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ABSTRACT

The purpose of this project is to teach learning and understanding of mathematics at the ninth grade level through the use of science experiments. This part of the program contains significant amounts of material normally found in a beginning algebra class. The material should be found useful for classes in general mathematics as a preparation for enrollment in algebra the following term. In particular, the loaded beam experiment introduces negative numbers, opposites, absolute values and addition of signed numbers. The number generator experiment yields ordered pairs; when graphed, the equation of a line and its slope are determined. The falling sphere experiment gives the same kind of data but also requires the fitting of a "best" straight line. The quadratic function is approached through three experiments: the wick, horizontal metronome, and oscillating spring. Finally, the idea of tangents and slope of a curve are developed through the inclined plane, the lens, and floating magnet - with need found for translation of axes. Included in the Teacher's Commentary are background information, discussion of activities and exercises, and answers to problems. (RH)

SCHOOL
MATHEMATICS
STUDY GROUP

MATHEMATICS THROUGH SCIENCE

PART III: AN EXPERIMENTAL APPROACH TO FUNCTIONS

TEACHERS' COMMENTARY

(revised edition)

SMSG

MATHEMATICS THROUGH SCIENCE

Part III: An Experimental Approach to Functions

Teachers' Commentary

(revised edition)

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PREFACE

The original version of Mathematics Through Science, Part III, prepared in summer, 1963, was tried out in the spring semester of the 1963-1964 school year by 30 teachers in 16 school system with 2,009 ninth grade students. From teacher and student comments and evaluations a thoroughgoing revision has been prepared by a writing team in summer, 1964. The basic tenets originally formulated for Mathematics Through Science have been adhered to. An attempt has been made, however, to provide a reading level in terms of the expectancy for grade nine. New experiments have been devised to substitute for earlier ones which occasioned difficulties for teachers and students.

Part III Revised contains significant amounts of material normally to be found in a beginning algebra course. It is believed that Part III Revised may be found useful for classes in general mathematics as a preparation for enrollment in algebra the following term. Mathematics Through Science endeavors to break a "lock-step" in mathematics education. It seeks to open doors to students upon a new domain of ideas and applications. Thus students may gain mathematical knowledge and skills plus some understanding of scientific investigations and principles.

In particular, the loaded beam experiment introduces negative numbers, opposites, absolute values and addition of signed numbers. The number generator experiment yields ordered pairs, from which graph, equation of a line and its slope are determined. The falling sphere experiment gives the same kind of data but also requires the fitting of a "best" straight line. The quadratic function is approached through three experiments: the wick, horizontal metronome, and oscillating spring. Finally, the idea of tangents and slope of a curve are developed through the inclined plane, the lens, and floating magnet—with need found for translation of axes.

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EQUIPMENT LIST

PART III

Sources for equipment in the following experiments are indicated below and are coded at the right of each item.

(1) Scientific supply (i.e., Cenco or Welch, etc.)

(2) Hardware store

(3) Stationery store

(4) Variety store

(5) Home

Chapter 1

AN EXPERIMENTAL APPROACH TO THE REAL NUMBERS

1. The Loaded Beam - Students are to work in groups. Each group should have the following equipment:

   1 meter stick - (1)

   1 15-inch flexible wooden ruler - (4)

   1 3-inch C clamp - (2)

   1 plastic pulley with mounting rod (approximately 2-inch diameter) - (1), Cenco No. 75660

   1 set of weights (10, 20, 20, 50, 100, 200, 200, 500 grams) - (1)

   1 ring stand support - (1), Cenco No. 72002-2, 72175-3

   1 right angle clamp - (1), Cenco No. 12264

   1 spool heavy thread or nylon cord - (4)

Chapter 2

AN EXPERIMENTAL APPROACH TO LINEAR FUNCTIONS

1. Real Number Generator - students are to work in groups. Each group should have the following equipment:

   1 $\frac{1}{2}$-inch diameter threaded rod, 12 inches long - (2)

   1 hex nut and washer threaded for rod - (2)

   1 12-inch ruler

   2 empty Scotch tape holders - (5)

   1 tube metal cement - (2), i.e., Miracle Brite Magic Adhesive, or similar

   1 roll masking tape

2. Seesaw Experiment - Teacher demonstration with student help.

   1 meter stick - (1)

   1 set of weights (10, 20, 20, 50, 200, 200 grams) - (1)

   2 pulleys - (1) or nails

   1 spool nylon thread - (4)

   1 balance support with knife edge clamp - (1), Cenco No. 75560 or triangular block of wood

Chapter 3

THE FALLING SPHERE

1. The Falling Sphere - students are to work in small groups. Each group should have the following equipment:

   1 glass cylinder or jar at least 8 inches high

   1 steel ball bearing, about $\frac{1}{8}$-inch diameter - (bicycle shop)

   1 small horseshoe magnet - (2)

1 12-inch ruler, also calibrated in centimeters - (3)

1 bottle Karo Syrup (white) - (grocery)

4 paper strips, about 1" × 10" - (5)

1 roll cellophane tape - (1) enough for entire class

1 metronome - (school)

Chapter 4

AN EXPERIMENTAL APPROACH TO NONLINEAR FUNCTIONS

1. The Wick - students are to work in small groups. Each group should have the following equipment:

   1 500-ml beaker - (1)

   1 package paper clips

   1 roll 1-inch wide chromatography paper - (1)

2. The Horizontal Metronome - the number of student groups depends on the number of sets of equipment available. Each set should contain the following equipment:

   1 hacksaw blade, high speed (molybdenum steel), 12" long, $\frac{1}{2}$" wide, 0.025" thick - (2)

   2 $\frac{1}{2}$-lb squares, plumber's lead or solder - (2)

   1 clamp-base vise, 2"-wide jars - (2)

   1 sweep second-hand watch - (5), or stopwatch - (1)

   4 sheets frosted acetate ($8\frac{1}{2}$" × 11") - (engineering supply)

3. The Oscillating Spring - the number of student groups depends on the number of sets of equipment available. Each set should contain the following equipment:

   1 set of hook weights (100, 200, 200, 500) - (1)

   1 window-shade roller spring - (can be obtained from a shade shop), or the equivalent - (2)

   4 sheets frosted acetate ($8\frac{1}{2}$" × 11")

1 sweep second-hand watch - (5), or stopwatch - (1)

1 roll masking tape — should be sufficient for many groups

1 dowel, hardwood ($\frac{1}{2}$" × 36")

Chapter 5

ANALYSIS OF NONLINEAR FUNCTIONS

1. The Inclined Plane - the number of student groups depends on the number of sets of equipment available. Each set should contain the following equipment:

   1 aluminum angle (8-ft length, $\frac{3}{4}$" sides), "Reynold's Do-It-Yourself Aluminum", No. 7A (2)

   1 aluminum angle (4-ft length, $\frac{3}{4}$" sides), "Reynold's Do-It-Yourself Aluminum", No. 7A - (2)

   1 stopwatch, $\frac{1}{2}$-second divisions - (1), or wristwatch with sweep second hand

   1 meter stick - (1)

   1 billiard ball (approximately $2\frac{5}{16}$" diameter), or smooth croquet ball - (sports supplier)

   1 pound of plastolene clay

2. The Simple Lens - the number of student groups depends on the number of sets of equipment available. Each set should contain the following equipment:

   1 positive lens, focal length 8 inches or less - (4)

   1 meter stick - (1)

   1 pound of plastolene - (4)

   1 straight pin

   1 flashlight - (5, 2)

   1 roll adding machine tape - (3), one roll for entire class

3. The Floating Magnet - the number of student groups depends on the number of sets of equipment available. Each set should contain the following equipment:

1 aluminum knitting needle (size 9) - (4)

1 set hook weights (10, 20, 20, 50, 100 grams) - (1)

1 board to mount weights ($\frac{1}{4}$-inch hole to accomodate $\frac{7}{32}$-inch knitting needle) - (5)

4 circular ceramic magnets, with center hole $\frac{7}{32}$-inch diameter. These magnets are taken from magnetic kitchen hooks - (4)

1 epoxy glue - (4) (enough for entire class)

1 centimeter ruler

Chapter 1

AN EXPERIMENTAL APPROACH TO THE REAL NUMBERS

1.1 Introduction

In this chapter we use a "Loaded Beam" to develop the negative numbers. The experimental results will give an intuitive understanding of absolute value and the addition of real numbers. The number line will be extended to include the negative numbers and the operation of addition over the real numbers will be developed. The number line will also be used to extend the property of ordering for all real numbers.

From a coordinate system on a line, we will move to the real number plane. A coordinate system for the plane will be developed and we will learn to associate each point of the plane with an ordered pair of real numbers.

The mathematics developed in this chapter is very similar to that developed in some of the early chapters of Algebra I or General Mathematics. For this reason it would be wise for the teacher to make an early evaluation of the material to determine the appropriate timing for the study of this chapter.

If the student already has a good understanding of the mathematics developed in this chapter the chapter may be omitted.

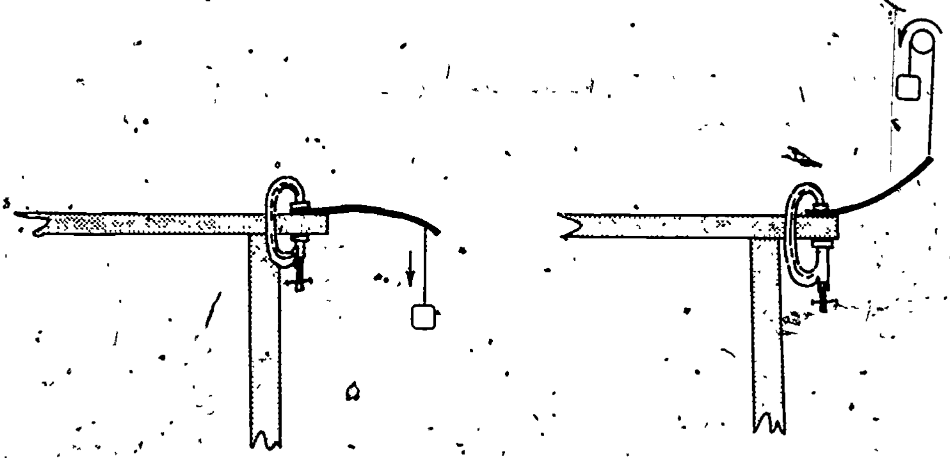
1.2 The Loaded Beam

Each group should have the following equipment:

1 meter stick - (1)
1 15-inch flexible wooden ruler - (4)
1 3-inch C clamp - (2)
1 plastic pulley with mounting rod (approximately 2-inch diameter) - (1), Cenco No. 75660
1 set of weights (10, 20, 20, 50, 100, 200, 200, 500 grams) - (1)
1 ring stand support - (1), Cenco No. 72002-2, 72175-3
1 right angle clamp - (1), Cenco No. 12264
1 spool heavy thread or nylon cord - (4)

The essential feature of this experiment is the linear behavior of the position of the end of the beam as the load changes both in size and direction.

One of the physical variables in this experiment is the length of the beam. The 15-inch ruler should be clamped so that there is at least a 12-inch overhang. If the C-clamp is not placed at the end of the desk, the beam will have different lengths for "upward" and "downward" loads (Figure 1).



(a) (b)

Figure 1

A piece of wood placed over the ruler will prevent this and assure the student a beam of constant length (Figure 2).

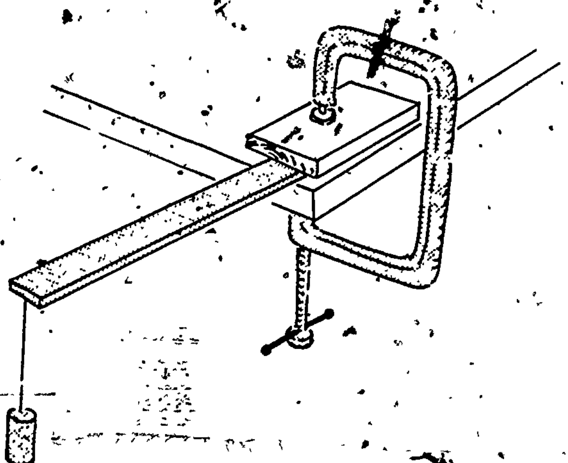


Figure 2

When the pulley arrangement is set up for upward deflections, care must be taken so that the string is approximately perpendicular to the beam (Figure 3). If the string makes an angle with the beam, then only part of the force applied to the beam will be used in bending the beam. One component of the force will be "compressing" or "stretching" the beam (Figure 4). If this happens then the upward deflection for a 100 gram load (for example) will not agree with the downward deflection for the same load.

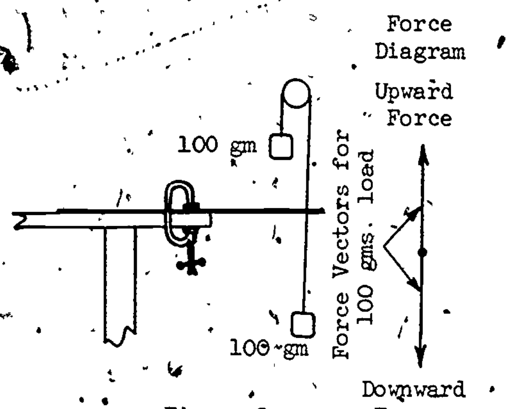


Figure 3

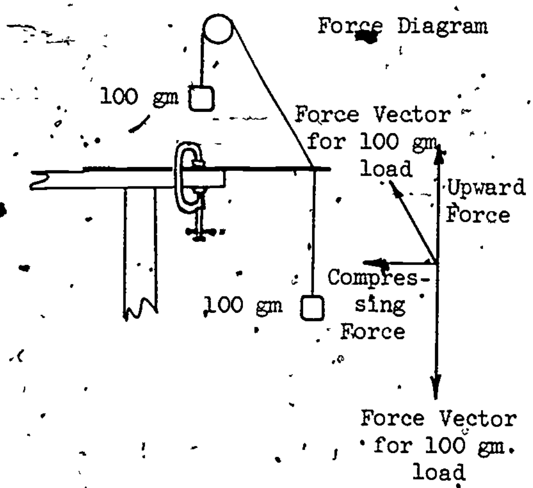


Figure 4

Students may work in groups of six or eight, but each student should have an opportunity to observe the apparatus at close quarters and take some deflection readings of his own to get a good feel for the linear nature of the deflection and the additive properties associated with this deflection. The pulley should be tapped lightly before each reading to minimize errors caused by friction.

There are several substitutions of materials which may be made for the equipment used in the experiment. The beam may be a 15-inch flexible ruler, as suggested in the text, or several other things may be used. Among these are a wooden yardstick which may be supplied free of charge by your local hardware store, a wooden meter stick, a metal carpenter's rule, or $1'' \times \frac{1}{8}''$ aluminum stripping cut in appropriate lengths. Instead of the ring stand - pulley set-up you may make an L-shaped support from 2 × 4's and drive one or two large nails into the upright to replace the pulley. If the nails are used instead of the pulley, it would be best to make a special effort to get nylon string, since this would introduce the least amount of friction into the system. (See Figure 5.)

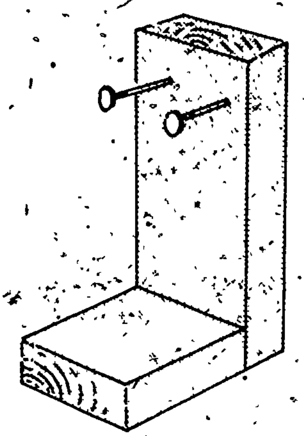


Figure 5

1.3 The Real Number Line

This section assumes that the student is already familiar with the set of positive numbers as displayed on a number line. The idea of a unit distance is emphasized and the negative numbers are developed within the setting of the experiment.

In general, we have taken the point of view that the student really has some experience with negative numbers. He is ready to label the points to the left of 0. We extend the numbers of arithmetic to the set of real numbers by attaching the negative numbers to the familiar numbers of arithmetic.

In graphing real numbers, the teacher should emphasize the fact that the number line which the student draws is only an approximation of an ideal number line. Consequently, any information which he deduces from his number line is only as accurate as his drawing.

An understanding of the Pythagorean Theorem is implicit in the method developed for locating $\sqrt{2}$ on the number line. However, this should not be allowed to distract from the main ideas of the section.

For the more capable student, the Pythagorean Theorem can be briefly discussed. The scheme for graphing $\sqrt{2}$ can then be extended to give a method for determining successively $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$ etc. Given the unit distance on the number line, let ℓ be a line parallel to the number line and one unit away from it. Construct the $\sqrt{2}$ as illustrated in the Figure 6.

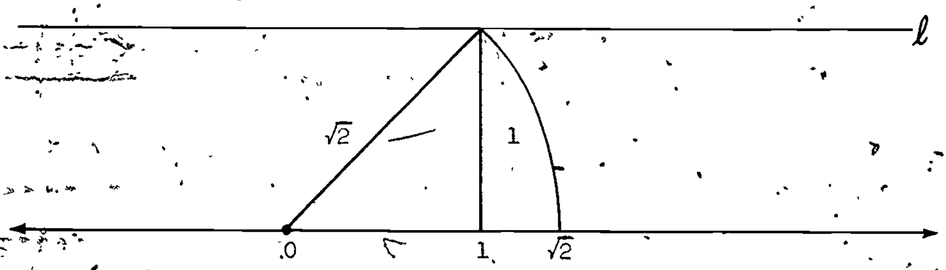


Figure 6

The perpendicular to the number line at $\sqrt{2}$ meets ℓ at a point A, and the circle with center O and radius OA meets the number line at $\sqrt{3}$ (Figure 7).

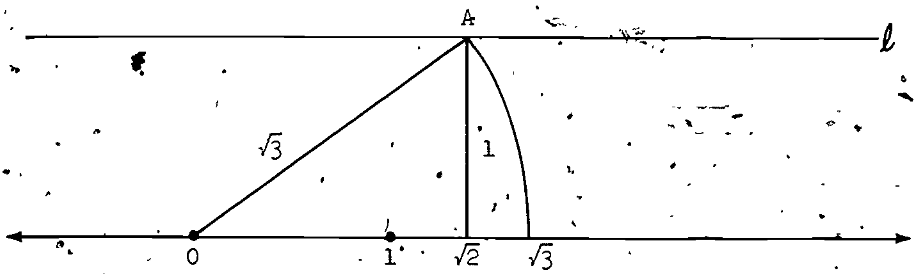


Figure 7

Applying the same technique to $\sqrt{3}$, we can locate $\sqrt{4}$. This process may be continued indefinitely (Figure 8).

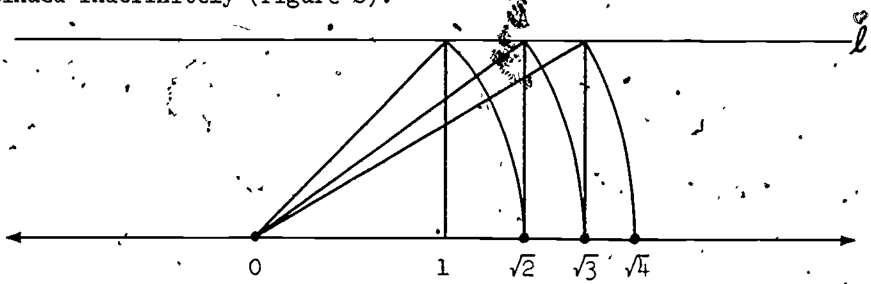


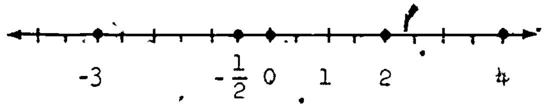
Figure 8

We want the student to realize that there are many points on the number line which do not have rational coordinates.

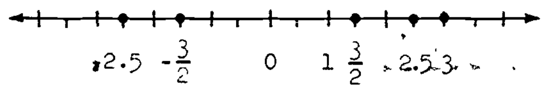
Exercise 1

1. For each of the following, construct a number line and determine the points whose coordinates are as follows:

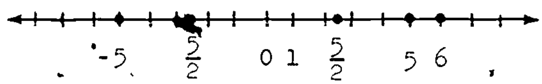
(a) $0, 4, 2, -\frac{1}{2}, -3$



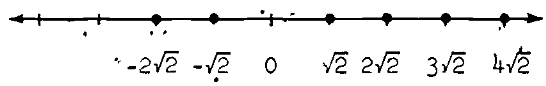
(b) $\frac{3}{2}, -\frac{3}{2}, 2.5, -2.5, 3$



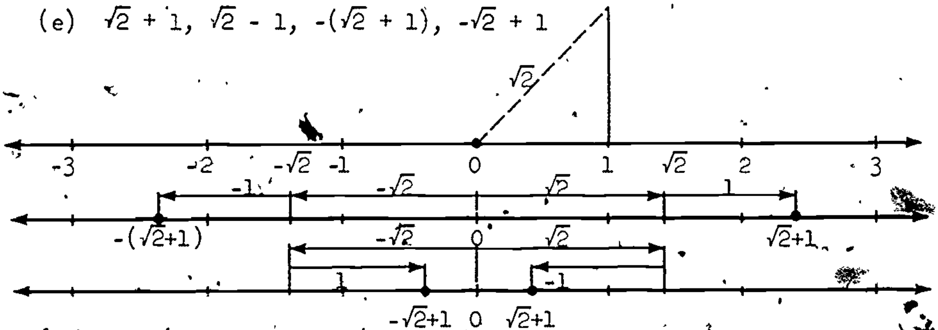
(c) $-5, -\frac{5}{2}, \frac{5}{2}, 5, 6$



(d) $\sqrt{2}, -\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, -2\sqrt{2}$



(e) $\sqrt{2} + 1, \sqrt{2} - 1, -(\sqrt{2} + 1), -\sqrt{2} + 1$



2. Arrange each set of three numbers given below in the order in which they would appear on the number line, reading from left to right.

(a) $10, 4, 6$ — $4 < 6 < 10$

(b) $4, 2, -4$ — $-4 < 2 < 4$

(c) $-1, -2, -3$ — $-3 < -2 < -1$

(d) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ — $\frac{2}{3} < \frac{3}{4} < \frac{4}{5}$

(e) $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ — $\frac{5}{4} < \frac{4}{3} < \frac{3}{2}$

(f) $\frac{7}{4}, \sqrt{3}, 1.73$ — $1.73 < \sqrt{3} < \frac{7}{4}$

(g) $-2.24, -\sqrt{5}, -\frac{9}{4}$ — $-\frac{9}{4} < -2.24 < -\sqrt{5}$

(h) $\sqrt{7}, -2.65, \frac{13}{5}$ — $-2.65 < \frac{13}{5} < \sqrt{7}$

3. Which of the following rational numbers is closest to $\sqrt{2}$? $\frac{577}{408}$

(a) $\frac{3}{2}$ $\qquad (\frac{3}{2})^2 = 2.25$

(b) $\frac{17}{22}$ $\qquad \frac{17}{22} < 1 \therefore (\frac{17}{22})^2 < 1$

(c) $\frac{7}{5}$ $\qquad (\frac{7}{5})^2 = 1.96$

(d) $\frac{99}{70}$ $\qquad (\frac{99}{70})^2 \quad 2.00019$

(e) $\frac{577}{408}$ $\qquad (\frac{577}{408})^2 \quad 2.00016$

1.4 Ordering the Real Numbers

The comparison property is also called the trichotomy property of order. Notice that it is a property of $<$; that is, given any two different numbers, they can be ordered so that one is less than the other. When the property is stated we must include the third possibility that the numerals name the same number. Hence, the name "trichotomy".

Although "$a < b$" and "$b > a$" involve different orders, these sentences say exactly the same thing about the numbers a and b. Thus, we can state a trichotomy property of order involving ">" as:

> For any number a and any number b, exactly one of these is true,
>
> $$a > b, \ a = b, \ b > a .$$

If, instead of concentrating attention on the order relation, we concentrate on the two numbers, then either "$a < b$" or "$a > b$" is true, but not both. Here we fix the numbers a and b and then make a decision as to which order relation applies. It is purely a matter of which we are interested in: the numbers or the order. The comparison property is concerned with an order.

The extension of the comparison property to the transitive property is an important extension of the ordering properties. Any attempt to illustrate this transitive property of $<$ with triples of integers is likely to be met with a vociferous "So what!" by your students. On the other hand, not only can this property be illustrated with fractions as in the text, but the student can also begin to appreciate its usefulness.

Exercise 2

1. Use appropriate properties to order each of the following pairs of numbers.

(a) 0, 56

$0 < 1$ and $1 < 56$, so $0 < 56$

(b) -7, 0

$-7 < -1 < 0$

(c) 33.3, $33\frac{1}{3}$

$33.3 < 33.33 < 33.333 \ldots = 33\frac{1}{3}$.

(d) -50, -100

$50 < 75 < 100$; so $50 < 100$ and $-100 < -50$

(e) $\frac{25}{26}, \frac{27}{26}$

$\frac{25}{26} < 1$ and $1 < \frac{27}{26}$ so $\frac{25}{26} < \frac{27}{26}$

(f) $\frac{2}{3}, \frac{667}{10,000}$

$\frac{2}{3} < .6667 < \frac{667}{10,000}$

(g) $-\frac{3}{4}, -.75$

$-\frac{3}{4} = -.75$

(h) $-\frac{5}{8}, -\frac{6}{8}$

$\frac{5}{8} < \frac{6}{8}$ and $-\frac{6}{8} < -\frac{5}{8}$

(i) $-\pi, -3.14$

$3.14 < 3.14159 < \pi$ and $-\pi < -3.14$

(j) $\sqrt{3}, -1.732$

$-1.732 < 0$ and $0 < \sqrt{3}$

$-1.732 < \sqrt{3}$

2. In the blanks below, use one of the symbols, =, <, >, to make a true sentence.

(a) $\frac{3}{5} \underline{>} -\frac{6}{10}$

(b) $\frac{3}{5} \underline{>} \frac{3}{6}$

(c) $-\frac{3}{5} \underline{<} -\frac{3}{6}$

(d) $\sqrt{2} \underline{<} \sqrt{3}$

(e) $-\frac{1}{3} \underline{>} -.666\ldots$

(f) $\frac{9}{16} \underline{>} \frac{1}{2}$

(g) $\sqrt{2} + 3 \underline{>} \sqrt{3} + 2$

(h) $\frac{1}{4} \underline{>} .125$

(i) $2\sqrt{5} \underline{<} 5\sqrt{2}$

(j) $-\frac{103}{13} \underline{<} -\frac{205}{26}$

3. Use the transitive property to determine the ordering of the following groups of three real numbers.

(a) $-\frac{1}{5}, \frac{3}{2}, 12$

$-\frac{1}{5} < \frac{3}{2} < 12$

(b) $\pi, -\pi, \sqrt{2}$

$-\pi < \sqrt{2} < \pi$

(c) 1.7, 0, -1.7

$-1.7 < 0 < 1.7$

(d) $-\frac{27}{15}, -\frac{3}{15}, -\frac{2}{15}$

$-\frac{27}{15} < -\frac{3}{15} < -\frac{2}{15}$

(e) $3^2, 4^2, (3 + 4)^2$

$3^2 < 4^2 < 7^2$

(f) $-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}$

$-\frac{1}{2} < -\frac{1}{3} < -\frac{1}{4}$

(g) $1 + \frac{1}{2}, 1 + (\frac{1}{2})^2, 1 + (\frac{1}{2})^3$

$1 + \frac{1}{8} < 1 + \frac{1}{4} < 1 + \frac{1}{2}$

4. State a transitive property for ">" and illustrate this property with Problem 3(a) and (b).

If a, b, c are real numbers, and if $a > b$ and $b > c$, then $a > c$.

(a) $12 > \frac{3}{2} > -\frac{1}{5}$

(b) $\pi > \sqrt{2} > -\pi$

5. Sandy and Bob are seated on opposite ends of a seesaw, and Sandy's end of the seesaw comes slowly to the ground. Harry replaces Sandy at one end of the seesaw, after which Bob's end of the seesaw comes to the ground. Who is heavier, Sandy or Harry?

$S > B$	Sandy is heavier than Bob
$B > H$	Bob is heavier than Harry
$S > H$	Sandy is heavier than Harry

1.5 Opposites

Your students have quite likely observed by now that, except for 0, the real numbers occur as pairs, the two numbers of each pair being equidistant from 0 on the real number line. Each number in such a pair is called the opposite of the other. To complete the picture, 0 is defined to be its own opposite. In locating the opposite of a given number on the number line, you may well want to use a compass to emphasize that the number and its opposite are equidistant from 0.

If x is a positive number, then $-x$ is a negative number. The opposite of any negative number x is the positive number $-x$, and $-0 = 0$. The student should not jump to the conclusion that when n is a real number, then $-n$ is a negative number; this is true only when n is a positive number.

In order to motivate the "ordering property for opposites",

For real numbers a and b,
if $a < b$, then $-b < -a$,

it would be well to consider several other pairs of numbers. For example, we could consider a pair of distinct positive numbers, a pair of distinct negative numbers, 0 and a positive number, and 0 and a negative number.

Exercise 3

1. Simplify each of the following expressions.

(a)	$-(4 + 2)$	-6	(g)	$-(2 + 5) + 15$	8
(b)	$-(-2.3)$	2.3	(h)	$-(7 - 10) - 3$	0
(c)	$-(42 + 0)$	-42	(i)	$-(3 \times 4) + (-3)$	-15
(d)	$-(3.6) - (2.4)$	-6.0	(j)	$-[-(-5)] + 5$	0
(e)	$-(42 \times 0)$	0	(k)	$-(-7) + [-(-7)]$	14
(f)	$-[-(-4)]$	-4	(l)	$-(3) + [-(-3)] - [-(-3)]$	-3

2. What kind of number is -x if x is positive?

   -x is a negative number.

   If x is negative?

   -x is a positive number.

   If x is zero?

   -x is zero.

3. What kind of number is x if -x is a positive number?

   x is a negative number.

   If -x is a negative number?

   x is a positive number.

   If -x is zero?

   x is zero.

4. (a) Is every real number the negative of some real number?

   Yes. Remember zero is its own negative.

   (b) Is the set of all negatives of real numbers the same as the set of all real numbers?

   Yes.

   (c) Is every opposite of a number a negative number?

   No. The opposite of a negative number is a positive number.

5. For each of the following pairs, determine which is the greater number.

(a)	2.97, -2.97	2.97 > -2.97	(e)	-370, -121	-121 > -370
(b)	-12, 2	2 > -12	(f)	0.12, 0.24	0.24 > 0.12
(c)	-358, -762	-358 > -762	(g)	0, -0	0 = -0
(d)	-1, 1	1 > -1	(h)	-0.1, -0.01	-0.01 > -0.1
			(i)	0.1, 0.01	0.1 > 0.01

6. Write true sentences for the following numbers and their opposites, using the relations " < " or " > ".

Example: For the numbers 2 and 7, $2 < 7$, and $-2 > -7$.

(a) $\frac{2}{7}, -\frac{1}{6}$ $\qquad -\frac{1}{6} < \frac{2}{7}$, and $\frac{1}{6} > -\frac{2}{7}$

(b) $\sqrt{2}, -\pi$ $\qquad -\pi < \sqrt{2}$ and $\pi > -\sqrt{2}$

(c) $\pi, \frac{22}{7}$ $\qquad \pi < \frac{22}{7}$ and $-\pi > -\frac{22}{7}$

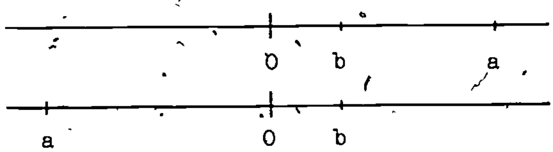
(d) $3(\frac{4}{3} + 2), \frac{5}{4}(20 + 8)$ $\qquad 3(\frac{4}{3} + 2) < \frac{5}{4}(20 + 8)$ and $-3(\frac{4}{3} + 2) > -\frac{5}{4}(20 + 8)$

(e) $-(\frac{8 + 6}{7}), -2$ $\qquad -(\frac{8 + 6}{7}) = -2$

(f) $-\{(3 + 17)0\}, -\{(5 + 0)3\}$ $\qquad -\{5 \times 3\} < 0$ and $0 < 5 \times 3$

7. Let us write "★" for the phrase "is farther from 0 than" on the real number line. Does "★" have the comparison property enjoyed by " > "; that is, if a and b are different real numbers, is it true that a ★ b or b ★ a but not both?

No. The relation a ★ b could look like either of the following figures.



where $a > b$

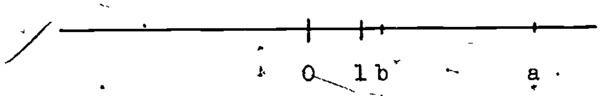
where $b > 0$

Does "a ★ b" have a transitive property?

Yes. If a is farther from zero than b and b is farther from zero than c, then a is farther from zero than c.

For which subset of the set of real numbers do "★" and " > " have the same meaning?

The set of all non-negative real numbers.



$a > b$

25

8. Translate the following English sentences into mathematical expressions, describing the variable used.

(a) The load on the beam is greater than 100 grams. What is the load?

$\ell > 100$

(b) The deflection of the beam was no more than 18 mm up. What was the deflection?

$p \geq -18$

(c) Paul hung 30 grams from the beam, but Jim added more than 60 grams to the load. What was the load?

$\ell > 30 + 60$ or $\ell > 90$.

9. Change the numerals "$-\frac{13}{42}$" and "$-\frac{15}{49}$" to forms with the same denominators. (Hint: First do this for $\frac{13}{42}$ and $\frac{15}{49}$.) What is the order of $-\frac{13}{42}$ and $-\frac{15}{49}$? (Hint: Knowing the order of $\frac{13}{42}$ and $\frac{15}{49}$, what is the order of their opposites?)

$$\frac{13}{42} \cdot \frac{49}{49} = \frac{637}{2058} \text{ and } \frac{15}{49} \cdot \frac{42}{42} = \frac{630}{2058} \text{ so } \frac{15}{49} < \frac{13}{42}, \text{ and } -\frac{13}{42} < -\frac{15}{49}.$$

Now state a general rule for determining the order of two negative rational numbers.

For any negative rational numbers -a and -b, if $a < b$ then $-b < -a$.

1.6 Absolute Value

The concept of the absolute value of a number is one of the most useful ideas in mathematics. We will find an immediate application of absolute value when we define addition and multiplication of real numbers.

The usual definition of the absolute value of the real number n is that it is the number $|n|$ for which

$$|n| = \begin{cases} n, & \text{if } n \geq 0 \\ -n, & \text{if } n < 0. \end{cases}$$

This is also the form in which the absolute value is most commonly used. On the other hand, since students seem to have difficulty with definitions of this kind, we define the absolute value of a number in such a way that it can be clearly pictured on the number line.

By observing that this "greater" of a number and its opposite is just the distance between the number and 0 on the real number line, we are able to interpret the absolute value "geometrically".

Exercise 4

1. Find the absolute values of the following numbers.

(a) $\lvert -7\rvert$	7	(f) $\lvert -[-(-3)]\rvert$	3
(b) $\lvert -(-3)\rvert$	3	(g) $\lvert 14 \times 0\rvert$	0
(c) $\lvert (6 - 4)\rvert$	2	(h) $\lvert (4 + 3) - 7\rvert$	0
(d) $\lvert -14 + 0\rvert$	14	(i) $\lvert -[-(-5)]\rvert$	5
(e) $\lvert -(10 - 8)\rvert$	2	(j) $\lvert -[5(3 - 2)]\rvert$	5

2. For a negative number x, which is greater, x or $\lvert x\rvert$?

   $\lvert x\rvert > x$, since all positive numbers are greater than any negative number.

3. Which of the following statements are true?

(a) $\lvert -7\rvert < 3$	false	(e) $-3 < 17$	true
(b) $\lvert -2\rvert \le \lvert -3\rvert$	true	(f) $-2 < -\lvert 3\rvert$	false
(c) $\lvert 4\rvert \ngtr \lvert 1\rvert$	false	(g) $\lvert\sqrt{16}\rvert > \lvert -4\rvert$	false
(d) $2 \nless \lvert -3\rvert$	false	(h) $\lvert -2\rvert^2 = 4$	true

4. Simplify each of the following.

(a) $\lvert 2\rvert + \lvert 3\rvert$	5	(j) $\lvert -2\rvert + \lvert -3\rvert$	5
(b) $\lvert -2\rvert + \lvert 3\rvert$	5	(k) $-(\lvert -3\rvert - 2)$	-1
(c) $-(\lvert 2\rvert + \lvert 3\rvert)$	-5	(l) $-(\lvert -2\rvert + \lvert -3\rvert)$	-5
(d) $-(\lvert -2\rvert + \lvert 3\rvert)$	-5	(m) $3 - \lvert 3 - 2\rvert$	2
(e) $\lvert -7\rvert - (7 - 5)$	5	(n) $-(\lvert -7\rvert - 6)$	-1
(f) $7 - \lvert -3\rvert$	4	(o) $\lvert -5\rvert \times \lvert -2\rvert$	10
(g) $\lvert -5\rvert \times 2$	10	(p) $-(\lvert -2\rvert \times 5)$	-10
(h) $-(\lvert -5\rvert - 2)$	-3	(q) $-(\lvert -5\rvert \times \lvert -2\rvert)$	-10
(i) $\lvert -3\rvert - \lvert 2\rvert$	1		

1.7 Addition of Real Numbers

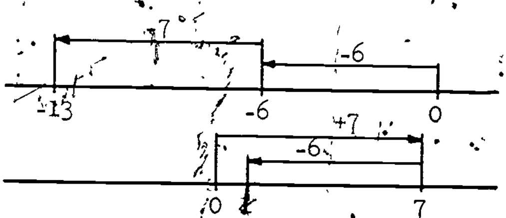
At this point we again return to the experiment to reinforce intuitive understanding of the operation of addition over the real numbers.

We have seen that the definition of addition of real numbers satisfies two of three requirements we make. It includes, as a special case, the familiar addition of numbers of arithmetic, and it agrees with our intuitive feeling for this operation as shown in working with loading the beam and with its deflection. The third requirement is that addition of real numbers have the same basic properties that we observed for addition of numbers of arithmetic. It would be awkward, for instance, to have addition of numbers of arithmetic commutative and addition of real numbers not commutative.

Notice that the commutative and associative properties were regarded as <u>axioms</u> for the numbers of arithmetic, and the operation of addition was regarded essentially as an <u>undefined</u> operation. For the real numbers, however, we have made a <u>definition</u> of addition in terms of earlier concepts. If our definition has been properly chosen, we should find that the properties can be <u>proved</u> as theorems.

Exercise 5

1. Perform the indicated additions on real numbers, using the number line to aid you.



(a) $(-6) + (-7)$ <u>-13</u>

(b) $(7) + (-6)$ <u>1</u>

(c) $(-9) + (5)$ <u>-4</u>

(d) $6 + (-4)$ <u>2</u>

(e) $(-8) + (8)$ <u>0</u>

(f) $(25) + (-73)$ <u>-48</u>

(g) $5\frac{1}{2} + 2\frac{1}{2}$ <u>8</u>

(h) $(-2) + (-7)$ <u>-9</u>

(i) $(-4.6) + (-1.6)$ <u>-6.2</u>

(j) $(-3\frac{1}{3}) + (2\frac{2}{3})$ <u>$-\frac{2}{3}$</u>

Parts (c) through (j) would be illustrated in a similar manner.

2. Tell in your own words what you do to the two given numbers to find their sum.

(a) $7 + 10$ Start at 7 on the number line and move 10 units to the right to get a sum of 17.

(b) $7 + (-10)$ Start at 7 on the number line and move 10 units to the left to get a sum of -3.

(c) $10 + (-7)$

(d) $(-10) + (-7)$

(e) $10 + 0$

(f) $0 + (-7)$

(g) $7 + |-10|$

(h) $|7| + (-10)$

(i) $|7 + (-10)|$

(j) $(-|-10|) + |7|$

Parts (c) through (j) would be described in a similar manner.

3. In each of the following, find the sum, first according to the definition and then by any other method you find convenient.

(a) $(-5) + 3$ $\underline{-2}$

(b) $(-11) + (-5)$ $\underline{-16}$

(c) $(-\frac{8}{3}) + 0$ $\underline{-\frac{8}{3}}$

(d) $2 + (-2)$ $\underline{0}$

(e) $18 + (-14)$ $\underline{4}$

(f) $12 + 7.4$ $\underline{19.4}$

(g) $(-\frac{2}{3}) + 5$ $\underline{4\frac{1}{3}}$

(h) $(-35) + (-65)$ $\underline{-100}$

4. In the course of a week the variations in mean temperature from the seasonal normal of 71 were -7, 2, -3, 0, 9, 12, -6. What were the mean temperatures each day?

Sun: $71 - 7 = 64$ Mon: $71 + 2 = 73$ Tues: $71 - 3 = 68$

Wed: $71 + 0 = 71$ Thurs: $71 + 9 = 80$ Fri: $71 + 12 = 83$

Sat: $71 + (-6) = 65$

What is the sum of their variations?

$$(-7) + 2 + (-3) + 0 + 9 + 12 + (-6) = 7$$

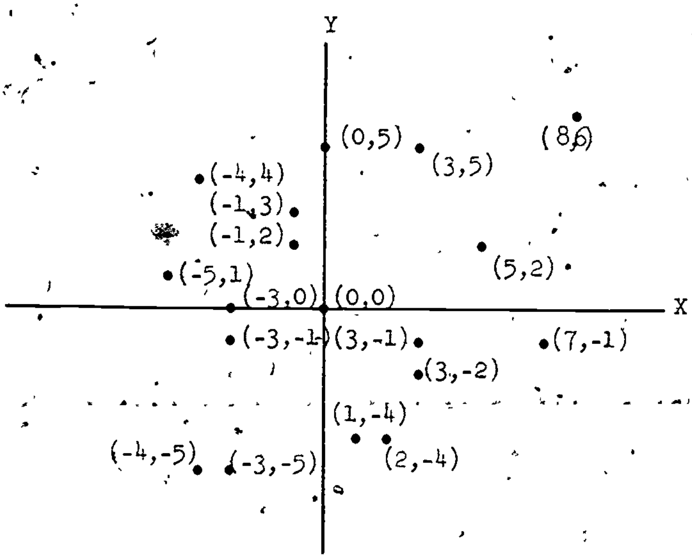
1.8 The Real Number Plane

You will notice that in the discussion of the coordinate system in a plane care was taken not to mention the x-axis or the y-axis. It was desired to give the student the feeling that the label attached to the horizontal axis and to the vertical axis would be dependent upon the sets which represent the domain and the range of the relation being graphed.

Exercise 6

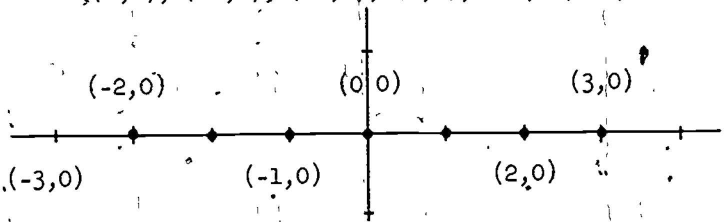
1. Plot the following ordered pairs of numbers, write the number of the quadrant or the position on an axis in which you find the point represented by each of these ordered pairs:

(a)	(3,5)	I	(j)	(3,-2)	IV
(b)	(-5,1)	II	(k)	(-3,-5)	III
(c)	(1,-4)	IV	(l)	(-1,3)	II
(d)	(-4,4)	II	(m)	(2,-4)	IV
(e)	(0,0)	origin	(n)	(5,2)	I
(f)	(0,5)	+ y-axis	(o)	(-3,0)	- x-axis
(g)	(-3,-1)	III	(p)	(-4,-5)	III
(h)	(7,-1)	IV	(q)	(-1,2)	II
(i)	(8,6)	I	(r)	(3,-1)	IV



2. (a) Plot on a coordinate plane the following set of points:

{(0,0), (-1,0), (-2,0), (2,0), (-3,0), (3,0)} .



(b) Do all the points in this set seem to lie on the same line?

Yes, they all lie on the horizontal axis.

(c) What do you notice about the vertical coordinate for each of the points?

All ordered pairs have 0 as the vertical coordinate.

3. (a) Plot the points in the following set:

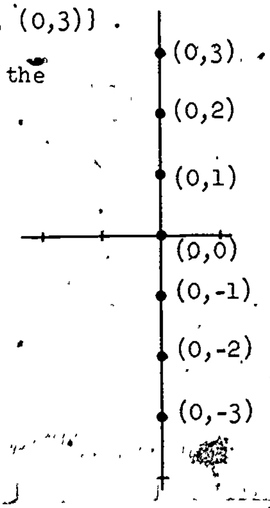
{(0,0), (0,-1), (0,1), (0,-2), (0,2), (0,-3), (0,3)} .

(b) Do all the points named in this set seem to be on the same line?

Yes, they all lie on the vertical axis.

(c) What do you notice about the horizontal coordinate for each of the points?

All ordered pairs have 0 as the horizontal coordinate.

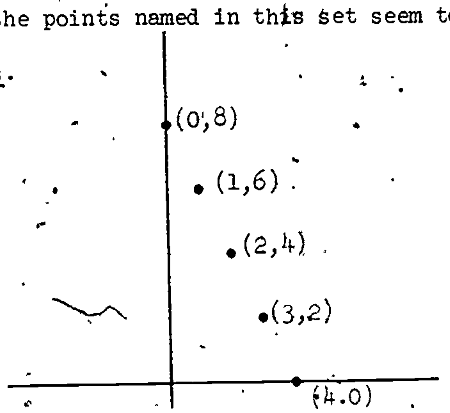


4. (a) Plot the points in the following set:

{(0,8), (1,6), (2,4), (3,2), (4,0)} .

(b) Do all the points named in this set seem to lie on the same line?

Yes.



<u>Sample Test Items</u>

1. Arrange each set of four numbers given below in the order in which they would appear on the number line, reading left to right.

   (a) $0, 4, -\frac{1}{2}, -3$

   (b) $4, \frac{7}{4}, -2, -\frac{9}{4}$

   (c) $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$

2. Use the transitive property to determine the ordering of the following groups of three real numbers.

   (a) $-\frac{1}{8}, -\frac{1}{9}, -\frac{1}{10}$

   (b) $\frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3$

   (c) $1.4, 1.41, 1.414$

3. Simplify the following expressions.

   (a) $-(-4.7)$

   (b) $-[-(-3)]$

   (c) $-(3 + 2) + 6$

   (d) $-(2 \times 3) + 6$

4. For each of the following pairs, determine which is the least number.

   (a) $2.4, 24$

   (b) $-0.01, -0.001$

   (c) $-2.09, -2.10$

   (d) $-315, -362$

5. Translate the following into mathematical expressions.

   (a) Bob hung 80 grams from the beam but Roy added more. What was the load?

   (b) The deflection of the beam was at least 8mm down. What was the deflection?

6. Which of the following statements are true?

   (a) $|-3| < |-2|$

   (b) $|-3|^2 = 9$

   (c) $5 \nless |-6|$

   (d) $|\sqrt{25}| < |-5|$

   (e) $-|-3| = -3$

   (f) $|-5| < |-6|$

7. Simplify each of the following:

(a) $-(|-3| + |5|)$
(b) $8 - (-5)$
(c) $|-7| - |3|$
(d) $-(|-2| - |4|)$
(e) $|-7| \times |-3|$
(f) $-(|-4| \times |-3|)$

8. In each of the following, find the indicated sum.

(a) $(-|-6| + |5|)$
(b) $(-15) + (-25)$
(c) $(-\frac{7}{4}) + (-\frac{5}{4})$
(d) $16 + (-12)$
(e) $|7 + (-12)|$
(f) $(-2.8) + (1.6)$

9. (a) Plot the points in the following set on a coordinate plane.

$\{(3,4), (-2,-1), (-4,-4), (0,1), (5,6)\}$

(b) Do all the points seem to lie on the same line?

(c) If not, what are the greatest number of points on one line?

Answers to Sample Test Items

1. (a) $-3, -\frac{1}{2}, 0, 4$
(b) $-\frac{9}{4}, -2, \frac{7}{4}, 2$
(c) $\frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}$

2. (a) $-\frac{1}{8} < -\frac{1}{9} < -\frac{1}{10}$
(b) $\frac{1}{2^3} < \frac{1}{2^2} < \frac{1}{2}$
(c) $1.4 < 1.41 < 1.414$

3. (a) 4.7
(b) -3
(c) 1
(d) 0

4. (a) 2.4
(b) -0.01
(c) -2.10
(d) -362

5. (a) $\ell > 80$
(b) $p \leq 8$

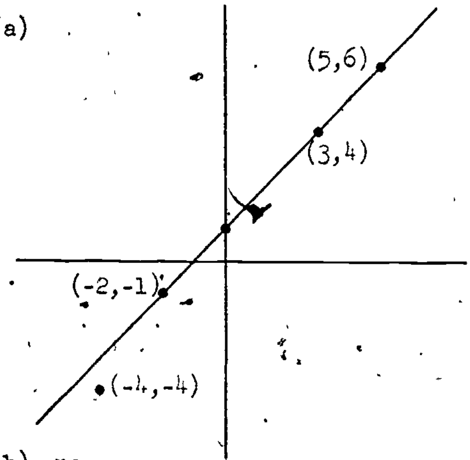
6. (b), (e) and (f) are true.

7. (a) -8
(b) 13
(c) 4
(d) 2
(e) 21
(f) -12

8. (a) -1 (c) -3 (e) 5

(b) -40 (d) 4 (f) -1.2

9. (a)



(b) no

(c) 4, all except (-4,-4) .

Chapter 2

AN EXPERIMENTAL APPROACH TO LINEAR FUNCTIONS

2.1 Real Number Generator

In Chapter 2, an effort is made to further motivate the understanding of the number line and the coordinate plane. The number generator has potential for aiding the understanding of a wide range of mathematical ideas. In this instance, an examination of many aspects of arithmetic is made.

The Number Generator Experiment may be done either in student groups or as a teacher demonstration. Equipment list for teacher demonstration follows:

1 threaded rod ($\frac{1}{2}$" × 12")
1 $\frac{1}{2}$" hex nut
1 $\frac{1}{2}$" inside diameter, $1\frac{1}{2}$" outside diameter washer
2 transparent tape holders
1 12" ruler
1 small roll of masking tape
$\frac{1}{2}$ lb of modeling clay
1 tube of adhesive for indicator (Miracle Brite Magic)

Take a one-foot piece of $\frac{1}{2}$" threaded rod with a fitting hex nut and washer. Glue the washer to the hex nut and thread the combination on the rod. Support the rod with two transparent tape holders and modeling clay. Mask the ruler and "glue" it on tape holders as illustrated in the student text.

In the manipulation of the number generator, the direction of rotation of the indicator is deliberately ignored so the "turns" are positive only. The faces of the hex nut are deliberately marked only on the zero and $\frac{1}{2}$-turn faces so that the direction of rotation will not be an obvious issue. If the hex nut were marked 0, 1, 2, 3, 4, 5, when you turn it the opposite direction would read 0, 5, 4, 3, 2, 1.

Exercise 1 should give the students a good review of the arithmetic of the rational numbers. The conversions of turns and face changes to numbers on the scale might stimulate a discussion which would make rational numbers more meaningful to the student.

Exercise 1

1. How many turns of the indicator are necessary to generate the following numbers?

(a) 3	30 turns	(d) -2.8	28 turns
(b) -4	40 turns	(e) 5.45	$54\frac{1}{2}$ turns
(c) 1.4	14 turns	(f) $1\frac{1}{3}$	$13\frac{1}{3}$ turns or 13 turns + 2 face changes

2. How many face changes of the hex nut from the zero point will generate the following numbers?

(a) 2	120	(i) $-2\frac{1}{10}$	126
(b) -3	180	(j) $\frac{5}{6}$	50
(c) $\frac{33}{60}$ or $\frac{11}{20}$	33	(k) $\frac{3}{5}$	36
(d) $\frac{5}{12}$	25	(l) $\frac{15}{3}$	300
(e) $\frac{5}{4}$	75	(m) $\frac{27}{6}$	270
(f) $-1\frac{1}{2}$	90	(n) $5\frac{1}{3}$	320
(g) $\frac{7}{12}$	35	(o) $7\frac{3}{10}$	438
(h) $-4\frac{11}{15}$	284		

3. What numbers would be generated by the following number of turns of the indicator?

(a) Right 35	3.5 or $3\frac{1}{2}$	(d) Left 42	-4.2 or $-4\frac{1}{5}$
(b) Left 15	-1.5 or $-1\frac{1}{2}$	(e) Right $17\frac{1}{2}$	1.75 or $1\frac{3}{4}$
(c) Right 95	9.5 or $9\frac{1}{2}$	(f) Left $2\frac{1}{3}$	-.233... or $-\frac{7}{30}$

$$10x = -2.333\ldots$$
$$x = -.233\ldots$$
$$9x = -2.1$$
$$x = \frac{-2.1}{9} = -\frac{21}{90} = -\frac{7}{30}$$

4. What numbers would be generated by the following number of face changes from the zero position?

(a) Right 90 $1\frac{1}{2}$ (d) Right 156 $2\frac{3}{5}$

(b) Left 45 $-\frac{3}{4}$ (e) Left 512 $-8\frac{8}{15}$

(c) Left 256 $-4\frac{4}{15}$ (f) Right 316 $5\frac{4}{15}$

2.2 Functions and Relations

At this point the student should arrange the data from Exercise 1, Problems 1 and 3, in "ordered pair" form with the number on the scale in the position of the first element and the number of turns of the indicator necessary to generate the second element. Then plot the ordered pairs on a coordinate plane as instructed in the text.

Since the unit distance on the two axes need not be the same, a typical graph may appear as follows (Figure 1).

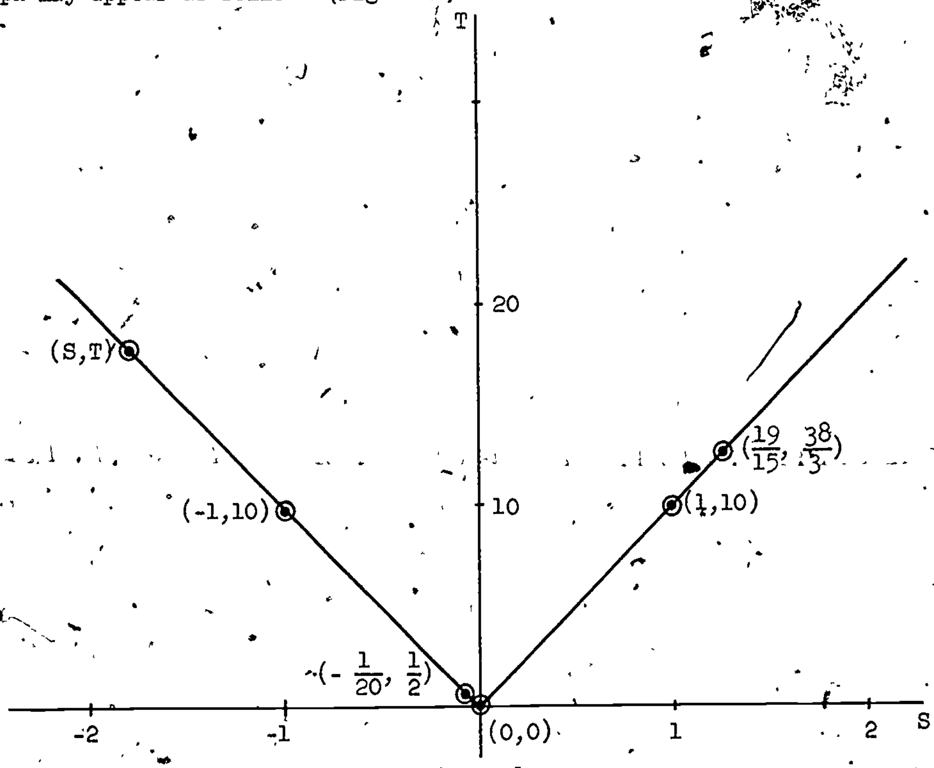


Figure 1

2.3 The Face-Scale Relation

Now return to Exercise 1, Problems 2 and 4. This time the student will prepare a new set of ordered pairs. The first element this time will be the number of face changes from a fixed position and the second element will be the number on the scale corresponding to the first. The graph in this instance may appear as Figure 2 after due consideration of continuity.

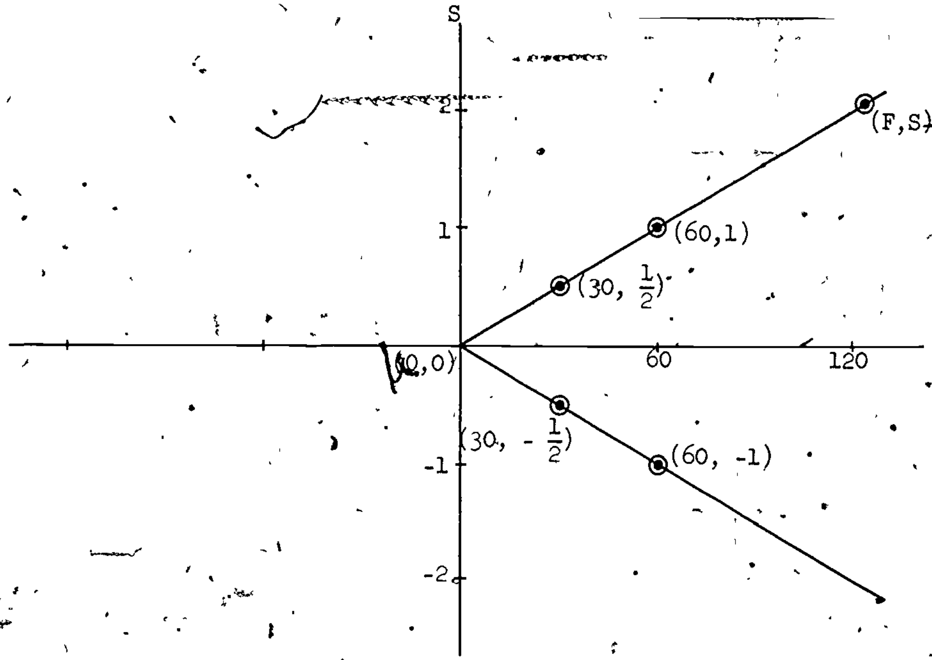


Figure 2

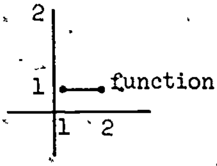
This section could produce a good discussion of the importance of definitions in mathematics.

It is possible some students will have difficulty with discrete and continuous relations. Many examples should be given such as counting the cars in a parking lot and the parking lot itself provided there are no "breaks" in it. Don't be formal or rigorous with your examples. Physical continuity is sufficient, so build on the student's intuition.

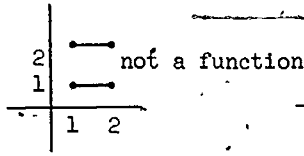
Exercise 2

1. Which of the graphs of the relations shown below are graphs of a function?

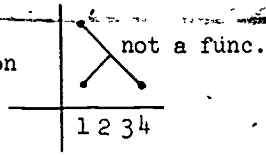
Examples:



(1)

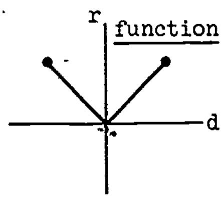


(2)

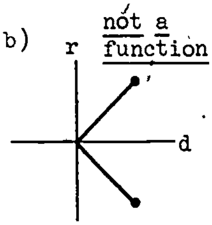


(3)

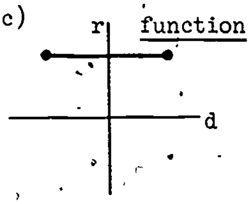
(a)



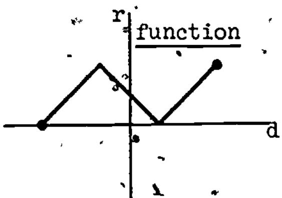
(b)



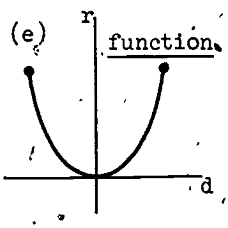
(c)



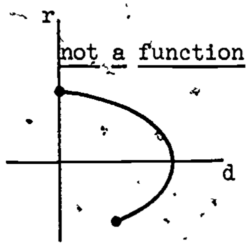
(d)



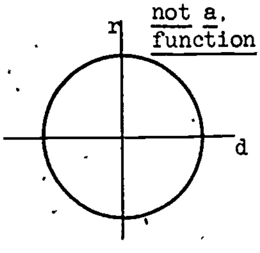
(e)



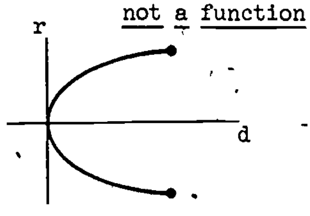
(f)



(g)

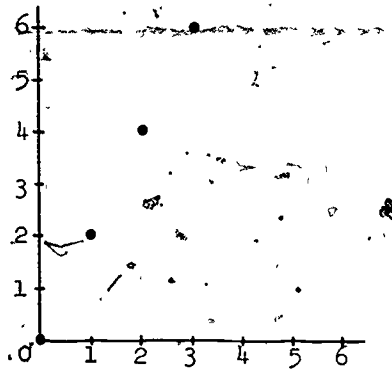


(h)



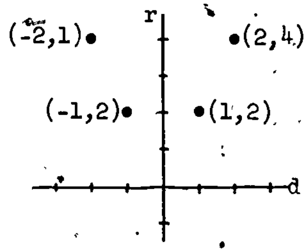
2. Graph the ordered pairs given below, state the domain and range, and tell if the relation is a function.

Example: {(0,0), (1,2), (2,4), (3,6)}



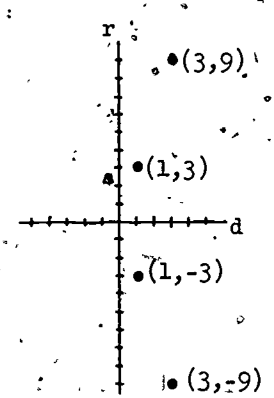
domain {0, 1, 2, 3}
range {0, 2, 4, 6}
relation is a function
(discrete)

(a) {(1,2), (-1,2), (-2,4), (2,4)}



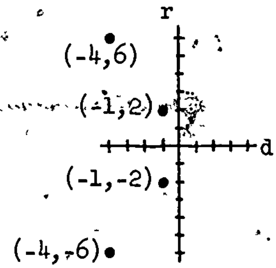
domain {-1, 1, -2, 2}
range {2,4}
relation is a function

(b) {(1,3), (1,-3), (3,9), (3,-9)}



domain {1,3}
range {-3, 3, -9, 9}
relation is not a function

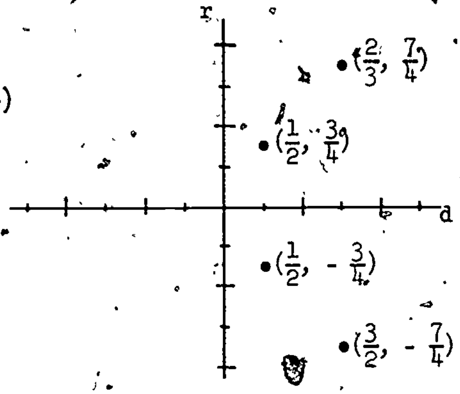
(c) {(-1,-2), (-1,2), (-4,-6), (-4,6)}



domain {-1, -4}
range {-2, 2, -6, 6}
relation is not a function

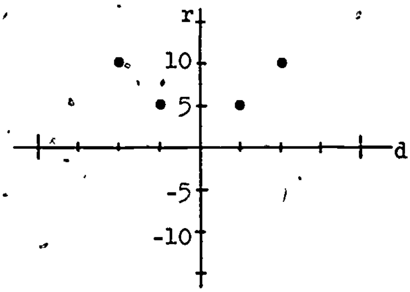
(d) $\{(\frac{1}{2}, \frac{3}{4}), (\frac{1}{2}, -\frac{3}{4}), (\frac{3}{2}, \frac{7}{4}), (\frac{3}{2}, -\frac{7}{4})\}$

domain $\{\frac{1}{2}, \frac{3}{2}\}$
range $\{\frac{3}{4}, -\frac{3}{4}, \frac{7}{4}, -\frac{7}{4}\}$
relation is not a function



(e) $\{(\frac{1}{4},5), (-\frac{1}{4},5), (\frac{1}{2},10), (-\frac{1}{2},10)\}$

domain $\{-\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}, \frac{1}{2}\}$
range $\{5, 10\}$
relation is a function



2.4 Seesaw Experiment and Multiplication of Numbers

Here the equipment may be varied. If you have the balance support, fine, but the prism-shaped block of wood will serve as a fulcrum satisfactorily.

The Seesaw Experiment may be done in either student groups or as a teacher demonstration.

1 meter stick
1 balance support, knife edge clamp or triangular block of wood
1 set of weights
2 pulleys, nails or rods
1 spool of nylon thread

When we introduce the seesaw in the manner outlined in the text, we are opening the door to equilibrium. However, we don't feel we want to open it too far. An excellent discussion of this whole problem is found in a variety of books:

Studies of Mathematics, Volume XI (SMSG), by George Polya, A. C. Vroman, Inc. (Section 2.2.5).

Science of Mechanics, by Ernst Mach, Open Court (Chapter 1).

Complete Works of Archimedes, translation by Heath, Dover or Great Books.

The discussion of multiplication could be expanded as in: First Course in Algebra (SMSG), Part I, Chapter 7, Yale University Press.

Exercise 3

1. Fill in the blanks:

   (a) The product of two positive numbers is a positive number.

   (b) The product of two negative numbers is a positive number.

   (c) The product of a negative and a positive number is a negative number.

   (d) The product of a real number and 0 is 0.

2. Calculate the following:

   (a) $(-\frac{1}{2})(-4)$ $\quad$ 2

   (b) $\left((-\frac{1}{2})(2)\right)(-5)$ $\quad$ 5

   (c) $(-\frac{1}{2})\left((2)(-5)\right)$ $\quad$ 5

   (d) $(-3)(-4) + (-3)(7)$ $\quad$ -9

   (e) $(-3)\left((-4) + 7\right)$ $\quad$ -9

   (f) $(-3)(-4) + 7$ $\quad$ 19

   (g) $|-3|(-4) + 7$ $\quad$ -5

   (h) $|3||-2| + (-6)$ $\quad$ 0

   (i) $(-3)\left(|-2| + (-6)\right)$ $\quad$ 12

   (j) $(-3)\left(|-2| + |-6|\right)$ $\quad$ -24

   (k) $(-0.5)\left(|-1.5| + (-4.2)\right)$ $\quad$ 1.35

3. Find the values of the following for $x = -2$, $y = 3$, $a = -4$.

   (a) $2x + 7y$ $\quad$ $2(-2) + 7(3) = 17$

   (b) $3(-x) + \left((-4)y + 7(-a)\right)$ $\quad$ $3(2) + \left((-4)(3) + 7(4)\right) = 6 + 16 = 22$

   (c) $x^2 + 2(xa) + a^2$ $\quad$ $(-2)^2 + 2(-2)(-4) + (-4)^2 = 4 + 16 + 16 = 36$

   (d) $(x + a)^2$ $\quad$ $\left(-2 + (-4)\right)^2 = (-6)^2 = 36$

   (e) $x^2 + \left(3|a| + (-4)|y|\right)$ $\quad$ $(-2)^2 + \left(3|-4| + (-4)|3|\right) = 4 + 0 = 4$

   (f) $|x + 2| + (-5)\left|(-3) + 2\right|$ $\quad$ $|-2 + 2| + (-5)\left|(-3) + 2\right| = 0 + (-5)|-1| = -5$

2.5 Slope

Numerous examples are given in the text to lead the student to develop a facility for computing slopes.

Exercise 4

1. Which of the following two ordered pairs determine a horizontal line, a vertical line and a line which is neither?.

(a)	(3,2), (5,2)	horizontal	(f)	(2,3), (2,2)	vertical
(b)	(0,0), (7,0)	horizontal	(g)	(562,10), (562,11)	vertical
(c)	(10,4), (4,10)	neither	(h)	(3,14), (6,28)	neither
(d)	(5,6), (6,7)	neither	(i)	(9,8), (9,1)	vertical
(e)	(2,8), (4,8)	horizontal	(j)	(0,8), (0,5)	vertical

2. For each of the following two ordered pairs, state the rise and the run for the lines determined by these points: $\frac{\text{rise}}{\text{run}}$.

(a)	(2,5), (4,8)	$\frac{3}{2}$	(f)	(763,763), (25,25)	$\frac{738}{738}$
(b)	(3,9), (2,1)	$\frac{8}{1}$	(g)	(8,7), (2,9)	$\frac{2}{-6}$
(c)	(8.5,7), (9,9)	$\frac{2}{0.5}$	(h)	(8,10), (0,10)	$\frac{0}{8}$
(d)	(20,10), (25,7)	$\frac{-3}{5}$	(i)	(3.7,12.6), (5.2,2.1)	$-\frac{10.5}{1.5}$
(e)	(5,3), (5,986)	$\frac{983}{0}$ undefined	(j)	$(\frac{3}{4}, \frac{5}{6})$, $(\frac{5}{4}, \frac{11}{6})$	$\frac{6/6}{2/4}$ or $\frac{1}{1/2}$

2.6 Absolute Value and Relation

The function we introduce whose equation is $T = 10|S|$ is related to the absolute value function whose equation is $y = |x|$. By considering the domain $S \geq 0$ we get $T = 10S$ and develop the idea of positive slope. Then considering the domain $S \leq 0$, we extend to negative slope by examining the graph of $T = -10S$. $T = 10|S|$ gives us an example with which to define function in the earlier section of this chapter. $y = |x|$ is an interesting function. It maps the real numbers into the non-negative real numbers. In contrast, the function $y = mx + b$, where m is defined and $\neq 0$, maps the real numbers into the real numbers.

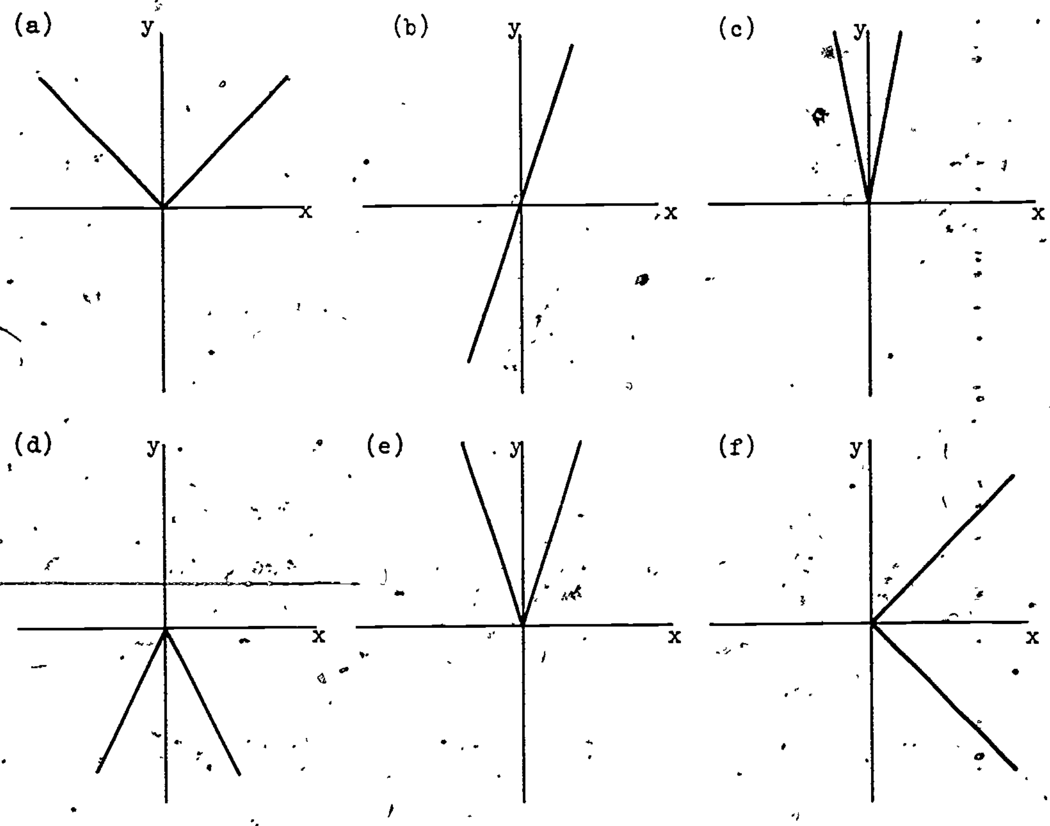
Exercise 5

1. Check the ordered pairs you obtained in the scale-turns relation to see if they satisfy either $T = 10S$ or $T = -10S$.

2. Check the ordered pairs you obtained in the faces-scale relation to see if they satisfy either $S = \frac{1}{60} F$ or $S = -\frac{1}{60} F$.

3. Graph each of the following.

   (a) $y = |x|$
   (b) $y = |-3|x$
   (c) $y = 5|x|$
   (d) $y = -2|x|$
   (e) $y = 3|-x|$
   (f) $|y| = x$

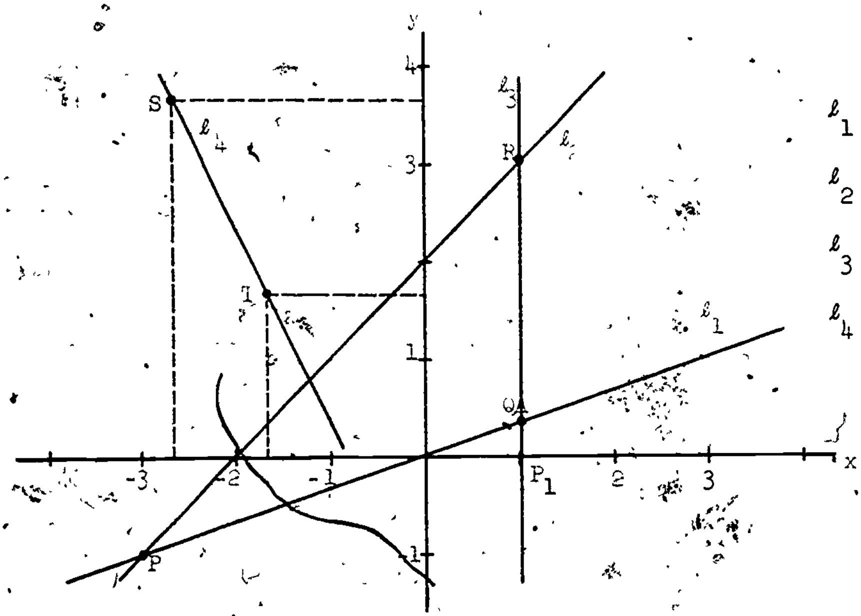


2.7 Slope-Intercept Form

The teacher may feel additional examples are necessary and many of the exercises earlier in the chapter could be utilized before the students try Exercise 6 (for example, Exercise 4).

Exercise 6

1. Calculate the slopes of lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4 in the figure below using in each case the two points indicated on the lines.



$\ell_1 = m_1 = \frac{1}{3}$

$\ell_2 = m_2 = 1$

$\ell_3 = m_3$ undefined

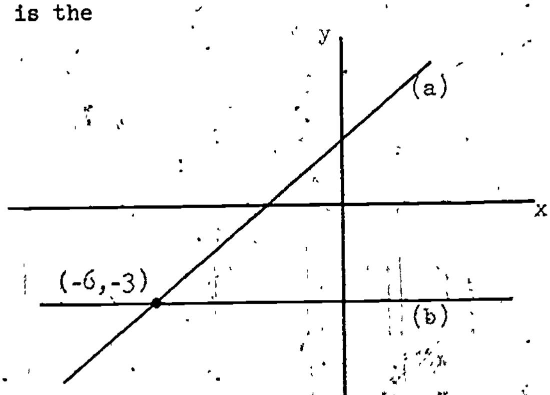
$\ell_4 = m_4 = -2$

2. What is the slope of a horizontal axis? a vertical axis?

0 ; undefined

3. With reference to a set of coordinate axes, select the point (-6,-3) and through this point
   (a) draw the line whose slope is $\frac{5}{6}$. What is the equation of this line?
   (b) draw the line through (-6,-3) which has a slope of zero. What is the equation of this line?

   (a) $y = \frac{5}{6}x + 2$
   (b) $y = -3$



4. Draw the following lines:

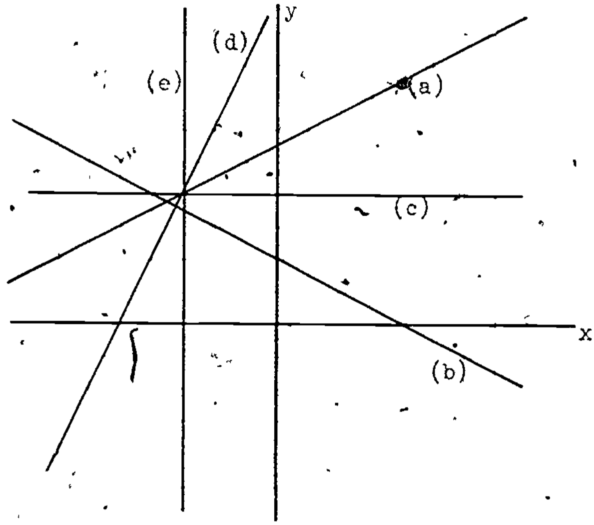
(a) a line through the point (-1,5) with slope $\frac{1}{2}$.

(b) a line through the point (2,1) with slope $-\frac{1}{2}$.

(c) a line through the point (3,4) with slope 0.

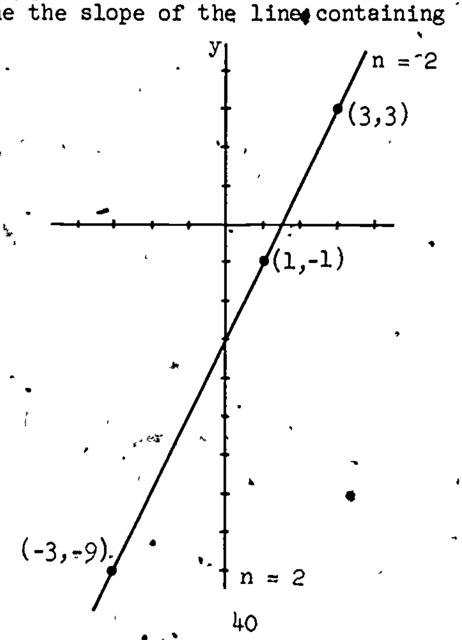
(d) a line through the point (-3,4) with slope 2.

(e) a line through the point (-3,-4) with slope undefined. (What type of line has no defined slope?) vertical



5. Consider the line containing the points (1,-1) and (3,3). Is the point (-3,-9) on this line? Yes

Hint: Determine the slope of the line containing (1,-1) and (3,3); then determine the slope of the line containing (1,-1) and (-3,-9).



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6. Write an equation of each of the following lines.

(a) The slope is $\frac{2}{3}$ and the y-intercept number is -2.

(The y-intercept number is the vertical coordinate of the point at which the line crosses the vertical axis. In this case the coordinates of the intercept are (0,-2).)

$$y = \frac{2}{3}x - 2$$

(b) The slope is $\frac{3}{4}$ and the y-intercept number is 0.

$$y = \frac{3}{4}x$$

(c) The slope is -2 and the y-intercept number is $\frac{4}{3}$.

$$y = -2x + \frac{4}{3}$$

(d) The slope is -7 and the y-intercept number is -5.

$$y = -7x - 5$$

7. What is the slope of the line containing the points (0,0) and (3,4)? What is the y-intercept number? Write the equation of the line.

$$m = \frac{4 - 0}{3 - 0} = \frac{4}{3}; \quad b = 0; \quad y = \frac{4}{3}x$$

8. Verify that the slope of the line which contains the points (-3,2) and (3,-4) is -1.

$$m = \frac{2 - (-4)}{-3 - 3} = \frac{6}{-6} = -1$$

If (x,y) is a point on this same line, the slope could be written as

$$m = \frac{y - 2}{x - (-3)} \quad \text{or} \quad \frac{y - (-4)}{x - 3}.$$

Show that both expressions for the slope give the same equation for the line.

$m = \frac{y - 2}{x + 3}$ or $m = \frac{y + 4}{x - 3}$	Since m = -1
$y - 2 = m(x + 3)$ $\quad y + 4 = m(x - 3)$	$y - 2 = -(x + 3)$ $\quad y + 4 = -(x-3)$
	$y = -x - 1$ $\quad y = -x - 1$

9. Write the equations of the lines through the following pairs of points. Use the method of Problem 8.

(a) (0,3) and (-5,2) $m = \frac{1}{5}$; $y = \frac{1}{5}x + 3$

(b) (5,8) and (0,-4) $m = \frac{12}{5}$; $y = \frac{12}{5}x - 4$

(c) (0,-2) and (-3,-7) $m = \frac{5}{3}$; $y = \frac{5}{3}x - 2$

(d) (5,-2) and (0,6) $m = -\frac{8}{5}$; $y = -\frac{8}{5}x + 6$

(e) (-3,3) and (6,0) $m = -\frac{1}{3}$; $y = -\frac{1}{3}x + 2$

(f) (-3,3) and (-5,3) $m = 0$; $y = 3$

(g) (-3,3) and (-3,5) m is undefined; $x = -3$

(h) (4,2) and (-3,1) $m = \frac{1}{7}$; $y = \frac{1}{7}x + \frac{10}{7}$

10. Graph each of the following:

(a) $y = \frac{3}{5}x + 8$

(b) $y = -\frac{8}{5}x - 12$

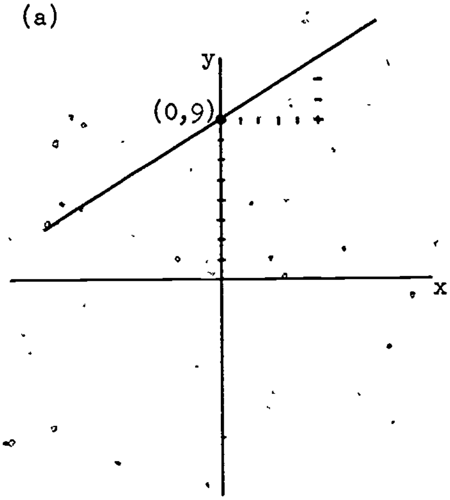
(c) $3x + 4y = 16$

(d) $y = |x| + 5$

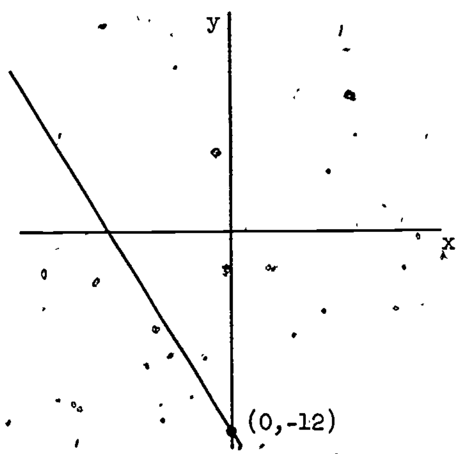
(e) $y = |x - 3|$

(f) $y = 2|x - 1| + 4$

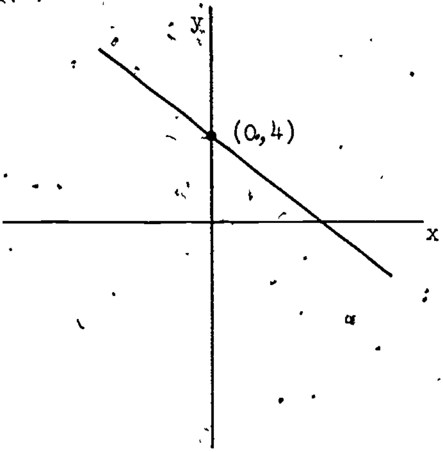
(a)



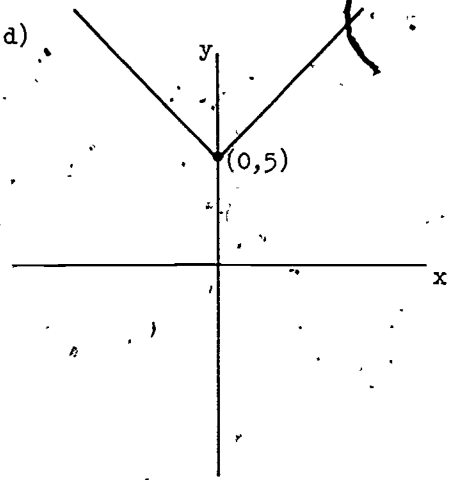
(b)



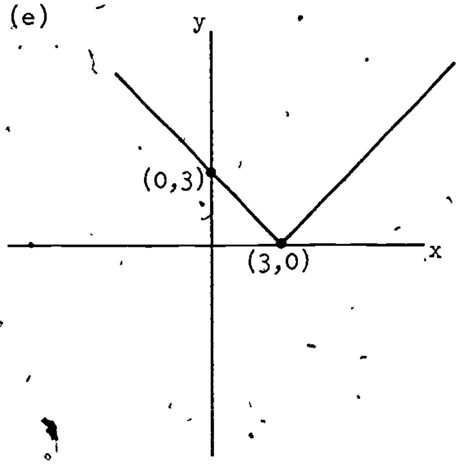
(c)



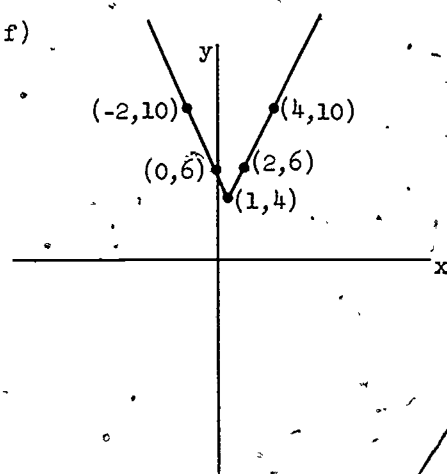
(d)



(e)

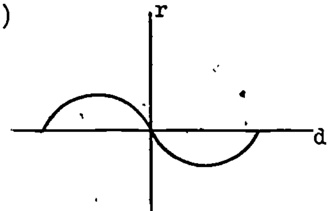


(f)

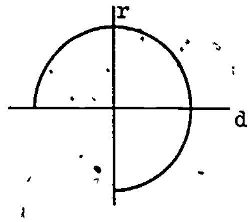


1. Which of the graphs of the relations shown below are graphs of functions?

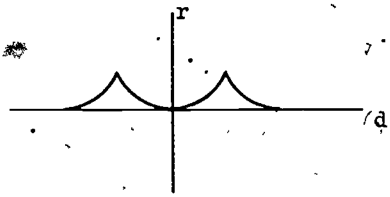
(a)



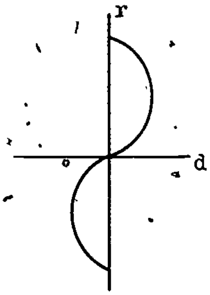
(b)



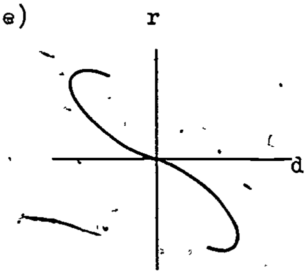
(c)



(d)



(e)



2. Graph the ordered pairs given below, state the domain and range and tell if the relation is a function.

(a) $\{(1,2), (3,5), (5,8), (-2,-3)\}$

(b) $\{(-1,-2), (0,0), (-1,2), (-2,4)\}$

3. Calculate the following:

(a) $(-3)(-\frac{1}{3})(5)$

(b) $(-2)(-6) + (-3)$

(c) $(-5)(-2) + 8$

(d) $|4||-3| + (-5)$

(e) $(-2)|-5| + 10$

(f) $(-1.5)(|2.5| + |-1.5|)$

4. Find the values of the following for $x = -1$, $y = 2$.

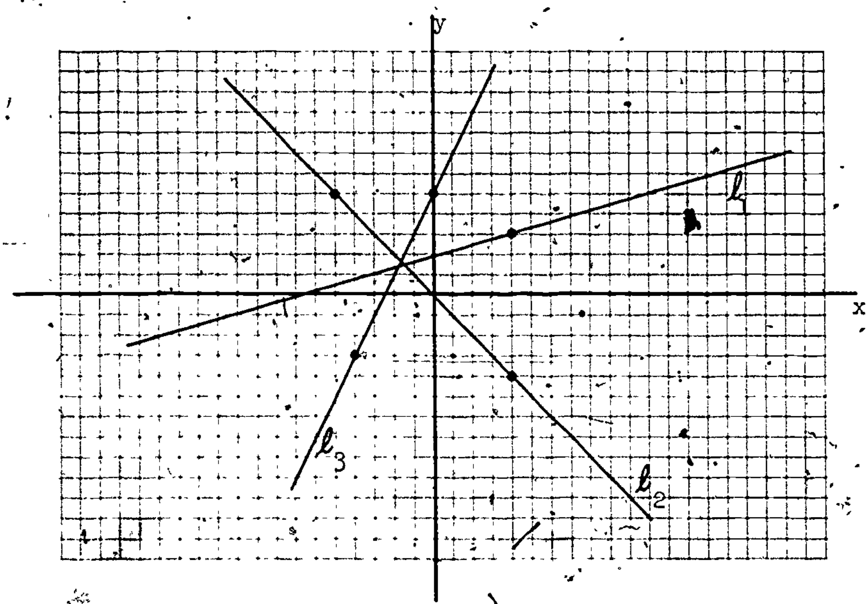
~~(a) $3x + 5y$~~

(b) $|x + 3| + |y - 3|$

(c) $(x)^2 + (y)^2$

(d) $(x + y)^2$

5. Calculate the slopes of lines $_1$, $_2$ and $_3$ using the two points indicated on the lines.



6. Draw the following lines:

(a) a line through the point (0,0) with slope $\frac{1}{2}$.

(b) a line through the point (5,-7) with slope $\frac{2}{3}$.

(c) a line through the point (0,9) with slope $-\frac{4}{5}$.

(d) a line through the point (1,5) with slope 7.

7. Write an equation of each of the following lines.

(a) the slope is $\frac{4}{7}$ and the y-intercept is 1.

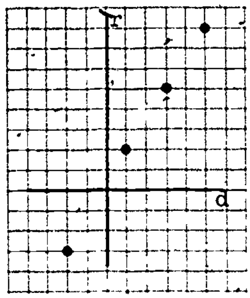
(b) the slope is $-\frac{3}{5}$ and the y-intercept is $\frac{4}{5}$.

(c) the slope is -3 and the y-intercept is -5.

Answers to Sample Test Items

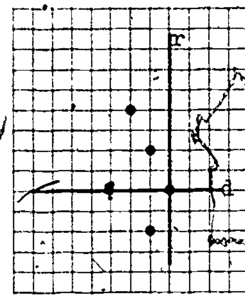
1. (a) a function
   (b) not a function
   (c) a function
   (d) not a function
   (e) not a function

2. (a)



domain {-2, 1, 3, 5}
range {-3, 2, 5, 8}
a function

(b)



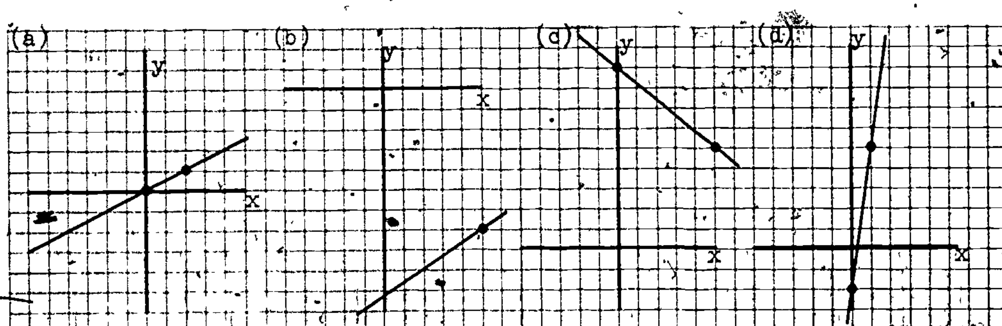
domain {-1, 0, -2}
range {-2, 0, 2, 4}
not a function

3. (a) 5
   (b) 9
   (c) -30
   (d) 7
   (e) 0
   (f) -6.0

4. (a) 7
   (b) 3
   (c) 5
   (d) 1

5. 1 : $m_1 = \frac{2}{7}$; 2 : $m_2 = -1$; 3 : $m_3 = 2$

6.



7. (a) $y = \frac{4}{7}x + 1$ (b) $y = -\frac{3}{5}x + \frac{4}{5}$ (c) $y = -3x - 5$

Chapter 3

THE FALLING SPHERE

3.1 The Falling Sphere

This experiment should be performed by groups of three or four students. The equipment needed by each group performing the experiment is as follows:

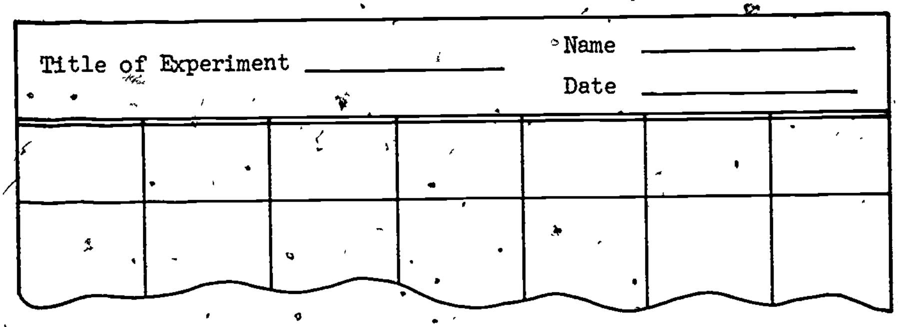
- 1 glass cylinder or jar at least 8 inches high
- 1 steel ball bearing, about $\frac{1}{8}$-inch in diameter
- 1 small horseshoe magnet
- 1 ruler with metric scale
- 2 rubber bands (small)

In addition you will need Karo Syrup (white) to fill each cylinder, a metronome or some other audible timer, and a supply of paper strips, about 1" × 10".

The purpose of this chapter is to review and extend some of the concepts of linear functions which were introduced in Chapter 2. The student will encounter the experimental situations from which the mathematics will arise. That is, he must do the experiment himself, measure the things which change, record the data in an orderly fashion, and examine it critically for whatever general relation it shows.

This experiment, as well as all others in this text, were performed by members of the writing team. You, as teachers, will be given the results of these experiments and that data we have collected. The student should be expected to find his own data and make his own analysis of it. With this method the results of each team of students may be different, but if their work is done carefully, the differences should not be too great. Even though these differences exist, the real need for graphing the data collected, and examining the graphs for definite relations, functions and equations should be apparent.

We suggest that a standard data sheet be used for all experiments (Figure 1). If possible, a supply of these data sheets should be dittoed for the use of your students during this course. In some of th later experiments we refer to the data and graphs of experiments already performed. If the data sheets and graphs are kept in some form of folder or punched for insertion in a ring-binder it will be easy to find the appropriate page when such a reference is made.



3.2 The Falling Sphere

In performing this experiment and all others, the students should follow directions carefully. Mana times the reasons for some directions become apparent as the experiment is performed. If you make a few trials of each experiment before the particular section is reached you will most likely be able to expand the directions and ease the students over some of the more difficult portions.

This experiment is an investigation of terminal velocity. However, only after the data has been graphed and analyzed is the idea of velocity introduced.

The Falling Sphere is actually a small steel ball bearing. When this bearing is dropped into a cylinder of Karo Syrup it reaches terminal velocity after a fall of a few millimeters and then continues with a constant velocity.

It would be advantageous to prepare a teacher demonstration to show how the factors mentioned in the text will influence the speed of the ball and why we must carefully control certain factors in performing an experiment.

(1) Cylinders (all the same size and shape) can be filled with various fluids, water, glycerine, a light oil, and Karo Syrup. Dropping ball bearings (all the same size) into each container should lead the students to reach conclusions as to the effect of the fluid on the velocity.

(2) Dropping ball bearings of different sizes into the same container filled with Karo Syrup will show the effect of the size of the "sphere" on the speed.

(c) Cylinders or jars with different size openings can be used to show the effect of the type of container on the terminal velocity. The cylinders should be filled with the same type of fluid (preferably the Karo Syrup) and the same size bearing used as falling spheres.

The effect of the diameter of the container will be very small if there is any noticeable effect at all.

(4) As an enterprising teacher, you will probably also want to demonstrate with a genuine pearl dropped in Prell Shampoo.

The student experiment should be performed with the same container and the same ball bearing for all trials. The fluid used should be white Karo Syrup. Four trials will be needed to provide the necessary data.

Affix a thin paper tape to the cylinder with two rubber bands (or tape). Fill the cylinder with Karo Syrup. Drop a small steel ball bearing into the fluid close enough to the tape to make fairly accurate marks on the tape at regular intervals. A small horseshoe magnet placed against the outside wall of the container will attract the bearing to the inside wall and allow you to position the bearing so that it will fall along one side of the tape.

A metronome is a convenient device for marking the equal time intervals. It may be helpful to adjust the metronome so that it makes a "clicking" sound every second and then mark the position of the ball for every other "click". After a few trials you will adjust to the beat of the metronome and be able to mark the position of the bearing quite accurately. The students should make a few practice trials to develop the necessary techniques. The magnet is used to retrieve the ball bearing after it has come to rest on the bottom of the cylinder. The tape for each trial must be saved for use in the next section of the text.

Should a metronome be difficult to obtain, it may be possible to borrow one for a short time. Make a tape recording of the metronome sounding at the correct intervals. This tape can be played loudly enough for the use of the entire class.

3.3 Tabulating Data

The tapes obtained from the four trials must now be measured. Each tape in turn should be fastened to the centimeter rule. The zero mark on the tape should coincide with a centimeter mark on the ruler (Figure 2 in students' text). Measure the distance from the zero mark to each mark on the tape. Since these distances are to be plotted on the coordinate plane, accuracy is important. The data collected by the writing team is recorded in Figure 2.

THE FALLING SPHERE EXPERIMENT				
Time t (seconds)	Trial 1 Distance d (millimeters)	Trial 2 Distance d (millimeters)	Trial 3 Distance d (millimeters)	Trial 4 Distance d (millimeters)
0	0	0	0	0
2	5.5	6.5	6.5	6.5
4	11.5	13.5	12.0	15.5
6	20.0	21.0	19.0	23.0
8	27.5	27.5	27.0	30.0
10	35.0	35.5	34.0	37.5
12	43.5	42.0	40.5	45.0
14	50.5	49.5	47.0	52.5
16	56.0	57.0	54.0	59.0
18	62.5	65.5	61.5	67.0

Figure 2

3.4 Analysis of Data

This section is a general one dealing with the handling of experimental data. The students should be made to realize they must check the domain and range of their relations before setting the scales of the coordinate axes. The axes do not have to have the same scales. For example, if we were to graph the data recorded in the experiment on the loaded beam, the load varied from 0 grams to 300 grams while the position of the end of the beam changed about 30 or 40 centimeters. If we were to graph this data, the scale on the horizontal axes would go from 0 to 300 while the scale on the vertical axis would go from 0 at no load to the deflection at maximum load.

By connecting points the student is inferring physical continuity. He is beginning to develop the physical model, but this does not imply any mathematical continuity. The decision on whether an experiment could yield intermedicate points must be based on phenomena being studied. It would be desirable for the teacher to mention a few examples of discontinuous physical situations. For example, there is a maximum height to which a ball bounces. If the height of bounce is related to the corresponding bounce number, physical continuity cannot be inferred. The graph is a set of distinct points which cannot be connected, for height cannot be inferred for parts of bounces. Half bounces cannot be associated with a maximum height.

3.5 Graphing the Experimental Data

Obtaining the "best straight line" involves making the assumption that there exists a line which best fits the data. Often it will not be too difficult to obtain a line which will satisfy the student. Do not put too much stress on this now. The student will soon have practice, and he should learn quickly. This line is an idealization of the graph of the physical data. This idealization does not represent any possible physical situation. It is purely an abstraction.

The slope can now be determined by the student. This slope will have a special significance. Note that the vertical distance is a measure of the fall of the ball bearing, and the horizontal distance is a measure of the corresponding interval of time. Therefore, the slope becomes

$$\frac{\text{measure of distance}}{\text{measure of time}}$$

or, distance per unit of time, which is velocity.

Since the data graphed determines a straight line, the velocity of the falling ball bearing is constant. This substantiates the statement that terminal velocity had been reached by the time of the initial reading.

Figure 3 is the graph of the data recorded in Figure 2.

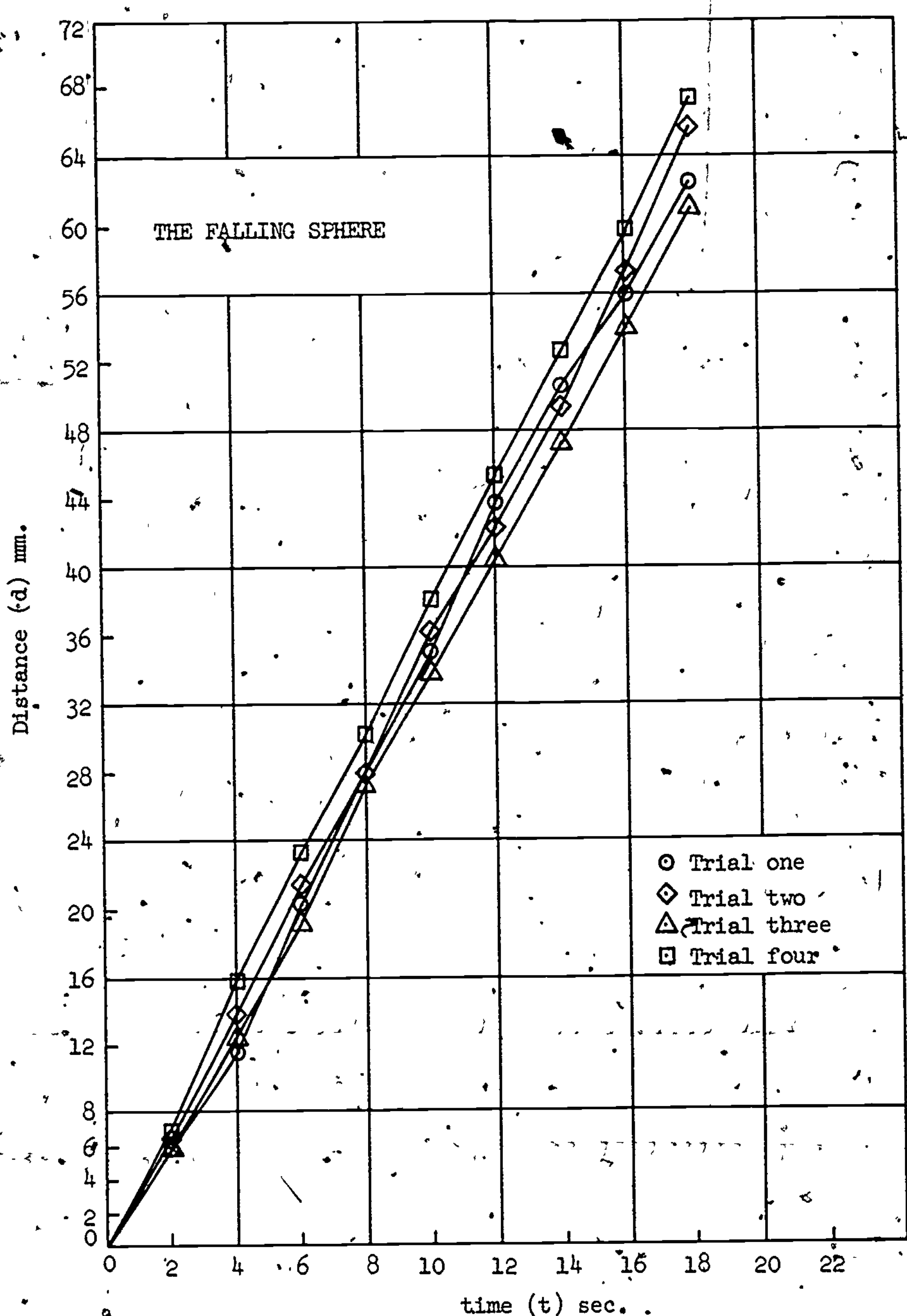

THE FALLING SPHERE
Distance (d) mm.
72
68
64
60
56
52
48
44
40
36
32
28
24
20
16
12
8
6
4
2
0
0
2
4
6
8
10
12
14
16
18
20
22
time (t) sec.
Trial one
Trial two
Trial three
Trial four

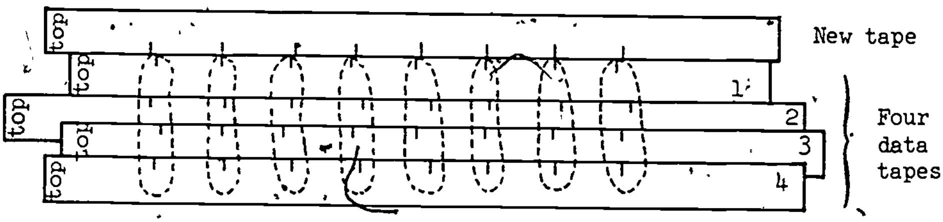
Figure 3

The appendix to this text gives a method for finding the "best fitting line" known as the "method of least squares". This method is probably too difficult to present to students but will be helpful to you as a teacher.

Exercise 1

The purpose of the first two problems in this exercise is to give alternate ways to find the best straight line from the data already collected. Since the data collected by different groups of students should vary there will be no standard solution for either problem. The teacher should, however, discuss the problems in detail to reinforce the students understanding of a "best straight line" and techniques of handling experimental data by taking averages.

1. Reproduce the "best straight line" you have drawn to represent the data of this experiment on a clean sheet of coordinate paper. Take the four pieces of paper tape used to mark the position of the ball and arrange them so that the zero marks are in line. On a clean fifth tape make a mark to indicate a "zero" position and align this mark with the other zero marks.

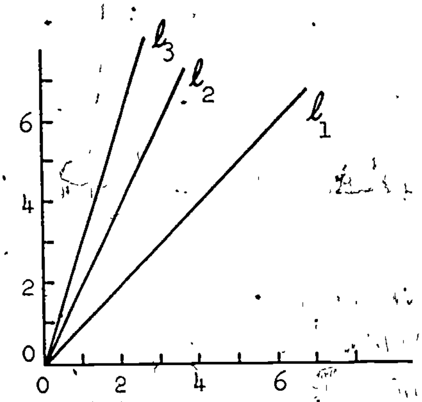


The other marks on your tapes will not be "in line", but should tend to center in groups. Make a mark on the clean tape to indicate your "guess" as to the position which best represents each vertical set of marks. Using the fifth tape as if it were a new trial, mark your measurements in the usual way, enter the data in your table, and graph the ordered pairs. Do these points come closer to forming a straight line than any of your four trial runs? How does this line compare with the "guess" you made from the "braid" arrangement?

2. From the data of your four trials find the average distance traveled by the ball in each time interval. To do this, add the distance each row of the trials in Table 1 and divide by the number of trials. Make a new column in your table, "Average Distance (mm)", and now plot average distance versus time on the same sheet of coordinate paper used for Problem 1. How close do these points come to forming a straight line? You now have three lines on this sheet of coordinate paper. The first is the "best straight line" from your original data, the second is the line obtained in Problem 1, and the third line is the one obtained by the process of

averaging. How do these three lines compare?

3. Draw the first quadrant using a scale of 1 second for each horizontal division and 1 millimeter for each vertical division. Draw a line which passes through the origin and has a slope of 1mm/sec; 2mm/sec; and 3 mm/sec. Label these lines.



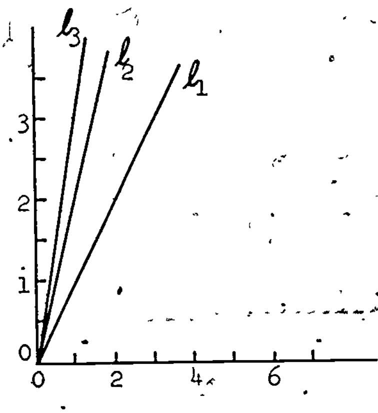
4. Repeat the above exercise with a horizontal scale of 1 second per division, but with a vertical scale of 0.5 millimeter per division. Are the two slopes the same?

The slopes in Problem 3 and Problem 4 are the same because slope is defined by

$$m = \frac{\text{vertical distance}}{\text{horizontal distance}}.$$

Hence, the slope of ℓ_1 in both cases is

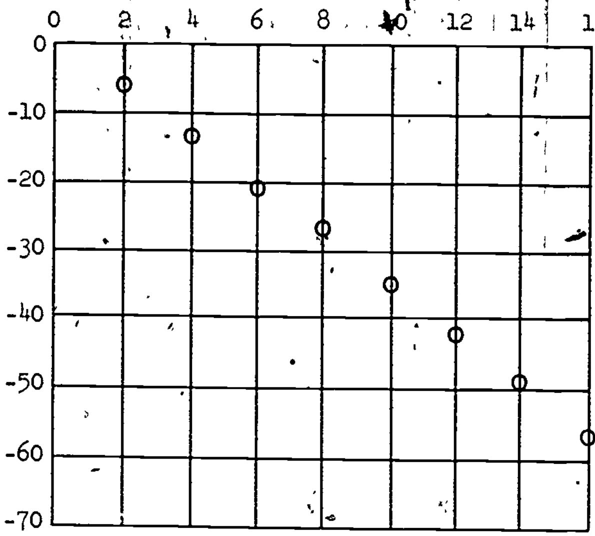
$$m = \frac{1}{1} = \frac{4}{4} = 1.$$



5. Draw a fourth quadrant on a sheet of coordinate paper. Use the same horizontal scale (in seconds) that you used to represent the data from the Falling Sphere Experiment. Make a negative distance scale (in millimeters) along the vertical axis. Note that this was the orientation of your scale when you performed the experiment. Plot the time-distance data from your experiment on this sheet and draw the "best" line. Calculate the slope. What is the significance of a negative velocity?

Table for No. 5

time	distance
0	0
2	-6.5
4	-13.5
6	-21.0
8	-27.5
10	-35.5
12	-42.0
14	-49.5
16	-57.0
18	-65.5



The slope is $m = -3.6$.
The velocity is in a
downward direction.

3.6 The Point-Slope Form

There are several forms for the equation of a straight line. In this section the point-slope form is developed. This form is reduced to the slope-intercept form.

The ordered pair for the point at which the graph of the line crosses the horizontal axis would have the form $(a,0)$. Now choose any arbitrary point on the line having coordinates (x,y). The slope, m, which has been described as $\frac{\text{vertical change}}{\text{horizontal change}}$ will be written as $\frac{y - 0}{x - a} = m$ from which we obtain the equation

$$y - 0 = m(x - a)$$

or

$$y = m(x - a)$$

Extend the discussion to generalize the equation. That is, assume the first point to be a general point with coordinates (c,d). Retain the second point as the general point with coordinates (x,y). We have now completely generalized the discussion. From the definition of slope we have the equation

$$\frac{y - d}{x - c} = m .$$

Now the equation becomes

$$y - d = m(x - c)$$

which is the general point-slope form.

The section concludes with the derivation of the slope-intercept form as a special case of the point-slope form.

Exercise 2

In Problems 1 and 2 the students will have to calculate the slope, first using the coordinates of the two points read from the graph. Then by the use of this slope and one of the given points together with an arbitrary point (x,y), the equation may be written in the point-slope form or in one of the special forms when it applies to the particular line. For example, for line ℓ_2 in Problem 1, the given points R and S have the coordinates (3,2) and (12,8). Hence the slope is

$$m = \frac{8 - 2}{12 - 3} = \frac{6}{9} = \frac{2}{3}.$$

Using the point (3,2) with $m = \frac{2}{3}$, the point-slope form gives

$$\frac{y - 2}{x - 3} = \frac{2}{3}$$

$$3y - 6 = 2x - 6$$

$$y = \frac{2}{3}x.$$

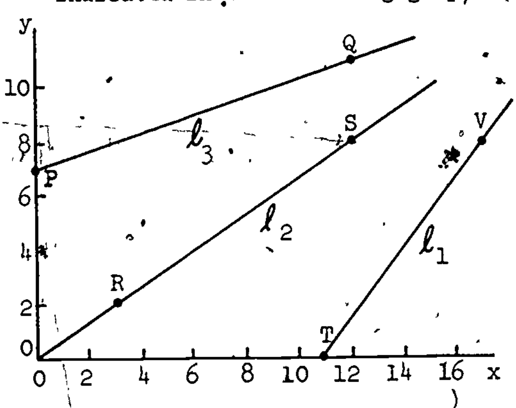
Using (12,8)

$$\frac{y - 8}{x - 12} = \frac{2}{3}$$

or

$$y = \frac{2}{3}x.$$

1. Write the equations of the lines ℓ_1, ℓ_2 and ℓ_3, using the two points indicated in the following graph.



ℓ_1 : $\frac{8 - 0}{17 - 11} = \frac{4}{3}$

$y = \frac{4}{3}x - \frac{44}{3}$

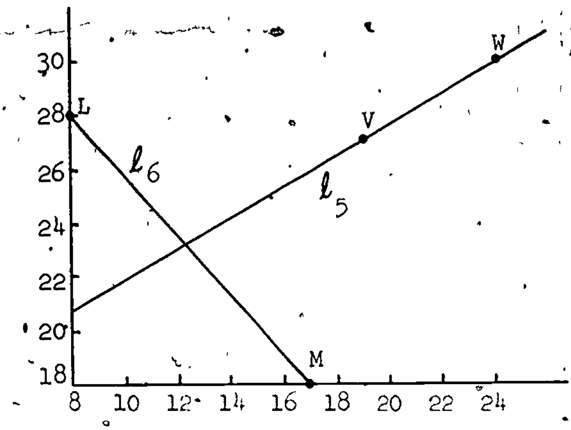
ℓ_2 : $\frac{8 - 2}{12 - 3} = \frac{2}{3}$

$y = \frac{2}{3}x$

ℓ_3 : $\frac{11 - 7}{12 - 0} = \frac{1}{3}$

$y = \frac{1}{3}x + 7$

2. Write the equations of the lines ℓ_5 and ℓ_6.



ℓ_5 : $\frac{30 - 27}{24 - 19} = \frac{3}{5}$

$y - 27 = \frac{3}{5}(x - 19)$

ℓ_6 : $\frac{8 - 28}{15 - 8} = -\frac{20}{7}$

$y - 28 = -\frac{20}{7}(x - 8)$

3. Find the x and y intercepts for lines ℓ_5 and ℓ_6. Do not extend the lines to obtain a graphical solution. Remember that the y-intercept is the point for which x = 0, and the x-intercept is the point where y = 0.

ℓ_5 : $y - 25 = \frac{3}{5}(x - 19)$

$y = \frac{3}{5}x - \frac{57}{5} + 25$

$y = \frac{3}{5}x + \frac{68}{5}$

if x = 0, then $y = \frac{68}{5}$

ℓ_6 : $y - 28 = -\frac{20}{7}(x - 8)$

$y = -\frac{20}{7}x + \frac{160}{7} + 28$

$y = -\frac{20}{7}x + \frac{356}{7}$

if x = 0, then $y = \frac{356}{7}$

4. Refer to your time-distance graph obtained in the Falling Sphere Experiment. Using a point not on the vertical axis together with the slope, find the equation to represent the best straight line. Show that this is equivalent to the equation obtained using the slope-intercept form.

The actual equation that each student derives will depend upon the particular set of data that the student has collected. Care should be taken to convince the student that the point-slope form and the slope-intercept form are equivalent.

5. The following equations are expressed in point-slope form. Solve each of these for y. State the slope of the line and the y-intercept in each case.

		m	y-intercept
$y - 6 = 3(x + 4)$	$y = 3x + 18$	3	18
$y + 2 = -2(x - 3)$	$y = -2x + 4$	-2	4
$y + 7 = \frac{5}{2}(x - 2)$	$y = \frac{5}{2}x - 8$	$\frac{5}{2}$	-8
$y - 0.5 = -4(x + 3.5)$	$y = -4x - 13.5$	-4	-13.5

6. Make a graph of the data obtained in the Loaded Beam Experiment, fit a "best" line and obtain an equation of this line using the slope-intercept form and the point-slope form.

See comment made for Problem 4.

3.7 Relations and Converses

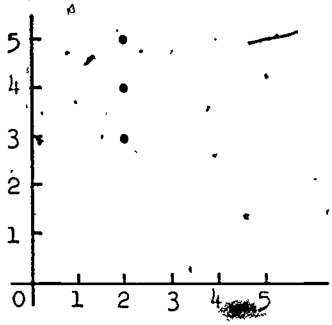
This section continues and extends the section on relations in Chapter 2. If necessary, you should review 2.2 before beginning this section. The concepts discussed in Sections 2.2, 3.7 and 3.8 will be used in later work in this text and in more advanced work in both science and mathematics. Every effort should be made to see that these sections are clearly understood.

Exercise 3

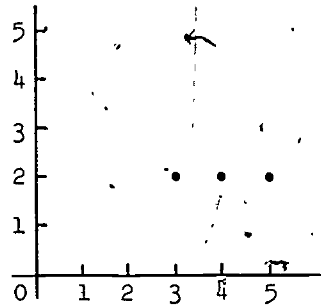
(a) Graph the ordered pairs given below, state the domain and range and tell if the relation is a function.

(b) In each case form the converse relation by interchanging the first and second elements of the ordered pairs. Graph the converse, state the new domain and range and tell if the converse is a function.

1. Q = {(2,3), (2,4), (2,5)}
(converse) = {(3,2), (4,2), (5,2)}

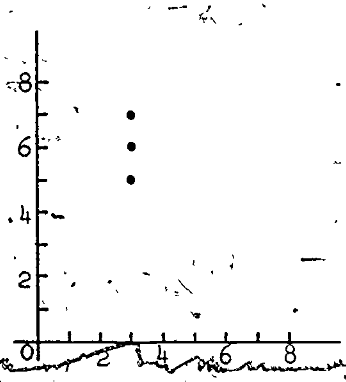


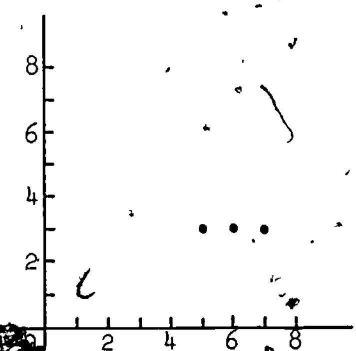
Converse



	Q	Converse
domain:	{2}	{3,4,5}
range:	{3,4,5}	{2}
	not a function	function

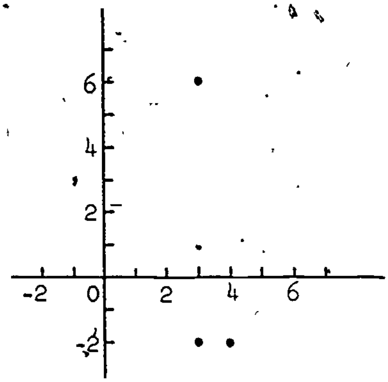
2. M = {(5,3), (6,3), (7,3)}
(converse) = {(3,5), (3,6), (3,7)}

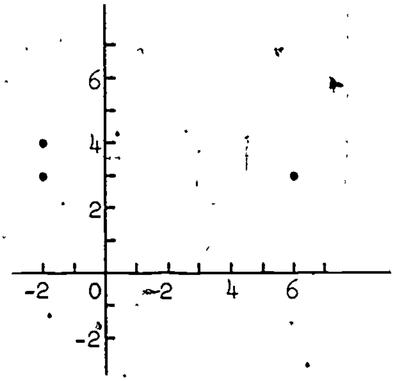




	M	Converse
domain:	{3}	{5,6,7}
range:	{5,6,7}	{3}
	not a function	function

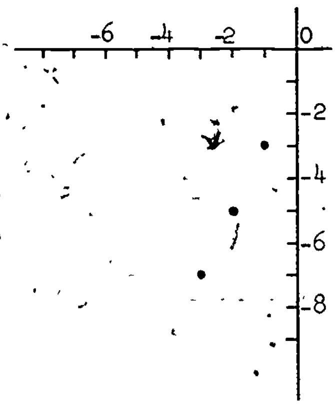
3. $N = \{(3,6), (3,-2), (4,-2)\}$
(converse) = $\{(6,3), (-2,3), (-2,4)\}$

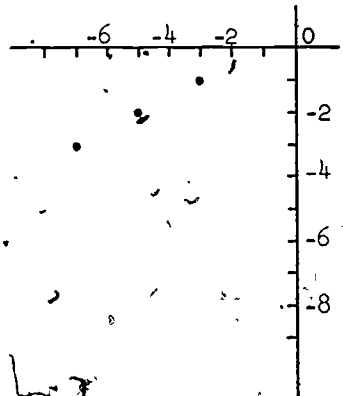




domain:	{3,4}	{-2,6}
range:	{-2,6}	{3,4}
	not a function	not a function

4. $P = \{(-1,-3), (-2,-5), (-3,-7)\}$
(converse) = $\{(-3,-1), (-5,-2), (-7,-3)\}$





domain:	{-1,-2,-3}	{-3,-5,-7}
range:	{-3,-5,-7}	{-1,-2,-3}
	function	function

3.8 Inverse Functions

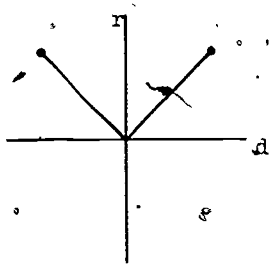
The definition of a function in the text tells when a given relation is a function. The graphical methods given for telling this may prove most useful at this point.

Exercise 4

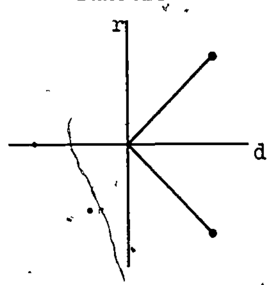
1. Refer to Exercise 2 in Chapter 2. For each of the graphs, check to see if the converse of the relation shown is a function. Are any of these relations one-to-one functions?

Problem 1

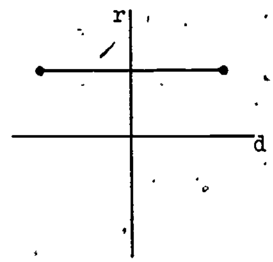
(a) converse is not a function



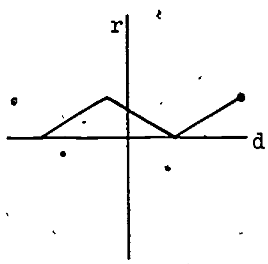
(b) converse is a function



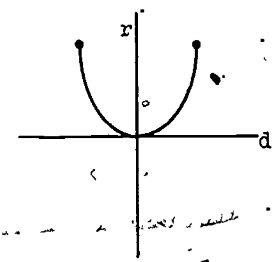
(c) converse is not a function



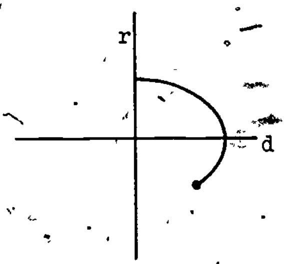
(d) converse is not a function



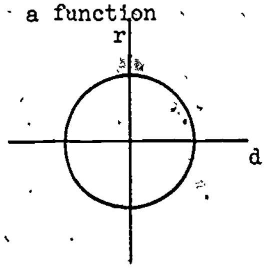
(e) converse is not a function



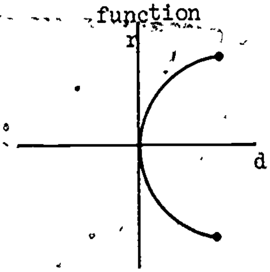
(f) converse is a function



(g) converse is not a function



(h) converse is a function



None of these relations is a one-to-one function.

Problem 2

(a) $\{(1,2), (-1,2), (-2,4), (2,4)\}$

Converse is not a function

(b) $\{(1,3), (1,-3), (3,9), (3,-9)\}$

Converse is a function

(c) $\{(-1,-2), (-1,2), (-4,-6), (-4,6)\}$

Converse is a function

(d) $\{(\frac{1}{2}, \frac{3}{4}), (\frac{1}{2}, -\frac{3}{4}), (\frac{3}{2}, \frac{7}{4}), (\frac{3}{2}, -\frac{7}{4})\}$

Converse is a function

(e) $\{(\frac{1}{4},5), (-\frac{1}{4},5), (\frac{1}{2},10), (-\frac{1}{2},10)\}$

Converse is not a function

None of these relations is a one-to-one function.

2. In the Falling Sphere Experiment, the data in the table forms a relation.

(a) What are the domain and the range of this relation?

The domain is the set of time values and the range is the set of distance values.

(b) Is this relation a function?

Yes.

3. Does the "best straight line" describe a function?

Yes.

4. Are the domain and range of the "best straight line" relation the same as the domain and range of the "data relation"? Explain.

Not necessarily, since the "data relation" is a discrete relation and the "best straight line" is a continuous relation.

5. Are the domain and range of the equation found to represent ~~the~~ "best straight line" the same as the domain and range of the best straight line relation?

The domain of the graph is a subset of the set of positive real numbers, while the domain of the linear equation is the set of all real numbers.

6. In the Falling Sphere Experiment we obtained the equation $d = mt$. Obtain the converse relation by algebraic means.

$t = \frac{1}{m}d$ where d is the new domain and t is the new range.

How might we have conducted the experiment to give the converse relation directly?

Pick distances along the container and, with a stopwatch, see how long it takes the falling sphere to travel that distance.

7. Is the converse relation obtained in the preceding exercise an inverse function? Explain.

Yes; every distance is traveled in exactly one time and every time has associated with it exactly one distance.

3.9 Graphical Translation of Coordinate Axes

Translating axes by the use of the sheet of frosted acetate (or other substitutes), is not difficult and will often help the student to simplify the equation of the line and the orientation of the graph. We will also use translation of axes in later experiments which will yield quadratic functions. If the acetate sheets are used, the teacher should draw a set of coordinate axes in the middle of each sheet. Use a heavy ball-point pen. Do not write any number or scales on the sheet. Students will write on these with a pencil since they can then be easily erased.

While it is possible to position the new axes in any manner whatsoever, (i.e., they can be translated and rotated) translation horizontally and vertically will be sufficient for our purposes.

The amount and direction of translation may be purely arbitrary but when we use this on graphs based on science experiments the physical situation usually suggests a "natural" location for the new axes.

Exercise 5

1. With reference to a set of coordinate axes, select the point (2,3) and through this point draw the line whose slope is $\frac{1}{2}$. What is the equation of this line?

$y - 3 = \frac{1}{2}(x - 2)$

Use your plastic overlay to obtain the new equation of this line when the origin is shifted:

(a) to the left 3 units

$y - 3 = \frac{1}{2}(x - 2 + [-3])$

$y - 3 = \frac{1}{2}(x - 5)$

(b) to the right 3 units

$y - 3 = \frac{1}{2}(x - 2 + 3)$

$y - 3 = \frac{1}{2}(x + 1)$

(c) 4 units upward

$y - 3 + 4 = \frac{1}{2}(x - 2)$

$y + 1 = \frac{1}{2}(x - 2)$

(d) 4 units downward

$y - 3 - 4 = \frac{1}{2}(x - 2)$

$y - 7 = \frac{1}{2}(x - 2)$

(e) to the left 3 units and up 4 units

$y - 3 + 4 = \frac{1}{2}(x - 2 - 3)$

$y + 1 = \frac{1}{2}(x - 5)$

(f) to the left 3 units and down 4 units

$y - 7 - 4 = \frac{1}{2}(x - 2 - 3)$

$y - 13 = \frac{1}{2}(x - 5)$

2. With reference to a set of coordinate axes, draw the line which passes through the points (1,7) and (7,5). What is the equation of this line?

$m = \frac{7 - 5}{1 - 7} = \frac{2}{-6} = -\frac{1}{3}$

$\frac{y - 7}{x - 1} = -\frac{1}{3}$ or $\frac{y - 5}{x - 7} = -\frac{1}{3}$

$y - 7 = -\frac{1}{3}$[illegible]) $\qquad y - 5 = -\frac{1}{3}(x - 7)$

Use your plastic overlay to obtain the new equation of this line when the origin is shifted:

(a) to the x-intercept

let $y = 0$

$-7 = -\frac{(x - 1)}{3}$

$-21 = -x + 1$

$x = 22$

$y - 7 = -\frac{1}{3}(x - 1 + 22)$

$y - 7 = -\frac{1}{3}(x + 21)$

$y - 7 = -\frac{1}{3}x - 7$

$y = -\frac{1}{3}x$

(b) to the y-intercept

let $x = 0$

$y - 7 = -\frac{1}{3}(0 - 1)$

$y - 7 = \frac{1}{3}$

$y = 7 + \frac{1}{3} = \frac{22}{3}$

$(y - 7 + \frac{22}{3}) = -\frac{1}{3}(x - 1)$

$y + \frac{1}{3} = -\frac{1}{3}x + \frac{1}{3}$

$y = -\frac{1}{3}x$

(c) to the point (4,6)

$$y - 7 + 6 = -\frac{1}{3}(x - 1 + 4)$$

$$y - 1 = -\frac{1}{3}(x + 3)$$

3.10 Algebraic Translation of Coordinate Axes

This section is purely a mathematical discussion of translation of the axes. Except for the better students, you should not expect your class to master this subject. Some of them may "discover" the algebraic procedure while doing the graphical translation. Graphical translation (Section 3.9) is the section to be stressed in the study of translation for this grade level. You may feel, however, that more drill similar to Exercise 2 will be needed. All you will need to provide this drill is to give the class equations of the sort 3x - 2y - 7 and ask them to translate the origin to various points like the y-intercept, the point (5,4) or (0,6), and so on.

To translate the axes of this equation to the point (5,4) write it in the form

$$y = \frac{3}{2}x - \frac{7}{2}$$

or

$$y - o = \frac{3}{2}\left(x + \left(\frac{7}{3}\right)\right) .$$

It is now in the point-slope form and

$$Y + (0) + K = m\left(X + \left(-\frac{7}{3}\right) + h\right)$$

where h and K are 5 and 4 respectively. This becomes

$$Y + 0 + 4 = \frac{3}{2}\left(X + \left(-\frac{7}{3}\right) + 5\right)$$

$$Y + 4 = \frac{3}{2}\left(X + \frac{8}{3}\right)$$

or

$$Y = \frac{3}{2}X .$$

Since the point (5,4) is on the line, you would expect to get equations in the form

$$Y = mX .$$

If, on the other hand, you want to translate to the point (5,-4), it is done in this manner:

$$Y + (0) + (-4) = \frac{3}{2}\left(X + \left(-\frac{7}{3}\right) + 5\right)$$

$$Y - 4 = \frac{3}{2}\left(X + \frac{8}{3}\right)$$

$$Y - 4 = \frac{3}{2}(X) + 4$$

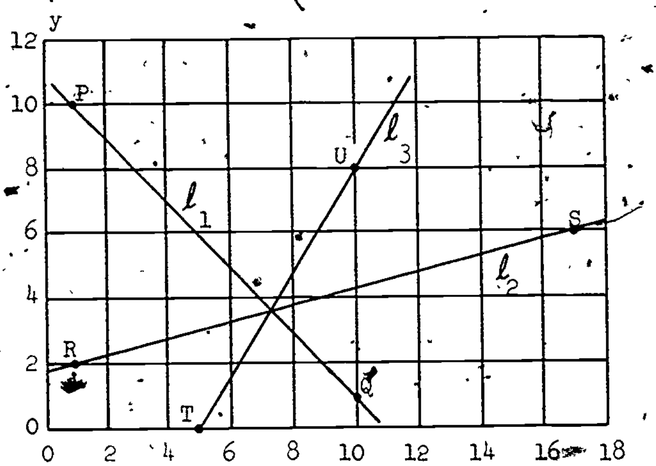
$$Y = \frac{3}{2}X + 8 .$$

Exercise 6

Refer to Exercise 5 for the algebraic solutions asked for in Exercise 6.

Sample Test Items

1. Write the equations of the lines ℓ_1, ℓ_2 and ℓ_3, using the two points indicated in each case.



2. The following equations are expressed in point-slope form. Solve each of these for y. State the slope of the line and the y-intercept in each case.

   (a) $y - 9 = 6(x + 7)$

   (b) $y + 4 = -3(x - 5)$

   (c) $y + 3 = \frac{3}{2}(x - 8)$

   (d) $y + 1.5 = -2(x - 4.5)$

3. (a) Graph the ordered pairs given below, state the domain and range and tell if the relation is a function.

   (b) Form the converse relation. Graph the converse, state the new domain and range and tell if the converse is a function.

   $\{(5,-5), (2,3), (0,0), (2,-3), (5,5)\}$

4. Converse of a linear function is a linear function. If $y = mx$ is the equation of the original function, then $y = \frac{1}{m}x$ is the equation of the converse. What conclusion can you draw about the graphs of a linear function and its converse?

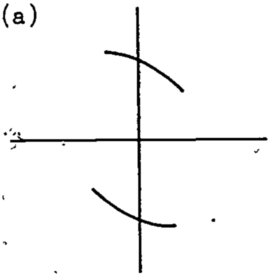
5. The graphs of various relations are sketched below. For each graph:

(i) Is the relation a function?

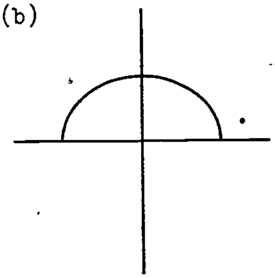
(ii) Is the converse relation a function?

(iii) Does the relation and its converse form a one-to-one function?

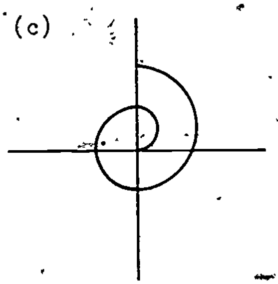
(a)



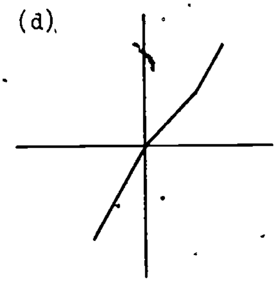
(b)



(c)



(d)



6. With reference to a set of coordinate axes, select the point (2,3) and through this point draw the line whose slope is $\frac{3}{2}$.

(a) What is the equation of this line?

(b) Use your plastic overlay to obtain the new equation of this line when the origin is shifted

(i) to the left 3 units

(ii) 2 units downward

(iii) 3 units to the left and 2 units downward.

7. Write the equation of the line which passes through the points (3,2) and (7,4) in point-slope form. Obtain the equation of this line algebraically when the origin has been translated

(a) to the right 4 units

(b) to the left 3 units and up 2 units.

Answers to Sample Test Items

1. ℓ_1: $m_1 = \frac{1 - 10}{10 - 1} = \frac{-9}{9} = -1$

$$y - 10 = -1(x - 1)$$
$$y - 10 = -x + 1$$
$$y = -x + 11$$

ℓ_2: $y = \frac{1}{4}x + \frac{7}{4}$

ℓ_3: $y = \frac{8}{5}x - 8$

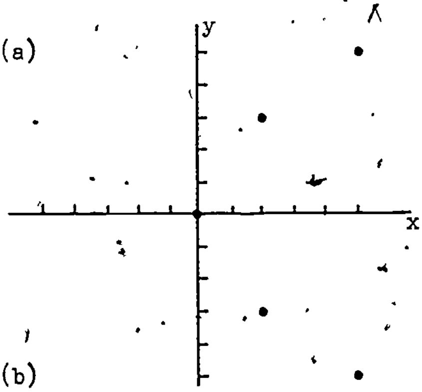
2. (a) $y = 6x + 51$ $\quad m = 6$; (0,51)

(b) $y = -3x + 11$ $\quad m = -3$; (0,11)

(c) $y = \frac{3}{2}x - 15$ $\quad m = \frac{3}{2}$; (0,-15)

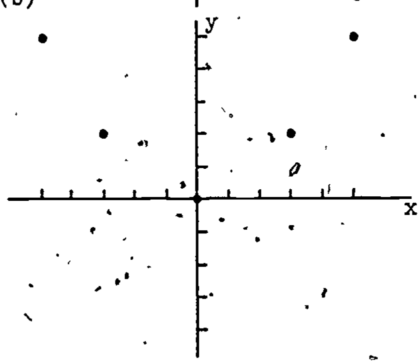
(d) $y = -2x + 7.5$ $\quad m = -2$; (0,7.5)

3. (a)



domain {0, 2, 5}
range {-5, -3, 0, 3, 5}
relation is not a function

(b)



domain {-5, -3, 0, 3, 5}
range {0, 2, 5}
converse is a function

4. The slope of the converse of the linear function is the reciprocal of the slope of the original function. The product of the slope of the original function and its converse is one.

5. (a) Relation is not a function.
Converse is a function.
Relation and converse do not form one-to-one functions.

(b) Relation is a function.
Converse is not a function.
Relation and converse do not form one-to-one functions.

(c) Relation is not a function.
Converse is not a function.
Relation and converse do not form one-to-one functions.

(d) Relation is a function.
Converse is not a function.
Relation and converse are one-to-one functions.

6. Equation of line

$$y - 3 = \frac{3}{2}(x - 2)$$

(i) $Y - 3 = \frac{3}{2}(x - 2 - 3)$

$Y - 3 = \frac{3}{2}(x - 5)$

(ii) $Y - 3 - 2 = \frac{3}{2}(x - 5)$

$Y - 5 = \frac{3}{2}(x - 5)$

(iii) $Y - 5 = \frac{3}{2}(x - 5)$

7. Slope $= \frac{4 - 2}{7 - 3} = \frac{1}{2}$

Equation

(a) $y - 2 = \frac{1}{2}(x - 3)$

$y - 2 = \frac{1}{2}(x - 3 + 4)$

$y - 2 = \frac{1}{2}(x + 1)$

(b) $y - 2 = \frac{1}{2}(x - 3)$

$y - 2 + 2 = \frac{1}{2}(x - 3 - 3)$

$y = \frac{1}{2}(x - 6)$

Chapter 4

AN EXPERIMENTAL APPROACH TO NONLINEAR FUNCTIONS

4.1 Introduction

The treatment of nonlinear functions follows naturally from the linear functions introduced in the preceding chapter. The work in this chapter assumes the development of linear functions as a prerequisite.

The first experiment in this chapter, the Wick Experiment, is designed to yield a definite nonlinear graph, and thus lead to a discussion of a "best curve" rather than a "best straight line".

In the second experiment, the Horizontal Metronome, the variables are selected so that they yield the quadratic form

$$y = Ax^2 + C .$$

This leads naturally into a consideration of the role played by the constants A and C. Both the rate of opening of the parabola and vertical translation are studied graphically.

In the third experiment of the chapter, the Oscillating Spring, the converse of the quadratic function is actually generated by the experiment. From this there follows a discussion of relations and their converses. Again the emphasis is on a graphical analysis and the development of the students' intuitive feel for functions. The last section of the chapter considers the horizontal translation of the parabola.

As in the earlier chapters of this text, the experiments are designed to motivate the students in a natural way toward the mathematical development.

4.2 The Wick

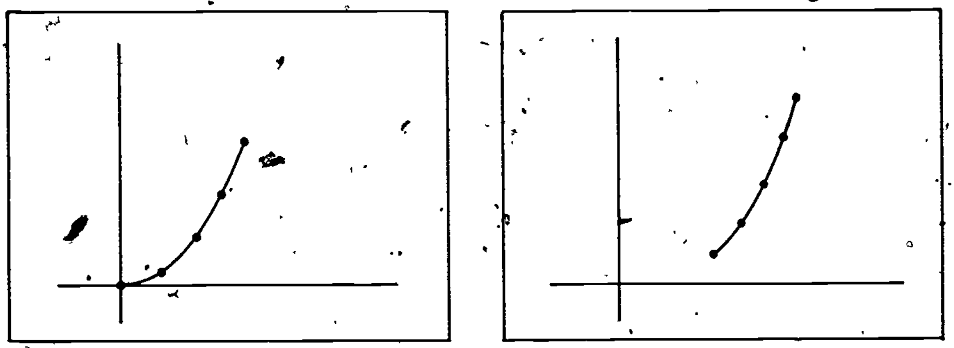
Since this experiment is the initial one in nonlinear relations, it should be performed in small groups of two or three students. The equipment needed for each group is:

- 1 500-cc beaker or drinking glass or quart jar, etc.
- 1 ruler with metric scale
- 1 wrist watch with sweep second hand

1 roll of cellophane tape

supply of strips of chromatography paper, approximately 15 cm long (4 or 5 for each group)

This experiment is designed to yield a definite nonlinear graph. The next two experiments give only a small portion of the curve and it is necessary to analyze the physical situation to realize that the data points should be connected by a curve rather than a straight line (Figure 1, (a) and (b)).



(a) The Wick Experiment (b) The Horizontal Metronome

Figure 1

The experimental setup is quite simple (Figure 2). The students should make two or three practice trials to develop the ability to place the wick into the water so that the zero mark is right at the surface of the water. One student should watch the water move up the wick and tell his partner as the water reaches each numbered point. The partner should record the time

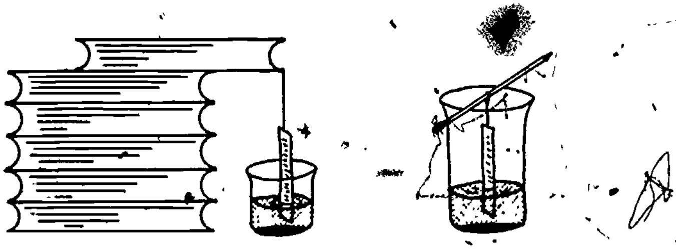


Figure 2

the wick is placed in the water and the time the water reaches each point. The students should alternate these roles for each trial.

4.3 The Physical Model

After the students have completed the plotting of the points represented by the distance-time relation, it is necessary to get some sort of line or curve for the data. It is immediately obvious that no straight line can fit the data, and the most realistic physical model is a smooth curve through or near the points. This best curve, through or near the points, represents a kind of physical idealization of the data. Moreover, physical continuity is assumed; that is, there would be a time for any pre-selected distance one would care to use. It is to be emphasized that at this point no mathematical idealization of the data has been made.

As compared to the procedure of drawing the best straight line through or near the points in the case of a linear function, the best curve through or near the points in the present case has an added feature. The ideal curve should be a smooth one. The smoothness criterion stems only from our physical intuition concerning the behavior of the variables in the experiment. It will be found later that the best mathematical model of the relation involved is also a smooth curve.

4.4 Mathematical Model

Once a smooth curve has been drawn, the student has the problem of finding a mathematical relation between distance and time. Exercise 1, which follows this section, is a very important set of problems. In these problems the student will see how some operation on one of the variables in a nonlinear relation may yield new ordered pairs which are linearly related.

Exercise 1

Each of the following problems consists of a set of ordered pairs of the form (x,y).

(a) Graph each set of ordered pairs. (Check the domain and range before setting scales on the x- and y-axes.)

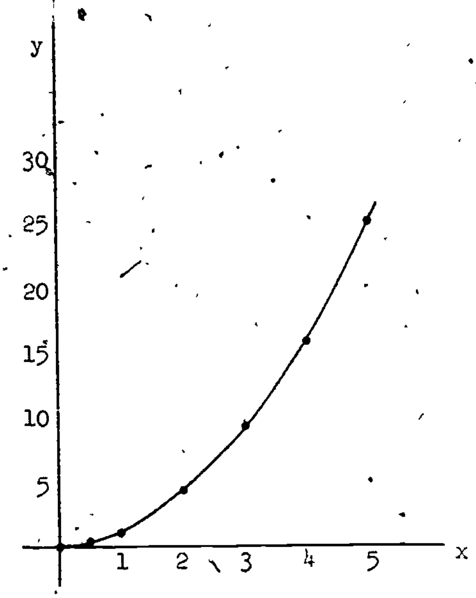
(b) Draw a smooth curve through the points.

(c) Form a new set of ordered pairs following the instructions given with each problem. (Problem 1 is partially completed as an example.)

(d) Graph this new set of ordered pairs on a new sheet of graph paper.

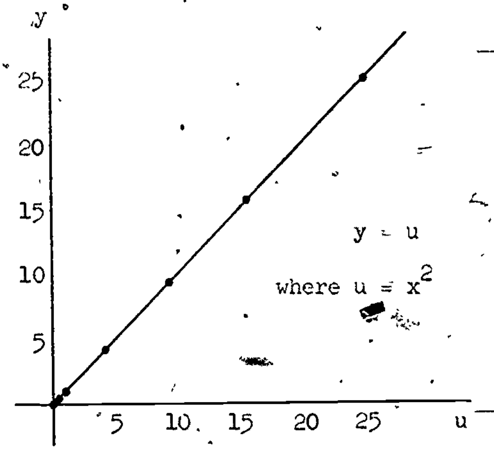
(e) In each case, part (d) should yield a straight line; find the equation of this line using the methods of Chapter 3.

1. $\{(0,0), (\frac{1}{2}, \frac{1}{4}), (1,1), (2,4), (3,9), (4,16), (5,25)\}$

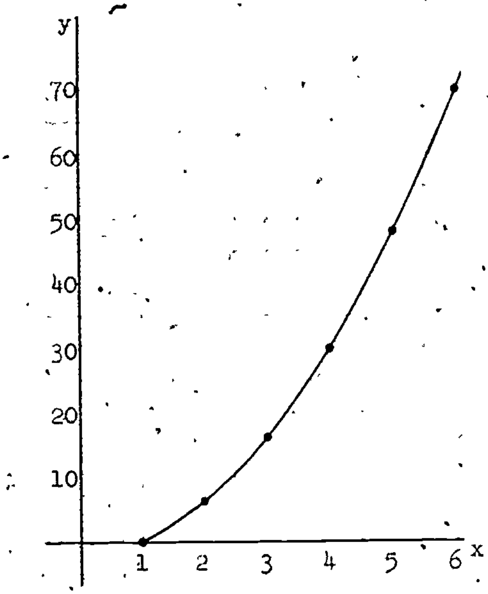


Form ordered pairs of the form (x^2,y).

$\{(0,0), (\frac{1}{4}, \frac{1}{4}), (1,1), (4,4), (9,9), (16,16), (25,25)\}$

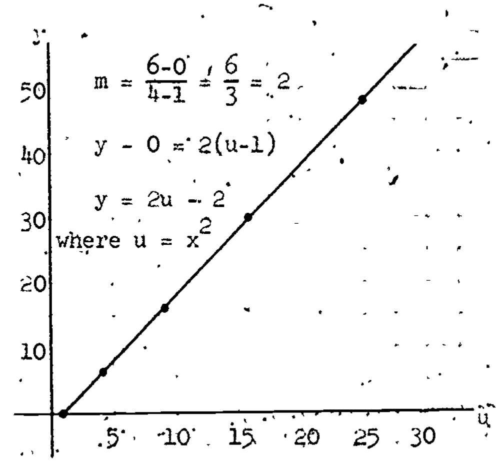


2. $\{(1,0), (2,6), (3,16), (4,30), (5,48), (6,70)\}$

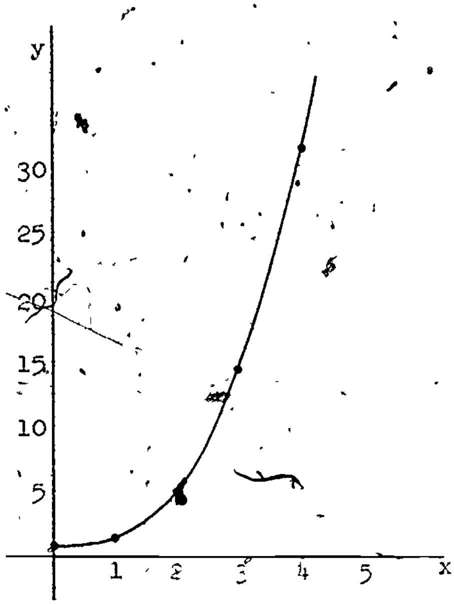


Form ordered pairs of the form (x^2,y).

$\{(1,0), (4,6), (9,16), (16,30), (25,48), (36,70)\}$

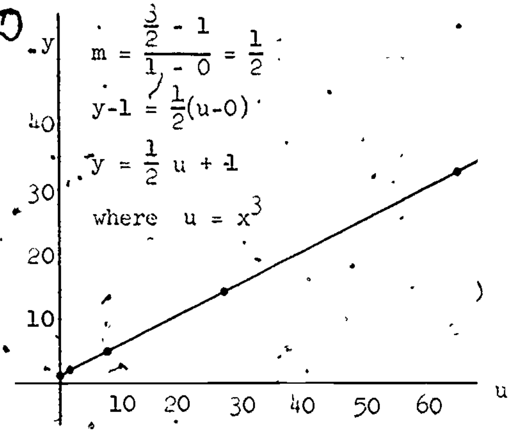


3. $\{(0,1), (1,1\frac{1}{2}), (2,5), (3,14\frac{1}{2}), (4,33)\}$

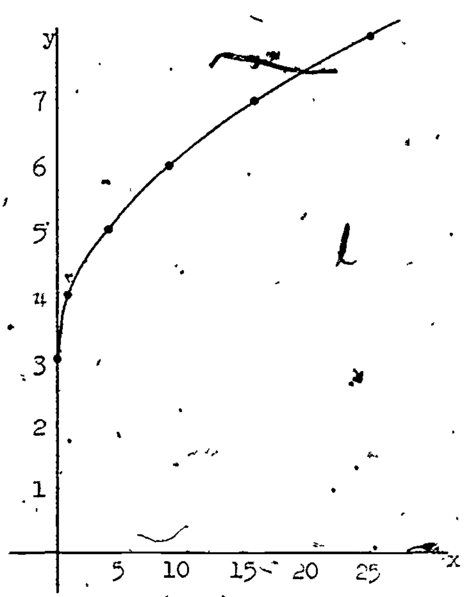


Form ordered pairs of the form (x^3,y).

$\{(0,1), (1,1\frac{1}{2}), (8,5), (27,14\frac{1}{2}), (64,33)\}$

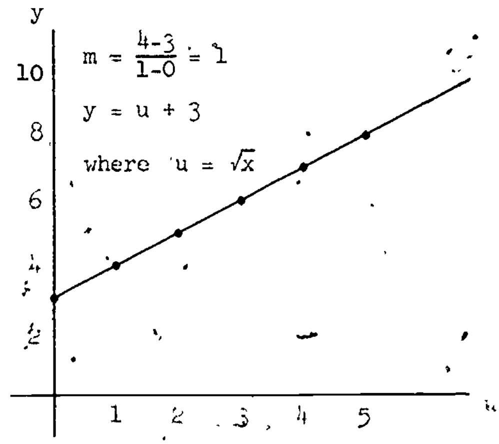


4. $\{(0,3), (1,4), (4,5), (9,6), (16,7), (25,8)\}$



Form ordered pairs of the form $(\sqrt{x},y)$.

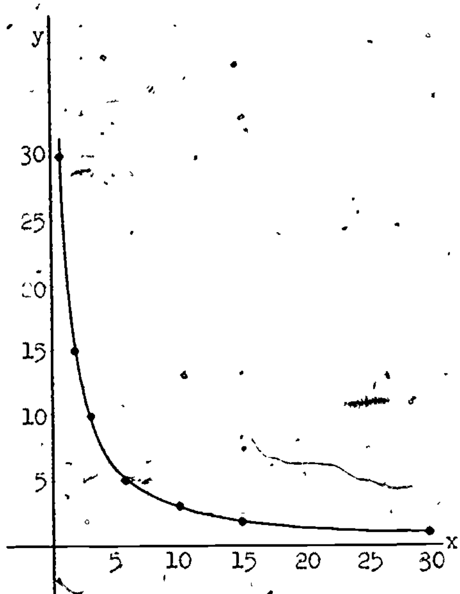
$\{(0,3), (1,4), (2,5), (3,6), (4,7), (5,8)\}$

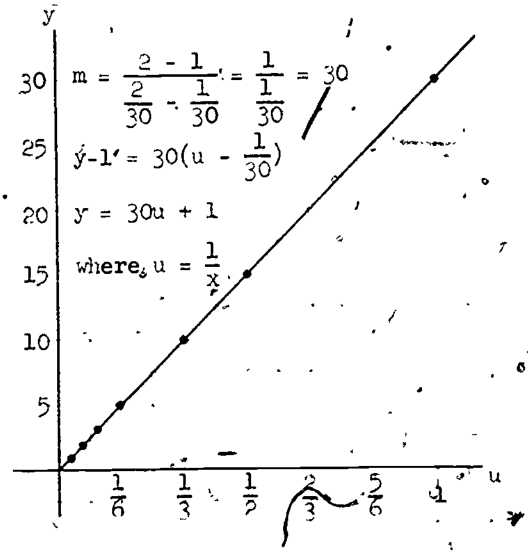


5. $\{(30,1), (15,2), (10,3), (6,5), (3,10), (2,15), (1,30)\}$

Form ordered pairs of the form $(\frac{1}{x}, y)$.

$\{(\frac{1}{30},1), (\frac{1}{15},2), (\frac{1}{10},3), (\frac{1}{6},5), (\frac{1}{3},10), (\frac{1}{2},15), (1,30)\}$





6. Using the set of ordered pairs (d,t) you obtained from the Wick Experiment, form and graph the ordered pairs:

(a) (d^2,t)

(b) (d,t^2)

(c) (d^3,t)

The graph of each will depend upon the particular set of data that the student collected.

Which of these gives data which is closest to a straight line?

The plot of (d^2,t) ordered pairs should most closely approximate the graph of a straight line.

4.5 The Horizontal Metronome Experiment

This experiment should be performed in groups of 4 or 5 students. The equipment needed for each group is:

1 high-speed hacksaw blade (molybdenum steel)

1 one-pound square of plumber's lead or solder

1 clamp base vise, 2-inch wide jaws

1 wrist watch with sweep second hand (stopwatch if available)

A vibrating elastic rod is used as the introductory experiment to quadratic relations. Any mass which is acted upon by a force proportional to its displacement will execute simple oscillatory motion. In this experiment, the lead mass is caused to move by the elastic forces of the rod. The student text calls for a stopwatch to measure the period. A stopwatch is convenient but it is also possible to make the time measurements with the sweep second hand of a wristwatch. The students may question the validity of timing many oscillations and dividing by the number of oscillations to get the period. It may be worthwhile to try this method for 10, 20 and 50 oscillations showing the period is the same for each case. This should make the method described in the student text seem reasonable.

As with most oscillating systems, the period is slightly dependent upon the amplitude for vibrations of large amplitude. The mass should be pulled aside just enough to ensure 50 oscillations. At long lengths (greater than 20 cm) the period-length relation does not follow a parabola. Care must be exercised in starting the motion of the system to prevent a "wobble" motion of the mass.

The text suggests that measurements of the period of the system be made for blade lengths ranging from 20 cm to 10 cm. You should have the students take data with as short a blade length as possible. The curvature of the parabola is most clearly evident at these short lengths. A table of values found with similar equipment is shown on the next page (Figure 3).

The students themselves should enter the column headings as they see the need for these particular headings. Once the class agrees upon the appropriate headings, however, uniformity is to be desired. There will be a need for only the first four columns at this point.

The students should, if at all possible, work in teams of four. In this way each student will follow closely all aspects of the experiment, and profit accordingly. This experiment loses its effectiveness when done as a student or teacher demonstration.

The display of the data on coordinate paper follows closely the procedures developed in the Work Experiment. Since the curvature of the parabola is not large, the students should be cautioned about plotting their data carefully. It is important that the student realize the "smooth curve" is

Length cm	Number of Oscillations	Time sec	Period sec	Length Squared cm^2	Calculated Curve Using $t - 0.1 = .0025d^2$ Periods			
20	50	58.0	1.16	400				
19	50	52.0	1.04	361				
18	50	45.5	0.91	324	0.91			
17	50	40.2	0.8	289				
16	50	36.2	0.72	256	0.74			
15	50	32.5	0.65	225				
14	50	28.9	0.58	196				
13	50	26.4	0.53	169				
12	50	23.0	0.46	144	0.46			
11	50	20.7	0.41	121				
10	50	19.0	0.38	100				
8					0.26			
4					0.14			
0					0.10			

Figure 3

much more realistic physically than connecting points by straight lines. You should also emphasize that the physical model does not extend beyond the set of points obtained in the experiment. That is, the domain is limited by the experiment. A plot of the data for this experiment is shown and a smooth curve has been drawn to represent the data (Figure 4).

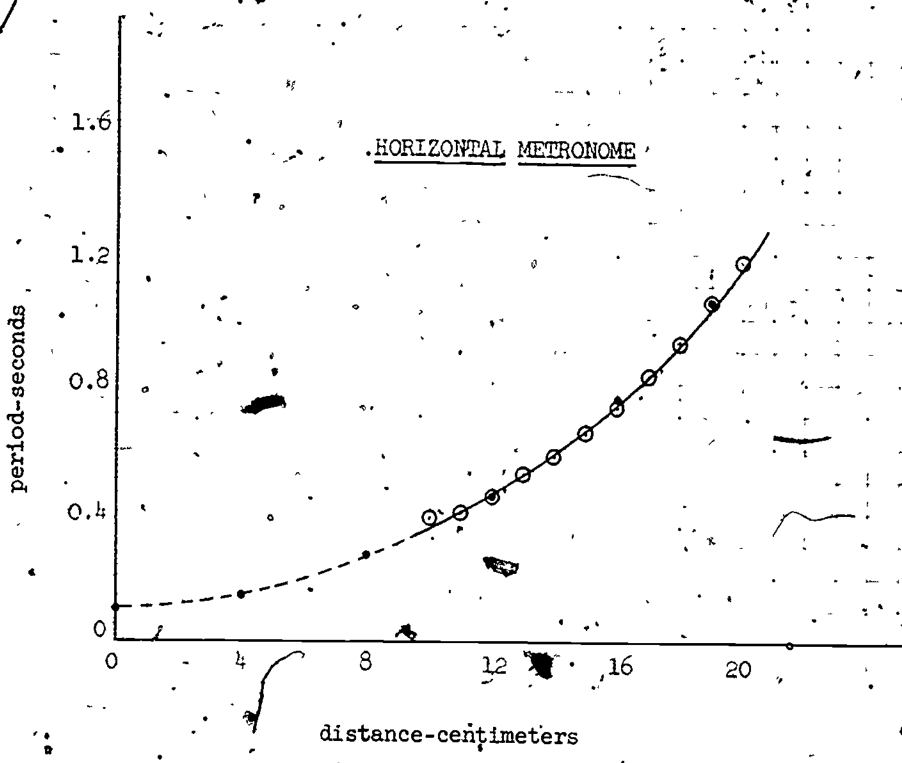


Figure 4

The students should carefully label both the horizontal and the vertical axes. Units must be specified for both. The distance values (d) should be plotted along the horizontal axis, since this is the domain of the relation. The period values (t) make up the range of the relation and should be plotted along the vertical axis.

At this point a review of Exercise 1 may prove valuable. The "best curve" for the Horizontal Metronome resembles the curve of Problem 1. In this problem forming ordered pairs (x^2,y) yielded a linear relation; hence, it seems logical to try ordered pairs of the form (d^2,t) in this case.

The "best" line drawn through the plot of the (d^2,t) relation again follows the procedure used in the linear chapter. Since the domain does not include zero, the student should be cautioned not to extend the straight line to the t-axis intercept. The point-slope form should be used to determine the equation of the straight line. When the domain has been extended, the intercept can be used to check the mathematical model.

A graph of the (d^2,t) relation, using the data from column 5 of the table, is shown in Figure 5. The equation obtained for this line is

$$t = 0.0025d^2 + 0.1$$

Using this equation calculated values were plotted (solid circles in Figure 4) on the original (d,t) graph. As with previous graphs, the straight lines will not be the same from group to group.

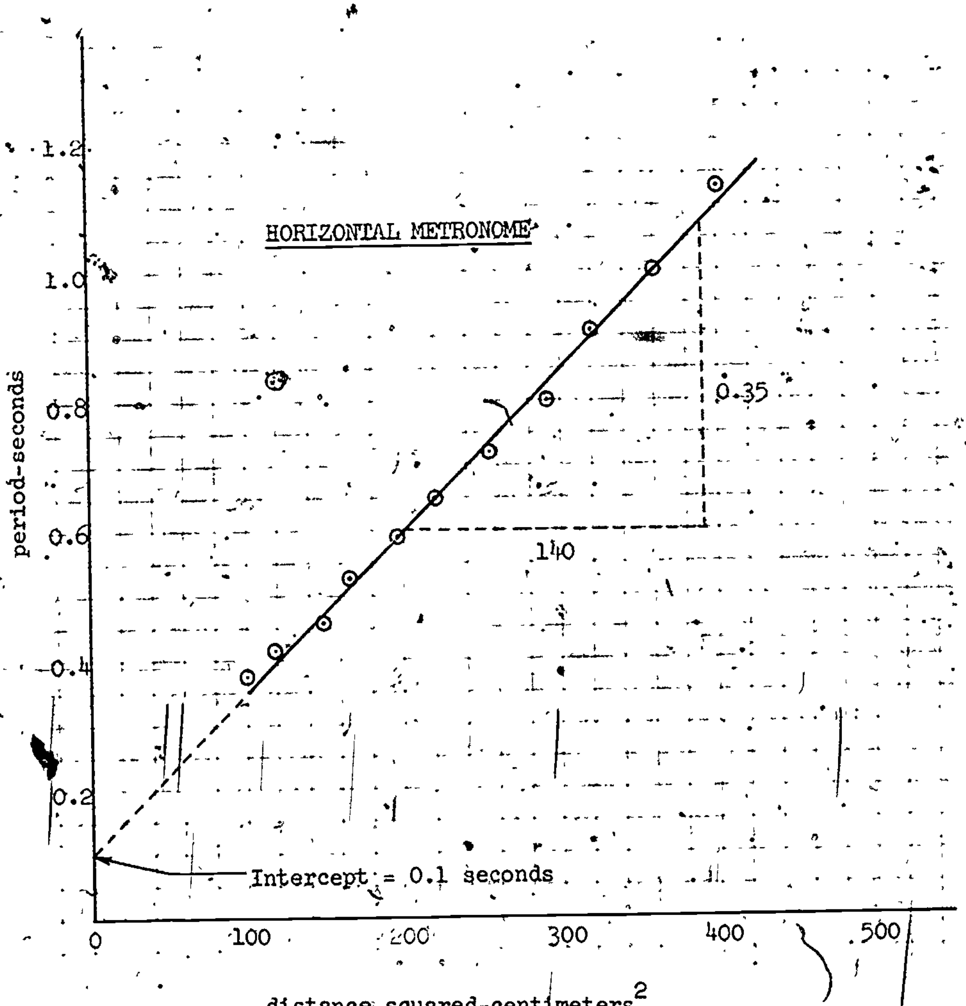


Figure 5

Exercise 2

1. The following equations describe various curves.

   (a) What ordered pairs would you form in each case to show a straight line graph?

   (b) Use the following numbers (-2, -1, 1, 2) from the domain of the given relation to form the predicted ordered pairs.

   (c) Plot these points and check to see if they fall in a straight line.

   (d) Write the linear equation for each graph.

Example: $y = 3(\frac{1}{x}) + 2$

Predict ordered pairs of the form $(\frac{1}{x}, y)$.

$y = 3(-\frac{1}{2}) + 2$

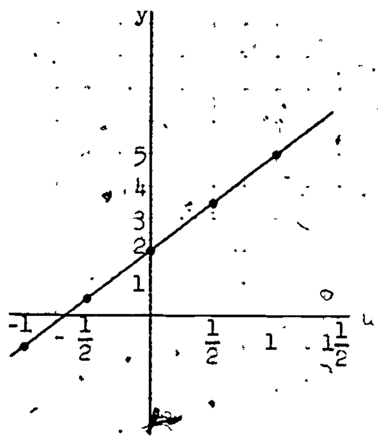
$= -\frac{3}{2} + 2 = \frac{1}{2}$ $\qquad (-\frac{1}{2}, \frac{1}{2})$

In a similar way, the following ordered pairs are calculated:

(-1,-1)

(1,5)

$(\frac{1}{2}, \frac{7}{2})$



Note: 0 cannot be used to form an ordered pair for this relation, since $\frac{1}{0}$ is undefined. This point (0,2) is missing from the graph since the ordered pair (0,2) is not in the relation.

The equation of the line is $y = 3U + 2$. The domain of U is all real numbers except 0, and the range of y is all real numbers except 2.

(a) $y = x^3 + 7$

(x^3, y)

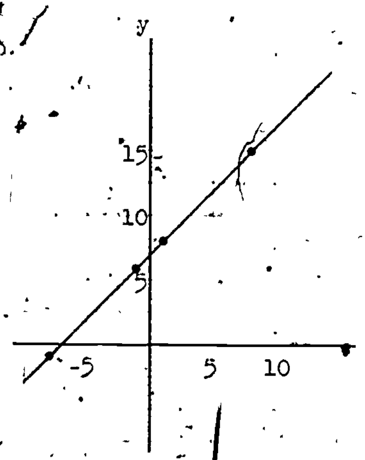
(-8,-1)

(-1,6)

(1,8)

(8,15)

$y = U + 7$



(b) $y - 3 = x^2 + 7$

(x^2, y)

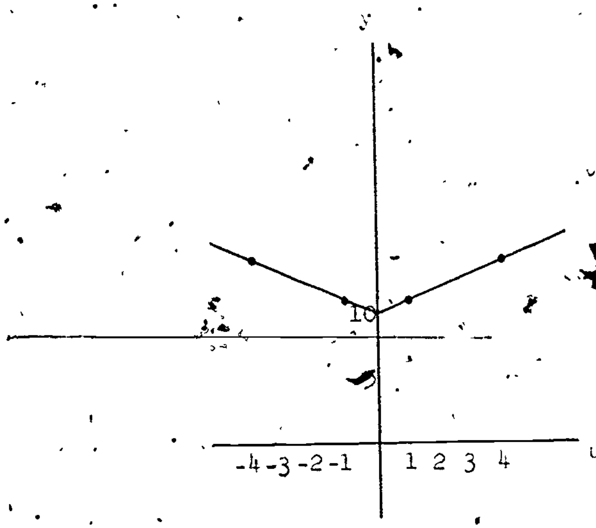
$(|4|, 14)$

$(|1|, 11)$

$(|-1|, 11)$

$(|-4|, 14)$

$y = |U| + 10$



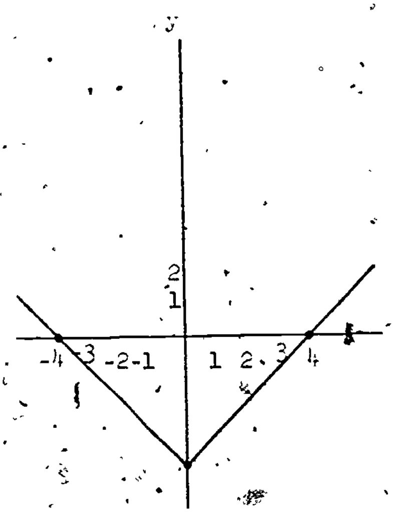
(c) $y = \frac{1}{x^2} - 4$

$(\frac{1}{x^2}, y)$

$(\frac{1}{4}, -\frac{15}{4})$

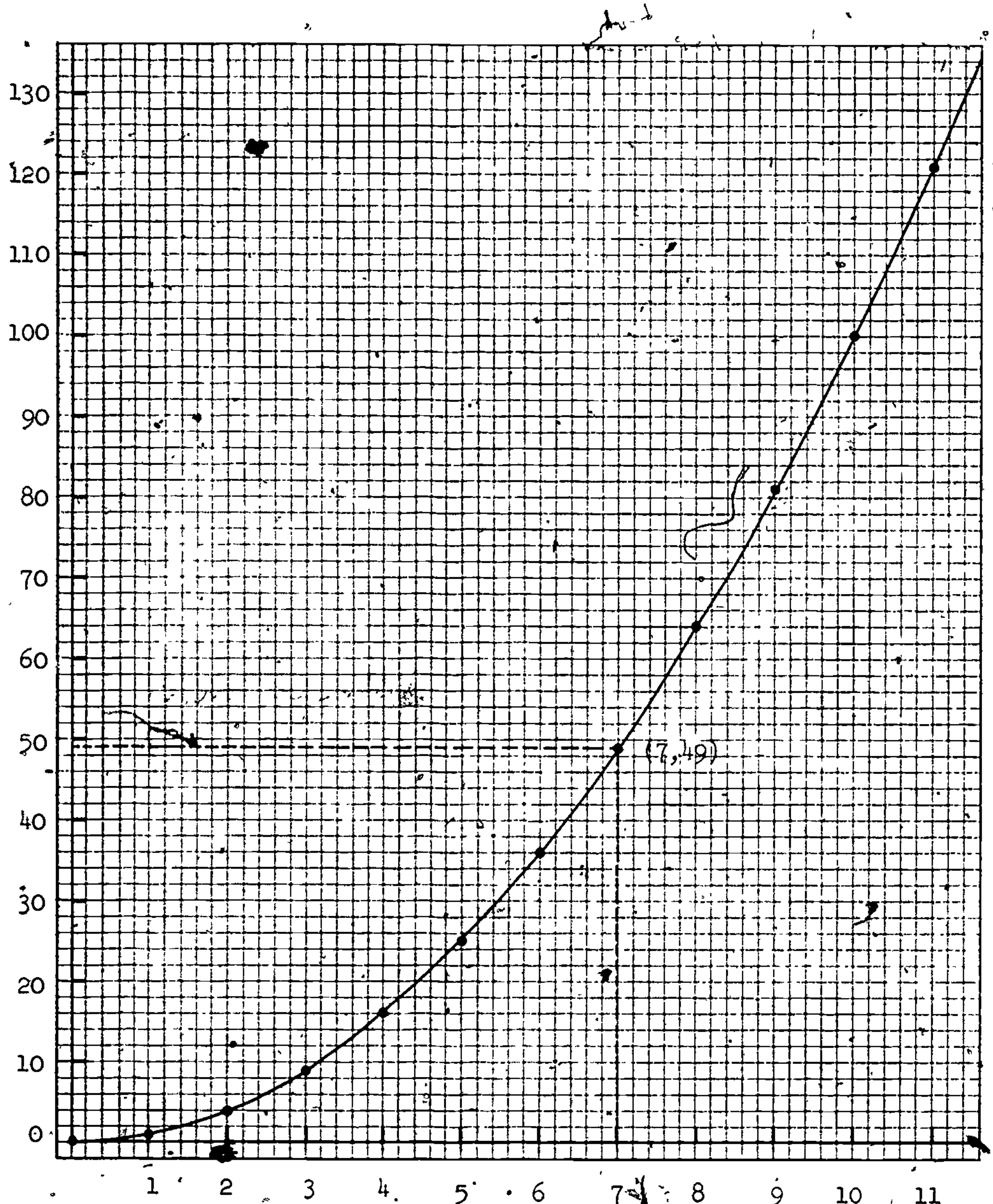
$(1, -3)$

$y = |U| - 4$



2. If you pick any point on the graph of $y = x^2$, the first element of the ordered pair will be the square root of the second element.

For example: To find $\sqrt{49}$, consider the ordered pair of the graph for which 49 is the second element (7,49). The first element of this ordered pair is 7 which is $\sqrt{49}$.



From the graph, obtain the following values.

(a)	$\sqrt{25}$	5	(f)	$\sqrt{8}$	2.8	(k)	$(10.8)^2$	117
(b)	$\sqrt{121}$	11	(g)	$\sqrt{116}$	10.8	(l)	$(3.2)^2$	10
(c)	$\sqrt{2}$	1.4	(h)	$\sqrt{74}$	8.6	(m)	$(5.5)^2$	30
(d)	$\sqrt{40}$	63	(i)	$(4.6)^2$	21	(n)	$(76)^2$	7.6×10^2
(e)	$\sqrt{68}$	8.2	(j)	$(9.4)^2$	88			$= 58 \times 10^2 = 5800$

Note: The student should not be expected to read the horizontal axis closer than the nearest 0.1 and the vertical axis closer than the nearest unit.

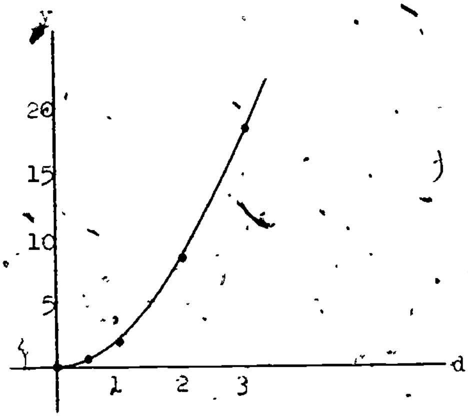
3. From your original graph of (d,t) pairs find the value of t corresponding to $d = 8.5$ cm. Using the equation you obtained to describe the distance-period relation calculate the period corresponding to a distance of 8.5 centimeters. Compare the two results.

From the experimental graph, the period corresponding to $d = 8.5$ cm is 0.28 sec.

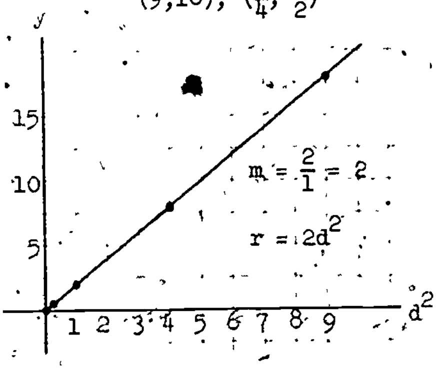
Using the equation $t = 0.0025\,d^2 + 0.1$, when $d = 8.5$ cm $t = 0.28$ sec.

4. Each of the following sets of ordered pairs (d,r) describes various curves.
   (a) Plot the points.
   (b) Draw the curve.
   (c) Form new ordered pairs of the form (d^2,r) and plot these points.
   (d) If the (d^2,r) ordered pairs form a linear relation, draw the straight line and find the equation of the line.

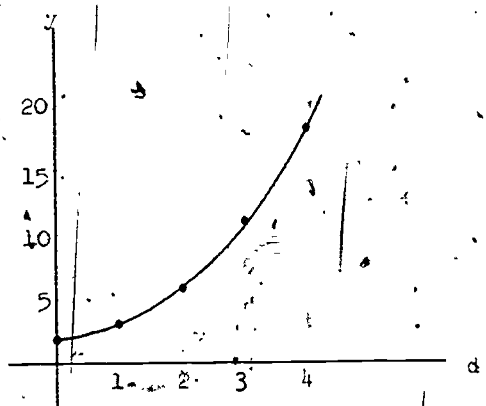
(a) $\{(0,0), (1,2), (2,8), (3,18), (\frac{1}{2}, \frac{1}{2})\}$



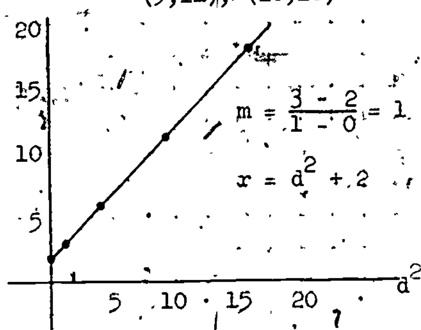
(d^2,r): $(0,0)$, $(1,2)$, $(4,8)$, $(9,18)$, $(\frac{1}{4}, \frac{1}{2})$



(b) $\{(0,2), (1,3), (2,6), (3,11), (4,18)\}$

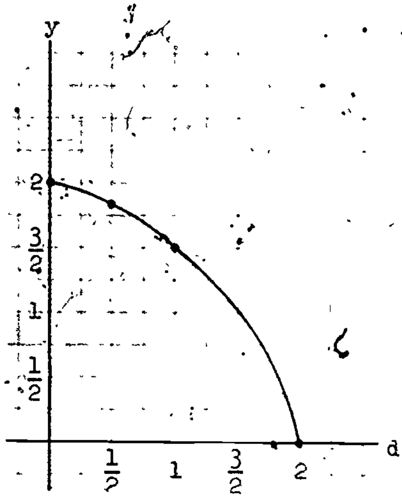


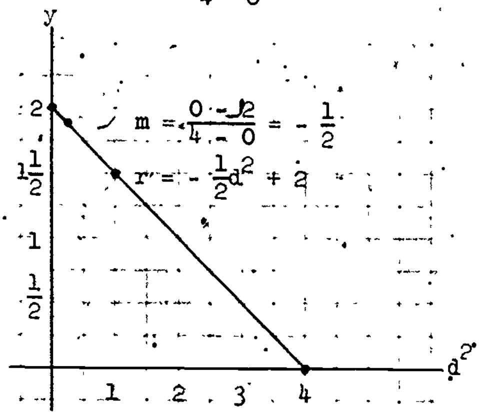
(d^2,r): $(0,2)$, $(1,3)$, $(4,6)$, $(9,11)$, $(16,18)$



(c) $\{(0,2), (2,0), (1, \frac{3}{2}), (\frac{1}{2}, \frac{15}{8})\}$

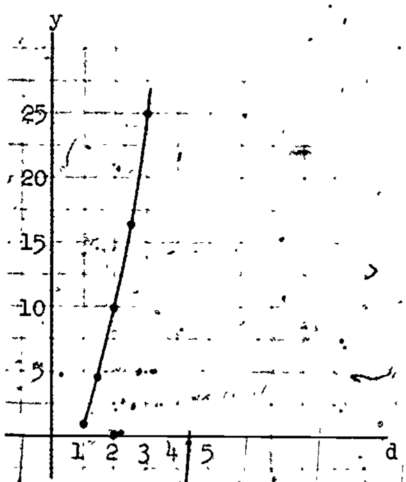
(d^2,r): $(0,2)$, $(4,0)$, $(1,\frac{3}{2})$, $(\frac{1}{4}, \frac{15}{8})$

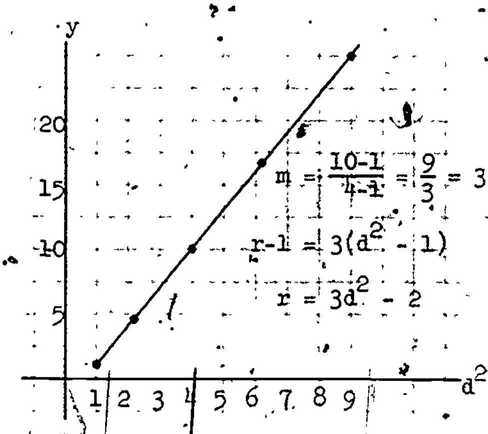




(d) $\{(1,1), (2,10), (3,25), (\frac{3}{2}, \frac{19}{4}), (\frac{5}{2}, \frac{67}{4})\}$

(d^2,r): $(1,1)$, $(4,10)$, $(9,25)$, $(\frac{9}{4}, \frac{19}{4})$, $(\frac{25}{4}, \frac{67}{4})$





5. If we consider the domain of d to include all positive real numbers, use your mathematical model to calculate the values of the period that correspond to the following values of d.

{Note: The equation from the text is used ($t = 0.0025\ d^2 + 0.1$). Student responses will differ from those given below.}

(a) $d = 50$ cm   <u>6.25 sec</u>   (c) $d = 500$ cm   <u>625 sec</u>

(b) $d = 100$ cm   <u>25 sec</u>   (d) $d = 1000$ cm   <u>2.5×10^3 sec</u>

4.6 The Parabola

In this section we study the graph of $y = Ax^2$, where A is any nonzero real number. We show how the graph can be obtained from the graph of $y = x^2$. The sign of A and the absolute value of A are the determining factors. If A is positive, the curve opens upward; if A is negative, the curve opens downward. If $|A|$ is small, the curve is relatively flat; if $|A|$ is large, the curve is steep. This can be proved, but it is probably best shown by examples. Figure 6 illustrates some of these facts.

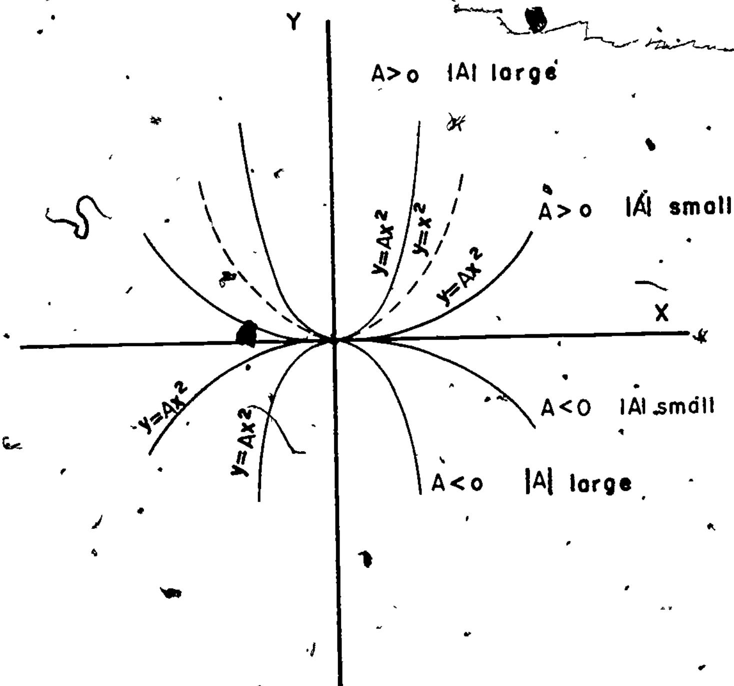


Figure 6

The student should be asked to make tables similar to those in the text for various values of A. They should plot these points and draw a smooth curve through them. The study of the significance of the coefficient A gives qualitative information, but does not enable a student to draw the graph.

In this section we show how the graph of $y = Ax^2 + C$ can be obtained from the graph of $y = Ax^2$. The graph of $y = Ax^2 + C$ is congruent to the graph of $y = Ax^2$ (have students use a sheet of onion-skin paper or frosted acetate to show the congruence). The graph is obtained from the latter by translating it upward or downward, depending on the value of C. The student needs to know how to use the number C to locate the graph of $y = Ax^2 + C$. The vertex of the curve is $|C|$ units above the graph of $y = Ax^2$ if C is positive, and $|C|$ units below the graph of $y = Ax^2$ if C is negative. Examples, rather than formal proofs should be stressed. The next figure shows some of these facts (Figure 7).

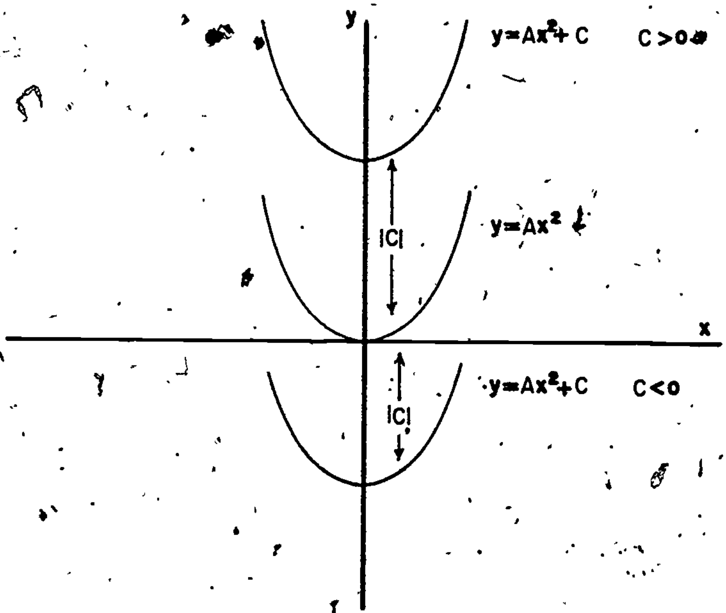


Figure 7

Exercise 3

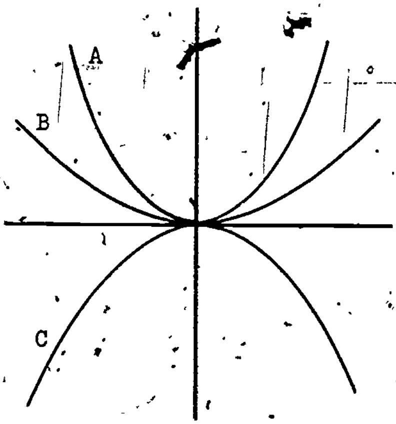
1. The three curves shown at the right are sketches of the graphs of:

$y = \frac{1}{2}x^2$

$y = -\frac{1}{3}x^2$

$y = \frac{1}{6}x^2$

Match each curve with the proper equation.



Curve A is the graph of $y = \frac{1}{2}x^2$

Curve B is the graph of $y = \frac{1}{6}x^2$

Curve C is the graph of $y = -\frac{1}{3}x^2$

2. Describe how the graph of $y = Ax^2$ differs from the graph of $y = x^2$ in each of the following cases:

(a) $A = 0$

The smaller the value of A, the flatter the graph of $y = Ax^2$, and for $A = 0$, it degenerates into the linear equation $y = 0$ which is the x-axis.

(b) $0 < A < 1$

All parabolas for $0 < A < 1$ are between the x-axis and the parabola $y = x^2$.

(c) $A > 1$

For $A > 1$ the parabolas $y = Ax^2$ are inside the parabola $y = x^2$.

(d) $A = -1$

For $A = -1$ the parabola has the same shape as $y = x^2$ reflected with respect to the x-axis.

3. Make a table of at least seven ordered pairs for each of the following equations. Use both positive and negative values of x. Draw all of the graphs on the same sheet of coordinate paper.

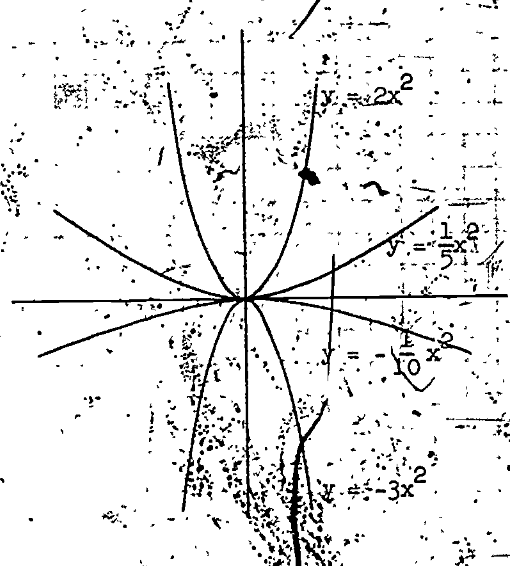
(a) $y = 2x^2$

(b) $y = \frac{1}{5}x^2$

(c) $y = -3x^2$

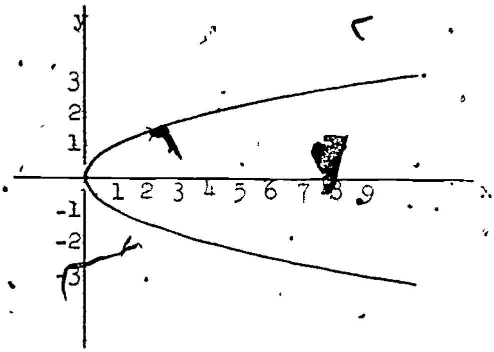
(d) $y = -\frac{1}{10}x^2$

x	$2x^2$	$\frac{1}{5}x^2$	$-\frac{1}{10}x^2$	$-3x^2$
3	18	$\frac{9}{5}$	$-\frac{9}{10}$	-27
2	8	$\frac{4}{5}$	$-\frac{2}{5}$	-12
1	2	$\frac{1}{5}$	$-\frac{1}{10}$	-3
0	0	0	0	0
-1	2	$\frac{1}{5}$	$-\frac{1}{10}$	-3
-2	8	$\frac{4}{5}$	$-\frac{2}{5}$	-12
-3	18	$\frac{9}{5}$	$-\frac{9}{10}$	-27



4. Plot the ordered pairs given below and draw a smooth curve through the points.

x	9	4	1	0	1	4	9
y	-3	-2	-1	0	1	2	3



Is this relation a function? No.

Is the converse of this relation a function? Yes.

Can you think of an equation to describe the relation? $x = y^2$.

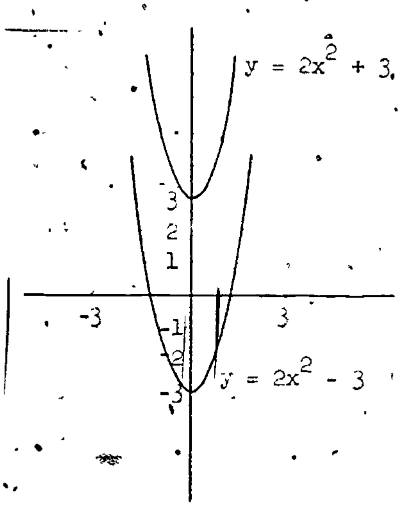
5. For each of the following pairs of equations below, plot the graphs using a single set of coordinate axes for each pair.

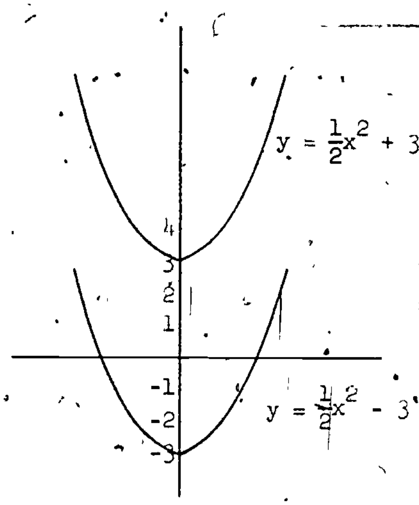
(a) $y = 2x^2 + 3$

$y = 2x^2 - 3$

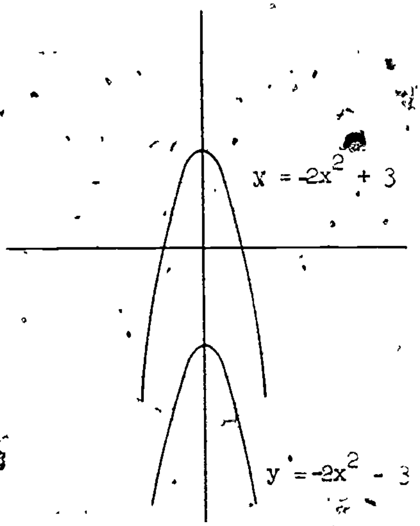
(b) $y = \frac{1}{2}x^2 + 3$

$y = \frac{1}{2}x^2 - 3$

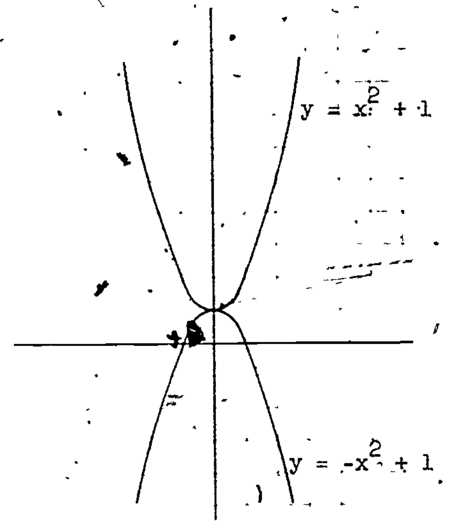




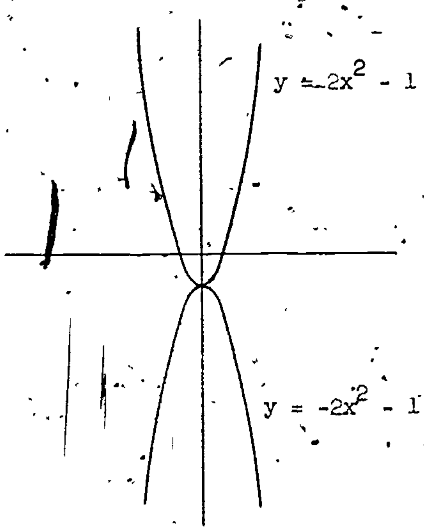
(c) $y = -2x^2 + 3$
$y = -2x^2 - 3$



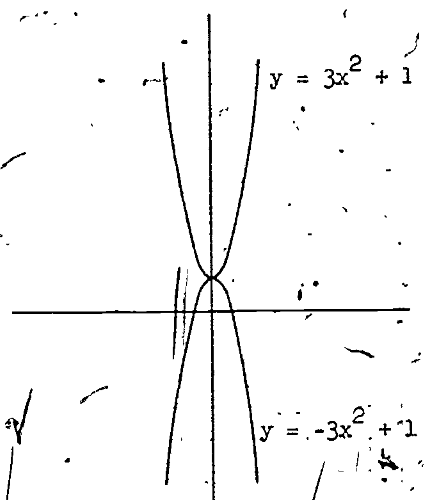
(d) $y = -x^2 + 1$
$y = x^2 + 1$



(e) $y = -2x^2 - 1$
$y = 2x^2 - 1$



(f) $y = -3x^2 + 1$
$y = 3x^2 + 1$



6. Which of the relations in Problem 5 have a minimum value and which have a maximum value? What are these values?

(a) Both have a minimum.

$y = 2x^2 + 3$ Min: (0,3)

$y = 2x^2 - 3$ Min: (0,-3)

(b) Both have a minimum.

$y = \frac{1}{2}x^2 + 3$ Min: (0,3)

$y = \frac{1}{2}x^2 - 3$ Min: (0,-3)

(c) Both have a maximum.

$y = -2x^2 + 3$ Max: (0,3)

$y = -2x^2 - 3$ Max: (0,-3)

(d) One minimum and one maximum.

$y = x^2 + 1$ Min: (0,1)

$y = -x^2 + 1$ Max: (0,1)

(e) One minimum and one maximum.

$y = 2x^2 - 1$ Min: (0,-1)

$y = -2x^2 - 1$ Max: (0,-1)

(f) One minimum and one maximum.

$y = 3x^2 + 1$ Min: (0,1)

$y = -3x^2 + 1$ Max: (0,1)

7. The following equations describe curves which are not parabolas. What ordered pairs would you form in each case to show a parabolic relation?

(a) $y = 2x^4 + 3$ (x^2, y)

(b) $y = x^6 - 2$ (x^3, y)

4.7 The Oscillating Spring Experiment

This experiment should be performed in groups of 4 or 5 students. The equipment needed for each group is:

1 spring (the spring from a window shade roller is excellent for this experiment)

1 $\frac{1}{2} \times 36$-inch hardwood dowel

1 wrist watch with sweep second hand
(stop watch if available)

1 100-gram mass

2 200-gram masses

1 500-gram mass

masking tape

For any spring, the extension of the spring is proportional to the force applied to it, if the force is not sufficient to exceed the so-called elastic limit for the spring.

In the present experiment, a spring is made to oscillate with a mass suspended from it. The period-mass relation is the most interesting one to pursue.

A suitable spring will oscillate for a time of several minutes with a reasonable amplitude. Because of this fact, one may measure the time interval for 50 complete oscillations with a stopwatch and obtain the period simply by dividing this time interval by 50.

It is good practice always to encourage suggestions from the students as to what the important variables in a given experimental situation might be. Eventually you will want to focus the attention of the class upon the period-mass relation.

The spring found inside a window shade roller is an excellent one for our purpose. These may be obtained at any shade shop, and are to be preferred over a commercial item such as those obtained from a scientific supply company. This spring may be loaded with a mass of about 1 kilogram. It is convenient to use masses of 100, 200, 300, ..., 1000 grams, thus giving ten data points. The students should be warned to pull the masses downward when starting the oscillation no farther than necessary to give a smooth oscillation that will persist at least 50 times.

As the spring-mass system is oscillating in an up and down motion, there will also be a slight pendulum swing to the system. It may be found that for a particular load on the spring, the period of oscillation of the spring will couple with the period of swing of the device considered as a pendulum. If this happens, you will note a change of "up and down" motion to "swinging" motion. This change from one form of oscillation to another is an interesting phenomenon in itself but definitely not a desirable one here. If one of the student groups reports this type of behavior, have them change the length of the spring and mass combination. Simply hang a "chain" of two or three paper clips to the spring and suspend the mass from the bottom clip.

The student's data sheet should appear similar to the one in Figure 8. This one gives data collected from the experiment as performed by members of the writing team. The student's data will, of course, differ from that shown.

Mass of load grams ℓ	Number of Oscillations	Time Sec	Period Sec T	Mass of load squared (grams)^2 ℓ^2	Period squared (sec)^2 T^2	
100	100	36	0.36	1×10^4	0.13	
200	100	51	0.51	4×10^4	0.26	
300	100	63	0.63	9×10^4	0.40	
400	100	73	0.73	16×10^4	0.52	
500	100	81	0.81	25×10^4	0.66	
600	100	87	0.87	36×10^4	0.76	
700	100	95	0.95	49×10^4	0.90	
800	100	100	1.00	64×10^4	1.0	
900	100	108	1.08	81×10^4	1.2	
1000	100	114	1.14	100×10^4	1.3	

Figure 8

4.8 The Physical Model

After the students have completed the plotting of the points represented by the mass-period (ℓ,t) relation, it will be necessary to get some sort of line or curve for the data. By now the student should be accustomed to this procedure and realize that the "best" curve through or near the points represents a kind of physical idealization of the data. Moreover, physical continuity is assumed; that is, there would be a period for any pre-selected mass one would care to suspend from the spring. It is to be emphasized that at this point, no mathematical idealization of the data has been made (Figure 9).

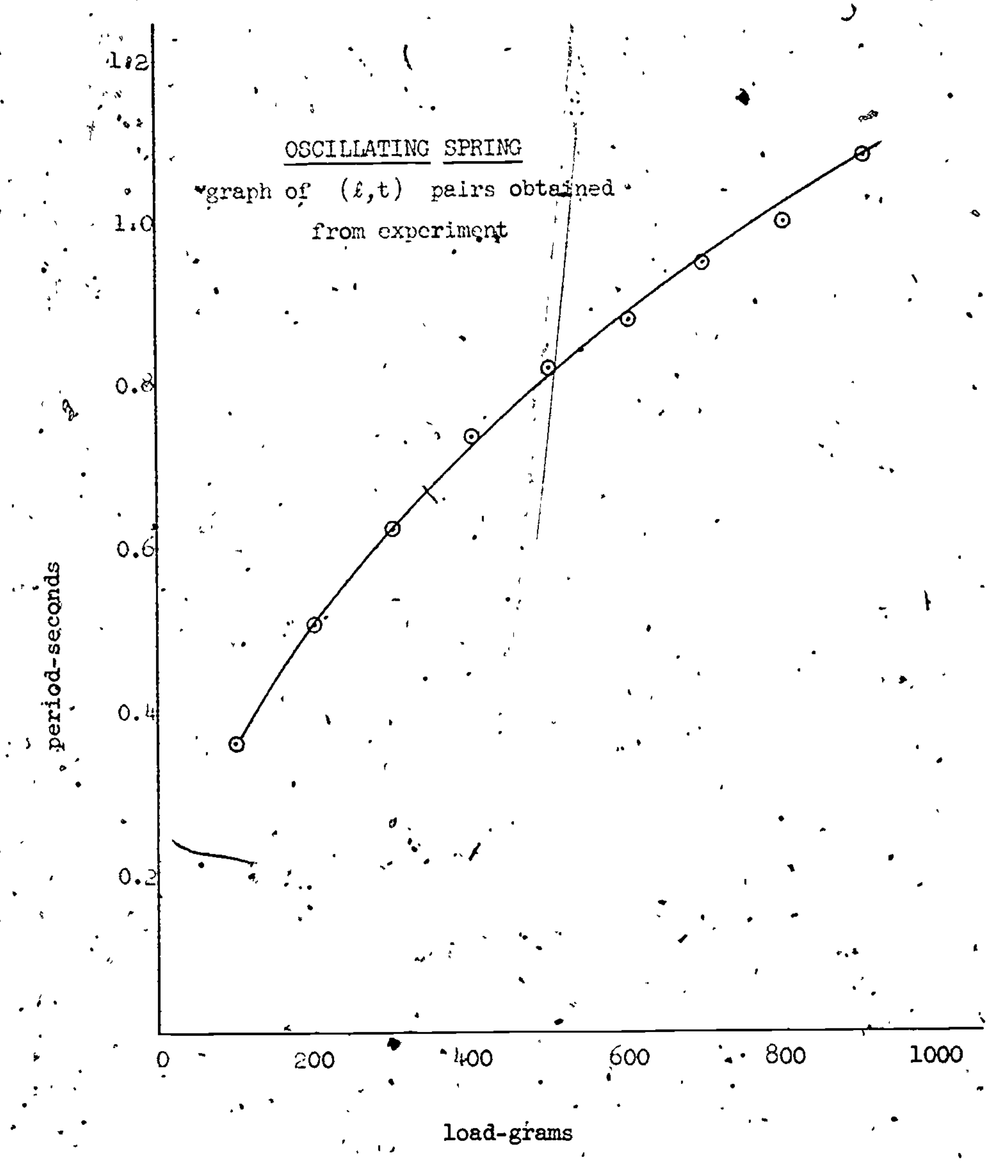


Figure 9

Once the smooth curve has been drawn, the student has the problem of getting some relation between ℓ and t which will be linear. This same problem was faced and solved in the Horizontal Metronome Experiment. In this experiment the student is allowed to follow this same procedure by calculating sets of ordered pairs in the form of (ℓ^2, t). However, when these are plotted, they do not lie in a straight line as indicated in the graph (Figure 10); that is, this particular procedure is not successful. This is not a bad experience for the student, because in all problems of this sort, the scientist searches for some relations which will be linear. Sometimes one scheme will work and sometimes another. This idea will be developed further as we go on with the experiments.

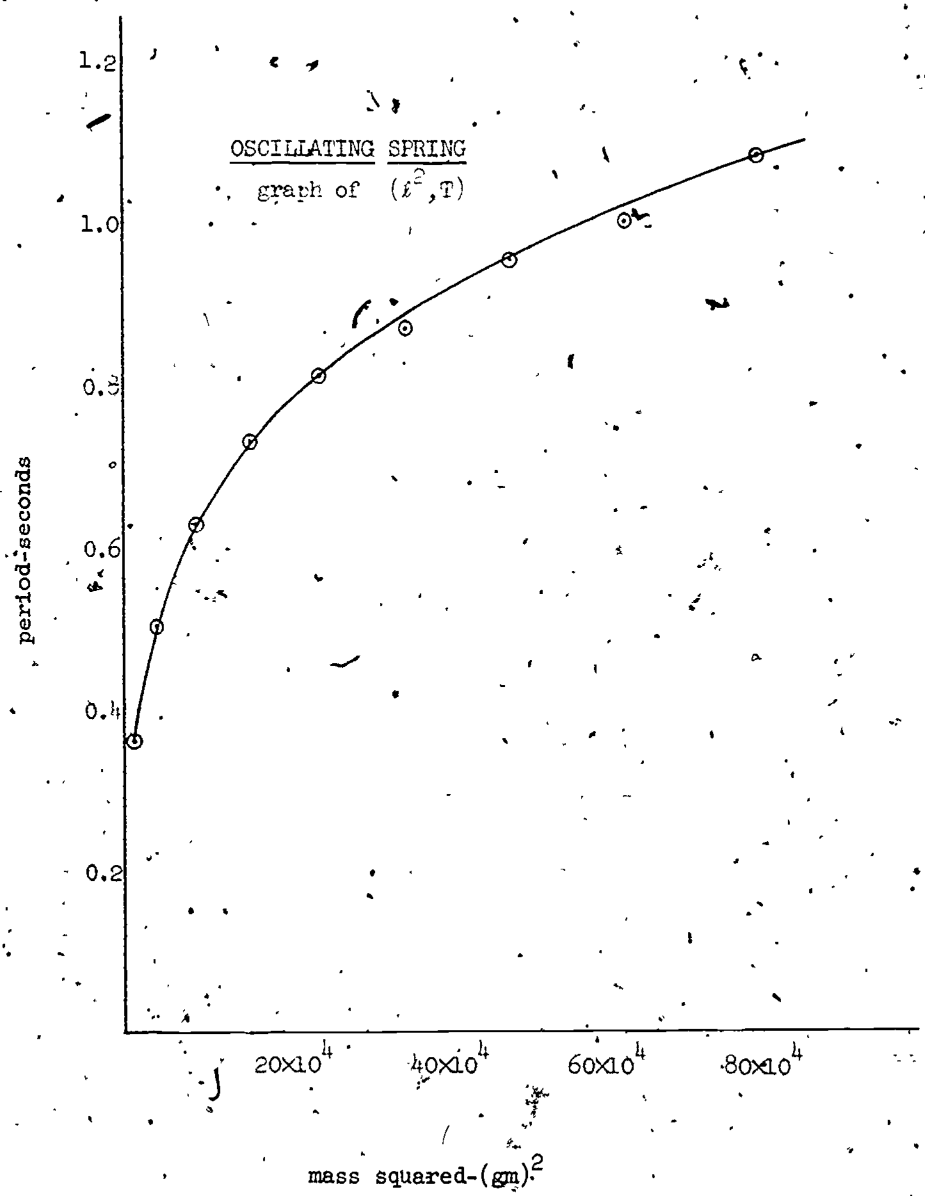


Figure 10

4.9 The Oscillating Spring's Converse Relation

The student must now start a search for some other scheme which may work in this situation. This is carefully discussed in the student's text and he first plots the converse of the (ℓ, t) relation because it then "looks" like the Metronome graph already discussed in this chapter. The graph of the converse appears in Figure 11.

The student then proceeds to calculate the values of ordered pairs (t^2, ℓ) and enter them as column 6 of his data sheet. These are already on the data sheet which is included in this commentary and the points are plotted on Figure 12. These are, of course, based on our data and the student's data and graph will be different.

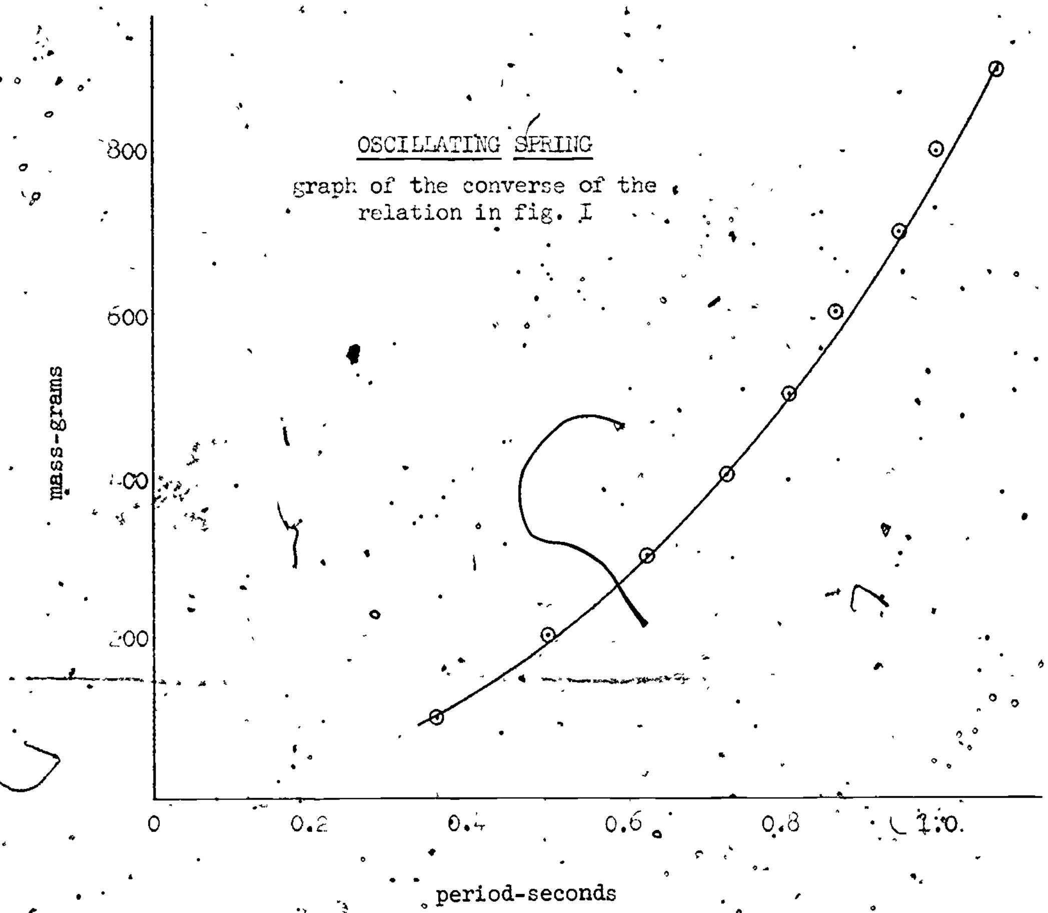


Figure 11

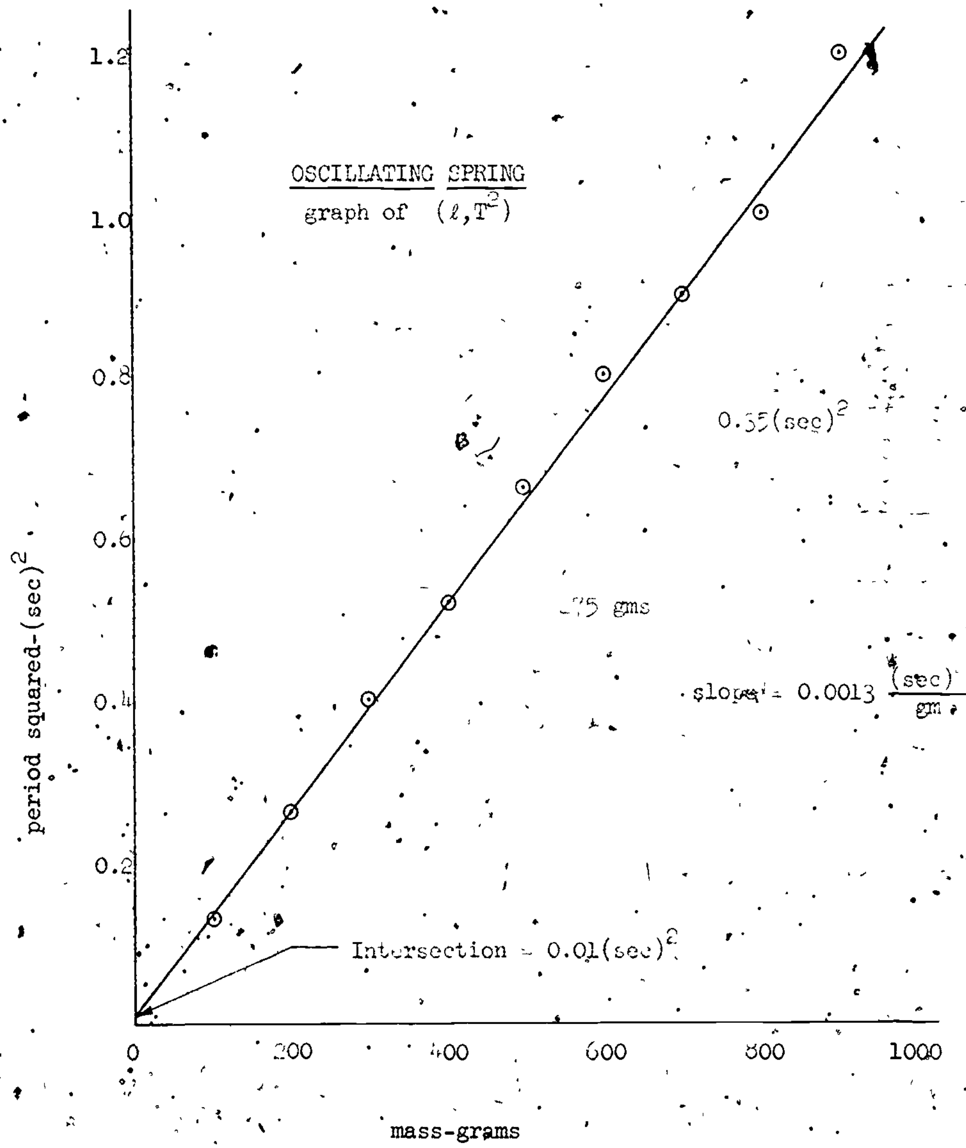


Figure 12

From previous experience with linear functions, the student should have no trouble in writing the equation of the line provided he is convinced that it is a line. This will be

$$\ell = mt^2 + b$$

and for the line on Figure 5, the equation will be

$$t^2 = 0.0013\ell + 0.01$$

To find whether this equation of the line actually represents the graph of the converse of the original data of Figure 4, the student must check for specific values of t and ℓ. He is instructed to do this by adding a new column to his data sheet headed "load (ℓ) in grams, calculated". The calculating is done by letting t equal the values which appear in the data table and finding ℓ from the equation above. For example, when t = 0.36 (first row)

$$(.36)^2 = 0.0013\ell + 0.01$$

$$\ell = 99.8$$

The experimental value of ℓ was 100.

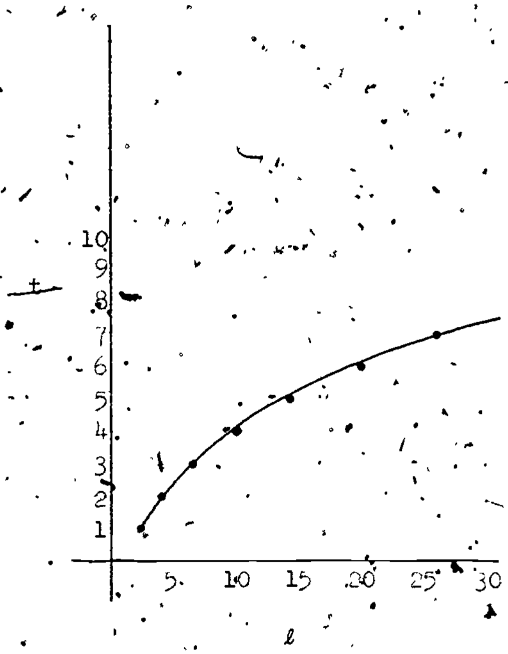
The fact that a line can be used to approximate the location of these points is all important. It is to be noted that a (0,0) point has not been tabulated simply because it was not an experimental point. If the question of the significance of this point did not arise before, it surely will now. The important point to make here is that the student should not assume that the line in this second graph should go through the origin. If a student's line runs through the origin, fine, but if it does not, that's all right, too. The student should not be burdened with the knowledge that a very small positive intercept will be obtained if his experiment has been done carefully.

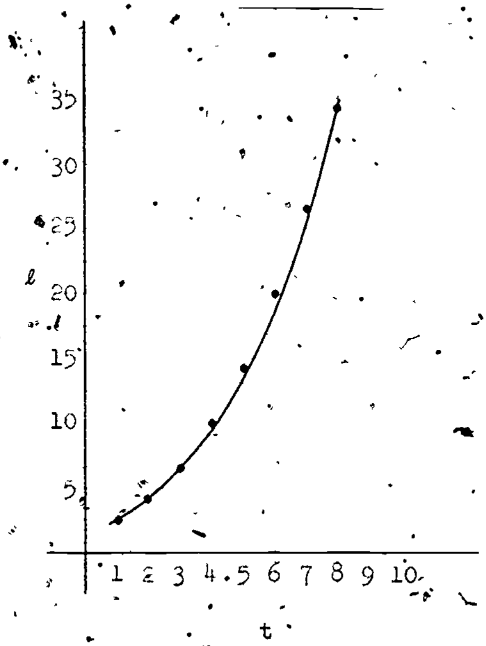
Exercise 4

The table at the right shows the experimental data for a new oscillating spring. The load (ℓ) in grams was fixed, and then the corresponding periods (t) in seconds were measured.

ℓ (grams)	t (sec)
2.5	1
4.0	2
6.5	3
10.0	4
14.5	5
20.0	6
26.5	7
34.0	8

1. Graph the relation and its converse on separate sheets of coordinate paper.





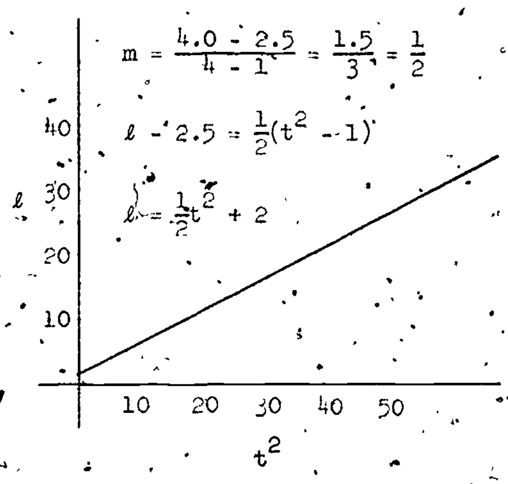
2. Graph the (t^2, ℓ) relation. Draw the "best" straight line and obtain the equation for ℓ.

t^2	ℓ
1	2.5
4	4.0
9	6.5
16	10.0
25	14.5
36	20.0
49	26.5
64	34.0

$$m = \frac{4.0 - 2.5}{4 - 1} = \frac{1.5}{3} = \frac{1}{2}$$

$$\ell - 2.5 = \frac{1}{2}(t^2 - 1)$$

$$\ell = \frac{1}{2}t^2 + 2$$



3. Use your equation obtained above to calculate values of the load in grams for each value of the period in the range of the experimental relation. Compare the calculated and experimental values of the load.

$\ell = \frac{1}{2}t^2 + 2$

$\ell = \frac{1}{2}(1) + 2 = 2.5$

$\ell = \frac{1}{2}(2)^2 + 2 = 4.0$

$\ell = \frac{1}{2}(3)^2 + 2 = 6.5$

$\ell = \frac{1}{2}(4)^2 + 2 = 10$

$\ell = \frac{1}{2}(5)^2 + 2 = 14.5$

$\ell = \frac{1}{2}(6)^2 + 2 = 20$

$\ell = \frac{1}{2}(7)^2 + 2 = 26.5$

$\ell = \frac{1}{2}(8)^2 + 2 = 34$

4.10 Relations and Converses

This section is intended to reinforce the idea of a relation and its converse.

Exercise 5

1. In the series of graphs shown in the accompanying figure pair them so that in each case you have a relation and its converse.

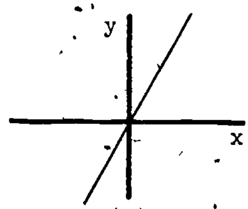
   (a) and (b) form converse relations.
   (c) and (d) form converse relations.
   (e) and (g) form converse relations.
   (f) and (h) form converse relations.
   (i) and (l) form converse relations.
   (j) and (k) form converse relations.

2. Which of the graphs in the figure represent functions?

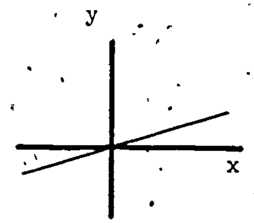
   See the labels for each group.

3. Which pairs of graphs obtained in Problem 1 represent one-to-one functions? (Note: If both a relation and its converse are functions, then these two relations are called one-to-one functions.)

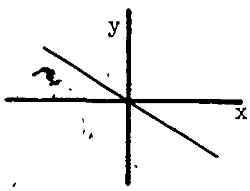
   (a) and (b) represent one-to-one functions.
   (c) and (d) represent one-to-one functions.
   (e) and (g) represent one-to-one functions.
   (f) and (h) represent one-to-one functions.



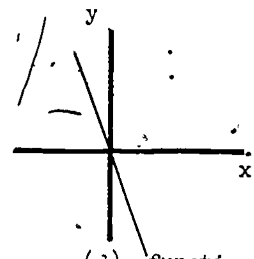
(a) function



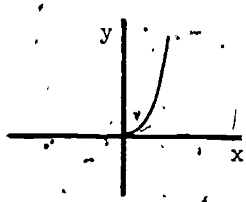
(b) function



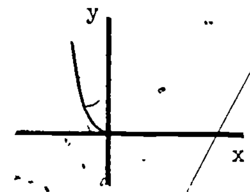
(c) function



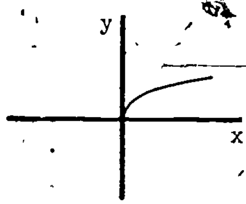
(d) function



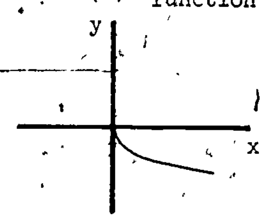
(e) function



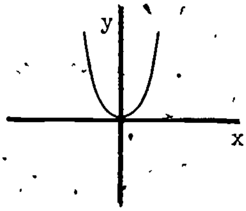
(f) function



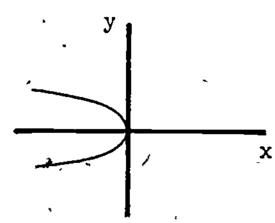
(g) function



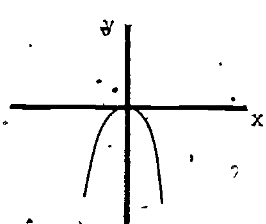
(h) function



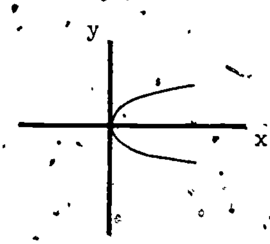
(i) function



(j) not a function



(k) function



(ℓ) not a function

4.11 Translation of the Parabola

In this section we show how the graph of $y = A(x - k)^2$ can be obtained from the graph of $y = Ax^2$. The graph of $y = A(x - k)^2$ is congruent to the graph of $y = Ax^2$ and is obtainable from the latter by translating it, either to the right or to the left depending on the value of k. The student needs to know how to use the number k to locate the graph of $y = A(x - k)^2$. The curve is $|k|$ units to the right of the graph of $y = Ax^2$ if k is positive and $|k|$ units to the left of this graph if k is negative. Figure 13 shows some of these facts. ($A < 0$ for these graphs.)

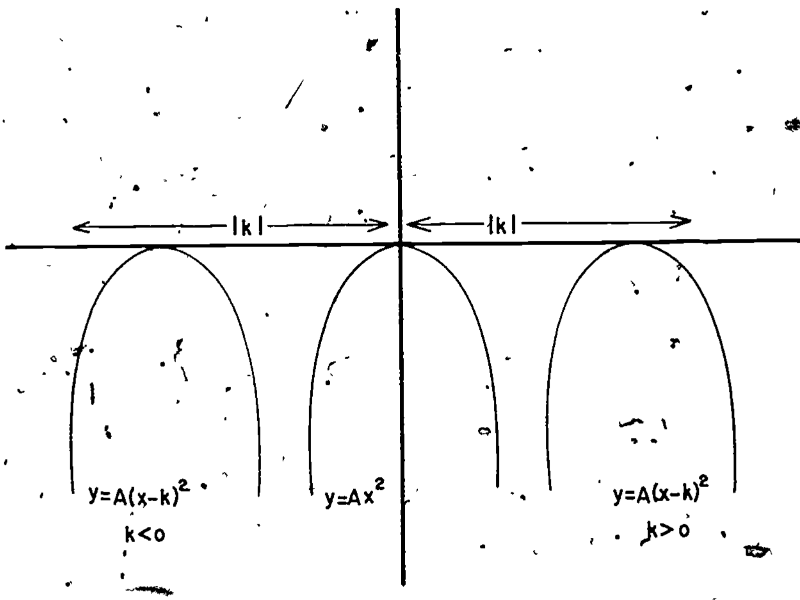


Figure 13

The last part of the section shows that the graph of $y = A(x - k)^2 + p$ is congruent to the graph of $y = Ax^2$ and can be obtained from this latter graph by translating it up or down depending on the value of p and right or left depending on the value of k. The student will need to know how to use the numbers k and p to locate the graph of $y = A(x - k)^2 + p$. The curve is $|p|$ units above the graph of $y = Ax^2$ if p is positive and $|p|$ units below this graph if p is negative, $|k|$ units to the right if k is positive, and $|k|$ units to the left if k is negative.

Here, too, examples rather than formal proofs should be stressed. Figure 14 shows some of these facts.

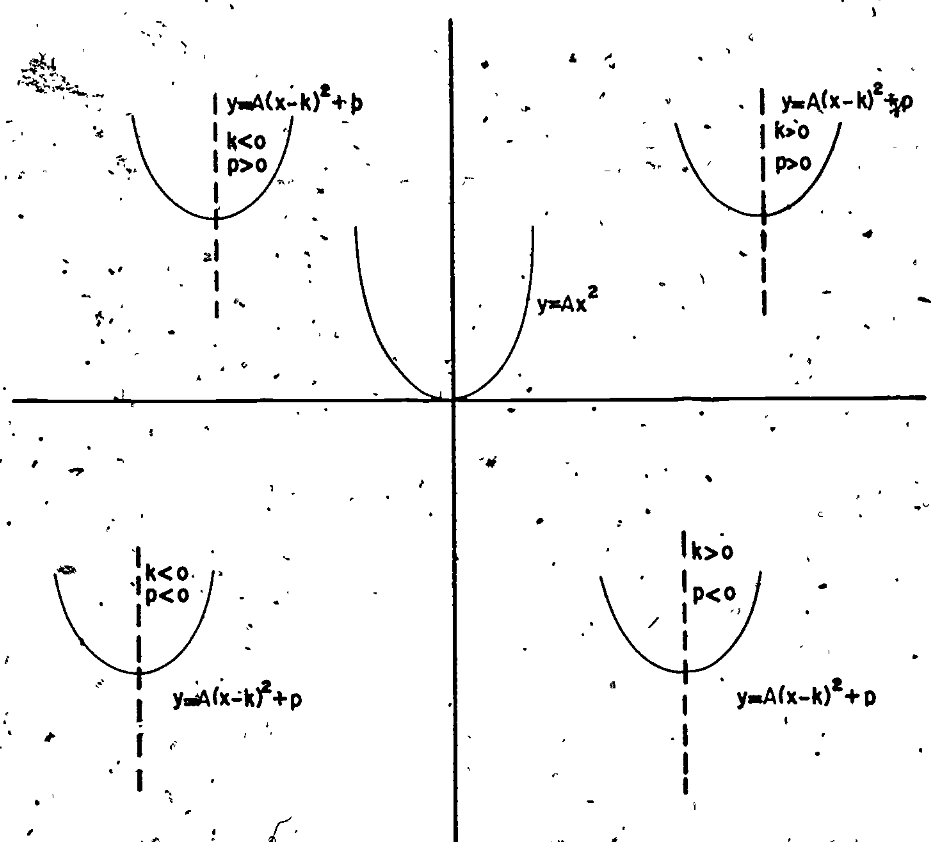


Figure 14

If your class has learned how to solve a quadratic equation by completing the square, the material found in Section 16-2 of SMSG First Course in Algebra will follow the material we have presented thus far.

Exercise 6

1. For each of the following, describe how you can obtain the graph of the first from the graph of the second equation.

   (a) $y = 3(x + 4)^2$; $y = 3x^2$

   You can obtain the graph of $y = 3(x + 4)^2$ by moving the graph of $y = 3x^2$ four units to the left.

   (b) $y = -2(x - 3)^2$; $y = -2x^2$

   You can obtain the graph of $y = -2(x - 3)^2$ by moving the graph of $y = -2x^2$ three units to the right.

(c) $y = -\frac{1}{2}(x + 1)^2$; $y = -\frac{1}{2}x^2$

You can obtain the graph of $y = -\frac{1}{2}(x + 1)^2$ by moving the graph of $y = -\frac{1}{2}x^2$ one unit to the left.

(d) $y = \frac{1}{3}(x + \frac{1}{2})^2$; $y = \frac{1}{3}x^2$

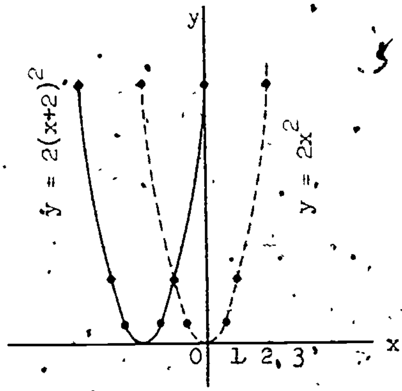
You can obtain the graph of $y = \frac{1}{3}(x + \frac{1}{2})^2$ by moving the graph of $y = \frac{1}{3}x^2$ one-half unit to the left.

2. Set up a table of at least seven ordered pairs of the relation below, and then draw its graph.

$y = 2(x + 2)^2$

x	-4	-3	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	0
y	8	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	8

You can obtain the graph of $y = 2(x + 2)^2$ by moving the graph of $y = 2x^2$ two units to the left.



3. Complete the following table of ordered pairs for the equation

$y = 2x^2 + 8x + 8$.

x	-5	-4	-3	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$
y	18	8	2	2	4.5	8	12.5

4. Draw the graph of the equation in Problem 3 and compare with the graph drawn in Problem 2.

Graphs drawn for Problems 2 and 3 are the same.

5. Compare the location of each of the following graphs (without drawing the graph) with the location it would have it if were in the form of $y = Ax^2$.

(a) $y = 3(x - 2)^2 - 4$

The graph of $y = 3x^2$ is moved two units to the right and four units downward.

(b) $y = -(x + 3)^2 + 1$

The graph of $y = -x^2$ is moved three units to the left and one unit upward.

(c) $y = \frac{1}{2}(x - 2)^2 - 2$

The graph of $y = \frac{1}{2}x^2$ is moved two units to the right and two units downward.

(d) $y = -2(x + 1)^2 + 2$

The graph of $y = -2x^2$ is moved one unit to the left and two units upward.

6. Find equations for the following parabolas.

(a) The graph of $y = x^2$ moved 5 units to the left and 2 units downward.

$$y = (x + 5)^2 - 2$$

(b) The graph of $y = -x^2$ moved 2 units to the left and 3 units upward.

$$y = -(x + 2)^2 + 3$$

(c) The graph of $y = \frac{1}{3}x^2$ moved $\frac{1}{2}$ unit to the right and 1 unit downward.

$$y = \frac{1}{3}(x - \frac{1}{2})^2 - 1$$

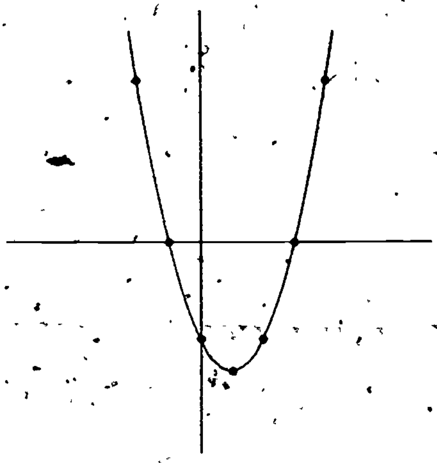
(d) The graph of $y = \frac{1}{2}(x + 7)^2 - 4$ moved 7 units to the right and 4 units upward.

$$y = \frac{1}{2}x^2$$

7. Set up a table of at least 7 ordered pairs for the relation below, and then draw its graph.

$$y = (x - 1)^2 - 4$$

x	y
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5



8. Set up a table of at least 7 ordered pairs for the following relation, and draw its graph.

$$y = x^2 - 2x - 3$$

Compare this graph with that drawn for Problem 7.

Problems 7 and 8 will have the same graph.

x	y
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5

Sample Test Items

TRUE-FALSE

1. All parabolas have a maximum or minimum point.
2. In the equation $y = Ax^2 + C$, if $C = 0$ the vertex of the parabola is at the origin.
3. In the equation $y = Ax^2 + C$, the absolute value of A tells how rapidly the graph of the parabola opens.
4. The (distance,time) relation discussed in the Wick Experiment graphed as a continuous line over the domain of all positive distances.
5. The physical limitations imposed by an experiment must always be imposed on the mathematics developed from the experiment.
6. In order to get a straight line graph from the relation $y = 5x^3 + 4$ you could graph (x^3,y) ordered pairs.
7. The vertex of the graph of the relation $y = A(x - h)^2 + k$ is $(+h,-k)$.
8. The number C in the relation $y = Ax^2 + C$ indicates a vertical translation of the graph of $y = Ax^2$.

MULTIPLE CHOICE

1. The vertex of the graph of $y = 3(x + 2)^2 - 5$ is:
   (a) (3,-5)
   (b) (12,5)
   (c) (2,-5)
   (d) (-2,-5)
   (e) (-4,-5)

2. The equation $y = 3(x + 2)^2 - 5$ is an example of
   (a) a linear equation
   (b) a parabola
   (c) a physical model
   (d) a quadratic equation
   (e) none of these

3. In the graph of $y = Ax^2 + C$, the magnitude of $|A|$ indicates

(a) a maximum or minimum point

(b) a horizontal translation

(c) the rate at which the graph opens

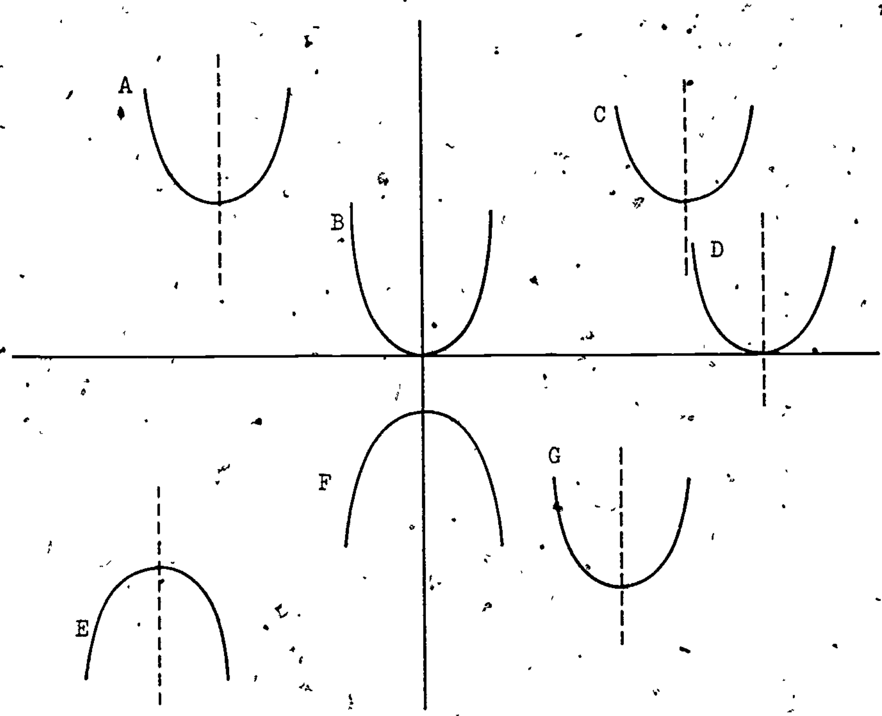
(d) a vertical translation

(e) none of these

MATCHING

The following figure shows the graphs of several equations of the form

$$y = A(x - k)^2 + p .$$



Which of the graphs, if any, satisfy the following conditions? If there is no graph pictured which satisfies a given condition, write the word NONE in the blank.

1. $A > 0$
   $k > 0$
   $p > 0$ ________

2. $A < 0$
   $k = 0$
   $p < 0$ ________

3. $A > 0$
   $k = 0$
   $p = 0$ ________

4. $A > 0$
   $k > 0$
   $p = 0$ ________

5. $A > 0$
   $k < 0$
   $p > 0$ ________

6. $A > 0$
   $k < 0$
   $p = 0$ ________

7. $A < 0$
   $k = 0$
   $p > 0$ ________

8. $A > 0$
   $k > 0$
   $p < 0$ ________

9. $A < 0$
   $k < 0$
   $p > 0$ ________

10. $A < 0$
    $k < 0$
    $p < 0$ ________

PROBLEMS

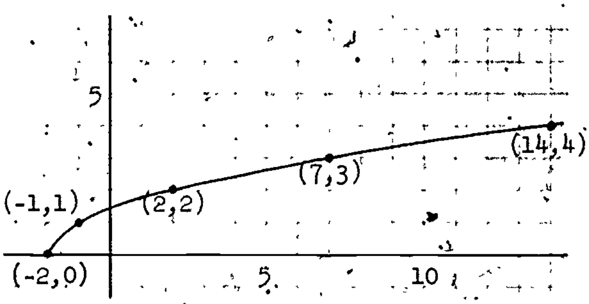
1. Using the set of ordered pairs of the form (x,y)

   $\{(0,4), (2,1), (3,2), (-1,-3), (-2,2)\}$

   (a) form the set of (x^2,y) ordered pairs
   (b) form the set of (x^3,y) ordered pairs
   (c) form the set of (x,y^2) ordered pairs
   (d) form the set of (x,y^3) ordered pairs

2. The graph displays a set of ordered pairs.

   (a) What form of ordered pairs would give a straight line graph?
   (b) Using the points whose coordinates are indicated in the graph, form the ordered pairs of the form which would give a straight line graph.



(c) Sketch the graph of the straight line, using the set of ordered pairs you just formed for part (b).

(d) Write the equation of the straight line which you graphed in part (c).

3. Draw a graph of each of the following relations.

(a) $y = x^2 + 1$

(b) $y = (x - 2)^2$

(c) $y = (x - 3)^2 - 4$

4. Find the coordinates of the vertex of the parabola that would result from the graph of each of the following equations. State whether each vertex is a maximum point, a minimum point, or neither.

(a) $y + 1 = 2x^2 + 1$

(b) $x = (y - 1)^2 + 2$

(c) $y + \frac{2}{3} x^2 = 4$

(d) $(x + 2)^2 - 3 = 3y$

(e) $y = \frac{9}{10} (x^2 + 1)$

(f) $y - x^2 + 2 = 0$

Answers to Sample Test Items

TRUE-FALSE

1. False
2. True
3. True
4. False
5. False
6. True
7. False
8. True

MATCHING

1. C
2. F
3. B
4. D
5. A
6. None
7. None
8. G
9. None
10. E

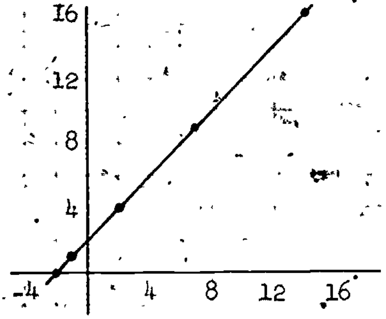
MULTIPLE CHOICE

1. (c)
2. (d)
3. (c)

PROBLEMS

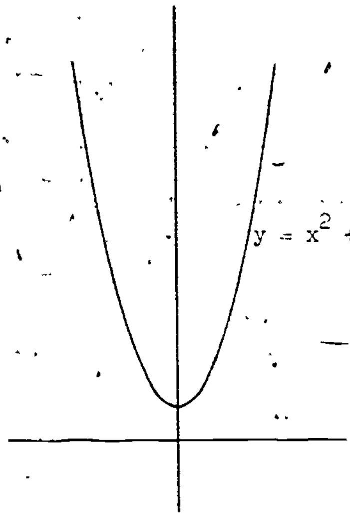
1. (a) {(0,4), (4,1), (9,2), (1,-3), (4,2)}
   (b) {(0,4), (8,1), (27,2), (-1,-3), (-8,2)}
   (c) {(0,16), (2,1), (3,4), (-1,9), (-2,4)}
   (d) {(0,64), (2,1), (3,8), (-1,-27), (-2,8)}

2. (a) (x,y^2)
   (b) (-2,0), (-1,1), (2,4), (7,9), (14,16)
   (c)

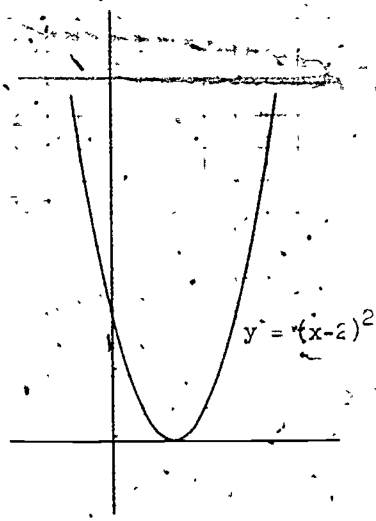


   (d) $y = x + 2$

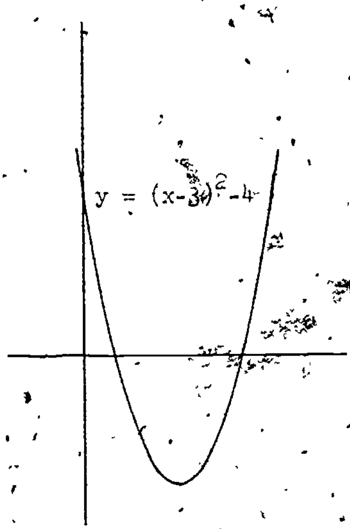
3. (a)



(b)



(c)



4. (a) Vertex: (0,0); Min.
   (b) Vertex: (2,1); Neither
   (c) Vertex: (0,4); Max.
   (d) Vertex: (-2,-1); Min.
   (e) Vertex: $(0,\frac{9}{10})$; Min.
   (f) Vertex: (0,-2); Min.

Chapter 5

ANALYSIS OF NONLINEAR FUNCTIONS

5.1 Introduction

There are three major topics developed in this chapter: the slope of a curve at a point, the hyperbola, and irregular curves. The slope of a curve at a point and its physical interpretation comes naturally from the analysis of a simple experiment. Although full development of the tangent line takes considerable mathematical sophistication, the fundamental idea can be seen quite easily. The student should obtain some insight into the importance of discussing curves by their slope.

It is also felt that when the student has mastered the previous chapters, his interest will be heightened by investigating more complex functions. The last two experiments provide such an introduction. In the first case a curve is studied which reverses the pattern of considering positive power relations. The reciprocal function and the graph of the hyperbola is easily obtained with the Lens Experiment. The properties of the hyperbola lend themselves to a discussion of curve sketching. The Floating Magnet Experiment should be an entertaining experiment to perform. Analysis of its behavior, however, is not simple. It will challenge, and hopefully lead to active discussions by, the students.

5.2 Inclined Plane

In this section the quadratic is developed by examining the motion of a ball rolling down an inclined plane. The method followed in developing the parabola is similar to the oscillating spring. The slope of the curve at a point is introduced and then shown to be equal to the velocity of the ball at that point.

The equipment necessary to perform this experiment is as follows:

1 aluminum rail (angle) 8 feet by $\frac{3}{4}$ inches

1 aluminum rail (angle) 4 feet by $\frac{3}{4}$ inches

1 lb. modeling clay

thread for leveling

billiard ball (although not advisable, a smooth croquet ball may be substituted for the billiard ball)

1 stopwatch or wristwatch with sweep second hand

Since the equipment takes a long running space, the number of groups performing the experiment will be limited. It is possible, however, to use the aisles and ends of the room to provide for a maximum number of participants. It is emphasized again that the involvement of students with the experiment is necessary.

The inclined plane is mounted with modeling clay to the tops of two adjacent desks or tables. It should be supported in at least three places. The clay provides a semi-rigid mounting which will allow minor adjustments to be made. A convenient angle for the rail to make with the horizontal is obtained by raising one end 1.5 inches above the other. The actual angle is not critical, but it should be kept small so the time for an eight-foot roll is about 7 seconds. The straightness of the rail is very important in the performance of the experiment. A thread stretched along the rail provides an accurate method of determining the straightness of the rail. Attach the thread to one end of the rail and pull it tight from the other end. The thread should lie close to one of the rail edges. By manipulating the modeling clay, any sag or hump can be removed from the rail.

The four-foot horizontal section is butted to the low end of the inclined rail. It is also supported by pieces of clay. The discontinuity at the joint should be kept as small as possible. To level the horizontal section, roll the ball along the rail and adjust it with the clay until the ball moves with a constant velocity in either direction. An alternate method would be to use a level. It is important that both rails be free of any nicks on their edges.

Mark the following distances on the rail starting from the lower end: 15, 30, 50, 100, 150, 200 and 240 centimeters. The horizontal rail is marked at 100 centimeters from the bottom of the incline.

The ball is released at the various marks and the time interval to the bottom is measured. Three trials are made for each length, and the times averaged. The measurements on the horizontal rail will not be used immediately, but it is very difficult to set the equipment up exactly the same way another day. For each of the above distances up the plane make measurement of the time taken for the ball to roll the hundred centimeters along the horizontal rail. This data is to be compared with the data on the incline so it is important that it be taken at the same time. The experimental data for a trial run is presented in Table 1.

Distance cm	Trial 1 Time Sec	Trial 2 Time Sec	Trial 3 Time Sec	Average Time Sec	Average Time Squared Sec^2	Time to Travel 100 cm Sec	Velocity cm/sec
15				1.9	3.6	5.8	17
30				2.6	6.8	4.7	21
50				3.4	11.6	3.6	28
100				4.6	21	2.6	38
150				5.7	32.5	2.2	46
200				6.7	45	1.9	53
240				7.3	53	1.6	62

Table 1

5.3 Analysis of the Experiment

The analysis of the data follows the procedure outlined in the oscillating spring. Distances on the plane are selected and, therefore, are elements of the domain. In this case, direct analysis will not yield a linear graph, so the converse set of ordered pairs is plotted. (See Figure 1.) The linear graph formed from (t^2, d) relation follows immediately in Figure 2. All distances are measured from the bottom of the inclined plane and, hence, the graph of the relation and its associated straight line pass through the origin. At time zero the distance travelled by the ball is zero. The origin should be used to select the "best" straight line. Since the parabola passes through the origin, the equation the student will obtain is of the form

$$d = \text{slope}\,(t^2)$$

$$d = At^2$$

From the straight line graph in Figure 2 the slope is $\frac{130}{28} \cong 4.6$. The mathematical model becomes

$$d = 4.6\,t^2 .$$

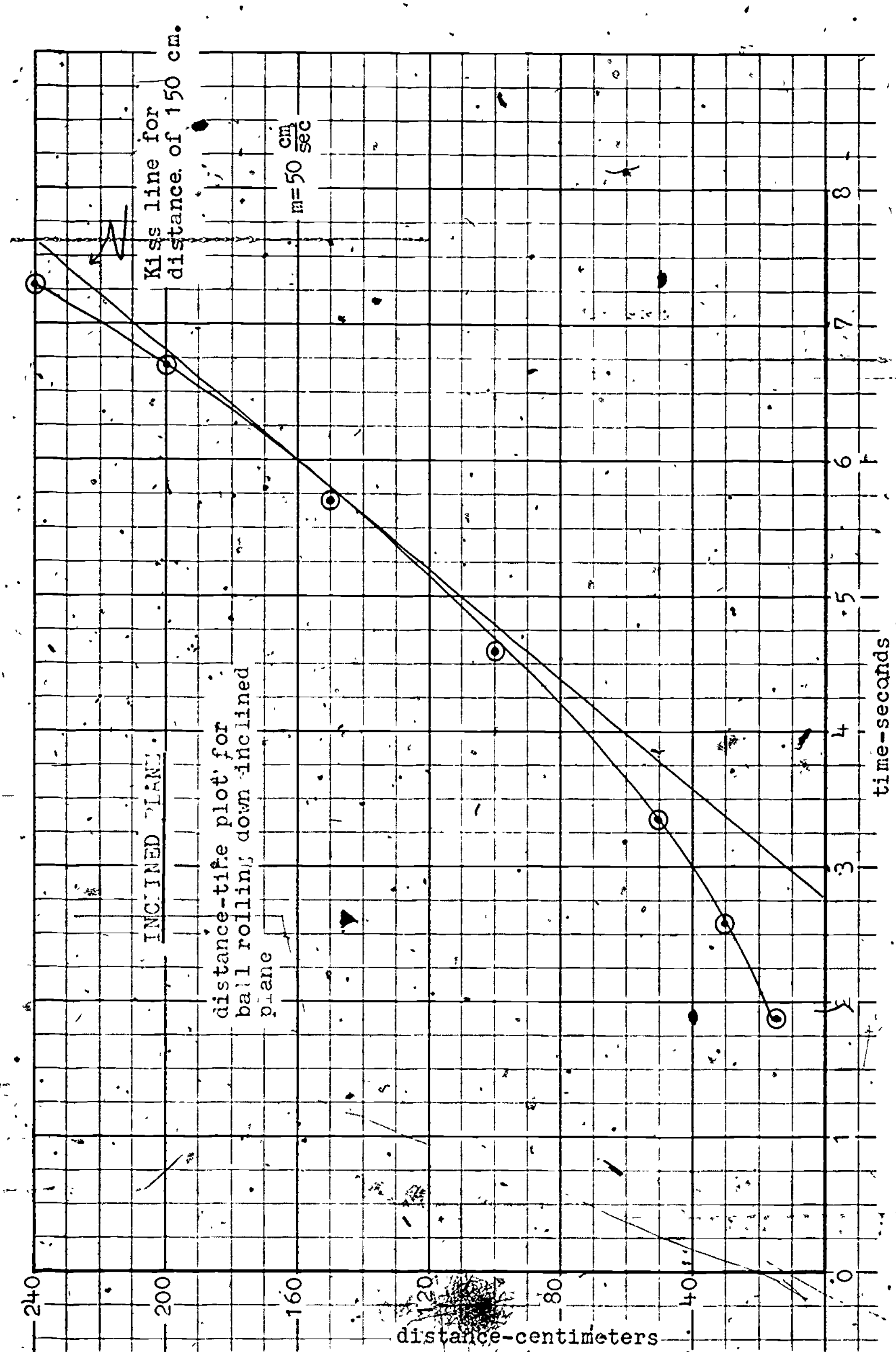

Kiss line for distance of 150 cm.
m=50 cm/sec
INCLINED PLANE
distance-time plot for ball rolling down inclined plane
time-seconds
distance-centimeters
240
200
160
120
80
40
0
1
2
3
4
5
6
7
8

Figure 1

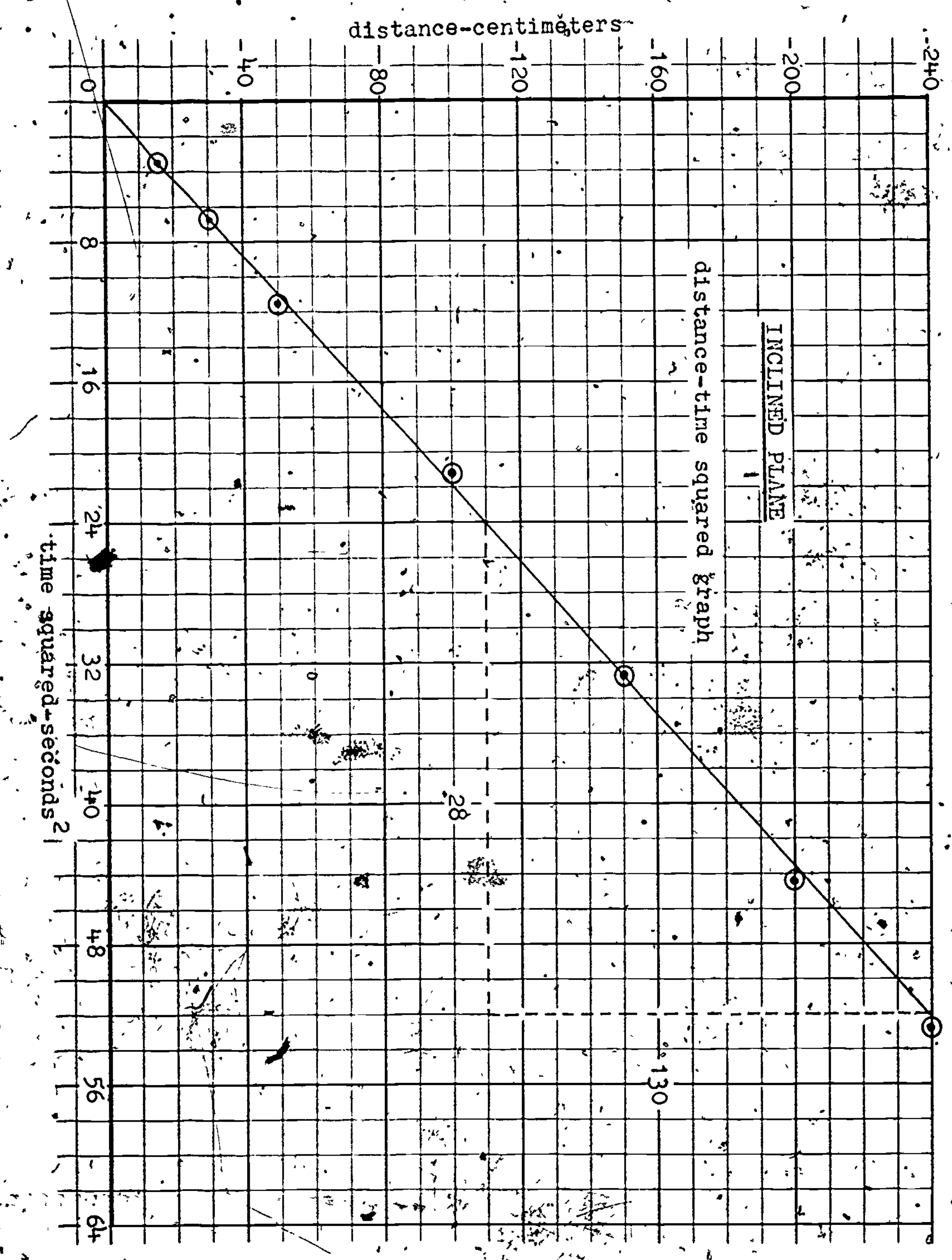

INCLINED PLANE
distance-time squared graph
distance-centimeters
0
40
80
120
160
200
240
time squared-seconds^2
8
16
24
32
40
48
56
64
28
130

Figure 2

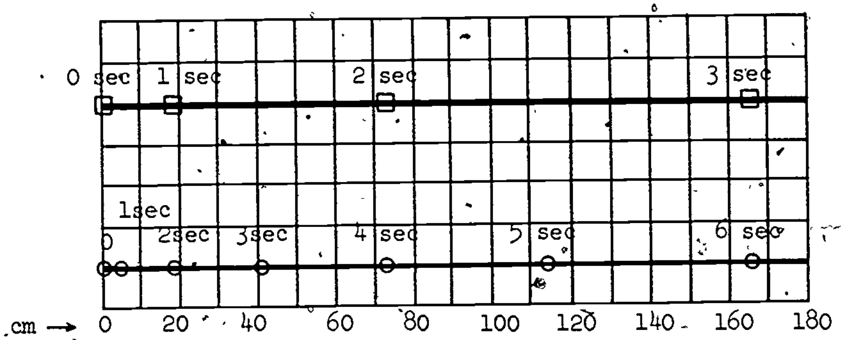
Exercise 1

1. Use the equation $d = At^2$. With your measured value of the coefficient A, calculate distance values that correspond to times of: 0, 1, 2, 3, 4, 5, 6, 7 seconds.

   distances computed from $d = 4.6\ t^2$:
   0, 4.6, 18.4, 41.4, 73.6, 115, 166, 205 centimeters

2. Draw a vertical line to represent the inclined plane. Starting at the top, mark to scale on a piece of graph paper the calculated positions of the ball along the inclined plane. Label these positions with the corresponding times.

   graphs for problems 2 and 4



3. On the drawing of the inclined plane in the exercise above, very carefully mark the position you think the ball will occupy at a time of 2.5 seconds. Using the equation, now calculate the position of the ball at this time. Compare this point with your estimated position. If the student's guess is very far off, try another point. The distance for 2.5 sec is 28.8 cm.

4. Multiplying your value of "A" by four will form a new equation. With this equation calculate distance values for times of 0, 1, 2, 3 seconds.

   The following calculations are made using the equation $d = 18.4\ t^2$: $d = 0$, 18.4, 73.6, 166 centimeters. The plot is represented by open squares. See answer to Problem 2.

5.4 Slope of a Curve at a Point

The concept of the slope of a curve at a point is an extremely important one. The primary purpose for performing the Inclined Plane Experiment has been to develop this concept. For this reason, some steps were deleted in the development of the parabola.

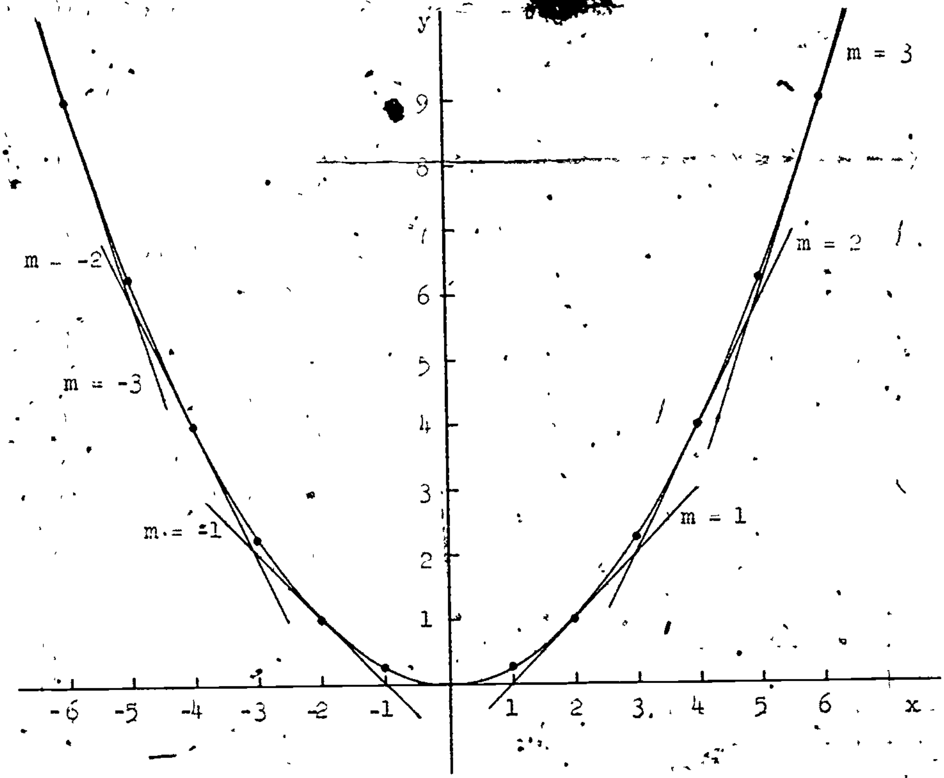
It is important to stress that the slope of a curve at a point is defined as the slope of the line which kisses the curve at that one point. The kiss line is the tangent line to the curve. With this definition in mind, the student can measure the slope of a kiss-line directly from the graph. In the present instance we have a time-distance relation. The slope of a kiss line on this graph is therefore a distance divided by a time. The physical unit of the slope is then the same as the physical unit of velocity, but this does not prove that the slope is a velocity. It is only reasonable to suppose that the slope might be interpreted as a velocity.

It is important to realize that the present work constitutes an opportunity to present a concept that many students will eventually encounter in calculus. With the present treatment, a student should find the tangent to a curve as developed in calculus much more easily handled.

Exercise 2

1. Carefully draw a graph of the parabola $y = \frac{1}{4} x^2$, using integrally-spaced values of x from -6 to +6 inclusive. Graphically find the slope of the parabola at the points for which x equals 6, 4, 2, 0, -2, -4, -6.

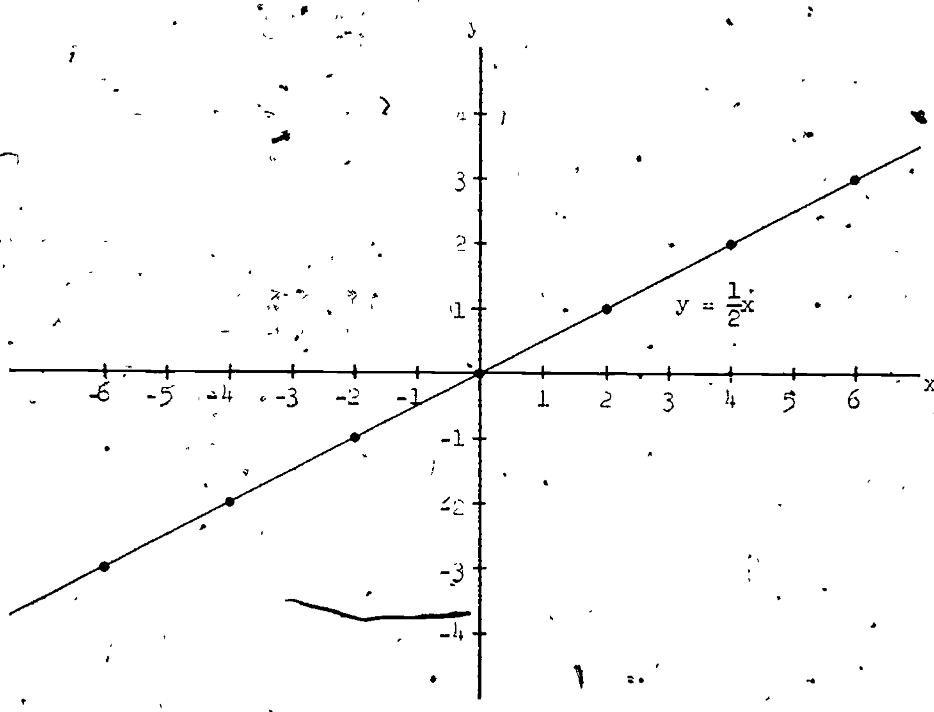
   The slope of the parabola $y = \frac{1}{4} x^2$ at the points for which x equals 6, 4, 2, 0, -2, -4 and -6 are, in order: 3, 2, 1, 0, -1, -2, -3. The graph that follows shows the constructions.



Exercise $y = \frac{1}{4}x^2$

2. The straight line is characterized by a constant slope whereas the quadratic has a continuously changing slope. It is possible to find the slope for many points on the curve, and hence, generate a new function which would consist of ordered pairs composed of slope and the elements from the domain.

From the slopes found in Problem 1, form a set of ordered pairs (x,slope). On a sheet of graph paper, draw coordinate axes and plot this set of ordered pairs. What conclusions can you draw about this new function? The ordered pairs following the sequence of Problem 1 are: (6,3), (4,2), (2,1), (0,0), (-2,-1), (-4,-2), (-6,-3). The graph is linear.



3. Compare the slope of the curve in Problem 2 with the coefficient of x^2 in Problem 1.

   The slope of the line is twice the coefficient of x^2. (This comes close to derivatives and the interrelation of curves.)

5.5 Experimental Measurement of the Slope

The experimental measurement of the velocity of the ball rolling down the inclined plane is an extremely important aspect of the experiment. The velocity can be directly determined by providing a 4-foot section of rail mounted horizontally and positioned at the end of the incline. Some care should be taken to place this section of rail horizontally. It should also be placed against the end of the incline so that the ball will move as smoothly as possible. The velocity of the ball will be approximately constant (disregarding friction) as it moves along this horizontal section. The horizontal velocity will depend upon the distance up the incline to the point from which the ball was released.

In the experiment, it is recommended that the ball be released 150 cm up the incline from its lower end. A distance of 100 cm is marked from the lower end of the incline along the horizontal section of rail. It is, of course, extremely important that the incline be set at precisely the same angle that was used in obtaining the original time-distance data. Otherwise, the measured velocity for the same release point would not be the same as it was before.

For a release point 150 cm up the incline, the ball was found to require 2.2 seconds to traverse the 100-cm distance along the horizontal section. The measured velocity is thus 46 cm/sec. This is to be compared with the slope of the time-distance relation taken at the point for which the distance is 150 cm. The kiss line for this point is shown on the graph of this relation. The measured slope is 50 cm/sec. Allowing for some experimental error, we can conclude that the two values are the same. It is important that the student measure the slope first, so that he has made a commitment concerning the velocity. He then later compares this with the directly measured value of the velocity.

This result is very important. We know that the directly observed velocity has the same value as the slope of the kiss line. Since the slope of the kiss line was defined to be the same as the slope of the curve at a point, the velocity of the ball is therefore the slope of the curve at a point.

5.6 The Simple Lens

This experiment will introduce a reciprocal function and the graph of a hyperbola. The first part of the experiment consists of a discussion of the focal length of a lens and how to find the location of focal points. The lens you obtain may have a focal length stamped on its container. However, you should not omit this preliminary material and the student should use the experimental value of the focal length obtained in the classroom.

The following equipment is necessary for each group performing the experiment:

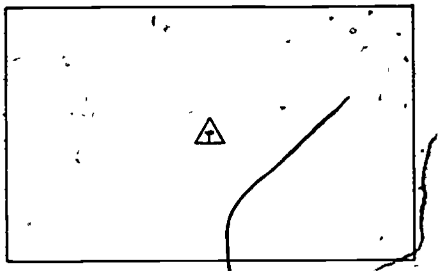
1 positive lens, focal length 8 inches or less
1 meter stick
1 flashlight or light source
1 object screen
1 2-meter strip of paper

1 focus or image screen

To obtain the focal length of the lens, mount the lens on the meter stick with modeling clay. If the room is darkened and the meter stick is pointed through an open window at some distant object, you can obtain a "picture" on a white card used as a screen. You probably will find that drawing the window shade part-way will cut out some of the stray light reaching the screen and hence give a better picture. Make at least three determinations of the distance from the lens to the card when the image is in focus. Most students at this level have not used a lens for this purpose and will find this procedure most interesting. Let a few students make determinations of the focal length. Average the values and use the average value throughout the experiment.

Use a piece of cardboard as an "object screen". Cut a small triangular hole with sides one inch, at the center of the cardboard. Insert a pin in the base of the triangle. (See Figure 3.)

A flashlight provides a very convenient light source, but any type of lamp will serve. The screen will prevent stray light from reaching the image screen.



Object Screen

Figure 3

Fasten the 2-meter tape to the floor. Place the lens halfway along the tape. Use modeling clay as a base and try to arrange the lens so that its center is at the same height from the floor as the head of the pin which is being used as the ob[illegible]

On both sides of the lens make a mark on the tape corresponding to the focal points. The focal points are located a distance from the lens equal to the focal length. Be careful not to move the lens once the focal points have been marked. All object and image distances will be measured from these points. Use the meter stick and mark each centimeter from the focal points to both ends of the tape.

At this point the student should examine some of the properties of the lens. Place the object about one focal length from the focal point. The image will be found at about the same distance from the other focal point. The class should observe as the object moves closer to the lens the image screen must move away from the lens to retain a sharp focus. Reverse the procedure and show the opposite effect. As the object moves away from the lens, the image moves closer. Place the object closer to the lens than the focal point. Have someone in the class try to find an image position. Do not spend more than a few minutes on the procedure but try to show that an image cannot be found on the screen when the object is inside the focal point. Emphasize the fact that for large object distances the image will not be closer to the lens than the focal point. It is a fact that we cannot place the object within the focal point and obtain an image on the screen. No matter what object distance we pick, the image screen will not be moved closer to the lens than the focal point. These are our reasons for making all measurements from the focal points.

To perform the experiment, place the object on one of the centimeter marks about three focal lengths from the focal point. Move the screen until you have a sharp image. Record object and image distance <u>from the focal points</u>. Move the object until the image is out of focus. Move the object at least two centimeters each time. Move the screen to regain focus and record the distances. If you continue in this manner until the image moves off the tape, you should have at least 15 data points.

The data and the graph for an actual experiment are reproduced in Table 2. A lens with a focal length of 21.4 cm was used. The initial distance of the object from the focal point was 65 cm and readings were taken until the object was 7 cm from the focal point. This gave 27 ordered pairs to plot. A graph of the set of ordered pairs appears in Figure 4.

The Simple Lens

Focal Length 21.4 cm

Object Distance d (Centimeters)	Image Distance d' (Centimeters)	$\frac{1}{d}$
65	8.3	.015
61	8.6	
58	9.0	
54	9.4	.018
52	9.6	
50	9.7	.020
48	9.8	
45	10.4	.022
43	11.1	
41	11.6	
39	12.1	
37	13.1	
35	13.7	.028
33	15.0	.030
31	15.8	
29	16.6	.031
27	18.2	
25	19.6	.040
23	21.1	
21	23.1	.048
19	26.2	
17	29.1	
15	33.8	.066
13	39.2	
11	46.6	
9	55.0	.111
7	69.6	

Table 2

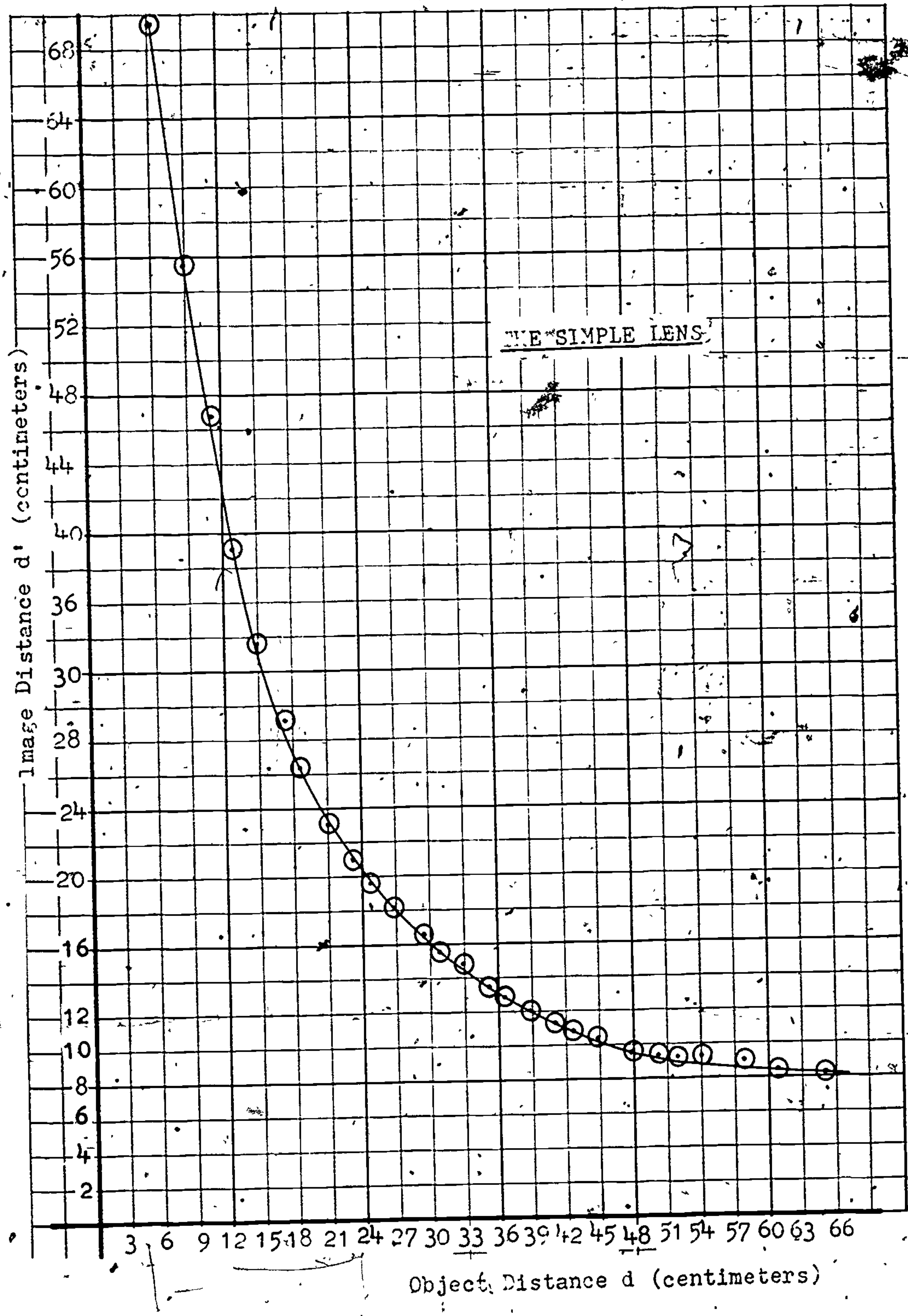

THE SIMPLE LENS
Image Distance d' (centimeters)
68
64
60
56
52
48
44
40
36
30
28
24
20
16
12
10
8
6
4
2
3 6 9 12 15 18 21 24 27 30 33 36 39 42 45 48 51 54 57 60 63 66
Object Distance d (centimeters)

Figure 4

Exercise 3

1. In the lens experiment what is the domain and what is the range?

   The domain is the set of all object distances.

   The range is the set of all image distances.

2. Does the graph of the relation (Figure 8) represent a function? Why?

   Yes, for every image position there exists only one object distance.

3. Would it be meaningful to pass a smooth curve through the plotted points? Why?

   Yes. For each intermediate position of the lens there will be a point at which there is an image.

4. Discuss the possibility of extending the graph of the curve to very large or very small object distances.

   For very large object distances the image position changes very little for changes in object position. It is difficult to accurately locate image positions. Image positions for objects close to the focal point are highly sensitive to the object position.

5.7 The Lens Relation

It must be understood that the experimental curve of the lens relation shown in Figure 4 cannot be represented by either a linear or quadratic relation. Certainly no movement or rotation in any way can result in a linear relation. Therefore, linear functions are easily ruled out. The student may find it more difficult to rule out quadratics, but the curve which results from this experiment cannot be a quadratic. Two points should make this obvious to the student. First, in relation to the axes, the curve does not behave as a quadratic. For large values of the domain, the values of the range are not large. In fact, the values of the range are very small. Also, for large values of the range the corresponding domain values are very small. Secondly, the curve cannot be a parabola even if it is rotated. The tangents of a parabola tend to become parallel to each other as we move out on each branch of the curve. The tangents of this curve tend to be at right angles to each other as we move away from the origin. The rejection of the quadratic relation will be more fully discussed in the Floating Magnet Experiment.

The lens relation requires a new approach in obtaining a linear relation. Previously other devices were used to develop the line. In this case it is necessary to use a reciprocal relation in order to determine the set of ordered pairs requisite for the linear relation.

The student's knowledge of the reciprocals of numbers should be sufficient for his use in this chapter. The ordered pairs $(\frac{1}{X}, X')$ will provide the necessary data.

When the $(\frac{1}{X}, X')$ relation is plotted, it is argued that the line should pass through the origin. The student should be made to understand that the origin is not a data point, but that the line approaches the origin as a limit by the kind of mathematical reasoning that has been employed. When X is very large $\frac{1}{X}$ is small. For these large values of X, X' is shown to be very small in Figure 4. Therefore, the extended line comes very close to the origin, as shown in Figure 5.

In forming the $(\frac{1}{X}, X')$ relation it is not necessary to calculate the reciprocals of all the object distances. Eleven points were used in the analysis of the data recorded in column 3 of Table 2. At least ten points should be used. If the points are selected starting with large values of object distances, there is no crowding of points at the lower end.

The graph which compares the experimentally obtained curve with the curve calculated from the student's equation should be one of great satisfaction for the student. The good match here indicates that the data for the object position-image position relation for the lens has been reasonably accurate. The beauty and symmetry of natural laws are another feature that may be stressed, together with the power and simplicity of the mathematical description.

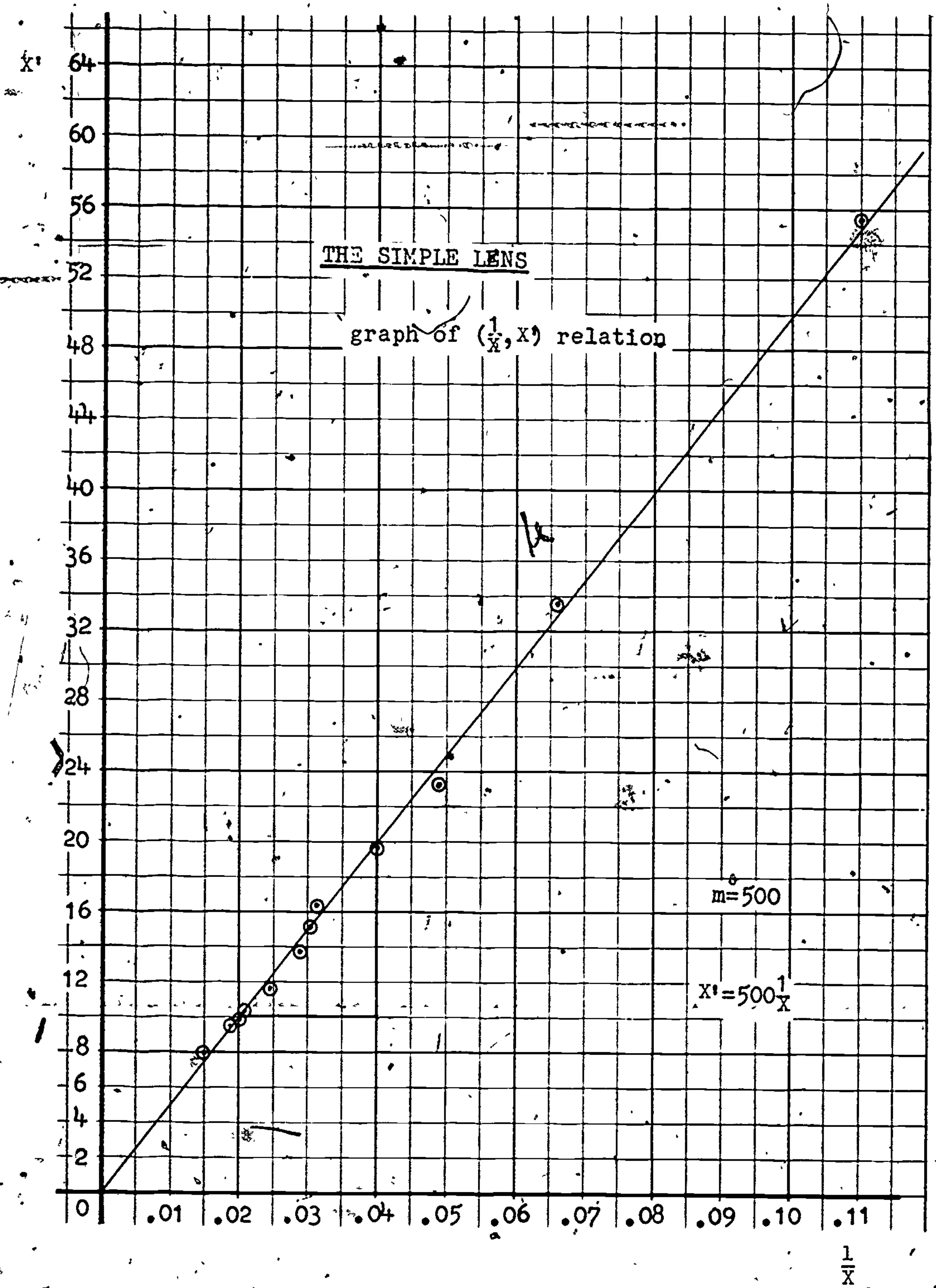
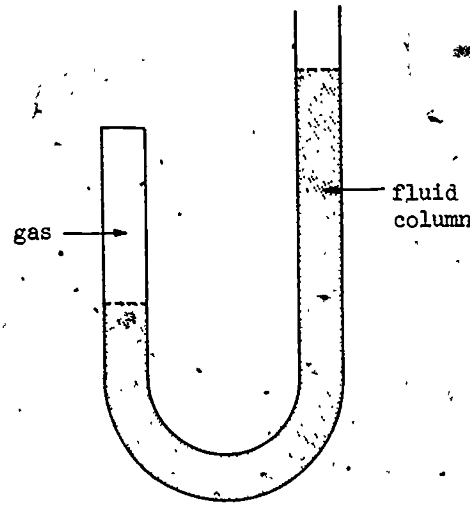

THE SIMPLE LENS
graph of ($\frac{1}{X}$, X') relation
X'
64
60
56
52
48
44
40
36
32
28
24
20
16
12
8
6
4
2
0
.01
.02
.03
.04
.05
.06
.07
.08
.09
.10
.11
$\frac{1}{X}$
m=500
X' = 500 $\frac{1}{X}$

Figure 5

Exercise 4

1. The following table contains data taken from an experiment with gases.

Pressure $\frac{lb}{in^2}$	Volume cm^3
4	169
5	135
10	68
12	56
15	45
18	38
20	34
25	25
30	23
35	19

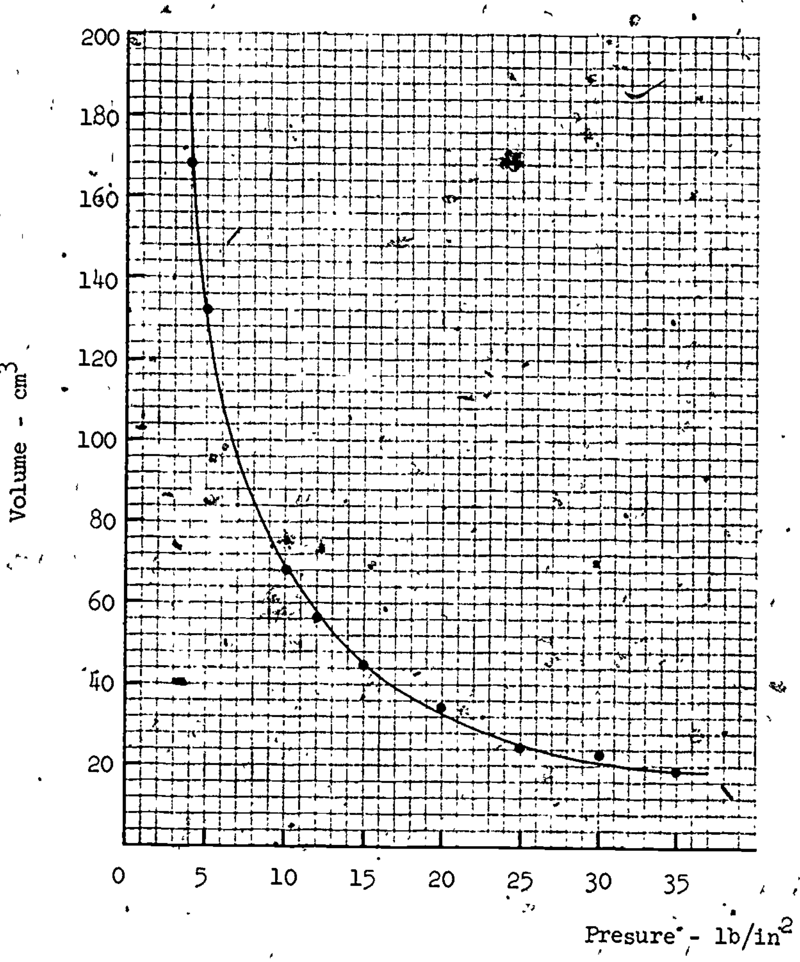


By raising and lowering the fluid column, different pressures can be exerted on the gas contained in the left portion of the table. As the fluid column is raised, the pressure is increased and the gas volume decreases.

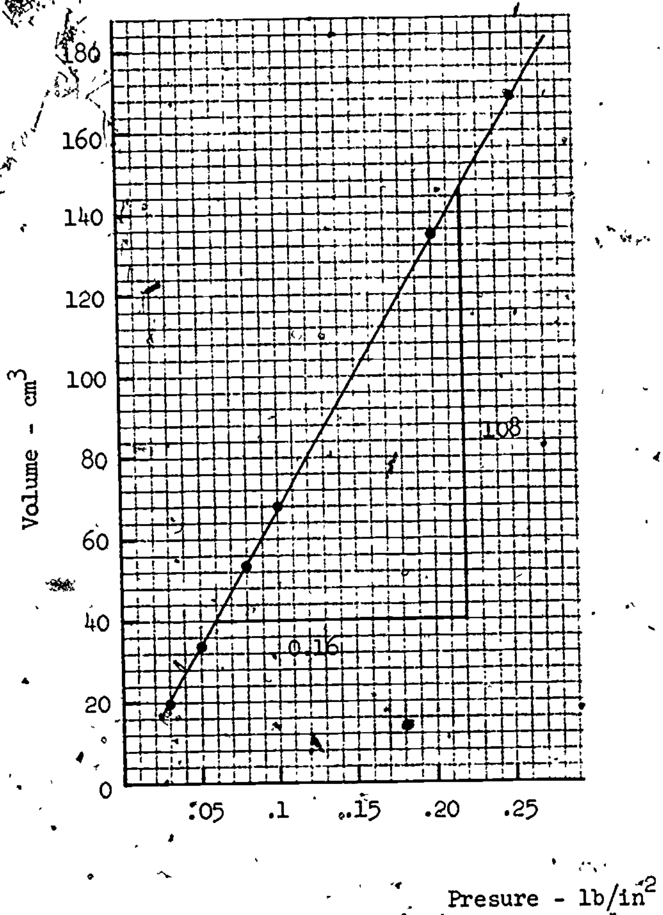
(a) Which elements of the table are the domain and which are the range?

Since the pressures are selected, they are the domain.
The volume elements are the range.

(b) On a coordinate plane, plot the ordered pairs from the table and construct a physical model.



(c) Form a new relation $(\frac{1}{p},V)$ and plot these new ordered pairs. The ordered pairs taken in the same sequence as the table are: (0.25,169), (0.20,135), (0.10,68), (0.083,56), (0.967,45), (0.056,38), (0.050,34), (0.040,25), (0.033,23), (0.029,19) .



(d) Using this information, find the mathematical model which best represents the data.

From the above graph the slope is $\frac{108}{.16} = 675$.

Using this value, the mathematical model becomes

$$V = 675 \cdot \frac{1}{P} \quad \text{or} \quad PV = 675$$

5.8 The Reciprocal Function

The preceding section developed the reciprocal relation for positive values of object distances. The present discussion logically extends the domain to negative values. It is true that negative object distance and negative image distance do have physical significance. However, this opens up the subject of virtual images and diverging light rays which will lead the student too far afield. The more direct approach of the mathematical extension to negative values is used.

Since the domain is extended to include all real numbers except zero, a number of points should be brought to the student's attention. Zero is excluded as an element of the domain because a number divided by zero is not defined over the real numbers. This exclusion of zero divides the relation into two parts. One portion lies in the first quadrant and the other in the third quadrant. Also, the curve exhibits a high degree of symmetry. Exchanging the domain and range does not change the graph. Therefore $y = x$ is a line of symmetry. Substituting $-x$ for x and $-y$ for y does not change the graph and therefore the curve is symmetric about the origin. Finally, the reciprocal relation is a function. Each element of the range corresponds to only one element in the domain.

In the text the phrases "direct variation" or "inverse variation" are not used. These terms have been avoided because of their wide abuse. Instead the terms linear function or reciprocal function are used since they denote the form of the equation. The material in this section has touched on only a few properties of the reciprocal function. In particular, it is hoped that the student will begin to see the symmetry that may arise in the graphs of functions.

Exercise 5

1. Does the range of the function $X' = \frac{f^2}{X}$ include the value $X' = 0$? Explain.

   The range of the function $X' = \frac{f^2}{X}$ does not include the value of $X' = 0$. No matter how large X becomes, the value of X' is always nonzero.

2. Does the simple lens equation $X' = \frac{f^2}{X}$, with the range and domain restricted to the values that can be obtained experimentally, represent a function if X and X' are interchanged? Why?

The simple lens relation $X' = \frac{f^2}{X}$ represents a function over the domain and range of the experimental relation, even if X and X' are interchanged. This property defines a 1:1 function.

3. The focal length of the lens found in many cameras is 50 cm. Calculate X' in centimeters for an object at a distance X of 1 meter; 10 meters; 1.5×10^8 meters (the distance to the moon); and 5.8×10^{10} meters (the distance to the sun). $X' = \frac{2500}{X(\text{cm})}$

The X' values are, in order: 25 cm, 2.5 cm, 1.7×10^{-7} cm, 4.3×10^{-10} cm.

4. In each of the relations I through V:

(a) For what value of x will the denominator become zero?

(b) Is it possible for x to be equal to zero?

(c) Find the value of y which corresponds to the following values of x: {-8, -3, 0, 1, 3, 4, 7}.

(d) Using the values just found, form ordered pairs of the form (x,y) and plot on the coordinate plane.

(e) Join the points with a smooth curve. Remember that there will be one number (part a) which is not in the domain of the relation.

(f) Is there any number which is not in the range of this relation? If so, what is it?

I. $y = \frac{10}{x - 2}$

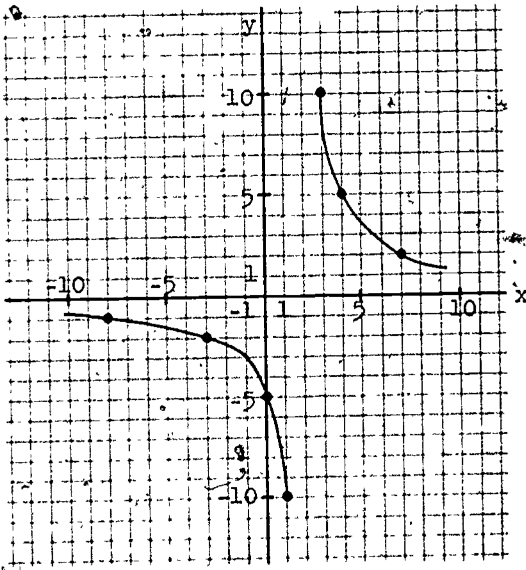
(a) x = 2

(b) yes

(c) {-1, -2, -5, -10, 10, 5, 2}

(d), (e) see graph

(f) yes. y = 0



II. $y - 5 = \frac{10}{x - 2}$

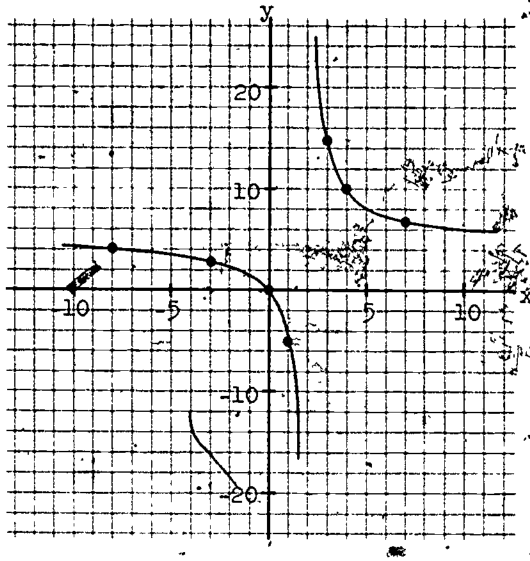
(a) x = 2

(b) yes

(c) {4, 3, 0, -5, 15, 10, 7}

(d), (e) see graph

(f) yes. y = 5



III. $y = \frac{-10}{x - 2}$

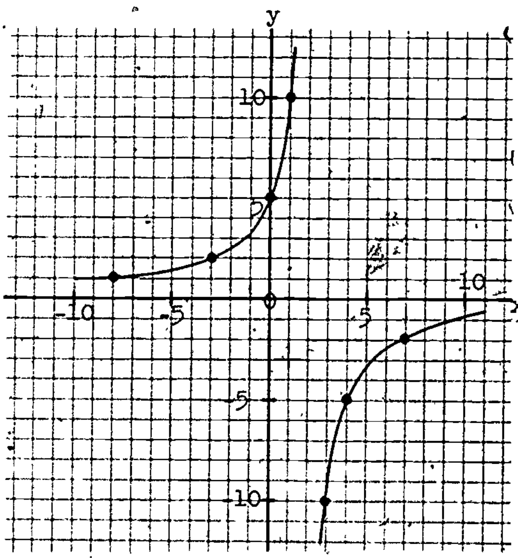
(a) x = 2

(b) yes

(c) {1, 2, 5, 10, -10, -5, -2}

(d), (e) see graph

(f) yes. y = 0



IV. $y - 5 = \frac{10}{2 - x}$

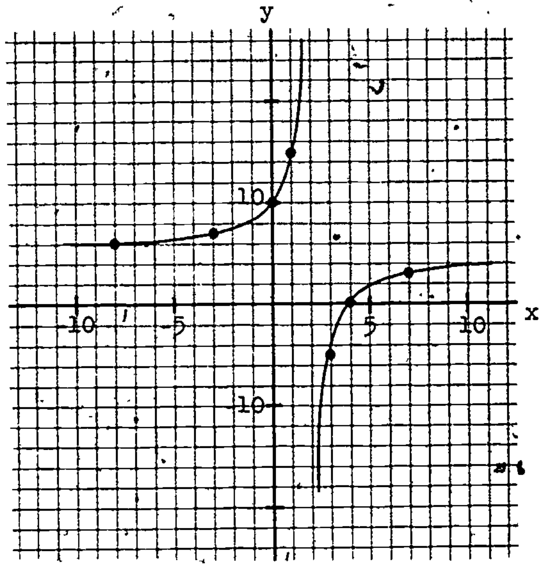
(a) $x = 2$

(b) yes

(c) {6, 7, 10, 15, -5, 0, 3}

(d), (e) see graph

(f) yes. $y = 5$



V. $y = \frac{10}{|x - 2|}$

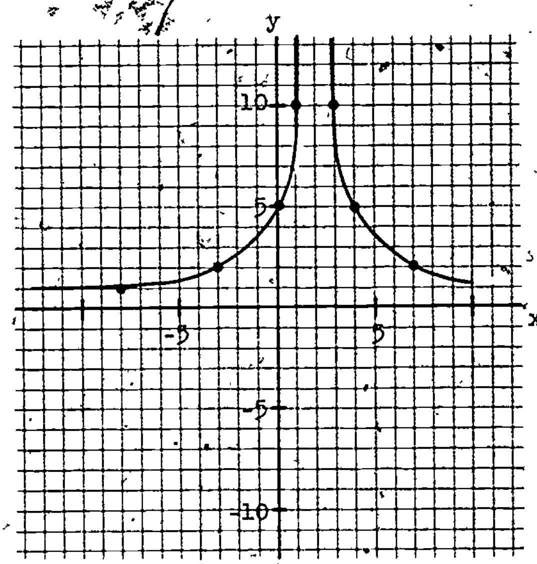
(a) $x = 2$

(b) yes

(c) {1, 2, 5, 10, 10, 5, 2}

(d), (e) see graph

(f) yes. $y = 0$



5.9 Translation of Axes

The translation of linear functions was introduced in Chapter 3. The method for translating the axes of the coordinate plane both vertically and horizontally was developed. This procedure was extended to parabolas in Chapter 4. In both cases the development included not only translation of the axes but also translation of the curve. Since it is physically significant, translation along both horizontal and vertical axes is performed on the reciprocal function.

A logical question arises as to the point from which measurements are taken in the lens experiment. Taking measurements from the focal point may seem somewhat artificial. The lens is probably a more logical starting point. Therefore the translation of axes by an amount f is introduced.

Since the translation is somewhat complex the student can analyze what is being accomplished by using the frosted acetate or onion-skin paper. Moving the acetate by the amount f in each direction will give a pictorial representation of the translation. A shift to the right a distance f for the object focal point puts it coincident with the lens. Likewise, a similar shift of f for the focal point on the image side puts it coincident with the lens. These two translations change the form of the equation and establish the lens as the starting point.

Exercise 6

1. Start with the equation $\frac{1}{X} + \frac{1}{X} = \frac{1}{f}$ whose significance is described in the text. Algebraically translate the axes to the right and upward by the amount f in each direction. Hint: Form the equation $\frac{1}{X + f} = \frac{1}{f} - \frac{1}{X' + f}$ and simplify.

   The translations in this problem are obtained by adding f to both object and image distances.

$$\frac{1}{X + f} + \frac{1}{X' + f} = \frac{1}{f}$$

$$\frac{1}{X + f} = \frac{1}{f} - \frac{1}{X' + f}$$

$$\frac{1}{X + f} = \frac{X'}{f(X' + f)}$$

$$f(X' + f) = X'(X + f)$$

$$f^2 = XX'$$

2. Algebraically solve the equation $\frac{1}{X} + \frac{1}{X'} = \frac{1}{f}$ for X'.

$$\frac{1}{X'} = \frac{1}{f} - \frac{1}{X} = \frac{X - f}{Xf}$$

$$X' = \frac{X'f}{X - f}$$

3. $y = \frac{3x}{x - 3}$ is a hyperbola in the form found in Problem 2. By how much and in what directions would one have to translate the axes to put it in the form $y = \frac{k}{x}$.

If one were to translate the axes to the right 3 units and upward 3 units the resulting equation could be of the correct form.

$$y + 3 = \frac{3(x + 3)}{(x - 3) + 3}$$

$$= \frac{3x + 3 \cdot 3}{x}$$

$$= 3 + \frac{3 \cdot 3}{x}$$

$$y = \frac{3 \cdot 3}{x}$$

4. Translate the axes used to describe the parabola $y = x^2 - 4x + 4$ so that the vertex of the parabola lies at the origin. By what amounts and in what directions did you translate the axes?

The parabola $y = x^2 - 4x + 4$ can be rewritten as

$$y = (x - 2)^2.$$

A horizontal translation of 2 units to the right gives

$$y = (x + 2 - 2)^2$$

or, $\quad y = x^2$,

a parabola whose vertex is at the origin.

5.10 Curve Sketching

This section introduces an important topic that is often omitted in a traditional treatment of algebra. The student should learn curve sketching at an early stage in his mathematical training. Although the topic is dealt with briefly here, many opportunities will arise in the future in which the student may practice curve sketching. This he should be encouraged to do.

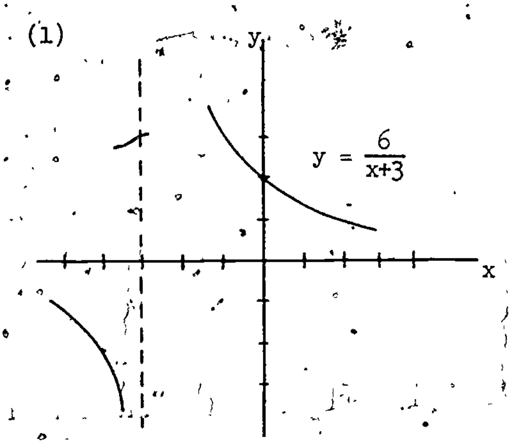
Mathematical insights are developed in this way that must otherwise be developed by the student on his own.

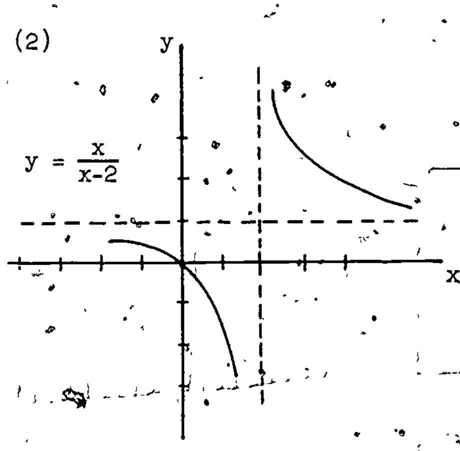
The seven steps discussed in curve sketching forms a continuous procedure for analysis of the function. The division into two parts in the text is for convenience in illustrating individual portions of the curve. Also the student will not be burdened with too many new ideas at once. After a little practice he should see the similarity in the operations performed in the two sections. Steps one and seven consider the equation for large positive and negative values of the elements of the domain. Steps two and six analyze the equation for small positive and negative values of the elements in the domain. Four discusses an intermediate behavior of the equation, and three and five the singularities. Here singularities mean any points of peculiar behavior of the equation. It is important that the student understand the needs and requirements of each of the seven steps in the analysis.

Exercise 7

Sketch the following relations for all possible values of x:

1. $y = \frac{6}{x + 3}$
2. $y = \frac{x}{x - 2}$
3. $y = x(x - 2)$
4. $x = y^2 - 2y + 1$
5. $y = 2(x + 1)$





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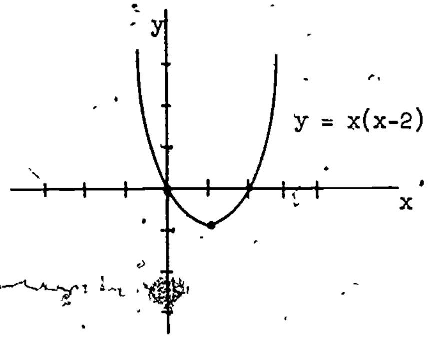

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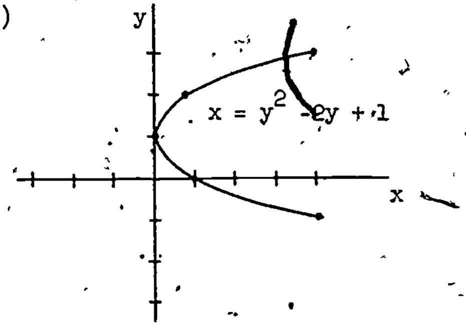
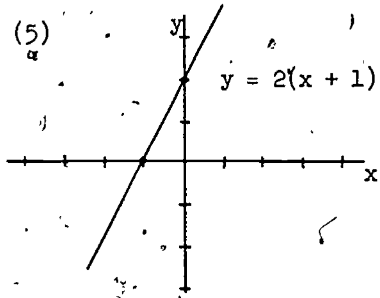


(5)



5.11 The Floating Magnet

In this section the student will encounter further work with reciprocal functions. In addition, however, a new and important aspect of mathematical relations is developed. The student will learn that it is not always possible to find a mathematical relation that will accurately represent the experimental results over the entire experimental domain and range. The complex behavior of some physical systems brings out the idea of "curve fitting".

It is found that a reciprocal relation describes the experimental results over a portion of the experimental curve, but fails to describe it elsewhere. This is analagous to the breakdown of a linear relation for describing a stretched spring or bent beam. When the spring is stretched too far, or the beam bent too much, the relation between load and stretching or bending ceases to be a linear one.

The Floating Magnet Experiment

The experiment with the "floating" magnets should be a most interesting one for the students to perform. This will provide good motivation for the mathematics that is developed. The equipment list which follows is needed for each group performing the experiment.

4 circular magnets each $\frac{1}{8}$ -inch thick (or, 2 circular magnets each $\frac{1}{4}$ -inch thick)

1 aluminum or wooden knitting needle (of a size to easily slip through holes in magnets)

1 two-tube set of epoxy glue

1 mounting board (with holes through which needles will slip easily)

1 kit of hooked weights, each kit to include:

- 1 ten-gram weight
- 2 twenty-gram weights
- 1 fifty-gram weight
- 1 one-hundred gram weight

1 centimeter rule

A supply of four-lines-to-the-inch quadrille coordinate paper

The caps of each knitting needle are left in place, but each needle is cut to a length of 3 to 4 inches. For the aluminum needles, one end of a stretched-out paper clip is glued a short way inside the cut end of the needle. This provides a hook at the other end to which the various weights may be suspended. Epoxy glue is recommended for this purpose. The magnets are then arranged in pairs to repel one another, slipped on to the knitting needle, and the entire device is then supported on a mounting board with the needle placed through the hole. The needle must pass freely through this hole as well as the magnet holes. In this position the bottom magnet system rests on the mounting board while the upper magnet system rides against the cap of the knitting needle and is suspended away from the lower system. By loading the end of the paper clip with weights, the separation between the pairs of magnets can be made smaller.

The data table and three graphs that follow represent the work actually performed with a floating magnet arrangement. This material is included only to serve as a guide and may perhaps represent the kind of work for which the student may strive.

No particular problems should be encountered by the students when performing this experiment. Be sure to instruct them to tap the needle gently prior to making a distance measurement. If the needle becomes "hung up" on the side of a magnet hole or on the one in the meter stick, the tapping will allow the needle to come to its proper position. Loads should be suspended on the needle until the separation distance between the magnets is approximately 1 mm.

The Floating Magnet

load (ℓ) in grams	distance (d) in mm	separation (s) in mm = 51.1 - d	$\frac{1}{s}$	selected s	calculated ℓ
0	35.0	16.1	.06	2 mm	198 gm
20	41.0	10.1	.10	4	88
40	44.2	6.9	.14	6	51.5
60	46.0	5.1	.20	8	33.0
80	47.4	3.7	.27	10	22.0
100	48.22	2.9	.34	12	14.7
140	49.0	2.1	.48	14	9.4
160	50.2	0.9	1.11	16	5.5

curve fitted to first four points

Table 3

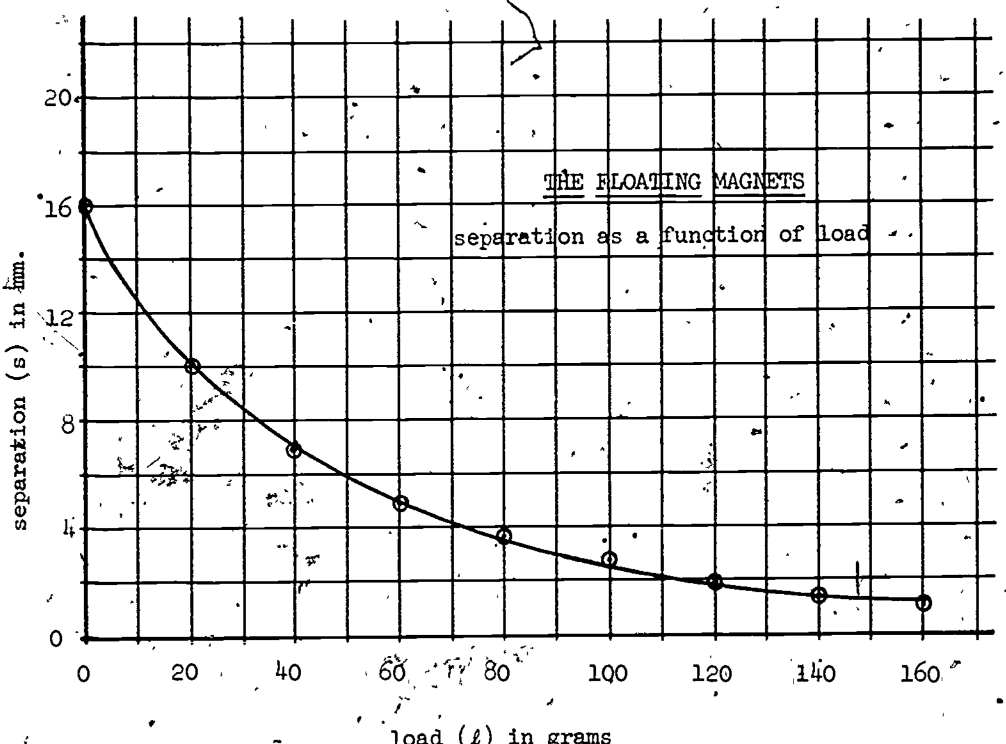


Figure 6

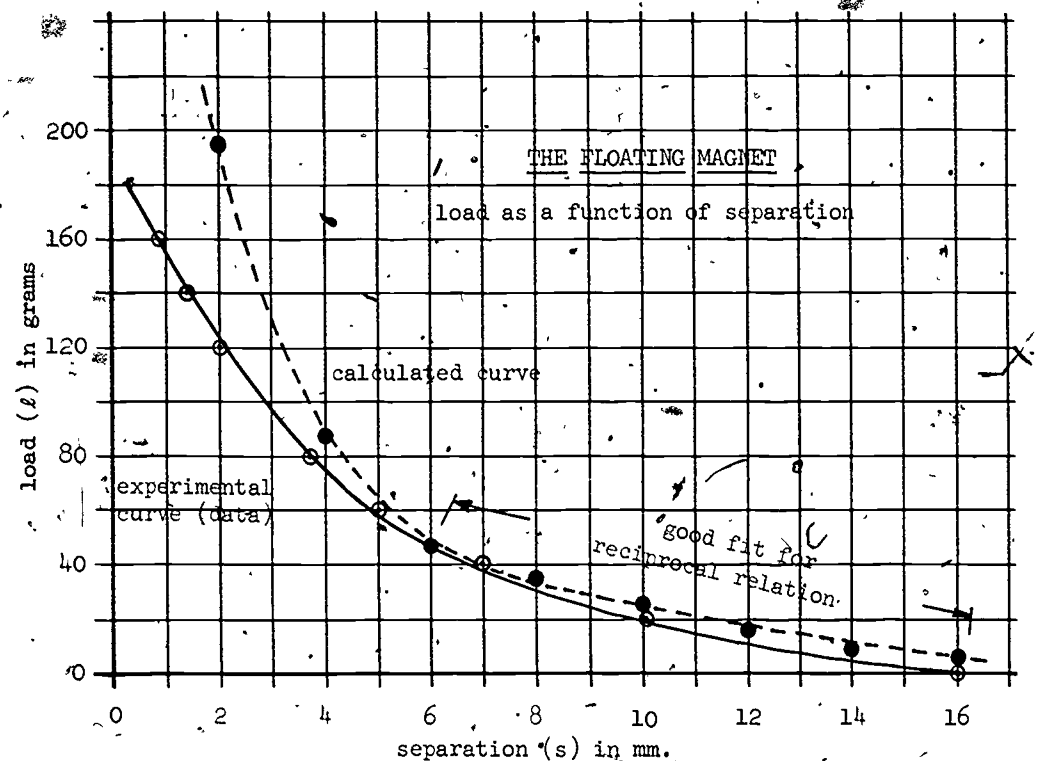


Figure 7

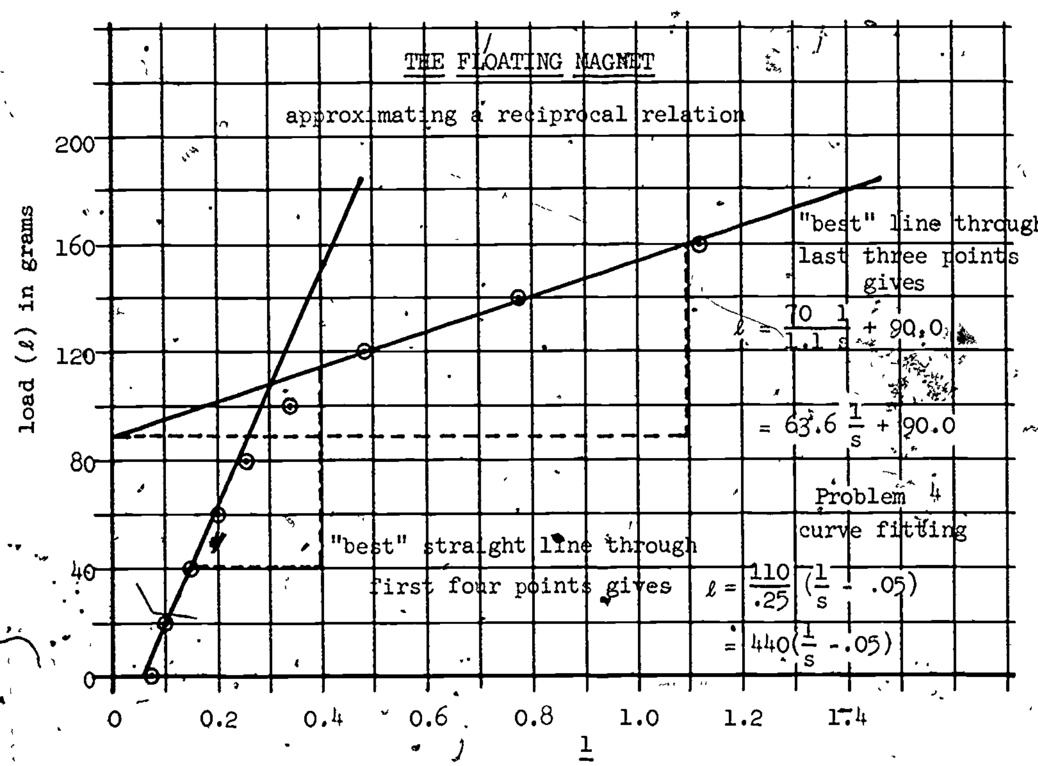


Figure 8

5.12 Search for a Mathematical Model

The relation evolved from the data of the Floating Magnet Experiment does not fit the pattern of the graphs in the student text Figure 18, (a), (b), (c). Students need to attempt to fit a pattern before reaching a conclusion that methods already developed do not work. The reciprocal relation in Figure 18(d) can then be analyzed. The next section will explore the quantitative behavior of this relation.

5.13 The Reciprocal Relation

The graph of Figure 6 shows the "best" curve through the data points, with separation distance (s) plotted as a function of the load (ℓ). This, of course, is the way the experiment was performed: loads were determined and separation distances resulted from these loads.

For reasons explained in the text, the converse of the experimental relation is graphed. This is shown in the graph in Figure 7. Since all relations have converses, there should be no problem here. Now that the converse relation has been plotted, this is the relation with which we will work from this point on. It will be instructive to the student to have graphed both the relation and its converse for he will then be able to see that there is very little difference between the two in this case.

Figure 8 exhibits the graph of load (ℓ) plotted against $\frac{1}{s}$. This graph is most interesting since it shows a decided kink. The first four points (small loads and small $\frac{1}{s}$ values) approximate a line. The equation that is obtained from this line is shown on the graph. The dashed lines indicate the right triangle from which the slope was measured. (Also drawn on this graph is the line that approximates the last three points, an exercise that will be referred to later.)

The points calculated from the straight line drawn through the first four points on the graph in Figure 8 are displayed on the graph in Figure 7. A calculated curve (dashed) is drawn through the calculated points. It is seen that this calculated curve "fits" the experimental curve for a range of loads from 0 to 60 grams.

Exercise 8

1. In the Floating Magnet Experiment we obtained the relation

$$\ell = m\left(\frac{1}{s} - c\right).$$

Algebraically obtain the converse of this relation. What separation distance does it predict for zero load?

To obtain the converse of $\ell = m\left(\frac{1}{s} - c\right)$, solve for s.

$$\frac{\ell}{m} = \frac{1}{s} - c$$

$$\frac{1}{s} = \frac{\ell}{m} + c = \frac{\ell + mc}{m}$$

$$s = \frac{m}{\ell + mc}$$

For zero load, the converse relation predicts a separation distance of $\frac{1}{c}$ $\left(\frac{m}{mc} = \frac{1}{c}\right)$. This value can be checked directly with the graph in Figure 8.

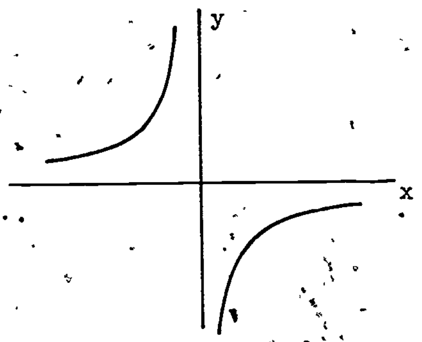
2. For a limited domain, the floating magnet function was found to be

$$\ell = m\left(\frac{1}{s} - c\right).$$

What is the unit of m? the unit of c?

The unit of m is gram mm. The unit of c is $\frac{1}{\text{mm}}$.

3. Sketch roughly the graph of $y = \frac{k}{x}$ for $k < 0$.



4. A particular reciprocal relation is $y = \frac{1}{x}$. Find the elements in the range that correspond to the following elements in the domain: 10^{-6}, 10^{-4}, 10^{-2}, 1, 10^{2}, 10^{4}, 10^{6}.

In order, the elements in the range of $y = \frac{1}{x}$ are: 10^{6}, 10^{4}, 10^{2}, 1, 10^{-2}, 10^{-4}, 10^{-6}.

5. For the relation of the previous problem, find the elements in the domain of the relation that correspond to the following elements in the range: $10^{-6}, 10^{-4}, 10^{-2}, 1, 10^{2}, 10^{4}, 10^{6}$.

In order, the elements of the domain are: $10^{6}, 10^{4}, 10^{2}, 1, 10^{-2}, 10^{-4}, 10^{-6}$.

6. Locate the x- and y-intercepts for the relation $y = \frac{k}{x}$ for $k > 0$.

There are no intercepts for the relation $y = \frac{k}{x}$ for any value of $k \neq 0$.

5.14 Curve Fitting

The student should easily understand that the reciprocal relation fits the experimental curve for only a limited portion of the graph. It is important to stress also that there may be no simple curve that can ever be found to fit certain experimental relations. These are mathematical "facts of life".

Since the initial attempt to fit the data succeeded for only a portion of the graph, a second reciprocal relation is fitted to the last three points. This second curve is developed in Problem 4. A few of your better students may be interested in plotting $\frac{1}{s^2}$ and $\frac{1}{s^3}$ against the load. A straight line portion of this graph can be obtained to give another type of "fit" to the experimental curve. It turns out, however, that these new relations give no better fit than was obtained before.

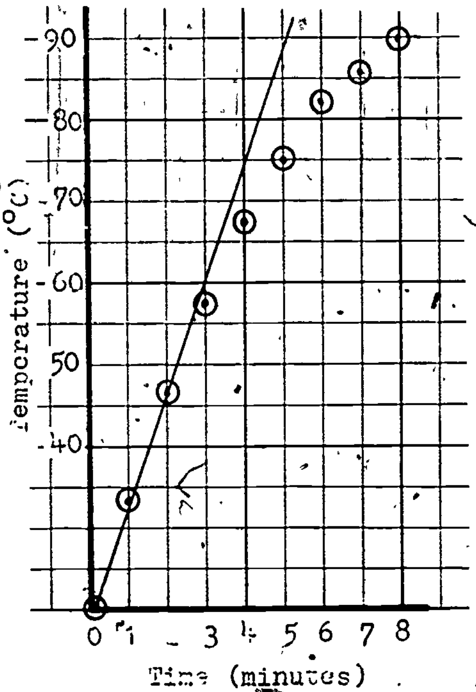
Exercise 9

A beaker of water was heated on a hot plate. The temperature of the water was recorded every minute and the following data was obtained.

Time (min)	Temp. (°C)
0	20
1	34
2	47
3	58
4	67
5	75
6	82
7	86
8	90

1. Graph the time-temperature relation. Over what range and domain would you say that the relation is a linear one?

The graph of the time-temperature relation appears at the right. The relation is linear over the domain from 0 to 2 or 3 minutes and over the range of temperatures from 20 to 50 or 60 degrees.



2. Draw your best straight line to represent the time-temperature relation for a restricted time domain. Find the equation that represents this line.

The equation that represents the line drawn is $C = 13.6t + 20$.

3. Use the equation obtained in Problem 2 to calculate temperatures for each of the 9 time readings. What is the error in temperature prediction at times of 1 min; 4 min; 7 min?

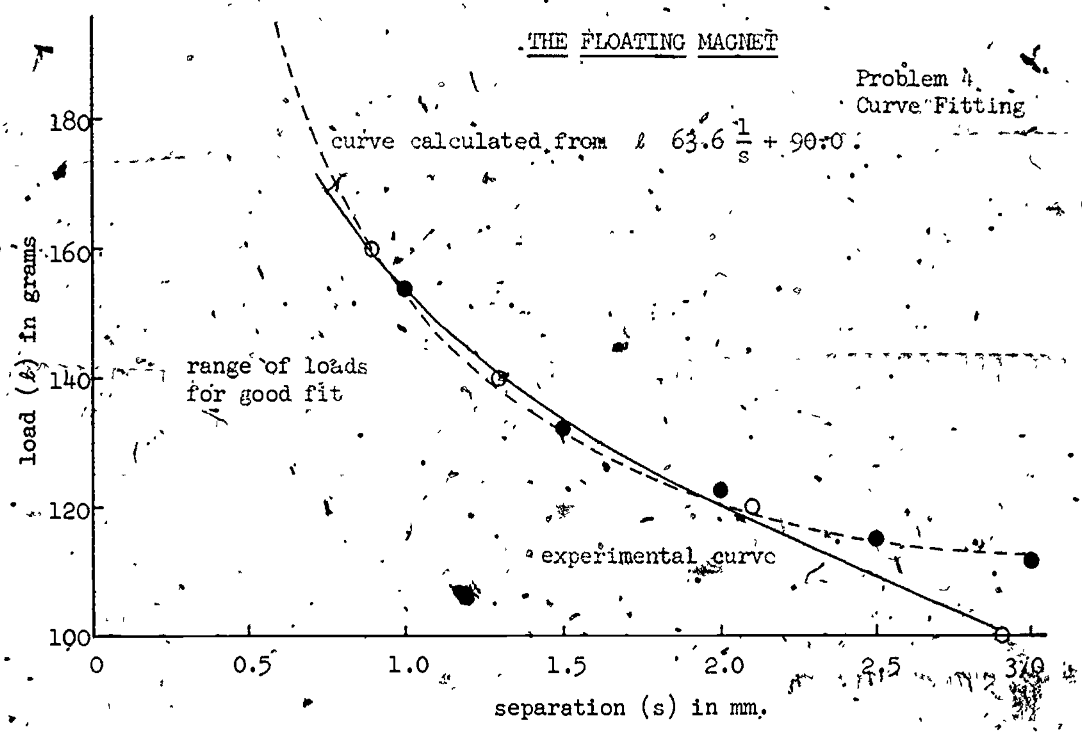
t (min)	0	1	2	3	4	5	6	7	8
temp (°C)	20	34	47	61	74	88	102	115	129

At times of 1, 4 and 7 minutes the errors in the temperature predictions from the equation are 0, 7 and 29 degrees respectively.

4. In the Floating Magnet Experiment you made a graph of the reciprocal of the separation distance ($\frac{1}{s}$) along the horizontal axis and the load (ℓ) along the vertical axis. Draw a best straight line through the points which represent loads of 120, 140 and 160 grams. Obtain the equation for this line. Calculate load values (ℓ) from this equation, selecting the best straight line through the load points for 120, 140 and 160 grams as shown on graph number 3. The equation of this line is $\ell = 63.6\frac{1}{s} + 90.0$. (See Figure 8.) The selected separations (s) and the corresponding loads (ℓ) are shown at the extreme right of the data table. The calculated curve is compared to the experimental curve on graph number 4. Both scales have been changed for this graph. A good fit is obtained for loads between 120 and 160 grams.

Problem 4 CURVE FITTING selected s	 calculated
0.5 mm	217 gm
1.0	154
1.5	132
2.0	122
2.5	115
3.0	111

See the following figure for a graph of this data.



Sample Test Items

1. (a) Select a set of ordered pairs that satisfy the following three equations.

   (b) Use these ordered pairs to construct the graphs of these equations.

   I. $y - 1 = 3(x - 1)$

   II. $y + 3 = \frac{1}{2} x^2$

   III. $y = \frac{4}{x - 2}$

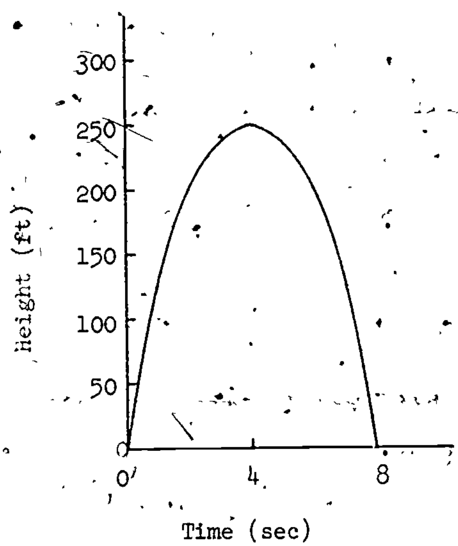
2. Using the graphs in the preceding question find the slopes of the curves at the following points. Draw the kissing line in each case.

   (a) (2,4) on Graph I.

   (b) (0,-3), (4,5) on Graph II.

   (c) (4,2) (0,-2), (2,100) on Graph III.

3. The following graph was drawn from information gathered in an experiment dealing with a ball thrown into the air. The height of the ball above the ground was plotted as a function of the time it took the ball to reach a definite height.



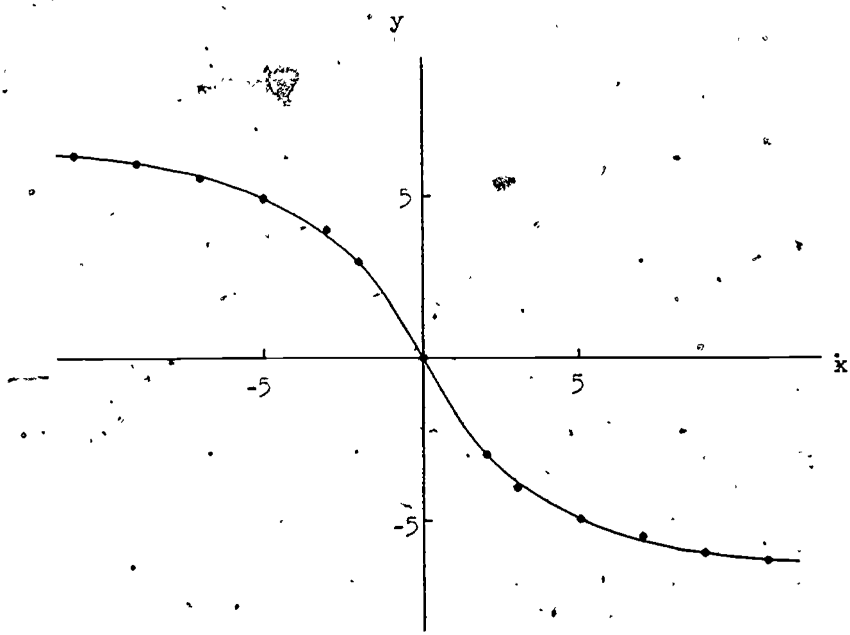
   (a) Does the graph describe a function? What is the domain and range of the relation described by the function?

   (b) Find the slope of the curve at the following points: (2,192), (4,256), (7,112), (8.1,-13.6). What is the physical meaning of the slope in each case?

4. Analyze and sketch $y = \frac{3}{x - 4}$.

5. Find the equation of the curve which is represented by the following ordered pairs: $\{(1,3), (\frac{1}{3},9), (\frac{1}{2},6), (3,1), (6,\frac{1}{2})\}$.

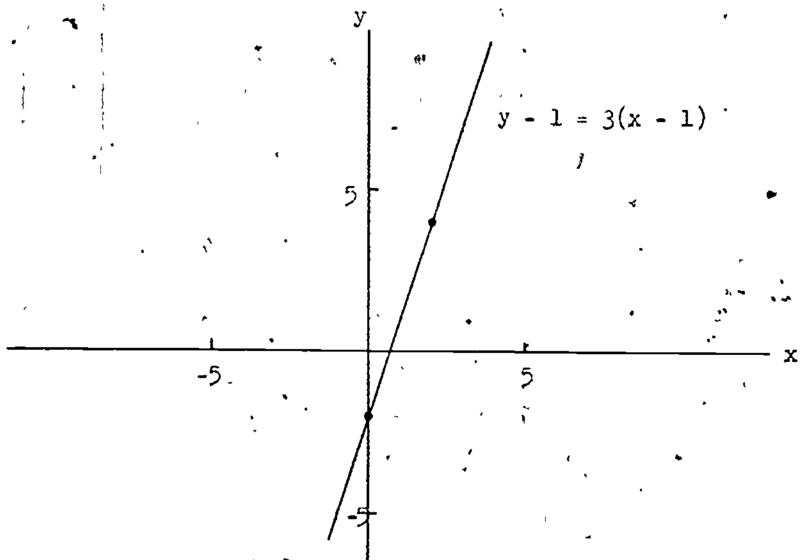
6. Draw "best" straight lines to represent the relation below for a restricted domain.

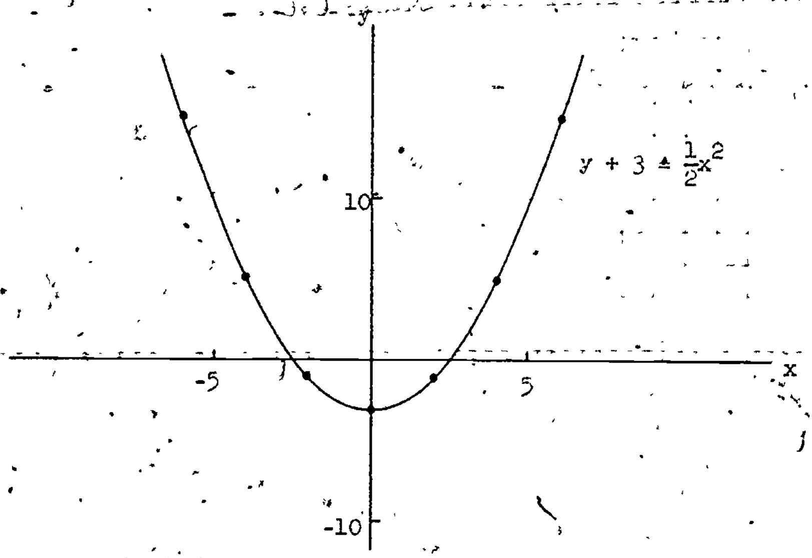


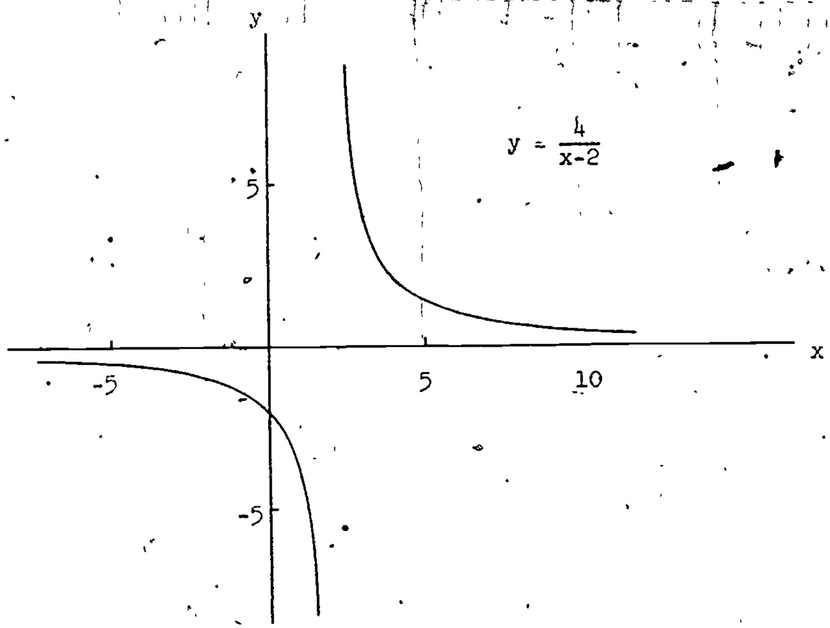
Over what domain and range would you say that the relation is a linear one?

Answers for Sample Test Items

1.







2. (a) Slope is everywhere the same and equal to 3.
   (b) At (0,-3) slope is 0.
   At (4,5) slope is 4.
   (c) At (4,2) slope is -1.
   At (0,-2) slope is -1.
   At (2,100) slope is undefined.

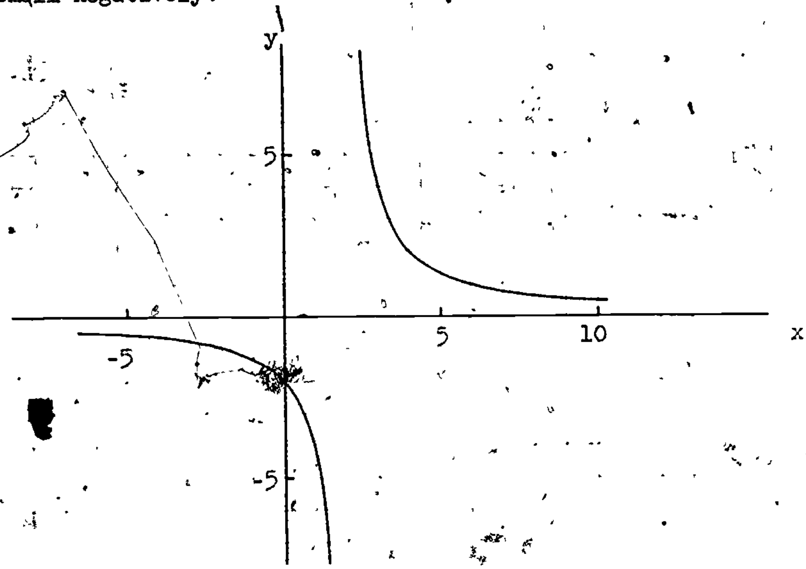
3. The equation of the parabola is: $S = 128t - 16t^2$.
   The slopes at the various points are:
   (2,192) slope is 64
   (4,256) slope is 0
   (7,112) slope is -96
   (8.1,-13.6) slope is 0 (ball has hit ground)

(1) For very large values of x the values of y are very small.
(2) As the value of x becomes smaller the value of y becomes larger.
(3) When x = 4, y is undefined.
(4) As x becomes smaller and less than 4 but remains positive, y becomes smaller negatively.

(5) When $x = 0$, $y = -\frac{3}{4}$.

(6) When x is equal to small negative values y is also small negatively.

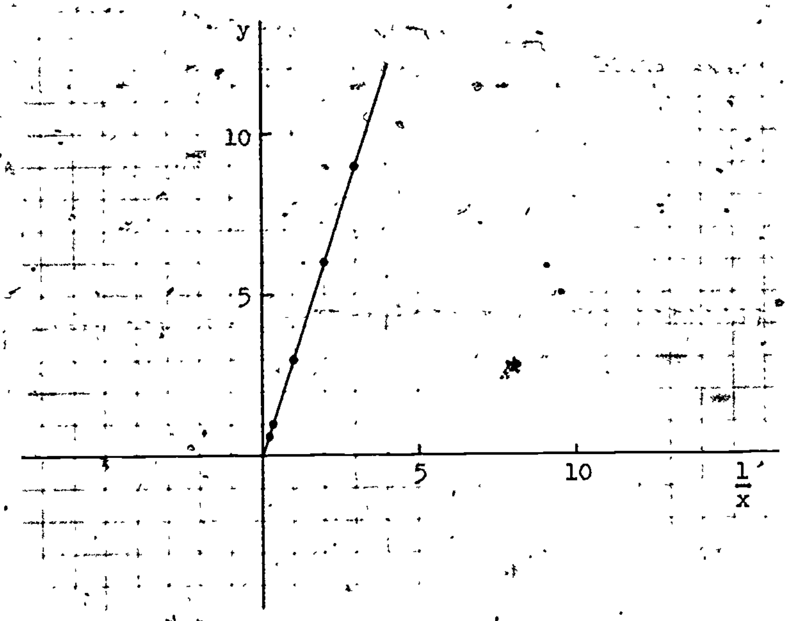
(7) For very large negative values of x, the values of y become very small negatively.



5. To find the mathematical model first determine the reciprocals of the domain, and then form the reciprocal relation of ordered pairs, (reciprocal of the domain,range). This gives

$$\{(1,3),\ (3,9),\ (2,6),\ (\tfrac{1}{3},1),\ (\tfrac{1}{6},\ \tfrac{1}{2})\}.$$

Then plot these points on a set of rectangular coordinates.



These points determine a straight line which appears to go through the origin. The slope of this line can be found by

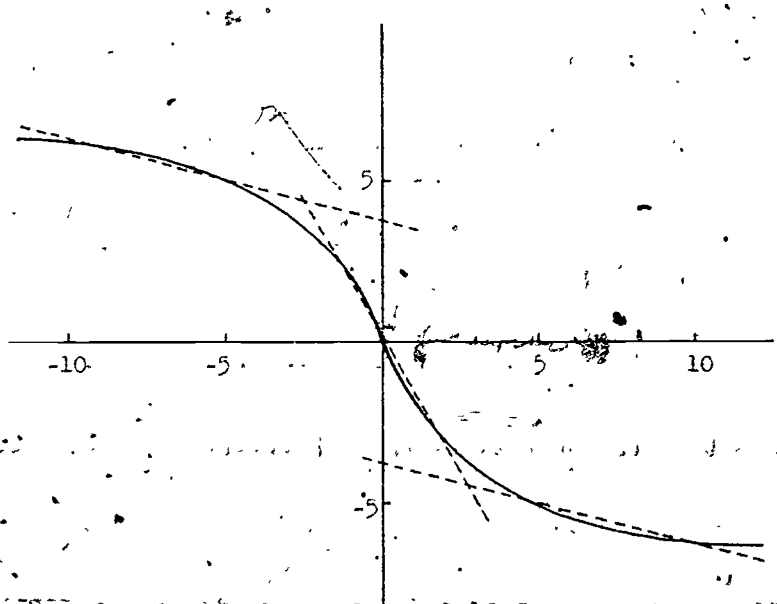
$$m = \frac{9-3}{4-2} = \frac{6}{2} = 3 .$$

Since the slope is $m = 3$, the desired equation is $y = \frac{3}{x}$.

6. Three possible "best" straight lines can be drawn. The domains and ranges are respectively:

domain: -11 to -5
-2 to +2
+5 to +11

corresponding ranges:
+5 to $+6\frac{1}{4}$
+3 to -3
-5 to $-6\frac{1}{4}$



Appendix A

THE BEST FITTING LINE

A.1 Experimental Data

In those experiments in which the data is supposed to satisfy a linear relation, the text tells the student to plot the data points and draw the straight line which seems to "best fit" the data. This is obviously an individual choice, but if the data is reasonably exact, most students will do a very good job of finding such a line.

In this appendix we wish to give you a little background into the mathematical treatment of the "best fitting line". Some of your students may realize that there must be some mathematical procedure for finding these lines and you should be ready to answer questions on this subject.

First, however, let us mention a few facts about experimental data. Some experiments produce "good" data; some produce "poor" data. When we say that an experiment produces "poor" data, we mean that there is a great deal of "scatter" in the numbers we obtain. Figure 1 shows the graphs of "good" data and "poor" data which might be obtained in actual experiments.

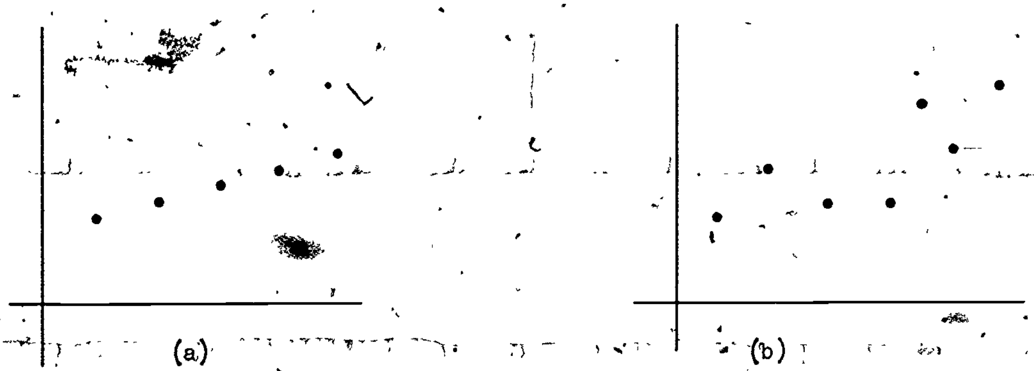


Figure 1

In Figure 1(a) it is easy to see what the best fitting line is, but in Figure 1(b), it is difficult. Scatter like this is often inherent in the experiment, but sometimes when such data is obtained, it is possible to improve the experimental conditions to try and obtain better values. Frequently, the best way to do this is to average the results of several runs. Here, however, we are interested in what can be done with data which has not been refined.

Suppose we have obtained a set of data such as shown in Figure 2.

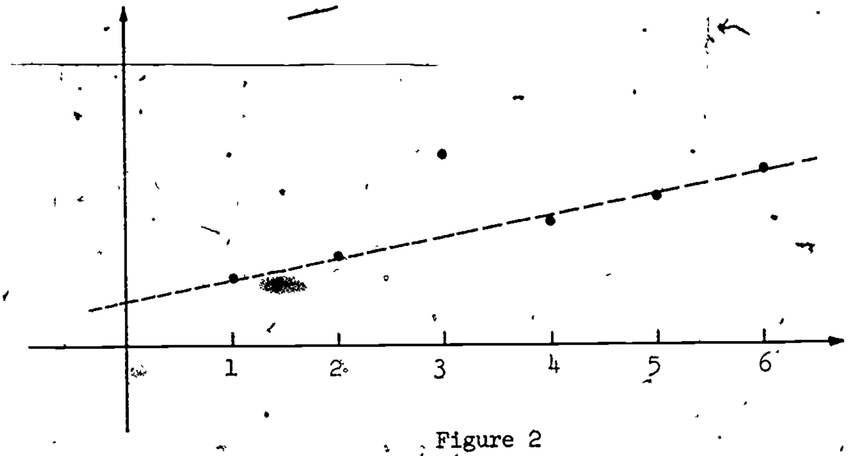


Figure 2

The dotted line seems to fit the data very well except for one point. What should we do about that point?

The answer depends on the experiment: Suppose that the data in Figure 2 represents the length of a spring plotted against the mass suspended from the spring. It would be impossible for the spring to shorten as we add more mass. Therefore, the third point must have been in error (possible a scale was read wrong). The point should therefore be dropped from the data.

On the other hand, there are types of experiments in which we cannot automatically conclude that this point is in error. In this case, the only thing to do is go back and check the experiment. Many important discoveries have been made in this way.

If it is impossible to go back to check the data point which is out of line, the best course is to discard it. However, a point should be discarded on this basis only when it is clearly out of line. The data in Figure 1(b) would be improved in appearance by discarding two or three of the points, but this would not be approved. The data points are scattered and none of them are clearly out of line.

A.2 The Method of Least Squares

Suppose that we have a number of data points. How do we decide what line comes closest to fitting these points?

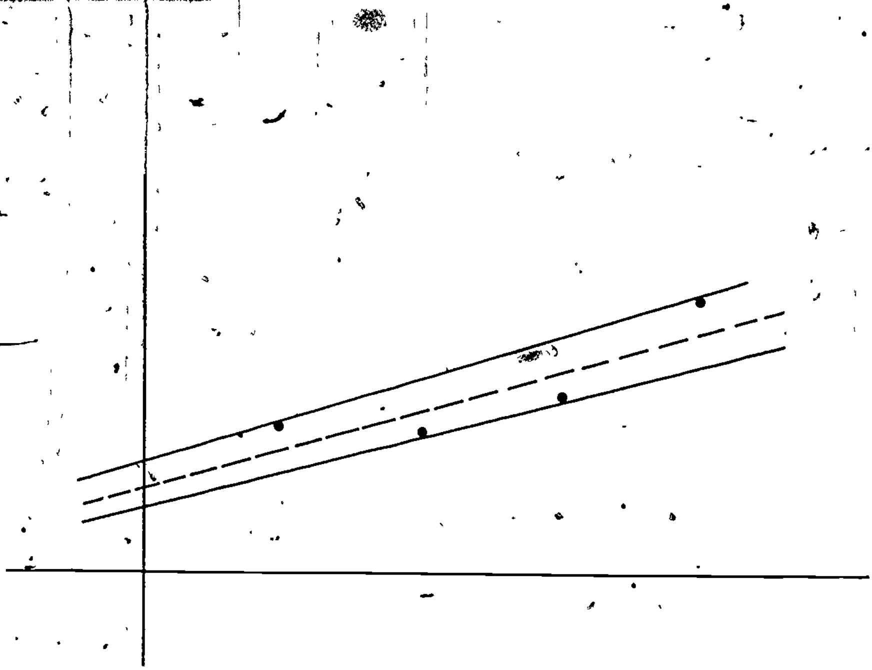


Figure 3

Look at Figure 3. Here we show four points. The dotted line must be a better fit to the data than either of the solid lines. We feel sure of this since the top line is "above" all the points, while the bottom line is "below" them. The best line must somehow thread its way through the points without being either above or below too many.

This condition alone is not enough to characterize the best fitting line. There are still too many lines which would satisfy this requirement.

In most experimental situations, the horizontal coordinate (x-coordinate) of the data points can be assumed to be known exactly (or at least with a higher degree of accuracy than the vertical coordinate). This is because this was the variable that was "controlled" in the experiment.

Suppose that we have a set of points with coordinates (x_1,y_1), (x_2,y_2), ...,

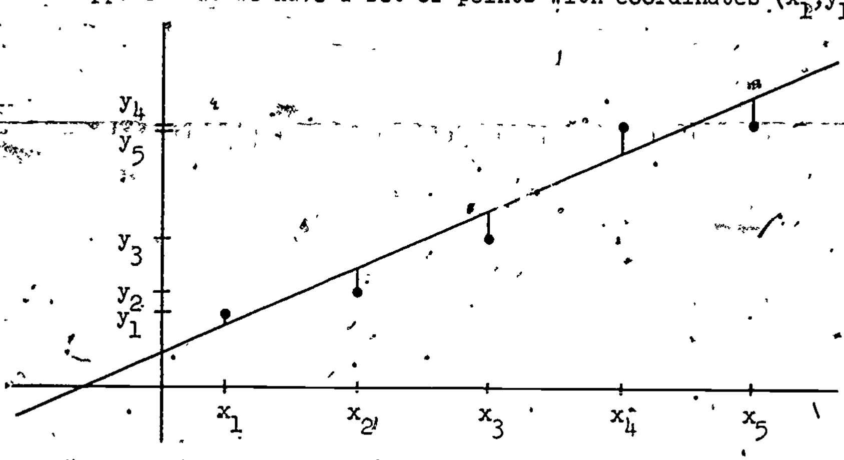


Figure 4

and that we plot these in the coordinate plane and draw in some line $y = mx + b$. At each of the x's there will be some vertical distance between this line and the corresponding data point. At x_1, for example, the line will contain the point $(x_1, mx_1 + b)$. The vertical distance between this point and the data point (x_1, y_1) is

$$(mx_1 + b - y_1) .$$

This will be positive if the line passes above this point and negative if it passes below.

It might seem that the best fitting line would be the one for which the sum of the absolute values of these differences is a minimum. Unfortunately, it is very difficult to find the line which satisfies this condition. It turns out to be much easier to find the line for which the sum of the squares of these vertical differences is a minimum. This is the so-called "method of least squares". The method of least squares can be used to find the best fitting line. It can also be applied to the problem of finding the best fitting parabola, or the best fitting line satisfying given conditions.

A.3 The Best Fitting Line

The best fitting line through an arbitrary set of data points must pass through the "center of gravity" of the points. That is, if the data is given by the ordered pairs (x_1, y_1), (x_2, y_2), ... , (x_n, y_n), and if we let $\bar{x}$ and $\bar{y}$ be the averages of the x_i and y_i coordinates,

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$$
$$\bar{y} = \frac{1}{n}(y_1 + y_2 + \dots + y_n) ,$$

then the best fitting line must pass through the point $(\bar{x}, \bar{y})$.

The equation of the best fitting line is then

$$y - \bar{y} = m(x - \bar{x})$$

where the slope m is given by

$$m = \frac{x_1y_1 + x_2y_2 + \dots + x_ny_n - n\overline{xy}}{x_1^2 + x_2^2 + \dots + x_n^2 - n\bar{x}^2}$$

(A proof of this is given in Section 2.6 .)

An example may help explain how this is done. We prepare a table of the values of x_i and y_i. Two more columns are added to this table. One is the values of x_i^2 for each line of the table and the second contains the values of $x_i y_i$. We then take the total of the values in each column.

x_i	y_i	x_i^2	$x_i y_i$	computed y_i
1	7	1	7	6.23
2	10	4	20	9.80
3	12	9	36	13.37
5	20	25	100	20.51
6	25	36	150	24.08
17	74	75	313	

$$\bar{x} = \frac{17}{5} = 3.4 \qquad n\,\bar{x}\,\bar{y} = 5 \cdot (3.4) \cdot (14.8) = 251.60$$

$$\bar{y} = \frac{74}{5} = 14.8 \qquad n\,\bar{x}^2 = 5 \cdot (3.4)^2 = 57.80$$

$$m = \frac{313 - 251.60}{75 - 57.80} = \frac{61.40}{17.20} = 3.57$$

The equation of the best fitting line is therefore

$$y - 14.8 = 3.57\,(x - 3.4)$$

or

$$y = 3.57x + 2.66$$

By using this equation together with the values of x_i, we can compute the "theoretical" values of y_i. These values have been listed in the last column of the table for comparison. Thus when $x = 5$ we find

$$y_i \text{ (theoretical)} = (3.57) \cdot 5 + 2.66 = 20.51$$

A.4 The Best Fitting Line Through the Origin

Often there are good reasons for assuming that the best fitting line through a set of points must pass through the origin. That is, that it must be of the form $y = mx$. It is then only necessary to determine the proper value of m needed to fit the data.

Suppose that the data values are given by the pairs $(x_1 y_1)$, (x_2, y_2), ..., (x_n, y_n). Then the slope of the best fitting line is given by

$$m = \frac{x_1 y_1 + x_2 y_2 + \ldots + x_n y_n}{x_1^2 + x_2^2 + \ldots + x_n^2} .$$

This same result is often written in the more compact form

$$m = \frac{\sum x_i y_i}{\sum x_i^2} ,$$

where the capital sigma symbol is used to indicate that the terms of this form are to be added together (the subscript i runs through the values 1 to n).

If we used the same data as given in the example of Section A.3, but assumed that the line must pass through the origin, we would obtain

$$m = \frac{313}{75} = 4.17$$

and, hence, the best fitting line would be

$$y = 4.17x .$$

If you plot these points and the two lines, you will see that this line does not fit the data nearly as well as the line found in Section A.3.

A.5 The Best Fitting Parabola

Suppose we have a set of data which we are sure should be related by a quadratic relation of the form

$$y = bx^2 ,$$

then we can determine the best value of b in the following way. If the data is given by the ordered pairs, (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), then the best value for b is given by

$$b = \frac{x_1^2 y_1 + x_2^2 y_2 + \ldots + x_n^2 y_n}{x_1^4 + x_2^4 + \ldots + x_n^4}$$

Notice that this formula only applies in the case of a parabola whose vertex is at the origin; that is, when the functional relation can be assumed to be of the form given above. The computations may be arranged in the following way:

x_i	y_i	x_i^2	x_i^4	$x_i^2 y_i$	computed y_i
1	2	1	1	2	2.96
3	25	9	81	225	26.64
5	82	25	625	2050	74.00
7	141	49	2401	6909	145.04
			3108	9186	

$$b = \frac{9186}{3108} = 2.96$$

The computed values of y_i shown in the last column are obtained from the formula $y = 2.96\,x^2$.

2.6 Proofs

Let $A > 0$ and consider the quadratic function of t :

$$F(t) = At^2 + Bt + C .$$

For what value of t is this expression a minimum? While this can be found easily with the help of calculus, it can be found also by elementary means. We merely have to complete the square in the quadratic expression to find

$$F(t) = At^2 + Bt + \frac{B^2}{4A} + C - \frac{B^2}{4A} \qquad (1)$$

$$= [A^{\frac{1}{2}}t + \frac{B}{2A^{\frac{1}{2}}}]^2 + C - \frac{B^2}{4A}$$

The first term in this last expression is always greater than or equal to zero (since it is squared) for any value of t . The minimum possible value of $F(t)$ is therefore attained when the squared term is equal to zero. This occurs when $t = -\frac{B}{2A}$. This is the value that gives the minimum of $F(t)$.

Now, let us use this result to prove some of the assertions made above. First, let us take one of the simpler cases, that of a straight line passing through the origin. Here we assume that we have the data (x_1,y_1), (x_2,y_2), ..., (x_n,y_n) and we wish to find the best fitting straight line in the form $y = mx$ to this data. We wish to minimize the sums of the squares of the differences $mx_i - y_i$. The expression we wish to minimize is

$$F(m) = \sum_{i=1}^{n} |mx_i - y_i|^2 \tag{2}$$

$$= \sum [m^2x_i^2 - 2mx_iy_i + y_i^2]$$

$$= \left(\sum x_i^2\right)m^2 - 2\left(\sum x_iy_i\right)m + \sum y_i^2 .$$

This is a quadratic function of m. If we compare it with equation (1), we see that the minimum value of the sums of the squares of the differences occurs when

$$m = \frac{-B}{2A} = \frac{-(-2\sum x_iy_i)}{2(\sum x_i^2)} = \frac{\sum x_iy_i}{\sum x_i^2} .$$

In a similar way, we can derive the best fitting quadratic expression to fit a given set of data. If we assume that $y = bx^2$, then we wish to minimize

$$F(b) = \sum_{i=1}^{n} [bx_i^2 - y_i]^2 \tag{3}$$

$$= \sum [b^2x_i^4 - 2bx_i^2y_i + y_i^2]$$

$$= \left(\sum x_i^4\right)b^2 - 2\left(\sum x_i^2y_i\right)b + \sum y_i^2 .$$

This is a quadratic expression in b. Again, comparing this with (1), we find that the best fitting parabola is given by setting

$$b = \frac{-B}{2A} = \frac{-(-2\sum x_i^2y_i)}{2\sum x_i^4} = \frac{\sum x_i^2y_i}{\sum x_i^4} .$$

Finally, let us look at the problem of finding the best fitting straight line of the form $y = mx + b$. Here we have two parameters which can be adjusted to minimize the sum of the squares of the differences. First, let us assume that m is fixed and try to find the best value of b. To do this, we set

$$F(b) = \sum [mx_i + b - y_i]^2 = \sum [b + (mx_i - y_i)]^2$$

$$= \sum [b^2 + 2b(mx_i - y_i) + (mx_i - y_i)^2] .$$

Since $\sum b^2 = \sum_{i=1}^{n} b^2$ is the sum of n identical terms, $\sum b^2 = nb^2$.
Therefore we find

$$F(b) = nb^2 + 2b \sum (mx_i - y_i) + \sum (mx_i - y_i)^2 .$$

To minimize this function of b, we set

$$b = \frac{-B}{2A} = \frac{-[2 \sum (mx_i - y_i)]}{2n} = -\frac{1}{n} \sum (mx_i - y_i) .$$

Using the definitions

$$\bar{x} = \frac{1}{n} \sum x_i , \quad \text{and} \quad \bar{y} = \frac{1}{n} \sum y_i ,$$

we find that

$$b = -m\bar{x} + \bar{y} .$$

Putting this into the equation for the line, we have

$$y = mx - m\bar{x} + \bar{y} .$$

This is equivalent to

$$y - \bar{y} = m(x - \bar{x}) .$$

Thus, we have shown that the best fitting line must pass through the point $(\bar{x},\bar{y})$. Next, we obtain the best value for m . To do this we wish to minimize

$$F(m) = \sum [m(x_i - \bar{x}) + \bar{y} - y_i]^2$$

$$= \sum [m(x_i - \bar{x}) - (y_i - \bar{y})]^2$$

$$= \sum [m^2(x_i - \bar{x})^2 - 2m(x_i - \bar{x})(y_i - \bar{y}) + (y_i - \bar{y})^2]$$

$$= m^2 \sum (x_i - \bar{x})^2 - 2m \sum (x_i - \bar{x})(y_i - \bar{y}) + \sum (y_i - \bar{y})^2 .$$

To minimize this, we merely follow the procedure used before and set

$$m = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

For computational purposes, the numerator and denominator of this expression can be simplified in the following way:

$$
\begin{aligned}
(x_i - \bar{x})(y_i - \bar{y}) &= \sum [x_i y_i - \bar{x} y_i - \bar{y} x_i + \bar{x}\bar{y}] \\
&= \sum x_i y_i - \bar{x} \sum y_i - \bar{y} \sum x_i + \sum \bar{x}\,\bar{y} \\
&= \sum x_i y_i - n\bar{x}\bar{y} - n\bar{x}\bar{y} + n\bar{x}\bar{y} \\
&= \sum x_i y_i - n\bar{x}\bar{y} .
\end{aligned}
$$

Also,

$$
\begin{aligned}
(x_i - \bar{x})^2 &= \sum [x_i^2 - 2\bar{x}x_i + \bar{x}^2] \\
&= \sum x_i^2 - 2\bar{x} \sum x_i + \sum \bar{x}^2 \\
&= \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 \\
&= \sum x_i^2 - n\bar{x}^2 .
\end{aligned}
$$