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ABSTRACT

This volume was prepared by the School Mathematics Study Group (SMSG) to help elementary teachers develop a sufficient subject matter competence in the mathematics of the elementary school program. Background material for related SMSG materials for grades four through eight are included. Chapters in the book are: (1) What is Mathematics; (2) Numeration; (3) Whole Numbers; (4) Rational Number System; (5) Coordinates and Equations; and (6) Real Numbers. The appendices include additional materials related to the topics and answers to questions and problems in the text. (RH)

SCHOOL
MATHEMATICS
STUDY GROUP

STUDIES IN MATHEMATICS
VOLUME VI

Number Systems

(preliminary edition)

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PREFACE

Mathematics is fascinating to many persons because of its utility and because it presents opportunities to create and discover. It is continuously and rapidly growing because of intellectual curiosity, practical applications and the invention of new ideas.

The many changes which have taken place in the mathematics curriculum of the junior and senior high schools have resulted in an atmosphere in which mathematics has found acceptance in the public mind as never before. The effect of the changes has begun to influence the elementary school curriculum in mathematics as well. It is generally accepted that the present arithmetic program in grades 1-6 will be substantially changed in the next five years.

The time is past when elementary teachers can be content that they have taught mathematics if they have taught pupils to compete mechanically. More and more mathematics will be taught in the elementary grades; one important criterion of the effectiveness of a school's program will be the extent to which pupils understand. If mathematics is taught by people who do not like, and do not understand, the subject, it is highly probable that pupils will not like, and will not understand, it as well.

The School Mathematics Study Group materials for grades 4, 5, and 6 contain sound mathematics, presented at such a level that it can be understood by children. Experimental classes have found the program stimulating, and teachers have been enthusiastic about the results. Teachers in the experimental programs have received the special services of a consultant to help them with the subject matter. It is not surprising that teachers who have been trained to teach arithmetic in the traditional sense need to study more mathematics if they are to have confidence in their ability to introduce new ideas. They must know far more than the students, and understand something of the later implications of the topics they teach.

This volume, Number Systems, and a companion volume, Intuitive Geometry, have been prepared by the School Mathematics Study Group to help elementary teachers develop a sufficient subject matter competence in the mathematics of the elementary school program. What kind of material should teachers at this level have? Is a course in calculus, or abstract algebra, or applications of arithmetic what they need? The opinion of the editors is that elementary teachers need a thorough discussion of all the materials they might teach in grades 4, 5, and 6, from a higher point of view, but in a very similar setting, presented in much the same way they would present it, so that they themselves might experience something of the joy of discovery and accomplishment in mathematics which they may expect from their own pupils.

With the ideas in mind which have been expressed in the paragraph above, the existing 7th and 8th grade SMSG course of study was decided upon as the content which would be of greatest benefit to elementary teachers. The material has been carefully edited, with the idea that it must now serve a much different purpose than that for which it was originally intended. You, as the mature individual, will be able to appreciate much that would escape junior high school students. Even though, in some cases, your technical ability may not be well-developed, you will be able to think critically and to make connections between what you know and new material. In some cases, you will be surprised to find that "new" ideas in mathematics are really only a new language for ideas you have known either implicitly or explicitly for years. While your ability to compute may be improved only slightly as you study this book, you will find that you understand many operations and concepts that were previously vague or even merely tricks used to get correct answers.

In elementary mathematics today, the properties of the numbers are considered to be as important as the actual computations with numbers. The nature of the operations with numbers is considered to be as important as the answers obtained. But in addition to the work with numbers, the ideas of geometry must also be taught. Both the applications of number in geometry (measurement) and the relationships between geometric elements independent of number help to form a foundation for the later study of geometry. The introduction of basic mathematical ideas in the elementary grades is the opportunity for which you are preparing yourself.

CONTENTS

Chapter 1

WHAT IS MATHEMATICS?

1-1. Mathematics as a Method of Reasoning

> "Once, on a plane, I fell into conversation with the man next to me. He asked me what kind of work I do. I told him I was a mathematician. He exclaimed, 'You are! Don't you get tired of adding columns of figures all day long?' I had to explain to him that this can best be done by a machine. My job is mainly logical reasoning."

Just what is this mathematics which many people are talking about these days? Is it counting and computing? Is it drawing figures and measuring them? Is it a language which uses symbols like a mysterious code? No, mathematics is not any one of these. It includes them but it is much more than all of these. Mathematics is a way of thinking, a way of reasoning. Some of mathematics involves experimentation and observation, but most of mathematics is concerned with deductive reasoning.

By deductive reasoning you can prove that from certain given conditions, a definite conclusion necessarily follows. In arithmetic you learned how to prove definite statements about numbers. If a classroom has 7 rows of seats with 5 seats in each row, then there are 35 seats in all. You know that this is true without counting the seats and without actually seeing the room.

Mathematicians use reasoning of this sort. They prove "if--then" statements. By reasoning they prove that if something is true, then something else must be true.

By logical reasoning you can often find all the different ways in which a problem can be solved. Sometimes you can show by reasoning that the problem has no solution. All the problems in Exercises 1-1 given below can be solved by reasoning. No calculations are needed, although you may find it helpful sometimes to draw a diagram.

Exercises 1-1

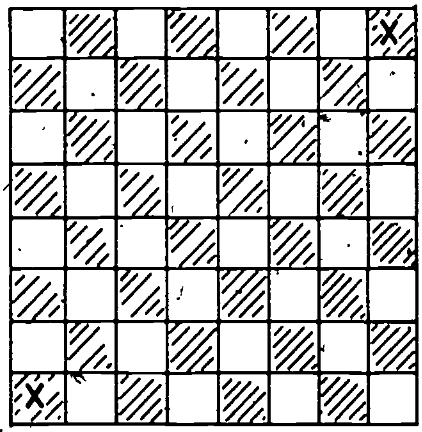
1. A 200-pound man and his two sons each weighing 100 pounds want to cross a river. If they have only one boat and it can safely carry only 200 pounds, how can they cross the river?
2. If the man in Problem 1 weighs 175 pounds and one of his sons weighs 125 pounds and the other 100 pounds, can they use the same boat? If not, what weight must the boat carry safely so that they can cross the river?
3. A farmer wants to take a goose, a fox, and a bag of corn across a river. If the farmer is not with them, the fox will eat the goose or the goose will eat the corn. If the farmer has only one boat large enough to carry him and one of the others, how does he cross the river?
4. Is it possible to measure out exactly 2 gallons using only an 8-gallon container and a 5-gallon container? The containers do not have individual gallon markings or any other markings.

*5. Three cannibals and three missionaries want to cross a river. They must share a boat which is large enough to carry only two people. At no time may the cannibals outnumber the missionaries, but the missionaries may outnumber the cannibals. How can they all get across the river using only the boat?

*6. Eight marbles all have the same size, color, and shape. Seven of them have the same weight and the other is heavier. Using a balance scale, how would you find the heavy marble if you make only two weighings?

*7. Suppose you have a checkerboard and dominoes. Each domino is just large enough to cover two squares on the checkerboard. How would you place the dominoes flat on the board in such a way as to cover all the board except two opposite corner squares?

Note: All the squares except the two squares in opposite corners are to be covered.
You may choose to leave the two white squares in opposite corners uncovered instead of the dark or shaded squares marked "x."



1-2. Deductive Reasoning

You can solve other types of problems by deductive reasoning. Suppose there are thirty pupils in a classroom. Can you prove that there are at least two of them who have birthdays during the same month? You can prove this in many ways without knowing the birthdays of any of them. One way is to reason like this. Imagine twelve boxes, one for each month of the year. Imagine also that you write each pupil's name on a slip of paper and then put the slip into the proper box. If no box had more than one slip of paper, then there could not be more than 12 names in all. Since there are 30 names, then at least one box must contain more than one name.

Mathematicians are always eager to prove the best possible result. Here, you can use the same method to prove that at least three members of the class have birthdays in the same month. If no box had more than two slips of paper, then there could not be more than 24 names in all. Since there are 30 names, then at least three pupils have birthdays in the same month. Each problem in the next set of exercises can be solved by reasoning of the above type.

Exercises 1-2

1. Assume you have a set of five pencils which you are going to distribute among four persons. Describe how some one of them will get at least two pencils.
2. (a) Would you have to give at least two pencils to some one person if you were distributing ten pencils among six persons?
   (b) What can you say if you are distributing a dozen pencils among five persons?
3. What is the least number of students that could be enrolled in a school so that you can be sure that there are at least two students with the same birthday?
4. What is the largest number of students that could be enrolled in a school so that you can be sure that they all have different birthdays?
5. There are five movie houses in a town. What is the smallest number of people that would have to go to the movies to make certain that at least two persons will see the same show?
6. In problem 5, what is the largest number of people that would have to go to the movies so that you could be sure that no two of them will see the same show?
7. If 8 candy bars are to be distributed among [illegible] boys, how many boys can receive three candy bars if each boy is to receive at least one bar?

8. In a class of 32 students various committees are to be formed. No student can be on more than one committee. Each committee contains from 5 to 8 students. What is the largest number of committees that can be formed?

*9. What is the answer to problem 8 if every student can be on either one or two committees?

1-3. From Arithmetic to Mathematics

Another way in which mathematicians and other scientist solve problems is to make experiments and observations. This method is called the experimental method. Can you think of scientific problems which have been solved in this way?

Many times mathematicians experiment to discover a general way of solving an entire set of problems. After the general method has been discovered, they try to prove that it is correct by logical reasoning.

The part of mathematics which you know best is arithmetic. Often in arithmetic you can obtain results by experiment and by reasoning which can save you a lot of hard work and time spent in calculation.

When Karl Friedrich Gauss, a famous mathematician, was about 10 years old, his teacher wanted to keep the class quiet for a while. He told the children to add all the numbers from 1 to 100, that is 1 + 2 + 3 + ... + 100. (Note: To save writing all the numbers between 3 and 100, it is customary to write three dots. This may be read "and so on.") In about two minutes Gauss was up to mischief again. The teacher asked him why he wasn't working on the problem. He replied, "I've done it already." "Impossible!" exclaimed the teacher. "It's easy," answered Gauss. "First I wrote:

1 + 2 + 3 + 4 + ... + 100, then I wrote the numbers in reverse order:

100 + 99 + 98 + 97 + ... + 1, then I added each pair of numbers:

101 + 101 + 101 + 101 + ... + 101.

When I added, I got one hundred 101's. This gave me 100 x 101 = 10,100. But I used each number twice. For example, I added 1 to 100 at the beginning, and then I added 100 to 1 at the end. So I divided the answer 10,100 by 2. The answer is $\frac{10,100}{2}$ or 5,050."

Who was Karl Friedrich Gauss? When did he live?

Exercises 1-3

1. Add all the numbers from 1 to 5, that is, 1 + 2 + 3 + 4 + 5, using Gauss's method. Can you discover another short method different from the Gauss method?
2. Can Gauss's method be applied to the problem of adding the numbers: 0 + 2 + 4 + 6 + 8?
3. By a short method add the odd numbers from 1 to 15, that is 1 + 3 + 5 + ... + 15.
4. This problem gives you a chance to make another discovery in mathematics.

	Add the numbers below:	Multiply the numbers below:
a.	1 + 3 = ?	2 × 2 = ?
b.	1 + 3 + 5 = ?	3 × 3 = ?
c.	1 + 3 + 5 + 7 = ?	4 × 4 = ?

   d. Look at the sums and the products on the right. What seems to be the general rule for finding the sums of numbers on the left?

   e. Apply your new rule to 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15. Check with your answer in Exercise 3.

5. Add the odd numbers: 7 + 9 + 11 + ... + 17.

6. Add the even numbers from 22 to 40: 22 + 24 + 26 + ... + 40.
7. Add all the numbers from 0 to 50: 0 + 1 + 2 + ... + 50.
8. Will Gauss's method work for any series of numbers whether you start with 0 or 1, or any other number?
*9. Add all the numbers from 1 to 200 by Gauss's method. Then add all the numbers from 0 to 200 by Gauss's method. Are the answers for these two problems the same? Why?
*10. Suppose Gauss's teacher had asked him to add the numbers: 1 + 2 + 4 + 8 + 16 + ... + 256. (Here you start with 1 and double each number to get the one which follows.) Is there any short method for getting the sum?
*11. If you have a short method for doing Problem 10, try this one. Add the numbers: 2 + 6 + 18 + ... + 486, where you start with 2 and multiply each number by 3 to get the one which follows. The answer is 728. Can you find a short method for getting the sum?

1-4. Kinds of Mathematics

Mathematicians reason about all sorts of puzzling questions and problems. When they solve a problem they usually create a little more mathematics to add to the ever-increasing stockpile of mathematical knowledge. The new mathematics can be used with the old to solve even more difficult problems. This process has been going on for centuries and the total accumulation of mathematics is far greater than most people can imagine. Arithmetic is one kind of mathematics. The trigonometry, algebra, and plane geometry you will study are other kinds.

Today there are more than eighty different kinds of mathematics. No single mathematician can hope to master more than a small bit of it. Indeed the study of any one of these eighty different branches would occupy a mathematical genius throughout his entire life.

Moreover, hundreds of pages of new mathematics are being created every day of the year -- much more than one person could possibly read in the same day. In fact, in the past 50 years, more mathematics has been discovered than in all the preceding thousands of years of man's existence.

Chance or Probability

One of the eighty kinds of mathematics which is interesting to mathematicians and also useful in present day problems is the study of chance or probability.

Many happenings in our lives are subject to uncertainty, or chance. There is a chance the fire alarm may ring as you read this sentence. There is a chance that the teacher may give a test today. It is difficult to predict exactly the chance of any of these events, although in such cases we would be satisfied to know that the chance is very small.

Many times, however, a person might want to calculate the chance more precisely in order to compare the chances of several alternatives. Mathematicians have been studying such questions for many years, and this kind of mathematics is called probability.

A number of such probability questions were answered for Walt Disney before Disneyland was built. When he considered setting up Disneyland, Disney wanted to know how big to build it, where to locate it, what admission to charge, and what special facilities to provide for holidays. He didn't want to take a chance on spending $17,000,000 to build Disneyland without knowing something of the probability of success.

What he really wanted answered was this type of question: If I build a certain type of facility, at this particular location, and charge so much admission, then what is the probability that I will make a certain amount of money?

Disney went to the Stanford Research Institute. There he talked with a group of mathematically trained people who are specialists in applying mathematical reasoning to business problems.

The people at Stanford first collected statistics about people--(their income, travel habits, amusement preferences, number of children, etc.). Combining this information by mathematical reasoning they predicted the probability that people would come to a certain location and pay a given price of admission. From reasoning like this they could predict the probability of having a successful Disneyland of a certain type at a given spot. Knowing the chance of success under given conditions, Disney was better able to decide how and where to build Disneyland and how much to charge for admission.

This example is typical of the way probability is often used to give an estimate of the degree of uncertainty of an event or the chance of success of a proposed course of action.

The following problems are mainly to give you some idea of what simple probability is about.

Exercises 1-4

1. To see how a mathematician might begin to think about probability, imagine that you toss two coins. There are four equally likely ways that the coins can come up:

First Coin	H	H	T	T
Second Coin	H	T	H	T

H is used to represent heads and T is used to represent tails. HH describes the event of both coins coming up heads. The probability of tossing 2 heads with 2 coins is 1 out of 4 or $\frac{1}{4}$. It is not possible to predict what will happen in any one toss, but it is possible to predict that if the two coins are tossed 100,000 times or so, then the two heads will come up about $\frac{1}{4}$ of the time. Try this experiment 100 times and tabulate your results. Tabulate other experiments of the whole class and see how many HH trials turn up in the total number of trials.

2. What is the probability that both coins show tails when two coins are tossed?
3. What is the probability that a head appears when one coin is tossed?
4. What is the probability of drawing the ace of spades from a full deck of 52 cards?
5. What is the probability of drawing some ace from a full deck of 52 cards?
6. What is the probability of throwing a die (one of a pair of dice) and having two dots come up?
7. There are four aces (from a playing card deck of 52 cards) to be dealt to four people. What is the probability that the first person who receives a card gets the ace of hearts?

*8. What is the probability of throwing a pair of ones (one dot on each die) with one pair of dice?

*9. What is the probability of having three heads come up if 3 coins are tossed? What is the probability of having exactly 2 heads turn up? At least 2 heads?

1-5 Mathematics Today

You are living in a world which is changing very rapidly. There are new medicines and vaccines. There are new ways to make business decisions. There are new ways of computing. And, there are hundreds of other new developments reported every day. The interesting thing is that mathematics and mathematicians have had a part in almost all of them.

In the telephone industry mathematics is used to design switching devices, so that when you dial a phone number you have a good chance of avoiding a busy signal. Mathematics has contributed especially in discovering better ways to send information over telephone wires or by wireless communication.

In the aircraft industry mathematics helps determine the best shape for an airplane or space ship, and how strong its construction must be. Another kind of mathematics predicts whether a plane will shake itself to pieces as it flies through stormy air at high speeds. Still different forms of mathematics help design the radio and radar devices used to guide the plane and to communicate with other planes and with airfields.

In almost all kinds of manufacturing, mathematics (the probability you studied in the last section) is used to predict the reliability of the goods manufactured. Many times the manufacturer must make a guarantee of reliable performance based on a mathematical prediction. If the mathematician makes a mistake the manufacturer loses money (and the mathematician may lose his job!).

Other kinds of new mathematics help business men decide how much to produce, how best to schedule production to avoid paying too much overtime, and where to build new plants in order to reduce shipping costs.

In the petroleum industry mathematics is used extensively in deciding how many oil wells to sink and where to drill to get the most oil from an oil field at the least cost. Mathematical techniques also help the gasoline manufacturer to decide how much gasoline of various qualities to refine from different crude petroleums.

In all these businesses and industries, in universities and government agencies as well, the mathematics of computing and the big new electronic computers are widely used.

Why is mathematics now used in so many areas? One reason is that mathematical reasoning, and the kinds of mathematics already developed, provide a precise way to describe complicated situations and to analyze difficult problems. The language of mathematics is expressed in shorthand symbols, all precisely defined, and used according to definite logical rules. This often makes it possible to study problems too complicated to visualize. Frequently, mathematical reasoning predicts the possibility or impossibility of a scientific experiment. Sometimes, the most useful answer a mathematician can find is to prove beyond question that the problem (or machine, or system, or experiment) being studied is impossible. The mathematical work may also show why the problem is impossible in the present form and may suggest a way to get around the difficulties.

1-6. Mathematics as a Vocation

Before World War II almost all mathematicians were employed as teachers ~~in~~ schools and colleges. Since then, the world of mathematics and the world of mathematicians have changed tremendously. Today there are more teachers of mathematics than ever before. In junior and senior high school there are about 50,000 people who teach mathematics. There are about 5,000 more teachers employed in colleges and universities. But now (1960), in business, industry, and government there are more than 20,000 persons working as mathematicians.

The Federal Government hires mathematicians in numerous agencies for many different assignments. Literally thousands of people work with computers and computer mathematics. Industries of all types are hiring mathematicians to solve complex mathematical problems, ~~to~~ help other workers with mathematical difficulties and even to teach mathematics to other employees.

These changes have been brought about by the revolutionary advances in science and technology which were talked about. These changes are still continuing.

1-7. Mathematics in Other Vocations

Many people who are not primarily mathematicians need to know a lot of mathematics, and use it almost every day. This has long been true of engineers and physicists. Now they find it necessary to use even more advanced mathematics. Every new project in aircraft, in space travel, or in electronics demands greater skills from the engineers, scientists, and technicians.

Mathematics is now being widely used and required in fields such as social studies, medical science, psychology, geology, and business administration. Mathematical reasoning and many kinds of mathematics are useful in all these fields. Much of the use of electronic computers in business and industry is carried on by people who must learn more about mathematics and computing in order to carry on their regular jobs. To work in many such jobs you are required to know a lot about mathematics. Merely to understand these phases of modern life, and to appreciate them enough to be a good citizen, you need to know about mathematics.

1-8. Mathematics for Recreation

Just as music is the art of creating beauty with sounds, and painting is the art of creating beauty with colors and shapes, so mathematics is the art of creating beauty with combinations of ideas. Many people enjoy mathematics as a fascinating hobby. Many people study mathematics for fun just as other people enjoy music or painting for pleasure.

For many thousands of years people have enjoyed working various kinds of problems. A good example of this is a problem concerning the Konigsberg Bridges. In the early 1700's the city of Konigsberg, Germany was connected by seven bridges. Many people in the city at that time were interested in finding if it were possible to walk through the city so as to cross each bridge exactly once.

From the diagram showing these bridges, can you figure out whether or not this can be done? You may be interested in knowing that a mathematician answered this question in the year 1736.

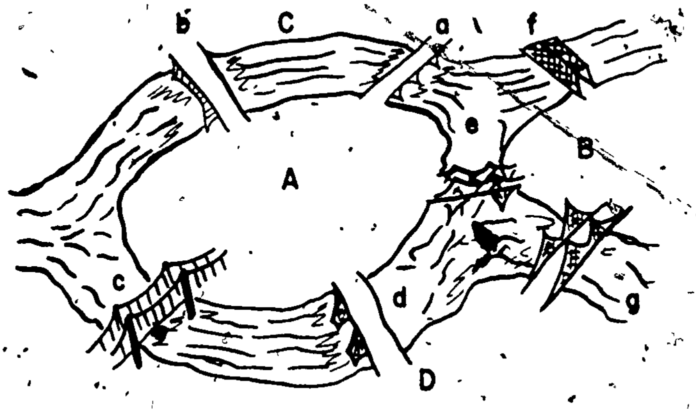


Figure 1-8a

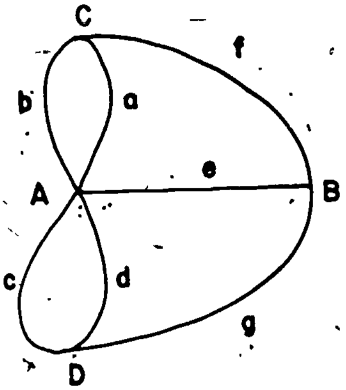


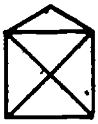
Figure 1-8b

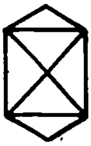
Figure 1-8b will help you see the various ways of walking through the city using the bridges to go from one piece of land to another. Use C in place of the piece of land to the north and D, the land to the south. A is the island and B is the land to the east. The lines leading from A, C, D, and B show routes across the bridges to the various parts of the city. The bridges are lettered a, b, c, d, e, f, g. At points B, C, and D three routes come together and at point A five routes meet.

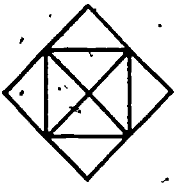
Several kinds of mathematics were developed by mathematicians as a result of their work on such thought-provoking problems as this one.

Exercise 1-8

*1. Of the three figures shown, two can be drawn without lifting your pencil or retracing a line, while the other cannot. Which two can be drawn without going over any lines twice and without crossing any lines?







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Chapter 2

NUMERATION

2-1. Caveman's Numerals

In primitive times, men were probably aware of simple numbers in counting, as in counting "one deer" or "two arrows". Primitive peoples learned to use numbers to keep records. Sometimes they tied knots in a rope, or used a pile of pebbles, or cut marks in sticks to represent the number of objects counted. A boy counting sheep would have  pebbles, or he might make notches in a stick, as . One pebble, or one mark in the stick would represent a single sheep. The same kind of record is made when votes in a class election are tallied as . When people began to make marks for numbers, by making scratches on a stone or in the dirt, or by cutting notches in a stick, they were writing the first numerals. Numerals are symbols for numbers. Thus the numeral "7" is a symbol for the number seven. Numeration is the study of how symbols are written to represent numbers.

As centuries passed, early people used sounds, or names, for numbers. Today a standard set of names for numbers is used. A rancher counting sheep compares a single sheep with the name "one," and 2 sheep with the name "two" and so on. Man now has both symbols (1, 2, 3, . . .) and words (one, two, three, . . .) which may be used to represent numbers.

Ancient Number Systems

One of the earliest systems of writing numerals of which there is any record is the Egyptian. Their hieroglyphic, or picture, numerals have been traced as far back as 3300 B.C. Thus, about 5000 years ago, Egyptians had developed a system with which they could express numbers up to millions. Egyptian symbols are shown on the following page.

Our Number	Egyptian Symbol	Object Represented
1	𓏤	stroke or vertical staff
10	𓎆	heel bone
100	𓍢	coiled rope or scroll
1000	𓆼	lotus flower
10000	𓂭	pointing finger
100,000	𓆐	burbot fish (or polliwog)
1,000,000	𓁨	astonished man

These symbols were carved on wood or stone. The Egyptian system was an improvement over the caveman's system because it used these ideas:

1. A single symbol could be used to represent the number of objects in a collection. For example, the heel bone represented the number ten.

2. Symbols were repeated to indicate other numbers. The group of symbols 𓍢𓍢𓍢 meant 100 + 100 + 100 or 300.

3. This system was based on groups of ten. Ten strokes make a heelbone, ten heelbones make a scroll, and so on.

The following table shows how Egyptians wrote numerals:

Our numerals	4	11	23	20,200	1959
Egyptian numerals	𓏤𓏤𓏤𓏤	𓎆𓏤	𓎆𓎆𓏤𓏤𓏤	𓂭𓂭𓍢𓍢	𓆼 𓍢𓍢𓍢 𓍢𓍢𓍢 𓍢𓍢𓍢 𓎆𓎆𓎆 𓎆𓎆 𓏤𓏤𓏤 𓏤𓏤𓏤 𓏤𓏤𓏤

About 2000 B.C., the Babylonians wrote on cuneiform tablets made of clay with a triangular stylus. With the stylus a mark 𒁹 was made on the clay to represent the number "one". By turning the stylus, they made this symbol 𒌋 for "ten". They combined these symbols to write numerals up to 59 as shown in the table below:

Our numerals	5	13	32	59
Babylonian numerals	𒁹𒁹𒁹 𒁹𒁹	𒌋𒁹𒁹𒁹	𒌋𒌋𒌋𒁹𒁹	𒌋𒌋𒌋 𒁹𒁹𒁹 𒁹𒁹𒁹 𒌋𒌋 𒁹𒁹𒁹

Numbers above 59 are difficult since the Babylonians developed a crude place value but had no zero symbol by which missing groups could be indicated. One and sixty are represented by the same symbol.

The Roman system was used for hundreds of years. There are still a few places at the present time where these numerals are used. Dates on cornerstones and chapter numbers in books are often written in Roman Numerals.

Historians believe that the Roman numerals came from pictures of fingers, like this: |, ||, |||, and ||||. The Romans then used a hand for five,  Gradually some of the marks were omitted, and they wrote V for five. Two fives put together made the symbol for ten, X. The other symbols were letters of their alphabet. The following table shows the other letters used by the Romans:

Our Numeral	1	5	10	50	100	500	1000
Roman Numeral	I	V	X	L	C	D	M

In early times the Romans repeated symbols to make larger numbers in the same way that the Egyptians had done many years before. Later, the Romans made use of a subtraction convention in order to represent a number with fewer symbols.

The values of the Roman symbols are added when a symbol representing a larger number is placed to the left in the numeral.

MDCLXVI = 1,000 + 500 + 100 + 50 + 10 + 5 + 1 = 1666.

DLXI = 500 + 50 + 10 + 1 = 561

When a symbol representing a smaller value is written to the left of a symbol representing a larger value, the smaller value is subtracted from the larger.

IX = 10 - 1 = 9.

XC = 100 - 10 = 90.

The Romans had restrictions on subtracting:

1. V, L and D (symbols representing numbers that start with 5) are never subtracted.
2. A number may be subtracted only in the following cases:
   - I can be subtracted from V and X only.
   - X can be subtracted from L and C only.
   - C can be subtracted from D and M only.

Addition and subtraction can both be used to write a number. First, any number whose symbol is placed to show subtraction is subtracted from the number to its right; second, the values found by subtraction are added to all other numbers in the numeral.

$$CIX = 100 + (10 - 1) = 100 + 9 = 109$$

$$MCMLX = 1{,}000 + (1{,}000 - 100) + 50 + 10$$
$$= 1{,}000 + 900 + 50 + 10 = 1960.$$

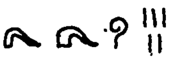
Sometimes the Romans wrote a bar over a number. This multiplied the value of the symbol by 1,000.

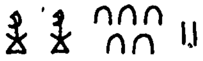
$\overline{X} = 10{,}000$, $\overline{C} = 100{,}000$, and $\overline{XXII} = 22{,}000$.

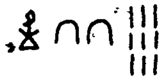
There were many other number systems used throughout history: the Korean, Chinese, Japanese, and Indian systems in Asia; the Mayan, Incan, and Aztec systems of the Americas; the Hebrew, Greek, and Arabian systems in the Mediterranean regions.

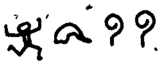
Exercises 2-1

1. Write the following numbers using Egyptian numerals:
   a. 19 b. 53 c. 666 d. 1960
2. The Egyptians usually followed a pattern in writing large numbers. However, the meanings of their symbols were not changed if the order in a numeral was changed. In what different ways can twenty-two be written in Egyptian notation?
3. Write our numerals for each of the following numbers:

a. 

b. 

c. 

d. 

4. Write the following numbers in Babylonian numerals:
   a. 9 b. 22 c. 51
5. Write our numerals for each of the following numbers:
   a.  b.  c. 
6. Write our numerals for each of the following numbers:
   a. XXIX b. LXI c. XC d. CV e. DCLXVI
   f. MCDXCII
7. Write the following numbers in Roman numerals:
   a. 19 b. 57 c. 888 d. 1690
   e. 15,000
8. Write each of these pairs of numbers in our notation, then add the results and change the answer to Roman numerals.
   a. MDCCIX and DCLIV
   b. MMDCXL and MCDVIII
9. Explain the difference in meaning of the Roman III and the decimal notation, 111.

2-2. The Decimal System

History and Description

All of the early number systems are an improvement over matching notches or pebbles. It is fairly easy to represent a number in any of them. It is difficult to use them to add and multiply. Some instruments, like the abacus, were used by ancient peoples to do arithmetic problems.

The way numerals are written in our number system was developed in India. Arab scholars learned about these numerals and carried them to Europe. Because of this, our numerals are called Hindu-Arabic numerals. It is interesting to note that most Arabs have never used these symbols. Because our system uses groups of ten, it is called a decimal system. The word decimal comes from the Latin word "decem", which means "ten".

The decimal system is used in most of the world today because it is a better system than the other number systems discussed in the previous section.

Long ago man learned that it was easier to count large numbers of objects by grouping the objects. The same idea is used today when a dime is used to represent ten pennies and a dollar is used to represent a group of ten dimes. Probably the reason that a numeral system is used in which objects are grouped in tens is that people have ten fingers. This accounts for the fact that the ten symbols are called "digits" when they are used as numerals. With the ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 any whole number can be written.

Other ways of grouping have been used. The Celts in Europe and the Mayans in Central America both used groups of twenty. Some Eskimo tribes use groups of five.

The decimal system uses the idea of place value to represent the size of a group. The size of the group represented by a symbol depends upon the position of the symbol or digit in a numeral. The symbol tells us how many of that group there are. In the numeral 123, the "1" represents one group of one hundred; the "2" represents two groups of ten, or twenty; and the "3" represents three ones, or three. This clever idea of place value makes the decimal system the most convenient system in the world.

Since grouping is by tens in the decimal system, its <u>base</u> is ten. Because of this, each successive (or next) place to the left represents a group ten times that of the preceding place. The first place tells us how many groups of one. The second place tells us how many groups of ten, or ten times one (10 × 1). The third place tells us how many groups of ten times ten (10 × 10), or one hundred; the next, ten times ten times ten (10 × 10 × 10), or one thousand, and so on. By using a base and the ideas of place value, it is possible to write any number in the decimal system using only the ten basic symbols. There is no limit to the size of numbers which can be represented by the decimal system.

To understand the meaning of the number represented by a numeral such as 123, you can add the numbers represented by each symbol. Thus 123 means (1 × 100) + (2 × 10) + (3 × 1), or 100 + 20 + 3. The same number is represented by 100 + 20 + 3 and by 123. When you write a numeral such as 123 you are using number symbols, the idea of place value, and base ten.

One advantage of our decimal system is that it has a symbol for zero. Zero is used to fill places which would otherwise be empty and might lead to misunderstanding. The numeral for three hundred-seven is 307. Without a symbol for zero it might be necessary to write 3-7. The meaning of 3-7 or 3 7 might be confused. The origin of the idea of zero is uncertain, but the Hindus were using a symbol for zero about 600 A.D., or possibly earlier.

The clever use of place value and the symbol for zero makes the decimal system one of the most efficient systems in the world. Pierre Simon Laplace (1749 - 1827), a famous French mathematician, called the decimal system one of the world's most useful inventions.

Expanded Numerals and Exponential Notation

The decimal system of writing numerals has a base ten. Starting at the units place, each place to the left has a value ten times as large as the place to its right. The first six places from the right to the left are shown below:

hundred thousand	ten thousand	thousand	hundred	ten	one
(10×10×10×10×10)	(10×10×10×10)	(10×10×10)	(10×10)	(10)	(1)

Often these values are written more briefly, by using an exponent above the 10. This exponent shows how many 10's are multiplied. Numbers that are multiplied are called factors. In this way, the values of the places are written and read as follows:

(10 × 10 × 10 × 10 × 10)	10^5	"ten to the fifth power"
(10 × 10 × 10 × 10)	10^4	"ten to the fourth power"
(10 × 10 × 10)	10^3	"ten to the third power"
(10 × 10)	10^2	"ten to the second power"

(10)	10^1	"ten to the first power"
(1)	1	"one"

In an expression as 10^2, the number 10 is called the base and the number 2 is called the exponent. The exponent tells how many times the base is taken as a factor in a product. 10^2 indicates (10×10) or 100. A number such as 10^2 is called a power of ten, and in this case it is the second power of ten. The exponent is sometimes omitted for the first power of ten; 10 is usually written, instead of 10^1. All other exponents are always written. Another way to write $(4 \times 4 \times 4)$ is 4^3, where 4 is the base, and 3 is the exponent.

How can the meaning of "352" be written with exponents?

$352 = (3 \times 10 \times 10) + (5 \times 10) + (2 \times 1)$

$= (3 \times 10^2) + (5 \times 10^1) + (2 \times 1)$. This is called expanded notation.

Exercises 2-2

1. Write each of the following using exponents.
   a. $3 \times 3 \times 3 \times 3 \times 3$
   b. $2 \times 2 \times 2 \times 2$
   c. $5 \times 5 \times 5 \times 5 \times 5 \times 5$
   d. 4×4
2. Five is used as a factor how many times in each of the following?
   a. 5^3
   b. 5^7
   c. 5^{10}
3. Write each of the following without exponents as $2^3 = 2 \times 2 \times 2$.
   a. 4^3
   b. 3^4
   c. 33^5
4. What does an exponent denote?
5. Write each of the following expressions as shown in the example: 4^3 means $4 \times 4 \times 4 = 64$.
   a. 3^3
   b. 5^2
   c. 4^4
   d. 2^5
   e. 6^2
   f. 7^3

6. Which numeral represents the larger number?
   a. 4^3 or 3^4 b. 2^9 or 9^2

7. Write the following numerals in expanded notation as shown in the example: $210 = (2 \times 10^2) + (1 \times 10^1) + (0 \times 1)$
   a. 468 c. 59.126
   b. 5324 d. 109,180

8. What is the relation between the exponent of a power of 10 and the zeros when that number is written in decimal notation?

9. Write the following numerals with exponents:
   a. 1000 c. 1,000,000
   b. 100,000 d. one hundred million

10. A mathematician was talking to a group of arithmetic students one day. They talked about a large number which they decided to call a "googol". A googol is 1 followed by 100 zeros. Write this with exponents.

*11. What is the meaning of 10^2? of 10^1? What do you think should be the meaning of 10^0?

2-3. Numerals in Base Seven

Some of the characteristics of decimal numerals may have escaped your notice simply because the numerals are so familiar. A study of a system of notation with a different base should increase your understanding of decimal numerals.

Let us see how to write numerals in base seven notation. This time it is necessary to work with groups of seven. Look at the x's below and notice how they are grouped in sevens with some x's left over.

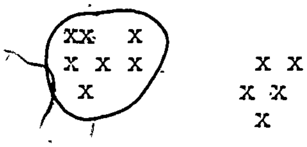


Figure 2-3-a

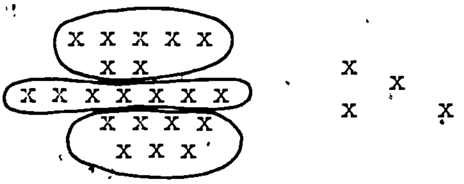


Figure 2-3-b

In figure 2-3-a there is one group of seven and five more. The numeral is written 15_{seven}. In this numeral, the I indicates that there is one group of seven, and the 5 means that there are five ones.

In figure 2-3-b how many groups of seven are there? How many x's are left outside the groups of seven? The numeral representing this number of x's is 34_{seven}. The 3 stands for three groups of seven, and the 4 represents four single x's or four ones. The "lowered" seven merely shows that the base is seven.

When grouping is in sevens the number of individual objects left can only be zero, one, two, three, four, five, or six. Symbols are needed to represent those numbers. Suppose the familiar 0, 1, 2, 3, 4, 5 and 6 are used for these rather than new symbols. As you will discover, no other symbols are needed for the base seven system.

If the x's are marks for days, 15_{seven} is a way of writing 1 week and five days. In our decimal system this number of days is "twelve" and is written "12" to show one group of ten and two more. The base name in our numerals is not written since everyone knows what the base is.

The name "fifteen" is not used for 15_{seven} because fifteen is 1 ten and 5 more. 15_{seven} shall be read as "one, five, base seven".

Notice that in base ten, one, two, three, four, five, six, seven, eight and nine are represented by single symbols. How is the base number "ten" represented? This representation, 10, means one group of ten and zero more.

With this idea in mind, think about counting in base seven. Try it yourself and compare with the following table, filling in the numerals from 21_{seven} to 63_{seven}. In this table the "lowered" seven is omitted.

Counting in Base Seven

Number	Symbol	Number	Symbol
one	1	one, four	14
two	2	one, five	15
three	3	one, six	16
four	4	two, zero	20
five	5	two, one	21
six	6	--------	--
one, zero	10	six, three	63
one, one	11	six, four	64
one, two	12	six, five	65
one, three	13	six, six	66

What would the next numeral after 66_{seven} be? Here you would have 6 sevens and 6 ones plus another one. This equals 6 sevens and another seven, that is, seven sevens. How could (seven)^2 be represented without using a new symbol? A new group is introduced, the (seven)^2 group. This number would then be written 100_{seven}. What does the number really mean? Go on from this point and write a few more numbers. What would be the next numeral after 666_{seven}?

Now you are ready to write a list of place values for base seven.

Place Values in Base Seven

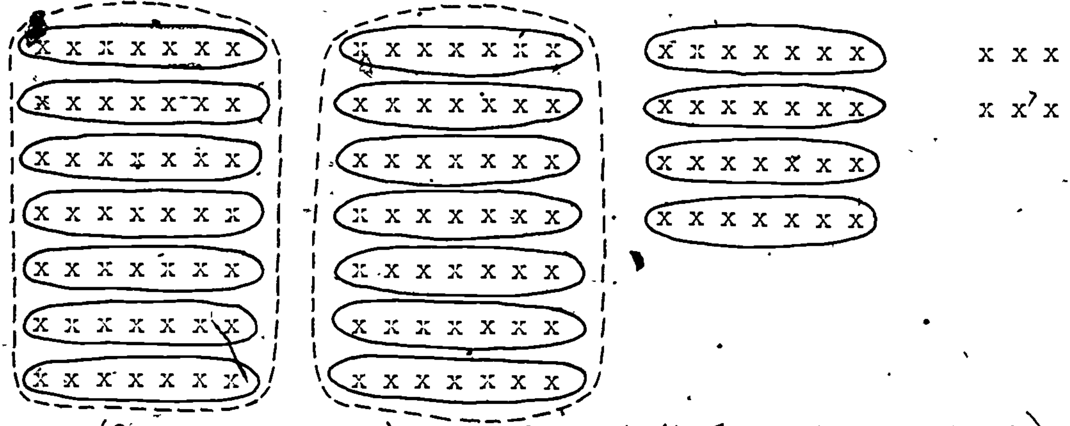
(seven)^5	(seven)^4	(seven)^3	(seven)^2	(seven)^1	(one)

Notice that each place represents seven times the value of the next place to the right. The first place on the right is the <u>one place</u> in both the decimal and the seven systems. The value of the second place is the <u>base</u> <u>times</u> <u>one</u>. In this case what is it? The value in the third place from the right is (seven × seven), and in the next place (seven × seven × seven).

What is the decimal name for (seven × seven)? It is necessary to use this (forty-nine) when changing from base seven to base ten. Show that the decimal numeral for (seven)^3 is 343. What is the decimal numeral for (seven)^4?

Using the chart above, you can see that $246_{seven} = (2 \times seven \times seven) + (4 \times seven) + (6 \times one)$.

The diagram shows the actual grouping represented by the digits and the place values in the numeral 246_{seven}:



(2 × seven × seven) (4 × seven) (6 × one)

If you wish to write the number of x's above in the decimal system of notation you may write:

$$\begin{aligned} 246_{seven} &= (2 \times 7 \times 7) + (4 \times 7) + (6 \times 1) \\ &= (2 \times 49) + (4 \times 7) + (6 \times 1) \\ &= 98 + 28 + 6 \\ &= 132_{ten} \end{aligned}$$

Regroup the x's above to show that there are 1 (ten × ten) group, 3(ten) groups, and 2 more. This should help you understand that $246_{seven} = 132_{ten}$.

Exercises 2-3

1. Group the x's below and write the number of x's in base seven notation:

a. x x x x x
x x x x x
x x x x
x x x x

b. x x x x x x x x x
x x x x x x x x x
x x x x x x x x x
x x x x x x x x x
x x x x x x x x x
x x x x x x x x x
x x x x x x x x

2. Draw x's and group them to show the meaning of the following numerals.

a. 26_{seven} b. 101_{seven}

3. Write each of the following numerals in expanded form and then in decimal notation.

   a. 33_{seven} b. 524_{seven}

4. Write the next consecutive numeral after each of the following numerals.

   a. 6_{seven} c. 666_{seven}

   b. 10_{seven} d. 1006_{seven}

5. What is the value of the "6" in each of the following numerals?

   a. 560_{seven} b. 605_{seven} c. 6050_{seven}

6. In the base seven system write the value of the fifth place counting left from the units place.

*7. What numeral in the seven system represents the number named by six dozen?

8. Which number is greater, 250_{seven} or 205_{ten}?

9. Which is smaller, 2125_{seven} or 754_{ten}?

10. A number is divisible by ten if a remainder of zero is obtained when the number is divided by ten.

    a. Is 30_{ten} divisible by ten? Why?

    b. Is 241_{ten} divisible by ten? Why?

    c. How can you tell by glancing at a base ten numeral whether the numer is divisible by ten?

*11. a. What would the phrase "a counting number is divisible by seven" mean?

     b. Is 30_{seven} divisible by seven? Why?

*12. State a rule for determining when a number written in base seven is divisible by seven.

You should see from problems 10-12 that the way of determining whether a number is divisible by ten depends on the system in which it is written. The rule for divisibility by ten in the decimal system is similar to the rule for divisibility by seven in the base seven system.

*13. Is 11_{seven} an even number or an odd number? Can you tell simply by glancing at the following which represent even or odd numbers?

12_{seven}, 13_{seven}, 14_{seven}, 25_{seven}, 66_{seven}.

What could you do to tell?

Here again a rule for divisibility in base ten will not work for base seven. Rules for divisibility seem to depend on the base with which you are working.

*14. On planet X-101 the pages in books are numbered in order as follows: 1, ∠, Δ, □, ⧅, ⊠, 1-, 11, 1∠, 1Δ, 1□, 1⧅, 1⊠, ∠-, ∠1, and so forth. What seems to be the base of the numeration system these people use? Why? How would the next number after ∠1 be written? Which symbol corresponds to our zero? Write numerals for numbers from □- to ⧅Δ.

*15. Find a rule for determining when a number written in base seven is divisible by two.

2-4. Computation in Base Seven

Addition

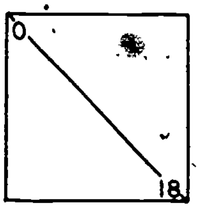
In the decimal, or base ten, system there are 100 "basic" addition combinations. In base seven, an addition table has only 49 basic combinations. Computation in base seven can be understood best by first making an addition table for base seven.

Exercises 2-4-a

1. Make a chart to show the basic sums when the numbers are written in base seven notation. Four sums are supplied to help you.

+	0	1	2	3	4	5	6
0							
1				4			
2					6		
3						11	
4							13
5							
6							

2. Draw a diagonal line from the upper left corner to the lower right corner of the chart as shown at the right.



   a. Is 3 + 4 the same as 4 + 3?
   b. How could the answer to part a) be determined from the chart?
   c. What do you notice about the two parts of the chart?
   d. What does this tell you about the number of <u>different</u> combinations which must be mastered?

e. Which would be easier, to learn the necessary multiplication combinations in base seven or in base ten? Why?

f. Find $4_{ten} + 5_{ten}$ and $4_{seven} + 5_{seven}$. Are the results equal; that is, do they represent the same number?

The answer to problem 2f is an illustration of the fact that a number is an idea independent of the numerals used to write its name. Actually, 9_{ten} and 12_{seven} are two different names, for the same number.

Do not try to memorize the addition combinations for base seven. The value in making the table lies in the help it gives you in understanding operations with numbers.

The table that you completed in problem 1 of the last set of exercises shows the sums of pairs of numbers from zero to six. Actually, little more is needed to enable us to add larger numbers. In order to see what else in needed, let us consider how to add in base ten. What are the steps in your thinking when you add numbers like twenty-five and forty-eight in the decimal notation?

$25 = 2$ tens $+ 5$ ones $= \longrightarrow 25$

$\underline{48} = \underline{4 \text{ tens} + 8 \text{ ones}} = \longrightarrow \underline{48}$

6 tens $+ 13$ ones $= 7$ tens $+ 3$ ones $= 73$

Try adding in base seven: $14_{seven} + 35_{seven}$

1 seven + 4 ones (You may look up the sums 5 + 4 and

$\underline{3 \text{ sevens} + 5 \text{ ones}}$ 3 + 1 in the base seven addition table.)

4 sevens + 12 ones = 5 sevens + 2 ones = 52_{seven}.

How are the two examples alike? How are they different? When is is necessary to "carry" (or regroup) in the ten system? When is it necessary to "carry" (or regroup) in the seven system?

Try your skill in addition on the following problems. Use the addition table for the basic sums.

42_{seven}	65_{seven}	32_{seven}	254_{seven}	435_{seven}	524_{seven}
$\underline{13}_{seven}$	$\underline{11}_{seven}$	$\underline{25}_{seven}$	$\underline{105}_{seven}$	$\underline{625}_{seven}$	$\underline{564}_{seven}$

The answers in order are 55_{seven}, 106_{seven}, 60_{seven}, 362_{seven}, 1363_{seven}, and 1421_{seven}.

Subtraction

How did you learn to subtract in base ten? You probably used subtraction combinations such as 14 - 5 until you were thoroughly familiar with them. You know the answer to this problem but suppose, for the moment, that you did not. Could you get the answer from the addition table? You really want to ask the following question "What is the number which, when added to 5, yields 14?"

The idea discussed above is used in every subtraction problem. One other idea is needed in many problems, the idea of "borrowing" or "regrouping". This last idea is illustrated below for base ten to find 761 - 283:

$$\begin{array}{lclcl} 7 \text{ hundreds} + 6 \text{ tens} + 1 \text{ one} & = & 6 \text{ hundreds} + 15 \text{ tens} + 11 \text{ ones} & = & 761 \\ \underline{2 \text{ hundreds} + 8 \text{ tens} + 3 \text{ ones}} & = & \underline{2 \text{ hundreds} + \ 8 \text{ tens} + \ 3 \text{ ones}} & = & \underline{283} \\ & & 4 \text{ hundreds} + \ 7 \text{ tens} + \ 8 \text{ ones} & = & 478 \end{array}$$

Now let us try subtraction in base seven. How would you find $6_{seven} - 2_{seven}$? Find $13_{seven} - 6_{seven}$. How did you use the addition table for base seven? Find answers to the following subtraction examples:

$$\begin{array}{ccccc} 15_{seven} & 12_{seven} & 11_{seven} & 15_{seven} & 13_{seven} \\ \underline{6}_{seven} & \underline{4}_{seven} & \underline{6}_{seven} & \underline{5}_{seven} & \underline{4}_{seven} \end{array}$$

The answers to these problems are 6_{seven}, 5_{seven}, 2_{seven}, 6_{seven}, and 6_{seven}.

Let us work a harder subtraction problem in base seven comparing the procedure with that used above:

$$\begin{array}{lclclcl} 43_{seven} & = & 4 \text{ sevens} + 3 \text{ ones} & = & 3 \text{ sevens} + 13 \text{ ones} & = & 43_{seven} \\ \underline{16_{seven}} & = & \underline{1 \text{ seven} \ + 6 \text{ ones}} & = & \underline{1 \text{ seven} \ + \ 6 \text{ ones}} & = & \underline{16_{seven}} \\ & & & & 2 \text{ sevens} + \ 4 \text{ ones} & = & 24_{seven} \end{array}$$

Be sure to note that "13 ones" above is in the seven system and is "one seven, three ones." If you wish to find the number you add to 6_{seven} to get 13_{seven}, how can you use the table to help you? Some of you may think of the number without referring to the table.

Practice on these subtraction examples:

$$\begin{array}{ccccc} 56_{seven} & 61_{seven} & 34_{seven} & 452_{seven} & 503_{seven} \\ \underline{14_{seven}} & \underline{35_{seven}} & \underline{26_{seven}} & \underline{263_{seven}} & \underline{140_{seven}} \end{array}$$

The answers are 42_{seven}, 23_{seven}, 5_{seven}, 156_{seven} and 233_{seven}.

Exercises 2-4-b

1. Each of the following examples is written in base seven. Add Check by changing the numerals to decimal notation and adding in base ten as in the example:

$$\begin{array}{ccc} \text{Base Seven} & & \text{Base Ten} \\ 16_{seven} & = & 13 \\ 23_{seven} & = & 17 \\ \overline{42}_{seven} & & \overline{30} \end{array}$$

Does $42_{seven} = 30$?

a. $\begin{array}{l} 25_{seven} \\ \underline{31_{seven}} \end{array}$ b. $\begin{array}{l} 56_{seven} \\ \underline{21_{seven}} \end{array}$

c. $160_{seven} + 430_{seven}$

d. $403_{seven} + 563_{seven}$

e. $6245_{seven} + 5314_{seven}$

f. $645_{seven} + 666_{seven}$

2. Use the base seven addition table to find:

a. $6_{seven} - 4_{seven}$ b. $11_{seven} - 4_{seven}$

c. $12_{seven} - 5_{seven}$

3. Each of the following examples is written in base seven. Subtract. Check by changing to decimal numerals.

   a. $\begin{array}{r} 10_{seven} \\ \underline{5_{seven}} \end{array}$ b. $\begin{array}{r} 65_{seven} \\ \underline{26_{seven}} \end{array}$

   c. $44_{seven} - 35_{seven}$

   d. $502_{seven} - 266_{seven}$

   e. $634_{seven} - 52_{seven}$

   f. $3451_{seven} - 2164_{seven}$

4. Show by grouping x's that:

   a. 4 twos = 11_{seven}

   b. 6 threes = 24_{seven}

Multiplication

In order to multiply, a table of basic multiplication facts is needed.

Exercises 2-4-c

1. Complete the multiplication table below for base seven. Suggestion: To find $4_{seven} \times 3_{seven}$ you could write four x's three times and regroup to show the base seven numeral. Better still, you might think of this as $3_{seven} + 3_{seven} + 3_{seven} + 3_{seven}$.

Multiplication. Base Seven

x	0	1	2	3	4	5	6
0							
1							
2					11	13	
3							
4				15			
5							
6							51

2. There are fewer entries in the base seven table than in the table for base ten. What does this fact tell you about the ease of learning multiplication in base seven?
3. Imagine the diagonal line drawn from the "×" sign to the lower right-hand corner of the last table.
   a. How are the entries above the diagonal line related to those below it?
   b. What fact does the observation of part a tell you about $3_{\text{seven}} \times 4_{\text{seven}}$?

Use the base seven multiplication table to find the following products.

$$\begin{array}{r} 52_{\text{seven}} \\ \times\ 3_{\text{seven}} \\ \hline \end{array} \qquad \begin{array}{r} 34_{\text{seven}} \\ \times\ 6_{\text{seven}} \\ \hline \end{array} \qquad \begin{array}{r} 421_{\text{seven}} \\ \times\ 4_{\text{seven}} \\ \hline \end{array} \qquad \begin{array}{r} 621_{\text{seven}} \\ \times\ 2_{\text{seven}} \\ \hline \end{array} \qquad \begin{array}{r} 604_{\text{seven}} \\ \times\ 5_{\text{seven}} \\ \hline \end{array}$$

The answers are 216_{seven}, 303_{seven}, 2314_{seven}, 252_{seven}, 31406_{seven}.

$$\begin{array}{r} 45_{\text{seven}} \\ \times\ 32_{\text{seven}} \\ \hline 123 \\ 201 \\ \hline 2133_{\text{seven}} \end{array}$$

Check the multiplication shown at the right and then answer the following questions. How do you get the entry 123 on the third line? How do you get the entry 201 on the fourth line? Why is the 1 on line 4 placed under the 2 on line 3? Why is the 0 on line 4 placed under the 1 on line 3? If you do not know why the entries on lines 3 and 4 are added to get the answer, you will study this more thoroughly later.

One way to check your work is to change the base seven numerals to base ten numerals as shown here:

$$\begin{array}{rll} 604_{\text{seven}} & = (6 \times 49) + (0 \times 7) + (4) = 294 + 4 = & 298_{\text{ten}} \\ \times\ 35_{\text{seven}} & = (3 \times 7) + (5) \qquad\quad = 21 + 5 = & \times\ 26_{\text{ten}} \\ \hline 4226 & & 1788 \\ 2415 & & 596 \\ \hline 31406_{\text{seven}} & = (3 \times 2401) + (1 \times 343) + (4 \times 49) + (0 \times 7) + (6) = & 7748_{\text{ten}} \end{array}$$

Division

Division is left as an exercise. You may find that it is not easy. Working in base seven should help you understand why some boys and girls have trouble with division in base ten. Here are two examples you may wish to examine. All the numerals within the examples are written in base seven. How can you use the multiplication table here?

Division in Base Seven

$$\begin{array}{rr} & 454_{seven} \\ 6_{seven} & \overline{)\,4053_{seven}} \\ & \underline{33}\phantom{00_{seven}} \\ & 45\phantom{0_{seven}} \\ & \underline{42}\phantom{0_{seven}} \\ & 33_{} \\ & \underline{33}_{} \end{array} \qquad \begin{array}{rr} & 2015_{seven} \\ 46_{seven} & \overline{)\,126125_{seven}} \\ & \underline{125}\phantom{000_{seven}} \\ & 112\phantom{00_{seven}} \\ & \underline{46}\phantom{00_{seven}} \\ & 335_{} \\ & \underline{332}_{} \\ & 3_{} \end{array}$$

Exercises 2-4-d

1. Multiply the following numbers in base seven numerals and check your results in base 10.
   - a. $14_{seven} \times 3_{seven}$
   - b. $63_{seven} \times 12_{seven}$
   - c. $56_{seven} \times 43_{seven}$
   - d. $3046_{seven} \times 24_{seven}$
   - e. $250_{seven} \times 341_{seven}$

*2. Divide. All numerals in this exercise are in base seven.
   - a. $6_{seven}\overline{)\,42_{seven}}$
   - b. $5_{seven}\overline{)\,433_{seven}}$
   - c. $4_{seven}\overline{)\,2316_{seven}}$
   - d. $21_{seven}\overline{)\,2625_{seven}}$

3. Write in expanded form:
   - a. 403_{seven}
   - b. 189_{ten}

4. Which of the numerals in Exercise 3 represents the larger number?

5. Add the following:
   - a. $52_{seven} + 14_{seven}$
   - b. $\begin{array}{r} 65_{seven} \\ \underline{25}_{seven} \end{array}$

c. 434_{seven}
$\underline{324_{seven}}$

6. Subtract the following:

a. 13_{seven}
$\underline{6_{seven}}$

b. $30_{seven} - 1_{seven}$

c. 402_{seven}
$\underline{35_{seven}}$

7. Rewrite the following paragraph replacing the base seven numerals with base ten numerals.

Louise takes grade 10_{seven} mathematics in room 234_{seven}. The book she uses is called <u>Junior High School Mathematics 10_{seven}</u>. It has 21_{seven} chapters and 1102_{seven} pages. There are 44_{seven} pupils in the class which meets 5_{seven} times each week for 106_{seven} minutes daily. 16_{seven} of the pupils are girls and 25_{seven} are boys. The youngest pupil in the class is 14_{seven} years old and the tallest is 123_{seven} inches tall.

2-5 Changing from Base Ten to Base Seven

You have learned how to change a number written in base seven numerals to base ten numerals. It is also easy to change from base ten to base seven. Let us see how this is done.

In base seven, the values of the places are: one, $seven^1$, $seven^2$, $seven^3$, and so on. That is, the place values are <u>one</u> and <u>the powers of seven</u>.

$$seven^1 = 7_{ten}$$
$$seven^2 = (7 \times 7) \text{ or } 49_{ten}$$
$$seven^3 = (7 \times 7 \times 7) \text{ or } 343_{ten}.$$

Suppose you wished to change 12_{ten} to base seven numerals. This time, <u>think</u> of groups of powers of seven instead of actually grouping marks. What is the largest power of seven which is contained in 12_{ten}? Is $seven^1$ the largest? How about $seven^2$ (forty-nine) or $seven^3$ (three hundred forty-three)?

You can see that only seven^1 is small enough to be contained in 12_{ten}.

When you divide 12 by 7 you have

$$\begin{array}{r} 1 \\ 7\overline{)12} \\ 7 \\ \hline 5 \end{array}$$

What does the 1 on top mean? What does the 5 mean? They tell us that 12_{ten} contains 1 seven with 5 units left over, or that $12_{\text{ten}} = (1 \times \text{seven}) + (5 \times \text{one})$. Thus $12_{\text{ten}} = 15_{\text{seven}}$.

Be sure you know which place in a base seven numeral has the value seven^2, the value seven^3, the value seven^4, and so on.

How is 54_{ten} regrouped for base seven numerals? What is the largest power of seven which is contained in 54_{ten}?

In 54_{ten} you have ___?___ × seven^2 + ___?___ × seven + ___?___ × one.

$$\begin{array}{r} 1 \\ 49\overline{)54} \\ 49 \\ \hline 5 \end{array}$$

You have (__1__ × seven^2) + (0 × seven) + (__5__ × one).
Then $54_{\text{ten}} = 105_{\text{seven}}$.

Suppose the problem is to change 524_{ten} to base seven numerals. Since 524_{ten} is larger than 343 (seven^3), find how many 343's there are.

$$\begin{array}{r} 1 \\ 343\overline{)524} \\ 343 \\ \hline 181 \end{array}$$

Thus 524 contains one seven^3 with 181 remaining, or $524 = (1 \times \text{seven}^3) + 181$, and there will be a "1" in the seven^3 place.

Now find how many 49's (seven^2) there are in the remaining 181.

$$\begin{array}{r} 3 \\ 49\overline{)181} \\ 147 \\ \hline 34 \end{array}$$

Thus 181 contains 3 49's with 34 remaining, or $181 = (3 \times \text{seven}^2) + 34$, and there will be a "3" in the seven^2 place.

How many sevens are there in the remaining 34?

$$7\overset{4}{\overline{)34}}$$
$$\underline{28}$$
$$6$$

Thus 34 contains 4 seven's with 6 remaining, or 34 = (4 × seven) + 6, and there will be a "4" in the sevens place.

What will be in the units place? You have:

$524_{ten} = (1 \times seven^3) + (3 \times seven^2) + (4 \times seven) + (6 \times one)$

$524_{ten} = 1346_{seven}$.

Cover the answers below until you have made the changes for yourself.

$10_{ten} = (1 \times seven) + (3 \times one) = 13_{seven}$

$46_{ten} = (6 \times seven) + (4 \times one) = 64_{seven}$

$162_{ten} = (3 \times seven^2) + (2 \times seven) + (1 \times one) = 321_{seven}$

$1738_{ten} = (5 \times seven^3) + (0 \times seven^2) + (3 \times seven) + (2 \times one)$
$= 5032_{seven}$.

In changing base ten numerals to base seven, first select the <u>largest</u> place value of base seven (that is, power of seven) contained in the number. Divide the number by this power of seven and find the quotient and remainder. The <u>quotient</u> is the first digit in the base seven numeral. Divide the remainder by the next smaller power of seven and this quotient is the second digit. Continue to divide remainders by each succeeding, smaller power of seven to determine all the remaining digits in the base seven numeral.

<u>Exercises 2-5</u>

1. Show that:

a. $50_{ten} = 101_{seven}$

b. $145_{ten} = 265_{seven}$

c. $1024_{ten} = 2662_{seven}$

2. Change the following base ten numerals to base seven numerals:

   a. 12 c. 218

   b. 36 d. 1320

Problems 3, 4, and 5 will help you discover another method for changing base ten numerals to base seven.

3. Divide 1958_{ten} by ten. What is the quotient? What is the remainder? Divide the quotient by ten. What is the new quotient? The new remainder? Continue in the same way, dividing each quotient by ten until you get a quotient of zero. How are the successive remainders related to the original number? Try the same process with $123,456,789_{ten}$. Try it with any other number.
4. Divide 524_{ten} by seven. What is the quotient? The remainder? Divide the quotient by seven and continue as in Exercise 3, except that this time divide by seven instead of ten. Now write 524_{ten} as a base seven numeral and compare this with the remainders which you have obtained.
5. Can you now discover another method for changing from base ten to base seven numerals?
6. In each of the examples below there are some missing numerals. Supply the numerals which will make the examples correct. Remember that if no base name is given, then the base is ten.

   a. Addition:
   $$\begin{array}{r} 675 \\ \underline{486} \\ ???? \end{array}$$

   b. Addition:
   $$\begin{array}{r} 8\ 9\ 4 \\ \underline{?\ ?\ ?} \\ 1\ 1\ 6\ 9 \end{array}$$

   c. Addition:
   $$\begin{array}{r} 4\ 3\ 2_{seven} \\ \underline{?\ ?\ ?_{seven}} \\ 1\ 4\ 1\ 6_{seven} \end{array}$$

   d. Addition:
   $$\begin{array}{r} 2\ 3\ 0\ 5_{seven} \\ \underline{?\ ?\ ?_{seven}} \\ 3\ 1\ 0\ 0_{seven} \end{array}$$

e. Addition:

$$\begin{array}{r} 2\,6\,4_{\text{seven}} \\ 3\,5\,2_{\text{seven}} \\ 1\,4\,0_{\text{seven}} \\ \hline \end{array}$$

f. Multiplication:

$$\begin{array}{r} 5\,1\,4_{\text{seven}} \\ \times \qquad ?_{\text{seven}} \\ \hline 2\,1\,4\,5_{\text{seven}} \end{array}$$

*g. Multiplication:

$$\begin{array}{r} ?\,?\,?_{\text{seven}} \\ \times \quad 5\,4_{\text{seven}} \\ \hline 3\,6\,2\,0\,1_{\text{seven}} \end{array}$$

2-6. Numerals in Other Bases

You have studied base seven numerals, so you now know that it is possible to express numbers in systems different from the decimal scale. Many persons think that the decimal system is used because the base "ten" is superior to other bases, or because the number ten has special properties. Earlier it was indicated that ten is probably used as a base because man has ten fingers. It was only natural for primitive people to count by making comparisons with their fingers. If man had had six or eight fingers, he might have learned to count by sixes or eights.

Our familiar decimal system of notation is superior to the Egyptian, Babylonian, and others because it uses the idea of place value and has a zero symbol, not because its base is ten. The Egyptian system was a tens system, but it lacked efficiency for other reasons.

Our decimal system uses ten symbols. In the seven system you used only seven symbols, 0, 1, 2, 3, 4, 5, and 6. How many symbols would Eskimos use counting in base five? How many symbols would base six require? A little thought on the preceding questions should lead you to the correct answers. Suggest how many symbols are needed for base twenty.

Write sixteen x's. Enclose them in groups of four x's. Can you write the numeral 16_{ten} in base four numerals? How many groups of four are there? Remember, you cannot use the symbol "4" in base four. A table of the powers of four in decimal numerals is shown below.

(four^3)	(four^2)	(four^1)	(one)
$(4 \times 4 \times 4)$	(4×4)	(4)	(1)
(64)	(16)	(4)	(1)

To write sixteen x's in base four you need $(1 \text{ group of four}^2) + (0 \text{ groups of four}) + (0 \text{ ones})$. That is, $16_{ten} = 100_{four}$.

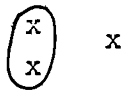
2-7. The Binary and Duodecimal Systems

There are two other bases of special interest. The base two, or binary, system is used by some modern, high speed computing machines. These computers, sometimes incorrectly called "electronic brains," use the base two. The twelve, or duodecimal, system is considered by some people to be a better base for a system of notation than ten.

Binary System

Historians tell of primitive people who used the binary system. Some Australian tribes still count by pairs, "one, two, two and one, two twos, two twos and one," and so on.

The binary system groups by pairs as is done with the three x's at the right. How many groups of two are shown? How many single x's are left? Three x's means 1 group of two and 1 one. In binary notation the numeral 3_{ten} is written 11_{two}.



Counting in the binary system starts as follows:

Decimal numerals	1	2	3	4	5	6	7	8	9	10
Binary numerals	1	10	11	100	101	110	111	?	?	?

How many symbols are needed for base two numerals? Notice that the numeral 101_{two} represents the number of fingers on one hand. What does 111_{two} mean?

$$111_{two} = (1 \times two^2) + (1 \times two^1) + (1 \times one) = 4 + 2 + 1 = 7_{ten}.$$

How would you write 8_{ten} in binary notation? How would you write 10_{ten} in binary notation? Compare this numeral with 101_{two}.

Modern high speed computers are electrically operated. A simple electric switch has only two positions, open (on) or closed (off). Computers operate on this principle. Because there are only two positions for each place, the computers use the binary system of notation.

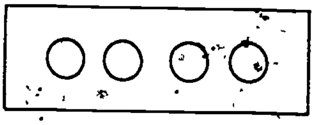


Figure 2-7a

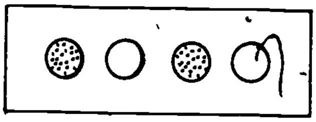


Figure 2-7b

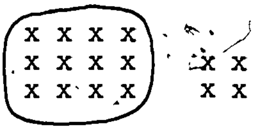
The drawing at the right is used to represent a computer. The four circles represent four lights on a panel, and each light represents one place in the binary system. When the current is flowing the light is on, shown in Figure 2-7b as ◍ . A ◍ is represented by the symbol "1". When the current does not flow, the light is off, shown by ○ in Figure 2-7b. This is represented by the symbol "0". The panel in Figure 2-7b represents the binary numeral 1010_{two}. What decimal numeral is represented by this numeral? The table at the right shows the place values for the first five places in base two numerals.

two^4	two^3	two^2	two^1	one
2x2x2x2	2x2x2	2x2	2	1
16	8	4	2	1

$$\begin{aligned} 1010_{two} &= (1 \times two^3) + (0 \times two^2) + (1 \times two^1) + (0 \times one) \\ &= (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1) \\ &= 10_{ten}. \end{aligned}$$

Duodecimal System

In the twelve, or duodecimal system, the grouping is by twelves. You frequently count in dozens, as with a dozen eggs, a dozen rolls, or a dozen pencils. Twelve dozen (12 × 12) is called a gross. Schools sometimes buy pencils by the gross.



The sixteen x's shown at the right are grouped as one group of twelve with four x's left. Written as a base twelve numeral,

$$16_{ten} = (1 \times twelve) + (4 \times one) = 14\ twelve.$$

Draw twenty-five x's on a sheet of paper. Draw circles around groups of twelve. How many groups of twelve are there? Are any x's left over? How would you write 25_{ten} in duodecimal notation?

$$25_{ten} = (2 \times twelve) + (1 \times one) = 21_{twelve}.$$

To write numerals in base twelve it is necessary to invent new symbols in addition to using the ten symbols from the decimal system. How many new symbols are needed? Base twelve requires twelve symbols, two more than the decimal system. You can use "T" for ten and "E" for eleven as shown in the table below:

Base ten	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Base twelve	0	1	2	3	4	5	6	7	8	9	T	E	10	11	12	?	?

Notice that "T" is another way of writing 10_{ten} and "E" is another way of writing 11_{ten}. Why is 12_{ten} written as 10_{twelve}? To write 195_{twelve} in expanded notation,

$$\begin{aligned} 195_{twelve} &= (1 \times twelve^2) + (9 \times twelve^1) + (5 \times one) \\ &= (1 \times 144) + (9 \times 12) + (5 \times 1) \\ &= 257_{ten}. \end{aligned}$$

Exercises 2-7

1. a. How many threes are there in 20_{three}?
   b. How many fours are there in 20_{four}?
   c. How many fives are there in 20_{five}?
   d. How many sixes are there in 20_{six}?

2. Write the following in expanded notation. Then write the base ten numeral for each as shown in the example.

   Example: $102_{five} = (1 \times 25) + (0 \times 5) + (2 \times 1) = 27$

   a. 245_{six} c. 1002_{three}
   b. 412_{five} d. 1021_{eight}

3. Write the following decimal numerals in bases six, five, and eight. Remember the values of the powers for each of these bases. Note the example:

   $7_{ten} = 11_{six} = 12_{five} = 13_{four} = 21_{three}$

   a. 11_{ten} b. 15_{ten}

4. Make a counting chart in base two for the numbers from zero to thirty-three.

Base Ten	1	2	3	4	. . .	33
Base Two	1	10	11		. . .	

5. Copy and complete the addition chart for base two shown at the right. How many addition facts are there?

Addition, Base Two

+	0	1
0		
1		

6. Using the same form as in Exercise 5, make a multiplication chart for base two. How many multiplication facts are there? How do the tables compare? Does this make working with the binary system difficult or easy? Explain your answer.

7. Write the following binary numerals in expanded notation and then in base ten notation.

   a. 111_{two} c. 10101_{two}

   b. 1000_{two} d. 11000_{two}

8. Write the following numerals in expanded notation and then in base ten notation.

   a. 111_{twelve} c. $47E_{\text{twelve}}$

   b. $3T2_{\text{twelve}}$ d. TOE_{twelve}

*9. Make up a place value system where the following symbols are used:

Symbol	Decimal Value	Name
0	0	do
1	1	re
∧	2	mi
≥	3	fa
10	4	re do

Write the numerals for numbers from zero to twenty in this system. Write the names in words using "do, re," etc.

*10. Using the symbols and scale from Problem 9, complete the addition and multiplication tables shown below.

+	0	1	∧	≥
0				
1				
∧				
≥				

×	0	1	∧	≥
0				
1				
∧				
≥				

11. What advantages and disadvantages, if any, do the binary and duodecimal systems have as compared with the decimal system?

12. Write the following in duodecimal notation.

   a. 425_{ten} b. 524_{ten}

*13. An inspector of weights and measures carries a set of weights which he uses to check the accuracy of scales. Various weights are placed on a scale to check accuracy in weighing any amount from 1 to 16 ounces. Several checks have to be made, because a scale which accurately measures 5 ounces may, for various reasons, be inaccurate for weighings of 11 ounces and more.

What is the smallest number of weights the inspector may have in his set, and what must their weights be, to check the accuracy of scales from 1 ounces to 15 ounces? From 1 ounce to 31 ounces.

*14. People who work with high speed computers sometimes find it easier to express numbers in the octal, or eight system rather than the binary system. Conversions from one system to the other can be done very quickly. Can you discover the method used?

Make a table of numerals as shown below:

Base ten	Base eight	Base two
1	1	1
2	2	10
5	5	101
7	?	?
15	?	?
16	?	?
32	?	?
64	?	?
256	?	?

Compare the powers of eight and two up to 256. Study the powers and the table above. $101{,}011{,}010_{two} = 532_{eight}$? Can you see why?

---

*15. There are a number of ways to change numerals written in other number bases to base ten notation. A student suggested this method:

Example A: To change 46_{twelve} to base ten notation.

Because there are 2 more symbols in base twelve, multiply (2 × 4) and add the result to 46_{ten}.

Does this method work for 46_{twelve}? Does it work for any two digit number written in base twelve?

Example B: To change 46_{seven} to base ten notation.

Because there are three fewer symbols, multiply (3 × 4) and subtract from 46_{ten}. Does the method work for 46_{seven}? Does it work for any two digit number written in base seven?

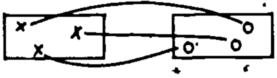
Chapter 3

WHOLE NUMBERS

3-1. Counting Numbers

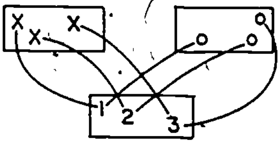
The counting numbers are the numbers used to answer the question "How many?" Primitive man developed the idea of number by the practice of matching objects, or things, in one set with objects in another set. When a man's sheep left the fold in the morning he could put a stone in a pile as each sheep went out. When the sheep returned in the evening he took a stone out of the pile as a sheep went into the pen. If there were no stones left in the pile when the last sheep was in the pen he knew that all the sheep had returned. Similarly, in order to keep count of the number of wild animals he had killed he could make notches in a stick -- one notch for each animal. If he were asked how many animals he had killed he could point to the notches in the stick. The man was saying that there were just as many animals killed as there were notches in the stick. The man was trying to answer the question "How many?" by making a one-to-one correspondence between the animals and the notches in the stick. He was also trying to answer the question "How many?" by making a one-to-one correspondence between the stones of the pile and the sheep of the flock. The one-to-one correspondence means that exactly one stone corresponded to each sheep and exactly one sheep corresponded to each stone. This says that the number of sheep was the same as the number of stones.

Some of us have learned the meaning of number in counting by using such one-to-one correspondences. When you look at various sets of objects as in the figure, you see that there is a certain property that these sets possess. This property may be described by saying that there are "just as many" marks in one set as in the other. A one-to-one correspondence between the sets can be shown by joining the



marks with strings, or paths. Each mark is joined to a mark of the other set. No marks are left over in either set and no mark is used twice. The correspondence shows that there are "just as many" marks in one set as in the other but it does not tell us "how many" there are in terms of a number.

Fortunately there is a standard set which can be used to tell us "how many" there are in each set. It also can be used to tell us that there are "just as many" in one set as in the other. This standard set is the set of counting numbers represented by the numerals 1, 2, 3, 4, 5, . . . In the figure each set of marks is put in a one-to-one correspondence with the set of numerals 1, 2, 3. The number of marks is the same as the number represented by the last numeral of the matching set. This kind of one-to-one correspondence between the marks and the set of numerals tells us that there are "just as many" in one set as in the other, and also tells us "how many" marks are in each set.



The method of using the counting numbers is such a natural one that the counting numbers are also called the "natural numbers." In this text they are called counting numbers. You may see them called "natural numbers" in other books.

Let us agree that the first counting number is 1. The set of all the counting numbers <u>and zero</u> is called the set of "whole numbers".

Exercises 3-1

1. A theatre owner wants to know how many people attended his theatre last night. He knows the first ticket was marked 27 and the last ticket was marked 81. How did he decide that 54 people attended? Was he correct?
2. If there is a one-to-one correspondence between the set of people in the room and the set of pairs of shoes in the room, then there is a two-to-one correspondence between the set of shoes in the room and the set of people in the room. List a few examples of two-to-one and four-to-one correspondences.

3. The following illustrates a one-to-one correspondence between the __________ numbers and the __________ numbers.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & \cdots \end{array}$$

3-2. Commutative Properties for Whole Numbers

If you have three apples in a basket and put in two more, then the number of apples in the basket is obtained by adding 2 to 3. You think of 3 + 2. If you started with two apples in the basket, and put in three more, then the number of apples in the basket is obtained by adding 3 to 2. You think of 2 + 3. In either case it is clear that there will be 5 apples in the basket. This may be written 2 + 3 = 3 + 2.

The arithmetic teacher read two large numbers to be added. One boy did not understand what his teacher said when she read the first number. He wrote the second number and then asked her to repeat the first number. When she read it again, he wrote it below the second number instead of above it. If all the students do the addition correctly, will the boy find the same sum as the students who heard all the dictation the first time?

The boy wrote: $\begin{array}{r} 2437 \\ \underline{6254} \end{array}$ The others wrote: $\begin{array}{r} 6254 \\ \underline{2437} \end{array}$

This idea which was just described is called the <u>commutative property of addition for whole numbers</u>. It means that the order in which two numbers are added does not affect the sum. The word <u>property</u> is used here in the usual meaning of the word -- it is something that is <u>characteristic of</u> the operation of addition:

3 added to 4 is 7 or 4 + 3 = 7,

4 added to 3 is 7 or 3 + 4 = 7.

Thus, you can write 4 + 3 = 3 + 4. This checks the commutative property of addition for these two whole numbers.

The commutative property of addition for whole numbers may be stated as:

Property 1. If a and b represent whole numbers then

$$a + b = b + a.$$

In the above example a is 4 and b is 3.

Multiplication is another operation which can be performed on numbers. Is there a commutative property of multiplication?

In learning the multiplication tables you learned that 7 [illegible] 5 = 35 and that $5 \times 7 = 35$. Similarly $9 \times 8 = 72$ and $8 \times 9 = 72$.

Property 2. If a and b represent whole numbers, then

$$a \times b = b \times a.$$

In mathematics it is often said that one number is greater than another. To simplify writing the phrase "is greater than", the symbol $>$ is used. So, to write "5 is greater than 3", merely write $5 > 3$. To indicate that "a is greater than b", write $a > b$. Similarly, the symbol $<$ means "is less than". Hence, $4 < 7$ is written for "4 is less than 7". Notice that each of these new symbols points toward the smaller of the two numbers being compared.

The symbol $\neq$ is used for "is not equal to". For example, $5 \neq 3$ and $4 \neq 0$.

In comparing three numbers such as 3, 6 and 11, $3 < 6 < 11$ or $11 > 6 > 3$ may be written. Note that the statement $3 < 6 < 11$ really stands for the two statements "3 is less than 6" and "6 is less than 11".

Exercises 3-2

1. Indicate whether each statement is true or false:

   a. $6 + 4 = 4 + 6$

   b. $13_{four} + 32_{four} < 32_{four} + 13_{four}$

   c. $6 < 7 < 14$

   d. $8 \div 2 = 2 \div 8$

   e. $45 \cdot 36 < 36 \cdot 45$

   f. $5 + 4 > 5 + 3$

   g. $315 + 462 = 462 + 315$

   h. $5 > 3 > 10$

   i. If $16 > 7$ and $7 > 5$ then $16 > 5$

2. Using the symbols =, <, and >, make the following statements true.
   - a. $23 \cdot 12 ? 12 \cdot 32$
   - b. $16 ? 9 ? 3$
   - c. $(3 \cdot 2) + 5 ? 5 + (3 \cdot 2)$
   - d. $86 \cdot 135 ? 135 \cdot 86$
   - e. $24 \div 3 ? 3 \div 24$
   - f. Given that a, b, and c are whole numbers: If $a > b$ and $b > c$, then $a ? c$.
3. Give the whole number or whole numbers which may be used in place of a to make the statements true.
   - a. $5 \cdot 7 = 7 \cdot a$
   - b. $2 \cdot a < 2 \cdot 1$
   - c. $3 \cdot a < 3 \cdot 2$
   - d. $2 + a < 2 + 7$
   - e. $7 \cdot 3 > a \cdot 5$
   - f. $a + 3 = 3 + a$
4. Do you think subtraction has the commutative property? To find out you must ask whether $a - b$ is equal to $b - a$ for all whole numbers a and b. If you can find at least one pair of whole numbers for which it is not true, then subtraction cannot have the commutative property.
5. Does division of whole numbers have the commutative property? Give an example which illustrates your answer.
6. Which of the following activities are commutative?
   - a. To put on a hat and then a coat.
   - b. To put on socks and then shoes.
   - c. To pour red paint into blue paint.
   - d. To close the hatch and dive the submarine.
7. Which of the defined operations below are commutative?
   - a. "D" means to find the sum of the first and twice the second. Example: $3 \text{ D } 5 = 3 + (2 \cdot 5)$ or 13.
   - b. "Z" means to find the sum of the first and the product of the first and the second. Example: $4 \text{ Z } 7 = 4 + (4 \cdot 7)$ or 32.
   - c. "F" means to find the product of the first and one more than the second. Example: $8 \text{ F } 0 = 8 \cdot 1$ or 8.
   - d. "Q" means to find three times the sum of the first and the second. Example: $8 \text{ Q } 5 = 3 \cdot (8 + 5)$ or 39.

8. Make up a defined operation as in Ex. 7 which is
   (a) commutative, (b) not commutative.
9. List some activities which are commutative and some which are not commutative.

3-3. Associative Properties for Whole Numbers

What is meant by $1 + 2 + 3$? Is $(1 + 2) + 3$ meant in which 1 and 2 are added and then 3 is added to the sum? Or is $1 + (2 + 3)$ meant in which 2 and 3 are added and then their sum is added to 1? Or, does it make any difference? You have seen that the order in which two numbers are added does not affect the sum (commutative property of addition). You can see that the way you group three numbers to add them does not affect the sum. For example,

$$(1 + 2) + 3 = 3 + 3 = 6 \text{ and}$$

$$1 + (2 + 3) = 1 + 5 = 6.$$

This idea of grouping the numbers differently without changing the sum is called the associative property of addition for whole numbers. This property may be used to make addition easier if the sum of one pair of three numbers is easier to find than the sum of another pair. If you are asked to add $12 + 4 + 2$ you might first add 12 and 4 and then add 2 to 16. Or you might think of first adding 4 and 2 and then adding 6 to 12. If you add each of the following by grouping the numbers differently you will be showing applications of the associate property.

$$7 + 9 + 11 = 7 + (9 + 11) = 7 + 20 = 27$$
$$12 + 7 + 33 = 12 + (7 + 33) = 12 + 40 = 52$$
$$97 + 53 + 100 = (97 + 53) + 100 = 150 + 100 = 250$$

The associative property can be used in finding the sum of 12 and 7. Perhaps you have always used it but did not call it by name. Notice how it can be used: $12 + 7 = (10 + 2) + 7 = 10 + (2 + 7) = 19$.

The associative property of addition for whole numbers may be stated as:

Property 3. If a, b and c represent any whole numbers

$$(a + b) + c = a + (b + c).$$

In everyday life you speak of "adding" or combining several things. Whether such combinations have the associative property depends on the things you combine. Is (gasoline + fire) + water the same as gasoline + (fire + water)?

The commutative property of addition means the order of any two numbers may be changed without affecting the sum. The associative property means that numbers may be grouped in pairs for the purpose of adding pairs of them without affecting the sum. Just as there is a commutative property for addition and multiplication, you might expect the associative property to belong to both operations.

What is meant by $2 \cdot 5 \cdot 4$? Does this mean $(2 \cdot 5) \cdot 4$ in which 2 is first multiplied by 5 and then 10 is multiplied by 4, or does this mean $2 \cdot (5 \cdot 4)$ in which 5 is first multiplied by 4 and then 2 is multiplied by 20? Both give the same answer and the conclusion is that either meaning can be given to $2 \cdot 5 \cdot 4$. This is true for any whole numbers.

Property 4. If a, b, and c represent any whole numbers,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Is there an associative property for subtraction? Perhaps you can answer the question by considering just one example. Try $10 - (6 - 4)$ which is $10 - 2$ or 8. But $(10 - 6) - 4 = 0$, so that $10 - (6 - 4)$ is not equal to $(10 - 6) - 4$. This shows that subtraction does not have the associative property. At first you may think that one example is not enough and that the property might hold if you used some other numbers. But, if the associative property is to hold for subtraction then it must hold for all whole numbers. Hence, by showing one set of three whole numbers for which the property is not true you know that it cannot be a property for all whole numbers.

Do you think the associative property holds for division? What does $16 \div 4 \div 2$ mean? It may mean $(16 \div 4) \div 2$, or it may mean $16 \div (4 \div 2)$. The first of these equals 2 and the second equals 8, so they are not equal to each other. This shows that division does not have the associative property.

These remarks about subtraction and division show us also that expressions like $10 - 6 - 4$ and $16 \div 4 \div 2$ do not have any meaning. Of course, the expressions, $(10 - 6) - 4$ and $10 - (6 - 4)$, do have meanings and they are different. Also, $(16 \div 4) \div 2$ and $16 \div (4 \div 2)$ make sense, but their meanings are different.

Exercises 3-3

1. Example: $(4 + 3) + 2 = 4 + (3 + 2)$

   Here, $(4 + 3) + 2 = 7 + 2 = 9$, and $4 + (3 + 2) = 4 + 5 = 9$. Show that the following are true in the above way. State the property illustrated in each problem.

   a. $(21 + 5) + 4 = 21 + (5 + 4)$

   b. $(9 \cdot 7) \cdot 8 = 9 \cdot (7 \cdot 8)$

   c. $436 + (476 + 1) = (436 + 476) + 1$

   d. $(57 \cdot 80) \cdot 75 = 57 \cdot (80 \cdot 75)$

2. a. Does $(10 - 7) - 2$ equal $10 - (7 - 2)$?

   b. Does $18 - (5 - 2)$ equal $(18 - 5) - 2$?

   c. What generalization can you make regarding the associative property of subtraction?

3. a. Does $(32 \div 8) \div 2$ equal $32 \div (8 \div 2)$?

   b. Does $(60 \div 30) \div 2$ equal $60 \div (30 \div 2)$?

   c. Place parentheses in $75 \div 15 \div 5$ so that it will equal 1.

   d. Place parentheses in $75 \div 15 \div 5$ so that it will equal 25.

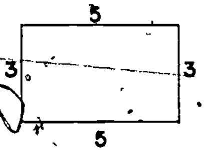
   e. Place parentheses in $80 \div 20 \div 2$ so that it will equal 8.

   f. Place parentheses in $80 \div 20 \div 2$ so that it will equal 2.

   g. What generalization can be made concerning the associative property and division?

3-4. The Distributive Property

In finding the perimeter of the top of a desk one pupil measured the length of each side in feet and found the measurements as shown in the diagram. Then he found the perimeter in feet by finding the sum 5 + 3 + 5 + 3 = 16. Another pupil said he thought that this was all right but that it was more work than necessary. He said he would add 5 and 3 and multiply their sum by 2. Will this give the same answer? A third pupil said she thought it would be better to multiply 5 by 2 and 3 by 2 and then add these two products. The second and third pupils may not have known the name of the principle they were using but it is useful and important. It is called the distributive property. In terms of the pupils' problem it states simply that



$$2 \cdot (5 + 3) = (2 \cdot 5) + (2 \cdot 3)$$

and

$$2 \cdot (5 + 3) = (2 \cdot 8)$$

You have been using this property in many ways for a long time. Consider, for example, $3 \cdot 13$ or $\begin{array}{r} 13 \\ \times\ 3 \\ \hline 39 \end{array}$. You were really using the distributive property because:

$$3 \cdot 13 = 3 \cdot (10 + 3) = (3 \cdot 10) + (3 \cdot 3) = 30 + 9 = 39.$$

Let us see how you use the distributive property in finding the product $9 \cdot 36$. You probably perform the multiplication about as follows:

$$\begin{array}{r} 36 \\ \times\ 9 \\ \hline 324 \end{array} \qquad \text{or} \qquad \begin{array}{rl} 36 & \\ \times\ 9 & \\ \hline 54 & (9 \times 6) \\ 270 & (9 \times 30) \\ \hline 324 & \end{array}$$

Do you see that the left example is a short way of doing the problem? You were really using the distributive property;

$$\begin{aligned} 9 \cdot 36 &= 9 \cdot (30 + 6) \\ &= (9 \cdot 30) + (9 \cdot 6) \quad \text{distributive property} \\ &= 270 + 54 \\ &= 324. \end{aligned}$$

The distributive property is also important in operations involving fractions. Let us find the product of 8 and $12\frac{1}{4}$. First, recall that $12\frac{1}{4}$ means $12 + \frac{1}{4}$. Then

$$\begin{aligned} 8 \cdot 12\tfrac{1}{4} &= 8 \cdot (12 + \tfrac{1}{4}) \\ &= (8 \cdot 12) + (8 \cdot \tfrac{1}{4}) = 96 + 2 \\ &= 98. \end{aligned}$$

The distributive property is:

Property 5. If a, b, and c are any whole numbers then

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

The distributive property is the only property of the three studied in this chapter which involves two operations, namely, addition and multiplication. This does not mean that any problem which involves these two operations is performed by using the distributive property. For example, $(3 \cdot 5) + 14$ means that the product of 3 and 5 must be found and then 14 added to the product: $(3 \cdot 5) + 14 = 15 + 14 = 29$.
However, $3 \cdot (5 + 14) = (3 \cdot 5) + (3 \cdot 14) = 15 + 42 = 57$.

Exercises 3-4

1. Show that the following are true by doing the indicated operations. Example: $3 \cdot (4 + 3) = (3 \cdot 4) + (3 \cdot 3)$

$$3 \cdot (4 + 3) = 3 \cdot 7 = 21$$
$$(3 \cdot 4) + (3 \cdot 3) = 12 + 9 = 21$$

   a. $12 \cdot (5 + \frac{1}{4}) = (12 \cdot 5) + (12 \cdot \frac{1}{4})$
   b. $(67 \cdot 48) + (67 \cdot 52) = 67 \cdot (48 + 52)$
   c. $(72 \cdot \frac{1}{2}) + (\frac{1}{2} \cdot 72) = 72 \cdot (\frac{1}{2} + \frac{1}{2})$.

2. Make each of the following a true statement illustrating the distributive property.
   a. $3 \cdot (4 + \quad) = (3 \cdot 4) + (3 \cdot 3)$
   b. $(2 \cdot 7) + (3 \cdot \quad) = (\qquad) \cdot 7$

3. Using the distributive property rewrite each of the following.
   Examples: (1) $5 \cdot (2 + 3) = (5 \cdot 2) + (5 \cdot 3)$
   (2) $(6 \cdot 4) + (6 \cdot 3) = 6 \cdot (4 + 3)$
   a. $15 \cdot (6 + 13)$
   b. $(5 \cdot 12) + (4 \cdot 12)$

4. Using the idea of the distributive property, the following examples can be rewritten.
   (1) $10 + 15$ as $(5 \cdot 2) + (5 \cdot 3)$ or $5 \cdot (2 + 3)$
   (2) $15 + 21$ as $(3 \cdot 5) + (3 \cdot 7)$ or $3 \cdot (5 + 7)$
   Use the distributive property to rewrite the following in a similar way.
   a. $27 + 51$
   b. $100 + 115$
   c. $30 + 21$

5. Which of the following are true?
   a. $3 + (4 \cdot 2) = (3 + 4) \cdot (3 + 2)$
   b. $3 \cdot (4 - 2) = (3 \cdot 4) - (3 \cdot 2)$
   c. $(4 + 6) \div 2 = (4 \div 2) + (6 \div 2)$
   d. $3 + (4 \cdot 2) = (3 \cdot 4) + (3 \cdot 2)$

*6. 45 can be written as $(40 + 5)$ and 23 can be written as $(20 + 3)$. Using the distributive property the product of 45 and 23 would be:

$(40 + 5) \cdot (20 + 3)$ or

$40 \cdot (20 + 3) + 5 \cdot (20 + 3)$ or

$(40 \cdot 20) + (40 \cdot 3) + (5 \cdot 20) + (5 \cdot 3)$

Check

$$\begin{array}{r} 45 \\ \times\ 23 \\ \hline 135 \\ 90 \\ \hline 1035 \end{array}$$

Completing the operations gives:

$800 + 120 + 100 + 15$ or 1035

Rewrite the following using the distributive property and check as above.

a. $64 \cdot 66$

b. $75 \cdot 75$

*7. Indicate which property was used in going from one line to the next.

a. $[(2 \cdot 5) + (3 \cdot 2)] + (5 \cdot 2) \cdot 7$

b. $[(5 \cdot 2) + (3 \cdot 2)] + (5 \cdot 2) \cdot 7$ commutative property for multiplication

c. $[(5 \cdot 2) + (3 \cdot 2)] + 5 \cdot (2 \cdot 7)$ "?"

d. $[(3 \cdot 2) + (5 \cdot 2)] + 5 \cdot (2 \cdot 7)$ "?"

e. $(3 \cdot 2) + [(5 \cdot 2) + 5 \cdot (2 \cdot 7)]$ "?"

f. $(3 \cdot 2) + [(5 \cdot 2) + 5 \cdot (7 \cdot 2)]$ "?"

g. $(3 \cdot 2) + [(5 \cdot 2) + (5 \cdot 7) \cdot 2]$ "?"

h. $(3 + [5 + (5 \cdot 7)]) \cdot 2$ "?"

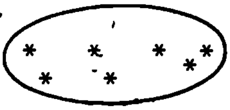
3-5. Sets and the Closure Property

When a word in mathematics is needed to talk about a collection, the word set will be used, as a set of numbers, a set of marks on the page, a set of stars in a diagram.

A set of numbers: 5, 36, 7, 8

A set of marks: ////////

A set of stars in a diagram:



Other examples of sets are: the set of coins in your pocket, the set of vowels in our alphabet, a set of chessmen, a set of cattle (you might say a herd of cattle), the set of cities in the U.S.A. which have a population of more than one million.

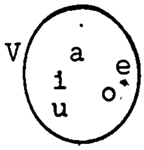
The counting numbers form a set. Remember that the counting numbers are 1, 2, 3, 4, 5, 6, ... where the three dots are used to indicate that the set of numbers continues indefinitely. There is no last number. N will be used to represent the set of counting numbers, and these numbers will be put within braces { } to indicate that they are the objects in the set which is designated by N. Hence, you may write

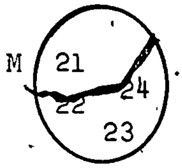
$$N = \{1, 2, 3, \ldots\}$$

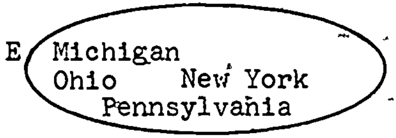
and read it "N is the set of counting numbers."

You may choose any capital letter to represent the set. If you have the set $S = \{1, 2, 3, 4, 5, 6, 7\}$ you may describe this by saying that S is the set of counting numbers from 1 to 7 inclusive, or S is the set consisting of the counting numbers less than 8.

A few more examples of sets and the abbreviated way of writing them will help make the concept clear. "V is the set of vowels in our alphabet" becomes "$V = \{a, e, i, o, u\}$". "M is the set of counting numbers which are greater than 20 and less than 25" becomes "$M = \{21, 22, 23, 24\}$". "E is the set of states in the U. S. A. which are touched by Lake Erie" becomes "E = {Michigan, Ohio, Pennsylvania, New York}".







In the first set the objects are letters, in the second the elements are numbers, in the third each element is a state. The word element is used for any object of a set. Thus, an element may be a letter, a number, a word, a cat, a marble or whatever is in the set.

Now the set of counting numbers will be used to help us understand another new idea for sets. This is the idea of closure. If any two counting numbers are added, the sum is a certain counting number. For example, $7 + 9 = 16$, $234 + 543 = 777$ and each sum is a counting number. If the sum of any two elements of a set is an element of the set, the set is closed under addition. Since the sum of any two counting numbers is a counting number, the set N of counting numbers is closed under addition. It must be emphasized here that any two means every two. The set $S = \{1, 2, 3, 4, 5, 6, 7\}$ is not closed under addition since two numbers can be found in the set whose sum is not in the set; e.g., $5 + 6 = 11$ and 11 is not in S. Is the set $M = \{21, 22, 23, 24\}$ closed under addition? Give the reason for your answer. Notice that if there is at least one pair of elements in M whose sum is not in M, then M is not closed under addition.

Closure deals with a property of sets under a given operation. The set need not be the counting numbers. The operation may not be addition. For example, let T be the set of all counting numbers ending in 0 or 5. This set is closed under multiplication. It is not closed under division since, for example, (20 ÷ 5) is not an element of T.

Exercises 3-5

1. Let Q = {1, 3, 5, 7, 9, 11, 13, . . .} be the set of all odd numbers.
   a. Is the sum of two odd numbers always an odd number?
   b. Is the set of odd numbers closed under addition?
2. Is the set of even numbers closed under addition?
3. Is the set of all multiples of 5 (5, 10, 15, 20, 25, etc.) closed under addition?
4. What is true of the sets of numbers in Exercises 1, 2, 3 under multiplication?
5. Are the following sets of numbers closed under addition?
   a. The set of counting numbers greater than 50?
   b. The set of counting numbers from 100 through 999?
   c. The set of counting numbers less than 48?
   d. The set of counting numbers whose numerals end in 0?
6. Are the sets of numbers in Exercise 5 closed with respect to multiplication?
7. Are all sets of counting numbers which are closed under addition also closed under multiplication? Why?
8. Are any of the sets of numbers in Exercise 5 closed under subtraction?
9. Are any of the sets of numbers in Exercise 5 closed under division?

3-6. Inverse Operations

Often you do something and then undo it. You open the door; you shut the door. You open the window; you close the window. One operation is the inverse of the other.

The inverse of putting on your coat is taking off your coat. The inverse operation of division is multiplication. The inverse operation of addition is subtraction.

Suppose you have \$220 in the bank and you add \$10 to it. Then you have \$220 + \$10 = \$230. Now undo this by drawing out \$10. The amount that remains is \$230 - \$10 = \$220. The athletic fund at your school might have \$1800 in the bank and after a game have \$300 more. Then the fund has \$1800 + \$300 or \$2100 in it. But the team needs new uniforms which cost \$300 so \$300 is withdrawn to pay for them. The amount left is \$2100 - \$300, or \$1800. These operations undo each other. Subtraction is the inverse of addition.

Of course, this idea could be expressed in more general terms. Let x represent the number of dollars originally in the bank. If the amount which is deposited is b, then $x + b = a$, where a represents the number of dollars that is now in the bank. How shall this operation be undone? From the number of dollars represented by a, subtract the number of dollars withdrawn, represented by b, and you have the number represented by x. Write $x = a - b$.

Notice that if a and b are whole numbers, and if $a > b$, then there is a whole number x so that $b + x = a$. Examples: If a is 17 and b is 10, then x is the whole number 7 so that $10 + 7 = 17$. When a is greater than b it is always possible to find x so that $a = b + x$. Can you make the same generalization if the above operation $b + x = a$, is changed to multiplication, $b \cdot x = a$? If you substitute 2 for b and 3 for a you will see that there is no <u>whole</u> number that can be substituted for x such that $2 \cdot x = 3$. If one substitutes certain numbers--for example, if $a = 20$ and $b = 4$--then there is a whole number that can be substituted for x such that $4 \cdot x = 20$.

In this example x must represent 5, since $4 \cdot 5 = 20$. You get the 5 by dividing 20 by 4.

If b is 6 and a is 24 then x must be 4 since $6 \cdot 4 = 24$.

If b is 5 and a is 40 then x must be 8 since $5 \cdot 8 = 40$.

If b is 3 and a is 30 then x must be 10 since $3 \cdot 10 = 30$.

In each example the number for x is found by dividing the number represented by a by the number represented by b. In general, if there is a counting number x that can be multiplied by a counting number b to get counting number a, then this number x can be found by dividing a by b. This is written as $b \cdot x = a$. Multiply x by b to obtain a. To undo the operation you must perform the inverse operation which means that you must divide a by b to obtain x: $b\overset{x}{\overline{)\,a}}$. The inverse operation of multiplying by b is dividing by b.

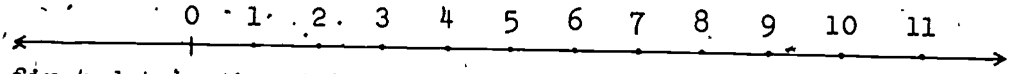
Exercises 3-6

1. Find, if possible, a whole number which can be used for x in each of the following to make it a true statement. If there is no whole number that can be used for x, then say there is none.

   a. $9 + x = 14$
   b. $4 + x = 11$
   c. $10 = x + 2$
   d. $8 + x = 11$
   e. $3 + x = x + 3$
   f. $x = 20 \div 4$
   g. $5 \cdot x = 30$
   h. $9 \cdot x = 0$
   i. $x = 3 \div 3$
   j. $11 \cdot x = 11$

3-7. Betweenness and the Number Line

How whole numbers are related may be shown with a picture. Select some point on a line as below and label it zero (0). Label



the first dot to the right of zero the first counting number and each dot after that to the right the succeeding counting numbers. This picture is often referred to as The Number Line. Any whole number is smaller than any of the numbers on the right side of it and greater than any of the numbers on its left. For example 3 is less than 5 and greater than 2. This may be written $2 < 3 < 5$, since 2 is less than 3 and 3 is less than 5. With the number line you can also determine how many whole numbers there are between any two whole numbers. For example, to find how many whole numbers there are between 6 and 11 you can look at the picture and count them. There are four of them, 7, 8, 9, and 10.

Exercises 3-7

1. How many whole numbers are there between:
   a. 25 and 25
   b. 28 and 25
   c. 26 and 25
   d. 114 and 25
   e. If a and b are whole numbers, and $a > b$, is the number of whole numbers between a and b:
      (1) $b - a$?
      (2) $(a - 1) + b$?
      (3) $a - (b + 1)$?
      (4) $(a - b) + 1$?
2. What is the whole number midway between:
   a. 7 and 13
   b. 9 and 13
   c. 17 and 19
   d. 17 and 27
3. Which of the following pairs of whole numbers have a whole number midway between them?
   a. 9, 17
   b. 19, 36
   c. a, b if a and b are even whole numbers
   d. a, b if a and b are odd whole numbers
   e. a, b if a is odd and b is even.

4. The whole numbers a, b and c are so located on The Number Line that b is between a and c, and $c > b$.
   a. Is $c > a$? Explain with a number line.
   b. Is $b > a$? Explain with a number line.
   c. Is $b < c$? Explain with words.
5. The whole numbers a, b, c and d are so located on The Number Line that b is between a and c and a is between b and d. What relation, if any, is there among b, c, and d?

3-8. The Number One

The number one is a special number in several ways. One is the smallest of our counting numbers. You may build any counting number, no matter how large, by beginning with 1 and adding 1's until you have reached the desired number. For example, to obtain the number five, you can begin with our special number 1 and repeat the addition of 1. $1 + 1 = 2$, $2 + 1 = 3$, $3 + 1 = 4$, $4 + 1 = 5$. There is no largest counting number.

In multiplication if you wish to obtain a different numeral for a number, you can multiply by a selected form of the special number 1. In this way you may get a different numeral, but it represents the same number. You may recall that in rewriting 4 as $\frac{8}{2}$, you were simply multiplying 4 by $\frac{2}{2}$. Of course, $\frac{2}{2}$ is our special number 1. Multiply $\frac{1}{3}$ by $\frac{3}{3}$ and get $\frac{3}{9}$; multiply $\frac{4}{5}$ by $\frac{2}{2}$ and get $\frac{8}{10}$. These are examples of multiplying by the number 1 in selected forms $\frac{3}{3}$, and $\frac{2}{2}$. This means that the new fractions are different in form from the original ones but they still represent the same number. The special number one when used as a multiplier makes the product identical with the multiplicand. Because the product of any counting number and one is the original counting number, the number 1 is called the "identity element" for multiplication.

Since division is the inverse operation of multiplication, is the number one also special in division? What happens if you divide any counting number by one? You do obtain the same counting number. But if you divide 1 by a counting number you do not get the counting number. For this reason the number one is not the identity element for division. A counting number multiplied by 1 is the same as 1 multiplied by the counting number. But a counting number divided by one is not the same as one divided by that counting number. If C represents any counting number these multiplication and division operations using the number 1 can be expressed in the following ways.

$$C \cdot 1 = 1 \cdot C;$$
$$C \div 1 = C;$$
$$C \div C = 1;$$
$$1 \div C \neq C \text{ if } C \neq 1.$$

You have learned to use 10^2 to mean $10 \cdot 10$; 10^3 to mean $10 \cdot 10 \cdot 10$; 10^6 to mean $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$. The "2", "3", "6" are called exponents. The exponents are small, but the numbers represented by 10^2, 10^3 and 10^6 are very large. If 1 is used in place of 10 this is not true. For $1^2 = 1 \cdot 1$; $1^3 = 1 \cdot 1 \cdot 1$; $1^6 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ and these are still the number 1. In fact 1^2 or 1^{200} or 1^{3056} is still 1.

Our discussion of the number one may be summarized briefly in the mathematical sentences below. Can you translate them into words? The letter C here represents any counting number.

a. $C = 1$ or $(1 + 1)$ or $(1 + 1 + 1)$ or ... etc.
b. $1 \cdot C = C$
c. $C \div 1 = C$
d. $C \div C = 1$
e. $1^C = 1$

Exercises 3-8

1. Can you get any counting number by the repeated addition or subtraction of 1 to or from any other counting number? Give an example to support your conclusion.
2. By the above process can you get a number that is not a counting number? Give an example to support your conclusion.
3. Robert said, "The counting numbers are not closed under the subtraction of ones but they are closed under the addition of ones." Show by an example what Robert meant.
4. Perform the indicated operations:
   - a. $(4 - 3)\overline{)876429}$
   - b. $1\overline{)976538}$
   - c. $897638 \cdot (5 - 4)$
   - d. $896758 \cdot \frac{4}{4}$
   - e. $3479 \cdot 1^{110}$
   - f. $97 \cdot x^6$ (if x is 1)
   - g. $1^7 \cdot (489 \div 489)$
   - h. $\frac{8}{8} \cdot 1^5 \div 1^4$

3-9. The Number Zero

Although zero is not included in the counting numbers, it is considered as one of the whole numbers. Most of the time it is used according to rules of the counting numbers, and in a sense it is used to count. If you withdraw all your money from the bank, you can express your bank balance with this special number zero. If you have answered no questions correctly, your test score may be zero. If there are no chalkboard erasers in the classroom, the number of erasers may be expressed by zero. In all these cases, no money in the bank, no correctly answered questions and no erasers, the zero indicated that there are no objects or elements in the set of objects being discussed. If there are no elements in the set, it is called an empty set.

The number zero is the number of elements in the empty set. In this sense, some persons say that zero means "not any." Others say it means "nothing" because there is nothing in the set. As you shall see, these are rather confused and limited concepts of zero.

On a very cold morning Paul was asked the temperature. After looking at the thermometer he replied, "zero." Did he mean there was "not any"? Did he mean "nothing"? No, he meant the top of the mercury was at a specific point on the scale called zero. Fred had an altimeter in his car so he could check the altitude as they drove in the Rocky Mountains. On one vacation trip they drove to the Salton Sea. On the way down Fred exclaimed, "Look, the altitude is zero!" When the altimeter indicates zero, it does not mean there is "nothing," it means there is a specific altitude which is called zero. It is just as specific and real as an altitude of 999 feet.

The sum of a counting number and one is always the next larger counting number. The sum of a counting number and zero is always the original counting number. For example, $4 + 0 = 4$. This fact might be expressed in symbols $C + 0 = C$ where C is any counting number. Or it might be expressed by saying that zero is the "identity element" for addition.

The difference between the same two natural numbers is the special number zero. For example, $4 - 4 = 0$. Did you notice that in this subtraction operation you do not get a counting number? To put the idea in more elegant language, you would say that the set of counting numbers is not closed under subtraction.

What could $3 \cdot 0$ mean? You might think of the number of chairs in 3 rooms if each room contains zero chairs. Thus, any number of rooms containing zero chairs would have a total of zero chairs. This idea might be expressed in symbols by writing $C \cdot 0 = 0$, where C is any counting number.

The product $0 \cdot 3$ is even more difficult to explain. But you do know by the commutative property for multiplication that $3 \cdot 0 = 0 \cdot 3$. You have seen that $3 \cdot 0 = 0$. Therefore, $0 \cdot 3 = 0$ because the commutative property for multiplication is true for all whole numbers. If a represents any whole number, $a \cdot 0 = 0 \cdot a = 0$. If a is zero $0 \cdot 0 = 0$.

There is a very important principle expressed in the above symbols, but it may not be seen at the first glance. Did you observe that if the product of two or more whole numbers is zero, then one of the numbers must be zero? For example, $4 \cdot 5 \cdot 0 = 0$. In mathematics you will use this fact frequently.

What could zero divided by 3 mean? If you have a room with zero chairs and divide the room into three parts, it could mean the number of chairs in each part of the room. With this meaning, $0 \div 3$ should be 0. If $3\overset{0}{\overline{)0}}$ then 0×3 should be zero, by the inverse operation.

If $\frac{0}{7} = 0$, what is $\frac{7}{0}$? Is $0\overline{)7}$ a counting number? Let us assume that $0\overline{)7}$ is equal to some number represented by N. This means that 7 is equal to zero times some number N. $(7 = 0 \times N)$. The product of any number by zero is zero, therefore, there is no number N that will equal $0\overline{)7}$. In more elegant language, you may say that $\frac{7}{0}$ is not the name of any counting number or zero. Therefore, this operation cannot be performed. A counting number cannot be divided by zero.

Could ~~zero~~ be divided by zero? In symbols the question is "$\frac{0}{0} = ?$". Or $0\overset{?}{\overline{)0}}$. If $0\overline{)0}$ equals some number n then by our definition of multiplication, $0 \times n = 0$. What numbers could replace n? Could n be 3? Of course, n could be any counting number or zero. Since $0\overline{)0}$ could be any whole number, the symbol $\frac{0}{0}$ has too many meanings. Therefore, you should remember that you cannot divide either a counting number or zero by zero.

Our discussion of the special number zero may be summarized briefly in the mathematical sentences below. State them in words if u and w represent any whole numbers and C represents any counting number.

a. $w + 0 = w$
b. $0 + w = w$
c. $w - 0 = w$
d. $0 \cdot w = 0$
e. $w \cdot 0 = 0$.
f. If $u \cdot w = 0$, then either u or w is zero or both are zero.
g. $0 \div C = 0$
h. $C \div 0$ has no meaning.

Exercises 3-9

1. Select the sumbols that represent zero:

a. $\frac{0}{4}$
b. $5 - 4$
c. $\frac{a}{0}$
d. $0 + 0$
e. $0 \cdot 4$
f. $4 \cdot 0$
g. $\frac{4 - 4}{2}$
h. $\frac{4}{2 - 2}$
i. $2 \cdot (4 + 6 + 0)$
j. $(2 \cdot 4) \div 0$
k. $\frac{4}{4}$

2. Can you find an error in any of the following statements?
a and b are whole numbers.

a. If $a \cdot b = 0$, a or $b = 0$
b. If $a \cdot b = 1$, a or $b = 1$
c. If $a \cdot b = 2$, a or $b = 2$
d. If $a \cdot b = 3$, a or $b = 3$
e. If $a \cdot b = C$, a or $b = C$

3-10. Primes

In this section there will be a discussion on how counting numbers can be expressed as products of other counting numbers. The number 1 is in a class by itself since every counting number is a multiple of 1; that is, every counting number is divisible by 1.

What are the multiples of 2 which are greater than 2? First write down the numbers, for instance, from 1 to 30 inclusive. The first multiple of 2 greater than 2 is 4; cross out the 4 and every second number after that. The list will then look like the following:

1	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~	11	~~12~~
13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~	21	~~22~~	23	~~24~~
25	~~26~~	27	~~28~~	29	~~30~~						

Do not cross out 2. The numbers above which are not crossed out are 1, 2, and the numbers less than 31 which are not multiples of 2. Let us continue the process. Our second step would be to go through the same table and cross out the multiples of 3 which are greater than 3. Then the table would look like this:

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~
13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~
25	~~26~~	~~27~~	~~28~~	29	~~30~~						

Here every third number beginning with 6 has been crossed out, but 3 has not been crossed out since that is the number whose multiples are being found. The next number not crossed out is 5. So for the third step cross out every fifth number after 5 (that is, beginning with 10). For the fourth and fifth steps, similarly cross out multiples of 7 and 11 which are greater than 7 and 11. Keep track of the multiples as indicated. Did you cross out any new numbers when you were considering multiples of 11? Would you cross out any new numbers if you considered multiples of 12? of 13?

From the way in which the table was constructed every number crossed out is a multiple of a smaller number different from 1. These numbers are called <u>composite numbers</u>.

<u>Definition</u>: A composite number is a counting number which is divisible by a smaller counting number different from 1.

The table is called the "Sieve of Eratosthenes" for the first 30 numbers. It is called a "sieve" because in it you have sifted out all the composite numbers less than 31.

Except for the number 1, the numbers of the Sieve of Eratosthenes which are not crossed out are called <u>prime numbers</u>.

<u>Definition</u>: A prime number is a counting number, other than 1, which is divisible only by itself and 1.

Since it eliminates the composite numbers, the Sieve of Eratosthenes is a good way of finding a list of all prime numbers up to a certain point. The composite numbers are sifted out. The prime numbers remain.

The number 1 is <u>not</u> included in the set of primes partly because it is divisible by itself only. There will be another stronger reason for this later on.

<u>Exercises 3-10</u>

1. Express each of the following counting numbers as a product of two smaller counting numbers or indicate that it is impossible to do this:

   a. 12 b. 31 c. 8 d. 35 e. 39 f. 41 g. 95

2. a. How many prime numbers are less than 50?
   b. How many prime numbers are less than 100?

3. a. By what number is 24 divisible?
   b. The number 24 is a multiple of what numbers?
   c. Are the two sets of numbers you have found in a and b the same? Why or why not?

4. Write 12 in all possible ways as a product of counting numbers greater than 1.

5. List the pairs of prime numbers less than 100 which have a difference of 2. How many are these? Such pairs are called <u>twin primes</u>.
6. Express each even number between 4 and 22 as a sum of two prime numbers. (An even number, recall, is one divisible by 2). Most mathematicians believe that every even number greater than 2 is the sum of two prime numbers but no one has been able to prove it.
7. Are there three numbers that might be called prime triplets?
8. What is the intersection of the set of prime numbers and the set of odd numbers less than 30?

3-11. Factors

The word "factor" is commonly used in mathematics. Though the term may be new to you, the idea is not. You know that $5 \times 2 = 10$. Instead of calling one of the numbers the multiplicand and the other one the multiplier, both of them are given the same name -- <u>factor</u>. Thus, 5 and 2 are factors of 10; 6 and 7 are factors of 42, since $6 \times 7 = 42$. Also, $42 = 2 \times 3 \times 7$; so 2, 3, and 7 are factors of 42.

<u>Example 1</u>: Write 12 as a product of factors.

$$12 = 2 \times 6,$$
$$\text{or } 12 = 3 \times 4,$$
$$\text{or } 12 = 2 \times 2 \times 3 = 2^2 \times 3$$

When you say "the factors" you mean "all the factors" of a number. For example, the number 6 has four factors, 1, 2, 3, and 6. The number one and the number itself are always factors of a number.

<u>Example 2</u>: Find the set of factors of 20...

The set of factors of 20 is {1, 2, 4, 5, 10, 20}.

The idea of factors is associated with multiplication. In mathematical symbols factor is defined in the following way:

<u>Definition</u>. If a, b, and c are whole numbers and if $ac = b$, then the number a is called a factor of b. (Under these conditions c is also a factor of b.)

Using the terms of the last section, 3 is a factor of 12 because 12 is divisible by 3. In the symbols of the definition, the number a is a factor of b if b is divisible by a.

The number 1 has only one factor, itself. Each prime number has exactly two factors, itself and 1.

<u>Definition</u>. If a counting number is written as a product of prime numbers, this product is called a <u>complete factorization</u> of the given number.

<u>Example 3</u>: Find a complete factorization of 72.

Method I

$72 = 8 \times 9$

$72 = (4 \times 2) \times (3 \times 3)$

$72 = (2 \times 2) \times 2 \times (3 \times 3)$

$72 = (2 \times 2 \times 2) \times (3 \times 3)$

Using exponents,

$72 = 2^3 \times 3^2$

Method II

Using continuing short division

2	72
2	36
2	18
3	9
	3

$72 = 2^3 \times 3^2$.

In the factorization of 72, $2^2 \times 3 \times 2$ is the same as $2^3 \times 3$ except for the order of the factors. In fact, a very fundamental property of the counting numbers is that there is only one way to write a complete factorization of any counting number except for the order in which the prime factors appear.

This property is given a special name:

<u>The Unique Factorization Property of the Counting Numbers</u>:

Every counting number greater than 1 can be factored into primes in only one way except for the order in which they occur in the product.

Here there is another reason for excluding 1 from the set of prime numbers. If 1 had been called a prime, then 5 could have been expressed as a product of primes in many different ways: 5 × 1, 5 × 1 × 1, 5 × 1 × 1 × 1, ... Here the product would not be unique except for the order in which the factors are written.

Exercises 3-11

1. List the set of factors for each of the following:
   a. 10 b. 15 c. 9 d. 18 e. 27 f. 24 g. 11
2. Write a complete factorization of:
   a. 10 b. 15 c. 9 d. 30 e. 45 f. 50 g. 13
3. According to our definition of factor, is zero a factor of 6? Is 6 a factor of zero? Explain your answers.
4. a. What factors of 20 do not appear in a complete factorization of 20?
   b. What factors of 72 do not appear in a complete factorization of 72?
5. Find a complete factorization of:
   a. 105 c. 64 e. 301
   b. 42 d. 345 f. 323

Definition. If a whole number is divisible by two it is an *even number*. If a whole number is not divisible by two it is an *odd number*.

6. Tell whether these numbers are odd or even:
   a. 3 × 2 × 6
   b. 128 · 37
   c. ~~3 × 3 × 7~~
   d. 3 · (4 + 7)
   e. 5 · (9 + 13)
7. Classify each of the following as odd or even:
   a. 11_{three} c. 33_{five}
   b. 12_{five} d. 101_{two}
8. From the results of Problem 7 would you say that divisibility is a property of a numeral or a property of a number? Explain your answer.

*9. Copy the following table for counting number N and complete it through $N = 30$.

N	Factors of N	Number of Factors	Sum of Factors
1	1	1	1
2	1,2	2	3
3	1,3	2	4
4	1,2,4	3	7
5	1,5	2	6
6	1,2,3,6	4	12
7	1,7	2	8
8	1,2,4,8	4	15

a. Which numbers represented by N in the table above have exactly two factors?

b. Which numbers N have exactly three factors?

c. If $N = p^2$ (where p is a prime number), how many factors does N have?

d. If $N = pg$ (where p and g are different prime numbers), how many factors does N have? What is the sum of its factors?

e. If $N = 2^k$ (where k is a counting number), how many factors does N have?

f. If $N = 3^k$ (where k is a counting number), how many factors does N have?

g. If $N = p^k$ (where p is a prime number and k is a counting number), how many factors does N have?

3-12. Divisibility

To find the factors of a number, you can always guess and try, but it is much easier if you can tell from looking at a number whether or not it has a given factor. From Chapter 2 or from Sieve of Eratosthenes it is clear that a number written in the decimal system is even if the last digit is even. At least this is true as far as the sieve you have constructed goes. Thus:

A counting number written in the decimal system is even if its last digit is one of 0, 2, 4, 6, 8. If its last digit is not one of these, it is odd.

Examples:

734 is an even number since its last digit is an even number.

If 734 is divided by 2 the remainder is 0.

391 is an odd number since its last digit is not an even number.

If 391 is divided by 2 the remainder is 1.

A counting number expressed in the decimal system is divisible by 5 if its last digit is 0 or 5. Otherwise it is not divisible by 5.

What about divisibility by 3? Can you tell by looking at the last digit? The first ten multiples of 3 are

0, 3, 6, 9, 12, 15, 18, 21, 24, 27.

Each of the possible last digits, 0,1,2,3,4,5,6,7,8, and 9, appears in this list. On the other hand, none of the following are divisible by 3 even though each of the possible last digits appears here also:

4, 7, 10, 13, 16, 19, 22, 25, 28, 31.

You can see, then, that you cannot tell whether a number is divisible by 3 by looking at the last digit.

But suppose you add the digits of the multiples of 3. For 12 you have 1 + 2 = 3; for 15 you have 1 + 5 = 6; for 18 you have 1 + 8 = 9. By this means you can form the following table:

Multiple of 3	0	3	6	9	12	15	18	21	24	27	30	33	36	39
Sum of digits	0	3	6	9	3	6	9	3	6	9	3	6	9	12

Multiple of 3	42	45	48	51	54	57	60	63	66	69	72
Sum of Digits	6	9	12	6	9	12	6	9	12	15	9

Can you make any statement that seems to be true about the sum of the digits for all multiples of 3? You will see that in each case the sum of the digits is divisible by 3. Furthermore, if you add the digits of any number that is not divisible by 3 (take 25 where the sum of the digits is 7), the sum of the digits is not divisible by 3. Can you see why this will be true for all numbers? See Problem 3 in the next set.

You may notice that every third sum of digits in the table on the previous page is divisible by 9 and every third multiple of 3 is divisible by 9. Hence there is the following test for divisibility by 9.

A number is divisible by 9 if the sum of its digits is divisible by 9. Otherwise it is not divisible by 9.

Examples:

8325 is divisible by 9 since 8 + 3 + 2 + 5 = 18 which is divisible by 9. If 8325 ÷ 9 the remainder is 0.

5762 is not divisible by 9 since 5 + 7 + 6 + 2 = 20 which is not divisible by 9. If 5762 ÷ 9 the remainder is 2.

It is important to notice that the tests for divisibility which have been given in this section depend on the number being written in the decimal system. For instance, the number 21 in the decimal system is written 30_{seven} in the system base seven. This number 30_{seven} is not even in spite of the fact that its last digit is zero. However, since 30_{seven} means (3 × seven) + 0, the fact that the last digit is zero tells us that the number is divisible by seven. If a number is written to the base seven it is very easy to tell whether or not it is divisible by seven; one merely looks to see if the last digit is zero.

The property of one number being a factor of another does not depend on the way it is written; for instance, seven is always a factor of twenty-one, no matter how it is written. But the tests for divisibility which are given here depend on the system of numeration in which the number is written.

Exercises 3-12

1. Find the smallest prime factor of each of the following:
   a. 115 b. 135 c. 321 d. 484 e. 539 f. 121
2. Can you give a test for divisibility of 6 in the decimal system?
3. Can you give a test for divisibility by 15 in the decimal system?
4. Which of the following numbers are divisible by 2:
   a. 1111_{ten} b. 1111_{seven} c. 1111_{six} d. 111_{three}
5. Suppose a number is written in the system to the base seven. Is it divisible by ten if its last digit is zero? Is it divisible by three if the sum of its digits is divisible by three?

*6. Answer the above questions for a system of numeration to the base twelve.

*7. Find a test for divisibility by 6 in a system of numeration to the base seven.

*8. Give a test for divisibility by 4 in the decimal system.

3-13. Greatest Common Factor

Consider the numbers 10 and 12. Both 10 and 12 are even numbers. They are both divisible by 2, or you may say that 10 and 12 are multiples of 2. Because 2 is a factor of 10 and is also a factor of 12, 2 is a "common factor" of 10 and 12.

All whole numbers are multiples of 1. Thus 1 is a common factor of the members of any set of whole numbers. Therefore, when you are looking for common factors you generally look for numbers other than 1.

Do the numbers 12 and 30 have any common factors? Writing the set of factors of 12 and the set of factors of 30 as shown at the right you see that there are several common factors. The numbers 1, 2, 3, and 6 are the common factors of 12 and 30.

Set of factors of 12 is {1,2,3,4,6,12}

Set of factors of 30 is {1,2,3,5,6,10,15,30}

Is 6 a factor of both 12 and 30? Referring to the earlier listing of these factors, you see that 12 and 30 have the common factors 1, 2, 3, and 6. How does 6 differ from the other common factors? It is the largest of the common factors of 12 and 30. Such a factor is called the "greatest common factor".

Definition: The greatest common factor of two whole numbers is the largest whole number which is a factor of each of them.

Generally, the greatest common factor is more useful in mathematics than other common factors. Therefore, you will be interested in the greatest common factor.

Exercises 3-13

1. Write the set of all factors for each of the following. List these carefully as you will use these sets in answering Problem 2 below.
   a. 6 c. 12 e. 16
   b. 8 d. 15 f. 21
2. Using your answers in Problem 1 above, write the set of common factors in each of the following cases:
   a. 6,8 c. 12, 15 e. 12,15,21
   b. 8,12 d. 6,8,12 f. 8,12,16
3. Write the set of all factors for each of the following.
   a. 19 c. 36 e. 45
   b. 28 d. 40 f. 72
4. Write the set of common factors for each of the following.
   a. 19,28 c. 28,40 e. 40,72
   b. 16,36 d. 36,45 f. 19,36,45
5. Find the greatest common factor in each of the following cases:
   a. 15, 25 c. 40, 48, 72
   b. 18, 30 d. 20, 50, 100
6. a. What is the greatest common factor of 6 and 6?
   b. What is the greatest common factor of a and a where a is any counting number?
7. a. What is the greatest common factor of 1 and 6?
   b. What is the greatest common factor of 1 and a where a represents any whole number?

8. Let a and b represent any two different whole numbers where $a < b$.
   a. Will a and b always have a common factor? If so, what is the factor?
   b. Let c represent a common factor of a and b. Can $c = a$? If so, give an example.
   c. Can $c = b$? If so, give an example.

9. Suppose 1 is the greatest common factor of three numbers. Must one of the three numbers be a prime number? If not, write a set of three composite numbers whose greatest common factor is 1.

10. In finding the greatest common factor for a set of numbers it is sometimes troublesome to write out all the factors. Try to find a shorter way of obtaining the greatest common factor. Assume that you are to find the greatest common factor of 36 and 45.
    a. Write a complete factorization of 36 and of 45. (List all of the prime factors of 36 and of 45).
       Example: $36 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^2 \cdot 3^3$
       $45 = ? \cdot ? \cdot ? \qquad = ? \cdot ?$
    b. What is the greatest common factor of 36 and 45?
    c. Compare the list of prime factors of 36 and 45 and the greatest common factor of 36 and 45. Can you see a shorter way of obtaining the greatest common factor?

11. Factor completely each number in the following sets and find the greatest common factor for each set of numbers.
    a. {24, 60}
    b. {36, 90}
    c. {72, 108}
    d. {24, 60, 84}
    e. {42, 105, 147}
    f. {165, 234}

*12. a. What is the greatest common factor of 0 and 6?
   b. What is the smallest common factor of 0 and 6?
   c. What is the smallest common factor for any two whole numbers?

*13. You have learned about operations with whole numbers; addition, subtraction, multiplication, and division. In this section you studied the operation of finding the greatest common factor. This is sometimes abbreviated G.C.F. For this problem only let us use the symbol " Δ " for the operation G.C.F. For any whole numbers, a and b and c,

$a \Delta b$ = G.C.F. for a and b

or $a \Delta c$ = G.C.F. for a and c

Example $12 \Delta 18 = 6$

$9 \Delta 15 = 3$

a. Is the set of whole numbers closed under the operation Δ ?

b. Is the operation Δ commutative; that is, does $a \Delta b = b \Delta a$?

c. Is the operation Δ associative, that is, does $a \Delta (b \Delta c) = (a \Delta b) \Delta c$?

3-14. Remainders in Division

The usual way of finding the answer to this division problem is shown below:

$$\begin{array}{r} 3 \text{ Remainder } 1 \\ 5\overline{)16} \\ \underline{15} \\ 1 \end{array}$$

To check the answer use the following idea:

$$16 = (5 \times 3) + 1.$$

In the division problem above, the 16 is called the dividend, the 5 is the divisor, the 3 is the quotient, and the 1 is the remainder.

Let's try another example. Divide 253 by 25.

$$\begin{array}{r} 25\,\overline{)\,253} \\ \underline{25} \\ 3 \end{array}$$

Does 253 = (25 × 10) + 3?

In division:

dividend = (divisor × quotient) + remainder

Using mathematical symbols, where

"a" represents the dividend,
"b" represents the divisor,
"q" represents the quotient,
"R" represents the remainder.

This division relation may be expressed as follows:

$$a = (b \cdot q) + R$$

Consider the following example in division:

$$\begin{array}{rl} 24 & \text{Remainder } 23 \\ 25\,\overline{)\,623} & \\ \underline{50} & \\ 123 & \\ \underline{100} & \\ 23 & \end{array}$$

You can write

$$623 = (25 \times 24) + 23.$$

This follows the general form:

dividend = (divisor × quotient) + remainder

or

$$a = (b \cdot q) + R$$

Exercises 3-14

1. Copy and complete the following table. Do this carefully as you will use the table in answering Question 2.

	DIVIDEND =	(DIVISOR ·	QUOTIENT) +	REMAINDER
EXAMPLE	9	4	2	1
a.	41	13	?	?
b.	59	?	5	9
c.	?	11	6	0
d.	77	?	3	17
e.	81	?	?	0

2. Use the table in Problem 1 in answering parts a, b, and c.
   a. Compare the divisor and quotient in each part. Does one of these always have the greater value in a division problem?
   b. Compare the quotient and dividend. Which one has the greater value, if the dividend and divisor are both counting numbers?
   c. Compare the divisor and the remainder. Which one always has the greater value in a division problem?
   d. Can the dividend be zero? If so, give an example.
   e. Can the divisor be zero? If so, give an example.
   f. Can the quotient be zero? If so, give an example.
   g. Can the remainder be zero? If so, give an example.
3. Using the table in Problem 1, answer the following questions.
   a. Can any whole number appear as a dividend? If not, give an example.
   b. Can any whole number appear as a divisor? If not give an example.
   c. Can any whole number appear as a quotient? If not give an example.
   d. Must the remainder always be some whole number? Explain.

4. Copy and complete the following table for the division relation.

$$a = (b \cdot q) + R$$

	a	b	q	R
a.	50	12	?	?
b.	100	?	?	0
c.	283	17	?	?
d.	630	?	25	5

5. Using the table above answer the following:
   a. Can R be greater than b? If so give an example.
   b. Can q be greater than b? If so, give an example.
   c. Can R be greater than the quotient q? If so, give an example.
   d. Can any whole number be a possible value of b? Explain.
   e. Can any counting number be a possible value of b? Explain.
   f. Can any whole number be a possible value of a? Explain.

6. Using the division relation, $a = (b \cdot q) + R$,
   where $R < b$, answer the following:
   a. If $b = 11$, describe the set of all possible remainders.
   b. If all the possible remainders in a division problem are the whole numbers less than 25, what is b?
   c. If $b = K$, which one of the following represents the number of all possible remainders.
      (K), $(K + 1)$, or $(K - 1)$.

*7. In Section 13, you learned how to find the greatest common factor of two numbers. By using the division relation there is another method for doing this.

EXAMPLE: Find the greatest common factor of 35 and 56.

(1) First, divide the larger number, 56, by the smaller number, 35.

(2) Second, divide the divisor, 35, by the remainder 21.

(3) Next, continue dividing the last divisor by the last remainder until the remainder is 0.

The last divisor used is the greatest common factor. The 7 is the greatest common factor of 35 and 56. Note that when 14 is divided by 7 the remainder is 0. The 7 is the last divisor used.

Using the above method, find the greatest common factor for each of the following pairs of numbers:

a. 124 and 836.
b. 336 and 812
c. 1207 and 1349

*8. Is there a one-to-one correspondence between the following set of numbers? If so, set up the correspondence.

a. The set of counting numbers from 1 through 11 and the set of whole numbers from 0 through 10.
b. The set of odd numbers between 50 and 80 and the set of even numbers between 17 and 47.
c. The set of all multiples of 3 which are less than 44 and the set of all multiples of 7 which lies between 100 and 200.

3-15. Least Common Multiple

You have already learned a great deal about multiples of numbers:

that all whole numbers are multiples of 1;
that even numbers {0, 2, 4, 6, ...,} are multiples of 2;
that {0, 3, 6, 9, ...,} are multiples of 3.

Similarly the multiples of any counting number can be listed.

The number 2 is an even number, and the number 3 is an odd number. Usually you do not think of such numbers as having much in common. Yet if you look at the set of multiples for 2 and the set of multiples of 3 you see that they do have something in common. Some of the multiples of 2 are also multiples of 3. For example, 6 is a multiple of both 2 and 3. There are many such numbers divisible by both 2 and 3. The set of these numbers is written as follows:

{6, 12, 18, 24, 30, ...,}

Definition: Numbers which are multiples of more than one number are called common multiples of those numbers. "Common" means belonging to more than one. Thus 6 and 12 are common multiples of 2 and 3.

Definition: The least common multiple of a set of counting numbers is the smallest counting number which is a multiple of each member of the set of given numbers.

Note that 0 is a common multiple for any set of whole numbers. However, in adding or subtracting fractions, 0 cannot be used as a common denominator. Can you write $\frac{1}{2}$ with a zero denominator? Because you cannot do so, you will be interested only in the least common multiple other than zero.

Suppose you wish to find the least common multiple of 12 and 18. First, list the sets of multiples of each:

Set of Multiples of 12: {0, 12, 24, 36, 48, 60, 72, 84, ...,}

Set of Multiples of 18: {0, 18, 36, 54, 72, ...,}

The set of common multiples of 12 and 18 is {0, 36, 72, 108, ...,}. The smallest counting number in this set is 36. Therefore, 36 is the least common multiple of 12 and 18.

Exercises 3-15

1. Write the least common multiple of the elements of each of the following sets.
   a. 6 and 8   b. 9 and 12
2. Find the least common multiple of the elements of each of the following sets.
   a. {2, 6, 7}   b. {8, 9, 12}
3. Find the least common multiple for each of the following sets:
   a. {4, 6}   c. {10, 12}
   b. {6, 9}   d. {10, 15, 30}
4. Answer the following:
   a. If c and d are composite counting numbers can c or d be the least common multiple? Write an example to explain your answer.
   b. If c and d are composite counting numbers, must c or d be the least common multiple? Write an example to your answer.

5. a. What is the least common multiple of 6 and 6?
   b. What is the least common multiple of 29 and 29?
   c. What is the least common multiple of a and a where a is any counting number?
6. a. What is the least common multiple of 1 and 6?
   b. What is the least common multiple of 1 and 29?
   c. What is the least common multiple of 1 and a where a represents any counting number?
7. a. If a and b are different prime numbers, can a or b represent the least common multiple of a and b?
   b. If a and b are different prime numbers how can you represent the least common multiple of a and b?
   *c. If a, b, and c are different prime numbers, what is the least common multiple of a, b, and c?
8. Study the following example. Try to discover a shorter way to determine the least common multiple.

EXAMPLE: To find the least common multiple of 4, 6, and 8:

(1) First, write a complete factorization for each number.

$4 = 2^2$ $6 = 2 \cdot 3$ $8 = 2^3$

(2) The least common multiple is $2^3 \cdot 3$ or 24.

(3) Note that $2^2 \cdot 2 \cdot 3 \cdot 2^3 = 192$ which is a common multiple of 4, 6, and 8, but not the least.

Now find the least common multiple of each set in the following parts.

a. 12, 16 c. 10, 14
b. 14, 16 d. 4, 5, 6

*9. a. Is there a greatest common multiple of 3 and 5? If so, write an example.
   b. Is there a greatest common multiple of any set of counting numbers?

*10. a. May 0 be considered as a multiple of zero? (Does $0 \times 0 = 0$?)

b. May 0 be considered as a multiple of six? (Does $6 \times 0 = 0$?)

c. May 0 be considered as a multiple of a if a is any whole number?

d. Assume the least common multiple was defined as "the smallest whole number" instead of "the smallest counting number". What would be the least common multiple for any set of counting numbers?

e. Using the correct definition for least common multiple, is there a least common multiple for any counting number and 0?

Review Exercises

1. Find the greatest common factor of the numbers in each of the following sets of numbers.

a.	{23, 43}	d.	{74, 146}
b.	{66, 78}	e.	{45, 72, 252}
c.	{39, 51}	f.	{44, 92, 124}

2. Find the least common multiple of the numbers in each of the sets of numbers in parts a. through f. in Problem 1.

3. a. Find the product of the members of each set of numbers in Problem 1.

b. Find the product of the greatest common factor and the least common multiple for each set of numbers in Problem 1. (Refer to your answers for Problem 1 and Problem 2.)

c. How do your answers for a and b compare?

4. Let a and b represent two counting numbers. Suppose that the greatest common factor of a and b is 1.

a. What is the least common multiple of a and b? Give an example to explain your answer.

b. Would your answer for part a. be true if you started with three counting numbers a, b, and c? (Remember, the greatest common factor is 1.) Give an example to explain your answer.

5. a. Can a prime number be even? Give an example to explain your answer.
   b. Can a prime number be odd? Give an example to explain your answer.
   c. How many prime numbers end with the digit 5?
   d. With the exception of two prime numbers, all primes end with one of four digits. Write the two primes which are exceptions.
   e. Write the other four digits which occur in the ones place for all primes other than the exception you found in part d.
6. Suppose the greatest common factor of two numbers is the same as their least common multiple. What must be true about the numbers? Give examples to explain your answer.
7. 112 tulip bulbs are to be planted in a garden. Describe all possible arrangements of the bulbs if they are to be planted in straight rows with an equal number of bulbs per row.
8. Two bells are set so that their time interval for striking is different. Assume that at the beginning both of the bells strike at the same time.
   a. One bell strikes every three minutes and the second strikes every five minutes. If both bells strike together at 12:00 o'clock noon, when will they again strike together?
   b. One bell strikes every six minutes and the second bell every fifteen minutes. If both strike at 12:00 o'clock noon, when will they again strike together?
   c. Find the least common multiple of 3 and 5; and of 6 and 15. How do these answers compare with parts a. and b.?

9. a. Can the greatest common factor of some whole numbers ever be the same number as the least common multiple of those whole numbers? If so, give an example.

   b. Can the greatest common factor of some whole numbers ever be greater than the least common multiple of those numbers? If so, give an example.

   c. Can the least common multiple for some whole numbers ever be less than the greatest common factor of those whole numbers? If so, give an example.

*10. a. Is it possible to have exactly four composite numbers between two consecutive primes? If so, give an example.

   b. Is it possible to have exactly five consecutive composite numbers between two consecutive primes? If so give an example.

*11. Given the numbers 135, 222, 783, 1065. <u>Without dividing</u> answer the following questions. Then check your answers by dividing.

   a. Which numbers are divisible by 3?
   b. Which numbers are divisible by 6?
   c. Which numbers are divisible by 9?
   d. Which numbers are divisible by 5?
   e. Which numbers are divisible by 15?
   f. Which numbers are divisible by 4?

*12. Ten tulip bulbs are to be planted so that there will be exactly five rows with four bulbs in each row. Draw a diagram of this arrangement.

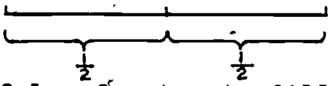
*13. Do you think there is a largest prime number? Can you find it or can you give a reason why you think there is no greatest one?

Chapter 4

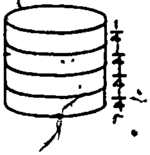
THE RATIONAL NUMBER SYSTEM

4-1. History of Fractions

Man has not always known about fractions. Historically, he introduced fractions when he began to measure as well as count. If he divided a piece of string into two parts of equal length, then each part was $\frac{1}{2}$ as long as the original string.



If he had to pour 4 cupfuls of water to fill a container,



then he said that the cup held $\frac{1}{4}$ the amount of water in the container.

The Egyptians worked with fractions. At first they used only unit fractions and the fractions, $\frac{2}{3}$ and $\frac{3}{4}$. Unit fractions are fractions with numerators of 1, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$. The Egyptians used the notation  for $\frac{1}{5}$, that is, the numeral for 5 with a special mark written over it. When they had to use other fractions, they expressed them in terms of unit fractions:

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$

$$\frac{15}{24} = \frac{1}{2} + \frac{1}{8}$$

The Rhind Papyrus (1700 BC), copied by the scribe Ahmes from an older document now lost, has a set of tables showing how to express fractions in terms of unit fractions.

The Babylonians usually used fractions with denominators of 60, 60^2 (3600), and 60^3 (216,000), etc. because the base of their system of notation was 60. Since our units of time are borrowed from the Babylonians, an hour is divided into sixtieths, called minutes, and a minute is divided into sixtieths, called seconds.

Roman children primarily learned about fractions with denominators of twelve. They did not have symbols for fractions but they did have names for fractions such as $\frac{1}{12}, \frac{2}{12}, \ldots$

Over the years many other notations were used. Our present notation with the fraction bar "—" came into general use in the 16th century.

A number may have several names. Several names or numerals for the same number are 6, VI, $2 \cdot 3$, $3 \cdot 2$. Some other names for this number are the fractions $\frac{6}{1}$, $\frac{12}{2}$, $\frac{18}{3}$, $\frac{24}{4}$. Each fraction is a name for a certain number. The fractions $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{5}{15}$, $\frac{6}{18}$ represent one number. The fractions $\frac{3}{2}$, $\frac{6}{4}$, $\frac{9}{6}$, $\frac{12}{8}$, $\frac{15}{10}$, $\frac{18}{12}$ represent another number. Such numbers are called <u>rational</u> <u>numbers</u>. Numerals for other rational numbers are $\frac{7}{1}$, $\frac{8}{2}$, $\frac{3}{4}$.

How do you check the division problem:

$$12 \div 3 = 4?$$

You multiply 3 by 4 to see if you get 12. When you divide 12 by 3 you are finding an answer to the question:

3 times what number equals 12?

or

$$3 \cdot ? = 12$$

It is better to use a letter such as "x" instead of "?" for the number, as in

$$3 \cdot x = 12.$$

If you replace x by 4, the number sentence,

$$3 \cdot x = 12$$

is true. Mathematicians usually write $\frac{12}{3}$ instead of 12 + 3. So they write, $\frac{12}{3} = 4$.

With the new symbol for division you can write,

$\frac{12}{3} = 4$ because $3 \cdot 4 = 12$,

$\frac{6}{2} = 3$ because $2 \cdot 3 = 6$

Because $\frac{12}{3} = 4$ you can replace 4 by $\frac{12}{3}$ in $3 \cdot 4 = 12$ and write

$$3 \cdot \frac{12}{3} = 12.$$

Similarly,

$$2 \cdot \frac{6}{2} = 6.$$

If $x = \frac{6}{2}$, then $2 \cdot x = 6$,

if $x = \frac{7}{3}$, then $3 \cdot x = 7$.

Class Discussion Problems.

1. (a) If $x = \frac{10}{2}$, $2 \cdot x$ is what number?
   (b) If $x = \frac{63}{9}$, $9 \cdot x$ is what number?
   (c) If $x = \frac{5}{2}$, $2 \cdot x$ is what number?
2. For each of the following give a fractional name for the number represented by x.
   (a) $2 \cdot x = 10$
   (b) $2 \cdot x = 5$
   (c) $9 \cdot x = 4$
   (d) $4 \cdot x = 13$

In general, if a and b are whole numbers, and b is not zero, $\frac{a}{b}$ is the number x for which $b \cdot x = a$. Because it is simpler, a symbol such as $b \cdot x$ is usually written bx, and $8 \cdot x$ is written $8x$. The multiplication symbol is still necessary in writing a symbol such as $8 \cdot 6$. Why? You have heard that division is the inverse of multiplication. Here this is used to change the question $\frac{a}{b} = ?$
to $b \cdot ? = a$.

A symbol " $\frac{a}{b}$ " where a and b are numbers, with b not zero, is called a fraction. If a and b are whole numbers, with b not zero, the number represented by the fraction, $\frac{a}{b}$, is called a rational number; any number which can be written in this form is called a rational number. For example, 0.5 represents a rational number because the same number can be written $\frac{1}{2}$. A fraction is a name for a rational number just as numeral is a name for a number. Different names for the same number are:

$$3 \ , \ \text{III} \ , \ \frac{6}{2} \ , \ \frac{9}{3} \ , \ \frac{63}{21} \ .$$

The names

$$\frac{6}{2} \ , \ \frac{9}{3} \ , \ \frac{63}{21}$$

are fractions.

Sometimes $\frac{a}{b}$ is a whole number. This happens when b is a factor of a, and only then.

Two fractions which represent the same number are called equivalent fractions.

Exercises 4-1

1. Give an example of each of the following kinds of numbers.
   (a) Counting number.
   (b) Whole number.
   (c) A whole number which is not a counting number.
   (d) A rational number which is not a whole number.
2. Which of the following represent rational numbers?
   (a) $\frac{3}{7}$ (c) 0.13
   (b) $\frac{5}{3}$ (d) 4
3. Copy and complete the following statements.
   (a) If $x = \frac{11}{3}$, then ___x = ___ .
   (b) If $x = \frac{63}{9}$, then ___x = ___ .
   (c) If $x = \frac{0}{5}$, then ___x = ___ .
   (d) If $x = \frac{123}{11}$, then ___x = ___ .

4. For Problem 3, in which cases is the number x a whole number? When x is a whole number, write it with a single digit.

5. Without dividing or ~~factoring,~~ decide which of the following statements are true. As an example, to show that $\frac{168}{21} = 8$, multiply 8 by 21 to see if you get 168.

   (a) $\frac{262}{17} = 16$

   (b) $\frac{744}{124} = 6$

   (c) $\frac{15251}{151} = 101$

*6. For each of the following, write a number sentence which describes the problem in mathematical language. Use x for the unknown number, and tell, in each case, for what it stands.

   Example: Sam's father is sawing a twelve-foot log into 6 equal lengths. How long will each piece be?

   Answer: If x is the length of each piece in feet, then $6 \cdot x = 12$.

   (a) If 12 cookies are divided equally among 3 boys, how many cookies does each boy receive?

   (b) Mr. Carter's car used 10 gallons of gasoline for a 160-mile trip. How many miles did he drive for each gallon of gasoline used?

   (c) If it takes 20 bags of cement to build a 30-foot walk, how much cement is needed for each foot of the walk?

   (d) Thirty-two pupils were divided into 4 groups of the same size. How many pupils were in each group?

   (e) A teacher has 12 sheets of paper to distribute evenly in a class of 24. How much paper will each pupil receive?

4-2. Properties of Rational Numbers

You have seen that the whole number 3 can be written, $\frac{3}{1}$, which shows that 3 is a rational number. In a similar way, you can show that each whole number is a rational number.

When you studied whole numbers you learned that the whole numbers had certain properties. Learning about rational numbers is made easier by knowing that the rational numbers have some of the same properties.

You remember that the sum of two whole numbers is always a whole number, and the product of two whole numbers is always a whole number. That is, if a and b are whole numbers, there is a whole number c for which $a + b = c$ and a whole number d for which $a \cdot b = d$. The set of whole numbers has the closure property for addition and multiplication.

The set of rational numbers also has the closure property for addition and multiplication. The sum of two rational numbers is a rational number. You know that $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$, $\frac{12}{4} + \frac{1}{4} = \frac{13}{4}$, $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, $2 + \frac{1}{4} = \frac{9}{4}$. The product of two rational numbers is a rational number. Notice that $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$, $\frac{2}{3} \cdot \frac{6}{7} = \frac{12}{21}$, $\frac{5}{1} \cdot \frac{1}{6} = \frac{5}{6}$. In precise language this can be stated:

1) The set of rational numbers is closed with respect to the operations of addition and multiplication.

You know that $3 + 4 = 4 + 3$ and $3 \cdot 4 = 4 \cdot 3$ because, for the whole numbers, addition and multiplication have the commutative property. These operations also have the commutative property for the rational numbers. You know that $\frac{2}{5} + \frac{3}{5} = \frac{3}{5} + \frac{2}{5}$, and $\frac{2}{5} \cdot \frac{3}{5} = \frac{3}{5} \cdot \frac{2}{5}$. In precise language this can be stated:

2) The operations of addition and multiplication for the rational numbers have the commutative property, that is:

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

You also remember that $5 + (3 + 4) = (5 + 3) + 4$, and $5 \cdot (3 \cdot 4) = (5 \cdot 3) \cdot 4$, because addition and multiplication have the associative property for the whole numbers. For the rational numbers also, these operations have the associative property. You know that $\frac{5}{3} + (\frac{1}{3} + \frac{2}{3}) = (\frac{5}{3} + \frac{1}{3}) + \frac{2}{3}$, and that $\frac{3}{1} \cdot (\frac{1}{3} \cdot \frac{6}{3}) = (\frac{3}{1} \cdot \frac{1}{3}) \cdot \frac{6}{3}$. In precise language, this can be stated:

3) The operations of addition and multiplication for the rational numbers have the associative property, that is:

$$a + (b + c) = (a + b) + c \text{ and } a(bc) = (ab)c$$

From the distributive property you know that you get the same result if you think of $5 \cdot 5 = 25$ as you do if you think of $5 \cdot 2 + 5 \cdot 3 = 10 + 15 = 25$. For the whole numbers, multiplication is distributive over addition. The distributive property also holds for the rational numbers. You have used this property for the rational numbers when you multiplied $4\frac{1}{5}$ by 5. In our symbols, $5 \cdot (4\frac{1}{5}) = 5 \cdot (4 + \frac{1}{5}) = (5 \cdot 4) + (5 \cdot \frac{1}{5}) = 20 + 1 = 21$. In precise language, this can be stated:

4) The operation of multiplication is distributive over addition for the rational numbers; that is:

$$a(b + c) = ab + ac$$

Among the whole numbers were two special numbers 1 and 0. These are also rational numbers.

5) Among the rational numbers are special numbers 0 and 1; 0 is the identity for addition and 1 is the identity for multiplication.

When it is said that 0 is the identity for addition it is meant, for instance, that $0 + 3 = 3 + 0 = 3$; that is, that adding zero to any number does not change it. This can be expressed in symbols as:

$$0 + a = a + 0 = a ,$$

no matter what rational number a is. Similarly when it is said that 1 is the identity for multiplication, it is meant, for instance, that $1 \cdot 5 = 5 \cdot 1 = 5$; that is, multiplying any number by 1 does not change it. This can be expressed in symbols as:

$$1 \cdot a = a \cdot 1 = a .$$

You can see that 1 is a rational number by writing it as the fraction $\frac{1}{1}$. To see that 0 is a rational number you should remember that 0 divided by any counting number is 0. If $x = \frac{0}{1}$, $1 \cdot x = 0$ and x must equal zero. In defining a rational number, $\frac{a}{b}$, it was said that b could not be zero. You can see the reason for this by seeing what happens to $\frac{5}{0}$.

If $x = \frac{5}{0}$, then $0 \cdot x = 5$. There is no number x for which $0 \cdot x = 5$ so there is no number $\frac{5}{0}$.

These five properties of the rational numbers furnish the reasons for some of the rules you state for fractions. These properties may be used to establish the fact that $\frac{3}{2} = \frac{15}{10}$.

If $x = \frac{3}{2}$, then $2x = 3$.

Since 2x and 3 are names for the same number,

$$5 \cdot (2x) = 5 \cdot 3 .$$

By the associative property,

$$(5 \cdot 2)x = 5 \cdot 3$$
$$10x = 15$$
$$x = \frac{15}{10}.$$

But x is a name for $\frac{3}{2}$, so

$$\frac{3}{2} = \frac{15}{10}$$

If you write the last equation,

$$\frac{3}{2} = \frac{5 \cdot 3}{5 \cdot 2},$$

you see that you would have arrived at the same fraction if you had multiplied the numerator and denominator of $\frac{3}{2}$ by 5. Generalizing, you will get

Property 1. If the numerator and denominator of a fraction are multiplied by the same counting number, the number represented is not changed. If the numerator and denominator are divided by the same counting number, the number represented is not changed.

You saw that $\frac{3}{2}$ and $\frac{15}{10}$ are fractions for the same number. Other names for this number are: $\frac{6}{4}$, $\frac{9}{6}$, $\frac{18}{12}$. By what numbers should you multiply the numerator and denominator of $\frac{3}{2}$ to get these fractions? Since in $\frac{3}{2}$ the numerator and denominator have no common factors except 1, this is called the simplest form of the fraction.

To find the simplest form of $\frac{72}{45}$, find the greatest common factor of 72 and 45, which is 9. Then

$$\frac{72}{45} = \frac{9 \cdot 8}{9 \cdot 5} = \frac{8}{5}.$$

You may prefer to take more steps and do it this way:

$$\frac{72}{45} = \frac{3 \cdot 24}{3 \cdot 15} = \frac{24}{15} = \frac{3 \cdot 8}{3 \cdot 5} = \frac{8}{5}.$$

To write the fraction $\frac{a}{b}$ in simplest form, find the greatest common factor k of a and b, where $a = kc$ and $b = kd$; then by Property 1,

$$\frac{a}{b} = \frac{kc}{kd} = \frac{c}{d}.$$

Exercises 4-2

1. Which of the following represent rational numbers?

   (a) 7

   (b) $\frac{9}{5}$

   (c) $3 + \frac{1}{2}$

   (d) 2.15

   (e) $\frac{0}{7}$

   (f) $\frac{5}{0}$

   (g) 0.0

2. Each of the following is true by one of the properties of rational numbers. Name the property in each case.

   (a) $\frac{2}{3} \cdot \frac{5}{6} = \frac{5}{6} \cdot \frac{2}{3}$

   (b) $\frac{5}{2}(\frac{3}{2} \cdot \frac{2}{5}) = (\frac{5}{2} \cdot \frac{3}{2}) \cdot \frac{2}{5}$

   (c) $5(2 + 3) = 5 \cdot 2 + 5 \cdot 3$

   (d) $\frac{1}{2}(4 + 6) = \frac{1}{2}(4) + \frac{1}{2}(6)$

   (e) $1 \cdot \frac{5}{6} = \frac{5}{6}$

   (f) $\frac{2}{3} \cdot \frac{5}{5} = \frac{2}{3}$

   (g) $0 + \frac{7}{8} = \frac{7}{8}$

   (h) $\frac{3}{2} + \frac{4}{5} = \frac{4}{5} + \frac{3}{2}$

   (i) $\frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) = (\frac{1}{2} + \frac{1}{3}) + \frac{1}{4}$

   (j) $5(6 \cdot 7) = (5 \cdot 6) \cdot 7$

3. What is the difference between 53 of the 67 equal parts of a rectangle, and one of the 67 equal parts of 53 rectangles, assuming that all rectangles considered are equal in size?

*4. Find five pairs of counting numbers which can be used as values for n and d to make the following number sentence true:

$$3n = 2d.$$

Compute $\frac{n}{d}$ in each case. What is the general rule?

*5. Use the number sentence $2x = 3$ to show that $\frac{3}{2} = \frac{1}{2} \cdot 3$. Hint: If $2x$ and 3 are names for the same number then $\frac{1}{2}(2x) = \frac{1}{2} \cdot 3$.

4-3. Reciprocals

You know that

$$2 \cdot \frac{1}{2} = 1, \quad 3 \cdot \frac{1}{3} = 1, \quad \frac{1}{4} \cdot 4 = 1, \quad \frac{1}{5} \cdot 5 = 1, \quad 31 \cdot \frac{1}{31} = 1.$$

Let us recall our definition of a rational number and see how these products are related to our definition.

a and b are whole numbers,
$b \neq 0$

$$bx = a.$$

$$x = \frac{a}{b}$$

$$\boxed{\begin{array}{l} \text{Let } a = 1 \\ \text{Let } b = 31 \end{array}}$$

1 and 31 are whole numbers,
$31 \neq 0$

$$31x = 1$$

$$x = \frac{1}{31}$$

You know that $\frac{1}{31}$ is the rational number by which you can multiply 31 to obtain 1.

Now consider the equation

$$bx = 1.$$

Then $\quad x = \frac{1}{b}$ (b is a counting number.)

and $\quad b \cdot \frac{1}{b} = 1.$

The number $\frac{1}{b}$ is called the reciprocal of b. Also, b is called the reciprocal of $\frac{1}{b}$. <u>If the product of two numbers is 1, the numbers are called reciprocals of each other</u>.

You have seen that counting numbers and numbers like $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{29}$ have reciprocals. Does $\frac{3}{4}$ have a reciprocal? Is there a number by which $\frac{3}{4}$ may be multiplied to get 1? You may know the answer from arithmetic.

You know that $\frac{3}{4} \cdot 4 = 3$ and $3 \cdot \frac{1}{3} = 1$, so

$$\frac{3}{4} \cdot \left(4 \cdot \frac{1}{3}\right) = \left(\frac{3}{4} \cdot 4\right) \cdot \frac{1}{3}$$

$$= (3) \cdot \frac{1}{3} = 1,$$

using the associative property.

You can see that if $\frac{3}{4}$ is multiplied by $4 \cdot \frac{1}{3}$ the product is 1. But $4 \cdot \frac{1}{3} = \frac{4}{3}$. Hence, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

What is the product $\frac{7}{8} \cdot \frac{8}{7}$? Why? You can see why the product is 1, by recalling that $\frac{8}{7} = 8 \cdot \frac{1}{7}$,

$$\frac{7}{8} \cdot \frac{8}{7} = \frac{7}{8} \cdot (8 \cdot \frac{1}{7}) = (\frac{7}{8} \cdot 8) \cdot \frac{1}{7} = 7 \cdot \frac{1}{7} = 1.$$

From your experience in multiplying numbers represented by fractions you probably know, without showing all the steps, that $\frac{8}{7} \cdot \frac{7}{8} = 1$. The steps show why this is a true sentence, making use of the properties of rational numbers that you know.

The examples lead us to the conclusion that $\frac{a}{b} \cdot \frac{b}{a} = 1$, provided neither a nor b is zero.

<u>Property 2. The reciprocal of the rational number $\frac{a}{b}$ is the rational number $\frac{b}{a}$, if $a \neq 0$ and $b \neq 0$.</u>

<u>Exercises 4-3</u>

1. Write the reciprocals of the rational numbers.

   (a) 11　　　　(c) $\frac{2}{7}$

   (b) $\frac{1}{5}$　　　　(d) $\frac{50}{3}$

2. In the following, letters represent rational numbers, all different from zero. Write the reciprocals.

   (a) m　(b) s　(c) $\frac{1}{c}$　(d) $\frac{r}{s}$　(e) $\frac{t}{w}$

3. Write the following as sentences involving multiplication, and find n in each.

   (a) $8 \div 7 = n$

   (b) $2 \div 11 = n$

   (c) $64 \div 36 = n$

4. Write the set of numbers consisting of the reciprocals of the members of the set, Q, where

$$Q = \{1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}\}.$$

5. (a) When is the reciprocal of a number greater than the number?
   (b) When is the reciprocal of a number less than the number?
   (c) When is the reciprocal of a number equal to the number?
   (d) If n is a counting number, can we correctly say that one of the following is always true?

   $$(1)\ n > \frac{1}{n},\quad (2)\ \frac{1}{n} > n,\quad (3)\ n = \frac{1}{n}$$

6. The product of what number and n is 1, where n is a number such that $2n = 19$?

7. (a) Find n if $8n = 36$.
   (b) Find q if $36q = 8$.
   (c) What can you say about the numbers n and q?

8. The population of Cary is 16,000 and of a neighboring city, Davis, 30,000.
   (a) The population of Davis is how many times the population of Cary?
   (b) The population of Cary is how many times the population of Davis?
   (c) What can you say about the answers for (a) and (b)?

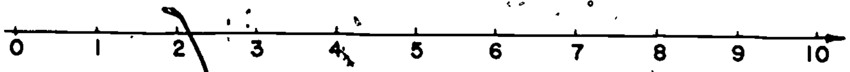
*9. If $14h = 5$, then $(\frac{1}{5})(14n) = (\frac{1}{5})(5)$. (a) Why? (b) Use the equation in (a) to find the number by which we can multiply n (given by $14n = 5$) to obtain 1.

*10. If $ax = b$ and $by = a$, and a and b are not zero, x and y are reciprocals. Why is this true?

4-4. Using The Number Line

Recall how the number line was constructed in a previous chapter You started with a line and selected a point which you called "0".

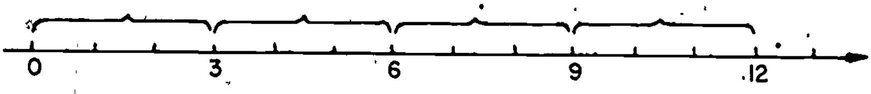
After selecting a unit of length, you marked off this distance on the ray, starting at 0 and extending to the right. After doing this again and again you can label the endpoints of the segments 1, 2, 3, and so on, and have the picture shown below:



The counting number at each point shows how many segments of unit length are measured off from 0 to that point.

To add 3 and 2 on the number line, start with 0 and lay off a distance of 3 equal units. Then measure off 2 more equal units to the right. The union of these two segments is that whose left end is at 0 and whose right end is at 5; the length of this segment is the sum of the lengths of the other two segments. Thus you have a picture on the line which shows the sum: $3 + 2 = 5$.

To multiply 3 by 4 on the number line you can start from 0 and lay off four segments of length 3 as shown below:



This also gives a means of division. If you divide the segment with endpoints 0 and 12 into four equal parts, each part will be of length 3. Hence the same picture can represent both

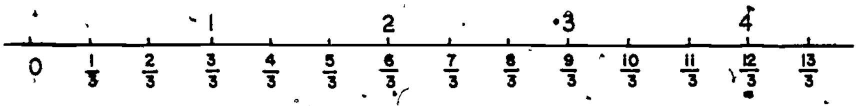
$$12 = 4 \times 3 \text{ and } \frac{12}{4} = 3.$$

If you think of 12 as designating 12 feet, the line shows that there are four yards in twelve feet and hence

$$\frac{12}{3} = 4, \text{ or, twelve-thirds is equal to four.}$$

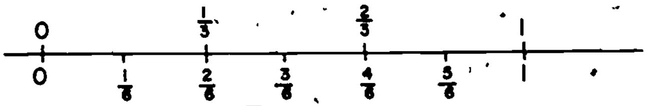
In words: "Twelve feet is equal to four yards" may also be written "twelve thirds-of-a-yard is equal to four yards," since one foot is one third-of-a-yard.

So far you have shown on the number line only segments whose lengths are whole numbers of units. You could just as well have considered fractions by labeling the points differently. For instance, you might have:



You could see from this just as well as from the previous number line that $12 \cdot (\frac{1}{3}) = 4$. Also you can see that $\frac{13}{3} = 4 + \frac{1}{3}$ from the same number line.

Recall that Property 1, expressed the fact that you can multiply the numerator and denominator of a fraction by a counting number without changing the number which it represents. If the denominator is small, you can show this on the number line. For instance, $\frac{1}{3} = \frac{2}{6}$ can be seen from the following:



Property 1 gives us a way to simplify certain fractions. It can also be used to show whether or not the numbers represented by two fractions are equal.

Example 1. Suppose you have the question:

$$\text{Is } \frac{6}{9} = \frac{8}{12}\ ?$$

This can be answered by finding the simplest form of each of the fractions as follows:

$$\frac{6}{9} = \frac{2\cdot 3}{3\cdot 3} = \frac{2}{3} \quad\text{and}\quad \frac{8}{12} = \frac{2\cdot 4}{3\cdot 4} = \frac{2}{3}.$$

Since each number is equal to $\frac{2}{3}$, the given numbers are equal to each other.

Example 2. Is $\frac{9}{15}$ equal to $\frac{14}{22}$? Now,

$$\frac{9}{15} = \frac{3}{5} \quad\text{and}\quad \frac{14}{22} = \frac{7\cdot 2}{11\cdot 2} = \frac{7}{11}.$$

Is $\frac{3}{5}$ equal to $\frac{7}{11}$? Whatever your answer is, can you give a reason? You found that it is easier to compare fractions if they have the same denominator. This suggests another method which you may use to answer the question, "Is $\frac{3}{5}$ equal to $\frac{7}{11}$?" You can rewrite the two fractions with a common denominator, that is, having the same denominator. This denominator must be a multiple of both 5 and 11. The least such denominator is 55, since this is the least common multiple of 5 and 11. Then

$$\frac{3}{5} = \frac{3\cdot ?}{5\cdot 11}$$

shows that since you multiplied the denominator by 11 to get 5 · 11, you must also multiply the numerator by 11, that is, you must also fill in the question mark with an 11. Thus

$$\frac{3}{5} = \frac{3\cdot 11}{5\cdot 11} = \frac{33}{55}$$

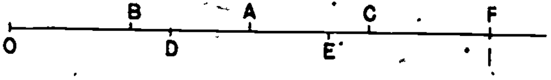
and similarly

$$\frac{7}{11} = \frac{7\cdot 5}{11\cdot 5} = \frac{35}{55}.$$

Therefore $\frac{9}{15}$ is not equal to $\frac{14}{22}$.

Exercises 4-4

1. (a) Consider the points labeled A, B, C, D, E, and F on the number line:



Give 3 different fraction names for each of the points A, B, C, D, E, and F as:

A: $\frac{1}{2}$, $\frac{2}{4}$, ___; etc.

(b) Is the rational number located at point B less than or greater than the one located at A? Explain your answer.

(c) Is the rational number located at C less than or greater than that at A? Explain your answer.

2. Interpret on the number line the following:

(a) $\frac{20}{5} = 4$ (b) $\frac{20}{4} = 5$ (c) $\frac{23}{5} = 4\frac{3}{5}$

3. Express each of the following as a whole number added to a rational number less than 1:

(a) $\frac{57}{39}$ (b) $\frac{137}{23}$

4. Show on the number line the equality: $\frac{2}{8} = \frac{3}{12}$

5. Show how to subtract on the number line for each of the following:

(a) 3 - 2 (b) 9 - 5

*6. In the Example, you found whether two fractions represented the same number by writing each of them as equivalent fractions with equal denominators. Could you also have found whether they represented the same number by writing each of them as equivalent fractions with equal numerators? Remember that two fractions were called "equivalent" if they represented the same number.

4-5. Multiplication and Division of Rational Numbers

You recall that $3 \cdot \frac{1}{3} = 1$ and $3 \cdot \frac{2}{3} = 2$. In general if a and b are whole numbers, where b is not zero,

$$b \cdot \frac{1}{b} = 1 \quad \text{and} \quad b \cdot \frac{a}{b} = a.$$

What does $\frac{5}{7} \cdot 2$ equal? You can find an answer if you can find a number x for which

$$x = \frac{5}{7} \cdot 2.$$

But then, x and $\frac{5}{7} \cdot 2$ are names for the same number, so that

$$7x = 7(\frac{5}{7} \cdot 2).$$

Using the associative property for multiplication

$$7x = (7 \cdot \frac{5}{7}) \cdot 2 = 5 \cdot 2 = 10$$

$$x = \frac{5 \cdot 2}{7} = \frac{10}{7}$$

You started with $x = \frac{5}{7} \cdot 2$, and you have shown that $x = \frac{5 \cdot 2}{7}$, so

$$\frac{5}{7} \cdot 2 = \frac{5 \cdot 2}{7}.$$

In general, if a, b, and c are whole numbers and b is not zero,

$$c \cdot \frac{a}{b} = \frac{c \cdot a}{b}.$$

Similarly, you can find a number x for which

$$x = \frac{2}{3} \cdot \frac{5}{7}.$$

Since x and $\frac{2}{3} \cdot \frac{5}{7}$ are names for the same number

$$3x = 3(\frac{2}{3} \cdot \frac{5}{7}).$$

Using the associative property,

$$3x = (3 \cdot \frac{2}{3}) \cdot \frac{5}{7}$$

$$3x = 2 \cdot \frac{5}{7}$$

$$3x = \frac{2 \cdot 5}{7}$$

But $3x$ and $\frac{2 \cdot 5}{7}$ are names for the same number. Thus

$$7 \cdot (3x) = 7\left(\frac{2 \cdot 5}{7}\right)$$

$$7 \cdot (3x) = 2 \cdot 5$$

Using the associative property for multiplication,

$$(7 \cdot 3)x = 2 \cdot 5.$$

$$x = \frac{2 \cdot 5}{7 \cdot 3} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$$

You started with $x = \frac{2}{3} \cdot \frac{5}{7}$, and you have shown that $x = \frac{2 \cdot 5}{3 \cdot 7}$, so

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7}$$

This example shows that:

To *multiply two rational numbers written as fractions*, you find a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.

In symbols, if $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers,

$$\blacktriangleright\ \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}.$$

You can use this property to reduce $\frac{12}{16}$ to simplest form, instead of Property 1, in this way

$$\frac{12}{16} = \frac{4 \cdot 3}{4 \cdot 4} = \frac{4}{4} \cdot \frac{3}{4} = 1 \cdot \frac{3}{4} = \frac{3}{4}.$$

You can even show Property 1 for the rational number $\frac{a}{b}$ this way if k is not zero:

$$\frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{k}{k} \cdot \frac{a}{b} = \frac{k \cdot a}{k \cdot b} = \frac{ka}{kb}.$$

In this chapter you learned that you could write $5 + 3$ as $\frac{5}{3}$. If a and b are any two numbers where b is not zero, you can write $\frac{a}{b}$ instead of $a + b$. If a and b are rational numbers, $\frac{3}{2}$ and $\frac{1}{2}$, for example, you can write

$$\frac{3}{2} + \frac{1}{2} = \cfrac{\frac{3}{2}}{\frac{1}{2}} .$$

Notice that the bar between $\frac{3}{2}$ and $\frac{1}{2}$ is longer than the other two bars.

When you studied inverse operations you learned that multiplication and division were inverse operations. This means that

$$\frac{8}{2} = 4 \text{ if } 2 \cdot 4 = 8$$

and only then. This same relation holds between multiplication and division if a, b, and x are rational numbers, and b is not zero. That is,

$$\frac{a}{b} = x \text{ if } bx = a$$

and only then. For example

$$\cfrac{\frac{3}{2}}{\frac{1}{2}} = x \text{ if } \frac{1}{2}x = \frac{3}{2}$$

and only then. Here $x = 3$ makes both statements true. Other names for x are $\frac{3}{1}$ and $\frac{6}{2}$.

As another example, look at

$$x = \cfrac{\frac{3}{2}}{\frac{5}{7}} .$$

How do <u>you</u> find x? You <u>can</u> find x without using rules for division.

$$x = \cfrac{\frac{3}{2}}{\frac{5}{7}} \quad \text{if} \quad \frac{5}{7}x = \frac{3}{2}$$

and only then. In the right hand equality $\frac{5}{7}x$ and $\frac{3}{2}$ are both names for the same number. Now multiply this number by $\frac{7}{5}$, the reciprocal of $\frac{5}{7}$, and use the associative property to get

$$\left(\frac{7}{5} \cdot \frac{5}{7}\right) \cdot x = \frac{7}{5} \cdot \frac{3}{2}$$

$$1 \cdot x = \frac{7}{5} \cdot \frac{3}{2}$$

$$x = \frac{7}{5} \cdot \frac{3}{2}$$

But at the beginning of the example you had

$$x = \frac{\frac{3}{2}}{\frac{5}{7}},$$

so

$$\frac{\frac{3}{2}}{\frac{5}{7}} = \frac{7}{5} \cdot \frac{3}{2}.$$

Using only rules you have had earlier it was shown in the example the reason for the following statement.

> To find the <u>quotient</u> of two rational numbers written as fractions you find the product of the numerator and the reciprocal of the denominator. In symbols:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}.$$

It is not even necessary to remember this statement about dividing one rational number by another if you divide in the following way.

$$\frac{\frac{3}{2}}{\frac{5}{7}} = \frac{\frac{3}{2} \cdot \frac{7}{5}}{\frac{5}{7} \cdot \frac{7}{5}} = \frac{\frac{3}{2} \cdot \frac{7}{5}}{1} = \frac{3}{2} \cdot \frac{7}{5}.$$

You merely have to look at the denominator and decide what number you must multiply it by to get the number 1, then multiply numerator and denominator by that number.

In an earlier chapter you learned that the whole numbers were not closed for the operation of division. In this section a method was shown for finding the quotient of any two rational numbers, a and b, b not zero. The operation of division does not allow dividing by zero. Is the set of rational numbers, with zero omitted, closed for the operation of division?

Exercises 4-5

1. Find each product as a fraction in simplest form. Before you multiply, express each part of the product as a single fraction. For example $\frac{3}{4} \cdot (2\frac{1}{2}) = \frac{3}{4} \cdot \frac{5}{2} = \frac{15}{8}$

   (a) $8 \cdot (2\frac{2}{3})$
   (b) $(3\frac{1}{3}) \cdot (2\frac{2}{3})$
   (c) $12 \cdot (5\frac{5}{6})$
   (d) $(2\frac{1}{5}) \cdot (3\frac{1}{3})$
   (e) $(5\frac{1}{4}) \cdot (\frac{2}{7})$
   (f) $(1\frac{1}{7}) \cdot (1\frac{1}{8})$

2. Find the number r for which the following statement is true.

$$\frac{3}{4} \cdot (\frac{1}{3} \cdot \frac{7}{8}) = (r \cdot \frac{1}{3}) \cdot \frac{7}{8}$$

3. On a road map 1 inch represents 10 miles. If the distance on this map from your house to school measures $1\frac{5}{8}$ inches, how many miles do you live from school?

4. Write each of the following quotients in simplest form. Factor numbers whenever it is to your advantage.

   (a) $\dfrac{\frac{9}{16}}{\frac{1}{4}}$
   (b) $\dfrac{\frac{1}{3}}{\frac{1}{4}}$
   (c) $\dfrac{\frac{9}{8}}{\frac{1}{2}}$
   (d) $\dfrac{\frac{12}{17}}{6}$
   (e) $\dfrac{\frac{12}{15}}{\frac{24}{30}}$
   (f) $\dfrac{16}{\frac{1}{4}}$

5. Find the following quotients. Express each answer as a whole number, or as a whole number plus a number smaller than 1.

(a) $\frac{1}{4} \div \frac{1}{8}$ (c) $\frac{2}{3} \div \frac{1}{4}$

(b) $\frac{2}{3} \div \frac{1}{12}$ (d) $5 \div \frac{2}{3}$

6. Find the following quotients in simplest form. Start by putting each fraction in simplest form before dividing.

(a) $\dfrac{\frac{1}{9}}{\frac{8}{8}}$ (c) $\frac{2}{3} \div \frac{6}{8}$

(b) $\frac{3}{11} \div \frac{5}{5}$ (d) $\frac{120}{60} \div \frac{7}{14}$

*7. Do you think division is commutative? Find each quotient to see if you are right.

(a) $\frac{2}{3} \div \frac{1}{2}$ (b) $\frac{1}{2} \div \frac{2}{3}$

*8. Do you think that division is associative? Find each quotient to see if you are right.

(a) $\frac{3}{2} \div (\frac{9}{4} \div \frac{7}{6})$ (b) $(\frac{3}{2} \div \frac{9}{4}) \div \frac{7}{6}$

*9. (a) $\frac{12}{10}$ is how many times as large as $\frac{12}{100}$?

(b) $\frac{4}{7}$ is how many times as large as $\frac{7}{4}$?

4-6. Addition and Subtraction of Rational Numbers

You have examined the elements of the set of rational numbers. You know that there are many names for the same rational number. You have used the operations of multiplication and division. Only two operations remain to be considered: addition and its inverse operation subtraction. Let us look at the addition operation first.

You are all familiar with the idea that $\frac{1}{3} + \frac{1}{3} = 2 \cdot \frac{1}{3} = \frac{2}{3}$; also $\frac{4}{3} = 4 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$. Continuing with other rational numbers, let us find a single fraction for:

$$\frac{2}{3} + \frac{4}{3}.$$

Using what you already know let us write this as:

$$\frac{2}{3} = 2 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{3}$$

and

$$\frac{4}{3} = 4 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}.$$

Then

$$\frac{2}{3} + \frac{4}{3} = (\frac{1}{3} + \frac{1}{3}) + (\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}) = 6 \cdot \frac{1}{3} = \frac{6}{3}$$

or you may write

$$\frac{2}{3} + \frac{4}{3} = (2 \cdot \frac{1}{3}) + (4 \cdot \frac{1}{3})$$

$$= (2 + 4)\frac{1}{3} = 6 \cdot \frac{1}{3}.$$

$$= \frac{6}{3},$$

using the distributive property to get the second line.

What is the result of adding $\frac{a}{b}$ to $\frac{c}{b}$, where a, b, and c are whole numbers and b is not 0? $\frac{a}{b} + \frac{c}{b}$ may be written $a \cdot \frac{1}{b} + c \cdot \frac{1}{b}$. By the distributive property this is equal to

$$(a + c) \cdot \frac{1}{b} = \frac{a + c}{b}.$$

If a, b, and c are whole numbers and b is not 0, then $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$. This may be read in words: The sum of two numbers whose fractions have the same denominator is the sum of the numerators divided by the common denominator.

Let us look at a more difficult addition example:

$$\frac{3}{4} + \frac{7}{10}.$$

The least common multiple of the denominators 4 and 10 is 20. You can write each fraction with a denominator 20. You recall that

$$\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20} \text{ and } \frac{7}{10} = \frac{7 \cdot 2}{10 \cdot 2} = \frac{14}{20}.$$

Then

$$\frac{3}{4} + \frac{7}{10} = \frac{15}{20} + \frac{14}{20} = \frac{15 + 14}{20} = \frac{29}{20}.$$

Also

$$\frac{3}{10} + \frac{7}{15} = \frac{3 \cdot 3}{10 \cdot 3} + \frac{7 \cdot 2}{15 \cdot 2} = \frac{9}{30} + \frac{14}{30} = \frac{23}{30}.$$

The sum of any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ may be found similarly. A common multiple of b and d is bd.

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d} = \frac{ad}{bd} \text{ and } \frac{c}{d} = \frac{b \cdot c}{b \cdot d} = \frac{bc}{bd}$$

Using what you know about adding rational numbers whose fractions have the same denominator you will have:

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad + bc}{bd}. \end{aligned}$$

Thus you may say:

If a, b, c, and d are whole numbers and b and d are not zero, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

In the following set of exercises you will see again that the commutative and associative properties of addition as well as the distributive property hold in the rational number system.

Exercises 4-6a

1. Find each of the following sums.

   (a) $4 + \frac{3}{8}$

   (b) $\frac{2}{5} + \frac{3}{10}$

   (c) $\frac{3}{10} + \frac{2}{5}$

   (d) $\frac{7}{8} + \frac{3}{16}$

   (e) $\frac{3}{16} + \frac{7}{8}$

2. Does it appear that the addition operation for rational numbers is commutative?

3. Find each of the following sums.

   (a) $(\frac{1}{5} + \frac{5}{2}) + \frac{2}{3}$

   (b) $\frac{1}{5} + (\frac{5}{2} + \frac{2}{3})$

   (c) $(\frac{12}{15} + \frac{6}{3}) + \frac{4}{6}$

   (d) $\frac{12}{15} + (\frac{6}{3} + \frac{4}{6})$

4. Does it appear that the addition operation for rational numbers is associative?

*5. (a) Is it true that $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$ for all rational numbers $\frac{a}{b}$ and $\frac{c}{d}$?

   (b) What property of addition is shown in Part (a)?

*6. (a) Is it true that $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$ for all rational numbers $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$?

   (b) What principle of addition is shown in Part (a)?

Keeping in mind both the commutative and associative properties of addition you can find the sum of $2\frac{1}{4}$ and $3\frac{5}{8}$ quite easily.

$$\begin{aligned} 2\tfrac{1}{4} + 3\tfrac{5}{8} &= (2 + \tfrac{1}{4}) + (3 + \tfrac{5}{8}) \\ &= (2 + 3) + (\tfrac{1}{4} + \tfrac{5}{8}) \quad \text{by commutative and} \\ &= (2 + 3) + (\tfrac{2}{8} + \tfrac{5}{8}) \quad \text{associative properties.} \\ &= 5 + \tfrac{7}{8} \\ &= 5\tfrac{7}{8} \end{aligned}$$

You could also work the preceding problem this way.

$$\begin{aligned} 2\tfrac{1}{4} + 3\tfrac{5}{8} &= \tfrac{9}{4} + \tfrac{29}{8} \\ &= \tfrac{18}{8} + \tfrac{29}{8} \\ &= \tfrac{47}{8} \\ &= 5\tfrac{7}{8} \end{aligned}$$

7. Find each of the following sums in two different ways:

(a) $4\frac{1}{8} + 3\frac{4}{8}$ (b) $8\frac{5}{16} + 1\frac{13}{16}$ (c) $2\frac{12}{32} + 14\frac{23}{32}$

8. Find the numbers:

(a) $\frac{0}{3} + \frac{7}{8}$ (b) $(3 + 5) \div (\frac{1}{3} + \frac{1}{5})$ (c) $\frac{a}{b} + \frac{c}{d}$

9. In each of the following examples, see if you can do the work in your head first. Then check by working out each example on paper.

(a) $1 \div (\frac{1}{3} \div \frac{1}{5})$ (b) $(1 \div \frac{1}{3}) \div \frac{1}{5}$

10. Based on your results in 9(a) and 9(b) would you say that division is associative?

11. In the magic square below, add the numbers in each column. Then, adding across, find the sum of the numbers in each row. Now add the numbers in each diagonal. (Top left corner to lower right corner, etc.)

$11\frac{1}{4}$	$2\frac{1}{2}$	$8\frac{3}{4}$
5	$7\frac{1}{2}$	10
$6\frac{1}{4}$	$12\frac{1}{2}$	$3\frac{3}{4}$

Since subtraction is so closely related to addition it is not necessary to go through all the steps in developing the ideas again. As with addition, if the fractions have the same denominators, then you can simply subtract the numerators and place this difference over the common denominator. For example:

$$\frac{12}{7} - \frac{9}{7} = \frac{12 - 9}{7} = \frac{3}{7}.$$

Since subtraction is the inverse operation of addition,

$$\frac{12}{7} - \frac{9}{7} = \frac{3}{7}$$

means the same as

$$\frac{9}{7} + \frac{3}{7} = \frac{12}{7}.$$

In symbols this relation of two rational numbers may be written

$$\frac{c}{b} - \frac{a}{b} = \frac{c - a}{b}$$ when b is not 0 and c is larger than or equal to a.

Remember that the rational numbers of this chapter, are not closed under subtraction.

If two fractions have different denominators, you can rewrite each of them with a common denominator. Let us use this example: $\frac{4}{5} - \frac{5}{7}$.

$$\begin{aligned}\frac{4}{5} - \frac{5}{7} &= (\frac{4}{5} \cdot \frac{7}{7}) - (\frac{5}{7} \cdot \frac{5}{5}) \\ &= \frac{28}{35} - \frac{25}{35} \\ &= \frac{28 - 25}{35} \\ &= \frac{3}{35}\end{aligned}$$

Check your answer by addition.

$$\frac{3}{35} + \frac{5}{7} = \frac{4}{5}.$$

As in the operation of addition of rational numbers this same procedure may be applied to the general case $\frac{c}{d} - \frac{a}{b}$.

$$\begin{aligned}\frac{c}{d} - \frac{a}{b} &= (\frac{c}{d} \cdot \frac{b}{b}) - (\frac{a}{b} \cdot \frac{d}{d}) \\ &= \frac{cb}{bd} - \frac{ad}{bd} \\ &= \frac{cb - ad}{bd}.\end{aligned}$$

Hence $\frac{c}{d} - \frac{a}{b} = \frac{cb - ad}{bd}$ if (cb) is greater than or equal to (ad) and b and d are not zero.

You now know how to subtract any two rational numbers when the result is a rational number. Look at this problem:

$$5\frac{2}{3} - 2\frac{1}{2}$$

$5\frac{2}{3}$ and $2\frac{1}{2}$ may be written in fractional form:

$$\begin{aligned} 5\frac{2}{3} - 2\frac{1}{2} &= \frac{17}{3} - \frac{5}{2} \\ &= \left(\frac{17 \cdot 2}{3 \cdot 2}\right) - \left(\frac{5 \cdot 3}{2 \cdot 3}\right) \\ &= \frac{34}{6} - \frac{15}{6} \\ &= \frac{19}{6} \\ &= 3\frac{1}{6} \end{aligned}$$

You probably learned a little different way of subtracting and would do this same problem in this way:

$$\begin{array}{rcr} 5\frac{2}{3} & = & 5\frac{4}{6} \\ -\,2\frac{1}{2} & = & -\,2\frac{3}{6} \\ \hline & & 3\frac{1}{6} \end{array}$$

In the subtraction problem below it will be convenient for us to rewrite rational numbers in fractional form. You have written the rational number $8\frac{1}{3}$ as $\frac{25}{3}$. Now let us write it in still another way. By using the associative law you may write $8\frac{1}{3} = 8 + \frac{1}{3} = (7 + 1) + \frac{1}{3} = 7 + (1 + \frac{1}{3}) = 7 + \frac{4}{3}$. Can you write out the reasoning, as is done above, for writing $7\frac{1}{4}$ as $6 + \frac{5}{4}$? For $12\frac{3}{8}$ as $11 + \frac{11}{8}$? For $13\frac{2}{3}$ as $12 + \frac{5}{3}$?

In the following problem:

$$8\frac{1}{3} - 2\frac{5}{6}$$

you will find it convenient to write $8\frac{1}{3}$ as $7\frac{4}{3}$ since $\frac{5}{6}$ is greater than $\frac{1}{3}$. But, $\frac{5}{6}$ is not greater than $\frac{4}{3}$

$$\begin{aligned}
8\frac{1}{3} - 2\frac{5}{6} &= (8 + \frac{1}{3}) - (2 + \frac{5}{6}) \\
&= (7 + \frac{4}{3}) - (2 + \frac{5}{6}) \\
&= (7 + \frac{4 \cdot 2}{3 \cdot 2}) - (2 + \frac{5}{6}) \\
&= (7 + \frac{8}{6}) - (2 + \frac{5}{6}) \\
&= (7 - 2) + (\frac{8}{6} - \frac{5}{6}) \\
&= 5 + \frac{3}{6} \\
&= 5\frac{3}{6} \\
&= 5\frac{1}{2}
\end{aligned}$$

You may have written this.

$$\begin{array}{rcrcr}
8\frac{1}{3} & = & 7\frac{4}{3} & = & 7\frac{8}{6} \\
-\ 2\frac{5}{6} & = & 2\frac{5}{6} & = & 2\frac{5}{6} \\
\hline
 & & & & 5\frac{3}{6} = 5\frac{1}{2}
\end{array}$$

Exercises 4-6b

1. Subtract. Express answers in simplest forms.

   (a) $\frac{14}{6} - \frac{11}{9}$

   (b) $(4\frac{1}{2} - 1\frac{1}{4}) - 2$

   (c) $16\frac{2}{3} - \frac{7}{3}$

   (d) $\frac{90}{100} - \frac{109}{1000}$

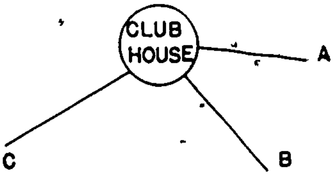
   (e) $(\frac{2}{3} \cdot \frac{5}{6}) - (\frac{2}{3} \cdot \frac{1}{6})$

   (f) $\frac{2}{3}(\frac{5}{6} - \frac{1}{6})$

   (g) Compare your results for e and f.

   (h) From your work in e, f, and g, do you think multiplication is distributive over subtraction?

*2. Vic, Mel, and Bob built a club house. Vic lives $\frac{2}{3}$ of a mile from the club house. Mel lives $\frac{5}{8}$ of a mile from the club house. Bob lives $\frac{7}{9}$ of a mile from the club house. In the diagram below, A represents the home of the boy nearest the club house, B, the next and C, the farthest. Which boy lives at A? at C? at B? How much farther is B from the club house than A? How much farther is C than A?



3. The length of a field is $30\frac{1}{2}$ rods. The width of the same field is $15\frac{3}{4}$ rods. Find the difference between the length and width of the field.

4. Mary is making a chintz apron that requires $1\frac{5}{8}$ yds. of material. She has $4\frac{3}{4}$ yds. of chintz. How much material will she have left?

5. Mr. Twiggs changed the price of potatoes in his store from $4\frac{1}{2}$¢ a pound to 3 pounds for 14¢.

   (a) Did he raise or lower the price?

   (b) How much was the increase or decrease per pound?

6. (a) Is the difference of $17\frac{1}{4}$ and $6\frac{3}{4}$ more or less than 11?

   (b) How can you tell without working out the example?

7. Is the number represented by a fraction changed if you subtract the same number from both numerator and denominator? Give an example.

*8. Fill in the blanks in the magic square below so that the sum of the numbers in each row is the same. Then find the sum of the numbers of each column and both diagonals.

$\frac{2}{3}$	$\frac{1}{12}$	$\frac{1}{2}$
	$\frac{5}{12}$	$\frac{7}{12}$
$\frac{1}{3}$		$\frac{1}{6}$

4-7. Ordering

You have had methods of finding whether two fractions represent the same number. It is often important to know which of two given unequal fractions represents the larger number. If, in one store you can get three apples for a dime, and in another two apples for a dime, it is easy to see that apples are cheaper in the first store since you get more for a dime there. This can also be seen by noticing that in the first store you get $\frac{3}{10}$ of an apple for a cent, and at the second you get only $\frac{2}{10}$ of an apple for a cent. You know that $\frac{3}{10}$ is greater than $\frac{2}{10}$, which can be written

$$\frac{3}{10} > \frac{2}{10} .$$

On the number line

$\frac{3}{10}$ occurs to the right of $\frac{2}{10}$.

The problem would be harder, if in the first store, you could get 3 apples for five cents and in the second store 8 apples for 13 cents. Then you would have to answer the question:

$$\text{Is } \frac{3}{5} > \frac{8}{13} ?$$

It would require careful drawing to answer this by use of the number line. There is a better method of finding the answer. It is the method of the example of section 4; that is, two fractions are found with equal denominators which represent the given numbers. Since the denominators are 5 and 13, the smallest denominator which can be used for both fractions will be the least common multiple of 5 and 13, which is 65. Then

$$\frac{3}{5} = \frac{3 \cdot 13}{5 \cdot 13} = \frac{39}{65} \quad \text{and} \quad \frac{8}{13} = \frac{8 \cdot 5}{13 \cdot 5} = \frac{40}{65}.$$

Since $\frac{40}{65} > \frac{39}{65}$ it is also true that $\frac{8}{13} > \frac{3}{5}$.

Either method shows that if a number line is divided into 65 equal divisions, the point representing $\frac{8}{13} = \frac{40}{65}$ lies to the right of the point representing $\frac{3}{5} = \frac{39}{65}$.

A fraction in which the numerator is greater than the denominator is often called an <u>improper fraction</u>. The number which such a fraction represents must be greater than 1. To see why this is so, consider the fraction $\frac{13}{11}$. This is greater than $\frac{11}{11}$ which is equal to 1. Thus $\frac{13}{11}$ is greater than 1. Similarly $\frac{3568}{3452}$ is greater than 1 since it is greater than $\frac{3452}{3452}$ which is equal to 1.

Exercises 4-7

1. For each of the following pairs of fractions, tell which represents the larger number:

   (a) $\frac{7}{12}$ and $\frac{2}{3}$ (b) $\frac{4}{5}$ and $\frac{13}{16}$ (c) $\frac{13}{15}$ and $\frac{13}{7}$

2. If the numerator of a fraction is greater than twice its denominator, show that the number which the fraction represents is greater than 2.

*3. If two fractions have equal denominators and if the numerator of the first is greater than the numerator of the second, then the number represented by the first is greater than the number represented by the second. Suppose two fractions have equal numerators; how can you tell by comparing the denominators which fraction represents the larger number? (Try a few pairs of fractions first to see how it goes.)

4-8. Ratios

A ratio is a comparison of two numbers by division. The numbers may be measures of physical quantities. The word, "per," indicates division. It is used to express the ratio of the measures of two physical quantities, such as miles and hours. Store prices provide additional examples. Prices relate value to amount such as $1.00 per pound, or $.59 per dozen. In each case, the per indicates a ratio, generally between two different kinds of quantities. This is sometimes called a rate. Notice further that the second quantity in each case represents the standard of comparison. A store charges "10 ¢ per comb"; one comb is the standard of comparison, and they want 10 ¢ in the cash register for each comb sold. Of course, this standard does not always represent a quantity of one. For example, cents per dozen, dollars per pair (for shoes), dollars per 1000 (for bricks).

Definition. The ratio of a number a to a number b $(b \neq 0)$ is the quotient $\frac{a}{b}$. (Sometimes this ratio is written $a:b$.)

In some examples ratios are all equal to one another. In such a situation, it can be said that the physical quantities measured by the numbers are proportional to one another. A proportion is a statement of equality of two ratios.

Then the statement

$$\frac{14}{21} = \frac{2}{3}$$

is a proportion.

You know from the first part of this chapter that

(a) $\frac{2}{3} = \frac{6}{9}$; (b) $\frac{3}{4} = \frac{24}{32}$; (c) $\frac{5}{6} = \frac{15}{18}$; (d) $\frac{3}{8} = \frac{375}{1000}$

In (a) $\frac{2}{3} = \frac{6}{9}$, you see that $2 \cdot 9 = 18$ and $3 \cdot 6 = 18$; thus $2 \cdot 9 = 3 \cdot 6$. In (b) $\frac{3}{4} = \frac{24}{32}$, you see that $3 \cdot 32 = 96$ and $4 \cdot 24 = 96$; thus $3 \cdot 32 = 4 \cdot 24$. Similarly, in (d) $\frac{3}{8} = \frac{375}{1000}$, you see that $3 \cdot 1000 = 8 \cdot 375$, since $8 \cdot 375 = 3000$. Check this property for (c) $\frac{5}{6} = \frac{15}{18}$.

If $\frac{a}{b} = \frac{c}{d}$ and $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{d}{d} = \frac{c}{d} \cdot \frac{b}{b}$$

$\frac{a}{b}$ is multiplied by $\frac{d}{d}$, which equals 1, and $\frac{c}{d}$ is multiplied by $\frac{b}{b}$.

$$\frac{ad}{bd} = \frac{cb}{db}$$

$$\frac{ad}{bd} = \frac{bc}{bd}$$

These products are obtained so that the two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ will be expressed as fractions with a common denominator.

Thus $ad = bc$.

If two fractions with the same denominator are names for the same rational number, their numerators are equal.

You have seen:

<u>Property 1</u>.

If $\frac{a}{b} = \frac{c}{d}$ and $b \neq 0$ and $d \neq 0$, then $ad = bc$.

This property may be useful to you in solving problems involving proportion. It is also true that if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$ provided $b \neq 0$ and $d \neq 0$. The proof of this property is left as a problem. (See Problem *9.)

Exercises 4-8

1. Some objects were measured and the data are recorded here. Copy and complete the table.

	Shadow length	Height	Ratio
Garage	3 feet	8 feet	
Clothes pole	36 inches		$\frac{3}{8}$
Tree	$7\frac{1}{2}$ feet	20 feet	
Flag pole		144 inches	$\frac{3}{8}$
Fence	$11\frac{1}{4}$ inches	30 inches	

2. In a class there are 30 students of whom 12 are girls.
   (a) What is the ratio of the number of girls to the total number of students in the class?
   (b) What is the ratio of the number of boys to the total number of students in the class?
   (c) What is the ratio of the number of girls to the number of boys?

3. In Problem *9 you are asked to show that $\frac{a}{b} = \frac{c}{d}$, if $ad = bc$ and $b \neq 0$ and $d \neq 0$. Use this property to find which of the following pairs of ratios are equal.
   (a) $\frac{10}{5}$, $\frac{20}{10}$
   (b) $\frac{6}{3}$, $\frac{16}{9}$
   (c) $\frac{48}{16}$, $\frac{42}{14}$

4. In each of the following, find x so that these will be true statements.

   (a) $\frac{20}{8} = \frac{x}{6}$

   (b) $\frac{14}{30} = \frac{x}{90}$

   (c) $\frac{x}{3} = \frac{75}{15}$

5. A cookie recipe calls for the following items.

1 cup butter	$1\frac{1}{2}$ cups flour
$\frac{2}{3}$ cup sugar	1 teaspoon vanilla
2 eggs	

   This recipe will make 30 cookies.

   (a) Rewrite the recipe by enlarging it in the ratio $\frac{3}{1}$.

   (b) How many cookies will the new recipe make now?

   (c) Suppose you wanted to make 45 cookies, how much would you need of each of the items listed above?

6. Use a proportion to solve these problems. Check your work by solving the problems by another method.

   (a) What is the cost of 3 dozen doughnuts at $.55 per dozen?

   (b) What is the cost of 12 candy bars at 4 for 15¢? (How could you state this price using the word "per"?)

   (c) What is the cost of 8500 bricks at $14 per thousand?

   (d) A road has a grade of 6%, which means that it rises 6 feet per 100 feet of road. How much does it rise in a mile? Find the answer to the nearest foot.

7. Common units to express speed are "miles per hour" and "feet per second." A motor scooter can go 30 miles per hour. How many feet per second is this? (The danger of of high speed under various conditions often can be realized better when the speed is given in feet per second.)

8. The following table lists pairs of numbers, A and B. In each pair the ratio of A to B is the number $\frac{6}{7}$. Complete the table.

	A	B	Ratio $\frac{A}{B}$	Ratio in simplest form
(a)	12	14	$\frac{12}{14}$	$\frac{6}{7}$
(b)		21		
(c)	30			
(d)		100		
(e)	100			

*9. Prove the property,

if $ad = bc$ and $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$.

When you have proved this statement, you can now say

if $\frac{a}{b} = \frac{c}{d}$, and $b \neq 0$ and $d \neq 0$, then $ad = bc$, and only then.

(Hint: If $ad = bc$, then $a = \frac{bc}{d}$. You can express $\frac{bc}{d}$ as $b \cdot \frac{c}{d}$.)

4-9. Percent

The word "percent" comes from the latin phrase "per centum", which means "by the hundred". If a paper with 90 of the answers correct has 100 answers, then 90 answers out of the 100 are correct. The ratio $\frac{90}{100}$ could be used instead of the phrase "90 percent" to describe the part of the answers which are correct. The word "percent" is used when a ratio is expressed with a denominator of 100.

$$90 \text{ percent} = \frac{90}{100} = 90 \cdot \frac{1}{100}.$$

For convenience the symbol, %, is used for the word "percent". This symbol is a short way of saying $\frac{1}{100}$.

$$\frac{16}{100} = 16 \cdot \frac{1}{100} = 16\,\% .$$

$$\frac{77}{100} = 7 \cdot \frac{1}{100} = \ ?\,\% .$$

$$? = 13 \cdot \frac{1}{100} = 13\,\% .$$

The number 2 can be written

$$\frac{2}{1} = \frac{200}{100} = 200\,\% .$$

In other words 200% means $200 \times \frac{1}{100} = \frac{200}{100} = 2.$

A class of 25 pupils is made up of 11 girls and 14 boys. The ratio of the number of girls to the number of pupils in the class can be expressed many ways. For instance:

$$\frac{11}{25} = \frac{22}{50} = \frac{33}{75} = \frac{44}{100} = \frac{55}{125} = \frac{66}{150} .$$

If you wish to indicate the percent of the class that is girls, which fraction gives the information most easily? Why? The ratio of the number of boys to the total number in the class may be written

$$\frac{14}{25} = \frac{c}{50} = \frac{d}{75} = \frac{56}{100} = \frac{e}{125} = \frac{f}{150} .$$

What numbers are represented by the letters c, d, e, f? Notice the two ratios $\frac{11}{25}$ (girls) and $\frac{14}{25}$ (boys). What is the sum of the two ratios? Find the sum of the two ratios $\frac{44}{100}$ and $\frac{56}{100}$. Express the two ratios and their sum as percents using the symbol, %. The entire class is considered to be 100% of the class because

$$\frac{25}{25} = \frac{100}{100} = 100\% .$$

<u>Any number</u> $\frac{a}{b}$ <u>can be expressed as a percent by finding the number</u> c <u>such that</u>

$$\frac{a}{b} = \frac{c}{100} = c \cdot \frac{1}{100} = c\,\% .$$

Exercises 4-9a

1. Write each of the following numbers as a percent.

(a) $\frac{1}{2}$ (b) $\frac{1}{5}$ (c) $\frac{3}{2}$ (d) $\frac{7}{5}$

2. Consider the following class.

Student	Hair Color
Robert	blond
Joan	brown
Mary	blond
John	red
Joe	brown
Betty	brown
Ray	blond
Don	brown
Margaret	red
David	brown

(a) What percent are boys?

(b) What percent are blonds?

(c) What percent are redheads?

(d) What percent are not redheads?

(e) What percent are brown-haired girls?

(f) What percent are redheaded boys?

3. The monthly income for a family is $400. The family budget for the month is shown.

Payment on the mortgage for the home	$ 80
Taxes	20
Payment on the car	36
Food	120
Clothing	48
Operating expenses	32
Health, Recreation, etc.	24
Savings, and Insurance	40

(a) What percent of the income is assigned to each item of the budget?

(b) What is one check on the accuracy of the 8 answers?

Data about business, school, -- are sometimes given in percent. It is often more convenient to refer to the data at some later time if they are given in percent than if they are given otherwise.

A few years ago the director of a camp kept some records for future use. Some information was given in percent, and some was not. The records gave the following items of information.

(1) There were 200 boys in camp.

(2) One hundred percent of the boys were hungry for the first dinner in camp.

(3) On the second day in camp 44 boys caught fish.

(4) One boy wanted to go home the first night.

(5) A neighboring camp director said "forty percent of the boys in my camp will learn to swim this summer. We shall teach 32 boys to swim."

From (1) and (2), how many hungry boys came to dinner the first day?

$$\frac{100}{100} \cdot 200 = 200$$

Of course you should know without computation that 100% of 200 is 200.

From (3), you can find the percent of boys who caught fish on the second day. The ratio of the number of boys who caught fish to the number of boys in camp is $\frac{44}{200}$. If you call x% the percent of the boys who caught fish, then

$$\frac{x}{100} = \frac{44}{200}$$

$$200x = 100 \cdot 44 \qquad \text{(Property 1.)}$$

$$x = \frac{100 \cdot 44}{200}$$

$$x = 22$$

Twenty-two% of the boys caught fish.

From (4), you can find the percent of the boys who were homesick. If this is $x\%$, then

$$\frac{x}{100} = \frac{1}{200}$$

$$200x = 100 \qquad \text{(Property 1.)}$$

$$x = \frac{100}{200}$$

$$x = \frac{1}{2}$$

Of the boys, $\frac{1}{2}\%$ were homesick. This may be read "one-half percent" or "one half of one percent." You may prefer to say "one half of one percent," since this emphasizes the smallness of it.

From the information in Item 5, the total number of boys in the second camp can be found. Call this number of boys x.

40% means $\frac{40}{100}$ of the group, and also refers to 32 boys.

You want to find x such that $\frac{40}{100} = \frac{32}{x}$.

$$\frac{40}{100} = \frac{32}{x}$$

$$40x = 3200 \qquad \text{(Property 1.)}$$

$$x = \frac{3200}{40}$$

$$x = 80$$

Exercises 4-9b

1. There are 600 seventh-grade pupils in a junior high school. The principal hopes to divide the pupils into 20 sections of equal size.
   - (a) How many pupils would be in each section?
   - (b) What percent of the pupils would be in each section?
   - (c) What number of pupils is 1% of the number of pupils in the seventh grade?
   - (d) What number of pupils is 10% of the number of pupils in the seventh grade?

2. One hundred fifty seventh-grade pupils come to school on the school bus.
   (a) What percent of the seventh-grade pupils come by school bus?
   (b) What percent of the seventh-grade pupils come to school by some other means?
3. In a section of 30 pupils, 3 were tardy one day.
   (a) What fractional part of that section was tardy?
   (b) What percent of that section was tardy?
4. One day a seventh-grade pupil heard the principal say, "Four percent of the ninth graders are absent today." A list of absentees for that day had 22 names of ninth-grade pupils on it. From these two pieces of information, the seventh-grade pupil discovered the number of ninth-grade pupils in the school. How many ninth-grade pupils are there?

4-10. Decimal Notation

You recall from a previous chapter that a base ten numeral, such as 3284, written as $3(10^3) + 2(10^2) + 8(10^1) + 4$ was said to be in expanded form. Written as 3284 the numeral is said to be in positional notation. In base ten this form is also called decimal notation. Each digit represents a certain value according to its place in the numeral. In the above example, the 3 is in thousands place, the 2 is in hundreds place and so on. The form for place value in base ten shows that the value of each place immediately to the left of a given place is ten times the value of the given place. Each place value immediately to the right of a given place is one tenth of the place value of the given place. Looking back to the number 3284 written in expanded form you will notice that reading from left to right the exponents of ten decrease.

Suppose you want to write 5634.728 in expanded form. You could write

$5(10^3) + 6(10^2) + 3(10^1) + 4 + 7(\ ?\ ) + 2(\ ?\ ) + 8(\ ?\ )$. The 4 is in unit's (or one's) place. The value of this place is $\frac{1}{10}$ of the value of the place before it. If you extend the numeral to the right of one's place and still keep this pattern, what will be the value of the next place? Of the next place after that one? To write this in expanded form to the right of unit's place as well as to the left of unit's place you will have:

$5(10^3) + 6(10^2) + 3(10^1) + 4 + 7(\frac{1}{10}) + 2(\frac{1}{10^2}) + 8(\frac{1}{10^3})$ where the first place to the right of one's place is the one-tenth's place or simply tenth's place.

The following chart shows the place values both to the left and to the right of the unit's place.

Place Value Chart

Hundred thousand	Ten thousand	Thousand	Hundred	Ten	Unit or one	Tenth	Hundredth	Thousandth	Ten-thousandth	Hundred-thousandth
100,000	10,000	1,000	100	10	1	0.1	0.01	0.001	0.0001	0.00001
10^5	10^4	10^3	10^2	10^1	10^0	$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$

The places to the right of one's place are usually referred to as the decimal places.

When the number 3284.569 is written in decimal notation the decimal point locates the one's place. The number is read "Three thousand two hundred eighty-four <u>and</u> five hundred sixty-nine thousandths" or "Three two eight four -- point -- five six nine.

Example 1.

Write $5(10^2) + 7(10^1) + 3(\frac{1}{10})$ in decimal notation.

Notice that the one's place was not written. Could one's place be written $0(10^1)$? Your answer 570.3 will help you answer this question.

Exercises 4-10

1. Write each of the following in decimal notation:

   (a) $6(10) + 5(1) + 8(\frac{1}{10}) + 7(\frac{1}{10^2})$

   (b) $4(10^2) + 3(10) + 6(1) + 1(\frac{1}{10}) + 9(\frac{1}{10^2})$

   (c) $5(10) + 2(\frac{1}{10}) + 4(\frac{1}{10^2})$

   (d) $4(\frac{1}{10}) + 8(\frac{1}{10^2}) + 3(\frac{1}{10^3})$

   (e) $2(\frac{1}{10^3}) + 6(\frac{1}{10^4})$

2. Write each of the following in expanded form:

   (a) 3.01
   (b) 0.0102
   (c) 0.10001
   (d) 30.03

3. Write each of the following in words: For example, 3.001 is written three and one thousandth.

   (a) 7.236　　(c) 1.0101

   (b) 0.004　　(d) 909.009

4. Write in decimal numeral form:
   (a) Three hundred and fifty-two hundredths
   (b) Five hundred seven ten-thousandths
   (c) Fourteen thousandths
   (d) Sixty and 7 hundredths
   (e) Thirty-two hundred-thousandths
   (f) Eight and nineteen thousandths

*5. Write 0.1_{two} in base ten.

*6. Write $\frac{1}{2}$ in the duodecimal system.

*7. Write $\frac{5}{6}$ in the duodecimal system.

*8. Change 10.011_{two} to base ten.

4-11. Operations with Decimals

Suppose you wish to add two numbers in decimal form; for example, 0.73 + 0.84. You know how to add: 73 + 84 = 157. How do you handle the decimal places? You may proceed as follows:

$$0.73 = 73 \times \frac{1}{100} \text{ and } 0.84 = 84 \times \frac{1}{100}.$$

Therefore,

$$\begin{aligned} 0.73 + 0.84 &= (73 \times \frac{1}{100}) + (84 \times \frac{1}{100}) \\ &= (73 + 84) \times \frac{1}{100} \\ &= 157 \times \frac{1}{100} = 1.57. \end{aligned}$$

Notice that the distributive property has been used here.

Suppose you wish to find the sum, 0.73 + 0.125. First rewrite the numbers as fractions:

$$0.73 = 73 \times \frac{1}{100} = 730 \times \frac{1}{1000} \text{ and } 0.125 = 125 \times \frac{1}{1000}.$$

Use the last form of 0.73, since then $\frac{1}{1000}$ appears in both products.

$$\begin{aligned} 0.73 + 0.125 &= (730 \times \frac{1}{1000}) + (125 \times \frac{1}{1000}) \\ &= (730 + 125) \times \frac{1}{1000} \\ &= 855 \times \frac{1}{1000} = 0.855. \end{aligned}$$

These examples can be handled more conveniently by writing one number below the other as follows.

$$\begin{array}{r} .73 \\ + \ .84 \\ \hline 1.57 \end{array} \qquad\qquad \begin{array}{r} 0.73 \\ + \ 0.125 \\ \hline 0.855 \end{array}$$

Notice that the decimal points are written directly under one another. This is because you will want to add the number in the $\frac{1}{10}$ place in the first addend to the number in the $\frac{1}{10}$ place in the second addend, and the number in the $\frac{1}{100}$ place in the first addend to the number in the $\frac{1}{100}$ place in the second, etc. Thus,

$$0.73 = \frac{7}{10} + \frac{3}{100} \text{ and } 0.84 = \frac{8}{10} + \frac{4}{100}$$

and therefore

$$\begin{aligned} 0.73 + 0.84 &= \frac{7 + 8}{10} + \frac{3 + 4}{100} \\ &= \frac{15}{10} + \frac{7}{100} \\ &= 1.57 \end{aligned}$$

Subtraction can be handled in the same way. For example,

$$\begin{array}{r} 0.84 \\ - \ 0.73 \\ \hline 0.11 \end{array} \qquad\qquad \begin{array}{r} 0.83 \\ - \ 0.74 \\ \hline 0.09 \end{array}$$

Exercises 4-11a

1. There are 16 ounces in 1 pound. Which is heavier, 7 ounces or 0.45 lb.?
2. In most of Europe, distances are measured in kilometers (remember that a kilometer is about 5/8 of a mile). The distance from city A to Paris is 37.5 kilometers and the distance from city B to Paris is 113.2 kilometers. How far in kilometers is it from city A to city B by way of Paris?
3. Suppose the three cities, A, B and Paris in the previous problem are on the same road and city A is between Paris and city B. Then how far in kilometers is it between cities A and B along this road?

*4. Find the value of the sum, in the decimal system, of the numbers 10.01_{two} and 1.01_{two}. First change into the decimal system and then add.

Suppose you wish to multiply two numbers in decimal form. For example, 0.3 × 0.25. You know how to multiply these numbers: 3 × 25 = 75. Just as before, you will write

$$\begin{aligned} 0.3 \times 0.25 &= 3 \times \frac{1}{10} \times 25 \times \frac{1}{100} \\ &= 3 \times 25 \times \frac{1}{10} \times \frac{1}{100} \\ &= 75 \times \frac{1}{1000} \\ &= 0.075 \end{aligned}$$

(1) How many digits are there to the right of the decimal point in 0.3?

(2) How many digits are there to the right of the decimal point in 0.25?

(3) What is the sum of the answers to (1) and (2)?

(4) How many digits are there to the right of the decimal point in 0.075?

(5) Compare the answers to (3) and (4).

Now multiply 0.4 × 0.25. What is your answer? Answer the five questions above, (1), (2), (3), (4), (5) for these numbers. Do the answers to (3) and (4) still agree?

Property 2: To find the number of decimal places in the product when two numbers are multiplied, add the number of decimal places in the two numerals.

You can consider now the problem of dividing one in decimal form by another, for example, $0.125 \div 0.5$.

The first step is usually to find a fraction whose denominator is a whole number and so that the new fraction is also a name for $0.125 \div 0.5$. In this case, start with $\frac{0.125}{0.5}$. Then, in order to replace the denominator by a whole number, multiply numerator and denominator by 10 to get $\frac{1.25}{5}$. Using fractions you could work it out like this:

$$\frac{1.25}{5} = \frac{1}{5} \times 1.25 = \frac{1}{5} \times \left(\frac{1}{100} \times 125\right) = \frac{1}{100} \times \left(\frac{1}{5} \times 125\right)$$

$$= \frac{1}{100} \times 25 = 0.25$$

But a much shorter way is to use the usual form for division.

```
    0.25
5 )1.25
   1 0
    .25
     25
```

Then:

(divisor)		(quotient)		(dividend)
5	×	0.25	=	1.25

and you can see that in the equality the number of decimal places (two) in the product is the sum of the number of decimal places in the members of the product (0 + 2) just as it is said in Property 2. Whenever the divisor is a whole number, the dividend and the quotient have the same number of decimal places. By placing the decimal point of the quotient directly above that of the dividend, you locate the decimal point of the quotient automatically in the correct place.

It is very easy to make a mistake in placing the decimal point of the answer and hence it is always a good plan to check by estimating the answer. For this example you can see that

$$\frac{0.125}{0.5} \text{ is about } \frac{0.1}{0.5} = \frac{1}{5} = 0.2,$$

which is reasonably close to our quotient.

Try a more complicated fraction: $\frac{5.313}{2.53}$. If you multiplied the denominator by 10 you would have 25.3 which still is not a whole number. But if you multiply by 100, it becomes 253, which is a whole number. The given fraction then is equal to $\frac{531.3}{253}$ and the division is performed in the usual way.

$$\begin{array}{r@{\,}l}
 & 2.1 \\
253 & \overline{)531.3} \\
 & \underline{506} \\
 & 25\ 3 \\
 & \underline{25\ 3}
\end{array}$$

Then 253 × 2.1 = 531.3 and you see again that the number of decimal places in 253 (which is zero) plus the number of decimal places in 2.1 (which is 1) is equal to the number of decimal places in 531.3 (which is 1).

You should check the following:

$$\frac{0.75}{0.2} = \frac{7.5}{2} = \frac{7.50}{2} = 3.75 .$$

Of course you cannot expect your divisions to "come out exactly" in all cases. Suppose you try to find the decimal expression for $\frac{2}{7}$. If you want the quotient to three decimal places, you will divide 2.000 by 7. If it were to be six places you would divide 2.000000 by 7, and so forth. Suppose you find it to six places:

$$\begin{array}{r@{\,}l}
 & 0.285714\ldots \\
7 & \overline{)2.000000} \\
 & \underline{1\ 4} \\
 & 60 \\
 & \underline{56} \\
 & 40 \\
 & \underline{35} \\
 & 50 \\
 & \underline{49} \\
 & 10 \\
 & \underline{7} \\
 & 30 \\
 & \underline{28} \\
 & 2
\end{array}$$

Exercises 4-11b

1. Find the following products.
   (a) 0.009×0.09
   (b) 0.0025×2.5
2. Find the following quotients.
   (a) $0.575 \div 0.4$
   (b) $2.04 \div 0.008$
3. Express the following numbers as decimals.
   (a) $\frac{3000}{8}$ (b) $\frac{300}{8}$ (c) $\frac{30}{8}$ (d) $\frac{3}{8}$ (e) $\frac{3}{80}$ (f) $\frac{3}{800}$
4. $\frac{.015 \times .0025 \times 2.5}{.05 \times .03} = ?$
5. About how many miles is a distance of 3.8 kilometers? (One kilometer is about $\frac{5}{8}$ of a mile.)

*6. Find the following product:

$$1.14_{\text{seven}} \times 2.4_{\text{seven}}$$

Decimal Expansion

Let us look more closely at rational numbers in decimal form. You recall that $\frac{1}{8}$ is the quotient of 1 divided by 8.

$$8\overline{)1.000}\quad 0.125$$

So $\frac{1}{8}$ and 0.125 are names for the same rational number.

Consider the rational number $\frac{1}{3}$. You all know the decimal numeral which names $\frac{1}{3}$. It is found by dividing 1 by 3.

$$3\overline{)1.000}\quad 0.333\ldots$$

Without performing the division do you know what digits will appear in each of the next 6 places? At some stage in the division do we get a remainder of zero?

The 3 dots indicate that the decimal numeral never ends.

Look again at the decimal numeral for $\frac{1}{8}$. Recall that it was obtained by dividing 1 by 8.

$$\begin{array}{r} 0.1250\ldots \\ 8\overline{)1.00000} \\ \underline{8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \\ \underline{0} \\ 0 \end{array}$$

After the first subtraction the remainder is 2.
The second remainder is 4.

The third remainder is 0.

The fourth remainder is 0. Can you predict the fifth and sixth remainders?

The division process usually ends at the stage where a zero remainder is obtained. However you could just as well continue dividing, getting at each new stage a remainder of zero, and a quotient of zero.

It is clear that once you get a remainder of zero every remainder thereafter will be zero. You could write $\frac{1}{8} = 0.125000\ldots$ just as you wrote $\frac{1}{3} = 0.333\ldots$, but this is seldom done. Therefore 0.125000... is a repeating decimal with 0 repeating over and over again. Likewise 0.333... is a repeating decimal with the digit 3 repeating.

The phrase <u>decimal expansion</u> means that there is a digit for every decimal place. Note that the decimal expansion of $\frac{1}{8}$ and the decimal expansion of $\frac{1}{3}$ are both repeating.

<u>Class Discussion Exercises</u>

Let us look at the decimal which names the rational number $\frac{1}{7}$.

$$\begin{array}{r} 0.142857142857\ldots \\ 7\overline{)1.000000000000} \end{array}$$

1. Can you tell, without performing the division, the digits that should appear in each of the next six places?
2. Is there a block of digits which continues to repeat endlessly? Let us place a horizontal bar over the block of digits which repeats. Thus $0.142857\overline{142857}\ldots$ uses the bar to mean that the same digits repeat in the same order and the 3 dots to mean that the decimal never ends.

3. Name $\frac{1}{11}$ by a decimal numeral.
4. How soon can you recognize a pattern?
5. Will there be a zero remainder if you continue dividing?
6. Does this decimal repeat? How should you indicate this?
7. Observe that the decimal repeats as the remainder repeats. Look at the procedure by which you found a decimal numeral for $\frac{1}{7}$.

$$\begin{array}{r l} & 0.142857142857\ldots \\ 7 & \overline{)1.000000000000} \\ & \underline{7} \\ & 30 \\ & \underline{28} \\ & 20 \\ & \underline{14} \\ & 60 \\ & \underline{66} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 10 \\ & \underline{7} \\ & 30 \\ & \underline{28} \\ & 20 \\ & \underline{14} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 1 \end{array}$$

After the first subtraction the remainder is 3.

The second remainder is 2.

The third remainder is 6.

The fourth remainder is 4.

The fifth remainder is 5.

The sixth remainder is 1.

The seventh remainder is 3. Is the seventh digit to the right of the decimal point the same as the first?

The eighth remainder is the same as second. Is the eighth digit to the right of the decimal point the same as the second?

Notice that the digits in this quotient begin to repeat whenever a number appears as the remainder for the second time.

8. Make similar observations when you divide 1 by 37 to find a decimal numeral naming $\frac{1}{37}$.

You may conclude that every rational number can be named by a decimal numeral which either repeats a single digit or a block of digits over and over again.

Exercises 4-11c

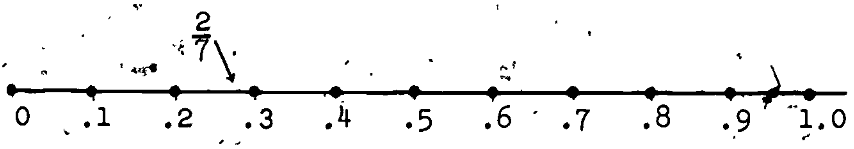
1. Write a decimal numeral (or decimal) for $\frac{1}{13}$.
    (a) How soon can you recognize a pattern?
    (b) Does this decimal end?
    (c) How should you indicate that it does not end?
    (d) Is there a set of digits which repeats periodically?
    (e) How should you indicate this?
2. Write the decimals for:
    (a) $\frac{1}{3}$
    (b) $\frac{7}{8}$
    (c) $\frac{1}{9}$

    See how soon you can recognize a pattern in each case. In performing the division watch the remainders. They may give you a clue about when to expect the decimal numeral to begin to repeat.
3. Write the decimals for:
    (a) $\frac{1}{11}$ (b) $\frac{3}{11}$ (c) $\frac{14}{11}$
4. Is it true that the number $0.6\overline{363}\ldots$ is seven times the number $0.0\overline{909}\ldots$?
5. Find the decimal numeral for the first number in each group and calculate the others without dividing.
    (a) $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}$
    (b) $\frac{1}{20}, \frac{3}{20}, \frac{11}{20}$.
    (c) $\frac{1}{1000}, \frac{111}{1000}, \frac{927}{1000}$,

---

4-12. Rounding

Suppose you wish only an approximate decimal value for $\frac{2}{7}$. You might, for instance, want to represent this on the number line where each segment represents 0.1 as below:



Where would $\frac{2}{7}$ be on this line? You know its decimal form is 0.285714... . Perhaps you can see immediately from this that $\frac{2}{7}$ must lie between 0.20 and 0.30. If this is not clear, look at it a little more closely. First:

0.285714... = 0.2 + 0.08 + 0.005 + (other decimals), and hence it is certainly greater than 0.2, the first number in the sum. Second, 0.2 is less than 0.3; 0.28 is less than 0.30; 0.285 is less than 0.300 and so forth. No matter how far the decimal is computed, what you get is always less than 0.3000000... Also you can see that $\frac{2}{7}$ is closer to 0.30 than to 0.20 since 0.25 is halfway between 0.20 and 0.30 and the number 0.285714... is between 0.25 and 0.30. So $\frac{2}{7}$ goes on the number line in about the place indicated by the arrow.

You could represent $\frac{2}{7}$ more closely on the number line if you divided each of the segments into ten parts. Then you would see that $\frac{2}{7}$ is between 0.28 and 0.29 but a little closer to 0.29. (You may want to write the reasons out in more detail than is done above, in order to make it clearer.) You would say that $\frac{2}{7}$ is 0.29 to the nearest hundredth. This is called "rounding $\frac{2}{7}$ to two places". What would $\frac{2}{7}$ be, rounded to three places? The answer is: 0.286.

This rounding is useful in estimating results. For instance, suppose you have to find the product: 1.34 × 3.56. This would be approximately 1 × 4 = 4 or, if you wanted a little closer estimate, you could compute: 1.3 × 3.6 = 4.7 approximately.

Rounding is also useful when you are considering approximations in percents. For instance, if it turned out to be true that about 2 out of 7 families have dogs, it would be foolish to carry this out to many decimal places in order to get an answer in percent. You would usually just use two places and say that about 29 % of all families have dogs, or you could round this still further and refer to 30 %.

Exercises 4-12

1. Round the following number to two places.
   - (a) 0.0351
   - (b) 0.0449
2. Round the following numbers to three places.
   - (a) 0.1599
   - (b) 0.00009
   - (c) 0,3249
3. Express the following numbers as decimals correct to one place.
   - (a) $\frac{7}{23}$
   - (b) $\frac{6}{23}$
   - (c) $\frac{2}{23}$
   - (d) $\frac{1}{23}$
4. (a) A piece of land is measured and the measurements are rounded to the nearest tenth of a rod (in other words, the measures are rounded to one decimal place.) The length, after rounding is 11.1 rods and width is 3.9 rods. Find the area rounded to the nearest tenth of a square rod.

   (b) Suppose that the length is 11.14 rods and the width is 3.94 rods rounded to the nearest hundredth of a rod. Find the area rounded to the nearest hundredth of a square rod. What is the difference between this answer and the previous one?

4-13. Percents and Decimals

You have learned that the number $\frac{51}{100}$ can be written as 51% and also as 0.51. Actually you can read both 0.51 and $\frac{51}{100}$ as "fifty-one hundredths". Here are three different expressions for the same number:

$$\frac{51}{100} = 51\% = 0.51 ,$$

which are read: "fifty-one hundredths is equal to fifty-one percent is equal to fifty-one hundredths".

Similarly

$$\frac{1}{2} = \frac{50}{100} = 0.50 = 50\% = 0.5 .$$

Also $65\% = 0.65 = \frac{65}{100} = \frac{13}{20}$,

and $\frac{3}{5} = \frac{60}{100} = 0.60 = 60\% = 0.6$.

Class Exercises 4-13

1. Express as percents:

   (a) $\frac{3}{25}$

   (b) 0.4

2. Express as decimals:

   (a) $\frac{75}{4}$

   (b) 3%

It is a little more difficult to express $\frac{1}{8}$ as a percent since 8 is not a factor of 100. You know that its decimal form is 0.125. Other ways of writing this number are: $\frac{12.5}{100}$ or 12.5%; also $\frac{12\frac{1}{2}}{100}$ or $12\frac{1}{2}\%$. Similarly, 0.375 is the same as 37.5% or $37\frac{1}{2}\%$ and may also be written as:

$$\frac{375}{1000} = \frac{25 \cdot 15}{25 \cdot 40} = \frac{15}{40} = \frac{3 \cdot 5}{8 \cdot 5} = \frac{3}{8} .$$

Thus 0.375 is equal to both $37\frac{1}{2}\%$ and $\frac{3}{8}$.

3. Express as percents:

   (a) $\frac{1}{125}$ (b) 0.475

4. Express as decimals:

   (a) $16\frac{1}{4}\%$ (b) $\frac{3}{16}$

How do you find the percent equivalent of $\frac{1}{3}$ whose decimal, 0.333... repeats endlessly? If you wish an approximate value you can round the decimal to two places and have:

$\frac{1}{3}$ is approximately equal to 0.33 or 33%.

An accurate name in percent for one-third can be found as follows:

$$\frac{1}{3} = \frac{\frac{1}{3}}{1} = \frac{\frac{1}{3} \cdot 100}{100} = \frac{33\frac{1}{3}}{100} = 33\frac{1}{3}\,\%$$

so that an accurate expression for $\frac{1}{3}$ is: $33\frac{1}{3}\,\%$.

5. Express approximately as percents:

   (a) $\frac{10}{9}$ (b) $\frac{2}{7}$

6. Express the above accurately as percents.

   How can $28\frac{4}{7}\%$ be expressed as a fraction?

$$28\frac{4}{7}\% = \frac{28\frac{4}{7}}{100} = \frac{\frac{200}{7}}{100} = \left(\frac{200}{7} \times \frac{1}{100}\right) = \frac{2}{7}.$$

*7. Express the following as fractions:

   (a) $25\frac{1}{4}\,\%$ (b) $125\frac{1}{2}\,\%$

8. Copy the following chart and fill in the missing names of numbers. The completed chart will be helpful to you in future lessons.

Fraction Simplest form	Hundred as denominator	Decimal	Percent
(a) $\frac{1}{8}$			
(b)	$\frac{300}{100}$		
(c)		0.375	
(d)			150%
(e)	$\frac{62.5}{100}$		
(f)		0.01	
(g) $\frac{7}{8}$			
(h)			100%
(i)	$\frac{16\frac{2}{3}}{100}$		
(j) $\frac{5}{6}$			
(k) $\frac{1}{9}$			
(l)	$\frac{60\frac{1}{2}}{100}$		
(m)		0.005	

9. Draw a number line and name points on it with the percents in Problem 8.

4-14. Applications of Percent

Percent is used to express ratios of numerical quantities in everyday experience. It is important for you to understand the notation of percent, and also, to be accurate in computing with numbers written as percents. Let us look at some examples in the use of percent.

Example 1. Suppose the annual income of a family is \$4560 and 32 % of the budget is allowed for food for the year. Then if x stands for the number of dollars allowed for food,

$$\frac{x}{4560} = \frac{32}{100}.$$

Using Property 1,

$$100x = 32(4560) = 145,920.$$

Hence

$$x = 1459.20$$

and the amount spent for food would be \$1459.20.

Example 2. The number 900 is what percent of 1500? Can you find the answer to this example without use of pencil? If x % stands for the percent which 900 is of 1500,

$$\frac{x}{100} = \frac{900}{1500}.$$

Again using Property 1,

$$1500\,x = 100(900) = 90000.$$

Hence

$$x = \frac{90,000}{1,500} = 60,$$

and 900 is 60 % of 1500.

Example 3. Suppose a family with an annual income of \$4560 rents a house for \$77.00 per month. What percent of the family income will be spent for rent? First, the rent for the year will be twelve times the monthly rent, that is \$924.00. If x % is the percent that 924 is of 4560,

$$\frac{x}{100} = \frac{924}{4560}$$

and, using Property 1 again,

$$4560x = 100(924)$$

$$4460x = 92,400$$

$$x = \frac{92,400}{4,560} = 20.3\ldots$$

Hence x % is about 20 % ; that is, about 20 % of the family income is spent for rent.

Exercises 4-14a

In the following problems it may be necessary to round some answers. Round the money answers to the nearest cent, and round percent answers to the nearest whole percent.

1. If the sales tax in a certain state is 4 % of the purchase price, what tax would be collected on the following purchases?
   (a) A dress selling for $17.50
   (b) A bicycle selling for $49.50
2. In a junior high school there are 380 seventh-grade pupils, 385 eighth-grade pupils, and 352 ninth-grade pupils.
   (a) What is the total enrollment of the school?
   (b) What percent of the enrollment is in the seventh grade?
   (c) What percent of the enrollment is in the eighth grade?
   (d) What percent of the enrollment is in the ninth grade?
   (e) What is the sum of the numbers represented by the answers to (b), (c), and (d)?
3. Mr. Martin keeps a record of the amounts of money his family pays in sales tax. At the end of one year he found that the total was $96.00 for the year. If the sales tax rate is 4 %, what was the total amount of taxable purchases made by the Martin family during the year?

*4. A certain store gives a 10% discount for cash and a 5% discount for purchases made on Mondays. That is, if a customer purchased an article priced at $100 and paid cash it would cost him $100 - $10 = $90 . Then if the day of his purchase were Monday he would get a further 5% discount, which would make the net price $85.50 since 5% of 90 is 4.50 and

90 - 4.50 = 85.50 .

Suppose the 5% discount on $100 had been computed first and the 10% second, would the final net price be the same? Would these two ways of computing the final net price give the same result for an article priced at $200? Why?

*5. In a certain store each customer pays a sales tax of 2% and is given a 10% discount for cash. That is, if a customer purchased an article priced at $100 and paid cash it would cost him $90 plus the sales tax or $91.80, since 2% of $90 is $1.80 . Suppose the sales tax were computed on $100 and then the 10% discount allowed, would the resulting net cost be the same? Why or why not?

*6. A customer in the store of Problem *4 added the discounts and thought that since he was paying cash for an article on Monday, he should receive 15% discount. If this were the case he would have paid $85. for the article priced at $100, instead of the $85.50. How should the shopkeeper have worded his notice of discounts to make it clear that he had in mind the calculation given in Problem *4?

Percents of Increase and Decrease

Percent is used to indicate an increase or a decrease in some quantity. Suppose that Central City had a population of 32,000 (rounded to the nearest thousand) in 1950. If the population increased to 40,000 by 1960, what was the percent of increase?

40,000
- 32,000
8,000 (actual increase)

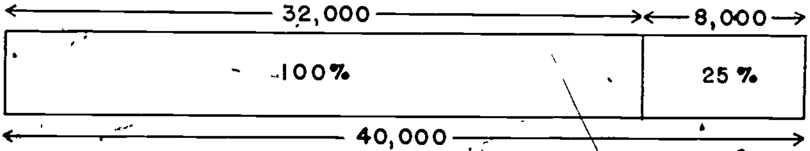
$$\frac{8,000}{32,000} = \frac{x}{100}$$

$$x = \frac{800,000}{32,000}$$

$$x = 25$$

There was an increase of 25%.

Notice that the percent of increase is found by comparing the actual increase with the earlier population figure.



The 40,000 was made up of the 32,000 (100%) plus the increase of 8,000 (25%). So the population of 40,000 in 1960 was 125% of the population of 32,000 in 1950.

Suppose that Hill City had a population of 15,000 in 1950. If the population in 1960 was 12,000, what was the percent of decrease? If x represents the percent of decrease, then

15,000
- 12,000
3,000 actual decrease

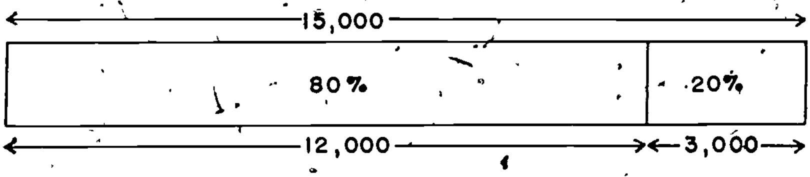
$$\frac{3,000}{15,000} = \frac{x}{100}$$

$$x = 100 \cdot \frac{3,000}{15,000}$$

$$x = 20$$

There was a decrease of 20%.

Notice that the population decrease also is found by comparing the actual decrease with the earlier population figure.



The 12,000 is the difference between 15,000 (100%) and the decrease of 3,000 (20%). So the population in 1960 of 12,000 was 80% of the population of 15,000 in 1950.

Exercises 4-14b

In each problem, 1 through 5, compute the percent of increase or decrease. If necessary, round percents to the nearest tenth of a percent.

1. In a junior high school the lists of seventh-grade absentees for a week numbered 29, 31, 32, 28, 30. The next week the five lists numbered 22, 26, 24, 25, 23.
   (a) What was the total number of pupil-days of absence for the first week?
   (b) What was the total for the second week?
   (c) Compute the percent of increase or decrease in the number of pupil-days of absence.
2. On the first day of school a junior high school had an enrollment of 1050 pupils. One month later the enrollment was 1200. What was the percent of increase?
3. One week the school lunchroom took in $450. The following week the amount was $425. What was the percent of decrease?
4. During 1958 a family spent $1490 on food. In 1959 the same family spent $1950 on food. What was the percent of increase in the money spent for food?

*5. During 1958 the owner of a business found that sales were below normal. The owner announced to his employees that all wages for 1959 would be cut 20%. By the end of 1959 the owner noted that sales had returned to the 1957 levels. The owner then announced to the employees that the 1960 wages would be increased 20% over those of 1959.
   (a) Which of the following statements is true?
      (1) The 1960 wages are the same as the 1958 wages.
      (2) The 1960 wages are less than the 1958 wages.
      (3) The 1960 wages are more than the 1958 wages.
   (b) If your answer to part (a) is (1), justify your answer. If your answer to part (a) is (2) or (3) express the 1960 wages as a percent of 1958 wages.

*6. In an automobile factory the number of cars coming off the assembly line in one day is supposed to be 500. One week the plant operated normally on Monday. On Tuesday there was a breakdown which decreased the number of completed cars to 425 for the day. On Wednesday operations were back to normal.
   (a) What was the percent of decrease in production on Tuesday compared with Monday?
   (b) What was the percent of increase in production on Wednesday compared with Tuesday?

7. (a) A salesman gets a 6% commission on an article which he sells for $1000. How much commission does he get?
   (b) A bank gets 6% interest per year on a loan of $1000. How much interest does the bank get?
   (c) The tax on some jewelry is 6% . How much tax would one have to pay on a pearl necklace worth $1000.?
   (d) Can you find any relationship among (a), (b), and (c) ?

8. (a) An article is sold at a 5% discount. If the stated price is $510, what did it sell for?
   (b) A small town had a population of 510 people on January first, 1958. The population decreased 5% during the following twelve months. What was its population on January first, 1959?
   (c) On a loan of $510, instead of charging interest, a bank loaned it on a discount of 5%; that is, gave the customer $510 minus 5% of $510 at the beginning of the year with the understanding that the amount of $510 would be paid back at the end of the year. How much did the customer receive at the beginning of the year?
   (d) Can you find any relationship among (a), (b), and (c) ?

9. What was the interest rate in problem 8 (c) above?

10. (a) Mr. Brown paid $210 in gasoline taxes during a year. If the tax on gasoline is 31%, how much did he spend on gasoline?
    (b) Mr. Smith made a down payment of $210 on a washing machine. If this was 31% of the total cost, what did the washing machine cost?
    (c) In a certain town 31% of the population was children. If there were 210 children, what was the population of the town?
    (d) On an article priced at $210, a merchant made a profit of 31%. What was the amount of his profit in dollars?
    (e) Can you find any relationship among (a), (b), (c), and (d)?

*11. The income tax collector looked at the income tax return of Mr. Brown mentioned in Problem 10 (a). He inquired to find that Mr. Brown drove a Volkswagen which would go about 30 miles on a gallon of gasoline. He also found that Mr. Brown could walk to work.
    (a) If gasoline cost $.30 a gallon (including tax), how many gallons did Mr. Brown buy? (Use information from Problem 10 (a).)
    (b) How far could he drive with this amount of gasoline?
    (c) What would be the average per day?
    (d) Why did the tax collector question Mr. Brown's return?

4-15. Large Numbers

The population of a city of over a million inhabitants might have been given as 1,576,961, but this just happened to be the sum of the various numbers compiled by the census takers. It is certain that the number changed while the census was being taken, and it is probable that 1,577,000 would be correct to the nearest thousand. For this reason there is no harm in rounding the original number to 1,577,000. In fact, for most purposes, you would merely say that the population of the city is "about one and one-half million", which could be written also:

Population of city $\approx$ 1,500,000.

The symbol $\approx$ is used to mean "is approximately equal to".

There are other ways of writing this number. One million can be written: 1,000,000 or $(10 \times 10 \times 10 \times 10 \times 10 \times 10)$ or 10^6. The notation $10 \times 10 \times 10 \times 10 \times 10 \times 10$ is sometimes read "the product of six tens". The exponent 6 indicates the number of tens used as factors in the product. You could also get the exponent by counting the number of zeros in the numeral 1,000,000.

The number 1500 can be expressed in several ways, 150×10 or 15×100 or $1.5 \times 1000 = 1.5 \times 10^3$. Similarly, 325 can be written as 32.5×10 or 3.25×10^2. In each of these two examples the last numeral is of this form:

(<u>a number between 1 and 10</u>) $\times$ (<u>a power of 10</u>).

In the first case it is 1.5×10^3 and in the second case it is 3.25×10^2. Write each of the following in this form.

Example: $4037 = 4.037 \times 10^3$

(a) 859 (c) 4832.59

(b) 7623 (d) 9783.6

When 1,500,000 is written as 1.5×10^6, it is said to be written in <u>scientific notation</u>.

(a) Is 15×10^5 in scientific notation? Why, or why not?

(b) Is 3.4×10^7 in scientific notation? Why, or why not?

(c) Is 0.12×10^5 in scientific notation? Why, or why not?

<u>Definition</u>. A number is expressed in scientific notation if it is written as the product of a number between 1 and 10 and the proper power of ten. Also a number is expressed in scientific notation when it is written just as a power of ten.

Both 1.7×10^3 and 10^4 are in <u>scientific notation</u>.

Note that $146{,}000 = 1.46 \times 10^5$ could also be written as

1.460×10^5 or 1.4600×10^5.

Exercises 4-15a

1. Write the following in scientific notation:

   (a) 1,000 (d) $10^2 \times 10^7$

   (b) $10^1 \times 10^1 \times 10^1 \times 10^7$ (e) 10×10^5

   (c) $10 \times 10 \times 10 \times 10$ (f) 10,000,000

2. Write the following in scientific notation:

   (a) 6,000 (d) 78,000

   (b) 678 (e) 600×10

   (c) 459,000,000 (f) 781×10^7

3. Write a numeral for each of the following in a form which does not use an exponent or indicate a product:

   (a) 3×10^4 (c) 436×10^6

   (b) 5.00×10^7 (d) 17.324×10^5

4. Round off each of the following to the nearest hundred. Express the rounded number in scientific notation.

   (a) 645 (d) 70,863

   (b) 93 (e) 600,000

   (c) 1233 (f) 5,362,449

Calculating with Large Numbers

Not only is scientific notation shorter in many cases but it makes certain calculations easier. Suppose you want to find the value of the product: $100 \times 1,000$. The first factor is the product of two tens. The second is a product of three tens, so, you have $100 = 10^2$ and $1000 = 10^3$.

Then,

$$\begin{aligned} 100 \times 1000 &= 10^2 \times 10^3 \\ &= (10 \times 10) \times (10 \times 10 \times 10) \\ &= 10 \times 10 \times 10 \times 10 \times 10 \\ &= 10^5 \end{aligned}$$

Hence $10^2 \times 10^3 = 10^5$. Notice that the exponent 5 is the sum of the exponents, 2 and 3.

Suppose you wish to find the product of 93,000,000 and 10,000. In scientific notation, this would be:

$$(9.3 \times 10^7) \times 10^4 = 9.3 \times (10^7 \times 10^4)$$ By which property?

$$= 9.3 \times 10^{11}$$

Now try a more difficult example:

$$93{,}000{,}000 \times 11{,}000 =$$

$$\left.\begin{aligned} &= (9.3 \times 10^7) \times (1.1 \times 10^4) \\ &= (9.3 \times 1.1) \times (10^7 \times 10^4) \end{aligned}\right\}$$

Note: The order of the factors has been changed by using the associative and commutative properties of multiplication.

$$= 10.23 \times 10^{11}$$

$$\doteq 1.023 \times 10^{12}$$

Distances to the stars are usually measured in "light years". A light year is the distance that light travels in one year. This is a good way to measure such distances. If they were expressed in miles, the numbers would be so large that it would be difficult to write them, much less understand what they mean.

Exercises 4-15b

1. It has been determined that light travels about 186,000 miles per second. In Parts (a) to (d) below, do not perform the multiplication, just indicate the product (an example of an indicated product is $2.4 \times 10 \times 56 \times 10^4$).

   Using 186,000 miles per second as the speed of light,

   (a) How far would light travel in 1 minute?

   (b) How far would light travel in 1 hour?

   (c) How far would light travel in 1 day?

   (d) How far would light travel in 1 year?

   (e) Find the number written in Part (d) and show that when "rounded off" it is 6.3×10^{12}.

   (f) The number written in Part (e) is about $6\frac{1}{2}$ ___?___.

   (g) Why is the number written in Part (d) not the exact number of miles that light travels in one year? Try to give two reasons.

2. Multiply, and express your answer in scientific notation:

   (a) $10^4 \times 3.5 \times 10^9$ (c) $7 \times 3 \times 10^5$

   (b) $300 \times 10^5 \times 20$ (d) $9.3 \times 10^7 \times 10 \times 10^6$

3. Multiply, and write your answer in scientific notation:

   (a) $9{,}000{,}000 \times 70{,}000$ (c) $25{,}000 \times 186{,}000$

   (b) $125 \times 8{,}000{,}000$ (d) $1100 \times 5 \times 200{,}000$

4. The distance around the earth at the equator is about 25,000 miles. In one second electricity travels a distance equal to about 8 times that around the earth at the equator. About how far will electricity travel in 10 hours?

5. The earth's speed in its orbit around the sun is a little less than seventy thousand miles per hour. About how far does the earth travel in its yearly journey around the sun?

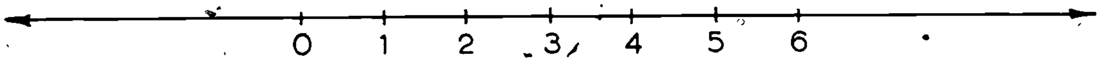
Chapter 5

COORDINATES AND EQUATIONS

5-1 The Number Line

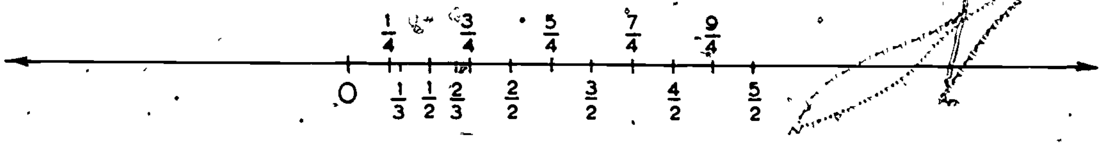
The idea of number is an abstract one. The development of a good number system required centuries in the civilization of man. To help understand numbers and their uses, many schemes have been used. One of the most successful of these ways to picture numbers is the use of the number line.

You can think of the line in the drawing below



as extending indefinitely in each direction. You can choose any point of the line and label it 0. Next, choose another point to the right of 0 and label it 1. This really determines a unit of length from 0 to 1. Starting at 0, lay off this unit length repeatedly toward the right on the number line. This determines the location of the points corresponding to the counting numbers 2, 3, 4, 5, $\cdots$.

The number $\frac{1}{2}$ is associated with the point midway between 0 and 1. By laying off this segment of length one-half unit over and over again, the additional points corresponding to $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ are determined. Next, by using a length which is one-third of the unit segment and measuring this length successively to the right of zero, the points $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \cdots$ are located. Similarly, points are located to the right of 0 on the number line corresponding to fractions having denominators 4, 5, 6, 7, $\cdots$. Some of these are shown in the following figure.



By this natural process there is associated with each rational number a point on the line. Just one point of the line is associated with each rational number. There is thus a one-to-one correspondence between these rational numbers and some of the points of the line. You can speak of the point on the number line corresponding to the number 2 as the point 2. Because of this one-to-one correspondence between number and point, each point can be named by the number which labels it. This is one of the great advantages of the number line. It allows us to identify points and numbers and helps us use geometric points to picture relations among numbers.

Remark: You might think that this one-to-one correspondence assigns a number to every point on the line, to the right of 0. This is far from true. In fact, there are many, many more points unlabeled than labeled by this process. These unlabeled points correspond to numbers like π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, which are not rational numbers. Later you shall study more about such numbers.

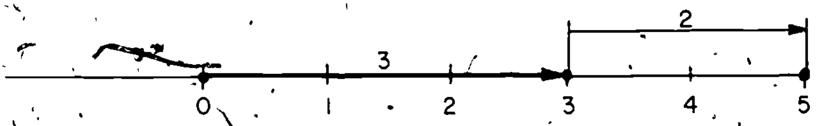
Properties of the Number Line

The number line locates numbers by means of points in a very natural way. The construction of the number line locates the rational numbers in order of increasing size. Hence you can always tell where a number belongs on the line. The larger of two numbers always lies to the right on a number line. Thus: $5 > 3$ (5 is greater than 3), and on the number line 5 lies to the right of 3. A number greater than 3 corresponds to a point located to the right of 3. Since $2 < 4$, 2 lies to the left of 4. You can easily check the relative positions of numbers such as 3, 0, $\frac{5}{4}$, 1, $\frac{4}{3}$, $\frac{3}{2}$. Once you have located the corresponding points on the line it is easy to tell at a glance whether one number is greater than another or less than another.

The point corresponding to 0 is chosen as a point of reference and called the origin. The half-line extending to the right from the origin along the number line is called the positive half-line. Any number which is greater than zero lies on this positive half-line and is called a positive number. In particular, the counting numbers 1, 2, 3, 4 ··· are called the positive integers.

Addition on the Number Line

Addition of two numbers can easily be pictured on the number line. To add 2 to 3, start at 3 and move 2 units to the right. In this way the operation $3 + 2 = 5$ is represented by a motion along the number line. The motion ends at the point corresponding to the sum.



You may also think of the number 3 as determining an arrow (or directed line segment) starting at 0 and ending at 3. To represent the addition of 2 to 3, simply draw an arrow of length 2 to begin at 3 instead of at 0. The arrow (directed line segment) representing 3 + 2 thus begins at 0 and ends at 5. To avoid confusion, these arrows are frequently indicated slightly above the number line

Exercises 5-1

1. For each of the following numbers draw a number line. Use one inch as the unit of length. Locate a point of origin on the line, and then locate the point corresponding to the number. Just above the number line, draw the corresponding arrow.

   (a) 4 (b) $\frac{1}{2}$

   (c) $\frac{17}{8}$ (d) $2\frac{1}{4}$

2. Represent each of the following sums by means of three arrows on a number line.

   (a) $1 + 6$ (c) $\frac{3}{8} + \frac{9}{8}$

   (b) $\frac{1}{2} + \frac{3}{2}$ (d) $\frac{3}{4} + 1.25$

3. Locate the following numbers on a number line and determine which is the largest in each set.

   (a) $1, \frac{9}{8}, \frac{3}{2}$

   (b) $\frac{3}{4}, \frac{8}{8}, 0.875$

4. Locate on a number line the midpoints of the following segments.

   (a) From 0 to 2. (c) From $\frac{1}{2}$ to $\frac{7}{2}$.

   (b) From $\frac{1}{8}$ to $\frac{5}{8}$. (d) From 2 to 6.5.

5. Use a diagram representing addition by means of arrows on the number line to show that $2 + 3 = 3 + 2$. What property of addition does this illustrate?

6. Using arrows to represent addition on the number line, show that

$$(2 + 3) + 1 = 2 + (3 + 1).$$

What property of addition does this illustrate?

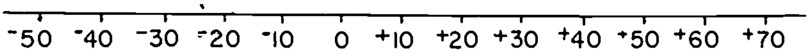
7. Think of a way to represent the product $3 \cdot 2$ by means of arrows on the number line. Try it also for $5 \cdot 2$ and $6 \cdot \frac{1}{4}$.

8. How would you show that $2 \cdot 3 = 3 \cdot 2$ by means of arrows on the number line? What property of multiplication does this illustrate?

5-2 Negative Rational Numbers

In the preceding discussion of number line there is a very serious gap. The points to the left of zero were not labeled. Only the half-line from the origin in the positive direction was used. To suggest how to label these points (and why you want to!), let us look at the familiar example of temperature.

A number line representing temperature, such as you find on a thermometer, often looks like this.

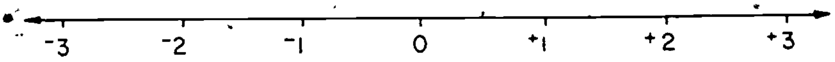


Temperature in Degrees Fahrenheit

Here temperatures less than zero are represented by numbers to the left of the origin and designated by the symbol " $^-$ ". Temperatures greater than zero are identified with the sign " $^+$ ". Thus, $^-10$ refers to a temperature of 10 degrees below zero (to the left of zero), and $^+10$ refers to a temperature of 10 degrees above zero (to the right of zero). Actually, above zero and below zero seem more natural terms to use when the scale is vertical.

This idea of distance (or of points) along a line on opposite sides of a fixed point occurs frequently in our ordinary tasks. Think how often you speak of distances to the left or to the right, locations north or south of a given point, altitudes above or below sea level, longitudes east or west, or the time before or after a certain event. In each of these situations, there is a suggestion of points located on opposite sides of a given point (or number), or distances measured in opposite directions from a given point (or number). All of them suggest the need for a number line which uses points to the left of the origin as well as points to the right of the origin.

The natural way to describe such a number line is easy to see. You will start with the number line for positive rationals which you have already used. Using the same unit lengths, measure off distances to the left of zero as shown below:



Locate $^-1$ as opposite to $^+1$ in the sense that it is 1 unit to the left of zero. Similarly $^-2$ is opposite to $^+2$, $^-(\frac{1}{4})$ is located opposite to $^+\frac{1}{4}$, $^-(\frac{5}{2})$ is opposite to $^+\frac{5}{2}$, etc. These "opposite" numbers, corresponding to points to the left of zero, are called <u>negative numbers</u>. Each negative number lies to the left of zero on the number line and corresponds to the opposite positive number. This direction "to the left" is called the negative direction.

Negative numbers are denoted as $^-1$, $^-2$, $^-(\frac{1}{4})$, $^-(\frac{3}{2})$, $^-(\frac{9}{8})$ etc., by use of the raised hyphen. You can read ($^-2$) as "negative two." This symbol " $^-$ " tells us that the number is less than zero (lies to the left of zero). Sometimes to emphasize that a number is positive (greater than zero), the symbol " $^+$ " is written in a raised position as in $^+2$, $^+(\frac{3}{2})$, etc.

The new numbers that have been introduced by this process are the negative rational numbers. The set consisting of positive rational numbers, negative rational numbers, and zero, is called the rational numbers.

The special set of rational numbers which consists of the positive integers, the negative integers and zero is called the set of integers. This set is frequently denoted as:

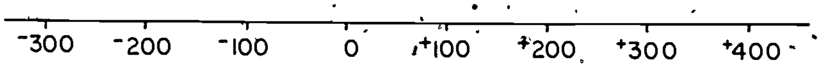
$$I = \{\cdots, {}^-4, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, 4, \cdots\}.$$

Note that the set of integers consists of only the counting numbers and their opposites together with zero.

Examples of the Use of Negative Numbers

The negative numbers are as real and as useful as the positive numbers you have used before. In fact you have used them many times without calling them negative numbers. Their special usefulness is in denoting the idea of "opposite" or "oppositely directed" which was mentioned.

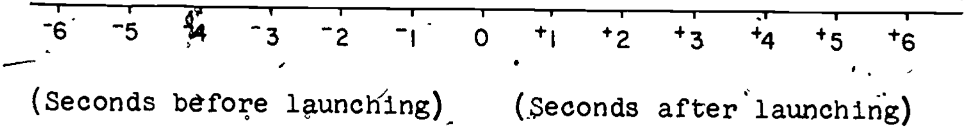
Let us use positive numbers to denote distance east of Chicago. The negative numbers will denote distances west of Chicago. A number line like the one below



Distance from Chicago in miles

can therefore be used to plot the position of an airliner flying an East-West course passing over Chicago. For an airliner flying a North-South course over Chicago, how could you interpret this number line?

The time before and after the launching of a satellite can be indicated on a number line like the following:



Note that tne number line you use need not be placed horizontally. If you speak of altitude above sea level as positive and altitude below sea level as negative, it may seem more natural to use a number line in the vertical position.

To represent business profits and losses, a vertical line is more convenient. A higher position in the line seems naturally to correspond to greater profit.

Exercises 5-2

1. Locate on the number line points corresponding to the following numbers.

   (a) $^{-}8$ (d) $\frac{4}{8}$

   (b) $^{-}(\frac{7}{4})$ (e) 1.5

   (c) $1\frac{3}{4}$ (f) $^{-}(\frac{1}{2})$

   Are there any pairs of "opposites" on this list?

2. Sketch the arrows determined by the following rational numbers.

   (a) 6 (d) $\frac{12}{8}$

   (b) $^{-}4$ (e) $^{-}(\frac{20}{8})$

   (c) $^{-}5.5$ (f) $\frac{3}{4}$

3. Arrange the following numbers in the order in which they appear on the number line: $^{-}4$, $\frac{1}{4}$, $^{-}(\frac{7}{4})$, $\frac{5}{8}$, $^{-}6$, $^{-}(\frac{3}{8})$, $\frac{3}{4}$.
   Which is the largest? Which is the smallest?

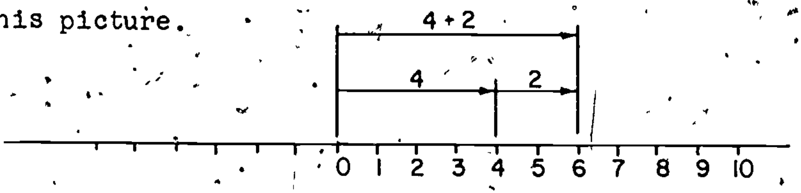
4. How could you represent the following quantities by means of positive and negative numbers?
   (a) A profit of $2000; a loss of $6000.
   (b) An altitude of 100 ft. above sea level; an altitude of 50 ft. below sea level.
   (c) A distance of 2 miles East; a distance of 4 miles West.

5. The elevator control board of a department store lists the floors as B 2, B 1, G, 1, 2, 3. Here G refers to ground level and B 1, B 2 denote basement levels. How could you use positive and negative numbers to label these floors?

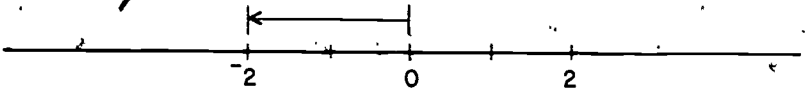
6. Draw a number line indicating altitudes from $^{-}1,000$ ft. to $^{+}10,000$ ft. Use intervals of 1,000 ft. Locate altitudes of $^{-}800$ ft., $^{+}100$ ft., $^{+}2500$ ft., $^{-}500$ ft.

5-3 Addition of Rational Numbers

You saw that the addition of two positive numbers is easily represented on the number line. On the number line, the sum $4 + 2$ is represented by the point 2 units beyond 4. Note that in adding a ~~positive~~ number to another positive number, you always move to the right (in the positive direction) along the number line. So this process of addition can be described by saying that in adding 2 to 4, you start at 4 and move 2 units to the right, or 2 units in the positive direction. You saw that a convenient way to represent this process is by means of arrows (directed line segments) of appropriate length. Thus, the sum $4 + 2$ corresponds to this picture.

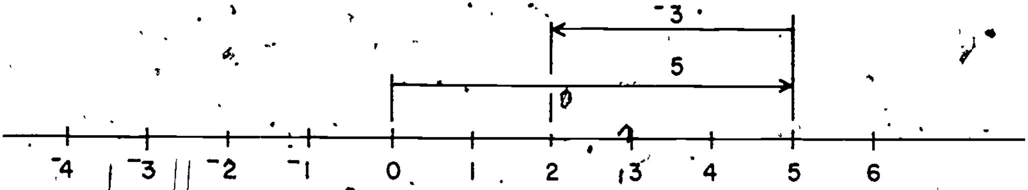


Think now of our construction for the negative numbers in the number line. Remember that $^-2$ is the opposite of 2. To say that $^-2$ is the opposite of 2 means that $^-2$ is the same distance from 0 as 2 but in the negative direction as shown here:



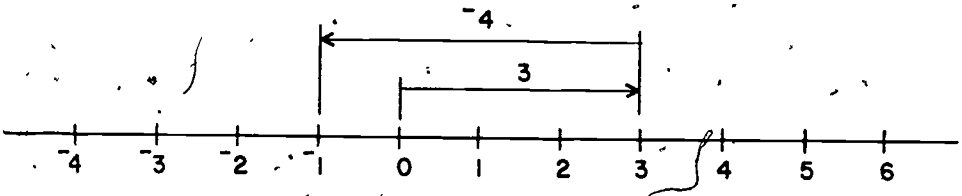
The arrow associated with $^-2$ is 2 units in length and specifies the negative direction as indicated in the sketch. How would you sketch $^-4$; $^-(\frac{1}{2})$; $^-{}^-3$; $^-(\frac{5}{3})$?

What would the sum $5 + (^-3)$ mean? Using directed arrows, you can find the point corresponding to $5 + (^-3)$ by starting at 0, moving 5 units in the positive direction and then 3 units in the negative (opposite) direction. Thus, $5 + (^-3) = 2$, as shown in the following sketch:



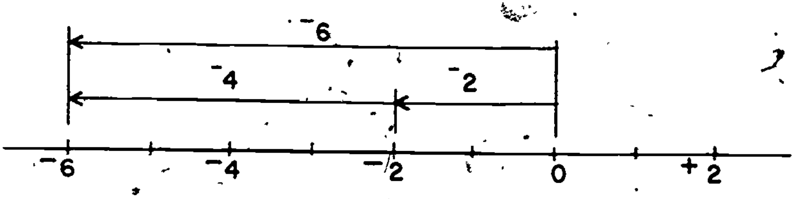
Here, $^-3$, is associated with an arrow of length 3 units directed in the negative direction. In adding $(^-3)$ to 5, you simply draw the arrow for $^-3$ as originating at 5 (that is, beginning at the end of the arrow corresponding to 5).

To add 3 and $^-4$, draw a sketch like the following:



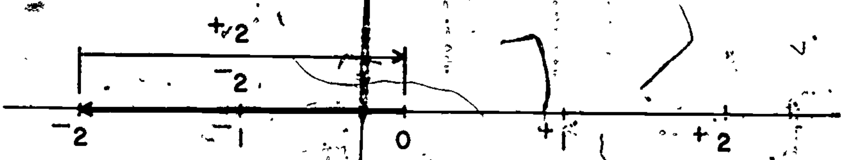
Thus, $3 + (^-4) = {}^-1$. Find the sum $2 + (^-5)$ in the same way.

Consider the sum $(^-2) + (^-4)$. Here the arrows are both in negative direction. You can see from a sketch that $(^-2) + (^-4) = (^-6)$.



In the same way, find the sums: $(^-3) + (^-2)$; $(^-1) + (^-6)$; $(^-6) + (^+2)$.

One property of special interest is illustrated by the sum $(^-2) + (^+2)$.



Here $^-2$ corresponds to an arrow of length 2 units in the negative direction. Adding $(^-2)$ and (2) corresponds to moving 2 units to the left from 0 and then 2 units back to 0. Thus $(^-2) + (2) = 0$. Check that $(^-1) + 1 = 0$, $3 + (^-3) = 0$, $(^-8) + 8 = 0$.

You can see by the use of the number line that the addition of numbers, whether positive or negative, is really very simple. You need only keep in mind the location of the numbers on the number line to carry out the operation. It is seen that:

When both numbers are positive the sum is positive,

as in $4 + 2 = 6$;

and, when both numbers are negative the sum is negative,

as in $(^-4) + (^-2) = {^-6}$.

When one number is positive and one number is negative, it is the number farther from the origin which determines whether the sum is positive or negative.

For example:

In $(^-4) + 2 = {^-2}$,

the sum is negative because the $^-4$, which is farther from zero than 2, is negative.

In $4 + (^-2) = 2$,

the sum is positive, because the 4, which is farther from zero than 2, is positive.

Another way of saying this is, the arrow of greater length determines whether the sum is positive or negative. In fact, this one rule also works in the case when both numbers are positive or when both numbers are negative. Notice that in cases like $(^-2) + 2$ and $3 + (^-3)$, the arrows are of equal lengths but opposite in direction. In these cases the sum will be zero.

Exercises 5-3

1. Find the following sums and sketch, using arrows on the number line.

   (a) $9 + (^-5)$

   (b) $10 + (^-7)$

   (c) $5 + (^-10)$

   (d) $(^-12) + 7$

2. Supply the missing numbers in each of the following statements so that each statement will be true.

   (a) $3 + (^-3) =$

   (b) $(\ \ ) + (^-4) = 0$

   (c) $(-75) + 74 =$

   (d) $14 + (^-2) =$

3. Obtain the sum in each of the following problems. Sketch the addition by means of arrows on the number line.

   (a) $25 + (^-6)$

   (b) $(^-5) + (^-7)$

   (c) $(-8) + 11$

   (d) $(^-6) + 9$

4. Supply a negative number in each blank space so that each sum will be correct.

   (a) $7 + (\ \ ) = 2$

   (b) $10 + (\ \ ) = {}^-1$

   (c) $4 + (\ \ ) = (^-2)$

   (d) $(^-4) + (\ \ ) = (^-10)$

5. A company reports income for the first six months of a year as follows:

January	$5000	profit	April	$1000	profit
February	$2000	profit	May	$4000	loss
March	$6000	loss	June	$3000	loss

(a) How could you represent these income figures in terms of positive and negative numbers?

(b) What is the total income for the six-month period?

(c) What is the total income for the first three months of the year?

(d) What is the total income for the four-month period, March, April, May and June?

6. In four successive plays from scrimmage, starting at its own 20 yard line, Franklin High makes

a gain of 17 yards, then
a loss of 6 yards, next
a gain of 11 yards, and finally
a loss of 3 yards.

(a) Represent the gains and losses in terms of positive and negative numbers.

(b) Where is the ball after the fourth play?

(c) What is the net gain after the four plays?

7. (a) Think of a way to represent the product $3(^{-}2)$ by means of arrows on the number line. Try it also for:

(b) $5(^{-}1)$.

(c) $2 \cdot {}^{-}(\frac{1}{4})$.

Inverse Elements under Addition

Recall that $^+2 + (^-2) = 0$. This sentence says that $(^-2)$ is the number which when added to $^+2$ yields 0. You saw previously that 0 is the identity element under the operation of addition. Any two numbers with sum 0 are said to be inverse elements under addition. Hence $(^-2)$ is the inverse element corresponding to $^+2$ under the operation of addition. Therefore $(^-2)$ shall be called the additive inverse of $^+2$. Likewise $^+2$ is the additive inverse of $(^-2)$ Taken together, the elements $^+2$ and $(^-2)$ are called additive inverses.

Class Exercises 5-3a

1. Find the additive inverse of each of the following numbers.

   7, $(^-9)$, 11, $(^-12)$, $(^-6)$, 15, $(^-20)$, 0, $(^-\frac{2}{3})$, $\frac{4}{9}$, $^-(\frac{7}{8})$, $\frac{30}{31}$.

2. Which of the following pairs are additive inverses?

   (a) $^+20$; $^+20$

   (b) $(^-5)$, $(^-5)$

   (c) $.5$, $^-(\frac{1}{2})$

   (d) $\frac{^-1}{2}$, $\frac{2}{4}$

On the number line you can see that any number and its additive inverse will be represented by arrows of the same length and opposite direction. When added, these "opposite" arrows of equal length always give 0.

In the addition of a positive and a negative number, it was noted that the larger arrow determines whether the sum is positive or negative. The length of the arrow for the sum can be obtained by the picture of additive inverses. For example, in the sum $5 + (^-2) = 3$ you may write

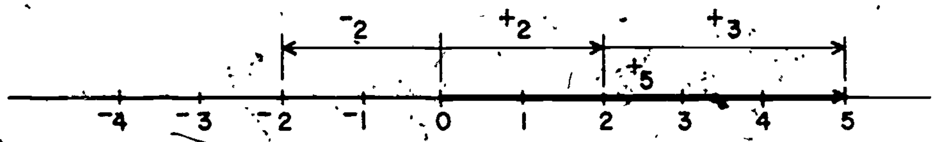
$$5 + (^-2) = 3 + 2 + (^-2)$$

by introducing the additive inverse of the number represented by the smaller arrow. Then,

since $2 + (^-2) = 0$ you have

$$5 + (^-2) = 3 + 2 + (^-2) = 3 + 0 = 3.$$

Note that the other arrow of the two into which 5 is separated represents the sum 3.



This procedure is a general one, as the following examples illustrate.

$$8 + (^-7) = 1 + 7 + (^-7) = 1 + 0 = 1$$

$$(^-8) + 7 = (^-1) + (^-7) + 7 = {}^-1 + 0 = {}^-1$$

$$19 + (^-26) = 19 + (^-19) + (^-7) = 0 + (^-7) = {}^-7$$

In each case, the additive inverses add up to zero, and the remaining number is the sum.

Class Exercises 5-3b

1. Sketch the arrows corresponding to the numbers in the above three examples. In each case, determine the arrow corresponding to the sum.

2. Perform the following additions by introducing the additive inverse for the smaller arrow, and sketch the operation on the number line.

   (a) $^{+}10 + (^{-}5)$

   (b) $8 + (^{-}6)$

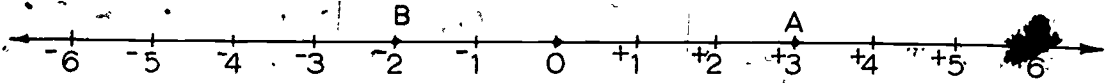
   (c) $(^{-}4) + 3$

   (d) $(^{-}13) + 9$

5-4 Coordinates and Graphs

Coordinates on a Line

Let us consider the number line from a different point of view. As you have seen, a rational number can always be associated with a point on the number line. The number associated in this way with a point of the line is called a coordinate of the point. In the drawing below, the number zero is associated with the reference point called the origin.



Point A is denoted by the number $(^{+}3)$. Point B is denoted by $(^{-}2)$. $A(^{+}3)$ is written to mean that A is the point with coordinate $^{+}3$. Likewise $B(^{-}2)$ means that B is the point with coordinate $(^{-}2)$ on the line.

Recall that every positive rational number is associated with a point on the positive half-line. Every negative rational number corresponds to a point on the negative half-line. The coordinate that is assigned to a point in this way tells us two things. It tells us the <u>distance</u> from the origin to the point. It also tells us the <u>direction</u> from the origin to the point.

Exercises 5-4a

1. Draw a segment of a number line 6 inches in length and place the origin at its mid-point. Mark off segments of length one inch. On the line locate the following points:

$$A(^{-}1),\ B(\tfrac{5}{2}),\ C(1),\ T(0),\ L(^{-}\tfrac{3}{2}),\ P(^{-}2).$$

2. (a) In Problem 1, how far is it in inches between the point labeled T and the point labeled L?
   (b) between P and B?
   (c) between L and B?
   (d) from the origin to A?

3. Using a number line with 1 inch as the unit of length, mark the following points:

$$R(\tfrac{1}{3}),\ S(\tfrac{5}{6}),\ D(^{-}\tfrac{3}{2}),\ F(0),\ E(^{+}\tfrac{3}{2}).$$

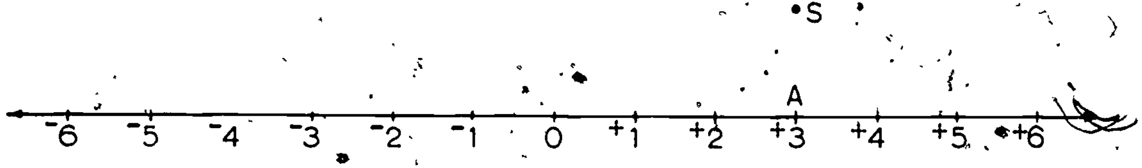
4. If the line segment in Problem 3 were a highway and it was drawn to a scale where 1 inch represents 1 mile, how far in miles is it between these points on the highway:
   (a) F and R?
   (b) D and E?

5. Draw a number line in a vertical instead of horizontal position. Mark your number scale with positive numbers above the origin and negative numbers below the origin. Label points to correspond with the rational numbers 0, 1, 2, 3, $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$.

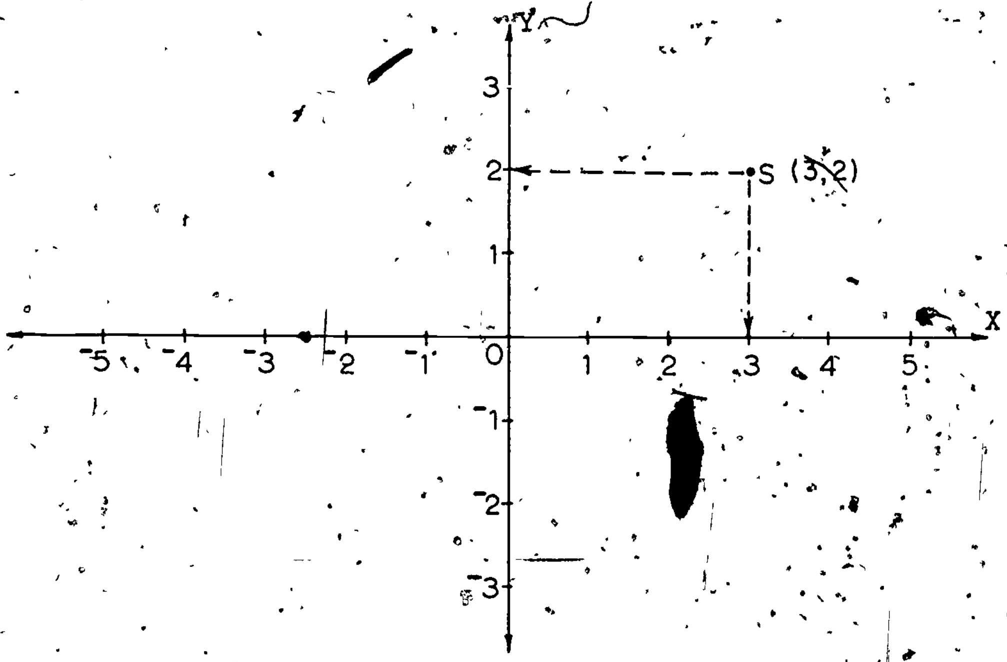
Coordinates in the Plane

Recall from your previous work in this chapter that number lines can be drawn vertically as well as horizontally.

You have learned that a single coordinate locates a point on the number line. A point like S below is not on the number line and cannot be located by a single coordinate. However, you can see that



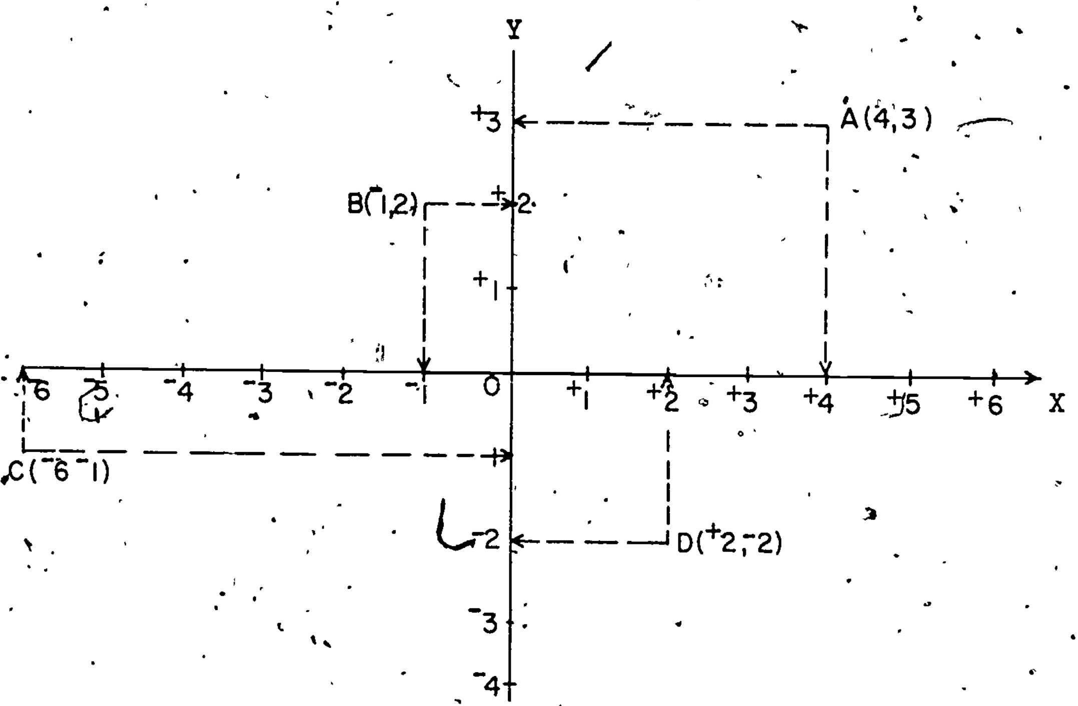
S is directly above the point A($^{+}3$). To locate point S, draw a vertical number line perpendicular to the horizontal number line and intersecting it at the origin. Your drawing should look like this:



The horizontal number line is called the X-axis and the vertical number line is called the Y-axis. When both number lines are referred to, they are called the axes.

To determine the coordinates of point S look at the diagram above. Draw a line segment from point S perpendicular to the X-axis. It intersects the X-axis at $(^{+}3)$. Now draw a perpendicular from point S to the Y-axis at $(^{+}2)$. Point S has an x-coordinate of $(^{+}3)$ and a y-coordinate of $(^{+}2)$, which is written as $(^{+}3, {}^{+}2)$. Parentheses are used and the x-coordinate is always written before the y-coordinate.

In the diagram below, observe how the coordinates of points A, B, C, and D were located.



Thus, in general, P(x, y) represents the point P in terms of its coordinates. This may be done for any point P in the coordinate plane. This system of coordinates is called a rectangular system because the axes are at right angles to each other and distances of points from the axes are measured along perpendiculars.

from the points to the axes. Each ordered pair of rational numbers is assigned to a point in the coordinate plane. Locating and marking the point with respect to the X-axis and the Y-axis is called <u>plotting the point</u>.

The idea of a coordinate system is not new to you. When you locate a point on the earth's surface, you do so by identifying the longitude and latitude of the point.

Note that the order in which you write these numbers is important. Suppose you were giving directions to help a friend locate a certain place in a city laid out in rectangular blocks (streets at right angles to each other). You tell him to start at the center of the city, go 3 blocks east and two blocks north. Would this be the same as telling him to go 2 blocks east and 3 blocks north? Of course not! Do you see why it is important to be careful with the order when writing a pair of coordinates?

<u>Exercises 5-4b</u>

1. Given the following set of ordered pairs of rational numbers, locate the points in the plane associated with these pairs.

   $\{(4,1),\ (1,0),\ (0,1),\ (2,4),\ (4,4),$

   $({}^-1,{}^-1),\ ({}^-3,3),\ (4,{}^-3),\ ({}^-5,3),\ (0,\ {}^-5),\ ({}^-6,0)\}.$

2. (a) Plot the coordinate pairs in the following set:

   $A = \{(0,0),\ ({}^-1,0),\ ({}^+1,0),\ ({}^-2,0),\ ({}^+2,0),\ ({}^-3,0),({}^+3,0)\}$

   (b) Do all of the points named by Set A seem to lie on the same line?

   (c) Does the condition $y = 0$ describe the line containing the points of Set A?

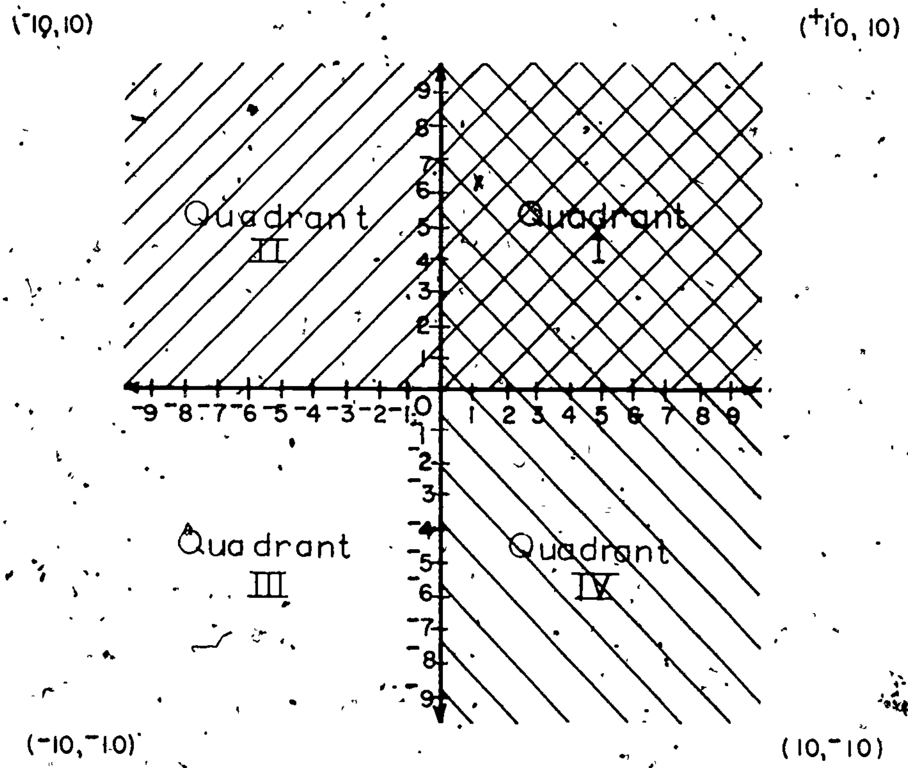
3. (a) Plot the coordinate pairs in the following set:

   $B = \{(0,0), (0,^-1), (0,^+1), (0,^-2), (0,^+2), (0,^-3), (0,^+3)\}$

   (b) Do all of the points named by Set B seem to lie on the same line?

   (c) Does the condition $x = 0$ describe the line containing the points of Set B?

Did you notice that the half planes above and below the X-axis intersect the half planes to the right and to the left of the Y-axis? These intersections are called <u>quadrants</u> and are numbered in a counter-clockwise direction with Quadrant I being the intersection of the half plane above the X-axis and the half plane to the right of the Y-axis. This quadrant does not include the points on the positive X-axis or positive Y-axis, nor does it include the origin.



Points in the intersection set of these two half planes are in the first quadrant or Quadrant I. The intersection of the half plane above the X-axis and half plane to the left of the Y-axis is Quadrant II. Quadrant III is the intersection of the half plane below the X-axis and the half plane to the left of the Y-axis. Quadrant IV is the intersection of the half plane to the right of the Y-axis and the half plane below the X-axis. Note that the coordinate axes are not a part of any quadrant.

The numbers in an ordered pair may be positive, negative, or zero, as you have noticed in the exercises. Both numbers of the pair may be positive. Both numbers may be negative. One may be positive and the other negative. One may be zero, or both may be zero.

Where are all the points for which both numbers in the ordered pair are positive? Will they all be in the same quadrant? How can you tell?

Where are all the points for which both numbers of the ordered pair are negative? Show this by plotting some points. Learn to predict in which quadrant the point lies if you know its coordinates.

Class Exercises 5-4a

1. Given the following ordered pairs of numbers, write the number of the quadrant in which you find the point represented by each of these ordered pairs.

Ordered Pair

(a) $(3, 5)$ (b) $(1, {}^{-}4)$ (c) $({}^{-}4, 4)$ (d) $({}^{-}3, {}^{-}1)$

(e) $(8, 6)$ (f) $(7, {}^{-}1)$ (g) $({}^{-}3, {}^{-}5)$

2. (a) Both numbers of the ordered pair of coordinates are positive. The point is in Quadrant ____________.

(b) Both numbers of the ordered pair of coordinates are negative. The point is in Quadrant ________.

(c) The x-coordinate of an ordered pair is negative and the y-coordinate is positive. The point is in Quadrant________.

(d) The x-coordinate of an ordered pair is positive and the y-coordinate is negative. The point is in Quadrant________.

3. (a) The x-coordinate of an ordered pair <u>is</u> zero and the y-coordinate <u>is</u> <u>not</u> zero. The point lies on which axis?

(b) The x-coordinate of an ordered pair <u>is</u> <u>not</u> zero and the y-coordinate <u>is</u> zero. The point lies on which axis?

(c) <u>Both</u> coordinates of an ordered pair <u>are</u> zero. The point is located where?

Exercises 5-4c

1. (a) Plot the points of set $L = \{A(^+2, ^+1), B(^+2, ^+3)\}$.

(b) Use a straight edge to join A to B. Extend line segment AB.

(c) Line AB seems to be parallel to which axis?

2. (a) Plot the points of set $M = \{A(^+2, ^+3), B(^+5, ^+3)\}$.

(b) Use a straight edge to join A to B. Extend line segment AB.

(c) Line AB seems to be parallel to which axis?

3. (a) Plot the points of set $N = \{A(0,0), B(^+2, ^+3)\}$

(b) Join A to B. Extend line segment AB.

(c) Is line AB parallel to either axis?

4. (a) Plot the points of set $P = \{A(^+4, ^+4), B(^+2, 0)\}$

(b) Join A to B. Extend line segment AB.

(c) Plot the points of set $Q = \{C(^+6, ^+3), D(0, ^+1)\}$

(d) Join C to D. Extend line segment CD.

(e) What is the intersection set?

5. (a) Plot the points of set $R = \{A(0,0), B(^+6,0), C(^+3, ^+4)\}$ on the coordinate plane.

(b) Use a straight edge to join A to B, B to C, C to A.

(c) Is the triangle (1) scalene, (2) isosceles or (3) equilateral?

6. (a) Plot the points of set S {A($^{+}2$, $^{+}2$)), B($^{-}2$, $^{+}2$), C($^{-}2$, $^{-}3$), D($^{+}2$, $^{-}3$)}.

   (b) Use a straight edge to join A to B, B to C, C to D and D to A.

   (c) Is the figure a square?

   (d) Draw the diagonals of the figure.

   (e) The coordinates of the point of intersection of the diagonals seem to be______ ____.

7. (a) Plot the points of set T = {A($^{+}2$, $^{+}1$), B($^{+}3$, $^{+}3$), C($^{-}2$, $^{+}3$), D($^{-}3$, $^{+}1$)}.

   (b) Use a straight edge to join A to B, B to C, C to D and D to A.

   (c) What is the name of the quadrilateral formed?

   (d) Draw the diagonals of quadrilateral ABCD.

   (e) The coordinates of the point of intersection of the diagonals seem to be ____________.

Graphs

Consider the following statement:

$$y = 4$$

Draw a pair of coordinate axes and label them. Locate a point (0,4) in the plane. Locate the five other points: ($^{-}4$,4), (2,4), ($^{-}2$,4), (5,4), and (7,4). The statement y = 4 describes these ordered pairs of numbers since the y-coordinate in each pair is 4. Are there other points in the plane with a y-coordinate of 4? Draw a line containing the set of points described by this statement y = 4. Are the x-coordinates of all these points equal?

The graph of the set of points described by the statement $y = 4$ lies on the line 4 units above and parallel to the X-axis.

Class Exercise 5-4b

1. What statement describes the following set of points in the coordinate plane?
   (a). The set of points with y-coordinate 3.
   (b) The set of points with y-coordinate $^-3$.
   (c) The set of points with y-coordinate $^-4$.
   (d) The set of points with y-coordinate $\frac{1}{2}$.

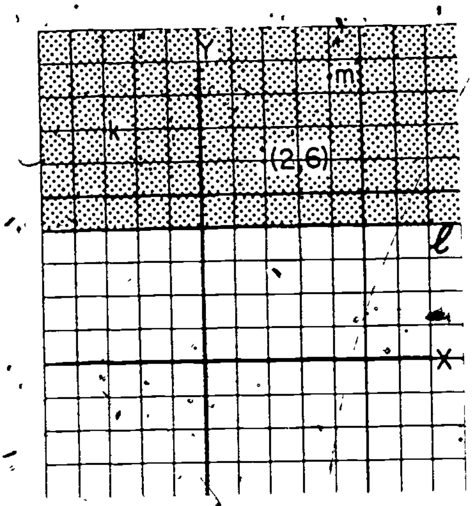
   Sketch the graph of the set of points described by each of the statements listed above.

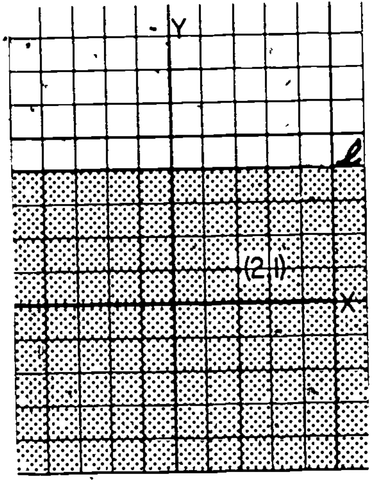
2. Sketch the graph of the set of points described by each statement listed:

(a)	$x = 2$	(d)	$y = 2\frac{1}{3}$
(b)	$y = 6$	(e)	$y = {}^-1\frac{1}{2}$
(c)	$x = 1\frac{1}{2}$	(f)	$x = {}^-3$

Now let us consider these statements:

(a) $y > 4$ (b) $y < 4$





In diagram (a) line ℓ, you recall, is the graph of the set of points described by the statement $y = 4$. Choose a point (2,6). Does the statement $y = 4$ describe this ordered pair? Since the y-coordinate in this ordered pair is greater than 4, the statement $y > 4$ describes it. Are there other points in the plane with y-coordinates greater than 4? Locate two other points, k and m, with y-coordinates greater than 4. Are these points above the line ℓ? Yes, they are in the shaded region which is one of the half planes determined by the line $y = 4$.

The graph of the set of points described by the statement $y > 4$ lies in the half-plane above the line 4 units above and parallel to the X-axis.

In the shaded part of diagram (b) are located points for which the coordinates satisfy the statement $y < 4$. Locate point (2,1) in this region. Since the y-coordinate is less than 4, the statement $y < 4$ describes it. Locate other points in the plane with y-coordinates less than 4. Are these points in the half plane below line ℓ? Try other points in the half plane below line ℓ to see if they satisfy the statement $y < 4$.

The graph of the set of points described by the statement $y < 4$ lies in the half plane below the line 4 units above and parallel to the X-axis.

Exercises 5-4d

1. Sketch the graph of the set of points selected by each statement below. Use different coordinate axes for each graph.

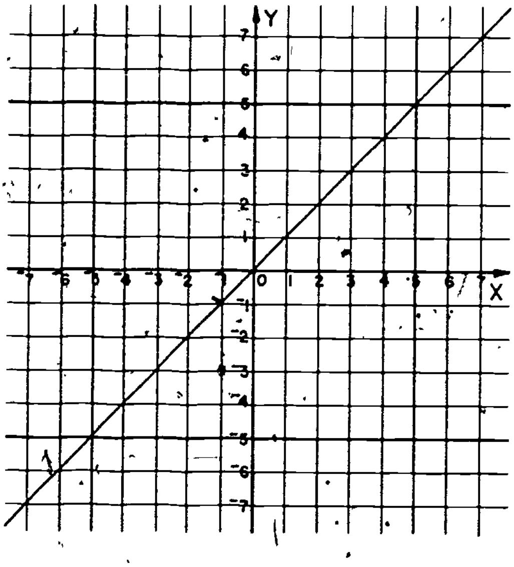
(a)	$y = {}^{+}2$	(g)	$x = {}^{-}3$
(b)	$y > {}^{+}2$	(h)	$x > {}^{-}3$
(c)	$y < {}^{+}2$	(i)	$x < {}^{-}3$
(d)	$x = 3$	(j)	$y = {}^{-}2$
(e)	$x > 3$	(k)	$y > {}^{-}2$
(f)	$x < 3$	(l)	$y < {}^{-}2$

Let us look at the statement $y = x$. Using the points named by the coordinate pairs of set Q, sketch the graph of $y = x$,

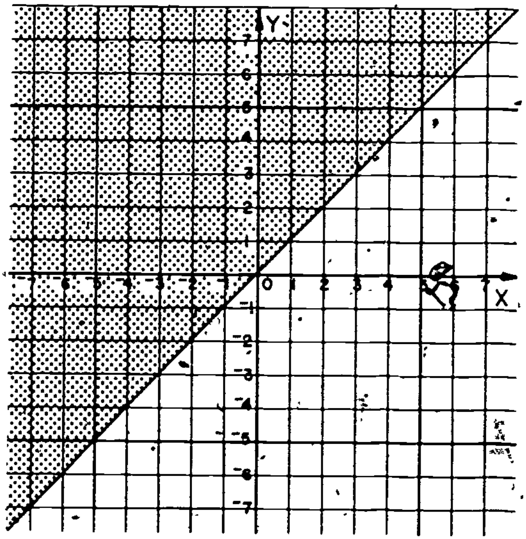
$$Q = \{(0,0), (1,1), ({}^{-}1,{}^{-}1), (2,2), ({}^{-}2,{}^{-}2), (3,3), ({}^{-}3,{}^{-}3)\}$$

Does it seem to be true for every point on the sketch that the y-coordinate is equal to the x-coordinate of the same point?

$y = x$



$y > x$



How does the graph of $y > x$ differ from the graph of $y = x$? Is it true that the set containing coordinate pairs with the y-coordinate greater than the corresponding x-coordinate is the set described by the statement $y > x$?

Exercises 5-4e

Use a different set of coordinate axes for each graph.

1. (a) Find four points different from the points in Q, described by the statement $y = x$.
   (b) Draw the graph of $y = x$.
2. (a) Sketch the graph of $y < x$.
   (b) Construct a set R containing the coordinate pairs of four points described by the statement $y < x$.
3. Describe the differences between the graphs of $y = x$, $y > x$, and $y < x$.

5-5 Multiplication, Division and Subtraction of Rational Numbers

Some of the cells in the table below have been filled from our knowledge of arithmetic. Also the property that the product of a negative number and 0 is 0 has been used.

	$^-2$	$^-1$	0	1	2	3
$^-2$			0			
$^-1$			0			
0	0	0	0	0	0	0
1			0	1	2	3
2			0	2	4	6
3			0	3	6	9

Now to complete the table, observe, for example, as you go up in the right hand column that each number is 3 less than the number below it. This column shall be referred to as the "3 column." Thus, the "3 column" would become

¯6
¯3
0
3
6
9

Similarly the "3 row" would become

¯6	¯3	0	3	6	9

Applying this notion to the remainder of the cells, the table would be completed as shown below.

	¯2	¯1	0	1	2	3
¯2	4	2	0	¯2	¯4	¯6
¯1	2	1	0	¯1	¯2	¯3
0	0	0	0	0	0	0
1	¯2	¯1	0	1	2	3
2	¯4	¯2	0	2	4	6
3	¯6	¯3	0	3	6	9

In particular, you can see that the top row, which is the "$^-2$ row," is completed as shown here:

4	2	0	$^-2$	$^-4$	$^-6$

In other words, if you are to keep the property of multiplication of counting numbers, which you recalled earlier for multiples of 2 and 7, you must accept the following products:

$$(^-2)\cdot(^-2) = 4 \quad \text{and} \quad (^-2)\cdot 3 = {}^-6.$$

You should notice similar results in other parts of the table. For each column and each row, the difference between two consecutive entries is a fixed amount. As far as this table is concerned, the product of two negative numbers is a positive number, and the product of a negative and a positive number (in either order) is a negative number. These conclusions are actually correct for all positive and negative rational numbers. It should be clear, however, that this result has not been proved, nor has it been given as part of a definition. Only one reason has been shown why the conclusions are plausible.

In the exercises your attention will be called to other ways which should make our conclusions about products of positive and negative numbers seem appropriate.

Exercises 5-5a

1. Look at the large multiplication table which was completed in this section. In which rows do the products increase as you move to the right?

2. Using the same table, in which columns do the products decrease as you move down?
3. Give, in correct order, the products of 7 and the integers from $^-4$ to 6.
4. Give, in correct order, the products of $^-4$ and the integers from $^-5$ to 5.
5. Complete the following table. If it helps you see the pattern, add appropriate columns on the right, or add appropriate rows at the bottom of the table.

	$^-3$	$^-2$	$^-1$	0	1	2
$^-5$						
$^-4$						
$^-3$						
$^-2$						
$^-1$						
0						
1						

6. Illustrate the commutative property of multiplication for:

   (a) $^-2$ and 1 (b) $^-3$ and 0 (c) $^-4$ and 5 (d) 15 and $^-6$.

7. Illustrate the associative property of multiplication for numbers: $^-2$, $^-1$, and 5.

8. Illustrate the distributive property of multiplication over addition for the sets of numbers, using the last two numbers in each set as a sum.

   (a) $^{-}4, (3+8)$ (b) $^{-}2, (^{-}3+6)$ (c) $^{-}10, (^{-}8+^{-}1)$

9. Find the products:

   (a) $^{-}4 \cdot 0$
   (b) $^{-}4 \cdot 2$
   (c) $^{-}4 \cdot 5$
   (d) $8 \cdot {}^{-}3$
   (e) $17 \cdot {}^{-}2$
   (f) $49 \cdot {}^{-}5$
   (g) $^{-}6 \cdot {}^{-}9$
   (h) $^{-}6 \cdot 8 \cdot {}^{-}12$
   (i) $^{-}10 \cdot {}^{-}8 \cdot {}^{-}(\frac{2}{3})$
   (j) $^{-}(\frac{4}{3}) \cdot {}^{-}(16) \cdot {}^{-}(\frac{3}{4})$

10. Find the products:

    (a) $(^{-}1) \cdot 4$ (b) $(^{-}1) \cdot 5$ (c) $(^{-}1) \cdot 11$ (d) $8 \cdot (^{-}1)$ (e) $77 \cdot (^{-}1)$

11. Show the use of the number line in finding the products:

    (a) $3 \cdot {}^{-}2$ (b) $5 \cdot {}^{-}2$ (c) $4 \cdot {}^{-}3$.

12. After losing 2 yards on each of 3 successive plays, a football team is on its own 30-yard line.

    (a) At what position was the team before these losses?
    (b) If you express a loss of 2 yards with the number $^{-}2$, the position of the team before the 3 losses is given by $30 - 3(^{-}2)$. Why do you subtract $3(^{-}2)$?

13. Interpret the following expressions in terms of losses or gains of a football team.

    (a) $47 + 4(^{-}5)$.
    (b) $15 - 2(^{-}3)$.

14. Find n in each of the following equations.

(a) $3n = {}^-36$ (d) ${}^-3n = 30$

(b) $5n = {}^-75$ (e) ${}^-2n = {}^-8$

(c) ${}^-2n = 10$ (f) ${}^-6n = {}^-12$

15. In the following problems in multiplication put a number in the parentheses so that the statements will be correct.

(a) $(\ ) \cdot 6 = {}^-12$ (e) $({}^-5) \cdot (\ ) = {}^-20$

(b) $5 \cdot (\ ) = {}^-15$ (f) $11 \cdot (\ ) = {}^-110$

(c) $({}^-10) \cdot (\ ) = 100$ (g) $({}^-1) \cdot (\ ) = 1$

(d) $({}^-5) \cdot (\ ) = 20$ (h) $({}^-7) \cdot (\ ) = 0$

16. Find the products:

(a) $({}^-6) \cdot ({}^-10)$ (e) $({}^-4) \cdot ({}^-10)$

(b) $({}^-3) \cdot ({}^-4)$ (f) $4 \cdot ({}^-10)$

(c) $\frac{5}{2} \cdot -6$ (g) $({}^-10) \cdot 4$

(d) $-(\frac{21}{3}) \cdot {}^-6$ (h) $({}^-6) \cdot ({}^-7)$

Division of Rational Numbers

You know that if $3 \cdot n = 39$, then $n = 13$, since $3 \cdot 13 = 39$. Also, in the definition of rational numbers you call $\frac{39}{3}$ (or $39 \div 3$) the rational number n for which $3 \cdot n = 39$.

$3 \cdot n = 39$.

Let us apply these methods in division of rational numbers involving positive and negative numbers.

Find n if $2n = {}^{-}18$.

We know $2\,({}^{-}9) = {}^{-}18$

Hence, $n = {}^{-}9$ or $(\frac{-18}{2})$

Also ${}^{-}18 \div 2 = {}^{-}9$.

In this section, division will be discussed only as the operation which is the inverse of multiplication. To find ${}^{-}8 \div {}^{-}2$, you can think

$$
\begin{aligned}
{}^{-}8 \div {}^{-}2 = n \quad \text{or} \quad {}^{-}2 \cdot n &= {}^{-}8 \\
n &= 4, \text{ since } {}^{-}2 \cdot 4 = {}^{-}8. \\
{}^{-}8 \div {}^{-}2 &= 4.
\end{aligned}
$$

The question, "What is 16 divided by ${}^{-}4$?", is the same as the question, "By what number can ${}^{-}4$ be multiplied to obtain 16? You know, ${}^{-}4 \cdot {}^{-}4 = 16$. Hence

$$16 \div {}^{-}4 = {}^{-}4.$$

Which of the following are true statements?

(a) ${}^{-}63 \div {}^{-}9 = 7$ (d) ${}^{-}2 \cdot {}^{-}(\frac{3}{2}) = 3$

(b) $45 \div {}^{-}5 = 9$ (e) ${}^{-}2 \div 3 = {}^{-}(\frac{3}{2})$

(c) ${}^{-}8 \cdot {}^{-}13 = 104$ (f) $3 \div {}^{-}(\frac{3}{2}) = {}^{-}2$

You should be able to show that all of these are true statements except (b) and (e).

Before starting to do the exercises, study the following and be sure that you know why they are true statements.

$^{-}7 \cdot {}^{-}5 = 35$ $\quad$ $35 \div {}^{-}7 = {}^{-}5$ $\quad$ $35 \div {}^{-}5 = {}^{-}7$

$^{-}7 \cdot {}^{-}(\frac{5}{7}) = 5$ $\quad$ $^{-}35 \div {}^{-}7 = 5$ $\quad$ $^{-}35 \div 5 = {}^{-}7$

$7 \cdot \frac{5}{7} = 5$ $\quad$ $5 \div 7 = \frac{5}{7}$ $\quad$ $5 \div \frac{5}{7} = 7$

$7 \cdot {}^{-}(\frac{5}{7}) = {}^{-}5$ $\quad$ $^{-}5 \div 7 = {}^{-}(\frac{5}{7})$ $\quad$ $^{-}5 \div {}^{-}(\frac{5}{7}) = 7$

What is the reciprocal of $^{-}(\frac{4}{3})$? You know that $^{-}(\frac{4}{3})$ and n are reciprocals if

$$^{-}(\tfrac{4}{3}) \cdot n = 1$$

Since $\qquad (\frac{4}{3}) \cdot (\frac{3}{4}) = 1$

and $\qquad {}^{-}(\frac{4}{3}) \cdot {}^{-}(\frac{3}{4}) = 1,$

we have $\qquad n = {}^{-}(\frac{3}{4})$

therefore, $^{-}(\frac{3}{4})$ is the reciprocal of $^{-}(\frac{4}{3})$.

Exercises 5-5b

1. Find the products:

(a) $^{-}4 \cdot 7$

(b) $^{-}4 \cdot {}^{-}3$

(c) $2 \cdot {}^{-}6$

(d) $^{-}(\frac{3}{4}) \cdot 4$

(e) $^{-}10 \cdot {}^{-}(\frac{2}{5})$

(f) $^{-}(\frac{5}{6}) \cdot {}^{-}(\frac{12}{25})$

2. Find the quotients:

(a) $^{-}28 \div 7$ (d) $^{-}3 \div 4$

(b) $12 \div {}^{-}3$ (e) $4 \div {}^{-}(\frac{2}{5})$

(c) $^{-}12 \div 2$ (f) $\frac{2}{5} \div {}^{-}(\frac{5}{6})$

3. Find r so that these sentences will be true statements.

(a) $5\,r = {}^{-}10$ (d) $2\,r = \frac{3}{8}$

(b) $^{-}2\,r = 6$ (e) $3\,r = {}^{-}(\frac{3}{2})$

(c) $^{-}3\,r = {}^{-}21$ (f) $({}^{-}6)r = {}^{-}(\frac{18}{3})$

4. Write the reciprocals of each number in P:

$$P = \{6,\ \frac{3}{2},\ 1,\ \frac{5}{6},\ {}^{-}1,\ {}^{-}(\frac{3}{2}),\ {}^{-}(\frac{7}{3})\}$$

5. Divide in each of the following:

(a) $\frac{^{-}18}{^{-}9}$ (e) $\frac{30}{^{-}6}$

(b) $\frac{^{-}25}{5}$ (f) $\frac{30}{6}$

(c) $\frac{^{-}30}{6}$ (g) $\frac{0}{^{-}6}$

(d) $\frac{^{-}30}{^{-}6}$

6. Find n if:

(a) $^{-}2 \div 3 = n$ (b) $2 \div {}^{-}3 = n$

7. Write $^{-}(\frac{2}{3})$ as a quotient in two ways.

8. Find n if

   (a) $7\ n = {}^{-}6$ (b) $^{-}7\ n = 6$

9. Write two sentences, using n, in which $n = {}^{-}(\frac{7}{6})$ would make the sentence a true statement.

10. Find n for each of these equations.

   (a) $(^{-}25)n = {}^{-}92$ (c) $^{-}4\ n = {}^{-}(\frac{4}{3})$

   (b) $(^{-}92) \div (^{-}25) = n$ (d) $^{-}(\frac{4}{3}) \div (^{-}4) = n$

11. Complete the statements:

   (a) If a and b are positive or negative integers, then $\frac{a}{b}$ is the rational number x for which ____________.

   (b) If $\frac{a}{b}$ is a rational number then $\frac{a}{b}$ is positive if a and b are ____________________.

   (c) If $\frac{a}{b}$ is a rational number then $\frac{a}{b}$ is negative, if either a or b is ____________ and the other is ____________.

Subtraction of Rational Numbers

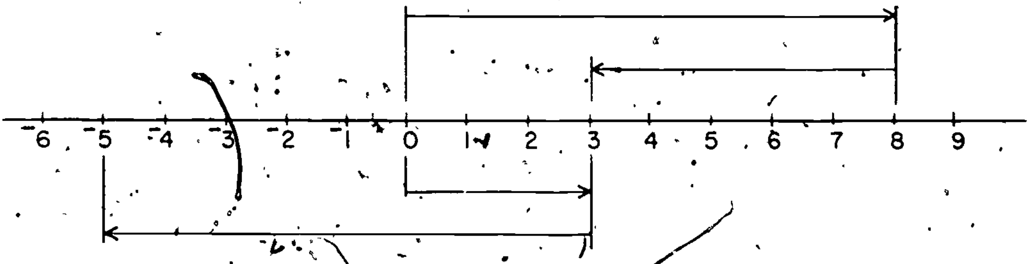
In arithmetic you learned, for example, that if $7 : 4 = 11$, then $11 - 7 = 4$ and $11 - 4 = 7$. Or if $23 + 17 = 40$, then $40 - 17 = 23$ and $40 - 23 = 17$. Using letters for positive numbers you make use in arithmetic of the property:

If $a + b = c$, then $c - a = b$ and $c - b = a$.

This property will be used for negative numbers as well. For example, it will be said:

If $8 + {}^{-}5 = 3$, then $3 - 8 = {}^{-}5$ and $3 - {}^{-}5 = 8$.

It may help you to understand this if you refer again to the number line.



The arrows above the number line show $8 + {}^{-}5 = 3$. The arrows below the number line show $3 - 8 = {}^{-}5$. In the exercises you will be asked to suggest a way of showing $3 - ({}^{-}5) = 8$, using the number line. This should be easier to do after you have had a little practice with subtraction of negative numbers, using the "additive" point of view.

To find $({}^{-}7) - 4$, you can think of the number which added to 4 gives $({}^{-}7)$. Since $4 + ({}^{-}11) = {}^{-}7$, you know that $({}^{-}7) - 4 = {}^{-}11$. Let us consider some other examples.

$3 + {}^{-}2 = 1$ and $1 - 3 = {}^{-}2$ and $1 - ({}^{-}2) = 3$

${}^{-}4 + {}^{-}5 = {}^{-}9$ and ${}^{-}9 - ({}^{-}4) = {}^{-}5$ and ${}^{-}9 - ({}^{-}5) = {}^{-}4$.

Again, to find $({}^{-}15) - 8$, think of the number which added to 8 gives $^{-}15$. The sum of 8 and $^{-}23$ is $^{-}15$. Thus

$$({}^{-}15) - 8 = {}^{-}23.$$

Suppose two boys had been playing a game and at 9:10 P.M. the scores were:

Henry	Jack
5	$^{-}2$
$^{-}1$	0
3	4
$^{-}4$	$^{-}1$
2	$^{-}2$
5	$^{-}1$

Then they recalled that they had agreed to stop playing at 9 o'clock. In order to keep this agreement the boys decided not to count the scores on the last round. Thus,

Henry's score would be $5 - 2 = 3$, and

Jack's score would be $^{-}1 - {}^{-}2 = 1$.

In order to subtract $(^{-}2)$ from Jack's score the boys thought of the number which added to $^{-}2$ is $^{-}1$. This number is 1. $[^{-}2 + 1 = {}^{-}1.]$ You can check this result by noticing that the sum of all of Jack's scores, except the last is 1.

Exercises 5-5c

1. Add the numbers in each set.

   (a) $^{-}2$, 5

   (b) 7, $^{-}7$

   (c) $^{-}5$, $^{-}2$

   (d) $^{-}4$, 1, 8

   (e) $^{-}2$, $^{-}3$, 15

   (f) 21, $^{-}6$, $^{-}7$

   (g) $^{-}8$, $^{-}13$, $^{-}24$

   (h) $^{-}7$, 50, $^{-}110$

   (i) $^{-}23$, $^{-}19$, 14

2. Find the sum of $^{-}(\frac{3}{4})$ and $(\frac{7}{2})$ and write two equations involving subtraction which can be obtained from this sum.

3. Find x in the following:

(a) $(^{-}5) + 2 = x$

(b) $(^{-}3) + x = 8$

(c) $8 + x = {}^{-}3$

(d) $x + (^{-}4) = 11$

(e) $^{-}(\frac{2}{3}) + {}^{-}(\frac{3}{2}) = x$

(f) $^{-}(\frac{2}{3}) + x = {}^{-}(\frac{13}{6})$

(g) $^{-}(\frac{1}{2}) + x = \frac{9}{2}$

(h) $x + \frac{4}{3} = {}^{-}(\frac{11}{4})$

4. Supply the missing number in each case.

(a) $8 + 5 + (\quad) = 8$

(b) $^{-}6 + (^{-}3) + (\quad) = 6$

(c) $(^{-}11) + 6 + (\quad) = {}^{-}11$

(d) $(^{-}11) + (^{-}6) + (\quad) = {}^{-}11$

(e) $(^{-}3) + (\quad) + (^{-}8) = (^{-}8)$

(f) $(^{-}3) + (^{-}7) + (\quad) = {}^{-}7$

5. Suggest a way of subtracting $(^{-}8)$ from 3 making use of the number line.

6. What are the additive inverses of

(a) 10 (b) $^{-}100$ (c) $\frac{1}{2}$ (d) $\frac{7}{9}$ (e) $^{-}(\frac{8}{5})$ (f) $^{-}(\frac{49}{51})$

7. Explain why subtracting 2 is the same as adding the additive inverse of 2.

8. Perform the following subtractions.

(a) $(^-10) - (^-3) =$

(b) $4 - 6 =$

(c) $16 - 12 =$

(d) $8 - (^-2) =$

(e) $(^-8) - 2 =$

(f) $(^-8) - (^-2) =$

(g) $(^-9) - 2$

(h) $9 - (^-3) =$

(i) $7 - (5) =$

(j) $7 - (^-5) =$

(k) $2 - 9 =$

(l) $2 - (^-9) =$

(m) $3 - 10 =$

(n) $3 - (^-10) =$

9. Complete the table for $y = 2x - 3$.

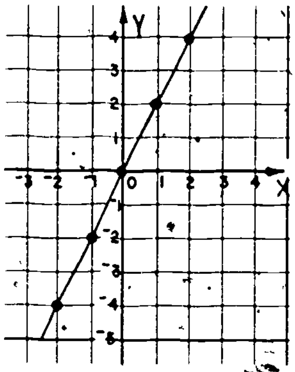
x	$^-1$	0	1	2	3	4
y						

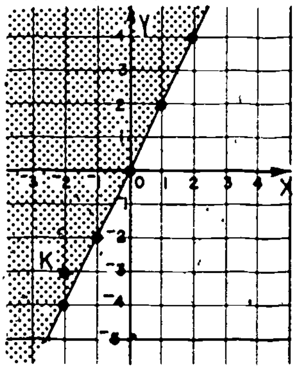
10. Complete the table for $y = {^-2}x - (^-1)$.

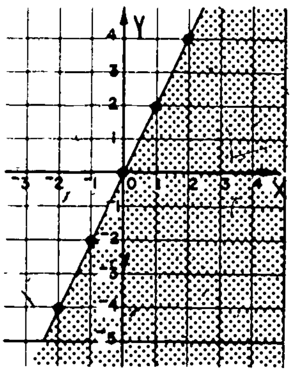
x	$^-2$	$^-1$	0	1	2	3	4
y							

Graphs of Other Relations

You now have had experience with the four operations on the positive, negative and zero rational numbers. You have had an introduction to sets of ordered pairs of rational numbers and graphs of a few simple conditions. Some other relations and their graphs will now be investigated.







The 5 points marked on the graph (a) at the left, above, have as coordinates elements of the set:

$$T = \{(0,0),\ (^-1,^-2),\ (^-2,^-4),\ (1,2),\ (2,4)\}.$$

The condition, $y = 2x$, describes the ordered pairs of numbers which are elements of the set, T. This is true since the y-coordinate is two times the x-coordinate in each pair.

In set U the y-coordinate is also two times the x-coordinate in each pair.

$$U = \{(^-3,^-6),\ (^-0.5,^-1),\ (0.5,1),\ (1.5,3),\ (2.5,5)\}.$$

Without marking in your book, locate these points on the graph (a) You will find that these 5 points also are on the line drawn in the graph. The graph could be said to contain the union of the set of points given by the number pairs in sets T and U. Can you say that a point will be on this line, if its coordinates satisfy the condition, or relation $y = 2x$?

In the shaded part of graph (b), above, are located the points for which the coordinates satisfy the condition $y > 2x$. The shaded region is one of the half-planes determined by the line designated by $y = 2x$. Let us select a point in this region, for example, K. The coordinates of K are $(^-2, ^-3)$. If you substitute these numbers in $y > 2x$, you will have

$$^-3 > 2(^-2) \text{ or } ^-3 > ^-4.$$

This is a true statement and thus the point $(^-2, ^-3)$ satisfies the condition $y > 2x$. Try other points in the half plane above the graph of $y = 2x$, and see if the coordinates of these points satisfy the condition, $y > 2x$.

Graph (c) above is the graph of the other half plane determined by $y = 2x$. The coordinates of points in the half plane satisfy the condition $y < 2x$. Check a few of these points to see if the condition is satisfied.

Consider the following condition:

$$y = 3x + 2.$$

If $x = 0$, then $y = (3)(0) + 2$ or $y = 2$

If $x = 1$, then $y = (3)(1) + 2$ or $y = 5$

If $x = ^-1$, then $y = (3)(^-1) + 2$ or $y = ^-1$ (Check this)

If $x = 2$, then $y = (3)(2) + 2$ or $y = 8$

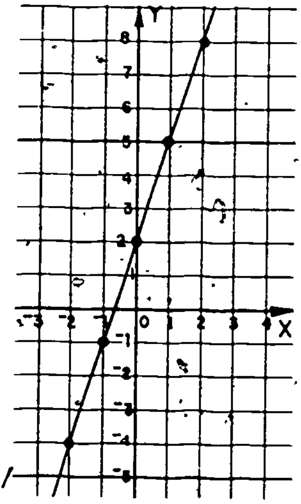
If $x = ^-2$, then $y = (3)(^-2) + 2$ or $y = ^-4$ (Check this)

Set T below contains some of the ordered pairs of rational numbers described by $y = 3x + 2$.

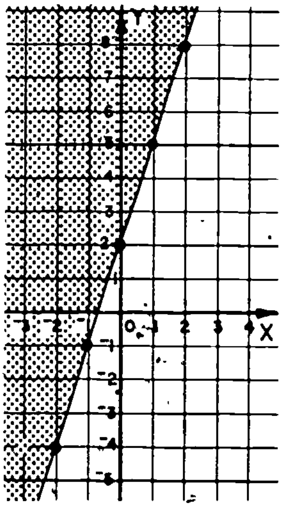
$$T = \{(0,2), (1,5), (^-1,^-1), (2,8), (^-2,^-4)\}.$$

The ordered pairs in T are plotted in graph (a), $y = 2x + 3$, below. Do these points all seem to lie on the same line?

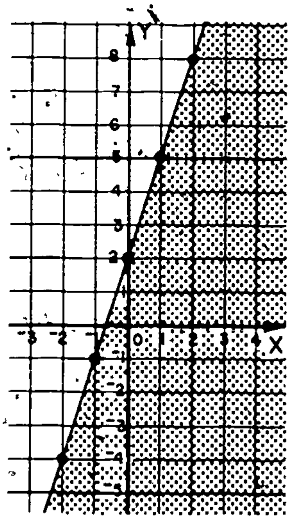
(a) $y = 3x + 2$



(b) $y > (3x + 2)$



(c) $y < (3x + 2)$



The graphs, (b) $y > 3x + 2$ and (c) $y < 3x + 2$, are graphs of the two half planes determined by the line given by $y = 3x + 2$. In Problems 6 and 7 you will be asked questions about these graphs.

Consider the following condition:

$$y = {}^{-}2\,x + ({}^{-}3)$$

If $x = {}^{-}2$, then $y = ({}^{-}2)({}^{-}2) + ({}^{-}3) = 4 + ({}^{-}3)$ or $y = 1$

If $x = {}^{-}1$, then $y = ({}^{-}2)({}^{-}1) + ({}^{-}3) = 2 + ({}^{-}3)$ or $y = {}^{-}1$

If $x = 0$, then $y = ({}^{-}2)\,(0) + ({}^{-}3) = 0 + ({}^{-}3)$ or $y = {}^{-}3$

If $x = 1$, then $y = ({}^{-}2)\,(1) + ({}^{-}3) = {}^{-}2 + ({}^{-}3)$ or $y = {}^{-}5$

If $x = 2$, then $y = ({}^{-}2)\,(2) + ({}^{-}3) = {}^{-}4 + ({}^{-}3)$ or $y = {}^{-}7$

Set P below contains some of the ordered pairs of rational numbers described by $y = {}^{-}2\,x + ({}^{-}3)$.

$$P = \{({}^{-}2, 1), ({}^{-}1, {}^{-}1), (0, {}^{-}3), (1, {}^{-}5), (2, {}^{-}7)\}$$

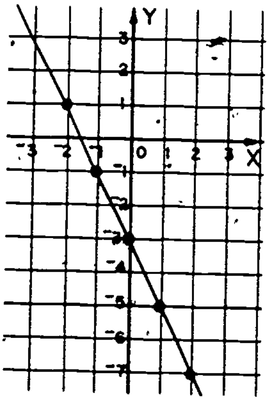
The number pairs in P are plotted on the graph (a) $y = {}^{-}2\,x + ({}^{-}3)$ below.

Sometimes coordinates of points, which satisfy a relation, as a table are displayed. In this example the table would be

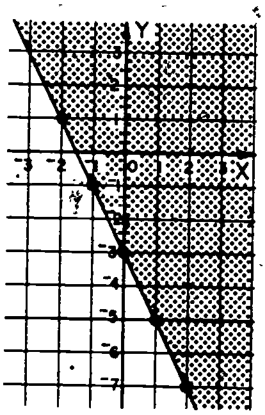
x	${}^{-}2$	${}^{-}1$	0	1	2
$y = {}^{-}2x + {}^{-}3$	1	${}^{-}1$	${}^{-}3$	${}^{-}5$	${}^{-}7$

As with the other examples in this section, the graph of the condition $y = {}^{-}2\,x + {}^{-}3$ is displayed with the graphs of the half planes determined by the line associated with $y = {}^{-}2\,x + {}^{-}3$.

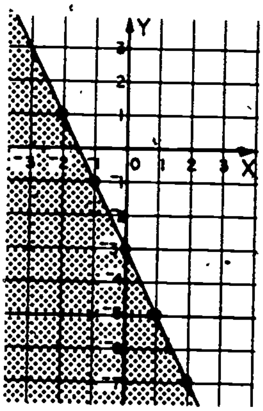
(a) $y = {}^{-}2x + ({}^{-}3)$



(b) $y > [{}^{-}2x + ({}^{-}3)]$



(c) $y < [{}^{-}2x + ({}^{-}3)]$



Exercises 5-5d

1. Given the relation, $y = 3\,x$, find five ordered pairs of numbers which satisfy this condition.
2. Plot the ordered number pairs found in Problem 1 and draw a straight line through the points.
3. Draw the graph of the relation $y > 3\,x$.

4. Sketch the graph of $y < 3x$.

5. Do the graphs of $y = 3x$, $y > 3x$, and $y < 3x$ seem to include all the points in the coordinate plane? Why?

6. Give the coordinates of 5 points in the graph of the condition $y > 3x + 2$. (See this graph in this section.)

7. On which of the two graphs, $y > 3x + 2$ or $y < 3x + 2$, are the following points?

   (a) (1,10) (b) ($^-5$,6) (c) ($^-2$,0) (d) (0,0) (e) ($^-6$,$^-6$)

8. Draw careful graphs of the following conditions:

   (a) $x = {}^-5$
   (b) $y = 2x - 1$
   (d) $y = {}^-(\frac{1}{2})x$
   (e) $y > x - 2$

9. Plot the following set of ordered pairs

   $S = \{(0,0), (1,1), (^-1,1), (2,4), (^-2,4), (3,9), (^-3,9)\}$
   $(^+4,16), (^-4,16)$

   (a) Do they all seem to lie on a straight line?

   (b) Lightly sketch a curve containing the points plotted.

   (c) Is it true that the y-coordinate of each point is the square of the x-coordinate of the corresponding point?

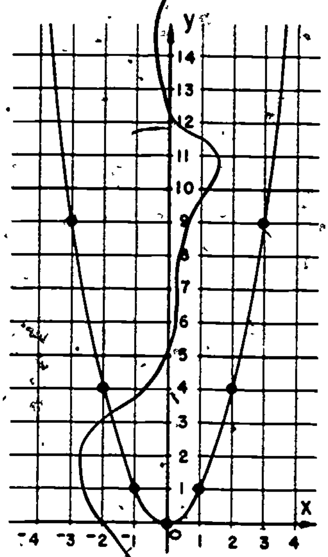
   (d) Does the condition $y = x^2$ describe the points named by Set S?

   (e) Construct a set T containing ordered pairs described by $y = x^2$ which are not contained in set S.

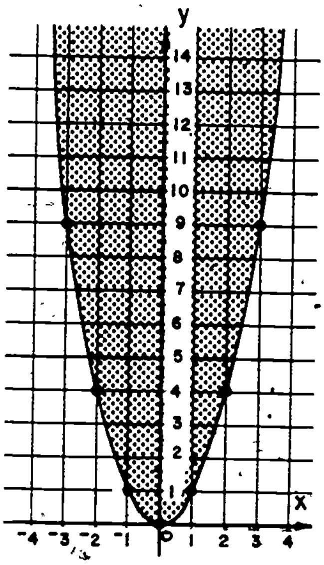
   (f) Does the curve sketched in answer to (b) contain the graph of $y = x^2$?

Study carefully the diagrams below. They contain the graphs of (a) $y = x^2$ (b) $y > x^2$ (c) $y < x^2$.

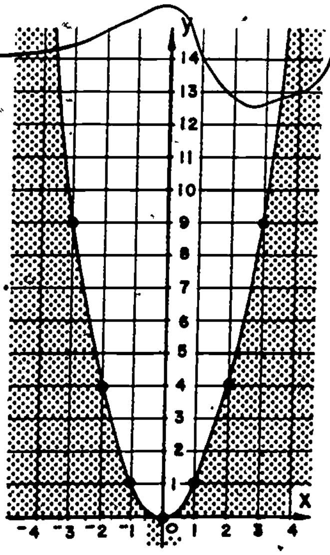
(a) $y = x^2$



(b) $y > x^2$



(c) $y < x^2$



5-6 Finding the Unknown

Suppose you are trying to find a certain number. Let us call the number x. Mathematicians often use letters like "x," "y," "v," and so on to represent unknown numbers. You are given the following clue:

$$x + 5 = 7.$$

In words you may say that 7 is 5 more than the unknown number. In this example can you find the unknown number?

Sometimes the unknown number is not so easy to find. For example suppose you were given this problem:

> Tom bought a ticket for a football game. Altogether he paid \$1.10 (or 110 cents), including the tax. If the cost of the ticket is \$1.00 more than the amount of the tax, what is the amount of tax on the ticket?

Again, you are given certain clues. You must use the clues carefully if you are to find the correct answers. To help you find the amount of the tax you may use the clues to write a number sentence. If x represents the amount of the tax, a correct number sentence is,

$$x + (x + 100) = 110.$$

In words the number sentence states that the amount of the tax, x (in cents), added to the number of cents in the cost of the ticket, x + 110, is equal to the total charge in cents for the ticket, 110. The correct answer is 5 cents, or x = 5. The correct price of the ticket is \$1.05. The amount of the tax is \$.05, and \$1.05 - \$.05 = \$1.00.

In both of the problems above the clues were used to write number sentences. Each clue was a statement about numbers. Some of these numbers were known and some unknown. Since the verb in each of these sentences was the "equals" sign, such number sentences are called equations. When you are finding what number x represents, it is said that you are solving an equation for the unknown x (or whatever letter you are using to represent the unknown).

Equations are used in many ways in many different fields. You can solve equations to find the currents in an electrical network when you know the voltages and the resistances. You can solve equations in order to design airplanes or space ships. You can solve equations in order to find out what is happening in a cancer cell.

You can also use equations to predict the weather. Now methods for predicting the weather very accurately are known. The only trouble is that these methods require the solution of about a thousand equations with the same number of unknowns. Even with the best of the modern high speed computers, it would take two weeks to compute the prediction of tomorrow's weather. Therefore, the meterologists make many approximations. They simplify the equations in such a way that they can compute the prediction in a short enough time. They will be able to make better predictions when there are more efficient ways to solve many equations with many unknowns.

Our progress in many fields of knowledge depends on finding better methods for solving equations. Many leading mathematicians are working on such problems. When you finish this chapter you should see that equation solving is not a lucky hit-or-miss activity which depends on trial and error.

Exercises 5-6

In Problems 1-4 below, use your knowledge of arithmetic to find the value of the unknown in each of the equations so that the equations will be true statements.

1. Find the value of the unknown in each of the following equations:

(a) $x + 3 = 5$ (d) $m + 25 = 31$

(b) $y + 5 = 12$ (e) $s + 17 = 42$

(c) $k + 13 = 15$ (f) $t + 10 = 5$

2. Find the value of the unknown in each of the following equations.

(a) $x - 7 = 2$ (d) $x - 3 = 6$

(b) $y - 5 = 5$ (e) $p - 15 = {}^-1$

(c) $n - 9 = 2$ (f) $x - 5 = 3$

3. Find the value of the unknown in each of the following equations.

(a) $4b = 12$ (d) $9m = 72$

(b) $3a = 9$ (e) $13x = {}^-13$

(c) $5w = 35$ (f) $7y = {}^-56$

4. Find the value of the unknown in each of the following equations.

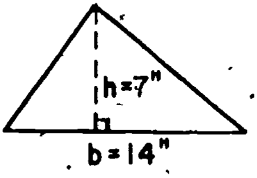
(a) $\frac{n}{3} = 2$ (d) $\frac{d}{9} = 2$

(b) $\frac{a}{4} = 4$ (e) $\frac{h}{{}^-3} = 5$

(c) $\frac{k}{8} = {}^-2$ (f) $\frac{{}^-s}{{}^-3} = {}^-7$

5. A formula for finding the perimeter of a rectangle is $p = 2\ell + 2w$. Find the perimeter of a rectangle whose length is 7 feet and whose width is 4 feet.

6. Use the formula $A = \frac{1}{2}bh$ to find the number of square units of area in the triangle shown at the right.



7. What is the area of a square whose length is 15 inches? Use $A = s^2$ as the formula for the area.

8. A formula used in finding simple interest is written $i = prt$, where

   i is the interest in dollars,
   p is the principal (or amount borrowed),
   r is the rate (or per cent) of interest per year,
   t is the time in years.

   Find the simple interest for a bank loan of $750 for 3 years at 6% interest.

9. To find the circumference of a circle, the formula $c = 2\pi r$ may be used. Find the circumference of a circle whose radius is 10 inches. (Use $\frac{22}{7}$ or 3.14 for π).

10. Find the area of the floor of a circular room whose radius is 13 feet. The formula is $A = \pi r^2$. (Use $\frac{22}{7}$ or 3.14 for π).

5-7 Number Phrases and Number Sentences

In talking to people or when writing, you use sentences. In mathematics sentences are written about numbers. A sentence about numbers is often written in the form of an equation, such as,

$$x + 7 = 9.$$

This sentence about numbers says,

> "If seven is added to a certain number x the result is nine."

You are familiar with numbers, such as 9. The number 9 is a part of the number sentence above. Another part of the sentence is $x + 7$. These expressions, $x + 7$ and 9, are not sentences. They are parts of sentences. Such expressions are called <u>phrases</u>.

Phrases are not sentences. A phrase does not make a statement. In a sentence about numbers a phrase represents a number. A phrase that describes or represents a number is called a number phrase.

A number phrase may represent one specific number. For example, the number phrases

$$3 + 5, \quad 9, \quad \frac{16}{2}, \quad \text{III} + \text{V}, \quad \text{and} \quad 10$$

represent specific numbers. In each of these examples, the value of the number phrase is known, or it can be determined.

What number is represented by $x - 4$? you cannot determine the number represented by $x - 4$ unless you know the value of x. Thus $x - 4$ may have many different values. Number phrases which do not represent a specific number are called open phrases. You may think of an open phrase as one whose value is "open" to many possibilities. Examples of open phrases are,

$$x - 4, \quad 7y, \quad 2 + z, \quad \frac{B}{4}.$$

To solve problems by using equations you must be able to translate the clues given in the problem into an equation. To do this you must express the numbers in the problems as number phrases. In the previous section, you used the number sentence,

$$x + 5 = 7.$$

Is the value of 7 known? What about $x + 5$? Is 7 an open phrase. What about $x + 5$?

To work with number phrases you must also be able to translate the phrase into words. The phrase $x + 5$ may be translated as "the number x increased by five." Can $7x$ be translated as "seven times the number x?".

Sometimes pupils are confused because an open phrase such as $x + 7$ may have many different translations. For example, other translations are:

"The number x added to seven,"
or "the number x increased by seven,"
or "the sum of x and seven,"
or "seven more than the number x."

However, all of the translations have the same mathematical meaning. Furthermore, all of the English translations mean the same as "x + 7." With practice you will learn to understand the different ways of expressing a number phrase.

Exercises 5-7a

1. Translate each of the following number phrases into symbols.
   (a) The sum of x and 5.
   (b) The number x decreased by 3.
   (c) The product of 8 and x.
   (d) One fourth of the number x.
   (e) The number x increased by 10.
   (f) The number 7 multiplied by x.
   (g) The number which is 11 subtracted from x.
   (h) The number x divided by 2.

2. For each one of the number phrases in Problem 1, find the number represented by the phrase if $x = 12$ in each part.

3. Translate each of the following number phrases into words:
   (a) $x + 1$
   (b) $x - 3$
   (c) $2x$
   (d) $\frac{18}{x}$
   (e) $4x$
   (f) $^{-}6 + x$

4. Find the number represented by each of the number phrases in Problem 3 if $x = 6$.

5. Find the number represented by each of the number phrases in Problem 3 if $x = {}^{-}2$.

6. The unknown number is not always represented as x. Translate each of the following number phrases into symbols using the letter of each part as the unknown number. For example, in Part (a) use "a" as the unknown number.

   (a) The sum of six and a number.
   (b) Eight times a number.
   (c) Eight times a number and that amount increased by 1.
   (d) Three subtracted from eight times a number.
   (e) The amount represented by eight times a number divided by 4.
   (f) Two times a number; then increase that amount by 3.
   (g) Five multiplied by the sum of a number and 2.
   (h) Ten less than seven times a number.

7. Find the number represented by each of the number phrases in problem 6 if the unknown number is $^{-}3$.

8. Translate each of the following number phrases into words. Write the word "number" to represent the unknown number in each phrase.

   Example: $y + 3$ A number increased by three.

   (a) $2n + 5$
   (b) $6 - 3q$
   (c) $(b-1)\,7$
   (d) $\frac{5-d}{2}$
   (e) $15 + 2w$

9. Find the number represented by $2n + 5$ for each of the following values:

   (a) $n = 5$
   (b) $n = {}^{-}5$
   (c) $n = 0$
   (d) $n = {}^{-}1$

10. Find the number represented by $6 - 3q$ for each of the following values:

   (a) $q = 0$
   (b) $q = {}^{-}1$
   (c) $q = {}^{+}1$
   (d) $q = 5$

Number Sentences

Consider the following sentences:

"The sum of 8 and 7 is 15."
"The number twelve has six factors."
"$x + 3 = 8$."
"$4 + 5 = 3 \cdot 3$."
"$3 < 2 + 4$."
"$2^2 > 2$."

Do you agree that these are all sentences? They are examples of number sentences. Each of them consists of two number phrases connected by a verb. What are the verbs in these sentences? The first two are easy to find. The word "is" is the verb in the first sentence, and the word "has" is the verb in the second. What is the verb in the sentence "$x + 3 = 8$"? Perhaps you have never thought of "=" as a verb in a sentence. What are the verbs in the remaining sentences?

The three most common verbs in number sentences are "=", "<", and ">", but there are others. "Six is a factor of twelve" is a number sentence, and it is sometimes written "$6 \mid 12$." The symbol "$\mid$" is another "number verb" meaning "is a factor of".

One word of caution: You would not use the name "number sentence" for such vague statements as,

"100,000,000 is a very large number,"

or

"$123,456 \times 654,321$ is hard to find."

These sentences involve more than statements about numbers. They involve our reactions to numbers. You will be interested only in sentences which are purely about numbers and their relationship with other numbers.

Some sentences are true sentences. For example,

"$4 + 5 = 3 \cdot 3$," and

"The sun sets in the west,"

are true. A sentence need not be true, however. The sentences

"$3 > 2 + 4$," and

"Abraham Lincoln was the first president of the United States,"

are not true sentences. Consider the sentences,

"Jimmy was at Camp Holly all day yesterday," and

"$x + 3 = 8$."

Are they true? Are they false? You may answer, "I don't know. Which Jimmy do you mean? To what number does "x" refer? These sentences are neither true nor false, because they contain words or symbols which do not refer to only one thing. "Jimmy" can mean any boy with that name, and "x" can stand for any number. You might look at the camp records and say that if "Jimmy" means Jimmy Mills of Denver, the first sentence is true, but if it means Jimmy Shultz of Cincinnati, then it is false. The second sentence is true if $x = 5$, but it is false if $x = 6$ or if x is any number other than 5.

What can you say about the truth of the following sentences?

"$13 - x = 7$."

"George was the first president of the United States."

"$3 + x = x + 3$."

"If Jimmy was at Camp Holly all day yesterday, then he was not at home at that time."

These sentences are similar in that each contains a word or symbol which can refer to any one of many objects. Do you see any difference between the first two sentences and the second two? Can the first two sentences be true? Can the first two sentences be false? Can either of the last two sentences ever be false?

Suppose a number sentence involves a symbol like "x" or "y." If the symbol can refer to anyone of many numbers the sentence is called an <u>open sentence</u>. It is not necessarily a true sentence. It is not necessarily a false sentence. It leaves the matter open for further consideration. Look at this open sentence:

$$x + 7 = \frac{10}{x - 2}.$$

It is composed of three parts: a verb, "=", and two open phrases, "$x + 7$" and "$\frac{10}{x - 2}$." The open sentence states that for a certain number x these two open phrases represent the same number. Can you discover such a number x ? Can you find more than one? Try some numbers. After working for a while you might say, "The sentence is true if $x = 3$ or $x = {}^{-}8$, but it is false if x is any other number." The numbers 3 and ${}^{-}8$ are called solutions of the open sentence. The set $\{3, {}^{-}8\}$ is called the set of solutions (or solution set) of the open sentence.

When you find the entire set of solutions of an open sentence, you can say that you have solved the sentence. An equation is a particular kind of number sentence. It is a number sentence which involves the verb "=". Hence to solve an equation means to find its entire set of solutions. The set of solutions of an equation may contain one member or it may contain several members. It might even be the empty set.

Is this sentence an equation?

"$x - 4 > 7$."

What is the verb in the sentence above? Is it "="? Since the verb is not "=", the sentence is not an equation. You might say that the sentence indicates that the two phrases, $x-4$ and 7 are not equal. Such a sentence is called inequality.

Can you determine the set of solutions for the inequality $x - 4 > 7$? How large must the number x be in order for the inequality to be true? Is $5-4 > 7$? Is $7 - 4 > 7$? Is $12 -4>7$? Do you see that "$x-4 > 7$" is true if x is any number greater than 11? Also, "$x-4 >7$" is false for any other value of x. Thus the set of solutions of the inequality is the set of all numbers which are greater than 11.

Exercises 5-7b

1. Translate each one of the following number sentences into symbols.
   - (a) The number x increased by 5 is equal to 13.
   - (b) The number 3 subtracted from x is equal to 7.
   - (c) The product of 8 and x is equal to 24.
   - (d) When x is divided by 4 the quotient is 9.
   - (e) Ten more than the number x is 21.
   - (f) The number 7 multiplied by x is equal to $^{-}35$.
   - (g) The number 11 subtracted from x is $^{-}5$.
   - (h) The number 6 less than x is 15.

2. For each one of the equations you wrote in Problem 1, find the set of solutions by using your knowledge of arithmetic.

3. Translate each one of the following number sentences into symbols.
   - (a) The number x increased by 2 is greater than 4.
   - (b) The number 5 multiplied by x is less than 10.
   - (c) The result of dividing x by 7 is greater than 2.
   - (d) Three less than the number x is greater than 6.
   - (e) The number x decreased by 5 is less than 13.
   - (f) The product of $^{-}3$ and the number x is greater than $^{-}9$.

4. For each one of the inequalities you wrote in Problem 2, use your knowledge of arithmetic to find the set of solutions.

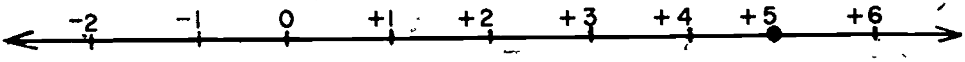
5. Translate each one of the following number sentences into words. Use the term "a number" or "a certain number" to represent the unknown number.
   - (a) $y + 2 = 5$
   - (b) $z + (^-3) = 7$
   - (c) $2a = {}^-10$
   - (d) $h - 5 > 9$
   - (e) $5m < 15$
   - (f) $7 - k = 2$
   - (g) $d - 3 < 4$
   - (h) $\frac{w}{3} > 9$
   - (i) $k - 7 = {}^-2$
   - (j) $\frac{c}{^-30} = 6$

6. Using your knowledge of arithmetic, find the set of solutions for each of the number sentences in Problem 5.

A picture of a solution set using the number line can be drawn. Consider the following example for the open sentence,

$$x + 3 = 8.$$

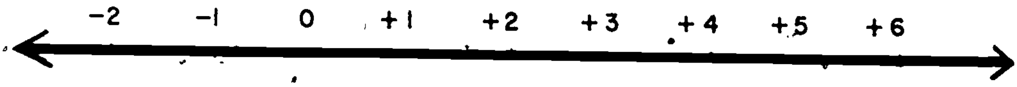
This open sentence has only one solution, 5. The set of solution is $\{5\}$. On the number line this set of solutions can be represented as shown below:



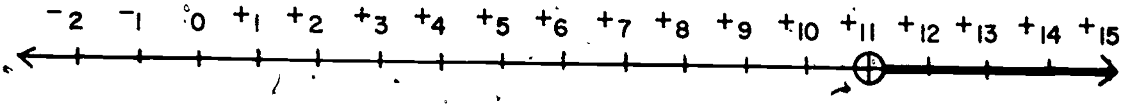
Since the only solution for this equation is 5, a solid "dot," or circle, is marked on the number line to correspond with the point for 5. No other mark is put on the drawing.

How can you draw a picture of the set of solutions for this sentence, $x + 3 = 3 + x$?

First, find the set of solutions. Try $x = 6$. Does $6 + 3 = 3 + 6$? Try $x = 13$. Does $13 + 3 = 3 + 13$? Try any other number. What property of addition tells us that this sentence is true no matter what number we use for x? The set of solutions for this equation is the set of all numbers. This solution set is represented on the number line by drawing a heavy dark line along the entire number line as shown below.

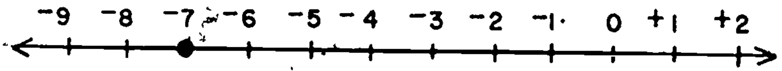


Earlier in this chapter the inequality $x - 4 > 7$ was discussed. The set of solutions for this inequality is the set of all numbers which are greater than 11. This set of solutions is represented on the number line as shown below:



The number "11" is not in the set. This is indicated by an open circle on the point corresponding to 11 on the number line. The part of the number line to the right of the 11 is shaded showing that all points to the right of 11 are in the set of solutions.

Consider the equation $11 + x = 4$. What is the set of solutions? Try some numbers. Recall what you have learned about negative numbers. What is $11 + (^-7)$? Is $(^-7)$ a solution? Can you find any other solutions? You should not be able to do so. The set of solutions for this equation is $\{^-7\}$. This is represented on the number line as shown here:



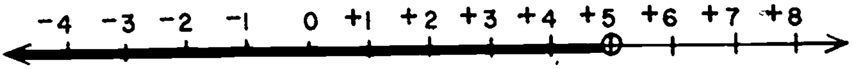
Why is a "solid" circle drawn at the point corresponding to $^-7$? Why is no other point indicated on the number line?

For what numbers is the inequality written below a true sentence?

$$x - 4 < 1$$

Try some numbers. You should find that the set of solutions is the set of all numbers less than $^+5$. On the number line this is represented as an "open" circle at the point corresponding to $^+5$ and a heavy black line drawn along all points of the number line which lie to the left of $^+5$.

Here is the drawing:



The sets of solutions for different number sentences may be different. Some of the solution sets contain only one member. Such sets may be represented by a single filled circle on a drawing of the number line. The circle drawn at the point which corresponds to the number in the set of solutions. If the set of solutions is the set of all numbers, you may draw a heavy, dark line along the entire number line. In this case, the solution set is represented by the entire number line. The sets of solutions for inequalities are represented by a part of the number line. The inequalities that were discussed were all represented by half lines on the number line. An open, or empty, circle was used to indicate a point not included in the set of solutions.

Exercises 5-7c

1. Using your knowledge of arithmetic, find the set of solutions for each one of the following number sentences.

   (a) $x + 2 = 6$
   (b) $4 + x = 0$
   (c) $2x = 6$
   (d) $3x < 3$
   (e) $x - 4 > 1$
   (f) $\frac{x}{5} = {}^-1$
   (g) $2x < 10$
   (h) $4 - x > 1$

2. For each one of the number sentences in Problem 1 represent the set of solutions on a number line.

3. Using your knowledge of arithmetic, find the set of solutions for each one of the following number sentence.

   (a) $x + 1 = 1 + x$
   (b) $y - 1 > 0$
   (c) $1 - b > 0$
   (d) $a + 2 = 1 + a$
   (e) $3w = {}^-15$
   (f) $14 + x = 13$
   (g) $13 - x = 14$
   (h) $\frac{2}{x} = {}^-1$

4. For each one of the number sentences in Problem 3 show the set of solutions on a number line.

5. Sometimes an equation or an inequality is only part of a sentence. Just as you can build longer sentences out of shorter ones by using such words as "and," "or," and "but," you can join number sentences together to make longer ones. Such sentences are called <u>compound sentences</u>.

   Consider the compound number sentence

   "$x - 4 < 7$ and $x - 1 > 0$."

   In order to be a solution of this sentence, a number x must be a solution of both the sentence "$x - 4 < 7$" and the sentence "$x-1 > 0$."

   The elements of the solution set of the sentence are the numbers which are in both the solution set of "$x -4 < 7$" and the solution set of "$x - 1 > 0$."

   The set of solutions of "$x - 4 < 7$" is the set of all numbers less than 11.

   The set of solutions of "$x -1 > 0$" is the set of all numbers greater than 1.

   What is the set of solutions of the compound sentence? Show this set on a number line.

6. Is the set of solutions for the compound sentence in Problem 5 the intersection of the sets of solutions for the two inequalities or is it the union of the two sets of solutions?

7. For each of the following compound sentences find the set of solutions.

   (a) $x - 2 < 7$ and $x + 4 > 6$

   (b) $x - 3 = 6$ and $x - 3 > 6$

   (c) $2x > 6$ and $\frac{x}{2} < 3$

8. For each of the compound sentences in Problem 7 represent the set of solutions on the number line.

9. (a) Find the set of solutions for

$$x^2 = 9.$$ (There are two possible solutions).

(b) Represent the set of solutions for Part (a) on the number line.

10. (a) Find the set of solutions for

$$x^2 < 9$$

(b) Represent the set of solutions for Part (a) on the number line.

11. (a) Find the set of solutions for the following compound sentence:

$$x + 7 = 6 \quad \text{or} \quad 2x - 1 = 5.$$

(Note: in mathematics "or" means either the first or the second or both the first and the second).

(b) Represent the set of solutions for Part (a) on the number line.

12. (a) Find the set of solutions for

"$x - 1 = 4$ or $x - 1 > 4$"

(This compound sentence is sometimes abreviated "$x - 1 \geq 4$")

(b) Show the set of solutions for Part (a) on the number line.

13. Find the set of solutions for

"$x < 10$ or $x - 9 > 0$."

In writing sentences in mathematical language you will be using letters to represent the unknown numbers. You should first decide what the letter will represent. Then you should write your sentence. For example, write in symbols the following sentence:

Mary, who is 14 years old, is five years older than her brother.

Let "y" represent the number of years in Mary's brother's age.

Then, since Mary is five years older, $y + 5$ is Mary's age.

The sentence in symbols is, $14 = y + 5$.

Remember, you should always explain what the letter represents before you write your sentence in symbols. By solving the equation for y, you can determine the brother's age.

Exercises 5-7d

1. You are to write each of the following sentences in symbols. First describe what number the letter represents. Then write the sentence.

   (a) John, who is 10, is four years younger than his sister who is y years old.

   (b) Steve bought m model plane kits costing 25 cents each and paid 75 cents.

   (c) Sam's age seven years from now will be 20 since his age now is b.

   (d) The number of inches in f feet is 72.

   (e) The number of yards in f feet is 5.

   (f) Ann, who is n years old, was 3 years old ten years ago.

   (g) The number of cents in d dollars is 450.

   (h) A baby sitter charges x cents per hour before midnight and y cents per hour after midnight. She earned 350 cents while sitting from 8 P. M. — 2 A. M..

2. Translate each of the following sentences into an equation or inequality. First describe what number the letter represents.
   (a) If 3 dollars is added to twice the money Dick has, the result is less than 23.
   (b) At a certain speed the plane will travel more than 500 miles in 2 hours.
   (c) If 1 is added to twice Susie's age the result is 19.
   (d) Janet's father drove from Mill City to Dover, a distance of 240 miles, at an average speed of 40 miles per hour. How long did it take for the drive?
   (e) If Sally earns 65 cents per hour for baby sitting, how much will she earn in 5 hours?

3. In each of the following:
   (I) Describe the number represented by the letter you select.
   (II) Translate the problem into an equation.
   (III) Solve the equation.
   (a) In ten years Mr. Smith will be forty years old. How old is he now?
   (b) If I give you $5.00, you will have $12.00. How much do you have now?
   (c) Vera is two times as tall as her brother. She is 64 inches tall. How tall is he?
   (d) Paul was 14 years old in 1958. In what year was he born?
   (e) The area of a certain triangle is 35 square inches. The length of the base is 7 inches. What is the length of the altitude?
   (f) Twenty percent of a number is 10. What is the number?

(g) A store sells bicycles at a 20% discount. If a bicycle sells for $28, what was the original price?

(h) After traveling 120 miles a driver still had to go $\frac{5}{8}$ of his journey. How many miles were there in his complete journey?

5-8. Finding Solution Sets

You have seen that often you wish to find the solution set of an equation or an inequality. In most of the problems you have met, the set of solutions may be discovered by inspection or by trial.

Perhaps a method can be found which will make equations easier to solve. An easy equation will be used to illustrate the method, and then the method will be tried out on a harder one.

How did you solve the equation

$$x + 3 = 7?$$

Perhaps you thought, "If I add $^-3$ to $x + 3$, I get x, so I must add $^-3$ to 7. If I do, I get 4, so x must be 4." If you thought that, you were already using this method, and you were using a very important property of "equals."

Suppose "a" and "b" represent the same number. For instance, a might be $5 + 4$ and b might be $3 + 6$. What can you say about the numbers $a + 5$ and $b + 5$? Could they be different from each other, or must they be the same? If "a" and "b" both represent 9 then $a + 5$ and $b + 5$ must both be 14.

The general principle can be expressed like this:

Addition Property: If you have two equal numbers and add a third number to each of them, the resulting numbers are equal.

In symbols, if a, b, and c are numbers, and $a = b$, then

$$a + c = b + c \quad \text{and} \quad c + a = c + b.$$

Can you discover a subtraction property of "equals"?

How can you use the addition property to solve the equation

$$x + 3 = 7?$$

The solution would be easy to find if the "3" were not there. If you use the addition property, adding the number ${}^{-}3$, you have:

$$\text{If} \quad \overbrace{x + 3}^{a} = \overbrace{7}^{b}$$

$$\text{then} \quad \overbrace{(x + 3)}^{a} \overset{+}{\ } \overbrace{+{}^{-}3}^{c} = \overbrace{7}^{b} \overset{+}{+} \overbrace{{}^{-}3}^{c}$$

By the associative property and the fact that $3 + {}^{-}3 = 0$, You can see that $(x + 3) + {}^{-}3 = x + (3 + {}^{-}3)$ and $x + (3 + {}^{-}3) = x$. Therefore,

$$\text{if} \quad (x + 3) + {}^{-}3 = 7 + {}^{-}3$$
$$\text{then} \quad x = 7 + {}^{-}3.$$

Since $7 + {}^{-}3 = 4$, you see that

$$x = 4.$$

You have found, using the addition property, that if the equation $x + 3 = 7$ has a solution, then the solution must be 4. you still do not know that 4 is a solution. Let us see.

$$\text{If} \quad x = 4,$$
$$\text{then} \quad x + 3 = 4 + 3,$$
$$\text{so} \quad x + 3 = 7.$$

Thus 4 is a solution. Since you saw first that there cannot be any other solution, it follows that 4 is the only solution of the equation. In other words, the solution set is $\{4\}$.

Notice that in the process of solving the equation you have done two things. You first proved a uniqueness statement, showing that there is only one possible number for a solution. Then you proved an existence statement, showing that this number is a solution.

In a previous section you learned how to find $3 + (^-4)$. It is $^-1$. Also, $3 - 4$ is $^-1$. Thus, $3 + (^-4) = 3 - 4$. Perhaps this is the way you find $3 + (^-4)$ in your head; by thinking of $3 - 4$. Similarly $7 + (^-5) = 2$, and $7 - 5 = 2$, so $7 + (^-5) = 7 - 5$. What is $^-3 + (^-2)$? What is $^-3 - 2$? Do they both equal $^-5$? What do you add to 4 to obtain $4 - 2$? You add $^-2$. This is true for any number: To obtain $a - b$ you may <u>add</u> ^-b to a.

If a and b are numbers, then $a - b = a + (^-b)$.

Using this fact you can solve the equation

$$x - 3 = 8$$

by writing it as

$$x + (^-3) = 8.$$

Then you can apply the addition property as you have in the preceding examples. What number would you add to solve this last equation? Why?

<u>Exercises 5-8a</u>

1. (I) Use the addition property to find the only possibilities for solutions of the following equations. (II) Then show that the number <u>is</u> a solution.

Example: $x + (^-3) = 11$

I) $(x + (^-3)) + 3 = 11 + 3$ by the addition property, adding 3.

$(x + (^-3 + 3)) = 14$ by the associative property of addition.

$x = 14$ since $^-3 + 3 = 0$.

II) If $x = 14$ then $x + (^-3) = 14 + (-3) = 11$, so 14 is a solution.

(a) $x + 5 = 6$

(b) $x + 6 = 5$

(c) $x + {^-7} = 7$

(d) $x - 7 = {^-7}$

(e) $t + 6 = {^-13}$

(f) $4 = s + 3$

(g) $^-2 = {^-4} + x$

(h) $x + \frac{3}{2} = 10$

(i) $y - \frac{3}{2} = \frac{5}{2}$

(j) $u + 14 = \frac{9}{5}$

(k) $\frac{13}{7} = 1 + x$

(l) $x + {^-(\frac{4}{9})} = {^-(\frac{7}{13})}$

2. Apply the addition property to these equations, adding the indicated number, and write the resulting equation.

Example: $3x + 4 = 5$ (add $^{-}4$)

$(3x + 4) + {}^{-}4 = 5 + {}^{-}4$ by the addition property.

$3x + (4 + {}^{-}4) = 1$ by the associative property.

The resulting equation is: $3x = 1$.

(a) $2x + 5 = 10$ (add $^{-}5$)
(b) $3x + 10 = 5$ (add $^{-}10$)
(c) $5x + 2 = {}^{-}2$ (add $^{-}2$)
(d) $10x + {}^{-}1 = 9$ (add 1)
(e) $2u + 1 = 11$ (add $^{-}1$)

3. (a) What number do you add (using the addition property) to solve $x + 3 = 2$?

(b) What number do you add (using the addition property) to solve $x + ({}^{-}7) = 4$?

(c) What is the relation between 3 and $^{-}3$ relative to addition?

(d) What is the relation between 7 and $^{-}7$ relative to addition?

4. Solve the following equations. (Remember that "solving an equation" includes showing that any possible solution you find <u>is</u> a solution.)

(a) $x + 3 = 11$
(b) $x + {}^{-}6 = \frac{4}{3}$
(c) $x - \frac{9}{2} = \frac{5}{2}$
(d) $2x - 7 = x$ (First add ^{-}x)
(e) $^{-}x = 7 - 2x$
(f) $\frac{1}{2}x + {}^{-}2 = 1.5 - \frac{1}{2}x$

You are now ready to try some harder equations. How would you solve the equation

$$2x + 1 = 11?$$

Let us not try too much at a time! Before you ask what x is, first ask an easier question: What is 2x? You answered this question in Problem 2 of the last set of exercises. The addition property tells us that

if $2x + 1 = 11$

then $2x = 11 + {}^{-}1$, by the addition property adding ${}^{-}1$,

or $2x = 10$.

Now that you have found 2x, you can try solving the new equation for x. This is a new kind of equation which you can solve using another important property of "equals."

Suppose "a" and "b" represent the same number again, but suppose you multiply a and b by 5 instead of adding 5. Could 5a and 5b be two different numbers if a and b are the same?

<u>Multiplication Property</u>: If you have two equal numbers and multiply them by a third number not 0, the results are equal. If a, b, and c are numbers and $a = b$, then

$$ac = bc \quad \text{and} \quad ca = cb.$$

Can you discover a division property of "equals"? Be careful! Remember that you cannot divide by zero.

If you apply the multiplication property to the equation

$$2x = 10,$$

what number do you want to use as a multiplier? You wish to "eliminate" the 2 so $\frac{1}{2}$, the reciprocal of 2, would be a good choice. Using the multiplication property with $\frac{1}{2}$ as a multiplier, you will have:

$$\text{If} \quad \overbrace{2x}^{a} = \overbrace{10}^{b}$$

$$\text{then} \quad \overbrace{\tfrac{1}{2}}^{c}\cdot\overbrace{(2x)}^{a} = \overbrace{\tfrac{1}{2}}^{c}\cdot\overbrace{10}^{b}.$$

Applying the associative property of multiplication, you have $\frac{1}{2}(2x) = (\frac{1}{2} \cdot 2)x = 1 \cdot x = x$. You all know that $\frac{1}{2} \cdot 10 = 5$,

$$\text{so,} \qquad x = 5.$$

You have found that if $2x + 1 = 11$ has a solution, then it must be 5 (a *uniqueness* statement). You still have to show that 5 *is* a solution (an *existence* statement).

$$\begin{aligned} \text{If } & x = 5 \\ \text{then } & 2x = 2 \cdot 5 = 10 \\ \text{and } & 2x + 1 = 10 + 1 \\ \text{or } & 2x + 1 = 11. \end{aligned}$$

Thus 5 *is* a solution.

You have used two properties, the addition property to find $2x$ and the multiplication property to find x. This process will make it possible to solve without difficulty even the hardest of the equations which were examined at the beginning of this section.

Class Exercises 5-8

1. Indicate which property, the addition or the multiplication, is used in solving the following equations.

(a) $x + 10 = 22$	(f) $14 - x = 0$
(b) $6.2 + x = 1.12$	(g) $\frac{1}{2} x = 17$
(c) ${}^-2 + x = {}^-10$	(h) $18 + y = 8.6$
(d) $5 x = 15$	(i) $u + 6 = 5 + 3$
(e) $6 = \frac{x}{18}$	(j) $19 = 6 - y$

2\. What property is used, and how is it used, to get the the second equation from the first?

Example: (1) $2x + 4 = 7$
(2) $2x = 3$; addition property, adding $^{-}4$.

(a) (1) $2(y + 4) = 8$
(2) $y + 4 = 4$

(b) (1) $1.6 = 4y$
(2) $0.4 = y$

(c) (1) $\frac{2(m + 5)}{3} = 6$

(d) (1) $\frac{2}{5}x = 10$
(2) $\frac{1}{5} x = 5$

(e) (1) $(0.3m) - 7.2 = 5$
(2) $(3m) - 72 = 50$

(f) (1) $5x - 2 = 3x + 6$
(2) $2x - 2 = 6$

3\. Use the properties as indicated on the following equations.

Example: $3y - 2 = 7$ Addition property with (2).

Answer: $(3y - 2) + 2 = 7 + 2$
$3y + (^{-}2 + 2) = 9$
$3y = 9$

(a)	$7 = 3x + 1$	Addition property with $(^{-}1)$.
(b)	$6 = 3w$	Multiplication property with $(\frac{1}{3})$.
(c)	$\frac{t}{2} - 1.7 = {}^{-}1.3$	Multiplication property with (2).
(d)	$b = \frac{x}{18}$	Multiplication property with (18).
(e)	$0.14 + x = 5.28$	Addition property with $(^{-}0.14)$.
(f)	$5x - 7 = 2x$	Addition property with $(^{-}2x)$.

Exercises 5-8b

1. Solve the following equations by using the properties of "equals." Give your reason for each step.

(a) $2x + 1 = 7$ (c) $\frac{t}{2} - 3 = {}^{-}4$

(b) $y - 2 = 6$ (d) $3x - 5 = {}^{-}4$

2. Solve the following equations.

(a) $x + 3 = 5$ (e) $y - 3 = 5$

(b) $3 + y = {}^{-}5$ (f) $2w - 3 = 5$

(c) $2v + 3 = 5$ (g) $2t - 11 = 5t + 1$

(d) $3 + 2m = {}^{-}5$ (h) $15 - 5w = 2w + 1$

3. (a) In solving the equation $9x = 27$ what number would you use as a multiplier?

(b) In solving the equation $\frac{1}{3}x = 4$ what number would you use as a multiplier?

(c) In solving the equation $\frac{4}{5}x = \frac{1}{2}$ what number would you use as a multiplier?

(d) What is the relation between 9 and $\frac{1}{9}$, relative to multiplication?

(e) What is the relation between $\frac{4}{5}$ and $\frac{5}{4}$, relative to multiplication?

4. Which of the two properties (multiplication and addition) of "=" are also true of "<?" Replace "=" with "<" in each property and tell whether it is still true or not. If it is not true, give examples with numbers in which it is false.

5. In solving an equation such as $3x + 1 = 9$, you have learned to use the addition property first (to find $3x$) and the multiplication property second (to find x). Sometimes you will find it best to reverse the order in which you use these properties. Solve the following equations by using the multiplication property first.

(a) $4(x + 1) = 12$

(b) $7(x-2) = 13$

(c) $\frac{x + 9}{3} = 5$

(d) $0.6(x - 0.3) = 0.2$

(e) $\frac{3x + 4}{2} = 7$

(f) $\frac{4x + 1}{0.12} = 3$

Chapter 6

REAL NUMBERS

6-1 Review of Rational Numbers

In your study of mathematics you have used several number systems. Probably the first of these was the system of counting numbers. You learned that this number system has various properties. Other number systems have some of these properties; some of these properties are not found in any number systems other than the system of counting numbers. Let us recall a few familiar properties. Each counting number has an immediate successor, the next larger counting number. The sucessor of the counting number n is $n + 1$. With one exception, each counting number n has an immediate predecessor, the next smaller counting number $n - 1$. What is the one exception here? The system of counting numbers has a smallest element (what is it?) but no largest element. The system of counting numbers is closed under the operations of addition and multiplication, but not under the operations of subtraction and division.

In an earlier chapter you studied the system of integers which contains the set of counting numbers (now called positive integers). For each positive integer a there is an opposite number ^-a. The opposites of positive integers are called negative integers. If a is a counting number, then $a + {}^-a = 0$. The systems of integers is closed under the operations of addition, multiplication, and subtraction, but not under division.

The set of integers is contained in another set of numbers which is called the set of rational numbers. As you know, the set of integers, is adequate for many purposes, such as reporting the population of a country, the number of dollars you have (or owe), the number of vertices in a triangle, and so on. The integers alone are not suitable for many purposes, particularly for the process of measurement. If only the integers could be used for measuring, names for subdivisions of units would have to be invented. This is done to some extent; instead of saying $5\frac{1}{3}$ feet, 5 feet 4 inches is sometimes said. But a different name for a subdivision of an inch is not used. Instead, $7\frac{1}{4}$ inches, or 7.25 inches is written using rational numbers which are not integers. If only the integers could be used, you could never speak of $3\frac{1}{2}$ quarts, or 2.3 miles, or 0.0001 inch.

Recall that a rational number may be named by the fraction symbol "$\frac{p}{q}$", where p and q are integers, and $q \neq 0$.

Just as there is a negative integer which corresponds to each positive integer (or counting number), there is a negative rational number which corresponds to each positive rational number.

You already may have observed the familiar properties for rational numbers, which may be summarized as follows:

Closure: If a and b are rational numbers, then $a + b$, $a \cdot b$ (more commonly written ab), and $a - b$ are rational numbers; $\frac{a}{b}$ is a rational number if $b \neq 0$.

Commutativity: If a and b are rational numbers, then $a + b = b + a$, and $a \cdot b = b \cdot a$ $(ab = ba)$.

Associativity: If a, b, and c are rational numbers, then $a + (b + c) = (a + b) + c$, and $a(bc) = (ab)c$.

Identities: There is a rational number zero such that if a is a rational number, then $a + 0 = a$.

There is a rational number 1 such that $a \cdot 1 = a$.

Distributivity: If a, b, and c are rational numbers, then $a(b + c) = ab + ac$.

Additive inverses: If a is a rational number, then there is a number (^{-}a) such that $a + (^{-}a) = 0$.

Multiplicative inverses: If a is a rational number and $a \neq 0$, then there is a number b such that $ab = 1$.

Order: If a and b are different rational numbers, then either $a > b$, or $a < b$.

Exercise 6-1

1. Is there a smallest negative integer? A largest one?
2. If n represents a negative integer, what represents the next larger one? the next smaller one?
3. Is the set of negative integers closed under the operation of
   (a) addition?
   (b) subtraction?
   (c) multiplication?
   (d) division?
4. What is the multiplicative inverse of $^{-}(\frac{7}{4})$?
5. What is another name for "multiplicative inverse"?
6. How can you tell whether two fractions represent the same rational number?
7. What are three other names for the rational number $\frac{5}{7}$?
8. Look at each statement below and tell which of the properties listed for rational numbers it illustrates.
   (a) $^{-}(\frac{3}{4}) + \frac{5}{6} = \frac{1}{12}$, and $\frac{1}{12}$ is a rational number.
   (b) $\frac{5}{8} + 0 = \frac{5}{8}$
   (c) $1 \cdot {}^{-}(\frac{3}{4}) = {}^{-}(\frac{3}{4})$
   (d) $^{-}(\frac{3}{4}) \cdot {}^{-}(\frac{5}{8}) = {}^{+}\frac{15}{32}$, and ${}^{+}\frac{15}{32}$ is a rational number.

(e) $\frac{2}{3} \cdot (\frac{1}{3} + \frac{1}{2}) = (\frac{2}{3} \cdot \frac{1}{3}) + (\frac{2}{3} \cdot \frac{1}{2})$

(f) $^-(\frac{5}{8}) \cdot {}^-(\frac{1}{3}) = {}^-(\frac{1}{3}) \cdot {}^-(\frac{5}{8})$

(g) $\frac{11}{10} + (\frac{3}{10} + \frac{7}{10}) = (\frac{11}{10} + \frac{3}{10}) + \frac{7}{10}$

9. Express each of the following in the form $\frac{p}{q}$ or $^-(\frac{p}{q})$, where p and q are counting numbers.

(a) $17\frac{1}{2}$

(b) $\frac{5}{^-7}$

(c) $^-4 + \frac{1}{3}$

(d) $^-0.35$

(e) 10

(f) 17.03

10. What is the additive inverse of each of the following?

(a) $^-28$

(b) 756

(c) $^+3\frac{1}{7}$

(d) $^-(\frac{176}{5})$

11. Complete the statement, "The simplest name for a rational number written in the form $\frac{a}{b}$ is the one in which a and b have no common factor except ___________".

12. The only rational number that does not have a reciprocal is the number $\frac{p}{q}$ when p is ___________.

13. Arrange the following rational numbers in order. List the smallest one first.

$$\frac{4}{7}, \frac{3}{8}, 0.41, \frac{7}{16}, \frac{2}{5}, {}^-4, {}^-(\frac{2}{3}), 0$$

*14. Find the average of the two rational numbers $^-8$ and $^+4$.

*15. Is it always possible to find the average of two integers and have the average be an integer? Explain.

6-2 Density of Rational Numbers

One of the observations you have made about the integers is that every integer is preceded by a particular integer, and is followed by a particular integer. The integer which precedes $^{-}8$ is $^{-}9$, and the integer which follows 1005 is 1006. In other words, if n is an integer, then its predecessor is $(n - 1)$, and its successor is $(n + 1)$. There are no integers between $(n - 1)$ and n or between n and $(n + 1)$.

This means that on the number line there are wide gaps between the points which correspond to the integers. There are many points between the points corresponding to n and $(n + 1)$.

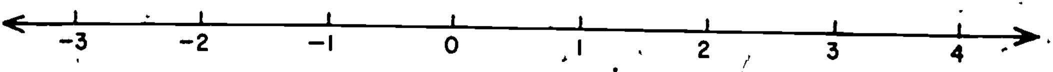


Figure 6-1

Now consider all the rational numbers, and the points on the number line which correspond to them. Such points are called rational points. On the number line below are shown the rational points between $^{-}3$ and 4 which may be named by the fractions with denominators 2, 3, 4, and 6.

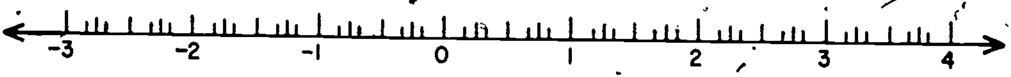
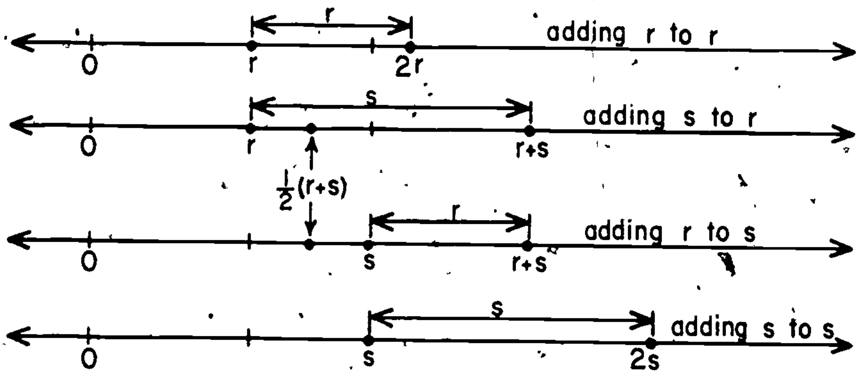


Figure 6-2

Consider two positive rational numbers r and s, with $r < s$. Then consider what happens when r and s are added to each of these numbers.



You see that $2r < r + s < 2s$. Taking half of each you get $r < \frac{1}{2}(r + s) < s$. It is not difficult to show that $r < \frac{1}{2}(r + s) < s$, even if r is negative or r and s are both negative. You might try to prove this yourself using the number line, if you wish. The number $\frac{1}{2}(r + s)$ is the <u>average</u> of the numbers r and s. You have observed, then, that the average of two rational numbers is between these numbers. On the number line what point do you suppose corresponds to the average of two numbers? It is the mid-point of the segment determined by the two numbers. If r and s are rational numbers, is $\frac{1}{2}(r + s)$ a rational number? What properties of the rational number system tell us that it is?

To summarize: <u>The mid-point of the segment joining two rational points on the number line is a rational point corresponding to the average of the two numbers.</u>

The mid-point of the segment joining the points for $\frac{1}{2}$ and $\frac{1}{3}$ is the point corresponding to the number $\frac{5}{12}$, since

$$\frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{3}{6} + \frac{2}{6}\right) = \frac{1}{2}\,\frac{5}{6} = \frac{5}{12}.$$

The mid-point of the segment joining the points for $\frac{1}{8}$ and $\frac{1}{7}$, is the point corresponding to the number $\frac{15}{112}$, since

$$\frac{1}{2}\left(\frac{1}{7} + \frac{1}{8}\right) = \frac{1}{2}\left(\frac{8}{56} + \frac{7}{56}\right) = \frac{15}{112}.$$

By finding the average in this manner it is possible to find rational numbers between each pair of consecutive numbers represented in the row of fractions below.

$$\frac{0}{1}\quad \frac{1}{8}\quad \frac{1}{7}\quad \frac{1}{6}\quad \frac{1}{5}\quad \frac{1}{4}\quad \frac{2}{7}\quad \frac{1}{3}\quad \frac{3}{8}\quad \frac{2}{5}\quad \frac{3}{7}\quad \frac{1}{2}\quad \frac{4}{7}\quad \frac{3}{5}\quad \frac{5}{8}\quad \frac{2}{3}\quad \frac{5}{7}\quad \frac{3}{4}\quad \frac{4}{5}\quad \frac{5}{6}\quad \frac{6}{7}\quad \frac{7}{8}\quad \frac{1}{1}$$

If these new fractions are inserted in the row, the row would begin

$$\frac{0}{1},\ \frac{1}{16},\ \frac{1}{8},\ \frac{15}{112},\ \frac{1}{7},\ \frac{13}{84},\ \ldots$$

If you found all the new fractions in this row which could be found in this way, there would be 43 fractions between $\frac{0}{1}$ and $\frac{1}{1}$. This process could be continued indefinitely. You could find points between $\frac{0}{1}$ and $\frac{1}{16}$, between $\frac{1}{16}$ and $\frac{1}{8}$, and so on. You could find as many rational numbers as you wish between 0 and 1 by taking averages, averages of averages, and so on indefinitely.

The discussion above suggests an important property of the rational numbers. This is the property of <u>density</u>: <u>Between any two distinct rational numbers there is a third rational number</u>. On the number line, this means that the number of rational points on any segment is unlimited; no matter how many points on a very small segment have been named, it is possible to name as many more as you please. There is no "next" rational point to the left or right of a given rational point.

<u>Exercises 6-2</u>

1. Are the integers dense? That is, is there always a third integer between any two integers? Illustrate your answer.
2. Is there a smallest positive integer? a largest?
3. Is there a smallest negative integer? a largest?
4. Is there a smallest positive rational number? a largest negative rational number?
5. Think of the points for 0 and $\frac{1}{100}$ on the number line. Name the rational point P which is halfway between 0 and $\frac{1}{100}$. Name the point halfway between the point P and 0; between the point P and $\frac{1}{100}$.
6. In the same way, find three rational numbers between $\frac{1}{20}$ and $\frac{1}{10}$.
7. Think of the segment with end-points $\frac{1}{1000}$ and $\frac{2}{1000}$. Show a plan you could follow to name as many rational points as you please on this segment. Use your plan to name at least five points.

6-3 Decimal Representations For The Rational Numbers

It is often very helpful to be able to express rational numbers as decimals. When it is necessary to compare two rationals that are very close together, converting to decimal form makes the comparison easier. The decimal form is particularly helpful if there are several rational numbers to be arranged in order. For example, consider the fractions $\frac{13}{25}$, $\frac{27}{50}$, $\frac{3}{8}$, and $\frac{9}{20}$ and their corresponding decimals 0.52, 0.54, 0.375, and 0.45. It is much easier to order the numbers when they are written in decimal form.

Some rational numbers are easily written in decimal form. You know how to write, by inspection,

$$\frac{1}{2} = 0.5, \quad \frac{1}{4} = 0.25, \quad \frac{1}{8} = 0.125, \quad \frac{1}{5} = 0.2, \quad \frac{1}{25} = 0.04$$

$$\frac{1}{125} = 0.008, \quad \text{and also} \quad \frac{17}{2} = 8.5, \quad 5\frac{3}{4} = 5.75, \quad \frac{175}{10} = 1.75.$$

For other rational numbers, a decimal expression may not be as obvious but you can always obtain it by the usual process of division. For example

$$\frac{1}{3} = 0.33333 \ldots$$

$$\frac{8}{3} = 2.6666666 \ldots$$

$$\frac{1}{7} = 0.142857142857142857 \ldots$$

$$\frac{1}{13} = 0.07692307692307 \ldots$$

$$\frac{1}{11} = 0.09090909 \ldots$$

$$\frac{123}{14} = 8.7857142857142 \ldots$$

The examples that have been discussed seem to suggest that the decimal expansions for rational numbers either terminate (like $\frac{1}{2} = 0.5$) or repeat (like $\frac{1}{3} = 0.3333333 \ldots$). What would be a reasonable way to study such decimal expansions? Since you have used the division of numerator by denominator to obtain a decimal representation, you might study carefully the process which you carry out in such cases.

Consider the rational number $\frac{7}{8}$. If you carry out the indicated division you would write

```
     .875
  8)7.000
    6 4
      60        remainder  6
      56
       40       remainder  4
       40
        0       remainder  0
```

In dividing by 8, the only remainders which can occur are 0, 1, 2, 3, 4, 5, 6, and 7. The only remainders which did occur were 6 at the first stage, then 4 and finally 0. When the remainder 0 occurs, the division is exact. Division is exact if after some stage the process of division continues to produce only zero remainders and zero quotients. Such a decimal is often spoken of as a terminating decimal.

What about a rational number which does not have a terminating decimal representation? Suppose you look at a particular example of this kind, say $\frac{2}{13}$. The process of dividing 2 by 13 proceeds like this:

```
       0.153846153
   13)2.000000000       remainder
       1 3
         70                7
         65
          50               5
          39
          110             11
          104
            60             6
            52
             80            8
             78
              20           2
              13
               70          7
               65
                50         5
                39
                110       11
                etc.
```

Here the possible remainders are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Not all the remainders do appear, but 7, 5, 11, 6, 8, 2 occur first in this order. At the next stage in the division the remainder 7 re-occurs and the sequence of remainders 7, 5, 11, 6, 8, 2 occurs again. In fact the process repeats itself again and again. The corresponding sequence of digits in the quotient --153846--will therefore occur periodically in the decimal expansion for $\frac{2}{13}$. This type of repeating decimal is sometimes referred to as a periodic decimal.

In order to write such a periodic decimal concisely and without ambiguity it is customary to write

$$0.153846153846153846\ldots \quad \text{as} \quad 0.\overline{153846}\ \ldots$$

The bar (vinculum) over the digit sequence 153846 indicates the set of digits which repeats. Similarly, 0.3333... is written as $0.\overline{3}\ldots$. If it seems more convenient you can write 0.3333... as $0.3\overline{3}\ldots$ or $0.33\overline{3}\ldots$, and $0.\overline{153846}\ldots$ as $0.153846\overline{153846}\ldots$.

The method which has been discussed is quite a general one and it can be applied to any rational number $\frac{a}{b}$. If the indicated division is performed then the only possible remainders which can occur are 0, 1, 2, 3, ... (b - 1). It is necessary to look only at the stages which contribute to the digits that repeat in the quotient. These stages usually occur after the zeros begin to repeat in the dividend. If the remainder 0 occurs, the decimal expansion terminates at this stage in the division process. Actually, a terminating decimal expansion like 0.25 as 0.25000 ... or $0.25\overline{0}$ ..., may be written with a repeated zero to provide a periodic expansion. Note that a zero remainder may occur prior to this stage without terminating the process, for example,

```
    112.2
  5)561.0
    5          Remainder
    06             0
     5
     11            1
     10
      10           1
      10
       0           0
```

If 0 does not occur as a remainder after zeros are annexed to the dividend, then after at most (b - 1) steps in the division process one of the possible remainders 1, 2, ..., (b - 1) will reoccur and the digit sequence will start repeating.

You can see from this argument that any rational number has a decimal expansion which is periodic.

Exercises 6-3

1. Find decimals for these rational numbers. Continue the division until the repeating begins, and write your answer to at least ten decimal places.

   (a) $\frac{9}{4}$

   (b) $\frac{5}{24}$

   (c) $\frac{3}{35}$

   (d) $\frac{14}{37}$

   (e) $\frac{11}{909}$

2. Which of the following convert to decimals that repeat zero (terminate)?

   (a) $\frac{1}{2}$

   (b) $\frac{1}{3}$

   (c) $\frac{1}{4}$

   (d) $\frac{1}{5}$

   (e) $\frac{1}{6}$

   (f) $\frac{1}{7}$

   (g) $\frac{1}{8}$

   (h) $\frac{1}{9}$

   (i) $\frac{1}{10}$

   (j) $\frac{1}{11}$

   (k) $\frac{1}{12}$

   (ℓ) $\frac{1}{13}$

3. Write in completely factored form the denominators of those fractions that terminated in Problem 2.

4. Carry to six decimal places the following fractions.

(a) $\frac{1}{7}$ (d) $\frac{4}{7}$

(b) $\frac{2}{7}$ (e) $\frac{5}{7}$

(c) $\frac{3}{7}$ (f) $\frac{6}{7}$

6-4 The Rational Number Corresponding To A Periodic Decimal

You saw how to find by division the decimal expansion of a given rational number. But suppose you have the opposite situation, that is, you are given a periodic decimal. Does such a decimal in fact represent a rational number? How can you find out?

This problem can be approached by considering an example. Let us write the number 0.132132132132 ... and call it n, so that $n = 0.132132\overline{132} \ldots$. The periodic block of digits is 132. If you multiply by 1000 this shifts the first block to the left of the decimal point and gives the relation

$$1000n = 132.132132\overline{132} \ldots$$

Since $$n = 0.132132\overline{132} \ldots$$

You can subtract n form each side of the first equation to yield

$$999n = 132 \text{ so that}$$

$$n = \frac{132}{999}, \text{ or in simplest form,}$$

$$n = \frac{44}{333}.$$

You find by this process that $0.132132132\overline{132} \ldots = \frac{44}{333}$.

The example here illustrates a general procedure which mathematicians have developed to show that every periodic decimal represents a rational number. You see. therefore, that there is a one-to-one correspondence between the set of rational numbers, and the set of periodic decimals. It would be quite equivalent then for us to define the rational numbers as the set of numbers represented by all such periodic decimals.

Before you leave the subject of decimals there is one interesting fact about terminating decimals which will be discussed.

You saw that rationals like $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, $\frac{15}{8} = 1.875$, $\frac{397}{1000} = 0.397$, $\frac{692}{25} = 27.68$ all are represented by terminating decimals. How can you determine when this will be the case? If, for inspiration, you look at the rationals of this type which have been discussed, there is an obvious clue: The denominators seen to have only the prime factors 2 or 5, or both. (See Problem 3 in Exercises 6-3.)

Consider a rational number in which the denominator is a power of 2, such as $\frac{39}{2^4}$.

By multiplying by $1 = \frac{5^4}{5^4}$ you can write

$$\frac{39}{2^4} = \frac{39\cdot 5^4}{2^4\cdot 5^4} = \frac{39\cdot 5^4}{10^4} = \frac{39\cdot 625}{10{,}000} = \frac{24{,}375}{10{,}000} = 2.4375.$$

Similarly if you have a rational number in which the denominator is a power of 5, you can proceed as in the following example,

$$\frac{3}{3125} = \frac{3}{5^5} = \frac{3\cdot 2^5}{5^5\cdot 2^5} = \frac{3\cdot 32}{10^5} = \frac{96}{100{,}000} = 0.00096.$$

Quite generally, if you have any rational number with only powers of 2 and powers of 5 in the denominator, you can use the same technique. For example,

$$\frac{3791}{2^7\cdot 5^4} = \frac{3791\cdot 5^7\cdot 2^4}{(2^7\cdot 5^4)\cdot(5^7\cdot 2^4)} = \frac{3791\cdot 5^7\cdot 2^4}{(2^7\cdot 5^7)\cdot(5^4\cdot 2^4)} = \frac{3791\cdot 5^7\cdot 2^4}{10^7\cdot 10^4} = \frac{3791\cdot 5^7\cdot 2^4}{10^{11}},$$

and this gives a terminating decimal representation.

In order to establish a general fact of this kind suppose you ask the following question. What rational number $\frac{p}{q}$ (p and q assumed to have only 1, as a common factor) can be represented by $\frac{N}{10^k}$ where N is an integer?

Suppose

$$\frac{p}{q} = \frac{N}{10^k}.$$

Therefore $q \cdot N = p \cdot 10^k$.

This says that q divides the product of p and 10^k. But you assumed that p and q have only 1 as a common factor. Hence q must divide 10^k. But the only possible factors of $10^k = 2^k \cdot 5^k$ are multiples of powers of 2 and powers of 5.

Thus you have proved that a rational number r has a terminating decimal representation if and only if the denominator of r consists only of products of powers of 2 and powers of 5; that is, r must be of the form

$$r = \frac{A}{2^m \cdot 5^n}$$

Exercise 6-4

1. Express each of the following as a decimal.

   (a) $10 \times 0.999\overline{9} \ldots$

   (b) $100 \times 3.12\overline{12} \ldots$

   (c) $10 \times 0.004\overline{4} \ldots$

   (d) $1{,}000 \times 0.61345\overline{345} \ldots$

   (e) $10{,}000 \times 6.0123\overline{0123} \ldots$

2. Subtract in each of the following.

(a) $\begin{array}{r} 3128.99\overline{9}\ \ldots \\ \underline{312.89\overline{9}}\ \ldots \end{array}$

(b) $\begin{array}{r} 9.99\overline{9}\ \ldots \\ \underline{0.99\overline{9}}\ \ldots \end{array}$

(c) $\begin{array}{r} 162.162\overline{162}\ \ldots \\ \underline{0.162\overline{162}}\ \ldots \end{array}$

(d) $\begin{array}{r} 301.0101\overline{01}\ \ldots \\ \underline{3.0101\overline{01}}\ \ldots \end{array}$

(e) $\begin{array}{r} 1.23333\overline{3}\ \ldots \\ \underline{0.12333\overline{3}}\ \ldots \end{array}$

(f) $\begin{array}{r} 354.54\overline{54}\ \ldots \\ \underline{3.54\overline{54}}\ \ldots \end{array}$

(g) $\begin{array}{r} 27075.075\overline{075}\ \ldots \\ \underline{27.075\overline{075}}\ \ldots \end{array}$

(h) $\begin{array}{r} 416.4777\overline{7}\ \ldots \\ \underline{41.6477\overline{7}}\ \ldots \end{array}$

3. Write each of the following in the form $\frac{a}{b}$, where a and b are counting numbers.

(a) $\frac{4.11}{9}$ (b) $\frac{16.3}{99}$ (c) $\frac{1.03}{999}$

4. The first step in writing a rational number in decimal form as a fraction is to choose the power of 10 by which the original decimal number should be multiplied. For each of the following numbers N find the smallest number of the form 10^k (10, 100, 1000, etc.) so that $(10^k \cdot N) - N$ is a terminating decimal. Show this to be true.

Example:

$$\begin{array}{rl} N = & 1.324\overline{24}\ \ldots \\ 100N = & 132.424\overline{24}\ \ldots \\ N = & \underline{1.324\overline{24}}\ \ldots \\ 100N-N = & 131.10000\ \ldots \end{array}$$

(a) $0.55\overline{5}\ \ldots$

(b) $0.73\overline{73}\ \ldots$

(c) $0.901\overline{901}\ \ldots$

(d) $3.0233\overline{3}\ \ldots$

(e) $163.17\overline{7}\ \ldots$

(f) $672.42\overline{42}\ \ldots$

(g) $0.12345\overline{656}\ \ldots$

(h) $3.4100\overline{0}\ \ldots$

5. What rational numbers have these decimal expressions?

   (a) $0.09\overline{09}\ldots$
   (b) $0.111\overline{1}\ldots$
   (c) $0.055\overline{5}\ldots$
   (d) $0.123\overline{123}\ldots$
   (e) 0.1625
   (f) $0.166\overline{6}\ldots$
   (g) $5.125\overline{125}\ldots$
   (h) $10.045\overline{45}\ldots$

6. Write each denominator of the following numbers in completely factored form.

   (a) $\frac{7}{32}$
   (b) $\frac{47}{100}$
   (c) $\frac{5}{9}$
   (d) $\frac{13}{50}$
   (e) $\frac{12}{35}$
   (f) $\frac{21}{80}$
   (g) $\frac{71}{120}$
   (h) $\frac{1}{160}$

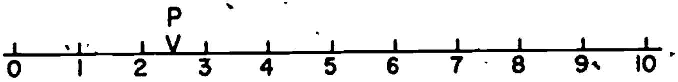
7. Which of the numbers in Problem 2 have decimals which repeat zero?

8. Assuming the a to have the value one (1) in the rational number $\frac{a}{b}$, what numbers between 63 and 101 may be b and give a terminating decimal expression for $\frac{a}{b}$?

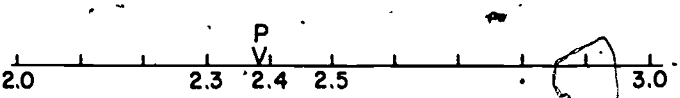
6-5 Rational Points on the Number Line

If you think of the rational numbers as specified by decimal representations, you can see immediately how to locate and how to order the corresponding points on the number line.

Consider for example the rational number $2.39\overline{614}\ldots$ and its place on the number line. The digit 2 in the units place tells us immediately that the corresponding rational point P lies between the integers 2 and 3 on the number line. Graphically then the first rough picture is this:



A more precise description is obtained by looking at the first two digits 2.3 which tell us immediately that P lies between 2.3 and 2.4. On the interval from 2 to 3, then, divided into tenths (and magnified ten times for easy comparison) P is found as shown below



If you continue the process of successively refining the location of P on the number line you will have a picture such as the following

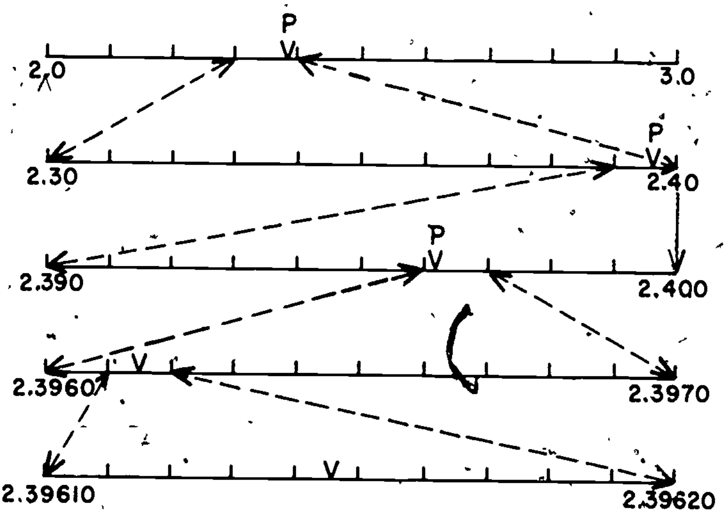
P (2.3...)

P (2.39...)

P (2.396...)

P (2.3961...)

P (2.39614...)



Location of point P corresponding to $2.39\overline{614}$ ...

From such a decimal representation for a rational number you can easily find how to locate the number to any desired degree of accuracy on the number line.

Moreover, given any two distinct rational numbers in this form it is a simple matter to tell by inspection which is larger and which is smaller, and which precedes the other on the number line.

If you think of locating the point $\frac{3}{7}$ carefully on the number line would you prefer to use $\frac{3}{7}$ or $0.\overline{428571}\ldots$? If you wish to compare $\frac{3}{7}$ with another rational, which form is easier to use, $\frac{3}{7}$ or $0.\overline{428571}\ldots$?

Exercises 6-5

1. Arrange each group of decimals in the order in which the points to which they correspond would occur on the number line. List first the point farthest to the left.

   (a) 1.379 1.493 1.385 5.468 1.372

   (b) $^-9.426$ $^-2.755$ $^-2.761$ $^-5.630$ 2.763

   (c) 0.15475 0.15467 0.15463 0.15475 0.15598

2. In Problem (1c), which points lie on the following segments:

   (a) The segment with endpoints 1 and 2?

   (b) The segment with endpoints 0 and 1?

   (c) The segment with endpoints 0.1 and 0.2?

   (d) The segment with endpoints 0.15 and 0.16?

   (e) The segment with endpoints 0.154 and 0.155?

3. Draw a 10 centimeter segment; label the endpoints 0 and 1, and divide the segment into tenths. Mark and label the following points:

   (a) 0.23 (b) 0.49 (c) 0.80 (d) 0.6 (e) 0.08 (f) 0.95

4. Arrange each group of rational numbers in order of increasing size by first expressing them in decimal form.

   (a) $\frac{3}{9}, \frac{4}{10}, \frac{17}{50}$ (c) $\frac{3}{7}, \frac{4}{9}$

   (b) $\frac{2}{3}, \frac{7}{10}, \frac{67}{100}$ (d) $\frac{152}{333}, \frac{415}{909}$

6-6 Irrational Numbers

You have learned many things about rational numbers. One of the most important is the density property; between any two distinct rational numbers on the number line there is a third rational number. This tells us that there are many rational numbers and rational points - very many of them. Moreover, they are spread throughout the number line. Any segment, no matter how small, contains infinitely many rational points. One might think that all the points on the number line are rational points. Let us locate a certain point on the number line by a very simple compass and straight edge construction. Perhaps this point will have a surprise for us.

a. Construct a number line and call it ℓ. Let A be the point zero and B be the point one.

b. At B, construct a ray m perpendicular to ℓ.

c. On m construct a line segment $\overline{BC}$, one unit long.

d. Draw segment $\overline{AC}$.

e. With A as center and radius AC, draw a circular arc which intersects ℓ. Call the point of intersection D.

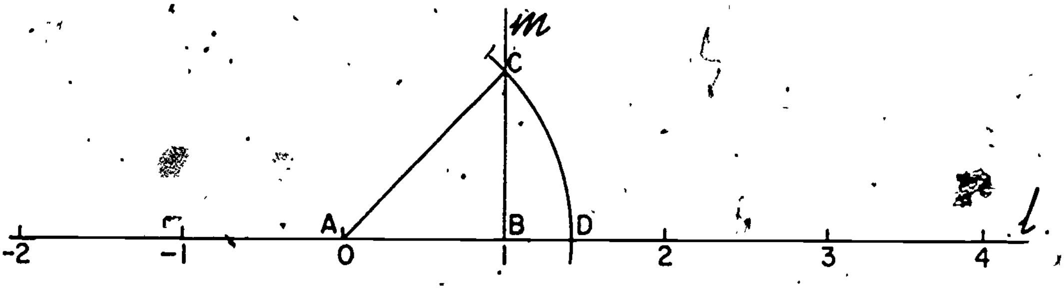


Figure 6-3.

Now consider two questions:

(1) To what number (if any) does point D correspond?

(2) Is this number a rational number?

Consider the first question, "To what number does point D correspond?" First find the length of $\overline{AC}$, since $\overline{AC}$ and $\overline{AD}$ have the same length. The unit of measure used will be the unit distance on the number line. In Figure 6-3, triangle ABC is a right triangle. The measure of $\overline{AB}$ is 1. The measure of $\overline{BC}$ is 1. You can use the Pythagorean property to find AC.

$$AC^2 = BC^2 + AB^2$$

$$AC^2 = 1^2 + 1^2$$

$$AC^2 = 2$$

The positive number whose square is 2 is defined as the <u>square root of 2</u> and is written $\sqrt{2}$.

Thus,

$$AC = \sqrt{2}, \text{ so}$$

$$AD = \sqrt{2}.$$

Therefore, the point D corresponds to the number $\sqrt{2}$. Is $\sqrt{2}$ a rational number? Is it the quotient of two integers, and can it be represented as a fraction $\frac{p}{q}$, in which p and q are integers and $q \neq 0$?

To answer this question, a line of reasoning which people very often use will be followed. This type of reasoning can be illustrated by the following conversation between a mother and her son. John was late from school. When his mother scolded him he tried to avoid punishment by saying that he had run all the way home. "No, you didn't run all the way," she said firmly. John was surprised and ashamed, and asked, "How did you know?" "If you had run all that way, you would have been out of breath," she said. "you are not out of breath. Therefore you did not run."

John's mother had used indirect reasoning. She assumed the opposite of the statement she wished to prove, and showed that this assumption led to a conclusion which could not possibly be true. Therefore her assumption had to be false, and the original statement had to be true.

It will be proved that $\sqrt{2}$ is not a rational number. Indirect reasoning will be used. It will be assumed that $\sqrt{2}$ is a rational number and then it will be shown that this assumption leads to an impossible conclusion.

Assume that $\sqrt{2}$ is a rational number. Then $\sqrt{2}$ can be written as $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Take $\frac{p}{q}$ in simplest form. <u>This means that p and q have no common factor except 1.</u>

If $\sqrt{2} = \frac{p}{q}$, then $2 = \frac{p^2}{q^2}$, and so $2q^2 = p^2$. Since p and q are integers, then p^2 and q^2 are also integers. If $p^2 = 2q^2$ then p^2 must be an even number. (An integer is even if it is equal to 2 times another integer.) Thus, $p \cdot p$ must be even. An odd number times an odd number is an odd number. (Do you remember why?) Thus, p must be even, and can be written as 2a, where a is an integer.

Then $p^2 = 2q^2$ may be written as $(2a)^2 = 2q^2$

$$\begin{aligned} \text{and} \quad (2a)\cdot(2a) &= 2q^2 , \\ \text{and} \quad 2\cdot(2a^2) &= 2q^2 \\ \text{and} \quad 2a^2 &= q^2 \end{aligned}$$

This tells us that q^2 is also an even number since it is equal to 2 times another integer. So q is also an even number.

Thus our assumption, that $\sqrt{2}$ is a rational number $\frac{p}{q}$ in simplest form, has led us to the conclusion that p and q both have the factor 2. This is impossible, since the simplest form for a fraction is the one in which p and q have <u>no</u> common factor other than 1. So the statement "$\sqrt{2}$ is a rational number" must be false.

Since the measure of segment $\overline{AD}$ in Figure 6-3 is $\sqrt{2}$, then $\sqrt{2}$ must be the number which corresponds to point D. It has been shown that $\sqrt{2}$ is not a rational number. Therefore, there is at least this one point on the number line which corresponds to some number which is not a rational number. In other words, even though the rational points are dense, the set of points on the number line contains more points than there are rational numbers.

A number like $\sqrt{2}$, which is not a rational number, is called an <u>irrational</u> <u>number</u>. The prefix "ir" changes the meaning of "rational" to "not rational."

Exercises 6-6

1. Construct a figure like Figure 6-3, and label point D "$\sqrt{2}$". Then use your compass to locate the point which corresponds to the number $^{-}(\sqrt{2})$, and label it.
2. Draw a number line, using a unit of the same length as the unit in Problem 1. Use the letter A for the point 0 and the letter B for the point 2. At B construct a segment perpendicular to the number line and 1 unit in length, and call it $\overline{BP}$. Draw $\overline{AP}$. What is the measure of segment $\overline{AP}$?
3. Use the drawing for Problem 2, and locate on the number line the points which correspond to $\sqrt{5}$ and $^{-}(\sqrt{5})$. Label the points.
4. Do you think $\sqrt{5}$ is a rational number or an irrational number? Why?
5. Using the same method as in Problems 2 and 3, locate the point $\sqrt{3}$. Can you work out a way to locate the point for $\sqrt{6}$? For $\sqrt{7}$?
6. Locate the points which correspond to these numbers:

   (a) $2\sqrt{2}$ (b) $3\sqrt{2}$ (c) $^{-}(3\sqrt{2}.)$

7. Do you think that $(2\sqrt{2})$ is a rational number or an irrational number?

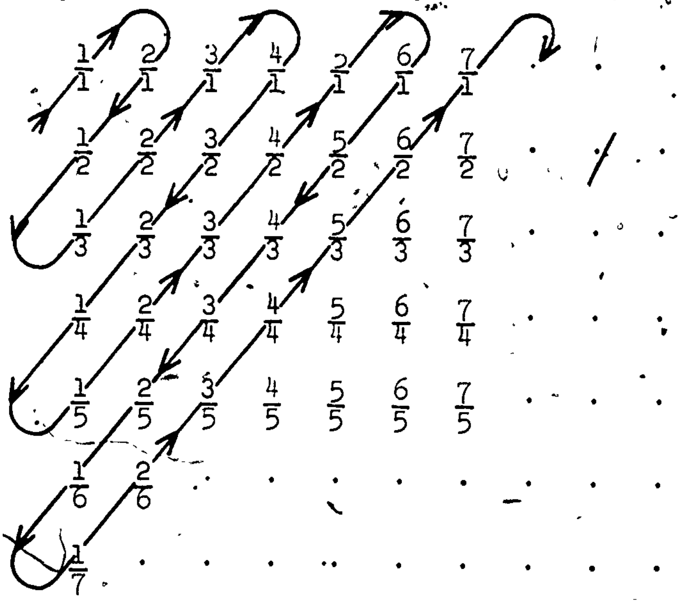
*8. Prove that $\sqrt{5}$ is an irrational number. (Use indirect reasoning very similar to the line of reasoning which was used to show that $\sqrt{2}$ is irrational. At one point you will have to know that if p^2 has 5 as a factor, then p also has 5 as a factor. Prove this simple fact. Before you try to prove that $\sqrt{5}$ is irrational, think of the unique factorization property of counting numbers. If the prime number 5 were not a factor of p then how could it be a factor of p^2?)

Enumerating the Rationals

In the preceding discussion it was proved that $\sqrt{2}$ is not a rational number. Moreover, it appears that there are many other numbers, such as $\sqrt{3}$ and $\sqrt{5}$ which are not rationals. If you think about the rationals and the irrationals a bit you can see how to write many, many irrationals. For example, every number of the form $\frac{a}{b}\sqrt{2}$, where $\frac{a}{b}$ is rational, will be irrational. Hence the set $\{\frac{a}{b}\sqrt{2}\}$ can be put into one-to-one correspondence with the set of rationals $\{\frac{a}{b}\}$. Yet the set $\{\frac{a}{b}\sqrt{2}\}$ is obviously only a very small part of the irrationals!

Indeed, you have suffered a great disillusionment - the rational numbers, despite being dense on the number line, actually leave empty more positions than they fill! An even worse shock to our intuition, perhaps is to find that a line segment whose length is not given by a rational number can be constructed so easily.

In fact, one of the really important distinctions between the rational number system and the system of irrationals is that you can show how to display all the rationals. One scheme is to proceed as follows. Write the array as follows.



You can find the positive rational $\frac{4}{5}$ in the 4th column at the fifth row. In what row and what column would you look for the rational $\frac{9}{10}$? For $\frac{8}{17}$? For $\frac{p}{q}$?

By following the snaky line in the above display a one-to-one correspondence can be shown between the set of positive rationals and the set of counting numbers like this:

Counting Numbers	1	2	3	4	5	6	7	3	. . .
	↕	↕	↕	↕	↕	↕	↕	↕	
Rational Numbers	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	. . .

In this listing of the rational numbers the snaky line has been followed but all fractions which are not in simplest form have been left out, because they are only other names for numbers already in our list. In this display $\frac{1}{2}$ is the 3rd rational number, $\frac{4}{1}$ is the 6th rational number, what is the 8th rational number? The 11th? If the above were continued, $\frac{5}{3}$ would be the what-th rational number?

When a one-to-one correspondence has been set up between a given set and the set of counting numbers (or a subset of the set of counting numbers) mathematicians say that the set has been "enumerated". Thus the set of positive rational numbers above has been "enumerated".

Georg Cantor (1845-1918) discovered in 1874 that the set of irrational numbers cannot be "enumerated" by any method. There are so many irrational numbers that it is <u>impossible</u> to set up a one-to-one correspondence between the set of these numbers and the set of counting numbers. No matter how you try to display irrational numbers some irrational numbers will always be left out - more than have been included, as a matter of fact. This is what is meant when it is said that the rational numbers leave more places empty on the number line than they fill.

If you are interested in learning more about this important phase of mathematics you might refer to <u>One Two Three . . . Infinity</u> by Geroge Gamow (pages 14-23). A brief but interesting history of Cantor's life can be found in <u>Men of Mathematics</u> by E. T. Bell (Chapter 29).

6-7 A Decimal Representation for $\sqrt{2}$

Numbers like $\sqrt{5}$ and $\sqrt{2}$ correspond to points on the number line, they specify lengths of line segments and they satisfy our natural notion of what a number is. Perhaps the most unusual aspect about $\sqrt{2}$ is the way it was defined: $\sqrt{2}$ is the positive number n which when squared yields 2, so that

$$n^2 = 2.$$

This differs from the previous way of defining numbers, since up to now this chapter has dealt mainly with integers and numbers defined as ratios of integers

In order to help us gain a better understanding of $\sqrt{2}$ a new way of describing $\sqrt{2}$ in terms of more familiar notions will be discussed. If, for example, $\sqrt{2}$ could be expressed as a decimal this would help us to compare it with the rational numbers. It would also tell us where to place it on the number line.

Let us think about the definition of the number $\sqrt{2}$, namely $(\sqrt{2})^2 = 2$. If you think of squaring 1 and 2 you will note immediately that

$$1^2 < (\sqrt{2})^2 < 2^2 \text{ and hence } 1 < \sqrt{2} < 2.$$

This says that $\sqrt{2}$ is greater than 1 and less than 2, but you already knew that. You might try a closer approximation by testing the squares of 1.1, 1.2, 1.3, 1.4, 1.5. A little arithmetic of this sort (try it!) leads us to the result

$$1.96 = (1.4)^2 < (\sqrt{2})^2 < (1.5)^2 = 2.25,$$

and therefore you conclude that $1.4 < \sqrt{2} < 1.5$.

The arithmetic involves a little more work at the next stage but you can see with a little more computation that

$$1.9881 = (1.41)^2 < (\sqrt{2})^2 < (1.42)^2 = 2.0164,$$

and therefore

$$1.41 < \sqrt{2} < 1.42.$$

If you try to extend the process further you will get at the next stage

$$1.414 < \sqrt{2} < 1.415.$$

You can see that this process can be continued as long as our enthusiasm lasts, and gives a better decimal approximation at every stage. If you continued to 7 place decimals you would find

$$1.4142135 < \sqrt{2} < 1.4142136.$$

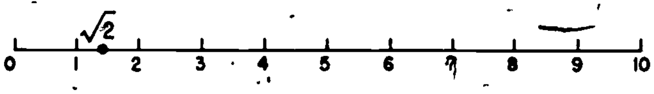
This is a very good approximation of $\sqrt{2}$, for $(1.4142136)^2 = 2.00000010642496$.

By the use of the defining property, $(\sqrt{2})^2 = 2$, then, you can find decimal approximations for $\sqrt{2}$ which are as accurate as you wish. You are led to write

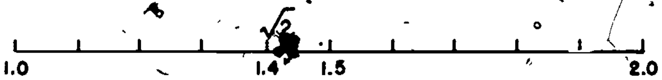
$$\sqrt{2} = 1.4142135 \ldots$$

where the three dots indicate that the digits continue without terminating, as the process above suggests.

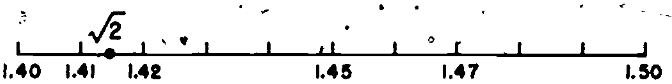
Geometrically the procedure you have followed can be described as follows in the number line. Looking first at the integers of the number line on the segment from 0 to 10, you saw that $\sqrt{2}$ would be between 1 and 2.



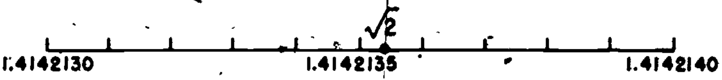
Enlarging our view of this segment (by a ten-fold magnification) you saw that $\sqrt{2}$ is on the segment with end-points 1.4 and 1.5



and again magnifying this picture, $\sqrt{2}$ lies within the interval (1.41, 1.42)



and so on till the 8th stage shows us that $\sqrt{2}$ lies between 1.4142135 and 1.4142136.



This process shows us how to read the successive digits in the decimal representation for $\sqrt{2}$. At the same time it gives a way to define the position of the point on the real line.

When you write the number $\sqrt{2}$ as 1.4142135 . . . it looks suspiciously like many rational numbers you have seen, such as

$$\frac{1}{3} = 0.3333333 \ . \ . \ . \qquad \text{and} \qquad \frac{1}{7} = 0.14285714 \ . \ . \ .$$

You pause to ask, how are they different and how can you tell a rational from an irrational number when you see only the decimal representations of the numbers?

The one special feature of the decimal representation of a rational number is that it is a periodic decimal. As you have seen, every periodic decimal represents a rational number. Then the decimal representation of $\sqrt{2}$ cannot be periodic, for $\sqrt{2}$ is irrational. You can be sure that as you continue to find new digits in the decimal representation

$$\sqrt{2} = 1.41412135 \ . \ . \ .$$

no group of digits will ever repeat indefinitely. You can only be certain that a decimal names a rational number when the period of the decimal is indicated, usually with a vinculum ($\overline{}$).

Exercises 6-7

1. Between what two consecutive integers are the following irrational numbers? (Write your answer as suggested for (a)).
   - (a) $\sqrt{30}$ $\left[\underline{\ ?\ } < \sqrt{30} < \underline{\ ?\ }\right]$
   - (b) $\sqrt{89}$
   - (c) $\sqrt{253}$
   - (d) $\sqrt{4280}$ (Hint: 4280 is 42.80×10^2, so begin estimating by thinking of $\sqrt{43 \cdot 10}$)
   - (e) $\sqrt{9315}$

2. Express (a), (b), and (c) as decimals to six places:
   (a) $(\sqrt{3})^2$
   (b) $(1.732)^2$
   (c) $(1.733)^2$
   (d) Find the difference between your answers for (a) and (b); find the difference between your answers for (a) and (c).
   (e) To the nearest thousandth what is the best decimal expression for $\sqrt{3}$?

Which of the numbers suggested is the better approximation of the following irrational numbers?

3. $\sqrt{3}$: 1.73 or 1.74
4. $\sqrt{15}$: 3.87 or 3.88
5. $\sqrt{637}$: 25.2 or 25.3

Find, to the nearest tenth, the nearest decimal expression for these irrational numbers:

6. $\sqrt{10}$ 7. $\sqrt{149}$ 8. $\sqrt{221}$

9. For what number n is $n^2 = 10$?
10. For what number n is $n^2 = 149$?

6-8 Irrational Numbers and the Real Number System

You have seen that all rational numbers have periodic decimal representations. You saw also that $\sqrt{2}$ is not rational and that it is represented by a non-periodic decimal. $\sqrt{2}$ is called an irrational number.

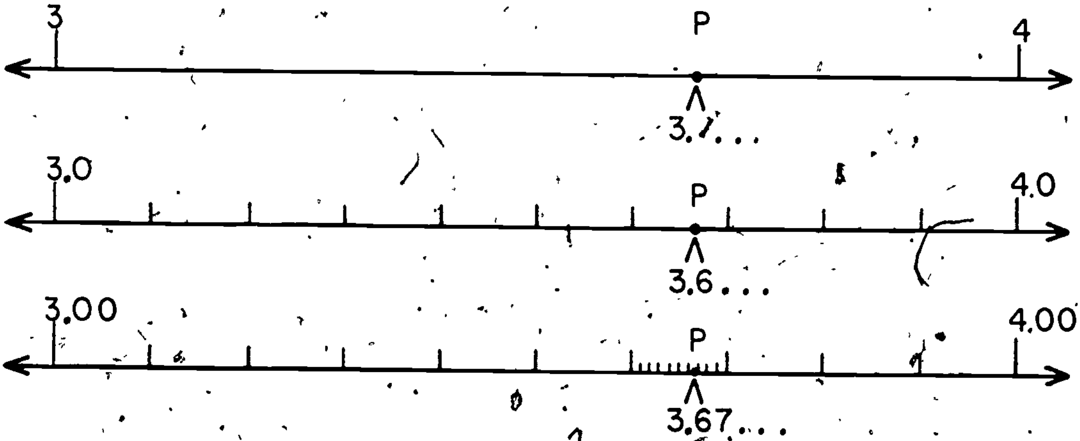
This decimal form is now used to define the set of irrational numbers. An irrational number is defined as any number with a non-repeating decimal representation.

The system composed of all rational and irrational numbers is called the real number system.

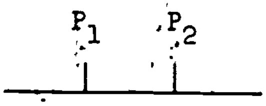
From this you see that any real number can be characterized by a decimal representation.

If the decimal representation is periodic the number is a rational number, otherwise the number is an irrational number.

With every point P on the real number line there is associated one and only one number of this form by a process of successive locations in decimal intervals of decreasing length. The drawings below illustrate the first few steps in finding the decimal corresponding to a point, P on the number line. Consider point P between 3 and 4.



Note that any two distinct points P_1 and P_2 will correspond to distinct decimal representations, for if they occur as



on the number line you need only subdivide the number line by a sufficiently fine decimal subdivision (tenths, hundredths, thousandths, etc.) to assure that P_1 and P_2 are separated by a point of subdivision.

Conversely, given any decimal, you have found how to locate the corresponding point of the real number line by considering successive rational decimal approximations provided by the number. (Remember how you started to locate the point 2.39614... in Section 5.)

Thus there is a one-to-one correspondence between the set of real numbers and the set of points on the number line.

The set of real numbers contains the set of rational numbers as a subset. You have learned that these rational numbers form a mathematical system with operations, addition and multiplication, and their inverses, subtraction and division. The same is true of the entire set of real numbers. You can add real numbers, rational or irrational, and you can multiply real numbers. The resulting number system has all of the properties of the rational number system. In addition it has one important property which the rational number system does not have. This will be discussed below.

The familiar properties which the real number system shares with the rational number system are listed first.

Property 1. Closure.

a) Closure under Addition. The real number system is closed under the operation of addition, i.e., if a and b are real numbers then $a + b$ is a real number.

b) Closure under Subtraction. The real number system is closed under the operation of subtraction (the inverse of addition), i.e., if a and b are real numbers then $a - b$ is a real number.

c) Closure under Multiplication. The real number system is closed under the operation of multiplication, i.e., if a and b are real numbers then $a \cdot b$ is a real number.

d) Closure under Division. The real number system is closed under the operation of division (the inverse of multiplication), i.e., if a and b are real numbers then $a \div b$ (when $b \neq 0$) is a real number.

The operations of addition, subtraction, multiplication, and division on real numbers display the properties which you have already observed for rationals. These may be summarized as follows:

Property 2. Commutativity.

a) If a and b are real numbers, then $a + b = b + a$.

b) If a and b are real numbers, then $a \cdot b = b \cdot a$.

Property 3. Associativity.

a) If a, b, and c are real numbers, then $a + (b + c) = (a + b) + c$.

b) If a, b, and c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Property 4. Identities.

a) If a is a real number, then $a + 0 = a$, i.e., zero is the identity element for the operation of addition.

b) If a is a real number, then $a \cdot 1 = a$, i.e., one is the identity element for the operation of multiplication.

Property 5. Distributivity. If a, b, and c are real numbers, then $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Property 6. Inverses.

a) If a is a real number, there is a real number (^-a), called the additive inverse of a such that $a + (^-a) = 0$.

b) If a is a real number and $a \neq 0$ there is a real number b, called the multiplicative inverse of a such that $a \cdot b = 1$.

Property 7. Order. The real number system is ordered, i.e, if a and b, are different real numbers then either $a < b$ or $a > b$.

Property 8. Density. The real number system is dense, i.e., between any two distinct real numbers there is always another real number. Consequently, between any two real numbers you can find as many more real numbers as you wish. In fact you can easily see that: 1) There is always a rational number between any two distinct real numbers, no matter how close. 2) There is always an irrational number between any two distinct real numbers, no matter how close.

The ninth property of the system of real numbers is one which is not shared by the rationals.

Property 9. Completeness The real number system is complete, i.e., to each point on the number line there corresponds a real number, and, conversely, to each real number there corresponds a point on the number line.

You saw that in the system of rationals there is no number $\sqrt{2}$ which when squared yields 2. However, in the real number system as defined, such a number is included.

If a and b are positive and $b = a^n$ we write $a = \sqrt[n]{b}$ (read " a is an nth root of b "). Since $\frac{a^n}{b^n} = \frac{a \cdot a \cdot a \ldots a}{b \cdot b \cdot b \ldots b}$ $= \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b} = \left(\frac{a}{b}\right)^n$, $\frac{a}{b}$ is an nth root of $\frac{a^n}{b^n}$. The number $\frac{a^n}{b^n}$ is called a perfect nth power. In general, if $\frac{c}{d}$ is a positive rational number written in such a way that c and d are counting numbers with no common factors other than 1, then $\frac{c}{d}$ is called a perfect nth power if and only if both c and d are nth powers of some counting numbers. The number $\frac{8}{27}$ is a perfect 3rd power, the number $\frac{32}{50}$ is a perfect 2nd power (since it can be written as $\frac{16}{25}$), but the number $\frac{5}{17}$ is not a perfect nth power for any n greater than 1.

It happens that the nth root of any positive rational number which is not itself a perfect nth power is an irrational number. This means that such numbers as $\sqrt{3}$, $\sqrt[3]{\frac{5}{17}}$, $\sqrt[4]{15}$, $\sqrt{\frac{2}{3}}$ are irrational numbers whereas $\sqrt{25}$, $\sqrt[4]{16}$, $\sqrt{\frac{81}{144}}$, $\sqrt[3]{8}$ are rational numbers. Hence, in the system of rationals you cannot hope to extract nth roots of any numbers which are not perfect nth powers. However, when the irrationals are included to form the real number system, nth roots of positive rational numbers can be found. Thus a very useful property of the real number system is:

The real number system contains nth roots, $\sqrt[n]{\frac{a}{b}}$ of all positive rational numbers $\frac{a}{b}$, $b \neq 0$.

This assures us that among the real numbers there can be found numbers as $\sqrt{3}$, $\sqrt{7}$, $1 + \sqrt{5}$, $\sqrt[3]{4}$, $\sqrt[5]{23}$ and any other nth roots of positive rational numbers.

In addition to irrational numbers which arise from finding roots of rational numbers there are many more irrational numbers which are called transcendental numbers. One example of a transcendental number is the number π which you have met in your study of circles. Recall that π is the ratio of the measure of the circumference of a circle to the measure of its diameter. It is surprisingly hard to prove that π is irrational, but it has been done. The decimal representation

$$\pi = 3.14159265 \ . \ . \ .$$

cannot be repeating. The number π is <u>not</u> $\frac{22}{7}$, although $\frac{22}{7}$ is a fair approximation to π. (Compare the decimal representation of $\frac{22}{7}$ with that of π.)

When you studied logarithms in high school, you were studying numbers which are almost all transcendental numbers. If N is any positive real number and x is the exponent such that

$$10^x = N$$

then it is said that x is the logarithm of N to the base 10. If N is a power of 10, say $N = 10^2$, then clearly $10^x = 10^2$, so 2 is the logarithm of 10^2 to the base 10. In such a case, the logarithm is a rational number. But for most numbers the logarithm will be a (transcendental) irrational number.

The trigonometric ratios, sine of an angle, and tangent of an angle are other expressions which usually turn out to be transcendental irrational numbers. These ratios are defined in a later chapter.

Exercises 6-8

1. Which of the following numbers do you think are rational and which irrational? Make two lists.
   - (a) $0.231\overline{231}\ldots$
   - (b) $0.23123112311123\ldots$
   - (c) $\frac{3\sqrt{2}}{7\sqrt{2}}$
   - (d) $\sqrt{7}$
   - (e) $0.78\overline{342}\ldots$
   - (f) $\frac{\pi}{2}$
   - (g) $\frac{3}{4}\sqrt{6}$
   - (h) $9 - \sqrt{3}$
   - (i) $0.7500\overline{0}\ldots$
   - (j) $\frac{58}{11}$
   - (k) $0.9595595559555559\ldots$
2. Write each of the rational numbers in Problem 1 as a decimal and as a fraction.
3. For each of the irrational numbers in Problem 1 write a decimal correct to the nearest hundredth.
4. (a) Make up 3 terminating decimals for rational numbers.
   (b) Make up 3 repeating decimals for rational numbers.
   (c) Make up 3 decimals for irrational numbers.

You have learned how to insert other rational numbers between two given rationals. Now that you have studied decimal representations for real numbers, you can see how to insert rational or irrational numbers between real numbers. Look at these decimals for two numbers a and b.

$$a = 4.219\overline{317}\ldots$$

$$b = 4.23401001000100001\ldots$$

These numbers are quite close together, but any decimal which begins 4.22... will be greater than a and less than b. You can then continue the decimal in such a way as to make it rational or or to make it irrational. For example, $4.225\overline{225}\ldots$ is rational and 4.225622566225666... is irrational.

5. (a) Write a decimal for a rational number between $2.3846\overline{846}\ldots$ and $2.369\overline{369}\ldots$

   (b) Write ~~a decimal~~ for an irrational number between the numbers in (a).

6. Write decimals for (a) a rational number and (b) an irrational number between 0.346019... and 0.342806...

7. Write decimals for (a) a rational number and (b) an irrational number between 67.283... and 67.28106006...

8. Do you think that the real number system contains square roots of all integers? Support your answer by an example.

9. An approximation which the Babylonians used for π was the interesting ratio $\frac{355}{113}$. How good an approximation is this? Is it as good as $\frac{22}{7}$?

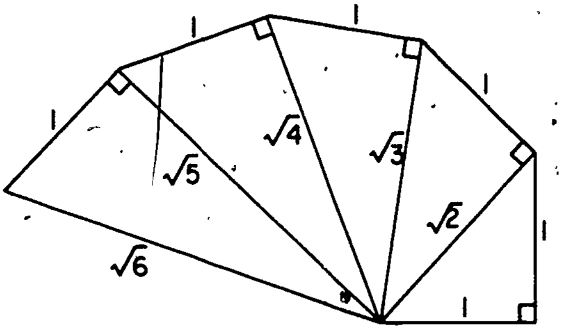
6-9 Irrational Numbers in the World Around Us

You see many examples of rationals every day -- the price of groceries, the amount of a bank balance, the rate of pay, the amount of a weekly salary, the grade on a test paper.

Although the irrationals have not been considered for very long, it is easy to see many examples which involve irrational numbers. For example, consider a circle of radius one unit. What is its circumference? Why, 2π units, of course. In fact, any circle whose radius is a rational number has a circumference which is irrational. Also, the circular closed region of radius r has an area, the measure of which is an irrational number (πr^2).

The volume of a circular cylinder is found by the formula $V = \pi r^2 h$ and its lateral surface area A by $A = 2\pi rh$ where h is the altitude of the cylinder. Here also the volume and area are given by irrational numbers if the radius r and altitude h are given as rationals.

Also, you will learn how to construct lengths of irrational measure by the following simple succession of right triangles:



Exercises 6-9

1. Which of the following numbers are rational and which are irrational?

   The number of units in:

   (a) the circumference of a circle whose radius is $\frac{1}{2}$ unit.

   (b) the area of a square whose sides are one unit long.

   (c) the hypotenuse of a right triangle whose sides are 5 and 12 units long.

   (d) the area of a square whose sides have length $\sqrt{3}$ units.

   (e) the volume of a cylinder whose height is 2 units and whose base has radius 1 unit.

   (f) the area of a right triangle with hypotenuse of length 2 units and equal sides.

2. With the use of the facts that $\sqrt{2} \approx 1.414$ and that $\sqrt{3} \approx 1.732$ show that $\sqrt{2} \cdot \sqrt{3} \approx \sqrt{6} \approx 2.449$.

3. When you begin to compute with irrational numbers you sometimes encounter relationships which look rather peculiar at first but which make perfect sense on closer inspection.

   Here are two examples:

   The multiplicative inverse of $\sqrt{2}$ is $\frac{1}{2}\sqrt{2}$.

   The multiplicative inverse of $(\sqrt{3} + \sqrt{2})$ is $(\sqrt{3} - \sqrt{2})$.

   *(a) Verify these assertions approximately by using the decimal approximations given in Problem 2.

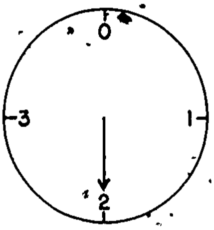
   *(b) Verify these assertions exactly by computing with the irrational numbers themselves.

*4. Find the radius of a circle whose circumference is 2. Give an approximate value for the radius. (Use 3.1416 for π).

Appendix

MATHEMATICAL SYSTEMS

A-1. A New Kind of Addition.



The sketch above represents the face of a four-minute clock. Zero is the starting point and, also, the end-point of a rotation of the hand.

With the model you might start at 0 and move to a certain position (numeral), and then move on to another position just like the moving hand of a clock. For example, you may start with 0 and move $\frac{2}{4}$ of the distance around the face. You would stop at 2. If you follow this by a $\frac{1}{4}$ rotation (moving like the hand of a clock), you would stop at 3. After a rotation of $\frac{2}{4}$ from 0 you could follow with a $\frac{3}{4}$ rotation. This would bring you to 1. The first example could be written 2 + 1 gives 3 where the 2 indicates $\frac{2}{4}$ of a rotation from 0, the + means to follow this by another rotation (like the hand of a clock), and the 1 means $\frac{1}{4}$ rotation, thus you arrived at the position marked 3 (or $\frac{3}{4}$ of a rotation from 0). The second example would be 2 + 3 gives 1 where the 2 and + still mean the same as in the first example and the 3 means a rotation of $\frac{3}{4}$. A common way to write this is:

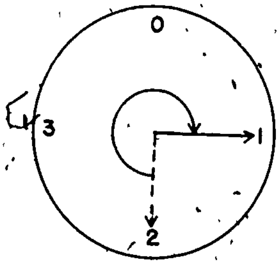
$$2 + 3 \equiv 1 \pmod{4}$$

which is read:

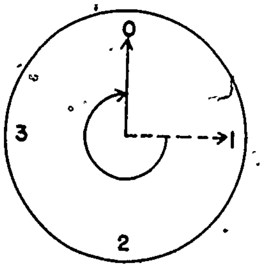
> Two plus three is equivalent to one (mod 4).

The (mod 4) means that there are four numerals: 0, 1, 2, 3 on the face of the clock. The + sign means what is described above - this is our new type of addition. The ≡ between the 2 + 3 and the 1 indicates that 2 + 3 and 1 are the same (that is, "equivalent") <u>on this clock</u>. This is called briefly "addition (mod 4)." Of course there are other possible notations which could be used but this is the usual one. The expression "(mod 4)" is derived from the fact that sometimes 4 is called "the modulus" which indicates how many single steps are taken before repeating the pattern.

Example. Find (2 + 3) + 3 (mod 4).



$2 + 3 \equiv 1 \pmod 4$



$1 + 3 \equiv 0 \pmod 4$

$(2 + 3) + 3 \equiv 1 + 3 \equiv 0 \pmod 4$

The following table illustrates some of the addition facts in the (mod 4) system.

(Mod 4)

+	0	1	2	3
0	0	1	2	
1			3	0
2				1
3		0		

A table of this sort is read by following across horizontally from any entry in the left column, for instance 2, to the position below some entry in the top row, such as 3 (see arrows). The entry in this position in the table is then taken as the result of combining the element in the left column with the element in the top row (in that order). In the case above you can write $2 + 3 \equiv 1 \pmod{4}$. Use the table to check that $3 + 1 \equiv 0 \pmod{4}$.

Example 3. Complete the following number sentences to make them true statements.

(a) $3 + 4 \equiv ? \pmod{5}$
The mod 5 system represented by the face of a clock should have five positions; namely, 0, 1, 2, 3, and 4. If you draw this clock you will see that $3 + 4 \equiv 2 \pmod{5}$ since the 3 means a rotation of $\frac{3}{5}$ from 0. This is followed by a $\frac{4}{5}$ rotation which ends at 2.

(b) $2 + 3 \equiv ? \pmod{5}$
$2 + 3 \equiv 0 \pmod{5}$. This is a $\frac{2}{5}$ rotation from 0 followed by a $\frac{3}{5}$ rotation which brings us to 0.

Exercises A-1

1. Copy and complete the table for addition (mod 4). Use it to complete the following number sentences:

(a) $1 + 3 \equiv ? \pmod{4}$
(c) $2 + 2 \equiv ? \pmod{4}$
(b) $3 + 3 \equiv ? \pmod{4}$
(d) $2 + 3 \equiv ? \pmod{4}$

2. Make a table for addition (mod 3) and for addition (mod 5).

3. Use the tables in Problem 2 to find the answers to the following:

(a) $1 + 2 \equiv ? \pmod{3}$
(c) $2 + 2 \equiv ? \pmod{3}$
(b) $3 + 3 \equiv ? \pmod{5}$
(d) $4 + 3 \equiv ? \pmod{5}$

4. Make whatever tables you need to complete the following number sentences.

   (a) $5 + 3 \equiv ? \pmod{6}$

   (b) $5 + 5 \equiv ? \pmod{6}$

   (c) $3 + 6 \equiv ? \pmod{7}$

   (d) $4 + 5 \equiv ? \pmod{7}$

   Note: be sure to keep all the tables you have made. You will find use for them later in this chapter.

5. Find a replacement for x to make each of the following number sentences a true statement.

   (a) $3 + x \equiv 2 \pmod{5}$

   (b) $x + 4 \equiv 3 \pmod{5}$

   (c) $x + 2 \equiv 0 \pmod{3}$

   (d) $4 + x \equiv 4 \pmod{5}$

6. Seven hours after eight o'clock is what time? What new kind of addition did you use here?

7. Nine days after the 27th of March is what date? What new kind of addition did you use here?

A-2. A New Kind of Multiplication.

Before considering a new multiplication let us look at a part of a multiplication table for the whole numbers. Here it is:

x	0	1	2	3	4	5	↓ 6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	8	10	12
3	0	3	6	9	12	15	18
4	0	4	8	12	16	20	24
→ 5	0	5	10	15	20	25	30
6	0	6	12	18	24	30	36

If you had this table and forgot what 5 times 6 is equal to, you could look in the row labeled 5 and the column labeled 6 and find the answer, 30, in the 5-row and 6-column (see arrows above). Of course it is easier to memorize the table since you use it so frequently, but if you had not memorized it, it might be a very convenient thing to have in our pocket for easy reference.

How would you make such a table if you didn't know it already? This would be quite easy if you could add. The first line is very easy — you write a row of zeros. For the second line you merely have to know how to count. For the third line you add 2 each time; for the fourth line add 3 each time, and so forth.

Now, if you use the same method, you can get a multiplication table (mod 4). First block it out, filling in the first and second rows and columns:

(Mod 4)

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
→ 2	0	2		
3	0	3		

There are just four blanks to fill in. To get the 2-row (indicated by the arrow above), add twos. Thus the third entry (which is 2×2) is $2 + 2 \equiv 0 \pmod 4$. Then our first three entries will look like this:

$$2 \mid 0 \quad 2 \quad 0$$

To get the fourth entry, add 2 to the third entry. Since $0 + 2 \equiv 2 \pmod 4$, the complete 2-row is now

$$2 \mid 0 \quad 2 \quad 0 \quad 2.$$

For the last row you will have to add threes. Here $3 + 3 \equiv 2 \pmod{4}$ and $3 + 2 \equiv 1 \pmod{4}$. The complete table is:

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

The last row, for instance, is obtained by adding 3 successively (mod 4).

Now consider one way in which this table could be used. Suppose a lamp has a four-way switch so that it can be turned to one of four positions: off, low, medium, high. You might let numbers correspond to these positions as follows:

off	low	medium	high
0	1	2	3

If the light were at medium and you flicked the switch three times, the light would be at the low position since $2 + 3 \equiv 1 \pmod{4}$. Suppose the light were off and three people flicked the switch three times each; what would be the final position of the light? The answer would be "low" since $3 \cdot 3 \equiv 1 \pmod{4}$ and the number 1 corresponds to "low."

Exercises A.-2

1. (a) Make a table for multiplication (mod 5).
   (b) Make a table for multiplication (mod 7).
   (c) Make a table for multiplication (mod 6).

   Note: Keep these tables for future use.

2. Complete the following number sentences to make them true statements. You may find the tables you constructed in Problem 1 useful.

(a) $3 \times 2 \equiv ? \pmod 5$ (d) $1 + (3 \times 4) \equiv ? \pmod 6$

(b) $3 \times 4 \equiv ? \pmod 6$ (e) $5 + (6 \times 5) \equiv ? \pmod 7$

(c) $6 \times 4 \equiv ? \pmod 7$

3. Find a replacement for x to make each of the following number sentences a true statement: (Draw the clocks which you need.)

(a) $5 \cdot 10 \equiv x \pmod{11}$ (c) $(3 \cdot 4) + 2 \equiv x \pmod 8$

(b) $7 \cdot 13 \equiv x \pmod{15}$ (d) $(4 \cdot 7) + 11 \equiv x \pmod{13}$

4. Form a table of remainders after division by 5, where the entry in any row and column is the remainder after the product is divided by 5. For instance, since the remainder is 1 when $2 \cdot 3$ is divided by 5, there will be a 1 in the 2-row and 3-column (see arrows). Complete the table:

	0	1	2	↓ 3	4
0	0	0	0	0	0
1	0	1	2	3	4
→ 2	0	2		1	3
3	0	3		4	
4	0	4	3		

5. Do you notice any relationship between the table of Problem 5 and another you have found? Can you give any reason for this? How can this be used to make the solution of some of the problems simpler?

*6. Use the multiplication table (mod 5) to find the replacement for x to make each of the following number sentences a true statement:

(a) $3x \equiv 1 \pmod 5$

(b) $3x \equiv 2 \pmod 5$

(c) $3x \equiv 3 \pmod 5$

(d) $3x \equiv 4 \pmod 5$

(e) $3x \equiv 0 \pmod 5$

*7. If it were (mod 6) instead of (mod 5) in the previous problem, would you be able to find x in each case? If not, which equivalences would give some value of x?

8. A jug of juice lasts three days in the Willcox family. One Saturday Mrs. Willcox bought six jugs which the family started using on the following day. What day of the week would it be necessary to purchase juice again? Answer this question using numbers (mod 7).

A-3. What is an Operation?

You are familiar with the operations of ordinary arithmetic--addition, multiplication, subtraction and division of numbers. In Section A -1, a different operation was discussed. You made a table for the new type of addition of the numbers 0, 1, 2, 3. This operation is completely described by the table that you made in Problem 1 of Exercises A -1. That is, there are no numbers to which the operation is applied except those indicated and the results of the operation on all pairs of these numbers are given. The table tells what numbers can be put together. For instance, the table tells us that the number 5 cannot be combined with any number in the new type of addition since "5" does not appear in the left column nor in the top row. It also tells us that $2 + 3 \equiv 1 \pmod 4$. Here are some other tables.

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(c)

□	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

(d)

⊙	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

(e)

Δ	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

In each case you have a set of elements: in (a) the set is $\{1, 2, 3, 4, 5\}$; in (d) it is $\{1, 2, 3\}$. You also have an operation: in (a) it is $+$; in (d) it is $\odot$. Finally you have the result of combining any two elements by mean of the operation; in (a) the results are 1, 2, 3, 4 or 5; in (c) they are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. All of these operations are called <u>binary operations</u> because they are applied to two elements to get a third. So far the elements have been numbers but you shall see later that they do not need to be.

The two elements which you combine may be the same one and the result of the operation may or may not be an element of the set but it must be something definite - not one of several possible things.

You are already familiar with some operations defined on the set of whole numbers.

Any two whole numbers can be added. Addition of 8 and 2 gives 10.

Any two whole numbers can be multiplied. Multiplication of 8 and 2 gives 16.

Addition and multiplication are two different operations defined on the set of whole numbers.

When an operation is described by a table, the elements of the set are written in the same order in the top row (left to right) and in the left column (top to bottom). Keeping the order the same will make some of our later work easier.

You must also be careful about the order in which two elements are combined. For example,

$$2 \square 1 = 5, \quad \text{but} \quad 1 \square 2 = 4.$$

For this reason, you must remember that when the procedure for reading a table was explained, it was decided to write the element in the left column first and the element in the top row second with the symbol for the operation between them. You must examine each new operation to see if it is commutative and associative. These properties have been discussed in previous chapters; they are briefly reviewed here.

An operation $+$ defined on a set is called <u>commutative</u> if, for any elements, a, b, of the set, $a + b = b + a$.

An operation $+$ defined on a set is called <u>associative</u> if any elements, a, b, c, of the set can be combined as $(a + b) + c$, and also as $a + (b + c)$, and the two results are the same: $(a + b) + c = a + (b + c)$.

Exercises A-3

1. Use the tables at the beginning of this section to answer the following questions:

   (a) $3 + 3 = ?$ if you use Table (a).

   (b) $3 + 3 = ?$ if you use Table (b).

   (c) $3 \square 2 = ?$

   (d) $2 \square 3 = ?$

   (e) $2 \odot 2 = ?$

   (f) $1 \odot 1 = ?$

   (g) $(2 \odot 3) \odot 3 = ?$

   (h) $2 \odot (3 \odot 3) = ?$

   (i) $(1 \square 1) \square 2 = ?$

   (j) $1 \square (1 \square 2) = ?$

   (k) $2 \Delta (3 \Delta 4) = ?$

   (l) $(2 \Delta 3) \Delta 4 = ?$

2. (a) Which of the binary operations described in the tables in this section are commutative?

   (b) Is there an easy way to tell if an operation is commutative when you examine the table for the operation? What is it?

3. How can you tell if an operation is associative by examining a table for the operation? Do you think the operations described in the tables in this section are associative?

4. Are the following binary operations commutative? Make at least a partial table for each operation. Which ones do you think are associative?

   (a) Set: All counting numbers between 25 and 75.
   Operation: Choose the smaller number.
   Example: 28 combined with 36 produces 28.

   (b) Set: All counting numbers between 500 and 536.
   Operation: Choose the larger number.

   (c) Set: The prime numbers.
   Operation: Choose the larger number.

   (d) Set: All even numbers between 39 and 61.
   Operation: Choose the first number.

   (e) Set: All counting numbers less than 50.
   Operation: Multiply the first by 2 and then add the second.

   (f) Set: All counting numbers.
   Operation: Find the greatest common factor.

   (g) Set: All counting numbers.
   Operation: Find the least common multiple.

   (h) Set: All counting numbers.
   Operation: Raise the first number to a power whose exponent is the second number.
   Example: 5 combined with 3 produces 5^3.

5. Make a table for an operation that has the commutative property.

6. Make up a table for an operation that does not have the commutative property.

The word "binary" indicates that two elements are combined to produce a result. (Note that the word "combine" is used with any operation, not just with addition.) There are other kinds of operations. A result might be produced from a single element, or by combining three or more elements. When you have a set and, from any one element of the set, you can determine a definite thing, it is said there is a "unary operation" defined on the set. If you combined three elements to produce a fourth it would be called a "ternary operation". One example of a ternary operation would be finding the G.C.F. of three counting numbers: e.g. the G.C.F. of 6, 8, and 10 is 2.

*7. Try to show a way of describing the following unary operation by some kind of a table?
Set: All the whole numbers from 0 to 10.
Unary Operation: Cube the number.
Example: Doing the operation to 5 produces $5^3 = 125$.

A-4. Closure.

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(c)

□	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

(d)

⊙	1	2	3
1	3	1	2
2	1	2	3
3	2	3	1

Study the Tables (a) and (b). In Table (a) the numbers in the body of the table resulting from performing the operation are the same numbers which were combined by the operation, (1, 2, 3, 4, 5). But in Table (b) the numbers in the body of the table resulting from performing the operation were different numbers from those combined (6, 8, 10, etc. instead of 3, 5, 7, 9). You have seen this kind of difference before, and there is a name for it. It has been said the set of whole numbers is "closed under addition" because if any two whole numbers are combined by adding them, the result is a whole number. In the same way the set {1, 2, 3, 4, 5} in Table (a) is closed under the new type of addition there since the results of the operation are again in the same set.

However, the set of odd numbers is not closed under addition since the result of adding two odd numbers is not an odd number. In the same way, in Table (b) the set 3, 5, 7, 9 is not closed under the operation of addition given there since the result is not one of the set {3, 5, 7, 9}.

Example 1:

The set of whole numbers: {1, 2, 3, 4} is not closed under multiplication because $2 \cdot 3 = 6$ which is not one of the set. Of course $1 \cdot 2 = 2$ is in the set but for a set to be closed the result must be in the set no matter what numbers of the set are combined.

Example 2:

The set of whole numbers is not closed under subtraction. For example, consider the two whole numbers 6 and 9. There are two different ways you can put these two numbers together using subtraction: 9 - 6 and 6 - 9. The first numeral, "9 - 6", is a name for the whole number 3, but the numeral "6 - 9" is not the name of any whole number. Thus, subtracting two whole numbers does not always give a whole number.

Example 3:

The set of counting numbers is not closed under division. It is true that $\frac{8}{2} = 8 \div 2$ is a counting number, but there is no counting number $\frac{9}{2}$. Can you give some other illustrations of closure, that is, sets closed under an operation and sets not closed under an operation?

Exercises A-4

1. Study again Tables (a) - (d) in this section. Which tables determine a set that is closed under the operation? Which tables determine a set that is not closed under the operation? How do you know?

2. Which of the sets below are closed under the corresponding operations?

   (a) The set of even numbers under addition.

   (b) The set of even numbers under multiplication.

   (c) The set of odd numbers under multiplication.

   (d) The set of odd numbers under addition.

   (e) The set of multiples of 5 under addition.

   (f) The set of multiples of 5 under subtraction.

   (g) The set $\{1, 2, 3, 4\}$ under multiplication (mod 5).

   (h) The set of counting numbers less than 50 under the operation of choosing the smaller number.

   (i) The set of prime numbers under addition.

   (j) The set of numbers whose numerals in base five end in "3" under addition.

3. Find the smallest set of counting numbers.

   (a) Closed under addition and containing 2.

   (b) Closed under multiplication and containing 2.

4. Is the set of all whole numbers closed under any of the following operations: multiplication, addition, subtraction, division? Give reasons.

5. Consider the numbers 1 and 2. If a set containing three numbers is to be closed under multiplication, it must contain $2 \cdot 2$, that is, 4. Then it must have $2 \cdot 4$ and $4 \cdot 4$. In fact it must have all the numbers:

   (S) 1, 2, 4, 8, 16, 32, ...,

   that is, all numbers expressible in the form 2^k where k is a natural number, as well as the number 1 itself. It is said that 2 "generates" a set of numbers, S, under multiplication.

   (a) Find the set generated by 7 under addition.
   (b) Find the set generated by 7 under multiplication.

6. Let S be the set determined by Table (d) in this section. Find the subset of S which is generated by 1 under $\odot$. Find the subset of S which is generated by 2 under $\odot$.

*7. What subset of the set of rational numbers is generated by 3? Is this set closed under division? (Is 3 in the set? Is $\frac{1}{3}$ in the set? Is $3 \div \frac{1}{3}$ in the set?) Does $(3 \div 3) \div 3 = 3 \div (3 \div 3)$? Is the division operation associative?

*8. If an operation defined on a set is commutative, must the set be closed under the operation?

*9. If an operation defined on a set is associative, must the set be closed under the operation?

*10. Make up a table for an operation defined on the set {0, 43, 100} so that the set is closed under the operation.

*11. Make up a table for an operation defined on the set {0, 43, 100} so that the set is not closed under the operation.

A-5. Identity Element; Inverse of an Element.

In the study of the number one in ordinary arithmetic, you observed that the product of any number and 1 (in either order) is the same number.

For instance

$$2 \times 1 = 2, \quad 1 \times 2 = 2, \quad 156 \times 1 = 156, \quad 1 \times 156 = 156.$$

For any number n in the arithmetic of rational numbers, $n \cdot 1 = n$ and $1 \cdot n = n$.

In the study of the number zero in the arithmetic of rational numbers you observed that the sum of 0 and any number (in either order) gave that same number; that is the sum of any number and 0 is the number. For instance

$$2 + 0 = 2, \quad 0 + 2 = 2, \quad 468 + 0 = 468, \quad 0 + 468 = 468$$

For any number n in ordinary arithmetic, $n + 0 = n$ and $0 + n = n$.

One is the identity for multiplication in ordinary arithmetic.
Zero is the identity for addition in ordinary arithmetic.

Suppose $*$ stands for a binary operation. Some possibilities for $*$ are the following:

1. If $*$ means addition of rational numbers, 0 is an identity element because $0 * a = a = a * 0$ for any rational number, a.
2. If $*$ means multiplication of rational numbers, 1 is an identity element because $1 * a = a = a * 1$ for any rational number.

3. If $*$ means the greater of two counting numbers, then $1 * 2 = 2$ because 2 is greater than 1; $1 * 3 = 3$ because 3 is greater than 1; $1 * 4 = 4$ since 4 is greater than 1, etc. In fact

$$1 * a = a \text{ and}$$
$$a * 1 = a$$

no matter what counting number a is. So 1 is the identity for this meaning of the operation $*$.

This could be stated formally as follows: If $*$ stands for a binary operation on a set of elements and if there is some element, call it e, which has the property that

$$e * a = a \text{ and}$$
$$a * e = a$$

for every element a of the set, then e is called an identity element of the operation $*$.

As another example consider the following table for an operation which is called #.

#	A	B	C	D
A	B	C	D	A
B	C	D	A	B
C	D	A	B	C
D	A	B	C	D

Is there an identity element for #? Could it be A? Is $A \# B = B$? (Read "A sharp B equals B"). Since, from the table $A \# B = C$, the answer to the question is "no" and you see that A cannot be the identity. Neither can B be the identity since A # B is not A. However, D is an identity for # , since

$$\begin{aligned} A \# D = D \# A &= A, \\ B \# D = D \# B &= B, \\ C \# D = D \# C &= C, \\ D \# D &= D. \end{aligned}$$

Compare the column under D with the column under the # . Compare the row to the right of #D with the row to the right of the # . What do you notice? Does this suggest a way to look for an identity element when you are given a table for the operation?

If you have an identity element then you may also have what is called an inverse element. If the operation is multiplication for rational numbers, the identity is 1 and the two rational numbers a and b are called inverses of each other if their product is 1, that is, if each is the reciprocal of the other.

Suppose the operation is addition (mod 4). Here 0 is the identity element and two numbers are called inverses if their sum is 0, that is, if combining the two numbers by the operation gives 0. To find inverses (mod 4) for addition, from the table:

0 is the identity

$2 + 2 \equiv 0 \pmod{4}$

$3 + 1 \equiv 0 \pmod{4}$

$1 + 3 \equiv 0 \pmod{4}$

(Mod 4)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Here 0 is its own inverse, 2 is its own inverse, and 3 and 1 are inverses of each other.

Definition. Two elements a and b are inverses (or either one is the inverse of the other) under a binary operation * with identity element e if a * b = e and b * a = e.

Exercises A-5a

1. Study tables (a) - (d) in Section A-3.

   (a) Which tables describe operations having an identity and what is the identity?

   (b) Pick out pairs of elements which are inverses of each other under these operations. Does each member of the set have an inverse?

2. For each of the operations of Problem 4, Exercises A.-3;

   (a) Does the operation have an identity and, if so, what is it?

   (b) Pick out pairs of elements which are inverses of each other under these operations.

   (c) For which operations does each element have an inverse?

*3. Can there be more than one identity element for a given binary operation?

You have just had an example of an additive inverse. If the operation is multiplication the inverse is called a multiplicative inverse.

Recall the (mod 5) multiplication.

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

How would you decide what elements of the set {0, 1, 2, 3, 4} have (multiplicative) inverses in this mathematical system? The identity for multiplication (mod 5) is 1. You would be looking for products which are the identity, so you should look for ones in the table. There are 4 ones in the table. They tell us that $1 \cdot 1 \equiv 1 \pmod 5$, $2 \cdot 3 \equiv 1 \pmod 5$, $3 \cdot \_\_\_ \equiv 1 \pmod 5$, and $4 \cdot \_\_\_ \equiv 1 \pmod 5$. (You supply the missing numbers.) Thus the multiplicative inverse of 2 in (mod 5) is 3. What is the multiplicative inverse of 3 in (mod 5)? of 4?

Division is called the inverse operation for multiplication because dividing by a number a has the same effect as multiplying by the multiplicative inverse of a.

Exercises A-5b

1. (a) Use the multiplication table for (mod 6) to find, wherever possible, a replacement for x to make each of the following number sentences a true statement:

$1 \cdot x \equiv 1 \pmod 6$ $4x \equiv 1 \pmod 6$

$2x \equiv 1 \pmod 6$ $5x \equiv 1 \pmod 6$

$3x \equiv 1 \pmod 6$

(b) Which elements of the set {0, 1, 2, 3, 4, 5} have multiplicative inverses in (mod 6)?

2. Remember that division is defined as the inverse operation for multiplication. Thus, in the arithmetic of rational numbers, the question "Six divided by two is what?" means, really, "Six is obtained by multiplying two by what?" Division (mod n) can be defined in this way.

$$6 \div 4 \equiv ? \pmod 5 \qquad \text{means}$$
$$(4)(?) \equiv 6 \pmod 5$$

Using the multiplication table (mod 5) find $3 \div a \pmod 5$ for $a = 1, 2, 3, 4$ and then find $3 \cdot \frac{1}{a} \pmod 5$, for these four values of a. Show that the results are the same for each value of a.

3. Using the table for addition (mod 5), call (-a) the additive inverse of a and show that $3 - a \pmod 5$ and $3 + (-a) \pmod 5$ are equal for each value of a ($a = 1, 2, 3, 4$).

4. In the arithmetic of rational numbers which of the following sets is closed under division?

   (a) $\{1, 2, \frac{1}{2}\}$

   (b) $\{1, 2, 2^2, 2^3, \ldots\}$

   (c) The non-zero counting numbers.

   (d) The rational numbers.

5. (a) Which of the following sets is closed under multiplication (mod 6)?

   {0, 1, 2, 3, 4, 5}, {2, 4}, {0, 1, 5}, {1, 5}, {5}?

   (b) Which of the sets in (a) contain a multiplicative inverse (mod 6) for each of its elements?

   (c) Which of the sets in (a) is closed under division (mod 6)?

6. (a) Which of the sets {A, B}, {C, D}, {B, C, D}, {A, D} is closed under the operation * defined by the table below?

*	A	B	C	D
A	A	A	A	A
B	A	A	B	B
C	A	B	D	C
D	A	B	C	D

For instance, {A, D} is closed under the operation because if we pick out that part of the table we have the little table

*	A	D
A	A	A
D	A	D

which contains only A's and D's. On the other hand the set {A, C} is not closed since its little table would be

*	A	C
A	A	A
C	A	D

Here the table contains a D, which is not one of the set {A, C}.

(b) Is there an identity for *? If so, what is it?

(c) Which of the sets in (a) has an inverse under * for each of its elements?

*(d) Which of the sets in (a) is closed under the inverse operation for *? (You might use the symbols $\frac{*}{*}$ for this operation, so that $a \frac{*}{*} b = ?$ means $b * ? = a$.)

A-6. What Is a Mathematical System?

The idea of a set has been a very convenient one in this book -- some use has been made of it in almost every chapter. But there is really not a great deal that can be done with just a set of elements. It is much more interesting if something can be done with the elements (for instance, if the elements are numbers, they can be added or multiplied). If you have a set and an operation defined on the set, it is interesting to find out how the operation behaves. Is it commutative? associative? Is there an identity element? Does each element have an inverse? The "behavior" of the arithmetic operations (addition, subtraction, multiplication and division) on numbers was discussed in Chapters 3 and 4. You have seen that different operations may "behave alike" in some ways (both commutative, for instance). This suggests that sets with operations defined on them are studied to see what different possibilities there are. It is too hard for us to list all the possibilities, but some examples will be given in this section and the next. These are examples of mathematical systems.

> Definition. A mathematical system is a set of elements together with one or more binary operations defined on the set.

The elements do not have to be numbers. They may be any objects whatsoever. Some of the examples below and in the next section are concerned with letters or geometric figures.

Here are some tables.

(a)

○	A	B
A	A	B
B	A	B

(b)

*	P	Q	R	S
P	R	S	P	Q
Q	S	R	Q	P
R	P	Q	R	S
S	Q	P	S	R

(c)

~	△	□	○	\
△	△	□	○	\
□	□	○	\	△
○	○	\	△	□
\	\	△	□	○

Exercises A-6

1. Which one, or ones, of the Tables (a), (b), (c) describes a mathematical system? Show that your answer is correct.

2. Use the tables above to complete the following statements correctly.

(a) B o A = ?

(b) Δ ~ ○ = ?

(c) \ ~ \ = ?

(d) A o B = ?

(e) Q * R = ?

(f) R * S = ?

(g) P] * R = ?

(h) □ ~ ○ = ?

(i) \ ~ □ = ?

(j) B o B = ?

(k) A o A = ?

(ℓ) S * S = ?

3. Which one, or ones, of the binary operations o, *, ~ is commutative? Show that your answer is correct.

4. Which one, or ones, of the binary operations o, *, ~ has an identity element? What is it in each case?

5. Use the tables above to complete the following statements correctly.

(a) P * (Q * R) = ?

(b) (P * Q) * R = ?

(c) P * (Q * S) = ?

(d) (P * Q) * S = ?

(e) (R * P) * S = ?

(f) R * (P * S) = ?

(g) Δ ~ (Δ ~ \) = ?

(h) (Δ ~ Δ) ~ \ = ?

(i) (○ ~ □) ~ Δ = ?

(j) ○ ~ (□ ~ Δ) = ?

6. Does either of the operations described by Table (b) or Table (c) seem to be associative? Why? How could you prove your statement? What would another person have to do to prove you wrong?

7. (a) In Table (c) what set is generated by the element □ ?

(b) In Table (b) what set is generated by the element P?

*8. For each of the following tables, tell why it does not describe a mathematical system.

(a)

*	1	2
1	1	1
2	1	

(b)

*	1	2
1	the product of 3 and 6	the sum of 2 and 4
2	a number between 3 and 8	0

(c)

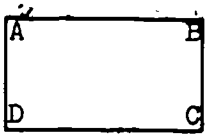
*	1	2
3	1	4
4	2	3

A-7. Mathematical Systems without Numbers.

In the last section there were some examples of mathematical systems without numbers in them. Suppose we want to invent one. What do you need?

You must have a set of things. Then, you need some kind of a binary operation -- something that can be done with any two elements of our set. It was found that the properties of closure, commutativity, associativity, etc. are very helpful in simplifying expressions. It would be nice to have some of these properties.

Let's start with a card. Any rectangular shaped card will do. It will be used to represent a closed rectangular region. Lay the card on your desk and label the corners as in the sketch. Now pick the card up and write the letter "A" on the other side (the side that was touching the desk) behind the "A" you have already written. Be sure the two letters "A" are back-to-back so they are labels for the same corner of the card. Similarly, label the corners B, C, and D on the other side of the card (be sure they're back-to-back with the B, C, and D you have already written.)

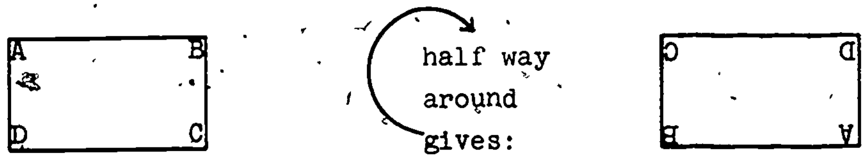


What set shall you take? Instead of numbers, take elements which have something to do with the card. Start with the card in the center of your desk and with the long sides of the card parallel to the front of your desk. Now move the card -- pick it up, turn it over or around in any way -- and put it back in the center of your desk with the long sides parallel to the front of your desk. The card looks just the same as it did before, but the corners may be labeled differently (a corner that started at the top may now be at the bottom; for instance). The position of the card has been changed, but the closed rectangular region looks as it did in the beginning. (The "picture" stays the same. Individual points may be moved.) The elements of our set will be these changes of position. Take all the changes of position that make the closed rectangular region look as it did in the beginning. (Long sides parallel to the front of the desk.) How many of these changes are there?

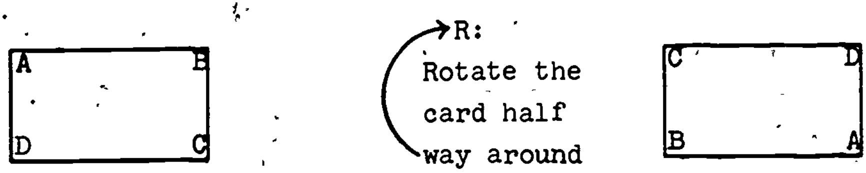
You may start with the card in some position which you will call the standard position. Suppose it looks like the figure below.

Leaving the card on your desk, rotate it half way around its center. A diagram of this change is:

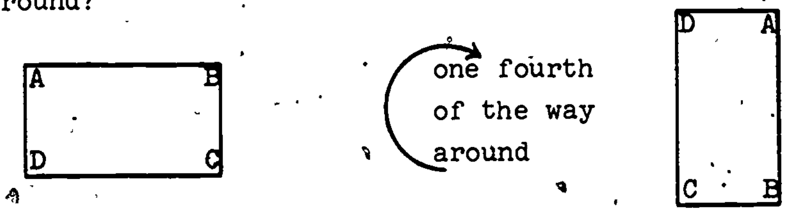
Standard Position



Since the letters "A", "B", etc. are only used as a convenience to label the different corners of the card, do not bother to write them upside down. The diagram below represents this change of position, and it will be called the change "R" (for rotation).

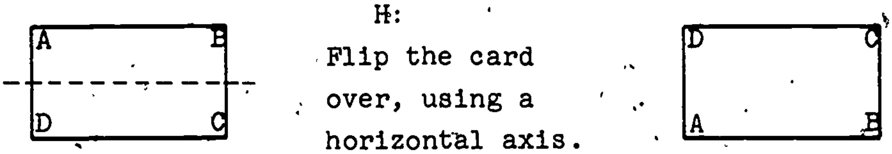


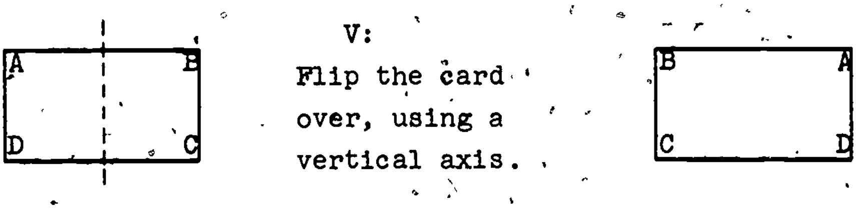
What would happen if the card were rotated one fourth of the way around?



Does the card look the same before and after the change? No, this change of position cannot be in our set, since the two pictures are quite different.

Are there other changes of position of the closed rectangular region which make it look the way it did in the beginning? Yes, youcan flip the card over in two different ways as shown by the diagrams below:

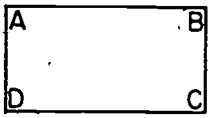

H:
Flip the card over, using a horizontal axis.


V:
Flip the card over, using a vertical axis.

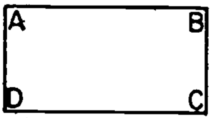
There is one more "change of position" - the "change" which is really no change at all, that, which leaves the card alone. This is called I and it is introduced for completeness sake.

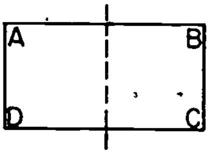
Now you have the set of elements; it is I, V, H, R.

Let us summarize what they are for easy reference:

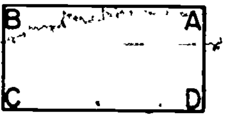


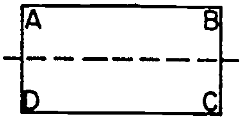
Element I:
Leave the card
in place.



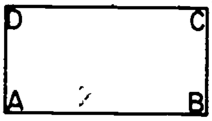


Element V:
Flip the card
over using a
vertical axis

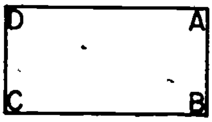




Element H:
Flip the card
over using a
horizontal axis



Element R:
Rotate the card
halfway around in
the direction
indicated



Recall the definition of a mathematical system. There were two requirements:

(a) A set of elements.

(b) One or more binary operations defined on the set of elements.

The set {I, V, H, R} satisfies the first condition. Now it is necessary to satisfy the second condition; you need an operation. What operation shall you use? How can you "combine any two elements of our set" to get a "definite thing"? If the set is to be closed under the operation, the "definite thing" which is the result of the operation should be one of the elements again.

Here is a way of combining any two elements of our set. You will do one of the changes AND THEN do the other one. You will use the symbol "ANTH" for this operation (perhaps you can think of a better one). Thus "H ANTH V" means flip the card over, using a horizontal axis, and then flip the card over, using a vertical axis." Start with the card in the standard position and do these changes to it. What is the final position of the card? Is the result of these two changes the same as the change R? What does "V ANTH H" mean? Try it with your card. Now you can fill in the table for the operation. Some of the entries are given in the table at the right.

ANTH	I	V	H	R
I	I	V		
V			R	H
H	H	R		
R			V	

Exercises A-7

1. Check the entries that are given in the table above and find the others. Use your card.
2. From your table for the operation ANTH, or by actually moving a card, fill in each of the blanks to make the equations correct.

   (a) R ANTH H = ?

   (b) R ANTH ? = H

   (c) ? ANTH R = H

   (d) ? ANTH H = R

(e) (R ANTH H) ANTH V = ?

(f) R ANTH (H ANTH V) = ?

(g) (R ANTH H) ANTH ? = V

(h) (R ANTH ?) ANTH V = H

(i) (? ANTH H) ANTH V = R

3. Examine the table for the operation ANTH.

(a) Is the set closed under the operation?

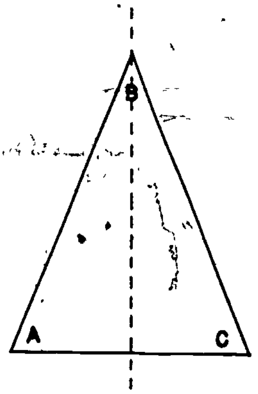
(b) Is the operation commutative?

(c) Do you think the operation is associative? Use the operation table to check several examples.

(d) Is there an identity element for the operation ANTH?

(e) Does each element of the set have an inverse under the operation ANTH?

4. Here is another system of changes. Cut a triangular card with two equal sides. Label the corners as in the sketch (both sides, back-to-back). The set for the system will consist of two changes. The first change, called I, will be: Leave the card in place. The second change, called F, will be: Flip the card over, using the vertical axis. F ANTH I will mean: Flip the card over, using the vertical axis, and then leave the card in place. How will the card look -- as if it had been left in place, I, or as if the change F had been done? What does I ANTH F mean? Does R ANTH I = F or does F ANTH I = I?



(a) Complete the table below:

ANTH	I	F
I		
F		

(b) Is the set closed under this operation?

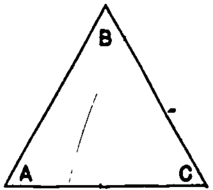
(c) Is the operation commutative?

(d) Is the operation associative? Are you sure?

(e) Is there an identity for the operation?

(f) Does each element of the set have an inverse under the operation?

5. Make a triangular card with three equal sides and label the corners as in the sketch (both sides, back-to-back). The set for this system will be made up of these six changes.



I: Leave the card in place.

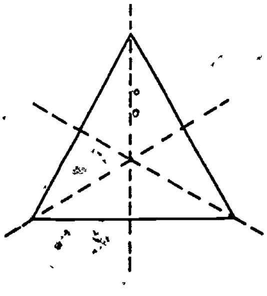
R: Rotate the card clockwise $\frac{1}{3}$ of the way around.

S: Rotate the card clockwise $\frac{2}{3}$ of the way around.

T: Flip the card over, using a vertical axis.

U: Flip the card over, using an axis through the lower right vertex.

V: Flip the card over, using an axis through the lower left vertex. Three of these will be rotations about the center (leave in place and two others). The other three will be flips about the axes. (Caution: the



axes are stationary; they do not rotate with the card. For example, the vertical axis remains vertical -- it would go through a different corner of the card after rotating the card one third of the way around its center.) Make a table for these changes. Examine the table. Is this operation commutative? Is there an identity change? Does each change have an inverse?

*6. Try making a table of changes for a square card. There are eight changes (that is, eight elements). What are they? Is there an identity element? Is the operation ANTH commutative?

A-8. The Counting Numbers and the Whole Numbers.

The mathematical systems that you have studied so far in this chapter are composed of a set and one operation. Examples are modular addition or multiplication and the changes of a rectangular or triangular card. A mathematical system given by a set and two operations would appear to be more complicated than these examples. However, as you may have guessed, ordinary arithmetic is also a mathematical system and you know that you can do more than one operation using the same set of numbers -- for examples, you can add and multiply.

To be definite, let us choose the set of rational numbers. This set, together with the two operations of addition and multiplication forms a mathematical system which was discussed in another chapter. Are there properties of this system which are entirely different from those you have considered in systems with only one operation? Yes, you are familiar with the fact that $2 \cdot (3 + 5) = (2 \cdot 3) + (2 \cdot 5)$. This is an illustration of the distributive property. More precisely, it illustrates that multiplication distributes over addition. The distributive property is also of interest in other mathematical systems.

Definition. Suppose you have a set and two binary operations, * and o, defined on the set. The operation * distributes over the operation o if a * (b o c) = (a * b) o (a * c) for any elements a, b, c, of the set. (And you can perform all these operations.) The exercises below contain examples which illustrate this definition.

In a mathematical system with two operations, there are the properties which were previously discussed for each of these operations separately. The only property which is concerned with both operations together is the distributive property.

Exercises A-8

1. Consider the set of counting numbers:
   (a) Is the set closed under addition? under multiplication? Explain.
   (b) Do the commutative and associative properties hold for addition? for multiplication? Give an example of each.
   (c) What is the identity element for addition? for multiplication?
   (d) Is the set of counting numbers closed under subtraction? under division? Explain.

   The answers to (a), (b), and (c) tell us some of the properties of the mathematical system composed of the set of counting numbers and the operations of addition and multiplication.

2. Answer the questions of Problem 1 (a), (b) (c) for the set of whole numbers. Are your answers the same as for the counting numbers?

3. (a) For the system of whole numbers, write three number sentences illustrating that multiplication distributes over addition.

   (b) Does addition distribute over multiplication? Try some examples.

4. The two tables below describe a mathematical system composed of the set {A, B, C, D} and the two operations * and o.

*	A	B	C	D
A	A	A	A	A
B	A	B	A	B
C	A	A	C	C
D	A	B	C	D

o	A	B	C	D
A	A	B	C	D
B	B	B	D	D
C	C	D	C	D
D	D	D	D	D

   (a) Do you think * distributes over o? Try several examples.

   (b) Do you think o distributes over *? Try several examples.

5. Answer these questions for each of the following systems. Is the set closed under the operation? Is the operation commutative? associative? Is there an identity? What elements have inverses?

   (a) The system whose set is the set of odd numbers and whose operation is multiplication.

   (b) The system whose set is made up of zero and the multiples of 3, and whose operation is multiplication.

   (c) The system whose set is made up of zero and the multiples of 3 and whose operation is addition.

(d) The system whose set is made up of the rational numbers between 0 and 1 (not including 0 and 1), and whose operation is multiplication.

(e) The system whose set is made up of the even numbers and whose operation is addition. (Zero is an even number.)

(f) The system whose set is made up of the rational numbers between 0 and 1, and whose operation is addition.

6. (a) In what ways are the systems of 5(b) and 5(c) the same

(b) In what ways are the systems of 5(a) and 5(b) different?

*7. Make up a mathematical system of your own that is composed of a set and two operations defined on the set. Make at least partial tables for the operations in your system. List the properties of your system.

*8. Here is a mathematical system composed of a set and two operations defined on that set.

Set: All counting numbers.

Operation *: Find the greatest common factor.

Operation o: Find the least common multiple.

(a) Does the operation * seem to distribute over the operation o? Try several examples.

(b) Does the operation o seem to distribute over the operation *? Try several examples.

9. Write the multiplication table (mod 8) and recall or write again the multiplication table (mod 5) which you found in Exercises 12-2.

10. Answer each of the following questions about the mathematical systems of multiplication (mod 5) and (mod 8).

(a) Is the set closed under the operation?

(b) Is the operation commutative?

(c) Do you think the operation is associative?

(d) What is the identity element?

(e) Which elements have inverses, and what are the pairs of inverse elements?

(f) Is it true that if a product is zero at least one of the factors is zero?

11. Complete each of the following number sentences to make it a true statement.

(a) $2 \times 4 \equiv ?\ (\text{mod } 5)$

(b) $4 \times 3 \equiv ?\ (\text{mod } 5)$

(c) $5^2 \equiv 1\ (\text{mod } ?)$

(d) $2^3 \equiv 0\ (\text{mod } ?)$

12. Find the products:

(a) $2 \times 3 \equiv ?\ (\text{mod } 4)$

(b) $2 \times 3 \equiv ?\ (\text{mod } 6)$

(c) $5 \times 8 \equiv ?\ (\text{mod } 7)$

(d) $3 \times 4 \times 6 = ?\ (\text{mod } 9)$

(e) $4^3 \equiv ?\ (\text{mod } 5)$

(f) $6^2 \equiv ?\ (\text{mod } 5)$

*(g) $6^{256} \equiv ?\ (\text{mod } 5)$

13. Find the sums:

(a) $1 + 3 \equiv ?\ (\text{mod } 5)$

(b) $4 + 3 \equiv ?\ (\text{mod } 5)$

(c) $2 + 4 \equiv ?\ (\text{mod } 5)$

(d) $4 + 4 \equiv ?\ (\text{mod } 5)$

14. (a) Find the values of $3(2 + 1)\ (\text{mod } 5)$ and $(3 \cdot 2) + (3 \cdot 1)\ (\text{mod } 5)$.

(b) Find the values of $4(3 + 1)\ (\text{mod } 5)$ and $(4 \cdot 3) + (4 \cdot 1)\ (\text{mod } 5)$.

(c) Find the values of $(3 \cdot 2) + (3 \cdot 4)\ (\text{mod } 5)$ and $3(2 + 4)\ (\text{mod } 5)$.

(d) In the examples of this problem is multiplication distributive over addition?

15. (a) Find the values of $3 + (2 \cdot 1) \pmod{5}$ and $(3 + 2) \cdot (3 + 1) \pmod{5}$.

(b) Find the values of $4 + (3 \cdot 1) \pmod{5}$ and $(4 + 3) \cdot (4 + 1) \pmod{5}$.

(c) Find the values of $(3 + 2) \cdot (3 + 4) \pmod{5}$ and $3 + (2 \cdot 4) \pmod{5}$.

(d) In the examples of this problem is addition distributive over multiplication?

*16. Find the quotients?

(a) $2 \div 3 \equiv ? \pmod{8}$

(b) $6 \div 2 \equiv ? \pmod{8}$

(c) $0 \div 2 \equiv ? \pmod{8}$

(d) $3 \div 4 \equiv ? \pmod{5}$

(e) $0 \div 2 \equiv ? \pmod{5}$

(f) $0 \div 4 \equiv ? \pmod{5}$

(g) $7 \div 3 \equiv ? \pmod{10}$

*(h) $7 \div 6 \equiv ? \pmod{8}$

17. Find the following; remember that subtraction is the inverse operation of addition.

(a) $7 - 3 \pmod{8}$

(b) $3 - 4 \pmod{5}$

(c) $3 - 4 \pmod{8}$

*(d) $4 - 9 \pmod{12}$

18. Make a table for subtraction mod 5. Is the set closed under the operation?

19. Find a replacement for x which will make each of the following number sentences a true statement. Explain.

(a) $2x \equiv 1 \pmod{5}$

(b) $3x \equiv 1 \pmod{4}$

(c) $3x \equiv 0 \pmod{5}$

(d) $3x \equiv 0 \pmod{6}$

(e) $x \cdot x \equiv 1 \pmod{8}$

(f) $4x \equiv 4 \pmod{8}$

20. In Problem 11 (d) and (f), find at least one other replacement for x which makes the number sentence a true statement.

A-9 Summary and Review.

A binary operation defined on a set is a rule of combination by means of which any two elements of the set may be combined to determine one definite thing.

A mathematical system is a set together with one or more binary operations defined on that set.

A set is closed under a binary operation if every two elements of the set can be combined by the operation and the result is always an element of the set.

An identity element for a binary operation defined on a set is an element of the set which does not change any element with which it is combined.

Two elements are inverses of each other under a certain binary operation if the result of this operation on the two elements is an identity element for that operation.

A binary operation is commutative if, for any two elements, the same result is obtained by combining them first in one order, and then in the other.

A binary operation is * associative if, for any three elements, the result of combining the first with the combination of the second and third is the same as the result of combining the combination of the first and second with the third.

$$a * (b * c) = (a * b) * c.$$

The binary operation * distributes over the binary operation o provided

$$a * (b \circ c) = (a * b) \circ (a * c)$$

for all elements a, b, c.

ANSWERS TO CHAPTER 1

Answers to Exercises 1-1; page 2:

1. Two sons cross; one returns. Father crosses; other son returns. Two sons cross.
2. No. They need a boat carrying 225 pounds. Solution as in 1 above.
3. Man takes goose and returns alone. He takes fox and returns with goose. He takes corn across river and returns alone to pick up goose.
4. Yes. This depends on the fact that $8x + 5y = 2$ has solutions in integers, such as $x = -1$, $y = 2$ and $x = 4$, $y = -6$. The first means that if you fill the 5-gallon jug twice and empty it once into the 8-gallon jug, you will have 2 gallons left. The second solution means that if you fill the 8-gallon jug four times and use it to fill the 5-gallon jug 6 times, you will have 2 gallons left. The first solution is best.
5. M_1 M_2 M_3 C_1 C_2 C_3

   M_1 C_1 cross river; M_1 returns.
   C_2 C_3 cross river; C_1 returns.
   M_1 M_2 cross river; M_1 C_2 return.
   M_1 M_3 cross river; C_3 returns.
   C_1 C_2 cross river; C_1 returns.
   C_1 C_3 cross river.
6. Balance the two groups of 3 marbles each. If they balance then it is only necessary to balance the remaining two marbles to find the heavy one. If the two groups of 3 marbles do not balance, take the heavier group. Of the 3 marbles in the heavier group balance any 2 marbles. If they balance, the remaining marble is the heaviest one. If the 2 marbles do not balance, the heaviest will be 1 of the 2 on the balance.

7. In solving the problem is it practical to try out all the possible ways the dominoes may be placed on the board? This would be difficult because there are more than 65,536 ways to cover the whole board. The solution may be found in another way:

   How many squares are there altogether on the board? (64)

   How many squares must be covered? (62)

   What is special about the two squares next to each other? (They are of different colors.)

   What is special about the two opposite corners? (They are the same color.)

Answers to Exercises 1-2; page 4:

1. O.K.
2. (a) Yes.
   (b) Someone gets at least 3 pencils.
3. 367. (The answer must take leap year into account.)
4. 1. Tell the class that the first two to enroll might be twins.
5. 6.
6. 1.
7. Only 1.
8. 6 committees.
9. 12 committees.

Answers to Exercises 1-3; page 6:

1. Another method is this: 2 + 4 = 3 + 3, 1 + 5 = 3 + 3.
   That is, the sum is the same as:
   3 + 3 + 3 + 3 + 3 = 5 × 3 = 15.
   This can be called the "averaging method."

2. Either method works. Gauss method: $\frac{8 \times 5}{2}$;

   Averaging method: $5 \times 4 = 20$.

3. $\frac{16 \times 8}{2} = 64$. Here there is an even number of quantities so that the "averaging method" must be modified to give 8 eight's or $8 \times 8 = 64$.

4. (a) 4.
   (b) 9.
   (c) 16.
   (d) The sum of the first "n" consecutive odd numbers equals the square of "n".
   (e) 64.

5. $\frac{24 \times 6}{2} = 72$; $6 \times 12 = 72$.

6. $\frac{62 \times 10}{2} = 310$; $31 \times 10 = 310$.

7. $\frac{50 \times 51}{2} = 1275$; $51 \times 25 = 1275$.

8. Yes, provided that the numbers are in arithmetic progression; that is, there is the same difference between each pair of adjacent numbers.

9. Yes. If you start with 1 there are 200 integers in the series giving us $\frac{(1 + 200)200}{2}$. If you start with 0 there are 201 integers in the series giving us $\frac{(0 + 200)201}{2}$. The products of like factors are equal. The method also may be used in a series if you select a number other than 1 or 0 as a starting point.

10. (a) If you add 1 to the quantity, the sum up to any number is equal to the next number. Hence, the sum plus 1 is equal to $2 \cdot 256 = 512$. Therefore, the sum is 511.

    (b) This is more in the spirit of Gauss:

    Sum = $1 + 2 + 4 + \ldots + 256$

    $2 \times$ sum = $2 + 4 + \ldots + 256 + 512$

    Subtracting: Sum = 511.

11. Sum = 2 + (6 + 18 + ... + 486)
    3 × sum = 6 + 18 + ... + 486 + 1458
    Subtracting: 2 × (the sum) = 1458 - 2 = 1456
    Sum, 728.

<u>Answers</u> <u>to</u> <u>Exercises</u> <u>1-4</u>; page 9:

2. One out of four or $\frac{1}{4}$.
3. One out of two or $\frac{1}{2}$.
4. One out of 52 or $\frac{1}{52}$.
5. Four out of 52 or $\frac{4}{52} = \frac{1}{13}$.

   The probability may be thought of as a ratio of

   $$\frac{\text{the number of possible favorable selections}}{\text{the total number of all possible selections}}.$$

   The pupils should be reminded that to say his chances are 1 out of 13 does <u>not</u> mean that he will necessarily draw an ace in the first 13 draws.
6. One out of six. A die has 6 sides and only one side has two dots.
7. There are four possibilities in all, only one of which is favorable. Hence the probability is $\frac{1}{4}$.
8. One out of 36. The more advanced students should reason that there are 36 possible combinations. A table may be constructed to show the possibilities. A possible diagram is the following.

Number of Possibilities	1	2	3	4	5	6	7	8	9	...	34	35	36
1st die	1	1	1	1	1	1	2	2	2	...	6	6	6
2nd die	1	2	3	4	5	6	1	2	3	...	4	5	6

   Since there is only one possible way of making two ones the probability is $\frac{1}{36}$.

9. The possibilities are easily enumerated.

Number of Possibilities	1	2	3	4	5	6	7	8
1st coin	H	H	H	H	T	T	T	T
2nd coin	H	H	T	T	H	H	T	T
3rd coin	H	T	H	T	H	T	H	T

For 3 heads to come up, the probability is $\frac{1}{8}$. For exactly 2 heads the probability is 3 out of 8 or $\frac{3}{8}$. For at least two heads the chance is 4 out of 8 or $\frac{1}{2}$. Note that this equals the probability of exactly two heads $(\frac{3}{8})$ plus the probability of 3 heads $(\frac{1}{8})$.

Answer to Exercise 1-8; page 15:

1. The first figure can be drawn if you start at either of the vertices where an odd number of segments come together. The second figure has no such vertices so it can be drawn by starting at any vertex. The third figure has four vertices where an odd number of segments come together so it cannot be drawn without lifting your pencil or retracing a line.

ANSWERS TO CHAPTER 2

Answers to Exercises 2-1; page 20:

1. (a) ∩∩∩|||
   (b) ∩∩∩∩∩|||
   (c) 𓍢𓍢𓍢𓍢𓍢𓍢
   ∩∩∩∩∩∩
   ||||||
   (d) 𓆼 𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢 ∩∩∩∩∩∩
   (e) 𓁨 𓂭𓂭𓂭 𓍢𓍢 ∩||||

2. ∩∩|| , ∩|∩| , ||∩∩ ,
   ∩||∩ , |∩∩| , |∩|∩

3. (a) 200,105　(c) 1029
   (b) 2052　(d) 1,100,200

4. (a) 𒁹𒁹𒁹
   𒁹𒁹𒁹
   𒁹𒁹𒁹
   (b) 𒌋𒌋𒁹𒁹　(c) 𒌋𒌋𒌋𒌋𒌋𒁹

5. (a) 15　(b) 37　(c) 55

6. (a) 29　(d) 105
   (b) 61　(e) 666
   (c) 90　(f) 1492

7. (a) XIX　(d) MDCXC
   (b) LVII　(e) $\overline{\text{XV}}$
   (c) DCCCLXXXVIII

8. (a) 1709
   654
   2363 = MMCCCLXIII
   (b) 2640
   1408
   4048 = MMMMXLVIII

9. In the Roman system, the symbol shows the number of units to be added while our system shows the number of groups and each group has a different number of units. The decimal system involves place value while the Roman system does not.

Answers to Exercises 2-2; page 24:

1. (a) 3^5 (c) 5^6

   (b) 2^4 (d) 4^2

2. (a) three

   (b) seven

   (c) ten

3. (a) $4 \times 4 \times 4$

   (b) $3 \times 3 \times 3 \times 3$

   (c) $33 \times 33 \times 33 \times 33 \times 33$

4. The exponent tells how many times the base is taken as a factor.

5. (a) $3 \times 3 \times 3 = 27$

   (b) $5 \times 5 = 25$

   (c) $4 \times 4 \times 4 \times 4 = 256$

   (d) $2 \times 2 \times 2 \times 2 \times 2 = 32$

   (e) $6 \times 6 = 36$

   (f) $7 \times 7 \times 7 = 343$

6. (a) 4^3 means 64. 3^4 means 81.

   (b) 2^9 means 512. 9^2 means 81.

7. (a) $(4 \times 10^2) + (6 \times 10^1) + (8 \times 1)$

   (b) $(5 \times 10^3) + (3 \times 10^2) + (2 \times 10^1) + (4 \times 1)$

   (c) $(5 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (2 \times 10^1) + (6 \times 1)$

   (d) $(1 \times 10^5) + (0 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (8 \times 10^1) + (0 \times 1)$

8. The exponent of the base "10" tells how many zeros are written to the right of the "1" when the numeral is written in the usual way.

9. (a) 10^3 (b) 10^5 (c) 10^6 (d) 10^8

10. 10^{100}.

is 1.)

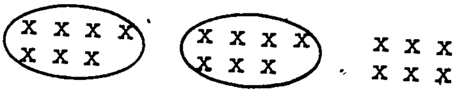
11. 100, 10, 1. Some discussion might be devoted to the meaning given to 10^0. This point need not be stressed at this time, but it will be used in later chapters.

Answers to Exercises 2-3, page 28:

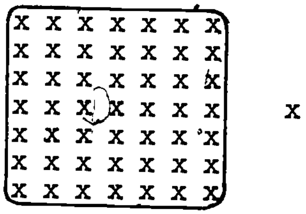
1. (a) 24_{seven}

   (b) 116_{seven}

2. (a)



   (b)



3. (a) (3 × seven) + (3 × one) = 24

   (b) (5 × seven × seven) + (2 × seven) + (4 × one) = 263

4. (a) 10_{seven} (c) 1000_{seven}

   (b) 11_{seven} (d) 1010_{seven}

5. (a) 560_{seven} The 6 means 6 sevens

   (b) 605_{seven} The 6 means 6(seven × seven)'s or 6(forty-nine)'s

   (c) 6050_{seven} The 6 means 6(seven × seven × seven)'s or 6(three hundred forty-three)

6. seven^4 or seven to the fourth power

7. 132_{seven}

8. 205_{ten}.

9. Neither. They are equal.

10. (a) Yes. $30_{ten} = (3_{ten} \times ten) + (0 \times 1)$

    (b) No. When 241_{ten} is divided by 10_{ten} there is a non-zero remainder.

    (c) If the units digit is zero the number is divisible by ten; otherwise it is not divisible by ten.

11. (a) It has a remainder of zero when divided by seven.

    (b) Yes. $(3 \times seven) + (0 \times 1)$ is divisible by seven; Remainder is 0.

12. A number written in base seven is divisible by seven when the units digit is zero.

13. 11_{seven} is even.
No; you cannot tell merely by glancing at the numerals. You could tell by converting each number to base ten. There is another method which is shorter and has much value in teaching divisibility ideas.

For example,

$$\begin{aligned} 12_{seven} &= (1 \times 7) + (2 \times 1) = 1 \times (6 + 1) + (2 \times 1) \\ &= (1 \times 6) + (1 \times 1) + (2 \times 1) \\ &= (1 \times 6) + \{(1 + 2) \times 1\} \end{aligned}$$

The first term is divisible by two but the second term is not divisible by two; hence the sum is not divisible by two. Note that the digit in the units place of the last expression in the display $1 + 2$, is the sum of the digits of 12_{seven} and this sum is not divisible by two. This is a general rule for base seven numerals.

14. They use seven symbols and seem to have a place value system with base seven. They appear to use |, ∠, △, □, ⧅, ⊠, for 1, 2, 3, 4, 5, 6 and — for zero. ∠∠ follows ∠| .

15. See discussion problem 13.

Answers to Exercises 2-4a; page 31:

1.

Addition, Base Seven

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

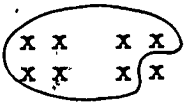
2. (a) Yes
   (b) By reading each result from the table and noting results.
   (c) Chart is symmetric with respect to the diagonal.
   (d) 55 different combinations; just a bit over half the total number of combinations.
   (e) In base seven because there are fewer.
   (f) They are equal since $9 = 12_{seven}$

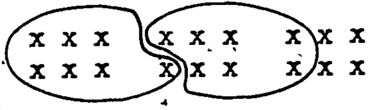
Answers to Exercises 2-4b ; page 34:

1. (a) 56_{seven}
   $(19 + 22 = 41)$
   (b) 110_{seven}
   $(41 + 15 = 56)$
   (c) 620_{seven}
   $(91 + 217 = 308)$
   (d) 1266_{seven}
   $(199 + 290 = 489)$
   (e) 14562_{seven}
   $(2189 + 1873 = 4062)$
   (f) 1644_{seven}
   $(327 + 342 = 669)$

2. (a) 2_{seven} (b) 4_{seven} (c) 4_{seven}

3. (a) 2
   $(7 - 5 = 2)$
   (b) 36
   $(47 - 20 = 27)$
   (c) 6
   $(32 - 26 = 6)$
   (d) 203_{seven}
   $(247 - 146 = 101)$
   (e) 552_{seven}
   $(319 - 37 = 282)$
   (f) 1254_{seven}
   $(1259 - 781 = 478)$

4. (a)


(b)


Answers to Exercises 2-4c; page 35:

1\.

Multiplication, Base Seven

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

2\. Since multiplication combination~~s are~~ needed only up to 6×6 instead of up to 9×9, multiplication is easier to learn in base seven than in base ten.

3\. (a) Both parts are alike.

(b) $3_{seven} \times 4_{seven} = 4_{seven} \times 3_{seven}$; this is an illustration of the fact that multiplication is commutative.

Answers to Exercises 2-4d; page 37:

1\. (a) 45_{seven}

(b) 1116_{seven}

(d) 2464_{seven}

(d) $106,533_{seven}$

(e) $125,150_{seven}$

2\. (a) 5_{seven}

(b) 62_{seven}

(c) 421_{seven} with a remainder of 2_{seven}.

(d) 123_{seven} with a remainder of 12_{seven}.

3. (a) $(4 \times 7 \times 7) + (0 \times 7) + (3 \times 1) = 199_{\text{ten}}$
   (b) $(1 \times 10 \times 10) + (8 \times 10) + (9 \times 1) = 189_{\text{ten}}$

4. 403_{seven}

5. (a) 66_{seven} (c) 1061_{seven}
   (b) 123_{seven}

6. (a) 4_{seven} (b) 26_{seven} (c) 334_{seven}

7. Room 123; book 7; 15 chapters; 394 pages; 32 pupils; 5 days; 55 minutes; 13 girls; 19 boys; 11 years old; 66 inches or 5 feet 6 inches tall.

Answers to Exercises 2-5; page 40:

1. (a) $50_{\text{ten}} = (1 \times \text{seven}^2) + (0 \times \text{seven}) + (1 \times \text{one})$
   $= 101_{\text{seven}}$

   (b) $145_{\text{ten}} = (2 \times \text{seven}^2) + (6 \times \text{seven}) + (5 \times \text{one})$
   $= 265_{\text{seven}}$

   (c) $1024_{\text{ten}} = (2 \times \text{seven}^3) + (6 \times \text{seven}^2) + (6 \times \text{seven}) + (2 \times \text{one}) = 2662_{\text{seven}}$

2. (a) 15_{seven}
   (b) 51_{seven}
   (c) 431_{seven}
   (d) 3564_{seven}

3. Q = 195 R = 8
   Q = 19 R = 5
   Q = 1 R = 9
   Q = 0 R = 1

4. $Q = 74$ $R = 6$
   $Q = 10$ $R = 4$
   $Q = 1$ $R = 3$
   $Q = 0$ $R = 1$

   $524_{ten} = 1346_{seven}$; the digits of the base seven numeral are the remainders which have just been obtained.

5. Divide by seven and continue to divide each quotient by seven. The digits in the numeral sought will be the remainders in order with the first remainder in the "one" place.

6. (a) 1161_{ten}
   (b) 275_{ten}
   (c) 654_{seven}
   (d) 462_{seven}
   (e) 1116_{seven}
   (f) 3
   (g) 462_{seven}

Answers to Exercises 2-7, page 46:

1. (a) two (b) two (c) two (d) two

2. (a) $(2 \times 36) + (4 \times 6) + (5 \times 1) = 101$
   (b) $(4 \times 25) + (1 \times 5) + (2 \times 1) = 107$
   (c) $(1 \times 27) + (0 \times 9) + (0 \times 3) + (2 \times 1) = 29$
   (d) $(1 \times 64) + (0 \times 16) + (2 \times 4) + (1 \times 1) = 73$

Other answers are acceptable, i.e., $(2 \times 6^2) + (4 \times 6^1) + (5 \times 1)$.

3.

	Base Ten	Base Six	Base Five	Base Eight
(a)	11	15	21	13
(b)	15	23	30	17

4.

Base ten	0	1	2	3	4	5	6	7	8	9	10
Base two	0	1	10	11	100	101	110	111	1000	1001	1010

Base ten	11	12	13	14	15	16	17	18	19	20
Base two	1011	1100	1101	1110	1111	10000	10001	10010	10011	10100

Base ten	21	22	23	24	25	26	27	28	29
Base two	10101	10110	10111	11000	11001	11010	11011	11100	11101

Base ten	30	31	32	33
Base two	11110	11111	100000	100001

5. Addition, Base two

+	0	1
0	0	1
1	1	10

There are only four addition "facts."

6. Multiplication, Base Two

×	0	1
0	0	0
1	0	1

There are only four multiplication "facts." The two tables are not alike, except that 0 + 0 and 0 × 0 both equal 0.

The binary system is very simple because there are only four addition and four multiplication "facts" to remember. Computation is simple. Writing large numbers, however, is tedious.

7. (a) $111_{two} = (1 \times two^2) + (1 \times two) + (1 \times one) = 7$

(b) $1000_{two} = (1 \times two^3) + (0 \times two^2) + (0 \times two) + (0 \times one) = (1 \times 2^3) = 8$

(c) $10101_{two} = (1 \times two^4) + (0 \times two^3) + (1 \times two^2) + (0 \times two) + (1 \times one) = (1 \times 2^4) + (1 \times 2^2) + (1 \times 1) = 21$

(d) $11000_{two} = (1 \times two^4) + (1 \times two^3) + (0 \times two^2) + (0 \times two) + (0 \times one) = (1 \times 2^4) + (1 \times 2^3) = 24$

8. (a) $111_{twelve} = (1 \times twelve^2) + (1 \times twelve) + (1 \times one) = (1 \times 144) + (1 \times 12) + (1 \times 1) = 157$

(b) $3T2_{twelve} = (3 \times twelve^2) + (T \times twelve) + (2 \times one) = (3 \times 144) + (10 \times 12) + (2 \times 1) = 554$

(c) $47E_{twelve} = (4 \times twelve^2) + (7 \times twelve) + (E \times one) = (4 \times 144) + (7 \times 12) + (11 \times 1) = 671$

(d) $TOE_{twelve} = (T \times twelve^2) + (0 \times twelve) + (E \times one) = (10 \times 144) + (11 \times 1) = 1451$

9. The new system is in base four.

Base ten.	New Base	New Base Names
0	0	do
1	$\vert$	re
2	$\wedge$	mi
3	$\geq$	fa
4	$\vert 0$	re do
5	$\vert\vert$	re re
6	$\vert\wedge$	re mi
7	$\vert\geq$	re fa
8	$\wedge 0$	mi do
9	$\wedge\vert$	mi re
10	$\wedge\wedge$	mi mi

Base ten	New Base	New Base Names
11	$\wedge\geq$	mi fa
12	≥ 0	fa do
13	$\geq\vert$	fa re
14	$\geq\wedge$	fa mi
15	$\geq\geq$	fa fa
16	$\vert 00$	re do do
17	$\vert 0\vert$	re do re
18	$\vert 0\wedge$	re do mi
19	$\vert 0\geq$	re do fa
20	$\vert\vert 0$	re re do

10.

+	0	1	∧	≥
0	0	1	∧	≥
1	1	∧	≥	10
∧	∧	≥	10	11
≥	≥	10	11	1∧

×	0	1	∧	≥
0	0	0	0	0
1	0	1	∧	≥
∧	0	∧	10	1∧
≥	0	≥	1∧	∧1

11. The binary system is extremely simple in computation. Large numbers are tedious to write.

The duodecimal system may be used conveniently to represent large numbers. Twelve is divisible by 1, 2, 3, 4, 6 and 12 while ten is divisible only by 1, 2, 5 and 10. The twelve system requires more computational "facts" which will increase difficulties in memorizing tables of addition and multiplication. Twelve is used in counting dozens, gross, etc., and in some of the common measures of length.

12. (a) $2E5_{\text{twelve}}$

(b) 378_{twelve}

13. Five weights; 1 oz., 2 oz., 4 oz., 8 oz., may be used to check any weight up to 15 ounces. By adding a 16 oz. weight, any weight up to 31 ounces may be checked.

14. People who work with computers often use the base eight. To change from binary to octal and back is simple with the help of the table:

Binary	Octal
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7

For example, you have

$$2000_{ten} = 011,111,010,000_{two} = 3720_{eight}$$

Note the grouping of numerals by threes in the binary numeral. The sum of the place values of digits in each group results in the octal numeral. Hence

$$011 = (1 \times 2) + (1 \times 1) = 3$$

$$111 = (1 \times 4) + (1 \times 2) + (1 \times 1) = 7, \text{ etc.}$$

15. The method works for base twelve and base seven. It will also work for other bases. For bases larger than ten, add. For bases less than ten, subtract.

Example in Base Six:

44_{six} multiplying: $(4 \times 4) = 16$

$44 - 16 = 28_{ten}$

Example in Base Fifteen:

$46_{fifteen}$ multiplying: $(4 \times 5) = 20$

$46 + 20 = 66_{ten}$

ANSWERS TO CHAPTER 3

Answers to Exercises 3-1; page 52:

1. He subtracted 27 from 81. No, the answer should be 55.

2.

Two-to-one correspondence	Four-to-one correspondence
(a) Feet to people	(a) Tires to cars, wagons
(b) Ears, eyes, arms, hands etc., to people.	(b) Legs to dogs, horses cows

3. The following illustrates a one-to-one correspondence between the counting numbers and the even numbers.

Answers to Exercises 3-2; page 54:

1. Parts a, c, f, g, i are true
   Parts b, e, h, d are false

2. (a) $23 \cdot 12 < 12 \cdot 32$
   (b) $16 > 9 > 3$
   (c) $(3 \cdot 2) + 5 = 5 + (3 \cdot 2)$
   (d) $86 \cdot 135 = 135 \cdot 86$
   (e) $24 \div 3 > 3 \div 24$
   (f) $a > c$

3. (a) 5
   (b) 0
   (c) 0, 1
   (d) 0, 1, 2, 3, 4, 5, 6
   (e) 0, 1, 2, 3, 4
   (f) any whole number

4. No, subtraction does not have the commutative property.

Answers to Exercises 3-3; page 58:

1. (a) $(21 + 5) + 4 = 21 + (5 + 4)$ Associative property of addition
   $(21 + 5) + 4 = 26 + 4 = 30$
   $21 + (5 + 4) = 21 + 9 = 30$

   (b) $(9 \cdot 7) \cdot 8 = 9 \cdot (7 \cdot 8)$ Associative property of multiplication
   $(9 \cdot 7) \cdot 8 = 63 \cdot 8 = 504$
   $9 \cdot (7 \cdot 8) = 9 \cdot 56 = 504$

(c) $436 + (476 + 1) = (436 + 476) + 1$ Associative property of addition
$436 + (476 + 1) = 436 + 477 = 913$
$(436 + 476) + 1 = 912 + 1 = 913$

(d) $(57 \cdot 80) \cdot 75 = 57 \cdot (80 \cdot 75)$ Associative property of multiplication
$(57 \cdot 80) \cdot 75 = (4560) \cdot 75$
$57 \cdot (80 \cdot 75) = 57 \cdot 6000 = 342,000$

2. (a) No. (b) No. (c) There is no associative property of subtraction, or the associative property of subtraction does not hold.

3. (a) No.
(b) No.
(c) $(75 \div 15) \div 5 = 1$
(d) $75 \div (15 \div 5) = 25$
(e) $80 \div (20 \div 2) = 8$
(f) $(80 \div 20) \div 2 = 2$
(g) The associative property does not hold for division.

Answers to Exercises 3-4; page 60:

1. (a) $12 \cdot 5\frac{1}{4} = 63$
$60 + 3 = 63$
(b) $3216 + 3484 = 6700$
$67 \cdot 100 = 6700$
(c) $36 + 36 = 72$
$72 \cdot 1 = 72$

2. (a) $3 \cdot (4 + \underline{3}) = (3 \cdot 4) + (3 \cdot 3)$
(b) $(2 \cdot 7) + (3 \cdot 7) = (\underline{2} + \underline{3}) \cdot 7$

3. (a) $(15 \cdot 6) + (15 \cdot 13)$ (b) $12 \cdot (5 + 4)$

4. (a) $(3 \cdot 9) + (3 \cdot 17) = 3 \cdot (9 + 17)$
(b) $(5 \cdot 20) + (5 \cdot 23) = 5 \cdot (20 + 23)$
(c) $(3 \cdot 10) + (3 \cdot 7) = 3 \cdot (10 + 7)$

5. The following parts are true: b, c.
The following parts are false: a, d.

*6. (a) $(60 + 4)\cdot(60 + 6) = 60\cdot(60 + 6) + 4\cdot(60 + 6)$
$= (60\cdot60) + (60\cdot6) + (4\cdot60) + (4\cdot6)$
$= 3600 + 360 + 240 + 24$
$= 4224$

(b) $(70 + 5)\cdot(70 + 5) = 70\cdot(70 + 5) + 5\cdot(70 + 5)$
$= (70\cdot70) + (70\cdot5) + (5\cdot70) + (5\cdot5)$
$= 4900 + 350 + 350 + 25$
$= 5625$

7.

From (a) to (b)	Commutative property of multiplication
From (b) to (c)	Associative property of multiplication
From (c) to (d)	Commutative property of addition
From (d) to (e)	Associative property of addition
From (e) to (f)	Commutative property of multiplication
From (f) to (g)	Associative property of multiplication
From (g) to (h)	Distributive property

Answers to Exercises 3-5; page 64:

1. (a) No. The sum of 2 odd numbers is always an even number.

   (b) No.

2. Yes, since the sum of 2 even numbers is always an even number.

3. Yes. Since each of the numbers in the set is a multiple of 5, the sum of any two numbers in the set is a multiple of 5.

4. Each set is closed under multiplication.

5. (a) Yes

   (b) No. For example $500 + 501 = 1001$ and 1001 is not in the set.

   (c) No. For example, $3 + 47 = 50$, and 50 is not in the set.

   (d) Yes. If the numerals of 2 numbers end in 0, then the sum of the numbers ends in 0.

6. (a) Yes. (b) No. (c) No. (d) Yes.

7. Yes. Multiplication is an abbreviated process for addition.

8. No.

9. No.

Answers to Exercises 3-6; page 66:

1. (a) 5
   (b) 7
   (c) 8
   (d) 3
   (e) Any whole number
   (f) 5
   (g) 6
   (h) 0
   (i) 1
   (j) 1

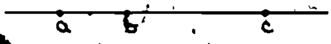
Answers to Exercises 3-7; page 67:

1. (a) None.
   (b) 2
   (c) None.
   (d) 88
   (e) 3 is answer. The answer is either $a - (b + 1)$ or $(a - b) - 1$ or $(a - 1) - b$.

2. (a) 10
   (b) 11
   (c) 18
   (d) 22

3. (a), (c), (d).

4. (a) Yes. (b) Yes. (c) Yes.



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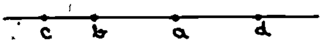


Either of the two situations is possible. The diagrams indicate that b is between c and d regardless of whether $c < d$ or $d < c$.

Answers to Exercises 3-8; page 70:

1. You can get any counting number by the repeated addition of 1 to another counting number if the number you wish to get is larger than the counting number to which you add.

   You can get any counting number by the repeated subtraction of 1 from another counting number if the number you wish to get is smaller than the counting number from which you subtract.

2. Yes. 1 - 1 = 0; 3 - 1 - 1 - 1 = 0. Zero is not a counting number.

3. The successive addition of 1's to any counting number will give a counting number. But, the successive subtraction of 1's from any counting number will become 0 if carried far enough.

4. (a) 876429 (c) 897638 (e) 3479 (g) 1

   (b) 976538 (d) 896758 (f) 97 (h) 1

Answers to Exercises 3-9; page 73:

1. The symbols in the following parts represent zero: (a), (d), (e), (f), (g), (o), (q), (s).

2. The error is in the generalization to c in part (e). If a · b = c, a or b does not need to be c. Example: 2 · 2 = 4. This exercise shows the error that may be made by making a generalization on a few cases.

Answers to Exercises 3-10; page 75:

1. (a) 12 = 3 × 4 or 2 × 6 (e) 39 = 3 × 13

   (b) 31 is prime (f) 41 is prime

   (c) 8 = 2 × 4 (g) 95 = 5 × 19

   (d) 35 = 5 × 7

Of course in all these cases the two factors may be written in reverse order.

2. (a) There are 15 prime numbers less than 50.

   (b) There are 25 prime numbers less than 100.

3. (a) The number 24 is divisible by 1, 2, 3, 4, 6, 12 and 24.

   (b) 24 is a multiple of 1, 2, 3, 4, 6, 12 and 24

   (c) 24 is a multiple of each of the sets of numbers in (a) and (b) since being a "multiple of" is the same as being "divisible by".

4. $12 = 2 \times 6 = 6 \times 2 = 3 \times 4 = 4 \times 3 = 2 \times 2 \times 3 = 2 \times 3 \times 2$
   $= 3 \times 2 \times 2$

5. 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43, 59 and 61, 71 and 73. There are eight such pairs.

6. $4 = 2 + 2$; $6 = 3 + 3$; $8 = 5 + 3$; $10 = 3 + 7 = 5 + 5$;
   $12 = 5 + 7$; $14 = 3 + 11 = 7 + 7$; $16 = 3 + 13 = 5 + 11$;
   $18 = 5 + 13 = 7 + 11$; $20 = 3 + 17 = 7 + 13$; $22 = 3 + 19$
   $= 5 + 17 = 11 + 11$.

7. Yes; 3, 5, 7. This is the only set because at least one of any set of three consecutive odd numbers is divisible by 3.

8. 3, 5, 7, 11, 13, 17, 19, 23, 29.

Answers to Exercises 3-11; page 78:

1. (a) 1, 2, 5, 10
   (b) 1, 3, 5, 15
   (c) 1, 3, 9
   (d) 1, 2, 3, 6, 9, 18
   (e) 1, 3, 9, 27
   (f) 1, 2, 3, 4, 6, 8, 12, 24
   (g) 1, 11

2. (a) $2 \cdot 5$
   (b) $3 \cdot 5$
   (c) $3 \cdot 3$ or 3^2
   (d) $2 \cdot 3 \cdot 5$
   (e) $3 \cdot 3 \cdot 5$ or $3^2 \cdot 5$
   (f) $2 \cdot 5 \cdot 5$ or $2 \cdot 5^2$
   (g) 13

3. Zero is not a factor of six since there is not a number which, when multiplied by zero, gives a product of six. 6 is a factor of zero since the product of six and zero is zero. Thus the definition is satisfied.

4. (a) 1, 4, 10, 20
   (b) 1, 4, 6, 8, 9, 18, 24, 36, 72

5. (a) $3 \cdot 5 \cdot 7$
   (b) $2 \cdot 3 \cdot 7$
   (c) 2^6
   (d) $3 \cdot 5 \cdot 23$
   (e) $7 \cdot 43$
   (f) $17 \cdot 19$

6. (a) Even
   (b) Odd
   (c) Odd
   (d) Odd
   (e) Even

7. (a) 11_{three} is four; it is even
   (b) 12_{five} is seven; it is odd.
   (c) 33_{five} is eighteen; it is even.
   (d) 101_{two} is five; this is odd.

8. Divisibility is a property of number. It is the <u>number</u> which is divisible by another number. The numeral is a way of writing the number. In base ten, a numeral which represents an even number ends with an even number; in base five this is not necessarily so. This is illustrated in Exercise 7.

9. Some of the items in the table will be:

	Factors of N	Number of Factors	Sum of Factors
18	1,2,3,6,9,18	6	39
24	1,2,3,4,6,8,12,24	8	60
28	1,2,4,7,14,28	6	56

(a) 2,3,4,7,11,13,17,19,23,29 (the prime numbers)

(b) 4,9,25 (the squares of prime numbers)

(c) Three: 1,p and p^2

(d) Four: 1,p,g,pg. The sum is $1 + p + g + pg$

(e) The factors are: $1, 2, 2^2, 2^3, \ldots, 2^k$. There are $k + 1$ of them.

(f) The factors are: $1, 3, 3^2, 3^3, \ldots, 3^k$. There are $k + 1$ of them.

(g) If $N = p^k$, the factors are $1, p, p^2, p^3, \ldots, p^k$. There are $k + 1$ of them.

Answers to Exercises 3-12, page 82:

1. (a) 5 (b) 3 (c) 3 (d) 2 (e) 7 (f) 11

2. A counting number will be divisible by 6 if and only if it is divisible by both 2 and 3. Hence the test is that it must be even and the sum of its digits must be divisible by 3.

3. If a number is divisible by 15 it must be divisible by both 3 and 5, and conversely. Hence the test is that its last digit must be one of 0 and 5 and the sum of its digits must be a multiple of 3.

4. (a) Since the last digit is odd, the number is odd.

   (b) The number is 390 in the decimal system and hence is even. You might notice that the number is even since it can be written in the form $7^3 + 7^2 + 7 + 1$ which is the sum of an even number of odd numbers.

(c) Here ~~the~~ number in the decimal system is 259 which is odd. You might notice that ~~it~~ may be written $6^3 + 6^2 + 6 + 1$ which is the sum of 1 and three even numbers; hence is odd.

(d) Here the number in the decimal system is 40 which is even; or it may be shown even by the same kind of argument as in part b.

5. If a number is written in the system to the base seven, its last digit is zero if and only if it is divisible by seven, but it need not be divisible by ten. To test divisibility by 3 write the first few multiples of three in the system to the base seven as follows:

Number to the base seven	3	6	12	15	21	24	30	33	36	42
Sum of the digits	3	6	3	6	3	6	3	6	12*	6

*Notice that 12 is also written in the system to the base seven. Here when the first digit increases by 1, the second digit decreases by 4 giving a net decrease of 3. Hence the same test for divisibility by 3 works both in the decimal system and in the system to the base seven.

6. If a number is written in the number system to the base twelve and has zero as its last digit it must be divisible by 12 but need not be divisible by 10. It will be divisible by 3 if its last digit is one of 0,3,6,9. This may be shown in the same way that you tested for divisibility by 5 in the decimal system, since the pattern 3,6,9,0 repeats in the sequence of multiples of 3 written to the base twelve.

7. A number written to the base seven will be divisible by 6 if the sum of its digits is divisible by 6. This is apparent from the table given for Exercise 5 if you notice that every other sum of digits is even.

8. A number written in the decimal system will be divisible by 4 if <u>one</u> of the following holds:

   (a) the last digit is one of 0,4,8 and the tens digit is even.

   (b) the last digit is 2 or 6 and the tens digit is odd.

   This can be seen from the pattern in which the multiples of 4 fall. Also, since any multiple of 100 is divisible by 4, you could also say that a number represented by the last two digits is divisible by 4.

<u>Answers to Exercises 3-13</u>; page 83:

1. (a) {1, 2, 3, 6} (d) {1, 3, 5, 15}
   (b) {1, 2, 4, 8} (e) {1, 2, 4, 8, 16}
   (c) {1, 2, 3, 4, 6, 12} (f) {1, 3, 7, 21}

2. (a) {1, 2} (d) {1, 2}
   (b) {1, 2, 4} (e) {1, 3}
   (c) {1, 3} (f) {1, 2, 4}

3. (a) {1, 19}
   (b) {1, 2, 4, 7, 14, 28}
   (c) {1, 2, 3, 4, 6, 9, 12, 18, 36}
   (d) {1, 2, 4, 5, 8, 10, 20, 40}
   (e) {1, 3, 5, 9, 15, 45}
   (f) {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}

4. (a) {1} (d) {1, 3, 9}
   (b) {1, 2, 4} (e) {1, 2, 4, 8}
   (c) {1, 2, 4} (f) {1}

5. (a) 5 (c) 8
   (b) 6 (d) 10

6. (a) 6 (b) a

7. (a) 1 (b) 1

8. (a) Yes, 1

(b) Yes, c = 3, a = 3; b = 6 or 9 or 12, etc.

(c) No; the greatest common factor can never be greater than the smallest member of the set of numbers used.

9. No. {6, 10, 15}

10. (a) $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$

$45 = 3 \cdot 3 \cdot 5 = 3^2 \cdot 5$

(b) The greatest common factor is 3^2 or 9.

11. (a) $24 = 2^3 \cdot 3, \quad 60 = 2^2 \cdot 3 \cdot 5$

G.C.F. = $2^2 \cdot 3 = 12$

(b) $36 = 2^2 \cdot 3^2, \quad 90 = 2 \cdot 3^2 \cdot 5$

G.C.F. = $2 \cdot 3^2 = 18$

(c) $72 = 2^3 \cdot 3^2, \quad 108 = 2^2 \cdot 3^3$

G.C.F. = $2^2 \cdot 3^2 = 36$

(d) $24 = 2^3 \cdot 3, \quad 60 = 2^2 \cdot 3 \cdot 5; \quad 84 = 2^2 \cdot 3 \cdot 7$

G.C.F. = $2^2 \cdot 3 = 12$

(e) $42 = 2 \cdot 3 \cdot 7, \quad 105 = 3 \cdot 5 \cdot 7, \quad 147 = 3 \cdot 7^2$

G.C.F. = $3 \cdot 7 = 21$

(f) $165 = 3 \cdot 5 \cdot 11, \quad 234 = 2 \cdot 3^2 \cdot 13$

G.C.F. = 3

12. (a) 6 $(6 \times 0 = 0)$ (b) 1 (c) 1

13. (a) Yes (b) Yes (c) Yes

Answers to Exercises 3-14; page 87:

1.	Dividend	Divisor	Quotient	Remainder
(a)			3	2
(b)		10		
(c)	66			
(d)		20		

There are several possible answers for (e).

(e)		9	9	
		3	27	
		1	81	
		27	3	
		81	1	

2. (a) No.

   (b) The dividend is greater than the quotient in this case.

   (c) The divisor must always be greater than the remainder.

   (d) Yes. $0 \div 3 = 0$ or $0 = 3 \cdot 0 + 0$.

   (e) No. $3 \div 0$ is impossible because there is no number which when multiplied by 0 gives 3, with a remainder less than the divisor.

   (f) Yes. $0 \div 3 = 0$
   or $3 \div 5$ may be considered as giving a quotient of 0 with remainder 3. This might be the answer to the question: "How many $5 shirts can you buy with $3?"

   (g) Yes. $6 \div 6 = 1$ with 0 remainder
   $6 \div 2 = 3$ with 0 remainder

3. (a) Yes

   (b) No. It is impossible to divide by 0.

   (c) Yes.

   (d) Yes. A remainder of 0 is a whole number.

4.

	a	b	q	R
(a)			4	2
(b)		1	100	
		2	50	
		4	25	
		5	20	
		10	10	
(c)			16	11
(d)		25		

There are many possible answers as indicated here. By the commutative property the reverse order for each of these is also an acceptable answer.

5. (a) No

(b) Yes, $(16 \div 2 = 8,\ 6 = 2,\ q = 8)$

(c) Yes, $(200 \div 75 = 2$ and Remainder 50 $\therefore q = 2,\ R = 50)$.

(d) No. The divisor, b, may be any whole number except zero. Division by zero is impossible.)

(e) Yes (counting numbers do not include 0).

(f) Yes. (a may be 0, then $q = 0$. or, a may be any other number. However, q is not a counting number if $a < b$.

6. (a) The members of the set of all remainders are the whole numbers less than eleven.

(b) 25

(c) K

7. (a) (1) $836 \div 124 = 6$ and Remainder 92

(2) $124 \div 92 = 1$ and Remainder 32

(3) $92 \div 32 = 2$ and Remainder 28

(4) $32 \div 28 = 1$ and Remainder 4

(5) $28 \div 4 = 7$ and Remainder 0

The G.C.F. is 4

(b) G.C.F is 28

(c) G.C.F is 71

8. (a) Each of the sets has eleven elements (or members). One suggested correspondence is given here:

0	1	2	3	4	5	6	7	8	9	10
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
1	2	3	4	5	6	7	8	9	10	11

(b) Each of the sets {51, 53, 55, ..., 79} and {18, 20, 22, ..., 46} has 15 elements. There are many different ways in which a 1-1 correspondence can be set up between the two sets. Three of the correspondences are given below.

51 ⟷ 18	51 ⟷ 46	51 ⟷ 20
53 ⟷ 20	53 ⟷ 44	53 ⟷ 22
55 ⟷ 22	55 ⟷ 42	55 ⟷ 24
. .	. .	. .
. .	. .	. .
. .	. .	. .
79 ⟷ 46	79 ⟷ 18	77 ⟷ 46
		79 ⟷ 18

(c) Each of the sets {3, 6, 9, ..., 42} and {105, 112, 119, ..., 196} has 14 elements. There are many ways in which a 1-1 correspondence can be set up between the two sets.

Answers to Exercises 3-15; page 90:

1. (a) The least common multiple of 6 and 8 is 24.
   (b) The least common multiple of 9 and 12 is 36.

2. (a) 42 (b) 72

3. (a) 12 (c) 60
   (b) 18 (d) 30

4. (a) Yes 8 is the L.C.M. of 4 and 8.
   (b) No. The L.C.M. of 8 and 9 is 72.

5. (a) 6 (b) 29 (c) a

6. (a) 6 (b) 29 (c) a

7. (a) No; what is the L.C.M. of 2 and 3?

(b) For two different prime numbers a and b, the L.C.M. is the product of the two numbers a · b.

(c) For three different prime numbers, a, b and c, the L.C.M. is the product of the three numbers, a · b · c.

8. (a) 48 (b) 112 (c) 70 (d) 60

9. (a) No. (b) No.

10. (a) Yes. (d) 0.
(b) Yes. (e) No. Zero is not a counting number.
(c) Yes.

<u>Answers to Review Exercise</u>

1. (a) 1 (d) 2
(b) 6 (e) 9
(c) 3 (f) 4

2. (a) 6 (g) 989
(b) 24 (h) 858
(c) 14 (i) 663
(d) 75 (j) 5402
(e) 36 (k) 2520
(f) 105 (l) 31,372

3.

(a) Product of the numbers in Problem 1	(b) Product of the G.C.F. and L.C.M. of the numbers in Problem 1
(a) 23 × 43 = 989	(a) 1 × 989 = 989
(b) 66 × 78 = 5148	(b) 6 × 858 = 5148
(c) 39 × 51 = 1989	(c) 3 × 663 = 1989
(d) 74 × 146 = 10,804	(d) 2 × 5402 = 10,804
(e) 45 × 72 × 252 = 22,680	(e) 9 × 2520 = 22,680
(f) 44 × 92 × 124 = 501,952	(f) 4 × 31,372 = 155,488

(c) It is always true that the product of <u>two</u> numbers is equal to the product of their G.C.F. and their L.C.M. This can be seen from the following example

$$\text{Let} \quad r = 2^3 \times 5 \times 7 \quad \text{and} \quad s = a \times 5^2 \times 13$$

To get the G.C.F. take the product of the primes occuring in both raised to the smaller power: $2 \times$

To get the L.C.M. take the product of the primes raised to the larger power:

$$2^3 \times 5^2 \times 7 \times 13$$

Then $rs = 2^3 \times 5 \times 7 \times 2 \times 5^2 \times 13.$

and G.C.F. times L.C.M. $= 2 \times 5 \times 2^3 \times 5^2 \times 7 \times 13.$

One product is the same as the other except that the members are rearranged.

This is not true for three or more numbers.

4. (a) The product $a \cdot b$

example (1) The G.C.F. of 3 and 5 is 1.
The L.C.M. of 3 and 5 is 15.

example (2) The G.C.F. of 4 and 9 is 1.
The L.C.M. of 4 and 9 is 36.

(b) No. Not necessarily.
The G.C.F. of 2, 3 and 4 is 1.
The L.C.M. of 2, 3 and 4 is $2^2 \cdot 3$, or 12.
It is not $2 \cdot 3 \cdot 4$.

5. (a) 2 is a prime number. It is the only even prime number.

(b) All primes except 2 are odd, i.e. 3, 5, ..., .

(c) One. Only the prime number 5 has an ending in 5. All other numbers ending in 5 are multiples of 5, i.e., 15, 25, 35, ..., .

(d) 2 and 5.

(e) 1, 3, 7, and 9.

6. The two numbers are the same. For example: The L.C.M. of 7 and 7 is 7.

7. (a)

Rows	Bulbs per row
1	112
2	56
4	28
8	14
16	7

(Bulbs and rows may be interchanged.)

8. (a) 3, 6, 9, 12, 15--first bell,
5, 10, 15--second bell.
They strike together again in 15 minutes.

(b) 6, 12, 18, 24, 30--first bell,
15, 30--second bell.
They strike together again in 30 minutes.

(c) 15 is the least common multiple of 3 and 5.
30 is the least common multiple of 6 and 15.

9. (a) Yes. The G.C.F. of 6 and 6 is 6.
The L.C.M. of 6 and 6 is 6.

(b) No. The G.C.F. of the members of a set of numbers can never be greater than the largest number of the set of numbers because a factor of a number is always less than a multiple of the number unless the multiple is zero.

(c) No. The L.C.M. of the members of a set of numbers can never be less than the largest member of the set of numbers.

The least common multiple of two numbers is at least as big as the larger of the two numbers (since the L.C.M. is a multiple of the larger number). The greatest common factor of two numbers is no larger than the smaller of the two numbers (since the G.C.F. is a factor of the smaller number). If the least common multiple and greatest common factor are equal, the larger and smaller number must also be equal.

Illustrate with 4 and 6, locating the L.C.M. and G.C.F. on the number line; then use 4 and 4.

10. (a) No. It is not possible to have exactly four numbers between two odd numbers. Between any two odd primes there is always an odd number of numbers. If they are consecutive odd primes all the numbers between would have to be composite.

(b) Yes. For example, between 23 and 29 there are exactly 5 composite numbers; 24, 25, 26, 27, 28.

11. (a) 135, 222, 783, and 1065 are all divisible by three.

(b) 222 is the only number divisible by six.

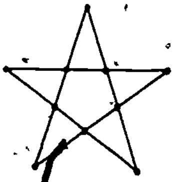
(c) 135 and 783 are divisible by nine.

(d) 135 and 1065 are divisible by five.

(e) 135 and 1065 are divisible by fifteen.

(f) None of the numbers are divisible by four.

12. The pattern is a five-pointed star.



13. There is no greatest prime number.

To show there is no largest prime number, it will be shown that if p is any prime, there is another prime larger than p. Denote by M the product of all the primes less than or equal to p: $M = 2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p$.

Then $M + 1$ is certainly larger than p and $M + 1$ has at least one prime factor (it may be a prime). But $M + 1$ does not have any of the primes 2, 3, 5, 7, ..., as a factor since division by any of these primes leaves a remainder of 1. Thus all the prime factors of $M + 1$ are larger than p, and hence p is not the largest prime. Since p was an arbitrary prime, there is no largest prime.

ANSWERS TO CHAPTER 4

Answers to Exercises 4-1; page 98:

1. (a) 1, 2, 3, etc.

   (b) 0, 1, 2, 4, etc.

   (c) 0

   (d) Numbers such as $\frac{5}{3}$, $\frac{4}{11}$

2. All are rational numbers.

3. (a) $5 \cdot x = 1$

   (b) $4x = 4$

   (c) $3x = 11$

   (d) $9x = 63$

   (e) $5x = 0$

   (f) $11x = 123$.

   There are other correct forms for the answers.

4. (b) 1 (d) 7 (e) 0

5. (a) false.

   (b) true

   (c) true

6. (a) If x is the number of cookies each boy should receive, then $3x = 12$.

   (b) If x is the number of miles Mr. Carter drove on each gallon of gasoline, then $10x = 160$.

   (c) If x is the number of bags of cement needed for each foot of walk, then $30x = 20$.

   (d) If x is the number of pupils in each group, then $4x = 32$.

   (e) If x is the number of sheets of paper each pupil receives, then $24x = 12$.

Answers to Exercises 4-2; page 104:

1. All except f.

2. (a) Commutative for multiplication
   (b) Associative for multiplication
   (c) Distributive
   (d) Distributive
   (e) Identity for multiplication
   (f) Identity for multiplication
   (g) Identity for addition
   (h) Commutative for addition
   (i) Associative for addition
   (j) Associative for multiplication

3. No difference in area.

4.

n	d
2	3
4	6
6	9
8	12
10	15

etc.

$\frac{n}{d} = \frac{2}{3}$

5. If $2x$ and 3 are names for the same number then

$$\frac{1}{2}(2x) = \frac{1}{2} \cdot 3$$

$$\left(\frac{1}{2} \cdot 2\right)x = \frac{1}{2} \cdot 3$$

$$1 \cdot x = \frac{1}{2} \cdot 3$$

$$x = \frac{1}{2} \cdot 3$$

But if $2x = 3$ then $x = \frac{3}{2}$. There are three names for the same number x, $\frac{3}{2}$, and $\frac{1}{2} \cdot 3$. Since $\frac{3}{2}$ and $\frac{1}{2} \cdot 3$ are names for the same number, $\frac{3}{2} = \frac{1}{2} \cdot 3$.

Answers to Exercises 4-3; page 106:

1. (a) $\frac{1}{11}$
   (b) 5
   (c) $\frac{7}{2}$
   (d) $\frac{3}{50}$

2. (a) $\frac{1}{m}$
   (b) $\frac{1}{s}$
   (c) c
   (d) $\frac{s}{r}$
   (e) $\frac{w}{t}$

3. (a) $7n = 8, \quad n = \frac{8}{7}$
   (b) $11n = 2, \quad n = \frac{2}{11}$
   (c) $36n = 64, \quad n = \frac{16}{9}$

4. Set of reciprocals is $\{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}\}$.

5. (a) If the number is between 0 and 1, the reciprocal is greater.
   (b) If the number is greater than 1, the reciprocal is less.
   (c) The reciprocal of 1 is 1.
   (d) One of these sentences, and only one, is true for every counting number.

6. $\frac{2}{19}$

7. (a) $\frac{9}{2}$
   (b) $\frac{2}{9}$
   (c) They are reciprocals of each other.

8. (a) $\frac{15}{8}$
   (b) $\frac{8}{15}$
   (c) They are reciprocals of each other.

*9. (a) If $14n$ and 5 are names of the same number, then $\frac{1}{5}(14n)$ and $\frac{1}{5}(5)$ are names of the same number.

*10. $x = \frac{b}{a}$ and $y = \frac{a}{b}$

$$\frac{b}{a} \cdot \frac{a}{b} = \frac{b}{a} \cdot a \cdot \frac{1}{b} = b \cdot \frac{1}{b} = 1$$

Or in a more sophisticated way,

$$ax = b \qquad by = a$$

$$(ax)(by) = b \cdot a$$

$$(ab)(xy) = ab$$

$$xy = \frac{ab}{ab} = 1$$

You may wish to think of $xy = w$, to have

$$(ab)(xy) = ab$$

$$(ab)w = ab$$

$$w = \frac{ab}{ab} = 1$$

$$xy = 1$$

Answers to Exercises 4-4; page 111:

1. (a)
   A: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$
   B: $\frac{1}{4} = \frac{2}{8} = \frac{3}{12}$
   C: $\frac{3}{4} = \frac{6}{8} = \frac{9}{12}$
   D: $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$
   E: $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$
   F: $\frac{1}{1} = \frac{2}{2} = \frac{3}{3}$

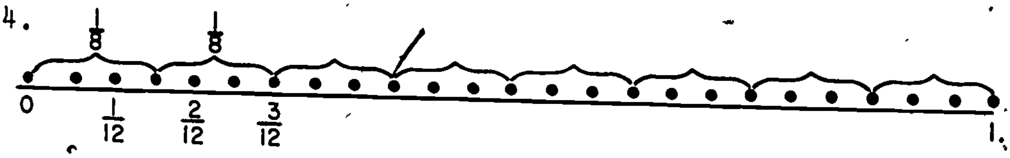
   (b) The rational number located at point B is less than the rational number located at point A. Because point B is to the left of point A on the number line.

   (c) Point A is to the left of point C. Therefore the rational number located at C is greater than the one located at A.

2. Here there are at least two interpretations for each. For example, in (a) one could exhibit a number line with 20 divisions and note that if it is divided into 5 equal parts, each will be 4 units long. Also, the same divisions could be labelled as fifths and then the line would be four units long. It is important that you see both.

3. (a) $\frac{57}{39} = 1\frac{16}{39}$ (b) $\frac{137}{23} = 5\frac{22}{23}$.

4.



5. (a) Here one measures three units to the right from the 0 point, then moves two units to the left from the point 3, arriving at the point 1 which is equal to 3 - 2.

(b) Similar.

Notice that here negative numbers are not involved. It is merely the idea that if you move to the right to add, you move in the opposite direction to subtract.

*6. One can certainly find whether two fractions are equal by changing them to fractions with equal numerators and examining their denominators. There are times when this method is easier than the other method, for instance, in examples like: Is $\frac{3}{11} = \frac{3}{10}$? Is $\frac{3}{11} = \frac{6}{23}$?

Answers to Exercises 4-5; page 116:

1. (a) $\frac{64}{3}$ (d) $\frac{22}{3}$
   (b) $\frac{80}{9}$ (e) $\frac{3}{2}$
   (c) 70 (f) $\frac{9}{7}$

2. $\frac{3}{4}$

3. $16\frac{1}{4}$

4. (a) $\frac{9}{4}$ (d) $\frac{2}{17}$
   (b) $\frac{4}{3}$ (e) 1
   (c) $\frac{9}{4}$ (f) 64

5. (a) 2 (c) $2\frac{2}{3}$
   (b) 8 (d) $7\frac{1}{2}$

6. (a) $\frac{1}{9}$ (c) $\frac{8}{9}$
   (b) $\frac{3}{11}$ (d) $\frac{9}{10}$

7. (a) $\frac{7}{9}$ (b) $\frac{4}{7}$

8. (a) $\frac{4}{3}$ (b) $\frac{3}{4}$

9. (a) 10 times as large
   (b) $\frac{16}{49}$ times as large

Answers to Exercises 4-6a; page 120:

1. (a) $4\frac{3}{8}$ or $\frac{35}{8}$ (d) $\frac{17}{16}$ or $1\frac{1}{16}$
   (b) $\frac{7}{10}$ (e) $\frac{17}{16}$ or $1\frac{1}{16}$
   (c) $\frac{7}{10}$
2. Yes.
3. (a) $3\frac{11}{30}$ or $\frac{101}{30}$ (c) $\frac{52}{15}$ or $3\frac{7}{15}$
   (b) $\frac{101}{30}$ or $3\frac{11}{30}$ (d) $\frac{52}{15}$ or $3\frac{7}{15}$
4. Yes.

*5. (a) Yes. (b) The commutative property.

*6. (a) Yes. (b) The associative property.

7. (a) $7\frac{5}{8}$ (c) $17\frac{3}{32}$
   (b) $10\frac{2}{16}$ or $10\frac{1}{8}$
8. (a) $\frac{7}{8}$ (c) $\frac{ad + bc}{bd}$
   (b) 15
9. (a) $\frac{3}{5}$ (b) 15
10. No.
11. Sums of columns, rows, and diagonals are $22\frac{1}{2}$.

Answers to Exercises 4-6b; page 125:

1. (a) $1\frac{1}{9}$ or $\frac{10}{9}$ (e) $\frac{4}{9}$
   (b) $1\frac{1}{4}$ or $\frac{5}{4}$ (f) $\frac{4}{9}$
   (c) $14\frac{1}{3}$ or $\frac{43}{3}$ (g) They are equal.
   (d) $\frac{791}{1000}$ or 0.791 (h) Yes.

*2. A Mel's home

B Vic's home

C Bob's home

B is $\frac{3}{72}$ or $\frac{1}{24}$ of a mile farther than A.

C is $\frac{11}{72}$ of a mile farther than A.

3. $14\frac{3}{4}$ rods

4. $3\frac{1}{8}$ yard

5. (a) raised (b) $\frac{1}{6}$¢ per pound

6. (a) Less

(b) If the difference were exactly 11, the larger number would be $6\frac{3}{4} + 11$ or $17\frac{3}{4}$. Since $17\frac{1}{4} < 17\frac{3}{4}$, the difference is less than 11.

7. In general, yes, unless the number subtracted is zero, or the original fraction has a value of 1.

8.

$\frac{2}{3}$	$\frac{1}{12}$	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{5}{12}$	$\frac{7}{12}$
$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{6}$

The sum in each row, each column and each diagonal is $\frac{5}{4}$.

Answers to Exercises 4-7; page 128:

1. (a) $\frac{2}{3} = \frac{8}{12}$ and hence $\frac{2}{3} > \frac{7}{12}$.

(b) $\frac{4}{5} = \frac{64}{80}$ and $\frac{13}{16} = \frac{65}{80}$. Hence, $\frac{13}{16} > \frac{4}{5}$.

(c) $\frac{13}{5} = \frac{91}{35}$ and $\frac{13}{7} = \frac{65}{35}$. Hence, $\frac{13}{5} > \frac{13}{7}$.

You might find the answer more quickly by noticing that the numerators are equal and hence, the larger fraction is the one with the smaller denominator, since the denominator denotes the number of equal parts into which the numerator is divided. See Problem 3.

2. You might answer this question by giving several examples showing the following which is shown in letters: $a \geqslant 2b$ implies that $\frac{a}{b} > \frac{2b}{b} = 2$.

*3. If two fractions have equal numerators, the fraction with the smaller denominator represents the larger number. A satisfactory reason at this stage should be that the denominator denotes the number of equal parts into which the numerator is divided; if the number of parts is smaller, each part must be larger. It is too early to discuss manipulations with inequalities.

<u>Answers to Exercises 4-8</u>; page 131:

1.

3	8	$\frac{3}{8}$
36	96	$\frac{3}{8}$
$7\frac{1}{2}$	20	$\frac{3}{8}$
54	144	$\frac{3}{8}$
$11\frac{1}{4}$	30	$\frac{3}{8}$

2. (a) Ratio of number of girls to total is $\frac{2}{5}$.

   (b) Ratio of number of boys to total is $\frac{3}{5}$.

   (c) Ratio of number of girls to number of boys is $\frac{2}{3}$.

3. The following ratios are equal:

   (a), (c),

4. (a) $x = 15$ (c) $x = 15$

   (b) $x = 42$

5. In the ratio 3:1 we have:

   3 cups butter
   2 cups sugar
   6 eggs
   $4\frac{1}{2}$ cups ~~flour~~
   3 teaspoons vanilla

   It will now make 90 cookies.

   To make 45 cookies you would rewrite the recipe in the ratio $1\frac{1}{2}:1$

   $1\frac{1}{2}$ cups butter
   1 cup sugar
   3 eggs
   $2\frac{1}{4}$ cups flour
   $1\frac{1}{2}$ teaspoons vanilla

6. (a) $1.65

   (b) 45 cents (45 cents per dozen)

   (c) $119

   (d) 316.8 feet or 317 to the nearest foot.

7. 44 feet per second

8. (a)	12	14	$\frac{12}{14}$	$\frac{6}{7}$
(b)	18	21	$\frac{18}{21}$	$\frac{6}{7}$
(c)	30	35	$\frac{30}{35}$	$\frac{6}{7}$
(d)	$85\frac{5}{7}$	100	$\frac{85\frac{5}{7}}{100}$	$\frac{6}{7}$
(e)	100	$116\frac{2}{3}$	$\frac{100}{116\frac{2}{3}}$	$\frac{6}{7}$

Answers to Exercises 4-9a; page 135:

1. (a) 50%
   (b) 20%
   (c) 150%
   (d) 140%

2. (a) 60%
   (b) 30%
   (c) 20%
   (d) 80%
   (e) 20%
   (f) 10%

3. (a) Mortgage 20%; taxes 5%; car 9%; food 30%; clothing 12%; operating 8%; health and recreation 6%; savings and insurance 10%.
   (b) The 8 answers total 100%.

Answers to Exercises 4-9b; page 137:

1. (a) 30
   (b) 5%
   (c) 6
   (d) 60

2. (a) 25%
   (b) 75%

3. (a) $\frac{1}{10}$
   (b) 10%

4. 550 hint: $\frac{22}{x} = \frac{4}{100}$

Answers to Exercises 4-10; page 140:

1. (a) 65.87 (d) 0.483
   (b) 436.19 (e) 0.0026
   (c) 50.24

2. (a) $3(1) + 1(\frac{1}{10^2})$ (c) $1(\frac{1}{10}) + 1(\frac{1}{10^5})$
   (b) $1(\frac{1}{10^2}) + 2(\frac{1}{10^4})$ (d) $3(10) + 3(\frac{1}{10^2})$

3. (a) Seven and two hundred thirty-six thousandths.
   (b) Four thousandths.
   (c) Three hundred sixty and one hundred one thousandths
   (d) One and one hundred one ten-thousandths.
   (e) Nine hundred nine and nine thousandths.
   (f) Three and forty-four ten-thousandths.

4. (a) 300.52 (d) 60.07
   (b) 0.0507 (e) 0.00032
   (c) 0.014 (f) 8.019

*5. 0.5

*6. 0.6_{twelve}

*7. $0.T_{\text{twelve}}$

*8. $10.011_{\text{two}} = 1(2) + 1(\frac{1}{2^2}) + 1(\frac{1}{2^3}) =$

$$= 2 + \frac{1}{4} + \frac{1}{8} = 2 + \frac{2}{8} + \frac{1}{8} =$$

$$= 2\frac{3}{8} = 2.375.$$

Answers to Exercises 4-11a; page 143:

1. $\frac{7}{16} = 0.4375$, and therefore, 0.45 lb. is greater than 7 ounces.
2. 150.7 kilometers.
3. 75.7 kilometers.

*4.
$$\begin{array}{r} 10.01_{two} \\ \underline{1.01_{two}} \\ 11.10_{two} \end{array} \qquad 11.1_{two} = 1(2^1) + 1 + \frac{1}{2} = 3\frac{1}{2} \text{ or } 3.5$$

Answers to Exercises 4-11b; page 146:

1. (a) 0.00081 (b) 0.00625
2. (a) 1.4375 (b) 255.
3. 375, 37.5, 3.75, 0.375, 0.0375, 0.00375.
4. 0.0625
5. About 2.4 miles

*6. 3.102_{seven} (In the multiplication table, base seven, you have:

$$4_{seven} \times 4_{seven} = 22_{seven}, \text{ and}$$

$$2_{seven} \times 4_{seven} = 11_{seven}.$$

Answers to Questions in Class Discussion.

1. 1, 4, 2, 8, 5, 7.
2. Yes.
3. $0.09\overline{09}09\ldots$
4. After the first subtraction.

5. No.

6. Yes; by a bar.

8. 0.270$\overline{270}$ ...

Answers to Exercises 4-11c; page 149:

1. 0.0769230$\overline{769230}$ ...
   - (a) After the 5th subtraction:
   - (b) No.
   - (c) By dots.
   - (d) Yes.
   - (e) By a bar.

2. (a) 0.3$\overline{3}$ ... (c) 0.1$\overline{1}$ ...
   - (b) 0.8750$\overline{0}$ ...

3. (a) 0.09$\overline{09}$ ... (c) 1.27$\overline{27}$ ...
   - (b) 0.27$\overline{27}$ ...

4. Yes.

5. (a) 0.2, 0.4, 0.8
   - (b) 0.05, 0.15, 0.55
   - (c) 0.001, 0.112, 0.927

Answers to Exercises 4-12; page 151:

1. (a) 0.04 (b) 0.04
2. (a) 0.160 (c) 0.325
   (b) 0.000
3. (a) 0.3 (c) 0.1
   (b) 0.3 (d) 0.0
4. (a) 43.30 sq. rd.
   (b) 43.89 sq. rd.
   0.59 sq. rd. difference

Answers to Class Exercise 4-13; page 152:

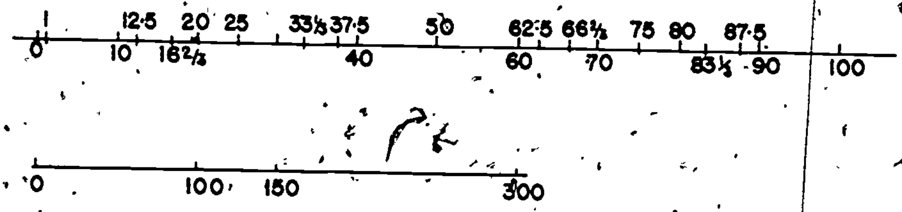
1. (a) 12% (b) 40%
2. (a) 18.75 (b) 0.03
3. (a) 0.8% (b) $47\frac{1}{2}$% or 47.5%
4. (a) 0.1625 (b) 0.1875

   Note that $0.16\frac{1}{4}$ and $0.18\frac{3}{4}$ we avoid since they are misleading.
5. (a) 111% (b) 29%
6. (a) $111\frac{1}{9}$% (b) $28\frac{4}{7}$%
7. (a) $\frac{101}{400}$ (b) $\frac{251}{200}$

   Of course many other answers are possible: For instance, (b) could be $\frac{125.5}{100}$.

8. (a)	$\frac{12\frac{1}{2}}{100}$	0.125	$12\frac{1}{2}$% or 12.5%
(b)	3	3	300%
(c)	$\frac{3}{8}$	$\frac{37\frac{1}{2}}{100}$	$37\frac{1}{2}$% or 37.5%
(d)	$\frac{3}{2}$	$\frac{150}{100}$	1.5%
(e)	$\frac{5}{8}$	0.625	62.5% or $62\frac{1}{2}$%
(f)	$\frac{1}{100}$	$\frac{1}{100}$	1%
(g)	$\frac{87.5}{100}$	0.875	87.5% or $87\frac{1}{2}$%
(h)	1	$\frac{100}{100}$	1
(i)	$\frac{1}{6}$	$0.1\overline{6}\overline{6}$ ...	$16\frac{2}{3}$%
(j)	$\frac{83\frac{1}{3}}{100}$	$0.83\overline{3}$ ...	$83\frac{1}{3}$%
(k)	$\frac{11\frac{1}{9}}{100}$	$0.11\overline{1}$ ...	$11\frac{1}{9}$%
(l)	$\frac{121}{200}$	0.605	$60\frac{1}{2}$%
(m)	$\frac{1}{200}$	$\frac{0.5}{100}$	0.5%

9.



Answers to Exercises 4-14a; page 156:

1. (a) \$0.70 (b) \$1.98

2. (a) 1117 (d) 32%

   (b) 34% (e) 100%

   (c) 34%

3. \$2400.

*4. The final net price would be the same for both methods of computation. After a number of examples, most pupils would see that the answer for a price of \$200 would be twice that for \$100, and similarly, for any multiple of \$100. If you had studied some algebra it could be shown as follows:

Computing it the first way you would have

$$P - (0.10)P - (0.05)\ [P - (0.10)P] =$$
$$P - (0.10)P - (0.05)\ P + (0.05)(0.10)P.$$

Computing it the other way you would have

$$P - (0.05)P - (0.10)\ [P - (0.05)P] =$$
$$P - (0.05)P - (0.10)P + (0.10)(0.05)P.$$

*5. If the sales tax were computed on \$100, the price would be \$102 before the discount, and \$91.80 after, just as before. This may seem a little strange since in the latter case, the discount is on the larger amount and the percentage of discount is larger than that of the tax. The same arguments based on examples could be used here, as were used in the previous problem. The algebra would look like this: By the first computation, the final amount would be:

$$P - (0.10)P + (0.02)\ [P - (0.10)P] =$$
$$P - (0.10)P + (0.02)P - (0.02)(0.10)P$$

and by the second it would be:

$$P + (0.02)P - (0.10)\ [P + (0.02)P] =$$
$$P + (0.02)P - (0.10)P - (0.10)(0.02)P.$$

*6. Under the computations of Problem 4, the net price is $85\frac{1}{2}\%$ of the original price, no matter what the original price is. This means that the discount is really $14\frac{1}{2}\%$ for this way of computing. The shopkeeper might explain that the discounts were to apply successively, that is, one after the other. As you have seen, it does not matter which is applied first.

Answers to Exercises 4-14b; page 159:

1. (a) 150 (c) decrease of 20%
   (b) 120

2. 14.3% or $14\frac{2}{7}\%$

3. 5.6% or $5\frac{5}{9}\%$

4. 30.9%

5. (a) The 1960 wages are less than the 1958 wages.
   (b) 96%

   Suggestion: Start with 1958 wages of $100 per week. You might be interested in finding what percent of increase from 1959 to 1960 will bring the wages back to $100. 25%.

6. (a) 15% (b) 17.6%

7. (a) $60 (c) $60
   (b) $60

   The object of this problem is to show that though the wording of the problems is different, the mathematics is the same.

8. (a) $484.50 (c) $484.50
(b) $484.50

Again the object is to show that the mathematics is the same in all cases.

9. 5.3 %.

10. (a) $677.42 (c) $677.42
(b) $677.42 (d) $ 65.10

By this time you might expect the answers to be the same. It will be fine if you can do this with discrimination. Since here, (d) is quite different, you may learn to proceed with caution.

11. (a) 2,258 gal.
(b) 67,742 mi.
(c) About 186 miles per day.
(d) It is unlikely that a man who walks to work would drive 186 miles a day.

Answers to Exercises 4-15a; page 163:

1. (a) 10^3 (d) 10^9
(b) 10^{10} (e) 10^6
(c) 10^4 (f) 10^7

2. (a) 6×10^3 (d) 7.8×10^4
(b) 6.78×10^2 (e) 6×10^3
(c) 4.59×10^8 (f) 7.81×10^9

3. (a) 30,000 (c) 436,000,000
(b) 50,000,000 (d) 1,732,400

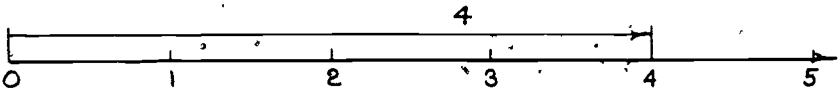
4. (a) $600 = 6 \times 10^2$ (d) $70900 = 7.09 \times 10^4$
   (b) $100 = 10^2$ (e) $600,000 = 6 \times 10^5$
   (c) $1200 = 1.2 \times 10^3$ (f) $5,362,400 = 5.3624 \times 10^6$

Answers to Exercises 4-15b; page 164:

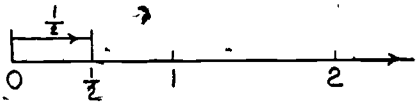
1. (a) $60 \times 186,000$
   (b) $60 \times 60 \times 186,000$
   (c) $24 \times 60 \times 60 \times 186,000$
   (d) $365 \times 24 \times 60 \times 60 \times 186,000$
   (e) Do not "round off" until multiplications have been performed; if "rounding off" takes place earlier the result becomes less accurate. Before rounding it is 6,307,200,000,000.
   (f) trillion
   (g) The speed of light used is the result of rounding an approximation. The number of days in a year (365) is an approximation also.
2. (a) 3.5×10^{13} (c) 2.1×10^6
   (b) 6×10^8 (d) 9.3×10^{14}
3. (a) 6.3×10^{11} (c) 4.65×10^9
   (b) 10^9 (d) 1.1×10^9
4. 7.2×10^9 miles
5. About 6.132×10^8 miles or 613,200,000 miles.

Answers to Exercises 5-1; page 170:

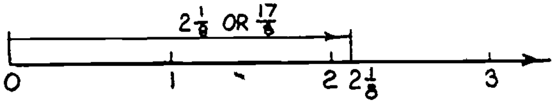
1. (a)



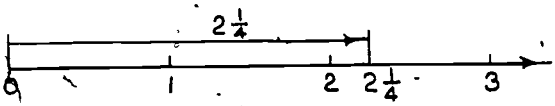
(b)



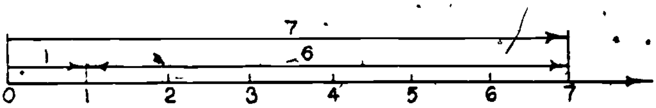
(c)



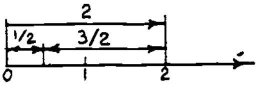
(d)



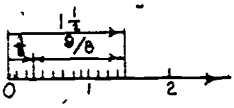
2. (a)



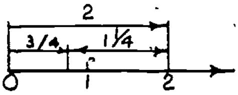
(b)



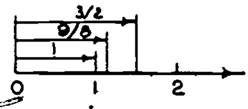
(c)



(d)

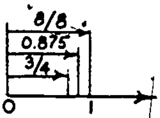


3. (a)



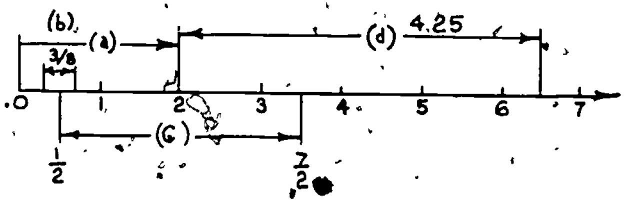
$\frac{3}{2}$ is the largest number in the set.

(b)



$\frac{8}{8}$ is the largest number in the set.

4.



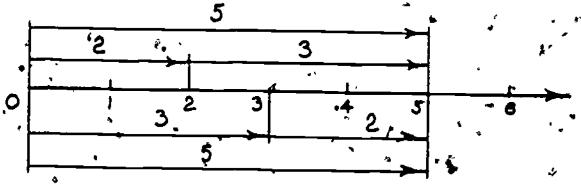
(a) [0,2], mid-point is 1

(b) [$\frac{1}{8}$, $\frac{5}{8}$], mid-point is $\frac{3}{8}$

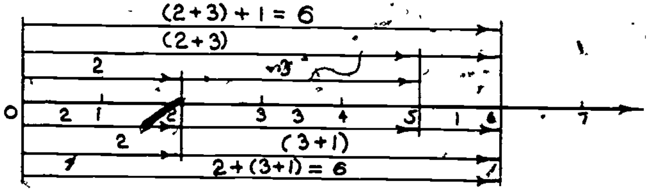
(c) [$\frac{1}{2}$, $\frac{7}{2}$], mid-point is 2

(d) [2, 6.5], mid-point is 4.25

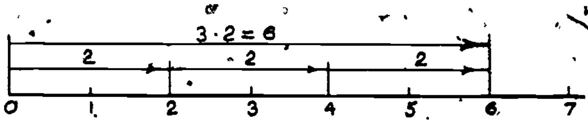
5.



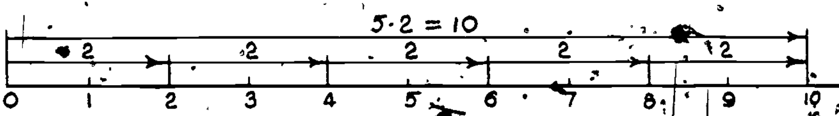
6.



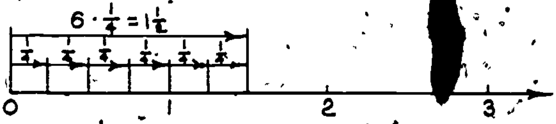
7. (a)



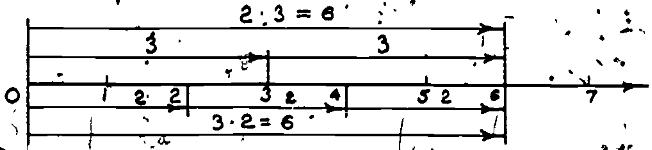
(b)



(c)



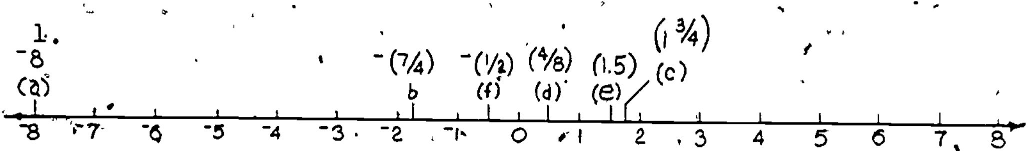
8.



This illustrates the commutative property of multiplicatio

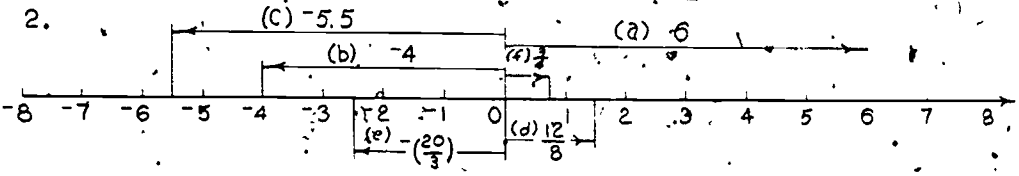
Answers to Exercises 5-2; page 174:

1.



yes $^-(\frac{7}{4})$ and $1\frac{3}{4}$ $\frac{4}{8}$ and $^-(\frac{1}{2})$

2.



3. $^-6$, $^-4$, $^-(\frac{7}{4})$, $^-(\frac{3}{8})$, $\frac{1}{4}$, $\frac{5}{8}$, $\frac{3}{4}$

largest is $\frac{3}{4}$

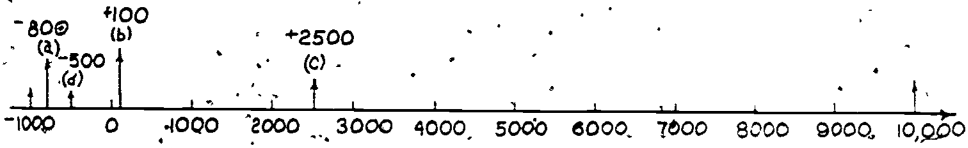
smallest is $^-6$

4. (a) $^+2000$ (b) $^+100$ (c) $^+2$

$^-6000$ $^-50$ $^-4$

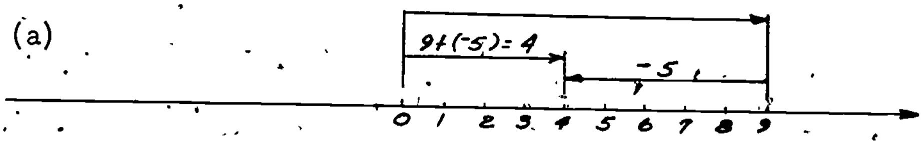
5. $^-2$, $^-1$, 0; $^+1$, $^+2$, $^+3$

6.

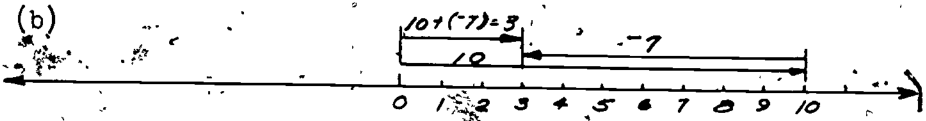


Answers to Exercises 5-3; page 179:

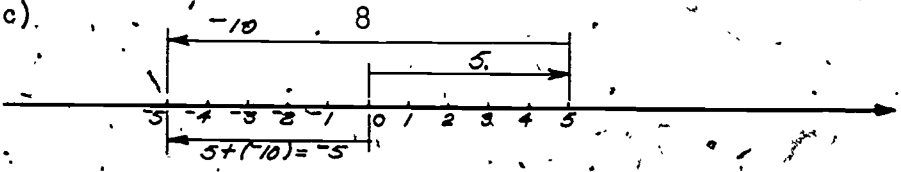
1. (a)



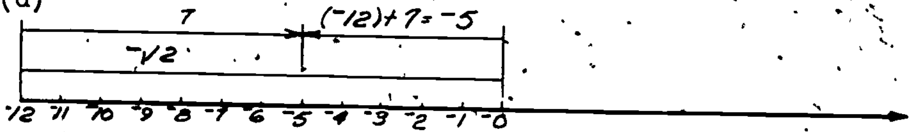
(b)



(c)



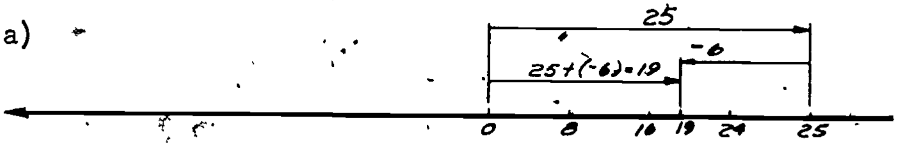
(d)



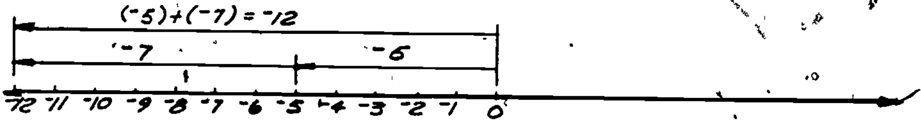
2. (a) 0 (c) $^-1$

(b) 4 (d) 12

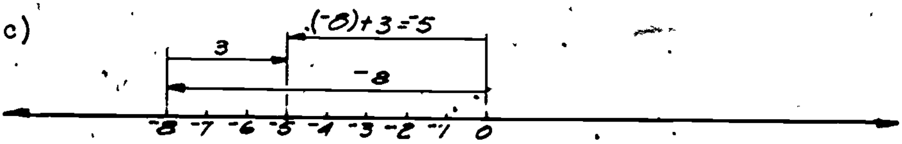
3. (a)



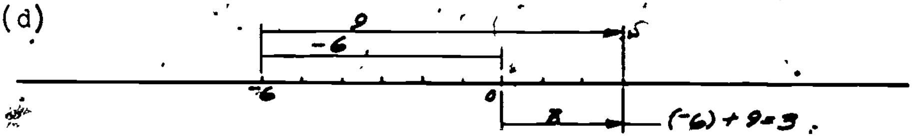
(b)



(c)



(d)



4. (a) $^-5$ (c) $^-6$

(b) $^-11$ (d) $^-6$

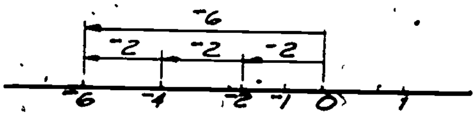
5. (a)

January	$^+5000$	April	$^+1000$
February	$^+2000$	May	$^-4000$
March	$^-6000$	June	$^-3000$

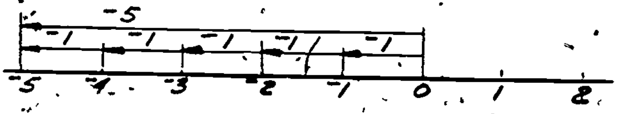
(b) $^-5000$ or \$ 5000 loss
(c) $^+1000$ or \$ 1000 profit
(d) $^-12,000$ or \$ 12,000 loss

6. (a) $^+17,\ ^-6,\ ^+11,\ ^-3$
(b) $20 + 17 = 37$
$37 + (^-6) = 31$
$31 + 11 = 42$
$42 + (^-3) = 39$
ball is on the 39 yard line
(c) $17 + (^-6) + 11 + (^-3) = 19$ The net gain is 19 yards.

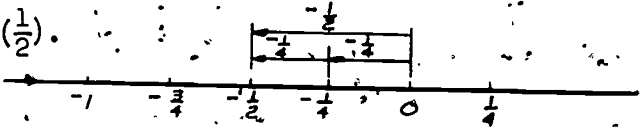
7. (a) $3(^-2) = {}^-6$



(b) $5(^-1) = {}^-5$



(c) $2 \cdot {}^-(\frac{1}{4}) = {}^-(\frac{1}{2})$.



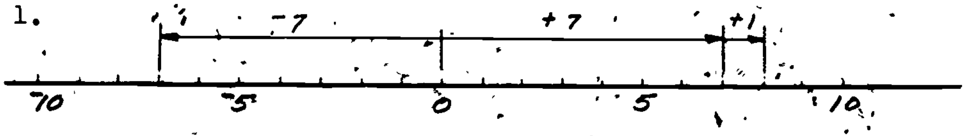
Answers to Class Exercises 5-3a; page 181:

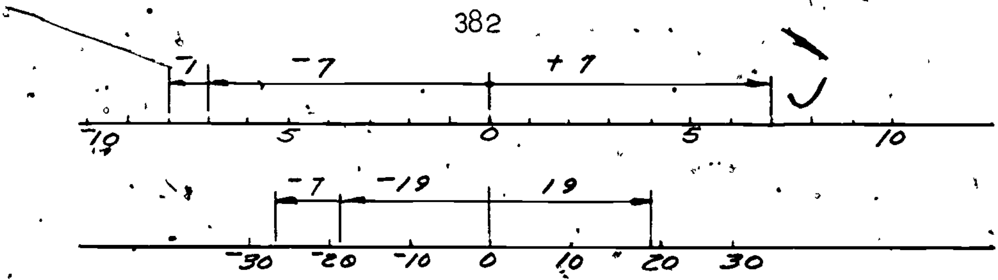
1. $^-7,\ 9,\ ^-11,\ 12,\ 6,\ ^-15,\ 20,\ 0,\ \frac{2}{3},\ ^-(\frac{4}{9}),\ \frac{7}{8},\ ^-(\frac{30}{31})$

2. The pairs in (c) and in (d) are additive inverses.

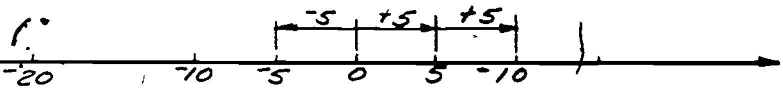
Answers to Class Exercises 5-3b; page 183:

1.

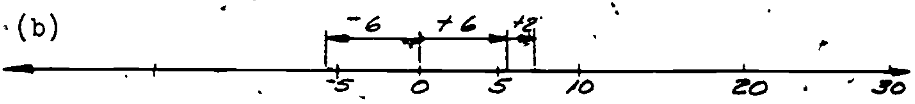




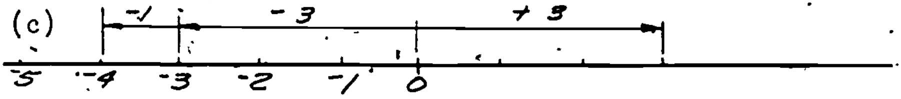
2. (a)



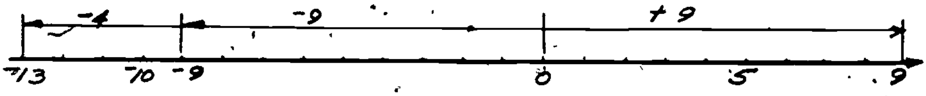
(b)



(c)

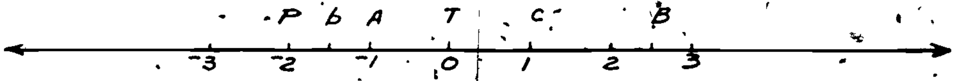


(d)



Answers to Exercises 5-4a; page 184:

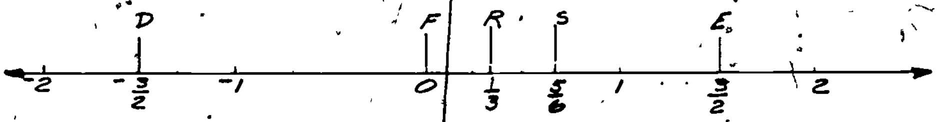
1.



2. (a) $1\frac{1}{2}$" (c) 4"

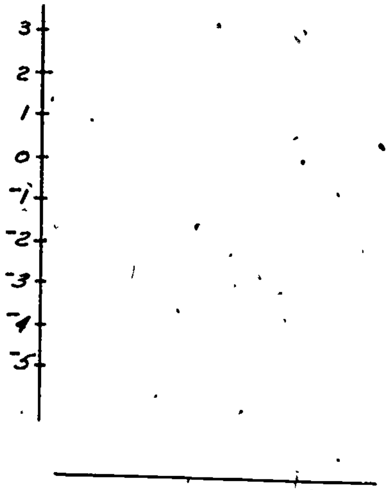
(b) $4\frac{1}{2}$" (d) 1"

3.



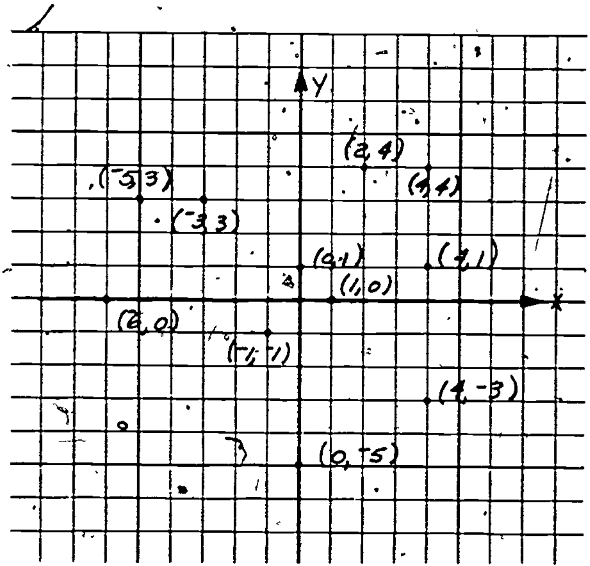
4. (a) $\frac{1}{3}$ mile (b) 3 miles

5.

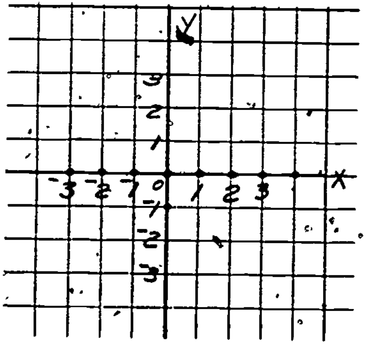


Answers to Exercises 5-4b; page 187:

1.



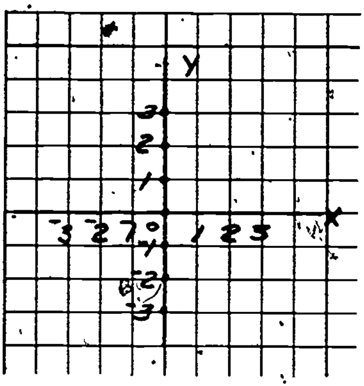
2. (a)



(b) yes

(c) yes

3. (a)



(b) yes

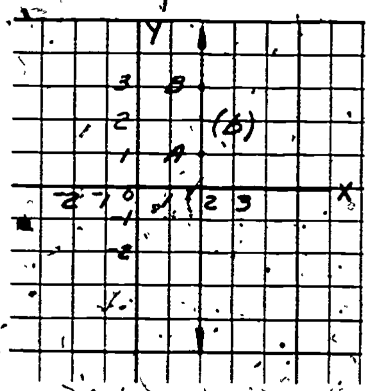
(c) yes

Answers to Class Exercises 5-4a; page 189:

1. (a) I
   (b) IV
   (c) II
   (d) III
   (e) I
   (f) IV
   (g) II
2. (a) I
   (b) III
   (c) II
   (d) III
3. (a) Y-axis
   (b) X-axis
   (c) At the origin

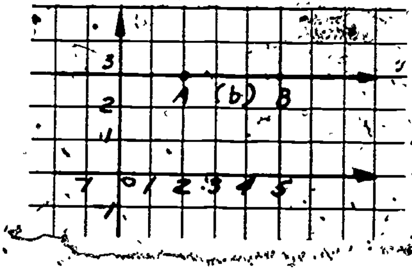
Answers to Exercises 5-4c; page 190:

1.



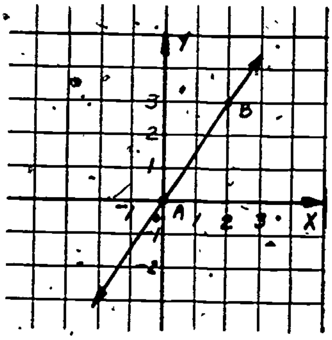
(c) the Y-axis

2.



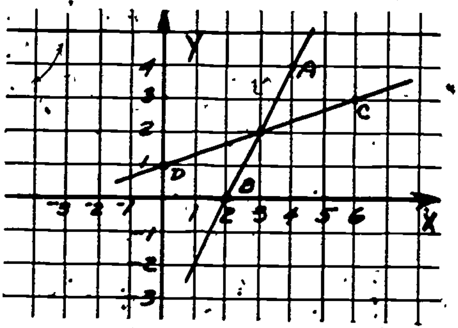
(c) the X-axis

3.

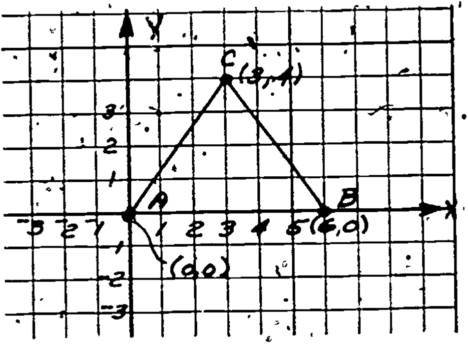


(c) neither axis.
AB is an oblique line.

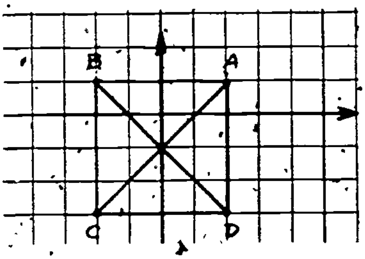
4.



(e) {(3,2)}

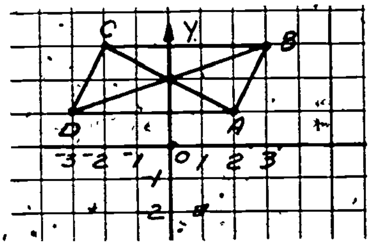
5. 

(c) isosceles

6. 

(c) yes

(e) (0, $^-1$)

7. 

(c) parallelogram

(e) (0, 2)

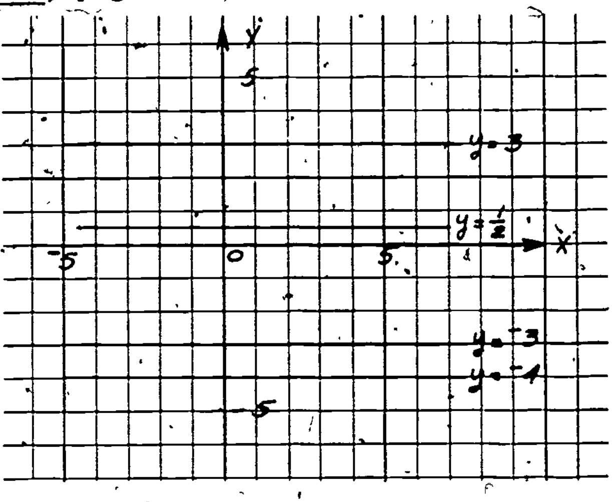
Answers to Class Exercises 5-4b; page 192:

1. (a) $y = 3$

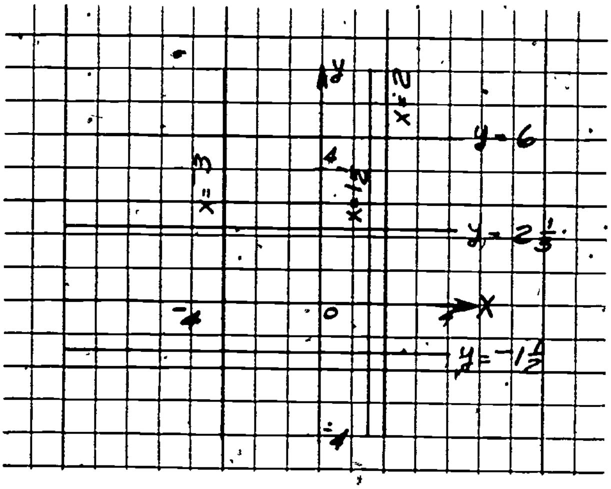
   (b) $y = {}^-3$

   (c) $y = {}^-4$

   (d) $y = \frac{1}{2}$

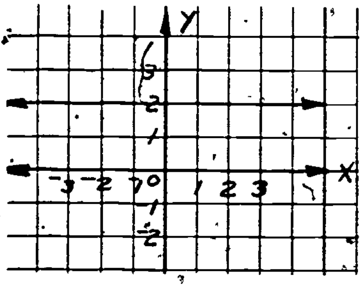


2.

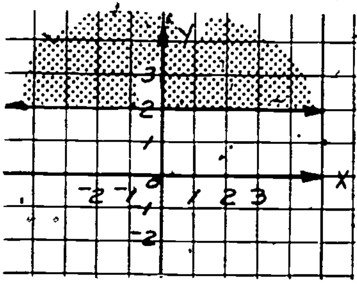


Answers to Exercises 5-4d; page 194:

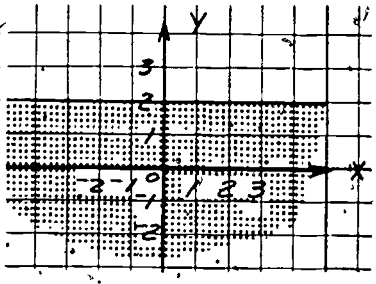
1. (a) $y = 2$



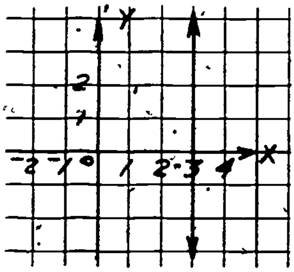
(b) $y > 2$



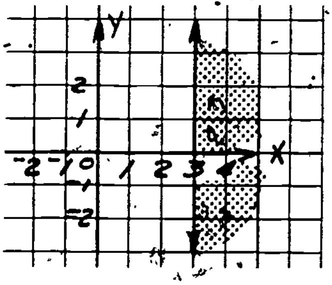
(c) $y < 2$



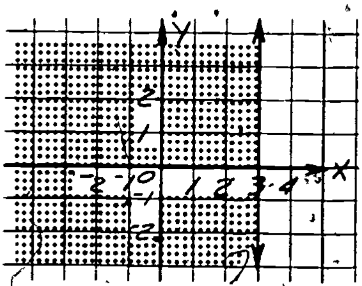
(d) $x = 3$



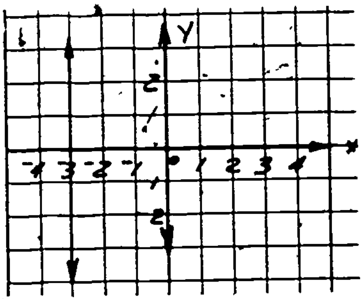
(e) $x > 3$



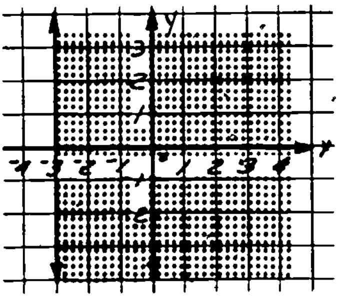
(f) $x < 3$



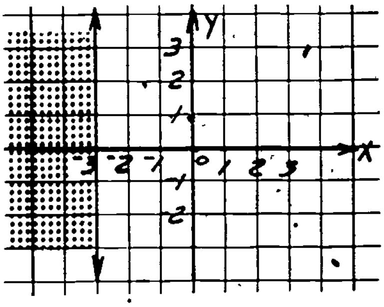
(g) $x = {}^-3$



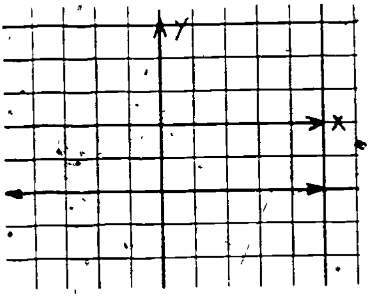
(h) $x > {}^-3$



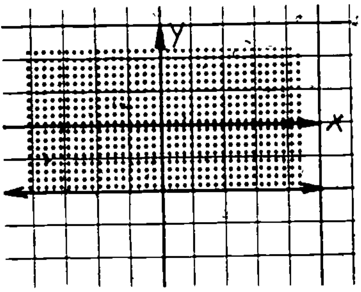
(i) $x < {}^-3$



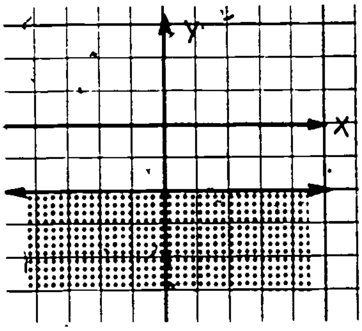
(j) $y = {}^-2$



(k) $y > {}^-2$



(l) $y < {}^-2$

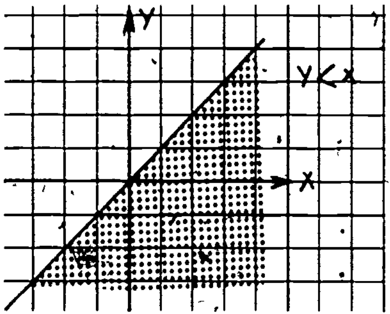


Answers to Exercises 5-4e; page 195:

1. (a) Possible points include $(4, 4)$, $(^-4, ^-4)$, $(5, 5)$ $(5, ^-5)$; $(\frac{1}{2}, \frac{1}{2})$, $(^-\frac{1}{2}, ^-\frac{1}{2})$.

   (b) The graph of $y = x$ is in the text.

2. (a)



   (b) A possible set R is:

   $R : \{(0, ^-1), (1, 0), (^-1, ^-3), (2, 1)\}$

3. The graph of $y = x$ is a line which passes through the origin and makes an angle of 45° with the x-axis.

   The graph of $y > x$ is the half-plane above and to the left of $y = x$. The graph of $y < x$ is the half-plane below and to the right of $y = x$.

Answers to Exercises 5-5a; page 197:

1. The products increase in the 1-row, 2-row, and 3-row as we move to the right.
2. The products decrease, as we move down, in the $^-2$ column and the $^-1$ column.
3. $^-28$, $^-21$, $^-14$, $^-7$, 0, 7, 14, 21, 28, 35, 42
4. 20, 16, 12, 8, 4, 0, $^-4$, $^-8$, $^-12$, $^-16$, $^-20$
5.

	$^-3$	$^-2$	$^-1$	0	1	2	
$^-5$	15	10	5	0	$^-5$	$^-10$	
$^-4$	12	8	4	0	$^-4$	$^-8$	
$^-3$	9	6	3	0	$^-3$	$^-6$	
$^-2$	6	4	2	0	$^-2$	$^-4$	
$^-1$	3	2	1	0	$^-1$	$^-2$	
0	0	0	0	0	0	0	
1	$^-3$	$^-2$	$^-1$	0	1	2	

6. (a) $(^-2)(1) = {}^-2$, $(1)(^-2) = {}^-2$, therefore $(^-2)(1) = (1)(^-2)$

   (b) $(^-3)(0) = 0$, $(0)(^-3) = 0$, therefore $(^-3)(0) = (0)(^-3)$

   (c) $(^-4)(^-5) = 20$, $(^-5)(^-4) = 20$, therefore $(^-4)(^-5) = (^-5)(^-4)$

   (d) $(15)(^-6) = {}^-90$, $(^-6)(15) = {}^-90$, therefore $(15)(^-6) = (^-6)(15)$

7. $(^-2)[(^-1)(5)] = (^-2)(^-5) = 10$, $[(^-2)(^-1)](5) =$
$(2)(5) = 10$, therefore $(^-2)[(^-1)(5)] = [(^-2)(^-1)](5)$.

8. (a) $^-4(3 + 8) = (^-4)(3) + (^-4)(8) = (^-12) + (^-32) = {}^-44$
$^-4(3 + 8) = {}^-4(11) = {}^-44$
therefore $^-4(3 + 8) = {}^-4(11)$

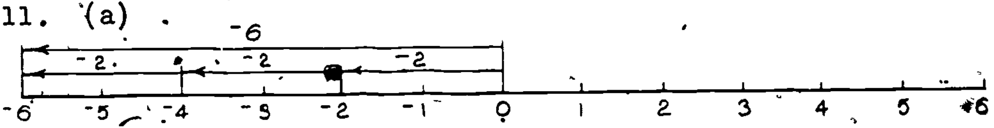
(b) $^-2(^-3 + 6) = (^-2)(^-3) + (^-2)(6) = 6 + (^-12) = {}^-6$
$^-2(^-3 + 6) = (^-2)(3) = {}^-6$
therefore $^-2(^-3 + 6) = (^-2)(3)$

(c) $^-10[(^-8) + (^-1)] = (^-10)(^-8) + (^-10)(^-1) = 80 + 10 = 9$
$^-10[(^-8) + (^-1)] = {}^-10(^-9) = 90$
therefore $^-10[(^-8) + (^-1)] = (^-10)(^-9)$

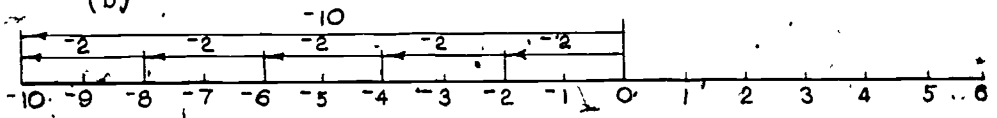
9. (a) 0 (e) $^-34$ (i) 903 (m) $^-66$
(b) $^-8$ (f) $^-245$ (j) 0.84 (n) $^-(\frac{160}{3})$
(c) $^-20$ (g) 54 (k) $^-60$ (o) $^-16$
(d) $^-24$ (h) 600 (l) 576

10. (a) $^-4$ (b) $^-5$ (c) $^-11$ (d) $^-8$ (e) $^-77$

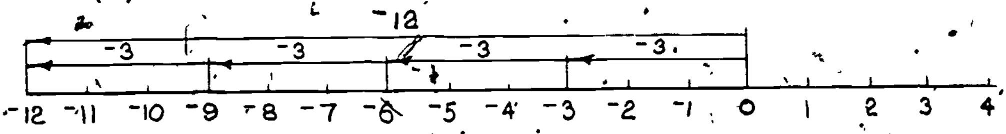
11. (a)



(b)



(c)



12. (a) 36 yard line

(b) In order to ascertain the initial position of the two teams you have to undo the addition of $3(^-2)$.

13. (a) A football team has the ball on the 47 yard line. It loses 5 yards on each of 4 successive plays. Its new position is ___?___.

(b) After losing three yards on two successive plays a football team is on the 15 yard line. At what position was the team before these losses?

14. (a) $^-12$ (c) $^-5$ (e) 4
(b) $^-15$ (d) $^-10$ (f) 2

15. (a) $^-2$ (d) $^-4$ (f) $^-10$
(b) $^-3$ (e) 4 (g) $^-1$
(c) $^-10$

16. (a) 60 (d) 42 (g) $^-40$
(b) 12 (e) 40 (h) 42
(c) 15 (f) $^-40$

Answers to Exercises 5-5b; page 202:

1. (a) $^-28$ (c) $^-12$ (e) 4
(b) $^+12$ (d) $^-3$ (f) $\frac{2}{5}$

2. (a) $^-4$ (c) $^-6$ (e) $^-10$
(b) $^-4$ (d) $^-(\frac{3}{4})$ (f) $^-(\frac{12}{25})$

3. (a) $r = {^-2}$ (c) $r = 7$ (e) $r = {^-(\frac{1}{2})}$
(b) $r = {^-3}$ (d) $r = \frac{3}{16}$ (f) $r = 1$

4. $P' = \{\frac{1}{6}, \frac{2}{3}, 1, \frac{6}{5}, {^-1}, {^-(\frac{2}{3})}, {^-(\frac{3}{7})}\}$

5. (a) 2 (e) $^-5$
(b) $^-5$ (f) 5
(c) $^-5$ (g) 0
(d) 5

6. (a) $n = \frac{(^-2)}{3} = {^-(\frac{2}{3})}$, (b) $n = \frac{2}{(^-3)} = {^-(\frac{2}{3})}$

7. $\frac{(^{-}2)}{3} = \frac{2}{(^{-}3)}$

8. (a) $n = \frac{^{-}6}{7} = {}^{-}(\frac{6}{7})$, (b) $n = \frac{6}{^{-}7} = {}^{-}(\frac{6}{7})$

9. $(^{-}6)\ n = 7$, $6n = {}^{-}7$

10. (a) $n = (\frac{92}{25}) = 3\frac{17}{25}$ (c) $n = \frac{1}{3}$

(b) $n = \frac{92}{25} = 3\frac{17}{25}$ (d) $n = \frac{1}{3}$

11. (a) $bx = a$
 (b) either both positive or both negative
 (c) negative, positive

Answers to Exercises 5-5c; page 206:

1. (a) 3 (d) 5 (g) $^{-}45$
 (b) 0 (e) 10 (h) $^{-}67$
 (c) $^{-}7$ (f) 8 (i) $^{-}28$

2. $\frac{11}{4}$, $\frac{11}{4} - {}^{-}(\frac{3}{4}) = \frac{7}{2}$, $\frac{11}{4} - \frac{7}{2} = {}^{-}(\frac{3}{4})$

3. (a) $x = {}^{-}3$ (e) ${}^{-}(\frac{13}{6})$

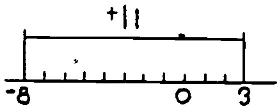
(b) $x = 11$ (f) ${}^{-}(\frac{3}{2})$

(c) $x = {}^{-}11$ (g) 5

(d) $x = 15$ (h) ${}^{-}(\frac{49}{12})$

4. (a) $^{-}5$ (d) 6
 (b) 3 (e) 3
 (c) $^{-}6$ (f) 3

5.



If one wishes to jump from $^{-}8$ to $^{+}3$ one has to travel in a positive direction and one must cover 11 units
$^{+}3 - (^{-}8) = {}^{+}11$.

6. (a) $^{-}10$ (c) $^{-}(\frac{1}{2})$ (e) $\frac{8}{5}$
(b) 100 (d) $^{-}(\frac{7}{9})$ (f) $^{-}\frac{49}{51}$

7. If $a - b = c$, and $a + (^{-}b) = c$, then $a - b = a + (^{-}b)$ (the transitive property of equality.)

8. (a) $^{-}7$ (e) $^{-}10$ (i) 2 (ℓ) 11
(b) $^{-}2$ (f) $^{-}6$ (j) 12 (m) $^{-}7$
(c) 4 (g) $^{-}11$ (k) $^{-}7$ (n) 13
(d) 10 (h) 12

9.

x	$^{-}1$	0	1	2	3	4
y	$^{-}5$	$^{-}3$	$^{-}1$	1	3	5

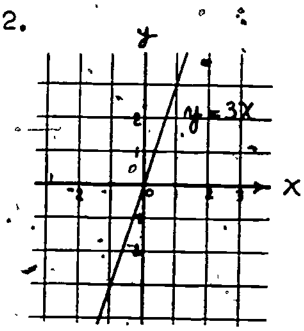
10.

x	$^{-}2$	$^{-}1$	0	1	2	3	4
y	5	3	1	$^{-}1$	$^{-}3$	$^{-}5$	$^{-}7$

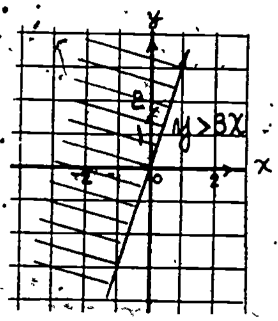
Answers to Exercises 5-5d; page 213:

1. sample set $W = \{(0,0), (1,3), (^{-}1,3), (2,6), (^{-}2,^{-}6)\}$

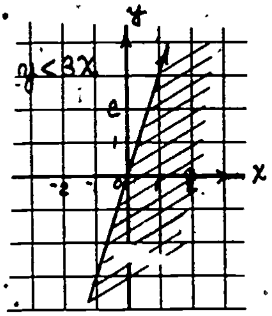
2.



3.



4.



5. Yes. All points in the plane will be included in the graph since for each value of x, y is allowed to be less than $3x$, greater than $3x$, or equal to $3x$. Note: Strictly speaking these graphs will not contain <u>all</u> points of the plane unless you have the <u>real</u> numbers to use as coordinates.

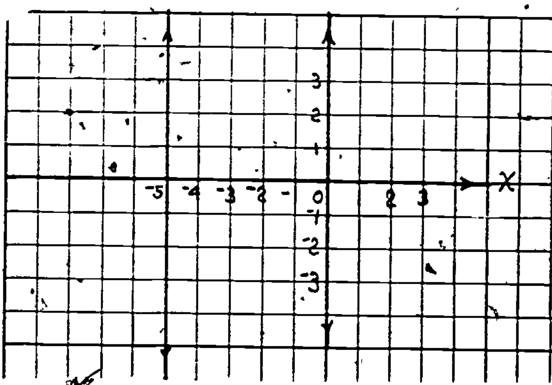
6. Sample points in the graph of $y > 3x + 2$ might be: $(0,3)$, $(1,6)$, $(2,9)$, $(3,11)$, $(4,14)$, $(^-1,^-2)$, $(^-2,^-5)$, $(^-3,^-8)$.

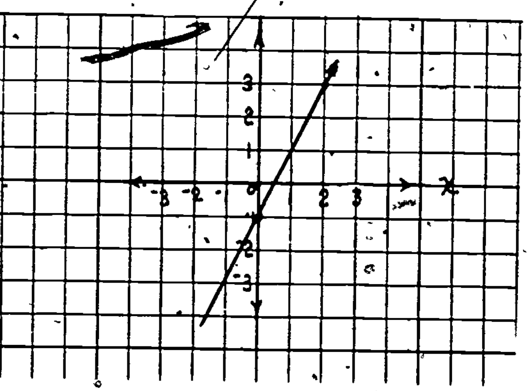
7. (a) $y > 3x + 2$ (d) $y < 3x + 2$

(b) $y > 3x + 2$ (e) $y > 3x + 2$

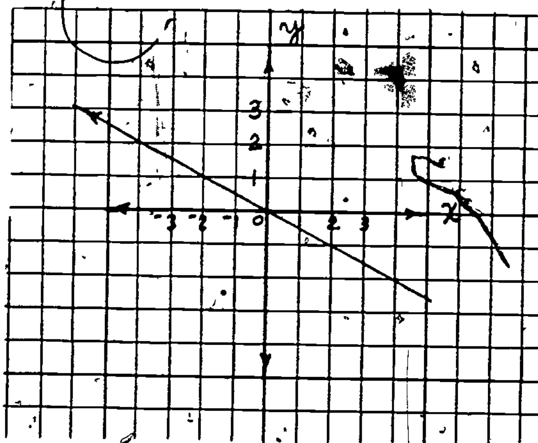
(c) $y > 3x + 2$

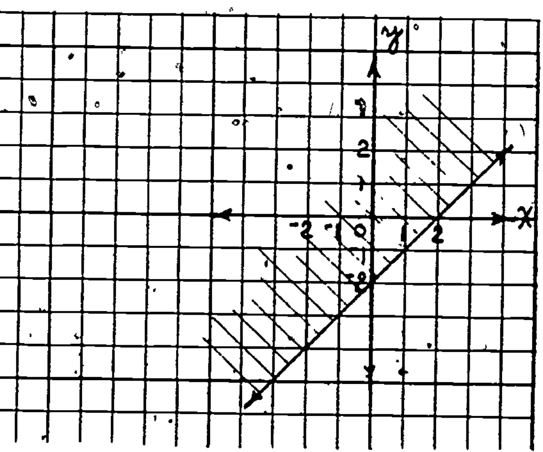
8. (a) $x = {}^-5$ (b) $y = 2x - 1$





(c) $y = {}^-(\frac{1}{2})x$ (d) $y > x - 2$





9. (a) no
(b) See diagram (a) in the text
(c) yes
(d) yes
(e) Samples (5,25), ($^{-}5$,25), (6,36), etc.
(f) yes

Answers to Exercises 5-6; page 217:

1. (a) $x = 2$
(b) $y = 7$
(c) $k = 2$
(d) $m = 6$
(e) $s = 25$
(f) $t = {}^{-}5$

2. (a) $x = 9$
(b) $y = 10$
(c) $n = 11$
(d) $x = 9$
(e) $p = 14$
(f) $x = 8$

3. (a) $b = 3$
(b) $a = 3$
(c) $w = 7$
(d) $m = 8$
(e) $x = {}^{-}1$
(f) $y = {}^{-}8$

4. (a) $n = 6$
(b) $a = 16$
(c) $k = {}^{-}16$
(d) $d = 18$
(e) $n = {}^{-}15$
(f) $s = 21$

5. $p = 22$ (feet)

6. $A = 49$ (square inches)

7. $A = 225$ (square inches)

8. $i = 135$ (dollars)

9. $c = 62\frac{6}{7}$ (inches) or 62.8 (inches)

10. $A = 531\frac{1}{7}$ (square feet) or 531.66 (square feet)

Answers to Exercises 5-7a; page 221:

1. (a) $x + 5$
(b) $x - 3$
(c) $8x$
(d) $\frac{1}{4}x$ or $\frac{x}{4}$
(e) $x + 10$
(f) $7x$
(g) $x - 11$
(h) $\frac{x}{2}$

2\. (a) 17
(b) 9
(c) 96
(d) 3
(e) 22
(f) 84
(g) 1
(h) 6

3\. Students will undoubtedly write different translations for each of these phrases. You should be primarily concerned with correct order of numbers.

(a) The number x increased by 1.
(b) The number x decreased by 3.
(c) The number 2 multiplies by x.
(d) The number 18 divided by x.
(e) The number 4 multiplied by x.
(f) The number $^{-}6$ added to x.

4\. (a) 7
(b) 3
(c) 12
(d) 3
(e) 24
(f) 0

5\. (a) $^{-}1$
(b) $^{-}5$
(c) $^{-}4$
(d) $^{-}9$
(e) $^{-}8$
(f) $^{-}8$

6\. (a) $6 + a$
(b) $8b$
(c) $8c + 1$
(d) $8d - 3$
(e) $\frac{8e}{4}$
(f) $2f + 3$
(g) $5(g + 2)$
(h) $7h - 10$

7\. (a) 3
(b) $^{-}24$
(c) $^{-}23$
(d) $^{-}27$
(e) $^{-}6$
(f) $^{-}3$
(g) $^{-}5$
(h) $^{-}31$

8. (a) Five more than two times a certain number.
   (b) Three times a certain number and the result subtracted from six.
   (c) A number decreased by one and the result multiplied by 7.
   (d) Five decreased by a number and the quantity divided by two.
   (e) Fifteen increased by two times a number

9. (a) 15
   (b) $^{-}5$
   (c) 5
   (d) 3

10. (a) 6
   (b) 9
   (c) 3
   (d) $^{-}9$

Answers to Exercises 5-7b; page 226:

1. (a) $x + 5 = 13$
   (b) $x - 3 = 7$
   (c) $8x = 24$
   (d) $\frac{x}{4} = 9$
   (e) $x + 10 = 21$
   (f) $7x = {}^{-}35$
   (g) $x - 11 = {}^{-}5$
   (h) $x - 6 = 15$

2. (a) $\{8\}$
   (b) $\{10\}$
   (c) $\{3\}$
   (d) $\{36\}$
   (e) $\{11\}$
   (f) $\{^{-}5\}$
   (g) $\{6\}$
   (h) $\{21\}$

3. (a) $x + 2 > 4$
   (b) $5x < 10$
   (c) $\frac{x}{7} > 2$
   (d) $x - 3 > 6$
   (e) $x - 5 < 13$
   (f) $3x > {}^{-}9$

4. (a) The set of all numbers greater than 2
   (b) The set of all numbers less than 2
   (c) The set of all numbers greater than 9
   (d) The set of all numbers greater than 14
   (e) The set of all numbers less than 18
   (f) The set of all numbers greater than $^{-}3$.

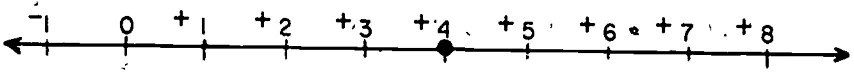
5. (a) Two more than a certain number is equal to five.
   (b) The sum of a certain number and negative three is seven.
   (c) Two multiplied by a number is equal to negative ten.
   (d) A number decreased by five is greater than nine.
   (e) The product of five and a number is less than fifteen.
   (f) Seven decreased by a number is two.
   (g) Three subtracted from a number is less than four.
   (h) A number divided by 3 is greater than nine.
   (i) A number decreased by seven is equal to negative two.
   (j) A number divided by $^-30$ is equal to six.

6. (a) {3}
   (b) {10}
   (c) {$^-5$}
   (d) The set of all numbers greater than 14
   (e) The set of all numbers less than 3
   (f) {5}
   (g) The set of all numbers less than 7
   (h) The set of all numbers greater than 27
   (i) {5}
   (j) {$^-180$}

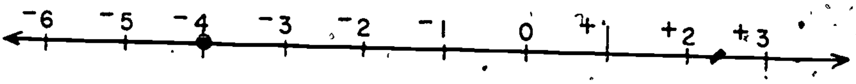
<u>Answers to Exercises 5-7c</u>; page 229:

1. (a) {4}
   (b) {$^-4$}
   (c) {3}
   (d) {all numbers less than 1}
   (e) {all numbers greater than 5}
   (f) {$^-5$}
   (g) {all numbers less than 5}
   (h) {all numbers less than 3}

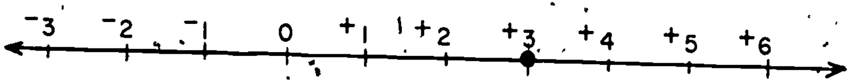
2. (a)



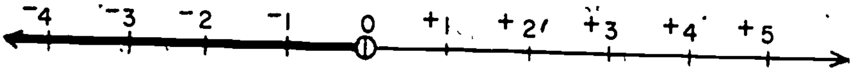
(b)



(c)



(d)



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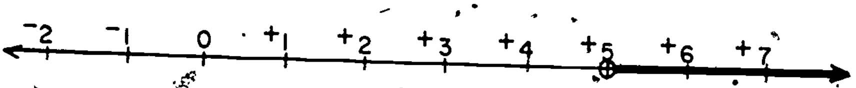
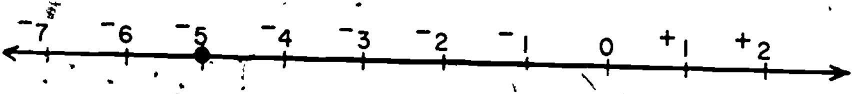


(f)



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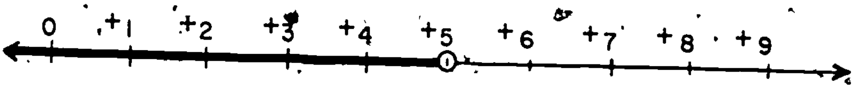

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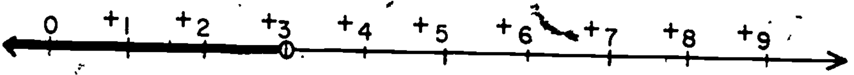


3. (a) {all numbers}

(b) {all numbers greater than 1}

(c) {all numbers less than 1}

(d) The empty set. (The sentence has no solutions.)

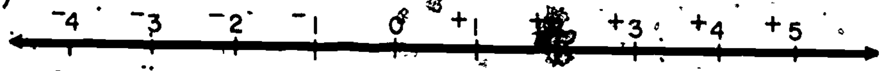
(e) {$^{-}5$}

(f) {$^{-}1$}

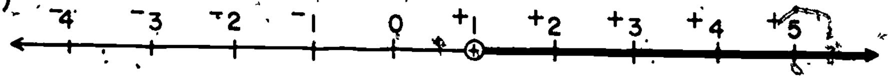
(g) {$^{-}1$}

(h) {$^{-}2$}

(a)



(b)



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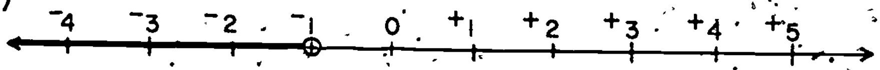

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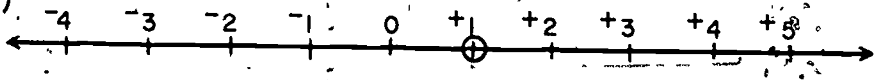
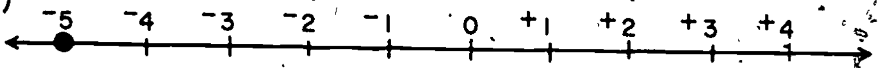
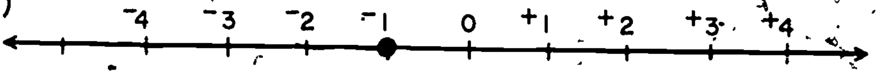


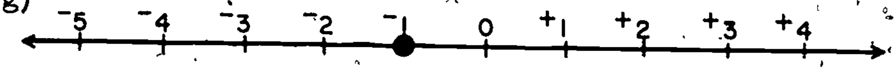
(e)



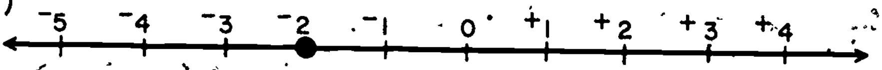
(f)



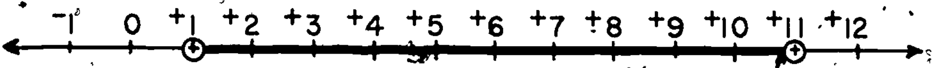
(g)



(h)



5. The set of all numbers between 1 and 11.



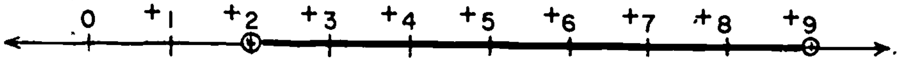
6. The intersection

7. (a) The set of all numbers less than 9 and greater than 2.

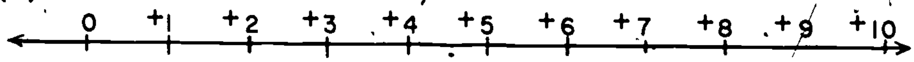
(b) The empty set.

(c) All numbers greater than 3 and less than 6.

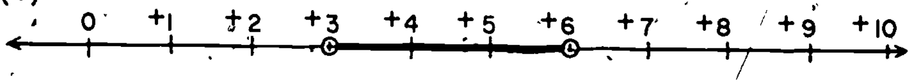
8. (a)



(b)

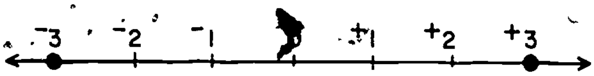


(c)



9. (a) $\{3, {}^-3\}$

(b)



10. (a) All numbers greater than $^-3$ and less than 3.

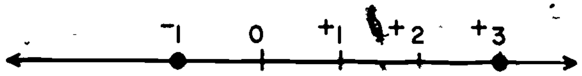
(b)

11. (a) For $x + 7 = 6$ $x = {}^-1$

For $2x - 1 = 5$ $x = 3$

The set of solutions is $\{{}^-1, 3\}$

(b)

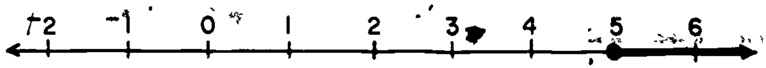


12. (a) For $x - 1 = 4$ $x = 5$

For $x - 1 > 4$ x is all numbers greater than 5.

The set of solutions is 5 and all numbers greater than 5.

(b)



13. The solution set of "$x < 10$" is all numbers less than 10
The solution set of "$x - 9 > 0$" is all numbers greater than 9.

Every number is in one of these two sets (or both). Thus the solution set of the compound sentence is the set of all numbers.

Answers to Exercises 5-7d; page 232:

1. (a) y is the number of years in his sister's age.

$$10 = y - 4$$

(b) m is the number of model plane kits.

$$25m = 75$$

(c) b is the number of years in Sam's age now.

$$b + 7 = 20$$

(d) f is the number of feet.

$$12f = 72$$

(e) f is the number of feet.

$$\frac{f}{3} = 5$$

(f) n is the number of years in Ann's age now.

$$n - 10 = 3$$

(g) d is the number of dollars.

$$100d = 450$$

(h) x is the number of cents earned per hour before midnight.
y is the number of cents earned per hour after midnight.

$$4x + 2y = 350$$

2. (a) d is the number of dollars that Dick has.

$3 + 2x < 23$

(b) r is the number of miles per hour that the plane travels.

$2r > 500$

(c) a is the number of years in Susie's age.

$1 + 2a = 19$

(d) x is the number of hours it takes to drive from Mill City to Dover.

$240 = 40x$ or $\frac{240}{40} = x$

(e) y is the number of cents that Sally will earn in 5 hours.

$y = 5 \cdot 65$

3. (a) I a is the number of years in Mr. Smith's age now.
II $a + 10 = 40$
III $a = 30$

(b) I x is the number of dollars you have now.
II $x + 5 = 12$
III $x = 7$

(c) I b is the number of inches in brother's height.
II $2b = 64$
III $b = 32$

(d) I p is the number of the year in which Paul was born.
II $p + 14 = 1958$
III $p = 1944$

(e) I h is the number of inches in the length of the altitude.
II $\frac{1}{2} \cdot 7h = 35$
III $h = 10$

(f) I y is the number
II $\frac{20}{100}y = 10$
III $y = 50$

(g)	I	p is the number of dollars in the original pric
	II	$p - \frac{20}{100}p = 28$ or $\frac{80}{100}p = 28$
	III	$p = 35$
(h)	I	x is the number of miles in the complete journe
	II	$120 + \frac{5}{8}x = x$
	III	$x = 320$

Answers to Exercises 5-8a, page 236:

1\. (a) (I)

$x + 5 = 6$	
$(x + 5) + {}^-5 = 6 + {}^-5$	addition property, adding $^-5$.
$x + (5 + {}^-5) = 1$	associative property
$x = 1$	$5 + {}^-5 = 0$

(II) If $x = 1$ then $x + 5 = 1 + 5 = 6$, so 1 is a solution.

(b) (I)

$x + 6 = 5$	
$(x + 6) + {}^-6 = 5 + {}^-6$	addition property, adding $^-6$.
$x + (6 + {}^-6) = {}^-1$	associative property
$x = {}^-1$	$6 + {}^-6 = 0$

(II) If $x = {}^-1$ then $x + 6 = {}^-1 + 6 = 5$, so $^-1$ is a solution

(c) (I)

$x + {}^-7 = 7$	
$(x + {}^-7) + 7 = 7 + 7$	addition property, adding 7.
$x + ({}^-7 + 7) = 14$	associative property
$x = 14$	${}^-7 + 7 = 0$

(II) If $x = 14$, then $x + {}^-7 = 14 + {}^-7 = 7$, so 14 is a solution.

(d) (I)

$x - 7 = {}^-7$	
$x + {}^-7 = {}^-7$	property of additive inverse
$(x + {}^-7) + 7 = {}^-7 + 7$	addition property, adding 7
$x + ({}^-7 + 7) = 0$	associative property
$x = 0$	${}^-7 + 7 = 0$

(II) If $x = 0$, then $x - 7 = 0 - 7 = 0 + {}^-7 = {}^-7$, so 0 is a solution.

(e) (I) $t + 6 = {}^-13$

$(t + 6) + {}^-6 = {}^-13 + {}^-6$ addition property, adding $^-6$

$t + (6 + {}^-6) = {}^-19$ associative property

$t = {}^-19$ $6 + {}^-6 = 0$

(II) If $t = {}^-19$, then $t + 6 = {}^-19 + 6 = {}^-13$, so $^-19$ is a solution.

(f) (I) $4 = s + 3$

$4 + {}^-3 = (s + 3) + {}^-3$ addition property, adding $^-3$

$1 = s + (3 + {}^-3)$ associative property

$1 = s$ $3 + {}^-3 = 0$

(II) If $s = 1$, then $s + 3 = 1 + 3 = 4$, or $4 = s + 3$, so 1 is a solution.

(g) (I) ${}^-2 = {}^-4 + x$

$4 + {}^-2 = 4 + ({}^-4 + x)$ addition property, adding 4

$2 = (4 + {}^-4) + x$ associative property

$2 = x$ $4 + {}^-4 = 0$

(II) If $x = 2$, then ${}^-4 + x = {}^-4 + 2 = {}^-2$, or ${}^-2 = {}^-4 + x$, so 2 is a solution.

(h) (I) $x + \frac{3}{2} = 10$

$(x + \frac{3}{2}) + {}^-(\frac{3}{2}) = 10 + {}^-(\frac{3}{2})$ addition property, adding ${}^-(\frac{3}{2})$

$x + \left(\frac{3}{2} + {}^-(\frac{3}{2})\right) = \frac{17}{2}$ associative property

$x = \frac{17}{2}$ $\frac{3}{2} + {}^-(\frac{3}{2}) = 0.$

(II) If $x = \frac{17}{2}$, then $x + \frac{3}{2} = \frac{17}{2} + \frac{3}{2} = \frac{20}{2} = 10$, so $\frac{17}{2}$ is a solution.

(i) (I) $y - \frac{3}{2} = \frac{5}{2}$

$y + {}^{-}(\frac{3}{2}) = \frac{5}{2}$

$\left(y + {}^{-}(\frac{3}{2})\right) + \frac{3}{2} = \frac{5}{2} + \frac{3}{2}$ addition property, adding $\frac{3}{2}$

$y + \left({}^{-}(\frac{3}{2}) + \frac{3}{2}\right) = \frac{8}{2} = 4$ associative property

$y = 4$ ${}^{-}(\frac{3}{2}) + \frac{3}{2} = 0$

(II) If $y = 4$, then $y - \frac{3}{2} = 4 - \frac{3}{2} = \frac{8}{2} - \frac{3}{2} = \frac{5}{2}$, or $y - \frac{3}{2} = \frac{5}{2}$, so 4 is a solution.

(j) (I) $u + 14 = \frac{9}{5}$

$(u + 14) + {}^{-}14 = \frac{9}{5} + {}^{-}14$ addition property, adding ${}^{-}14$

$u + (14 + {}^{-}14) = \frac{9}{5} + {}^{-}(\frac{70}{5}) = {}^{-}(\frac{61}{5})$ associative property

$u = {}^{-}(\frac{61}{5})$ $14 + {}^{-}14 = 0$

(II) If $u = {}^{-}(\frac{61}{5})$, then $u + 14 = {}^{-}(\frac{61}{5}) + 14 = {}^{-}(\frac{61}{5}) + \frac{70}{5} = \frac{9}{5}$, or $u + 14 = \frac{9}{5}$, so ${}^{-}(\frac{61}{5})$ is a solution.

(k) (I) $\frac{13}{7} = 1 + x$

${}^{-}1 + \frac{13}{7} = {}^{-}1 + (1 + x)$ addition property, adding ${}^{-}1$

$\frac{6}{7} = ({}^{-}1 + 1) + x$ associative property

$\frac{6}{7} = x$ ${}^{-}1 + 1 = 0$

(II) If $x = \frac{6}{7}$, then $1 + x = 1 + \frac{6}{7} = \frac{7}{7} + \frac{6}{7} = \frac{13}{7}$, or $\frac{13}{7} = 1 + x$, so $\frac{6}{7}$ is a solution.

(1) (I) $x + {}^-(\frac{4}{9}) = {}^-(\frac{7}{13})$

$\left(x + {}^-(\frac{4}{9})\right) + \frac{4}{9} = {}^-(\frac{7}{13}) + \frac{4}{9}$ addition property, adding $\frac{4}{9}$

$x + \left({}^-(\frac{4}{9}) + \frac{4}{9}\right) = {}^-(\frac{63}{117}) + \frac{52}{117} = {}^-(\frac{11}{117})$ associative property

$x = {}^-(\frac{11}{117})$ ${}^-(\frac{4}{9}) + \frac{4}{9} = 0$

(II) If $x = {}^-(\frac{11}{117})$, then $x + {}^-(\frac{4}{9}) = {}^-(\frac{11}{117}) + {}^-(\frac{4}{9}) =$
${}^-(\frac{11}{117}) + {}^-(\frac{4}{9}) = {}^-(\frac{11}{117}) + {}^-(\frac{52}{117}) = {}^-(\frac{63}{117})$ and
${}^-(\frac{63}{117}) = {}^-(\frac{7 \cdot 9}{13 \cdot 9}) = {}^-(\frac{7}{13})$. Thus
$x + {}^-(\frac{4}{9}) = {}^-(\frac{7}{13})$, and ${}^-(\frac{11}{117})$ is a solution.

2. (a) $2x + 5 = 10$
$(2x + 5) + {}^-5 = 10 + {}^-5$ addition property
$2x + (5 + {}^-5) = 5$ associative property
$2x = 5$

(b) $3x + 10 = 5$
$(3x + 10) + {}^-10 = 5 + {}^-10$ addition property
$3x + (10 + {}^-10) = {}^-5$ associative property
$3x = {}^-5$

(c) $5x + 2 = {}^-2$
$(5x + 2) + {}^-2 = {}^-2 + {}^-2$ addition property
$5x + (2 + {}^-2) = {}^-4$ associative property
$5x = {}^-4$

(d) $10x + {}^-1 = 9$
$(10x + {}^-1) + 1 = 9 + 1$ addition property
$10x + ({}^-1 + 1) = 10$ associative property
$10x = 10$

(e) $2u + 1 = 11$
$(2u + 1) + {}^-1 = 11 + {}^-1$ addition property
$2u + (1 + {}^-1) = 10$ associative property
$2u = 10$

3. (a) ${}^{-}3$

(b) 7

(c) 3 and ${}^{-}3$ are additive inverses of each other (or inverses under addition).

(d) 7 and ${}^{-}7$ are additive inverses of each other (or inverses under addition).

4. (a) $x + 3 = 11$
$(x + 3) + {}^{-}3 = 11 + {}^{-}3$ addition property
$x + (3 + {}^{-}3) = 8$ associative property
$x = 8.$
If $x = 8$, then $x + 3 = 8 + 3 = 11$, so 8 is a solution.

(b) $x + {}^{-}6 = \frac{4}{3}$
$(x + {}^{-}6) + 6 = \frac{4}{3} + 6 = \frac{4}{3} + \frac{18}{3}$ addition property
$x + ({}^{-}6 + 6) = \frac{22}{3}$ associative property
$x = \frac{22}{3}.$
If $x = \frac{22}{3}$, then $x + {}^{-}6 = \frac{22}{3} + {}^{-}6 = \frac{22}{3} + {}^{-}(\frac{18}{3}) = \frac{4}{3}$, so $\frac{22}{3}$ is a solution.

(c) $x - \frac{9}{2} = \frac{5}{2}$
$x + {}^{-}(\frac{9}{2}) = \frac{5}{2}$
$\left(x + {}^{-}(\frac{9}{2})\right) + \frac{9}{2} = \frac{5}{2} + \frac{9}{2}$ addition property
$x + \left({}^{-}(\frac{9}{2}) + \frac{9}{2}\right) = \frac{14}{2} = 7$ associative property
$x = 7.$
If $x = 7$, then $x - \frac{9}{2} = 7 - \frac{9}{2} = \frac{14}{2} - \frac{9}{2} = \frac{5}{2}$, so 7 is a solution.

(d) $2x - 7 = x$

$^{-}x + (2x - 7) = x + {}^{-}x$ addition property

$({}^{-}x + 2x) - 7 = 0$ associative property

$({}^{-}1 + 2)x - 7 = 0$ distributive property

$x - 7 = 0$

$(x + {}^{-}7) + 7 = 0 + 7$ addition property

$x + ({}^{-}7 + 7) = 7$ associative property

$x = 7.$

If $x = 7$, then $2x - 7 = 14 - 7 = 7 = x$ so 7 is a solution.

(e) $^{-}x = 7 - 2x = 7 + {}^{-}2x$

$^{-}x + 2x = (7 + {}^{-}2x) + 2x$ addition property

$^{-}x + 2x = 7 + ({}^{-}2x + 2x)$ associative property

$({}^{-}1 + 2)x = 7 + ({}^{-}2 + 2)x$ distributive property

$x = 7.$

If $x = 7$, then $7 - 2x = 7 - 14 = {}^{-}7 = {}^{-}x$, or $^{-}x = 7 - 2x$, so 7 is a solution.

(f) $\frac{1}{2}x + 2 = 1.5 - \frac{1}{2}x = 1.5 + {}^{-}(\frac{1}{2})x$

$\frac{1}{2}x + (\frac{1}{2}x + 2) = \frac{1}{2}x + \left({}^{-}(\frac{1}{2}x) + 1.5\right)$ addition property and commutative property of addition.

$(\frac{1}{2}x + \frac{1}{2}x) + 2 = \left(\frac{1}{2}x + {}^{-}(\frac{1}{2})x\right) + 1.5$ associative property

$(\frac{1}{2} + \frac{1}{2})x + 2 = \left(\frac{1}{2} + {}^{-}(\frac{1}{2})\right)x + 1.5$ distributive property

$x + 2 = 1.5$

$(x + 2) + {}^{-}2 = 1.5 + {}^{-}2$ addition property

$x + (2 + {}^{-}2) = {}^{-}.5$ associative property

$x = {}^{-}.5.$

If $x = {}^{-}.5$, then $\frac{1}{2}x + 2 = \frac{1}{2}({}^{-}.5) + 2 = {}^{-}.25 + 2 = 1.75$, and $1.5 - \frac{1}{2}x = 1.5 - \frac{1}{2}({}^{-}.5) = 1.5 + {}^{-}(\frac{1}{2})({}^{-}.5) = 1.5 + 2.5 = 1.75$. Thus, $\frac{1}{2}x + 2 = 1.75 = 1.5 - \frac{1}{2}x$, and $x = {}^{-}.5$ is a solution.

Answers to Class Exercises 5-8; page 239:

1. (a) Addition

   (b) Addition

   (c) Addition

   (d) Multiplication

   (e) Multiplication

   (f) Addition (or Addition $({}^{-}14)$ and Multiplication $({}^{-}1)$).

   (g) Multiplication

   (h) Addition

   (i) Addition

   (j) Addition (or Addition $({}^{-}6)$ and Multiplication $({}^{-}1)$).

2. (a) Multiplication property, multiplying by $\frac{1}{2}$.

   (b) Multiplication property, multiplying by $\frac{1}{4}$.

   (c) Multiplication property, multiplying by 3.

   (d) Multiplication property, multiplying by $\frac{1}{2}$.

   (e) Multiplication property, multiplying by 10.

   (f) Addition property, adding ${}^{-}3x = {}^{-}(3x)$.

3. (a) $7 + {}^{-}1 = (3x + 1) + {}^{-}1$

   $6 = 3x + (1 + {}^{-}1)$

   $6 = 3x.$

   (b) $\frac{1}{3} \cdot 6 = \frac{1}{3}(3w)$

   $2 = (\frac{1}{3} \cdot 3)w$

   $2 = w.$

   (c) $(\frac{t}{2} - 1.7) \times 2 = ({}^{-}1.3) \times 2$

   $\frac{t}{2} \times 2 - (1.7) \times 2 = {}^{-}2.6;$ Distributive propert

   $t - 3.4 = {}^{-}2.6.$

(d) $b = \frac{x}{18}$

$18b = 18(\frac{x}{18})$

$18b = (18 \cdot \frac{1}{18})x$

$18b = x.$

(e) $^-0.14 + (0.14 + x) = 5.28 + {}^-0.14$

$(^-0.14 + 0.14) + x = {}^-0.14 + 5.28$

$x = 5.14$

(f) $5x - 7 = 2x$

$^-(2x) + (5x - 7) = {}^-(2x) + 2x$

$(^-2x) + 5x) - 7 = 0$

$(^-2x + 5x) - 7 = 0$

$(^-2 + 5)x - 7 = 0;$ distributive property

$3x - 7 = 0.$

Answers to Exercises 5-8b; page 241:

1. (a)

$2x + 1 = 7$	
$(2x + 1) + {}^-1 = 7 + {}^-1$	Addition property (Ad. P.)
$2x + (1 + {}^-1) = 6$	Associative property (As. P. A.)
$2x = 6$	$1 + {}^-1 = 0$
$\frac{1}{2}(2x) = \frac{1}{2} \cdot 6$	Multiplication property (M. P.)
$(\frac{1}{2} \cdot 2)x = 3$	Associative property of multiplication (As. P. M.)
$x = 3.$	$\frac{1}{2} \cdot 2 = 1$

If $x = 3$, then $2x + 1 = 2 \cdot 3 + 1 = 6 + 1 = 7$, so $2x + 1 = 7$. Thus 3 is a solution, the only solution. The solution set is $\{3\}$.

(b) $y - 2 = 6$; same as $y + {}^-2 = 6$

$(y + {}^-2) + 2 = 6 + 2$	Ad. P.
$y + ({}^-2 + 2) = 8$	As. P. A.
$y = 8.$	${}^-2 + 2 = 0.$

If $y = 8$, then $y - 2 = 8 - 2 = 6$, so 8 is a solution. The solution set is $\{8\}$.

(c) $\frac{t}{2} - 3 = {}^-4$; same as $\frac{t}{2} + {}^-3 = {}^-4$

$(\frac{t}{2} + {}^-3) + 3 = {}^-4 + 3$	Ad. P.
$\frac{t}{2} + ({}^-3 + 3) = {}^-1$	As. P. A.
$\frac{t}{2} = {}^-1$	${}^-3 + 3 = 0$
$(\frac{t}{2}) \cdot 2 = {}^-1 \cdot 2$	M. P.
$t(\frac{1}{2} \cdot 2) = {}^-2$	As. P. M.
$t = {}^-2.$	$\frac{1}{2} \cdot 2 = 1$

If $t = {}^-2$, then $\frac{t}{2} - 3 = {}^-(\frac{2}{2}) - 3 = {}^-1 - 3 = {}^-1 + {}^-3 = {}^-4$, so $\frac{t}{2} - 3 = {}^-4$, and ${}^-2$ is a solution. The solution set is $\{{}^-2\}$.

(d) $3x - 5 = {}^-4$; same as $3x + {}^-5 = {}^-4$

$(3x + {}^-5) + 5 = {}^-4 + 5$	Ad. P.
$3x + ({}^-5 + 5) = 1$	As. P. A.
$3x = 1$	${}^-5 + 5 = 0.$
$\frac{1}{3}(3x) = \frac{1}{3} \cdot 1$	M. P.
$(\frac{1}{3} \cdot 3)x = \frac{1}{3}$	As. P. M.
$x = \frac{1}{3}.$	$\frac{1}{3} \cdot 3 = 1$

If $x = \frac{1}{3}$, then $3x - 5 = 3 \cdot \frac{1}{3} - 5 = 1 - 5 = {}^-4$, so $3x - 5 = {}^-4$. Thus $\frac{1}{3}$ is a solution. The solution set is $\{\frac{1}{3}\}$.

2. (a) $x + 3 = 5$

$(x + 3) + {}^-3 = 5 + {}^-3$

$x + (3 + {}^-3) = 2.$

$x = 2.$

If $x = 2$, $x + 3 = 2 + 3 = 5$ so $x + 3 = 5.$

Solution set = $\{2\}$.

The remaining parts of Problem 2 should be solved by following the pattern of 2(a). Since you are familiar with this pattern only the solution sets will be listed hereafter.

(b) The solution set is $\{{}^-8\}$.

(c) The solution set is $\{1\}$.

(d) The solution set is $\{{}^-4\}$.

(e) The solution set is $\{8\}$.

(f) The solution set is $\{4\}$.

(g) The solution set is $\{{}^-4\}$.

(h) The solution set is $\{2\}$.

3. (a) $\frac{1}{9}$ (d) They are inverses of each other, or reciprocals.

(b) 3 (e) They are inverses of each other, or reciprocals.

(c) $\frac{5}{4}$

4. <u>Addition Property</u>: If a, b, and c are numbers, and $a < b$, then $a + c < b + c$ and $c + a < c + b$.
This is <u>true</u>.

<u>Multiplication Property</u>: If a, b, and c are numbers, and $a < b$, then $ac < bc$ and $ca < cb$. <u>This is false</u>!
Example: $2 < 4$, but $2 \cdot {}^-1 = {}^-2$, $4 \cdot {}^-1 = {}^-4$, and ${}^-2 < {}^-4$ is false!
The multiplication property for "<" is true if the condition, $c > 0$, is added to the statement.

5. (a) $4(x + 1) = 12$

$\frac{1}{4}\left(4(x + 1)\right) = \frac{1}{4} \cdot 12$

$(\frac{1}{4} \cdot 4)(x + 1) = 3$

$x + 1 = 3$

$(x + 1) + {}^{-}1 = 3 + {}^{-}1$

$x + (1 + {}^{-}1) = 2$

$x = 2.$

If $x = 2$, then $4(x + 1) = 4(2 + 1) = 4 \cdot 3 = 12.$

Thus, $4(x + 1) = 12.$

Solution set: $\{2\}$.

(b) $7(x - 2) = 13$

$\frac{1}{7}\left(7(x - 2)\right) = \frac{1}{7} \cdot 13$

$(\frac{1}{7} \cdot 7)(x - 2) = \frac{13}{7}$

$x - 2 = \frac{13}{7}$

$(x + {}^{-}2) + 2 = \frac{13}{7} + 2$

$x + ({}^{-}2 + 2) = \frac{27}{7}$

$x = \frac{27}{7}$

If $x = \frac{27}{7}$, then $7(x - 2) = 7(\frac{27}{7} - 2) = 7 \cdot (\frac{13}{7}) =$

$(7 \cdot \frac{1}{7}) \cdot 13 = 13$, so $7(x - 2) = 13.$

Solution set: $\{\frac{27}{7}\}$.

(c) $\frac{x + 9}{3} = 5$

$3 \cdot (\frac{x + 9}{3}) = 3 \cdot 5$

$(3 \cdot \frac{1}{3})(x + 9) = 15$

$x + 9 = 15$

$(x + 9) + {}^{-}9 = 15 + {}^{-}9$

$x + (9 + {}^{-}9) = 6$

$x = 6$

If $x = 6$, then $\frac{x + 9}{3} = \frac{6 + 9}{3} = \frac{15}{3} = 5.$

Solution set: $\{6\}$.

(d) $.6(x - .3) = .2$

$\frac{1}{.6} = \frac{1}{.6} \cdot \frac{10}{10} = \frac{10}{6}$ so,

$\frac{10}{6}\left(.6(x - .3)\right) = \frac{10}{6} \cdot .2$

$(\frac{10}{6} \cdot \frac{6}{10})(x - \frac{3}{10}) = \frac{10}{6} \cdot \frac{2}{10} = \frac{1}{3}$

$x - \frac{3}{10} = \frac{1}{3}$

$\left(x + {}^-(\frac{3}{10})\right) + \frac{3}{10} = \frac{1}{3} + \frac{3}{10}$

$x + \left({}^-(\frac{3}{10})\right) + \frac{3}{10} = \frac{10}{30} + \frac{9}{30}$

$x = \frac{19}{30}$.

If $x = \frac{19}{30}$, then $.6(x - .3) = \frac{6}{10}(\frac{19}{30} - \frac{3}{10}) =$ $\frac{6}{10}(\frac{19 - 9}{30}) = \frac{6}{10} \cdot (\frac{10}{30}) = \frac{6 \cdot 10}{30 \cdot 10} + \frac{6}{30} = \frac{2}{10} = .2$. Thus $.6(x - .3) = .2$. Solution set: $\{\frac{19}{30}\}$.

(e) $\frac{3x + 4}{2} = 7$

$2(\frac{3x + 4}{2}) = 2 \cdot 7$

$(2 \cdot \frac{1}{2})(3x + 4) = 14$

$3x + 4 = 14$

$(3x + 4) + {}^-4 = 14 + {}^-4$

$3x + (4 + {}^-4) = 10$

$3x = 10$

$\frac{1}{3}(3x) = \frac{1}{3} \cdot 10$

$(\frac{1}{3} \cdot 3)x = \frac{10}{3}$

$x = \frac{10}{3}$

If $x = \frac{10}{3}$, $\frac{3x + 4}{2} = \frac{3 \cdot \frac{10}{3} + 4}{2} = \frac{10 + 4}{2} = \frac{14}{2} = 7$.

Thus $\frac{3x + 4}{2} = 7$. Solution set: $\{\frac{10}{3}\}$.

(f). $\frac{4x + 1}{.12} = 3$

$.12(\frac{4x + 1}{.12}) = .12 \cdot 3$

$(.12 \cdot \frac{1}{.12})(4x + 1) = .36$

$4x + 1 = .36$

$(4x + 1) + {}^{-}1 = .36 + {}^{-}1$

$4x + (1 + {}^{-}1) = {}^{-}.64$

$4x = {}^{-}.64$

$\frac{1}{4}(4x) = \frac{1}{4}({}^{-}.64)$

$(\frac{1}{4} \cdot 4)x = {}^{-}.16$

$x = {}^{-}.16$

If $x = {}^{-}.16$, $\frac{4x + 1}{.12} = \frac{4({}^{-}.16) + 1}{.12} = \frac{{}^{-}.64 + 1}{.12} =$

$\frac{.36}{.12} = \frac{.12 \cdot 3}{.12} = 3$. Thus $\frac{4x + 1}{.12} = 3$.

Solution set: $\{{}^{-}.16\}$.

ANSWERS TO CHAPTER 6

Answers to Exercises 6-1; page 245:

1. No, Yes, $^-1$

2. $N + 1$, $N - 1$

3. (a) Yes (c) No
   (b) No (d) No

4. $^-(\frac{4}{7})$

5. Reciprocal

6. If they are the same when written in simplest form. If the numerators are equal when the denominators are equal.

7. $\frac{10}{14}$, $\frac{15}{21}$, $\frac{20}{28}$, etc.

8. (a) Closure under addition.
   (b) Identity for addition.
   (c) Identity for multiplication.
   (d) Closure under multiplication.
   (e) Distributive property of multiplication over addition.
   (f) Commutative property of multiplication.
   (g) Associative property of addition.

9. (a) $\frac{35}{2}$ (d) $^-(\frac{7}{20})$
   (b) $^-(\frac{5}{7})$ (e) $\frac{10}{1}$
   (c) $^-(\frac{11}{3})$ (f) $\frac{1703}{100}$

10. (a) $^+28$ or 28 (c) $^-(3\frac{1}{7})$ or $^-(\frac{22}{7})$
    (b) $^-756$ (d) $^+\frac{176}{5}$ or $\frac{176}{5}$

11. One

12. Zero

13. $^-4$, $^-(\frac{2}{3})$, 0, $\frac{3}{8}$, $\frac{2}{5}$, 0.41, $\frac{7}{16}$, $\frac{4}{7}$

14. $^-2$

15. No. The average of an odd integer and an even integer is never an integer.

    $\frac{1}{2}(^-3 + 8) = 2\frac{1}{2}$

Answers to Exercises 6-2; page 249:

1. No. There is no integer between -5 and -4.

2. Yes, 1 is the smallest. No largest positive integer.

3. No smallest negative integer. -1 is the largest.

4. No. No.

5. $\frac{1}{200}$. $\frac{1}{400}$. $\frac{3}{400}$.

6. $\frac{3}{40}$, $\frac{5}{80}$ or $\frac{1}{16}$, $\frac{7}{80}$.

7. One such plan would be to use some other point than the halfway point. For example, the point $\frac{1}{3}$ of the way from $\frac{1}{1000}$ to $\frac{2}{1000}$ is $(\frac{1}{1000} + \frac{1}{3000})$ or $\frac{4}{3000}$. The next named points would be $(\frac{1}{1000} + \frac{1}{9000})$ or $\frac{10}{9000}$; $(\frac{1}{1000} + \frac{1}{27,000})$ or $\frac{28}{27,000}$; $(\frac{1}{1000} + \frac{1}{81,000})$ or $\frac{82}{81,000}$; $(\frac{1}{1000} + \frac{1}{243,000})$ or $\frac{244}{243,000}$.

Answers to Exercises 6-3; page 253:

1. (a) 2.2500000000... (d) 0.0378378378...
   (b) 0.2083333333... (e) 0.0121012101...
   (c) 0.0857142857...
2. a, c, d, g, i.
3. (a) 2 (g) 2·2·2
   (c) 2·2 (i) 2·5
   (d) 5
4. (a) 0.142857 (d) 0.571428
   (b) 0.285714 (e) 0.714285
   (c) 0.428571 (f) 0.857142

Answers to Exercises 6-4; page 256:

1. (a) $9.9\overline{9}\ldots$ (d) $613.45\overline{345}\ldots$
   (b) $312.12\overline{12}\ldots$ (e) $60123.0123\overline{0123}\ldots$
   (c) $0.04\overline{4}\ldots$
2. (a) $2816.10\overline{0}\ldots$ or 2816.1 (e) $1.110\overline{0}\ldots$ or 1.1
   (b) $9.0\overline{0}\ldots$ or 9 (f) $351.0\overline{0}\ldots$ or 351
   (c) $162.0\overline{0}\ldots$ or 162 (g) $27048.\overline{00}\ldots$ or 27048
   (d) $298.0\overline{0}\ldots$ or 298 (h) $374.830\overline{0}\ldots$ or 374.83
3. (a) $\frac{411}{900}$ (c) $\frac{103}{99900}$
   (b) $\frac{163}{990}$
4. (a)

$$\begin{aligned}10n &= 5.5\overline{5}\ldots\\ \underline{\quad n} &= \underline{0.5\overline{5}\ldots}\\ 10n - n &= 5.0\overline{0}\ldots\end{aligned}$$

(b)

$$\begin{array}{r}
100n = 73.73\overline{73}\ldots \\
n = 0.73\overline{73}\ldots \\
\hline
100n - n = 73.0\overline{0}\ldots
\end{array}$$

(c)

$$\begin{array}{r}
1000n = 901.901\overline{901}\ldots \\
n = 0.901\overline{901}\ldots \\
\hline
1000n - n = 901.0\overline{0}\ldots
\end{array}$$

(d)

$$\begin{array}{r}
10n = 30.23\overline{3}\ldots \\
n = 3.023\overline{3}\ldots \\
\hline
10n - n = 27.210\overline{0}\ldots
\end{array}$$

(e)

$$\begin{array}{r}
10n = 1,631.77\overline{7}\ldots \\
n = 163.17\overline{7}\ldots \\
\hline
10n - n = 1,468.60\overline{0}\ldots
\end{array}$$

(f)

$$\begin{array}{r}
100n = 67,242.42\overline{42}\ldots \\
n = 672.42\overline{42}\ldots \\
\hline
100n - n = 66,570.0\overline{0}\ldots
\end{array}$$

(g)

$$\begin{array}{r}
100n = 12.3456\overline{56}\ldots \\
n = 0.1234\overline{56}\ldots \\
\hline
100n - n = 12.22220\overline{0}\ldots
\end{array}$$

(h)

$$\begin{array}{r}
10n = 34.100\overline{0}\ldots \\
n = 3.410\overline{0}\ldots \\
\hline
10n - n = 30.690\overline{0}\ldots
\end{array}$$

5. (a) $\frac{1}{11}$

(b) $\frac{1}{9}$

(c) $\frac{1}{18}$

(d) $\frac{41}{333}$

(e) $\frac{1625}{10000} = \frac{65}{400} = \frac{13}{80}$

(f) $\frac{1}{6}$

(g) $\frac{5120}{999}$

(h) $\frac{221}{22}$

6. (a) $32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ or 2^5
   (b) $100 = 2 \cdot 2 \cdot 5 \cdot 5$ or $2^2 \cdot 5^2$
   (c) $9 = 3 \cdot 3$ or 3^2
   (d) $50 = 2 \cdot 5 \cdot 5$ or $2 \cdot 5^2$
   (e) $35 = 5 \cdot 7$
   (f) $80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$ or $2^4 \cdot 5$
   (g) $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$ or $2^3 \cdot 3 \cdot 5$
   (h) $160 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$ or $2^5 \cdot 5$

7. The numbers in Parts a, b, d, f, h.

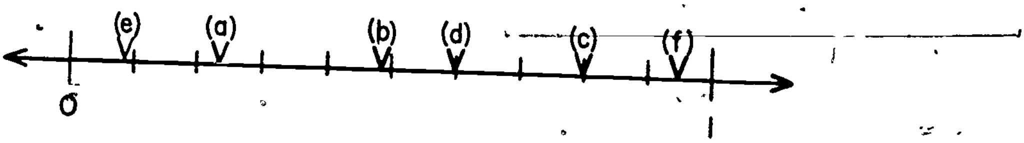
8. 64 and 80.

<u>Answers to Exercises 6-5; page 260:</u>

1. (a) 1.372 1.379 1.385 1.493 5.468
   (b) $^{-}9.426$ $^{-}5.630$ $^{-}2.765$ $^{-}2.763$ $^{-}2.761$
   (c) $^{-}0.15475$ 0.15463 0.15467 0.15475 0.15598

2. (a) none
   (b) all but $^{-}0.15475$
   (c) 0.15475 0.15467 0.15463
   (c) all but $^{-}0.15475$
   (d) all but $^{-}0.15475$

3.



4. (a) $\frac{17}{50} = 0.340\overline{0}\ldots$, $\frac{3}{9} = 0.3\overline{3}\ldots$, $\frac{4}{10} = 0.40\overline{0}\ldots$

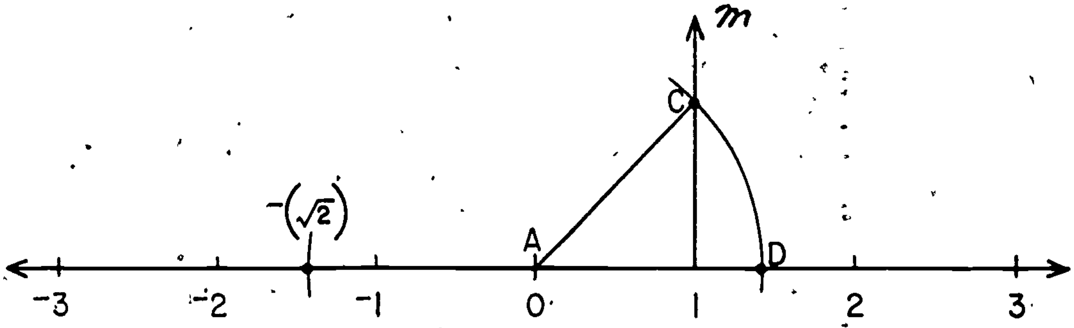
(b) $\frac{2}{3} = 0.6\overline{6}\ldots$, $\frac{67}{100} = 0.670\overline{0}\ldots$, $\frac{7}{10} = 0.70\overline{0}\ldots$

(c) $\frac{3}{7} = 0.\overline{428571}\ldots$; $\frac{4}{9} = 0.4\overline{4}\ldots$

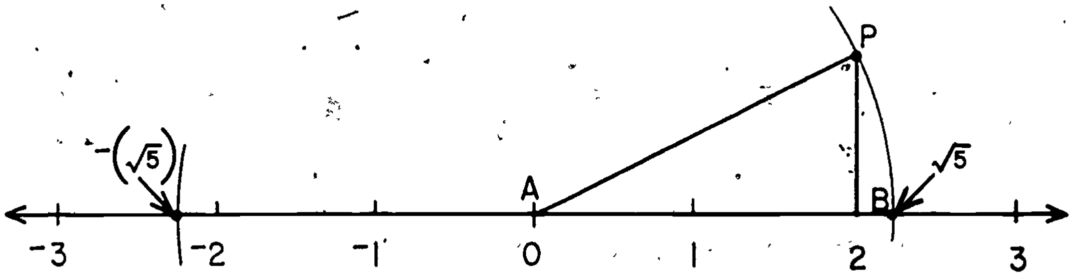
(d) $\frac{152}{333} = 0.456\overline{456}\ldots$, $\frac{415}{909} = 0.4565\overline{4565}\ldots$

Answers to Exercises 6-6; page 264:

1.



2.

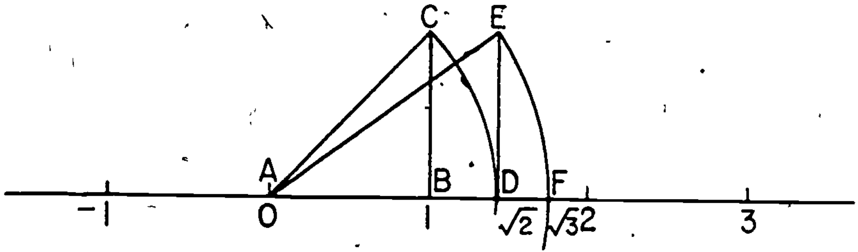


The measure of $\overline{AP}$ is $\sqrt{5}$.

3. See diagram for Problem 2.

4. Irrational. There is no rational number $\frac{a}{b}$ such that $(\frac{a}{b})^2 = 5$.

*5.



AB = 1 , BC = 1 , So AC = $\sqrt{2}$. Make AD = AC , so point D represents $\sqrt{2}$.

Draw DE perpendicular to AD , DE = 1 . Then $AE^2 = (\sqrt{2})^2 + 1^2$ and AE = $\sqrt{3}$. Make AF = AE so point E corresponds to $\sqrt{3}$.

Similarly $\sqrt{6}$ can be found by using a right triangle with sides of lengths $\sqrt{5}$ and 1; $\sqrt{7}$ by using a right triangle with sides of lengths $\sqrt{6}$ and 1 .

6. Lay off the distance $\sqrt{2}$ twice to the right of 0 ; 3 times to the right of 0 ; three times to the left of 0 .

7. Irrational.

*8. First show the following:

I. If a number is divisible by 5 , it can be expressed as $5 \cdot k$ where k is an integer.

II. If a number is not divisible by 5 , it can be expressed as $5n + R$, with $0 < R < 5$, R an integer. (Division Algorithm, Volume 1, Chapter 5)

III. If n is divisible by 5, then n^2 is divisible by 5.
$[(5k)^2 = 25k^2 = 5(5k^2)$. $5k^2$ is an integer by the closure property of integers under multiplication.]

IV. If n is not divisible by 5, then n^2 is not divisible by 5. If n is not divisible by 5, it may be expressed as $5k + 1$, or $5k + 2$, or $5k + 3$, or $5k + 4$.
$(5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1$. $\therefore$ it is not divisible by 5.
$(5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4$.
$\therefore$ it is not divisible by 5.
$(5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$.
$\therefore$ it is not divisible by 5.
$(5k + 4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1$.
$\therefore$ it is not divisible by 5.

V. If n is divisible by 5, and $\sqrt{n}$ is an integer, then $\sqrt{n}$ is divisible by 5.
Use an indirect argument.
Assume $\sqrt{n}$ is not divisible by 5. Then by (4) above, n is not divisible by 5. But n is divisible by 5. Therefore $\sqrt{n}$ is divisible by 5.
Now the proof follows the same form as the proof that 2 is not a rational number.

1. Assume $\sqrt{5}$ is a rational number. Then $\sqrt{5} = \frac{p}{q}$, p and q integers and $q \neq 0$. Let $\frac{p}{q}$ be the simplest expression for $\sqrt{5}$, so p and q have no common factors other than 1;
2. $5 = \frac{p^2}{q^2}$.
3. $\therefore 5q^2 = p^2$.
4. q^2 is an integer.

5. $\therefore p^2$ is divisible by 5 .

6. $\therefore p$ is divisible by 5 .

Let $p = 5k$, k an integer.

7. $5q^2 = (5k)^2$ or $25k^2$.

8. $q^2 = 5k^2$.

9. k^2 is an integer .

10. $\therefore q^2$ is divisible by 5 .

11. $\therefore q$ is divisible by 5 .

12. By statements 6 and 11 , both p and q have the factor 5 . But this contradicts the condition in Statement 1. Therefore the assumption that $\sqrt{5}$ is a rational number is false.

13. $\therefore$ $\sqrt{5}$ is not a rational number.

Answers to Exercises 6-7; page 270:

1. (a) $5 < \sqrt{30} < 6$.
   (b) $9 < \sqrt{89} < 10$.
   (c) $15 < \sqrt{253} < 16$.
   (d) $65 < \sqrt{4280} < 66$.
   (e) $96 < \sqrt{9315} < 97$.

2. (a) 3
   (b) 2.999824
   (c) 3.003289
   (d) .000176 .003289
   (e) 1.732

3. $(1.73)^2 = 2.9929$ $(1.74)^2 = 3.0276$ 1.73 is the better

4. $(3.87)^2 = 14.9769$ $(3.88)^2 = 15.0544$ 3.87 is the better

5. $(25.2)^2 = 635.04$ $(25.3)^2 = 640.09$ 25.2 is the better

6. 3.2

7. 12.2

8. 14.9

9. $n \approx 3.2$

10. $n \approx 12.2$

Answers to Exercises 6-8; page 277:

1.

Rational	Irrational
(a) $0.231\overline{231}\ldots$	(b) $0.23123112311123\ldots$
(c) $\frac{3\sqrt{2}}{7\sqrt{2}} = \frac{3}{7}$	(d) $\sqrt{7}$
(e) $0.78\overline{342}\ldots$	(f) $\frac{\pi}{2}$
(i) $0.750\overline{0}\ldots$	(g) $\frac{3}{4}\sqrt{6}$
(j) $\frac{58}{11}$	(h) $9 - \sqrt{3}$
	(k) $0.959559555955559\ldots$

Note that in (b) it is intended that the number of ones continues to increase.

Similarly, in (k) it is intended that the number of fives continues to increase.

2. (a) $0.231\overline{231}\ldots = \frac{231}{999} = \frac{77}{333}$

(c) $\frac{3\sqrt{2}}{7\sqrt{2}} = \frac{3}{7} = 0.428571\overline{428571}\ldots$

(e) $0.78\overline{342}\ldots = \frac{78264}{99900} = \frac{2174}{2775}$

(i) $0.750\overline{0}\ldots = \frac{3}{4}$

(j) $\frac{58}{11} = 5.27\overline{27}\ldots$

3.

Irrational Number	Nearest Hundredth
(b) $0.23123112311123\ldots$	0.23
(d) $\sqrt{7}$	2.65
(f) $\frac{\pi}{2}$	1.57
(g) $\frac{3}{4}\sqrt{6}$	1.84

(h) $9-\sqrt{3}$ 7.27

(k) 0.959559555955559... 0.96

Some possible answers for 4 , 5 , 6 , 7 .

4. (a) $.45632\overline{0}\ldots$ $8.27\overline{0}\ldots$ $.75\overline{0}\ldots$

   (b) $.45\overline{45}\ldots$ $12.762\overline{762}\ldots$ $.063\overline{63}\ldots$

   (c) .450450045... .123112311123... .565665666...

5. (a) $2.375\overline{375}\ldots$ (b) 2.370370037000...

6. (a) $.34\overline{4}\ldots$ (b) .34563445634456...

7. (a) 67.282... (b) 67.2829202002 0002...

8. No. There are no square roots of the negative integers in the real number system.

9. $\frac{355}{113} = 3.1415929\ldots$

   $\frac{22}{7} = 3.1428571\ldots$

   $\pi = 3.1415926\ldots$

   $\frac{355}{113}$ is a much better approximation to π than is $\frac{22}{7}$.

Answers to Exercises 6-9; page 280:

1. (a) irrational - circumference is π units.

   (b) rational - area is 1 square unit.

   (c) rational - hypotenuse is 13 units.

   (d) rational - $\left(\sqrt{3}\right)^2 = 3$.

   (e) irrational - volume is 2π cubic units.

   (f) rational - (each side is $\sqrt{2}$ units).

2. $(1.414\ldots) \times (1.732\ldots) \approx 1.414 \times 1.732 \approx 2.449$

3. (a) $\sqrt{2} = (1.414\ldots)$ $\frac{1}{2}\sqrt{2} = (0.707\ldots)$

$\sqrt{2} \times \frac{1}{2}\sqrt{2} = (1.414\ldots) \times (0.707\ldots)$

$\approx 1.414 \times 0.707 = 0.999698$

$\sqrt{3} + \sqrt{2} = (1.732\ldots) + (1.414\ldots) \approx 3.146$

$\sqrt{3} - \sqrt{2} = (1.732\ldots) - (1.414\ldots) \approx 0.318$

$(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} - \sqrt{2}) \approx (3.146) \times (0.318)$

$= 1.000428$

(b) $\sqrt{2} \times \frac{1}{2}\sqrt{2} = \frac{1}{2}(\sqrt{2})^2 = \frac{1}{2} \cdot 2 = 1$

$(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} - \sqrt{2}) = (\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} + {}^{-}\sqrt{2})$

$= (\sqrt{3} + \sqrt{2}) \cdot \sqrt{3} + (\sqrt{3} + \sqrt{2}) \cdot {}^{-}\sqrt{2}$

$= (\sqrt{3})^2 = (\sqrt{2} \cdot \sqrt{3}) + {}^{-}(\sqrt{3} \cdot \sqrt{2}) + {}^{-}(\sqrt{2})^2$

$= 3 + 0 + {}^{-}2$

$= 1$

4. Using 3.1416 for π the radius $\approx \frac{1}{3.1416}$

$\approx 0.318309\ldots$

ANSWERS TO APPENDIX

Answers to Exercises A-1; page 283:

1.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(a) 0 (c) 0

(b) 2 (d) 1

2. (mod 3)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(mod 5)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

3. (a) 0 (c) 1

(b) 1 (d) 2

4. (mod 6)

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(mod 7)

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

(a) 2 (c) 2

(b) 4 (d) 2

5. (a) 4 (c) 1
   (b) 4 (d) 0

6. Seven hours after eight o'clock is five o'clock. This is addition (mod 12).

7. Nine days after the 27th of March is the fifth of April. This is addition (mod 31) since there are 31 days in March.

Answers to Exercises A-2; page 286:

1. (a) (mod 5)

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(b) (mod 6)

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(c) (mod 7)

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

2. (a) 1 (d) 1
   (b) 0 (e) 0
   (c) 3

3. (a) 6 (c) 6
   (b) 1 (d) 0

4. (mod 5)

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

5. The table is identical with the multiplication Table (mod 5). Dividing a whole number by 5 and retaining the remainder yields the same results as those obtained by subtracting the greatest multiple of five contained in a given number and retaining the remainder. It may be easier to divide and retain the remainder.

*6. (a) $x = 2$ (d) $x = 3$

(b) $x = 4$ (e) $x = 0$

(c) $x = 1$

*7. (a) impossible

(b) impossible

(c) $x = 1$, $x = 3$, $x = 5$

(d) impossible

(e) $x = 0$, $x = 2$, $x = 4$

8. Wednesday

Answers to Exercises A-3; page 291:

1. (a) 1
   (b) 6
   (c) 8
   (d) 7
   (e) 2
   (f) 3
   (g) 1
   (h) 1
   (i) 8
   (j) Not possible; $1 \square 2 = 4$ and $1 \square 4$ is not defined since 4 does not appear in the top row.
   (k) 3
   (l) 3

2. (a), (b), (d), (e). The table must be symmetric about the diagonal from upper left to lower right.

*3. There is no short-cut method; to prove associativity each triple of elements must be combined in the two ways and the corresponding results must be equal. The operations of Tables (a) and (d), (e) are associative; those of Tables (b) and (c) are not.

4. The operation symbols are omitted in the following tables:

(a)

	26	27	28	...	74
26	26	26	26	...	26
27	26	27	27	...	27
28	26	27	28	...	28
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
74	26	27	28	...	74

Commutative: Yes
Associative: Yes

(b)

	501	502	503	...	535
501	501	502	503	...	535
502	502	502	503	...	535
503	503	503	503	...	535
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
535	535	535	535	...	535

Commutative: Yes
Associative: Yes

(c)

	2	3	5	7	11	...
2	2	3	5	7	11	...
3	3	3	5	7	11	...
5	5	5	5	7	11	...
7	7	7	7	7	11	...
11	11	11	11	11	11	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Commutative: Yes
Associative: Yes

(d)

	40	42	44	...	60
40	40	40	40	...	40
42	42	42	42	...	42
44	44	44	44	...	44
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
60	60	60	60	...	60

Commutative: No
Associative: Yes

(e)

	1	2	3	...	49
1	3	4	5	...	51
2	5	6	7	...	53
3	7	8	9	...	54
.	.	.	.	...	.
.	.	.	.	...	.
49	99	100	101	...	147

Commutative: No
Associative: No
(Try the triple
1, 2, 3.)

(f)

	1	2	3	4	5	6	...
1	1	1	1	1	1	1	...
2	1	2	1	2	1	2	...
3	1	1	3	1	1	3	...
4	1	2	1	4	1	2	...
5	1	1	1	1	5	1	...
6	1	2	3	2	1	6	...
.	.	.	.	.	.	.	...
.	.	.	.	.	.	.	...
.	.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(g)

	1	2	3	4	5	...
1	1	2	3	4	5	...
2	2	2	6	4	10	...
3	3	6	3	12	15	...
4	4	4	12	4	20	...
5	5	10	15	20	5	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Commutative: Yes

Associative: Yes

(h)

	1	2	3	4	...
1	1	1	1	1	...
2	2	4	8	16	...
3	3	9	27	81	...
4	4	16	64	256	...
.	.	.	.	.	...
.	.	.	.	.	...
.	.	.	.	.	...

Commutative: No

Associative: No

(Try the triple

2, 1, 3;

$(2^1)^3 = 8 \neq 2 = 2^{(1^3)}$

5. Many answers are possible, of course. The only requirement is that the table be symmetric about the diagonal from upper left to lower right (and that each place in the table be filled uniquely so that the table does describe an operation).

*	1	2	3
1	X	Y	Z
2	Y	P	Q
3	Z	Q	R

6. Many answers are possible, of course. The only requirement is that the table must not be symmetric about the diagonal from upper left to lower right (and that each place in the table be filled uniquely so that the table does describe an operation).

*	1	2	3
1	X	P	Z
2	P	R	Y
3	Q	P	Q

*7.

x	x^3
0	0
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729
10	1000

<u>Answers to Exercises A-4</u>; page 295:

1. The sets of (a) and (d) are closed under the corresponding operations (all the entries in the table appear in the left column and in the top row); those of (b) and (c) are not closed (some entries in Tables (b) and (c) do not appear in the left column and in the top row).

2. (a) Closed
   (b) Closed
   (c) Closed
   (d) Not closed
   (e) Closed
   (f) Not closed 15 - 35 cannot be performed
   (g) Closed
   (h) Closed
   (i) Not closed: 3 + 7 is not a prime
   *(j) Not closed: 3 + 3 = 11 (base 5)

3. (a) {2, 4, 6, ..., 2k, ...} where k is a counting number.
   (b) $\{2, 2^2, 2^3, \ldots, 2^k, \ldots\}$ where k is a counting number.

5. (a) {7, 14, 21, ..., 7k, ...} where k is a counting number.
   (b) $\{7, 7^2, 7^3, \ldots, 7^k, \ldots\}$ where k is a counting number.

6. (a) $1 \odot 1 = 3$, $(1 \odot 1) \odot 1 = 3 \odot 1 = 2$.
   $[(1 \odot 1) \odot 1] \odot 1 = 2 \odot 1 = 1$.
   If you continue the operation $\odot$, you will generate the same set again. Hence, the set {1, 2, 3} is the sub-set of S generated by 1 under the operation $\odot$.

   (b) $2 \odot 2 = 2$, $(2 \odot 2) \odot 2 = 2$.
   $[(2 \odot 2) \odot 2] \odot 2 = 2 \odot 2 = 2$.
   It is clear that the subset of S generated by 2 under the operation $\odot$ is the subset {2}.

*7. $\{3, (3 \div 3), (3 \div 3) \div 3, [(3 \div 3) \div 3] \div 3, \ldots\}$ or $\{3, 1, \frac{1}{3}, \frac{1}{9}, \ldots\}$.

Yes; 3 and $\frac{1}{3}$ are in the subset of rationals generated by 3 under division. No; $3 \div \frac{1}{3}$ or 9 is not in this subset. Therefore, the set is not closed under division and, hence, it cannot be associative.

8. No;

9. Yes;

10. Many answers are possible, of course. The only requirement is that each entry in the table belong to the set $\{0, 43, 100\}$ and that each place in the table be filled uniquely so that the table does describe an operation.

*	0	43	100
0	0	0	0
43	43	0	43
100	0	43	0

11. Many answers are possible, of course. The only requirement is that at least one entry in the table must *not* be an element of the set $\{0, 43, 100\}$ (and that each place in the table be filled uniquely so the table does describe an operation).

*	0	43	100
0	0	0	43
43	43	1	0
100	2	0	43

Answers to Exercises A-5a; page 300:

1. (a) In Table (a), the identity is 5.
   In Table (d), the identity is 2.

   (b) In Table (a), the inverse of 1 is 4; of 2 is 3; of 5 is 5.
   In Table (b), no element has an inverse.
   In Table (c), no element has an inverse.
   In Table (d), the inverse of 1 is 3; of 2 is 2.

   Each member of the sets for Tables (a) and (d) has an inverse. The operations described by Tables (b) and (c) do not have identities so no inverses can exist.

2. (a)

Operation	Identity
(a)	74
(b)	501
(c)	2
(d)	None
(e)	None
(f)	None
(g)	1
(h)	None

   (b) The only inverses are those listed below.

(a)	74 is the inverse of 74.
(b)	501 is the inverse of 501.
(c)	2 is the inverse of 2.
(g)	1 is the inverse of 1.

   (c) None.

3. No; if there are two identities (P and Q) for a given operation, then consider the result when P is combined with Q. Since Q is an identity, the result must be P. But since P is also an identity, the result must be Q. Thus, P and Q must be the same element since each equals the result of combining P and Q.

Answers to Exercises A-5b; page 301:

1. (a) $1x \equiv 1 \pmod{6}$, $x = 1$
   $2x \equiv 1 \pmod{6}$, not possible
   $3x \equiv 1 \pmod{6}$, not possible
   $4x \equiv 1 \pmod{6}$, not possible
   $5x \equiv 1 \pmod{6}$, $x = 5$

   (b) 1, 5. Each is its own inverse.

2. (mod 5)

a	$3 \div a$	$3 \cdot \frac{1}{a}$
1	$3 \div 1 \equiv 3$	$3 \cdot 1 \equiv 3$
2	$3 \div 2 \equiv 4$	$3 \cdot 3 \equiv 4$
3	$3 \div 3 \equiv 1$	$3 \cdot 2 \equiv 1$
4	$3 \div 4 \equiv 2$	$3 \cdot 4 \equiv 2$

3. (mod 5)

a	$3 - a$	$3 + (-a)$
1	$3 - 1 \equiv 2$	$3 + 4 \equiv 2$
2	$3 - 2 \equiv 1$	$3 + 3 \equiv 1$
3	$3 - 3 \equiv 0$	$3 + 2 \equiv 0$
4	$3 - 4 \equiv 4$	$3 + 1 \equiv 4$

4. (a) no (b) no (c) no
   (d) yes, except division by zero.

5. (a) {0, 1, 2, 3, 4, 5}
   (b) {1, 5}, {5}
   (c) {2, 4}, {1, 5}, {5}

6. (a) {A, B}, {C, D}, {A, D}
   (b) yes, D (d) {C, D}
   (c) {C, D}

Answers to Exercises A-6; page 304:

1. Each one of Tables (a), (b), (c) describes a mathematical system.
   For Table (a), the set is {A, B}; the operation is o.
   For Table (b), the set is {P, Q, R, S}; the operation is *.
   For Table (c), the set is {Δ, □, ○, \}; the operation is ~ .

2. (a) A
   (b) ○
   (c) ○
   (d) B
   (e) Q
   (f) S
   (g) P
   (h) \
   (i) Δ
   (j) B
   (k) A
   (l) R

3. The operation o is not commutative, since Table (a) is not symmetric.
   The operations * and ~ are both commutative, since both Tables (b) and (c) are symmetric.

4. There is no identity element for the operation o. There is no element e, such that both of the equations A o e = A and B o e = B are correct.
   The element R is the identity element for the operation *. The row of Table (b) with "R" in the left column is the same as the top row, and the column with "R" at the top is the same as the left column.
   The element Δ is the identity element for the operation ~ . The first row and column of Table (c) are the same as the top row and left column respectively.

5. (a) S
   (b) S
   (c) R
   (d) R
   (e) Q
   (f) Q
   (g) \
   (h) \
   (i) \
   (j) \

6. Each of the operations * and ~ seems to be associative since, in each of the cases you have tried, the corresponding expressions are equal. To prove the operations are associative, you would have to examine all cases and show that the corresponding expressions are equal. To prove an operation is not associative, a person would have to find one example where the corresponding expressions are not equal.

8. (a) The element 2 cannot be combined with 2 by the operation * (that is, 2 * 2 is not defined).

   (b) 2 * 1 is not uniquely defined. Many results are possible when 2 and 1 are combined.

   (c) The set given by this table is {1, 2, 3, 4}. But it is not possible to combine every pair of elements (e.g. 3 and 3). There is no operation <u>defined on the set</u>.

<u>Answers to Exercises A-7</u>; page 310:

1.

ANTH	I	V	H	R
I	I	V	H	R
V	V	I	R	H
H	H	R	I	V
R	R	H	V	I

2. (a) V
   (b) V
   (c) V
   (d) V
   (e) I
   (f) I
   (g) I
   (h) I
   (i) I

3. (a) Yes

(b) Yes

(c) Yes, the operation is associative. A proof would require that 64 cases be checked. Each pupil should check two or three; do not attempt to check all cases.

(d) Yes. I is the identity.

(e) Yes. Each element is its own inverse.

4. (a)

ANTH	I	F
I	I	F
F	F	I

(b) Yes

(c) Yes

(d) Yes. All cases can be checked (there are 8 cases in all).

(e) Yes. I is the identity element.

(f) Yes. Each element is its own inverse.

5.

ANTH	I	R	S	T	U	V
I	I	R	S	T	U	V
R	R	S	I	U	V	T
S	S	I	R	V	T	U
T	T	V	U	I	S	R
U	U	T	V	R	I	S
V	V	U	T	S	R	I

The operation is <u>not</u> commutative (R ANTH T $\neq$ T ANTH R) I is the identity element. Each of I, T, U, V is its own inverse element; R and S are inverses of each other.

*6. Notation:

I: Leave the square in place.

R_1: Rotate clockwise $\frac{1}{4}$ of the way around.

R_2: Rotate clockwise $\frac{1}{2}$ of the way around.

R_3: Rotate clockwise $\frac{3}{4}$ of the way around.

H: Flip the square over, using a horizontal axis.

V: Flip the square over, using a vertical axis.

D_1: Flip the square over, using an axis from upper left to lower right.

D_2: Flip the square over, using an axis from lower left to upper right.

Note: It was suggested that a square card not be used. This problem is included to show why such a suggestion was made.

ANTH	I	R_1	R_2	R_3	H	V	D_1	D_2
I	I	R_1	R_2	R_3	H	V	D_1	D_2
R_1	R_1	R_2	R_3	I	D_2	D_1	H	V
R_2	R_2	R_3	I	R_1	V	H	D_2	D_1
R_3	R_3	I	R_1	R_2	D_1	D_2	V	H
H	H	D_1	V	D_2	I	R_2	R_1	R_3
V	V	D_2	H	D_1	R_2	I	R_3	R_1
D_1	D_1	V	D_2	H	R_3	R_1	I	R_2
D_2	D_2	H	D_1	V	R_1	R_3	R_2	I

I is the identity element. The operation is not commutative (R_1 ANTH H $\neq$ H ANTH R_1).

Answers to Exercises A-8; page 314:

1\. (a) Since the sum of two counting numbers is always another counting number and the product of two counting numbers is always a counting number, the set is closed under addition and multiplication.

(b) Both the commutative property and the associative property hold for addition and multiplication.

Examples: Commutative: $2 + 3 = 3 + 2$; $4 \times 6 = 6 \times 4$

Associative: $3 + (4 + 7) = (3 + 4) + 7$; $3 \times (6 \times 8) = (3 \times 6) \times 8$.

(c) There is no identity element for addition. The identity element for multiplication is 1; for every counting number n, $n \cdot 1 = n = 1 \cdot n$.

(d) The counting numbers are not closed under subtraction or division.

2\. (a) The set of whole numbers is closed under addition and multiplication.

(b) Both operations are commutative and associative.

(c) There is an identity element for addition. It is zero; for any whole number n, $n + 0 = n = 0 + n$. The number 1 is the identity element for multiplication.

The answers are the same as for 1 (a), (b), (c) except that there is an identity element for addition in the whole number system and not in the counting number system.

3. (a) Three examples are: $2(3 + 4) = (2 \cdot 3) + (2 \cdot 4)$; $5(7 + 10) = (5 \cdot 7) + (5 \cdot 10)$; $1(1 + 1) = (1 \cdot 1) + (1 \cdot 1)$.

   (b) Addition does not distribute over multiplication; for example, $2 + (3 \cdot 4) = 14 \neq 30 = (2 + 3) \cdot (2 + 4)$.

4. (a) Yes, here are 3 illustrations that $*$ distributes over $\circ$:

   $A * (B \circ C) = A = (A * B) \circ (A * C)$
   $B * (B \circ B) = B = (B * B) \circ (B * B)$
   $C * (B \circ D) = C = (C * B) \circ (C * D)$

   (b) Yes, here are 3 illustrations that $\circ$ distributes over $*$.

   $A \circ (B * C) = A = (A \circ B) * (A \circ C)$
   $B \circ (B * B) = B = (B \circ B) * (B \circ B)$
   $C \circ (B * D) = D = (C \circ B) * (C \circ D)$

5. (a) Closed; commutative; associative; 1 is the identity; only the number 1 has an inverse.

   (b) Closed; commutative; associative; no identity; no inverses.

   (c) Closed; commutative; associative; 0 is the identity; only the number 0 has an inverse.

   (d) Closed; commutative; associative; no identity; no inverses.

   (e) Closed; commutative; associative; 0 is the identity; only the number 0 has an inverse.

   (f) Not closed; commutative; not associative; no identity; no inverses.

6. (a) Both sets are closed under the operations. Both operations are commutative and associative. Both systems involve the same set.

   (b) The system 5(a) has an identity and 5(b) does not. Also, the sets are different in these two systems.

*7. Many results are possible, of course.

*8. (a) Yes. You are asked to consider the two expressions a * (b o c) and (a * b) o (a * c), and find whether or not they are always equal. For example, using a = 8, b = 12, c = 15,
   8 * (12 o 15) = 8 * 60 = 4..
   (8 * 12) o (8 * 15) = 4 o 1 = 4.

   (b) Yes. You are asked to consider the two expressions a o (b * c) and (a o b) * (a o c), and find whether or not they are always equal. For example, using a = 8, b = 12, c = 15,
   8 o (12 * 15) = 8 o 3 = 24
   (8 o 12) * (8 o 15) = 24 * 120 = 24.

9. (mod 5)

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(mod 8)

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

10. (a) Mod 5: Yes; mod 8: Yes

(b) Mod 5: Yes; mod 8: Yes

(c) Mod 5: Yes; mod 8: Yes

(d) Mod 5: 1; mod 8: 1

(e) Mod 5: 1 and 4 are their own inverses; 2 and 3 are inverses of each other; 0 has no inverse. Mod 8: Only 1, 3, 5, 7 are inverses; each is its own inverse.

(f) Mod 5: Yes; mod 8: No. $2 \times 4 \equiv 0 \pmod 8$, $4 \times 2 \equiv 0 \pmod 8$, $4 \times 4 \equiv 0 \pmod 8$, $4 \times 6 \equiv 0 \pmod 8$, $6 \times 4 \equiv 0 \pmod 8$.

11. (a) 3

(b) 2

(c) 6, 8, 12, 24

(d) 4, 8

12. (a) 2

(b) 0

(c) 5

(d) 0

(e) 4

(f) 1

(g) 1. Any power of 6 ends in 6.

13. (a) 4

(b) 2

(c) 1

(d) 3

14. (a) 4, 4

(b) 1, 1

(c) 3, 3

(d) yes

15. (a) 0, 0

(b) 2, 0

(c) 0, 1

(d) no

16. (a) 6

(b) 3, 7

(c) 0, 4

(d) 2

(e) 0

(f) 0

(g) 9

(h) Not defined in this system.

17. (a) 4; What number added to 3 gives 7?
    (b) 4
    (c) 7
    *(d) 7

18.

-	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

The set is closed under subtraction mod 5.

19. (a) 3, 8, 13 and others (add 5)
    (b) 3, 7, 11 and others (add 4)
    (c) 0 and all multiples of 5 of the form 5K, K is a counting number.
    (d) Any even number
    (e) 3, or any odd number greater than 3.
    (f) 1, 3, 5 and so on (all odd numbers).

20. (d) Any even number
    (f) 1, 3, 5, 7, 9, 11, 13 and so on (all odd numbers)
