in the product.

$$(3 \times 10^7) \times (3 \times 10^8)\ \mathrm{m} = (3 \times 3) \times (10^7 \times 10^8)\ \mathrm{m}$$

Multiplying the coefficients gives us $3 \times 3 = 9$. To see how to multiply 10^7 by 10^8 let us write out the factors of 10. The product $10^7 \times 10^8$ becomes

$$\underbrace{\overbrace{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}^{10^7} \times \overbrace{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}^{10^8}}_{10^{15}}$$

Since in the product, 10 is a factor 7+8 times we can write

$$10^7 \times 10^8 = 10^{7+8} = 10^{15}.$$

Putting everything together we have

$$(3 \times 10^7) \times (3 \times 10^8)\ \mathrm{m} = (3 \times 3) \times (10^7 \times 10^8)\ \mathrm{m} = 9 \times 10^{15}\ \mathrm{m}$$

What we did in this example can be done with any two numbers in exponential notation. To find their product we multiply their coefficients and multiply the powers of ten. The exponent of a product of powers of ten equals the sum of their individual exponents.

Here is another example:

$$(8 \times 10^4) \times (3 \times 10^5) = 24 \times 10^9$$

44. Multiply:

(a) $10^2 \times 10^6$

(b) $10^3 \times 10^3$

(c) $10^3 \times 10^6$

45. Multiply:

(a) $5 \times (17 \times 10^4)$

(b) $4 \times (56.1 \times 10^4)$

(c) $(2 \times 10^4) \times (3 \times 10^2)$

(d) $(4.8 \times 10^2) \times (6 \times 10^5)$

(e) $(8.95 \times 10^4) \times (7.2 \times 10^5)$

(f) $(5.82 \times 10^3) \times 10^2$

46. Write the following numbers in standard notation by changing the coefficient to standard notation and multiplying the powers of ten.

Example: $426 \times 10^4 = 4.26 \times 10^2 \times 10^4 = 4.26 \times 10^6$

(a) 733×10^3

(b) 80.5×10^5

(c) 520.1×10^6

47. Calculate the following products and write the answers in standard notation.

(a) $(2 \times 10^4) \times (6 \times 10^5) \times (9 \times 10^{11})$

(b) $(3.5 \times 10^5) \times (0.6 \times 10^8) \times (2.0 \times 10^{17})$

(c) $(5 \times 10^7) \times (35 \times 10^{19}) \times (4 \times 10^8)$

Suppose we want to divide 10^5 by 10^2. Just as with multiplication of powers of ten we start by writing out the factors of ten.

$$\frac{10^5}{10^2} = \frac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10} = 10 \times 10 \times 10 = 10^3$$

Note that the exponent 3 in the answer is the difference of the exponents 5 and 2 in the quotient. Let us look at another example:

$$\frac{10^6}{10^4} = \frac{10 \times 10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10 \times 10} = 10 \times 10 = 10^2$$

These examples suggest that the exponent of a quotient of powers of ten equals the difference of the exponent of the numerator and the exponent of the denominator.

We can generally use this property when dividing numbers in exponential notation.

Consider, for example, 8×10^5 divided by 2×10^2:

$$\frac{8 \times 10^5}{2 \times 10^2} = \frac{8}{2} \times \frac{10^5}{10^2} = 4 \times 10^3$$

Therefore, to divide numbers in exponential notation we find the quotient of the coefficients and the quotient of the powers of ten. The answer is the product of the resulting coefficient and power of ten.

48. Divide:

(a) $10^9 \div 10^6$

(b) $\frac{10^5}{10}$

(c) $10^7 \div 10^6$

(d) $\frac{10^8}{10^4}$

49. Write the following numbers in standard notation.

Example: $0.6 \times 10^5 = 0.6 \times 10 \times 10^4 = 6 \times 10^4$

(a) 0.8×10^6

(b) 0.05×10^8

(c) 0.32×10^2

50. Divide and write the answers in standard notation.

(a) $\frac{4 \times 10^8}{2 \times 10^5}$

(b) $(4.75 \times 10^7) \div (2.5 \times 10^3)$

(c) $(6.21 \times 10^4) \div (3 \times 10^2)$

(d) $\frac{3 \times 10^6}{1.5 \times 10^4}$

(e) $\frac{4.536 \times 10^8}{8.1 \times 10^5}$

51. What is

(a) $\frac{1}{2}$ of (7.56×10^{10})?

(b) $\frac{1}{4}$ of (6.32×10^5)?

(c) $\frac{1}{10}$ of (32.5×10^5)?

52. Evaluate the following expressions.

(a) $\dfrac{(4.2 \times 10^3) \times (5 \times 10^{10})}{3 \times 10^8}$

(b) $\dfrac{(8 \times 10^5) \times (2.5 \times 10^8)}{16 \times 10^8}$

(c) $\dfrac{(3.12 \times 10^5) \times (4.1 \times 10^3)}{4 \times 10^2}$

53. What errors were made in calculating the following products?

(a) $(2 \times 10^3) \times (3 \times 10^5) = 6 \times 10^{15}$

(b) $2 \times (8 \times 10^3) = 16 \times 10^6$

(c) $(3.1 \times 10^6) \times (2 \times 10^6) = 6.2 \times 10^6$

54. What errors were made in calculating the following quotients?

(a) $(8 \times 10^{10}) \div (4 \times 10^5) = 4 \times 10^2$

(b) $(45 \times 10^5) \div (5 \times 10^3) = 9 \times 10^8$

(c) $(32.4 \times 10^8) \div (4 \times 10^2) = 8.1 \times 10^4$

SECTION 7 SIGNIFICANT DIGITS IN PRODUCTS AND IN QUOTIENTS

Consider a rectangle of 4.7 m by 5.68 m. In this rectangle the length is known to two significant digits whereas the width is known to three significant digits. How many significant digits should be reported for the area? Specifically, how many significant digits are there in the product 4.7×5.68?

Consider another question, this one involving division. The distance from the earth to the sun is 1.49×10^{11} m; light travels 3.00×10^{8} m in one second (sec). How long does it take light from the sun to reach the earth? To answer the problem we calculate the quotient $\frac{1.49 \times 10^{11}}{3.00 \times 10^{8}}$. Here the distance from the earth to the sun and the distance light travels in one second are given to three significant digits. To how many significant digits should we carry out the division?

No calculations can produce results that are more accurate than the data used in the calculations. For multiplication and division, it is good to remember a simple rule of thumb: The number of significant digits in the product or the quotient of two numbers is not more than the number of significant digits in the less accurate of the two numbers.

The less accurate number in the first question has only two significant digits. Therefore, the result should be reported to only two significant digits. That is, the area of the rectangle is 27 m^2 instead of 26.696 m^2.

In the second question both numbers are given to three significant digits. Therefore, the time it takes sunlight to reach the earth is 4.97×10^2 sec and not 4.96666×10^2 sec.

In practice we usually carry out divisions to one more digit than we need and then we round off. For multiplication, on the other hand, we do the entire multiplication before rounding.

55. In 1972, 8.8 million passenger cars were sold in the United States. Assume the average length of a passenger car to be 5 meters, and calculate the length of a caravan made up of all these new cars. If the caravan started at your city, to what other city would it extend?

56. In the film 2001, a spaceship traveled from our moon to one of the moons of Jupiter. Let us suppose this distance is about 2×10^9 km and that the ship could travel 3×10^4 km each hour. How much time will it take the spaceship to make this journey?

57. The total number of checks that pass through the Federal Reserve banks each day is 4×10^6, and their total value is $\$4.2 \times 10^9$. What is the average dollar value of each check?

58. MacDonald's claims to have sold 15 billion hamburgers over a twenty-five-year period.

(a) If every person in your school ate 3 hamburgers a day, how many hamburgers would be eaten in a year?

(b) How many years would it take for them to eat 15 billion hamburgers?

59. Measured in number of books, the largest university library in the United States is at Harvard University. It has 9×10^6 books. About how many meters of shelving do you think are in the library?

60. The diameter of the earth is about 1.3×10^7 m.

(a) Calculate the volume of a cube-shaped box in which the earth could fit.

(b) The actual volume of the earth is about one-half that of the box that could fit around it. What is the approximate volume of the earth in m^3?

(c) The volume of the sun is 1.3×10^6 times the volume of the earth. What is the approximate volume of the sun in m^3?

4. SIGNED NUMBERS

SECTION 1 NUMBERS LESS THAN ZERO

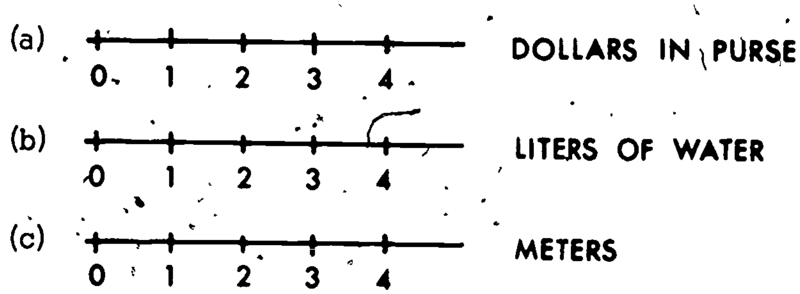
[?]

1. For each box in the following sentences decide what numbers you can fill in that make sense.
   (a) Jennifer has ☐ cats.
   (b) Bruno went to the movies ☐ times.
   (c) There are ☐ windows in the room.

2. For the boxes in the sentences below what numbers can be filled in that you think make sense? Are there any numbers that make sense in these sentences that would not make sense in any of the sentences in Question 1?
   (a) Maria has ☐ dollars in her purse.
   (b) The bucket contains ☐ liters of water.
   (c) The car moved a distance of ☐ meters.

All the boxes in the sentences in the preceding questions have one property in common: The smallest number that makes sense in any of the boxes is zero. If we put the numbers on a line that make sense in the boxes in Question 2, they will all be on one side of the zero (Figure 1).

Figure 1



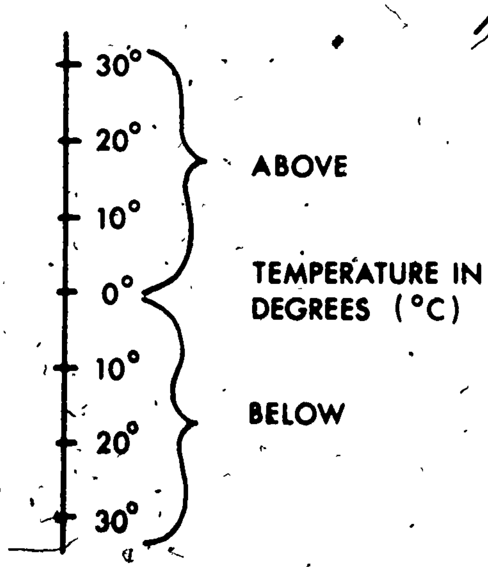
There are, however, many real situations in which numbers on <u>both</u> sides of zero make sense. One example is illustrated in Figure 2. As we all know, a temperature may be 10°C <u>above</u> zero or 12°C <u>below</u> zero. No matter where we are, the temperature rises as we move up the temperature line. For example, we can see that

$$20^{\circ} \text{ above zero} < 30^{\circ} \text{ above zero}$$

and

$$0^{\circ} < 10^{\circ} \text{ above},$$

Figure 2



Similarly, as the temperature drops we see on the temperature line that

$$10^\circ \text{ below} < 0^\circ$$

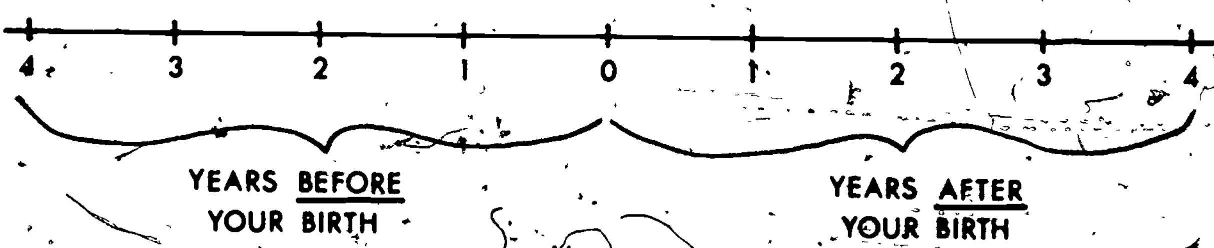
and

$$20^\circ \text{ below} < 10^\circ \text{ below}$$

(Indeed, we all know that it is colder when the thermometer reads 20° below than when it reads 10° below.)

Now we will consider another example of a real situation in which numbers make sense on both sides of the zero. Assume that you have a sister who was born 4 years <u>before</u> you and a brother who was born 3 years <u>after</u> you. If you want to put the numbers on a line with the zero corresponding to the year you were born, the line would extend on both sides of the zero. See Figure 3.

Figure 3



You can also see on this years line that a person born 1 year after you was born <u>before</u> the person born 2 years after you; that is,

$$1 \text{ year after} < 2 \text{ years after}$$

Similarly, someone born 3 years before you was born <u>before</u> the person born 2 years before you; that is,

$$3 \text{ years before} < 2 \text{ years before}$$

On any number line representing time, an event that occurred later than another is always to the right of an event that occurred earlier.

In mathematics, which we use in all kinds of situations, it is customary to call the numbers on the number line to the right of zero <u>positive</u> and those to the left of zero <u>negative</u>. When a number line is drawn vertically the positive numbers are placed above zero and the negative numbers below zero.

We will be denoting a positive number by a + sign and a negative number by a - sign on the left side of the number. +6 is read as "positive six," and -1.5 is read as "negative one and five-tenths." Positive and negative numbers together are called <u>signed numbers</u>.

3. Suggest some other examples of numbers on both sides of zero that state real situations.

4. Represent the following facts on the same number line, or, in this case, time axis.

   (a) The Korean war started 9 years after the United States entered World War II.

   (b) World War II started 2 years before the United States entered it.

   (c) World War I ended 23 years before the United States entered World War II.

   (d) The United States sent troops to Vietnam 20 years after the United States entered World War II.

5. On a number line plot the following milestones in the history of aviation and space travel. Consider the date of the first human space traveler to be zero.

1927	Charles Lindbergh makes first solo airplane flight across the Atlantic Ocean.
1957	First man-made satellite, Sputnik I, is launched.
1958	First U.S. satellite is launched.
1961	First human space traveler (Yuri Gagarin).
1962	First American orbits earth (John Glenn).
1963	First woman space traveler (Valentina Tereshkova).
1965	Man "walks in space."
1966	First soft landing is made on moon's surface by unmanned spacecraft.
1967	American astronauts are killed on launch pad by fire.
1969	First man walks on the moon (Neil Armstrong).

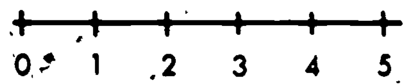
6. The elevations of cities and towns are expressed in meters above or below sea level. Plot the following data on an altitude line.

City or Location	Elevation in Meters
Boise	820
Death Valley	-86
Denver	1610
Jericho	-400
Jerusalem	760
Kansas City	230
Mexico City	2240
New York	17
Philadelphia	30
Tiberias	-210

SECTION 2 - SIGNED NUMBERS AND VECTORS

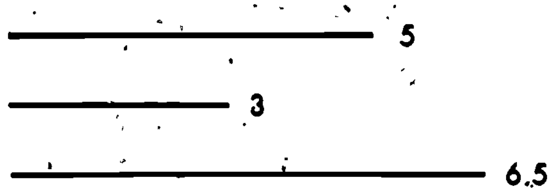
When we deal only with natural (unsigned) numbers, we can use the line segment between the number and zero to represent the number itself. For example, in Figure 4 the segment between 0 and 4 can stand for the number 4.

Figure 4



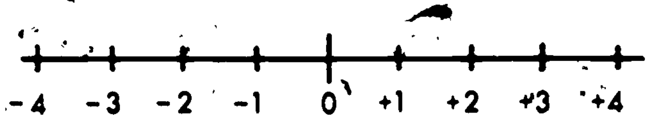
As shown in Figure 5, we can also use such segments to represent the numbers without the number line.

Figure 5



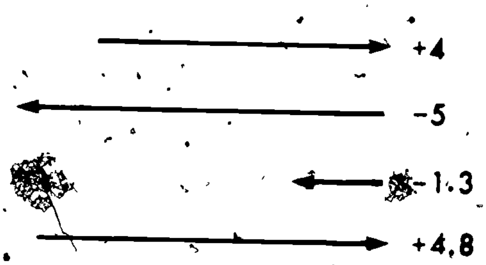
However, when we extend the number system to include both positive and negative numbers, we can no longer use a segment for a simple number. The reason is that the length of the segment, say, between 0 and +3 is the same as the distance between 0 and -3 (see Figure 6).

Figure 6



To distinguish between +3 and -3, we can turn the segments into arrows. Arrows pointing to the right will stand for positive numbers, and arrows pointing to the left will stand for negative numbers. The length of the arrows will represent the size or the <u>magnitude</u> of the number (see Figure 7).

Figure 7



A quantity that has a magnitude and a direction is called a <u>vector</u>. Vectors have many applications, but our immediate interest is in using them to represent signed numbers.

?

7. Refer to the number line in Figure 6 to draw vectors representing the following numbers.

   (a) -1.5

   (b) +5.2

   (c) -0.8

   (d) -3.9

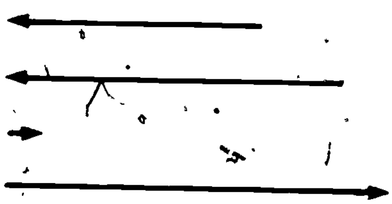
8. In the following vectors, 1 cm stands for 1 unit. What signed numbers do these vectors describe?

   (a)

   (b)

   (c)

   (d)



SECTION 3 ADDING SIGNED NUMBERS

How can we describe the addition of two positive numbers in terms of their corresponding vectors? Let us look at an example:

$$(+3) + (+2) = (+5)$$

In this case we put the signed numbers inside parentheses to separate them from the addition symbol. To do the problem using corresponding vectors we can draw the following diagram (Figure 8).

$$\begin{array}{r} (+3) \\ +\ \underline{(+2)} \\ (+5) \end{array}$$

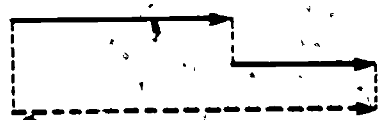


Figure 8

A translation of Figure 8 into words reads as follows:

Draw the vectors that describe the two numbers you want to add tail to head. The vector that starts at the tail of the first and ends at the head of the second describes the sum of the two numbers.

This method will give the right answer for any two positive numbers you choose so long as you draw the vectors to scale. Another example is shown in Figure 9.

$$\begin{array}{r} (+3.0 \times 10^4) \\ +\ \underline{(+1.5 \times 10^4)} \\ (+4.5 \times 10^4) \end{array}$$

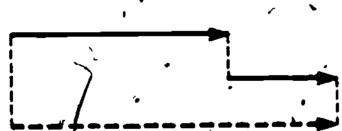
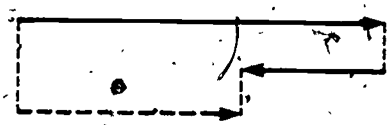


Figure 9

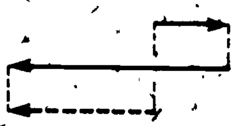
We will be defining the addition of signed numbers, whether positive or negative, by the tail-to-head addition of their corresponding vectors. Here are some examples in which one or both numbers are negative (Figure 10).

$$+\begin{array}{r}(+5)\\ \underline{(-2)}\\ ?\end{array}$$



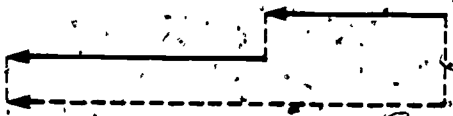
therefore $+\begin{array}{r}(+5)\\ \underline{(-2)}\\ (+3)\end{array}$

$$+\begin{array}{r}(+1)\\ \underline{(-3)}\\ ?\end{array}$$



therefore $+\begin{array}{r}(+1)\\ \underline{(-3)}\\ (-2)\end{array}$

$$+\begin{array}{r}(-2.5)\\ \underline{(-3.5)}\\ ?\end{array}$$



therefore $+\begin{array}{r}(-2.5)\\ \underline{(-3.5)}\\ (-6.0)\end{array}$

Figure 10

9. Use vectors to add the following signed numbers.

(a) $+\begin{array}{r}(+6)\\ \underline{(-4)}\end{array}$

(b) $+\begin{array}{r}(-5)\\ \underline{(-7)}\end{array}$

(c) $+\begin{array}{r}(+5)\\ \underline{(-7)}\end{array}$

(d) $+\begin{array}{r}(+2.5)\\ \underline{(-4.6)}\end{array}$

(e) $+\begin{array}{r}(-3.7)\\ \underline{(+4.2)}\end{array}$

10. Find the following sums.

(a) + $\begin{array}{r} -2.8 \\ \underline{-4.0} \end{array}$

(b) + $\begin{array}{r} +4.0 \\ \underline{-2.8} \end{array}$

(c) + $\begin{array}{r} +3.0 \text{ cm} \\ \underline{+1.7 \text{ cm}} \end{array}$

(d) + $\begin{array}{r} +3.0 \text{ m} \\ \underline{-1.7 \text{ m}} \end{array}$

(e) + $\begin{array}{r} -2.6 \text{ km} \\ \underline{-1.0 \text{ km}} \end{array}$

(f) + $\begin{array}{r} -1.7 \text{ m} \\ \underline{+5.0 \text{ m}} \end{array}$

11. Choosing suitable scales for your vectors, find the following sums.

(a) + $\begin{array}{r} +1.3 \times 10^4 \\ \underline{+3.5 \times 10^4} \end{array}$

(b) + $\begin{array}{r} -2.7 \times 10^3 \\ \underline{-7.6 \times 10^3} \end{array}$

(c) + $\begin{array}{r} -3.1 \times 10^6 \\ \underline{+5.5 \times 10^6} \end{array}$

(d) + $\begin{array}{r} -5.2 \times 10^5 \\ \underline{+3.7 \times 10^5} \end{array}$

Adding signed numbers as vectors is really what you do when you use a calculator. Entering the first number with its sign corresponds to drawing the first vector. Pushing the "+" key corresponds to saying that the next vector will be placed tail to head. Entering the second number corresponds to drawing the second vector. Finally, pushing the "=" key corresponds to measuring the length and noting the direction of the final vector.

On a calculator a number without a sign always stands for a positive number; that is, +3 = 3. Negative numbers always carry the minus sign. So that a negative number can be entered into the calculator, the calculator must have a key called "change sign" or, as on some calculators, a key labeled +/-.

If a calculator is available to you, use it to do Questions 9-11.

From the foregoing questions you may have already concluded that you can add signed numbers without drawing vectors. However, the vectors can help you remember the following simple rules that you use when adding the numbers on paper or in your head.

1. When you add two numbers of the same sign, you add their magnitudes. The sign of the sum is the same as the sign of the two numbers.
2. When you add two numbers of different signs, you subtract the number of the smaller magnitude from the number of the larger magnitude. The sign of the sum is the same as the sign of the number of the larger magnitude.

12. By sketching or imagining the vectors that correspond to each pair of signed numbers, determine which number is closest to their sum.

(a) $(+46.7) + (-155.8)$ is about $\begin{cases} +200 \\ -100 \\ +110 \end{cases}$

(b) $(-0.65) + (-1.50)$ is about $\begin{cases} -2.0 \\ +2.1 \\ -0.9 \end{cases}$

(c) $(-612) + (+2258)$ is about $\begin{cases} +2900 \\ +1600 \\ -1700 \end{cases}$

13. Find the following sums. Use sketches when helpful.

(a) $+\begin{array}{r} -2.5 \\ \underline{-1.7} \end{array}$

(b) $+\begin{array}{r} +4.3 \\ \underline{-1.2} \end{array}$

(c) $+\begin{array}{r} -4\frac{8}{10} \\ \underline{+2\frac{3}{10}} \end{array}$

(d) $+\begin{array}{r} -3\frac{1}{2} \\ \underline{-4\frac{3}{10}} \end{array}$

(e) $+\begin{array}{r} -2.3 \\ \underline{+3.7} \end{array}$

14. Find the following sums.

(a) $(+3.5) + (-4.6) + (-7.9)$

(b) $(-7.2 \times 10^5) + (+6.3 \times 10^5) + (-11.2 \times 10^5)$

(c) $(-0.52) + (-1.60) + (+3.56)$

SECTION 4 TOTAL AND AVERAGE

Suppose that a person made the following weekly deposits in his savings account: \$5.00, \$4.50, \$6.50, and \$3.00. What was his average deposit over the four-week period? To find the answer you would first calculate the total deposit by adding all the deposits:

$$\text{Total deposit} = 5.00 + 4.50 + 6.50 + 3.00 = \$19.00$$

Then you would divide by four:

$$\text{Average deposit} = \frac{\$19.00}{4} = \$4.75$$

This way of finding an average deposit will also work if during some weeks the person withdrew money. We simply consider the withdrawal to be a negative deposit. For example, a withdrawal of \$8.00 would be considered as a deposit of \$(-8.00). This way the rule for finding an average deposit would be the same as before: First find the total deposit by adding all the weekly deposits, and then divide by the number of weeks to find the average.

For example, suppose that during a five-week period a person made the following deposits:

(-8.00), (+16.00), (+4.00), 0, (-10.00)

Then, the average weekly deposit is

$$\frac{(-8.00) + (+16.00) + (+4.00) + 0 + (-10.00)}{5} = \$0.40$$

This method of finding averages works for all signed numbers whether they describe deposits, temperatures, or other quantities. For example, the average of 20, -10, 30, 40, -60 is

$$\frac{20 + (-10) + 30 + 40 + (-60)}{5} = 4$$

?

15. The temperature in Ski City was reported to be 28°F, 12°F, -3°F, 9°F, -11°F, 12°F, and 0°F at 2 p.m. on each day of a certain week. What was the average temperature in Ski City for 2 p.m. that week?

16. During a football game, Jim Nameless carries the ball and gains the following yardage:

    +22, -3, -8, +12, +44, +2, +5, +13, -2, -17

    In this list losses are listed as negative gains.

    (a) What was Jim's total gain?

    (b) What was Jim's average gain?

17. A tuning dial on a radio can be turned all the way around a number of times both clockwise (+) and counterclockwise (-). The dial is turned according to the following sequence: $+1\frac{1}{2}$ turns, $-\frac{3}{4}$ turn, +2 turns, $-1\frac{1}{4}$ turns, $-\frac{3}{4}$ turn, $+\frac{1}{4}$ turn, and $-2\frac{1}{4}$ turns.

(a) What is the sum of these signed numbers?

(b) What is the meaning of the sum?

18. What are the following sums?

(a) (0.100000) + (0.010000) + (0.001000) + (0.000100) + (0.000010) + (0.000001)

(b) (0.100000) + (-0.010000) + (0.001000) + (-0.000100) + (0.000010) + (-0.000001)

19. A group of students estimated a certain length to the nearest 0.1 cm. They later measured the length and compared their result with their guesses. If they estimated too high, the errors were recorded as positive. Errors on the low side were recorded as negative. The data (all given in cm) were as follows:

Estimating Errors

+1.2	-0.3	+0.1	+0.6	0
+0.3	-0.7	0	+0.3	-0.1

(a) What was the average error?

(b) What was the average of the errors that were too high?

(c) What was the average of the errors that were too low?

20. (a) Write down five numbers.

(b) Find their average.

(c) Find how far above or below the average each of the five numbers is.

(d) Now average the signed numbers you found in (c).

(e) Do parts (a) through (d) for another set of five numbers. Is your answer to part (d) the same as before? Can you think of a reason why?

SECTION 5. MULTIPLYING AND DIVIDING SIGNED NUMBERS

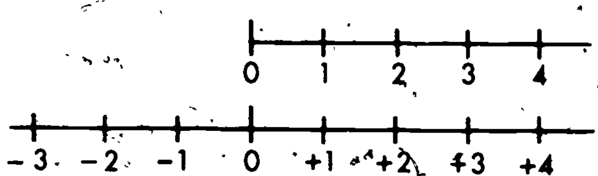
What should be the sign of the product of two signed numbers? To answer this question we will look to our experience with natural numbers for ideas.

When we extended the number line to include negative numbers, we decided to put the positive numbers where the natural numbers were (Figure 11).

Figure 11

Original number line

Extended number line



As we know, the product of two natural numbers is a natural number. Therefore it makes sense to define the product of two positive numbers as a positive number: For example, $(+2) \times (+3) = +6$. In practice, we hardly ever bother to write the + signs; we simply take them for granted.

What can we learn from natural numbers about a product of a positive number and a negative number, say, $(+3) \times (-4)$? We can look at the product of the natural numbers 3×4 as a repeated addition:

$$3 \times 4 = 4 + 4 + 4 = 12$$

And so it makes sense to define the product $(+3) \times (-4)$ as

$$(-4) + (-4) + (-4) = -12$$

Thus

$$(+3) \times (-4) = -12$$

The sign of the product of two numbers should not depend on whether the numbers are integers or not. Therefore, the product of any positive and negative numbers should be negative. For example,

$$\left(+\frac{1}{2}\right) \times \left(-\frac{3}{4}\right) = -\frac{3}{8}$$

$$(+1.2) \times (-0.60) = -0.72$$

21. For natural numbers the order in which two numbers are multiplied does not affect the result. This is known as the commutative law. For example, $2 \times 3 = 3 \times 2$. If we want this law to hold for signed numbers, what must be the sign of the product $(-2) \times (+3)$?

22. Multiply the following.

(a) $(+4) \times (-7)$

(b) $(-5) \times (+6)$

(c) $(+2 \times 10^3) \times (-4 \times 10^6)$

(d) $(+0.03) \times (+0.4)$

(e) $(+0.7) \times (-8.1 \times 10^4)$

23. Find the following products.

(a) $\left(+\frac{2}{3}\right) \times \left(+\frac{1}{4}\right)$

(b) $\left(+\frac{3}{10}\right) \times \left(-\frac{2}{10}\right)$

(c) $\left(-\frac{7}{8}\right) \times \left(+\frac{1}{2}\right)$

(d) $\left(+1\frac{1}{3}\right) \times \left(-2\frac{1}{2}\right)$

(e) $\left(+\frac{9}{4}\right) \times \left(-\frac{2}{5}\right)$

24. Calculate the following.

(a) $(+30) \times \left(+\frac{1}{6}\right)$

(b) $(+42) \times \left(-\frac{1}{7}\right)$

(c) $(-50) \times \left(+\frac{1}{2}\right)$

(d) $(+63) \times \left(-\frac{1}{9}\right)$

25. For natural numbers we can always replace division by a suitable multiplication: For example, $5 \div 4 = 5 \times \frac{1}{4}$. If we want signed numbers also to have this property, what must be the sign of the quotient of a positive number and a negative number?

26. Find the following quotients.

(a) $(+27) \div (+3)$

(b) $(+28) \div (-7)$

(c) $(-0.066) \div (+0.11)$

(d) $(+3 \times 10^7) \div (-2 \times 10^3)$

(e) $\frac{(-42 \times 10^8)}{(+6 \times 10^2)}$

(f) $\frac{(-150)}{(+0.3)}$

27. Find the average of each of the following sets of numbers.

(a) +6, -7, -15, +8, -21

(b) +5.2, -0.12, +1.05, -15.2, -6.2

(c) -7, +5, 0, 6, 3, -8, -12, 0, 0, 9

28. Suggest a quick way of finding the average of each of the following sets of numbers.

   (a) -5.2, -5.2, -5.2, +3.6, +3.6, +3.6, +3.6, -1.4, -1.4, -1.4

   (b) +0.068, +0.068, +0.068, -0.41, -0.41, 0, 0, 0, -0.25, -0.25

E. So far we have established the following rules for multiplying signed numbers:

> The product of two positive numbers is a positive number.
>
> The product of a positive number and a negative number is a negative number.

What should be the sign of the product of two negative numbers? To answer this question, we shall again look at what we know about natural numbers.

You can verify that $2 \times (3 + 5) = (2 \times 3) + (2 \times 5)$. This is an example of the <u>distributive law</u>. The law holds true for natural numbers, and it makes sense that it should also apply to signed numbers. For example,

$$(+2) \times \Big((+5) + (-3)\Big) \text{ should equal } \Big((+2) \times (+5)\Big) + \Big((+2) \times (-3)\Big)$$

Let us check if this is indeed true:

$$(+2) \times \Big((+5) + (-3)\Big) = (+2) \times (+2) = +4$$

and

$$\Big((+2) \times (+5)\Big) + \Big((+2) \times (-3)\Big) = (+10) + (-6) = +4$$

Our example illustrates that when we follow the two rules to multiply signed numbers, the distributive law holds true.

Now let us consider the product

$$(-2) \times \Big((+5) + (-3)\Big) = (-2) \times (+2) = -4$$

Applying the distributive law to this case, we get

$$-4 = \Big((-2) \times (+5)\Big) + \Big((-2) \times (-3)\Big)$$

We know that

$$(-2) \times (+5) = -10$$

Therefore,

$$-4 = -10 + \Big((-2) \times (-3)\Big)$$

This is only possible if

$$(-2) \times (-3) = +6$$

You can repeat this reasoning with any three signed numbers you choose. The product of two negative numbers has to be positive, if the distributive law is to hold true.

Caution: Do not make your own rule that says simply "two minuses make a plus." This is true for products. The sum of two negative numbers is indeed negative.

29. Calculate the following.

(a) $(-4) \times (-5)$

(b) $(-4) \times \left(-\frac{1}{3}\right)$

(c) $(-4) + (-5)$

(d) $\left(-\frac{1}{3}\right) + \left(-\frac{2}{3}\right)$

30. Find the following products.

(a) $(-24) \times \left(+\frac{1}{4}\right)$

(b) $(-35) \times \left(-\frac{1}{7}\right)$

(c) $(-2.3) \times (-5)$

(d) $(-0.04) \times (+0.3)$

31. Suggest a reason why the quotient of two negative numbers should be a positive number.

32. Do the following calculations.

(a) $\frac{-25}{-50}$

(b) $\frac{-0.06}{+0.3}$

(c) $\frac{+0.56}{-70}$

(d) $\frac{60}{0.2}$

(e) $\frac{-500}{0.25}$

(f) $\frac{-70}{-1400}$

SECTION 6 SUBTRACTING SIGNED NUMBERS

In Section 3 we learned to describe the addition of signed numbers in terms of their corresponding vectors. We shall now follow a similar method to learn how to subtract signed numbers.

Let us start with two positive numbers (Figure 12).

Figure 12

A translation of Figure 12 into words reads as follows:

Draw the vectors that describe the numbers <u>tail to tail</u>. The vector that starts at the <u>head</u> of the second vector and ends at the <u>head</u> of the first vector describes the difference of the two numbers.

Here is another example (Figure 13).

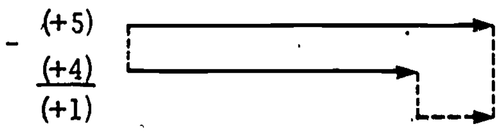


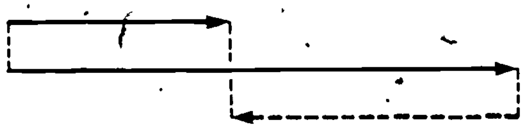
Figure 13

Another way to look at Figure 13 is to note that the answer vector (+1) is what you have to add to the second vector (+4) to get the first (+5).

We will now define the subtraction of any signed number by the tail-to-tail subtraction of their corresponding vectors. Consider an example in which both numbers are positive but the second one is larger than the first (Figure 14)

Figure 14

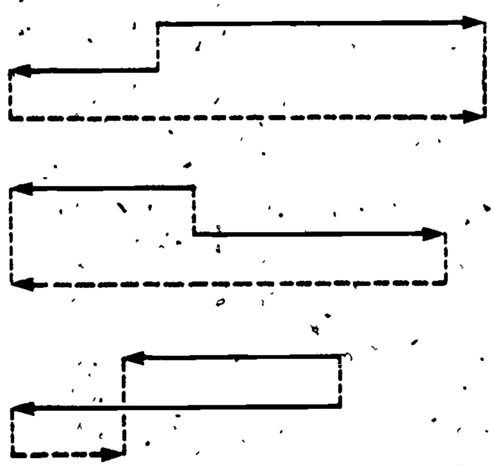
$$-\ \begin{array}{c}(+3)\\ \underline{(+7)}\\ ?\end{array}$$



$$\text{therefore}\quad -\ \begin{array}{c}(+3)\\ \underline{(+7)}\\ (-4)\end{array}$$

Figure 15 illustrates the subtractions in which at least one of the numbers is negative.

Figure 15



$$-\ \begin{array}{c}(+4.5)\\ \underline{(-2.0)}\\ ?\end{array}\qquad \text{therefore}\quad -\ \begin{array}{c}(+4.5)\\ \underline{(-2.0)}\\ (+6.5)\end{array}$$

$$-\ \begin{array}{c}(-2.5)\\ \underline{(+3.5)}\\ ?\end{array}\qquad \text{therefore}\quad -\ \begin{array}{c}(-2.5)\\ \underline{(+3.5)}\\ (-6.0)\end{array}$$

$$-\ \begin{array}{c}(-3.0)\\ \underline{(-4.5)}\\ ?\end{array}\qquad \text{therefore}\quad -\ \begin{array}{c}(-3.0)\\ \underline{(-4.5)}\\ (+1.5)\end{array}$$

33. Draw vectors to subtract the following signed numbers.

(a) $-\begin{array}{r}(+2)\\ \underline{(-6)}\end{array}$

(b) $-\begin{array}{r}(-4)\\ \underline{(+5)}\end{array}$

(c) $-\begin{array}{r}(+4)\\ \underline{(+7)}\end{array}$

(d) $-\begin{array}{r}(-6)\\ \underline{(-3)}\end{array}$

(e) $-\begin{array}{r}(-7)\\ \underline{(+2)}\end{array}$

34. Find the following differences.

(a) $-\begin{array}{r}(+2.5)\\ \underline{(-4.0)}\end{array}$

(b) $-\begin{array}{r}(-3.2)\\ \underline{(-6.6)}\end{array}$

(c) $-\begin{array}{r}(-5.3)\\ \underline{(+7.8)}\end{array}$

(d) $-\begin{array}{r}(-6.5)\\ \underline{(-7.2)}\end{array}$

(e) $-\begin{array}{r}(+1.8)\\ \underline{(+5.3)}\end{array}$

35. Choosing suitable scales for your vectors, find the following differences.

(a) $\begin{array}{r} (+5 \times 10^3) \\ - \ \underline{(+2 \times 10^3)} \end{array}$

(b) $\begin{array}{r} (-6 \times 10^7) \\ - \ \underline{(-8 \times 10^7)} \end{array}$

(c) $\begin{array}{r} (-4 \times 10^4) \\ - \ \underline{(+8 \times 10^4)} \end{array}$

(d) $\begin{array}{r} (+6.5 \times 10^9) \\ - \ \underline{(+7.0 \times 10^9)} \end{array}$

Just as in the case of addition, you can subtract signed numbers without drawing vectors to scale. Nevertheless, a vector sketch can help you apply the following simple rules when subtracting signed numbers in your head.

1. When you subtract two numbers of the same sign you subtract their magnitudes. The sign of the result can be seen from a sample vector sketch.
2. When you subtract two numbers of opposite signs you add their magnitudes. Again, the sign of the answer can be seen from a rough sketch.

36. By sketching or imagining the vectors that correspond to each pair of signed numbers, determine which number is closest to their difference.

(a) $(-25.6) - (+150.8)$ is about $\begin{cases} +125 \\ -175 \\ -125 \end{cases}$

(b) $(+0.85) - (+2.2)$ is about $\begin{cases} +1.4 \\ -1.4 \\ -3.1 \end{cases}$

(c) $(+521) - (-350)$ is about $\begin{cases} +900 \\ -200 \\ +200 \end{cases}$

37. Find the following differences.

(a) $(-1.5) - (-2.6)$

(b) $(+7.6) - (-1.9)$

(c) $\left(+8\frac{1}{2}\right) - (+12)$

(d) $(-25) - (+54)$

(e) $(+0.05) - (+0.007)$

(f) $(-0.6) - (-1.8)$

Describing signed numbers by vectors helped us in two ways: (1) to define the addition and subtraction of signed numbers, and (2) to check whether an answer is reasonable.

However, once you have had enough practice, you may want to add and subtract signed numbers without drawing vectors. In this case, the following rules are worth remembering.

Adding two signed numbers

If the signs are the same, add the magnitudes.

If the signs are different, subtract the magnitudes.

In each case, the sign of the sum is the sign of the number that has the larger magnitude.

Subtracting two signed numbers

If the signs are the same,
subtract the magnitudes.

If the signs are different,
add the magnitudes.

If the number that has the larger magnitude is first, the sign of the difference is the sign of the number with the larger magnitude.

If the number that has the larger magnitude is second, the sign of the difference is the opposite of the sign of the number with the larger magnitude.

38. Choose numbers and use vector sketches to illustrate the rules for each of the following.

(a) addition of two numbers that have the same signs

(b) addition of two numbers that have opposite signs

(c) subtraction of two numbers that have the same signs

(d) subtraction of two numbers that have opposite signs

39. Follow the rules to do the problems below.

(a) $-\begin{array}{r}+5\\ \underline{+3}\end{array}$ (d) $+\begin{array}{r}+3\\ \underline{-5}\end{array}$

(b) $+\begin{array}{r}-5\\ \underline{-3}\end{array}$ (e) $-\begin{array}{r}-7\\ \underline{+6}\end{array}$

(c) $-\begin{array}{r}-3\\ \underline{+5}\end{array}$ (f) $-\begin{array}{r}-7\\ \underline{-6}\end{array}$

40. (a) $(-4.7) + (+6.8)$

(b) $(-3.6) - (+5.7)$

(c) $(-1.8) + (-6.9)$

(d) $(+5.3) + (-1.4)$

(e) $(+8.3) - (+11.7)$

SECTION 7 CHANGE AND PERCENT CHANGE

41. The height of a child was 132 cm on January first and 138 cm on June first of the same year. By how much did the child's height change during the five months?

42. (a) The price of a clock radio was first \$32.95 and then \$49.95. What was the change in the price of the radio?

(b) The price of a calculator was first \$18.30 and then \$15.75. By how much did the price of the calculator change?

The question "by how much has something changed" is answered by subtracting the earlier value from the later value. With signed numbers the difference between the later and earlier values also makes sense, no matter what numbers are used. Here are some additional examples:

> Mildred's weight was first 129 pounds and then 124 pounds. The change in her weight was $124 - 129 = -5$ lbs.

Joshua's weight was first 124 pounds and then 129 pounds. His weight changed by 129 - 124 = +5 lbs.

In every case we take the later value and subtract the earlier from it.

[?]

43. The table below lists 10 a.m. and 2 p.m. temperatures in degrees Farenheit on the same day at different locations. By how much did the temperature change at each place?

Location	Temperature in °F	
	10 a.m.	2 p.m.
Los Angeles, California	45	53
Little Rock, Arkansas	15	8
Bismarck, North Dakota	-15	-12
Albany, New York	- 8	-13
Cincinnati, Ohio	- 5	+ 7

44. The following chart shows the batting average of Joe Whiff at the end of each month last season.

April	.253	August	.242
May	.271	September	.257
June	.275	October	.281
July	.269		

(a) Represent this data in a bar graph.

(b) During which month did Joe's average improve the most? By how much did it improve?

(c) During which month did Joe suffer the worst slump? How big was his slump?

You often hear or see statements such as "The price of milk went up 12%" or "Radios 20% off." Such statements also express change, although they do not tell us by how many dollars the price changed. They give the percent change — that is, the ratio of the change in price to the old price, expressed in hundredths.

The statement about milk means

$$\frac{\text{Change in price}}{\text{Old price}} = +0.12 = 12\%$$

Suppose that the old price were 84¢, then

$$\begin{aligned}\text{Change in price} &= +12\% \text{ of } 84¢ \\ &= +0.12 \times 84¢ \\ &= 10¢\end{aligned}$$

And the new price becomes

$$\begin{aligned}\text{New price} &= \text{old price} + \text{change} \\ &= 84¢ + 10¢ \\ &= 94¢\end{aligned}$$

When there is a drop in price, the change is negative. The statement about the radio says

$$\frac{\text{Change in price}}{\text{Old price}} = -0.20 = -20\%$$

Suppose the old price were 18.00 dollars. Then

$$\begin{aligned}\text{Change in price} &= -20\% \text{ of } 18.00 \text{ dollars}\\ &= -0.20 \times 18.00 \text{ dollars}\\ &= -3.60 \text{ dollars.}\end{aligned}$$

Again,

$$\begin{aligned}\text{New price} &= \text{old price} + \text{change}\\ &= 18.00 \text{ dollars} + (-3.60 \text{ dollars})\\ &= 14.40 \text{ dollars.}\end{aligned}$$

45. Because of increased prices, the cost of a $25.00 radio went up 15%.

    (a) What was the change in the cost of the radio?

    (b) What was the cost of the radio after the increase?

46. Martha noticed a pocket calculator listed at $17.88 that was on sale at 30% off.

    (a) What was the percent change in price?

    (b) What was the change in price?

    (c) What was the sale price of the calculator?

47. A decorative candle listed at $4.90 is on sale at 20% off.

    (a) Find the price change.

    (b) Find the sale price.