DOCUMENT RESUME

ED 143 530 SE 023 008

AUTHOR	Anderson, R. D.; And Others
TITLE	Mathematics for Junior High School, Supplementary Units. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	60
NOTE	122p.; For related document, see SE 023 009; Contains occasional light and broken type
EDRS PRICE	MF-$0.83 HC-$6.01 Plus Postage.
DESCRIPTORS	*Algebra; *Geometry; *Instructional Materials; Junior High School Students; Mathematics; Mathematics Education; Number Concepts; Secondary Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

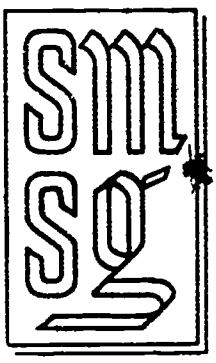
This document provides supplementary chapters for junior high school students studying SMSG or SMSG-type mathematics. Chapters include: (1) Sets; (2) Special Figures in Project Geometry; (3) Repeating Decimals and Tests for Divisibility; (4) Open and Closed Paths; (5) Finite Differences; (6) Recent Information on Primes; and (7) Games. Each chapter includes background information, discussion of the topic, and exercises. (RH)

SCHOOL MATHEMATICS STUDY GROUP

MATHEMATICS FOR JUNIOR HIGH SCHOOL

SUPPLEMENTARY UNITS

(revised edition)



MATHEMATICS FOR JUNIOR HIGH SCHOOL

SUPPLEMENTARY UNITS

(revised edition)

Prepared under the supervision of the Panel on 7th and 8th Grades of the School Mathematics Study Group:

R. D. Anderson, Louisiana State University

J. A. Brown, University of Delaware

Lenore John, University of Chicago

B. W. Jones, University of Colorado

P. S. Jones, University of Michigan

J. R. Mayor, American Association for the Advancement of Science

P. C. Rosenbloom, University of Minnesota

Veryl Schult, Supervisor of Mathematics, Washington, D.C.

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

TABLE OF CONTENTS

SUPPLEMENTARY UNIT 1

SETS

1-1. Introduction

You already are familiar with the word "set." A set of dishes is a collection of dishes. A set of dominoes is a collection, or group, of dominoes. In mathematics we use the word "set" to speak about any collection of any kind of thing. In your classroom there is a set of persons. There is also a set of noses in the room.

The language of sets is very useful in describing all sorts of situations. How is the set of pupils in your class related to the set of boys in the class? Compare the number in each of the following three sets:

the set of pupils in your class,
the set of boys in your class, and
the set of girls in your class.

The following three sets are related in a different way:

the set of redheads,
the set of baboons, and
the set of redheaded baboons.

In this chapter we are going to study relations between sets, and ways in which we can combine sets to obtain new ones. We shall find it convenient to invent some new words and symbols.

1-2. Sets, Their Members and Their Subsets

Sets and Their Members

When we speak of a set as a collection of things, we do not mean that the things are all together in one place or time. The set of all living women is a widely distributed set. You will

meet members of this set all over the world. The set of all presidents of the United States has as members George Washington and Dwight D. Eisenhower, among others. Name other members of this set.

The "things" may not be objects which you can touch or see. The set of all Beethoven symphonies does not contain any concrete objects. You may have heard some of its members. The set of all school orchestras in the United States is a set whose members are themselves sets of pupils. The set of classes in your school is another set whose members are sets. It is different from the set of all students in classes in your school. Which of these sets has more members: The set of students in your school or the set of classes in your school?

Sometimes we define a set by listing its members. Your teacher might appoint a committee to be in charge of the mathematical exhibits in your class. She may say, "The members of the Exhibits Committee shall be Lenore, Muriel, Dick and Al."

We often name a set which is defined in this way by listing names of its members and enclosing them in braces:

Exhibits Committee = {Lenore, Muriel, Dick, Al}.

We sometimes call the members of a set "elements of the set." You are an element of the set of mathematics students.

We use the symbol "$\in$" (Greek letter epsilon) to mean "<u>is a member of</u>." Thus we can express the fact that Lenore is on the committee by writing

Lenore $\in$ Exhibits Committee.

We could state the definition of the committee like this:

$x \in$ Exhibits Committee if and only if x represents Lenore or x represents Muriel or x represents Dick or x represents Al.

Another way to describe a set is to state the membership requirements. These are conditions that something must satisfy in

order to be in the set. The set of persons in your classroom has a very simple membership requirement. The object x is in the set if x is a person in your classroom, and only then. The set of common multiples of 4 and 6 is the set of all numbers, x, for which it is true that x is a multiple of 4 and x is a multiple of 6. You might imagine each object in the universe applying for membership in this set. If the object is not even a whole number, then we throw it out immediately. If it is a whole number, we divide it by 4. If the remainder is zero, we then divide the number by 6 and see whether 6 is a factor. If x passes this test, too, then x gets its membership card in the set. If it fails any of the tests, we reject it.

Property

You begin to see that for a particular set to be clearly defined there must be some scheme or device for determining whether or not a given element is in the set. Usually a set is described in terms of some property, or properties, which its elements have in common. For example, the set C may be thought of as the pupils in your lass. The common property is that each element is a member of your class. Again, you may consider set B as the set of boys in your class. The elements of this set contain two properties in common: (1) the elements are all in your class, and (2) the elements are all boys. Sometimes a set is described simply by enumerating the elements.

We could say the set, Lenore, Muriel, Dick and Al. This would form a set even if they were not on the same committee. We can describe a large set by listing a few members if it can be done so that there is no doubt as to whether or not an object is in the set. 0, 1, 2, 3, ... 100 describes the set of whole numbers from 0 through 100. Some sets with a limitless number of elements can be described by listing a few elements; in this case, also, there must be no doubt as to whether or not an object is in the set. An example of a limitless set is: 0, 2, 4, 6 ... 2n ... . What are the common properties of this set?

Exercises 1-2-a

1. List a common property or properties of the elements of each of the following sets:

   (a) {Sue, Jane, Dorothy, Mildred}.

   (b) {Washington, Jackson, Eisenhower}.

   (c) {1, 3, 5, 7, 9, 11}.

   (d) {12, 24, 36, 48}.

2. Translate the following mathematical sentences into English.

   (a) Tom $\in$ {Carl, Jim, Tom, Robert}.

   (b) 6 $\in$ {0, 2, 4, 6, 8, 10, ...}.

   (c) If X $\in$ {Tom, Carl, Bob, Jim} then X represents Tom, or X represents Carl, or X represents Bob, or X represents Jim.

3. Which of the following are true?

   (a) 4 $\in$ {3, 7, 10, 4}.

   (b) lion $\in$ {baboon, tiger, dog, lion}.

   (c) If X is a multiple of 6, then X $\in$ {8, 14, 17, 28}.

   (d) If X is a counting number, then X $\in$ {1, 2, 3, 4, 5, 6, ...}.

   (e) Washington, D. C. $\in$ {Alabama, Alaska, Arizona, ..., West Virginia, Wisconsin, Wyoming}.

4. List the members of the following sets:

   (a) The set of X such that X is a factor of 12 and 30.

   (b) The set of X such that X plays a violin, or X plays the viola, or X plays the cello.

   (c) The set of X such that X is a whole number.

   (d) The set of X such that X is one of the U. S. Presidents since 1930.

Subsets

Consider the set of major league baseball teams in New York in 1959. This set has one member, the New York Yankees Baseball Club. Its one member is itself a set, among whose members are Mickey Mantle and Yogi Berra. The set whose only member is a certain object is not the same as that object. The symbol {3} is a name for the set whose only member is 3.

The set of players on the New York Yankees team is a subset of the set of baseball players. Every member of the team is a baseball player. In symbols, we write: If $X \in$ Yankees, then $X \in$ the set of baseball players.

You have been introduced to a new word: that of subset. Let us consider another example. Suppose in a class of 25 pupils there are 3 pupils whose first name begins with "S." You can then say that these 3 pupils form a subset of the class. Again, consider the set of even counting numbers: 2, 4, 6, 8, 10, ... This set can be considered as a subset of the counting numbers: 1, 2, 3, 4, 5, 6, ...

Suppose the set of pupils in your class whose first names begin with "S" is {Sam, Susan, Sally}. Subsets of this set may be listed as follows: {Sam}, {Susan}, {Sally}, {Sam, Susan}, {Sam, Sally}, {Susan, Sally} and {Sam, Susan, Sally}. Sometimes we say that a set is a subset of itself.

Definition:

A set R is a subset of a set S if every element of R is an element of S.

It is necessary, at times, to talk about the relationship of a set, or the relationship of a set to another set. We say, for example, that the set of even counting numbers (which is a subset of the counting numbers) is contained in the set of counting numbers. To write this in mathematical language we use the "$\subset$", which is read "is contained in." You can now write: {2, 4, 6, 8, ...} $\subset$ {1, 2, 3, 4, 5, 6, ...}. Sometimes the symbol "$\supset$"

is also used. This is read " contains." You can now also write:

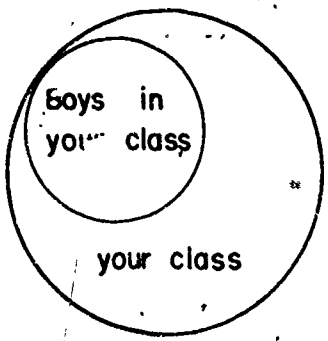
$$\{1, 2, 3, 4, 5, \ldots\} \supset \{2, 4, 6, 8, \ldots\},$$

which reads: The set of counting numbers contains the set of even counting numbers. Let the set of your class be called "C" and the set of boys in your class be called "B". You can then write:

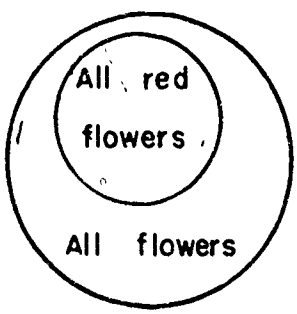
$$B \subset C, \quad \text{or}$$

$$C \supset B$$

You may be helped in this study by use of diagrams. A mathematician always draws figures or diagrams when possible. The diagrams used below are called "Venn" diagrams. Consider again the example $B \subset C$. We sketch the following:



This illustrates that the set of boys in your class is contained in the set of your class. Again:



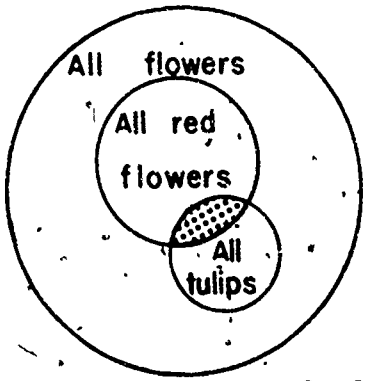
illustrates that the set of all red flowers is contained in the set of all flowers. Let the set of all red flowers be called R and the set of all flowers be called F. The relationship of

R and F can then be written as:

$$R \subset F, \quad \text{or}$$

$$F \supset R.$$

Consider the following Venn diagram:



This diagram indicates that the set of all red flowers belongs to the set of all flowers. It also indicates that the set of all tulips belongs to the set of all flowers. Let the set of all tulips be called T. The above relationship may now be expressed as:

$$R \subset F, \quad \text{and}$$

$$T \subset F.$$

What can you say about the relationship of set R and set T? You would certainly have to say that some tulips are red and are thus contained in the set R, but you certainly cannot say that $T \subset R$ is true. Give some thought to this situation for a while.

Exercises 1-2-b

1. Translate the following mathematical sentences into English:

   (a) If $X \in$ {Red flowers}, then $X \in$ the set of all flowers.

   (b) $M \subset N$, and $N \supset M$.

   (c) $\{1, 3, 5, 7, 9, 11 \ldots\} \subset \{1, 2, 3, 4, 5, 6, \ldots\}$.

2. Write all possible subsets of the set: {4, 5, 6}.

3. Translate the following English sentences into mathematical sentences.

   (a) The set {12, 20, 32} is contained in the set of all whole numbers.

   (b) The set of the Great Lakes contains the set of Lake Huron and Lake Michigan.

   (c) The set of {Hoover, Truman} is contained in the set of all U.S. presidents since 1920.

4. Draw a Venn diagram to illustrate the following:

   (a) The set of the Hudson and Ohio Rivers is contained in the set of all rivers in the United States.

   (b) The set of tigers, lions, and baboons is contained in the set of all animals.

   (c) The set of 16, 36, and 40 is contained in the set of all counting numbers which are multiples of 4.

   (d) The set of 6, $\frac{1}{2}$, $\frac{3}{8}$ is contained in the set of all rational numbers.

5. Which of the following are true and which are false?

   (a) {Al, Tom} $\supset$ {Al, Bob, Jack, Tom}.

   (b) {Sam, Sue} $\subset$ {Slim, Tom, Bob, Sally}.

   (c) The set of all yellow roses is contained in the set of all yellow flowers.

   (d) {28, 56, 112} $\subset$ the set whose elements are multiples of 4 and also of 7.

6. Given three sets A, B, and C. If $A \supset B$ and $B \supset C$, does $A \supset C$? Illustrate your answer with a Venn diagram.

1-3. Operations with Sets

Union

Suppose the set: {Bill, Jim, Tom, Sam} are the boys of a class who play in the band. Call this set B. Let the set: {Sam, Tom, Carl} be the boys in the same class who have red hair. Call this set R. Now if we combine these two sets we would get the set: {Bill, Carl, Jim, Tom, Sam}. This would be the set consisting of all elements which belong to set B, or to set R, or to both sets. We call this the union of two sets. The symbol used is: "$\cup$". We can now write:

$$\{\text{Bill, Jim, Tom, Sam}\} \cup \{\text{Sam, Tom, Carl}\} = \{\text{Bill, Carl, Jim, Tom, Sam}\}.$$

If we call the union of these two sets C, then you can write:

$B \cup R = C$, and it is read: B union R equals C.

The combining of two sets in this manner is called an operation. Before working some problems let us consider another matter which was introduced by writing $B \cup R = C$.

Equality of Sets

We say that two sets are equal if and only if each element of one is also an element of the other. Suppose we have two sets A and B: If $A \subset B$ and $B \subset A$ then we can say $A = B$. For example, suppose that in your class there are only four redheaded pupils which we shall call set R, and furthermore, these four redheaded pupils are the only ones having their birthdays in January, which we shall call set J. We can write:

$$R \subset J \text{ and } J \subset R, \text{ hence } R = J.$$

Consider again: $B \cup R = C$. If we can write $(B \cup R) \subset C$ and $C \subset (B \cup R)$, then we can say: $B \cup R = C$. After some thought you should see that this is a true statement. Instead of saying that two sets are equal, we sometimes say they are identical. This is a good expression since we can say that two sets are equal if

and only if every element of each is an element of the other.

Properties

1. Consider again the two sets, B and R. Do you suppose that

$$B \cup R = R \cup B?$$

Let us investigate:

$$B \cup R = \{Bill, Jim, Tom, Sam\} \cup \{Sam, Tom, Carl\}$$
$$= \{Bill, Carl, Jim, Tom, Sam\}.$$
$$R \cup B = \{Sam, Tom, Carl\} \cup \{Bill, Jim, Tom, Sam\}$$
$$= \{Bill, Carl, Jim, Tom, Sam\}.$$

You see, then, that $B \cup R = R \cup B$. Does this recall to you what you learned about the "commutative" property? With a little thought on the union concept, you should see that for any two sets M and N, $M \cup N = N \cup M$, and the commutative property is true for sets under the operation of union.

2. Do you think the following is true?

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

Let $A = \{1, 2, 3\}$; $B = \{1, 4\}$; $C = \{2, 5, 6\}$.

Then: $A \cup (B \cup C) = \{1, 2, 3\} \cup \{1, 2, 4, 5, 6\}.$
$= \{1, 2, 3, 4, 5, 6\},$

and $(A \cup B) \cup C = \{1, 2, 3, 4\} \cup \{2, 5, 6\},$
$= \{1, 2, 3, 4, 5, 6\}.$

You see, then, that in our example: $A \cup (B \cup C) = (A \cup B) \cup C$. This should recall to mind the associative property. With some thought you should see that under the operation of union the associative property is true for sets.

Exercises 1-3-a

1. (a) If set M = {Red, Blue, Green} and set N = {Blue, Yellow, White}, find M $\cup$ N.

   (b) Is M $\cup$ N = N $\cup$ M? Why?

2. Let A be the set of even counting numbers; B the set of odd counting numbers; and C the set of all counting numbers.

   (a) Is A $\cup$ B = C? Why?

   (b) Is A $\subset$ C? Why?

   (c) Is A $\subset$ B? Why?

   (d) Is A $\cup$ B = B $\cup$ A? Why?

   (e) Does B $\supset$ A? Why?

   (f) Draw a Venn diagram to illustrate B $\subset$ C.

   (g) Is A = B? Why?

3. Given three sets R, S, and T.

   (a) Is (R $\cup$ S) $\cup$ T = R $\cup$ (S $\cup$ T) = T $\cup$ (R $\cup$ S)? Why?

   (b) Suppose (R $\cup$ S) $\subset$ T and T $\subset$ (R $\cup$ S), then is R $\cup$ S = T? Why?

4. Let C be the set of pupils in your class, S be the set of pupils in your school, and X be the only redheaded pupil in your class. Discuss the following as to whether or not they are true.

   (a) X $\in$ S

   (b) C $\subset$ S

   (c) C = S

   (d) S $\subset$ C

   (e) X $\in$ C

   (f) S $\supset$ C

   (g) Is X a subset of C? Of S?

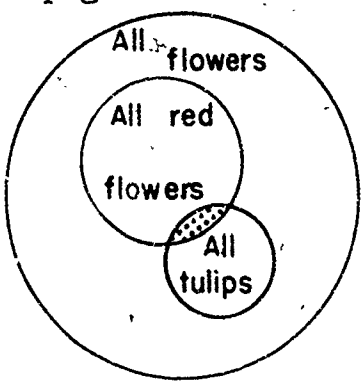
   (h) Is C a subset of S?

5. (a) Consider two concentric circles. Let X be the set of points within a circle whose radius is 4 units and Y be the set of points within a circle whose radius is 2 units. Draw a Venn diagram to show: X $\cup$ Y.

(b) Is $X \subset Y$, or $Y \subset X$? After giving your answer complete the statement: _____ is a subset of _____.

Intersection

Another operation with sets is that of intersection. Do you recall this operation from Chapter 4 of the Seventh Grade text? You no doubt remember that the symbol for intersection is "$\cap$". Consider sets A and B. If we now write: $A \cap B$, it is read "A intersection B." The intersection of two sets is the set of all elements which belong to both sets. For example, let set A be {Tom, Sue, Carl, Joan}, and set B be {Sam, Sue, Tom, Sally}. Then $A \cap B = \{Sue, Tom\}$. Do you remember the following Venn diagram we had several pages back?



You remember a question was raised about the relationship of R and T, where R was the set of all red flowers and T was the set of tulips. You can now see that the shaded part of the diagram is $R \cap T$. This situation presents us with another set which we have not mentioned. Are there any yellow tulips in set R?

Null Set

At times we have a set which is said to be empty. Such a set is sometimes called the "null set." For example, the set of yellow tulips contained in the set of all red flowers is an example of a null set. Suppose there are no redheaded pupils in your class then the set of redheaded pupils in your class is a null set.

Another example is the set all voters who have their legal residence in Washington, D. C. We shall use the symbol "ϕ" (the Greek letter phi, pronounced "fee") to designate the null set. We say that ϕ is a subset of every set.

Properties

1. Given two sets M and N: Is it true that $M \cap N = N \cap M$? Let M be $\{1, 2, 3, 4\}$ and N be $\{3, 4, 5, 6\}$, then $M \cap N = \{3, 4\}$ and $N \cap M = \{3, 4\}$. In view of your previous study you are led to see that the commutative property applies under the operation of intersection of sets.

2. In a similar manner, given three sets R, S, and T, it can be shown that the associative property holds. We would then have: $R \cap (S \cap T) = (R \cap S) \cap T$. Select an example of your own and see if you get a true result.

3. Are you reminded of anything by the following, where R, S and T are three sets?

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$$

Let $R = \{1, 2, 7\}$, $S = \{1, 3, 4\}$ and $T = \{2, 3, 5\}$.

Then $R \cup (S \cap T) = \{1, 2, 7\} \cup (\{1, 3, 4\} \cap \{2, 3, 5\})$

$= \{1, 2, 7\} \cup \{3\}$

$= \{1, 2, 3, 7\}$

and $(R \cup S) \cap (R \cup T) =$

$= (\{1, 2, 7\} \cup \{1, 3, 4\}) \cap (\{1, 2, 7\} \cup \{2, 3, 5\})$

$= \{1, 2, 3, 4, 7\} \cap \{1, 2, 3, 5, 7\}$

$= \{1, 2, 3, 7\}.$

This illustrates the distributive property of union with respect to intersection of sets. In working with sets we have two forms of this property. We have just studied one form: namely, $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$. The other form is: $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$, which is the distributive property, of intersection with respect to union of sets. This is

somewhat different from what you studied in working with the counting numbers in Chapter 3. There was only one form of the distributive property: namely, multiplication with respect to addition.

Exercises 1-3-b

1. Given the three sets: A = {boy, girl, chair}, B = {girl, chair, dog} and C = {chair, dog, cat}.
   (a) Find $A \cap B$.
   (b) Show that $A \cap C = C \cap A$.
   (c) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
   (d) Show that $A \cap (B \cap C) = C \cap (A \cap B)$.
2. (a) Where ϕ represents the null set, and H is any other set, is the following true? $\phi \cup H = H \cup \phi$. Explain your answer.
   (b) Is $\phi \cap H = H$? Explain your answer.
   (c) Under the operation of union of sets, what name may be applied to ϕ? (Hint: compare zero in addition and ϕ in union of sets.)
3. Let R represent the set of points on the segment $\overline{AB}$, and S represent the set of points on another segment $\overline{CD}$.
   (a) If $R \cap S = \phi$, then what is true about the two line segments?
   (b) If $R \cap S \neq \phi$, then what is true about the two line segments?
4. Are there any similarities in properties between the symbols "$\cup$" and "$\cap$", and the symbols "+" and "·"? Explain your answer.
5. Draw a Venn diagram to illustrate the intersection set of all members of the band in your school and all the pupils in your class.

6. Show by use of a figure the intersection set of two intersecting circular regions. Shade the intersection set.

7. (a) Let E be the set of even counting numbers: {2, 4, 6, 8, ···}. What must be the set F so that $E \cup F = C$, when C is the set of all counting numbers?

   (b) What is $E \cap F$?

8. Given two sets A and B:

   (a) If $A \subset B$, is it true that $A \cup B = B$? Explain your answer.

   (b) If $A \subset B$, is it true that $A \cap B = A$? Explain.

1-4. Order, One-to-One Correspondence

Order

In many situations the order in which we write the elements of a set is immaterial. For example, set A: {Bill, Tom, Sam}, can be written as {Tom, Sam, Bill}, or as {Sam, Bill, Tom}, just as well as in the original. Under our definition of equality, all three of these sets are equal. At times, however, the order is important. For example, the name William Thomas is not the same as Thomas William. If we wrote these two names as a set: {William, Thomas}, then under our present framework, we could just as well write the set as: {Thomas, William}, and the two sets would be equal, or identical. An ordered set is one wherein there is an element which is the first term, another element which is a second term, and so on. When we wish to indicate that the elements of a set are ordered, we shall use parentheses ( ), instead of braces { }. If we now write the set composed of the elements Thomas, William in the form: (Thomas, William) it is not equal to the set: (William, Thomas), because the set is ordered with the element Thomas in the first position and the element William in the second position. A set of two elements written in this manner

is sometimes called an ordered pair. You had some contact with ordered pairs when you made graphs in Chapter 13 of the Seventh Grade text and in Chapters 1 and 2 of the Eighth Grade. A set such as: (a, b, c) may be referred to as an ordered triple. This idea may be extended to many more than 3 elements. For example, the ordered set of the first n counting numbers: (1, 2, 3, 4, 5, 6, ..., n), would give us an ordered set of n elements where n may be any counting number. This idea will be used in the section on Counting.

Ordered pairs are very useful in many branches of mathematics. When you study a course called Analytical Geometry, you will deal with ordered pairs such as (1, 4), (6, 2), (12, 15).

Consider the set of people in line before the box office of a theater. Is order important in this situation? If you should try to move ahead of someone already in line, you would be made to understand, rather quickly, the importance of order in this case. There are people who consider order important enough to take a bed roll and sleep near a box office, so as to be well up in a line when the office opens. Some baseball fans do this for the World Series. Can you think of other similar situations?

As you know, the following is a true statement:

$$\{1, 2, 3\} = \{1, 3, 2\}.$$

On the other hand, $(1, 2, 3) \neq (1, 3, 2)$, because these are ordered sets.

One-to-one Correspondence

One basic study of sets deals with the comparison of two or more sets to see whether or not the numbers are equally numerous. This is done by matching the elements of the sets. In the opening pages of Chapter 2 in the Seventh Grade text, you read that in the long ago a shepherd probably kept account of his sheep by having a notched stick - a notch for each sheep and a sheep for each notch. With this arrangement he could tell whether

or not any sheep were missing by comparing, or matching, the set of notches with the set of sheep. If all sheep were present, we could say there was a one-to-one correspondence between the set of sheep and the set of notches.

Consider your class. Suppose there is the same number of seats in your classroom as there are pupils in your class. When all the pupils are present, then the set of seats and the set of pupils are in one-to-one correspondence. In other words, the two sets are equally numerous. If all pupils are present and seated in their assigned seats, then your teacher can tell at a glance that there is perfect attendance for the day. Without much more than a glance she can tell how many are absent, if some are not present. How does she do this? What can you say with respect to one-to-one correspondence of the following:

1. {1, 2, 3, 4}; {0, X, Λ, V}; {A, B, C, D}.
2. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10};
   {a, b, c, d, e, f, g, h, i, j}.
3. {the number of fingers on one hand};
   {the number of symbols in a base five system};
   {the number of players on a boys' basketball team}.

In each of the 3 groups there is a one-to-one correspondence between the elements of one set and the elements of each of the other sets.

We are now in a position to state a general principle with respect to sets and one-to-one correspondence as follows:

> Given two sets A and B. These two sets are said to be in one-to-one correspondence if we can pair, or match, the elements of A and B such that each element of A pairs with one and only one element of B, and in the same matching process each element of B pairs with one and only one element of A. This principle may be stated more precisely in the following way:

Let A and B be sets. There is a one-to-one correspondence between A and B if there exists a collection H of ordered pairs with the following properties:

1. The first term of each pair of H is an element of A,
2. The second term of each pair of H is an element of B,
3. Each element of A is a first term of exactly one pair of H,
4. Each element of B is a second term of exactly one pair of H.

In Problem 2 above let A be the set

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

and B the set

{a, b, c, d, e, f, g, h, i, j}.

The set H would look like this:

{(1,a), (2,b), (3,c), (4,d), (5,e), (6,f), (8,g) (9,i), (10,j)}

Unless the concept of order is to be taken into consideration, the matching process may be done in more than one way. Consider set A: {Bill, Tom, Sam}, and set B: {Ann, Jane, Susan}. Since these sets have only three elements, we can see at a glance that there is a one-to-one correspondence between them. The matching process, however, can be done in six ways. Two of them are as follows:

Set A	Set B
Bill	Ann
Tom	Jane
Sam	Susan

Set A	Set B
Bill	Ann
Tom	Susan
Sam	Jane

Figure 1-4

The symbol "$\longleftrightarrow$" or "$\updownarrow$" simply means, for example, that Bill is matched with Ann, and Ann is matched with Bill.

Let us consider the elements of these two sets again, and write the sets as follows:

A: (Bill, Tom, Sam), B: (Ann, Jane, Susan).

The notation indicates the two sets are now ordered. Of course, we can still match the elements in six ways. If, however, we want to preserve the order, the elements can be matched in only one way as follows:

Bill $\longleftrightarrow$ Ann
Tom $\longleftrightarrow$ Jane
Sam $\longleftrightarrow$ Susan.

Equivalence

You remember when we talked about the equality of sets, we said that two sets were equal, or identical, if and only if every element of each is an element of the other. For example,

$$\{1, 2, 3\} = \{1, 3, 2\}$$

because the two sets contain the same elements. The concept of one-to-one correspondence introduces a new concept of equality, that of equivalence. We say that two sets which are in one-to-one correspondence are equivalent. We shall indicate this fact by using the symbol "$\longleftrightarrow$", which was used in matching the elements of sets. For example:

$$\{\text{Bill, Tom, Sam}\} \longleftrightarrow \{\text{Ann, Jane, Susan}\}.$$

Again, given two sets A and B, if we write: $A \longleftrightarrow B$, we mean that there is a one-to-one correspondence between the elements of A and the elements of B.

Exercises 1-4

1. Construct tables similar to those of Figure 1-4 to show the additional four ways in which the two sets may be matched.

2. (a) How many elements are in each set in Problem 1?

   (b) In how many tables is Bill $\longleftrightarrow$ Ann?

   (c) What arithmetic operation on the answers to parts (a) and (b) gives the total number of ways in which the members of Set A can be matched with the members of Set B?

3. Use the sets C = {1, 2, 3, 4} and D = {a, b, c, d} for this problem.

   (a) Match each of the elements of Set C with an element of Set D keeping 1 $\longleftrightarrow$ a in all cases.

   (b) How many such matchings are possible?

   (c) In how many matchings is 2 $\longleftrightarrow$ a? Do not make additional tables.

   (d) In how many matchings is 3 $\longleftrightarrow$ a?

   (e) In how many matchings is 4 $\longleftrightarrow$ a?

   (f) How many possible matchings are there between sets C and D?

   *(g) Write a rule to find how many matchings are possible between two equivalent sets.

4. Determine whether the following are true or false. Use examples to illustrate your answers.

   (a) Identical sets are also equivalent.

   (b) Equivalent sets are also identical.

   (c) Equivalent sets may be identical.

   (d) Identical sets are never equivalent.

5. Construct a matching table for the following sets so that order will be preserved: (1, 2, 3, 4, 5, 6), (x, y, t, a, b, c).

6. Suppose you buy a carton of a dozen eggs. Is it necessary to count the eggs in order to tell whether or not you have a dozen? Why?

7. Given two sets x and y. If $x \subset y$ and $y \subset x$, can we say that the two sets are in one-to-one correspondence? Explain.

8. Are there more points on an arc of a circle than on its subtended chord? Explain your answer.

1-5 The Number of a Set and Counting

Given the sets: {1, 2, 3, 4} and {0, 1, $\wedge$, V}. You notice that there is a one-to-one correspondence between them. In addition you see that the sets are composed of 4 elements. In fact, any two sets which are in one-to-one correspondence have the same number of elements. Sets, however, will vary in the number of elements which they contain. This may vary all the way from zero, the null set, to an infinity of elements. The word "infinity" is not new to you, because you will remember that there are an infinite number of points on a line, or again, an infinite number of whole numbers. A set containing an infinite number of elements is called an infinite set; otherwise, the set is called a finite set. Since sets vary in the number of elements they contain, we can; then, assign a number to a set. We can only assign the same number, however, to those sets which have a one-to-one correspondence between them. In this discussion we shall consider only finite sets.

When we wish to talk about the number of a set we shall use the following notation: n(A). This is read: "the number of set A." More briefly it is at times read: "n of A."

For the sets:

$$\{1, 2, 3, 4\} \text{ and } \{0, 1, \bigwedge, V\},$$

we can now write:

$$n(\{1, 2, 3, 4\}) = n(\{0, 1, \bigwedge, V\}).$$

Counting

The use of the counting numbers: (1, 2, 3, 4, 5, 6, 7, ...), gives us a basic sequence which we may consider as the numbers of finite sets. Every counting number, then, may be considered as the number of the set of all counting numbers up to and including it.

Counting can be considered as a method of matching between any finite set and a subset of the counting numbers. Let us designate the set of counting numbers as C. Further, let us label the subsets of C as C_1, C_2, C_3, ... ; where $C_1 = \{1\}$, $C_2 = \{1, 2\}$, $C_3 = \{1, 2, 3\}$, and so on. As an example, let us count the set A composed of {Sam, Carl, Tom, Jack}.

Set A: {Sam, Carl, Tom, Jack}

Set C: {1, 2, 3, 4, 5, 6, 7, ...}.

By matching you see that set A matches with subset C_4 of the set C. Since $n(C_4) = 4$, then $n(A) = 4$.

Consider set A: {1, 2, 3, 4}, and set B: {5, 6, 7}, which are said to be disjoint. Two sets are said to be disjoint if they contain no elements in common. Now do you remember the expression $A \cup B$? Applying the operation we get a new set: {1, 2, 3, 4, 5, 6, 7}. Upon matching this new set with C, you note that it is C_7. So $n(A \cup B) = n(C_7) = 7$. Let us consider the problem through another example: Given the disjoint sets, M: {a, b, c, d}, and N: {e, f, g}. Now $M \cup N$ = {a, b, c, d, e, f, g}. Upon matching this new set with C, you notice that it is also C_7. Hence we have: $n(M \cup N) = n(C_7) = 7$.

Do you now notice that the number of the union of the two disjoint sets may be considered as the sum of the number of the sets?

Exercises 1-5

1. What is the number name of the following sets?

   (a) {1, 2, 3, 4, 5, 6}.

   (b) {a, b, c, d}.

   (c) {bird, dog, cat, chair, horn}.

   (d) {1, X , *, □, V, Λ }.

   (e) Which of the above sets have the same number?

2. Suppose a set R matches subset T of another set S. What can you say about the number of R in relationship to the number of S?

3. Considering only finite sets, if set M matches set N, and set N matches set R, what is the relationship of set M to set R?

4. How does the number of the set of automobiles being driven at this moment compare with the number of the set of their steering wheels?

5. By matching the sets C_{12} and C_7, show that $7 < 12$.

6. Given two sets A: {Bob, Sue, Tom, Joe} and B: {cat, dog, chair}. Find the set $A \cup B$. Now match the union of these sets with C and determine the number of the union set.

7. Given the two disjoint sets M: {1, 2, 3, 4}, and N: {5, 6, 7, 8, 9}. Find $M \cup N$ and determine $n(M \cup N)$ by comparing it with C.

BIBLIOGRAPHY

1. Allendoerfer, C. B. and C. O. Oakley, PRINCIPLES OF MATHEMATICS, McGraw-Hill Book Co., New York, 1955. Chapter 5.

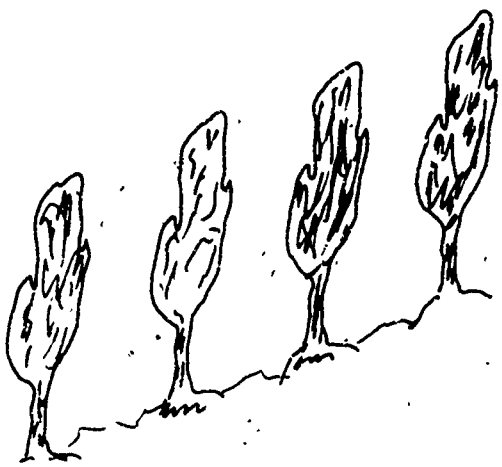
2. Kemeny, J. G., J. L. Snell, and G. L. Thompson. FINITE MATHEMATICS, Prentice-Hall, Inc., Englewood Cliffs, N. J. 1956. Chapter 2.

SUPPLEMENTARY UNIT 2

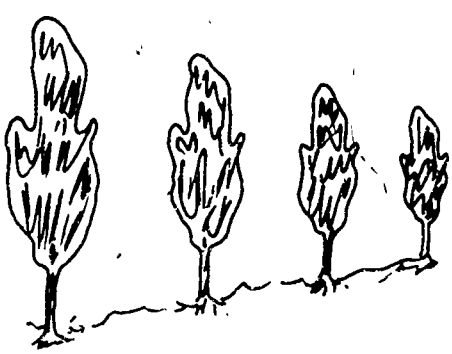
SPECIAL FIGURES IN PROJECTIVE GEOMETRY

2-1. Geometry and Art

In a certain park there is a row of poplar trees. They are evenly spaced, and all the same size and shape. Two boys wanted to draw a picture of them. The first said, "I know that these trees are all the same size. I know that there is the same distance between any two adjacent ones. This is how I will draw them."



The other said, "The trees further off look smaller to me, and even though I know they are not smaller I will draw them as I see them." Which of their pictures do you like better?



The second boy used the idea of perspective. This is a very important idea in art if we are interested in drawing things the

way they really look to us. It is the idea used in giving depth to a picture.

Of course, not all artists have wanted to do this. In ancient Egyptian art, for example, it was the rule to draw the pharaoh larger than anyone else in a picture, and the sizes of other people were made to depend on their importance.

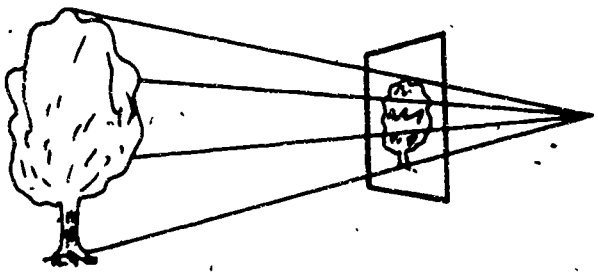
Not until the end of the Middle Ages did artists make serious systematic efforts to understand perspective. At that time they became greatly interested in learning the rules that would help them picture realistically the world about them. This period, which historians call the Renaissance, was a time of great development in science and learning as well as art. It was a time of new ideas and of a new interest in understanding the laws of nature. It was a time of experiment.

One of the artists of this period was Leonardo da Vinci. Though we remember him best for his paintings, he had a wide range of interests. Among other things he tried to design a way man could fly. He believed that a knowledge of science and mathematics is an essential tool for the artist.

An artist who did a great deal of work in developing rules of perspective was Albrecht Dürer. In some of his drawings we can see the way in which he studied these problems. You can find examples of them in Mathematics in Western Culture, by Morris Kline. This book contains many other pictures you will also find interesting.

A mathematician, Girard Desargues, wrote a book about the ideas of geometry that would be useful in connection with the study of perspective. He was the originator of what is called projective geometry.

The word "projective" can be understood if we think about drawing a picture. In drawing a tree, you can think of a line extending from each point you see to your eye. Each line intersects the

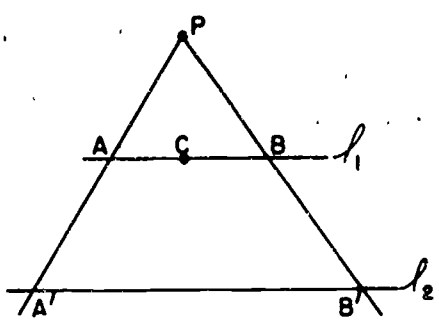


plane of your canvas in a point. The points in the picture thus match the points of the tree that we see. A geometer says that the picture (the set of points) on the canvas is a projection of the set of points of the tree.

Here is another example that will help you understand the sort of problems that occur in projective geometry. Suppose there is a triangular rose bed in a garden. Suppose an artist draws this rose bed several times. Perhaps he draws it first as seen from a point in the garden. Next he draws it as seen from the top of a high tower. Perhaps he tries other locations as well. He will find that in his pictures the rose bed is always triangular. He will find, however, that the triangle has different shapes depending on where he stands. He has discovered: The projection of a triangle is a triangle. Later we will see another discovery that can be made about this situation.

Projective Geometry in a Plane --
One-to-one Correspondences of Point Sets

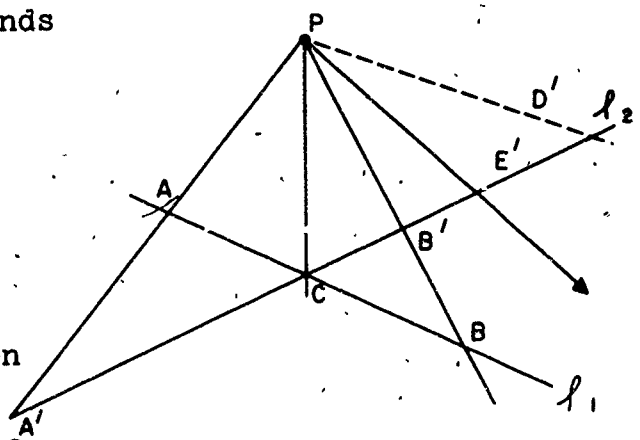
In this figure, lines ℓ_1 and ℓ_2 are parallel. Lines drawn from point P intersect lines ℓ_1 and ℓ_2. One such line intersects ℓ_1 in A and ℓ_2 in A'. Another intersects ℓ_1 in B and ℓ_2 in B'. The figure gives us a way of



matching the points on ℓ_1 with the points on ℓ_2. To find the point on ℓ_2 that matches C, for example, we would draw the line through C and P. The point where it intersects ℓ_2 is the point that matches C.

This matching of one set (the points on ℓ_1) with another set (the points on ℓ_2) is called a one-to-one correspondence, as we know. We have found a one-to-one correspondence between the points on ℓ_1 and the points on ℓ_2. (Of course, if we used some other point in place of P we would find another one-to-one correspondence between the points on ℓ_1 and those on ℓ_2. The two point sets can be matched in many different ways.)

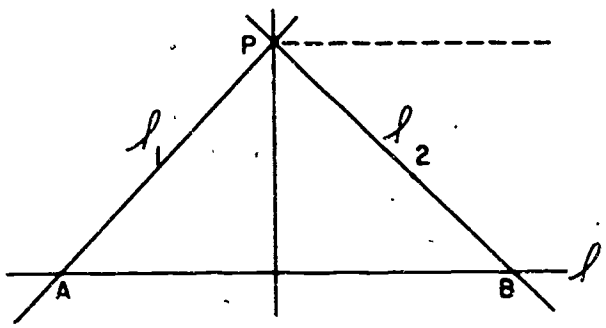
Did you wonder why we chose parallel lines for ℓ_1 and ℓ_2? Let us see what would happen if we did not. In the next figure ℓ_1 and ℓ_2 are not parallel. We can still draw lines through P cutting ℓ_1 and ℓ_2. Point A on ℓ_1 corresponds to point A' on ℓ_2. Point B corresponds to B'. Point C is a special point. It belongs to both the set of points on ℓ_1 and the set of points on ℓ_2. A line through P that intersects ℓ_1 in C also intersects ℓ_2 in C. In the correspondence between points on ℓ_1 and points on ℓ_2 the point C matches itself.



It looks as though we have once again a one-to-one correspondence between the points on ℓ_1 and the points on ℓ_2. But we need to stop and think very carefully. We need to remember that there is one line through P that is parallel to ℓ_1. Suppose this line (the dotted line in the figure) intersects ℓ_2 in the point D'. D' is a point on ℓ_2, but our system does not give any point on ℓ_1 that matches it. Points on ℓ_2 that are very close to D' match points that are very far out on ℓ_1. E' is one such point.

There is also a line through P that is parallel to ℓ_2. So there is also a point on ℓ_1 that has no matching point on ℓ_2. We have discovered: Our system gives us a way of matching each point except one on ℓ_1 with a point on ℓ_2, and of matching each point except one on ℓ_2 with a point on ℓ_1.

Let us consider another example. This figure shows some of the elements of the set of lines through P. Each of the lines through P in the figure intersects the line ℓ in a point. The figure shows a way of matching elements of the set of lines through P with elements of the set of points on ℓ. The line ℓ_1 matches the point A. The line ℓ_2 corresponds to point B.



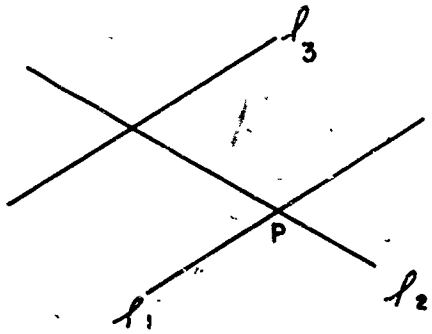
Again, however, we need to be careful. There is one line through P that is parallel to ℓ. This line does not have a matching point on ℓ. We see that: To each point on ℓ corresponds a line through P. To each line through P <u>except one</u> there corresponds a point on ℓ.

The Idea of Ideal Points

These examples will help you understand an idea that is very useful in projective geometry. It is the idea of an <u>ideal point</u> on a line.

In projective geometry we do not use the term "parallel lines." Another way of saying "two lines are parallel" is "two lines intersect in an ideal point." We think of each line as containing one and only one ideal point, as well as the usual points we are accustomed to thinking about. We also assume each ideal point is

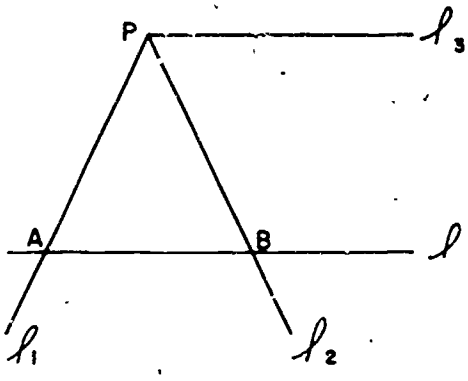
on a line. Actually each ideal point is on a set of lines, which are "parallel." In order to be quite clear, we can call the usual points <u>real points</u>. When we adopt this new language, we can say that <u>any</u> two lines in a plane meet in a point of some sort. In the figure, ℓ_1 and ℓ_2 meet in the real point P. ℓ_1 and ℓ_3 meet in an ideal point. Formerly we would have said they are parallel. The two statements mean the same thing.



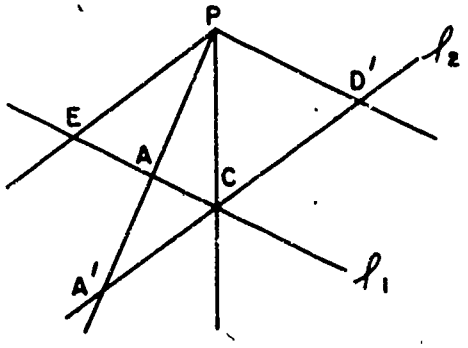
In our new language, the set of all points on a line is made up of all the real points of the line and, in addition, the ideal point of the line.

Let us use this new vocabulary to describe the one-to-one correspondences which we have already studied. As we do so, we will find that it is a very convenient language for describing these situations.

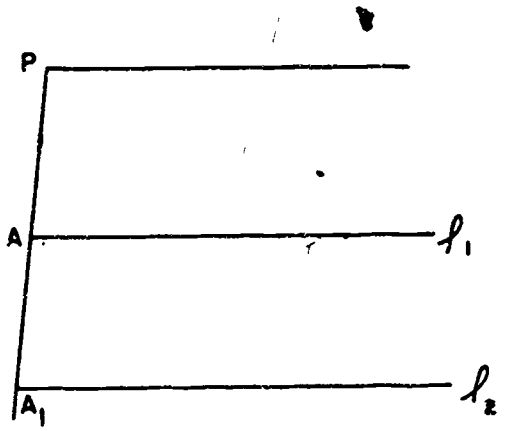
In this figure we can now say that there is a one-to-one correspondence between the set of <u>all</u> lines through P and the set of <u>all</u> points on ℓ. Line ℓ_1 corresponds to the real point A. Line ℓ_2 corresponds to the real point B. Line ℓ_3 we now say, intersects line ℓ in the <u>ideal</u> point of ℓ. It corresponds to the ideal point on ℓ.



In this figure we can now say that there is a one-to-one correspondence between the set of all points on ℓ_1 and the set of all points on ℓ_2. The real point A on ℓ_1 corresponds to the real point A' on ℓ_2. The real point C belongs to the set of all points on ℓ_1 and to the set of all points on ℓ_2. It corresponds to itself. The point D' on ℓ_2 corresponds to the ideal point on ℓ_1. The point E on ℓ_1, corresponds to the ideal point on ℓ_2. (Remember that we now say that each line contains an ideal point. The line through P and E intersects ℓ_2 in the ideal point.)



In this figure ℓ_1 and ℓ_2 intersect in an ideal point. There is a one-to-one correspondence between the set of all points on ℓ_1 and the set of all points on ℓ_2. The line through P and A intersects ℓ_1 and ℓ_2 in corresponding real points. The line through P parallel to ℓ_1 and ℓ_2 intersects ℓ_1 and ℓ_2 in an ideal point. This ideal point is an element of the set of all points on ℓ_1. It is also an element of the set of all points on ℓ_2. It corresponds to itself in the one-to-one correspondence.



We have introduced the idea of ideal point so that every pair of lines intersects in a point, that is, "two lines determine a point." What about the statement, "Two points determine a line," by which we mean that there is exactly one line through any two points? This is certainly true in the geometry that we are used to, that is, for two real points. But is it still true for pro-

jective geometry? We have three cases to consider.

Case 1. Two real points A and B determine a line. This is true in projective geometry just as it is true in the geometry we know.

Case 2. If A is an ideal point and B is a real point, then A and B determine a line. To see why this is so, let ℓ_1 be some line through A. Then, we know from our familiar geometry that there is exactly one line through B parallel to ℓ_1. Call this line ℓ_2. Then ℓ_1 and ℓ_2 will intersect in some ideal point of ℓ_1 which must be A since ℓ_1 has only one ideal point. We have thus shown that ℓ_2 is the only line through A and B. There is one line through A and B and there is only one.

Case 3. Suppose A and B are two ideal points. To take care of this case we define an "ideal line" which is the set of all ideal points.

As a result we have:

> In Projective Geometry not only do two points determine a line but two lines determine a point.

We do not have to distinguish between ideal and real points in this statement. This symmetrical arrangement is very convenient.

The language of ideal points is probably new to you. Like any new language, it may seem difficult until one is accustomed to it. The examples illustrate its advantages. When we use the idea of ideal points we do not have to consider parallel lines as exceptions to our descriptions.

You will understand better how the idea of ideal points originated if you think about railroad tracks. When we draw railroad tracks we draw them as though they come together far away. The idea of ideal point is suggested by the way parallel lines sometimes appear to meet when we draw objects in perspective.

Of course, if you are building a railroad track the idea of ideal points is not useful at all. When we build railroad tracks

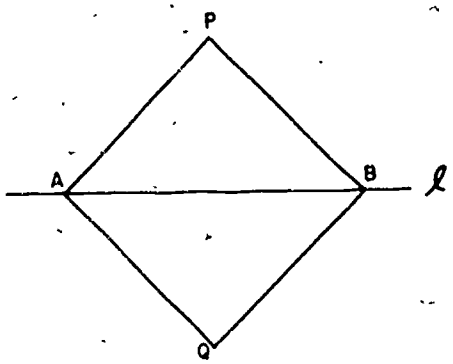
we need to know, for example, that lengths of the parts of the ties that lie between the tracks are all the same. The idea of length is studied in metric geometry. Metric geometry uses the idea of measurement. Projective geometry does not; this is why we say that projective geometry is non-metric.

You may feel that ideal points seem unnatural. But you should remember that all points, lines, and planes are ideas. They are ideas that are developed because they are interesting and useful for some purpose.

Exercises 2-1

1. Draw two parallel lines. Call them ℓ_1 and ℓ_2. Mark a point P between them. By drawing lines through P, find a one-to-one correspondence between the points on ℓ_1 and the points on ℓ_2. Label the points in your drawing, and name three pairs of corresponding points.

2. Mark points P and Q. Draw a line ℓ, as in the figure. The figure shows a way of matching the set of lines through P with the set of lines through Q. To the line through P and A corresponds the line through Q and A. The line through P and B is matched with the line through Q and B. In this way we can find a _____ _____ between the set of lines through P and the set of lines through Q. Draw three other pairs of lines illustrating this statement.

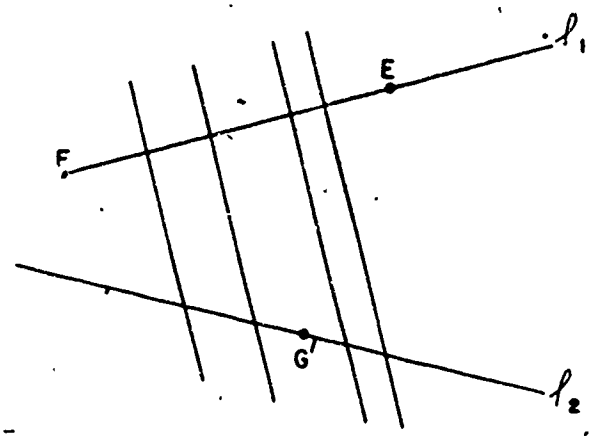


3. In Exercise 2, is there a line which belongs both to the set of lines through P and the set of lines through Q?

4. In Exercise 2, which line through P corresponds to the line through Q parallel to ℓ ? This line through P intersects ℓ in an _____.

5. Explain the meaning of the following statement: If P is any real point and ℓ is any line not passing through P, there is exactly one line which passes through P and through the ideal point on ℓ.

6. In this figure four of the lines are parallel.



(a) Four of the lines intersect in an

_____

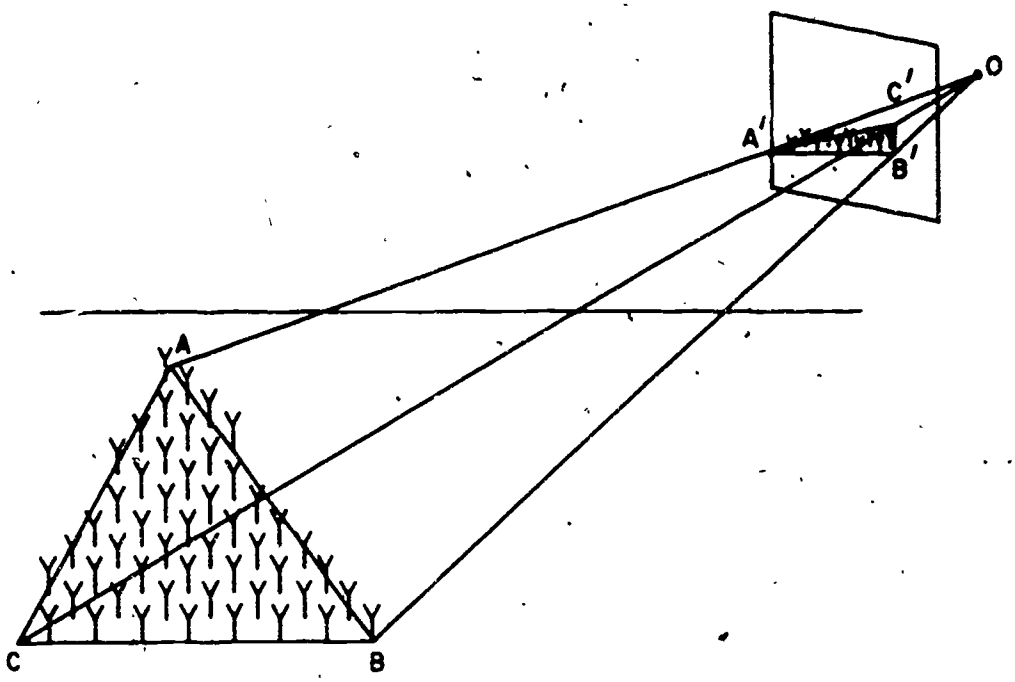
_____.

(b) The figure shows a system for finding a one-to-one correspondence between the points of ℓ_1 and the points ℓ_2. Find the points corresponding to E, F, and G'.

2-2. Desargues' Theorem

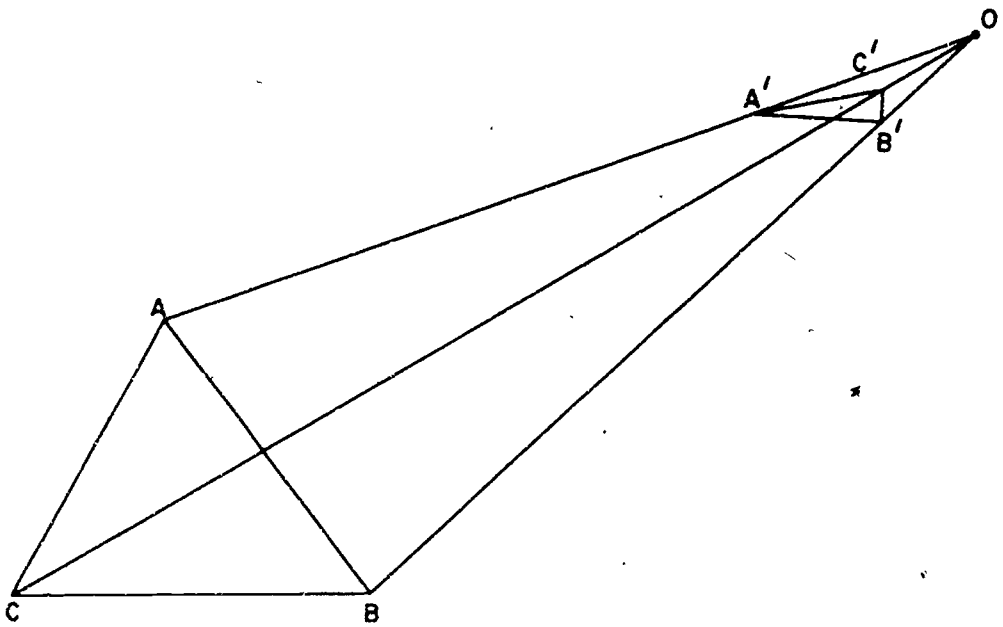
One of the most interesting ideas in projective geometry is that contained in Desargues' Theorem. In order to understand it, let us think again about a situation we considered earlier. Let us think about an artist who is drawing a triangular rose bed. Suppose that he is drawing his picture as he sees it from a tower high above a garden. On the following page there is a sketch that shows the two triangles -- the boundary of the rose

bed and the picture of it on his canvas. Each point on the rose bed is matched with a point on the canvas triangle.



In the sketch the vertices of the rose bed are called A, B, and C. In the artist's picture, the matching vertices are labeled A', B', C'. The three lines joining matching vertices all meet in point O -- the eye of the artist. The two triangles are said to be in perspective.

We can draw two triangles in the same plane that are in perspective. In the following figure two such triangles have been drawn. Again, the vertices of one triangle are matched with the vertices of the other. Again, the lines joining corresponding vertices meet in a point.

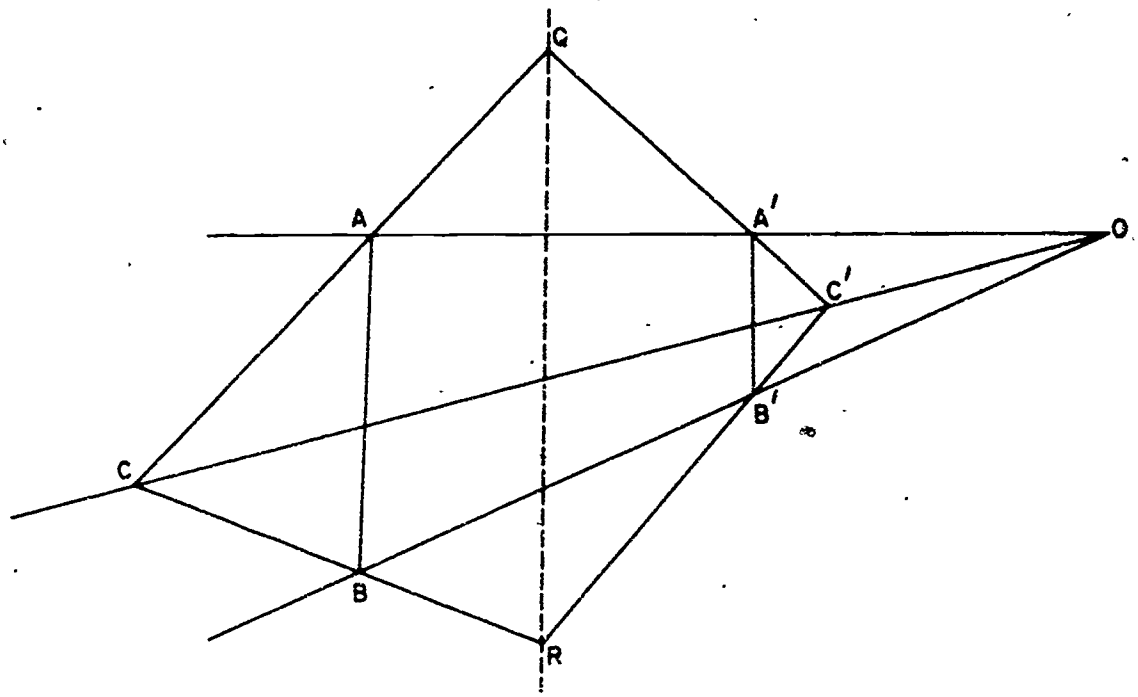


Exercise 2-2

Copy this figure carefully. Extend $\overline{AB}$ and $\overline{A'B'}$ until they intersect. Do the same thing with $\overline{AC}$ and $\overline{A'C'}$. Do the same thing with $\overline{BC}$ and $\overline{B'C'}$. You have found three intersection points. Label them P, Q, and R. Do you notice anything about these three points? They should all lie on the same line.

A boy said, "I wonder whether this will always be true if I extend the sides of a pair of triangles in perspective." He tried it several times. It appeared to be true each time. Of course it was sometimes difficult to be sure, because he needed to extend the lines a long way to find the intersection points. He decided, however, that it was probably always true that the three points of intersection were on the same line.

"But what about this figure?" asked another boy. "In my triangles, $\overline{AB}$ and $\overline{A'B'}$ have the same direction. When I extend them I get parallel lines. There is no real point of intersection."



"I notice something about the figure you have drawn, though," the first boy replied. "Those two lines are parallel to the line through Q and R. I think that this is another place where the idea of ideal point might be useful. We could say that the three points of intersection are all on the same line, but now one of the points is an ideal point."

He was right. If it is true that

(a) two triangles are in perspective, and

(b) each pair of corresponding sides, extended, has a point of intersection,

then the three points of intersection all lie on the same line.

Other cases involving pairs of "parallel" lines can conveniently be described by the idea of ideal point.

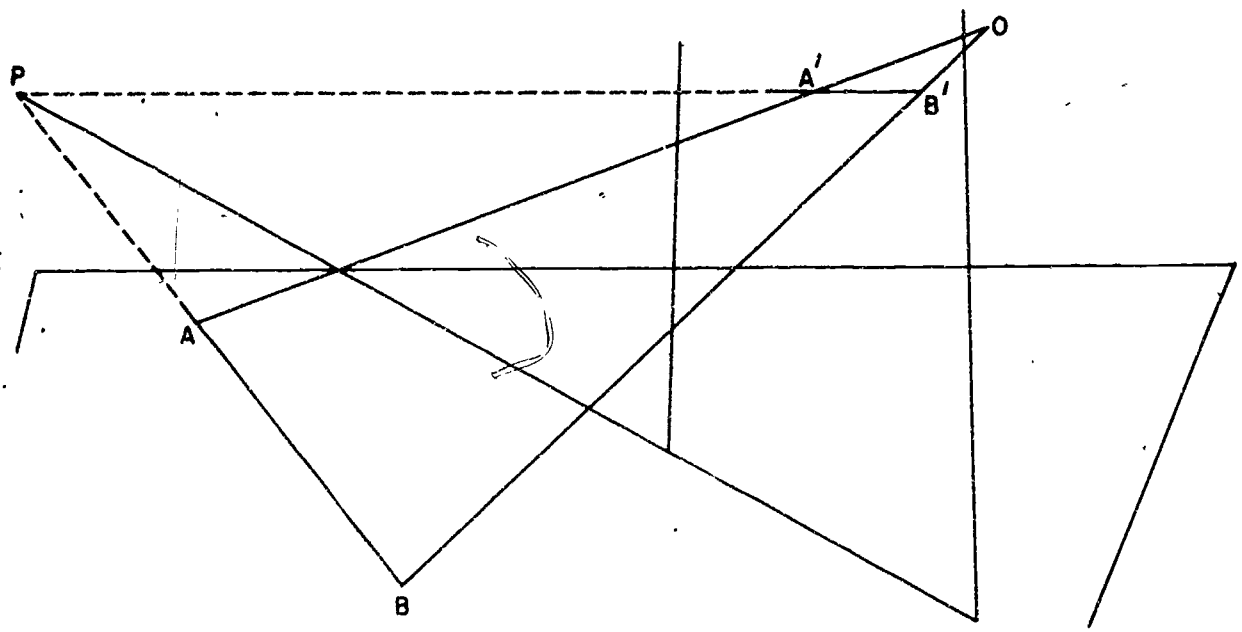
Of course, the second boy was not satisfied with leaving the matter at this. He wondered why the three intersection points all were on the same line. Perhaps you wonder too. If you do, you will be interested in knowing the way we prove that the points are always on a line. A proof makes us sure the statement is true -- a good proof also makes us understand better the reason.

Let us again think about the garden and the picture. Let us suppose that:

(a) the plane of the garden and the plane of the picture are not parallel (this is the way we drew the figure).

(b) none of the pairs of corresponding sides have the same direction.

Look at the line through A and A' and the line through B and B'. This figure will help you see the lines.



These two lines intersect at O. When we have a pair of intersecting lines, we can think about the plane they both lie in. The line through A and B is in this plane; so is the line through A' and B'. We supposed that these lines did not have the same direction. We know that two lines in the same plane that do not have the same direction meet, so we can be sure that these lines meet. P, of course, is the point where they meet.

Now let us think about where P is. P is on the line through A and B. This line is on the plane of the garden. So P must be on the plane of the garden. P is also on the line through A' and B', which is on the plane of the canvas. So P is also on the plane of the canvas. Now we can put these two facts together and say: P is on the intersection of two planes -- the plane of the canvas and that of the garden. The intersection of these two planes is a line.

Now we have proved that P is on the line of intersection of a certain pair of planes. We can prove in precisely the same way that the line through B and C and the line through B' and C' meet in a point, which we might label Q. We can also prove, by exactly the same reasoning as that used in the case of P, that Q is on the line of intersection of the plane of the canvas and the plane of the garden. Then we can reason the same way about the point R, the point of intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$.

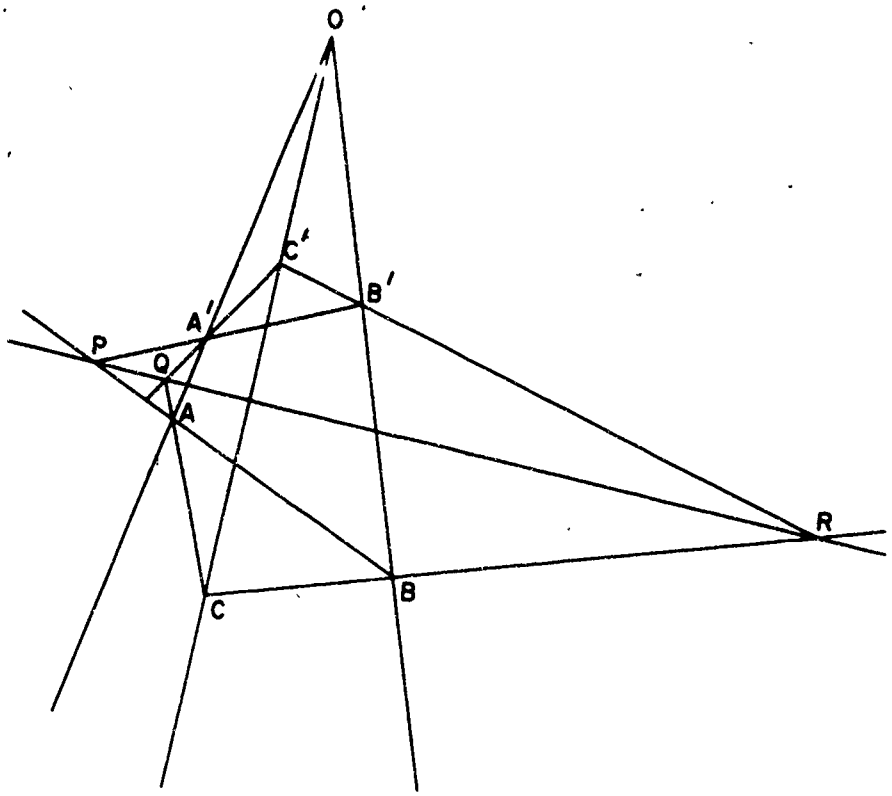
So we can see that P, Q, and R all lie on the same line -- the line where our two planes intersect.

Now we have proved our fact for two triangles that are in different (and not parallel) planes.

It is more difficult to prove that it is true when the two triangles are in the same plane. We can see, however, that if we took a picture of the garden and the canvas, we would really have two triangles in perspective in the same plane, and that the points of intersection of the pairs of corresponding sides of

corresponding sides of the triangles would all be on a line. When you are more familiar with the use of geometric reasoning in rather complicated figures, you should not find it difficult to use this idea in constructing a complete proof.

2-3. Points and Lines in Desargues' Theorem



In the figure we see that there are 10 main points: the vertices of the two triangles, the point O, and the three intersection points P, Q, and R. There are also 10 main lines: the sides of the triangle extended, the lines through corresponding vertices of the triangles, and the line on which lie P, Q, and R. By checking the figure you can see that --

(a) through each of the labelled points there are three of the special lines, and

(b) on each of the special lines there are three labelled points.

The figure for Desargues' theorem could be used for a very "democratic" committee diagram, where, by "democratic" we mean that in certain respects each committee member is treated like every other one. We could let each of the ten points correspond to a person and each of the ten lines correspond to a committee. If a certain point is on a certain line, then the corresponding person would be on the corresponding committee. Then

1. Each committee has three members and each person is on three committees.
2. Each pair of committees has no more than one person in common and each pair of persons is on no more than one committee.
3. Each committee has exactly one person in common with six other committees and each person is on a committee with six other persons.

Exercises 2-3

1. Draw several figures illustrating Desargues' theorem.
2. One of the remarkable aspects of the figure for Desargues' theorem is that each point and each line play exactly the same role. For example, we might think of A as the "beginning" point in place of O and one triangle could be taken to be COB. Since the third point on $\overleftrightarrow{AC}$ is Q, the third point on $\overleftrightarrow{AO}$ is A', and the third point on $\overleftrightarrow{AB}$ is P, the second triangle must be QA'P. Then the points of intersection of corresponding sides of the two triangles should be on a line. Find the line.
3. Follow through the steps in Exercise 2 starting with the point P.

4. The following converse of Desargues' theorem also holds: If ABC and $A'B'C'$ are two triangles and if the points P, Q, R defined as the intersections of the pairs $\overleftrightarrow{AB}$, $\overleftrightarrow{A'B'}$; $\overleftrightarrow{AC}$, $\overleftrightarrow{A'C'}$; $\overleftrightarrow{BC}$, $\overleftrightarrow{B'C'}$ lie on a line, then $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ are concurrent. Draw a figure which shows this.

5. (Brainbuster) Designate seven points by the numbers: 1, 2, 3, 4, 5, 6, 7. Call the set of three points 1, 2, 4 a "line ℓ_1" and so on according to the following table:

Line	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
Points	1,2,4	2,3,5	3,4,6	4,5,7	5,6,1	6,7,2	7,1,3

Show that each point lies on three lines. Is it true that each pair of points determines a line? Is it true that each pair of lines determines a point? Draw a figure which shows this. (You cannot make all the lines straight and one will have to jump over another.)

SUPPLEMENTARY UNIT 3

REPEATING DECIMALS AND TESTS FOR DIVISIBILITY

3-1. Introduction

This unit is for the student who has studied a little about repeating decimals, numeration systems in different bases, and tests for divisibility (casting out the nines, for instance) and would like to carry his investigation a little further, under guidance. The purpose of this monograph is to give this guidance; it is not just to be read. You will get the most benefit from this material if you will first read only up to the first set of exercises and then without reading any further do the exercies. They are not just applications of what you have read, but to guide you in discovery of further important and interesting facts. Some of the exercises may suggest other questions to you. When this happens, see what you can do toward answering them on your own. Then, after you have done all that you can do with that set of exercises, go on to the next section. There you will find the answers to some of your questions, perhaps, and a little more information to guide you toward the next set of exercises.

The most interesting and useful phase of mathematics is the discovery of new things in the subject. Not only is this the most interesting part of it, but this is a way to train yourself to discover more and more important things as time goes on. When you learned to walk you needed a helping hand, but you really had not learned until you could stand alone. Walking was not new to mankind -- lots of people had walked before -- but it was new to you. And whether or not you would eventually discover places in your walking which no man had ever seen before, was unimportant. It was a great thrill when you first found that you could walk, even though it looked like a stagger to other people. So, try learning to walk in mathematics. And be independent -- do not accept any more help than is necessary.

3-2. Casting out the Nines

You may know a very simple and interesting way to tell whether a number is divisible by 9. It is based on the fact that a number is divisible by 9 if the sum of its digits is divisible by 9; also the sum of the digits of a number is divisible by 9, if the number is divisible by 9. For instance, consider the number 156782. The sum of its digits is 1 + 5 + 6 + 7 + 8 + 2 which is 29. But 29 is not divisible by 9 and hence the number 156782 is not divisible by 9. If the second digit had been a 3 instead of 5, or if any of the digits to the right of 5 had been 2 less, the number would have been divisible by 9 since the sum of the digits would have been 27 which is divisible by 9. The test is a good one because it is easier to add the digits than to divide by 9. Actually we could have been lazy and instead of dividing 29 by 9, use the fact again, add 2 and 9 to get 11, add the 1 and 1 to get 2 and see that since 2 is not divisible by 9, then the original six digit number is not divisible by 9.

Why is this true? Merely dividing the given number by 9 would have tested the result but from that we would have no idea why it would hold for any other number. We can show what is happening by writing out the number 156,782 according to what it means in the decimal notation:

$$1 \times 10^5 + 5 \times 10^4 + 6 \times 10^3 + 7 \times 10^2 + 8 \times 10 + 2 =$$

$$1 \times (99999 + 1) + 5 \times (9999 + 1) + 6 \times (999 + 1) +$$

$$7 \times (99 + 1) + 8 \times (9 + 1) + 2.$$

Now by the distributive property, $5 \times (9999 + 1) = (5 \times 9999) + (5 \times 1)$ and similarly for the other expressions. Also we may rearrange the numbers in the sum since addition is commutative. So our number 156,782 may be written

$$1 \times (99999) + 5 \times (9999) + 6 \times (999) +$$

$$7 \times (99) + 8 \times 9 + (1 + 5 + 6 + 7 + 8 + 2).$$

Now 99999, 9999, 999, 99, 9 are all divisible by 9, the products involving these numbers are divisible by 9 and the sum of these products is divisible by 9. Hence the original number will be divisible by 9 if (1 + 5 + 6 + 7 + 8 + 2) is divisible by 9. This sum is the sum of the digits of the given number. Writing it out this way shows that no matter what the given number is, the same principle holds.

Exercises 3-2

1. (a) Test each of the numbers, 226843, 67945, 427536, and 45654 by the above method for divisibility by 9.

   (b) For any numbers in part (a) that are not divisible by 9, compare the remainders when the number is divided by 9 and when the sum of the digits is divided by 9.

   (c) From part (b) try to formulate a general fact that you suspect is true. Test this statement with a few more examples.

2. Choose two numbers. First, add them, divide by 9 and take the remainder. Second, divide each number by nine and find the sum of the remainders; divide the sum by 9 and take the remainder. The final remainders in the two cases are the same. For instance, let the numbers be 69 and 79. First, their sum is 148 and the remainder when 148 is divided by 9 is 4. Second, the remainder when 69 is divided by 9 is 6 and when 79 is divided by 9 is 7; the sum of 6 and 7 is 13, and if 13 is divided by 9, the remainder is 4. The result is 4 in both cases. Why are the two results the same no matter what numbers are used instead of 69 and 79? Would a similar result hold for a sum of three numbers? (Hint: write 69 as $7 \times 9 + 6$.)

3. If in the previous exercise we divided by 7 instead of 9, would the remainders by the two methods for division by 7 be the same? Why or why not?

4. Suppose in Exercise 2 we considered the product of two numbers instead of their sum. Would the corresponding result hold? That is, would the remainder when the product of 69 and 79 is divided by 9 be the same as when the product of their remainders is divided by 9? Would this be true in general? Could they be divided by 23 instead of 9 to give a similar result? Could similar statements be made about products of more than two numbers?

5. Use the result of the previous exercise to show that 10^{20} has a remainder of 1 when divided by 9. What would its remainder be when it is divided by 3? By 99?

6. What is the remainder when 7^{20} is divided by 6?

7. You know that when a number is written in the decimal notation, it is divisible by 2 if its last digit is divisble by 2, and divisible by 5 if its last digit is 0 or 5. Can you devise a similar test for divisibility by 4, 8, or 25?

8. In the following statement, fill in both blanks with the same number so that the statement is true:
A number written in the system to the base twelve is divisible by_____if its last digit is divisible by_____. If there is more than one answer, give the others, too. If the base were seven instead of twelve, how could the blanks be filled in? (Hint: one answer for base twelve is 6.)

9. One could have something like "decimal" equivalents of numbers in numeration systems to bases other than ten. For instance, in the numeration system to the base seven, the septimal equivalent of $5(\frac{1}{7}) + 6(\frac{1}{7})^2$ would be written $.56_7$. Just as the decimal equivalent of $5(\frac{1}{10}) + 6(\frac{1}{10})^2$ would be written $.56_{10}$ in the decimal system. The number .142857142857 ... is equal to $\frac{1}{7}$ in the decimal system and in the system to the base seven would be written $.1_7$. On the other hand, $.1_{10} = (.04620462 \ldots)_7$. What numbers would have terminating septimals in the numeration system to the base 7? What would the septimal equivalent of $\frac{1}{5}$ be in the system to the base 7? (Hint: remember that if the only prime factors of a number are 2 and 5, the decimal equivalent of its reciprocal terminates.)

10. Use the result of Exercise 3 to find the remainder when $9 + 16 + 23 + 30 + 37$ is divided by 7. Check your result by computing the sum and dividing by 7.

11. Use the results of the previous exercises to show that $10^{20} - 1$ is divisible by 9, $7^{108} - 1$ is divisible by 6.

12. Using the results of some of the previous exercises if you wish, shorten the method of showing that a number is divisible by 9 if the sum of its digits is divisible by 9.

13. Show why the remainder when the sum of the digits of a number is divided by 9 is the same as the remainder when the number is divided by 9.

3-3. Why Does Casting Out the Nines Work?

First let us review some of the important results shown in the exercises which you did above. In Exercises 2, you showed that to get the remainder of the sum of two numbers, after division by 9, you can divide the sum of their remainders by 9 and find its remainder. Perhaps you did it this way (there is more than one way to do it; yours may have been better). You know in the first place that any natural number may be divided by 9 to

get a quotient and remainder. For instance, if the number is 725, the quotient is 80 and the remainder is 5. Furthermore, $725 = (80 \times 9) + 5$ and you could see from the way this is written that 5 is the remainder. Thus, using the numbers in the exercise, you would write $69 = 7 \times 9 + 6$ and $79 = 8 \times 9 + 7$. Then $69 + 79 = (7 \times 9) + 6 + (8 \times 9) + 7$. Since the sum of two numbers is commutative, you may reorder the terms and have $69 + 79 = (7 \times 9) + (8 \times 9) + 6 + 7$. Then, by the distributive property, $69 + 79 = [(7 + 8) \times 9] + 6 + 7$. Now the remainder when $6 + 7$ is divided by 9 is 4 and $6 + 7$ can be written $(1 \times 9) + 4$. Thus $69 + 79 = [(7 + 8 + 1) \times 9] + 4$. So, from the form it is written in, we see that 4 is the remainder when the sum is divided by 9. It is also the remainder when the sum of the remainders, $6 + 7$, is divided by 9.

Writing it out in this fashion is more work than making the computations the short way but it does show what is going on and why similar results would hold if 69 and 79 were replaced by any other numbers, and, in fact, we could replace 9 by any other number as well. One way to do this is to use letters in place of the numbers. This has two advantages. In the first place it helps us be sure that we did not make use of the special properties of the numbers we had without meaning to do so. Secondly, we can, after doing it for letters, see that we may replace the letters by any numbers. So, in place of 69 we write the letter a, and in place of 79, the letter b. When we divide the number a by 9 we would have a quotient and a remainder. We can call the quotient the letter q and the remainder, the letter r. Then we have

$$a = (q \times 9) + r$$

where r is some whole number less than 9. We could do the same for the number b, but we should not let q be the quotient since it might be different from the quotient when a is divided by 9. We here could call the quotient q' and the remainder r'. Then we would have

$$b = (q' \times 9) + r'.$$

Then the sum of a and b will be

$$a + b = (q \times 9) + r + (q' \times 9) + r'.$$

We can use the commutative property of addition to have

$$a + b = (q \times 9) + (q' \times 9) + r + r'$$

and the distributive property to have

$$a + b = [(q + q') \times 9] + r + r'.$$

Then if $r + r'$ were divided by 9, we would have a quotient which we might call q'' and a remainder r''. Then $r + r' = (q'' \times 9) + r''$ and

$$\begin{aligned} a + b &= [(q + q') \times 9] + (q'' \times 9) + r'' \\ &= [(q + q' + q'') \times 9] + r''. \end{aligned}$$

Now r'' is a whole number less than 9 and hence it is not only the remainder when $r + r'$ is divided by 9 but also the remainder when $a + b$ is divided by 9. So as far as the remainder goes, it does not matter whether you add the numbers or add the remainders and divide by 9.

The solution of Exercise 4 goes the same way as that for Exercise 2 except that we multiply the numbers. Then we would have

$$\begin{aligned} 69 \times 79 &= (7 \times 9 + 6) \times (8 \times 9 + 7) \\ &= [(7 \times 9) \times (8 \times 9 + 7)] + 6 \times (8 \times 9 + 7) \\ &= (7 \times 9 \times 8 \times 9) + (7 \times 9 \times 7) + (6 \times 8 \times 9) + (6 \times 7). \end{aligned}$$

The first three products are divisible by 9 and by what we showed in Exercise 2, the remainder when 69×79 is divided by 9 is the same as the remainder when $0 + 0 + 0 + 6 \times 7$ is divided by 9. So in finding the remainder when a product is divided by 9, it makes no difference whether we use the product or the product of the remainders.

If we were to write this out in letters as we did the sum, it would look like this:

$$a \times b = (q \times 9 + r) \times (q' \times 9 + r')$$
$$= (q \times 9 \times q' \times 9) + (q \times 9 \times r') + (r \times q' \times 9) + (r \times r').$$

Again each of the first three products is divisible by 9 and hence the remainder when $a \times b$ is divided by 9 is the same as when $r \times r'$ is divided by 9.

We used the number 9 all the way above, but the same conclusions would follow just as easily for any number in place of 9, such as 7, 23, etc. We could have used a letter for 9 also but this seems like carrying it too far.

There is a shorter way of writing some of the things we had above. When letters are used, we usually omit the multiplication sign and write ab instead of $a \times b$ and $9q$ in place of $9 \times q$. Hence the last equation above could be abbreviated to

$$ab = qq'9 \times 9 + qr'9 + rq'9 + rr'$$

or

$$ab = 9 \times 9qq' + 9qr' + 9rq' + rr'.$$

But this is not especially important right now.

So let us summarize our results so far: The remainder when the sum of two numbers is divided by 9 (or any other number) is the same as the remainder when the sum of the remainders is divided by 9 (or some other number). The same procedure holds for the product in place of the sum.

These facts may be used to give quite a short proof of the important result stated in Problem 13, of Exercises 3-2. Consider again the number 156,782. This is written in the usual form:

$$(1 \times 10^5) + (5 \times 10^4) + (6 \times 10^3) + (7 \times 10^2) + (8 \times 10) + 2.$$

Now from the result stated above for the product, the remainder when 10^2 is divided by 9 is the same as when the product of

the remainders 1×1 is divided by 9, that is, the remainder is 1. Similarly 10^3 has a remainder $1 \times 1 \times 1$ when divided by 9 and hence 1. So all the powers of ten have a remainder 1 when divided by 9. Thus, by the result stated above for the sum, the remainder when 156,782 is divided by 9 is the same as the remainder when $(1 \times 1) + (5 \times 1) + (6 \times 1) + (7 \times 1) + (8 \times 1) + 2$ is divided by 9. This last is just the sum of the digits. Writing it this way it is easy to see that this works for any number.

Now we can use the result of Problem 13 of Exercises 3-2 to describe a check called "casting out the nines" which is not used much in these days of computing machines, but which is still interesting. Consider the product 867×934. We indicate the following calculations:

867	sum of digits: 21	sum of digits: 3
934	sum of digits: 16	sum of digits: 7
Product:	809,778	Product: $3 \times 7 = 21$

Sum of digits: $8 + 0 + 9 + 7 + 7 + 8 = 39$
Sum of digits: $3 + 9 = 12$
Sum of digits: $1 + 2 = 3$ Sum of digits: $2 + 1 = 3$.

Since the two results 3 are the same, we have at least some check on the accuracy of the results.

Exercises 3-3

1. Try the method of checking for another product. Would it also work for a sum? If so, try it also.
2. Explain why this should come out as it does.
3. If a computation checks this way, show that it still could be wrong. That is, in the example given above, find an incorrect product that would still check.

4. Given the number $(5 \cdot 7^5) + (3 \cdot 7^4) + (2 \cdot 7^3) + (1 \cdot 7^2) + (4 \cdot 7) + 3$. What is its remainder when it is divided by 7? What is its remainder when it is divided by 6? by 3?

5. Can you find any short-cu' in the example above analogous to casting out the nines?

6. In a numeration system to the base 7, casting out what number would result corresponding to that in the decimal system when nines are cast out?

7. The following is a trick based on casting out the nines. Can you see how it works? You ask someone to pick a number -- it might be 1678. Then you ask him to form another number from the same digits in a different order -- he might take 6187. Then you ask him to subtract the smaller from the larger and give you the sum of all but one of the digits in the result. (He would have 4509 and might add the last three to give you 14.) All of this would be done without your seeing any of the figuring. Then you would tell him that the other digit in the result is 4. Does the trick always work?

One method of shortening the computation for a test by casting out the nines, is to discard any partial sums which are 9 or a multiple of 9. For instance, if a product were 810,645, we would not need to add all the digits. We could notice that $8 + 1 = 9$ and $4 + 5 = 9$ and hence the remainder when the sum of the digits is divided by 9 would be $0 + 6$, which is 6. Are there other places in the check where work could have been shortened? We thus, in a way, throw away the nines. It was from this that the name "casting out the nines" came.

By just the same principle, in a numeration system to the base 7 one would cast out the sixes, to the base 12 cast out the elevens, etc.

3-4. Divisibility by 11

There is a test for divisibility by 11 which is not quite so simple as that for divisibility by 9 but is quite easy to apply. In fact, there are two tests. We shall start you on one and let you discover the other for yourself. Suppose we wish to test the number 17945 for divisibility by 11. Then we can write it as before

$$(1\cdot10^4) + (7\cdot10^3) + (9\cdot10^2) + (4\cdot10^1) + 5.$$

The remainders when 10^2 and 10^4 are divided by 11 are 1. But the remainders when 10, 10^3, 10^5 are divided by 11 are 10. Now 10 is equal to 11 - 1. $10^3 = 10^2$ (11 - 1), $10^5 = 10^4$ (11 - 1). That is enough. Perhaps we have told you too much already. It is your turn to carry the ball.

Exercises 3-4-a

1. Without considering 10 to be 11 - 1, can you from the above devise a test for divisibility by 11?
2. Noticing that 10 = 11 - 1 and so forth as above, can you devise another test for divisibility by 11?

We hope you were able to devise the two tests suggested in the previous exercises. For the first, we could group the digits and write the number 17945 as $(1 \times 10^4) + (79 \times 10^2) + 45$. Hence the remainder when the number 17945 is divided by 11 should be the same as the remainder when 1 + 79 + 45 is divided by 11, that is, 1 + 2 + 1 = 4. (2 is the remainder when 79 is divided by 11, etc.) This method would hold for any number.

The second method requires a little knowledge of negative numbers (either review them or, if you have not had them, omit this paragraph). We could consider - 1 as the remainder when 10 is divided by 11. Then the original number would have the same remainder as the remainder when $1 + [7(-1)^3] + 9 + [4(-1)] + 5$ is divided by 11, that is, when 5 - 4 + 9 - 7 + 1 is divided by 11. This last sum is equal to 4 which was what

we got the other way. By this test we start at the right and alternately add and subtract digits. This is simpler than the other one.

Exercises 3-4-b

1. Test several numbers for divisibility by 11 using the two methods described above. Where the numbers are not divisible, find the remainders by the method given.
2. In a number system to the base 7, what number could we test for divisibility in the same way that we tested for 11 in the decimal system? Would both methods given above work for base 7 as well?
3. To test for divisibility by 11 we grouped the digits in pairs. What number or numbers could we test for divisibility by grouping the digits in triples? For example we might consider the number 157892. We could form the sum of 157 and 892. For what numbers would the remainders be the same?
4. Answer the questions raised in Exercise 3 in a numeral system to base 7 as well as in numeral system to base 12.
5. In the repeating decimal for $\frac{1}{9}$ in the decimal system there is one digit in the repeating portion; in the repeating decimal for $\frac{1}{11}$ in the decimal system, there are two digits in the repeating portion. Is there any connection between these facts and the tests for divisibility for 9 and 11? What would be the connection between repeating decimals and the questions raised in Exercise 3 above?
6. Could one have a check in which 11's were "cast out"?
7. Can you find a trick for 11 similar to that in Exercise 1 above?

3-5. Divisibility by 7

There is not a very good test for divisibility by 7 in the decimal system. (In a numeration system to what base would there be a good test?) But it is worth looking into since we can see the connection between tests for divisibility and the repeating decimals. Consider the remainders when the powers of 10 are divided by 7. We put them in a little table:

n	1	2	3	4	5	6	7
Remainder when 10^n is divided by 7	3	2	6	4	5	1	3

If you compute the decimal equivalent for $\frac{1}{7}$ you will see that the remainders are exactly the numbers in the second line of the table in the order given. Why is this so? This means that if we wanted to find the remainder when 7984532 is divided by 7 we would write

$$(7 \times 10^6) + (9 \times 10^5) + (8 \times 10^4) + (4 \times 10^3) + (5 \times 10^2) + (3 \times 10) + 2$$

and replace the various powers of 10 by their remainders in the table to get

$$(7 \times 1) + (9 \times 5) + (8 \times 4) + (4 \times 6) + (5 \times 2) + (3 \times 3) + 2.$$

We would have to compute this, divide by 7 and find the remainder. That would be as much work as dividing by 7 in the first place. So this is not a practical test but it does show the relationship between the repeating decimal and the test.

Notice that the sixth power of 10 has a remainder of 1 when it is divided by 7. If instead of 7 some other number is taken which has neither 2 nor 5 as a factor, 1 will be the remainder when some power of 10 is divided by that number. For instance, there is some power of 10 which has the remainder of 1 when it is divided by 23. This is very closely connected with the fact that the remainders must from a certain point on, repeat. Another way of expressing this result is that one can form a

number completely of 9's like 99999999, which is divisible by 23.

Exercise 3-5

Complete the following table. In doing this notice that it is not necessary to divide 10^{10} by 17 to get the remainder when it is divided by 17. We can compute each entry from the one above, like this: 10 is the remainder when 10 is divided by 17; this is the first entry. Then divide 10^2, that is, 100 by 17 and see that the remainder is 15. But we do not need to divide 1000 by 17. We merely notice that 1000 is 100×10 and hence the remainder when 1000 is divided by 17 is the same as the remainder when 15×10, or 150 is divided by 17. This remainder is 14. To find the remainder when 10^4 is divided by 17, notice that 10^4 is equal to $10^3 \times 10$ and hence the remainder when divided by 17 is the same as when 14×10 is divided by 17, that is 4. The table then gives the remainders when the powers of 10 are divided by various numbers.

	3	7	9	11	13	17	19	21	37	101	41
1	1	1	1			1					
10^1	1	3	1			10					
10^2	1	2	1			15					
10^3	1	6	1			14					
10^4	1	4	1			4					
10^5	1	5	1			6					
10^6	1	1	1			9					
10^7	1		1			5					
10^8	1		1			16					
10^9	1		1			7					
10^{10}	1		1			2					
10^{11}	1		1			3					
10^{12}	1		1			13					
10^{13}	1		1			11					
10^{14}	1		1			8					
10^{15}	1		1			12					
10^{16}	1		1			1					

Find what relationships you can between the number of digits in the repeating decimals for $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, etc. and the pattern of the remainders. Why does the table show that there will be five digits in the repeating portion of the decimal for $\frac{1}{41}$? Will there be some other fraction $\frac{1}{?}$ which will have a repeating decimal with five digits in the repeating portion? How would you find a fraction $\frac{1}{?}$ which would have six digits in the repeating portion?

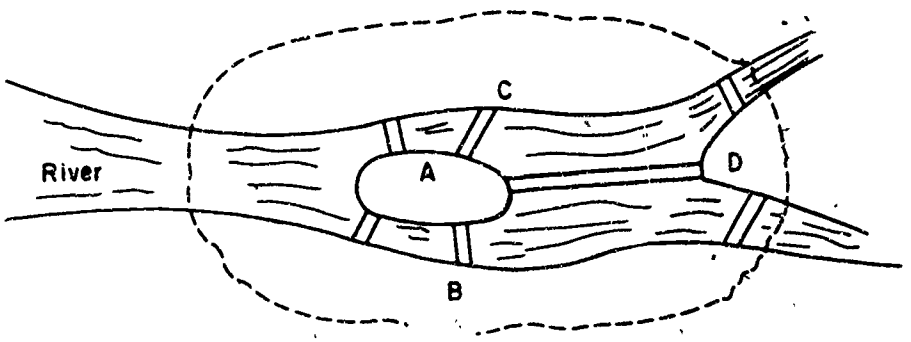
If you wish to explore these things further and find that you need help, you might begin to read some book on the theory of numbers. Also there is quite a little material on tests for divisibility in "Mathematical Excursions" by Miss Helen Abbott Merrill, Dover (1958).

SUPPLEMENTARY UNIT 4

OPEN AND CLOSED PATHS

4-1a. The Seven Bridges of Königsberg

A river flows through the city of Königsberg, Germany. (The city was taken over by the Russians following World War II and is now named Kaliningrad.) There is an island in the middle of the river which passes through the city. To the east of the island the river divides into two branches. There are seven bridges connecting the island and the different parts of the mainland as shown in the drawing below.



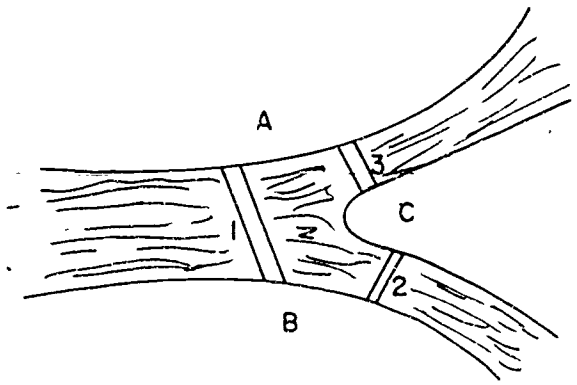
During the 1700's, a favorite pastime of the residents of Königsberg was to take walks through the city, following routes that led over each of the seven bridges. An interesting game developed whereby they tried to follow a path that led to all parts of the city in such a way that each bridge was crossed only once. And so the people of Königsberg amused themselves on their Sunday walks, but no one discovered a path that led to all parts of the city, passing over each bridge only once.

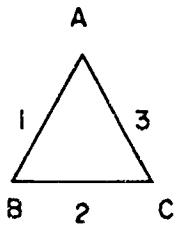
After the great Swiss mathematician, Euler, became court mathematician for Frederick the Great, a delegation from Königsberg came to him with the problem of the bridges. Now

you may not think that such a problem would interest a great mathematician, but Euler gave the delegation some help with the problem.

Euler decided that the exact shape of the different parts of the city did not matter. It would be simpler to think of the problem if the parts of the city were represented by points and the bridges by lines.

Let us think of another city, shown in the drawing below. The parts of the city are shown as points and the bridges by line segments in the figure on the right.





We can call a route from one part of the city to another a path. The path can be described by using a sequence of letters and numbers. For example,

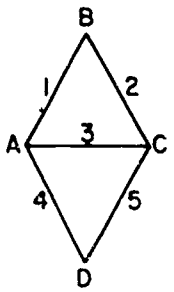
A 1 B 2 C 3 A

is a path starting at point A, following "bridge 1" to point B, then crossing "bridge 2" to C, and so on.

Note that the path described above starts at point A and ends at point A. Also, the path touches each vertex, but in doing so, the path crosses (or passes through) each segment only once. Such a path is called a closed path. Any path which goes through each vertex, passing through each segment only once but does not return to the starting point is called an open path.

Look at the drawing shown at the right. The path

A 1 B 2 C 3 A 4 D 5 C



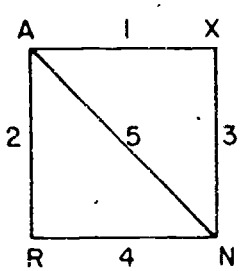
starts at A, passes through every vertex and passes through each segment only once. But it ends at point C. The path described in this sequence is an open path.

Euler asked, " Can we write a sequence of letters and numbers in which each number appears just once?" The men from Königsberg were amazed. "Of course!" They exclaimed. "After we draw the diagram, it is really very simple now that you have explained it. If we had only thought of looking at the problem in this way, we could have solved it ourselves." They went home and tried to finish the problem. Do you think you can solve the problem now?

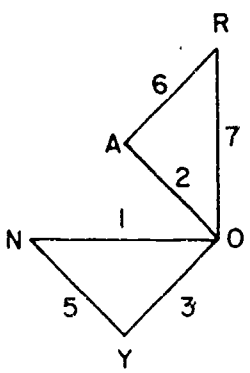
Exercises 4-1a

1. For each of the following drawings, start at A and describe a path. Tell whether the path is open or closed.

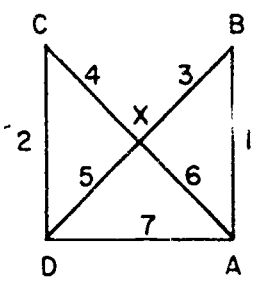
(a)



(b)

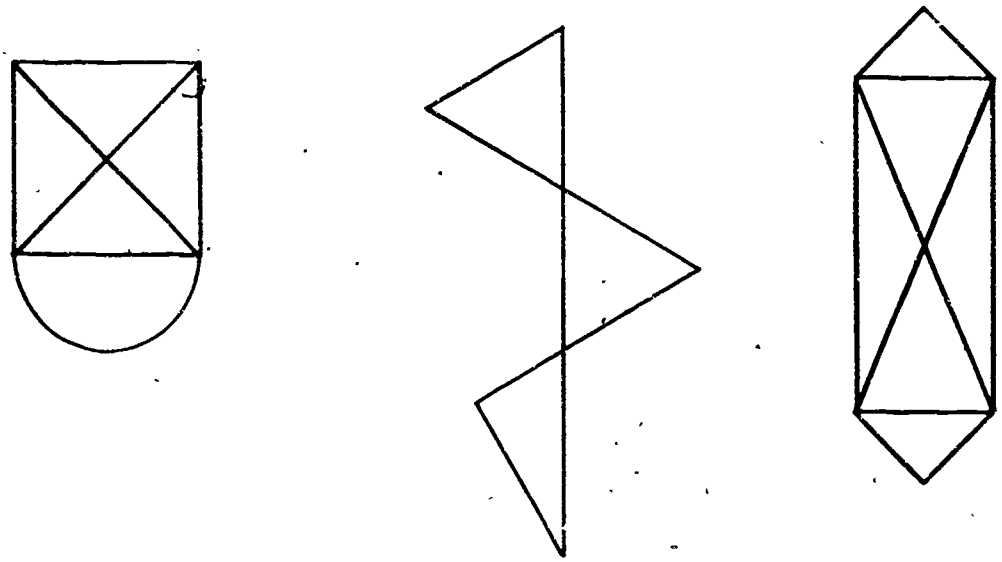


(c)



2. Make a copy of each of the following drawings. Label the vertices and the segments and tell whether there is an open path or a closed path.

(a) (b) (c)



3. Can you join the nine points shown below, starting at one point and drawing exactly five line segments without lifting your pencil tip from the paper or tracing over the same line segment twice?

4. Sometimes there is neither an open path or a closed path for a diagram. For each of the following determine whether a path is possible. If one is possible, label the vertices and the segments and describe the path.

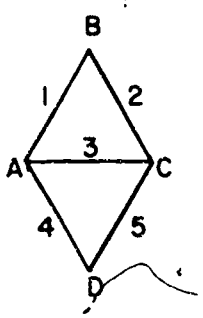
(a) (b) (c) (d)

5. How did Euler solve the problem of the Königsberg bridges? If you don't know, read the next section.

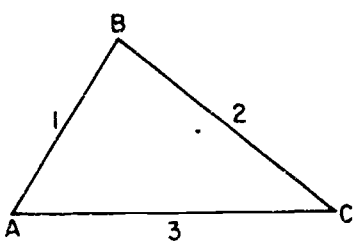
4-1b. The Solution

The following day the men came back to Euler and said, "We have been thinking about the problem, but we still cannot seem to solve it. There must be some simple idea which we have overlooked. If you could just get us started on the right track, we are sure we can solve it ourselves."

Euler replied, "All right. Let us look at the following drawing. There is a path which goes over each bridge once and only once. How can we describe the path?"



One of the men said, "One possible path is A 1 B 2 C 3 A 4 D 5 C." "But there is also a path in this drawing," said another. (The drawing is shown at the right.) "You can follow the sequence A 1 B 2 C 3 A."

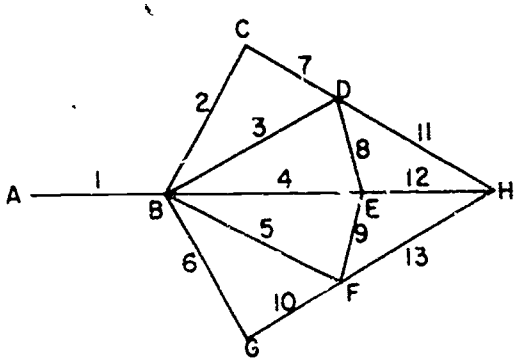


Euler replied, "Look at these two paths again. Examine them carefully. What comes before each letter except the first?" "A number," one answered. "This corresponds to a bridge leading to the point."

"What comes after each number except the last?" Euler asked. "A number, of course. There is also a bridge leading away from the point."

"How many bridges are there for each time the path goes through a point?" he asked. "Two bridges. We come into the point on one bridge, but we must use another bridge to go away from the point. For each time a letter appears in the path, except at the beginning or end, there are two numbers for these two bridges."

Euler suggested, "Let us call all points of the path, except for the two endpoints, inner points. Then for each inner point of the path there are two bridges. Suppose the point B appears three times as an inner point of the path. For instance, look at this diagram,



and the path A 1 B 2 C 7 D 3 B 4 E 9 F 5 B 6 G 10 F 13 H 12 E 8 D 11 H. How many bridges are connected to B?"

"Six," answered the men from Königsberg.

"How did you get that?" asked Euler.

"We simply multiplied the number of times the point appears by 2, the number of bridges connected with the point at each appearance."

"Will this always work?" Euler continued.

"Yes, for every inner point of the path."

"What kind of number do you get when you multiply some number by 2?" Euler asked again.

"Obviously, an even number." The men from Königsberg looked at each other, pleasantly surprised. "Then the total number of bridges leading to or from any inner point of the path must be even. Anyone could see that!"

"What about the endpoints, the first and the last point?"

They thought for a moment. "Let us see. There is a bridge leading from the first point. Then every other time the path goes through this point, there are two bridges. So the total number of bridges connected to the first point is one more than an even number. In other words, it is an odd number. The same is true of the last point."

Euler questioned them further. "Are you sure? Must the first point be different from the last point?"

They smiled. "Of course not. Thanks for reminding us not to overlook that possibility. If the path is closed, that is, if it comes back to the starting point, then that point will be like any inner point of the path. Then the number of bridges to or from that point must be even."

Euler suggested, " It might be a good idea to summarize what you have figured out so far."

They said, "All right. If the path is closed, then there is an even number of bridges connected to each point. If the path is open, then each of the two endpoints must have an odd number of bridges. Each of the inner points is connected to an even number of bridges. Now that we think of it, the problem is absurdly simple."

The men from Königsberg bent over the diagram and began counting. "The point C is connected to bridges 1, 2, and 3, the point D to bridges_____, the point A to bridges _____, and the point B to bridges_____. There are _____points connected to an odd number of bridges and_____ points connected to an even number of bridges. Is a closed path possible?_____(Yes, or no?) Is an open path possible?_____

(Yes, or no?) Such an easy problem, after all!" (Fill in the blanks yourself.)

After thanking Euler, the merry gentlemen from Königsberg went home. On the way, one of them said, "I don't see why Euler has such a great reputation. We really worked out every step of the problem ourselves. All Euler did was to suggest how to look at the problem and ask the right questions." His companions nodded and replied, "Yes, the problem was really so elementary that any child could have solved it."

What do you think?

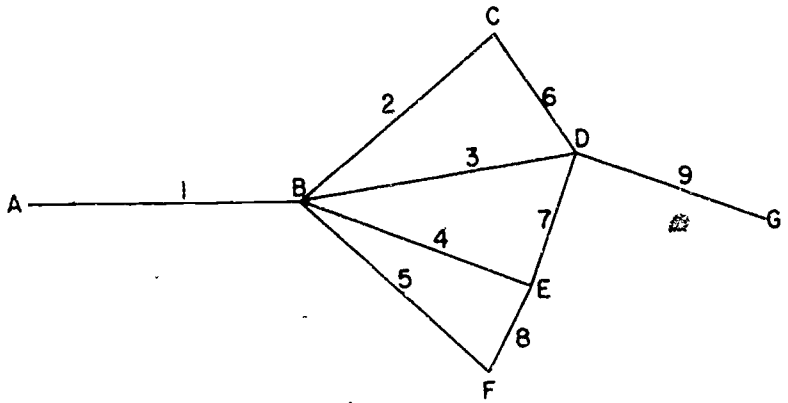
Exercises 4-1b

1. (a) For each diagram list the points which are connected to an even number of bridges.

   (b) List the points connected with an odd number of bridges.

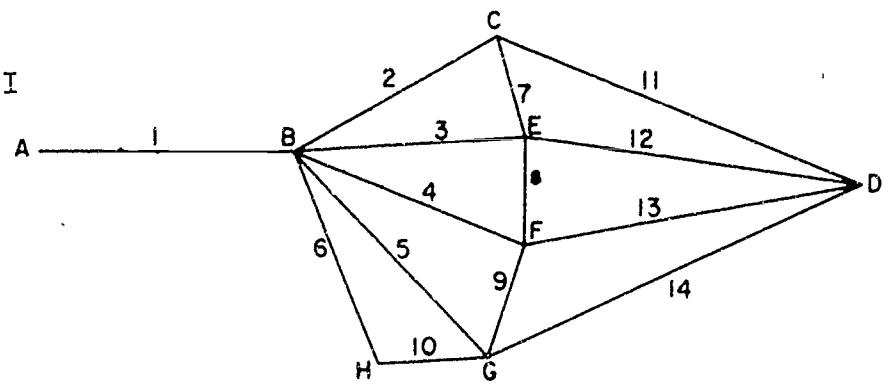
   (c) How many points of each kind are there in each diagram?

I

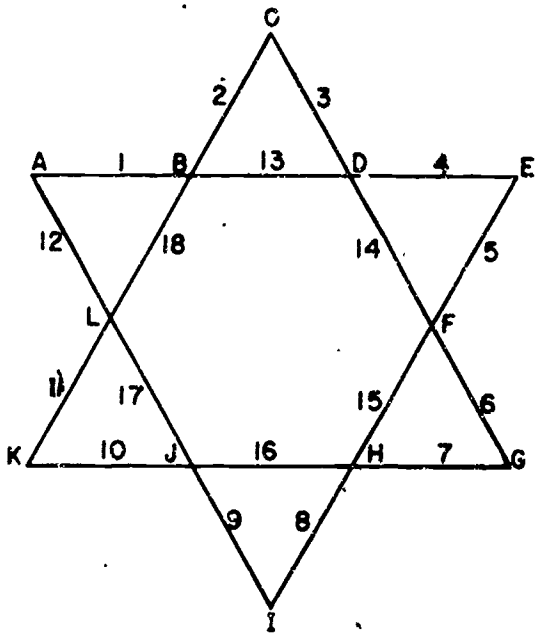


NOTE: II(on next page)

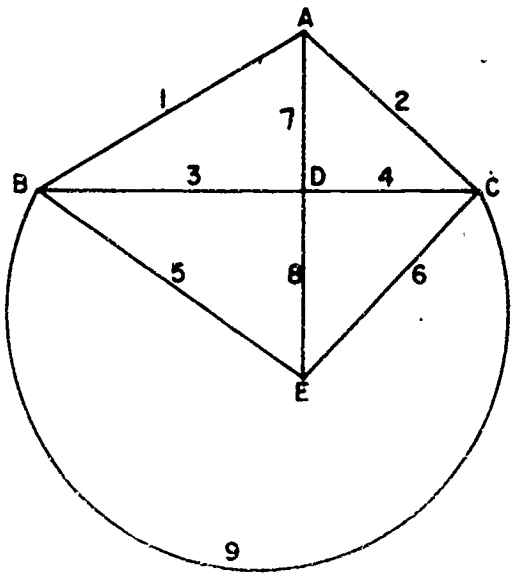
III



II


C
2
3
A
1
B
13
D
4
E
12
18
14
5
L
F
11
17
15
6
K
10
J
16
H
7
G
9
8
I

IV


A
1
7
2
B
3
D
4
C
5
8
6
E
9

2. (a) In which diagrams is it impossible to find a closed path which goes over every bridge just once?

   (b) In which diagrams is it impossible to find an open path of this kind?

3. For each of the diagrams where it might be possible to have a path going over each bridge exactly once, look for such a path. If you do find a path, describe it by a sequence of letters and numbers.

4. For each of these diagrams find a closed path starting at the point B which goes over each bridge just once, and which goes over the largest possible number of bridges.

5. In the upper figure on page 63 there are four paths from A to C which go over each bridge exactly once. One is given on page 63 and another is given by the sequence A 4 D 5 C 2 B 1 A 3 C. Find the other two.

4-2. What Happens if There Is a Path?

A drawing of a set of points and bridges, in which each point has at least one bridge attached to it, we will call a diagram. The points are called vertices (singular: vertex) of the graph. A vertex is called even if an even number of bridges are connected to it. Otherwise the vertex is called odd. A path is called closed if its last vertex is the same as its first vertex. Otherwise the path is called open. Notice that we are using the word "diagram" in a special way in this chapter.

By using the same reasoning that the men from Königsberg used, with Euler's help, you can prove the general statements:

Theorem 1. If there is in a diagram a closed path which goes over each bridge just once, then every vertex is even. If there is an open path of this kind, then there are two odd vertices, and all the rest are even.

(A theorem is a statement proved by logical reasoning.)

Exercises 4-2

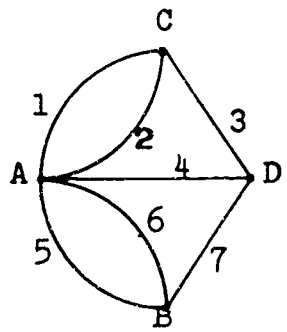
1. In the diagrams of Exercises 4-1-b, name the odd and the even vertices. How many odd vertices are there in each diagram? Does there seem to be a general principle?

2. State a general principle about the number of odd vertices in any diagram which seems to be true in all cases. Draw five more diagrams, and test whether your statement is true in each case. Compare your results with those of your classmates.

In any such diagram you may classify the vertices more precisely according to the number of bridges connected with each one. The number of bridges leading to or from a vertex we shall call the degree of the vertex. In the figure vertex A is of the 5th degree, whereas the others are of degree 3.

3. For each of the diagrams you have drawn, make a table showing the number of vertices of each degree, like this:

Degree	Number of Vertices
1	
2	
3	
4	
5	
etc.	See Problem 4



How is the total number of vertices related to the numbers in the right hand column?

4. Call the total number of vertices in a diagram V. Let V_1 be the number of vertices of degree 1, V_2 the number of degree 2, etc. (The numbers V_1, V_2, ... , are the numbers in the right hand column in the above table.) Express the relation between V and the numbers V_1, V_2, etc. as an equation.

5. Take any diagram. Label the bridges with numbers and the vertices with letters. List all pairs consisting of a vertex and a bridge connected to it. In the figure above pairs are named:

A1, A2, A4, A5, A6, B5, B6, B7, C1, C2, C3, D3, D4, D7 .

6. In Exercise 5, in how many pairs does a given bridge occur? How is the number of pairs related to the number of bridges? Let B be the number of bridges. Give a formula for the number of pairs in terms of B.

7. In Exercise 5, in how many pairs does a given vertex of degree 3 occur? In how many pairs does a given vertex of degree k occur? What is the total number of pairs in which a vertex of degree 3 occurs? What is the total number of pairs in which a vertex of degree k occurs?

8. Give a formula for the total number of pairs in Exercise 5 in terms of the numbers V_1, V_2, V_3, ...

9. Give a formula for the total number of odd vertices in terms of V_1, V_2, V_3, ...

10. Let U be the total number of odd vertices. Give a formula for the number $(2 \cdot B) - U$ in terms of V_1, V_2, V_3, etc.

11. Can you use the formula in Exercise 10 to prove the principle you discovered in Exercise 2?

---

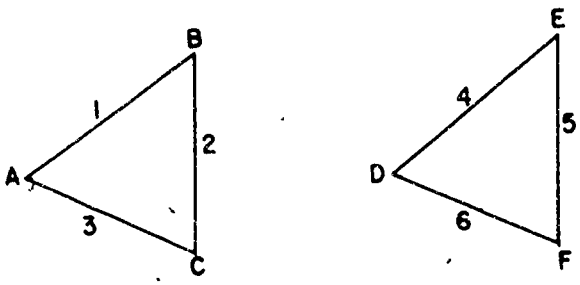
4 3. When Can You Be Sure That There Is a Path?

According to Theorem 1, if there is a closed path in a diagram which goes over each bridge exactly once, then a certain thing is true. This is a necessary condition that there be such a path in a diagram. If a diagram does not satisfy this condition, namely that all its vertices are even, then we are sure that there is no closed path going over each bridge just once.

Is this condition sufficient? If all the vertices are even does there exist a path of this kind in the diagram? Examine all the diagrams you have drawn so far. Find the ones which have only even vertices. Can you find, in each one of these a closed path going over each bridge once and only once? Can you draw a counter-example, a diagram with only even vertices in which there is no such path?

Does it seem as though the condition that the diagram have no odd vertices is sufficient? Compare your conclusions with those of your classmates before you read further.

Look at this diagram



Are there any odd vertices? Can you find a path which goes over every bridge just once? In fact, is there any path which goes over both bridges 1 and 4?

The trouble with this diagram is that it is made up of two separate figures. There is no use looking for a path which goes over every bridge unless the figures are connected. We say that a diagram is connected if every vertex can be joined to any other vertex by a path. In the figure above vertex A can be joined to B and C, but not to any of the other vertices.

It turns out that if a connected diagram has no odd vertices, then there is a closed path which goes over every bridge exactly once. We shall lead you to discover the proof in two stages.

Theorem 2. If a diagram has no odd vertices, then through every vertex there is a closed path which doesn't go over any bridge twice.

Proof: Suppose Q_1 is a vertex of the diagram. Find the longest path (measured by the number of bridges in it) which starts at Q_1 and doesn't go over any bridge more than once. Suppose, for example, that this path has 7 bridges in it. We could describe the path roughly like this:

$$Q_1Q_2Q_3Q_4Q_5Q_6Q_7Q_8.$$

Here the subscripts simply help us name the vertices. For example, Q_2 is the second vertex. We did not bother to write the numbers of the bridges between the names of the vertices. Now suppose Q_8 is not the same as Q_1. Is this path open or closed? Is Q_8 an inner point or an endpoint of this path? What do you know about the number of bridges connected to an endpoint of a path? What was assumed about the total number of bridges connected to any point of the diagram? Can this path contain all the bridges connected to Q_8?

If not, then there is at least one more bridge in the diagram, connected to Q_8 but not in this path. If we go over this bridge, too, then we will have a path

$$Q_1Q_2Q_3Q_4Q_5Q_6Q_7Q_8Q_9$$

starting at Q_1 with 8 bridges. This contradicts our assumption that the longest path, starting at Q_1, in the diagram has only 7 bridges.

Since we got into a contradiction by assuming that Q_8 was not the same as Q_1, then this assumption must be false. Therefore, Q_8 is the same as Q_1, so this is a closed path through Q_1 which doesn't go over any bridge twice.

Now you are ready for the second stage:

Theorem 3. If a connected diagram has only even vertices, then there is a closed path going over every bridge just once.

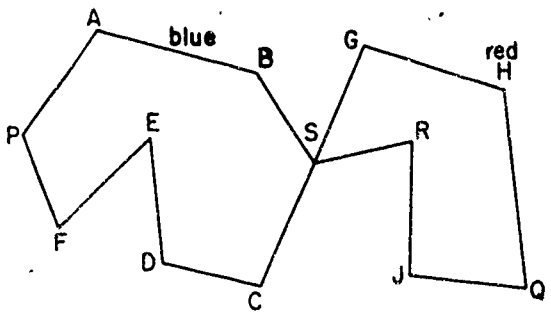
Proof: Suppose you look at the longest such path in the diagram. Color the bridges and vertices of this path blue. If this path does not contain every bridge, then color in red all bridges which are not in this path. We are going to assume that there is a red bridge, and see what follows. We claim that there is a purple vertex, that is one colored both blue and red.

To see this, take any red bridge and some blue vertex P. Since the diagram is connected, there is a path joining either vertex, say Q_1 of the given red bridge with the vertex P. Look at the last red bridge in this path. Suppose it leads from the vertex

R to the vertex S. Since this bridge is red, then S is colored red. But the next bridge in the path is blue. Therefore, S is also blue. So S is purple.

Now look at the diagram made up of the red bridges, which we can call simply the red diagram. Since the blue path is closed, there is an even number of blue bridges connected to each of its vertices. Since the total number of bridges connected to any vertex of the original diagram is even, that leaves an even number of red bridges (possibly 0) connected to any vertex.

Therefore in the red diagram there is an even number of bridges connected to each vertex. We can apply Theorem 2 to the red diagram. Hence there is a closed path in the red diagram through the purple vertex S. We have then a picture like this:

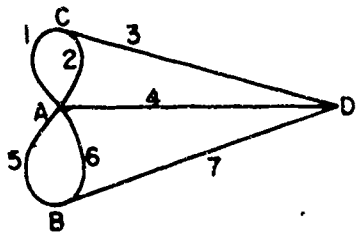


Then the path P A B S G H Q J R S C D E F P is a closed path which doesn't go over any bridge more than once. This path is longer than the blue path. This is a contradiction since the blue path was supposed to be the longest such closed path in the diagram.

We got into trouble by assuming that the blue path did not contain all the bridges. Therefore, it does contain all of them. So the blue path is the one we were looking for.

Exercises 4-3

1. The drawing at the right is a diagram of the Bridges of Königsberg.



   (a) Is there a closed path for the diagram?

   (b) Is there an open path for the diagram.

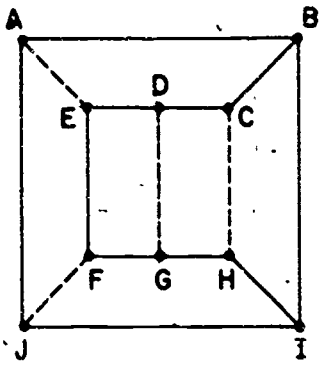
2. (Brainbuster) Prove that if a connected diagram has 2 odd vertices and all the rest even, then there is an open path which goes over each bridge exactly once.

4-4. Hamiltonian Paths

When the men from Königsberg asked Euler to help them with their problem, they probably expected him to "write out" a solution. Did he find a solution? He did show them that it could be proved that neither a closed or an open path could be found from the Bridges of Königsberg. In a sense, then, this was a solution.

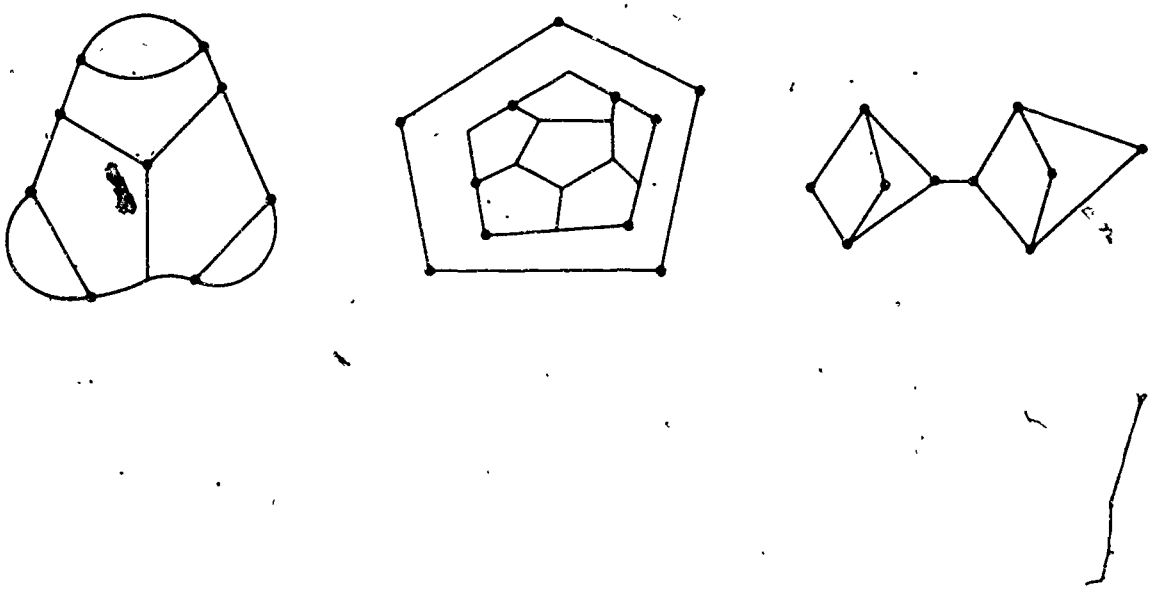
Are there any more such problems? There is one, and it seems so simple that one would think a solution could be found. Yet, no one knows the answer as yet. This problem deals with Hamiltonian Paths. Because the first problem of this type was solved by the great Irish mathematician Sir William Rowan Hamilton, the paths were named after him. A Hamiltonian path is a diagram in which a closed path goes through each vertex without going over each "bridge" more than once. A Hamiltonian path does not have to go over every "bridge" however.

The figure at the right shows a Hamiltonian path. The path follows the sequence of letters A B C D E F G H I J and then returns to A. The dotted lines represent bridges which are not in the Hamiltonian path.



The problem is, that a necessary and sufficient condition for a diagram to contain a Hamiltonian path is not known. One way for a person to become famous is to find an answer to questions such as this one. Perhaps that person might be you. We hope you have lots of fun trying.

Study the figures in the next drawing. Try to determine which of the figures contains a Hamilton path.



SUPPLEMENTARY UNIT 5

FINITE DIFFERENCES

5-1. Arthmetic Progressions

Suppose we look at a few interesting sets of numbers to begin with, and take differences of successive numbers:

Table I

1	2	3	4	5	6	...	n	(n+1)	...
	1	1	1	1	1	...	1		...

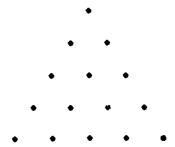
Between each successive pair of numbers and on the line below it we write the difference:

$$2 - 1 = 1,\ 3 - 2 = 1,\ 4 - 3 = 1,\ \ldots\ .$$

It begins to be monotonous after a while. Why did we have the number n? It was just to indicate any number (n stands for "any"). The next number after n would be (n + 1) since in this "sequence" you get each number by adding 1 to the number before. (When we have a set of numbers in some order, we call it a "sequence.") What would be the next one after (n + 1)? What would be the one before n? You should read this unit with a pencil and sheet of paper at hand so that you may answer these questions as they occur. You may also have questions of your own which you would like to try to answer.

There is nothing especially strange about the differences being 1 since one was added each time to get the next entry. Could you write a sequence in which all the differences are 2's or 3's or any other number? Any sequence for which the difference between successive numbers is the same every time is called an arithmetic progression.

Let us look back to the numbers of Table I. There is a connection with the game of ten pins or bowling. Look at the triangle of dots below:



If we omitted the last line we would have the usual arrangement of ten pins in a bowling alley. If there were just one row we would have one dot, if two rows, 3 dots, if three rows, 6 dots; etc. These numbers of dots are called "triangular numbers." We write these in a table:

Table II

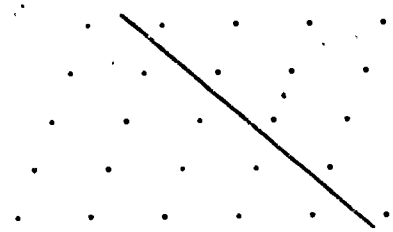
Triangular numbers:	1	3	6	10	15	21	28	...
Differences:		2	3	4	5	6	7	...

If we compare this table with Table I we can notice a number of interesting things. The first entries in the two tables are each 1. The second entry in Table II is the sum of the first two entries in Table I, the third entry in Table II is the sum of the first three entries in Table I, etc. The tenth entry in Table II would be the sum of the first ten entries in Table I. We could also say that the n-th entry in Table II (we do not yet have a formula for it) is the sum of the first n entries in Table I.

Another thing we notice in comparing the two tables is that the differences in the second line of Table II are the same as the entries in the first line of Table I except for the first one. Why is this so? Of course if we had written in Table II a third line giving the differences for the second line we would have had a succession of 1's as before.

Now we could find the sum of the first ten numbers in Table I by adding them - this would give us the tenth entry in the first

line of Table II, but this would be rather tedious. There is an interesting little trick that will give us our result with less effort. Suppose we form another triangle of dots like that above, turn it upside down and fit it carefully next to the one already written. Then we would have a figure like:



In this picture we have 5 rows with 6 dots in each row, which gives $5 \times 6 = 30$ dots in all. Hence the number of dots in the first triangle would be $\frac{1}{2} \times 30 = 15$, which is the fifth triangular number. If we wanted the 20th triangular number we would have a triangle of 20 rows. If we make another triangle of dots and place it as we did for the smaller triangle, we would have 20 rows with 21 dots each and hence 20×21 dots in the two triangles together, which implies that in each triangle there would be

$$\frac{1}{2} \times 20 \times 21$$

dots. So the 20th triangular number is 210, which is the same as the sum of the numbers 1, 2, 3, ... up to and including 20.

By this means we can find in the same manner the number of dots in any triangular array of this kind, that is, we can find any triangular number. Let us write a few:

40th triangular number: $\frac{1}{2} \times 40 \times 41 = 820$

100th triangular number: $\frac{1}{2} \times 100 \times 101 = 5050$

120th triangular number: $\frac{1}{2} \times 120 \times 121 = 7260.$

In each case we take the product of $\frac{1}{2}$, the number and 1 more than the number. We can get a formula by letting n stand for

the number and say that

$$\text{the n-th triangular number is } \frac{1}{2} \times n \times (n + 1).$$

Then we would get the above three values by letting $n = 40$, $n = 100$, $n = 120$. Other ways of writing $\frac{1}{2} \times n \times (n + 1)$ are

$\frac{1}{2}n(n + 1)$, $\frac{n}{2}(n + 1)$, or $\frac{n(n + 1)}{2}$

We could also get this result without any reference to dots by the use of an idea that is suggested by the triangles we drew Suppose we wanted the 20th triangular number. Then we could take the sum twice in two different orders:

$$1 + 2 + 3 + 4 + \ldots + 17 + 18 + 19 + 20$$

$$20 + 19 + 18 + 17 + \ldots + 4 + 3 + 2 + 1.$$

The sum of each column is 21, there are 20 columns and hence the sum of the numbers in the two rows is 20×21. The sum in each row is one-half of this. We could do this for any number in place of 20 and one way of showing this would be to write it out using n for the number in place of 20 or whatever number we had. It would look like this:

$$1 + 2 + 3 + 4 + \ldots + (n - 1) + n$$

$$n + n\text{-}1 + n\text{-}2 + n\text{-}3 + \ldots + 2 + 1.$$

The sum of each column is $n + 1$ and there are n columns. Hence the sum of all the numbers in the two rows is $n(n + 1)$, and half this is the sum for each row: $\frac{1}{2}n(n + 1)$.

We shall find still another way to get this sum in the next section.

Exercises 5-1

1. (a) Write the first twenty numbers in the sequence starting with 15 for which the differences are all 1's.

   (b) Write the numbers as $14 + 1$, $14 + 2$, and so on.

(c) Write the sum by writing the 14's first and then the others as: $14 + 14 + \ldots + (1 + 2 + \ldots)$

(d) This can be written as: $(\underline{?} \cdot 14) + (\frac{1}{2} \cdot 20 \cdot ?)$

(e) The sum written with one numeral is_____.

2. Follow the procedure of Problem 1 with the sequence starting with 142. Use ten numbers in the sequence and the difference of one. (The answer for (e) should be 1465)

3. What is the formula for the sum of the first n numbers in a sequence with the difference one? Let us agree to use b for the number that is one less than the first number in the sequence.

4. Write a sequence of numbers for which the difference is always 2. Begin with 13. What would be the sum of the first 20 numbers in this sequence? Note that $(2 + 4 + 6 \ldots)$ can be written as $2 \cdot (1 + 2 + 3 \ldots)$.

5. What is the formula for the sum of the first n numbers in a sequence with the difference 2?

6. Consider the formula: $2n + 7$ (remember that $2n$ means $2 \times n$). When $n = 1$, $2n + 7$ is $(2 \cdot 1) + 7 = 9$; when $n = 2$, $2n + 7$ is $(2 \cdot 2) + 7 = 11$, etc. We can form a table of values:

n:	1	2	3	4	5	6
2n + 7:	9	11	13	15	17	19

Carry this table out for the next three values of n. Use the numbers 9, 11, 13, ... as the first row of a table and then write below this row a row of differences. Do you notice any relationship between the formula and these differences.

7. Do the same as in Problem 6 for the formula $3n + 7$ and for $2n + 6$.

8. What would be the differences for the numbers defined by the formula $5n + 7$?

9. Write the first 20 odd numbers. Can you find their sum without just adding them? Can you guess what a formula for the sum of the first n odd numbers would be? Use either the trick at the end of Section 1 or the answer to Problem 5.

10. Give a formula for the sum of the first n - 1 numbers in Table I.

11. Find a formula for the sum of the following:

$$1, 1 + d, 1 + 2d, \ldots, 1 + nd.$$

12. Give a formula for the sum of the following:

$$1, 1 + d, 1 + 2d, \ldots, 1 + (n - 1)d.$$

13. Find a formula for the sum of the same sequence as in the previous problem except that 1 is replaced by b.

14. Suppose the first two numbers in a table are

7 and 12.

Write a table starting with these two numbers for which the first differences are all the same, that is, in which the numbers on the first row are in an arithmetic progression.

15. Write a table of numbers in an arithmetic progression in which the first two entries are 7 and 5 in that order.

16. If you have any two numbers instead of 7 and 12, or 7 and 5, could you make a table starting with the two given numbers in which the numbers of the first row form an arithmetic progression? Give reasons.

5-2. More Sequences

Now form a table of the squares of the integers. Recall that the square of 3 is 9 since $3 \cdot 3 = 9$, the square of 5 is 25 since $5 \cdot 5 = 5^2 = 25$, etc. We call them "squares" or "square numbers" because if we wrote our dots in squares instead of

triangles, as previously we would have the following sequence of squares:

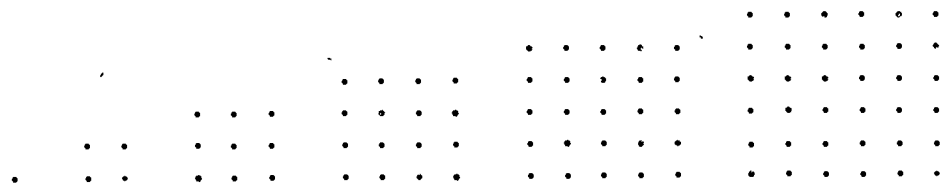


Table III

1	4	9	16	25	36	49	... n^2	$(n + 1)^2$
	3	5	7	9	11	13	... w	
		2	2	2	2	2	... 2	

Notice that the numbers here in the second row are in an arithmetic progression and that the differences in the third row are all 2's. We call the numbers in the second row of such a table "first differences" and those in the third row "second differences." What would be the n-th term in the second row, that is, the entry where w is? (w stands for "what.") This should not be hard to find since it is the difference of the two numbers above it. It is just

$$(n + 1)^2 - n^2.$$

Before getting a simpler expression for this difference of two squares, let us see how it goes for some of the numbers. Just to write $36 - 25 = 11$ is not especially enlightening. But suppose we write it as

$$6^2 - 5^2 = (5 + 1)^2 - 5^2.$$

If we use the distributive property several times we have:

$$\begin{aligned}(5 + 1)^2 &= (5 + 1)\cdot(5 + 1) = 6(5 + 1)\\ &= (6\cdot5) + (6\cdot1) = (5 + 1)\cdot5 + (5 + 1)\cdot1\\ &= 5^2 + (1\cdot5) + (5\cdot1) + 1\\ &= 5^2 + (2\cdot5) + 1.\end{aligned}$$

And thus

$$6^2 - 5^2 = 5^2 + (2\cdot5) + 1 - 5^2 = (2\cdot5) + 1.$$

In just the same way we could show that

$$7^2 - 6^2 = 6^2 + (2\cdot6) + 1 - 6^2 = (2\cdot6) + 1.$$

(Try it and see.) So, putting n in place of 5 or 6 or whatever number, we have

$$(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1.$$

We could write this in words: The difference between the squares of two successive integers is 1 more than twice the smaller one. For instance: $121^2 - 120^2 = (2\cdot120) + 1 = 241$. This is a much simpler computation than squaring both numbers and taking the difference. This can also be shown using diagrams of dots in squares, but this is left as an exercise.

This shows that the last entry in the second row of Table III should be $2n + 1$. We might check this: when n is 1, $2n + 1$ is 3; when n is 2, $2n + 1$ is 5, etc.

The numbers in the second row are in an arithmetic progression. If you look carefully, you will see that each number in the first row is 1 more than the sum of the numbers to the left of it in the row below. Why is this so? Another way of saying this is that the fifth number in the first row is the sum of the first five odd numbers, the sixth number in the first row is the sum of the first six odd numbers, etc. What would be the sum of the first 20 odd numbers? What is the sum of the first n odd numbers?

We can use this to get the formula for the sum of the first n counting numbers in still another way. Start with the sum of the $(n + 1)$ odd numbers

(1) $1 + 3 + 5 + 7 + \ldots + (2n + 1) = (n + 1)^2 = n^2 + 2n + 1.$
Subtract 1 from extreme left and extreme right.

(2) $3 + 5 + 7 + \ldots + (2n + 1) = n^2 + 2n$

Notice that 3 is the value of $2n + 1$ when $n = 1$, 5 is the value of $2n + 1$ when $n = 2$, etc. Then we can write the left side of equation (2) as follows:

$$\left((2\cdot1) + 1\right) + \left((2\cdot2) + 1\right) + \left((2\cdot3) + 1\right) + \ldots + (2n + 1)$$

If we write this in a different order, using the commutative property, we have

$$2\cdot1 + 2\cdot2 + 2\cdot3 + \ldots + 2n + (1 + 1 + 1 + \ldots + 1)$$

where there are n 1's in the parentheses. Then, from the distributive property, this can be written

$$2(1 + 2 + 3 + \ldots + n) + n$$

If we substitute this for the left side of equation (2) we get the equation:

$$2(1 + 2 + 3 + \ldots + n) + n = n^2 + 2n.$$

Subtract n from both sides to get

$$2(1 + 2 + 3 + \ldots + n) = n^2 + n.$$

Finally, if we divide both sides by 2 we have

$$1 + 2 + 3 + \ldots + n = \frac{1}{2}(n^2 + n) = \frac{1}{2}n(n + 1)$$

which is the formula we had before for the n-th triangular number.

This is, of course, a much harder way to find the sum of the first n counting numbers than by the other methods, but it does suggest a means of finding the sum of the squares. Let us try to find the sum of the squres of the counting numbers by considering a table of their cubes and the differences. Try it.

Table IV

1	8	27	64	125	216	...	n^3	$(n + 1)^3$	...
	7	19	37	61	91	...		w	
		12	18	24	30	...			
			6	6	6				

Notice that here it is the second differences which form an arithmetic progression and the third differences which are all the same.

The second row should be connected somehow with the square of the counting numbers. To get a clue for this connection, we must determine the formula for the last term in the second row, which we have called w. This is just

$$(n + 1)^3 - n^3.$$

Observe that

$$(n + 1)^3 = (n + 1)\cdot(n + 1)^2$$

We found previously that $(n + 1)^2 = n^2 + 2n + 1$, so

$$\begin{aligned}(n + 1)^3 &= (n + 1)\cdot(n^2 + 2n + 1)\\ &= n\cdot(n^2 + 2n + 1) + 1(n^2 + 2n + 1)\\ &= n^3 + 2n^2 + n + n^2 + 2n + 1\\ &= n^3 + 3n^2 + 3n + 1.\end{aligned}$$

Thus

$$(n + 1)^3 - n^3 = (n^3 + 3n^2 + 3n + 1) - n^3 = 3n^2 + 3n + 1.$$

To check this, let us form a little table of values:

n	1	2	3	4
$3n^2 + 3n + 1$	7	19	37	61

which checks with the second row of Table IV.

From the this we are now going to work out the following formula for the sum of the first n squares:

$$s = \frac{2n^3 + 3n^2 + n}{6}$$

If you find the algebra too difficult, you can just assume the formula and go on to the exercises after checking the formula for a few values of n

To get the formula first notice that in Table IV, $8 = 1 + 7$, $27 = 1 + 7 + 19$, $64 = 1 + 7 + 19 + 37$, etc. Each number in the first row after the 1 is 1 more than the sum of the numbers in the second row and to the left of it. That is, $(n + 1)^3$ is 1 plus the sum of the numbers in the second row through w, which is $3n^2 + 3n + 1$. Hence we have the following equation:

(3) $1 + 7 + 19 + 37 + \ldots + (3n^2 + 3n + 1) = (n + 1)^3$ or,

(4) $7 + 19 + 37 + \ldots + (3n^2 + 3n + 1) = (n + 1)^3 - 1$

From our work above we see that the right side of this equation is equal to

$$(n^3 + 3n^2 + 3n + 1) - 1 = n^3 + 3n^2 + 3n,$$

and the left side may be written using $n = 1, 2, 3 \ldots$ in $3n^2 + 3n + 1$ as

$$(3 \cdot 1^2 + 3 \cdot 1 + 1) +$$
$$(3 \cdot 2^2 + 3 \cdot 2 + 1) +$$
$$(3 \cdot 3^2 + 3 \cdot 3 + 1) +$$
$$\ldots\ldots\ldots$$
$$(3n^2 + 3n + 1).$$

Notice the squares of the numbers from 1 to n in the first column and the numbers from 1 to n in the second column. The last number in each line is 1. So if we add by columns we have, using the distributive property:

$$3 \times (1^2 + 2^2 + 3^2 + \ldots + n^2) +$$
$$3 \times (1 + 2 + 3 + \ldots + n) +$$
$$(1 + 1 + 1 + \ldots + 1),$$

where in the last line there are n 1's. We have called s the sum of the squares of the first n counting numbers; we know that the sum of the first n integers is $\frac{1}{2}(n^2 + n)$ and the sum of the

n 1's is n. Hence the expression can be abbreviated to

$$3s + 3\cdot\frac{1}{2}(n^2 + n) + n,$$

which is what the left side of (3) reduces to. If we equate it to what we found above for the right side we have:

$$3s + \frac{3}{2}(n^2 + n) + n = n^3 + 3n^2 + 3n$$

or

$$3s + \frac{3}{2}n^2 + \frac{3}{2}n + n = n^3 + 3n^2 + 3n.$$

Since $\frac{3}{2}n + n = \frac{5}{2}n$ $\quad 3s + \frac{3}{2}n^2 + \frac{5}{2}n = n^3 + 3n^2 + 3n.$

Subtracting $\frac{3}{2}n^2 + \frac{5}{2}n$ from both sides of the equal sign we get

$$3s = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \quad \text{or} \quad \frac{2}{2}n^3 + \frac{3}{2}n^2 + \frac{1}{2}n.$$

Finally if we multiply both sides by $\frac{1}{3}$ we have the formula

$$s = \frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

which is what we stated above.

You should check this for the first two or three values of n.

Exercises 5-2

1. (a) Using dots arranged in square patterns as shown at the beginning of this section, show that $(5 + 1)^2 - 5^2 = (2\cdot5) + 1$.

   (b) Using the same idea explain how to show that $(n + 1)^2 - n^2 = 2n + 1$. We can show $(n + 1)$ dots this way

   $\overbrace{\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot}^{n+1}$

2. Find a formula for the sum of the squares of the first n even integers. (Begin by writing these squares as $(2\cdot1)^2$, $(2\cdot2)^2$, $(2\cdot3)^2$, ... What is the n-th number in this sequence?)

3. Find a formula for the sum of the squares of the first n odd integers. (Begin by writing these squares as $(2\cdot1 - 1)^2$, $(2\cdot2 - 1)^2$, $(2\cdot3 - 1)^2$, ... Notice that $(2k - 1)^2 = 4k^2 - 4k + 1$ for $k = 1, 2, 3, \ldots$)

4. Given the numbers 4, 7, 12, can you form a table beginning with these numbers in which the first differences are in an arithmetic progression?

5. Answer the same question as that in Problem 4 but with the numbers 4, 7, 12 replaced by 10, 5, 11 in that order.

6. Given any three counting numbers, could a table be constructed having the given numbers as the first three entries in order and for which the first differences would be in an arithmetic progression? Give reasons for your answers.

*7. Find a formula for the sum of the first n cubes of counting numbers, that is, for 1, 8, 27, 64, etc.

5-3. Finding Formulas that Fit

By the methods we used in the previous sections we could find formulas for the sums of cubes, fourth powers, fifth powers and so on but the computations and algebra become more and more difficult. It is time we tried something else.

We can use some of the same methods to find formulas to fit some tables of values. Suppose we had the sequence of numbers:

3 7 11 15 19 ...

and we wanted a formula that would fit these values. We could form a table and take the first differences

Table V

3	7	11	15	19
	4	4	4	4

These differences are all the same, that is, the numbers in the first row are in an arithmetic progression. (Of course the next value might not be, but we are only trying to find a formula which fits the given vlaues.) From this we might guess that the formula for the numbers in the first row would be of the form: $b + an$ for some numbers b and a. Suppose we try it to see if it works. Then the n-th and $(n + 1)$st entries would be

$$b + an \quad \text{and} \quad b + a(n + 1)$$

and their difference would be

$$b + a(n + 1) - b - an = b + an + a - b - an = a$$

which is the difference. Since all the differences are 4, it follows that a must be 4 and our formula becomes

$$b + 4n.$$

Now when n is 1, $b + 4n$ must be the first entry, that is

$$b + 4 = 3$$

which means that b must be $^{-}1$ and hence the formula seems to be

$$4n + {}^{-}1 \quad \text{or} \quad 4n - 1$$

If we try this for various values of n we see that it works and this indeed fits the five entries in the first row of the table.

Actually we could see that this would have to work if the numbers are in an arithmetic progression, once we have fixed b so that the first entry fits the formula; for, whatever b is, the numbers in the first row would be

$$b + 4 \qquad b + (2 \cdot 4) \qquad b + (3 \cdot 4)$$

and the differences are all 4's.

Really we have proved more than we set out to do. We have the

Theorem: If the first differences of a table of values are all the same, call them a, then the numbers form an arithmetic progression and the formula for the n-th term is

$$b + an$$

where b is so chosen that a + b is the first number in the table.

By means of this theorem we could get a formula to fit any table of values in an arithmetic progression, that is, in which the first differences are all equal. What about tables in which this is not the case? In order to explore this, suppose we test the tables for a few formulas to see if we can make some guesses.

Table for $n(n + 2) = n^2 + 2n$

n	1	2	3	4	5	...
n(n + 2)	3	8	15	24	35	...
first differences	5	7	9	11		

Here the first differences form an arithmetic progression. (You should check these values and compute a few more.)

Table for $n(n + 1)(n + 2)$

n	1	2	3	4	5	6 ...
n(n + 1)(n +2)	6	24	60	120	210	336 ...
First differences	18	36	60	90	126 ...	
Second differences	18	24	30	36 ...		

Notice that $n(n + 1)(n + 2)$ is the product of three successive integers beginning with n. Here it is the second differences which are in an arithmetic progression. This would give us a way of computing the values of $n(n + 1)(n + 2)$ successively, assuming that the second differences are in an aritmetic progression no matter how far one goes in the table. For instance, the next second difference would be $42 = 36 + 6$, the next first

difference would be $126 + 42 = 168$ which means that the next entry in the line above would be $336 + 168 = 504$. To check this we see that $504 = 7 \times 8 \times 9$. (Notice that every number after the first line in the table is divisible by 6. Why is this so?)

Try one more table:

Table for $n(n + 1)(n + 2)(n + 3)$

n	1	2	3	4	5	6
$n(n + 1)(n + 2)(n + 3)$	24	120	360	840	1680	3024
First differences	96	240	480	840	1344	
Second differences	144	240	360	504		
Third differences	96	120	144			

Here it is the third differences that are in an arithmetic progression. Notice that every number after the first row is divisible by 24. Why is this so?

Before going further, you should try out a few for yourself.

Exercises 5-3-a

1. Find tables of values for each of the following formulas and compute first, second, third differences:

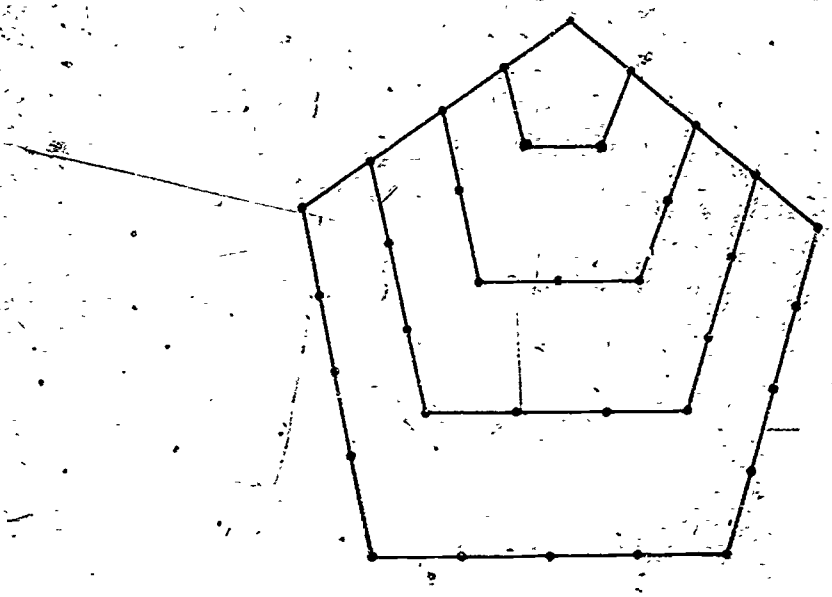
   (a) $n^2 + 3n + 2$

   (b) $\frac{n^3 - n}{6}$

   (c) $n^3 + n$

2. Suppose you computed a table for the formula: $n^4 - n^2$ and computed the first, second, etc. differences. Guess how soon you would come to an arithmetic progression. Then check it to find out.

Now we can come back to the problem of trying to find formulas that fit certain tables. In the beginning we considered triangular numbers and a little later, square numbers. What would "pentagonal numbers" be? (You remember that a pentagon is a five-sided figure-sided figure - the shape of the Pentagon in Washington.) Consider the following figure which is a set of pentagons:



We call 1 the first pentagonal number and 5 the next. In the next pentagon there will be 3 dots on a side and we add three sides with a total of $3 + 3 + 3 - 2 = 7$ dots. (We subtract 2 for the vertices which we have counted twice.) The next time we would add $4 + 4 + 4 - 2$, or 10 dots. Each time we add three more than we did the previous time. In this way we get the following table of pentagonal numbers:

Table VI

	1	5	12	22	35
first differences		4	7	10	13
second differences			3	3	3

From this and our previous experience it looks as if the formula which would fit this table would be of the form

$$an^2 + bn + c$$

for a proper choice of the numbers a, b, and c. Let us see if this will work out. Then the n-th and (n + 1)st terms would be

$$an^2 + bn + c \quad \text{and} \quad a(n + 1)^2 + b(n + 1) + c$$

and their difference would be

$$a[(n + 1)^2 - n^2] + b[(n + 1) - n] + c - c.$$

We have already found that $(n + 1)^2 - n^2 = 2n + 1$ and hence the difference is

$$a(2n + 1) + b = 2an + (a + b).$$

Now this has to be equal to the first difference formula. So the second differences must all be 2a. We see from the table that these differences are all 3. Thus $2a = 3$ and $a = \frac{3}{2}$. The formula for the first difference in the table must be $3n + (\frac{3}{2} + b)$ and $(\frac{3}{2} + b)$ must be 1 to have it give the number 4 when $n = 1$. So we have $\frac{3}{2} + b = 1$ and $b = -(\frac{1}{2})$.

Hence the formula for the numbers in the first line of Table VI, the pentagonal numbers, should be

$$\frac{3}{2}n^2 - \frac{1}{2}n + c$$

for a proper choice of c. Putting $n = 1$ in the formula and setting it equal to the first entry, 1, in the table, we get

$$1 = (\tfrac{3}{2}\cdot 1^2) - (\tfrac{1}{2}\cdot 1) + c = 1 + c$$

which shows that c must be zero. So our formula for the n-th term in the first row of Table VI seems to be

$$\frac{3}{2}n^2 - \frac{1}{2}n = \frac{3n^2 - n}{2}$$

and trying values for n will show you that it does indeed fit. Furthermore it has to fit since the first differences are fixed and they determine the entries on the first line after the first entry is given.

Exercises 5-3-b

1. Find formulas which fit each of the following tables of values:

   (a) 2 7 12 17 22 27

   (b) 5 19 43 77 121

   (c) 8 20 38 62 92

2. What kind of a formula do you think would fit the following table of values:

   2 10 30 68 130 222?

3. Have you ever noticed cannon balls piled in a triangular pyramid on an old battlefield? There might be a little pile with 3 in a triangle on the bottom and 1 on top of it, giving 4 in all. If there were three tiers, the triangle on the ground would have 6, plus the four above, would be 10. If there were four tiers, there would be 10 on the bottom with a total of 20 in the pile. These numbers are called pyramidal numbers and are

   1 4 10 20 35 ...

   Can you discover any relationship between them and the triangular numbers?

4. Suppose there is a table of values in which the third differences form an arithmetic progression. Can you gues what sort of a formula would fit the numbers of the table?

5. There is a famous theorem that every integer can be expressed as the sum of three or fewer triangular numbers. Try it out:

$1 = 1$, $2 = 1 + 1$, $3 = 3$, $4 = 1 + 3$, $5 = 1 + 1 + 3$, ...,

$14 = 1 + 3 + 10$, etc.

Notice that the numbers 5 and 14 actually need to have three triangular numbers in the sum. The theorem also says that every integer which is positive can be expressed as the sum of four or fewer square numbers, five or fewer pentagonal numbers, etc. You might be interested in trying this out. The proof is very difficult.

6. There are some sets of numbers that have the property that no row of differences, no matter how far you go, form an arithmetic progression. Two such sets are

(a) $2 \quad 2^2 \quad 2^3 \quad 2^4 \quad \ldots \quad 2^n \quad \ldots$

(b) $1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad \ldots$

where in the second sequence each number is the sum of the previous two. Show that no matter how many differences you take, no set will form an arithmetic progression.

7. We know from Problem 13 in Section 1, that any two given numbers may be used to start an arithmetic progression. Why does this show that no matter what two numbers you may name, I can find a formula like: $an + b$, which has these two numbers as values for $n = 1$ and $n = 2$?

8. Look at Problem 6 in Section 2 and see if you can answer the following question: Given any three numbers, Can we find a formula like

$$an^2 + bn + c$$

which will have the given numbers as values when $n = 1$, $n = 2$, $n = 3$?

9. What kind of a formula do you think would fit any set of four values? Can you draw any general conclusions?

SUPPLEMENTARY UNIT 6

RECENT INFORMATION ON PRIMES

6-1. Robinson's Results

We are going to report to you on results published by Professor Raphael M. Robinson, of the University of California at Berkeley, in October, 1958, issue of the Proceedings of the American Mathematical Society. This will give you some idea of how research mathematicians are applying high-speed computers to solve problems about primes.

Robinson's note is based on calculations carried out during 1956 and 1957 on the SWAC (Standards Western Automatic Computer) at the University of California in Los Angeles.

To obtain an idea of the meaning of this work, let us think for a moment about the problem of finding out whether a given number n is a prime. According to the definition of a prime, we must find out whether n is divisible by some smaller number other than 1. The most obvious method is to divide n by the numbers 2, 3, 4, ..., up to $n - 1$. If any of these numbers divide evenly into n, then n is not a prime. If none of these divisions come out evenly, then n is a prime. This method requires $n - 2$ divisions. If n is about 10^{100}, and if each division requires .001 of a second, then this would take about 10^{97} seconds. How many seconds are there in a year? About how many years would this take?

We could shorten the work very much if we think a little. If n is not a prime, then n can be expressed as a product of two smaller numbers:

$$n = a \cdot b.$$

If a is the smaller of these factors, then n is at least $a \cdot a = a^2$.

$$n \geq a^2.$$

Therefore, if n is not a prime, then it is divisible by some number a whose square is at most n. To test whether n is a prime, it is enough to divide n by the numbers $2, 3, \ldots,$ up to the largest number whose square is no larger then n. If $n \leq 1{,}000{,}000$, then we do not have to try any divisons greater than $1{,}000$, since $1{,}000^2 = 1{,}000{,}000$. Thus to see whether $999{,}997$ is a prime, we only need to divide by $2, 3, \ldots, 999$. By this method we only need 998 divisions instead of $999{,}995$ divisions in the previous method.

If n is about 10^{100}, then this method requires only about 10^{50} divisions, for $10^{50} \cdot 10^{50} = 10^{100}$. If each division takes $.001$ of a second, how many years would it take by this method to test whether n is a prime?

If we wish to test really large numbers, we must look for better methods so that we can obtain the answers in a reasonable time. Therefore, mathematicians try to find special classes of numbers which have special properties which enable us to reduce the work even more.

For example, a great deal of work has been done on numbers which are one less than a power of 2. We may represent such numbers in the form

$$n = 2^m - 1.$$

If $m = 2$, then $n = 2^2 - 1 = 4 - 1 = 3$, which is a prime. If $m = 4$, then $n = 2^4 - 1 = 16 - 1 = 15$, which is not a prime. If m is not a prime, then n cannot be a prime. But m may be a prime without n being a prime.

Exercises 6-1

1. Make a table for $n = 2^m - 1$, up to $m = 20$:

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14-20
n	1	3	7	15	31									

2. Test the statements:

(a) If m is divisible by 2, then n is divisible by 3.

(b) If m is divisible by 3, then n is divisible by 7.

(c) If m is divisible by 5, then n is divisible by 31.

(d) What is the general law?

6-2. Proth's Theorem

Robinson reports on numbers which are one more than a small multiple of a power of 2, that is, numbers of the form

$$n = (k \cdot 2^m) + 1,$$

where k is a small odd number.

He and his group tested for primeness all numbers of this form with $k < 100$ and $m < 512$ as well as a few larger numbers. First they divided by all numbers less than 10,000; and for $k \leq 7$ they tried divisors up to 100,000. After eliminating all small factors in this way, they then applied a theorem stated by Proth in 1878. Let us see if we cannot get some idea of what Proth's theorem says and how it is used without trying to examine all of the details.

Proth's theorem gives a method of testing numbers of the form $n = (k \cdot 2^m) + 1$ for primeness provided the counting number k' is odd and less than 2^m. We can avoid much of the complication of the statement of Proth's theorem if we restrict ourselves to the case where k is not divisible by 3. Thus we may use

$$k = 1, 5, 7, 11, 13, 17, \ldots$$

$$m = 1, 2, 3, 4, 5, 6, 7, \ldots$$

and we are able to test the numbers $n = (k \cdot 2^m) + 1$ for primeness. For these numbers, n, Proth's theorem states that:

n <u>is prime if and only if it is a factor of</u>

$$3^{\frac{n-1}{2}} + 1.$$

Does this look mysterious to you? It is likely that it does, because you are not a mathematician. It would very probably look a bit mysterious even to a mathematician if he didn't happen to be familiar with the special techniques which are needed for a proof of this particular theorem. If you will accept, however, our word that it is a true theorem (and a great many very respectable mathematicians will testify to its being true) then it should not be hard to see what it <u>says</u> and how it is <u>used</u>.

In the first place, what does $3^{\frac{n-1}{2}} + 1$ mean? The expression $\frac{n-1}{2}$ is being used as an exponent. The number n we are using here is odd. (Why? What is the form of n?) Thus $n - 1$ is even, so that $\frac{n-1}{2}$ is a counting number. Thus $3^{\frac{n-1}{2}} + 1$ is just one more than 3 raised to a counting number power. To test n for primeness we need only find this number and then divide it by n. If this division comes out even then n is a prime; otherwise n is a composite.

What numbers can we test for primeness by this method? Let us list a few of them in a table and then apply the test to some of them. Fill in the blank spaces in the table on the following page. Remember that Proth's theorem requires that $0 < k < 2^m$, and that we have restricted ourselves to numbers k which are <u>not</u> divisible by 3.

k	m	$n = (k \cdot 2^m) + 1$
1	1	3
1	2	5
1	3	9
5	3	
7	3	57
1	4	
5	4	81
7	4	113
11	4	
13	4	209
1	5	33

k	m	$n = (k \cdot 2^m) + 1$
5	5	161
7	5	225
11	5	
1		65
	6	321
	5	417
	5	545
		2,817
		17,409
17		2,177
	10	7,169
		10,241

Now let us see how the test works for a few of these numbers. To refresh our memories we restate it here:

If $n = (k \cdot 2^m) + 1$ where $0 < k < 2^m$ and k is not divisible by 3 then n is prime if and only if it is a factor of

$$3^{\frac{n-1}{2}} + 1.$$

Example 1: Let $k = 1$ and $m = 2$ so that $n = 5$. (Look it up in the table.) We are testing 5 for primeness. In this case $\frac{n-1}{2}$ is $\frac{4}{2}$ or 2, so

$$3^{\frac{n-1}{2}} + 1 = 3^2 + 1 = 9 + 1 = 10.$$

Is n a factor of $3^{\frac{n-1}{2}} + 1$? Is 5 a factor of 10? Yes, it is, so the test tells us that 5 is a prime. Does this check with what you already know?

Example 2: Let $k = 1$ and $m = 3$ so that $n = 9$. (Look it up.) We divide

$$3^{\frac{n-1}{2}} + 1 = 3^4 + 1 = 81 + 1 = 82$$

by 9. The division does not come out even, so the test tells us that 9 is not a prime. Does this check with what you already know about 9?

Example 3: If $k = 1$ and $m = 6$ then what is n? The table should tell you that $n = 65$. If it does not, work it out again. $\frac{n-1}{2}$ is 32, then, so

$$3^{\frac{n-1}{2}} + 1 = 3^{32} + 1 = 1,853,020,188,851,842.$$

We would have to divide this number by 65 to continue the test. It would not be worth the effort, however, since we can easily recognize that 65 has 5 as a factor, and is therefore not a prime.

Example 4: Let $k = 7$ and $m = 4$ so that $n = (k \cdot 2^m) + 1 =$ 113. In this case the number $3^{\frac{n-1}{2}} + 1 = 3^{66} + 1$ is 9 times the square of 1,853,020,188,851,842 plus 1. If you are ambitious you may calculate this number and divide it by $n = 113$. The division will come out even if you do your work correctly, so what do you conclude about 113?

Examples 3 and 4 should convince us of one thing. Proth's theorem is not well suited for testing large numbers for primeness by hand calculation. Large computers, however, are constructed expressly to make calculations of the order of the ones which discouraged us above. And they do them quickly! On the SWAC the time for the test was no more than $1\frac{1}{2}$ minutes as long as $m < 512$. For m about 1000 and $k = 3$, 5, or 7 the test took about 7 minutes. The number $n = (7 \cdot 2^{1000}) + 1$ is larger than 10^{300}.

Compare 7 minutes with the time it would take the machine to test 10^{300} for primeness by trying all possible factors. Earlier in this section you got some idea of this time for numbers of the order of 10^{100}.

For $k = 1$ the test had previously been carried out for all $m < 8192$, and the only primes of this form which have been found are the cases:

$$m = 0,\ 1,\ 2,\ 4,\ 8,\ \text{and}\ 16.$$

The largest new prime discovered by this work is the case $k = 5$, $m = 1947$:

$$n = (5\ 2^{1947}) + 1.$$

If you wish to estimate this number, first notice that

$$10^3 = 1000 < 2^{10} = 1024.$$

Therefore we have

$$2^{1947} > 2^{1940} = (2^{10})^{194} > (10^3)^{194} = 10^{582}.$$

Therefore n has more than 582 digits. On the other hand, notice that

$$2^{13} = 8096 < 10^4.$$

Therefore we have

$$n < 1 + (8 \cdot 2^{1947}) = 1 + (2^3 \cdot 2^{1947})$$

$$= 1 + 2^{1950} = 1 + (2^{13})^{150}$$

$$< 1 + (10^4)^{150} = 1 + 10^{600}.$$

Consequently n has no more than 600 digits.

Remember that by using the theorem of Proth, this prime was discovered by a single division taking a matter of minutes. By using either of the cruder methods discussed before at least 10^{291} divisions would have been necessary. How long would this have taken at the rate of a thousand divisions per second?

This number is the fourth largest prime known at present The larger ones are the numbers

$$n = 2^m - 1$$

with $m = 3217$, 2281, and 2203. The latter two were reported by Robinson in the Proceedings of the American Mathematical Society in 1954. The largest one was reported early in 1958 by H. Riesel in Mathematical Tables and Aids to Computation (page 60).

Example 5: Estimate the number of digits in each of three primes.

*Perhaps you would be interested in the general statement of Proth's theorem. For numbers $n = (k \cdot 2^m) + 1$ with k divisible by 3 the important difference in the test for primeness is that the number $3^{\frac{n-1}{2}} + 1$ must be replaced by a new number. The number to use is of the form

$$a^{\frac{n-1}{2}} + 1$$

where a is a counting number which may have to be chosen differently for different values of k and m. The condition which a must satisfy will be found in the statement of Proth's theorem.

Theorem: Let $0 < k < 2^m$ and $n = (k \cdot 2^m) + 1$. Suppose a is a counting number which has the property: no sum of a and a multiple of n is a perfect square. (Alternative: the sum of a and a multiple of n is never a perfect square.)

Then n is a prime if and only if it is a factor of

$$a^{\frac{n-1}{2}} + 1.$$

The condition which a must satisfy is rather a strange one. It would seem that it might be difficult to find a number which

satisfies it in some cases. We could never find such a number by any number of trial operations, for the condition which a must satisfy involves a statement about all multiples of n. We may reject some choices of a on the basis of a single calculation, though. If $k = 3$, and $m = 2$ so that $n = 3 \cdot 2^2 + 1 = 13$ then would $a = 4$ do? No, because $117 + a = 117 + 4 = 121$ is a perfect square, and 117 is a multiple of $n = 13$. To find a number a, which we can be sure will fit the condition for a given n, then, we will have to use reasoning. We will have to reason that, for a certain number a, no matter how many multiples of n we try, adding a will never give a perfect square. Mathematicians know enough about numbers so that finding such a number is not a very difficult problem. As you may have guessed from the discussion above, it is possible to show that whenever k is not divisible by 3 the number $a = 3$ satisfies the condition of the theorem. Once we have found the right number a to go with n we can avoid the many tedious calculations necessary to test a large number for primeness. Instead of dividing n by all prime numbers whose squares are less than n, we need only perform one calculation. We simply try the division

$$(a^{\frac{n-1}{2}} + 1) \div n;$$

if it comes out even n is a prime, if not, n is not a prime.

SUPPLEMENTARY UNIT 7

GAMES

What would you say if you were asked how a war, a business, and a game are alike? There is at least one similarity. In each of these someone is constantly making decisions. Officers in the armed forces must decide what strategy to follow in the coming battle. Many decisions are made by the player of a game. A businessman must make decisions in managing his business. Can we do more than just guess at the right decision?

Walt Disney recently constructed Disneyland. He had to make many decisions involving how big it should be, where it should be, what prices should be charged, etc. He went to the Stanford Research Institute and asked the mathematicians to help him make these decisions. The mathematicians helped him figure out what decisions to make in order to make the biggest profit.

Game theory is a new branch of mathematics. Game theory can be used to determine the best strategy to follow in making a decision. John von Neumann invented game theory in 1928. He applied these ideas to business decisions and game strategy. You will first learn how to use game theory in finding the best strategy to follow in a game.

7-1 Strategy

What is strategy? Strategy is a plan you follow to gain something you want. Usually, strategy means the plan for a coming battle in a war. We will use the word strategy to refer to a plan leading to any decision. Strategy will mean a complete plan that cannot be upset by "enemy" action or Nature. Your opponent is your "enemy". Let's talk about strategy in games. The first group of games will be very simple.

Tom and Jim are going to play a game. Each boy wants to win, of course. He must be careful or he might be fooled by the other boy. These are the rules.

1. Each boy writes on a piece of paper either the letter "A" or the letter "B" without letting the other see. Then they compare what they have written.
2. If both wrote A, Tom wins 6 points.
   If both wrote B, Tom wins 4 points.
   If one wrote B and the other A, Tom wins 5 points.
3. After six turns, the boys change roles and Jim receives the points according to Rule 2.
4. The winner is the boy who has the most points after each boy has had 6 turns.

The following diagram, called a payoff matrix, shows the points for each combination of ways in which the letters could have been written. (A rectangular arrangement of numbers is called a matrix).

		Jim's Choice A	Jim's Choice B	Minimum payoff for the row
Tom's Choice	A	6	5	5
	B	5	4	4
Maximum payoff for the column		6	5	

The number in each box shows the number of points that Tom will win for each combination of choices.

Example: Tom writes "B" and Jim writes "B" - 4 points. Jim will allow Tom as few points as possible so Tom must determine the smallest, or minimum payoff he would win for each of his choices. We write the minimum payoff to the right of each row of the payoff matrix. The minimum number of points Tom will win by writing "A" is 5. The minimum for the choice "B" is 4. We

see that the minimum payoff for "A" is greater than the minimum payoff for "B". Tom will always win at least 5 points by writing "A".

For each choice by Jim there is a certain greater, or maximum, payoff for Tom. Jim will try to make Tom's payoff as small as possible, but he knows Tom will try to win the maximum number of points each time. Jim must determine the maximum payoff for each of his choices. List the maximum payoff below the column for each choice. For which choice is the maximum payoff the lesser? The lesser of these maxima occurs when Jim writes "B" and is equal to 5. The maximum number of points Tom can win when Jim writes "B" is smaller than the maximum payoff when Jim writes "A". Jim will never lose more than 5 points if he writes "B".

To summarize, we have the following:

The minimum payoff for Tom if he calls

(a) A is 5.
(b) B is 4.

The greater of these minimum payoffs is 5.

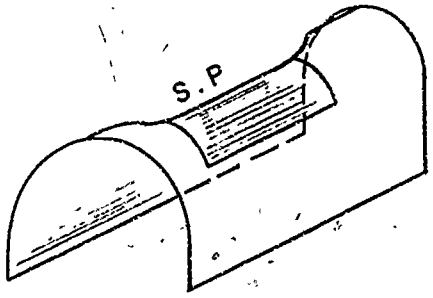
The maximum number of points Jim could lose if he calls

(a) A is 6.
(b) B is 5.

The lesser of tnese maximum payoffs is 5.

In this game notice that the greater of the minimum payoffs to Tom is equal to the lesser of the maximum losses of Jim. When this happens, we say that the game has a saddle-point.

Think of the shape of a saddle for a horse. The lines running from front to back dip down in the middle, or we could say they have a minimum point. The lines going across the saddle from left to right rise up to a maximum point in the middle. The greatest of the minimum points and the least of the maximum points are both at the center of the saddle. This is the point we call the



the saddle-point. When a game has a saddle point, the two choices which give the saddle point number are the best choices; and the best strategy for each boy is to choose that letter all of the time. Therefore the best strategy for Tom is always to write "B". This would become a very boring game. If Jim followed this strategy and Tom did not, Jim would always win.

Exercises 7-1

Work out the following problems. (The rules are the same as above except that the numbers of points that Tom wins are changed to the values given in the matrices below).

A.

Tom's choice	Jim's choice: A	Jim's choice: B
A	7	4
B	3	2

1. What is the minimum payoff for Tom if he chooses
   (a) A?_____
   (b) B?_____
2. Which is the greater of these minimum payoffs?
3. What is the maximum number of points Jim could lose if he chooses
   (a) A?_____
   (b) B?_____
4. Which is the lesser of these maximum payoffs?
5. Does the lesser of the maximum payoffs equal the greater of the minimum payoffs? Does this game have a saddle-point?
6. What is the best strategy for each boy?

B. Jim's choice

Tom's choice	A	B
A	5	8
B	6	7

1. What is the greater of the minimum payoffs Tom can win?
2. What is the lesser of the maximum payoffs Jim can lose?
3. Does the lesser of the maximum payoffs equal the greater of the minimum payoffs? Does this game have a saddle-point?
4. What is the best strategy for each boy?

C. Jim's choice

Tom's choice	A	B
A	2	4
B	8	6

1. What is the greater of the minimum payoffs Tom can win?
2. What is the lesser of the maximum payoff Jim can lose?
3. Does the lesser of the maximum payoffs equal to the greater of the minimum payoffs? Does this game have a saddle-point?
4. What is the best strategy for each boy.

D. In this game, each boy can write "A", "B", or "C". The payoff tells how many points Tom receives for each combination.

Jim's choice

Tom's choice	A	B	C
A	10	2	8
B	7	6	9
C	11	4	3

1. What is the minimum number of points Tom would win if he chooses
   (a) "A"?_____
   (b) "B"?_____
   (c) "C"?_____
2. Which is the greatest of these minimum payoffs?
3. What is the maximum number of points Jim will lose if he chooses
   (a) "A"?_____
   (b) "B"?_____
   (c) "C"?_____
4. Which is the least of these maximum payoffs?
5. Does the least of the maximum payoffs equal the greatest of the minimum payoffs? Does this game have a saddle-point?
6. What is the best strategy for each boy?

7-2. Business Strategy

These same methods can be used for making decisions in business. The primary concern of a business is how to make a good profit. In other words, each decision should result in a payoff which brings the maximum (that is, greatest) possible profit. From a pay-off matrix a business man could find the best strategy to follow. It is more difficult to set up a pay-off matrix for a business decision. Before you can do this you must learn some business terms.

Profit or Loss

Mr. James owns a bicycle shop. He buys a bike for $40 and sells it for $65. The difference between the selling price and the cost is the margin.

margin = selling price - cost

What is the margin for the bike?

Mr. James has many expenses in his business. He must pay wages, rent, taxes and utilities. These are called operating expenses. The difference between the margin and the operating expense is the profit.

profit = margin - expenses.

If Mr. James' expenses for selling this bike total $15, what is his profit?

What would happen if the expenses were greater than the margin? If this happens Mr. James will lose money. Mr. James would have a negative profit. If Mr. James' expenses had equaled $30 for that bike what would be the profit?

Example:

In a dime store, merchandise which cost $400 was sold for $525. What is the margin?

selling price	$525
- cost	$400
margin	$125

If the operating expenses equaled $80, what is the profit?

margin	$125
- expenses	80
profit	$ 45

Work the following problems:

Problems: Find the quantity asked for in each of the following problems.

Exercises 7-2

Dime Store

In a dime store, merchandise costing $250 and was sold for $340.

1. What is the margin?

The operating expenses totaled $60.

2. What is the profit?
3. If the operating expenses are $100, what is the profit?

Ice Cream Man

The ice cream supply for one day is bought for $25 and sold to customers for $48.

4. What is the margin?

The gas for the truck costs $4.00 per day.
The license costs 50¢ per day.
The insurance on the truck costs .40¢ per day.
Repairs and upkeep for the truck cost 50¢ per day.

5. What is the total operating expense per day?
6. What is the profit per day?

Theatre

In one week
720 tickets were sold at 25¢ a piece.
400 tickets were sold at 50¢ apiece.

7. What is the total price paid for all of the tickets?

Rent (per week)	$ 70
Rent for films	$ 50
Wages for one week	$120
Electricity (per week)	$ 30

8. What is the total operating expense for one week?
9. What is the profit for one week?

Gasoline - Service Station

The gas station sold 780 gallons of gas at 22¢ per gallon one day.

10. What is the total selling price?
11. If the gas cost 16¢ per gallon, what is the total margin?
12. The operating costs total $24.50, what is the profit?
13. At a later date, the station operator sold only 350 gallons of gas but had the same operating costs. Find the profit.

7-3 Pay-Off Matrix for a Business Decision

A vendor at the football game can sell either coffee or popcorn (but not both.) In warm weather the vendor can sell 300 cups of coffee. If coffee costs 5¢ and sells for 10¢ what is the margin per cup? What is the margin on 300 cups of coffee? In warm weather the vendor can sell 400 boxes of popcorn at 10¢ per box. What is the total margin if each box costs 5¢?

The following matrix will show the profits. (assume that there are no operating expenses:) Think of the weather as the vendor's opponent in a game and imagine that the weather is trying to defeat the vendor.

		Weather	
		warm	cold
Vendor's order	coffee	$15	
	popcorn	$20	

If the vendor sells coffee in cold weather, he sells 700 cups of coffee.

What would be the margin? What is the profit?

The vendor will sell 600 boxes of popcorn if it is cold outside.

What is the margin? What is the profit?

Place these numbers in the correct boxes. The vendor wants to make the most profit that is possible. Since he cannot be sure of the weather, he must determine his best strategy: Is there a saddle-point? What is his best strategy? You should have found that the vendor should always order popcorn. He will always make at least $20.

It should be emphasized that this is all on the assumption that the weather is completely unreliable. For instance, if the vendor could be sure that there would be more cold days than hot days, he should always serve coffee. We make the assumption for this problem that there is no such assurance about the weather.

Exercises 7-3

A. Set up a pay-off matrix for the following vendor and determine his best strategy.

The vendor can sell either hot dogs, soda pop or ice cream. Which one should he sell?

	Number Sold	Selling Price	Cost	Margin	Operating Cost	Profit
Hot Dogs	hot day - 150	15¢ each	11¢ each		$2.00	
	cold day - 250	15¢ each	11¢ each		$2.00	
Soda Pop	hot day - 350	10¢ each	6¢ each		$2.00	
	cold day - 275	10¢ each	6¢ each		$2.00	
Ice Cream	hot day - 500	10¢ each	5¢ each		$2.00	
	cold day - 180	10¢ each	5¢ each		$2.00	

Enter the profit for each item in the correct box of the following matrix.

Weather

	Hot	Cold
Hot dogs		
Soda pop		
Ice Cream		

B. Suppose a man is selling ice cream and coffee at the fair. His business is affected by the weather, which is sometimes hot and sometimes cold. His opponent is the weather. He has to order ahead of time but coffee sells well if the weather is cold, and ice cream sells well, if the weather is hot. The vendor must find the best strategy in order to make the maximum profit.

Assume the coffee costs 4 cents a cup and sells for 10 cents. The ice cream costs 5 cents a bar and sells for 10 cents. On a hot day the vendor will sell 400 ice cream bars and 200 cups of coffee. If his operating expense is $3 per day what is his profit on a hot day? On a cold day the vendor sells 500 cups of coffee and 200 ice cream bars. What is his profit on a cold day?

The following kind of chart may help you find the profit.

	Ice Cream	Coffee
Sold on cold day		
Total Receipts		
Ordered for cold day		
Total costs		
Margin		
Operating expense		$3.00
Profit		

Note: Assume that the vendor cannot store anything overnight so anything left at the end of the day is thrown away. For example, if the vendor orders for a cold day and it is hot, he will sell the 200 ice cream bars and only 200 cups of coffee. 300 cups of coffee would be wasted.

There are, therefore, three other charts necessary:

1. When he orders for a hot day and it is hot.
2. When he orders for a hot day and it is cold.
3. When he orders for a cold day and it is hot.

Fill in the following pay-off matrix (Make your own)

		Weather	
		Hot	Cold
Ordered for:	Hot day		
	Cold day		

Determine the maximum and minimum values in the matrix. Is there a saddle-point? When there is no saddle-point, a single call is not the best strategy. In order to find the best strategy you must know something about the laws of chance. You will study the laws of chance in more advanced work.

BIBLIOGRAPHY

1. Kemeny, J. G., J.L. Snell, and G.L. Thompson, FINITE MATHEMATICS, 1956, pp. 266-306, Prentice-Hall, Englewood N.J. (Other references are given here).

2. Williams, J. D., THE COMPLEAT STRATEGYST. 1950, McGraw Hill New York. This is a charming little book written, at least in its beginnings, for the person with little mathematical background.