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ABSTRACT

This document was developed to provide some perspectives on the use of case studies and other clinical approaches in mathematics education. A large portion of the paper contrasts several research strategies, discussing in detail some of the procedures and results of three distinctive types of clinical studies. The perspectives and the premises of Erlwanger, Piaget, and several Soviet studies are considered in relation to the techniques and outcomes which result from their work. Then the uses of clinical research methods and results in the classroom are discussed. Finally, references which readers may find useful in additional exploration of clinical approaches are cited. (MS)

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MATHEMATICS EDUCATION REPORTS

ON CLINICAL STUDIES IN MATHEMATICS EDUCATION

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Mathematics Education Reports

Mathematics Education Reports are being developed to disseminate information concerning mathematics education documents analyzed at the ERIC Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education. Reports in each of these categories may also be targeted for specific sub-populations of the mathematics education community. Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, the Conference Board of the Mathematical Sciences, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed by the editor.

At a meeting in April 1976, the Advisory Board of the journal Investigations in Mathematics Education recommended that a document be developed which would provide some perspectives on the use of case studies and other clinical approaches in mathematics education. The ERIC Center for Science, Mathematics, and Environmental Education agreed to sponsor the preparation and publication of such a document. Professor Easley was contacted--and the result is in your hands.

A large portion of this paper contrasts several research strategies, discussing in some detail the procedures -- and some of the results -- of three distinctive types of studies. The perspectives and the premises of Erlwanger, Piaget, and some Soviet studies are considered in relation to the techniques and outcomes which result from their work. Then the uses of clinical research in the classroom are discussed--both the use of research results and of research methods.

Finally, Professor Easley cites references which readers may find useful in further exploring clinical studies. The result is a document which is unique in providing information to consider as alternative patterns of research are sought.

ERIC/SMEAC is pleased to make this publication available to mathematics educators and others interested in clinical approaches.

Marilyn N. Suydam
Editor

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ON CLINICAL STUDIES IN MATHEMATICS EDUCATION*

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In recent years there has been an upsurge of case studies of individual pupils in mathematics programs and an increased interest in Piaget's clinical studies of children's thinking and other types of high-inference investigations. Such studies appear to violate the canons of research traditionally advocated by specialists in measurement and statistics, and many persons report they don't know how to evaluate clinical studies or how to use them in advancing their own understanding of the field. This monograph will discuss general conceptual and methodological issues that appear to block adequate evaluation and use of clinical studies and will illustrate these issues with detailed discussion of representatives of three major kinds or schools of clinical studies in mathematics education. The last two sections will discuss uses of clinical studies in mathematics teaching and introduce a general bibliography of studies known to this reviewer.

Contrasting Research Strategies

In a recent paper (Campbell, 1975), Donald T. Campbell reverses the position he took against individual case studies in earlier papers such as Campbell and Stanley (1966), Campbell (1961), and Raser, Campbell, and Chadwick (1970). Then, he had argued that studies in which a single group or an individual is studied only once "have such a total absence of control as to be of almost no scientific value" (Campbell and Stanley, 1966). Now he concedes that case studies may indeed have scientific value and that he had "overlooked a major source of discipline." An alert social scientist engaged in a clinical study generates dozens of predictions and expectations that are tested by his or her observations, and he or she is unlikely to retain a theory or interpretation unless most of the predictions it generates are supported by a large number of key data. In this way, then, a case study (or a clinical study) is seen as more like a pattern-matching task rather than as focusing on a single prediction and observational testing, as in most conventional experimental research. It is clear also that practitioners of case study methods tend to use the data gathered to help them shape a new theory or select among general theories rather than to test a theory chosen in advance. They seek to discover the natural processes occurring rather than to determine the distribution of an association or a process in a population.

Campbell (1975) suggests that case studies should tabulate those hypothetical consequences of the theories held in advance that were

*The author is indebted to many of his collaborators and students considerably beyond the sources indicated. While it is impossible to list them all, Beth Dawson, Bob Davis, Bob Stake and Marilyn Suydam deserve special attention.

supported or refuted, and the reasons for the step-wise changes that were made in theory during the study. Stake (1976), advocates that a case study should be written in such a way as to carry the reader vicariously through the experiences that the case writer had. Since case study writers are often paying less attention to the common theories than the emerging picture, this could mean that the readers have to keep their own counts of predictions supporting or refuting their own modifications of their own developing theory. What is clear from the literature on case study methodology (and confirmed by my own experience in training case study researchers) is that a skeptical and detached attitude on the part of the investigator usually results in rather dramatic changes in his or her preconceptions and produces an awareness of preconceptions he or she was not aware of having. When a case or clinical study seems to confirm preconceived ideas, I then tend to become suspicious that the study was superficially done and that very few consequences of those ideas were actually tested against the data generated in the study. So I think that Campbell's suggestions are good ones in the sense that they would help the inexperienced investigator follow the significant advances in a case study and distinguish between those observations that make advances and those that do not. This does not preclude also following Stake's advice in doing the writeup. In fact, it should help alert readers to make "tables" of their own preconceptions.

Whether a study simply tests hypotheses that were already formulated and seriously entertained, or undertakes to develop a substantial advance in theory, however, is not the only issue that divides the two contrasting strategies of research. A second major issue is that of generalizability of results. Experimentalists feel that they can generalize their findings from an experiment to the population as a whole because they have drawn an adequate random sample from the population about which a hypothesis speaks.* Clinical researchers feel that they can generalize from a study of a single case to some other individual cases because they have seen a given phenomenon in one situation in sufficient detail and know its essential workings to be able to recognize it when they encounter it in another situation.** On the conventional strategy for promoting generalizability, methodological canons are also being challenged. It should be recalled that the theory of inferential statistics gives us the probability that, in a series of experiments with a random variable, the result obtained could have arisen by chance alone. It does not speak to uncontrolled observations encountered in the classroom or in daily life. However, Guttman (1953) contended that sampling theory alone does not solve the problems of prediction and external validity, even within a series of random experiments. This led Cronbach et al. (1972) to develop a theory of generalizability that resolves some of the problems of this sort. However, those authors did not directly address generalizability beyond the conditions of a research design, although they do point to ways of estimating the accuracy of possible generalizations. A major contribution of their work is to distinguish the different domains from which a set of

*This presupposes some distribution law--typically a normal distribution of construct variability in the population.

**This presupposes that phenomena occur in natural types each of which is based on an underlying structure.

observations may be considered to have been drawn and the probability values that attach to inferences from the data collected to these different domains. (For example, reliability of observational instruments cannot be defined apart from a given domain.) Later Cronbach (1975) recommended something approaching the more serious case study when he wrote (after reviewing the methodological problems of aptitude-treatment interaction studies):

> Instead of making generalization the ruling consideration in our research, I suggest what we reverse our priorities: An observer collecting data in one particular situation is in a position to appraise a practice or proposition in that setting, observing effects in context. In trying to describe and account for what happened, he will give attention to whatever variables were controlled, but he will give equally careful attention to uncontrolled conditions, to personal characteristics, and to events that occurred during treatment and measurement. As he goes from situation to situation, his first task is to describe and interpret the effect anew in each locale, perhaps taking into account factors unique to that locale or series of events.

Cronbach's description of the research strategy he now recommends is reminiscent of a statement by McLean and Stanley (1964) that "experimentation and analysis have more in common with an exciting treasure hunt than with a cut-and-dried mathematics exercise." Both statements point to the possibility of seeing common processes underlying the various kinds of research reporting, and the realization that the advancement of understanding is a human enterprise which involves creative processes beyond the capacity of automated data processors alone.

Unfortunately, due to the traditions involved in reporting both experimental and case or clinical studies, it is the exciting generation of ideas and revision of expectations that is usually left out and which readers have to supply for themselves. The positivistic movement in philosophy, in quest of objectivity, attempted to rid science of intuitive and human elements, but it is fading in the light of the realization that a purely mechanical process stifles the very creative impulses needed for the advancement of knowledge. Having read of experiments demonstrating so-called Hawthorne and Pygmalion effects, we may sometimes fear our all-too-human tendencies toward self-fulfilling prophecies and want to put on the double blinders of the rigorous test of a new drug. But we also have the option of opening our eyes wider and observing ourselves as expectant participants in and expectant observers of instructional situations. This is the more promising effect of clinical studies, for blinders generally eliminate far more helpful information than the misinformation they clean up.

For example, subjects sometimes try to say what they think the interviewer wants to hear. An experimentalist tries to disguise his or her intent. The clinical interviewer can ask more questions. Serious efforts are made by Piaget, for instance, to discriminate between answers children may give "at random," answers given to please the interviewer, and answers based on convictions of various kinds, involving various degrees of spontaneity, etc. (Piaget, 1929). Clearly, in interpreting a clinical interview protocol, not every datum is to be taken as just as valuable as every other, and other data must be brought to bear to make choices among competing hypotheses (Easley, 1974). In dealing with such problems, clinical or case studies again make possible a more direct and more intricate connection between theory and data than does the experimental method with its formal, decision-theory apparatus. It is well to note that, in the natural sciences, the term "experiment" is not usually defined so narrowly as in the books on experimental design, and reports on research do not follow such rigid canons as are common in experimental psychology and education. (See Driessen and Derbyshine, 1966, and Walberg, 1966.) In education, it might seem that theoretical training and background is less needed to read research reports than in the natural sciences. To read most experimental reports, however, a knowledge of statistics is required. To read clinical or case studies, prior statistical training seems to be little demanded. This may contribute to the feeling that they are less scientific. However, as I shall attempt to show in considering three distinctive types of studies, appropriate kinds of theoretical knowledge are needed to make sense of each type.

Each type has a theoretical or practical perspective from which the clinical researcher starts, and an attempt will be made to point out what kind of changes in perspective the researcher is forced to make by the events of the clinical study themselves. It is regrettable, as Campbell points out, that few writers of case or clinical studies keep such a tally for the reader, although a reader who is alert for such changes will often find evidence of them. However, following Stake's suggestion that the audience of the case study needs to be able to relive the experiences of the investigator to some extent, we urge readers of the next three sections to keep their own lists of any assumptions which might be challenged by the kind of clinical data discussed. However, the real test would be to read the original studies. It should be emphasized that clinical studies have a powerful contribution to make to professional knowledge; however, the contribution does not lie in summaries or conclusions but in communicating new ways of seeing through thoughtful reading.

In the first two examples, Erlwanger's study of Benny's conception of arithmetic, and Piaget and his colleagues' study of the conservation of continuous quantity of liquids or plastic substance, the aim is a kind of common-sense epistemology--the view that in order to communicate a certain sense to someone a well-chosen set of examples is required. Many teachers have long known from practice that conventional ideas about mathematical knowledge are suspect, and it is only the research community in mathematics education that tends to keep them alive. In the third example, from recent Soviet studies of mathematics learning and teaching, it will appear that epistemology is of little interest and that the practices of the best teachers are what are being tested. Where they fail, imaginative remedies are sought, but no general assumptions about learning are either being tested or developed. Only after some familiarity with examples of such major types of clinical studies can we address the question: Of what use are these studies to teachers or teacher educators in mathematics?

In writing this monograph, I am responding to an historical phenomenon--the growth in attention given to particular studies--and I cannot defend all clinical methods nor recommend them in the abstract to practical educators. After years of trying to teach clinical observation and interviewing I can only make this general prediction: A few readers will find something of value in what others have found; somewhat more readers will find clinical methods useful if they carry them out themselves, oriented to problems of personal interest; but many more are likely to find that such studies only reveal individual variations in learning of no general value and of little interest. A few readers will discover that they open the doors to understanding the human mind. Proceed at your own risk!

Erlwanger's Studies of Children's Conceptions of Mathematics

Stanley Erlwanger's doctoral dissertation (1974) contains six case studies of children's conceptions of arithmetic which are being published serially in the Journal of Children's Mathematical Behavior by the Curriculum Laboratory of the University of Illinois at Urbana-Champaign. These studies exemplify a growing number of other studies of children's arithmetic in that journal, in the Arithmetic Teacher, and in other publications. They are more directly related to school curricula than Piaget's studies and more detailed in documentation than the Soviet studies--the two other major types to be considered in this monograph. Erlwanger's studies have also attracted considerable attention and generated at least two different interpretations, as, for example, can be seen by comparing my own editorial notes in Erlwanger (1973) with those of Davis in Erlwanger (1975). Davis assumed that "mathematical understanding" means "correct understanding" according to current views (not, for example, prehistorical counting, numerology, the Pythagorean theory of rationals, or even the Newtonian theory of infinitesimals), and attempts to use artificial intelligence procedures as models of thought. I accepted the child's statements carefully listened to as the authority for what his understanding is and looked for natural biological processes to account for them. So one should expect each to choose a different response to this remark by Wittgenstein:

> It is clear that we can make use of a mathematical work for a study in anthropology. But then one thing is not clear:--whether we ought to say: 'This writing shows us how operating with signs was done among these people,' or 'This writing shows us what parts of mathematics these people had mastered (Wittgenstein, 1956, p. 97e).

We cannot hope to resolve such controversy here, for each reader brings his own preconceptions of the nature of mathematics and of ways of improving mathematics education. Erlwanger notes a similar contrast in pointing out that recent reforms in mathematics education have been "directed at increasing children's understanding of mathematics," and that these reforms emphasized unifying ideas by introducing certain new content and refining symbolism. Then he cites Brownell (1944a), Buswell (1949), and Ginsburg (1972b) in support of the alternative goal of discovering how arithmetic is understood from the child's point of view. Presumably, both of these theoretical perspectives are opposed to the behaviorists' view that arithmetic is a set of responses correlated with a set of stimuli, which seems to characterize the Individually Prescribed Instruction (IPI) mathematics program which Erlwanger's subjects were studying in school.

In his introductory chapter and in the first case study (Benny), Erlwanger declares his allegiance with Buswell's view that meanings in arithmetic are what children think and not necessarily what adults think.

Benny, a 12-year-old, had recently completed several of the IPI exercises on decimal fractions. While he admitted that he usually didn't get them right on the first try, he claimed that, after having his answers checked, he could figure out what was wanted on a particular page. (His teacher supported this claim, indicating that he was a hard worker and one of the best mathematics students in the fifth grade.) In explaining his work on one problem, Benny says:

> Wait, I'll show you something. This is (has?) a key. If I ever get this one (i.e., 2 + .8)...actually, if I put 2 8/10, I get it wrong. Now down here, if I had this example (i.e., 2 + 8/10), and I put 1.0, I get it wrong. But really they're the same, no matter what the key says.

and later,

> If I did 2 + .3, that will give me a decimal; that will be .5. If I did it in pictures (i.e., physical models) that will give me 2.3. If I did it in fractions like this (i.e., 2 + 3/10), that will give me 2 3/10. (Erlwanger, 1973)

So he knows the answer that we consider the right answer, another one we don't accept, and still another answer that we might accept but the key doesn't. Since his task is to put down the one answer that the key accepts, he feels he has to explore all possibilities, and since the key accepts some answers that he doesn't understand physically, he doesn't take the physical way as a final authority. It reminds me a little of these queries of Wittgenstein's about whether we agree on what agreement means:

> Would it be possible that people should go through one of our calculations today and be satisfied with the conclusions, but tomorrow want to draw quite different conclusions, and other ones again on another day?
>
> Why isn't it imaginable that it should regularly happen like that: that when we make this transition one time, the next time, 'just for that reason,' we make a different one, and therefore (say) the next time the first one again?... It might be called a need for variety...Are our laws of inference eternal and immutable?' (Wittgenstein, 1956, p. 45e).

Now many mathematics educators have argued that once a child has understood the physical basis of an operation, he would always be able to perform it in a meaningful way. We see that, while Benny is not alternating between what he calls the decimal way, the physical way, and the fractions way, as Wittgenstein's remarks might fancifully suggest, he is not consistently using just one way, which is the main point of Wittgenstein's questioning the assumption that consistency is only natural. In fact, for Benny's situation, in which the key arbitrarily results in his getting marked wrong for responses which make sense to him, the more different ways he has to try, the more appropriate becomes a trial-and-error strategy. This he makes very clear in his perception of what is necessary.

This pattern, which is amply supported throughout the case study, also eliminates another common assumption made in teaching arithmetic--that right answers represent correct understanding and wrong answers represent either misunderstanding (if frequent--for IPI, if more than 20% of the test questions of a given type) or carelessness (if infrequent). We see that Benny's three ways of interpreting decimal fractions could result in his getting a page 100% wrong by any arithmetical standards (2 + .8 = 1.0), or 100% right (by what he calls "in pictures"), or 100% wrong-by-the-key (2 + .8 = 2 8/10) but right-by-the-definition-of-decimal-fractions. One might think he could more easily find the pattern that answers right-by-physical-models trivially reduce to answers that are right-by-the-key. However, it is clear that he has not found that generalization--after all, what is "trivial"? One is ultimately led to the conclusion that the different styles of notation (e.g., .8 vs. 8/10) are not the only thing he pays attention to. Clearly, he also pays attention to meanings (e.g., he knows that .8 means 8/10 although he also knows that only one of such equivalences match the key). So, one must entertain the hypothesis that, sometimes, Benny responds to questions in terms of meanings (physical pictures or equivalences) and sometimes in terms of the patterns of the symbols. So we reject the hypothesis that symbols are always meaningless or always meaningful, and that may be the key to why he cannot discover the pattern that physical meanings ("in pictures") always give right answers. Clearly, the other ways do not always give wrong answers. Recall that such different ways are described by Benny in the discussion of a given problem. They appear to enter his mind briefly while facing a given problem and not to function consistently as exclusive ways of thinking for a whole set of problems or for a week at a time. We can also reject, on the evidence of Benny's testimony, the common view that children usually know when they don't understand how to do a problem. Note that the way we state that assumption implies the awareness of a single, canonical procedure for doing a problem. Benny clearly cannot tell whether he knows the right way to do a problem until he gets his paper checked. It might be said, in contrast, that Benny feels more or less the same about different ways of dealing with a problem. From his perspective, not ours, understanding how to do a problem is having a way to proceed come to mind. In Piagetian terms, this is assimilating (with or without accommodation) the problem to a general schema of some sort which simultaneously directs the procedure and gives a sense of recognition or of understanding.

Piaget's Clinical Studies of the Concepts of Number and Physical Quantity

Having seen in the last section how the interpretations of Erlwanger's case studies are related to fundamental ideas about the nature of mathematical knowledge and learning, it should come as no surprise that Piaget's clinical studies actually have the development of such epistemological ideas as their goal. Piaget calls this thrust of his work "genetic epistemology" because he is developing a theory of the origins of mathematical and scientific knowledge, which relates the history of Western science to the development of quantitative and logical ideas in the individual (see Easley, 1977).

His theory is more intricate by far than the more familiar psychological theories of learning by conditioning. For example, in reconstructing the detailed processes and sequences of children's thought, he makes use of fundamental modern mathematical concepts (like transformation groups or Boolean Algebra, instead of the numerical mathematics of probability or measurements.) Educators and psychologists who attempt to operationalize Piaget's stages of intellectual development miss the main point. As Leahy (1974) points out, Piaget and Szeminska's book, The Child's Conception of Number (1952), contains a far more detailed quasi-mathematical discussion of the way in which children develop concepts of continuous and discontinuous quantities than has been considered, or even understood, by most Anglo-American psychologists. The problem is in part that few people are prepared for the way in which Piaget uses modern mathematical concepts to model children's early sensory-motor development and their later development of such traditional mathematical concepts as number and continuous quantities (like amount of liquid to drink, weight, length, area, and volume). In part, the problem is also that Piaget's theory of knowledge is non-empirical. That is, he does not accept the prevalent view that knowledge arises from the imprinting of information from the environment on the organism's highly plastic storage system. Instead, he believes that the "reflexes" which organize the neo-nate's sensory-motor system selectively assimilate aspects of the environment to which they can respond, and thereby provide a functional interpretation for these aspects. In the process of the interaction of the organism with its environment, new syntheses, enlargements, and refinements of these structures are produced. In short, the organism constructs a knowledge of its environment from its own prior organizations, and can only know what it can assimilate to its own organizing structures. This radical constructivism faces numerous questions: Why is the pathway of intellectual development so similar in individuals who are raised in very different environments? Why is the newly emergent organizing structure in a child not immediately available for use in the whole range of situations to which most adults readily apply it? Why do many striking examples of intuitive concepts that Piaget describes in young children resemble concepts which have only recently been recognized as powerful tools by mathematicians and scientists; e.g., the nonpermanent object, the displacement group, conservation of momentum and energy, a concept of speed not equivalent to the ratio of distance and time, and, with somewhat older children, atoms, action, and spatial concepts of gravity? Are these merely read into the behavior of children because of their importance in Western thought*, or do they represent part of a universal repository of concepts which adults can draw on and formulate in their specialized constructions?

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*If so, how does that differ from ethologists reading into various postures of animal behavior humanoid concepts such as "threat," "appeasement," "begging," "displacement," and on a longer time scale the "bonding" of a pair of birds? From the new science of ethology, we may learn that the question is not whether it is "read in" but whether doing so makes a useful theory, which raises implicitly another problem, namely how the theory can be used.

Mathematics educators, one might think, would be extremely interested in a theory that uses concepts of "new math" to explain how children, independently of any schooling, construct concepts of "old math." But no one seems to have taken the effort to work out a detailed exposition in conventional mathematical form of more than a few fragments of the theory Piaget proposes, and few have even tried to discuss the implications of its mathematical basis for understanding the nature of language, mathematics, and culture. As Papert suggested, the problem may lie very much in the novel uses to which Piaget puts logical and other mathematical concepts. Also, his theory conflicts strongly with the prevailing view that children's mathematical concepts are ideas which must have been taught to them or which they must have "caught" from their cultural environment, rather than ideas which they developed themselves by interacting mainly with their physical environment (Papert, 1963).

Among the best known of those who attempt to summarize Piaget's theory and criticize it is Jerome Bruner. His attempt to express the theory is representative of the difficulty educators and psychologists have with Piaget's theory. He says, for example:

> The achievement of this stage (conservation) permits the child to perform additional operations, operations conceived by Piaget to be organized in the form of mathematical groupings. One of these groupings involves the multiplying of relations. One such form of the multiplying of relations is called compensation. That is, the child when confronted with a tall, thin beaker and a short, fat one filled with liquid to a lower level supposedly multiplies 'greater height' by 'lesser width' and comes out with 'equal quantity' (Bruner, 1966, pp. 184-185).

In order to compare this statement with the original theory of Piaget and Szeminska and also to pave the way for considering more recent developments in the theory, we need to explore briefly the mathematical ideas involved. The term "groupings" refers to structures Piaget defined which bear some relation to a group, as defined in modern Algebra. They are not, however, closed structures, for operations on the elements of the group are limited to "neighboring" elements rather than defined across the entire set of elements. As Witz (1969) showed, grouping I is a semi-lattice and establishes a partial-order relation on classes of objects. Another grouping partially orders relations among objects and another orders two relations at once, giving rise to what Piaget calls "multiplication of relations."

The term "multiplying of relations" is an important process in much of Piaget's conceptualizing of children's thinking, but Bruner seems to be saying that the multiplication of relations can take the form of a compensation which permits the child to compute the equality of a quantity of liquid before and after the transformations of its shape by pouring it into a taller, narrower cylinder. Piaget and Szeminska, however, are quite explicit in their statement that this is impossible. It is not that they are looking for a logically sufficient argument for conservation, such as might be had from a formula for the volume of a cylinder or from an axiom of the incompressability of liquid. On the contrary, they have found that a metrical concept of volume does not appear until children reach the age of about 11 or 12, and they are trying to explain why, at around 6 years of age, most children acquire an unshakable belief in the conservation of the amount of liquid (e.g., amount to drink or amount of substance itself). This appears to be a nonempirical concept, in the narrow sense that such children couldn't have generalized from experiments with weight (sensed kinesthetically or even by means of scales) since at this age they still don't believe in the conservation of weight, or length, or any other relevant "property" except possibly the number of discrete objects. Instead of a logical proof or an empirical generalization, what Piaget and Szeminska are seeking to explain is the development of the system of concepts which have the psychological effect for the child of making the conservation of substance over transformations of shape a self-evident, necessary principle. In order to have this function, as Witz (1969) points out, it is necessary to connect the mathematical structures with the situation in which the clinical interview is conducted. In particular, we must demonstrate the application of these concepts to the conditions of the task in which the conservation of liquids (or other continuous substance) is investigated. We shall take, however, an approach to this task that should be more familiar to mathematics educators than Piaget's own.

Although relations of various kinds (numerical, logical, functional, geometrical, etc., as well as familial, political, and other kinds of social relations) were familiar objects of discourse for hundreds if not thousands of years, in the early 1920s, Norbert Weiner, later the founder of cybernetics, proposed a new definition of "relation" in terms of the concepts of set theory. His definition, that a binary relation is any set of ordered pairs of elements which are all members of a given set, proved to be exceedingly useful to mathematicians in various branches of the subject, fitting in well with the growing interest in both logic and set theory following the impressive work of such men as Cantor, Boole, Hilbert, and Frege. One important distinction that emerges from this definition is the distinction between the subset of the given set consisting of all first components of the ordered pairs belonging to a relation and the subset of the second components. Starting from this definition, relations may be divided into four types based on the type of correspondence between the set of first components (domain) and the set of second components (range) of the ordered pairs in a relation. Some relations have a one-to-one correspondence, e.g., "present monogamous husband of," which maps the domain of monogamously married women into the range of monogamously married men, or "integer successor of," which maps the domain consisting of all integers into itself. (Domain and range are identical here.) Some relations are many-to-one, such as "square of", which maps the domain of all integers into the range of square integers. (Thus (-2,+4) and (+2,+4) both belong to this relation.) These first two types, together, constitute the kind of relations called functions. Other relations are one-to-many, such as "offspring of", which maps the domain of parental couples into the range of people. The fourth kind of relation consists of many-to-many relations, such as "brother of" or "sister of" and "greater than" or "smaller than."

One special kind of many-to-many relations, consisting of all the sets of ordered pairs that can be formed between a first set taken as the domain and a second set taken as the range, is called the "Cartesian product" of the two sets. For example, the Cartesian product of {1,2,3} and {4,5,6} is {(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)} (see Figure 1).

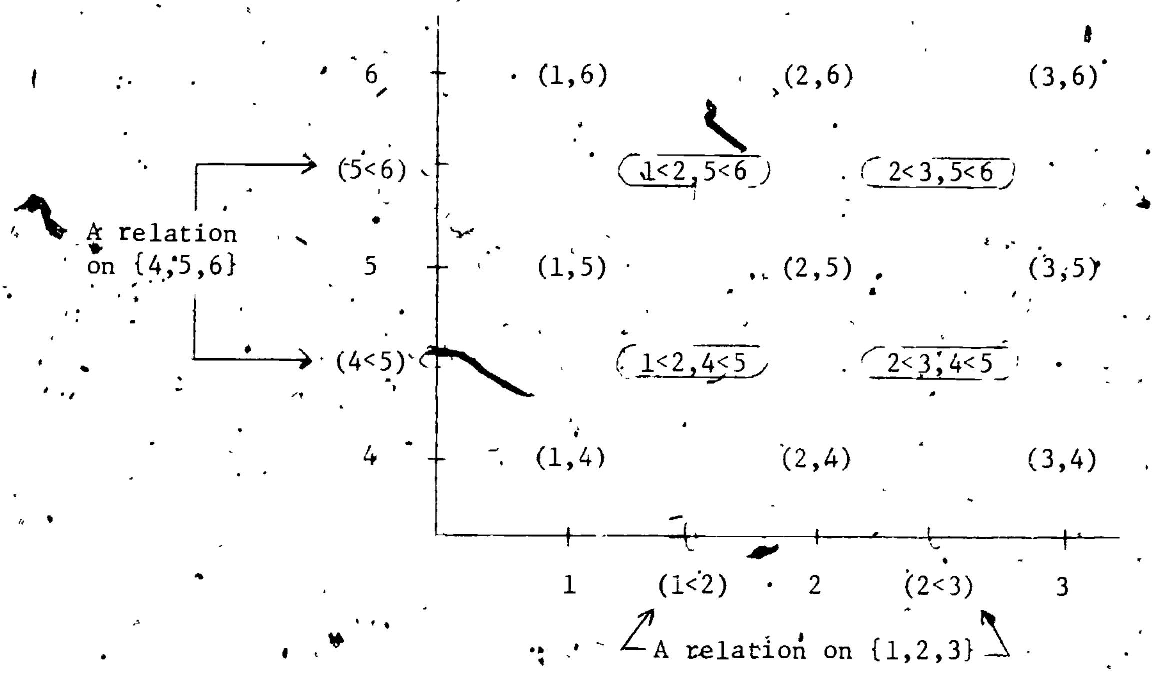

Figure 1. The Cartesian product of two sets {1,2,3} x {4,5,6} and (in the ovals) the Cartesian product of two asymmetrical relations defined on those sets.

The Cartesian product of two sets of ordered pairs is a set of ordered pairs whose components are all the ordered pairs formed by choosing first members from the first set of ordered pairs and second members from the second set, i.e., it is a set of ordered pairs of ordered pairs. This is what the Cartesian product of two relations is, since a relation is a set of ordered pairs. Does this concept adequately represent the result of what Piaget and Szeminska and others from Geneva refer to as "multiplication of relations"? With Weiner's definition, it cannot be otherwise. However, Weiner belongs to the positivistic tradition which Piaget and his colleagues roundly reject.

The Genevans refer to "logical multiplication" as a general operation* and

*Piaget and Szeminska (1952, p. 244) define logical multiplication as expressing "the fact that two or more attributes are considered simultaneously." Inhelder and Piaget (1964, p. 178) explain that such a "simple multiplication" is abstracted from "complete multiplication", i.e., complete cross-classification.

"multiplication of relations" as a specialized form of the former. If we choose to represent by a Cartesian product the result of "logical multiplication," we should note that logical multiplication is not commutative since for any two sets A and B, where A is not identical with B, A x B is not the same as B x A, for the order of all the ordered pairs in the product would be reversed. The reader can also convince himself or herself that forming Cartesian products of nonidentical sets is not associative either. Why then would mathematicians use for this set a term like "product", and why would Piaget use for the operation a term like "multiplication"? The answer is that, if cardinal numbers are defined as Russell did, by a process of grouping together all sets whose elements can be placed in 1-1 correspondence (thus defining the number two, for example, as the set of all doubles), then the product of any two cardinal numbers A and B is the cardinal number C of the Cartesian product of two sets whose cardinal numbers are A and B. While Piaget and Szeminska reject Russell's definition of cardinal number, they seem to accept the idea of multiplication of sets which Russell's definition employs (they call it "logical multiplication").

Notice how the concepts of "logical multiplication" and "Cartesian product" differ from the common definition of a "product" in many elementary school textbooks as the sum resulting from so many repetitions of the addition of a given number to itself and of "multiplication" as repeated addition. The idea of repeated addition enables children to fill in the multiplication tables and check results of the multiplication algorithm by another method, but it is of no help in understanding, for example, how areas and volumes are found by multiplication, or how velocity multiplied by time yields distance traveled. (It is surely from applications of multiplication like these that a "magical" view of arithmetic frequently arises and that children become unable to decide, in a word problem, which operation to perform on which pair of numbers.) With set models of line segments, one can construct by logical multiplication set models of areas, volumes, and other physical quantities, and even make meaningful the multiplication by transcendental numbers like pi. According to Piaget, however, the abstract theory needed to spell out these conceptions explicitly is implicitly developed by all children in their out-of-school thinking about their environment. The problem is that they are too often unable to connect their intuitively developed notions of quantity with the explicit arithmetic of school.

Where Piaget explicitly breaks off with Russell and the extreme formalism of much logical theory of the foundations of mathematics is not in terms of the structures and concepts of sets and logic but with the linguistic bases of those ideas proposed by formalist and logicist philosophers of mathematics. He says that logical forms do not arise from language but from the coordination of actions, and "the essential notions which characterize modern mathematics are much closer to the structures of 'natural' thought than are the concepts used in traditional mathematics" (Piaget, 1975).

For example, he argues that the way in which actions at the sensory-motor level are coordinated by assimilation schemes has a relation with the laws of the logic of sets. "Two schemes can be coordinated or disassociated (union). One can be partially nested in the other (inclusion), or only have a part in common with the other (intersection); ... and once a scheme imposes a goal on an action it is contradictory [exclusion] for the subject to go in the opposite direction" (Piaget, 1975). He not only argues that the more contemporary concepts of logic have their basis in the psychology of young children's action, but that it is through the progressive organization of action schemes that such logical notions as these are strengthened and combined in various ways to form the concepts of traditional mathematics. Number then is not a primitive concept that is easily understood, or easily related to counting. It is a complex synthesis of a number of separate but more fundamental ideas. Furthermore, the concepts of continuous quantity of liquid, amount of a plastic material, length, weight, and volume of solid objects also develop in a systematic order from, and by the natural functioning of, the same fundamental logico-mathematico operations which, in turn, were ultimately derived from the primitive logic of the interaction of assimilation schemes.

Piaget (1967), in response to Bruner (1966), as we can now understand, strongly objected to any identification of the multiplication of relations with conservation of quantity. But Piaget and Szeminska (1952) had made their objection explicit long before. They wrote then:

> It is obvious, however, that even if the operation of logical multiplication of relations were carried through by the child of this stage (nonconservation) it would not suffice for the construction of conservation of the whole quantity unless the height and width were simply permutated. A column of liquid whose height increases and whose width diminishes with respect to another column may be greater, equal, or less in volume than the other (p. 16).

In the light of Piaget and Inhelder (1974) and Piaget, Inhelder, and Szeminska (1960), in which the concept of volume is sharply distinguished from amount of substance, I suspect that this reference to "volume" is one the authors would now like to change. With this distinction in mind, and recalling that there are no instruments or procedures with which to measure "amount of substance," we may be puzzled by the next two sentences following the two just quoted (Piaget and Szeminska, 1952):

> In order to be certain that there is equality, the intensive graduation must be completed by an extensive quantification, i.e., it must be possible to establish a true proportion, and not merely a qualitative correlation, between the gain in height and the loss in width. In other words, there must be partition of some kind to supplement the coordination (p. 16).

How is extensive quantification with true proportion possible for a child who has no way of assigning numbers to amounts of substance?

From what we have seen earlier, the multiplication of relations bears no structural resemblance to a proportional (non-linear) compensation between the metrical quantities, height and width. Such compensation would be represented by continuous graphs (relations of infinite cardinality) shown in Figure 2.

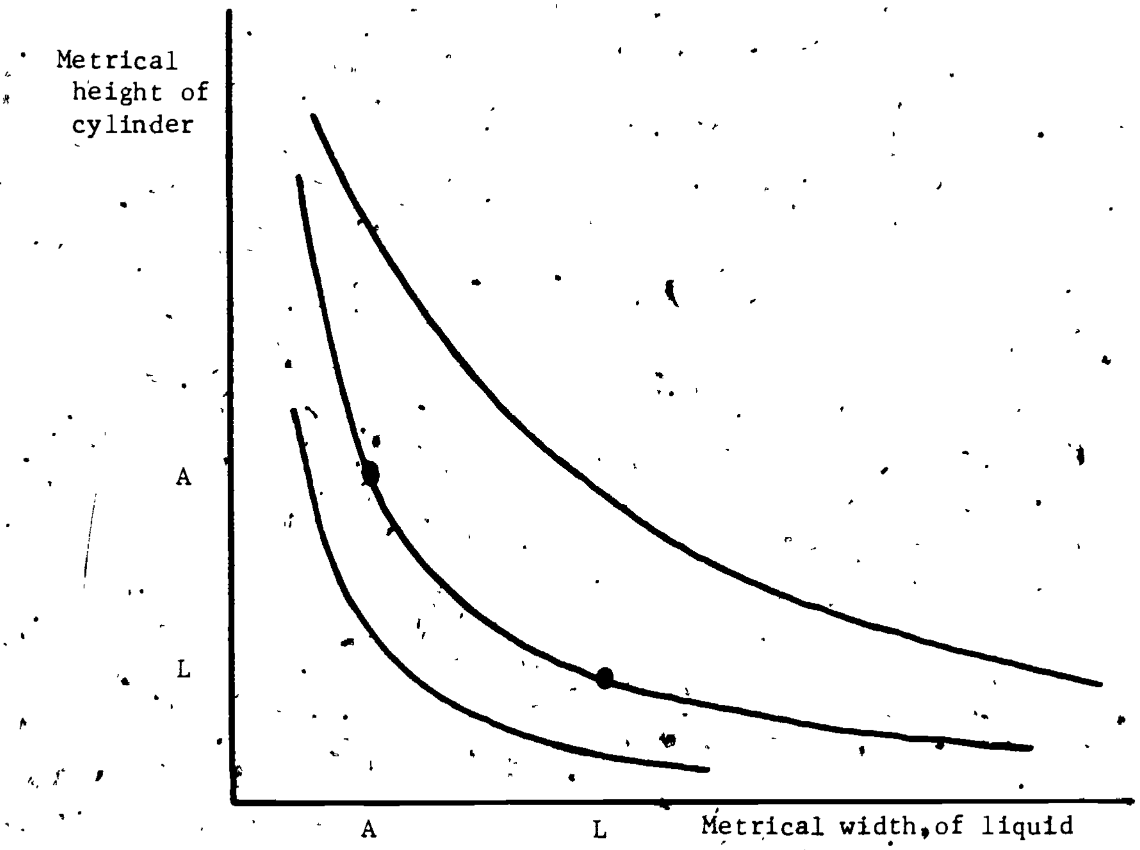

Figure 2. Lines of equal amount of liquid in cylindrical containers.

Such a graph depends on measurements of length, although it does not presuppose a measurement of volume, even though the formula for the volume of a cylinder might be the most familiar way to derive it for us who know it. However, conservation of length does not appear until about age 9, so the mystery still remains of how true proportion between length and width can be conceived by the child before he has achieved conservation of length.

We need an example from Piaget's clinical interview reports that would show us how children use the more modern logico-mathematical concepts they have developed themselves. Here is an excerpt from an interview with Edi, aged 6 years and 4 months (Piaget and Szeminska, 1952, pp. 15-16):

(Glass A was 1/5 filled--see Figure 3)

1. Int.: Pour as much orangeade into this one (1) as there is there (A).

2. Edi: (Filled L to the same level as that in A.)

3. Int.: Is there the same amount to drink?

4. Edi: Yes.

5. Int.: Exactly the same?

6. Edi: No.

7. Int.: Why not?

8. Edi: That one (A) is bigger.

9. Int.: What must you do to have the same amount?

10. Edi: Put some more in (filling L).

11. Int.: Is that right?

12. Edi: No.

13. Int.: Who has more?

14. Edi: Me (pouring some back). No, the other one has more (A). (He continued to add more and then pour some back, without reaching a satisfactory conclusion).

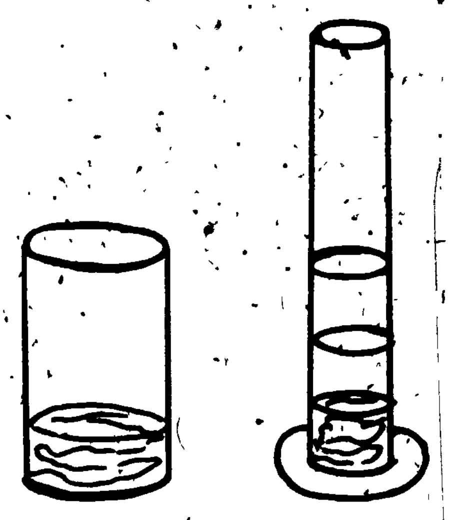

Figure 3. Two vessels used in investigating continuous quantity.

Of Edi's responses Piaget and Szeminska (1952, p. 16) wrote as follows: (Line numbers from protocol excerpt above have been added in the quotation.)

> ...the child begins ... by filling the narrow glass L to the same level as the wide glass A (2) ... he then discovers, by comparing the two columns of equal height, that one (A) is wider than the other (L) and decides that the first glass (A) contains more because it is bigger (8). Thus a second relation, that of width, is explicitly brought into the picture and 'logically multiplied' with that of the levels. In order to arrive at equality, the child pours a little more liquid into glass L (10) thus proving the reality of this multiplication of relations.

At 14, pouring some back, he continues to relate less width to greater height. That is, a pair of elements from the height relation are combined with a pair of elements from width, first transforming the level in one direction and then in the other.

To obtain a rigorous solution to the question posed to Edi, an algebraic relation between height and width is required (Figure 2). However, the concept of metrical quantity of liquids, developing before such measurements are made, requires an equivalence between parts of the liquid before and after transformation of shape. Piaget describes the typical partition of the quantity of liquid into two parts, the first of which has the same shape and same dimensions before and after the pouring transformation (marked "part unchanged" in Figure 4) and the second of which is transformed in shape. Once the child recognizes that the second part is transformed from a wide crescent to a narrow cylinder (Figure 4), the compensation of

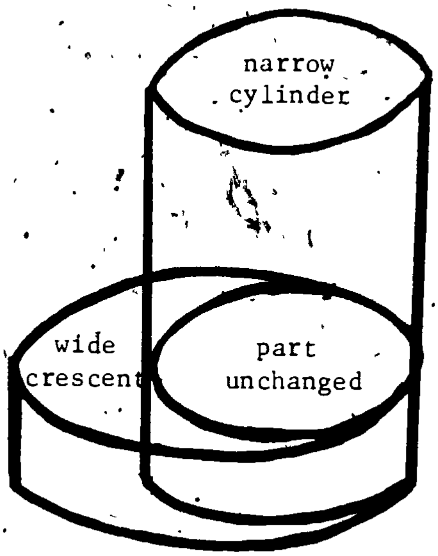

Figure 4. The partition of a quantity of liquid, before and after transformation of shape, leading to extensive quantity and conservation of quantity.

the two dimensions is assured. The concept of an extensive, additive quantity, the Genevans argued, is thereby established for amount of substance. Evidence of the recognition of this compensation, however, was rarely found in the interviews I have conducted with children, and Piaget and Szeminska give only one example. However, as the biologist needs only one flower to explain the reproductive processes of a plant, so this one example serves to show how it is possible that this compensation idea emerges and that conservation of amount of deformable substance (liquids or plasticine-like materials) over transformations of shape becomes evident.

This extensive quantitative idea does not, however, immediately transfer to conservation of weight, and the techniques of liquid measurement are not automatically grasped. Furthermore, conservation of volume of solids by displacement of liquids does not appear until after 10 or 11 years of age. These quantities continue to conflict sharply with kinesthetic systems involved in feeling the weight of something and in pushing aside water (thus generating a volume by the movement of a surface). Kinesthetic ~~deep~~ structures (Witz and Easley, 1971) tend to restrict the applicability of the operational schema of continuous quantity until about the age of 12-14 (see Piaget, 1974, p. 3).

One further development of Piaget's theory (1976) was stimulated by the recent development of the mathematical concept of category (McLane and Birkoff, 1967) as a new unifying concept. Since categories arise from morphisms which, in turn, arise from systematic correspondences, Piaget was recently motivated to look for uses of correspondences in the development of intelligence. Among other things, he and his colleagues considered whether establishing clear correspondences between parts of a piece of plasticine as it undergoes a transformation in shape would assist in the development of the concept of extensive, additive quantity and conservation of that quantity over transformations of shape. When pieces were broken off a ball of plasticine and added one-at-a-time to the ends of what turns out to resemble a sausage (see Figure 5), children of about 4 to 5 years

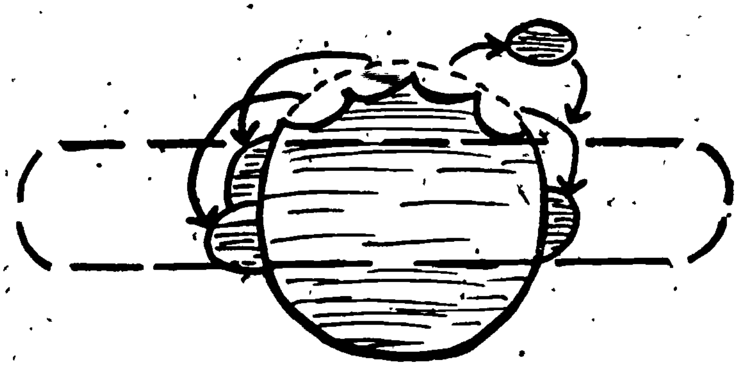

Figure 5. A ball of plasticine being transformed bit-by-bit into a "sausage" to demonstrate the correspondence of parts.

old knew immediately that the amount of plasticine had not been changed. (This recent experimental result had been foreshadowed in a footnote in Piaget and Szeminska, 1952, p. 24). However, it must have been a great surprise (contradicting much of what Piaget had hypothesized about the natural limits to accelerating intellectual development (Piaget, 1964)) when Inhelder, Blanchet, Sinclair, and Piaget recently found (1975) that several repetitions of this process of carrying out the transformation of a piece of plasticine led to true conservation of substance several years earlier than had previously been observed. Summarizing the implications of these experiments, they wrote (1975):

> The first of these implications is that one finds nothing more at the end of a movement than what was removed at the beginning of the movement. This then implies a compensation between what is additive at the point of arrival and what is subtractive at the point of departure... It is this compensation which was facilitated by the techniques of sections 1 and 6 [involving demonstrations to the subject] but which is spontaneously produced when the subjects, pulling a ball [of plasticine] out into a sausage, realize (belatedly) that it is getting thinner: They understand ~~then~~, on the one hand, that they have done nothing more than displace pieces non-delimitedly ("all we did was to lengthen it") from which, on the other hand, they understand the compensation between what is added in length and what is lost elsewhere (in thickness). This compensation (explicitly formulated by a number of subjects), whose appearance seems mysterious since it is not based on any measure, turns the other way around as soon as the changes in shape are understood as a result of simple displacements (therefore without "productions")...* (Inhelder, Blanchet, Sinclair, and Piaget, 1975; trans. J. and R. Easley.)

Here, we see real progress has been made in clarifying the mystery of a metrical quantity which cannot be measured. Compensation, and therefore conservation, come from a partition and a correspondence linking parts before, during, and after the transformation of shape. So it is the visual and conceptual tracking of the parts created by the partition that generates the certitude children feel about a quantity they can't measure or compute. The authors continue:

> This important point remains to be made, that the compensations and extensive and additive identities inherent in commutability** do not suppose at all, when they are applied to a continuum, an effective preexisting partition. The displacements, once understood as simple changes of position leaving invariant the quantity while changing the forms, imply the representation of a possible partition, ~~if~~ only between the parts displaced and those that remain in place; and that suffices to generate spatial compensations, extensive identities ("We didn't take anything away or add anything," which confirms the psychological status of the identity operation +X -X = 0), and additivity; for partitive

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*This recent observation was foreshadowed in Piaget and Szeminska (1952, p. 24).
**A generalization of commutativity without reference to left-right, temporal, or logical order.

addition; even when it operates on pieces which are not delimited but which are delimitable, and even before all measure or construction of units, is as operational as the union of sets or numerical addition (Inhelder, Blanchet, Sinclair and Piaget, 1975; trans. J. and R. Easley).'

We thus see that a form of addition, not logical multiplication, is the key precursor of continuous quantity. In the same study, these authors examine the relation between conservation of continuous and discontinuous quantities and find the cognitive structures in both cases to be isomorphic. In the case of conservation of number (Piaget and Szeminska, 1952), earlier discussed in terms of the coordination of two continuous quantities, the overall length of a row of counters and the density of the counters in a row (see Figure 6), is now seen to involve correspondence.*

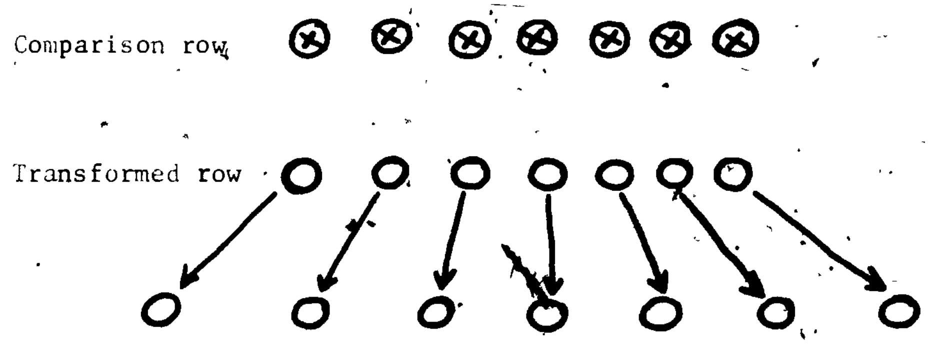

Figure 6. Transformations of length and density of a row of counters in a conservation of number clinical interview.

What is clear for educators from this selective review of developments in a much more extensive literature, is that the introduction into the mathematics curricula of sets, Cartesian products, relations, and functions in the name of new mathematics has not at all been applied to assist in the development of the concept of extensive metrical quantities. These are still assumed to be given operationally (in the sense of Percy Bridgman, not that of Piaget) by the procedures of measurement. However, the feasibility for assisting in and not derailing the natural psychological development of the concept of quantity, revealed in recent Genevan work, has very great significance for educators (Piaget, 1973).

*See a foreshadowing of this result in Piaget and Inhelder (1971), Ch. 8, S. 4).

Piaget's theory of the development of conservation of substance may seem too elaborate for the phenomena it explains. However, when we look at the operational stages through which children go in the attainment of the full, three-dimensional metrical concept of volume (Piaget, Inhelder and Szeminska, 1960, Ch. 14), we see that he makes very ample applications of the distinctions between logical and numerical multiplication, between asymmetrical relations and additive quantities, and between partitioned and nonpartitioned space. Briefly, the substages he finds in this development are:

IIA. Nonconservation of volume together with one-dimensional comparisons.

IIB. Intermediate responses.

IIIA. Logical multiplication of relations together with conservation limited to interior volume.

IIIB. First appearance of metrical relations.

IV. Mathematical multiplication of three-dimensional measurements together with conservation of true volume.

These stages cover children from 5 to 13 years of age and cover the full period of concrete operations plus the early portion of the period of formal operations. Reading the excerpts from clinical interview protocols reported in that chapter, together with the discussion of the intricacies of the evolving system of measurement and reasoning, is sobering indeed for anyone who ever thought that, in teaching children to measure with a measuring cup or graduated cylinder, he had taught the concept of metrical volume. To illustrate the difference between interior and exterior volume, we quote the excerpt given for Jui (age 9 years and 6 months, substage IIIA). He was asked to construct houses out of cubical blocks of unit measure equivalent in size to a model but with a different base, and to predict how much they would increase the level when placed in a basin of water.

Jui: Thinks that a block of 3 x 1 x 12 when compared with another of 3 x 3 x 4 "takes up less room that way."

Int.: Well, is there the same amount of room inside the house or not?

Jui. Yes, because there are always the same number of bricks whether the house is put upright or lengthwise.

Int.: How does that affect the amount of space which it takes up in the water?

Jui: It takes up more room when it's lying down.
(Piaget, Inhelder and Szeminska, 1960, p. 376)

Other studies of the displacement of water by objects immersed in it (Piaget and Inhelder, 1974; Piaget, 1974, pp. 88-89) confirm that the amount the water rises is seen as due to the weight of the object immersed rather than to its volume, i.e., the space it contains. This fact Piaget attributes to the degree that "bodies are understood to be made up of particles..." (Piaget, 1974, p. 94) and these particles are arranged in a "tight" way that ensures that the active weight and the passive being weighed down of these particles are distributed uniformly throughout the body by a kind of mutual interaction of the particles. Obviously, the same kind of consideration must be given to the composition of the water if its rising is to be seen as purely an effect of volume.

Here, particularly, we see a breakdown of the artificial barriers that separate mathematics and science in schools. But, most important, we can see that Piaget's research is not going to support any general policies as to when certain topics should be introduced in the curriculum or how the classroom should be arranged. There seems to be no escaping the implications he has carefully spelled out that teachers would have to understand rather well the process of cognitive development and listen to and observe children carefully so as to grasp with reasonable accuracy what kind of mental operations they are bringing to bear on a given task. Then they could help children individually and in groups explore their environment in ways that stretch their cognitive systems without penalizing them for having slower development than other classmates. (See Denis-Prinzhorn and Grize, 1966.)

Soviet Studies of Mathematics Learning and Teaching

The recent translations of studies of mathematics teaching and learning by Soviet psychologists are not only interesting for the contrast they provide with the more familiar American and European studies, but also they may enable us to gain an insight into the causes of our own educational and psychological history. Rather than think of learning as behavior change and teaching as the manipulation of stimuli, Soviet psychologists studying mathematics have been interested in problem solving and reasoning. For example, Krutetskii says:

> In conformity with the basic tenet of Soviet psychology that one must study abilities within the activity for which the abilities are being studied, and on the basis of an analysis of this activity, we believed that experimental problems should, as a rule, correspond to the nature of a pupil's mathematical activity (1976, p. 89).

He also refers to the view of Engels that "the numerical relations and spatial forms of the real world are the object of study in mathematics" (Krutetskii, 1976, p. 86). While this does not mean that algebraic relations and geometric theorems are relegated to a minor role in favor of statistical graphs and anatomical drawings (nothing could be further from the truth), the extreme forms of Platonism and formalism which underlie modern mathematics programs in the U.S. are not in evidence in these studies.

In effect, the curriculum as it is, and long has been, conceived by teachers, parents, students, and administrators is accepted as the starting point of research. Improvements are sought within it, not by replacement of it. However, improvements within the traditional curriculum and methods of teaching are not sought by attempting to conceptualize mathematical understanding differently or by putting aside the judgments teachers make, replacing their natural perceptivity with mechanically made decisions, as has occurred in many of the reforms and innovations in the U.S.A. Rather, it is assumed that teachers can tell which students understand correctly and which ones do not, and will understand what is discovered or invented in the way of improvements in pedagogy. The USSR may be the last major country preserving a close relation between the academicians and the school practitioners. (England has just decided that all teachers colleges have to be university affiliated to be allowed to continue, and those that cannot find a sponsoring university must close. It is feared that this may result in the same separation between schools and academicians as we now find in this country.)

When Krutetskii attempts to isolate the components of mathematical ability for the purposes of investigating their processes, understanding them better in order to assist in their development, he searches the literature and comes up with the following list:

> An ability to shorten the reasoning process, to think in curtailed structures.
>
> An ability to reverse a mental process (to transfer from a direct to a reverse train of thought).
>
> Flexibility of thought--an ability to switch from one mental operation to another.
>
> A mathematical memory. It can be assumed that its characteristics also arise from the specific features of the mathematical sciences, that this is a memory for generalizations, formalized structures, and logical schemes.
>
> An ability for spatial concepts, which is directly related to the presence of a branch of mathematics such as geometry (especially the geometry of space) (1967, p. 88).

These abilities are conceived as determined by the nature of mathematics more than by the nature of human thought processes. Krutetskii is not engaged in Piaget's task, trying to discover in children's thought the seeds of mathematical thought in general, i.e., how it was possible for man to have created mathematics. He is not attempting to apply a general learning theory, such as operant conditioning, or to define these abilities in behavioral terms. Nor is he adopting a general theory of the nature of mathematics or mathematical proof such as that deriving from Boole and

Frege. Each classical branch of school mathematics, arithmetic, algebra, geometry, is accepted as given and as generally understood by those who teach it. There is no attempt at reductionism, reducing either mathematics learning or mathematics itself to some simpler system of elements. The terms of these abilities are such as teachers who have mastered the branch of mathematics they teach would understand.

From this point of view, efforts to isolate general abilities derive from a kind of pointless, abstract psychology. He develops and studies intercorrelations among tests for each ability, but as Kilpatrick and Wirszup complain in the introduction, "He never shows, by including all the tests in one analysis, that the groups he has formed are associated with different factors" (Krutetskii, 1976, p. xv). What would be the point of doing that, except to sort pupils out in general ability groups which would do violence to their feelings of individual worth. A mathematics teacher's job is clearly to identify specific mathematical weaknesses and work for their improvement in individual pupils. So mathematical abilities are naturally defined within that conception. To illustrate this point more precisely, note how Krutetskii describes reversibility of mental processes:

> In the first place, it is the establishing of two-way (or reversible) associations (bonds) of the type A⟷B, as opposed to one-way bonds of the type A⟶B, which function only in one direction.
>
> In the second place, it is the reversibility of the mental process in reasoning, thinking in a reverse direction from the result or the product to the initial data ... But the specific ways that the thought travels can differ greatly.
>
> Thus we shall speak of direct and reverse bonds. A sequence of thought from A to E, say, will be regarded as a direct bond; a thought sequence going in the opposite direction (from E to A) will be regarded as a reverse bond (1976, p. 287).
>
> The overwhelming majority of the average pupils coped with solving the reverse problems without special exercises...
>
> As for the incapable pupils ... a direct bond can be securely anchored in them, but a reverse bond is not produced without special exercises. <u>Here we are discussing correct reverse bonds</u> ... For them the question does not even arise whether the reverse (theorem, course or reasoning) is correct in the given instance (1976, p. 289) (underlining added).

It is clear that, although Soviet psychologists may discuss bonds formed by arbitrary associations, in learning mathematics an evaluative process must always enter in to be useful. Hence, the term "bond" is stretched from its limited, reductionist meaning to a much richer meaning in which the process of reversing is not divorced from the process of judgment of correctness. This is an adaptation of psychological theory to the educational situation as defined by teachers that would not be made by many Western psychologists. Their view would tend to be that the practice and beliefs of teachers should be modified to conform to psychological discoveries, and that objectivity is the chief criterion, not usefulness. The question of whether most teachers can modify their views and practices to conform to even one of the psychological or theoretical positions without years of intensive retraining has to be answered before one can judge the merit of the Soviet strategy in educational psychology. But, at least, it can help us reflect on the disappointments of our own ventures, or in the remoteness of a Piagetian system.

Lest the reader be left with the impression that no substantial changes are being proposed in the traditional ways of teaching mathematics in Soviet schools, let me review, briefly, an example of the kind of changes some Soviet writers are proposing. Chetverukhin (1971) reviews the difficulties that teachers have had in teaching Euclid's Elements of Geometry in the development of sufficient visual imagination to permit an understanding of theorems and proofs. Pointing out the inadequacies of merely showing three-dimensional models of geometric objects, Chetverukhin tries out ways of exploring what geometrical abilities children have at various ages that might be developed. By asking children to draw familiar three-dimensional objects from memory, he was able to show that three-dimensional and complex mechanical aspects are perceived early and the ability to represent them accurately can be taught. In a task suitable for older secondary students, drawing sections of a ring (circular torus), either alone or in a rectangular box, proved challenging. The most difficult section, perpendicular to the plane of the torus and just grazing the hole, very few even at the post-secondary level could draw accurately except those in engineering courses. A major defect noticed was the tendency to use stereotyped forms, e.g., circles and rectangles in standard positions. He recommends strongly, on the basis of his study, that the teaching of geometry should be strongly supported by spatial concepts developed through such means as constructive and projective drawings for surfaces of rotation and perspective drawings of standard figures in arbitrary positions.

It is extremely interesting to speculate what research in mathematics education in this country would become if the mathematical understandings of teachers and the learning problems they identify in their pupils were to be taken as the starting point for detailed clinical studies of teacher-pupil interactions in order to arrive at helpful procedures, devices, and materials. This idea is not far from what Buswell, Brownell, and others were doing in the 1940s to make arithmetic meaningful (Weaver and Suydam, 1972), but progress was slow and changes in society and the intellectual climate of teachers colleges and the universities into which they were assimilated or converted were rapid. Teacher educators, in order to make their way in the university system of rewards, sought sophisticated intellectual systems to apply to the task of advancing public education. One prominent source was statistics; another was the structures of the various disciplines. Differences among philosophers of mathematics, for example, tended to spawn different recommendations for reform of mathematics education, the implication being that the traditional curriculum had no academically interesting structure and that a new structure had to be provided.

Another source of an intellectual system for work in mathematics education was psychological learning theory. More recently, cybernetics, artificial intelligence, the Piagetian theory of the development of the intellect, and task analysis, have also been employed for the redesign of parts of the mathematics curriculum. With most reform programs there are associated particular master teachers who personally developed the art of applying the particular intellectual system involved. Meanwhile, textbook publishers and editors took what they thought teachers would be able to use and packaged a mixture of new ideas with the old in the most attractive format they could muster, and schools struggled on with that necessary subject few ever claim to understand. The various forays of university-based mathematics educators seem to have had at best a superficial impact. Schools were possibly too busy with other matters: the civil rights movement, the drug culture, and, more subtly, with a set of rising expectations on the part of students and parents that individual interests, abilities, disabilities, and ethnic backgrounds would be accommodated, and furthermore that the school experience would be fun.

It is pointless to speculate on what might have been, if teachers colleges had remained separate from universities, and practical improvement in lessons had been the prime consideration and the application of theories of all sorts to reforming school teaching had been considered bourgeoise or even decadent. I don't believe that American eduation has lost its talent for practical advancement, but it does appear that the breach between people who work in schools and the academicians in universities has widened alarmingly. Despite this pessimistic note, however, the practical uses of the various kinds of clinical studies in mathematics education that have been exemplified by the foregoing sections do offer a substantial ray of hope. To this problem we now turn our attention.

The Uses of Clinical Research in the Mathematics Classroom

From the point of view of traditional types of research, the title of this section is ambiguous, because it could refer to the uses of research results or to the uses of research methods in the classroom. Both interpretations are necessary. From the perspectives of the clinical research reviewed above, there are no results which can be applied without at the same time engaging in further use of clinical methods, and the methods benefit greatly from what others have found. Traditionally, results are generally stated in terms of relationships between variables (usually some form of co-variation in some population) and clinical research usually results in the recognition of processes or mechanisms of some kind that explain why things work in a particular way. To be sure, mechanisms and processes vary in many ways. A clock mechanism can be measured in many details--the number and size of its cog wheels, the number and size of the teeth on each wheel, the type and period of its pendulum, etc., but these various measurements and the ways they covary are secondary because their meaning depends on understanding how a gear train works and how an escapement works. When the clocksmith sees what is wrong and what is needed to fix it, he usually has not depended on measuring any of those things. However, when a mechanism is hidden, as in a chemical reaction, or in a disease of a complex organism, the measurements of variables are essential to determine which of the many known mechanisms possible is involved at a given time. However, even when there are no cog wheels to be seen, the discovery of which of the possible mechanisms is operating demands a mental process quite different from studying patterns of covariation. Instead of looking at n-dimensional spaces, one thinks of structures, whether they be systems of atoms or systems of ideas or organ systems or systems of conceptions, as in Erlwanger's treatment of Benny.

In the case of Piaget the mechanism assumed in most of his later work on the development of intelligence in school-age children is based on the analogy between what mathematicians call structures and what biologists call structures. He supposes that mathematical structures underlie children's thinking about quantity and causality and that they go through progressive differentiation and specialization like the structures of the body. It is his hypothesis, in fact, that mathematical structures are biological structures of the species that does mathematics. Then, extending his analogy further, he proposes that these mathematical structures carry on a process of assimilation and accommodation like other biological structures. Few elementary school teachers will be applied mathematicians of the sort who might make use of a mathematical theory of mathematical learning. Most teachers may be happier with the more practical-minded Soviets who take abilities as recognized by teachers and attempt to differentiate them more clearly from each other and speculate about how they are formed and how they may be developed--not as general psychological variables, but as contextual perceptions.

Clinical researchers of any of these broad points of view, and others as well, will see classroom events more clearly because of their research, for they are likely to notice the kinds of structures they have seen before. They are also likely to notice variations from familiar structures that may make a difference in their functioning, as in a clockworks, the variations in a given structure are secondary, given by the structure, but important. In this process, no rules emerge that don't have lots of exceptions. The teacher must watch and listen and try to understand what the pupil is doing and saying, and take the pupil's words and actions seriously, not just as mischievous or sloppy, though occasionally they may be that. Any such hypotheses, as well as hypotheses about the pupil's substantive ideas of mathematics, need to be checked carefully from time to time. Jumping to conclusions is necessary to get started, but all such conclusions must also be challenged vigorously.

It is common to hear the objection that clinical understanding is idiosyncratic and not reliable enough to serve as a basis for a science or technology of education. This objection arises from a conservative methodology which hopes to reduce the process of scientific advance to mechanical procedures. However, physicists and mathematicians often have given expression to the great dependence that the advancement of science has on human freedom and creativity. Consequently, I am prone to take more seriously the objection of Herman Weyl, a mathematician who contributed greatly to the development of modern physics. Weyl argues that, although our knowledge of another person's mind is of indisputable adequacy for many intimate and complex acts of communication, it is nevertheless limited in contrast to the mathematical way of constructing theory developed during the past three centuries. He says:

...,the scope of the understanding from within appears practically fixed by human nature once for all, and may at most be widened a little by the refinement of language, especially the language in the mouth of the poets. Understanding, for the very reason that it is concrete and full, lacks the freedom of the 'hollow symbol' (1964, pp. 580-581).

In the case we are talking about of a teacher trying to understand his or her pupils by empathy and being responsive to them, the problem is not, as many teachers fear, that they will differ too much from what other teachers might say in their efforts to understand the same pupils, for that problem can be solved by careful examination. The limitation of direct personal understanding of another human being, or of oneself, is that there is not enough variability and too much stereotyping.

Whittaker (1975) describes how she encourages primary school teachers of mathematics to interview their pupils using Piagetian tasks, not as a means of assessing progress nor even as a basis for curriculum planning and decisions, but simply as a way of getting to know the children's minds. This seems to be Piaget's recommendation, too (Piaget and Duckworth, 1973). The advantage his tasks seem to have for this purpose is that they are safe from correction, once the teacher knows that children go through different stages in terms of the kinds of answers they are likely to give. An unacceptable answer from an adult point of view need arouse little concern, for the next year or so will certainly bring the child to the point where he or she has a more acceptable answer. The child knows, too, if the interview is conducted properly, that he or she can say what he or she thinks and give his or her own reasons. It is harder for a teacher to get such honesty from many pupils with regard to the specific subject matter being studied, and yet the Piagetian interviews cover ideas that are fundamentally related to the subject matter of mathematics.

It is certainly not necessary for a teacher of school mathematics to know Piaget's mathematical theory of the development of mathematical concepts. There are plenty of very successful teachers who never heard of it. And it is not necessary for teachers to have other interviewers come in and check how their students are conceiving of mathematics, for again successful teachers exist who don't do this. It does seem to me, however, that without some kind of clear channel of communication, in which every pupil can tune in to the teacher when he or she is feeling lost, there will be far too many pupils who are lost. Mathematics is traditionally a hard subject, and many people feel that it is no disgrace to be "dumb in math," but there seems to be no good reason for it. The most probable reason that a sizeable percentage of pupils have difficulty in mathematics classes at all levels of schooling is that the best students set up a private mode of communication with the teacher (often picking up the teacher's own words quickly) and those who don't immediately catch on to this language feel left out and stop trying to understand in their own way what is going on. They just try to do enough to get by or else they actively rebel against the teacher's requests.

Clinical interviews can help reestablish communication, through alternative channels, in alternative terminology, and can reduce the feeling that there is just one precise way in which the work is to be done. Teachers and pupils alike need the confidence that comes from occasionally finding out why something was hard for someone to understand. The point of view here is that each particular child is the authority on what he or she thinks and what makes sense to him or her, not Piaget, not any researcher, no matter how deeply the researcher has probed or how broadly he or she has sampled. Even if one knew that 95% of all fourth graders did not accept conservation of volume (a far more positive piece of information than any authority would support), it could not confute a particular child who plainly says that no matter how a piece of plasticine is deformed or arranged, it will displace just as much water. Statistical summaries can at best only alert the alert teacher to particular possibilities, and actually conducting clinical interviews can teach one better how to recognize these possibilities than any psychometric or interview schedule that could be published.

Conducting clinical interviews provides training in the interpretation of human thought and can, if desired, draw into the process the most sophisticated kinds of intellectual tools one possesses, but basically it depends on one person's attempt to understand another. The published studies, many of which will be listed and categorized in the next section, can only provoke one to try different kinds of tasks that might not have occurred spontaneously and to try probing for different kinds of ideas that one would not have thought to look for otherwise. They do not teach how to teach. That is something each teacher began learning at his or her mother's knee and continues to learn throughout a lifetime, for teaching is one of the most common forms of human interaction there is and its specialized organization in schools is the source of as much hindrance to its natural development as it is a help. I see many possibilities for the development of teaching by teachers, and clinical studies have a potential for helping many teachers achieve major advances in their ability to help children who otherwise wouldn't be reached.

The Literature of Clinical Studies in Mathematics Education

This section will list documents the reader may find useful in following up on the types of issues that have been introduced in preceding sections. It will follow the sequence of those sections.

The literature on the methodology of clinical and case studies relevant to mathematics education is limited indeed. Bang (1966) reviews the development of Piaget's <u>méthode clinique</u>. In addition to Stake (1976), Campbell (1975), Denis-Prinzhorn and Grize (1966), and Piaget (1929), which have already been mentioned, Witz (1973) is a key paper which introduces the methodology of analyzing a nonstructured interview in terms of the guiding conceptions of younger subjects. It takes the reader through the complex decision-making process in an analysis of an interview with a 4-1/2 year-old child working with a beam-type balance. Response latencies are used to support the decision about framework boundaries. Knifong (1971) analyzes four other such interviews with children of about the same age in a similar way, but also developing intra-framework structure. Easley (1974) develops the diagram of Witz' structural paradigm into a system for accounting for two different-sized elements of an interview protocol. This technique is further developed by Clement (1977). Vinh Bang (1966) traces the history of the clinical interview in Piaget's work. Piaget (1967) makes pointed suggestions on conservation interviews. The specific issues that divide experimentalists and more clinical approaches are treated in Easley (1966, 1967a, 1967b, 1971, 1973, 1974, and 1977). An entirely different methodological approach to protocol analysis is found in the work of Newell and Simon. See for example, Newell and Simon (1972) in which a computer simulation of human problem solving is presented; Paige and Simon (1972) for a discussion of solving word problems in algebra; and Simon (1975) for a general review.

Erlwanger's dissertation (1974), in addition to providing counter-examples to many conventional assumptions, is a rich source of ideas about children's responses to rather highly programmed situations. For example, the second case study concerns Mat, who differs from the subject of the first study, Benny, (Erlwanger, 1973, 1975) primarily in the extreme dependency that he has on the text and in the absence of any struggle or resentment against the lack of consistency from one page of his work to the next. It provides a powerful lesson in how mis-educative dutifully following directions can be. In addition to the discussion of Benny by editors in footnotes already alluded to [See Davis (1973) for his interpretation of the Benny case], other clinical studies of children attempting to solve arithmetic problems include Steinberg and Anderson (1973), Davis (1975a,b), Davis and Greenstein (1969), Jencks and Peck (1972, 1975), Leiser (1974), Peck and Jencks (1974), Witz and Albert (to appear). Earlier work along the same vein is reported from Germany (Fruedenthal, personal communication), Wagenshein (1965, 1970), and the studies earlier alluded to by Brownell (1944a) and Buswell (1926, 1949). Other studies by Brownell (for example, 1928, 1944b, 1947, 1956) constitute an important part of this literature.

An interesting study which illustrated a creative method of structural analysis is Witz and Albert (1975, to appear). It is a heuristic analysis of part of an interview on arithmetic with a very competent 12-year-old boy which moves to successively deeper levels of mental process, arriving finally by a boot-strapping process at the conception of an "irradiated state" of mind in which clarity is impressively evident.

I proposed (Easley, 1964) an operationalization of a particular application of one of Piaget's mathematical structures of the formal stage, the INRC group. Witz (1969) proposed an axiomatization of grouping I, and Grize (1969) discussed the axiomatization of Piagetian structures. Kamara and Easley (1976) review methodological issues concerning cross-cultural studies of cognitive development. Witz (1970) presented an axiomatization of an alternative structural analysis. However, on the whole, the methodology of mathematical modeling of cognitive structures in school mathematical subjects does not have a well developed literature. A more general approach to human cognition is described in Witz and Easley (to appear). For other points of view on Piagetian interviews see Smock (1973), Ennis (1975), Steffe (1973), and Lunzer (1973).

In addition to Piaget and Szeminska's (1952) classic work on number and quantity, and the more readable book on geometry (Piaget, Inhelder and Szeminska, 1960), the new translation (Piaget and Inhelder, 1974) fills a gap that has existed too long in English translations of Genevan studies of the development of quantities. The most elegant mathematical theory of the development of mathematical concepts, though much criticized, is Piaget and Inhelder (1956). Another new English translation (Piaget and Inhelder, 1975) on the origins of chance in children's thinking is most welcome, for it is one of the most clearly written of what usually turns out to be rather heavy tomes. For studies of logical multiplication and conservation learning, see Inhelder and Piaget (1964), Inhelder, Sinclair, and Bovet (1974), and Piaget (1977).

The chief resource today for Soviet studies in mathematics learning is the impressive fourteen-volume series edited by Kilpatrick and Wirszup (1969-75). I have certainly only just begun to sample this storehouse in such studies as the two by Bogolyubov (1972a, 1972b) on solving arithmetic problems and Chetverukhin's (1971) study of spatial concepts and imagination. Krutetskii's (1976) highly readable account of both clinical and correlational studies (also edited by Kilpatrick and Wirszup) is another source of major importance for the clarity of his philosophical position. I have also found help in developing my perspective on Soviet studies articles by Gal'perin (1969) and Gal'perin and Talyzina (1961). For general background, the text on the teaching of mathematics, to which some of the Soviet research studies can be related, is Khinchin (1968).

Robert Karplus and others have conducted a number of studies of tests on logical and proportional reasoning that invite comparison with Piaget's clinical data. Five studies, Karplus and Karplus (1970), Karplus and Peterson (1970, 1972), Karplus, Karplus, and Wollman (1974), and Wollman and Karplus (1974) have been recently reviewed by Travers and Easley (1976). Fuller, Karplus, and Lawson (1977) summarize this work and present their educational interpretation from the point of view of conventional physics.

Papers on the teaching of mathematics which discuss the uses of clinical studies include Bauersfeld (1967), Easley and Witz (1972), which use clinical data to criticize parameterized forms of individualization, Easley and Zwoyer (1975), Whittaker (1975), O'Brien (1974, 1975), Piaget (1975), Polya (1962, 1965), Piaget and Duckworth (1973), Skemp (1971), and Weaver and Suydam (1972). A rather different interpretation is placed on clinical research and teaching by Copeland (1974). [For a critical review see Easley (1971).] Anderson (1965) also takes a different view of what it means for a child to learn mathematical reasoning.

For a discussion of teachers' points of view on the teaching of mathematics, another subject which can be profitably studied by clinical methods, see Shirk (1973) and Busis, Chittenden, and Amarel (1976). Since one cannot avoid asking the question, "What is mathematics?", when engaged in examining the thinking of individuals, it is useful to have reference to such readable accounts as Lakatos (1963) and Halmos (1968). For more detailed documentation of rather similar views see Polya (1954). We may appropriately close this too brief review on the teaching of mathematics by the question raised in Halmos' paper:

> In painting and in mathematics there are some objective standards of good--the painter speaks of structure, line, shape, and texture, where the mathematician speaks of truth, validity, novelty,

generality--but they are relatively the easiest to satisfy. Both painters and mathematicians debate among themselves whether these objective standards should even be told to the young--the beginning [student] may misunderstand and overemphasize them and at the same time lose sight of the more important subjective standards of goodness... [Mathematics] is a creative art because mathematicians create beautiful new concepts; it is a creative art because mathematicians live, act, and think like artists; and it is a creative art because mathematicians regard it so (Halmos, 1968, p. 389).

For a discussion of a creative type of elementary school mathematics--as viewed by a mathematician--see Bauersfeld (1967). For a more teacher-oriented view of creativity on the part of elementary school children see Whittaker (1975). Driver (1973), however, shows that the original thinking of 12-year-old children with quantities can regularly lead them to conclusions considered wrong or at least very surprising by their teacher just because they have such a different conception of the quantities themselves and the mechanical way they interact. What we have to admit is that children engaged in creative thinking about quantities (both discrete and continuous) will take years to achieve the standards set by objective measurement, calculation, and logic. But it is a sad commentary on the mathematical ideas of teachers if we conclude, with Copeland (1974), that meaningful instruction in mathematics cannot occur until children have achieved the standard concepts. For, as Halmos (1968) says, there are "more important standards of goodness." We take these to include (for teachers and children alike) curiosity, analogy, symmetry, surprise, and a conviction that things ought to fall into place, into some natural order. When these fail to happen, school mathematics becomes a burden, a mysterious language, and worst of all a humiliation.

Summary

In this review of clinical studies in mathematics education, rather than attempting a complete survey and classification of studies, we have attempted to address the concerns of those who, not used to clinical or case studies, find it difficult to evaluate or even follow the literature. To this end, we have examined examples of three different kinds of studies which are of current interest, and we have tried to draw out the conceptual or theoretical issues in which their authors are involved. This has provided evidence to support Campbell's claim, discussed in our brief methodological section, that case studies often test many more theoretical hypotheses than is the case with experimental studies; the investigators, however, bent on finding an explanation, often neglected to mention the many hypotheses discarded and the evidence on which they were discarded.

Erlwanger's study of Benny, as we have indicated, raised the question as to how the different ways of thinking Benny used in arithmetic could be integrated--a question which Wittgenstein had raised in his studies of the foundations of mathematics. While we have no clear answers, we now have an important question that has been too long ignored--not just in various individualized programs in mathematics, but in mathematics education generally. In our review of Piaget's efforts to understand the origins of quantitative ideas in the cognitive development of children, we saw that sophisticated, modern mathematical concepts can provide theoretical tools for understanding processes that mathematics educators have taken for granted, such as the development of the concept of amount of liquid or other deformable substance. In our brief look at a few of the Soviet studies of teaching and learning mathematics, we saw that Soviet investigators, unlike European or American mathematics educators, have accepted the traditional goals and practices of teachers as given and not attempted to replace them, except to provide clear, practical solutions to problems identified by the teachers themselves.

In sum, instead of overriding concern for objective and general formulations of knowledge, clinical and case studies of these kinds show a primary concern for understanding and helping teachers understand and deal with the phenomena of children thinking quantitatively.

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