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ABSTRACT

This book is designed to illustrate how one general method of calculus is used in many different sciences and how different methods of calculus have furthered the development of essentially one field of science. The material is written so that it could serve as a math-science supplement for many courses. Chapters included are: (1) Introduction; (2) Growth, Decay, and Competition; and (3) Geometrical Optics and Waves. The Introduction contains suggestions for teaching, additional readings, and sequence of materials. (RH)

SCHOOL MATHEMATICS STUDY GROUP

Studies in Mathematics

VOLUME XV

Calculus and Science

By Victor Twersky

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PREFACE

Problems from the sciences that are injected into mathematics texts (to serve as vehicles for the development of theory or as examples for the use of theory) often exhibit the mathematical sciences as short-order menus and applied mathematics as the corresponding cook-books. We sought to counter this in the SMSG 12th grade text Calculus. In addition to exploiting applications to motivate the development of the subject and to illustrate some of its special consequences, E.G. Begle (Director of SMSG), A.A. Blank (Chairman of the Calculus team), and I decided that it would be of interest to attempt a systematic development of applications. As part of this program, I prepared one chapter (9: Growth, Decay and Competition) to show how essentially one general method of the calculus is used in many different sciences, and another (15: Geometrical Optics and Waves) to show how different methods of the calculus have furthered the development of essentially one field of science.

In the present volume, I have modified these two chapters to make them independent of the original text, so that they may serve as a math-science supplement to other programs. As such, the present material forms part of the series started by Volumes X and XI of the SMSG Studies in Mathematics, volumes based on the lectures for high school mathematics teachers given by Max M. Schiffer (Applied Mathematics in the High School, Volume X) and by George Polya (Mathematical Methods in Science, Volume XI). In an attempt to provide similar motivation for the audience addressed by Schiffer and by Polya, this volume begins with an introduction suggesting the inclusion of math-science material at the secondary level; much of this appeared in the Calculus (1-3: The Scope of Calculus, or in the Teachers Commentary), and in both books it helps prepare the reader for subsequent material.

Although this volume consists largely of personal contributions to the Calculus, I am very much aware of the stimulus and help that I received from my colleagues, particularly from A. A. Blank (the "Al" of Chapter 2), and from F. L. Elder, M. S. Klamkin, C. W. Leeds III, M. A. Linton, Jr., I. Marx, R. Pollack and H. Weitzner. I am also very pleased to acknowledge that the work would not have been attempted in any other environment than that created by Ed Begle and SMSG.

Victor Twersky

University of Illinois at Chicago Circle
Department of Mathematics, December, 1966

Chapter 1

INTRODUCTION

How did you hear about calculus?
How did you hear about Helen of Troy?
And have you yet heard the story about Al?

1. The Scope of Calculus.

Calculus is the study of the derivative and the integral, the relationship between these concepts and their applications. The great advance which takes the calculus beyond algebra and geometry is based on the concept of limit. The basic limit procedure of the differential calculus is typified by the problem of finding the slope of a curve; the basic limit procedure of the integral calculus is typified by the problem of finding the area enclosed by a curve. The slope is found as a derivative, the area as an integral, and superficially these appear to be unrelated. But there is only one calculus: derivative and integral are complementary ideas. If we take the slope of the graph of the area function, we are brought back to the curve itself. If we take the area under the graph of the slope function, we find the original curve again. The limit concept, in its guises of derivative and integral, together with this inverse relation between the two, provides the fundamental framework for the calculus.

The derivative and the integral may be interpreted geometrically as slope and as area, but these are only two among a wide range of interpretations. We usually begin with slope and area in order to introduce parts of the subject in an intuitive, geometrical way. But although an intuitive geometrical introduction is useful and suggestive, it must necessarily be based on very familiar steps. The steps are so familiar that it may not seem that they could lead ultimately to entirely new methods for solving completely different problems than those encountered in earlier courses. We should therefore stress that the concepts of derivative and integral are universal, and their incorporation into a calculus, a system of reckoning, enables us to solve significant problems in all branches of science. We can set the stage for a systematic development of the subject, and emphasize the universality of the concepts by mentioning new kinds of problems at the very start of the course. Then in addition to solving problems that are primarily vehicles for the development of theory or for the illustration of techniques, problems which do not begin to suggest the full scope of the subject, the course itself should also include significant applications of the calculus. Problems for which no methods of the earlier mathematical courses are particularly helpful can be found in very familiar contexts — problems that range from the spreading of rumors to the colors of the rainbow — and their very familiarity serves to emphasize the novelty of the methods.

No methods of the earlier courses help answer questions such as: How did you first hear about calculus? How did you first hear about Helen of Troy? To frame such questions mathematically, we must first isolate some essential features: Some stories spread like fires; others die out. If the story is too dull, nobody bothers to repeat it. But if the story is good, some of the people who hear it (and remember it) pass it along. Starting from these ideas, how far could one get by pre-calculus methods?

The same concepts that are basic to the spreading of stories are also basic to the processes of forgetting and learning. So many of the facts and procedures we stuffed into our heads and never used afterwards seem to have vanished. Others, that we met repeatedly and actually worked with have become so much a part of ourselves that we feel we have always known them. Our first exposures to these facts may not have taken, but repeated encounters in different contexts finally left their mark. (We take this as the theme of Chapter 2.)

Can we construct a mathematical description for the way that stories spread — the way that we learn, and forget? Starting from appropriate assumptions (the mathematical model) we can discuss some limited aspects of such processes with the aid of elementary calculus. Such processes illustrate a broad class of phenomena whose unifying features are the basic mathematical models for growth, decay, and competition. Besides helping to describe the spreading of rumors, and learning or forgetting, these same mathematical models serve to clarify our observations of radioactive decay, the attenuation of sunlight by a cloudy sky, the progress of chemical reactions, the growth of bacterial colonies, or the spread of disease through a city. In each of these situations, the essential feature is that the amount of some quantity is changing (with respect to time, or distance, or whatever) at a rate proportional to the amount already present. A process of this sort can be mathematically described by a certain type of equation (a differential equation) whose solutions, at least in the simplest cases, are combinations of exponential functions.

Other processes of nature change in a cyclic or periodic way; they repeat in identical form each year, each second, or every inch. The planetary motions, the tides, the harmonious chords of music, the propagation of x-rays through crystals — even the colors of oil films on water — all depend on periodic phenomena. For such processes, the rate of change of the rate of change of some quantity is (negatively) proportional to the quantity itself, and the mathematical model leads to a different class of differential equations whose solutions, in the simplest cases, are combinations of trigonometric functions.

With the calculus we may also investigate more complicated natural processes that involve a combination of growth or decay with some sort of cyclic behavior. We may also solve much simpler problems. How much time will it take to drive 300 miles if you start at a speed of 20 miles per hour, but increase your speed by 2 miles per hour for each hour of driving? At what angle should you throw a ball for it to travel as far as possible? In what directions with respect to the sun are the rainbow colors strongest?

These problems and many others which the calculus solves involve rates of change, this is the province of the differential calculus. A second broad variety of questions is concerned with totality — the summing of small effects, this is the province of the integral calculus. By recording your speed during a long trip, with many accelerations and decelerations, can you calculate how far the trip has taken you? If we know how a single drop of ink spreads on a blotter, can we predict what happens if we spill the whole bottle? Starting with a simple source of radiation, can we predict the total radiation from an extended distribution of such sources? Knowing the way a single droplet of water perturbs a ray of sunlight, can we determine how much light reaches us on a cloudy day from the entire overcast sky?

Such summation or integration problems are closely related to the rate-of-change or differentiation problems: the total effects result from an addition of small variations. Therefore we do not study separately an integral calculus and a differential calculus. We study a calculus comprising both differentiation and integration, and each aspect helps us to understand and apply the other.

Most of the applications of calculus emphasize the effects of variation or summation. "Calculus" was tailor-made to treat such problems. Except for the simplest problems of this type, the methods of arithmetic, geometry, and algebra are inadequate, and even for the simpler problems the methods of calculus are the more efficient or the more suggestive.

The calculus was invented to treat problems of physics. In turn, to the growth of the calculus, and to the developments it led to in the larger branch of mathematics known as analysis, we owe much of the progress in the physical sciences and modern engineering, and more recently in the biological and social sciences. The concepts and operations introduced by the calculus provide the right language, the right tools, even the right reflexes for the major part of the applications of mathematics to the sciences.

We should therefore emphasize applications in a general course, not just to show that calculus provides useful methods concepts for the sciences, but because so much of the calculus was developed to solve specific problems. We should show the student how the effort to solve physical problems led to methods of the calculus, how the attempt to make the best use of these methods and to understand their full scope and limitations led to the development of the calculus as an independent study, and how the products of this study in turn led to deeper insight into the original problems. We should show that science enriches mathematics by providing significant problems and concrete difficulties, and that mathematics enriches science by providing solutions of problems, and system and organization. We should balance accounts of physical motivations of mathematical procedures with mathematical formulations of physical phenomena, and match concept for concept that each has acquired from the other.

An ideal general course in calculus would maintain a balance of topics and of viewpoints that would meet the requirements of students who will become mathematicians, others who will become scientists primarily interested in applications, and still others for whom mathematics will become simply one of many deep intellectual experiences during their education. For the student of science, a fluent intuitive grasp of mathematical methods is a primary need, for the student of mathematics, a careful deductive development is essential. These different viewpoints conflict on occasion, but they also supplement each other, and both the scientist and the mathematician gain by a deeper appreciation of both these views and of their interrelations.

Historically, the replacement of an intuitive basis for the calculus (the method of infinitesimals) by a careful logical structure (the method of limits) marked a vital phase in the development of mathematics. This phase is far from complete. We are still attempting to learn how to combine intuition in approaching new problems with the effective use of logic, not only to temper and to verify our intuition, but to permit generalizations of broader applicability. Today, most mathematicians appreciate the essential roles of both intuitive and deductive procedures, not only for creating mathematics, but for learning it and for teaching it, and it is particularly important to stress their interplay in introductory courses.

2. Systematic Approaches to Calculus-Science Programs.

The primary purpose of a modern introductory course in Calculus is a systematic development of fundamental theory that stresses the novel features of calculus and indicates its relations to the earlier arithmetic, algebra, and geometry, as well as to the later analysis. The initial level of rigor should be appropriate for the students, and the problems that motivate the development of the subject, and the exercises that illustrate some of its special consequences or develop manipulative skill, should be meaningful to the students. After some grounding in theory and with some facility in techniques, we should attempt some systematic development of applications of the calculus to the sciences.

To serve up applications only as isolated problems and exercises provides the student with an indigestible stew of unrelated ingredients. To exhibit the mathematical sciences as short-order menus and applied mathematics as the corresponding cook book recipes is criminal. To shuffle together menus and recipes and rules of cooking without mnemonic guides yields only an unteachable and unlearnable mess. When we teach calculus, let us teach calculus. And when we consider applications within a systematic development of the basic concepts of the calculus, let us attempt to do so in a systematic way as concrete steps in the ascent.

There are two different systematic approaches we could follow. We could pick one general method of the calculus and show how it is used in many different sciences, or we could pick one general topic of science and show how different methods of the calculus have furthered its development. Because these two procedures are essentially different, and because both are indicative of the way applied mathematicians and mathematical scientists work in practice, we follow both procedures.

Chapter 2 (Growth, Decay, and Competition) illustrates the first approach. It develops mathematical models and shows how they are used in various sciences. To stress the generality of the mathematics and that the equations are completely independent of any scientific discipline, the models are introduced to describe the spreading of stories. The equations are then applied to radioactive decay, electric circuits, attenuation of sunlight and the color of the sky, propagation of nerve impulses, forgetting and learning, chemical reactions, and to sociology. This chapter stresses that despite the marked differences in the classes of phenomena studied in the various sciences, there are a number of processes that are fundamental to all. The same mathematical structures, and consequently the same underlying concepts, are encountered again and again in different contexts of nature. The equations are the same, but the functions and variables represent different measureable quantities with different names that change from science to science.

Chapter 3 (Geometrical Optics and Waves) illustrates the second systematic approach to applications. It follows a sequence of physical concepts (axioms called "laws of nature") and shows how various methods of the calculus supplement each other in revealing the implicit consequences. It starts with very restrictive laws, weakens them, and thereby generalizes the development. The procedure is "quasi-axiomatic." The laws contain undefined terms (as do the axioms of a mathematical system), and implicit restrictions on the associated observational and computational techniques (essentially as in mathematics where axioms are usually stated in an implicit context). The development suggests the heuristic search for first principles, and shows that physics, like mathematics, is cumulative: basic concepts thousands of years old are first accepted as laws and then exhibited as special consequences of a more general set of axioms. The chapter considers geometrical propagation and reflection (Euclid, Hero, caustics, shadows, edge diffraction, eikonals), refraction (Snell, Fermat, rainbows, mirages), energy conservation (Kepler-Lambert), waves and superposition (Huygens, Newton, Young, Fraunhofer, Rayleigh-Born), Rayleigh's theory for the color of the sky, Fresnel diffraction and the method of stationary phase, and some features of the complete mathematical model for scattering.

In the course of Chapter 2 we introduce many special labels from different sciences. However, these are not essential to the mathematics, and we rarely elaborate on them: instead we emphasize, that as far as the present development is concerned, they are merely different temporary names for the same mathematical components. The primary aim is to illustrate how one mathematical model based on calculus covers a variety of different phenomena. On the other hand in Chapter 3, we seek to present a sequence of physical concepts in a mathematical way (the structured development of major topics is an essential mathematical feature of this chapter) and to provide the physical motivation for several mathematical techniques that are basic to wave physics. We introduce these techniques in easy contexts where the results can be obtained directly, and then apply them (some in considerable depth) in more complicated situations.

Both Chapters can be fitted structurally in a calculus program, not only to illustrate many elementary concepts and methods, but to help provide motivation for subsequent developments. The first, which uses only elementary procedures to solve the equations that arise, provides a lead-in to a systematic discussion of elementary differential equations and techniques of integration (and it serves this purpose for Chapter 10 of the SMSG Calculus); similarly for the second, and multiple integrals, line and surface integrals, partial differential equations, etc.

The two chapters on mathematics and science are very different. Both attempt systematic approaches, but the first uses mathematics as the guiding thread, and the second uses science as the thread. The chapters supplement each other in indicating the ways that mathematics and science interact, and may suggest crossing threads of a fabric. The two specific threads we follow intersect at Rayleigh's theory for the color of the sky: in Chapter 2, it is a special case of a general attenuation process; in Chapter 3, a special case of a general scattering process. The chapters together may suggest that mathematics and science provide the threads that give structure to our concept of nature.

3. Style, Rigor, Notation, and All That.

Parts of Chapter 2 (which is a self-contained revision of Chapter 9 of the SMSG Calculus) are written in a relatively light vein in an attempt to break the monotony of the constant iteration of the basic material. It is easy to read (at least by 12th grade students), and the model for story spreading picks up the threads of Section 1.1 on "How did you hear about calculus? How did you hear about Helen of Troy? . . ."; etc. When we convert the equations to a model for forgetting and memorizing isolated facts) we pick up other threads. About half of the applications mentioned in Section 1.1 are covered in the next chapter, and many of the others are covered in the final chapter. The size of the space or the complexity of the equations allotted to the different disciplines touched on in Chapter 2 is not meant to correspond to their importance or to the writer's preferences (except for story spreading).

If Chapter 2 represents a set of hors d'oeuvres with a common flavor, then Chapter 3 (which is much harder) represents a substantial dinner with a structured sequence of dishes. This material (in somewhat different form) was initially assembled to provide the introductory lectures for the writer's graduate courses in Scattering Theory in the Mathematics Departments of Stanford University and of the Technion-Israel Institute of Technology. The different backgrounds of the students (mathematics, physics, and electrical engineering) called for a general elementary survey to introduce fundamental concepts and terms, and this also served to provide motivation for the mathematical model of scattering, for considering certain classes of problems, and for the development of both analytic and heuristic procedures. The present version, Chapter 3, is in general more elaborate; it was amplified initially to serve as a larger vehicle for calculus techniques (as Chapter 15 of the SMSG Calculus), and then revised to make it more self-contained. However, the final portion of Section 3.9 is largely a stop-gap for a development that should lead into Green's theorem, Maxwell's equations, the acoustic equations, and Schrödinger's equation.

Chapter 3, which attempts to give an individual esthetically complete picture of a portion of wave physics, is hindered because it has only a quite limited number of physical concepts and mathematical methods available for the development. We proceed heuristically in some cases (sometimes the rigorous procedure is mentioned parenthetically), and in other cases we visualize experiments and anticipate refinements in attempts to provide some substitute for the intuition that comes with first hand experience. It is hard to be both simple and honest, and to avoid using implicitly the more complete results that are available. Attempts have been made to weed out blunders and oversights, but the forms of various sections are still only provisional versions of what the writer would like them to be. In any case, much of the mathematical development is informal, and some discussions (probably many more than intended) are no more than suggestive.

As for notation, the following warning for the writer's material was included (by another member of the team) in the Calculus: Teachers Commentary (p. 655):

"In scientific usage there are often notational ambiguities which we have not eliminated. When we say N is a function of t we mean that there exists a function $\nu: t \rightarrow N$, not that N itself is a function. Careless paraphrase of this expression leads to the use of N(t) for $\nu(t)$, a particularly bad but common notation in science. While at first encounter, such ambiguities may distress the student, he may take comfort in the realization that comprehension of this and future scientific material he may read can be enhanced by less-cumbersome — if less precise — notation."

A less specific warning on notation that is also appropriate was written by the late I. Marx and perturbed by the writer in 1964:

"When a function f is defined by a formula such as $f(x) = x^2(72 - 4x)$, we distinguish between the function f, an assignment of values to all x in the domain of f, and the values f(x), the numbers assigned. Similarly, in geometry we distinguish between a point P and the coordinates (3,-2) of the point, and between a curve C and the equation $(x + 1)^2 = 2(y - 2)$ of the curve. However, for brevity and where confusion is unlikely, we shall omit certain words (which should be included for mathematical accuracy) with the understanding that they are implicit. We shall shorten 'the function f with values $f(x) = x^2(72 - 4x)$' to 'the function $f(x) = x^2(72 - 4x)$'; 'the point P with coordinates (3,-2)' to 'the point (3,-2)'; and 'the curve C with equation $(x + 1)^2 = 2(y - 2)$' to 'the curve $(x + 1)^2 = 2(y - 2)$'. In reading such phrases, we keep in mind that the mathematical object and its presentation by a formula or set of numbers are quite distinct: the representation is used to replace the object in order to make the material easier to read. Of course, such abbreviations are usually used conversationally by professional mathematicians, but they avoid them when they write for mathematics journals: they avoid them, not by putting in all the extra words (which would make the article absolutely unreadable), but by leaving out almost all words (which makes the article only conditionally unreadable)."

4. Collateral Reading and References.

Those who may not yet have read the preceding books of the SMSG math-science series should read the excellent and lively volumes by

Max M. Schiffer, Applied Mathematics in the High School, Vol. X of SMSG Studies in Mathematics, 1963;

George Polya, Mathematical Methods in Science, Vol. XI of SMSG Studies in Mathematics, 1963.

Part of the following discussions of population growth and Euclidean optics is based on Volume X (which assumes no calculus, and uses difference equations instead of differential equations). Both volumes are remarkable for their content, style, and for the insight they convey.

Practically everything in the following chapters is included in Calculus, Student Text plus Teachers Commentary (SMSG Calculus Team, Chairman A.A. Blank), SMSG, 1966.

Some of this text's alternative versions of the present material may well prove more palatable, and some of the results that are simply stated in the present volume were assigned as problems in the Calculus with worked solutions included in the Teachers Commentary. This text also includes H. Weitzner's Chapter on mechanics and oscillatory phenomena, many additional problems that extend the utility of the present chapters, and much additional material of interest to math-science programs.

Many of the physics topics touched on in Chapter 2 are covered in a good general college text on physics that uses calculus:

F.W. Sears and Mark W. Zemansky, University Physics, Addison-Wesley Press, Inc., Mass., 1952.

More results on radioactive disintegration are given by

Henry Semat, Atomic Physics, Rinehart and Co., Inc., New York, 1946.

Chemical reactions are discussed in detail by

S. Glasstone, Physical Chemistry, Ch. XI, D. Van Nostrand Co., Inc., New York, 1940.

The model for learning is based on

H. Von Foerster, Quantum Theory of Memory, Transactions of Sixth Conference on Cybernetics, pp. 112-134, Josiah Macy, Jr. Foundation, New York, 1956.

The Section on propagation of nerve impulses is based on the work of Blair and Rashevsky as discussed by

N. Rashevsky, Mathematical Biophysics, Ch. XXIII, University of Chicago Press, Illinois, 1948,

and the discussion of sociology was derived from

N. Rashevsky, Mathematical Theory of Human Relations, Principia Press, Indiana, 1947.

As for Chapter 3, an exciting, non-mathematical book on natural visual phenomena is that by

M. Minnaert, Light and Color in the Open Air, Dover Publications, Inc., New York, 1954,

and a detailed historical development of optics is given by

Ernst Mach, The Principles of Physical Optics, Dover Publications, Inc., New York, 1953.

A good general college text on geometrical and wave optics is

F. A. Jenkins and H. E. White, Fundamentals of Optics, McGraw-Hill Book Co., Inc., New York, 1957.

A detailed discussion of the wave fronts for reflection from a concave hemisphere is given by

R. W. Wood, Physical Optics, p. 54ff, The MacMillan Co., New York, 1934.

Detailed discussions of various aspects of wave physics are given by

R. B. Lindsay, Mechanical Radiation, McGraw-Hill, New York, 1960.

Photographs of the caustics of edge waves (or edge rays) are given by

J. Coulson and G. G. Becknell, Reciprocal Diffraction Relations Between Circular and Elliptical Plates, Physical Review XX, pp. 594-600 (1922); An Extension of the Principle of the Diffraction Evolute and Some of Its Structural Detail, Physical Review XX, pp. 607-612 (1922).

The discussion of Rayleigh scattering is based on papers number 8 and 247 of

Lord Rayleigh, Scientific Papers, Dover Publications, Inc., New York, 1964.

The extension of geometrical optics to diffraction and to other "physical optics" phenomena is given by

J. B. Keller, A Geometrical Theory of Diffraction, pp. 27-52, Proc. Symp. Appl. Math. Vol. 8, Amer. Math. Soc., Providence, R.I., 1958.

Introductory discussions to mathematical aspects of wave phenomena involving more than one scatterer are included in

V. Twersky, Multiple Scattering of Waves and Optical Phenomena, Journal of the Optical Society of America 52, pp. 145-171 (1962);

in particular, Section 1.2 on scattering by two small obstacles could be read after the discussion of one obstacle in Section 9 of Chapter 3.

Chapter 2

GROWTH, DECAY, AND COMPETITION

1. Introduction.

We recognize that the mature areas of today's sciences have undergone long historical development to reach their present systematic deductive stage. There have been earlier stages of observation and classification of superficially related phenomena, of measurement and the collection of data, of seeking detailed interrelations, of isolating essential factors, of creating new concepts to link the more obscure interrelations, of proposing "laws of nature" to summarize and generalize observations, of testing such laws to determine the class of phenomena to which they apply, of inventing more general more abstract laws, and of predicting less obvious phenomena. Cycles of observation, speculation, and verification mark our attempts to systematize our knowledge of nature. We exploit yesterday's science to invent today's tools to discover the science of tomorrow.

Mathematics plays its role on every stage of the development of a science into a deductive system — from the initial classification of related phenomena, to the search for the least number of fundamental principles on which they depend — from the establishment of laws of nature which serve essentially as axioms for a deductive system to the unfolding of the consequences implicit in such mathematical models of natural phenomena. Science deals with phenomena — with observations, predictions, and experiments on nature. Special areas of science require special equipment, and special measurement techniques. All areas require mathematical thinking, mathematical tools, and mathematical models.

All the sciences, from astronomy to zoology, use mathematical models for complicated phenomena observed in nature. To construct a model, we isolate the effects that appear to be fundamental, and we define relevant variables, parameters, and functions. As suggested by our observations and measurements, we seek appropriate equations for the dependence of the functions on the essential variables; for completeness, we may have to introduce auxiliary conditions that specify, for example, the initial values of the functions and of the variables at the start of a process. The solutions of the equations subject to the auxiliary restrictions may then be compared with additional measurements to determine their domain of applicability in nature.

We make observations, we create models, we make predictions; we make more observations, more models, more predictions, we day-dream and jump to conclusions; we seek to verify our guesses, and keep the very few that pass the tests. By such means, by a mixture of measurements, mathematics, and mysticism we seek to "understand" what is going on around us. If we can predict and describe a process and relate it to analogous processes that we know about, we are content — for a while.

Despite the marked differences in the classes of phenomena studied in the various sciences, there are a number of processes that are fundamental to all. The same mathematical structures, and consequently the same underlying concepts, are encountered again and again in different contexts of nature. The equations are the same, but the functions and variables represent different measureable quantities with different names that change from science to science. The stages and settings are very different, and the overall plots vary; but the subplots are routine, the actors go through the same motions, and only the names of the characters are changed.

Let us consider a class of processes that is common to all the sciences, processes that involve such notions as "growth," "decay," and "competition." We can construct mathematical models for such processes by using either very general abstract mathematical language, or by using the special language of one of the special sciences. To show complete impartiality, we do neither. Instead we use the language for story telling.

We begin with a story — a story about stories. Then we show how the same story can be told again and again and again . . . .

2. The Spread of a Story: Model for Growth.

Once upon a time (t_0) I told a number (N_0) of friends a story about my good friend Al. Months later (time t) someone came up and asked, "Did you hear the one about Al?" Since I had started the whole business, I didn't have to listen. Instead I asked myself, "How many $[N(t)]$ people have now heard the story about Al?"

How many people know the story about Al? Good stories spread, and this was a good one; the number of people that know it grows with time. The number $N(t)$ of people who know it at time t should be proportional to the original number N_0 that were told the story at time t_0 — to the N_0 storytellers that couldn't keep a good thing to themselves. The older the story, the more people know it. Therefore $N(t)$ increases with the length of time $t - t_0$ that the story has been circulating, as well as with the number of people available to spread it. If $N(t)$ know the story at time t, how many $N(\tau)$ know it at a slightly later time τ? Since $N(\tau) - N(t)$ must vanish if either $N(t)$ or $\tau - t$ vanishes, it is plausible to expect that the number $N(\tau) - N(t)$ of people who learn the story in the small time interval $\tau - t$ is directly proportional to both $N(t)$ and to the interval $\tau - t$. We accept these ideas as the initial assumptions, and express them mathematically in the form

$$(1) \qquad N(\tau) = N(t) + AN(t)[\tau - t], \qquad N(t_0) = N_0,$$

where A is a positive constant — the growth coefficient. (Have we left out anything? Sure. We'll talk about that later.)

Accepting (1) as an adequate model for the change in N over a small time interval still does not tell us how $N(t)$ is related to the initial value N_0. To determine this, we let the time interval approach zero and thus replace (1) by a differential equation, and then integrate over time to obtain $N(t)$ in terms of N_0.

From (1) we have

$$(2) \qquad \frac{N(\tau) - N(t)}{\tau - t} = AN(t).$$

If we discount the fact that friends come in integral packages (usually) and go to the limit as τ approaches t, we obtain

$$(3) \qquad \frac{dN(t)}{dt} = AN(t), \qquad N(t_0) = N_0.$$

Equation (3) states that the instantaneous rate of change of N is proportional to N: this is the basic equation for growth. Later on, we will also consider the case where A is negative; with A negative we have the basic equation for decay. (If A is zero, then N is a constant, and there is nothing to talk about — neither for potential story-tellers nor for us.)

In order to express $N(t)$ explicitly in terms of N_0 we integrate

$A = \frac{1}{N(t)} \frac{dN(t)}{dt}$ from $t = t_0$ to $t = t_1$, i.e., $\int_{t_0}^{t_1} A\,dt = \int_{t_0}^{t_1} \frac{1}{N(t)} \frac{dN(t)}{dt}\,dt$,

and obtain

$$(4) \qquad \log N(t_1) - \log N(t_0) = \log\left[\frac{N(t_1)}{N(t_0)}\right] = A(t_1 - t_0).$$

Expressing both sides of the equation in terms of exponentials, we have for $t_1 = t$,

$$(5) \qquad N(t) = N(t_0)e^{A(t - t_0)} = N_0 e^{A(t - t_0)}.$$

As a check, we differentiate (5), and verify directly that

$$(6) \qquad \frac{dN(t)}{dt} = AN_0 e^{A(t - t_0)} = AN(t)$$

in accord with (3).

For convenience in all that follows we take $t_0 = 0$ as the initial time. Thus (5) becomes

$$(7) \qquad N(t) = N_0 e^{At}, \qquad N_0 = N(0),$$

where t is the time that has elapsed since the start of the process.

From Equation (5), we see that $N \to \infty$ as $t \to \infty$ (i.e., N increases beyond any bound as t approaches infinity) which is not realistic for what we know about storytelling (and other growth processes). Later on we consider a more realistic model. The present model is incomplete and should be restricted to moderately small time interval t.

We have told a story about stories to get to (7). Now that we have (7), we recognize that the result has other interpretations and that the analysis has other applications. Equation (7) provides an elementary model for the growth of timber and vegetation, the growth of populations (people, bacteria), the growth of money in banks (generous banks where they credit the interest to the capital instantaneously), the growth of a substance in the course of a chemical reaction, and so on.

We can now answer such questions as:

If I tell 2 people the story at $t = 0$, and if the number $N(t)$ that knows it grows at a rate proportional to $A = 1$ per day, then how much is $N(t)$ at time $t = 7$ days? The answer from (7) is $2e^7$ or approximately 2193; thus more than 2000 people know the story a week after I started to spread it.

If I deposit \$10 at 5% interest per year and the bank adds the interest to the original amount continuously, then when will it reach \$20? For $A = \frac{5}{100}$ it follows from (4) that $\log\left(\frac{20}{10}\right) = \left(\frac{5}{100}\right) \cdot t$. Thus $t = 20 \log 2 = 20(0.693\cdots) \approx 13.9$ years.

In the following we refer to an equation, say (3), of this section by writing (2:3), etc.

3. Radioactive Disintegration: Decay with Time.

The same considerations that led us to our simple model for growth apply equally to the analogous model for decay (negative growth). We take a negative constant of proportionality $-A$ in (2:3) to correspond to $N(t)$ decreasing in time, and apply

$$(1) \qquad \frac{dN(t)}{dt} = -AN(t); \qquad N(0) = N_0; \qquad N(t) = N_0 e^{-At}$$

to the problem of radioactive decay. Different radioactive substances disintegrate at different rates corresponding to different values of the <u>decay coefficient</u> A. It is convenient to express the coefficient in terms of the <u>half-life</u> of the substance, the time it takes half of the initial amount of substance to disappear. (Why not the whole-life?) If τ is the half-life, then from (1) we have

$$\frac{N(t)}{N_0} = e^{-A\tau} = \frac{1}{2}$$

so that

$$\tau = \frac{-\log\frac{1}{2}}{A} = \frac{\log 2}{A} \approx \frac{0.693}{A}. \qquad (2)$$

Half of the material N_0 will be left at time τ, one-quarter will be left at time 2τ, etc. When will it all be gone? We see from (1) that in order for N to approach 0, we require that t approach infinity; we have $N \sim 0$ for all A (all substances) as $t \sim \infty$, and this is why the whole-life is a useless measure.

Let us consider a specific example. The half-life of radium is about 1600 years, and the corresponding decay coefficient A is

$$A \approx \frac{0.693}{1600} \approx 0.000433 \text{ per year}.$$

If we start with some given amount (N_0) and wait a hundred years, we get $\log\frac{N}{N_0} \approx -0.0433$, and consequently $N \approx 0.958N_0$ is the amount left. Thus only 4.2% disappears in one hundred years.

The basis for applying (1) to radioactivity is statistical, i.e., it holds in the sense of an average. Although the physical process is governed by probability, and we cannot tell when any one atom will disintegrate, it is quite useful to determine the mean life-time per atom. We start with N_0 atoms at $t = 0$ and end up with 0 atoms as t approaches infinity, and we are interested in the average length of time that an atom exists.

If n_1 atoms disappear at time t_1, and n_2 atoms at time t_2, etc., where

$$t_1 < t_2 < \cdots t_k$$

and $\sum_{i=1}^{k} n_i = N_0$, then the mean life-time of an atom is the average value

$$\frac{1}{N_0}\sum_{i=1}^{k} t_i n_i .$$

If the total number of atoms present in the interval (t_{i-1}, t_i) is N_i, and N_{i+1} is the number present in the interval (t_i, t_{i+1}), then

$$n_i = N_i - N_{i+1},$$

and the mean becomes

$$\frac{1}{N_0} \sum_{i=1}^{k} t_i (N_i - N_{i+1}).$$

Treating the relation between N and t as though it were continuous (essentially as for (2:3)), we define the mean life-time as

(3) $$T = -\frac{1}{N_0} \int_{N_0}^{0} t \, dN = \frac{1}{N_0} \int_{0}^{N_0} t \, dN .$$

We now seek the relation between the mean life-time T per atom and the decay coefficient A.

From (1), we have

(4) $$dN(t) = -AN(t)dt = -N_0Ae^{-At} dt .$$

Using (4), we change the integration variable in (3) to t, and the limits N_0 and 0 to the corresponding values $t = 0$ and ∞:

(5) $$T = A \int_0^\infty te^{-At} dt .$$

To evaluate the integral in (5), we regard it as the form $\int u \, dv$ (with $u = t$ and $dv = e^{-At}dt$) and integrate by parts:

$$\int_0^\infty te^{-At}dt = \left[\frac{te^{-At}}{-A} - \frac{e^{-At}}{A^2}\right]_0^\infty = \frac{1}{A^2} .$$

Substituting in (5), we obtain

(6) $$T = \frac{1}{A} .$$

Thus the mean life-time is the reciprocal of the decay coefficient.

We have a model for simple radioactive decay. What is left after an atom disintegrates? Many things, including "daughter" atoms which can also disintegrate. Later on we talk about some mother-daughter relations, and discuss the decay of both populations.

The simple decay model we have been considering also describes essential features of many other phenomena. As another example within the same mathematical structure, we need change only the names of the characters in order for the results to apply to the molecules of air in the room. Suppose that N_0 is the total number of molecules present, N(t) is the number that have not had collisions by time t, and that the mean time between collisions is T. Then $\frac{N(t)}{N_0}$, the probability that any one molecule goes for time t without a collision, follows directly from (1) and (6):

(7) $$\frac{N(t)}{N_0} = e^{-t/T} .$$

Same equation as previously, but the characters are now playing different physical roles, and, of course, the overall plot is quite different. Were we to continue the present story we would require much additional structure. If the mean velocity of the molecules is v, then $L = Tv$ is called the mean free path — the average distance a molecule travels between collisions — but this path leads to statistical mechanics (which bases the physical properties of matter on the motions of molecules) and would take us too far afield.

Electrical circuits offer several examples of decay processes. The fundamental roles are played by charge q, current $I = \frac{dq}{dt}$, and voltage or electromotive force V. Simple circuit components (capacitors or condensers, resistors, inductors or coils) are characterized by constants: capacity $C = \frac{q}{V}$, resistance $R = \frac{V}{I}$, and inductance $L = V/\frac{dI}{dt}$. If a condenser at voltage V discharges across a resistor it initiates a process described by

(8) $$\frac{dV}{dt} = \frac{-V}{CR},$$

and consequently

(9) $$V(t) = V_0 e^{-t/CR} = \frac{q_0}{C} e^{-t/CR},$$

where V_0 is the initial voltage on the condenser and q_0 the initial charge. A current I flowing through a circuit consisting of a coil and a resistor satisfies

(10) $$\frac{dI}{dt} = -\frac{R}{L} I,$$

and consequently,

(11) $$I = I_0 e^{-Rt/L}$$

where I_0 is the initial value of the current.

4. The Colors of the Sky: Attenuation with Distance.

The model of Section 3 for a process that decays with time can also be applied for processes that attenuate (weaken) with distance. Let us relabel the variable of (3:2) and call it "distance," and write it as "x" to keep the new role in view. We thus have

(1) $$\frac{dN(x)}{dx} = -AN(x)\,, \qquad N(0) = N_0\,; \qquad N(x) = N_0 e^{-Ax}$$

which we call the attenuation equation.

The attenuation of the earth's atmosphere with altitude is described approximately by (1) with x as the height above the earth's surface, $N(x)$ the number of molecules per unit volume (the density) of atmosphere at height x, and N_0 the density at ground level. There are different kinds of molecules in our atmosphere with different masses m, so that we should introduce m as a physical parameter and write

(2) $$N(m,x) = N_0(m)e^{-A(m)x}$$

for each constituent. If one introduces more structure into the model, it turns out that $A = ma$, where a is independent of m (but depends on temperature, and the acceleration of gravity).

As an alternative setting for (1), visualize a narrow beam of particles incident on the face of a medium of more massive particles as in Figure 1.

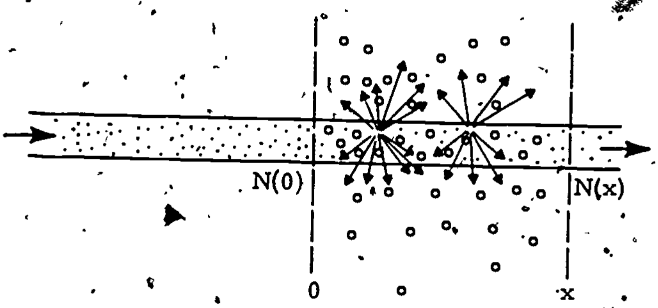


FIGURE 1

There are N_0 particles per unit volume of the incident beam and nothing happens to them until they encounter the medium that starts at $x = 0$. Then as the beam penetrates, its lighter particles hit the heavier ones of the medium and go off in other directions: particles of the incident beam are lost to other directions by scattering. (Visualize a column of children trying to charge through a crowd of milling adults.) Thus $N(x)$, the density of particles in the incident beam at a distance x within the medium is less than N_0; this attenuation is governed by the scattering coefficient per unit length A. (For other processes, density along the beam may attenuate because its particles combine with the heavier ones of the medium.)

The principal characters in the above are particles — loose characters that can stand for electrons, protons, children, etc. If we now relabel N as energy density per unit volume or intensity, then the same plot also holds for light-rays, x-rays, γ-rays, and all other kinds of waves meeting appropriate obstacles. For the particles, we mentioned one physical parameter, the mass m, and spoke of particles being scattered by more massive ones. For waves, the appropriate physical parameter is the wavelength λ — e.g., the distance between adjacent crests of equally spaced ripples on a lake. The longest wavelengths associated with visible light give the sensory impression called "red," and they are about twice as long as the shortest of the wavelengths associated with visible "blue." From blue light to ultra-violet to x-rays to γ-rays we go to shorter and shorter electromagnetic wavelengths, from red light to infrared to microwaves to radio waves we go to longer electromagnetic wavelengths. We can also talk about sound waves, water waves, and even the waves of "probability amplitude" associated with electrons, neutrons, and other fundamental particles.

With N for intensity, (1) in terms of an appropriate A describes the attenuation of a beam of sunlight penetrating a cloud or a layer of fog, etc. We can use (1) to determine the thickness of lead shields to be used with dental x-ray equipment or with a nuclear reactor to reduce stray radiation to a tolerable value. We could talk in greater detail on any of the above, but instead let us talk about something more colorful.

Let us consider Rayleigh's theory for the color of the sky. The essential feature of sunlight is that it is made up of light of different colors from red to blue (the visible spectrum) with associated wavelengths λ_r to λ_b such that (approximately)

(3) $$\lambda_r = 2\lambda_b \,.$$

The wavelength λ of an intermediate color (orange, yellow, green) satisfies $\lambda_r > \lambda > \lambda_b$. Rayleigh showed that when a beam of light of wavelength λ is scattered by the molecules of the earth's atmosphere (mainly nitrogen and oxygen), the intensity $N(\lambda,x)$ along the beam is governed approximately by (1) with

(4) $$A(\lambda) = \frac{C}{\lambda^4}$$

where C is independent of λ. (In the chapter on optics and waves we discuss this more fully.)

From (4) and (3) we have

(5) $$\frac{A(\lambda_b)}{A(\lambda_r)} = \frac{\lambda_r^4}{\lambda_b^4} = \frac{(2\lambda_b)^4}{\lambda_b^4} = 16$$

and consequently

(6) $$\frac{N(\lambda_b, x)}{N_0(\lambda_b)} = e^{-A(\lambda_b)x} = e^{-16A(\lambda_r)x} = \left[\frac{N(\lambda_r, x)}{N_0(\lambda_r)}\right]^{16}.$$

Thus the blue component of white light is 16 times more strongly attenuated than the red. A beam of white sunlight reddens with penetration into the earth's atmosphere because it loses its blue component more rapidly than its red. The blue that is lost from the sunbeams by scattering gives the sky its blue color in directions away from the sun. The direct beams from the overhead sun are still relatively white because they have not lost that much blue. The reddening of the direct beams is best seen when the sun is low on the horizon and its rays traverse maximum distance through the scattering atmosphere; the clouds in the path of these rays are bathed in red. Such colored effects and other scattering phenomena arising from water drops, dust particles, and other impurities in the atmosphere are more fully discussed in poetry courses.

5. Mother-Daughter Relations: Birth and Decay.

As mentioned previously, when a radioactive atom (the mother atom) disintegrates it may give rise to a daughter atom which can also disintegrate. Let us now consider such mother-daughter relations.

Suppose we have N_1 mother atoms which decay at the rate

(1) $$\frac{dN_1}{dt} = -A_1N_1, \qquad N_1(0) = N_{10}.$$

The rate at which the mothers decay equals the rate at which the daughters are created. But the daughters also decay on their own. If N_1 mothers with birth coefficient A_1 give rise to N_2 daughters with decay coefficient A_2, the rate of change of the number of daughters is given by

(2) $$\frac{dN_2}{dt} = A_1N_1 - A_2N_2, \qquad N_2(0) = 0.$$

Equations (1) and (2) form a pair of simultaneous equations for determining N_1 and N_2.

Let us first consider a limiting case such that the mothers decay very slowly compared to the daughters, i.e., the decay coefficient of the first is much smaller than that of the second, $A_1 \ll A_2$. This corresponds, for example, to the behavior for the pair radium-radon. For radium mothers, the half-life is approximately 1600 years: $A_1 = \frac{1}{1600}$ per year. The radon daughters have a half-life of about 4 days:

$$A_2 \approx \frac{1}{4\text{ days}} = \frac{360\text{ days}}{\text{year}} \cdot \frac{1}{4\text{ days}} = \frac{90}{\text{year}}.$$

Thus $A_2 \approx 144{,}000A_1$, and we may take the number N_1 of mothers constant for the purpose of obtaining a first approximation of $N_2(t)$:

Thus we regroup the terms in (2), replace t by t_1, and integrate from $t_1 = 0$ to $t_1 = t$,

$$-A_2 \int_0^t \frac{dN_2}{A_1N_1 - A_2N_2} = -A_2 \int_0^t dt_1,$$

subject to the approximation that N_1 is constant. We obtain

$$\log K(A_1N_1 - A_2N_2) = -A_2t,$$

and

$$K(A_1N_1 - A_2N_2) = e^{-A_2t},$$

where the integration constant K is to be determined from the initial conditions at t = 0. There are no daughters at t = 0:

$$K(A_1N_1 - 0) = e^{-A_2t}\Big|_{t=0} = 1,$$

so that

$$K = \frac{1}{A_1N_1}.$$

Consequently, the number of daughters at time t is given approximately by

(3) $$N_2 = \left(\frac{A_1}{A_2}\right)N_1(1 - e^{-A_2t}).$$

If A_2t is very large, then N_2 approximates $\frac{A_1N_1}{A_2}$, i.e., the number of daughter atoms approaches a fixed fraction of the relatively inert mother substance. (This is called long term or secular equilibrium.) What does this mean? It corresponds, for example, to the case where N_2 is a gas (such as radon) in a closed container, and a situation where just as much N_2 is created (from N_1) as is destroyed by radioactive decay. The birth rate of N_2 equals its death rate, so that $\frac{dN_2}{dt}$ is zero; our result as t approaches infinity in (3) is thus the same as that obtained by equating (2) to zero. Equilibrium corresponds to

(4) $$\frac{dN_2}{dt} = 0; \quad N_2 = \left(\frac{A_1}{A_2}\right)N_1.$$

Before continuing the development let us feedback the above to our earlier discussion of electrical circuits (3:8). Equation (2) with N_1 held constant is also the mathematical relation for the current I in an electrical circuit having resistance R, inductance L, and external voltage V:

(5) $$\frac{dI}{dt} = \frac{V}{L} - \frac{R}{L}I.$$

If V is constant, and if $I = 0$ at $t = 0$, then the solution of (5) for the growth of current in time is obtained by inspection of (2) and (3):

(6) $$I = \frac{V}{R}\left(1 - e^{-Rt/L}\right).$$

If $V = 0$ and $I = I_0$ at $t = 0$, then (5) reduces to (3:10).

Now back to mothers and daughters, and let the mothers also decay in (2). To take this into account, we substitute the solution of (1), i.e., $N_1(t) = N_{10}e^{-A_1t}$ into (2) and obtain

(7) $$\frac{dN_2}{dt} + A_2N_2 = A_1N_{10}e^{-A_1t}.$$

We solve for N_2 by using e^{A_2t} as the "integrating factor." We note that

$$\frac{d}{dt}\left(N_2e^{A_2t}\right) = \frac{dN_2}{dt}e^{A_2t} + N_2\frac{d}{dt}e^{A_2t} = \left(\frac{dN_2}{dt} + A_2N_2\right)e^{A_2t}.$$

To exploit this, we multiply both sides of (7) by e^{A_2t} to obtain

$$\frac{d}{dt}\left(N_2e^{A_2t}\right) = A_1N_{10}e^{A_2t - A_1t},$$

and then replace t by t_1 and integrate from $t_1 = 0$ to $t_1 = t$:

$$N_2e^{A_2t}\Big|_0^t = A_1N_{10}\int_0^t e^{A_2t_1 - A_1t_1}dt_1 = \frac{A_1N_{10}(e^{A_2t - A_1t} - 1)}{A_2 - A_1}.$$

Since $N_2(0) = 0$, we thus have

(8) $$N_2(t) = \frac{A_1}{A_2 - A_1}N_{10}\left(e^{-A_1t} - e^{-A_2t}\right),$$

which reduces to (3) if $A_1 \ll A_2$, and $A_1t \approx 0$. In distinction to the approximation (3), the present complete form N_2 of (8) vanishes both for $t \sim 0$ and $t \sim \infty$; consequently N_2 must have a maximum at a specific value of t.

If we differentiate (8) with respect to t we obtain

(9) $$\frac{dN_2}{dt} = \frac{A_1N_{10}}{A_2 - A_1}\left(-A_1e^{-A_1t} + A_2e^{-A_2t}\right).$$

This vanishes, and N_2 has a maximum, when

$$A_1e^{-A_1t} = A_2e^{-A_2t}, \quad \frac{A_1}{A_2} = e^{(A_1 - A_2)t}.$$

From the logarithmic form, we obtain

(10) $$t = \frac{\log A_1 - \log A_2}{A_1 - A_2}$$

as the time when the number of daughters is largest. The maximum value of daughters is $N_2 = N_{10}e^{-A_2t} = \left(\frac{A_1}{A_2}\right)N_{10}e^{-A_1t} = \left(\frac{A_1}{A_2}\right)N_1$ as in (4).

6. Neural Processes: Stimulation and Decay.

One method of studying the excitation and propagation of nerve impulses is based on stimulating a nerve fiber electrically. Characteristic measurable effects are produced in response to an electrical stimulus V (the voltage associated with a direct current, the discharge of a condenser, or an alternating current), provided that V is greater than a threshold value V_e — the minimum value of V that is just sufficient to cause the effects. A simple model (introduced by Blair on the basis of experimental data) describes the onset of the effects in terms of a local latency (also called the "local excitatory function") N(t) such that

(1) $$\frac{dN(t)}{dt} = KV(t) - AN(t),$$

where K is the growth of the latency per second per unit stimulus, and A is its decay coefficient. Thus the growth of N increases with the magnitude of the stimulus and decreases with N. (The function N may represent the difference between the concentration of an exciting ion at an electrode while V is applied and its concentration for V = 0.) If N(t) reaches (or exceeds) a threshold value N_e, then the nerve becomes excited (and a characteristic wave of physical-chemical changes with an associated electric potential propagates along the fiber).

The simplest application of (1) is to the situation

(2) $$N(0) = 0, \quad V(t) = V = \text{constant},$$

which corresponds to the application of a constant stimulus at time t = 0. By inspection of (5:2) and (5:3), the solution of (1) and (2) is

(3) $$N = \frac{KV}{A}\left(1 - e^{-At}\right).$$

Thus as $t \sim \infty$, we see that N approaches its largest value $N_{max} = \frac{KV}{A}$. Consequently excitation will occur if

(4) $$N_{max} = \frac{KV}{A} \geq N_e,$$

or equivalently if the stimulating voltage satisfies

(5) $$V \geq \frac{AN_e}{K} = V_e,$$

where V_e is the threshold stimulus mentioned previously. (The value V_e is known as the rheobase, the threshold or liminal value of the constant voltage necessary for excitation.)

Assuming that $V > V_e$ (so that excitation must occur), then the nerve becomes excited at the time t_e corresponding to N of (3) reaching the threshold value

(6) $$N_e = \frac{KV}{A}\left(1 - e^{-At_e}\right),$$

or, equivalently,

(7) $$t_e = \frac{1}{A}\log\frac{V}{V - V_e}$$

which is the latent period that elapses between the establishment of the constant stimulus and the release of excitation. If $V < V_e$, no real value of t_e exists. If $V = V_e$, then $t_e \sim \infty$; however, this is an inconvenient length of time for measurement purposes. A more convenient measure is the value of t_e corresponding to $V = 2V_e$:

(8) $$\tau = \frac{\log 2}{K} \approx \frac{0.693}{K}.$$

This is known as the chronaxie τ—the latent time before excitation for the case of a stimulating voltage equal to twice the threshold value.

What have we been doing in the above? —essentially changing the names of the characters introduced for radioactive decay and showing how a few simple ideas link parts of biology, physics, and chemistry. Let us now generalize the mathematical development to nonconstant values of V in (1).

If V is a function of time, we solve (1) in terms of $V = V(t)$ by proceeding essentially as for (5:7): we transpose AN to the left side, multiply by e^{At} to obtain $\left(\frac{dN}{dt} + AN\right)e^{At} = \frac{d}{dt}(Ne^{At}) = KV(t)e^{At}$, integrate the left side explicitly, and regroup to obtain

(9) $$N = e^{-At}\left\{N_0 + K\int_0^t V(t_1)e^{At_1}dt_1\right\}.$$

If $N(0) = N_0 = 0$, and V is a constant, then (9) reduces to (3).

If we stimulate the process by discharging a condenser of initial charge q, capacity C, and resistance R, then as in (3:9),

(10) $$V(t) = \left(\frac{q}{C}\right)e^{-t/CR}.$$

Substituting in (9) and integrating, we obtain for $N_0 = 0$,

(11) $$N = \frac{KqR}{(CRA - 1)}[e^{-t/CR} - e^{-At}]$$

which is simply (5:8) with different labels. Thus the excitation function N has a maximum when

(12) $$t = \frac{CR}{(1 - CRA)}\log\left(\frac{1}{CRA}\right).$$

If the maximum value of N is precisely the threshold value, then t of (12) is the corresponding latent time from onset of stimulus V to release of a wave of activity in the nerve. The corresponding initial voltage $V(0) = \frac{q}{C}$ is the threshold initial voltage of the condenser for excitation to occur. If the maximum does not equal the threshold, then we relate the condenser's characteristics to the threshold by equating (11) to N_e and using $V_e = \frac{AN_e}{K}$ to eliminate K.

If we stimulate the process by a sinusoidal alternating current, then the applied voltage is

(13) $$V(t) = V_0 \sin \omega t ,$$

where V_0 is the constant amplitude. Substituting in (9) for $N(0) = 0$ we have

(14) $$N(t) = e^{-At} KV_0 \int_0^t e^{At_1} \sin \omega t_1 dt_1 .$$

To evaluate the integral in (14), we regard it as the form $\int u dv$ (with $u = e^{At}$ and $dv = \sin \omega t\, dt$) and integrate by parts:

$$\int e^{At} \sin \omega t\, dt = \frac{-e^{At} \cos \omega t}{\omega} + \frac{A}{\omega} \int e^{At} \cos \omega t\, dt .$$

This doesn't look as if it will get us anywhere, but let us handle the new integral by again integrating by parts (with $dv = \cos \omega t\, dt$):

$$\int e^{At} \sin \omega t\, dt = \frac{-e^{At} \cos \omega t}{\omega} + \frac{A}{\omega^2} e^{At} \sin \omega t - \frac{A^2}{\omega^2} \int e^{At} \sin \omega t\, dt .$$

Transposing the integral from the right-hand side to the left and dividing through by $1 + \frac{A^2}{\omega^2}$ we obtain

$$\int e^{At} \sin \omega t\, dt = \frac{-\omega e^{At} \cos \omega t + Ae^{At} \sin \omega t}{\omega^2 + A^2} .$$

Consequently the solution of (14) is

(15) $$N(t) = \frac{KV_0}{\omega^2 + A^2} (A \sin \omega t - \omega \cos \omega t + \omega e^{-At}) .$$

The exponential term of (15) is significant only for small values of t. As t increases, e^{-At} becomes negligible and (15) reduces to

(16) $$N(t) \approx \frac{KV_0}{\omega^2 + A^2} (A \sin \omega t - \omega \cos \omega t).$$

This periodic approximation has equally spaced extrema in time, which occur when

(17) $$\frac{dN}{dt} = \frac{KV_0 \omega}{\omega^2 + A^2} (A \cos \omega t + \omega \sin \omega t) = 0 , \quad t = \frac{1}{\omega} \tan^{-1}\left(\frac{-A}{\omega}\right).$$

Substituting these values of t into (16), we find that the maxima of N equal

(18) $$N_{max} = \frac{KV_0}{(\omega^2 + A^2)^{1/2}} .$$

If we equate N_{max} to the threshold value N_e, then V_0 of (18) corresponds to the threshold value of the amplitude of the sinusoidal stimulus, say V_{0e}; however, the result $V_{0e}^2 \propto \omega^2 + A^2$ has only very limited validity in nature.

7. A More Realistic Story About Stories: Bounded Growth.

Let us return to our model for the spreading of stories (or of diseases, or of ink blots), and introduce more structure. Previously we assumed that the rate of change of the number that knew the story at time t was proportional only to the number itself:

$$(1) \qquad \frac{dN(t)}{dt} = AN(t), \qquad N(0) = N_0.$$

This is all right as far as it goes, but it ignores the fact that there is an upper bound (say N_m) on the number available to hear it: there is a finite number of people on earth, some don't talk your language, some don't talk at all, and most never listen. Furthermore, although we may tell the same person the same story a dozen times, each listener should be counted only once.

In view of these considerations, we replace (1) by

$$(2) \qquad \frac{dN}{dt} = AN\frac{(N_m - N)}{N_m}, \qquad N(0) = N_0,$$

where $N = N(t)$ know the story at time t and are available to spread it, and $N_m - N$ do not know the story, have good hearing, and are enthusiastic listeners and potential gossips. The factor $\frac{(N_m - N)}{N_m}$ is the ignorant fraction of the available population. Dividing both sides of (2) by N_m, we introduce $f = \frac{N}{N_m}$ as the fraction that know the story, and work with

$$(3) \qquad \frac{df}{dt} = Af(1 - f), \qquad f(0) = f_0 = \frac{N_0}{N_m}.$$

Our original model (1) gave $N \sim \infty$ as $t \sim \infty$. What does the present model give? We expect that $f \sim 1$ as $t \sim \infty$, i.e., that eventually everyone knows the story. (Even this model is far from complete, but at least this kind of result is acceptable.) From (3) we see that $\frac{df}{dt} \sim 0$ as $f \sim 1$, i.e., that f stops changing when everyone knows the story; from the discussion for (5:4) we may surmise that $\frac{df}{dt} \sim 0$ as $t \sim \infty$, but let us solve (3) and see the details.

From (3), we write $\int \frac{df_1}{f_1(1 - f_1)} = \int A dt_1$, where f_1 and t_1 are dummy variables. We decompose the integrand into "partial fractions." Since

$$\frac{1}{f(1-f)} = \frac{1}{f} + \frac{1}{1-f},$$

we obtain

$$\int\left[\frac{1}{f_1} + \frac{1}{1-f_1}\right]df_1 = \log \frac{f_1}{1-f_1}\bigg|_{f_0}^{f} = \log\frac{f}{1-f} - \log\frac{f_0}{1-f_0} = At.$$

Thus solving for f, we have

(4) $$f = \frac{f_0 e^{At}}{1 + f_0(e^{At} - 1)}.$$

If t is small, then the denominator approximates unity and $f \approx f_0 e^{At}$, in accord with the simplified model (1). On the other hand if t is large, we rewrite (4) as

(5) $$f = \frac{f_0}{f_0 + (1 - f_0)e^{-At}}$$

from which we see that $f \sim 1$ as $t \sim \infty$.

The above model indicates some of the essentials but it is still incomplete. However, it is good enough to show that although you may still not have heard the story about Al (see Appendix), you should by now have heard about the calculus — or at least about Helen of Troy.

8. Population Problems: Growth and Competition.

A more general equation which includes (7:1) and (7:3) as special cases is

(1) $$\frac{dN}{dt} = AN - BN^2, \qquad N(0) = N_0.$$

This is called the logistics equation. We still call A the growth coefficient, and we may call B the braking coefficient because the term $-BN^2$ slows the growth. The equation of unchecked growth, $\frac{dN}{dt} = AN$, permits $N \sim \infty$ as $t \sim \infty$, but Equation (1) does not. What bound does (1) impose on N? As in Section 7, we see that $\frac{dN}{dt} = 0$ when $A = BN$: the corresponding value $N = \frac{A}{B}$ must be the equilibrium value which N approaches as $t \sim \infty$.

At one time, essentially $\frac{dN}{dt} = AN$ was used as a model for the growth of populations of different countries, and this led to dire predictions as to the fate of mankind (law of Malthus). Then, essentially (1) was introduced (by Verhulst) and appropriate

A's and B's for various countries were obtained from their earlier census records; the projected growth curves were remarkably accurate (at least for all countries except Verhulst's — Belgium). The buildup of population growth arising from A has been interpreted as due to cooperation between people, and the slow-down associated with B as due to competition between people for limited resources. The competition interpretation is quite plausible: if p is the probability that a person wants a particular thing, then p^2 is the probability that two persons want it simultaneously; if there are N persons, then there are $\frac{N(N-1)}{2}$ competing pairs, and the total probability of competition is $\frac{p^2N(N-1)}{2}$; incorporating the linear term into the growth term of the differential equation, we may take BN^2 as a plausible measure of the simultaneous desire or of the competitive urge. However, the reason for regarding AN as a measure of coöperation is not clear. A probability interpretation similar to that for BN^2 indicates that AN corresponds to N persons acting quite independently of each other, this may well be as close to coöperation as one can expect from a group, and SMSG authors have therefore taken this as the guiding principle for preparing their textbooks.

Let us solve (1) by the same procedure we used for (7:3). The steps are essentially the same, and we get

$$\log \frac{N}{1 - CN} - \log \frac{N_0}{1 - CN_0} = At, \qquad C = \frac{B}{A},$$

and consequently

(2) $$N(t) = \frac{N_0 e^{At}}{1 + N_0(e^{At} - 1)\frac{B}{A}}.$$

If t is small, then the denominator is approximately $1 + N_0tB \approx e^{N_0tB}$, and (2) reduces to

(3) $$N \approx N_0 e^{t(A-BN_0)} = N_0 e^{t(A-D)},$$

where $D = BN_0$ is introduced as an abbreviation. Thus for small t, the result has the same form as for the simple model in terms of the growth coefficient A - D. On the other hand, if $t \sim \infty$, then the limit of (2) is $\frac{A}{B}$:

(4) $$N \sim \frac{A}{B} = \frac{AN_0}{D},$$

in accord with the remark after (1) that $N = \frac{A}{B}$ represents the long term equilibrium value.

9. Quizzes and Nonsense: Forgetting and Learning.

The previous sections also provide a simple model for forgetting and learning, at least of unconnected chains of nonsense syllables invented by psychologists for test purposes. Thus (as proposed by Von Foerster) we consider

(1) $$\frac{dN}{dt} = -AN + BN(N_0 - N), \quad N(0) = N_0,$$

where N_0 is the initial number of items memorized (dates, telephone numbers, unconnected theorems, etc.), A is a forgetting coefficient, and BN_0 is a memorization coefficient. The notion behind (1) is that your head is originally filled with N_0 "carriers" of information; some (AN) carriers just lose their information forever; some (BN) lose information in the sense that they pass information on to the empty $N_0 - N$ carriers.

Integrating (1) by partial fractions, or by comparison with (7:1) and (7:2) (the present (1) is (7:2) with a new growth coefficient $BN_0 - A$), we write the solution of (1) as

(2) $$\frac{N(t)}{N_0} = \frac{D - A}{D - Ae^{-(D-A)t}}, \quad D = N_0B.$$

The limit of $\frac{N(t)}{N_0}$ for $t \sim \infty$, the remembrance R (as defined by Von Foerster) depends critically on the magnitude of $\frac{D}{A}$. If $D > A$, then

(3) $$R = \frac{D - A}{D} = 1 - \frac{1}{D/A}, \quad \left(\frac{D}{A} > 1\right).$$

On the other hand, if $D \leq A$, then

(4) $$R = 0, \quad \left(\frac{D}{A} \leq 1\right).$$

Thus the remembrance of things past is zero for $\frac{D}{A} \leq 1$, and then increases towards unity as $\frac{D}{A}$ increases from unity.

10. Chemical Reactions: Multicomponent Processes.

Suppose we have a chemical substance with initial concentration C (gram-molecules per unit volume) which is reacting in time with something unimportant and plentiful to form another substance with concentration $N(t)$. The rate of change of N is proportional to the concentration of the original substance at time t, i.e., to $C - N(t)$:

(1) $$\frac{dN}{dt} = A(C - N), \quad N(0) = 0,$$

where N is the concentration of the new substance and A is called the reaction rate.

Equation (1), which is known as the law of mass action, is essentially the special case of (5:2) for A_1 much smaller than A_2; by inspection of (5:3) the solution (i.e., the concentration of the solution) is

(2) $$N = C(1 - e^{-At}).$$

Equivalently, Equation (1) is a shifted version of the simplest decay equation (3:1); substituting $M = C - N$ in (1), we obtain

$$\frac{dM}{dt} = -AM, \qquad M(0) = C - N(0) = C,$$

which leads directly to (2) for $N = C - M$.

From (2), we see that if $t = 0$, then $N = 0$; if $t \sim \infty$, then $N \sim C$, so that all of the original substance eventually reacts. We may isolate A in the form

(3) $$A = \frac{1}{t}\log\frac{C}{C - N(t)}$$

which is used in chemistry courses to determine A by measuring C, N, and t.

In a bimolecular reaction, we have two different substances with initial concentrations C_1 and C_2 which react at a rate determined by A to produce a third substance whose concentration is $N(t)$:

(4) $$\frac{dN}{dt} = A(C_1 - N)(C_2 - N), \qquad N(0) = 0.$$

This is just another variation of the logistics equation (8:1). The solution can be obtained from the previous ones, or directly:

$$\int\frac{dN}{(C_1 - N)(C_2 - N)} = \frac{1}{C_1 - C_2}\int\left[\frac{1}{C_2 - N} - \frac{1}{C_1 - N}\right]dN = \frac{1}{C_1 - C_2}\log\frac{C_1 - N}{C_2 - N} = At + K,$$

where

$$K = \frac{\log C_1/C_2}{C_1 - C_2}$$

follows from $N = 0$ at $t = 0$. Thus

(5) $$A = \frac{1}{t(C_1 - C_2)}\log\frac{C_2(C_1 - N)}{C_1(C_2 - N)},$$

and

(6) $$N = C_1\frac{1 - e^{(C_1-C_2)At}}{1 - (C_1/C_2)e^{(C_1-C_2)At}}.$$

The case $C_1 = C_2 = C$ may be obtained from the limit of (6) as C_1 approaches C_2. Equivalently, we start with

(7) $$\frac{dN}{dt} = A(C - N)^2, \quad N(0) = 0,$$

and integrate:

$$\int \frac{dN}{(C - N)^2} = \frac{-1}{C - N} + K = At.$$

The constant equals $K = -\frac{1}{C}$, and therefore

(8) $$A = \frac{1}{t} \frac{N}{C(C - N)}, \quad N = \frac{C^2At}{1 + CA}.$$

The equation for opposing unimolecular and bimolecular reactions has the form

(9) $$\frac{dN}{dt} = A(C - N) - BN^2, \qquad N(0) = 0.$$

We do not discuss this case but merely reduce it to a previous form. Thus we introduce

(10) $$D_1 = -\frac{1 + K}{2\frac{B}{A}}, \quad D_2 = -\frac{1 - K}{2\frac{B}{A}}, \quad K = \sqrt{1 + 4\frac{CB}{A}}.$$

in order to rewrite (9) as

(11) $$\frac{dN}{dt} = -B(D_1 - N)(D_2 - N).$$

We now have the form (4) with the previous A, C_1, C_2 replaced by $-B$, D_1, D_2, and the corresponding results may be written down by inspection.

We could go on to higher-order reactions of the form

(12) $$\frac{dN}{dt} = (C_1 - N)(C_2 - N)(C_3 - N) \ldots,$$

but we must finish the story.

11. Sociology: The End.

Now we could rehash everything. We could change the names of the characters in the previous equations and talk about profound sociological problems. Instead we introduce a more general model for the growth of populations, one which includes practically all of our previous equations as special cases, and scarcely talk at all.

In Section 8, the growth of a population of N individuals was described by

(1) $$\frac{dN}{dt} = AN - BN^2, \quad N(0) = N_0,$$

where A is the growth coefficient, and B is the braking coefficient. Let us now introduce more structure. We may write $A = \alpha - \beta$, where α is the birth coefficient (the birth rate per individual) and where β is one of two death coefficients.

If we assume that the population is confined to an area S, then we may write the other death coefficient as $\frac{C}{S}$, i.e., the death rate per individual $\frac{CN}{S}$, increases as S decreases or as N increases (no room to live). Thus the total death rate is $\left(\beta + \frac{CN}{S}\right)N$. Using $\gamma = \frac{C}{S}$ instead of B (merely for esthetic reasons) we rewrite (1) as

(2) $$\frac{dN}{dt} = \alpha N - (\beta + \gamma N)N.$$

A more general model (considered by Rashevsky) is that for the growth of a population consisting of two types of individuals with different birth and death characteristics. The total population is

(3) $$N = N_1 + N_2,$$

and N_1 and N_2 are specified by the simultaneous equations

(4) $$\begin{aligned} \frac{dN_1}{dt} &= \alpha_{11}N_1 + \alpha_{12}N_2 - \left[\beta_1 + \gamma_1(N_1 + N_2)\right]N_1, \\ \frac{dN_2}{dt} &= \alpha_{21}N_1 + \alpha_{22}N_2 - \left[\beta_2 + \gamma_2(N_1 + N_2)\right]N_2, \end{aligned}$$

where the α's, β's and γ's are all constants. The terms proportional to α represent the contributions of the two groups to the birth rates; the death rates that depend on γ_i (with $i = 1$ or 2) depend not only on N_i, but also on the total population $N_1 + N_2 = N$. The system of Equations (4) generalizes practically all the other equations considered previously in this chapter.

We do nothing with (4), but as a set of exercises you could obtain all the previous equations that we considered and that follow from (4) under suitable restrictions. Talk about active individuals and passive individuals; talk about active and passive disobedience; talk about social aggregates, freedom, crime, war, propaganda, etc. Write a book about it and call it War and Peace.

. . .

What have we tried to illustrate with this chapter? As you apply mathematics to the various sciences, you soon discover that at a fundamental level there appear to be only a few different kinds of processes going on: the same mathematical structure arises again and again in different contexts of nature. We have seen that the equations are the same, and only the names of the functions and variables change from science to science. The stages and settings are very different, and the overall plots vary; but the subplots are routine, the actors go through the same motions, and only the names of the characters are changed.

APPENDIX

THE STORY ABOUT AL

The Director of SMSG has ruled that the story about Al may be disseminated only by word of mouth.

Chapter 3

GEOMETRICAL OPTICS AND WAVES

1. Introduction.

There can be wonder and excitement in following a thread of mathematics through several sciences and recognizing their kinship in the concepts they share. There can be deeper wonder and greater excitement in following a single scientific thread through the cumulative concepts that trace its evolution.

Let us start from "laws of nature" that were isolated only after long years of observation, speculation, and verification, and show how various methods of calculus supplement each other in revealing the consequences implicit in these laws. The particular laws are limited in their domains of applicability in nature, and correspond to suitably restricted classes of observations. As we progress along the scientific thread, we trace part of the development from the early very special laws of geometrical optics to the modern very general laws that constitute the mathematical model for wave theory. We use optical terminology (light rays, mirrors, etc.) but the concepts we consider are basic to light, sound, and all wave physics.

We start with very restrictive laws, weaken them, and thereby enlarge the domain of the subject. The procedure we follow is "quasi-axiomatic." The "laws" or "axioms" we list contain undefined terms as well as implicit restrictions on both the observational and computational procedures that are associated with them. (The axioms of a mathematical system also contain undefined terms, and are also usually stated in an implicit context, but the situation is more obvious in science.)

We follow a thread that suggests the heuristic search for first principles on the basis of limited initial data, and the testing of principles by the new data they predict. We show that science, like mathematics, is cumulative. Basic concepts never die, but are exhibited differently at different evolutionary stages. A two thousand year old concept that was once accepted as a law — an axiom for a deductive development — is now a special restricted consequence of today's set of axioms. The concept lives not only within the new laws and as an intuitive guide for their exploration, but it still has a life of its own within an appropriately restricted domain — a domain whose boundary is today determined analytically by the new laws, instead of empirically.

We do not attempt to define "light," anymore than we would attempt to define "L:" L is for light, L is for life, L is for laugh, L is part of the ABC of communicating by symbols that represents perhaps man's most profound abstraction. Light is part of the ABC of physics.

We define neither light, nor its various subjective attributes: we start with "let there be light," and introduce various mathematical constructs that delineate certain measurable properties we associate with light. We are aware that you are reading with the aid of light, and that although this page is at present practically fully illuminated, you can change the situation by closing the book. You have seen landscapes by sunlight with no trace of structure to the flood of illumination, but you have also seen dancing spots of light traced by shafts of sunlight on a tree-shaded path, and beams of light entering darkened rooms through narrow cracks of slightly open doors; or you may have become aware of straight line characteristics revealed in floods of sunlight by the shadows that they cast. You have handled light sources such as electric lights, flashlights, candles, and have seen the stars as distant sources of light. You have seen the image of your face in a mirror, and the fractured appearance of things (fingers, silverware, etc.) partially submerged in water.

The most primitive constructs for such situations are geometrical, and they were introduced by the geometer Euclid (about 300 years before the current era). Euclid represented light as something "propagating" (traveling) along "rays" (straight lines) and as being "reflected" (thrown back in a special way) when it encountered a mirror. A geometry of rays and the idea that light travels at different speeds in different transparent materials serve also to account for the "refraction" (breaking or bending) of a ray passing from, say, air to water.

Why introduce the idea of traveling? The candle that is consumed as it gives forth light, and the monthly electric bill make clear that something is being used up to provide the light. We transform energy from some other form to the form associated with light, and the rays we consider are guides for the flow of energy.

In the next two Sections we use the calculus only to consider the geometry of rays — mostly straight rays, but also some curved ones; mostly familiar effects, but not necessarily familiar interpretations: we introduce a signed ray (a "shadow forming ray") to account for shadows, and some of the rays may split into many ("diffraction") to describe some aspects of what occurs when light strikes a sharp edge. Sections 2 and 3 deal with geometry, so that we need not mention energy flow until we associate a magnitude with a ray (as we do in Section 4). However, for the sake of the physical content, it should always be kept in mind that these rays, in some sense, are the directions for energy flow. In what sense is energy flow associated with light? We do not answer this, but merely provide an analogy that is appropriate for most aspects of visible phenomena.

There are two familiar ways in which energy travels: if you and a friend are in a swimming pool and you splash him, you shower him with water drops, each drop carrying some amount of energy from you to him, alternatively, if you plunge your hand into the water, much of the energy of the effort travels to him via a wave on the water's surface. As another pair of examples, you can attract his attention with energy in a packaged form by throwing a ball at him, or you can reach him with energy traveling as a sound wave by shouting at him.

Simple observations on propagation, reflection, refraction, and scattering of beams of light can be interpreted either as energy traveling as a stream of particles or as a collimated wave; the rays (Sections 2, 3, 4, 7) are either the trajectories for the particles, or the normals to the wave fronts (Section 5). For the more complex phenomena that we may observe readily, only the wave interpretation is adequate: these involve "interference" of light beams (Sections 6, 8, 9). Two beams of light may intersect and produce additive effects in the region of intersection but then emerge from this region unaltered in form: streams of waves show this characteristic, but streams of particles do not in general (i.e., some particles end up going in other directions than the original ones). For still more complex phenomena, some aspects are described more simply by a wave analog and others by a particle analog: Which is light? Wave or particle? Neither. Light is light. However, since we are more familiar with particles and waves (practically the A and the B of the ABC of physics) we may exploit these better known entities in learning and teaching about light.

2. Geometrical Optics.

2.1 Euclid's Principles.

Early observations of light sources (sun, star, lamps) and of the reflections of such sources and objects in smooth surfaces (water, polished metal) suggested that many phenomena could be described in terms of two "laws" of nature. We call these Euclid's principles of propagation [E1] and of reflection [E2]:

> [E1]: light travels along straight lines (rays);
>
> [E2]: when a ray is incident on a smooth plane surface, the incident ray, the reflected ray, and the normal to the surface all lie in the same plane, and the two rays make equal angles on the opposite sides of the normal.

Figure 2-1(a) illustrates [E2]; it shows the plane of incidence in which a ray from a source (S) reaches the observation point (P) via reflection at the intersection point (I) on the mirror; the rays are at angles α with the surface normal ($\hat{N}$). In Figure 2-1(b), we see that the reflected ray (I to P) is the extension of the mirror image (S' to I) of the incident ray (S to I).

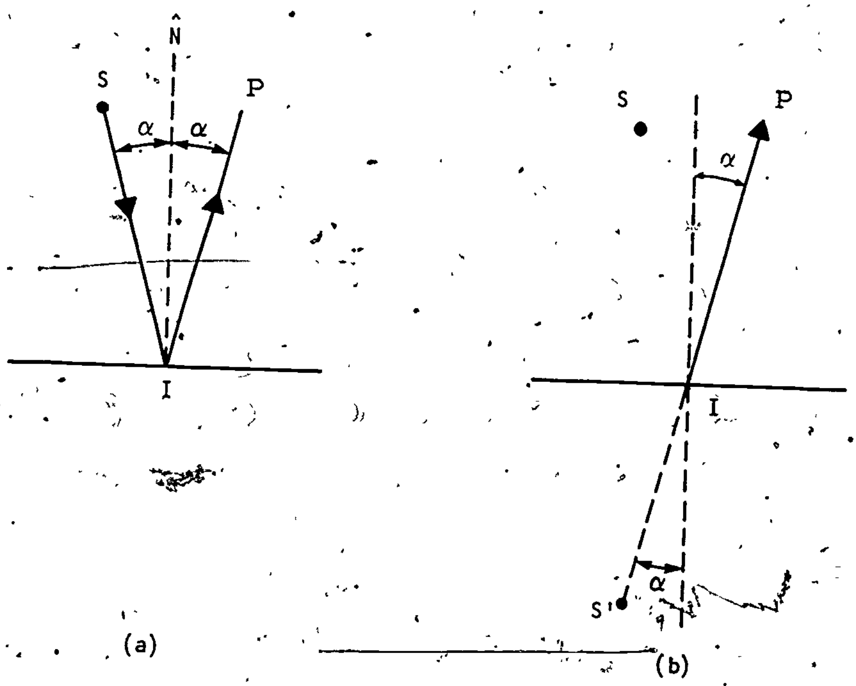


FIGURE 2-1

If a set of rays diverging from a source S is reflected from a plane mirror as in Figure 2-2(a), the corresponding set of reflected rays appear to originate from the image source S' as in Figure 2-2(b). Thus as far as the reflected set of rays (reflected ray system) is concerned, we may replace the mirror in Figure 2-2(b) by the source S' and reduce a reflection problem specified by [E2] to a propagation problem specified by [E1]. (This image method was essentially introduced by Heron or Hero several hundred years after Euclid.)

We regard [E1] as defining geometrical propagation in a uniform medium, and [E2] as defining geometrical reflection from smooth planes. These cover the situations of Figure 2-1 and 2-2 as well as more complicated reflection problems.

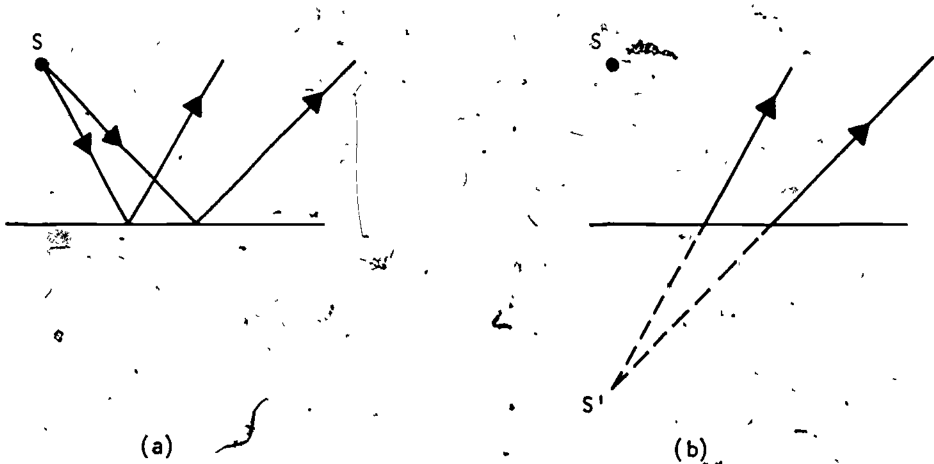
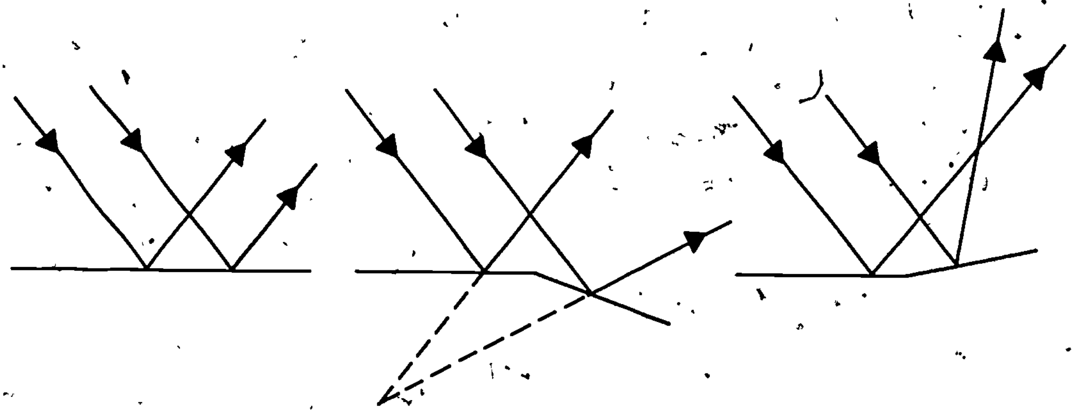


FIGURE 2-2

A set of parallel rays incident on a planar reflector as in Figure 2-3(a) is reflected as a set of parallel rays. If we regard this reflector as consisting of two hinged planes, and swing one away from the source as in Figure 2-3(b), then the reflected rays are said to diverge; if instead, we swing the plane towards the source, then the reflected rays converge as in Figure 2-3(c). In Figure 2-3(c), the reflected rays intersect, while in Figure 2-3(b), their extensions "behind" the mirrors intersect; the first (c) is called a real intersection, and the second (b) is called a virtual intersection. In either case, the reflected rays appear to originate at the intersection.

If we have a set of rays incident on a complex reflector consisting of many planar portions, then we can determine the reflected set by applying [E2]. Equivalently, once we have determined the intersections (real or virtual) of the set of reflected rays we have reduced the problem to one specified by [E1]. It is of particular interest to determine the intersections of rays reflected from curved mirrors. But before we consider curved mirrors, we introduce a more general law of nature than [E1].



(a) (b) (c)

FIGURE 2-3

2.2 Hero's Principle.

Euclid's principles describe the essentials of many directly observable phenomena (and also suggest applications not found freely in nature). However, the principles are very restrictive, and their description is very wordy. The restriction to plane mirrors is removed and the description is compressed by the more general principle of Hero:

[H]: a ray follows the shortest path between points.

Before applying [H] to more general situations than covered by [E], we use the calculus to show that [E] follows from [H]. Since, by definition, a straight line is the shortest path between points, we see that [H] covers [E1] directly. Similarly in considering [E2] we need not discuss wriggly paths. We consider Figure 2-4(a), and seek the shortest path between S and P via a point J on the surface of the mirror. From the start we take J in the plane through S and P that is perpendicular to the mirror: any displacement of J perpendicular to this plane will clearly lengthen the component paths L_1 and L_2. We introduce the lengths and angles of Figure 2-4(b) and seek the smallest value of $L = L_1 + L_2$ (as required by [H]) and show that this corresponds to $\gamma = \alpha$ (as required by [E2]).

In order for

(1) $$L = L_1 + L_2 = \sqrt{h_1^2 + x^2} + \sqrt{h_2^2 + (d - x)^2}$$

to be a minimum for fixed h_1 and h_2, and J on the surface of the mirror, we require

(2) $$\frac{dL}{dx} = 0 .$$

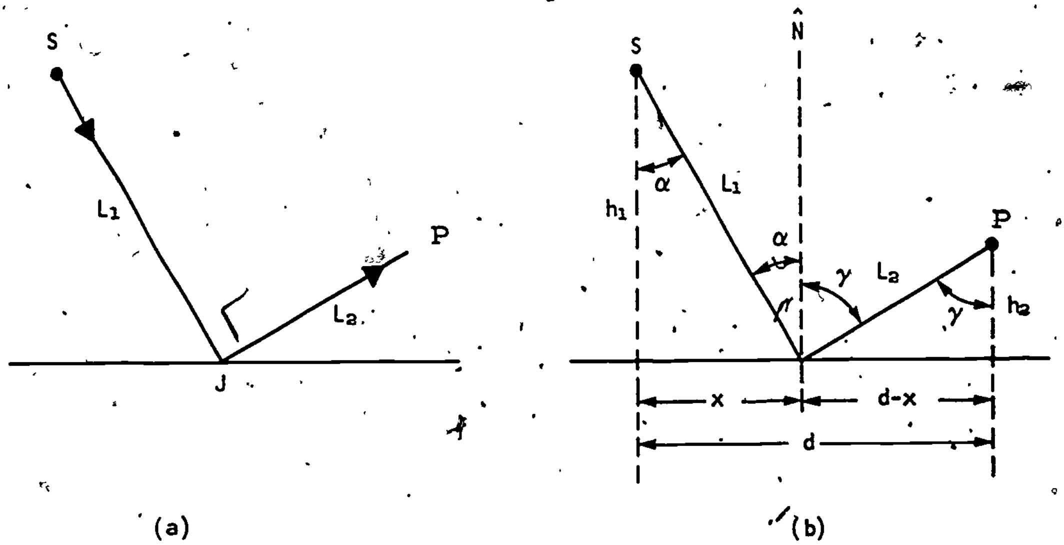


FIGURE 2-4

Thus

$$\frac{dL}{dx} = \frac{x}{\sqrt{h_1^2 + x^2}} - \frac{(d-x)}{\sqrt{h_2^2 + (d-x)^2}} = \frac{x}{L_1} - \frac{d-x}{L_2} = \sin\alpha - \sin\gamma = 0 ,$$

and consequently $\sin\gamma = \sin\alpha$ as in [E2].

Equation (2) by itself states only that L is an extremum, or equivalently that the ray path is <u>stationary for first order variations</u> of L. However, since

$$\frac{d^2 L}{dx^2} = \frac{\cos^2\alpha}{L_1} + \frac{\cos^2\gamma}{L_2} > 0 ,$$

we see that L is in fact a minimum. (Of course Hero did not use the calculus: he used the image principle and geometry to show that $\gamma = \alpha$ corresponds to the shortest path.)

It is clear from the above that applying [H] to reflection from a point on a curved surface is equivalent to using [E2] for reflection from the tangent plane at the point (and practical applications prior and subsequent to [H] have been based on [E2] plus the "tangent plane approximation"). Figure 2-5(a) shows reflection of a ray from a point on the concave side of a reflector, and Figure 2-5(b) shows the corresponding reflection from the convex side; note the relations of the directions. Similarly for a set of rays incident on a curved mirror, we can construct the reflected set (or equivalently, their intersections) by geometry; note that both situations in Figures 2-5(c) and (d) give rise to the same intersection point (real for c and virtual for d).

There are reflection situations that are not covered by [H] but are covered by [E2] and the tangent plane approximation. For example, if we consider a source (S) on the circumference of a circle and a diametrically opposed observation point (P),

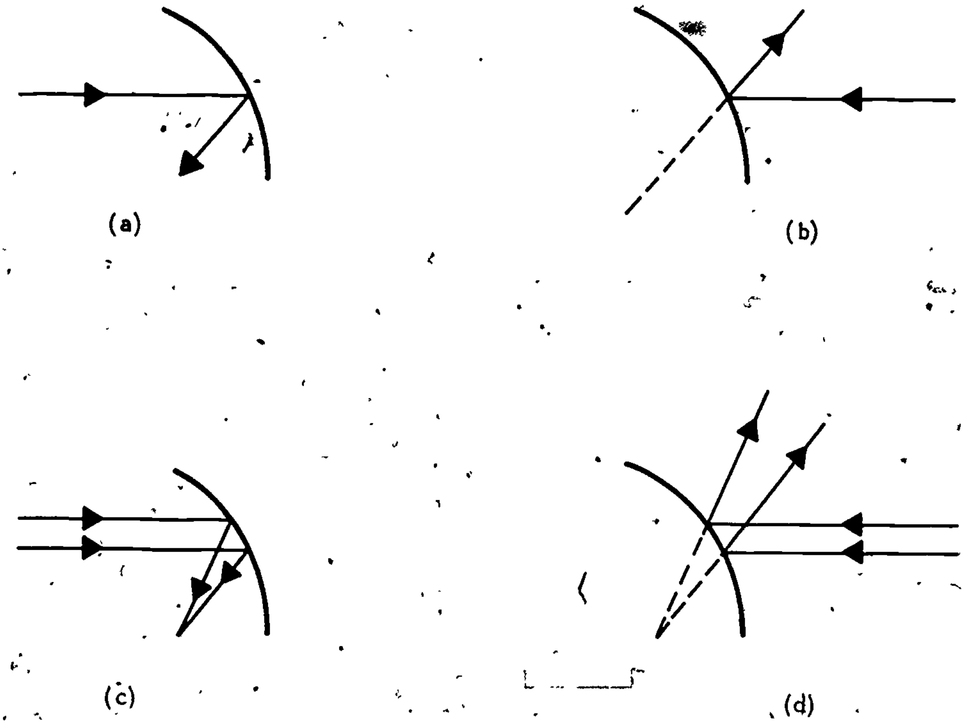


FIGURE 2-5

then the geometrically reflected ray from S to P via a point I on the circumference as in Figure 2-6(a), is the <u>longest</u> of all such paths (as is clear geometrically). For an arbitrary point on the circumference, we may write $SI = SP\cos\theta$ and $PI = SP\sin\theta$, so that $L = SP(\cos\theta + \sin\theta)$, and (2) in the form $\frac{dL}{d\theta} = SP(-\sin\theta + \cos\theta) = 0$ gives $\theta = \frac{\pi}{4} = \frac{\pi}{2} - \alpha$ as in the figure; but for this case

$$\frac{d^2L}{d\theta^2} = -SP(\cos\theta + \sin\theta) = -2\,SP\cos\frac{\pi}{4} < 0,$$

so that L is a maximum. There are also situations of interest covered by (2) for which the second derivatives vanish. For all such cases, independently of the second

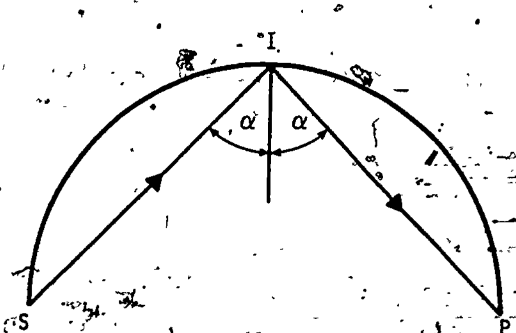


FIGURE 2-6(a)

derivative, the first derivative of the path length is zero, i.e., the ray path is stationary for first order variations. To cover all such situations, we replace [H] by the more general principle

[H']: a ray follows a stationary path.

We may distinguish two classes of curved reflectors and illustrate the essentials for the case of a concave cylindrical mirror. The simpler class corresponds to a "small aperture" mirror as in Figure 2-6(b); for this case the semi-aperture of the

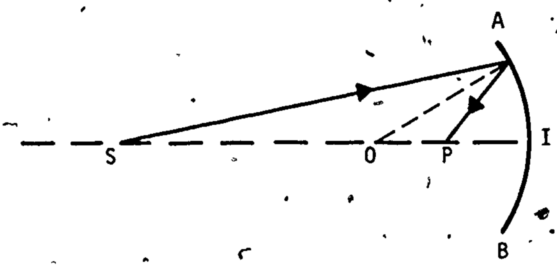


FIGURE 2-6(b)

mirror AB/2 is very small compared to its radius of curvature a. If I is the center point on the mirror, and S and P lie on the mirror's normal at I, then from the law of reflection, it follows that to a first approximation the rays from S reflected at all points of the mirror go through the point P such that

$$\frac{1}{SI} + \frac{1}{PI} = \frac{2}{a} .$$

In particular if SI ~ ∞, then the situation corresponds to an incident set of parallel rays as in Figure 2-6(c), and we obtain simply

$$PI = \frac{a}{2} .$$

Thus for the small-aperture mirror all reflected rays intersect at $\frac{a}{2}$ (the focus).

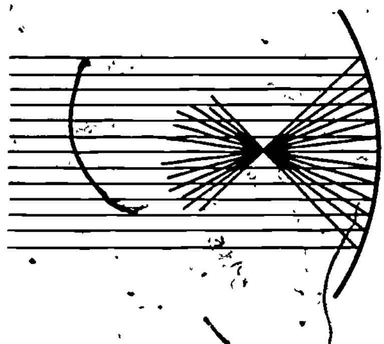


FIGURE 2-6(c)

The more general problem of reflection from a mirror with arbitrary sized aperture is illustrated in Figure 2-6(d) for a set of rays from a source on the axis of a

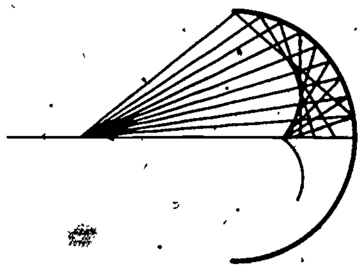


FIGURE 2-6(d)

semicircular mirror, Figure 2-6(b) shows only to the situation in the vicinity of the axis. If we rotate these figures around their symmetry axis, the situations correspond to reflection from portions of spherical mirrors. If the distance of the source from the reflector in Figure 2-6(d) becomes infinite (parallel set of rays incident), then the envelope of the reflected rays (the locus of intersections of neighboring rays) is an epicycloid (to be discussed subsequently); the cusp of this curve is at $\frac{a}{2}$.

The envelope of the rays is called a caustic; we may deal with a caustic surface, a caustic line, or a caustic point; the last is also called a focus. For specific sets of rays incident on specific curved reflectors we could determine the caustics geometrically; however a geometrical procedure is usually too tedious. Instead, we apply the calculus to a quite general situation, and determine the caustic that specifies the reflected field for a parallel set of rays incident on a cylindrical mirror. We restrict attention to the plane perpendicular to the cylinder's generator, so that the problem is essentially two dimensional, and derive the corresponding line caustic. Since the reflected rays are tangent to the caustic, once we know the caustic we specify the reflected field by means of [E1].

2.3 Caustics.

We consider a set of rays parallel to the x-axis incident on a two-dimensional mirror. For each incident ray, we could determine the corresponding reflected ray geometrically. Instead, as the initial step for a subsequent development, we consider an observation point $P(x, y)$ on the same side of the reflector as the source, and apply [H] to relate the incident ray that strikes the mirror at $I(\xi, \eta)$ to the reflected ray through P; see Figure 2-7. We could specify the point I and the incident ray that strikes it in terms of the parameter of arc length along the curve; however, it is more convenient to use the angle that the incident ray makes with the normal at I as

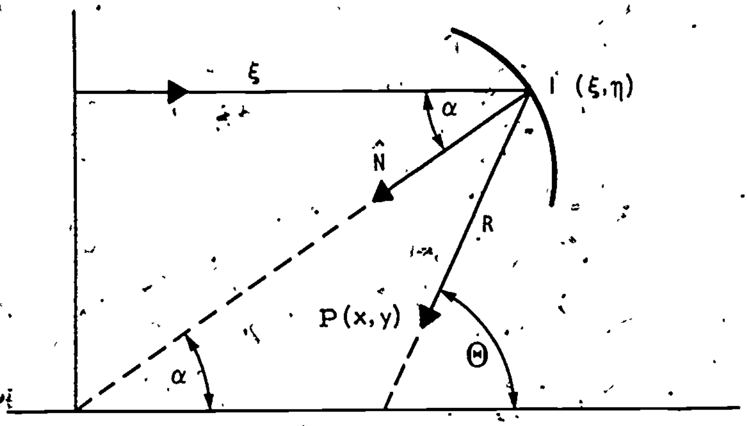


FIGURE 2-7

the parameter, the angle α such that $\tan\alpha$ is the slope of $\hat{N}$. The length of the incident ray measured from the y-axis is ξ; the length of the reflected ray from I to P is $R = \sqrt{(\xi - x)^2 + (\eta - y)^2}$, and its inclination to the x-axis is Θ.

The total length from the y-axis via I to P equals

(3) $$L = \xi + R = \xi + \sqrt{(\xi - x)^2 + (\eta - y)^2}\,.$$

Differentiating, we have

(4) $$L' = \xi' + \frac{\xi'(\xi - x) + \eta'(\eta - y)}{R} = \xi'[1 + \cos\Theta] + \eta'\sin\Theta$$

where the prime indicates differentiation with respect to α. Using [H], essentially as for (2), we equate L' to zero to obtain

(5) $$\frac{-\xi'}{\eta'} = \frac{\sin\Theta}{1 + \cos\Theta} = \frac{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}{2\cos^2\frac{\Theta}{2}} = \tan\frac{\Theta}{2}\,,$$

Thus from the chain rule we have

(6) $$\tan\frac{\Theta}{2} = \frac{-\xi'}{\eta'} = \frac{-d\xi}{d\eta}\,.$$

Since $\frac{d\eta}{d\xi}$ is the slope of the tangent of the reflector at the point of incidence, $\frac{-d\xi}{d\eta}$ is the slope of the normal $\hat{N}$, and equals $\tan\alpha$. Consequently

(7) $$\frac{-\xi'}{\eta'} = \tan\frac{\Theta}{2} = \tan\alpha\,,$$

from which

(8) $$\Theta = 2\alpha\,,$$

as could have been obtained directly from [E] on inspection of Figure 2-7.

The equation of the reflected ray arising from the ray incident at an angle α with $\hat{N}$ is

(9) $$\eta - y = (\xi - x)\tan \Theta = (\xi - x)\tan 2\alpha ,$$

which we rewrite as

(10) $$g(\alpha) = (\xi - x)\tan 2\alpha - (\eta - y) = 0 .$$

This specifies the set of reflected rays corresponding to a set of incident parallel rays. The parameter α describes not only the curve of the reflector $[\xi(\alpha), \eta(\alpha)]$, it also picks out the ray incident at ξ, η and the corresponding reflected ray (10) that reaches x, y. The point of intersection of two neighboring rays $g(\alpha) = 0$ and $g(\alpha + \Delta\alpha) = 0$, corresponding to α and $\alpha + \Delta\alpha$, is determined by $g(\alpha) = 0$ and $\frac{g(\alpha + \Delta\alpha) - g(\alpha)}{\Delta\alpha} = 0$. In the limit $\Delta\alpha \to 0$, the point of intersection of the rays falls on the envelope and is specified by the simultaneous equations

(11) $$g(\alpha) = 0, \quad g'(\alpha) = 0 .$$

Differentiating (10), we obtain

(12) $$g' = \xi' \tan 2\alpha + \frac{2(\xi - x)}{\cos^2 2\alpha} - \eta' ,$$

and with (7) we eliminate $\eta' = \frac{-\xi'}{\tan \alpha}$:

(13) $$g' = \xi' \left(\tan 2\alpha + \frac{1}{\tan \alpha}\right) + \frac{2(\xi - x)}{\cos^2 2\alpha} .$$

Since $\tan 2\alpha \tan \alpha + 1 = \frac{1}{\cos^2 2\alpha}$, we reduce (13) to

(14) $$g' = \frac{1}{\cos^2 2\alpha}\left(\frac{\xi' \cos 2\alpha}{\tan \alpha} + 2(\xi - x)\right) .$$

Thus, since (11) requires $g'(\alpha) = 0$, the x-coordinate of the point on the envelope is

(15) $$x = \xi + \frac{\xi' \cos 2\alpha}{2 \tan \alpha} ,$$

which we may rewrite in various equivalent forms, e.g.,

(16) $$x = \xi - \frac{\eta'}{2} \cos 2\alpha .$$

We obtain the corresponding y-coordinate by using (16) to eliminate $\xi - x$ from (10): thus $g = \frac{\eta'}{2} \cos 2\alpha \tan 2\alpha - (\eta - y) = 0$, and consequently

(17) $$y = \eta - \frac{\eta'}{2} \sin 2\alpha .$$

Equations (15) and (17) (from which we could eliminate α) specify the caustic curve, the envelope of the reflected rays (the locus of the intersections of neighboring rays). Given a specific reflector we can use its parametric representation to eliminate ξ and η, and thereby determine the caustic explicitly. We illustrate this for two simple reflectors, the parabola and semicircle.

Parabola. We consider a set of rays incident on the parabola

(18) $$\eta^2 = -4p\xi$$

as in Figure 2-8. The parametric equations of the parabola in terms of α, where

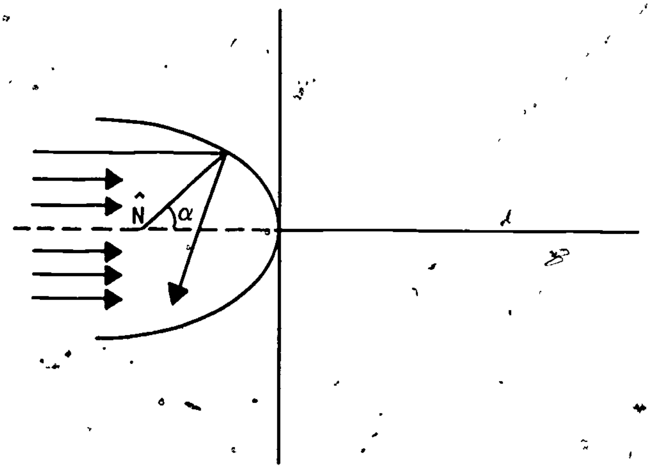


FIGURE 2-8

$\tan\alpha$ is the slope of the normal, are

(19) $$\eta = 2p\tan\alpha, \qquad \xi = -p\tan^2\alpha,$$

and consequently

(20) $$\eta' = \frac{2p}{\cos^2\alpha}, \qquad \xi' = -p\,\frac{2\tan\alpha}{\cos^2\alpha}.$$

Using these expressions for η and η' in (17), we have

(21) $$y = 2p\tan\alpha - \frac{p}{\cos^2\alpha}\sin 2\alpha = 2p\left(\tan\alpha - \frac{\sin\alpha}{\cos\alpha}\right) = 0.$$

Similarly, we use the corresponding expressions for ξ and ξ' in (15) to obtain

(22) $$x = -p\tan^2\alpha + \left(\frac{-p\,2\tan\alpha}{\cos^2\alpha}\right)\left(\frac{\cos 2\alpha}{2\tan\alpha}\right) = \frac{-p}{\cos^2\alpha}(\sin^2\alpha + \cos^2\alpha) = -p.$$

Thus the envelope of the reflected rays is

(23) $$x = -p, \quad y = 0,$$

i.e., the focus of the parabola as in Figure 2-9. The focusing property of the

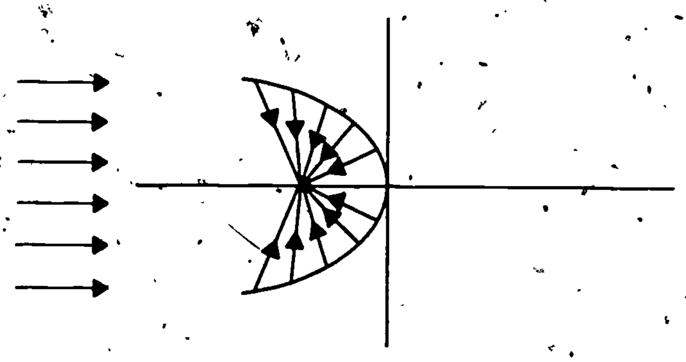


FIGURE 2-9

parabola accounts for its many applications (as telescope mirrors, microwave and sonic "dishes," etc.) for collecting practically parallel radiation (the rays from very distant sources) by reflecting the incident rays to a small appropriate detector placed at its focus. Similarly, parabolic reflectors are used for the inverse problem of converting the radiation from a source at the focus into a parallel beam of rays.

The above example is practically trivial in that (23) could have been obtained by much simpler procedures than the one we followed. In the next example we follow essentially the same procedure to obtain a far less obvious result.

Semicircle. The parametric equations of a circle of radius a for the problem of Figure 2-10 are

(24) $$\eta = a \sin \alpha , \quad \xi = a \cos \alpha ,$$

and consequently

(25) $$\eta' = a \cos \alpha , \quad \xi' = -a \sin \alpha ,$$

From (17) we then obtain for the caustic

(26) $$y = a \sin \alpha - a \cos \alpha \frac{\sin 2\alpha}{2} = a \sin^3 \alpha ,$$

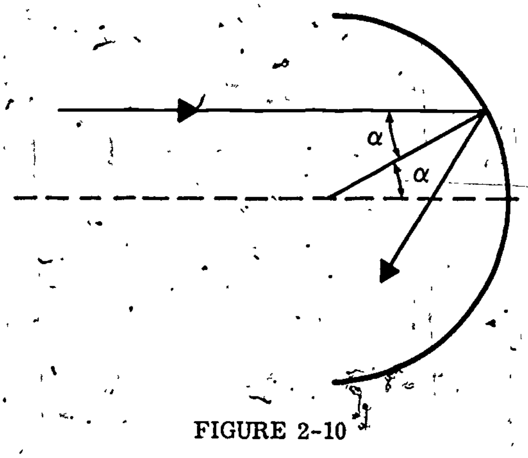


FIGURE 2-10

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and from (15),

(27) $$x = a\cos\alpha - \frac{a\sin\alpha\cos 2\alpha}{2\tan\alpha} = \frac{a\cos\alpha}{2}(1 + 2\sin^2\alpha).$$

Squaring and adding (26) and (27), we obtain

(28) $$\frac{4}{a^2}(x^2 + y^2) = 1 + 3\sin^2\alpha = 1 + 3\left(\frac{y}{a}\right)^{\frac{2}{3}},$$

the equation of the epicycloid traced by a point on a circle of radius $\frac{a}{4}$ rolling on the outside of a fixed circle of radius $\frac{a}{2}$.

The cusp or focal point of the caustic is at $x = \frac{a}{2}$, $y = 0$; this corresponds to $g'' = 0$, and occurs at $\alpha = 0$. The rays incident near the center of the mirror ($\alpha \approx 0$) are known as paraxial rays of "small aperture" mirror theory; only these give rise to reflected rays that appear to originate at the cusp $\frac{a}{2}$. [For parallel rays incident on the parabola, the entire caustic consists of the point focus; similarly for a source at one focus of an ellipse, all reflected rays go through the other focus (hence the label).]

Virtual Caustics. In the above we considered reflection from concave mirrors; for such cases the reflected rays intersect and the caustics are real in the sense defined in Section 2.2. Similarly for incidence on a convex reflector the extensions of the reflected rays behind the reflector intersect on a virtual caustic. The identical caustic curve specifies reflection from either side of the mirror. Figure 2-11 shows

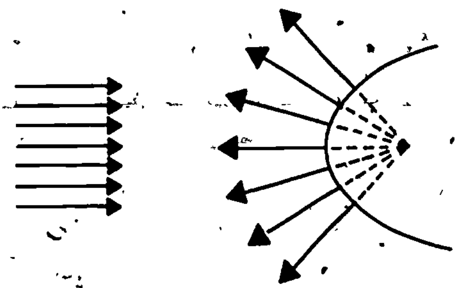

FIGURE 2-11

the situation for incidence on a convex parabolic reflector, and Figure 2-12 shows the analogous situation for a semicircle. Figure 2-13 shows the geometrical method of constructing the epicycloidal caustic of the semicircle.

Since the caustic of Figure 2-13 is the envelope of the set of extended reflected rays, it is tangent to all members of the family. From the figure we see that the distance from the mirror along the ray extension to its point of tangency with the caustic equals $\frac{a}{2}\cos\alpha$. [Without the geometrical construction, the result follows on subtracting from $\frac{a}{2\cos\alpha}$ (the total ray extension from cylinder surface to x-axis) the

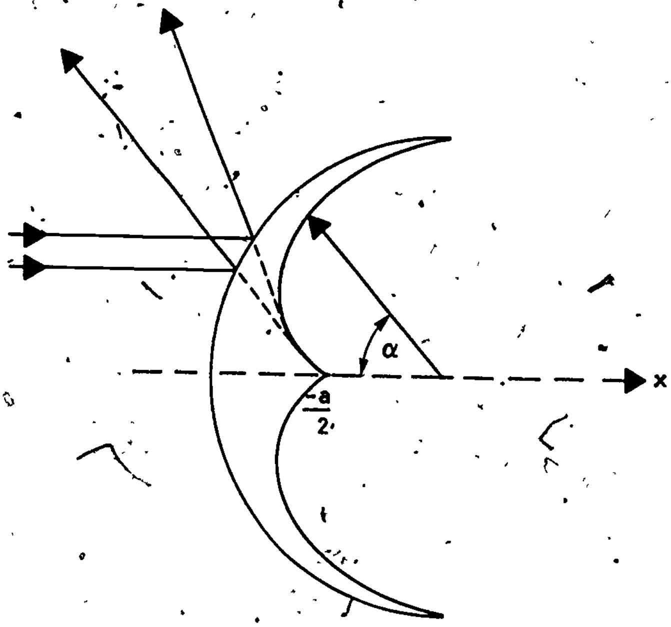

FIGURE 2-12

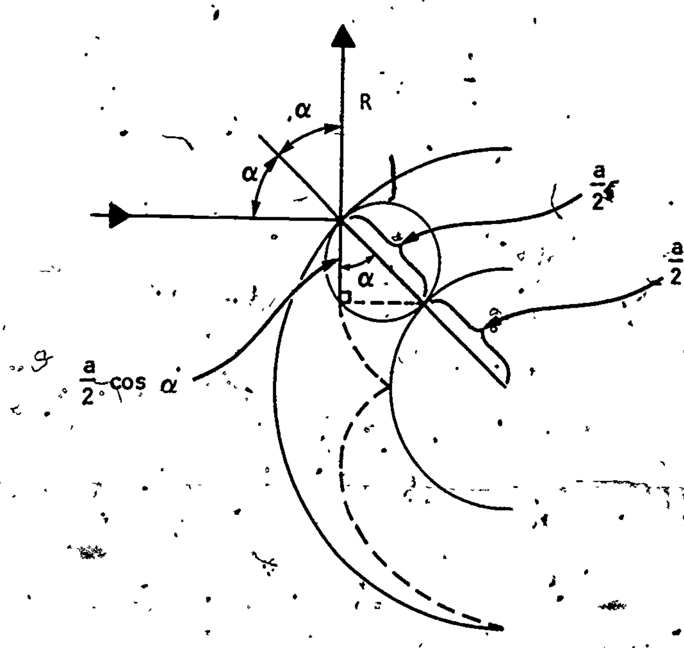

FIGURE 2-13

value $\frac{y}{\sin 2\alpha} = \frac{a \sin^3 \alpha}{2 \cos \alpha \sin \alpha}$ (the length of extension between caustic and x-axis.) Thus neighboring reflected rays of real length R (where R is the distance from the mirror) appear to diverge from a source (their point of intersection) at a distance $R + \frac{a}{2}\cos \alpha$ along their extension.

Since the reflected rays are tangent to the caustic, we may treat the caustic as the evolute (the locus of the centers of curvature) of a system of curves which are orthogonal to the rays. These curves, the involutes of the caustic, are called the eikonals or eikonal curves in ray theory; the radius of curvature at a point P on such a curve equals $R + \frac{a}{2}\cos\alpha$. The rays (the orthogonal trajectories of the eikonal curves) are tangent to the caustic and normal to the eikonals, and this provides a geometrical construction for the eikonals: they are traced by the points of a taut string as it unwinds from the caustic.

2.4 Shadows.

In the discussion of (3)ff we restricted consideration to an observation point P lying on the same side of the reflector as the source (the "lit side" of the reflector). If we drop this restriction, we obtain an additional solution of L' = 0 with L' as given in (4), i.e.,

(29) $$L' = 0 \quad \text{if } \Theta = \pi,$$

where the geometry is shown in Figure 2-14. Thus in addition to the geometrically

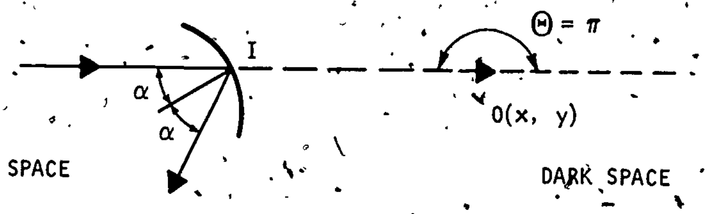

FIGURE 2-14

reflected ray shown in Figure 2-7, we see from L' = 0 (i.e., from [H']) that the incident ray also gives rise to another ray — one traveling along the original direction of incidence. Were the reflector absent, we would interpret this ray as the incident ray itself (i.e., the situation of [E1]). However, we insist on the presence of the reflector and seek a physically significant interpretation of the rays corresponding to (29).

When we interrupt a broad beam of light by a mirror, we notice essentially two effects: because of the mirror, there is not only some light observed in a region of space outside of the original beam, but there is also some light missing from a region of space originally filled by the beam before we inserted the obstacle. Were we interested solely in the original beam, then we might simply say that some of the light has been "bent" from its original direction (reflected) and let it go at that. However in order to ultimately specify the full effect of the obstacle analytically, we assign it a more positive role. We say that the incident rays "excite" the obstacle to produce not only the set of reflected rays but also a set of shadow forming rays parallel to the "missing" incident rays in the dark region of space. It is these shadow forming rays that we read into (29); these must cancel the incident rays on the "dark side" of the mirror to "create" the geometrical shadow. (This idea of shadow forming rays may be hard to reconcile with mental images of the reflection of rays based on a ball bouncing off a wall. However, were we interested in specifying the total effect of the wall in the ball-wall problem, we could also do so in terms of reflected balls and shadow forming balls.)

To make the role of the obstruction more explicit (and to set the stage for our subsequent discussion of scattering), we introduce a symbolic representation for the rays. We represent the effect of an incident ray by E_i, of the geometrically reflected ray by E_g, and of the shadow forming ray by E_s. We represent the total effect E_t corresponding to a ray E_i incident on a reflector as

(30) $$E_t = E_i + E: \quad E = \begin{cases} E_g & \text{in lit space} \\ E_s & \text{in dark space.} \end{cases}$$

Thus in the lit space the total effect is $E_t = E_i + E_g$ as shown by the two rays on the left hand side of Figure 2-14. On the other hand in the dark space we have $E_t = E_i + E_s$ corresponding to the dashed ray on the right hand side of Figure 2-14; in order that E_t represent the physical situation of the geometrical shadow, i.e., in order that E_t vanish, we require

(31) $$E_s = -E_i .$$

We take (31) as a supplementary assumption to [H']: the first solution ($\Theta = 2\alpha$) of $L' = 0$ accounts for geometrical reflection (and we subsequently determine a magnitude to be assigned to such rays); the second solution ($\Theta = \pi$) plus (31) accounts for shadow formation.

The symbol E in (30) represents the <u>scattered</u> part of the total effect $E_t = E_i + E$. This is the part of E_t that we may regard as originating at the obstacle to E_i, or as <u>outgoing</u> from the obstacle.

If we consider a system of parallel rays incident on a convex semicircular cylinder (or equivalently on a full circular cylinder), then the corresponding scattered ray system (reflected plus shadow forming rays) is as sketched in Figure 2-15.

The family of curves perpendicular to these rays is the corresponding infinite set of eikonals. Figure 2-16 plus its reflection in the x-axis shows several of these curves. These curves may be obtained geometrically from the caustics (the caustic for the shadow forming rays is the point at $x = -\infty$), or by constructing the normals of Figure 2-15 geometrically, or analytically. We consider an analytical derivation in a following section. At larger and larger distances from the scatterer the eikonals of Figure 2-16 become more and more circular.

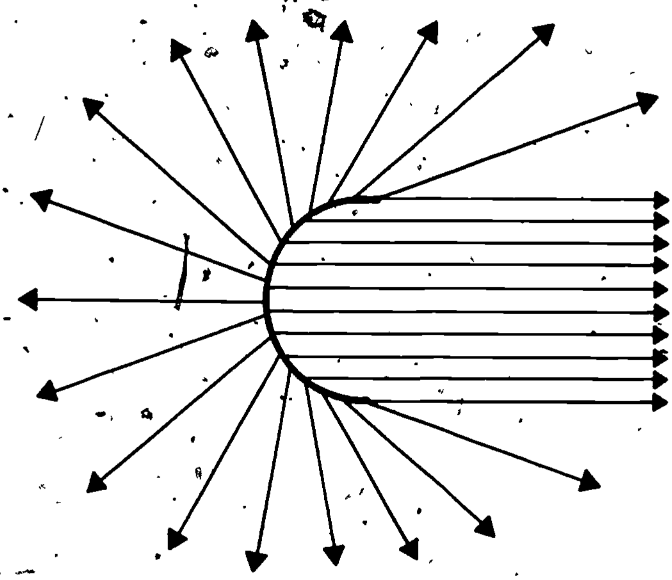

FIGURE 2-15

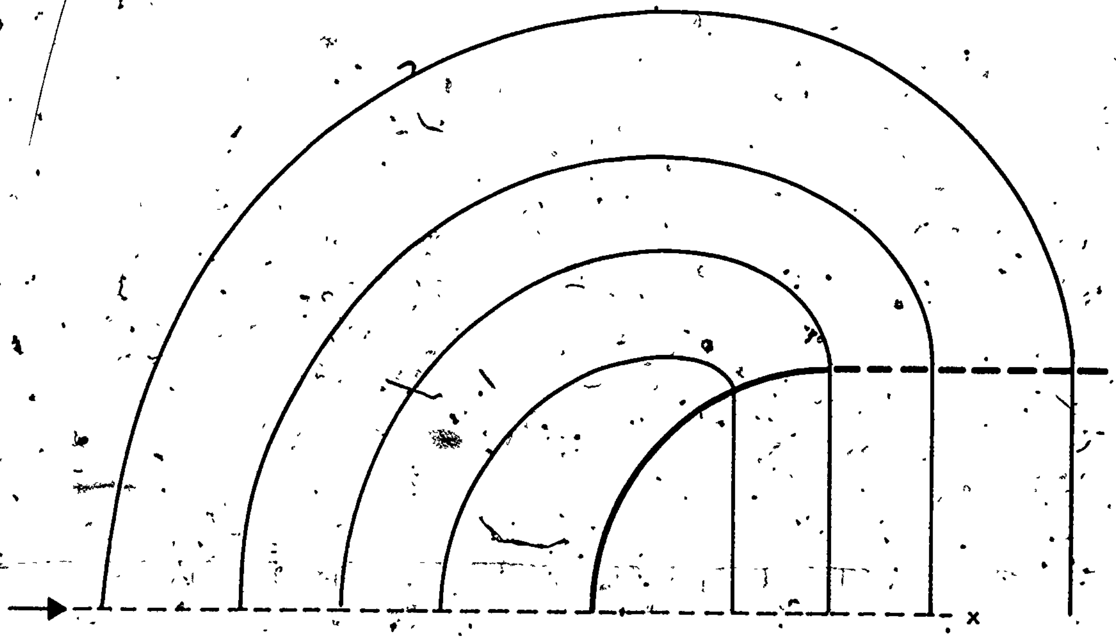

FIGURE 2-16

2.5 Edge Diffracted Rays.

There are additional sets of rays implicit in Hero's principle, and their utility has been shown by the recent investigations of J. B. Keller. In particular we consider edge diffracted rays arising when a ray is incident on a sharp edge (which "breaks up" or "diffracts" an incident ray). In order to motivate introducing such rays, let us review the preceding material.

We have discussed reflected rays and shadow forming rays, and we saw in connection with the semicircular cylinder that both kinds of rays were required to obtain a complete coverage of space by scattered rays (or equivalently to obtain closed scattered eikonals). However if the scatterer is a strip as in Figure 2-17(a), such rays

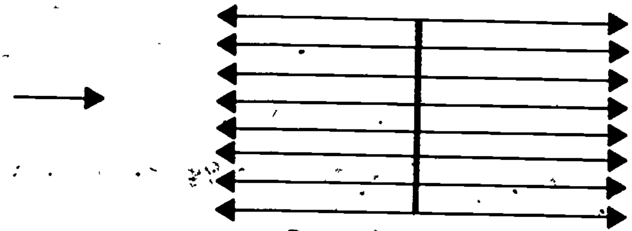

FIGURE 2-17(a)

alone do not cover space, which implies that the scatterer's influence is restricted to the two directions shown in the figure. To construct a scattered ray system that covers all space, we introduce the edge diffracted rays of Figure 2-17(b); these rays are included in [H], i.e., an incident ray striking the edge is diffracted to P via the shortest path.

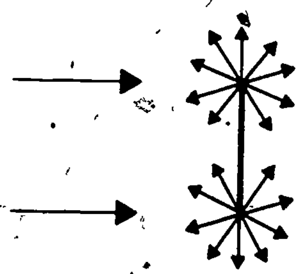

FIGURE 2-17(b)

From Figure 2-17(a) and 2-17(b), we see that there are essentially three different cases that arise for a fully illuminated strip; these correspond to the three different observation points of Figure 2-17(c). An observation point at P_1 receives two

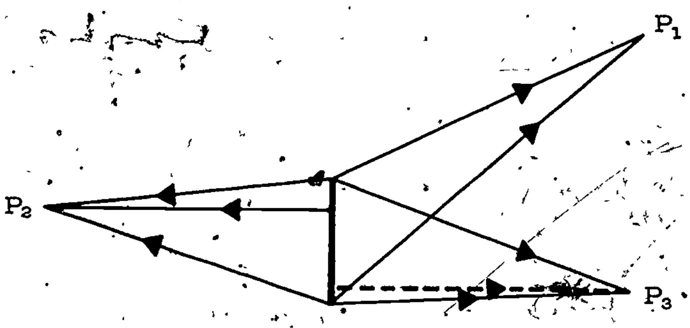

FIGURE 2-17(c)

diffracted rays; P_2 receives one reflected ray and two diffracted rays; P_3 receives one shadow-forming ray and two diffracted rays. In a subsequent section we show that the magnitude (of energy flow) associated with a diffracted ray is in general much smaller than the magnitude of the other rays in Figure 2-17(c). If we assume this result for present purposes, we neglect the diffracted rays in the regions corresponding to P_2 and P_3 and obtain the scattered ray system of Figure 2-17(d); this figure

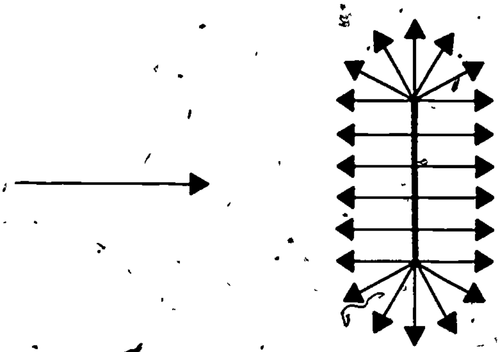

FIGURE 2-17(d)

shows only the "strongest" scattered ray at each observation point. A corresponding eikonal curve normal to the rays of Figure 2-17(d), is shown in Figure 2-17(e), and it is clear that such surfaces become more circular with increasing distance from the scatterer.

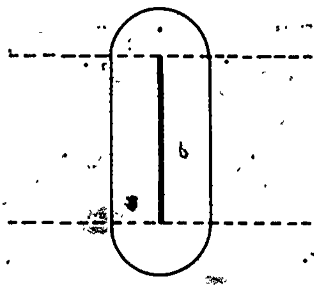

FIGURE 2-17(e)

The various rays of Figure 2-17 correspond only to the scattered ray system, i.e., to the effects in space arising from something that obstructs the incident rays; this figure does not take into account that the observation point is also reached by an incident ray. In particular, as discussed for equations (30) and (31), the incident rays and shadow-forming rays cancel in the shadow region corresponding to P_3. Thus the net effect in the shadow region must arise from the edge diffracted rays as in Figure 2-18; such effects have been discussed in detail by J. B. Keller. (Bright areas in the shadow region of obstacles with very regular edges were first commented on by Grimaldi, 1613-1663.)

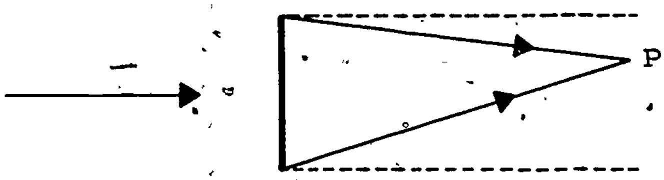

FIGURE 2-18

For present purposes, we consider only the caustic of the edge rays for the analogous problem of a disc. Thus if a parallel set of rays is normally incident on a circular disc as in Figure 2-19, each point of the edge gives rise to a "full fan of rays"

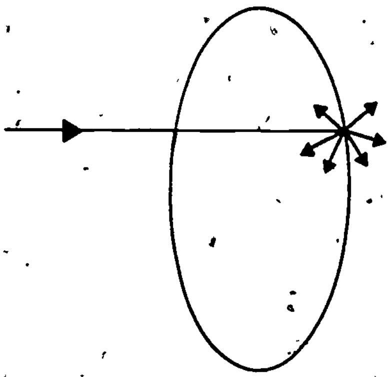

FIGURE 2-19

normal to the edge at that point. An off-axis observation point receives edge rays only from two points of the circumference on the disc, i.e., from the diametrically opposite points cut by the plane containing the observation point and the disc's axis. However, a point on the axis of the disc receives edge rays from the entire circumference: the axis is a caustic of the edge rays. Thus the center of the shadow of a normally illuminated circular disc should show a bright spot, the Arago bright spot, or Poisson bright spot (as predicted originally about 1800 via a wave argument — a big argument).

For the circular disc, the line caustic of the edge rays is the envelope of the planes normal to the edge of the disc. For a disc of general shape (an arbitrary planar scatterer) normal to the parallel incident rays, the corresponding caustic of the edge rays is a cylindrical surface, the envelope of the planes normal to the edge. Since two such planes intersect in a line normal to the disc, the caustic cylindrical surface generated by the lines of intersection is also normal to the plane of the disc. The cross section of the caustic cylinder cut by the plane of the disc (or as viewed on a screen in the disc's shadow), is the line envelope of normals to the edge in the plane of the disc; it is the evolute of the edge.

In particular for an elliptic edge

(32) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

the equation of the evolute, the four-cusped curve sketched in Figure 2-20, is

(33) $$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}} .$$

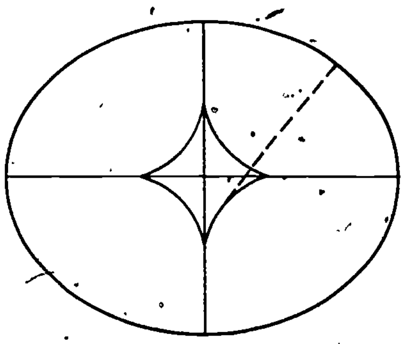

FIGURE 2-20

To derive (33) we write a point on the ellipse parametrically as

(34) $$\xi = a \cos \varphi , \quad \eta = b \sin \varphi .$$

The corresponding normal through x, y is specified by

(35) $$g(\varphi) = \frac{ax}{\cos \varphi} - \frac{by}{\sin \varphi} - a^2 + b^2 = 0 ,$$

and the derivative with respect to φ gives

(36) $$g'(\varphi) = \frac{ax}{\cos^3 \varphi} + \frac{by}{\sin^3 \varphi} = 0 .$$

Substituting (36) in (35) to eliminate either x or y, we see that

(37) $$\frac{ax}{\cos^3 \varphi} = \frac{-by}{\sin^3 \varphi} = a^2 - b^2 .$$

Consequently the locus of the normals is

(38) $$x = \frac{a^2 - b^2}{a} \cos^3 \varphi , \qquad y = \frac{a^2 - b^2}{b} \sin^3 \varphi .$$

Eliminating φ from (38), we obtain the required result (33). [Note that for the geometrically reflected rays we started with a set of lines (the rays), determined their caustic, and then identified the caustic as the evolute for a set of involutes (the eikonal curves). For the present case, however, we started with an involute (the edge of the disc) and determined the corresponding evolute, the envelope of its normals (the locus of the centers of the circles tangent to the involute, the locus of the centers of curvature).]

If we visualize an experiment in which we start with a circular disc and gradually convert it to one of elliptical cross section (or equivalently if we rotate the circular disc so that it is no longer perpendicular to the incident rays), then on a screen normal to the direction of incidence the bright spot changes to the four cusped evolute of the ellipse. Such caustic sections were photographed by Coulson and Becknell in 1922.

The associated scale factor: In the above we discussed rays reflected from surfaces and rays diffracted by edges. The edge rays as in Figure 2-17(b) are drawn radially outward from a point on the line representing the edge, but the scattered rays of Figure 2-15 for the cylinder are not radial. If we visualize the cylinder becoming narrower and narrower we might expect on the basis of our remarks for edge rays that in the limit the ray system of Figure 2-15 could be represented as a set of radial lines as in Figure 2-21.

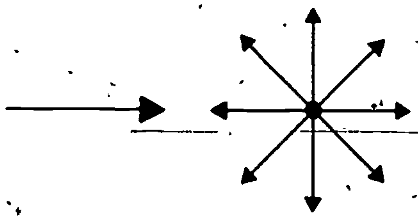

FIGURE 2-21

The situations of both Figures 2-15 and 2-21 are covered by [H'], and both correspond to scattering by a circular cylinder. In order to distinguish them we must associate a scale factor for length with a ray. To do so, we could assume that in addition to the geometrical property assigned to a ray by [H'], a ray of light (of a single color) has an associated length λ that is independent of the length of the ray path. We could then distinguish the two different scattering situations for the cylindrical obstacle of Figures 2-15 and 2-21 as follows: the ray system of Figure 2-15 corresponds to a very large cylinder $a >> \lambda$, and the ray system of Figure 2-21 corresponds to a very small cylinder $a << \lambda$.

The existence of an associated length might have been guessed (from Grimaldi's experiments on light diffracted into shadow regions) but was not. We show subsequently that the required scale factor emerges naturally as part of a more general model for such phenomena. We mention the matter now partly in anticipation, but primarily to stress that the present model is incomplete.

3. Snell's Law, Fermat's Principle.

In the preceding sections we considered a set of rays incident on reflecting surfaces, and used [H] to determine the reflected set of rays. We now extend the development to partially transparent surfaces and consider in addition a set of transmitted rays. A transmitted ray does not lie in general along the extension of the corresponding incident ray, but makes an appropriate angle with the ray extension (e.g., as in Figure 3-1); this kind of "break" in the ray path is called refraction.

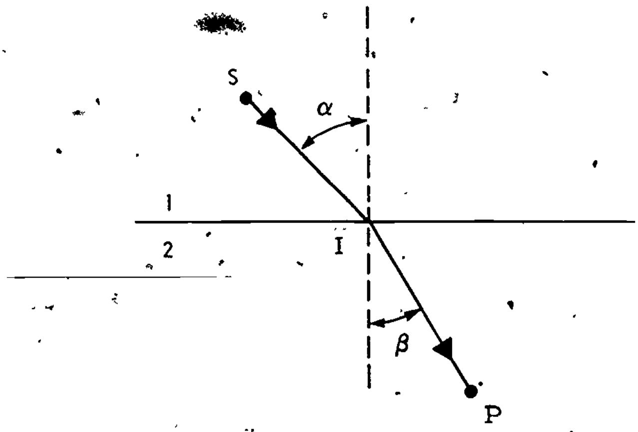

FIGURE 3-1

Observations and studies of the broken appearance of a rod partially immersed in water, and of a beam of light traveling partly in air and partly in water, go back to Euclid and Ptolemy (second century of this era), but the complete description of such effects was first given by Snell (1591-1626). As the appropriate analog of [E] for reflection, we have Snell's law of refraction:

[S]: A ray incident on the smooth plane interface between two transparent media gives rise (in addition to the reflected ray) to a refracted ray on the other side of the interface. The incident ray, the refracted ray, and the surface normal lie in the same plane, and the two rays are on opposite sides of the normal. The sine of the angle (β) that the refracted ray makes with the normal is proportional to the sine of the angle (α) of incidence.

From [S], we specify the direction of the refracted ray by

(1) $$\mu_2 \sin\beta = \mu_1 \sin\alpha,$$

or equivalently by

(2) $$\mu \sin\beta = \sin\alpha .$$

The constants μ_1, and μ_2 are called the indices of refraction, and $\mu = \frac{\mu_2}{\mu_1}$ is called the relative index of refraction. The situation is shown in Figure 3-1 for $\mu_1 < \mu_2$ (as assumed in all that follows): the ray travels from S to P via a point I on the interface. The μ's are physical constants which specify the essential physical property of the media for the topic at hand, they may be measured experimentally, and we assume they are known. In particular for light (yellow light) passing from air to water we have $\frac{\mu_1}{\mu_2} \approx \frac{3}{4}$.

We may apply [S] to such problems as a point source above or below an air-water interface. In particular the caustic for the system of refracted rays can be found by the method of Section 2. If we consider a point source under water $\left(\mu = \frac{4}{3}\right)$ and the rays for which $\sin\beta < \frac{3}{4}$, we can show that the virtual caustic for the rays refracted into air is the evolute of an ellipse, and that the eikonals are parallels of an ellipse. (An object under water, viewed along different directions from above, appears to lie along the corresponding rays tangent to this virtual caustic.)

Fermat assumed that in a given medium light travels with a velocity v inversely proportional to the index of refraction $\left(v = \frac{c}{\mu}\right.$, where c is the velocity of light in $\left.\text{vacuum}\right)$ and rewrote (1) as

$$(3) \qquad \frac{\sin\alpha}{v_1} = \frac{\sin\beta}{v_2}, \qquad v_1 = \frac{c}{\mu_1}, \qquad v_2 = \frac{c}{\mu_2}.$$

He then derived (3) from the following minimum principle called Fermat's Principle:

[F]: A ray takes the least time to travel between two points.

If the total ray path consists of two straight lines L_1 and L_2 in two media with velocities equal to v_1 and v_2 respectively, then the corresponding travel times are $t_i = \frac{L_i}{v_i}$ with $i = 1,2$; from [F] we see that $t_1 + t_2$ must be a minimum. Fermat's principle [F] not only replaces the clumsy [S] (the way [H] replaced [E]), it also includes [H] as the special case where $v_1 = v_2$ and the points S and P are on the same side of the interface.

We now use [F] and the geometry of Figure 3-2 to derive [S], essentially as we used [H] to derive [E]. The time taken to go the distance L_1 from S to I at a velocity v_1 is $t_1 = \frac{L_1}{v_1}$, and, similarly, $t_2 = \frac{L_2}{v_2}$ is the travel-time between I and P at velocity v_2 in medium 2. Thus [F] requires that

$$(4) \qquad \frac{L_1}{v_1} + \frac{L_2}{v_2} = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (d-x)^2}}{v_2}$$

be a minimum. Differentiating (4) with respect to x and equating the result to zero i.e.,

$$\frac{x}{v_1\sqrt{h_1^2 + x^2}} - \frac{d-x}{v_2\sqrt{h_2^2 + (d-x)^2}} = \frac{\sin\alpha}{v_1} - \frac{\sin\beta}{v_2} = 0,$$

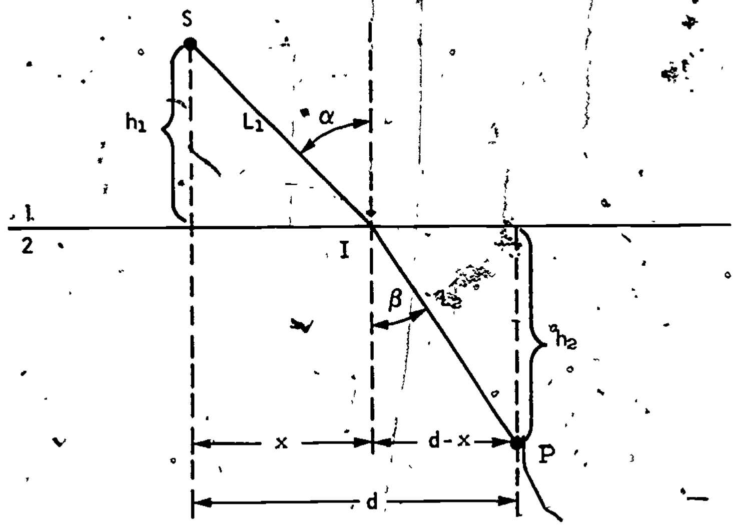

FIGURE 3-2

we obtain [S] in the form (1), (2) or (3):

(5) $$\sin\alpha = \frac{v_1}{v_2}\sin\beta = \frac{\mu_2}{\mu_1}\sin\beta = \mu\sin\beta\ .$$

If $\mu = 1$, and S and P are both in medium 1, then (5) reduces to [E].

It is clear from our discussion of the replacement of [H] by [H'] in Section 2, that we should also generalize [F] by replacing least time by stationary time. Equivalently, if we define the optical path length to be μL, then as the analog of [H'] we take

[F']: a ray follows the stationary optical path between points.

Unlike [S], we may use [F'] and (5) for refraction at curved interfaces. Thus we could now consider the refraction analogs of the reflection problems we considered previously. For example, we could determine the caustic for the two-dimensional problem of a set of parallel rays incident nose-on along the axis of a convex semicircle of radius a "capping" a strip (where the semicircle-strip region is characterized by μ), or the three-dimensional analog of incidence along the axis of a hemispherically capped rod. For this case, the caustic of the rays refracted at the semicircular interface can be obtained by essentially the method of the previous section; the cusp of the caustic lies on the axis at a distance $\mu a/(\mu-1)$ from the interface, so that e.g., for $\mu = 4/3$ the cusp is at 4a. Similarly we can obtain the caustic for the rays that undergo two refractions for incidence on a circle (or sphere); for this case the cusp is on the axis at a distance $(3\mu-2)a/2(\mu-1)$ from the first interface, e.g., for $\mu = 4/3$, the cusp is at 3a. (Would there be a shadow? Have you ever illuminated a cylindrical glass of water with a flashlight?)

Rainbow caustics. Newton (1719) showed that white light could be regarded as made up of light of different colors, each specified by a different value of some physical parameter (say ω), and that in general the relative index of refraction between two media depended on color, $\mu = \mu(\omega)$. Thus a ray of white light incident at an angle α on an interface may be treated as a set of coincident rays of different colors (ω) each being refracted a different angle $\beta(\omega)$ as determined by the corresponding index of refraction $\mu(\omega)$. Consequently, a single ray of incident white light becomes a fan of colored rays (the spectrum) on refraction, the different colors appearing at angles β determined by

$$\sin\beta(\omega) = \frac{\sin\alpha}{\mu(\omega)}. \quad (6)$$

For yellow light incident on an air-water interface we have $\mu = \frac{4}{3}$; for the colors red through yellow on to blue, $\mu(\omega)$ increases through $\frac{4}{3}$, and consequently $\sin\beta(\omega)$ decreases from red to blue as sketched in Figure 3-3.

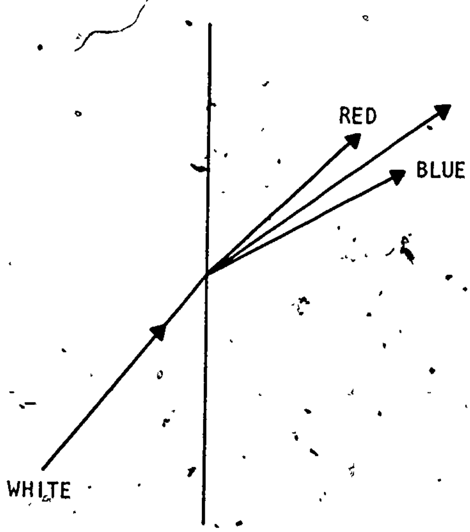

FIGURE 3-3

Relation (6) is strikingly exhibited in the rainbow formed by sunlight incident on spherical water drops. In the following we use the methods of calculus to determine the angles of the primary rainbow and secondary rainbow for circular cylinders and spheres.

A ray incident on a transparent circle (such as a cylinder of water in air) gives rise to an infinite number of rays. Some of these are shown in Figure 3-4; initially we consider the ray p. If a system of parallel rays is incident on the cylinder, then we deal with incident rays making all angles α (from 0 to 90°) with the cylinder's normal, and to each corresponds a different $p(\alpha)$. We want to show that in the vicinity

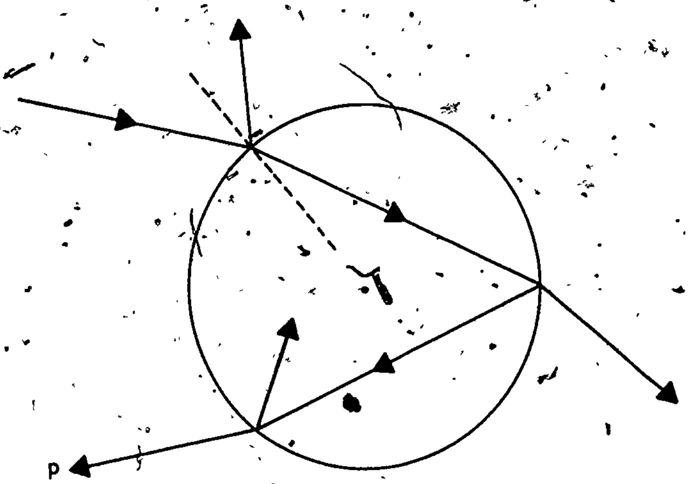

FIGURE 3-4

of some particular value of the angle α (say α_s) the rays $p(\alpha_s)$ will have a caustic (will be "focused"), or equivalently that the angle of emergence of $\bar{p}$ has a stationary value corresponding to α_s. ...

The primary rainbow corresponds to rays that have undergone two refractions and one internal reflection as shown in Figure 3-5. We now show that the angle φ (the angle between the emergent ray and the incident ray) has a stationary value φ_s and express φ_s in terms of the relative index μ.

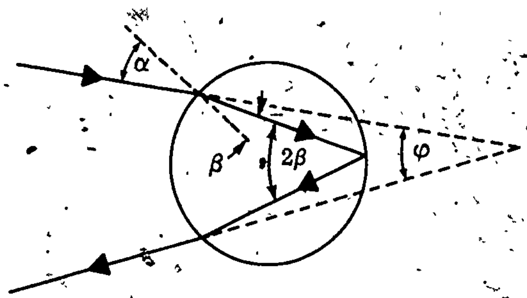

FIGURE 3-5

From the figure, we have:

(7) $$\frac{\varphi}{2} = 2\beta - \alpha .$$

Equating $\frac{d\varphi}{d\alpha}$ to zero we obtain

(8) $$2\frac{d\beta}{d\alpha} = 1 .$$

In addition, by differentiating the law of refraction $\mu \sin\beta = \sin\alpha$, we have

(9) $$\mu \cos\beta \, \frac{d\beta}{d\alpha} = \cos\alpha \, ,$$

so that (8) and (9) yield

(10) $$\mu \cos\beta = 2\cos\alpha \, .$$

Thus from (10) and (5), we obtain

(11) $$3\cos^2\alpha_s = \mu^2 - 1 \, ,$$

which determines the stationary value α_s of the angle of incidence, and consequently the corresponding values of β_s and φ_s. In particular,

(12) $$\sin\frac{\varphi_s}{2} = \frac{1}{\mu^2}\left[\frac{4-\mu^2}{3}\right]^{\frac{3}{2}} .$$

For yellow light, $\mu(\omega) \approx \frac{4}{3}$, and consequently $\varphi_s \approx 42°$; for the colors red through blue, the corresponding values of φ_s decrease through 42°.

This result for a cylinder also holds for a sphere, and is therefore basic to the rainbow formed when sunlight illuminates a region of air containing many water drops. For one water sphere, if the sun is in back of you and you can see the ray through P of Figure 3-5, the colored rays will be at about 42° with respect to the direction of incidence (in the plane of the sun, the drop, and your head). If the sunlight illuminates a large number of drops over a very large volume of space, then you will see the familiar rainbow arc.

For the <u>secondary rainbow</u>, corresponding to two internal reflections, we deal with the geometry of Figure 3-6. We now have

(13) $$\frac{\pi}{2} - \frac{\varphi}{2} = 3\beta - \alpha \, .$$

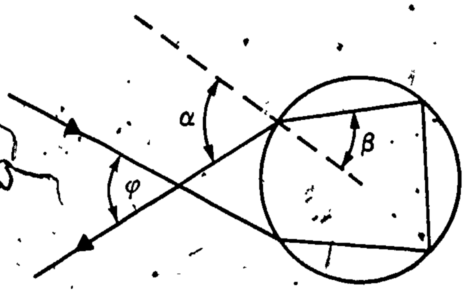

FIGURE 3-6

Differentiating (13), and using (9) and (5), we obtain

(14) $$8 \cos^2 \alpha_s = \mu^2 - 1,$$

and consequently

(15) $$\sin \frac{\varphi_s}{2} = \frac{\mu^4 + 18\mu^2 - 27}{8\mu^3}.$$

For $\mu = \frac{4}{3}$, we have $\varphi_s \approx 51°$. More generally, for n internal reflections, we have

(16) $$\cos^2 \alpha_s = \frac{(\mu^2 - 1)}{n(n+2)}.$$

Using the fact that $\mu(\omega)$ increases as the colors go from red to blue, one can describe the appearance of the primary and secondary arcs in space and account for the different orders of the colors in the two cases.

Stratified Medium: If we apply the law of refraction to a ray traveling through a set of parallel slabs as in Figure 3-7, such that each slab has a different index of refraction, we obtain

(17) $$\mu_0 \sin \theta_0 = \mu_1 \sin \theta_1 = \mu_2 \sin \theta_2 = \cdots = \text{constant} = c.$$

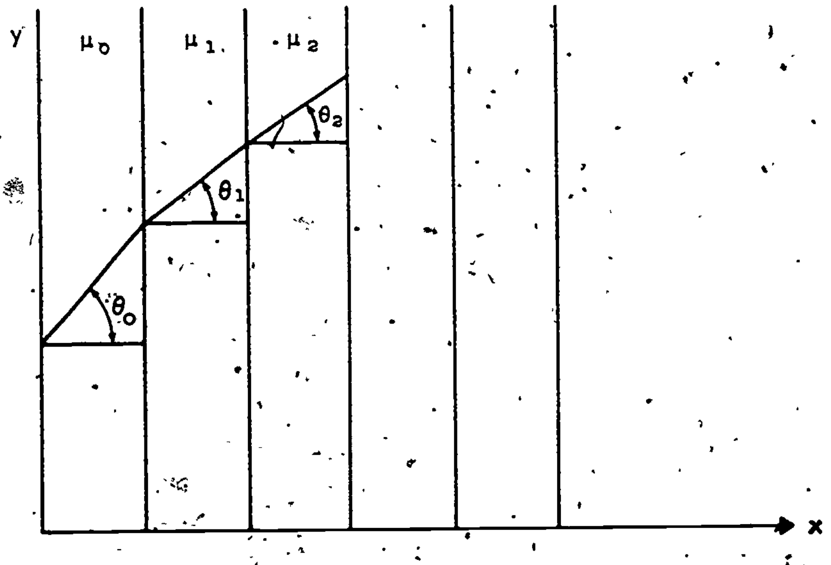

FIGURE 3-7

Similarly for the limiting case of a continuum whose index of refraction is solely a function of x, we have

(18) $$\mu(0) \sin \theta(0) = \mu(x) \sin \theta(x) = c.$$

Using $\frac{dy}{dx} = \tan\theta = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}}$, we obtain

(19)
$$\frac{dy}{dx} = \frac{\frac{c}{\mu}}{\sqrt{1 - \left(\frac{c}{\mu}\right)^2}},$$

from which

(20)
$$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{\mu}{c}\right)^2 - 1}}.$$

Integrating (20) between 0 and x, we obtain

(21)
$$y - y_0 = \int_0^x \frac{d\xi}{\sqrt{\left(\frac{\mu}{c}\right)^2 - 1}}$$

Thus in terms of $\mu(x)$ we have derived an equation to specify the set of rays that start at 0, y_0 and arrive at x, y.

As an illustration, we assume

(22)
$$\mu(x) = \frac{1}{1 + bx},$$

where b is an assigned parameter. To evaluate the integral (21) in terms of (22), we change the variable to φ, such that

(23)
$$c(1 + b\xi) = \sin\varphi,$$

and rewrite (21) as

(24)
$$y - y_0 = \frac{1}{cb} \int_{\sin^{-1} c}^{\sin^{-1} c(1 + bx)} \sin\varphi \, d\varphi.$$

Thus on integration, we obtain

(25)
$$\left[x + \frac{1}{b}\right]^2 + \left[y - y_0 + \frac{\sqrt{1 - c^2}}{cb}\right]^2 = \frac{1}{(cb)^2},$$

i.e., the equation of a circle of radius $\frac{1}{cb}$ whose center is located at $-\frac{1}{b}, y_0 - \frac{\sqrt{1 - c^2}}{cb}$.

Rotating the coordinate frame of Figure 3-7 (for convenience in the following application to rays in the atmosphere), we show ray paths in Figure 3-8 for (25) with $b < 0$ and with $b > 0$.

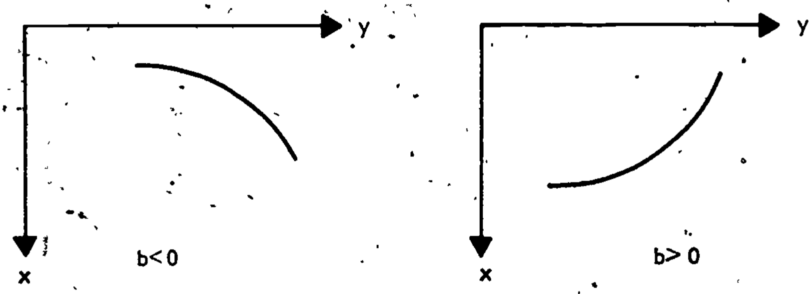

FIGURE 3-8

The above results serve to account for mirages. Normally the density of the atmosphere decreases gradually with increasing altitude; the index μ, which depends primarily on the density, also decreases gradually. However, over a cold extended surface the density and μ may decrease rapidly with height. An object on the surface may then be seen at large distances by means of down-curving rays as in Figure 3-9

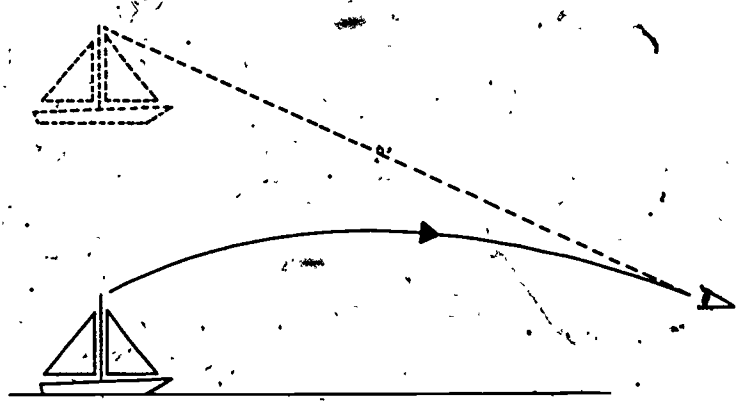

FIGURE 3-9

(in which the curvature is greatly exaggerated). The eye sights along the angle of the ray's arrival, and one imagines that the ship lies along the line-of-sight. On a much larger scale and with normal decrease of μ with altitude, Figure 3-9 accounts for our seeing the sun by refraction after it has passed below the horizon.

A more common mirage occurs over a hot extended surface when the density and μ first increase and then decrease with increasing height. For such cases the eye may see the object by an upcurving ray as well as by a straight ray as in Figure 3-10

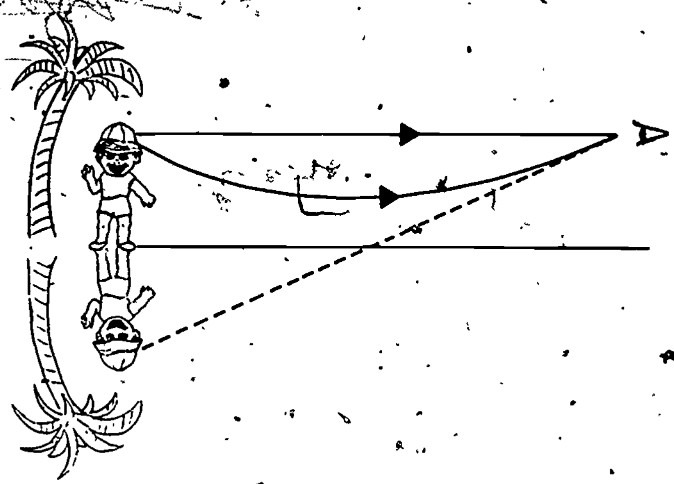

FIGURE 3-10

(in which the curvature is again greatly exaggerated). In this situation the eye sees mirror images; since this is reminiscent of reflection on water, one also imagines that a water surface is present.

4. Kepler-Lambert Principle.

In Section 2, we assumed Hero's principle [H'] that the ray path be stationary, and used the calculus to reveal some of the implicit physics. Except for the discussion of shadows, we did not associate a magnitude with the rays. We now do so, and then supplement [H'] with an energy principle or flux principle. We introduce a flux density $F = |F|$ as a measure of the energy flow per second through unit area normal to a ray; indicating the direction of a ray by a unit vector $\hat{R}$, we call $F\hat{R}$ the flux vector.

Kepler in 1604 (by a mixture of mysticism, insight, and some observations of light sources) proposed the inverse square law for the flux density associated with a source of light. He argued essentially as follows: If a steady source (one not varying with time) is emitting rays uniformly in all directions, then the total associated flux (total energy per second) passing through any spherical surface centered on the source (as in Figure 4-1) is a constant, then, since the surface of a sphere increases as the

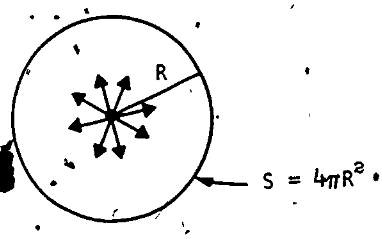

FIGURE 4-1

square of its radius R, the flux density F must be proportional to $\frac{1}{R^2}$. (Visualize the source as something like a steady omnidirectional water faucet.) Equivalently, since

(1) $$\int F(R)dS = F(R)\int dS = F4\pi R^2 = C, \text{ a constant},$$

it follows that

(2) $$F(R) = \frac{C}{4\pi R^2}.$$

Lambert (1760) generalized (1) by taking the component of the flux vector $F\hat{R}$ normal to a surface as the measure of the energy flow. Thus if S is any surface enclosing any steady source, and if $\hat{N}$ is the outward unit normal on S, then from the work of Kepler and Lambert it follows that:

[KL]: $$\int_S F\hat{R} \cdot \hat{N} dS = \int_S F \cos\theta\, dS = \text{constant} = C,$$

where θ is the angle between the ray direction $\hat{R}$ and the surface normal $\hat{N}$ as in Figure 4-2.

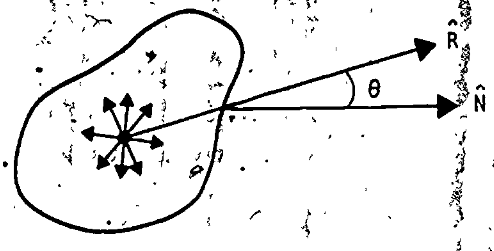

FIGURE 4-2

Equation (1) is the special case of [KL] corresponding to a uniform point source at the center of a sphere; for this case F depends only on R, and $\hat{R}$ is parallel to $\hat{N}$. If we take the constant in (1) to equal unity, then the corresponding form of (2) is the flux density for a unit point source:

(3) $$F(R) = \frac{1}{4\pi R^2},$$

Equation (3) corresponds to uniform radiation in three-dimensions.

We may also apply [KL] to determine the flux density for a unit source radiating uniformly in only two dimensions, i.e., to obtain F(R) for a unit line source. Thus we consider an extended source along the z-axis emitting rays uniformly in perpendicular xy-planes as in Figure 4-3. We apply [KL] for $C = 1$, and S equal to a coaxial

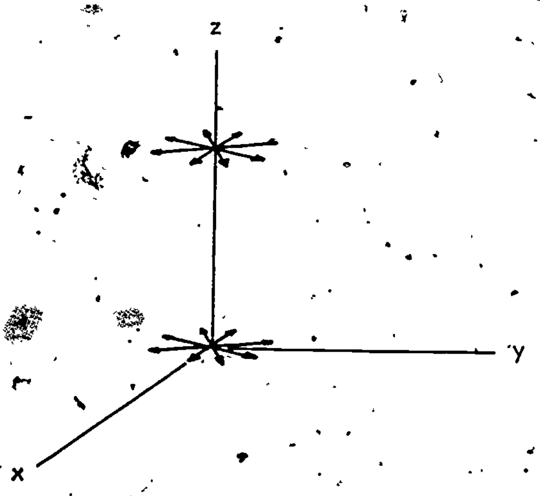

FIGURE 4-3

right circular cylinder having unit length along z and radius R as in Figure 4-4.

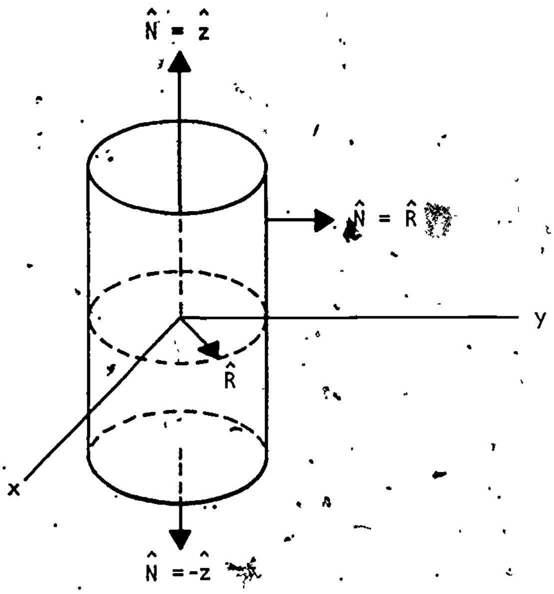

FIGURE 4-4

The [KL] integral vanishes over the flat caps of the cylinders at $z = \pm 1/2$: for these pieces, we see that $\hat{R}$ is perpendicular to $\hat{N} = \pm\hat{z}$ (where $\hat{z}$ is the unit vector $\frac{\vec{z}}{z}$), and consequently $\hat{R} \cdot \hat{N} = \pm\hat{R} \cdot \hat{z} = 0$. We are thus left with the integral over the circular wall (of height unity and radius R), for which $\hat{R} \cdot \hat{N} = \hat{R} \cdot \hat{R} = 1$:

(4) $$F(R) \cdot \int dS = F \cdot 1 \cdot 2\pi R = 1,$$

where we took C = 1 to correspond to a unit source. Thus from (4),

(5) $$F(R) = \frac{1}{2\pi R}$$

is the flux density for unit length of unit line source.

Similarly a planar source is defined as an infinite plane (say zy) emitting rays perpendicularly along $\hat{R} = \pm\hat{x} = \pm\frac{\vec{x}}{x}$. For this case we take S as a right cylinder as in Figure 4-5, with faces of unit area parallel to the source (and "enclosing" it). Since $\hat{R} \cdot \hat{N}$ vanishes except over these unit faces, [KL] for C = 1 gives

(6) $$F\int dS = F \cdot 2 = 1,$$

and consequently

(7) $$F = \frac{1}{2}$$

is the flux density for unit area of source.

It should be kept in mind that all the above equations are very special cases of [KL]. In general $F\hat{R}$ is a function of all coordinates, and [KL] applies for any closed surface enclosing any number of steady sources.

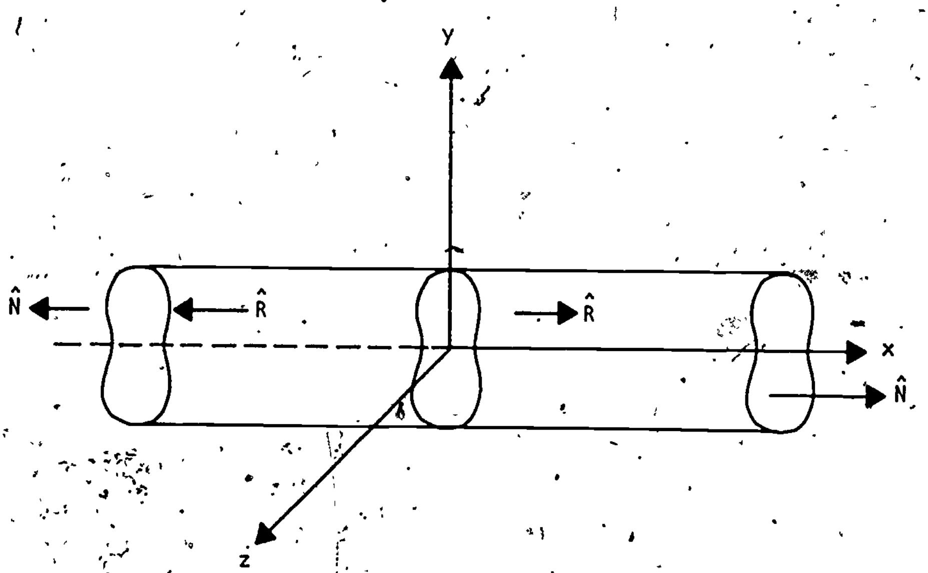

FIGURE 4-5

From [KL] it also follows that the integral over a surface S_o that does not enclose any sources must vanish:

(8) $$\int_{S_o} F\hat{R}\cdot\hat{N}dS = 0\ ,$$

i.e., the constant in [KL] is zero for a source-free region. (The source is outside the closed surface, so that whatever flows in through part of S_o flows out through another part.) We use this to define a pencil of rays (a narrow cone of rays) analytically.

Consider the capped tubular surface S_o of Figure 4-6 which encloses a set of

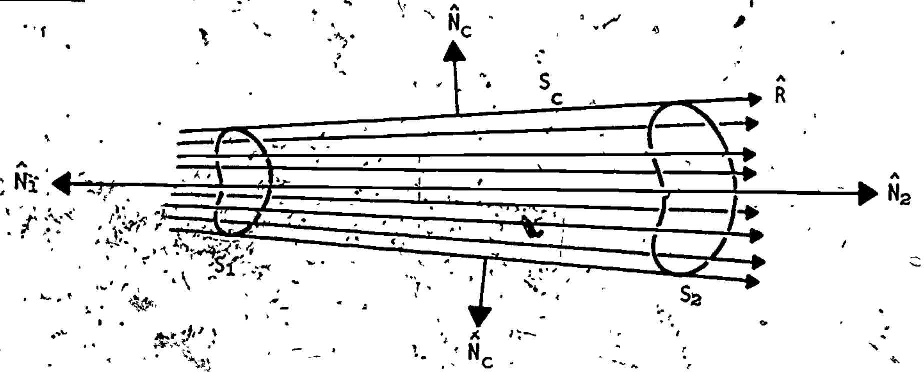

FIGURE 4-6

rays. The curved surface S_c is generated by the rays passing through the boundary curve of S_1, and the entrance and exit faces S_1 and S_2 are taken perpendicular to the rays, i.e., the faces are pieces of the eikonal surfaces discussed in Section 2. Thus if $\hat{N}_c$, $\hat{N}_1$, and $\hat{N}_2$ are the normals to S_c, S_1, and S_2, respectively, then $\hat{R} \cdot \hat{N}_c = 0$, $\hat{R} \cdot \hat{N}_1 = -1$, and $\hat{R} \cdot \hat{N}_2 = 1$. Applying (8) to $S_o = S_c + S_1 + S_2$, we see that the integral over S_c vanishes and we are left with

(9) $$\int_{S_1} F dS = \int_{S_2} F dS ,$$

where the integrals are over the entrance and exit faces of the tube. In general F varies from point to point on each face. However, except for special situations, we can take the faces small enough so that the variation of F over each is negligible, and approximate (9) by

(10) $$F_1 S_1 = F_2 S_2 .$$

The set of rays for which (10) holds is defined as a <u>pencil</u> of rays; the set is enclosed by a tube whose faces are portions of eikonals. In deriving (10), the "special" situations which we excluded are those where a face coincides with a focus or caustic. As discussed in Section 2, a focus corresponds to the intersection of many rays, so that a closely fitting tube enclosing such a set would narrow down to $S_2 = 0$; for such cases (10) is not a valid relation for F. However, such cases are still covered by (8) provided S_o does not intersect the caustic.

Let us apply (10) to the essentially two-dimensional problems of reflection from a cylinder with generator along z discussed previously in Section 2. Dropping the unessential z-coordinate (i.e., taking all pencils as having [illegible] height along z), we treat S of (10) as a small arc length equal to the local radius of curvature (ρ) times the small angle (ψ) subtended by S at the center of curvature (the origin of ρ), i.e.,

(11) $$S = \rho\psi .$$

Since the two faces in (10) are chosen as portions of eikonals (surfaces normal to the rays), and the centers of curvature are the limiting intersection of the common normals to the faces, we see that S_1 and S_2 have the same center of curvature. Thus $S_1/\rho_1 = S_2/\rho_2 = \psi$, and from (10) we obtain

(12) $$F_2 = F_1 \frac{S_1}{S_2} = F_1 \frac{\rho_1}{\rho_2} ,$$

which specifies the variation of the flux density with distance along rays.

We now consider the perfect (complete) reflection of a parallel pencil of rays of width S_0 and flux density F_0 from a convex curvilinear portion C_1 of a reflector as in Figure 4-7. The length of the eikonal of the corresponding reflected pencil is S_1 at C_1, and S_2 at a distance R from C_1. Perfect reflection means that no rays penetrate the reflector, i.e., the total incident flux is conserved by the process and passes through the terminal cap S_2. Thus (10) holds: $F_0 S_0 = F_1 S_1 = F_2 S_2$. Approximating the curves by their tangent lines, we have $S_1 = S_0$ (and [illegible]ch equals $C_1 \cos\alpha$,

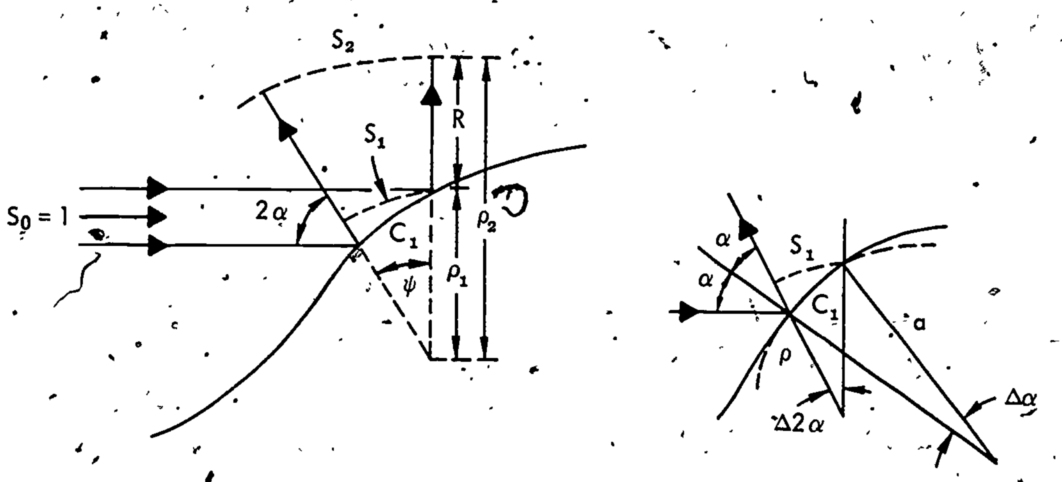

FIGURE 4-7

where α is the angle of the rays with the normal at C_1); consequently $F_1 = F_0$. Since C_1 is convex, $S_2 > S_1$ and it follows that $F_2 < F_1 = F_0$. To express F_2 explicitly in terms of F_0, we now use (12); we write the radii of curvature of the eikonals as $\rho_1 = \rho$ and $\rho_2 = \rho + R$, and obtain

(13) $$F_2 = F_1 \frac{\rho_1}{\rho_2} = \frac{F_0 \rho}{\rho + R}$$

where R is the distance along the reflected pencil, and $\rho + R$ is the total distance from the caustic (the locus of origins of the ρ's). Thus as discussed in Section 2, the rays that pass through S_2 appear to originate at their virtual intersection point (on the caustic) inside the reflector.

For the semicircular mirror of radius a (see Figure 2-13), we found previously that $\rho = \frac{a}{2} \cos\alpha$, where α is the angle of incidence with the surface normal; this also holds for reflection from a convex portion of a more general surface in terms of the radius of curvature a at the point of incidence. Thus for a <u>convex point</u> (i.e., a point on a convex portion of the mirror), the reflected flux density equals

(14) $$F = \frac{\frac{a}{2}\cos\alpha}{\frac{a}{2}\cos\alpha + R} F_0 ,$$

where we dropped the subscript 2. We may also rewrite the above as $F = \frac{F_0}{(1 + RQ)}$ where $Q = \frac{1}{\rho}$ is the curvature of the eikonal at the reflection point.

On the other hand, for reflection from a concave point, the caustic is real, and R and ρ are on the same side of the reflector as in Figure 4-8. For this case we replace ρ by $-\rho$ in (13) and (14), and obtain

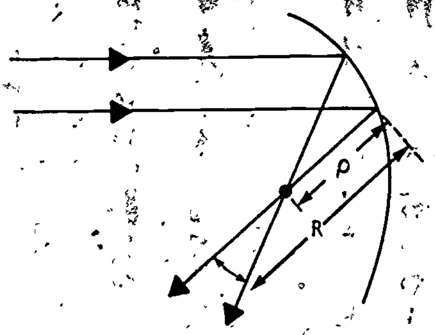

FIGURE 4-8

(15) $$F = \left|\frac{\rho}{\rho - R}\right| F_0 = \left|\frac{\frac{a}{2}\cos\alpha}{\frac{a}{2}\cos\alpha - R}\right| F_0 , \quad R \neq \rho = \frac{a}{2}\cos\alpha$$

where the absolute value is used because we defined F as a positive quantity. Equation (15) specifies F except on the caustic $R = \rho$, the special situation ($\underline{S} = 0$) excluded from the start when we introduced (10).

Equations (12) to (15) apply for two-dimensional problems. The discussion of the corresponding three-dimensional forms can also serve as a vehicle for some additional terminology on properties of surfaces. If the neighborhood of a point on a surface can be represented by functions all of whose derivatives exist at the point (regular at a point), then there are two orthogonal directions (the principal directions) on the surface for which the radii of curvature have maximum and minimum values; these are the principal radii of curvature ρ_a and ρ_b. In terms of ρ_a and ρ_b of the eikonal at the point of reflection, the analog of (13) is

(16) $$F = \frac{\rho_a \rho_b}{(R + \rho_a)(R + \rho_b)} F_0 ,$$

so that $\frac{F}{F_0}$ is essentially a product of two terms of the form in (13). Equation (16) holds if the scatterer is convex at the reflection point; the reflected tube diverges in both principal planes (the planes through the principal directions and through R). The extension of each ray within the reflector lies in general on two caustics; both caustics are virtual, and require negative values of R for their specification, i.e., they correspond to the vanishing of $-R + \rho_a$ and of $-R + \rho_b$. On the other hand, if the scatterer is concave at the reflection point, then ρ_a and ρ_b are negative and the caustics are real; the reflected tube is initially convergent in both planes. If the reflecting surface is convex in one plane and concave in the other (i.e., if it has a saddle point), then one caustic is real and one virtual; the reflected tube is then divergent in one plane, and initially convergent in the other. The first two cases (convex and concave) correspond to elliptic points of the surface, and the third to a hyperbolic point. Between these two classes of points lies the transition case of a parabolic point; at a parabolic point, one radius of curvature is infinite, and (16) reduces to (13) as obtained previously for two dimensions. If both ρ_a and ρ_b of (16) (or if ρ of (13)) become infinite, then we obtain the result for a plane reflector, $F = F_0$.

In addition to the above points, there are special points at which the principal directions are indeterminate: at an umbilical point the radii of curvature of any two normal sections (the curves cut from the surface by planes containing the normal) are equal. In general, the umbilical points of a surface are isolated points. However, we have already implicitly considered one surface all of whose points are umbilics, i.e., the plane scatterer for which $F = F_0$. The only other surface having this property is that of the sphere.

We can discuss reflection of parallel rays from a sphere of radius a by exploiting our analogous results for the circular cylinder. One caustic for the sphere is generated by rotating that for the circle around the axis of symmetry, i.e., $R = -\frac{a}{2}\cos\alpha$. In addition, there is a line caustic $R = -\frac{a}{2\cos\alpha}$ which arises from the rotational symmetry of the sphere: all rays incident on the sphere at an angle α (the rays in a circular tube) give rise to reflected rays whose extensions intersect the axis of symmetry at the same point $\frac{a}{2\cos\alpha}$ (after grazing the epicycloid) as in Figure 4-9. For

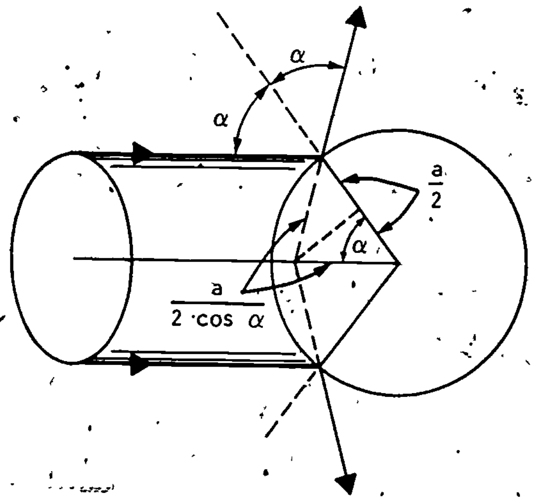

FIGURE 4-9

the circle in two dimensions we had only two such rays; for the sphere, we have a full ring. Thus in (16) the radii of curvature ρ_a and ρ_b of the eikonal at the reflection point equal $\frac{a}{2}\cos\alpha$ and $\frac{a}{2\cos\alpha}$, and we obtain

$$F = \frac{a^2}{4\left(R + \frac{a}{2\cos\alpha}\right)\left(R + \frac{a\cos\alpha}{2}\right)} F_0 \tag{17}$$

Flux and Path Length. A remarkable property of F as in (14) (remarkable only at the present stage of our development of a mathematical model for scattering) is that it can be rewritten in the form

$$F = \frac{F_0 a^2 \cos^2 \alpha}{R L''_H}, \tag{18}$$

where L''_H is the second derivative with respect to α of the general path length $\xi + R(\Theta) = L$ introduced in Equation (2.3), and where the subscript H indicates that we use the condition $L' = 0$, or $\Theta = 2\alpha$, as follows from Hero's principle. (We can replace L''_H/a^2 by the second derivative of L with respect to arc length along the reflector.) We may assure ourselves that (18) holds by retracing our derivation of the caustic in Section 2.3. Our equation $g(\alpha) = 0$ for a reflected ray corresponds to $L' = 0$, and our equation $g'(\alpha) = 0$ for the caustic of the reflected rays corresponds to $L''_H = 0$.

We mention this now to make more explicit that F becomes singular on the caustic $L''_H = g'(\alpha) = 0$, which indicates a limitation of our present esentially geometrical model for the propagation of light, and as a preview of a deeper relation between flux and path length that must hold for a more complete model.

Partially transparent surface: We can extend the present flux considerations for the case of a perfect reflector to the case of partially transparent media considered in Section 3, and obtain the corresponding reflected and transmitted fluxes when a pencil of rays is incident on the curved interface of two different optical media. At the present primitive stage of our model, we simply introduce a reflection factor $0 < P(\alpha) < 1$ as a multiplier for the values of the reflected flux (e.g., as in (14)) for the corresponding perfectly reflecting surface. Applying (8) to a pencil of rays incident on a plane interface (with S_0 "enclosing" the interface as in Figure 4-6), we then find that for the incident flux to equal the sum of that reflected and that transmitted we require that the geometrically transmitted flux be multiplied by the transmission factor $1 - P(\alpha)$.

Scattering Applications. In the above we applied [KL] to obtain the flux density for elementary sources (in one-, two-, and three-dimensions), and to determine the change of flux density along a ray in a pencil of varying cross section. We now extend our considerations of the elementary sources to the analogous scattering problems. We define the corresponding "elementary scatterers" by the previous stipulation that the total radiated flux equal unity and that it be distributed uniformly over the available directions; then we indicate generalizations. We do not solve any scattering problems explicitly, but exploit the previous development to introduce terms and general forms for subsequent use.

Thus if we have a set of parallel rays normally incident on a perfectly reflecting planar scatterer at $x = 0$, as in Figure 4-10, then from Section 2, the incident set of

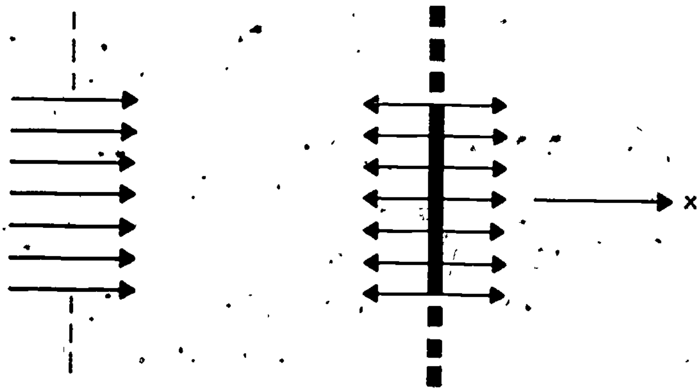

FIGURE 4-10

rays gives rise to a reflected set of rays and to a shadow forming set of rays. We may say that the incident rays have "excited" the plane and converted it to a source of radiation; we call the incident set the primary radiation and the scattered set (reflected plus shadow forming) the secondary radiation, and say that the plane has become a secondary source. We define an elementary planar scatterer as a secondary source fully analogous to the simple planar source considered in Figure 4-5 and Equations (6) and (7). (In a later section we consider analytically the specific problem that this corresponds to.) The essential feature of (7) is that the flux does not depend on distance. Similarly for a more general planar scatterer we write the scattered flux corresponding to the direction of incidence $\hat{x}$ as

(19) $$F = M(\hat{R}, \hat{x}) , \quad \hat{R} = \pm\hat{x} ,$$

where the direction of scattering $\hat{R}$ corresponds either to geometrical reflection, $\hat{R} = -\hat{x}$, or to forward scattering $\hat{R} = \hat{x}$. (For parallel rays incident on a perfect reflector, it turns out that M is the absolute square of E discussed in Section 2.4; if the incident flux density is unity, then M = 1.)

Similarly if we visualize rays incident perpendicularly on a fine cylinder as in Figure 4-11, and apply [H] essentially as for the discussion of edge diffracted rays

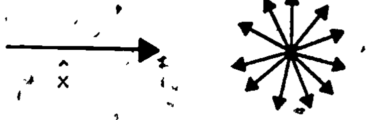

FIGURE 4-11

in Section 2, we see that the scattered set of rays travel radially outward from the scatterer. We define an elementary line scatterer as a secondary source fully analogous to the line source of Figures 4-3 and 4-4 and Equations (4) and (5), i.e., the total outgoing flux per unit length of scatterer is unity, and the scattered flux density per unit length is given by (5). Similarly for a more general line-like or cylindrical obstruction, F is inversely proportional to R but the flux density is no longer the same in all directions:

(20) $$F = \frac{M(\hat{R}, \hat{x})}{R},$$

where the direction of observation $\hat{R}$ may range over all values in the xy-plane.

Finally in three dimensions, we visualize a point scatterer excited by rays, and define a secondary point source analogous to that of Figure 4-2 and Equations (1) to (3). More generally, for an arbitrary scatterer in three dimensions, the analog of (20) is

(21) $$F = \frac{M(\hat{R}, \hat{x})}{R^2},$$

where again M depends only on directions and not on distance. The three functions M depend on various parameters, and their determination requires a more complete mathematical model than the present one. However, the forms (19), (20), and (21) give the appropriate dependence of F on R.

We are now in a position to further our discussion of the relative magnitudes of the different rays of Section 2. Thus parallel rays incident on a broad finite strip excite essentially two kinds of secondary sources: the body of the strip becomes a secondary planar source with reflected flux density equal to that incident, and the edges become secondary line sources with flux density specified in general by (20). The flux density of the rays geometrically reflected from a plane are independent of distance, but the flux density of the rays diffracted from the edges decreases in general as $\frac{1}{R}$ with increasing R. (The form (20) does not hold on or near a caustic of edge rays; we require a more complete scattering model in order to discuss magnitudes near caustics, not only of diffracted rays but of reflected and transmitted rays as well.)

5. Huygens' Principle.

In preceding sections we considered the reflection and refraction of a parallel set of rays. We started with Euclid's restrictive laws [E] of propagation and reflection, and then replaced [E] by the more general principle [H'] of Hero that the ray path (L) be stationary; using [H'] and the calculus we determined the caustics and foci of the rays reflected from curved surfaces. Similarly, to consider the set of rays transmitted through an interface of two transparent media (media specified by different indices of refraction μ) we started with the restrictive Snell's law [S] of refraction; and then replaced [S] (and [H']) by Fermat's more general principle [F'] that the travel time or optical path (μL) be stationary. Thus all our results on ray paths and their envelopes (caustics) are covered by the one principle [F'].

In the discussion of systems of rays in Section 2, we also introduced a system of eikonal curves (eikonal surfaces in three dimensions) that were perpendicular to the rays; in Figure 2-16, we sketched some of the eikonals for reflection of a set of parallel rays from a convex cylinder. From the remarks at the end of Section 2.4, we see that we can construct an eikonal curve of Figure 2-16 geometrically as the curve traced by the end of a taut string (taut against the epicycloid caustic curve of Figure 2-12) whose other end is fastened at the cusp. Thus if we measure length along the string, then each point of an eikonal is at the same distance from the cusp of the caustic.

In addition to [F'], we also used the Kepler-Lambert flux principle [KL] to associate a magnitude with the rays (the energy per second crossing unit area normal to a ray). We used [KL] to determine the flux for unit symmetrical sources (three-, two-, and one-dimensional), and to derive the change in flux density for a pencil of rays reflected from a curved surface.

Thus all our preceding discussion is covered by the two "laws of nature" [F'] and [KL] plus some of the implicit physics relevant to geometrical optics phenomena. The basic physics was contained in the two laws, the rest was mathematical manipulation based on a geometry of rays and some procedures of the calculus. As a preliminary to the introduction of additional structure into our mathematical model for the propagation of light, we now supplement our previous geometrical construction of the eikonals by an alternative construction called Huygens' principle. This principle by itself does not give us any new results but (and this is often much more significant) it gives us a new way of thinking about the results we have already obtained.

In Section 1 we mentioned the two familiar forms in which energy propagates: packaged around particles, or associated with waves. At the present stage of the development the flux involved in [KL] is in some sense guided along the geometrical rays. It is easy to visualize the rays as guide lines for very fine particles (a view held by the ancients, and refined by Newton — 1705), but we may also regard the rays as the normals of a system of wave surfaces (the eikonals).

Many individuals (Hooke, Euler, and others) regarded light as a wave motion in a special medium, but it was Huygens (1690) who introduced the subject as an analytical one. His intuition was based on the analogous two-dimensional problem of how disturbances travel on the surface of water. (Drops of water dripping off your fingers above the surface of still water create disturbances at their point of impact that then travel outward in circular ripples along the water surface.)

Huygens used the fact that light has a finite velocity of propagation v (as established experimentally by Römer, 1676) for the development of a wave theory of light. He assumed that in a given medium, light starting from an elementary source at time t_0 would spread as a spherical surface whose radius r(t) increased in time as $v(t - t_0)$.

If we start a light source at time t_0 and leave it turned on, the corresponding Huygens' wave surface is an outgoing spherical front — a discontinuous disturbance whose one-dimensional analog is shown in Figure 5-1. In this figure we plot a magnitude associated with the disturbance (say the flux density F introduced in Section 4, or a related function) as a function of time; at time $t_1 > t_0$, the wave front has moved a distance $v(t_1 - t_0)$, and it keeps advancing with increasing t. (The discontinuous function drawn in Figure 5-2 is called a Heaviside pulse.)

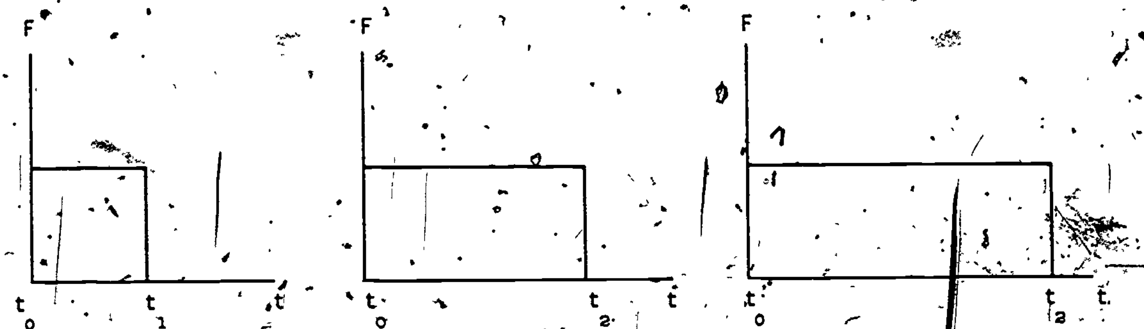

FIGURE 5-1

Starting with an advancing wave front (hereafter, the wave surface W) in three dimensions, Huygens regarded each point on the wave surface W as a new source of an elementary spherical wave (call it a wavelet, w) whose radius also increased in time proportionally to v. Thus if the original wave surface W is a sphere of radius

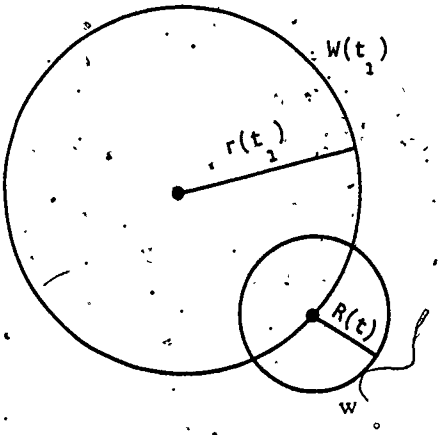

FIGURE 5-2

$r(t_1)$, the wavelet surface w spreads as a sphere of radius $R(t) = v(t - t_1)$; the two-dimensional analog is shown in Figure 5-2. To obtain the wave surface of the source's advancing wave front, Huygens prescribed

> [Hu]: to construct the wave surface $W(t_2)$ at time $t_2 > t_1$, regard the wave surface $W(t_1)$ at time t_1 as the locus of the centers of wavelets w of identical radius $R = v(t_2 - t_1)$, and take $W(t_2)$ as the outer envelope of the set of w's.

Figure 5-3, based on Figure 5-2, illustrates [Hu]. The essential notion is that if we assign an appropriate magnitude function to a wavelet, then only on the outward envelope of the set of w's (i.e., only on $W(t_2)$) do the magnitudes of the w's add up (reinforce) to give a significant overall effect.

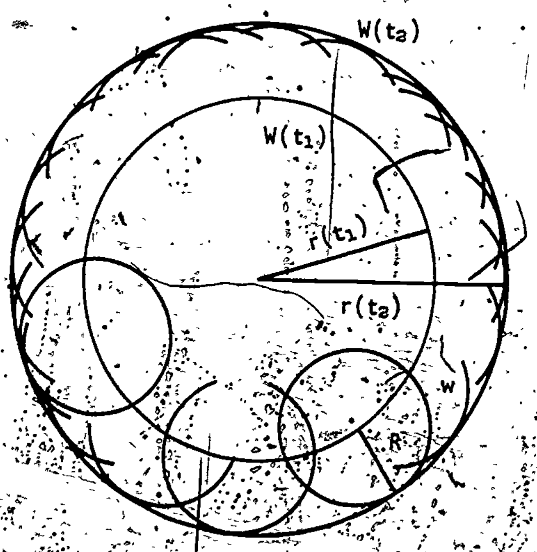

FIGURE 5-3

If a planar portion of a wave surface is incident on a reflecting surface, we can construct the reflected wave front by means of [Hu] as indicated in Figure 5-4. The

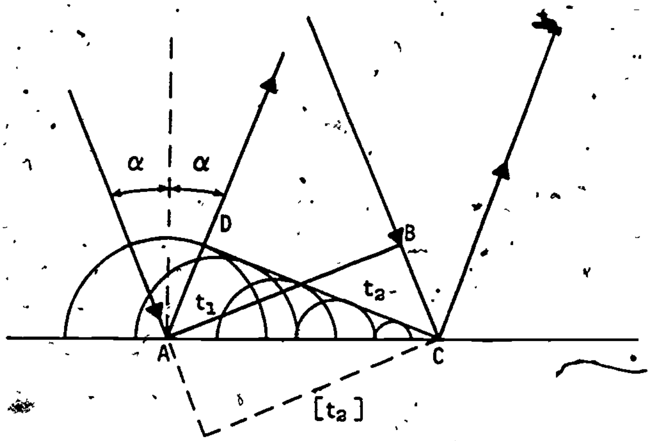

FIGURE 5-4.

figure shows the incident wave front at t_1 (plus the two rays or normals that bound it), and a dashed front at [t_2] to indicate where the front would have reached at time $t_2 > t_1$ in the absence of the reflector. The actual reflected front at time t (the image, shown unbroken) of the dashed front at [t_2] is the envelope of the wavelets generated by the incident front as it encountered the reflecting surface. The dashed front is also the wave front of the shadow forming rays discussed in Section 2. Figure 5-5 shows

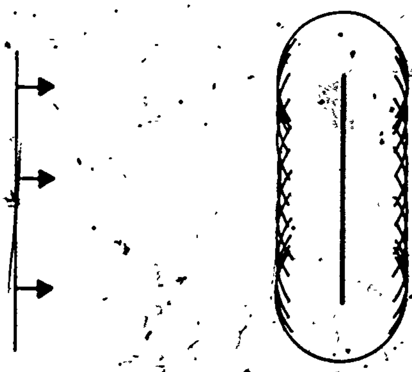

FIGURE 5-5

how Huygens' construction for scattering by a strip yields the closed scattered wave surface corresponding to the reflected plus shadow-forming plus diffracted rays of Figure 2-17(d); the result is of course simply the closed eikonal of Figure 2-17(e).

Similarly if the scattering surface is the interface between two different optical media specified by velocities v and v, we construct the transmitted portions of the wavelets to take into account that these portions are traveling at velocity v_1 instead of v, and then construct their envelope to obtain the refracted wave front as in Figure 5-6.

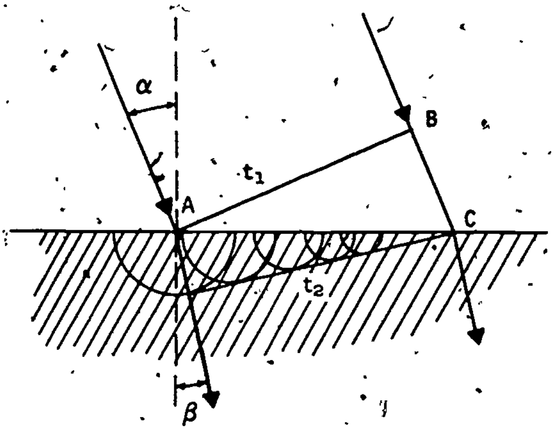

FIGURE 5-6

We have indicated that the Huygens' wave surfaces are simply the eikonal surfaces discussed in Section 2. We now apply [Hu] to reflection of a plane wave front (parallel ray system) by a perfectly reflecting convex semicircle and make this identification explicit. Since all waves in this problem move with the same velocity, all distances (L) traveled are proportional to time (t), so that we may work with either L or t; in order to exploit our previous figures and results, we work with distance L. The center of the circular scatterer in Figure 5-7 is at $x = 0$, $y = 0$. The corresponding incident wave is a plane wave front whose position at any time

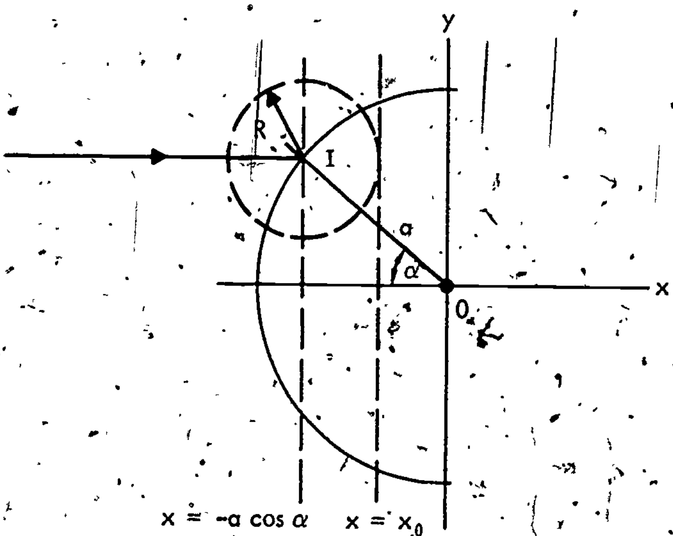

FIGURE 5-7

$t = t_0$ may be indicated by $x = x_0$; our reference time is $t = 0$, and our reference position is $x = 0$.

We treat the Huygens' construction for Figure 5-7 essentially as we did that of Figure 5-4. We construct wavelets of different radii at different points on the circular scatterer, the radius at a point being proportional to the time it would have taken the incident wave front to travel from that point to the plane $x = x_0$. Using Huygens' principle in this manner we may say that a point $I(\xi, \eta) = a(\alpha)$ of the scatterer, where $\xi = -a\cos\alpha < x_0$, under excitation by the wave front $x = -a\cos\alpha$ (see Figure 5-7) radiates a circular wavelet of radius $|\xi - x_0| = |a\cos\alpha + x_0|$; here $x = x_0$ is the present position of the incident wave front. The resultant wave front is the envelope of all such elementary wavelets. To construct a wave front geometrically, one draws enough such wavelets to enable their envelope to be sketched. Figure 5-8 shows the case $x_0 = 0$ (i.e., for the time when the incident front is at the origin) and Figure 2-16 of Section 2 shows additional curves for different values $x_0 > -\frac{a}{2}$; in each case, the straight portion of the curve corresponding to the shadow wave front is also the position that would have been reached by the incident front in the absence of the scatterer. The curves of Figure 2-16 can be constructed either by using the present procedure (circles centered on the scatterer) for different constants x_0, or by using the wave surface of Figure 5-8 as the locus of circles of identical radii and then drawing their outward envelope.

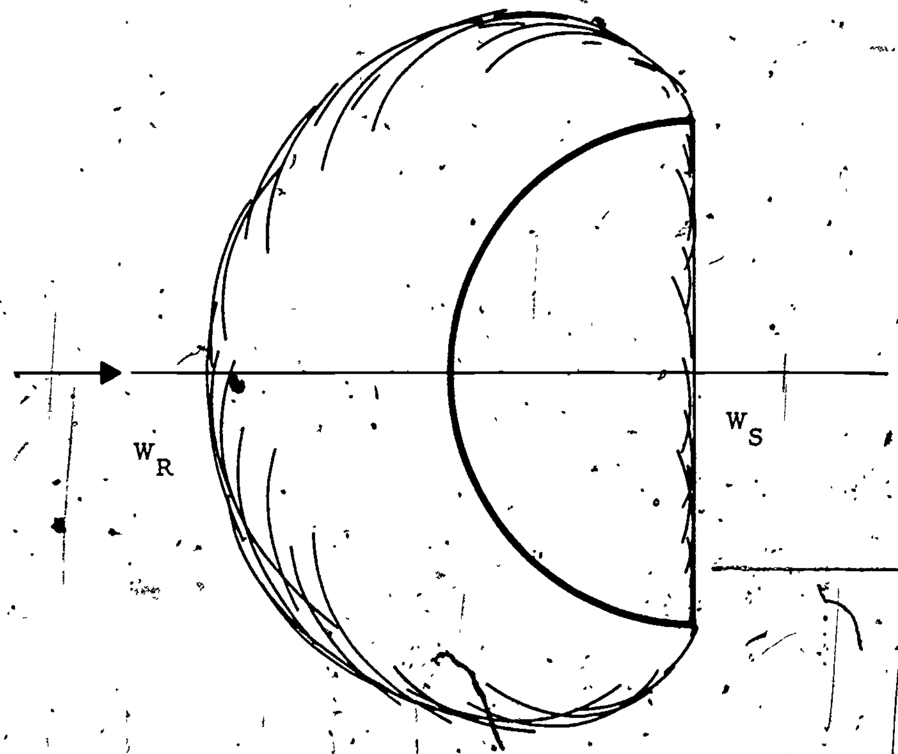

FIGURE 5-8

Analytically, we find the envelope of the family of circles by the same procedure we used in Section 2 to obtain the envelope of a set of straight lines. Thus if we take $x_0 = 0$, we have the equation of a Huygens' circlet

(1) $$(x + a\cos\alpha)^2 + (y - a\sin\alpha)^2 = a^2\cos^2\alpha .$$

The derivative with respect to α gives $y = a\sin\alpha - x\tan\alpha$, and substituting this expression for y into the equation of the circle (1) gives $x[x(1 + \tan^2\alpha) + 2a\cos\alpha] = 0$. Thus either

(2) $$x = -2a\cos^3\alpha \quad \text{and} \quad y = a\sin\alpha(1 + 2\cos^2\alpha) ; \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$$

or

(3) $$x = 0 \quad \text{and} \quad y = a\sin\alpha , \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2} .$$

The parametric equations (2) describe the envelope

(4) $$\frac{(x^2 + y^2)}{a^2} = 1 + 3\left(\frac{x}{a}\right)^{\frac{2}{3}}$$

so that the corresponding curve (W_R of Figure 5-8) is half of a two-cusped epicycloid (twice the size and rotated through 90 degrees, as compared with that for the rays shown in Figure 2-12). This portion of the wave front is generated by a point on a circle of radius $\frac{a}{2}$ rolling on the circle of radius a. The equations (3) specify the W_S portion of the envelope of Figure 5-8, which consists of a line segment of width $2a$ normal to the direction of incidence; this corresponds to the shadow forming wave.

Having one wave surface analytically as in (4) and (3), or graphically as in Figure 5-8, we can construct its normals (the rays of Section 2), and then obtain any other wave front on laying off a constant distance along the normals and joining the points. We can construct the evolute of the wave fronts (the caustic of the rays) and determine that $R + \frac{a}{2}\cos\alpha$ is the radius of curvature, where R is distance along the ray from the mirror; and, of course, we can "discover" the law of geometrical reflection by noting that at a given point, the reflected and incident wave normals make equal and opposite angles with the scatterer's normal.

From a "pure" wave view, in order to determine the scattered wave front when the incident front is at any position $x_0 > -a$, we use the wave surface of (4) and (3) derived for $x_0 = 0$ ($W = W_R + W_S$ of Figure 5-8) as the locus of the centers of circles of radius $|x_0|$, and again determine the envelope mechanically or analytically, i.e., we need not refer back to the surface of the scatterer. If $x_0 > -\frac{a}{2}$, we obtain the wave fronts shown in Figure 2-16 of Section 2. The point on a wave surface (corresponding to the incident front at x_0) at a distance $R = x_0 - a\cos\alpha$ from the mirror along a ray, may also be designated by the cylindrical coordinates r and θ as in Figure 5-9.

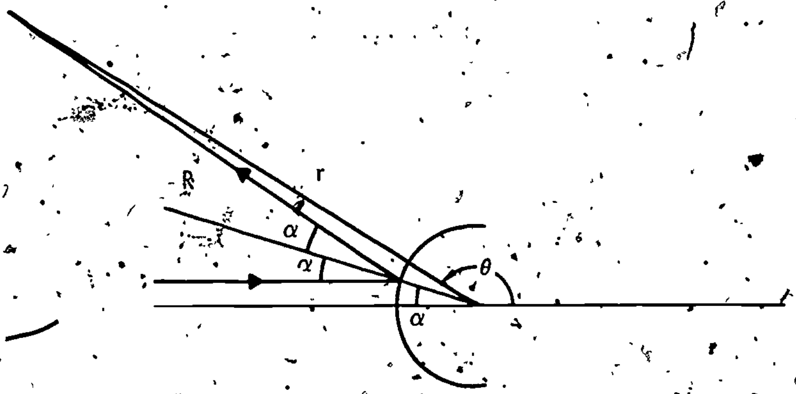

FIGURE 5-9

For very large values of x_0, we see that R and r are practically parallel and that $\theta \approx \pi - 2\alpha$; we have $r \approx R - a\cos\alpha \approx x_0 - 2a\cos\alpha \approx x_0 - 2a\left|\sin\frac{\theta}{2}\right|$, which corresponds to a wave from a source at $x = -\frac{a}{2}$ (the cusp of the virtual caustic). If we then neglect $a \ll r$, we obtain $r \sim x_0$ and the wave fronts approach circles centered on the origin of the mirror.

For incidence on the convex semicylinder, these wave fronts are real in the same sense as we spoke of real intersections for the rays of Section 2; for incidence on the concave semicylinder, the wave fronts of Figure 2-16 are virtual. The virtual wave fronts for incidence on the convex cylinder (the real ones for the concave case) are obtained for $x_0 < -a$; these are the curves of Figure 5-10(b) to (f) plus their images in the x-axis. If $-a < x_0 < -\frac{a}{2}$ as in Figure 5-10(a), then the wave system is part real (the part near the axis) and part virtual. For the sphere we obtain the virtual wave fronts by rotating the curves of Figure 5-10 around the x-axis.

In Figure 5-10(b) to (e), the straight lines correspond to the shadow forming wave, the curves intersected by the reflector suggest an edge wave, and the remaining curves suggest a wave outgoing from an origin at $x = -\frac{a}{2}$ (the geometric focus); the set of curves completely inside the reflector also includes the axial point $x = -\frac{a}{2}$ in Figure 5-10(f) which corresponds to $x_0 = -\frac{3a}{2}$. The most significant feature of the set of figures is that the locus of the cusps (which correspond to strong reinforcement of the wavelets) is the virtual caustic of the geometrical rays derived previously in Figure 2-12. The figures in reverse order, (f) to (a), illustrate that as x_0 increases from $-\frac{3a}{2}$ to $-\frac{a}{2}$, the inner cusps of the virtual wave system trace the epicycloidal virtual caustic of the reflected ray system; the outer cusps correspond to the virtual "point caustic" ($x = -\infty$) of the shadow forming rays.

If we construct the waves of Figure 5-10 by the procedure of Figure 5-7 (i.e., by means of wavelets of different radii centered on the scatterer), then we find that the "edge wave" is generated by wavelets originating relatively close to the edge of the

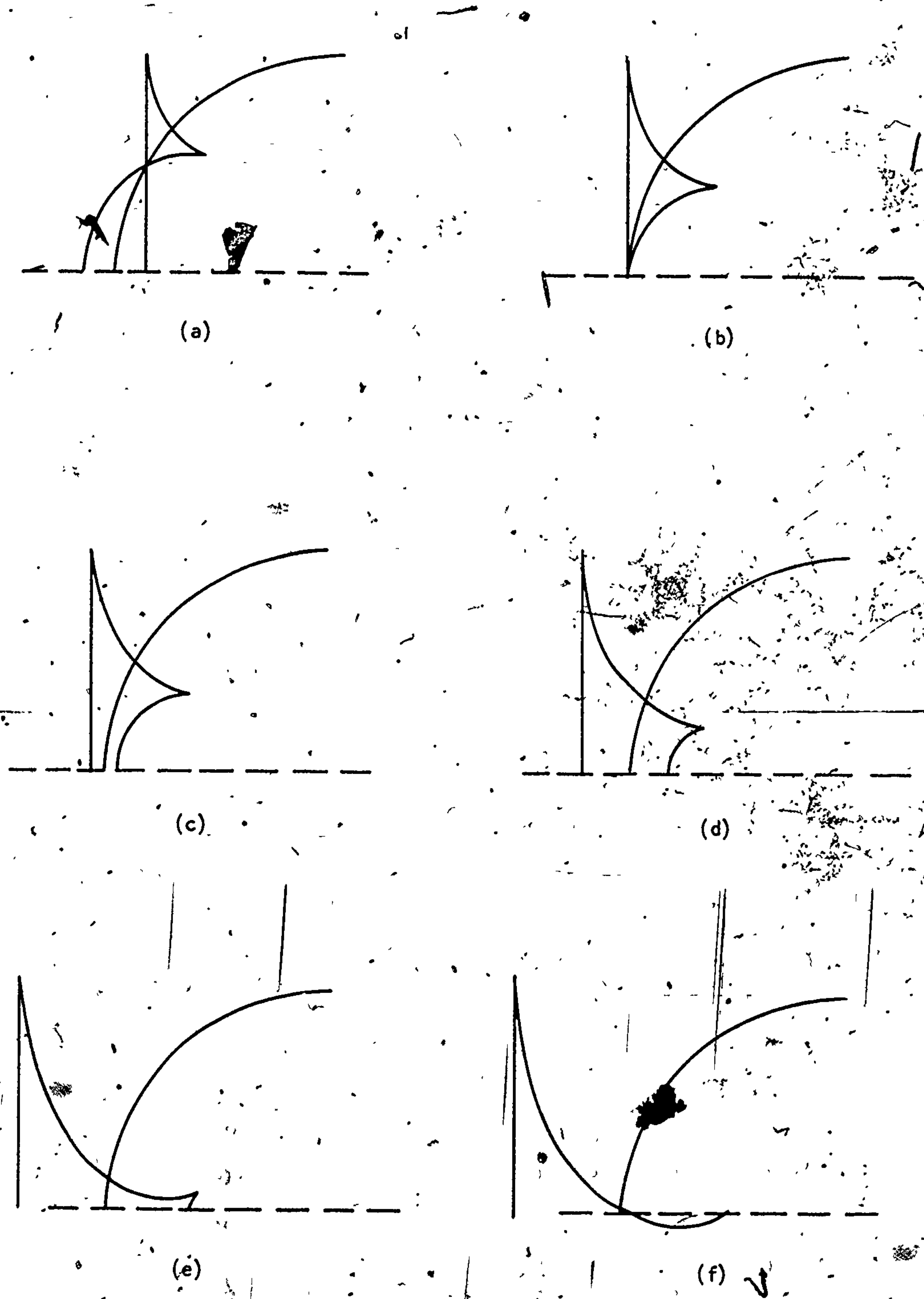

FIGURE 5-10

scatterer, and that the remaining curve (the "crater" generated by rotating the curve around the x-axis) is the envelope of the wavelets originating close to the axis. (All wavelets originating on the arc from the edge to the point (P) on the scatterer cut by the very edge wavelet contribute to the inner part of the edge wave, and the "crater wave" corresponds to wavelets on the arc between P and the axis.) Similarly if we construct the curves of Figure 5-10 by using wavelets of constant radii centered on the wave front of Figure 5-8, then the edge wave arises primarily from wavelets originating on the curve near the edge of the scatterer. If we invert the construction and generate the real wave fronts from a virtual front (e.g., Figure 5-10(d)), then the wavelets originating inside the reflector generate all but the straight line and "rounded corner" parts of Figure 2-16.

6. Periodic Waves.

In the preceding section we saw that [Hu] applied to a wave front gives directly the eikonals of ray theory. Huygens represented light essentially as an irregular sequence of isolated disturbances or pulses. The essential feature of the mathematical description of a wave pulse is shown in Figure 6-1, which represents some

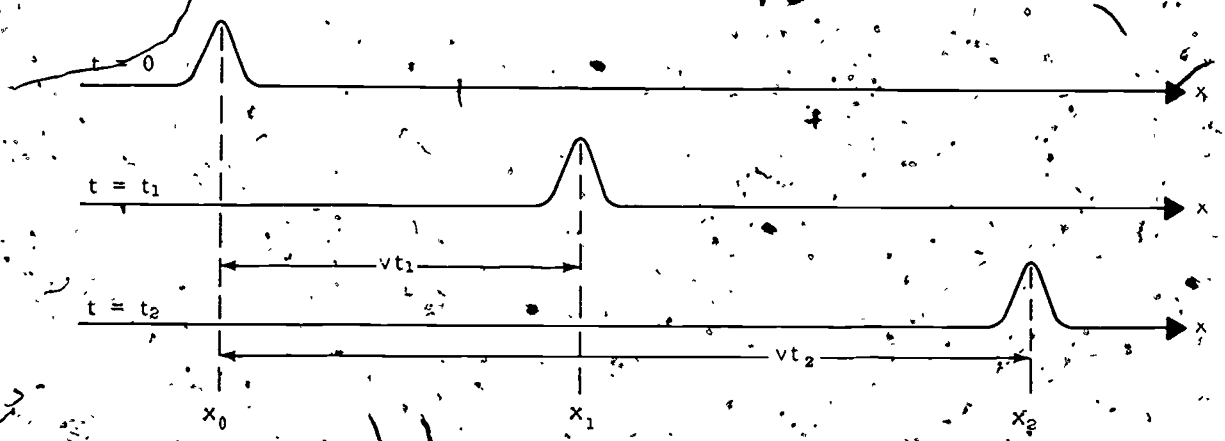

FIGURE 6-1

arbitrary disturbance propagating with velocity v along the x-axis. The significant aspect of Figure 6-1 is that the shape of the pulse does not change in time.

If we specify the pulse form at $t = 0$ by $y = f(x)$, then since the pulse form at any time t is obtained by the translation x to $x + vt$, the pulse form at time t is given by

$$(1) \qquad y = f(x - vt) .$$

Physically, we see that the function $f(x - vt)$ represents the unchanged disturbance moving along the x-axis (direction $\hat{x}$) with constant velocity v. Similarly a disturbance moving in the direction $-\hat{x}$ would be represented by $f(x + vt)$.

By itself [Hu] is merely another method for rederiving the results we obtained geometrically. However if we associate the idea of periodicity with Huygens' idea of waves, then we will have progressed quite far towards the full mathematical model we are developing.

Periodicity: Newton (1642-1727), by refracting a pencil of white light through a prism of glass, showed that a ray of white light could be regarded as made up of rays each having a single color (an idealization called monochromatic light), and that the relative index of refraction μ depended on color. We touched on this previously in our discussion of the rainbow when we worked with $\mu(\omega)$, with ω as the "color parameter." His studies on the colors obtained by illuminating thin transparent plates, essentially established that

[N]: monochromatic light is periodic with period dependent on ω.

Newton's picture of light as a stream of fine particles subject to periodic "fits" that followed each other at regular intervals is not appropriate for the visible phenomena he was familiar with, but the idea of periodicity related to color is as significant as Huygens' idea of waves.

Young (1801) combined Newton's idea of periodicity with Huygens' idea of waves, and regarded monochromatic light as made up of periodic waves.

If we rewrite (1) in the form f(p) with

(2) $$p = k(x - vt), \quad k = k(\omega)$$

where $k(\omega)$ (the "propagation constant") depends on color, and where p is called the phase of the wave, then Young's principle states

[Y]: monochromatic light can be represented by a wave that is a periodic function of the phase $p = k(x - vt)$.

Analytically, we express [Y] as

(3) $$f(p) = f(2\pi + p) = f(2\pi n + p); \quad n = 0, \pm 1, \pm 2, \cdots,$$

where the period of f is fixed at 2π, i.e., f has the same value each time its argument changes by 2π, the excess of p over an integral multiple of 2π gives position within the cycle, the base interval of length 2π.

If we add the constraint

(4) $$f(0) = A,$$

where A, the amplitude, is the maximum value of $|f|$, then the simplest wave-function satisfying (3) and (4) is the circular function

(5) $$f(p) = A\cos p = A\cos(k[x - vt]) \equiv u(x,t).$$

We may write

(6) $$k = \frac{2\pi}{\lambda}$$

where λ is the wavelength associated with light of a single color. If we increase x by Δx, then we increase p by $\Delta p = 2\pi \frac{\Delta x}{\lambda}$; each time x changes by the length λ we have $\frac{\Delta x}{\lambda} = 1$ and $\Delta p = 2\pi$, so that f(p) of (5) goes through a maximum and minimum in the process. The factor $kx = \frac{2\pi x}{\lambda}$ is a convenient dimensionless measure of distance for a monochromatic wave; it gives directly the phase change in units of 2π corresponding to traversing a distance x. Similarly, we may write $kvt = 2\pi \frac{t}{T}$ with

(7) $$kv = \frac{2\pi}{T}$$

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(7) $$kv = \frac{2\pi}{T}$$

as a dimensionless measure of time corresponding to the phase change in units of 2π for a time interval t. From (6) and (7) we write v, the phase velocity, as

(8) $$v = \frac{\lambda}{T}.$$

We are now in a position to interpret the parameters λ and $T = \lambda/v$ introduced in the above as well as the corresponding "color parameter" we have mentioned previously.

In Figure 6-2, we plot u of (5) versus x for fixed $t = t_0$, and in Figure 6-3, we

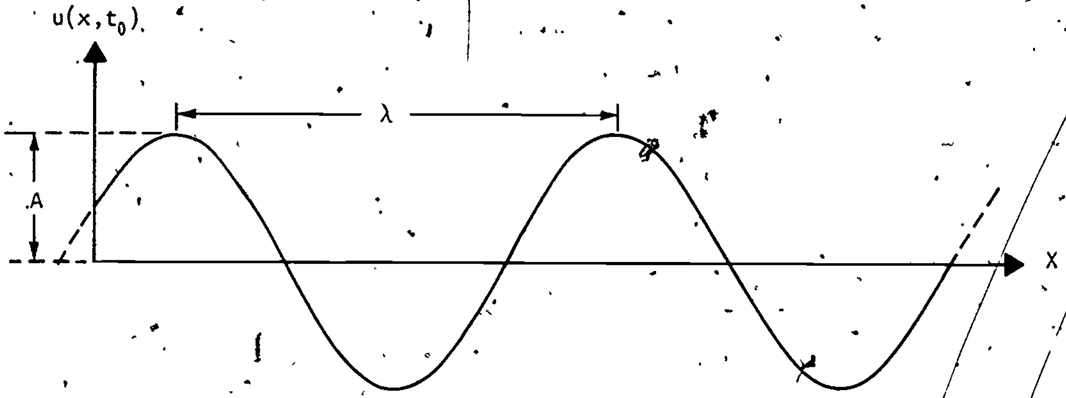

FIGURE 6-2

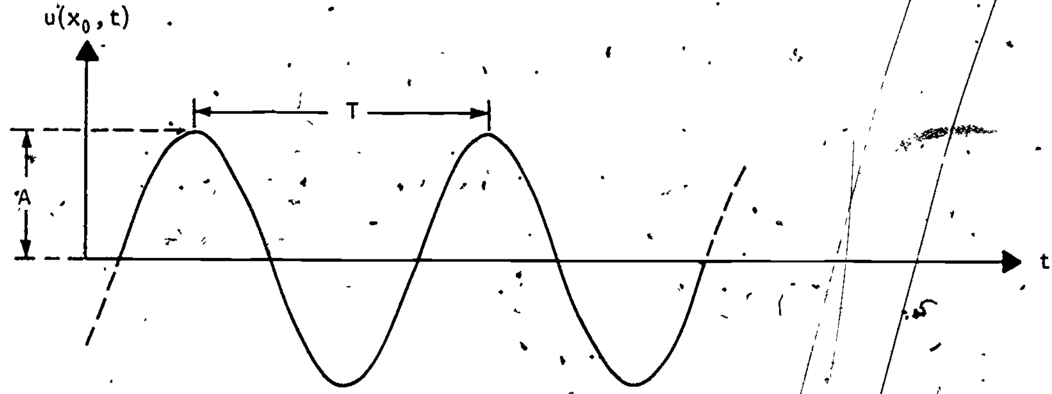

FIGURE 6-3

plot u of (5) versus t for fixed $x = x_0$. We see that at a fixed point in time, u is periodic in x; the wave form is repeated at intervals of length λ, which is why λ is called the <u>wavelength</u> or the space period. Similarly, at a fixed position on the x-axis, we observe that u is periodic in t: the wave form is repeated at time intervals T called the time period or simply the <u>period</u>.

At a given time t_0, we obtain the wave form of Figure 6-2. An instant later, $t_0 + \Delta t$, we obtain the same system of crests and valleys with each point of the wave shifted to $x + \Delta x$, such that $\left(\frac{x + \Delta x}{\lambda} - \frac{t_0 + \Delta t}{T}\right) = \frac{x}{\lambda} - \frac{t_0}{T}$. Thus we may visualize the wave as traveling in the direction $\hat{x} = \frac{\bar{x}}{\lambda}$.

The reciprocal of T is called the frequency (ν) of the source producing the wave and it is convenient to measure this frequency in units of 2π, i.e., to use $\omega = \frac{2\pi}{T}$ where ω is called the angular frequency. Thus we rewrite (5) as

(9) $$u = A \cos(kx - \omega t) ,$$

and we identify the color parameter ω as the angular frequency of the wave associated with light of a single color.

The angular frequency ω is a fixed characteristic of the source of the waves, and does not depend on the optical properties of the different media (characterized by different v) through which a wave passes, however, the wavelength $\lambda = 2\pi v/\omega$ does depend on the medium. In general the phase velocity v is a function of ω, so that waves of different frequencies travel with different velocities $v(\omega)$ in the same material. Equivalently, since the index of refraction is defined as inversely proportional to v, we may rephrase the above in terms of $\mu(\omega)$. Taking the development to equation (9) as applying to a medium with index of refraction $\mu = 1$ (free space or vacuum), we replace kx for the more general case by

(10) $$k\mu x \cong \frac{2\pi}{\lambda}\mu x = \frac{2\pi}{\lambda_\mu}x , \quad \lambda_1 = \lambda ,$$

where λ_μ (a function of ω and the material) is the wavelength in the optical medium defined by $\mu(\omega)$.

The wavelength λ is the scale factor we anticipated when we sought to distinguish between geometrical reflection and edge diffraction at the end of Section 2. We could have introduced much of the above structure into the ray picture by associating the idea of phase (periodicity) with a ray. However, the wave picture is in general more fruitful for the usual visible phenomena. For convenience in the following, we may use a mixed terminology with the rays understood as the corresponding wave normals.

Thus we say that if light of a single color travels a distance L in free space, its phase has changed by kL. Corresponding to the unit sources of Section 4, the phase at a distance R along a ray from the source differs from the phase at the source by kR. Similarly for the reflection problems of Section 2, the phase of the reflected ray at P in Figure 2-4a differs by $k(L_1 + L_2)$ from the phase at S, and the phase of the ray at P in Figure 2-7 relative to its phase at $x = 0$ is given by

$$p = k(\xi + R) = k\left(\xi + \sqrt{(\xi - x)^2 + (\eta - y)^2}\right) .$$

From the geometrical methods of constructing an eikonal (wave surface), we see that it is a curve (or surface) of constant phase (i.e., there is no phase difference between any two points on an eikonal), and we may label a particular eikonal by a particular value of p. Similarly for the elementary sources of Section 4, e.g., we may surround the point source of Figure 4-1 with a set of spherical surfaces of particular radii R_n corresponding to the particular phase differences kR_n.

Instead of working directly with cos p it is more convenient to carry out manipulations with

(11) $$e^{ip} = \cos p + i \sin p ,$$

and take the real part $\mathrm{Re}(e^{ip}) = \cos p$ when we want to exhibit the periodic behavior explicitly. If we represent (11) in the complex plane we obtain the vector diagram (Argand diagram) of Figure 6-4. As we progress along a ray (increase L), p

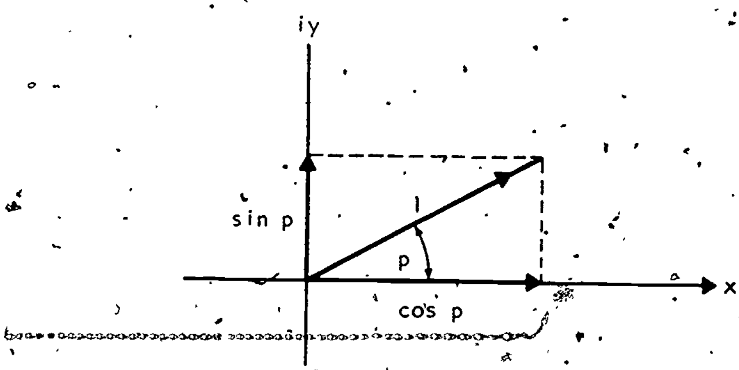

FIGURE 6-4

increases and the tip of the vector of unit length describes a circle of unit radius. The projection of the tip on the x-axis (the real axis) is the oscillatory function cos p; each time p increases by 2π (each time the tip describes a full circle), the x-projection goes through its maximum (+1) and minimum (-1) values. (The function e^{ip} is often called a phasor.) More generally, we work with

(12) $$f = Ae^{ip} ,$$

where the amplitude A is positive.

In subsequent applications we use the exponential form

(13) $$U = Ae^{i(kx-\omega t)} ,$$

such that u of (9) corresponds to Re U. We speak of (13) as a plane wave traveling in the $\hat{x}$ direction.

To tie in the present discussion with energy flux considerations of Section 4, we note that (in general) at distances from the source large compared to wavelength we may approximate A by a constant times $\sqrt{F}$, where F is the flux density introduced for the Kepler-Lambert principle. We write

(14) $$U \approx C\sqrt{F}\, e^{i(kL-\omega t)} ,$$

where L equals x or r, and where F in general depends on distance.

For the point source (or point scatterer) at the origin in three dimensions, we showed in Section 4 that $F = \frac{c}{r^2}$. We therefore write the corresponding wave as

(15) $$U \equiv C_3 \frac{e^{i(kr-\omega t)}}{kr} , \qquad r = \sqrt{x^2 + y^2 + z^2} ,$$

where we used the dimensionless kr (instead of r) in the denominator. Similarly for a line source (or line scatterer) along the z-axis, the wave corresponding to the flux density $F = \frac{c}{r}$ is

(16) $$U \approx C_2 \frac{e^{i(kr-\omega t)}}{\sqrt{kr}} , \qquad r = \sqrt{x^2 + y^2} .$$

In the same sense that we interpret (13) as a wave traveling along $\hat{x}$, we speak of (15) and (16) as waves traveling outwardly along $\hat{r} = \frac{\vec{r}}{r}$, or as outgoing waves. For the planar source (or planar scatterer) at x = 0 (one-dimensional case), F is independent of distance; the analog of (15) and (16) is

(17) $$U = \begin{cases} Ce^{i(kx-\omega t)} & \text{for} \quad x > 0 \\ Ce^{-i(kx+\omega t)} & \text{for} \quad x < 0 , \end{cases}$$

which we rewrite compactly as

(18) $$U = Ce^{i(k|x|-\omega t)} .$$

The explicit dependence of (15) and (16) on $\hat{r}$ and t facilitates qualifying [Y], which holds rigorously only for a plane wave, more generally, the waves are periodic in t, and only approximately periodic in r (or in p) because the denominators in (15) and (16) are not periodic. Thus an increase in r corresponds not only to an increase in phase but also to a decrease in magnitude, e.g., so that although $\frac{\cos(kr-\omega t)}{kr}$ has maxima at equal space intervals λ, the magnitudes of these maxima decrease with increasing r. However, such magnitude effects for $k \gg 1$ are insignificant for the problems at hand.

We should also qualify the preceding discussion of phase for the reflection problem by explicitly restricting it to perfect reflection. If the scatterer is partially transparent, then the ray reflected at an angle α with the surface normal undergoes in general an additional phase change $\delta(\alpha)$, which we add to the above p.

Interference: The concept of interference was introduced into wave physics by Young. We discuss interference subsequently in detail but mention it now to stress the most significant feature arising from associating a wave (or more specifically a phase) with light. The essentials are indicated in Figure 6-5 for scattering of a

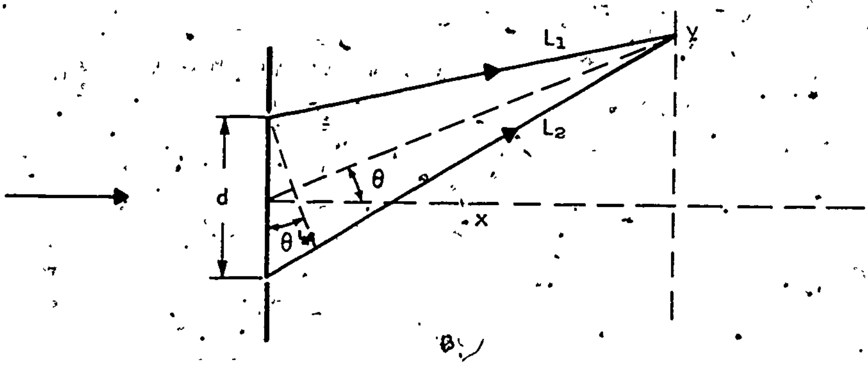

FIGURE 6-5

monochromatic plane wave by a screen containing two very narrow slits separated by $d \gg \lambda$. The waves (or rays with phase) that arrive at x, y ($x \gg d$) from the two slits have traveled different paths L_1 and L_2, and therefore differ in phase by

(19) $$\varphi = k(L_2 - L_1) \approx kd \sin\theta = \frac{2\pi d}{\lambda} \sin\theta .$$

Their magnitudes differ little, and we may write the resultant wave at x, y in the form

(20) $$U = U_1 + U_2 \approx W(1 + e^{i\varphi}) ,$$

where we have absorbed $e^{ikL_1 - i\omega t}$ and other factors into W. The corresponding energy flux density is proportional to

(21) $$F = |U|^2 = |W|^2 |1 + e^{i\varphi}|^2 = |W|^2 2(1 + \cos\varphi) .$$

Setting $|W|^2$ equal to unity, we show the essentials corresponding to

(22) $$F = |1 + e^{i\varphi}|^2 = 2(1 + \cos\varphi)$$

vectorially in Figure 6-6.

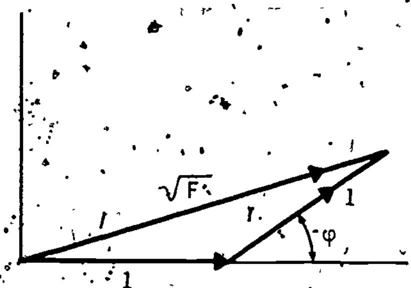

FIGURE 6-6

We see that if $\varphi = 0$ (i.e., $\theta = 0$ along the x-axis) then $F = 4$. (This corresponds essentially to a caustic of edge rays as discussed in Section 2; however, we now have much more structure for the description of light in the shadow region.) As we vary y, and the angle of observation $\theta \approx \frac{y}{x}$, the intensity F goes through a maximum value of 4 when

(23) $\varphi = 2n\pi, \quad n = 0, \pm 1, \pm 2 \ldots,$

and a minimum of zero when

(24) $\varphi = (2n + 1)\pi \, .$

This behavior is clear from (22), and more graphically from Figure 6-6: if $\varphi = 2n\pi$, then the vectors of unit length point in the same direction along a straight line and their resultant is 2; if $\varphi = (2n + 1)\pi$, then they point in opposite directions and cancel each other. The results for F with variation of φ are shown in Figure 6-7.

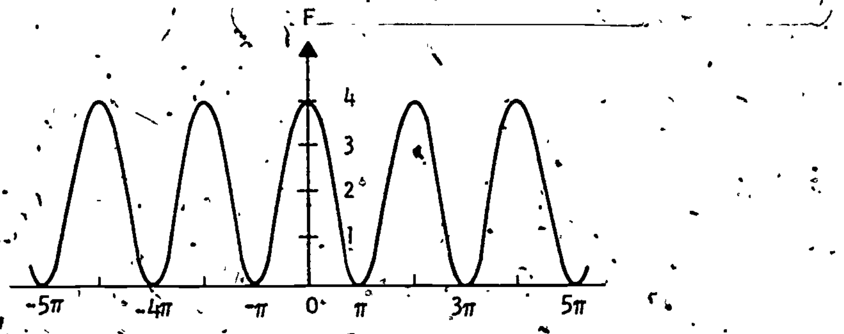

FIGURE 6-7

Thus for a monochromatic wave, (fixed λ), a parallel screen on the shadow side of the slit-screen (the dashed line at x in Figure 6-5) will show bright and dark bands: the bright "fringes" corresponding to $\varphi = 2n\pi$ are located on a screen at a distance x from the strip by

(25) $d \sin\theta \approx \frac{yd}{x} = 0, \lambda, 2\lambda, \ldots = n\lambda, \quad y = \frac{n\lambda x}{d},$

i.e., when the path difference is an integral number of wavelengths. Similarly the dark fringes corresponding to $\varphi = (2n + 1)\pi$ are located by

(26) $\frac{yd}{x} = \frac{\lambda}{2}, \frac{3\lambda}{2}, \ldots = \left(n + \frac{1}{2}\right)\lambda, \quad y = \left(n + \frac{1}{2}\right)\lambda\frac{x}{d} \, .$

We call (25) "constructive interference," and (26) "destructive interference."

If we use white light (a mixture of waves of different λ's), then along the axis $y = 0$, we obtain a white central fringe; however, from (25), the side fringes are displaced along y directly in proportion to λ and we therefore see bands of different colors. (Analogous phenomena in the shadow region of a wide strip were first noticed by Grimaldi.) Comparing (25) with measurements we find, from the displacement of the bands of light of different colors, that the wavelength for red light is about twice that of blue light, i.e.,

(27) $$\lambda_r \approx 2\lambda_b ,$$

and that the colors orange through yellow through green have wavelengths λ of length intermediate to that of red and blue.

In the following sections we consider several elementary applications to scattering phenomena of the Huygens-Newton-Young periodic wave theory of light. These applications are associated with Fraunhofer (1787-1826, an experimentalist), Fresnel (1788-1827, a theoretician), and Rayleigh (1842-1919, both).

Fraunhofer Diffraction by a Slit. We now apply the wave model to Fraunhofer diffraction of a plane wave by a slit of width $2a$ in a perfectly reflecting plane as in Figure 6-8. We take the origin at the center of the slit.

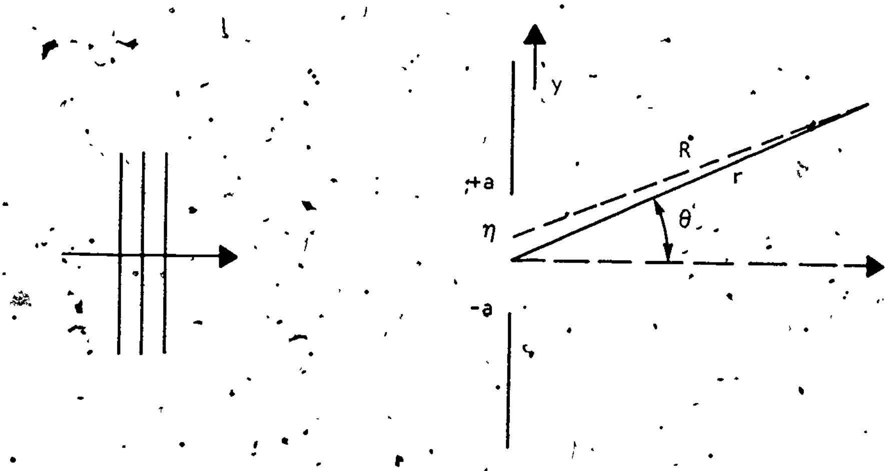

FIGURE 6-8

We write the incident wave as

(28) $$U_i = e^{i(kx-\omega t)}$$

and interpret (28) as a wave of unit flux density traveling in the x direction. Using [Hu] implicitly, we regard U_i as exciting wavelets in the plane of the aperture, and specify a wavelet originating at $x = 0$, $y = 0$ by the elementary outgoing wave form $C\frac{e^{i(kr-\omega t)}}{\sqrt{kr}}$ as in (16). Similarly for a wavelet originating at $0, \eta$ as in Figure 6-8 we use

(29) $$u = C\frac{e^{i(kR-\omega t)}}{\sqrt{kr}}, \quad R = \sqrt{r^2 - 2r\sin\theta + \eta^2}.$$

Every point of the line $y = \eta$, $x = 0$, for $-a \le \eta \le a$ (every line element of the slit) corresponds to a wavelet of the form (29). We represent the net effect of all such wavelets at a distant point $\vec{r}$ by the integral

$$U = \int_{-a}^{a} u(\eta)d\eta \ . \tag{30}$$

Restricting consideration to $r >> a$, we approximate R in the exponent by

$$R \approx r - \eta \sin\theta \ . \tag{31}$$

In the denominator we use simply $R \approx r$, because $|U|$ is much less sensitive to changes in the denominator than to changes of the phase. (From Figure 6-6 we see that a slight change of the magnitudes of the two vectors has little effect compared to a comparable change in the phase difference φ.) Thus (30) reduces to

$$U \approx C \frac{e^{i(kr-\omega t)}}{\sqrt{kr}} G(\theta) \ , \tag{32}$$

$$G(\theta) = \int_{-a}^{a} e^{ik\eta \sin\theta} d\eta \ , \tag{33}$$

i.e., U is an elementary cylindrical wave (source at the origin) as in (25), times a function of angles $G(\theta)$. (the scattering amplitude). Thus for $\theta = 0$ or π, we have

$$G(0) = G(\pi) = \int_{-a}^{a} d\eta = 2a \ , \tag{34}$$

where 2a is the width of the strip. For the other angles, we integrate the exponential and obtain

$$G(\theta) = \frac{e^{ika\sin\theta} - e^{ika\sin\theta}}{ik\sin\theta} = \frac{2i\sin(ka\sin\theta)}{ik\sin\theta} = 2a\left[\frac{\sin(ka\sin\theta)}{ka\sin\theta}\right] \equiv S\Gamma(\theta) = G(0)\Gamma(\theta) \ , \tag{35}$$

where S is the width of the strip, and Γ is an oscillatory function with zeros at $ka\sin\theta = n\pi$, $n = \pm 1, \pm 2, \cdots$. The limit of $\Gamma(\theta)$ for $\theta \to 0$ is 1.

Rayleigh-Born scattering by a sphere. As another illustration we consider Rayleigh-Born scattering by a sphere of radius a whose optical properties differ only very slightly from the free space in which it is imbedded (a "tenuous" scatterer). We use the geometry of Figure 6-9 with the sphere at the origin of the coordinates, and take

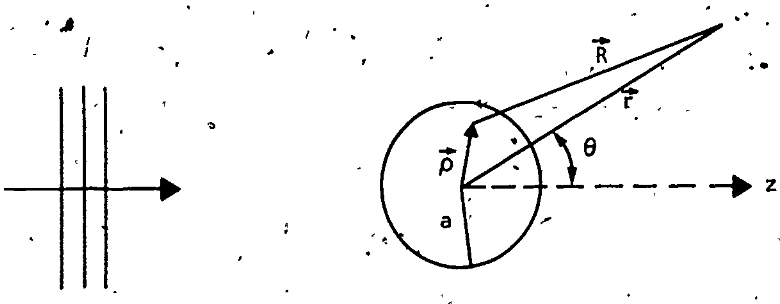

FIGURE 6-9

the plane wave

(36) $$U_i = e^{i(kz - \omega t)}$$

as the incident field.

We regard the sphere as made up of elementary sources of spherical wavelets, such that the source at the origin produces a wavelet $C \frac{e^{i(kr-\omega t)}}{kr}$ as in (15). The elementary source at the position $\vec{\rho} = (\xi, \eta, \zeta)$ excited by the incident field $U_i = e^{i(k\zeta - \omega t)}$ produces an effect at $\vec{r} = (x, y, z)$ described by

(37)
$$u = \frac{C}{kR} e^{ik(\zeta + R) - i\omega t},$$
$$R = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} = \sqrt{(r^2 + \rho^2) - 2\vec{r}\cdot\vec{\rho}},$$

where the phase is chosen to agree with that of U_i when $R = 0$. The net effect is represented by the volume integral of u over the sphere of radius a:

(38) $$U = \int u(\rho) dV(\vec{\rho}),$$

where dV is the volume element.

Essentially as for the slit problem, we restrict consideration to $r >> a$, and approximate R in the exponent of u by

(39) $$R \approx r - \hat{r}\cdot\vec{\rho}, \quad \hat{r} = \frac{\vec{r}}{r},$$

where $\hat{r}$ is a unit vector; in the denominator, we use simply $R \approx r$. Thus (39) becomes

(40) $$U \approx C \frac{e^{i(kr-\omega t)}}{kr} G(\theta),$$

(41) $$G(\theta) = \int e^{ik(\zeta - \vec{\rho}\cdot\hat{r})} dV,$$

i.e., U is an elementary spherical wave outgoing from the origin as in (15), times a scattering amplitude $G(\theta)$ that is independent of r but depends on the angle of observation θ (and, because of symmetry, on no other angle).

Since $\zeta = \vec{\rho}\cdot\hat{z}$, where $\hat{z} = \frac{\vec{z}}{z}$ is a unit vector, we may rewrite the scattering amplitude as

(42) $$G(\theta) = \int e^{ik\vec{\rho}\cdot(\hat{z}-\hat{r})} dV.$$

If $\hat{r} = \hat{z}$, i.e., in the forward scattered direction $\theta = 0$, we have $G(0) = \int dV = V$, where V is the volume of the scatterer. For arbitrary θ, the easiest way to integrate over the sphere is to introduce a new polar axis in the direction $\hat{z} - \hat{r}$.

From Figure 6-10 we see that since $\hat{z}$ and $\hat{r}$ have unit length, $\hat{z} - \hat{r}$ is a

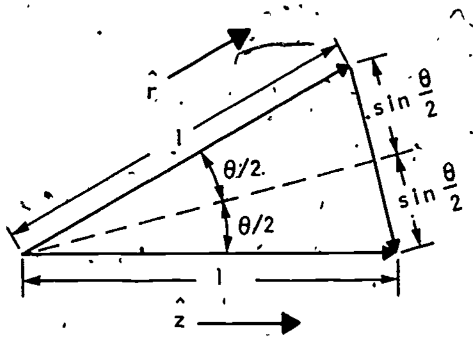

FIGURE 6-10

vector of length $2\sin\frac{\theta}{2}$. Thus we may write

(43) $$\hat{z} - \hat{r} = 2\sin\frac{\theta}{2}\,\hat{z}_0$$

where $\hat{z}_0$ is a unit vector along the axis z_0 of a new coordinate system. Consequently

(44) $$(\hat{z} - \hat{r})\cdot\vec{\rho} = \vec{\rho}\cdot\hat{z}_0\, 2\sin\frac{\theta}{2} = z_0\, 2\sin\frac{\theta}{2}$$

where $z_0 = \vec{\rho}\cdot\hat{z}_0$ is the projection of $\vec{\rho}$ along the new polar axis $\hat{z}_0$.

We now have

(45) $$G(\theta) = \int e^{i2kz_0\sin\frac{\theta}{2}}dV = \int e^{iKz_0}dV\,, \quad \therefore\ K \equiv 2k\sin\frac{\theta}{2}\,.$$

We introduce the polar coordinate system of Figure 6-11, so that we may replace

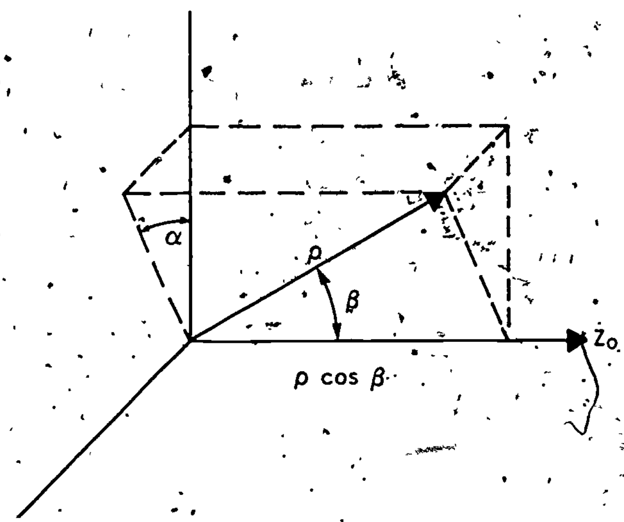

FIGURE 6-11

z_0 by $\rho \cos\beta$ and use $dV = d\alpha \rho^2 d\rho \sin\beta\, d\beta$ with ρ ranging from 0 to a, α from 0 to 2π, and β from 0 to π. Thus

(46) $$G(\theta) = \int_0^{2\pi} d\alpha \int_0^a \rho^2 d\rho \int_0^\pi e^{iK\rho\cos\beta} \sin\beta\, d\beta .$$

The integral over α yields 2π directly. The integral over β may be rewritten as

(47) $$\frac{1}{-iK\rho}\int_0^\pi d(e^{iK\rho\cos\beta}) = \frac{1}{-iK\rho}(e^{-iK\rho} - e^{iK\rho}) = \frac{2\sin K\rho}{K\rho}$$

Thus, we have reduced (46) to

(48) $$G(\theta) = 2\pi\int_0^a \rho^2 \frac{2\sin K\rho}{K\rho} d\rho = \frac{4\pi}{K}\int_0^a \rho \sin K\rho\, d\rho .$$

The remaining integral is simply the derivative of $\int \cos K\rho\, d\rho$ with respect to K:

(49) $$G(\theta) = \frac{-4\pi}{K}\frac{d}{dK}\int_0^a \cos K\rho\, d\rho = \frac{-4\pi}{K}\frac{d}{dK}\frac{\sin Ka}{K} = \frac{-4\pi}{K}\left[\frac{a\cos Ka}{K} - \frac{\sin Ka}{K^2}\right],$$

which we rewrite as

(50) $$G(\theta) = V3\left[\frac{\sin X}{X^3} - \frac{\cos X}{X^2}\right] \equiv VJ(\theta) = G(0)J(\theta), \quad V = \frac{4\pi a^3}{3}, \quad X \equiv 2ka \sin\frac{\theta}{2}$$

where V is the volume of the sphere. The value $G(0) = V$ obtained from (42) for $\hat{r} = \hat{z}$, $\theta = 0$, is also the limit of (50) for $X \to 0$. More generally, the result

(51) $$G(0) = \int dV = V,$$

which follows from (42) for $\hat{r} = \hat{z}$ (i.e., for forward scattering), holds for a scatterer of arbitrary shape subject to the present restriction that it be "tenuous."

From (40) and (50) we have

(52) $$U = C\frac{e^{i(kr-\omega t)}}{kr} VJ(\theta) ,$$

so that the corresponding flux density is proportional to V^2. If $a \ll \lambda$ (i.e., for a "small scatterer"), we have $J(\theta) \approx 1$, and consequently

(53) $$U \approx C\frac{e^{i(kr-\omega t)}}{kr} V .$$

In the next section, we consider an application of scattering which hinges essentially on the fact that $U \propto V/r$ for a variety of different scatterers.

7. Rayleigh Scattering.

In the present section, we digress from the development of the mathematical model for scattering in order to discuss a beautiful application to nature, this material serves as a supplement to Sections 4 and 6, and provides a more complete discussion of one of the topics touched on in Chapter 2. The development is based essentially on the flux principle [KL], and on the fact (determined by interference experiments, as discussed previously) that sunlight may be decomposed into light of different colors from red to blue with associated wavelengths λ_r to λ_b such that (approximately)

(1) $$\lambda_r = 2\lambda_b .$$

The wavelengths λ associated with the intermediate colors of the visible spectrum (orange, yellow, green) satisfy $\lambda_r > \lambda > \lambda_b$.

In 1871, Rayleigh developed a mathematical model to account for the blue color of the sky, and for the red color of clouds near dusk. The essential feature of Rayleigh's model is that when rays of different colors (different wavelength λ) are scattered by the molecules of the earth's atmosphere (mainly nitrogen and oxygen), the scatterers may be regarded as secondary sources in the sense of Equation (4:21) with flux density inversely proportional to λ^4, i.e.,

(2) $$F = \frac{C}{r^2\lambda^4} = |U|^2 F_0 ,$$

where r is distance from the scatterer, and C is independent of r and λ. Rayleigh used a more complete form of Equation (6.38) (one we consider in the last section of this chapter) to show that $U \propto \frac{V}{r\lambda^2}$ for various scatterers (e.g., broad ranges of relative index of refraction μ) provided their dimensions were small compared to λ, and also constructed a simplified intuitive derivation of (2).

The simplified derivation is based on dimensional analysis. Thus, if we divide the scattered flux density $F \propto |U|^2 = \frac{G^2}{r^2}$ by the incident value F_0, then the result is independent of the units in which we measure F: the ratio

(3) $$\frac{F}{F_0} = \frac{G^2}{r^2}$$

is simply a number and does not depend on units or dimensions (since F and F_0 are simply different values of the same physical quantity). For a scatterer of volume V whose dimensions are very small compared to λ, Rayleigh assumed

[R]: $$G = BV, \quad \frac{F}{F_0} = \frac{B^2V^2}{r^2}$$

where B is independent of the length dimensions of the scatterer. (In Equation (6:53) we considered a special case of [R], in Section 9 we consider an exception.) He could then obtain (2) from [R] by using dimensional analysis.

We indicate the length dimension involved in [R] as $\{L\}$ and consider the powers of L that enter the various terms. We have

(4) $$r^2 = \{L^2\}, \quad V^2 = \{L^6\}, \quad \frac{F}{F_0} = \{L^0\},$$

which state merely that since r is a length, r^2 must be the square of a length; since a volume is a length cubed, V^2 involves the sixth power of a length. The fact that the ratio $\frac{F}{F_0}$ is dimensionless (a "pure" number) means that B of [R] must satisfy

(5) $$B^2 = \{L^{-4}\}.$$

In view of the restriction on B mentioned for [R], the only length parameter available for (5) is λ. Thus, it follows that

(6) $$B^2 = \frac{g^2}{\lambda^4},$$

where g may depend on optical properties (through a relative index of refraction μ) and directions.

From (6) and [R] we have Rayleigh's inverse "fourth-power law"

(7) $$\frac{F}{F_0} = \frac{g^2 V^2}{r^2 \lambda^4}.$$

Substituting (1), we see that

(8) $$\frac{F(\lambda_b)}{F(\lambda_r)} = \frac{\lambda_r^4}{\lambda_b^4} = \frac{(2\lambda_b)^4}{\lambda_b^4} = 16.$$

Thus, if a beam of white light is incident on a small scatterer as in Figure 7-1, the

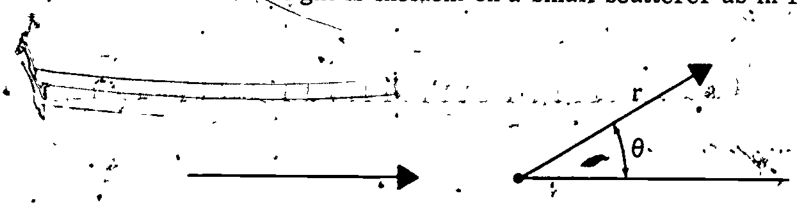

FIGURE 7-1

blue component of white light is scattered sixteen times as strongly as the red, i.e., the flux ratio of the blue and red components observed at an angle θ from the direction of incidence is given by (8).

Equation (7) specifies the scattered flux density at a point $r(\theta)$ as in Figure 7-1. The flux scattered through a spherical cap on a cone of half-angle α around the forward direction $\theta = 0$, as in Figure 7-2, is obtained by integration. The area of the cap, the portion of the sphere of radius r, is given by r^2 times the solid angle $\Omega(\alpha)$ the cap subtends at the origin (i.e., $r^2\Omega$ is the surface analog of arc length $r\theta$

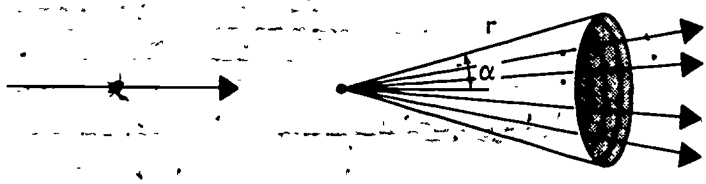

FIGURE 7-2

encountered in earlier two-dimensional problems). In terms of θ (the polar angle) measured from the direction of incidence $\hat{z}$, and in terms of the angle φ (the azimuthal angle) measured in a plane normal to $\hat{z}$, the area of the cap is given by

$$r^2\int_\alpha d\Omega \equiv r^2\int_0^{2\pi} d\varphi\int_0^\alpha \sin\theta\, d\theta = 2\pi r^2 (1 - \cos\alpha).$$

The corresponding flux through the cap may be written

(9) $$r^2\int_\alpha \frac{F}{F_0} d\Omega \equiv \sigma(\alpha).$$

Substituting (7) into (9) we obtain

(10) $$\sigma(\alpha) = \frac{V^2}{\lambda^4}\int_\alpha g^2 d\Omega ,$$

so that σ is independent of distance r. If $\alpha = \pi$, then the cap becomes a complete sphere; the value $\sigma(\pi)$ (called the "total scattering cross-section") is the total flux scattered in response to an incident plane wave of unit flux density. If the incident flux density is (say) I_1, then the total scattered flux is $I_1\sigma(\pi)$.

Let us now visualize an incident beam of rays flowing through a tube of cross-sectional area S as in Figure 7-3, and apply the Kepler-Lambert flux principle [KL]. The flux through S at z_1 is $I_1 S$, and from [KL] this equals the flux $I_2 S$ through the

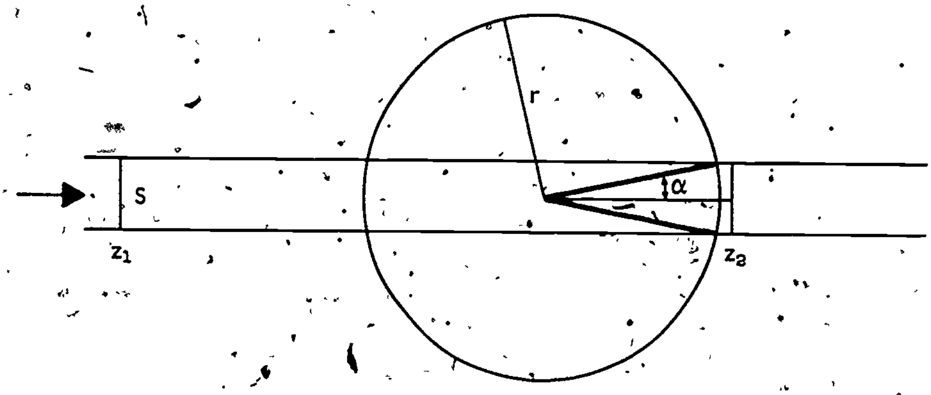

FIGURE 7-3

surface S at z_2 plus the flux through the incomplete spherical surface consisting of the sphere minus the cap α, i.e.,

(11) $$Q(\alpha) \equiv I_1 \int_{\pi-\alpha} \frac{F}{F_0} r^2 \, d\Omega \, , \qquad \int_{\pi-\alpha} d\Omega \equiv \int_0^{2\pi} d\varphi \int_\alpha^\pi \sin\theta \, d\theta \; ;$$

where the integral is over the incomplete spherical surface. Thus, from [KL],

(12) $$I_1 S = I_1 Q(\alpha) + I_2 S .$$

Regrouping terms, we have

(13) $$I_2 S = I_1 S - I_1 Q(\alpha)$$

which states simply that the flux in the beam of area S at z_2 equals the initial value $I_1 S$ minus that diverted to other directions by scattering out of the beam.

From (9), (10), and (11) we have

(14) $$Q(\alpha) = \sigma(\pi) - \sigma(\alpha) = \frac{V^2}{\lambda^4} \int_{\pi-\alpha} g^2 \, d\Omega \, ,$$

which we rewrite as

(15) $$Q = \frac{A}{\lambda^4}$$

in order to emphasize the dependence on λ. If we neglect $\sigma(\alpha)$ and approximate Q by the total scattering cross section $\sigma(\pi)$ (i.e., if we ignore the "hole" in the spherical surface), then the difference between the initial and final values of the flux along a parallel beam intercepted by a scatterer anywhere along the beam is

(16) $$I_2 S - I_1 S = -I_1 Q .$$

If there are N such scatterers in the beam, then under appropriate restrictions we may use (16) with the right-hand side multiplied by N to approximate the net effect:

(17) $$(I_2 - I_1) S = -N I_1 Q .$$

If there are n scatterers in unit volume in the geometry of Figure 7-4, then we have

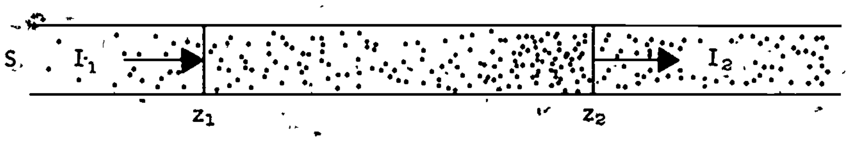

FIGURE 7-4

$N = nS(z_2 - z_1)$, and (17) reduces to

(18) $$I_2 - I_1 = -nQI_1 (z_2 - z_1) .$$

In the limit as z_1 approaches z_2, we obtain

(19) $$\frac{dI}{dz} = -nQI,$$

and consequently

(20) $$I = I_0 e^{-nQz},$$

with I_0 as the value at say $z = 0$.

From (36), we have $Q = \frac{A}{\lambda^4}$, so that

(21) $$I = I_0 e^{-n(A/\lambda^4)z}$$

which is the form considered in Chapter 2. Consequently $I(\lambda)$ decreases rapidly as λ decreases, and white light becomes (progressively with distance) relatively stronger in the red-yellow components because it is losing its blue components via scattering to other directions. Thus, as (7) accounts for the blue of the sky in directions other than towards the sun, (21) accounts for the redness of the clouds illuminated by sunlight near dusk. See the discussion of (6) in Section 4 of Chapter 2.

Equation (7) gives the scattered intensity for one scatterer excited by a wave of unit intensity. If one scatterer of a collection is at a distance z from the entrance face of the region of scatterers (as in Figure 7-5), then we multiply (7) by

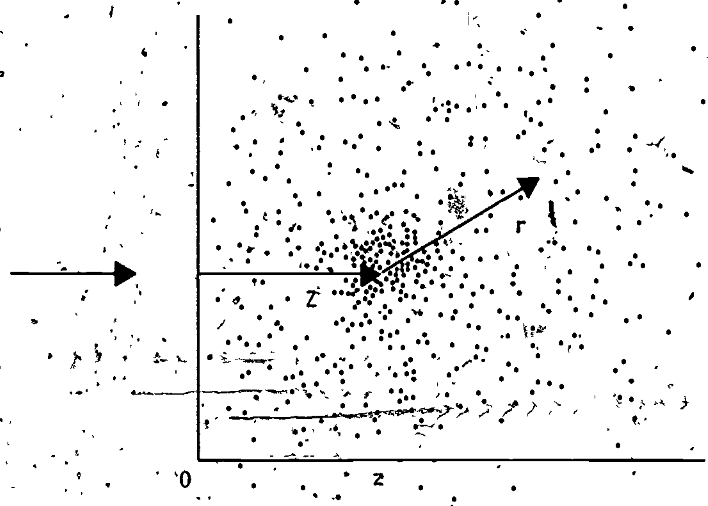

FIGURE 7-5

$I(z) = I_0 e^{-nQz}$ of (20) to account for the intensity loss of the excitation that reaches it. Similarly, if we observe a scattered ray from this scatterer at a distance r from its center, we incorporate an additional factor e^{-nQr} to account for the additional loss. Thus, the scattered intensity for one scatterer as in Figure 7-5, becomes

(22) $$F = \frac{g^2 V^2}{r^2 \lambda^4} I_0 e^{-n(\Lambda/\lambda^4)(z+r)}$$

where $z + r$ is the total ray path within the region of scatterers.

Let us rewrite (22) for $z + r = 1$ as

(23) $$F = \frac{B}{\lambda^4} e^{-D/\lambda^4}$$

a form that shows that F vanishes for both $\lambda \sim 0$ and $\lambda \sim \infty$, and has a maximum at a definite value of λ, say $\lambda = \Lambda$. Differentiating (23) with respect to λ, we obtain

(24) $$\frac{dF}{d\lambda} = \frac{dF}{d\lambda^{-4}} \frac{d\lambda^{-4}}{d\lambda} = Be^{-D\lambda^{-4}}(1 - \lambda^{-4}D) \left(-\frac{4}{\lambda^5}\right)$$

which vanishes for the wavelength

(25) $$\lambda^4 = D \equiv \Lambda^4$$

corresponding to a maximum scattered intensity

(26) $$F_\Lambda = \frac{B}{\Lambda^4} e^{-1} = \frac{B}{\Lambda^4 e}.$$

Dividing (23) by (26) we write the scattered intensity as

(27) $$F = F_\Lambda \frac{\Lambda^4}{\lambda^4} e^{1 - \Lambda^4/\lambda^4},$$

so that F is expressed in terms of the maximum value F_Λ and the corresponding wavelength Λ. This simple model applied to skylight gives a maximum wavelength in the blue-green region of the sun's spectrum.

8. Method of Stationary Phase.

We are now in a position to bridge the gap between the geometrical optics ray procedures of the early sections and the elementary wave procedures of Section 6. We do so initially by considering diffraction by a strip. We work with the wave forms of Equations (28), (29), and (30) of Section 6, and the geometry of Figure 8-1. Thus

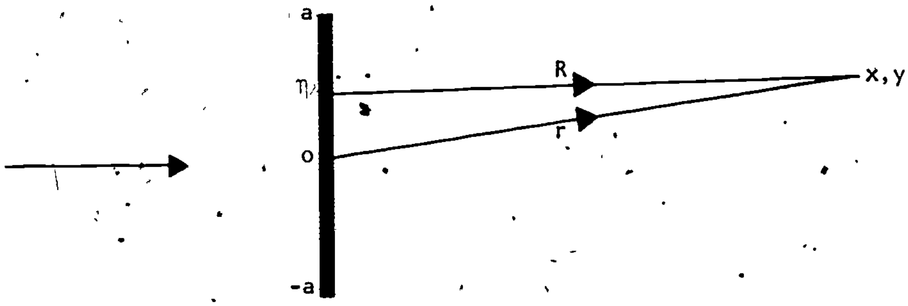

FIGURE 8-1

we take the incident wave proportional to

(1) $$U_i = e^{ikx} ,$$

the wavelet from a secondary line source on the strip as

(2) $$u = \frac{ce^{ikR}}{\sqrt{kR}} , \qquad R = \sqrt{x^2 + (y - \eta)^2} ,$$

and represent the net effect of the wavelets at r as the integral

(3) $$U = \int_{-a}^{a} u(\eta)\, d\eta = c \int_{-a}^{a} \frac{e^{ikR}}{\sqrt{kR}}\, d\eta .$$

We have suppressed the time factor $e^{-i\omega t}$ for brevity. The actual wave forms are obtained by multiplying the above by $e^{-i\omega t}$ and then taking the real part of the result.

The situation in Figure 8-1 is analogous to that of Figure 6-8, and if we assume $R \gg a = |\eta|_{max}$ we obtain the same forms as (32) to (35) of Section 6. Thus if we expand $\frac{R}{r}$ to first order in $\frac{\eta}{r}$ (as previously), we obtain $R \approx r - \eta \sin\theta$, and consequently the previous procedure yields

(4) $$U \approx \frac{ce^{ikr}}{\sqrt{kr}} \int_{-a}^{a} e^{ik\eta \sin\theta}\, d\theta$$
$$= \frac{ce^{ikr}}{\sqrt{kr}} \left[\frac{e^{ika\sin\theta} - e^{-ika\sin\theta}}{ik\sin\theta}\right] = \frac{ce^{ikr}}{\sqrt{kr}}\, 2a \left[\frac{\sin(ka\sin\theta)}{ka\sin\theta}\right] = \frac{ce^{ikr}}{\sqrt{kr}}\, 2a\,\Gamma(\theta) .$$

Thus except that the constant c may differ in the two cases, the present result (the Fraunhofer approximation for scattering or diffraction by a strip) is the same as that for the slit. (The relation of the result for the strip to that of the slit shown by the above is a special case of what is called Babinet's principle.)

We introduce $\gamma = ka \sin\theta$ to represent the phase difference between the rays from the center and from an edge to the observation point, and write $\Gamma = \frac{\sin\gamma}{\gamma}$ (the "Fraunhofer pattern factor"). The principal maxima of Γ correspond to $\gamma = 0$ (i.e., to the forward and back directions, $\theta = 0$ and π respectively). The secondary maxima of Γ occur at the zeros of $\tan\gamma - \gamma$, which are given approximately by $\gamma \approx 1.43\pi$, 2.46π, 3.47π and, for larger values, by $\gamma \approx \left(n + \frac{1}{2}\right)\pi$; for the first three of these zeros of $\frac{d\Gamma}{d\gamma}$, we have $\Gamma \approx -0.22, 0.13, -0.09$ respectively. The zeros of Γ correspond to $\gamma = n\pi$. The angular half-width of the principal maximum (obtained from the position of the first zero, $\gamma = ka \sin\theta = \pi$) is $\sin\theta \approx \theta = \frac{\pi}{ka} = \frac{\lambda}{2a}$.

The form (4) is restricted to $r >> a$. In order to consider situations where r and a are comparable in magnitude, or $r < a$, we use a different approximation for R in (2). Thus we now relax the requirement $r >> a$ and assume instead that we restrict the direction of observation to the neighborhoods of the forward scattered (the direction of incidence) and back scattered directions. More explicity, we assume $(y - \eta)^2 << x^2$, and use

$$R = \sqrt{x^2 + (y - \eta)^2} \approx x + \frac{(y - \eta)^2}{2x} \tag{5}$$

We use (5) in the exponent of (3), but in the denominator (essentially as for Equation (32) of Section 6) we use only the leading term $R \approx x$. Thus

$$U \approx \frac{ce^{ikx}}{\sqrt{kx}} \int_{-a}^{a} e^{ik(y-\eta)^2/2x} d\eta \tag{6}$$

The present integral describes Fresnel diffraction by a strip.

After we have analyzed the behavior of (6), we treat the analogous problem of scattering by a circular cylinder. A limiting case of the result we obtain will correspond to our geometrical optics results of Sections 2 and 4 in the form

$$U = \frac{ce^{ikL_H}}{\sqrt{RL''_H}} = C\sqrt{F}\, e^{ikL_H}, \tag{7}$$

where L_H is the ray path obtained previously by using Hero's principle [H']. L''_H is its second derivative with respect to a parameter along the circle, and F is the flux density obtained by the Kepler-Lambert principle [KL].

Our derivation of (7) from (3) will obviate the previous special assumptions. In particular, we will not have to assume Hero's "principle of the extremum path."

Statements such as "nature prefers an extremum," "nature abhors a vacuum," etc., may be useful aids to memory, but "nature" neither "prefers" nor "abhors," and such statements make our mysticism too explicit. We will show that Hero's principle [H'] is merely a consequence of an approximate evaluation of an integral.

The approximation procedure is known as the method of stationary phase. It was introduced by Kelvin as a mathematical method for approximating a class of integrals that come up frequently in wave problems. The integrals we are concerned with are of the form $U = \int u\,ds$, essentially as in (3).

Intuitively, the method is based on Young's concept of interference, and the essentials were discussed for Figure 6-6. A complex number $Ae^{i\varphi}$ may be represented as a vector of length A and direction angle (phase) φ on an Argand diagram, and the resultant of a set of numbers $A_n e^{i\varphi_n}$ is simply the vector sum $Te^{i\tau} = \Sigma A_n e^{i\varphi_n}$ shown in Figures 8-2 or 8-3. If the phase angles are all the same, then the elementary

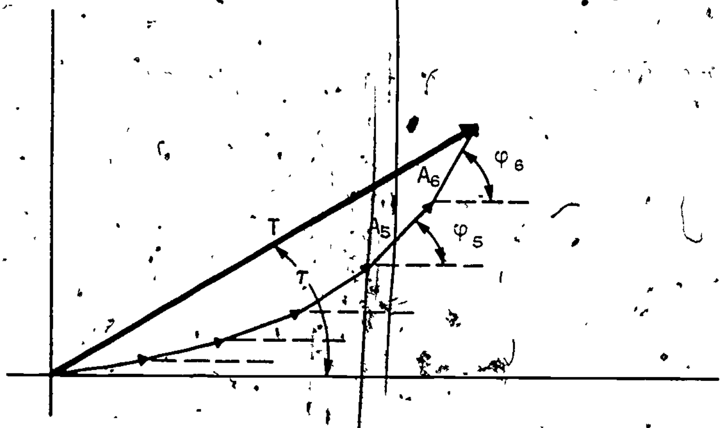

FIGURE 8-2

FIGURE 8-3

vectors all point along a straight line and $T = \Sigma A_n$: we say that the vectors reinforce each other, or that the elementary waves "interfere constructively." On the other hand, if the angles are such that the nose of the last vector ends up at the tail of the first to form a closed polygon, then $T = 0$; we say that the vectors cancel, or that the elementary waves "interfere destructively." In the situation shown in Figure 8-2, the phase φ_n changes slightly (varies slowly) with increasing n and the resultant magnitude T is close to its maximum value. In the situation shown in Figure 8-3, the variation of φ_n with n is large and T is small. The A_n and φ_n may depend on a parameter θ, and the magnitude $T(\theta)$ of the resultant,

(8) $$T(\theta)\, e^{i\tau(\theta)} = \sum_{n=1}^{N} A_n(\theta)\, e^{i\varphi_n(\theta)}$$

may assume any value between 0 and ΣA_n with variation of θ.

Similarly we may use the same idea for an integral of the form

(9) $$\int_{\eta_1}^{\eta_2} A(\eta;\theta)\, e^{i\varphi(\eta;\theta)}\, d\eta .$$

The integral of (4) is of the above form, and the series of maxima and minima shown by the resultant can be interpreted graphically by means of a vector diagram such as Figure 8-2. Similarly for the integral in (6).

In we can express φ in terms of a parameter θ, and then find that $\frac{d\varphi}{d\theta} = 0$ (and $\frac{d^2\varphi}{d\theta^2} \neq 0$) for some set of values of θ (say θ_s), then we say the <u>phase is stationary</u> at θ_s; for these values the situations are analogous to Figure 8-2 (not Figure 8-3), and we expect the resultants $T(\theta_s)$ to be large. Before discussing the general method mathematically, we first consider the strip problem without using the method explicitly. The results we obtain initially by relatively familiar procedures will provide a basis for introducing labels and concepts for our subsequent more general discussion.

<u>Fresnel Diffraction</u>: We may express (6) in terms of the tabulated <u>Fresnel integral</u>

(10) $$\mathcal{F}(\eta_1) = \int_0^{\eta_1} e^{i\pi\eta^2/2}\, d\eta = -\mathcal{F}(-\eta_1),$$

whose path in the complex plane (i.e., the trace of the point $\mathcal{F}(\eta_1)$ as a function of η_1) generates <u>Cornu's spiral</u> of Figure 8-4. The magnitude $|\mathcal{F}(\eta_1)| = T(\eta_1)$ is an oscillatory function of η_1. Figure 8-4 for (10) is a continuous analog of such cases of discrete vectors shown in Figures 8-2 and 8-3. Although the "elementary vectors" of Figure 8-4 form a curve of infinite length, the curve spirals in around the point $\frac{1}{2} + \frac{i}{2}$; the resultant $T = |\mathcal{F}|$ approaches the limit $\frac{1}{\sqrt{2}}$ as $\eta_1 \sim \infty$.

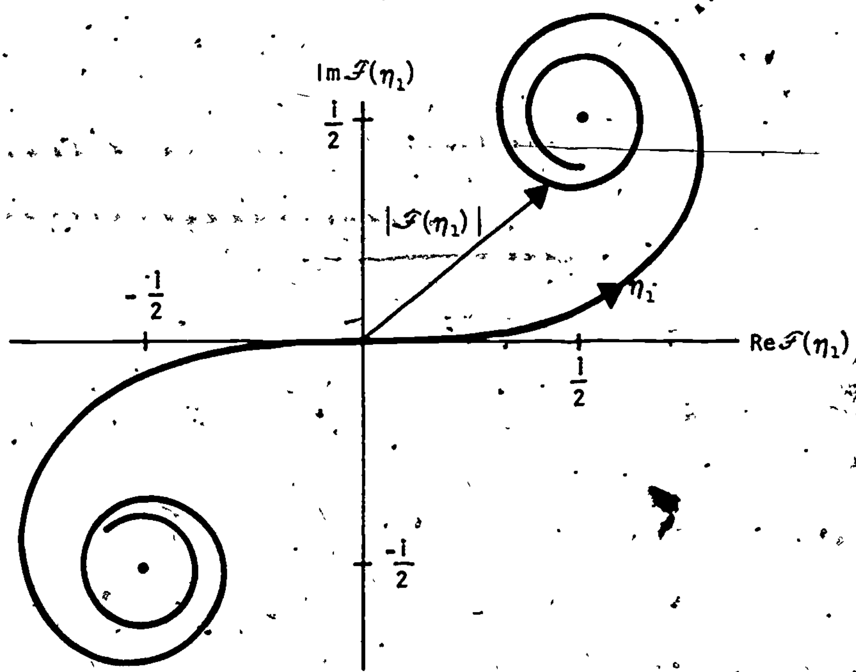

FIGURE 8-4

We now consider $\mathcal{F}(\eta)$ analytically. For large values of η_1, we use

(11) $$\mathcal{F}(\eta_1) = \left[\int_0^\infty - \int_{\eta_1}^\infty\right] e^{i\pi\eta^2/2}\,d\eta = \mathcal{F}(\infty) - \frac{1}{\pi i}\int_{\eta_1}^\infty \frac{1}{\eta}\,d\left(e^{i\pi\eta^2/2}\right) .$$

Integrating the second term by parts, we develop (11) as the series

(12) $$\mathcal{F}(\eta_1) = \mathcal{F}(\infty) + \frac{e^{i\pi\eta_1^2/2}}{i\pi\eta_1}\left[1 + \frac{1}{i\pi\eta_1^2} + \dots\right] .$$

Thus the error in using the leading term $\mathcal{F}(\infty)$ is proportional to $\frac{1}{\eta_1}$. The leading term itself, i.e.,

(13) $$\mathcal{F}(\infty) = \int_0^\infty e^{i\pi\eta^2/2}\,d\eta$$

can be determined as follows.

Let us consider first

(14) $$J = \int_0^\infty e^{-x^2}\,dx .$$

The square of J may be written as

(15) $$J^2 = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy .$$

Introducing polar coordinates $r^2 = x^2 + y^2$, $\tan\varphi = \frac{y}{x}$, we rewrite (15) as

(16) $$J^2 = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,dr\,d\varphi = \frac{\pi}{2}\int_0^{\infty} e^{-r^2} r\,dr .$$

Since $d(e^{-r^2}) = -2re^{-r^2}dr$, we obtain

(17) $$J^2 = \frac{\pi}{4}\int_0^{\infty} d(e^{-r^2}) = \frac{\pi}{4} .$$

Thus (14) equals

(18) $$J = \int_0^{\infty} e^{-x^2}dx = \sqrt{\frac{\pi}{4}} .$$

Similarly, we have

(19) $$\int_0^{\infty} e^{-Bx^2}dx = \sqrt{\frac{\pi}{B4}}$$

which is known as Laplace's integral. We treat the integral (13) heuristically by using $B = -\frac{i\pi}{2}$ to obtain

(20) $$\mathcal{F}(\infty) = \frac{1}{2}\sqrt{\frac{2}{-i}} = \frac{1}{1-i} = \frac{1+i}{2} = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}e^{i\pi/4}$$

which corresponds to the vector from the origin to the terminal point of the upper spiral in Figure 8-4. [The result may be verified by Cauchy's theorem. If we use $\eta = \zeta(1+i)/\sqrt{\pi}$ in (13), we obtain $\int e^{-\zeta^2}d\zeta(1+i)/\sqrt{\pi}$ where the path is along the 45° line in the complex ζ plane. We replace this path by the positive real axis plus the arc from 0 to 45° at infinity; the integral over the arc vanishes, and the integral over the real axis leads essentially to (18).]

On the other hand for small values of η, we expand the exponential of (10) as a series and integrate term by term

(21) $$\mathcal{F}(\eta_1) = \int_0^{\eta_1}\left[1 + \frac{i\pi\eta^2}{2} + \dots\right]d\eta = \eta_1 + \frac{i\pi\eta_1^3}{6} + \dots .$$

In terms of (10) we rewrite (6) as

(22) $$U = \sqrt{\pi}\,\frac{e^{ikx}}{k}\,cI ,$$

(23) $$I = \int_{\eta_-}^{\eta_+} e^{i\pi\eta^2/2}d\eta = \mathcal{F}(\eta_+) - \mathcal{F}(\eta_-) ,$$

$$\eta_{\pm} = \sqrt{\frac{k}{\pi x}}(\pm a - y) .$$

Before applying this result to the strip problem, let us first apply it to an "infinitely wide slit," and determine c for the Huygens' free-space wavelets that simply serve to regenerate the incident wave. For the limiting case $ka \sim \infty$, we have $\eta_{\pm} \sim \pm\infty$. Using the limiting values of $\mathcal{J}$ obtained from (10) and (20) we obtain

$$I \sim \mathcal{J}(\infty) - \mathcal{J}(-\infty) = 2\mathcal{J}(\infty) = \sqrt{2}\, e^{i\pi/4}, \tag{24}$$

and consequently

$$U \sim c\,\frac{\sqrt{2\pi}}{k}\, e^{i\pi/4} e^{ikx}. \tag{25}$$

But an "infinitely wide slit" means no obstruction, so that U of (25) must equal the incident wave e^{ikx}. Consequently, the unknown constant c of (2) and (22) for the elementary Huygens' sources is

$$c = \frac{k}{\sqrt{2\pi}}\, e^{-i\pi/4}. \tag{26}$$

More generally, if we are dealing with the secondary sources on a scatterer excited by the incident wave, we may write

$$c = \frac{k}{\sqrt{2\pi}}\, e^{-i\pi/4} g, \tag{27}$$

where g may depend on the material of the scatterer, and on directions. Thus we may rewrite (22) as

$$U = g\,\frac{e^{-i\pi/4}}{\sqrt{2}}\,[\mathcal{J}(\eta_+) - \mathcal{J}(\eta_-)]\, e^{ikx}, \qquad \eta_{\pm} = \sqrt{\frac{k}{\pi x}}\,(\pm a - y). \tag{28}$$

We now apply (28) to scattering by a strip as in Figure 8-1. There are essentially three different ranges of y that we consider.

In Figure 8-5 we specify three different ranges of y at a fixed value of x, say three different portions of a screen placed parallel to the strip. We will use (28), (12), and (21) to obtain explicit approximations of U for the three ranges of y corresponding to the braces shown in the figure. The range y_2 is centered on the geometrical projection of the edge of the strip (equivalently the neighborhood of the shadow boundary $y = a$), and the range y_1 includes much of the geometrical projection (the shadow) of the strip on the screen.

The range of y_1 corresponds to $\eta_+ >> 1$ and $-\eta_- >> 1$, and includes $y = 0$ as the special case $\eta_+ = -\eta_- >> 1$; the range of y_2 corresponds to $\eta_+ \approx 0$ and $-\eta_- >> 1$; the range of y_3 corresponds to $-\eta_+ >> 1$ and $-\eta_- >> 1$. If we replace x, y by $|x|$, $|y|$, then the results will apply not only for the three sets of points (x, y_i) in the first quandrant shown in Figure 8-5, but also to the sets obtained in the other

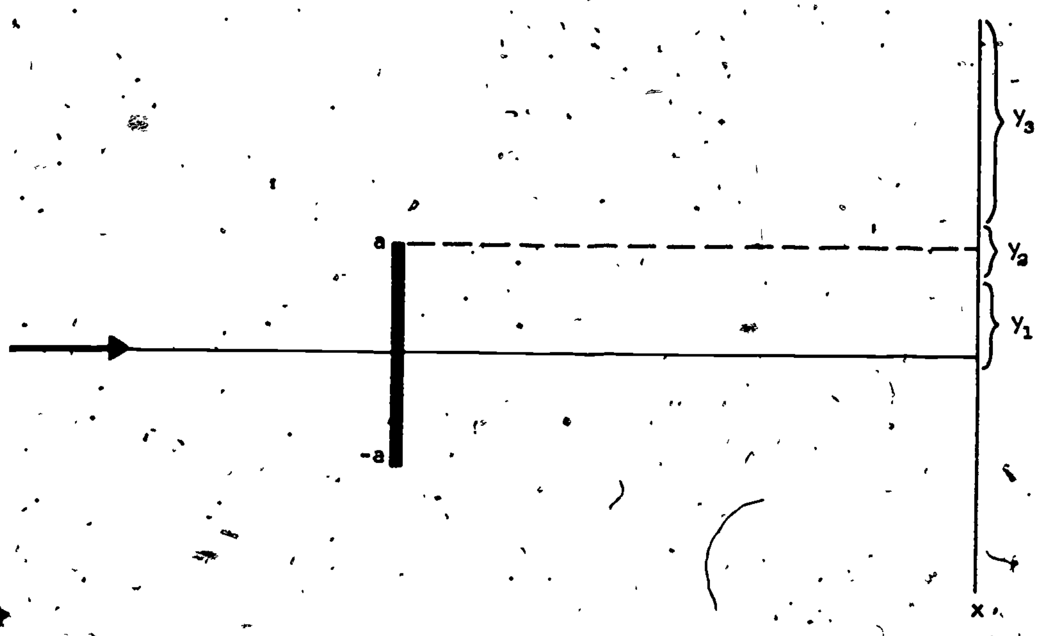

FIGURE 8-5

quadrants by reflecting the three given sets in the x-axis and in the y-axis. If, say, $\eta_+ = \eta_1 >> 1$, we approximate the integral by the first two terms of (12) obtained by using (20), i.e.,

$$\mathcal{F}(\eta_1) \sim \frac{e^{i\pi/4}}{\sqrt{2}} + \frac{e^{i\pi\eta_1^2/2}}{i\pi\eta_1}, \quad \eta_1 >> 1; \tag{29}$$

however, if η_1 is very small then we use only the leading term of (21),

$$\mathcal{F}(\eta_1) \approx \eta_1, \quad \eta_1 \approx 0. \tag{30}$$

At a point x, y_1 (or more generally at the four points $|x|, |y_1|$), we have $\pm\eta_\pm = \sqrt{\frac{k}{\pi|x|}}(a \pm y) >> 1$; substituting (29) into (28), we obtain

$$\begin{aligned} U &= g\frac{e^{ik|x|}}{\sqrt{2}}e^{-i\pi/4}[\mathcal{F}(\eta_+) + \mathcal{F}(-\eta_-)] \\ &\approx ge^{ik|x|}\left\{1 - \sqrt{\frac{|x|}{2\pi k}}e^{+i\pi/4}\left[\frac{e^{ik(a-y)^2/2|x|}}{a-y} + \frac{e^{ik(a+y)^2/2|x|}}{a+y}\right]\right\}. \end{aligned} \tag{31}$$

where g is not necessarily the same for the forward scattered direction $|x| = x$ as for the back scattered direction $|x| = -x$. Essentially as for (5), we require $|(a \pm y)/x| << 1$; subject to this we see that

$$U \sim ge^{ik|x|} \quad \text{as} \quad |x/k(a \pm y)^2| \sim 0. \tag{32}$$

Thus in this limit the strip is a one-dimensional secondary source. Taking into account that U is in general a function of direction, we write

(33) $$U_{\pm} \sim g_{\pm} e^{\pm ikx} \quad \text{for} \quad x \gtrless 0 .$$

In the forward direction, we require $g_+ = -1$ in order for a geometrical shadow to exist in the sense of Section 2, i.e., $U_+ \sim -e^{+ikx}$. Similarly for the case of a perfectly reflecting material, we require $|g_-| = 1$ in order that the ratio of the reflected to incident flux density ($|U_-/U_i|^2$) equal the previous result unity. Thus we have

(34) $$U_+ \sim -e^{+ikx}$$
$$U_- \sim g_- e^{-ikx} = e^{i\delta} e^{-ikx} \quad \text{for} \quad x < 0 .$$

where δ is a real number determined by the material of the strip (or, equivalently, by the boundary conditions). For present purposes we take $\delta = 0$, so that

(35) $$U_- \sim e^{-ikx} \quad \text{for} \quad x < 0 ;$$

physically this corresponds for example to a water wave or a sound wave incident on a rigid immovable strip, and also (subject to additional conditions) for the reflection of an electromagnetic wave from a metal strip.

We introduce the abbreviation

(36) $$u_e(Y) = -\sqrt{\frac{|x|}{2\pi k}}\, e^{i\pi/4}\, \frac{e^{ik[|x|+Y^2/2|x|]}}{Y} ,$$

so that we may rewrite (31) as

(37) $$U_{\pm} \approx \mp \left[e^{\pm ikx} + u_e(a-y) + u_e(a+y)\right] .$$

The present results apply for the geometry of Figure 8-6. From (5) we see that

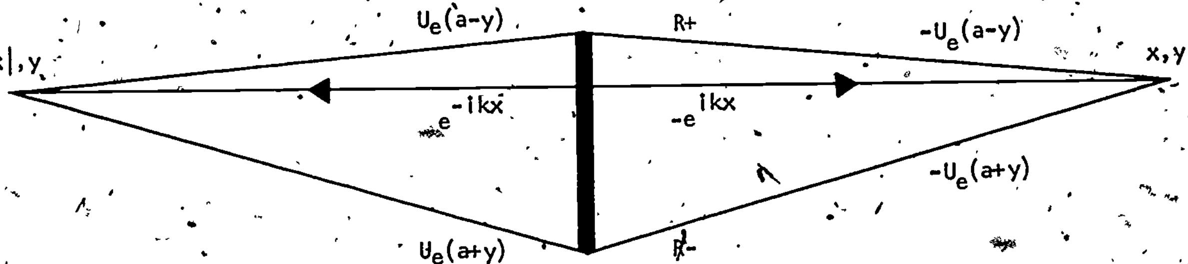

FIGURE 8-6

$R_{\pm} \approx x + (a \mp y)^2/2x$ so that the exponents of $u_e(a \mp y)$ are approximations of $kR_{\pm}$. The factors $kR_{\pm}$ are the phase changes of waves traveling from the edges of the strip 0, ±a to the observation point x, y so that we may interpret $u_e(a \pm y)$ as the edge waves corresponding to the edge rays discussed in Section 2. Thus the normals of the different waves of (37) shown as directions of propagation in Figure 8-6 are also the rays corresponding to such points as P_3 in Figure 2-17(c).

In the region of the geometrical shadow the total wave U_T is the sum of $U_i = e^{ikx}$ and U_+:

$$U_{T+} = U_i + U_+ = -u_e(a - y) - u_e(a + y) , \tag{38}$$

i.e., the shadow forming part of U_+ cancels the incident wave and we are left only with the edge waves or diffracted waves. Thus, corresponding to (38), since $|U_{T+}|^2 \propto \frac{1}{k} \div \frac{\lambda}{2\pi}$, a "perfect shadow" does not exist for non-vanishing λ, the diffracted field is small for relatively small x (in the immediate vicinity of the obstacle), but it increases in magnitude as x increases and the shadow "disappears" with increasing distance from the scatterer. (However, alternative forms are required for either $x \sim 0$ or $x \sim \infty$.) The field U_{T+} is oscillatory both in x and y. The flux density along the axis (y = 0), corresponding to $|U_{T+}|^2$ with

$$U_{T+} \approx \sqrt{\frac{2x}{\pi k a^2}}\, e^{i\pi/4} e^{ikR} , \qquad R = \sqrt{x^2 + a^2} \approx x + a^2/2x, \tag{39}$$

is a relative maximum, this is the analog of the Arago "bright spot" discussed for the disc. In the back-scattered region enclosed by projections of the strip edges parallel to -x, we have

$$U_{T-} = U_i + U_- = e^{ikx} + e^{-ikx} + u_e(a - y) + u_e(a + y) \tag{40}$$

where e^{-ikx} is the geometrically reflected wave.

The above results (31) to (40) are subject to two restrictions: the first, $|(y - a)/x| << 1$, enables us to use the approximations in (5), and restricts us to observation near the back and forward scattered directions; the second $k(a \pm y)^2/|x| >> 1$ is required in order to use approximation (29) for $\mathcal{F}$. The first bounds $|x|$ from below ($|x| >> a$), and the second from above. Together, the two restrictions also state that $ka = 2\pi a/\lambda >>> 1$ (i.e., the strip is very wide compared to the incident wavelength, and that y cannot be near ±a (i.e., y cannot approach the shadow lines or their analogs in the "lit region"). The corresponding edge wave u_e is the "near-caustic" form.

We cannot use (31) to (40) for a point $y \approx a$ in the range of y_2 of Figure 8-5. In that region although $-\eta_- = \sqrt{\frac{k}{\pi|x|}}(a + y) \approx \sqrt{\frac{k}{\pi|x|}}\,2a >> 1$, we see that $\eta_+ = \sqrt{\frac{k}{\pi x}}(a - y) \approx 0$. Thus, in the general form (28), we still use (29) for $\mathcal{F}(-\eta_-)$, but we must use (30) for $\mathcal{F}(\eta_+)$. Consequently at the four points $|x|$, $|y_2|$, we obtain

$$(41)\qquad U_{\pm} = \mp e^{ik|x|}\left[\sqrt{\frac{k}{2\pi|x|}}\,e^{-i\pi/4}(a-y) + \frac{1}{2} - \sqrt{\frac{|x|}{2\pi k}}\,e^{i\pi/4}\,\frac{e^{ik2a^2/|x|}}{2a}\right]$$

The second and third terms are of the form considered in (31). The first is a cylindrical wave, i.e., the wave of a line source on the edge — a "true" edge wave decreasing as $1/\sqrt{x}$ with increasing distance. The third term becomes negligible for very large ka, for which case the total field for $x > 0$ reduces to

$$(42)\qquad U_{T+} = U_i + U_+ \approx \frac{1}{2}e^{ikx} - H(kx)\frac{k(a-y)}{2}, \qquad H(kx) \equiv \left[e^{ikx}\sqrt{\frac{2}{\pi kx}}\,e^{-i\pi/4}\right].$$

Thus the field in the neighborhood of the shadow-line is half the incident wave plus a cylindrical wave corresponding to a line source with source strength proportional to $k(a-y)$. As $y \sim a$, we see that $U_T \sim \frac{1}{2}U_i$ linearly; this holds whether y approaches a from above from "above" or "below" in Figure 8-5. Similarly for $x < 0$, we have

$$(43)\qquad U_{T-} \approx U_i + U_- \approx e^{ikx} + \frac{1}{2}e^{-ikx} + H(k|x|)\,\frac{k(a-y)}{2}.$$

In the range of y_3 in Figure 8-5, we have $-\eta_{\pm} \gg 1$, and we again use (29) for both Fresnel integrals in (28). However, in contrast to (31), the scattered wave at the four points $|x|$, $|y_3|$, is given by

$$U_{\pm} \doteq \mp\frac{e^{ik|x|}}{\sqrt{2}}e^{-i\pi/4}\left[-\mathcal{J}(-\eta_+) + \mathcal{J}(-\eta_-)\right]$$

$$(44)\qquad \approx \mp e^{ik|x|}\sqrt{\frac{|x|}{2\pi k}}\,e^{i\pi/4}\left[+\frac{e^{ik(y-a)^2/2|x|}}{y-a} - \frac{e^{ik(y+a)^2/2|x|}}{y+a}\right]$$

$$\doteq \mp\left[u_e(a-y) + u_e(a+y)\right],$$

so that such points receive only the edge contributions of (37).

The above explicit approximations suffice for present purposes. A more complete discussion of the problem of the strip is given in introductory texts on optics in which the field at any point in space is usually computed graphically from Cornu's spiral Figure 8-4.

Thus up to moderately large $|x|$ the scattered wave is largely "confined" to the strip $|y| < a$ in the sense that within this strip we obtain the geometrically reflected and shadow forming waves. These two waves correspond to the waves scattered by an infinite plane, however, superimposed on these are the additional waves that we interpreted as edge waves. Because of the additional waves, the shadow is not "perfect;" careful observations (subject to the present restrictions on distance parameters) on the shadows of scatterers having very regular edges show a system of bright and dark bands parallel to the edges of the scatterer (a "fringe system").

We now determine the number and separation of such extrema that may be observed in the shadow region on a screen parallel to the strip.

We use

$$\frac{d}{dy}\int_0^{\zeta(y)} F(\eta)d\eta = F(\zeta)\frac{d\zeta}{dy}, \qquad \zeta = \zeta(y), \tag{45}$$

to differentiate U as given by (22) and (23), and obtain

$$\frac{dU}{dy} \propto \frac{d}{dy}\left[\mathcal{J}(\eta_+) - \mathcal{J}(\eta_-)\right] = e^{ik\eta_+^2/2}\frac{d\eta_+}{dy} - e^{i\pi\eta_-^2/2}\frac{d\eta_-}{dy}; \quad \frac{d\eta_+}{dy} = \frac{d\eta_-}{dy}. \tag{46}$$

The extrema of U, obtained from $dU/dy = 0$, correspond to

$$e^{i\pi\eta_+^2/2} = e^{i\pi\eta_-^2/2}. \tag{47}$$

Consequently the exponents must satisfy

$$\frac{\pi}{2}(\eta_+^2 - \eta_-^2) = -\frac{2k}{x}ay = 2n\pi; \quad n = 0, \pm 1, \cdots \tag{48}$$

The separation of extrema is thus

$$|y_{n+1} - y_n| = \Delta y = \frac{\pi x}{ka} \tag{49}$$

and there are N extrema, with N given by

$$N = \frac{2a}{\Delta y} = \frac{2ka^2}{\pi x} = \frac{4a^2}{\lambda x}, \tag{50}$$

in the geometrical projection of the slit. Thus the number of extrema increases with increasing strip width, or with decreasing wavelength, or decreasing axial distance.

Before continuing the main line of this section, we consider the range of very large $|x|$ excluded in the discussion based on (29). If $|x|$ becomes very large, so that $\frac{k(a \pm y)^2}{|x|} \approx 0$, then both $\eta_+ \approx 0$ and $\eta_- \approx 0$ in (28), and we approximate both Fresnel integrals by means of (21). Thus

$$\begin{aligned} U_\pm &\approx \mp\frac{e^{-i\pi/4}}{\sqrt{2}} e^{ik|x|}\left[\sqrt{\frac{k}{\pi|x|}}\right][a - y + a \mp y] \\ &= \mp\left\{\sqrt{\frac{2}{\pi k|x|}}\, e^{-i\pi/4} e^{ik|x|}\right\} ka = \mp H(k|x|)ka. \end{aligned} \tag{51}$$

Thus for this case, the scattered field is essentially that of a line source of strength ka. We derived this result via (5) which means that it is restricted to angles near the forward and back directions; comparing with (4), we see that (51) is merely the special case of (4) corresponding to $\theta = 0, \pi$, and that for other values of θ, the appropriate form of U at large distances is simply $\pm H(kr)ka\Gamma(\theta)$.

Method of Stationary Phase. We have discussed the preliminaries, and can now turn to the main topic of this section. In general, we consider an integral of the form

$$I = \int_{-a_-}^{a_+} G(x) e^{ikL(x)}\, dx, \quad k \gg 1, \tag{52}$$

where G is a slowly varying function of x compared to $e^{ikL(x)}$ in the sense that G changes only by a small fraction of itself when kL changes by 2π. If there exist one or more values of x for which $\frac{dL}{dx}$ vanishes, then the principal contributions to the value of the integral arise from the neighborhoods of the extrema (or stationary values) of L, elsewhere the contributions cancel through destructive interference as defined previously. We reiterate that the intuitive basis of this idea is the recognition that on an Argand diagram (as in Figures 8-2, 8-3, and 8-4), I is a sum of elementary vectors whose direction (essentially the phase kL) is in general a rapidly changing function of x, so that the resultant I is consequently small. However, if there exists a value of x for which $\frac{dL}{dx}$ vanishes, then the phase is stationary at this value and only slowly varying in its vicinity; the elementary vectors near this value are almost in phase and add to give a large result.

Proceeding analytically, the Taylor's expansion of L(x) around some value x_a is

$$L(x) = L(x_a) + L'(x_a)(x - x_a) + L''(x_a)(x - x_a)^2/2 + \cdots, \tag{53}$$

where $L'(x_a)$ means differentiate L(x) with respect to x and then set x equal to x_a; similarly for the second derivative L'', etc. If a value x_s exists for which $L'(x_s) = 0$, then

$$L(x) = L(x_s) + L''(x_s)\frac{(x - x_s)^2}{2} + \cdots. \tag{54}$$

Assuming that $L''(x_s) \neq 0$, and that the higher order terms are negligible, we keep only up to quadratic terms in the exponent of (52). We replace the slowly varying function G(x) by its value $G(x_s)$ at the stationary point (the point marking the center of the region in which the integrand contributes significantly), and work with

$$I \approx I_s = G(x_s) e^{ikL(x_s)} \int_{-a_-}^{a_+} e^{ikL''(x_s)(x-x_s)^2/2}\, dx$$

$$= G(x_s) e^{ikL(x_s)} \sqrt{\frac{\pi}{kL''(x_s)}} \left[\mathcal{F}(\eta_+) - \mathcal{F}(\eta_-)\right], \tag{55}$$

$$\eta_\pm = \sqrt{\frac{kL''(x_s)}{\pi}}\,(\pm a_\pm - x_s),$$

where $\mathcal{F}$ is the Fresnel integral as in (10)ff.

In particular, if $\pm\eta_{\pm} \sim \infty$, then $\mathcal{J} \simeq \mathcal{J}(\infty)$ of (20); we have $\mathcal{J}(\infty) - \mathcal{J}(-\infty) = \sqrt{2}\, e^{-i\pi/4}$, and consequently

(56) $$I_s \sim G_s e^{ikL_s + i\pi/4} \sqrt{\frac{2\pi}{kL''_s}} ,$$

where the subscript s indicates that the function is evaluated at the stationary point x_s. (If there is more than one stationary point, then I_s is a sum of such forms.) Compare (56) with (7).

Before applying (56), let us show that it is actually much larger than (52) for the case where the integral has no stationary point. We show that if there are no stationary values of L, then the integral (52) is only of the order $\frac{1}{k}$ as compared to I_s which (from (56)) is proportional to $\frac{1}{\sqrt{k}}$; since $k \gg 1$, I_s is therefore much larger. To see this we introduce L as the new integration variable:

(57) $$I = \int_{-a_-}^{a_+} G(x) e^{ikL(x)} dx = \int_{L(-a_-)}^{L(a_+)} \frac{G(L)}{L'} e^{ikL} dL = \frac{1}{ik} \int_{L(-a_-)}^{L(a_+)} \left[\frac{G(L)}{L'}\right] d\left(e^{ikL}\right) ,$$

and integrate by parts in order to develop I in powers of $\frac{1}{k}$:

(58) $$I = \frac{1}{ik}\left[\frac{G(L)}{L'} e^{ikL}\right]_{L(-a_-)}^{L(a_+)} + \frac{1}{k^2}\left[\frac{d}{dL}\left(\frac{G(L)}{L'}\right) e^{ikL}\right]_{L(-a_-)}^{L(a_+)} + \cdots .$$

Thus as long as L'/G does not vanish in the range $-a_-$ to a_+, the integral is only of order $\frac{1}{k}$ and is therefore much smaller than the stationary case I_s of (56).

As a first illustration, we apply (55) to the original integral (3) for the strip with the constant given by (27):

(59) $$U = g\sqrt{\frac{k}{2\pi}}\, e^{-i\pi/4} I , \quad I = \int_{-a}^{a} \frac{e^{ikR}}{\sqrt{R}} d\eta , \quad R = \sqrt{x^2 + (y-\eta)^2} .$$

Comparing with (52), we see that $G(\eta) = \frac{1}{\sqrt{R(\eta)}}$, and that $L(\eta) = R(\eta)$. Introducing φ as in Figure 8-7, we differentiate to obtain

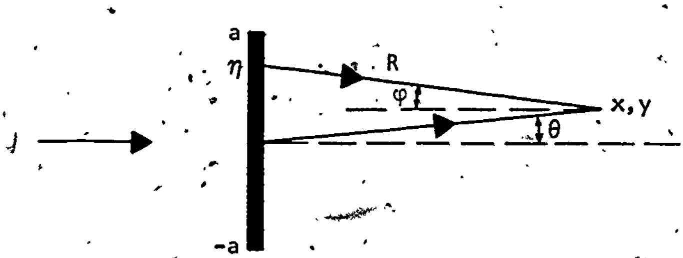

FIGURE 8-7

(60) $$\frac{dL}{d\eta} = L' = R' = \frac{\eta - y}{\sqrt{x^2 + (\eta - y)^2}} = \frac{\eta - y}{R} = \sin\varphi ,$$

$$(61)\quad R'' = \frac{1}{\sqrt{x^2 + (y-\eta)^2}} - \frac{(\eta - y)^2}{[x^2 + (y-\eta)^2]^{3/2}} = \frac{1}{R}\left(1 - \frac{(\eta - y)^2}{R^2}\right) = \frac{\cos^2\varphi}{R}.$$

The stationary values correspond to $R' = 0$:

$$(62)\qquad \eta_s = y\ ; \qquad \varphi_s = 0, \pi\ .$$

Consequently

$$(63)\qquad R_s = \pm x = |x|\ , \qquad |R_s''| = \frac{\cos^2\varphi}{R_s} = \frac{1}{R_s} = \frac{1}{|x|}\ .$$

Substituting into the integral of (59) via (55), we obtain

$$I = \int_{-a}^{a} \frac{e^{ikR}}{\sqrt{R}}\, d\eta \approx \frac{1}{\sqrt{R_s}}\, e^{ikR_s} \sqrt{\frac{\pi}{kR_s''}} \left[\mathcal{J}(\eta_+) - \mathcal{J}(\eta_-)\right] = \sqrt{\frac{\pi}{k}}\, e^{ik|x|} \left[\mathcal{J}(\eta_+) - \mathcal{J}(\eta_-)\right]$$

(64)

$$\eta_\pm = \sqrt{\frac{kR_s''}{\pi}}\,(\pm a - \eta_s) = \sqrt{\frac{k}{\pi |x|}}\,(\pm a - y)\ .$$

Introducing (64) into (59) yields the earlier form (28). The limiting form based on (56) gives $U = ge^{ik|x|}$ as previously, with $g = -1$ from our "shadow condition."

We may generalize the above directly to an arbitrary angle of incidence as in Figure 8-8. The incident plane wave may be written

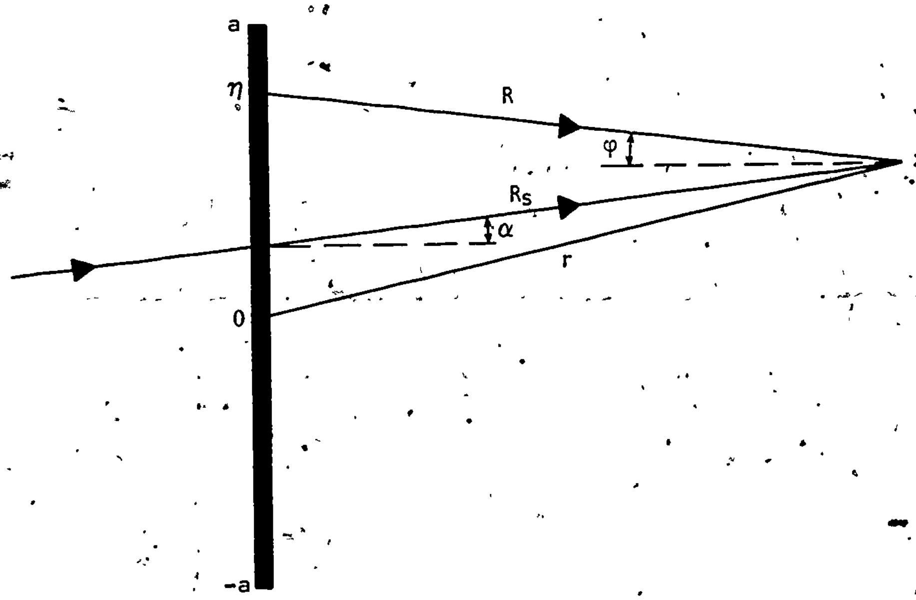

FIGURE 8-8

$$(65)\qquad U_i = e^{ikx\cos\alpha + iky\sin\alpha}\ ,$$

which is merely the form of (1) obtained by rotating the xy coordinate frame through an angle $-\alpha$. Since the phase of (65) is zero at the origin (the center of the strip), the wavelet originating at the origin has the same phase as obtained from (2). However, the wavelet originating at η is excited by $U_1(0, \eta) = e^{ik\eta \sin\alpha}$, so that its phase contains the additional term $k\eta \sin\alpha$. Thus instead of (2) we have

(66) $$u(\eta) = c \frac{e^{ikR + ik\eta \sin\alpha}}{\sqrt{kR}},$$

and we replace (59) by

(67) $$U = \int_{-a}^{a} c \frac{e^{ikR + ik\eta \sin\alpha}}{\sqrt{kR}} d\eta,$$

where the appropriate value of c will be determined from a limiting case.

Corresponding to (52), we take $G = c/\sqrt{kR}$ to be slowly varying, and differentiate the phase

(68) $$L = \eta \sin\alpha + \sqrt{x^2 + (\eta - y)^2},$$

with respect to η. We now have

(69) $$L' = \sin\alpha + \sin\varphi, \qquad L'' = \frac{\cos^2\varphi}{R}.$$

The stationary values correspond to

(70) $$L' = \sin\alpha + \sin\varphi = 0; \qquad \sin\varphi = -\sin\alpha; \qquad \varphi = -\alpha, \pi + \alpha,$$

which contains Euclid's principle of reflection and the principle of shadow formation. Since $\sin\varphi = \frac{\eta - y}{R} = \frac{\eta - y}{x} \cos\varphi$, we see from (70) that $y - \eta_s = |x| \tan\alpha$, $R_s = x \sec\varphi = |x| \sec\alpha$, and

(71) $$L_s = \eta_s \sin\alpha + x \sec\varphi_s = (y - |x| \tan\alpha) \sin\alpha + |x| \sec\alpha = y \sin\alpha + |x| \cos\alpha.$$

Similarly,

(72) $$L_s'' = \frac{\cos^2\alpha}{R_s}.$$

Substituting into (67) in terms of the limiting form (56) we obtain

(73) $$\begin{aligned} U &\sim \frac{c_s}{\sqrt{kR_s}} e^{\pm ikL_s + i\pi/4} \sqrt{\frac{2\pi}{k|L_s''|}} \\ &= e^{\pm ikx\cos\alpha + iky\sin\alpha} \frac{c_s}{k\cos\alpha} \sqrt{2\pi}\, e^{i\pi/4} \equiv U_\pm. \end{aligned}$$

Comparing U_+ with U_1 of (65), we determine c_s from the shadow condition $U_+ = -U_1$:

(74) $$c_s = -\frac{k\cos\alpha}{\sqrt{2\pi}} e^{-i\pi/4},$$

which differs from our earlier result by the additional factor $\cos\alpha$. Similarly the corresponding value of g for a perfect reflector as for (31) ff, is now replaced by $g\cos\alpha$.

Using the more general form corresponding to (55), we now have

$$U = g e^{ik|x|\cos\alpha + iky\sin\alpha} e^{-i\pi/4} \frac{1}{\sqrt{2}} \left[\mathcal{F}(\eta_+) - \mathcal{F}(\eta_-)\right],$$

$$\eta_\pm = \sqrt{\frac{k}{\pi|x|\sec\alpha}} \cos\alpha(\pm a + |x|\tan\alpha - y)$$

Circular Cylinder. Let us now apply the same method to consider scattering of the plane wave (1) by the convex cylinder as in Figure 8-9. The point $a(\varphi)$ on the

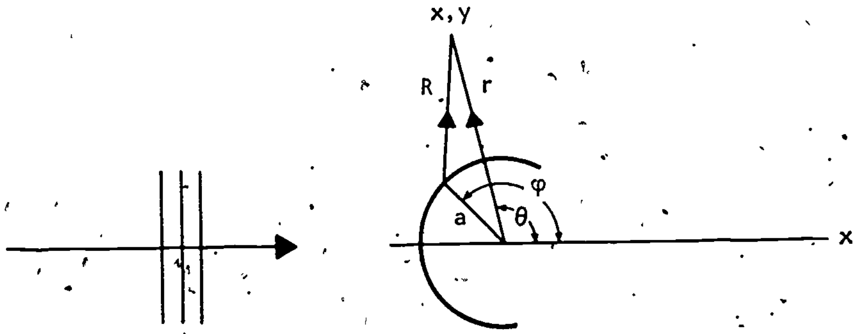

FIGURE 8-9

cylinder has the coordinates $a\cos\varphi$, $a\sin\varphi$, and the excitation at the point is $e^{ika\cos\varphi}$. We write the corresponding scattered field as the integral of u over the arc $a d\varphi$:

(75) $$U = \int_{\pi/2}^{3\pi/2} \frac{c}{\sqrt{kR}} e^{ik(R + a\cos\varphi)} a\,d\varphi .$$

We do not know c completely, but the result (74) and the corresponding form $g\cos\alpha$ suggest the generalization in which α (the angle of incidence with respect to the surface normal) is replaced by the present analog $\pi - \varphi$. Thus for convenience (and as can be justified with a more complete model) we use

(76) $$U = -ga\sqrt{\frac{k}{2\pi}} e^{-i\pi/4} I, \quad I = \int_{\pi/2}^{3\pi/2} \frac{e^{ik(R + a\cos\varphi)}}{\sqrt{R}} \cos\varphi\, d\varphi,$$

and we shall see that $\cos\varphi$ in the integrand is appropriate for both geometrical reflection and shadow formation. We consider only the range $0 < \theta < \pi$ explicitly; however, the results may be extended to all θ by introducing absolute values of the trigonometric functions (as required to preserve symmetry).

The phase of I is proportional to

(77) $$L = R + a\cos\varphi = \sqrt{r^2 + a^2 - 2ar\cos(\varphi - \theta)} + a\cos\varphi ,$$

and its derivative

(78) $$\frac{dL}{d\varphi} = L' = \frac{ar\sin(\varphi - \theta)}{R} - a\sin\varphi ,$$

vanishes for the two values $\varphi = \varphi_L$ or φ_D, such that

(79L) $$\frac{\sin(\varphi_L - \theta)}{R_L} = \frac{\sin\varphi_L}{r_L} ,$$

(79D) $$\frac{\sin(\varphi_D - \theta)}{R_D} = \frac{\sin(\pi - \varphi_D)}{r_D} .$$

For a given value of θ, the phase has only one stationary value: the value may correspond either to geometrical reflection as in Figure 8-10, or to forward scattering as in Figure 8-11. The first value applies for y in the "lit" region L in Figure 8-10; the second value, which yields the shadow forming rays, corresponds to y in the "dark" region D as in Figure 8-11. Equation (79L) corresponds to Euclid's principle of reflection, and (79D) to the prinoiple of shadow formation.

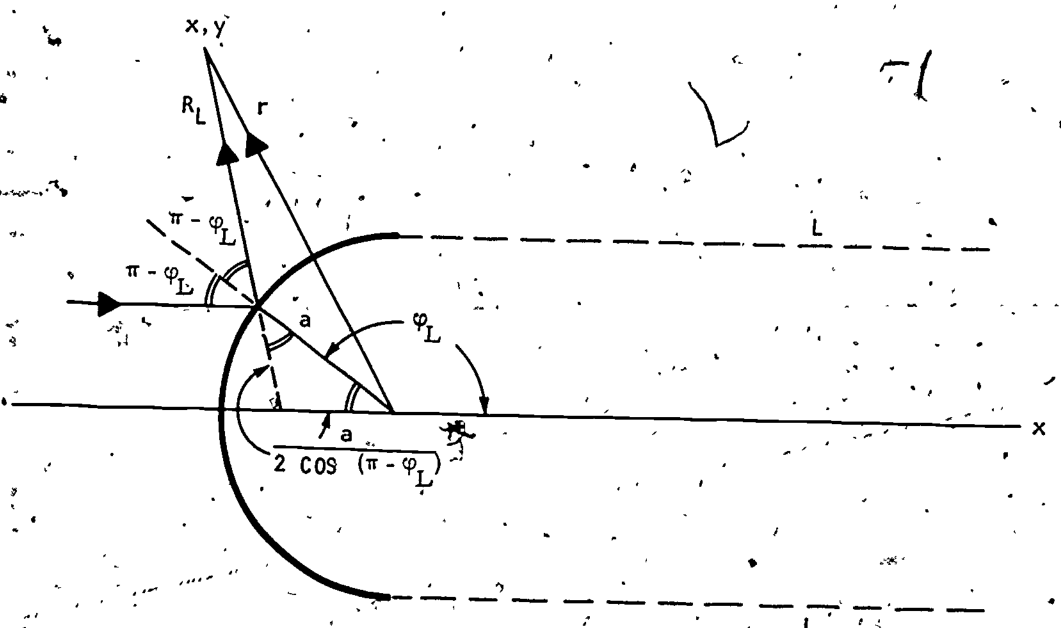

FIGURE 8-10

The second derivative of L equals

(80) $$\frac{d^2L}{d\varphi^2} = L'' = -\frac{a^2r^2\sin^2(\varphi - \theta)}{R^3} + \frac{ar}{R}\cos(\varphi - \theta) - a\cos\varphi ,$$

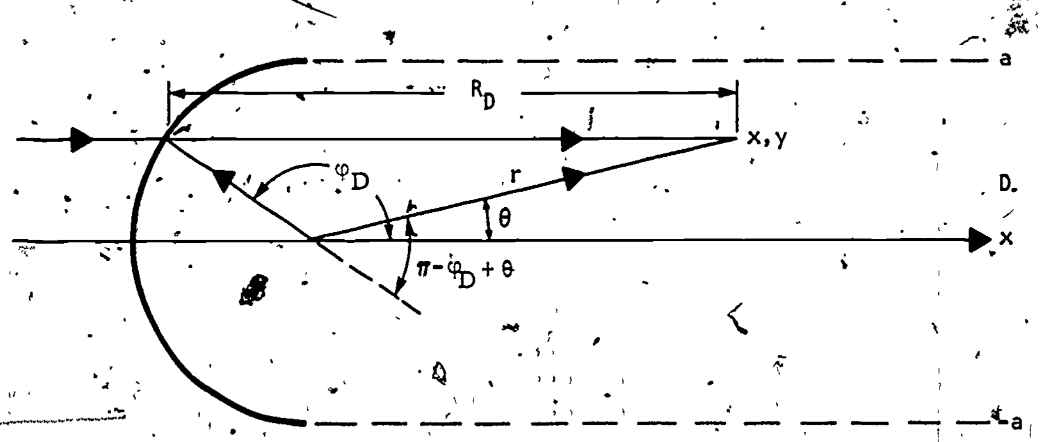

FIGURE 8-11

and substituting (79) leads to the special values of the stationary points. Thus for either case

(81) $$L_s'' = \frac{a}{R_s}\left[-a\sin^2\varphi_s + r\cos(\varphi_s - \theta) - R_s\cos\varphi_s\right],$$

where R_s and φ_s are the special values shown in either Figure 8-10 or 8-11. At a forward point, we see from Figure 8-11 that $r\cos(\pi - \varphi_D + \theta) + a = R_D\cos(\pi - \varphi_D)$; consequently, $r\cos(\varphi_D - \theta) - R_D\cos\varphi_D = a$, and (69) reduces to

(82D) $$L_D'' = \frac{a^2}{R_D}\cos^2\varphi_D .$$

On the other hand, at the reflection point, we see from Figure 8-10 that $r\cos(\varphi_L - \theta) = a + R_L\cos(\pi - \varphi_L) = a - R_L\cos\varphi_L$; consequently

(82L) $$L_L'' = \frac{a}{R_L}\left[a(1 - \sin^2\varphi_L) - 2R_L\cos\varphi_L\right]$$

$$= \frac{2a\cos(\pi - \varphi_L)}{R_L}\left[R_L + \frac{a}{2}\cos(\pi - \varphi_L)\right]$$

Similarly the stationary value of the phase function at a forward point equals

(83D) $$L_D = R_D + a\cos\varphi_D = R_D - a\cos(\pi - \varphi_D) = x .$$

Although we could eliminate R_L and φ_L in the reflected value, it is simpler to leave L_L in the original form

(83L) $$L_L = R_L - a\cos(\pi - \varphi_L) ;$$

here (for a given value of r and θ) R_L and φ_L are determined as in Figure 8-10, i.e., by noting which ray (or the corresponding value of y) of the incident wave front can reach $r(\theta)$ via reflection in the cylindrical surface.

Finally, the required slowly varying parts of the integrand of I evaluated at the stationary points are given by

$$G_s = \frac{\cos\varphi_s}{\sqrt{R_s}}, \qquad s = D, L. \tag{84}$$

Substituting (56) into (76), and then entering (84), we obtain

$$U_s = -ga\,G_s \frac{e^{ikL_s}}{\sqrt{L_s''}} = -ga\cos\varphi_s \frac{e^{ikL_s}}{\sqrt{R_s L_s''}}. \tag{85}$$

Thus for the perfect reflector, $g = 1$, we enter the above D-values in (85) and obtain the shadow forming wave

$$U_D = -e^{+ikx} = -U_i. \tag{86}$$

Similarly, the L-values give the geometrical reflected wave

$$U_L = \sqrt{\frac{a\cos\alpha}{2\left(R_L + \frac{a}{2}\cos\alpha\right)}}\, e^{ik(R_L - a\cos\alpha)}, \qquad \alpha = \pi - \varphi_L, \tag{87}$$

which we considered in Sections 2 and 4 from a geometrical basis. In particular, the flux density ratio F/F_0 of (4:14) is simply the present $|U_L|^2$, and similarly the previous (4:18) corresponds to $|U_s|^2$ of (85); the present forms are richer in that they make the ray path (L_s) as well as the caustic (L_s'') explicit.

The corresponding Fresnel approximations are obtained by using (55) instead of (56). We change variables to $\eta = a\sin\varphi$, $D_\varphi L = a\cos\varphi\, D_\eta L$, and obtain

$$\mathcal{U}_D = U_D \frac{e^{-i\pi/4}}{\sqrt{2}}\left[\mathcal{F}(\eta_+) - \mathcal{F}(\eta_-)\right], \qquad \eta_\pm = \sqrt{\frac{k}{\pi x}}(\pm a - y), \tag{88}$$

where U_D is given in (86); thus (88) is simply (28) for the range $x > 0$. Similarly, in terms of U_L of (87), we have

$$\mathcal{U}_L = U_L \frac{e^{-i\pi/4}}{\sqrt{2}}\left[\mathcal{F}(\eta_+) - \mathcal{F}(\eta_-)\right],$$

$$\eta_\pm = \sqrt{\frac{2k(R_L + \frac{a}{2}\cos\alpha)}{\pi R_L\, a\cos\alpha}}\,(\pm a - a\sin\alpha), \qquad \alpha = \pi - \varphi_L, \tag{89}$$

where $\mathcal{F}(\eta)$ is the Fresnel integral as in (10) ff.

<u>The field on caustics</u>: Although supplemented by phase considerations, equations (86) and (87) are still results of geometrical optics which we could construct piece-by-piece from the special "laws" of the earlier sections: [H'] gives the directions, [KL] the magnitudes, and the phases may be obtained from Newton's idea of periodicity. However we have now obtained these results essentially from the <u>single idea</u> of

periodic waves that evolved through the work of Huygens, Young and Fresnel into

[YF]: $$U = \int u\,ds\,,$$

which represents the scattered wave as an integral of elementary cylindrical waves over a line or an arc (as in (3), (67), and (75)), or of spherical waves over a surface or volume (as in (6:38)). Starting with [YF], we used mathematical procedures to approximate the integral, in particular, we obtained the general short-wavelength approximation (56), two of whose special cases are given by (86) and (87). Aside from superseding the earlier special laws, [YF] is applicable to many other phenomena than covered by [H'] and [KL], or by (56). We have already applied [YF] to obtain the Fraunhofer, Fresnel, and Rayleigh-Born scattering approximations, more of which are covered by (56). We now apply [YF] to supplement the short-wavelength form (56) by determining the magnitude of the field on a caustic, the case $L''_s = L''_H = 0$ excluded in (54) and in our discussion of [KL]. Thus, for example, the analog of (87) for reflection from the concave semicircle has a replaced by -a (see 4:15)), and does not hold on the caustic $L''_s = 0$, $R_L = -(a/2)\cos\alpha$. However, we do not require a special "law" to obtain a non-singular form, but merely a more appropriate approximation of [YF] than given by (56).

On a caustic, both L' and L'' vanish; in addition, on a cusp of a caustic (where the derivative of the equation of the caustic vanishes, since the curve changes direction) the third derivative $L^{(3)}$ must also vanish. Thus for such cases we can no longer approximate $L(x)$ by means of (54), i.e., we must keep additional terms in the Taylor series in order to obtain the first correction to $L(x_s)$.

Let us assume that the first $n-1$ derivatives of L at x_s vanish, and approximate $L(x)$ by the Taylor polynomial of n-th order:

(90) $$L(x) \approx L(x_s) + \frac{(x-x_s)^n}{n!}\left(\frac{d^n}{dx^n}L(x)\right)_{x=x_s} = L_s + (x-x_s)^n \frac{L_s^{(n)}}{n!}\,.$$

For this case, for infinite limits (corresponding essentially to (56)), we have

(91) $$I = \int_{-\infty}^{\infty} G(x)e^{ikL(x)}\,dx \sim G_s e^{ikL_s}\int_{-\infty}^{\infty} e^{ik(x-x_s)^n F^{(n)}/n!}\,dx\,.$$

If we introduce a new variable y through $y^n = k(x-x_s)^n L_s^{(n)}/n!$, then we may rewrite (91) as

(92) $$I = G_s e^{ikL_s}\left(\frac{n!}{kL_s^{(n)}}\right)^{\frac{1}{n}}\int_{-\infty}^{\infty} e^{iy^n}\,dy\,,$$

where the remaining integral (the "gamma function" integral) depends only on n.

Using (92) instead of (56) in (76), we obtain

(93) $$U \sim C_n e^{ikL_s} (k)^{\frac{1}{2} - \frac{1}{n}},$$

where we have suppressed practically everything but the dependence on $k = 2\pi/\lambda$. Thus away from a caustic, we have $n = 2$ and $|U|$ is independent of wavelength (e.g., as in (86) and (87)); this corresponds to "true geometrical optics." For the line caustic of the circular cylinder, we have $n = 3$, and since $R(\vec{r}') \approx a(\vec{r}')$ on the caustic $R = \frac{a}{2}\cos\alpha$, we see from (93) and from dimensional considerations that $U \propto (ka)^{\frac{1}{6}}$; similarly, for a cusp ($\alpha = 0$ for the circle), we have $n = 4$, and $U \propto (ka)^{\frac{1}{4}}$. More generally, since $\lambda \ll a$, we see that $k^{\frac{1}{2} - \frac{1}{n}}$ increases with increasing n and approaches $k^{\frac{1}{2}}$ (at which limit the phase is "completely" stationary). Thus U increases on caustics as the wavelength λ decreases; the field is always finite since the case of zero wavelength (the implicit assumption of conventional geometrical optics) is not physically realizable.

9. Mathematical Model for Scattering.

In previous sections we considered certain aspects of wave theory but based the development on several supplementary "laws of nature." In the present section we tie these special postulates together into a mathematical model for scattering.

Wave Equation: We considered the plane waves $\cos(\pm kx - \omega t)$, and, for convenience, worked with the real part of the corresponding exponentials:

(1) $$e^{\pm ikx - i\omega t} = e^{\pm ikx} \cdot e^{-i\omega t} = f(x)g(t) .$$

Equation (1) is a product of a function of x times a function t, each of which satisfies a second order differential equation:

(2) $$\frac{d^2 f(x)}{dx^2} = -k^2 f(x) , \quad k = \frac{2\pi}{\lambda} ;$$

(3) $$\frac{d^2 g(t)}{dt^2} = -\omega^2 g(t) = -k^2 v^2 g(t) , \quad \omega = kv = \frac{2\pi v}{\lambda} = 2\pi\nu .$$

As discussed previously, v is the phase velocity of the wave generated by a source vibrating at a frequency ν, and λ is the wavelength — the distance between crests.

The general form of (2) and (3) is

(4) $$\frac{d^2 F(y)}{dy^2} + \beta^2 F(y) = 0 ,$$

whose general solution equals

(5) $$C_1 \cos \beta y + C_2 \sin \beta y , \text{ or, equivalently, } D_1 e^{i\beta y} + D_2 e^{-i\beta y} ,$$

where the first constants (C_1, C_2) are linear combinations of the second (D_1, D_2). Thus in choosing the particular combinations that led to (1), we used some selection rules. We discuss these rules subsequently.

Now let us use the above to construct more general equations. Our attitude is the following. We know of phenomena that can be described by wave functions such as (1). Let us seek a general wave equation that yields (1) as well as more general wave forms. The more general waves may well correspond to phenomena not covered by (1).

From (2) and (3), we have

$$\frac{1}{f(x)} \frac{d^2 f(x)}{dx^2} = -k^2 , \quad \frac{1}{v^2 g(t)} \frac{d^2 g(t)}{dt^2} = -k^2 .$$

Subtracting one from the other, we obtain

(6) $$\frac{1}{f(x)} \frac{d^2 f(x)}{dx^2} - \frac{1}{v^2 g(t)} \frac{d^2 g(t)}{dt^2} = 0 ,$$

or equivalently

(7) $$g(t)\,\frac{d^2 f(x)}{dx^2} - \frac{f(x)}{v^2}\,\frac{d^2 g(t)}{dt^2} = 0 ,$$

where $\frac{d}{dx}$ operates only on $f(x)$, and $\frac{d}{dt}$ on $g(t)$. The present notation is awkward. We would like to combine $f(x)g(t)$ in a single form $E(x,t)$. To do so and preserve the idea that the differentiations with respect to x and t are independent, we introduce the notation $\frac{\partial}{\partial x}$ to represent differentiation with respect to x while t is fixed — partial differentiation; similarly for t. Thus, we rewrite (7) as

(8) $$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{v^2}\,\frac{\partial^2 E(x,t)}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{v^2\,\partial t^2}\right) E(x,t) = 0 .$$

This is called the wave equation. The wave functions of (1) are special solutions of (8) corresponding to periodic waves.

We generalize (8) to two spatial dimensions x, y by introducing an additional operation $\frac{\partial^2}{\partial y^2}$ into (8):

(9) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{v^2\,\partial t^2}\right) E(x,y,t) = 0 .$$

The plane waves $e^{\pm ikx\cos\alpha + iky\sin\alpha - i\omega t}$ that we considered in Section 8 are solutions of (9). Similarly for three spatial dimensions we introduce an additional operation $\frac{\partial^2}{\partial z^2}$ in (9). We write the general form as

(10) $$\left(\nabla^2 - \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2}\right) E(\vec{r},t) = \nabla^2 E(\vec{r},t) - \frac{1}{v^2}\,\frac{\partial^2 E(\vec{r},t)}{\partial t^2} = 0 ,$$

where

(11) $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

(which is also frequently written as Δ) is called Laplace's operator.

We may also recast (11) in polar coordinates r, θ, φ. In particular, the elementary spherical wave that we considered previously is the special solution of (10) that depends only on the magnitude r but is independent of θ and φ. The simpler equation for the elementary spherical wave

(12) $$E(r,t) = \frac{e^{ikr - i\omega t}}{r}$$

can be obtained by comparison with (1) and (8). Thus if we replace $E(x,t)$ by $rE(r,t)$ and $\frac{\partial^2}{\partial x^2}$ by $\frac{\partial^2}{\partial r^2}$, we obtain the corresponding equation for (12):

(13) $$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2}\right) rE(r,t) = 0 .$$

We may rewrite this directly in the form of (10):

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E}{\partial r}\right) - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0 \quad (14)$$

The general solution of (8) may be written

$$E(x,t) = F(vt - x) + G(vt + x) \quad (15)$$

where F and G are arbitrary twice-differentiable functions. Similarly, the general solution of (13) is

$$rE(r,t) = F(vt - r) + G(vt + r) . \quad (16)$$

The corresponding solution for the equation of the line source in two dimensions and of the general equation (10) cannot be expressed so simply. We mention the general solutions only to stress that the solutions corresponding to periodic waves are special cases.

Let us now ignore practically everything that led us to the wave equation (10). We accept (10) as fundamental and seek its periodic solutions. For completeness, we repeat the definitions of the fundamental parameters given in previous sections.

The time-periodic waves we considered correspond to solutions having the product form

$$E(\vec{r},t) = f(\vec{r})g(t) . \quad (17)$$

If we substitute (17) into (10), the "variables separate" in the sense that we obtain

$$\frac{1}{f(\vec{r})}\nabla^2 f(\vec{r}) = \frac{1}{v^2 g(t)}\frac{d^2 g(t)}{dt^2} \quad (18)$$

essentially as in (6). Since the left hand side of (18) is a function only of $\vec{r}$, and the right hand side only of t, each side must equal the same constant; call this constant $-k^2$. Thus (18) reduces to

$$\frac{d^2 g(t)}{dt^2} + k^2 v^2 g(t) = 0 , \quad (19)$$

$$\nabla^2 f(\vec{r}) + k^2 f(\vec{r}) = 0 , \quad (20)$$

where (20) is known as Helmholtz's equation.

Equation (19) is the form (4) we considered previously. Its solutions are the periodic functions in (5). Without any loss of generality, we pick

$$g(t) = e^{-ikvt} = e^{-i\omega t} , \quad \omega = kv \quad (21)$$

to work with. In equation (10), v is given as the velocity: the distance an element of the wave covers in unit time. From (21), we see that $g(t)$ is periodic in t, i.e., if the time t changes by multiples of the constant $T = \frac{2\pi}{\omega}$, then g is unaltered:

(22) $$g(t) = g(t + mT) \; ; \quad T = \frac{2\pi}{\omega} = \frac{1}{\nu} , \quad n = 1, 2, 3 \cdots .$$

Thus T is the periodicity of the wave in time, and ν (the frequency) is the number of times that g has the same value in unit interval of time. The analog in space, the wavelength $\lambda = vT = \frac{2\pi v}{\omega}$, is the distance covered by an element moving with velocity v for a time T. But from (21), we have $\frac{v}{\omega} = \frac{1}{k}$. Consequently $k = \frac{2\pi}{\lambda}$ is the relation between the "separation constant" (the propagation factor or wave number) and wavelength.

The space equation (20) is known as the reduced wave equation or Helmholtz's equation. The one dimensional case $\frac{d^2 f(x)}{dx^2} + k^2 f(x) = 0$ is given in (2), and the special case of the spherically symmetrical wave is implicit in (14), i.e., $\frac{d^2}{dr^2}(rf) + k^2 rf = 0 , \quad f = f(r)$.

The above equations specify propagation of waves in a medium whose properties are determined solely by v. For the periodic cases, once we fix the frequency factor ω, the corresponding wavelength in the medium is determined. If we are dealing with several such media specified by different velocities v_m ; $m = 0, 1, \cdots$, then we obtain the same wave equations with v replaced by v_m; the corresponding reduced wave equations for frequency factor ω involve $k_m = \frac{\omega}{v_m} = \frac{2\pi}{\lambda_m}$. To make full use of the earlier equations in v, we take $v_o = v$ = constant as a reference, and write

(23) $$v_m = \frac{v_o}{\mu_m} = \frac{v}{\mu_m} , \; \mu_o = 1 ,$$

where μ_m is the relative index of refraction; consequently $k_m = \mu_m k$. The corresponding space equation for a medium specified by μ, is

(24) $$(\nabla^2 + \mu^2 k^2) f(\vec{r}) = 0 ,$$

e.g.,

(25) $$\left(\frac{d^2}{dx^2} + \mu^2 k^2\right) f(x) = 0$$

for the one-dimensional case. If μ is independent of x, then the solutions of (25) are the forms (5) with $\beta = \mu k$.

<u>Conditions on the Solution</u>: All the problems we considered are described by functions $E(\vec{r})g(t) = E(\vec{r})e^{-i\omega t}$, where $E(\vec{r})$ is a particular solution of the reduced wave equation

[I]: $$(\nabla^2 + \mu^2 k^2) E(\vec{r}) = 0 .$$

The particular solution is determined by constraints that have been implicit in our development. The constraints are of two kinds:

[II]: restrictions on the solution at the scatterer's surface,

[III]: restrictions on the solution at large distances from the scatterer.

The additional constraints are necessary because the wave equation merely describes the local properties of the medium and how a wave travels from point to point. But what if the medium is discontinuous? e.g.,

{1}: suppose we have a glass of water and consider waves on the surface of the water bounded by the unyielding rim of the glass;

{2}: suppose we are in a boat on a very large lake and the boat is an obstacle for an incoming wave?

Cases {1} and {2} illustrate two essentially different kinds of wave problems we may be concerned with.

In {1} we deal with a bounded medium: we are given v, the shape of the boundary and constraints on the solution at the boundary, and then may seek to determine the forms and periods of the waves that can be maintained in such enclosed media. These are free vibration problems: the waves on a taut clothesline, the waves on the surface of a glass of water, the sound waves in a closed room, the electromagnetic waves in a metal cavity, etc., are illustrations, and analogous problems exist in the quantum theory of atomic states.

If the bounding surface is one that yields, then waves on the inside create waves on the bounding surface, and they may propagate in a region external to the surface. We may also set up vibrations on a surface and use the surface as a source of waves for the external medium, e.g., a vibrating drum head as a source of sound. All musical instruments, strings, drums, pipes are examples of "vibrator-radiator" systems for sound. (We have switched from talking about light to talking about sound and water waves; this is partly for convenience, but also to stress the fact that as far as wave physics goes there are analogous phenomena in all branches of science.)

In {2} we deal with a bounded object that represents an obstacle to a wave traveling in an essentially unbounded medium. We require conditions that tell us the shape and size of the obstacle, whether its surface is penetrable by waves, and, if so, then what is the medium inside its surface. Such boundary conditions or transition conditions specify the kind of discontinuity the obstacle represents in the imbedding medium. Depending on the phenomena we seek to model, we may require boundary conditions such as

[IIa] $E = 0$ on surface

or

[IIb] $\frac{\partial E}{\partial n} = 0$ on surface

where $\frac{\partial E}{\partial n}$ is the rate of change of E along the normal at a point on the surface. These conditions correspond to surfaces impenetrable to waves. If the surface is penetrable (partially transparent), then many phenomena correspond to the following. the waves outside the scatterer's surface travel in medium-1 and the wave functions satisfy $(\nabla^2 + \mu_1^2 k^2)E_1 = 0$; within the scatterer they travel in medium-2 and satisfy $(\nabla^2 + \mu_2^2 k^2)E_2 = 0$; at the surface E_1 and E_2 are related by the transition conditions

[IIc] $$E_1 = E_2 , \quad \frac{\partial E_1}{\partial n} = A \frac{\partial E_2}{\partial n} ,$$

where A is a supplementary physical constant. Thus in general, the wave problems we consider are specified by two physical constants (or "physical parameters") μ_2/μ_1 and A whose values must be assigned at the start.

Having [I] and [II], we complete the mathematical statement of the scattering problem by conditions at large distances from the scatterer [III]. These specify that we seek a solution consisting of essentially two terms: one term corresponds to the incident field, e.g., a plane wave

[IIIa] $$E_i = e^{ikx} ,$$

which is the space part of $e^{ikx - i\omega t}$; the other term, say E_s, corresponds to the outgoing wave radiated by the obstacle in response to E_i. In (6:40) for scattering by a tenuous sphere, we saw that the wave at large distances from the scatterer was the product of a function of directions and the elementary outgoing wave of a point source, $\frac{e^{ikr - i\omega t}}{r}$. Similarly, in (6:32) for scattering by a strip, the wave at large distances was proportional to that of a line source $\frac{e^{ikr - i\omega t}}{\sqrt{r}}$. The corresponding wave surfaces are symmetrical, and, in addition, we saw in Sections 2 and 5 that the eikonals corresponding to geometrical reflection from a semicircle and hemisphere, although complicated in shape in the vicinity of the obstacle, became more and more symmetrical with increasing distance. Similarly for a planar scatterer (6:17), the wave $e^{i(k|x| - \omega t)}$ is symmetrical in $|x|$. Suppressing the time dependence $e^{-i\omega t}$, we summarize all such cases by the statement

[IIIb] $$E_{sm} \sim g_m \frac{e^{ikr}}{r^{(m-1)/2}} \text{ as } r \sim \infty; \; m = 1, 2, 3$$

where r is measured from some point in the scatterer, and where g (called the scattering amplitude) is independent of r. Thus at large distances from the scatterer $(r \sim \infty)$, E_s reduces to the elementary symmetrical wave times a function of angles. The condition [IIIb] is a consequence of the weaker Sommerfeld radiation condition $\text{Lim}\, r^{(m-1)/2} \left\{ \frac{\partial}{\partial r} E_{sm} - ikE_{sm} \right\} = 0$ as $r \sim \infty$, and of a still weaker condition that the integral over a sphere of radius r of the absolute square of the function in braces approach zero as $r \sim \infty$. The condition for $r \sim \infty$ insures that we deal with outward radiation, and that the scatterer correspond to a source of waves (instead of sink of waves, and also rules out free vibrations as in {1}). Collectively we write the total field as

$$[III_m] \quad E = E_i + E_s ; \quad E_i = e^{ikx} ; \quad E_{sm} \sim g_m \frac{e^{ikr}}{r^{(m-1)/2}} ; \quad m = 1, 2, 3 ,$$

where i and s stand for incident and scattered respectively.

Equations [I], [II], and [III] constitute the mathematical model for scattering. They replace all the special principles we considered previously; they cover all the cases where the principles apply, and many additional ones as well. They incorporate the essential gross physics of the effects of an obstacle on the radiation from a source, and the cumulative fruits of two thousand years of evolution in providing a framework of well-posed problems with unique solutions. In summary, the wave equation [I] describes the local properties of the media, the surface conditions [II] take account of obstacles (interfaces, transition regions), and the conditions at large distances [III] specify that the field consists of a wave (E_i) from a primary source perturbed by a wave (E_s) outgoing from the obstacle. We can now seek analytically the redistribution of the radiation of a source arising from the presence of an obstacle.

Point Scatterer: As an elementary illustration let us consider the scattering of a plane wave e^{ikx} by a small sphere of radius a for the boundary condition [IIa], E = 0 at r = a. For the general case of a sphere of arbitrary radius a we would work with the complete solution of [I] for $\mu = 1$ subject to [III_3]. We would represent E_i and E_s in terms of angle-dependent functions and initially unknown constants, and then use [IIa] to determine the constants. However, the restriction $a \ll \lambda$, or equivalently,

(26) $$ka \approx 0$$

simplifies the problem. From the geometry of Figure 9-1, the incident field e^{ikr}

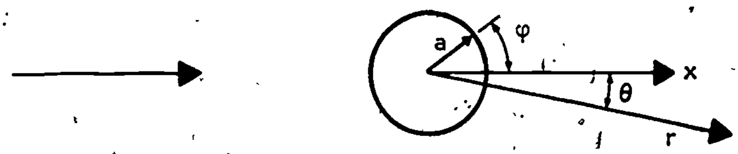

FIGURE 9-1

equals $e^{ika\cos\varphi}$ at the surface of the sphere; using the restriction (26), we have $e^{ika\cos\varphi} \approx 1$, so that we may work with the approximation

(27) $$E_i(a) = 1 .$$

Thus the exciting field at the surface is independent of angles, and the corresponding scattered wave must be similarly independent of angles: $E_s(r)$ is a solution of $\frac{d^2}{dr^2}(rE_s) + k^2 rE_s = 0$, and the only one satisfying [IIIb] at large distances is

(28)
$$E_s = C\frac{e^{ikr}}{r}$$

where C is a constant. At the surface of the scatterer $r = a$, we have $\frac{e^{ikr}}{r} \approx \frac{1}{a}$, and consequently the total field at $r = a$ is approximately

(29)
$$E(a) = E_i(a) + E_s(a) = 1 + \frac{C}{a}.$$

Applying the boundary condition $E(a) = 0$, we get

(30)
$$C = -a,$$

The scattered wave for $r > a$ is thus

(31)
$$E_s = -\frac{a}{r}e^{ikr}.$$

This corresponds physically, for example, to scattering of underwater sound by a small air-bubble (an exception to [R] of Section 7).

Slab Scatterer: As another example, let us consider scattering of a plane wave by a partially transparent slab as in Figure 9-2. The conditions on the problem are:

(32)
$$\left(\frac{d^2}{dx^2} + k^2\right)E_1 = 0, \quad |x| > a;$$

(33)
$$\left(\frac{d^2}{dx^2} + K^2\right)E_2 = 0, \quad K = \mu k; \quad |x| < a;$$

(34)
$$E_1 = E_2, \quad \frac{dE_1}{dn} = A\frac{dE_2}{dn}, \quad |x| = a.$$

(35)
$$E_1 = E_i + E_s: \quad E_i = e^{ikx}, \quad E_s \sim g\,e^{ik|x|} \text{ as } |x| \sim \infty.$$

From (35), we write

(36)
$$E_s = g_+e^{ikx}, \quad x > a; \quad E_s = g_-e^{-ikx}, \quad x < -a.$$

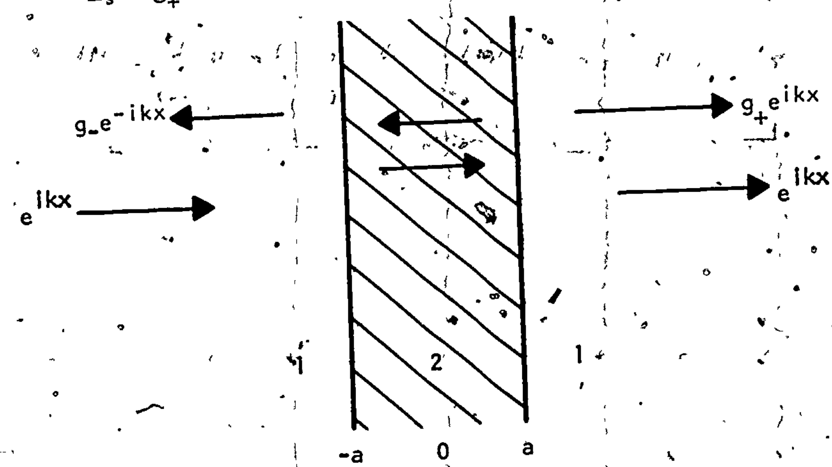

FIGURE 9-2

From (33), we take the most general solution in the form

$$E_2 = b_+ e^{iKx} + b_- e^{-iKx} . \tag{37}$$

We thus have four constants (g_+, g_-, b_+, b_-) to determine, and we do so by applying the surface conditions (34).

At $x = -a$, we get

$$e^{-ika} + g_- e^{ika} = b_+ e^{-iKa} + b_- e^{iKa} \tag{38}$$

$$k\left(e^{-ika} - g_- e^{ika}\right) = AK\left(b_+ e^{-iKa} - b_- e^{iKa}\right) ; \tag{39}$$

similarly, at $x = +a$,

$$b_+ e^{iKa} + b_- e^{-iKa} = (1 + g_+) e^{ika} \tag{40}$$

$$KA\left(b_+ e^{iKa} - b_- e^{-iKa}\right) = k(1 + g_+) e^{ika} . \tag{41}$$

Thus we have four algebraic equations for the four unknowns.

Solving these, and introducing the abbreviation

$$Q = \frac{Z - 1}{Z + 1} , \qquad Z = \frac{KA}{k} = \mu A , \tag{42}$$

we obtain

$$g_- = -Q \frac{e^{-i2ka}(1 - e^{i4Ka})}{1 - Q^2 e^{i4Ka}} \equiv R \tag{43}$$

$$g_+ + 1 = \frac{(1 - Q^2) e^{i(K-k)2a}}{1 - Q^2 e^{i4Ka}} \equiv T \tag{44}$$

where R and T are called the reflection and transmission coefficients. The corresponding internal field is

$$E_2 = (1 - Q) e^{i(K-k)a} \frac{\left[e^{iKx} + Q e^{iK(2a-x)}\right]}{1 - Q^2 e^{i4Ka}} . \tag{45}$$

Expanding the denominators in (43) and (44) enables us to interpret the solution in terms of multiple reflections inside the slab.

If we are dealing with a single interface at $x = 0$ as in Figure 9-3, then we

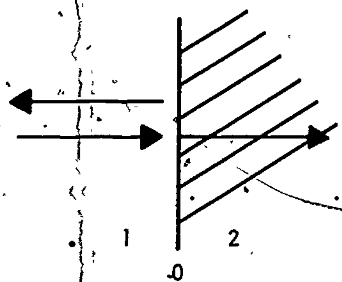

FIGURE 9-3

obtain simply

(46) $$E_1 = e^{ikx} - Qe^{-ikx}$$

(47) $$E_2 = (1 - Q)e^{iKx} .$$

The results and all the above may be generalized by inspection to an arbitrary angle of incidence α as in Figure 9-4. Thus in equations (32) and (33) we may replace

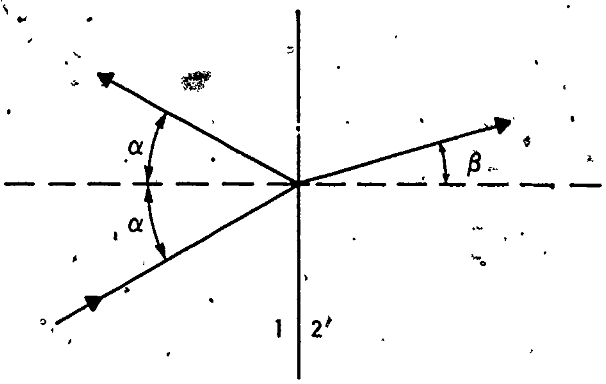

FIGURE 9-4

k^2 by $k^2 \cos^2 \alpha$ and K^2 by $K^2 \cos^2 \beta$ and obtain the same final functions in terms of the new constants $k \cos \alpha$ and $K \cos \beta$, e.g., (46) becomes

(48) $$e^{ikx \cos \alpha} - Q' e^{-ikx \cos \alpha} , \quad Q' = \frac{Z' - 1}{Z' + 1} , \quad Z' = \frac{KA \cos \beta}{k \cos \alpha} ,$$

and (47) becomes

(49) $$(1 - Q')e^{iKx \cos \beta} .$$

We may now multiply (48) and (49) by the same factor $e^{iky \sin \alpha}$. This converts (48) to

(50) $$E_1 = e^{ikx \cos \alpha + iky \sin \alpha} - Q' e^{-ikx \cos \alpha + iky \sin \alpha} = E_1(\alpha) - Q' E_1(\pi - \alpha) ,$$

where $E_1(\alpha)$ is a plane wave incident at an angle α, and $E_1(\pi - \alpha)$ is its mirror image in the plane $x = 0$. From (49), multiplication by $e^{iky \sin \alpha}$ gives

(51) $$(1 - Q')e^{iKx \cos \beta + iky \sin \alpha} .$$

If we require that

(52) $$k \sin \alpha = K \sin \beta , \quad \text{i.e.,} \quad \sin \alpha = \frac{K}{k} \sin \beta = \mu \sin \beta ,$$

then (51) equals

(53) $$E_2 = (1 - Q')e^{iKx \cos \beta + iKy \sin \beta}$$

which is a plane wave traveling at an angle β. Equation (52), which we recognize as "Snell's Law" [S], is thus an artifice for converting the solution of a one-dimensional problem to the corresponding two-dimensional solution; it insures that corresponding wave fronts match at the surface.

Integral representations: In Sections 6 and 8 we used Huygens' principle in order to represent the total scattered field as the integral of "wavelets" arising from elementary sources distributed over a surface or throughout a volume. To round out the previous intuitive discussion we should indicate how such forms follow from the present mathematical model [I], [II], and [III]. If we had available a theorem of Gauss (which relates certain surface and volume integral forms) we could prove that scattering functions E_s satisfying [I] and [III_m] can be represented in terms of elementary sources H as

$$E_{sm}(\vec{r}) = \int_S \left[H_m(k|\vec{r} - \vec{\rho}|)\partial_n E(\vec{\rho}) - E(\vec{\rho})\partial_n H_m(k|\vec{r} - \vec{\rho}|)\right] dS(\vec{\rho}) , \quad (54)$$

where $\vec{\rho}$ is a point on a surface $S(\vec{\rho})$ as in Figure 9-5 that encloses the scatterer but excludes the observation point $\vec{r}$, and where $\partial_n \equiv \frac{\partial}{\partial n}$ is the outward normal derivative. The function $E(\vec{\rho}) = E_1(\vec{\rho}) + E_s(\vec{\rho})$, the total field at $\vec{\rho}$, and its normal derivative, are weighting factors for the surface distribution of elementary sources $H_m(|\vec{r} - \vec{\rho}|)$ and $\partial_n H_m$ which radiate from $\vec{\rho}$ to $\vec{r}$.

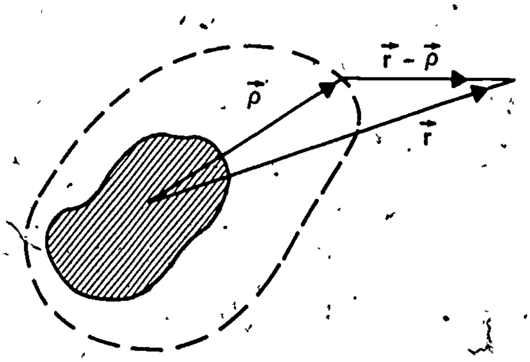

FIGURE 9-5

The elementary sources are essentially those we worked with in earlier sections. Thus, in three-dimensions (i.e., if all three space dimensions are significant in the problem), we require a point source

$$H_3(kR) = -\frac{e^{ikR}}{4\pi R}, \quad R = |\vec{r} - \vec{\rho}| = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} , \quad (55)$$

and in one-dimension, a planar source

$$H_1(kR) = \frac{e^{ikR}}{i2k}, \quad R = |x - \xi| . \quad (56)$$

In two-dimensions, for argument $kR >> 1$, the required line source approximates

$$(57) \qquad H_2(kR) \sim \frac{e^{-i\pi/4}}{4i}\sqrt{\frac{2}{\pi kR}}\, e^{ikR}, \qquad R = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

which is the form we worked with previously; the fact that (57) holds only for large values of the argument accounts for the restriction $kR >> 1$ that we mentioned for the strip and cylinder problems. For small values of kR, the elementary line source behaves quite differently:

$$(58) \qquad H_2(kR) \sim \frac{1}{2\pi}\ln kR, \quad kR \sim 0.$$

Its exact representation is given by

$$(59) \qquad H_2(kR) = \frac{1}{4i} H_0^{(1)}(kR),$$

where $H_0^{(1)}$ (which is known as Hankel's function of the first kind of order zero) is the special solution of (20) for two-dimensions corresponding to angle independent outgoing waves, i.e., it plays the same role in two-dimensions as $\frac{e^{ikR}}{R}$ and $e^{ik|x|}$ play in the other cases. If we specialize $S(\vec{\rho})$ to the surface of the scatterer itself, then we can use such surface conditions as [II] to obtain equations (integral equations) for the unknown values of $E_s(\vec{\rho})$ and $\partial_n E_s(\vec{\rho})$; for simple surfaces, the procedure is analogous to that we followed for the slab.

Although we will not prove (54), we will show how to obtain the approximate forms we worked with in earlier sections. Thus if we specialize $S(\vec{\rho})$ to the scatterer's surface and use the boundary condition [IIb] that $\partial_n E(\vec{\rho}) = 0$, we reduce (54) to

$$(60) \qquad E_s(\vec{r}) = -\int E(\vec{\rho})\partial_n H_m(k|\vec{r}-\vec{\rho}|)\,dS(\vec{\rho})$$

In particular, in two-dimensions and $k|\vec{r}-\vec{\rho}| >> 1$, we use (57) in (60) to obtain

$$(61) \qquad E_s(\vec{r}) \sim -\frac{e^{-i\pi/4}k}{4}\sqrt{\frac{2}{\pi k}}\int E(\vec{\rho})\frac{e^{ik|\vec{r}-\vec{\rho}|}}{\sqrt{|\vec{r}-\vec{\rho}|}}\hat{\rho}\cdot\hat{n}\,dS(\vec{\rho});\ \hat{\rho} = \frac{\vec{\rho}}{\rho},\ \hat{n} = \frac{\vec{n}}{n},$$

which is of the required form [IIIc]. If we knew the field $E(\vec{\rho})$ on the scatterer's surface, we could obtain the scattered field $E_s(\vec{r})$ by integration. If we do not know the field (and it is only for very simple shapes that $E(\vec{\rho})$ is known exactly), then we may seek heuristic physically motivated approximations.

In particular if the scatterer is very big compared to wavelength, then it is plausible to approximate $E(\vec{\rho})$ by elementary geometrical optics considerations. Following essentially Kirchhoff, one approximates the total field $E(\vec{\rho})$ on the "lit side" of the scatterer by twice the incident value E_i, and by zero on the "dark side." Thus if we substitute

$$(62) \qquad E(\vec{\rho}) \approx E_i(\vec{\rho}) \ \text{on lit side}; \quad E(\vec{\rho}) \approx 0 \ \text{on dark side}$$

into (61) we obtain the general case of the integrals we considered in Sections 6 and 8, i.e., for the strip with $dS(\vec{\rho}) = d\eta$ and the circular cylinder with $dS(\vec{\rho}) = a d\varphi$.

From (54) we could also construct the volume integral of elementary scatterers that we worked with previously for the case of a partially transparent sphere. First we specialize (54) to $\vec{\rho}$ on the scatterer and then use the transition conditions [IIc] to replace the external surface fields $E(\vec{\rho}) = E(k,\vec{\rho})$ and $\partial_n E(\vec{\rho})$ by the corresponding internal fields $E(K,\vec{\rho})$ and $A\partial_n E(K,\vec{\rho})$ where $K = k\mu$ is the internal wave number. We then use the same theorem of Gauss to convert the resulting surface integral to an integral over the volume of the scatterer. In particular for constant μ and $A = 1$, we would obtain

(63)
$$E_s = (k^2 - K^2)\int_V H_m(k|\vec{r} - \vec{\rho}|)E(K,\vec{\rho})dV(\vec{\rho})$$

where V is the volume of the scatterer. If we add E_i to both sides of (63) we obtain an integral equation for E which can be solved for simple shapes. We will not prove (63), but we will show how this rigorous result leads to the previous approximation (6:38).

For tenuous scatterers in the sense $K^2 = k^2\mu^2 \approx k^2$, Rayleigh replaced the unknown internal field $E(\rho)$ in (63) by the incident wave:

(64)
$$E(K,\rho) \approx E_i(k,\rho) = e^{ik\zeta}.$$

If we substitute (64) into (63) and specialize to three-dimensions by using (55), we obtain

(65)
$$E_s \approx \frac{k^2(\mu^2 - 1)}{4\pi}\int \frac{e^{ik|\vec{r} - \vec{\rho}|}}{|\vec{r} - \vec{\rho}|} e^{ik\zeta}dV(\vec{\rho})$$

which is the more complete version of the form we worked with previously in (6:38)ff. Equation (65) gives directly the result that the flux ($|E_s|^2$) is inversely proportional to λ^4 for various scatterers whose length dimensions are small compared to λ, and was used by Rayleigh prior to [R] of Section 7.

It should be stressed that such approximations as (62) and (64) are adequate only for limited ranges of the parameters. However, within their limitations, they provide useful and instructive explicit results for problems that cannot be solved rigorously.

In concluding this chapter, we should reiterate that we have covered merely a selected sequence of topics in wave physics. The wave equations that we "backed into" are, in general, generated in physics programs by operating on first order differential equations that relate the physical observables regarded as basic within a particular discipline, e.g., particle velocity and excess pressure in acoustics, electric and magnetic intensities in optics, radio, etc. The wave function E we have dealt with represents one of these different physical observables (or one of its Cartesian components), and may also stand for the probability density function of quantum mechanics. Similarly the media that we specified by an index of refraction μ and interface conditions represent quite different concepts in different disciplines, and involve different kinds of physical parameters, implicit continuity requirements on appropriate physically observable fields, etc. We have discussed neither the physics implicit in the above nor in the much more complex question of sources and the generation of fields. We began with let there be light, and followed a narrow thread of concepts.

Although we made "light" the theme for much of the development, we have not covered an essential aspect that distinguishes wave models for light from the models used for sound: light, and all electromagnetic waves, must also be characterized by polarization, this requires in general that we deal with vector wave functions with amplitudes perpendicular to the direction of propagation instead of the scalar functions we have considered. However, our discussion of light was in no sense meant to be comprehensive, and as stressed in the introduction of this chapter, there are many phenomena involving light that are not described by a wave model at all. In illustrating different applications of calculus, we have used light as a vehicle for an introduction to wave physics, not only because we have many visual experiences to draw on, but because the adequacy of a wave model for such phenomena was far from obvious to the early investigators (and not particularly obvious even to us without some careful observations). For water waves, the appropriateness of the mathematical model would have been clear from the start, and even for sound waves the intuition leads relatively directly from the visible waves on stringed instruments and on drum heads to waves in air. Thus in discussing light, we could introduce key topics leading to the development of the wave model essentially in their historical order, and thereby indicate the greater generality of the wave model, as well as the domains of applicability of the earlier special "laws of nature" that are now exhibited as consequences. However, the initial reservations that "light" is neither wave nor particle, and that only certain classes of phenomena involving light are adequately described by a wave model should not be lost sight of. Light is one of the most complex characters in the ABC of mathematical physics.

CODA

The preceding two chapters illustrate attempts at systematic approaches to applications of mathematics in science. Chapter 2 considers simple equations for growth and competition that arise again and again in superficially unrelated contexts of nature, it shows that phenomena and processes that occur in all the sciences are linked by one mathematical model. We select a narrow thread of mathematical methods and follow it through various sciences. Chapter 3 is quite different; there we follow science as a thread.. We select a narrow sequence of physical concepts leading from geometrical optics through wave physics and exhibit various methods of the calculus that further the development.

Thus our two chapters on mathematics and science are very different. They supplement each other in indicating the ways that mathematics and science interact. The first chapter follows a mathematics thread, the second a science thread, and the two together may suggest the crossing threads of a fabric. Our two threads intersect at Rayleigh's theory for the color of the sky: in Chapter 2, it is a special case of a general attenuation process; in Chapter 3, a special case of a general scattering process.

One thread suggests that mathematics intersects every science. The other thread suggests that every science intersects all of the mathematics. Together they may suggest that the interactions of mathematics and science are profound indeed. These are the threads of the fabric of our universe, the structure of our perception of nature. Mathematics and science are the very warp and woof of the universe our intellect has created.