DOCUMENT RESUME

ED 143 529 SE 023 007

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TITLE Mathematics for High School. Intermediate Mathematics, Part 1, Supplementary Unit I, The Development of the Real Number System. Revised Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 60
NOTE 95p.

EDRS PRICE MF-$0.83 HC-$4.67 Plus Postage.
DESCRIPTORS *Algebra; Arithmetic; *Instructional Materials; *Number Concepts; Secondary Grades; *Secondary School Mathematics; Supplementary Textbooks; *Textbooks
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is a supplementary unit to Mathematics for High School, Intermediate Mathematics, Part 1. In this publication, real numbers and rules for operating them are examined. The study begins by examining whole numbers and some of the properties of addition and multiplication of whole numbers. Most of the basic rules for algebra are developed from these properties. Included are background information, discussion of topics, exercises and student activities, and answers to exercises and activities. (RH)

SCHOOL MATHEMATICS STUDY GROUP

MATHEMATICS FOR HIGH SCHOOL

INTERMEDIATE MATHEMATICS (Part 1)

Supplementary Unit I

(THE DEVELOPMENT OF THE REAL NUMBER SYSTEM)

(revised edition)

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(THE DEVELOPMENT OF THE REAL NUMBER SYSTEM)

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PRINTED IN THE U.S.A.

THE DEVELOPMENT OF THE REAL NUMBER SYSTEM

1. Introduction

In the beginning of your study of arithmetic you learned about whole numbers; later you learned about fractions. In algebra you learned about negative numbers, and about irrational numbers. These numbers are all called real numbers.

As you were introduced to each kind of new number, you learned rules for operating with them, that is, for performing addition and multiplication and, where possible, subtraction and division. You learned the practical reasons why the new numbers were needed, and these reasons made some of the new operations seem natural and simple. Probably some of the rules, however, seemed arbitrary and mysterious. Are there some which still seem mysterious to you? Could one use different rules for operating with negative and fractional numbers? If not, why not? If there is only one "right" way of operating with these numbers is there any way you could find the "right" rules yourself?

In this chapter we are going to take a closer look at the real numbers and the rules for operating with them. We will see that there are just a few basic rules of algebra and that every other algebraic rule follows logically from these basic rules. We will show you that there are purely mathematical reasons for introducing each new kind of real and that the rules for operations with these numbers are the only ones possible if the operations are to have certain simple and familiar properties. We will actually discover these rules.

We will begin our study by examining the whole numbers and some of the properties of addition and multiplication of whole numbers. From these properties we will get most of our basic rules of algebra. We will not suppose that you know anything about other real numbers or operations with them.

2. Whole Numbers

The numbers first used by man, even before recorded history, were the numbers 1, 2, 3, 4, 5, ..., etc. which we call the natural numbers. Very early in man's history the number 0 was introduced. We call the natural numbers and 0 whole numbers. We are going to examine some of the properties of operations with whole numbers.

Exercises 2a. (Oral).

For each equation list all the whole numbers that satisfy it. If all whole numbers satisfy the equation the answer is all. If no whole number satisfies the equation the answer is none.

1. $x + 5 = 17$
2. $x / 2 = 3$
3. $A \cdot 6 + 3 = 33$
4. $33 / z = 11$
5. $33 / x = 4$
6. $(y + 2)(y + 1) = y \cdot y + 3 \cdot y + 2$
7. $x / 5 = 2$
8. $(2x - 1) / 13 = 1$
9. $z + 1 = z + 2$
10. $2 \cdot z + 6 = 6 + 2 \cdot z$
11. $x \cdot x = x$
12. $x + x = x$
13. $x + 19 = 21$
14. $x + 21 = 19$
15. $1 + x = x$
16. $1 + x = 1$
17. $x(x + 2) = x \cdot x + x \cdot 2$
18. $x + y = y + x$
19. $5x = x \cdot 5$
20. $2x + 10 = 2(x + 5) + 3$
21. $x - (y - z) = (x - y) + z$

Operations: What is an operation? An operation is a rule which assigns a number to certain pairs of numbers given in a definite order. We call a pair of numbers given in a definite order an ordered pair of numbers, and denote the ordered pair--first a, second b --by (a, b).

Addition: The first operation you learned, and still the one you consider the simplest and most fundamental is addition. Addition assigns to the ordered pair of numbers (a, b) the number $a + b$. What do you know about the addition of whole numbers? We are going to list some properties of addition which you use all the time.

The first and simplest property is that addition is always possible for whole numbers and the result is a whole number; that is, addition assigns a number to every ordered pair of whole numbers, and the number assigned is a whole number. For example, $4 + 3 = 7$, $112 + 2133 = 2245$, $19 + 0 = 19$ We say that the set (or collection, or class) of whole numbers is closed under addition, and refer to this property as the closure property of addition.

Closure property of addition: If a and b are whole numbers then $a + b$ is a whole number.

Exercise 2b. (For class discussion)

1. Is the set of all natural numbers closed under the operation:
   (a) addition?
   (b) subtraction?
   (c) multiplication?
   (d) division?
2. Is the set of all even natural numbers closed under:
   (a) addition?
   (b) multiplication?
3. Is the set of all odd natural numbers closed under:
   (a) addition?
   (b) multiplication?
4. Is the set of all natural numbers less than 10 closed under:
   (a) addition?
   (b) multiplication?

The second property of addition is that the order in which we add numbers does not affect the result; addition assigns the same number to the ordered pairs (a, b) and (b, a). For example, $5 + 2 = 2 + 5$. We call this property the commutative property of addition.

Commutative property of addition: For all a and b, $a + b = b + a$.

Addition assigns a number to a pair of numbers. If we wish to "add" three numbers given in a definite order, we must first add two of them, and then add the result to the third. We can group three numbers given in a definite order in two different ways, but the way in which we group them does not affect the result of addition; that is, $a + (b + c) = (a + b) + c$.
For example, with $a = 5$, $b = 6$ and $c = 9$, we have

$$a + (b + c) = 5 + (6 + 9) = 5 + 15 = 20$$
$$(a + b) + c = (5 + 6) + 9 = 11 + 9 = 20$$

We call this property the associative property of addition.

Associative property of multiplication: For all a, b, and c, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Multiplication: Let us turn now to the second fundamental operation, multiplication. Does it have properties similar to those of addition? Yes. The product of two whole numbers is a whole number; the order of factors does not affect the product; and the way in which we group numbers for multiplication does not affect the result.

Closure property of multiplication: If a and b are whole numbers then $a \cdot b$ is a whole number.

Commutative property of multiplication: For all a and b, $a \cdot b = b \cdot a$.

Associative property of addition: For all a, b, and c, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Distributive property: Are addition and multiplication related? The most important feature of their relationship is that we can multiply a number by the sum of two numbers, or we can multiply each of the numbers in the sum by the first number and then add-and we get the same result. Symbolically this is expressed by the equation $a(b + c) = ab + ac$. With $a = 5$, $b = 6$ and $c = 9$, for example, we have

$$5(6 + 9) = 5 \cdot 15 = 75$$
$$5 \cdot 6 + 5 \cdot 9 = 30 = 45 = 75$$

Because of the commutative property of multiplication we can also write $(b + c)a = ba + ca$. We describe both equations by saying that multiplication is distributive over addition.

Distributive property: For all a, b, and c, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Exercises 2c.

1. Indicate whether the following statements are false or true.

   If true state the property upon which your answer depends.

   (a) $6(4 + 5) = 6 \cdot 4 + 6 \cdot 5$ (a) __________

   (b) $6 + (4 \cdot 5) = (6 + 4) \cdot (6 + 5)$ (b) __________

   (c) $73 + 7 = 7 + 73$ (c) __________

   (d) $6 \cdot (7 + 4) = (6 \cdot 7) + 4$ (d) __________

   (e) $6 \cdot 19 = 19 \cdot 6$ (e) __________

   (f) $5 \cdot (20 \cdot 17) = (5 \cdot 20) \cdot 17$ (f) __________

   (g) $5 \cdot (3 \cdot 4) = (5 \cdot 3) \cdot (5 \cdot 4)$ (g) __________

   (h) $(5 + 7) + 6 = (7 + 5) + 6$ (h) __________

   (i) $(5 + 7) + 6 = 6 + (5 + 7)$ (i) __________

   (j) $(5 + 7) + 6 = 5 + (7 + 6)$ (j) __________

   (k) $3 \cdot 4 + 5 \cdot 6 = 4 \cdot 3 + 5 \cdot 6$ (k) __________

(l) $(12 + 88)\cdot(100 + 10) = (12 + 88)\cdot 100 + (12 + 88)\cdot 10$

(m) $3\cdot 7 + 8\cdot 2 = 8\cdot 2 + 3\cdot 7$ (l)_______ (m) _______

2. Explain why you get the same answer whether you add the column of figures up or down.

$$\begin{array}{r} 7 \\ 5 \\ \underline{3} \\ 15 \end{array}$$

The properties we have listed are certainly simple, but they are also <u>fundamental</u>. You may not realize that they are fundamental because most of the time you use them without thinking about them. To gain some appreciation of their importance, consider the fact that with addition and multiplication tables only for the numbers 1 through 9 and the rules $10\cdot 10 = 100$, $10\cdot 100 = 1000$, ... we can add and multiply any two whole numbers using simple procedures. How is this possible? What justifies the sum

$$\begin{array}{r} 23 \\ \underline{38} \\ 61 \end{array}$$

for example?

Using place value notation the associative and commutative properties of addition, and the distributive property, we proceed in the following way:

$23 + 38 = (2\cdot 10 + 3) + (3\cdot 10 + 8)$	Place value of digits
$= [(2\cdot 10 + 3) + (3\cdot 10)] + 8$	Associative property of addition
$= [(2\cdot 10) + (3 + 3\cdot 10)] + 8$	Associative property of addition
$= [(2\cdot 10) + (3\cdot 10 + 3)] + 8$	Commutative property of addition
$= [(2\cdot 10 + 3\cdot 10) + 3] + 8$	Associative property of addition
$= (2\cdot 10 + 3\cdot 10) + (3 + 8)$	Associative property of addition
$= (2 + 3)\cdot 10 + (3 + 8)$	Distributive property

The sums in parenthesis are known from addition tables. We have

$23 + 38 = 5\cdot10 + 11$ Addition table

$23 + 38 = 5\cdot10 + (10 + 1).$ Place value of digits

Finally using the associative and distributive properties again we have

$23 + 38 = (5\cdot10 + 10) + 1$ Associative property of addition

$23 + 38 = (5\cdot10 + 1\cdot10) + 1$ Since $10 = 1\cdot10$

$23 + 38 = (5 + 1)\cdot10 + 1$ Distributive property

$23 + 38 = 6\cdot10 + 1$ Addition table

$23 + 38 = 61$ Place value of digits.

Exercises 2d.

1. Use the properties for the natural numbers and the given definitions,

   $2 = 1 + 1, \quad 3 = 2 + 1, \quad 4 = 3 + 1,$

   to prove

   $2 + 2 = 4.$

2. One way of multiplying 32 by 23 without pencil or paper is: 20 times 32 is 640, 3 times 32 is 96 and the sum of 640 and 96 is 736. Explain why this gives the correct answer.

3. Assume that you know only addition and multiplication tables from the number 1 to 9, and that $10\cdot10 = 100$. Explain each step in multiplication

$$\begin{array}{r} 23 \\ 8 \\ \hline 184. \end{array}$$

   Which step corresponds to "carrying 2"?

4. Perform the indicated operations and give the reasons for each step in the operation.

(a) $13 + 25$	(e) 25×34
(b) $38 + 44$	(f) 12×100
(c) 16×1	(g) 123×100
(d) $86 + 35$	(h) 762×379

Multiplicative identity element: In the preceding, when we said 1·10, we used an important property of operations with whole numbers. This is the property of the number 1: $a \cdot 1 = 1 \cdot a = a$, for all a. The number 1 is the only number with this property. When there is only one element with a given property we will say it is unique. Because of its peculiar property the number 1 plays an important role. We shall call 1 the identity element for multiplication or multiplicative identity element.

Multiplicative identity element property: There is a unique element, 1, such that for all a, $a \cdot 1 = 1 \cdot a = a$.

Additive identity element: What is mean by an additive identity element? We mean a number x with the property that $a + x = a$ for all a. There is no additive identity among the natural numbers, but one is desirable even for the simple processes of arithmetic: it is for this reason that 0 is introduced. The number 0 has the property of an additive identity that $a + 0 = 0 + a = a$ for all a, and 0 is the only number with this property.

Additive identity element property: There is a unique element, 0, such that for all a, $a + 0 = 0 + a = a$.

Subtraction and division: We have not yet mentioned subtraction and division. The reason for this is that subtraction is defined in terms of addition and division is defined in terms of multiplication. To say that $x = b - a$ is to say that x is a solution of the equation $a + x = b$. Thus $2 = 5 - 3$ means that $3 + 2 = 5$, and $7 = 11 - 4$ means that $7 + 4 = 11$. We say that subtraction is the inverse of addition.

Definition of subtraction: $x = b - a$ means $a + x = b$

The symbol $b - a$ is read "b minus a," and is called a difference.

In the same manner, division is defined as the inverse of multiplication. $2 = 6 \quad 3$ means that $3 \cdot 2 = 6$ and $3 = 15/5$ means that $5 \cdot 3 = 15$. Generally $x = b/a$ means that x is the solution of the equation $a \cdot x = b$.

If $a = 0$, then for all values of x, $a \cdot x = 0$, so that if $b \neq 0$, there is no x such that $ax = b$. (The symbol $\neq$ is read "is not equal to.") However if $b = 0$, then any value of x satisfies the equation. To avoid this situation we exclude division by zero.

<u>Definition of division</u>: $x = \frac{b}{a}$ <u>means</u> $a \cdot x = b$ <u>and</u> $a \neq 0$.

The symbol $\frac{b}{a}$ (or b/a) is read "b over a," and is called a <u>fraction</u>.

The equations $a + x = b$ and $ax = b$, in terms of which subtraction and division are defined, are among the simplest equations of algebra. Yet, with these equations we run into trouble if, as we are assuming here, the only numbers we know are the natural numbers and zero. The trouble is that some of these equations may not have solutions. The equations

$$5 + x = 3 \quad , \quad 11 + x = 7$$
$$6 \cdot x = 3 \quad , \quad 15 \cdot x = 5$$

are examples of equations which do not have whole number solutions.

In the case of the equation $a + x = b$ we can say that the equation has a solution if b is larger than a or equal to a, and otherwise does not have a solution. In the case of the equation $ax = b$ we can say only that it has a solution if and only if a divides b, which is saying that the equation has a solution if and only if it has a solution which, you will agree, is not saying much. In either case we can make no <u>general</u> statement about the solutions of these equations. To be able to say anything about the solutions we have to know what specific numbers a and b represent.

An important feature of algebra is generality. An indication of this is the use of letters to stand for numbers, which is one feature which distinguishes algebra from arithmetic. One reason for the use of letters is that we wish to make statements about relations between numbers without having to specify the numbers. If we have to be specific in the case of the simplest algebraic equations the development of algebra is blocked at the beginning. The disire for a system of numbers in which the equations $a + x = b$ and $ax = b$ always have unique solutions is the mathematical reason for the creation of the new numbers which we will call integers and rational numbers.

Cancellation properties: Although we can make no general statement about the existence of solutions of the equations $a + x = b$ and $ax = b$ if a and b are whole numbers and x is required to be a whole number, we can make a general statement, about the uniqueness of solutions. We can say that if the equation $a + x = b$ has a solution it has only one solution. Another way of saying this is: If $a + x = a + y$, then $x = y$. For example, if

$$5 + x = 8, \text{ or}$$
$$5 + x = 5 + 3, \text{ then}$$
$$x = 3$$

We can make a similar statement about the equation $ax = b$. If $a \neq 0$ and if $ax = ay$, then $x = y$. For example, if

$$6x = 18, \text{ or}$$
$$6x = 6 \cdot 3, \text{ then}$$
$$x = 3$$

These statements express important properties of addition and multiplication; we will call them the cancellation properties of addition and multiplication. Because of the commutative properties of addition and multiplication we can also write: If $x + a = y + a$, then $x = y$; If $a \neq 0$ and $x \cdot a = y \cdot a$, then $x = y$.

<u>Cancellation property of addition</u>: <u>For all</u> a, <u>if</u> $a + x = a + y$ (<u>or if</u> $x + a = y + a$), <u>then</u> $x = y$.

<u>Cancellation property of multiplication</u>: <u>For all</u> a, <u>except</u> $a = 0$, <u>if</u> $a \cdot x = a \cdot y$ (or $x \cdot a = y \cdot a$), then $x = y$

<u>Exercises</u> 2e. (Oral)

Find the <u>whole number</u> solutions of the following equations. Indicate which equations have no whole number solutions. Also, state which cancellation properties are used in finding the solutions.

1. $x + 7 = 7$
2. $8x = 72$
3. $9 + x = 19$
4. $8 = 14 + x$
5. $2x = 0$
6. $3x = 2$
7. $x + 11 = 12$
8. $32 = 2x$
9. $13 = 3 + 2x$
10. $13 = 2 + 2x$
11. $9 + 2x = 93$
12. $28 + 3y = 1$

3. Number Systems

By a number system we shall mean a set of numbers for which addition and multiplication are defined and have all the properties listed in Section 2. The set of whole numbers is a number system; in later sections we will learn about others. We summarize the properties of a number system below.

Properties of a Number System

Closure	If a and b are in the system then $a + b$ is in the system. If a and b are in the system then $a \cdot b$ is in the system.
Commutative	For all a and b, $a + b = b + a$ For all a and b, $a \cdot b = b \cdot a$
Associative	For all a, b and c, $a + (b + c) = (a + b) + c$ For all a, b and c, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive	For all a, b and c, $a \cdot (b + c) = a \cdot b + a \cdot c$ $(b + c) \cdot a = b \cdot a + c \cdot a$
Identity element	There is a unique element 0, so that for all a, $a + 0 = 0 + a = a$ There is a unique element 1, so that for all a, $a \cdot 1 = 1 \cdot a = a$
Cancellation	For all a, if $a + x = a + y$, then $x = y$ For all a, if $x + a = y + a$ then $x = y$ For all a, except $a = 0$, if $ax = ay$ then $x = y$ For all a, except $a = 0$, if $xa = ya$, then $x = y$

An equation involving letters which is a true statement when the letters represent any numbers in a number system is called an identity equation, or briefly an identity, in the system. (The word identity here has no relation to the word identity in the phrases "additive identity, element" "multiplicative identity element.") The equations in the statements of the associative, commutative, and distributive properties are identities in every number system. Using these properties and the others stated above as axioms or postulates we can establish other statements, in particular identities, as theorems.

In proving theorems we will often use statements such as: if $a = b$ and $b = c$ then $a = c$; if $a = b$ and $c = d$ then $a + c = b + d$. Such statements are consequences of the meaning of equality and the meaning of operation. The equation $a = b$ means that the symbols a and b represent the same number. Thus, if a and b represent the same number and b and c represent the same number, then a and c represent the same number. If a and b represent the same number, and c and d represent the same number, then $a + c$ and $b + d$ represent the same number; namely the number assigned by addition to the ordered pair of numbers which can be represented either by (a, c) or by (b, d). When we use statements such as the two we have discussed we will say that we are using substitution statements.

As a simple illustration, consider the following

Theorem 1 For all a, b and c, $a + (b + c) = (c + a) + b$.

Proof:

$b + c = c + b$		Commutative property
$c + a = a + c$		Commutative property
$a + (b + c)$	$= a + (c + b)$	Substitution
	$= (a + c) + b$	Associative property
	$= (c + a) + b$	Substitution.

Theorems such as the preceding are necessary because addition and multiplication are defined only for pairs of numbers. We may use the symbol $a + b + c$ for the sum of three numbers only because of the associative property of addition. If addition did not have the associative property $a + b + c$ might mean $(a + b) + c$ or $a + (b + c)$ and these numbers would not be the same. Similarly we may write $a + b + c = c + a + b$, but only because of Theorem 1 and the associative property. There are many similar identities,

$$\begin{aligned} a + b + c &= c + a + b \\ &= c + b + a \\ &= b + c + a \\ &= b + a + c \\ &= a + c + b, \end{aligned}$$

and corresponding identities for multiplication. Instead of stating and proving all of these as theorems, we shall accept them, as proved, and say that we are using associative and commutative properties when we use them.

There are many statements in algebra which are actually theorems even though we don't usually call them theorems. In particular, every equation involving letters which is obtained when "simplifying," "multiplying out" etc. is an identity, and hence a theorem. The following examples illustrate this.

Example: Simplify $3x + 4(x + 7)$.

Solution:

$3x + 4(x + 7) = 3x + (4x + 28)$	Distributive property
$= (3x + 4x) + 28$	Associative property
$= (3 + 4)x + 28$	Distributive property
$= 7x + 28$	

Example: Multiply out $(a + b)^2$.

Solution:

$(a + b)^2 = (a + b)(a + b)$	
$= a(a + b) + b(a + b)$	Distributive property
$= (a\cdot a + a\cdot b) + (b\cdot a + b\cdot b)$	Distributive property
$= (a^2 + ab) + (ab + b^2)$	Commutative property
$= (a^2 + b^2) + (ab + ab)$	Associative and commutative properties
$= (a^2 + b^2) + (1\cdot ab + 1\cdot ab)$	Identity element property
$= (a^2 + b^2) + (1 + 1)\ ab$	Distributive property
$= a^2 + 2ab + b^2$	Associative and commutative properties

Exercises 3.

Simplify each of the following expressions, justifying each step as was done in the illustrative examples.

1. $2(5a + b) + 3(b + 2a)$
2. $(3x)(2y)$
3. $(x + 3)(2x + 3)$
4. $a + 3(a + 4)$
5. $x^2y + 5 + 3x^2y + 2$
6. $a(b + 2) + a(b + 2)$
7. $2a^2 + 3b^2 + 5b^2 + 7a^2$
8. $(6x + 9)(2x + 2) + 3x + 18$
9. $(x + a)(x + b)$
10. $(a + b)(x + y)$
11. $(ax + b)(cx + d)$

4. The Integers

We have defined subtraction in terms of the equation

$$a + x = b.$$

The statement $x = b - a$ means that x is a solution of this equation. But in the system of whole numbers this equation does not always have a solution, and hence $b - a$ does not always have meaning. For example, the equation $5 + x = 3$ does not have a solution in the system of whole numbers, and hence $3 - 5$ has no meaning in this system.

Thus, for the purposes of algebra it is desirable to enlarge our system of numbers so that the equation

$$a + x = b$$

always has a unique solution. To do this we join to the set of natural numbers and zero the set of negative integers--numbers represented by $-1, -2, -3, \ldots$ --and then define addition and multiplication in this new set, in such a way that all of the properties of a number system hold. The new system of numbers is called the system of integral numbers or integers. The system of integers is an extension of the system of whole numbers. By this we mean that the whole numbers are in the new system and that the operations of addition and multiplication applied to whole numbers yield the same result as in the system of whole numbers. Properties of the system of integers: As a number system, the system of integers has all the properties listed in Section 3. The additional property it possesses is the following.

Subtraction property: For all a and b, there is a unique x, such that $a + x = b$.

As examples consider the following equations which do not have solutions in the system of whole numbers

$$5 + x = 3 \quad , \quad 11 + x = 7.$$

In the system of integers these equations have the solutions -2 and -4, respectively.

<u>Exercises</u> 4a. (Oral)

Find all integer solutions of the following equations. If an equation has no integer solutions the correct answer is "none."

1. $t + 12 = 7$
2. $9z = 15$
3. $2x + 26 = 8$
4. $2s + 3 = -15$
5. $z + 3 = 2 + z$
6. $5x - 3 = 18$
7. $5x + 18 = 3$
8. $3y + 2(7 + 2y) = 19$.

Why is it necessary to introduce the negative integers and to operate with them as we do to get a system which has all the properties we desire? Let us forget about the integers now and just suppose that we have some number system which is an extension of the system of whole numbers and which has the subtraction property. We will show that the negative integers must be in this system and that the rules of operation with them must be the familiar rules of operation.

<u>Additive inverse elements</u>: According to the subtraction property the equation

$$a + x = 0$$

always has one and only one solution. For a given a we will call the unique solution of this equation the inverse of a for addition or the additive inverse of a, and denote it by $-a$. The property which defines $-a$ is therefore

$$a + (-a) = 0.$$

Observe that because of the commutative property of addition we have also,

$$-a + a = 0.$$

<u>Definition of additive inverse</u>: $-a$ <u>is the additive inverse of</u> a <u>means</u> $a + (-a) = -a + a = 0$.

For example, corresponding to the number 5 there is a number -5 with the property that

$$5 + (-5) = -5 + 5 = 0,$$

and $x = -5$ is the only number for which

$$5 + x = x + 5 = 0.$$

The symbol $-a$ is read "minus a", "negative a" or "opposite of a."

The additive inverses of the natural numbers are what we call the negative integers. As we shall show below $-(-a) = a$, so the additive inverses of the negative integers are the natural numbers. For this reason, the natural numbers will also be called the positive integers.

Let us prove the statement we just used.

Theorem 3: For all a, $-(a) = a$.

Proof: We have

$-a + [-(-a)] = 0$	Definition of additive inverse
$-a + a = 0$	Definition of additive inverse.
$-(-a) = a$	Subtraction property,

since by the subtraction property the equation $-a + x = 0$ has only one solution.

Now, using the equation $a + (-a) = 0$ which defines the negative integers and properties of a number system we can find the rules for operating with the integers.

Subtraction and additive inverses:

Consider the equation $a + x = b$. As in Section 2 we denote the solution of this equation by $b - a$. Now however the symbol always indicates a perfectly definite number, for by the subtraction property, the equation $a + x = b$ always has a unique solution. Thus, subtraction is an operation defined for all ordered pairs of integers.

We shall now establish an important connection between subtraction and additive inverses.

Theorem 4: For all a and b, $b - a = b + (-a)$

Proof:

$a + (-a) = 0$	Definition of additive inverse
$[a + (-a)] + b = 0 + b$	Substitution
$[a + (-a)] + b = b$	Additive identity element property

$a + (-a + b) = b$ Associative property

$a + [b + (-a)] = b$ Commutative property

so, $b - a = b + (-a)$ Definition of subtraction.

We shall refer to Theorem 4 as the <u>theorem on subtraction</u>. It states that a number is subtracted by adding its additive inverse. <u>Addition</u>: The theorem on subtraction is also part of the rule for adding positive and negative integers. For if b is larger than a the equation $a + x = b$ has a solution in the system of whole numbers. Since our new system is an extension of the system of whole numbers the solution of this equation, that is, $b - a$, in our new system must be the same whole number. Thus, for example, $5 + (-3) = 5 - 3 = 2$ and $17 + (-11) = 17 - 11 = 6$. To find the sum $b + (-a)$ when a and b are natural numbers and a is larger than b--for example, to find $3 + (-5)$ and $11 + (-17)$ -- we need another theorem, namely the following.

<u>Theorem</u> 5: <u>For all</u> a <u>and</u> b, $b + (-a) = -(a - b)$

<u>Proof</u>: The proof of this is left for the student as an exercise, (Exercises 4e, 13). For hints on procedure, see the proof of Theorem 6 below.

Using Theorem 5 we get, for example, $3 + (-5) = -(5 - 3) = -2$ and $11 + (-17) = -(17 - 11) = -6$. Thus, with Theorems 4 and 5 together, we can find the sum of two integers in every case in which one is positive and one is negative.

To complete the discussion of addition we need the following theorem, which enables us to find the sum of two negative integers.

<u>Theorem</u> 6: <u>For all</u> a <u>and</u> b, $-a + (-b) = -(a + b)$

Proof: We have

$a + (-a) = 0, \quad b + (-b) = 0$ Definition of additive inverse

$[a + (-a)] + [b + (-b)] = 0 + 0$ Substitution

$[a + (-a)] + [b + (-b)] = 0$	Additive identity property
$[a + b] + [(-a) + (-b)] = 0$	Associative and commutative property
$[a + b] + [-(a + b)] = 0$	Definition of additive inverse
$(-a) + (-b) = -(a + b)$	Subtraction property,

Since by the subtraction property the equation $(a + b) + x = 0$ has only one solution.

Using Theorem 6 we find for example $-3 + (-5) = -(3 + 5) = -8$, and $-11 + (-17) = -(11 + 17) = -28$. Thus, with Theorems 4, 5, and 6, and the rules for adding whole numbers, we can find the sum of any two integers. We can also use these theorems, and the Theorem on subtraction, and Theorem 3, to find the difference of any two integers, as indicated in the following examples.

Examples:

$2 - 3 = 2 + (-3)$	Theorem on subtraction
$= -(3 - 2$	Theorem 5
$= -1$	Substitution
$2 - (-3) = 2 + [-(-3)]$	Theorem on subtraction
$= 2 + 3$	Theorem 3 and substitution
$= 5$	Substitution
$-2 - 3 = -2 + (-3)$	Theorem on subtraction
$= -(2 + 3)$	Theorem 6
$= -5$	Substitution

<u>Exercises 4b</u>.

Perform the indicated operations.

1. $12 - (-12)$
2. $12 + 12$
3. $(-8) - (-4)$
4. $(-8) - 4$
5. $(-6) + (-2) + 3 + (-5) + 8$
6. $(116 + 88) + (-16)$
7. $7 \cdot a + (-a)$
8. $(-9 \cdot a) + (-7 \cdot a)$

9. $(10 - 12) + (x - 14) + (20 + x)$
10. $[1 + (-2)] - [(-6) + 3]$

Multiplication:

In a similar way we can prove two theorems which enable us to find the product of any two integers.

Theorem 7: For all a and b, $(-a) \cdot b = -(a \cdot b)$.

Proof: We have

$a + (-a) = 0$	Definition of additive inverse
$[a + (-a)] \cdot b = 0 \cdot b$	Substitution
$[a + (-a)] b = 0$	Theorem 1
$a \cdot b + (-a) \cdot b = 0$	Distributive property
$a \cdot b + [-(a \cdot b)] = 0$	Definition of additive inverse
$-(a \cdot b) = (-a) \cdot b$	Subtraction property

since by the subtraction property the equation $(a \cdot b) + x = 0$ has only one solution.

Examples: $(-5) \cdot 3 = -(5 \cdot 3) = -15$
$(-1) \cdot x = -(1 \cdot x) = -x$

Theorem 8: For all a and b, $(-a) \cdot (-b) = a \cdot b$

Proof: By the preceding theorem we have

$(-a) \cdot (-b) = -[a \cdot (-b)]$	Theorem 7
$= -[(-b) \cdot a]$	Commutative property
$= -[-(b \cdot a)]$	Theorem 7
$= b \cdot a$	Theorem 3
$= a \cdot b$	Commutative property

Examples: $(-5)(-3) = (5 \cdot 3) = 15$
$(-1)(-1) = (1 \cdot 1) = 1$

Exercises 4c.

Perform the indicated operations.

1. $5 \cdot (-2)$
2. $(-5) \cdot (-2)$
3. $(-2) \cdot 5 \cdot (-8)$
4. $(-3) + 2 \cdot (-6)$
5. $[(-3) + 2] \cdot (-6)$
6. $7\,(-93) + 7 \cdot 93$
7. $3\,(x - 1) + (x - 2)$
8. $(-2)(-3)(-1)$
9. $7\,(0)\,(-10)$
10. $4(a + b) + (a + b) + (-3)(a + b)$

We emphasize that the theorems we have proved in this section are true statements--and have been proved--for <u>any</u> <u>integers</u> a and b; they state <u>identities</u> in the system of integers. It is true, but irrelevant, that in most of our examples a and b have been positive integers.

<u>Examples:</u> Taking $a = -3$, $b = 2$ in Theorems 3 - 8 we obtain the following equations.

$-[-(-3)] = -3$	Theorem 3
$2 - (-3) = 2 + [-(-3)]$	Theorem 4
$2 + [-(-3)] = -(-3 - 2)$	Theorem 5
$[-(-3)] \cdot 2 = -[(-3) + 2]$	Theorem 6
$[-(-3)] \cdot 2 = -[(-3) \cdot 2]$	Theorem 7
$[-(-3)]\,[-2] = (-3) \cdot 2$	Theorem 8

The theorems we have proved thus far, together with the properties of a number system, constitute a basic set of identities in the system of integers, from which many other identities for the integers can be derived. In particular, we can use these basic identities to "simplify" expressions involving subtraction as well as addition and multiplication. This is so because the system of integers is closed under subtraction: subtraction is defined for every ordered pair of integers, and the result is an integer.

In the following examples, the equations we derive are identities in the system of integers. Instead of proceeding, one step at a time in the solution, we have combined several steps where this can be done without obscuring details. The students should fill in these details.

Example: Perform the indicated operation: $7 \cdot a - 3 \cdot a$

Solution:
$$\begin{aligned} 7 \cdot a - 3 \cdot a &= 7 \cdot a + [-(3 \cdot a)] \\ &= 7 \cdot a + (-3) \cdot a \\ &= [7 + (-3)] \cdot a \\ &= (7 - 3) \cdot a \\ &= 4 \cdot a \end{aligned}$$

Example: Simplify $(a + b)(a - b)$

Solution:
$$\begin{aligned} (a + b)(a - b) &= [a + b][a + (-b)] \\ &= a\,[a + (-b)] + b\,[a + (-b)] \\ &= a \cdot a + a \cdot (-b) + b \cdot a + b \cdot (-b) \\ &= a^2 - (ab) + (a \cdot b) - b^2 \\ &= a^2 - b^2 \end{aligned}$$

It should be noted that subtraction, like addition and multiplication, is defined only for a pair of numbers. Expressions involving three or more numbers without parentheses, such as $x - y + z$, may be used as abbreviations only when there is no danger of confusion. Thus, $x - y + z$ might mean $(x - y) + z$ or $x + (-y - z)$, and if we wish to use $x - y - z$ as an abbreviation we must show that

$$(x - y) + z = x + (-y + z)$$

for all x, y, and z. We have $x - y = x + (-y)$

$x - y = x + (-y)$ Theorem on subtraction

$(x - y) + z = [x + (-y)] + z$ Substitution

$(x - y) + z = x + (-y + z)$ Associative property.

Thus, there is no danger of ambiquity in using the expression $x - y + z$.

Summary: Let us look back at what we have done. We have assumed that there is a number system which is an extension of the system of natural numbers and which has the subtraction property. We defined the negative integers to be the additive inverses of the natural numbers. The natural numbers, zero, and the negative integers were then called the integers. We then discovered the rules for operating with these integers. These were the familiar rules, but they were not just stated or discovered by intuition. They were obtained as logical consequences of the properties of a number system and the subtraction property.

We have not said that the system of integers is the only system having the required properties. There are others. Our discussion shows that any such system must include the integers, and the system of integers is the smallest system having the required properties.

For reference we list below the properties, definition, and theorems formulated in this section.

Subtraction property: For all a and b, there is a unique x, such that $a + x = b$

Definition: Additive inverse-- $-a$ is the additive inverse of a means $a + (-a) = -a + a = 0$.

Theorem 3: For all a; $-(-a) = a$

Theorem 4: For all a and b, $b - a = b + (-a)$

Theorem 5: For all a and b, $b + (-a) = -(a - b)$

Theorem 6: For all a and b, $-a + (-b) = -(a + b)$

Theorem 7: For all a and b, $(-a) \cdot (b) = -(a \cdot b)$

Theorem 8: For all a and b, $(-a) \cdot (-b) = a \cdot b$

Exercises 4d.

Perform the indicated operations.

1. $(a + b) - 2a$
2. $3x + (-2x - 5y)$
3. $3x - (-2x + 5y)$
4. $[x + (x - y)] + 2(x - y)$
5. $-2(3x - 5y) + 10(x - y)$
6. $(a^2 - 2ab) - [(b^2 - 3ab) + (a^2 - b^2)]$
7. $(4x - 15z) - (3z - 5y) + (8x + 7y)$
8. $2x(3x - 7)$
9. $-2x^2(-5 + 2y)$
10. $(x + y)(x - 2y)$
11. $(3x + 4y)(2x - 5y)$
12. $3x^2(3 - x - 3x^2)$
13. $-2ab^2(-3ab)(5a)$
14. $(y - 5)(y^2 - 4y + 4)$
15. $(4c - 1)(c^2 + 5c - 6)$
16. $3x(x - 2y)(x + 2y)$
17. $(2x - 5y)^2$
18. $(2x - 3)(4x + 1)$
19. $-(2 - b)(2 + b) + (b - 2)(b + 2)$
20. $[3(a + b)]^2$
21. $(3a + b)(2a - 5b) + (5b - 2a)(a - 4b)$
22. $(3 + y)(2 + x)$
23. $(x + 4)(x^2 + 5x + 6)$
24. $[(a + b) - 2c][(a + b) + 2c]$
25. $[3x - (y - z)][3x - (y - z)]$

26. $[(2a + b) - 5][(2a + b) + 2]$

27. $[a + b + 3][r + s + 3]$

28. $(x - y + m - n)(x + y - m + n)$

29. $(2x - y + z)(2x - y - z)$

30. $(b^2 + 2b + 1)(b^2 - 2b - 1)$

31. $(2r - s - t)^2$

32. $(x - 2y)(x^2 + 2xy + 4y^2)$

33. $(x + y)(2x + 3y) - 4(y^2 - x^2) + (y - 6x)(y + x)$

34. $(x + 5)(x^2 - 5x + 25)$

35. $[(x + y) + 3][(x + y) - 3]$

36. $(a - b)(a + b) + 2b(b + a) - (a - b)^2$

Exercises 4e.

Prove the following identities.

1. $(x + y) - y = x$

2. $(x - y) + y = x$

3. $x - (y + z) = (x - y) - z$

4. $x - (y - z) = (x - y) + z$

5. $(x + y) - z = x + (y - z)$

6. $(x - y) - z = x - (y + z)$

7. $(x)(y)(-z) = -(xyz)$

8. $(-x)(y)(-z) = xyz$

9. $a(b - c) = ab - ac$

10. $(a - b)^2 = a^2 - 2ab + b^2$

11. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$

12. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$

13. $b + (-a) = -(a - b)$ (Theorem 5)

5 Factoring

Quite often we wish to represent an algebraic expression as a product. This is called factoring the expression. Factoring is useful, for example, in the simplification of fractions and in connection with the solution of equations.

In the simplest cases, an expression can be factored by using the distributive property one or more times.

Example: Factor $2x^2 + 3x$

Solution: By the distributive property

$$2x^2 + 3x = (2x + 3)x$$

There is no systematic procedure for factoring an expression; in fact, it generally is not possible to factor an expression. Essentially the only way one can factor an expression is by recognizing it is a product whose factors one knows. For this reason it is important to remember certain identities obtained by "mulitplying out" simple factors. The following identities in the system of integers are particularly useful.

For all a, b, and c $ab - ac = a(b - c)$
For all a, and b, $a^2 - b^2 = (a + b)(a - b)$
For all a and b, $a^2 + 2ab + b^2 = (a + b)^2$
For all a and b, $a^2 - 2ab + b^2 = (a - b)^2$
For all a, b and x, $x^2 + (a + b)x + ab = (x + a)(x + b)$
For all a and b, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
For all a and b, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

The proofs of these identities were given in the examples and exercises of the preceding sections.

Example: Factor $x^2 - 9$

Solution: $x^2 - 9$ is equal ~~to~~ $a^2 - b^2$ if $a = x$, and $b = 3$. Since $a^2 - b^2 - (a + b)(a - b)$ for all a and b, we have $x^2 - 9 = (x + 3)(x - 3)$.

<u>Example</u>: Factor $x^2 - 4x + 3$

Solution: $x^2 - 4x + 3$ will be equal to $x^2 + (a + b)x + ab$ if $ab = 3$ and $a + b = -4$. The only integers a, b which satisfy $ab = 3$ are $a = 3$, $b = 1$ and $a = -3$, $b = -1$. ($a = 1$, $b = 3$ and $a = -1$, $b = -3$ are also possible, but provide nothing new.) $a = -3$, $b = -1$, also satisfy $a + b = -4$. Thus with $a = -3$, $b = -1$, $x^2 - 4x + 3$ is equal to $x^2 + (a + b)x + ab$, and since $x^2 + (a + b)x + ab = (x + a)(x + b)$ for all a and b, we have

$$x^2 - 4x + 3 = [x + (-3)][x + (-1)]$$

$$x^2 - 4x + 3 = (x - 3)(x - 1)$$

Because of their usefulness in factoring the identities we have listed are sometimes called <u>factoring identities</u>. However any identity obtained by "multiplying out" several factors can be used as a factoring identity just as we have used those in our list.

We stated that factoring is important in connection with the solution of equations. This is a consequence of the following theorem.

<u>Theorem</u> 2:

If $a = 0$ or $b = 0$ then $a \cdot b = 0$

If $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Proof: To prove the first statementof the theorem suppose, say, that $b = 0$. Then

	$1 + 0 = 1$	Additive identity property
	$a(1 + 0) = a \cdot 1$	Substitution
	$a \cdot 1 + a \cdot 0 = a \cdot 1$	Distributive property
	$a + a \cdot 0 = a$	Multiplicative identity property
But,	$a + 0 = a$	Additive identity property
So,	$a \cdot 0 = 0$	Cancellation property of addition

and $a \cdot b = a \cdot 0 = 0$.

To prove the second statement suppose $a\,b = 0$: If $a = 0$ the theorem is proved. If $a = 0$ the theorem is proved. Suppose $a = 0$. The $a \cdot b = 0$, and by the first statement of the theorem, $a \cdot 0 = 0$. Thus by substitution $a \cdot b = a \cdot 0$. Since $a = 0$ it follows from the cancellation property of multiplication on that $b = 0$.

The two statements of Theorem 2 are usually combined in the single statement: "$a \cdot b = 0$ if and only if $a = 0$ or $b = 0$." In words: " product is zero if and only if one of its factors is zero." Notice that "or" is used, as is customary in mathematics with "or both" understood.

<u>Example</u>: Find all solutions of the equation $x^2 - 4x + 3 = 0$.

<u>Solution</u>: We showed in the preceding example that $x^2 - 4x + 3 = (x - 3)(x - 1)$ for all x. Suppose the given equation has a solution x. Then

$$x^2 - 4x + 3 = 0$$

$$(s - 3)(x - 1) = 0 ,$$

so that by the preceding theorem, either $x - 3 = 0$ or $x - 1 = 0$. If $x - 3 = 0$ then $x = 3$; if $x - 1 = 0$ then $x = 1$. Thus, if the equation has a solution it must be either 3 or 1. By substitution we verify that both 3 and 1 are solutions of the equation.

<u>Exercises</u> 5a.

Factor each of the following.

1. $ax + ay$
2. $5x^2y^3 + 30\,y^5$
3. $3bx - 6b^2y$
4. $14cd + 6ce - 2cf$
5. $y^2 - 25$
6. $49 - x^2$

5. $y^2 - 25$
6. $49 - x^2$
7. $9a^2 - 16b^2$
8. $4x^2 - 64$
9. $16 - 25a^2$

10. $x^2 + 4x + 4$
11. $4x^2 - 12xy + 9y^2$
12. $x^2 + 9x + 14$
13. $y^2 + 2y + 15$
14. $14 - 5w - w^2$

15. $y^3 + 27$
16. $x^3 - 64$
17. $c^9 + 1$
18. $27a^3 - 1$
19. $ac^3 - 64a$

20. $a(x + y) + b(x + y)$
21. $x(a - b) - y(a - b)$
22. $a^3 - a^2 - a + 1$
23. $(3x - y)^2 - 9u^2$
24. $9 - 6ab - a^2 - 9b^2$
25. $54a^2b^2 - 2a^2b^5$
*26. $x^6 - y^6$
27. $x^2 + 5x - 36$
28. $mx^2 - 12mx + 36m$
29. $a^4 - 16$
*30. $36 - 25x^2 + 4x^4$

31. $80a^2b^3 - 5b$

32. $a^{12} - 125c^{15}$

33. $mx + my + nx + ny$

34. $bx - by + cx - cy$

35. $t^3 - 5u - 5t + u^3$

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36. $x^2 - y^2 - x + y$

37. $8a^3 - 1$

38. $24 - 2y - y^2$

39. $9r^2 - 2s - s^2 - 1$

40. $w^2 - 11w + 24$

41. $9c^2 - 1$

42. $a^4 - 3a^2 - 4$

43. $y^3 + 2y^2 - 5y - 10$

44. $a^6 - 7a^3 - 8$

45. $x^2(a - b) + y^2(b - a)$

46. $64 - 16a + a^2$

47. $c^4 - 25$

48. $c^3 + d^3$

49. $27r^3 y + y$

50. $ab^2 - ay^2$

51. $81x^4 - 16y^2z^2$

52. $-3y^2 + 15y + 42$

53. $27a^3 + 8b^3$

54. $x^2 - 4x + 6xy - 24y$

55. $cx^4 - 2cx^2 - 8c$

56. $6m^2 - 73am + 12a^2$

57. $3r^3t + 192t^4$

58. $2w^2 + 3w - 10wb - 15b$

59. $64a^3b - 8b^4$

60. $8x^4y^2 - 20x^3y^2 - 12x^2y^2$

61. $3a^2 - 4a - 15$

62. $bx^2 + x^3 - by^2 - xy^2$

63. $1 + 49s^2 - 14s$

64. $18 - 45x - 8x^2$

65. $av - 2bv + 2cv - 7am + 14bm - 14cm$

66. $5k^2 + 28kw - 12w^2$

67. $12x^4 - 23x^2y^2 + 5y^4$

68. $x^3 - 9x^2y + 27xy^2 - 27y^3$

69. $4a^4 - 21a^2 - 25$

70. $(c - d)^3 - (a - 2b)^3$

71. $35(x^4 - 3xy) - 15(x^3y - 3y^2)$

72. $(x + y)^3 + (a + 3b)^3$

73. $6st - 9s^2 + r^2 - t^2 - 10r + 25$

74. $x^4 - 2x^2 + 1$

75. $(2x + 4)^4 - 18(2x + 4)^2 + 81$

Exercises 5b.

Find all solutions of the following equations. Check by substitution.

1. $y^2 - 7y + 12 = 0$

2. $x^2 + 7y - 18 = 0$

3. $x^2 - 3x - 10 = 0$

4. $x^2 + 2a - 24 = 8 - 2a$

5. $x^2 - 2x - 8 = -9$

6. $x^2 - x + 1 = 1$

7. $2x^2 + 3x + 1 = x^2 + 2x + 1$

8. $25x^2 - 100 = 0$

9. $2x^2 = x^2 + 9$

Exercises 5c.

1. "Multiply out" $(x + a)^2(x + b)$ and use the resulting identity to factor:
(a) $x^3 - 3x^2 + 4$; (b) $x^3 + 5x^2 + 7x + 3$;
(c) $x^3 + 3x^2 + 3x + 1$.

2. Write an equation in x which has $x = 1$, $x = 2$ and $x = 3$ as solutions, and has no other solutions.

6. Rational Numbers

In the system of integers the equation $a + x = b$ always has a solution, but it is still not true that the equation $a \cdot x = b$ always has a solution if $a \neq 0$. Since we have defined the quotient $\frac{b}{a}$ as the solution of the equation $a \cdot x = b$, it follows that $\frac{b}{a}$ does not always have meaning in the system of integers. For example, the equation $5 \cdot x = 3$ has no solution in the system of integers, and hence $\frac{3}{5}$ has no meaning in this system.

Again it is necessary to enlarge the number system. To do this we join to the set of integers the set of all fractions $\frac{b}{a}$ where a and b are integers and $a = 0$. We then define addition and multiplication in this new set. The new number system is called the system of rational numbers.

The system of rational numbers is an extension of the system of integers; that is, it contains the integers, and the operations of addition and multiplication applied to integers yield the same result as in the system of integers. Properties of the rational number system: The system of rational numbers has all the properties of the system of integers, namely, the properties of a number system and and the subtraction property. The additional property which the system of rational numbers possesses is the following.

Division Property: For all a and b, $a \neq 0$, there is a unique x such that $a \cdot x = b$.

As examples, consider the following equations which do not have solutions in the system of integers:

$$5 \cdot x = 3, \qquad -7 \cdot x = 11.$$

In the system of reational numbers these equations have the solutions $\frac{3}{5}$ and $-\frac{11}{7}$, respectively.

Now let us forget that we know the rational numbers and just suppose that we have some number system which is an extension of the system of integers and which has the properties of a number system, the subtraction property and the division property. We will show that the raticnal numbers must be in this system and find the rules for operating with them.

Exercises 6a

Use the definition of division and the division property to prove the following statements.

1. $\frac{a}{a} = 1 \qquad a \neq 0$
2. $\frac{a}{1} = a$
3. $\frac{0}{a} = 0 \qquad a \neq 0$
4. $\frac{a}{-1} = -a$
5. $\frac{2 \cdot a}{a} = 2 \qquad a \neq 0$

Multiplicative inverse elements: According to the division property the equation

$$a \cdot x = 1$$

has one and only one solution for each a except $a = 0$. For given $a \neq 0$ we call the solution of this quation, which is denoted by $\frac{1}{a}$, the multiplicative inverse of a. The property which defines $\frac{1}{a}$ is thus

$$a \cdot (\frac{1}{a}) = 1$$

Observe that because of the commutative property of multiplication we have also

$$(\frac{1}{a}) \cdot a = 1$$

Definition of multiplicative inverse: The multiplicative inverse of a is the number $\frac{1}{a}$ such that $(\frac{1}{a}) \cdot a = a(\frac{1}{a}) = 1$.

For example, corresponding to the number 5 there is a number $\frac{1}{5}$ with the property that

$$(\frac{1}{5}) \cdot 5 = 5 \cdot (\frac{1}{5}) = 1,$$

and $x = \frac{1}{5}$ is the only solution of the equation

$$x \cdot 5 = 5 \cdot x = 1.$$

The symbol $\frac{1}{a}$ is read "one over a" or "the reciprocal of a."

Division and multiplicative inverses: Consider the equation $a \cdot x = b$, $a \neq 0$. As in Section 2, we denote the solution of this equation by $\frac{b}{a}$. Now, however, the symbol always represents a definite number for by the division property, the equation $a \cdot x = b$ has a unique solution if $a \neq 0$. Thus in our new system division is an operation defined for all ordered pairs of numbers, except that division by 0 is not defined.

There is an important connection between division and multiplicative inverses similar to the connection between subtraction and additive inverses.

Theorem 9. For all a and b, $a \neq 0$, $\frac{b}{a} = b \cdot (\frac{1}{a})$

Proof:	$a(\frac{1}{a}) = 1$	Definition of multiplicative inverse
	$b[a(\frac{1}{a})] = b \cdot 1$	Substitution
	$b[a(\frac{1}{a})] = b$	Multiplicative identity element property
	$(ba)(\frac{1}{a}) = b$	Associative property
	$(ab)(\frac{1}{a}) = b$	Commutative property
	$a[b(\frac{1}{a})] = b$	Associative property
But	$a[\frac{b}{a}] = b$	Definition of division
So	$\frac{b}{a} = b(\frac{1}{a})$	Division property,

since by the division property the equation $a \cdot x = b$ has only one solution.

We shall refer to this theorem as the theorem on division. It states that division by a number is the same as multiplication by the multiplicative inverse of the number.

Equivalence of fractions: Before we go on to find the rules for addition and multiplication of fractions we have to make an important observation. The same number can be represented by two different fraction symbols

Theorem 10. If $cb \neq 0$, then $\frac{ca}{cb} = \frac{a}{b}$

Proof: Let $x = \frac{a}{b}$

Then $bx = a$	Definition of division
$c(bx) = ca$	Substitution
$(cb)x = ca$	Associative property
$x = \frac{ca}{cb}$	Definition of division

This is the important Equivalence Rule for fractions.

It is important to note that this rule can also be read from left to right (as can any equality) in the form

$$\text{If } bc \neq 0, \quad \frac{a}{b} = \frac{ca}{cb}$$

In words we can say: "If the numerator and denominator of a fraction are multiplied, or divided, by the same non-zero number the value of the fraction is unchanged."

Exercises 6b.

Use the equivalence rule for fractions to simplify:

1. $\frac{4}{28}$
2. $\frac{a^2}{ab}$
3. $\frac{9mny}{12m^2y^2}$
4. $\frac{3x - 9}{5x - 15}$
5. $\frac{a + b}{a^2 - b^2}$
6. $\frac{a^2 - 4}{2a^2 + 9a + 10}$
7. $\frac{2a^2 + a - 1}{6a^2 - 7a + 2}$
8. $\frac{3x^2 + xy}{x^2 + 3xy}$
9. $\frac{a^2 - b^2}{b^2 + ab - 2a^2}$
10. $\frac{ax + ay - bx - by}{am - bm - an + bn}$

Multiplication of fractions: The operation with fractions that is easiest to perform is multiplication.

Theorem 11. If $bd \neq 0$, then $(\frac{a}{b})(\frac{c}{d}) = \frac{ac}{bd}$.

Proof:

	Let $x = a/b$ and $y = c/d$	
Then	$bx = a$ and $dy = c$	Definition of division
	$(bx)(dy) = ac$	Substitution
	$(bd)(xy) = ac$	Associative and commutative properties
So	$xy = \frac{ac}{bd}$	Definition of division

This is the multiplication rule for fractions. In words we can say: "The product of two fractions is a fraction whose numerator is the product of the numerators of the original fractions, and whose denominator is the product of the denominators."

Addition of fractions: We will find the rule for addition of fractions in two steps, first for fractions with equal denominators and then for those with unequal denominators.

Theorem 12. If $b \neq 0$, then $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

Proof:

$\frac{a}{b} + \frac{c}{b} = a(1/b) + c(1/b)$	Theorem on division
$= (a + c)(1/b)$	Distributive property
$= \frac{a + c}{b}$	Theorem on division

This is the rule for addition of fractions with the same denominator. In words: "The sum of two fractions with the same denominator is a fraction with that same denominator and with a numerator equal to the sum of the numerators of the original two fractions."

Now consider any two fractions $\frac{a}{b}$ and $\frac{c}{d}$. According to the equivalence rule for fractions:

$$\frac{a}{b} = \frac{ad}{bd} \quad , \quad \frac{c}{d} = \frac{bc}{bd}$$

and hence

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} .$$

But now the fractions on the right have the same denominator so that we may use the preceding theorem to obtain

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

which is the general addition rule for fractions.

Theorem 13. If $bd \neq 0$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

You would find it very awkward to put this rule into words and, in fact, you will probably solve problems by the method we used to establish the rule, so we describe the procedure thus: "To add two fractions with different denominators, change them to equivalent fractions with the same denominator and use the rule for adding fractions with the same denominator."

When a and b are integers, $b \neq 0$, the fraction a/b represents a rational number. Taking a, b, c, d to be integers in the three preceding theorems we obtain the rules for adding and multiplying rational numbers. For example,

$$\left(\frac{-2}{3}\right) \cdot \frac{5}{7} = \frac{-2 \cdot 5}{3 \cdot 7} = \frac{-10}{21}$$

$$\frac{-2}{3} + \frac{5}{7} = \frac{-2 \cdot 7}{3 \cdot 7} + \frac{3 \cdot 5}{3 \cdot 7} = \frac{-14}{21} + \frac{15}{21}$$

$$= \frac{-14 + 15}{21} = \frac{1}{21}$$

We emphasize, however, that in these theorems a, b, c, d do not have to be integers, but may be any numbers in the system; in particular they may be numbers represented by fractions. This is true, indeed, for all the theorems we have stated in this and preceding sections, since in the proof of these theorems we used only properties which hold for all numbers in our system. This remark is important for the remainder of our discussion.

Exercises 6c.

Express as a single fraction and simplify:

1. $\frac{2}{3} \cdot \frac{7}{3}$
2. $\frac{4}{5} \cdot \frac{10}{8}$
3. $\frac{3x^3}{4y^2} \cdot \frac{5y}{x^2}$
4. $\frac{7a}{12b^3} \cdot \frac{20b^5}{35a^3}$
5. $\frac{(x+y)}{(x+2y)} \cdot \frac{(x-y)}{(2x+y)}$
6. $\frac{x^2-y^2}{x^3-y^3} \cdot \frac{x}{x+y}$
7. $\frac{x^2-2xy+y^2}{x^3-y^3} \cdot \frac{x^2+xy+y^2}{x-y}$
8. $\frac{2}{3} + \frac{4}{5}$
9. $\frac{2}{3} + \frac{5}{6}$
10. $\frac{3x}{4y} + \frac{y}{2x}$
11. $\frac{2a}{3b} + \frac{a}{b}$
12. $x+y + \frac{y^2}{x-y}$
13. $\frac{2x-3}{4-x} + \frac{x+11}{3x-12}$
14. $\frac{-3}{a+3} + \frac{3a-9}{a^2-9}$
15. $\frac{2a}{1+a^2} + \frac{-2}{a+1}$

<u>Division</u>: According to the theorem on division

$$\left(\frac{a}{b}\right) \Big/ \left(\frac{c}{d}\right) = \left[\left(\frac{a}{b}\right)\right]\left[1\Big/\left(\frac{c}{d}\right)\right],$$

that is, to divide by a fraction we multiply by its multiplicative inverse. We use the following theorem to find the multiplicative inverse of a fraction.

<u>Theorem</u> 14. <u>For all</u> a <u>and</u> b, $ab \neq 0$, $1\Big/\left(\frac{a}{b}\right) = \frac{b}{a}$

<u>Proof</u>: By definition, $1\Big/\left(\frac{a}{b}\right)$ is the unique solution of the equation $\left(\frac{a}{b}\right) \cdot x = 1$. We show by substitution that $x = \frac{b}{a}$ is a solution of this equation. Since the solution is unique we conclude that $1\Big/\left(\frac{a}{b}\right) = \frac{b}{a}$. The details are left as an exercise (Exercises 6g, #1).

Combining the theorem on division and the theorem we have just stated, we obtain

$$\frac{a}{b} \Big/ \frac{c}{d} = \left[\frac{a}{b}\right]\left[1\Big/\left(\frac{c}{d}\right)\right] = \frac{a}{b} \cdot \frac{d}{c},$$

and using the rule for multiplication of fractions we complete a proof of the following theorem.

Theorem 15. For all a,b,c,d, $bcd \neq 0$,

$$\left(\frac{a}{b}\right) / \left(\frac{c}{d}\right) = \frac{ad}{bc}$$

This theorem states a division rule for fractions. Usually, however, to divide fractions we follow the procedure with which we proved the theorem using the theorem on division and the theorem on the multiplicative inverse of a fraction.

Taking a,b,c,d to be integers in the division rule, we obtain a rule for dividing rational numbers. For example,

$$\frac{-2}{3} / \frac{5}{7} = \frac{-2}{3} \cdot \frac{7}{5} = \frac{-2 \cdot 7}{3 \cdot 5} = \frac{-14}{15} .$$

Again however, the theorems we have just proved hold when a,b,c,d represent any numbers in a system having the properties we have stated.

Subtraction: According to the theorem on subtraction, we have

$$\frac{a}{b} - \left(\frac{c}{d}\right) = \frac{a}{b} + \left(- \frac{c}{d}\right) .$$

Thus, to subtract fractions we have to be able to find the additive inverses of fractions. The following theorem enables us to do this.

Theorem 16. For all a and b, $b \neq 0$, $-\left(\frac{a}{b}\right) = \frac{-a}{b}$

Proof: By definition $-(a/b)$ is the unique solution of the equation $(a/b) + x = 0$. We verify by substitution that $x = (-a/b)$ is a solution of this equation. Since the equation has only one solution we conclude that $-(a/b) = (-a/b)$. The details are left for an exercise (Exercise 6g, #2).

Observe that we have

$$-\left(\frac{a}{b}\right) = \frac{a}{-b}$$

since

$$\frac{-a}{b} = \frac{(-1)\cdot a}{(-1)(-b)} = \frac{a}{-b}$$

The preceding theorem, the theorem on subtraction, and the addition rule, enable us to subtract any fractions, and in particular, to subtract rational numbers. For example,

$$\frac{2}{3}-\frac{3}{5}=\frac{2}{3}+\frac{-3}{5}=\frac{2\cdot 5}{3\cdot 5}+\frac{3\cdot(-3)}{3\cdot 5}$$

$$=\frac{10}{15}+\frac{-9}{15}=\frac{10+(-9)}{15}=\frac{1}{15}.$$

Exercises 6d.

Perform the indicated operations and express the result in its simplest form.

1. $2/3 \big/ 4/9$
2. $5/2/5$
3. $7a/b \big/ 3a/2b$
4. $\frac{xy^3}{yz} \Big/ x^2z$
5. $\frac{x^2-2x-15}{x^2-9} \Big/ \frac{12-4x}{x^2-6y+9}$
6. $(3+\frac{4}{5}) \big/ (\frac{2}{3}-1)$
7. $\dfrac{\frac{2}{3}-\frac{1}{5}}{\frac{1}{5}+\frac{1}{3}}$
8. $\dfrac{\frac{a}{b}+\frac{a}{c}}{ab+ac}$
9. $\frac{9}{10}-\frac{2}{5}$
10. $\frac{3y}{5}-\frac{2}{5y}$
11. $\frac{x+5}{x-5}-\frac{x-5}{x+5}$
12. $\frac{x+1}{x+2}-\frac{x+3}{x}$
13. $\frac{m^2+6m+9}{m-3}+3-m$
14. $\frac{3}{x+1}-\frac{2}{x-1}-\frac{5x+1}{x^2-1}$
15. $\frac{3}{x}+\frac{5}{1-2x}-\frac{2x-7}{4x^2-1}$

As a consequence of the subtraction and division properties, any equation of the form $ax+b=c$, $a \neq 0$, where a, b, and c are rational numbers, has a rational solution. Such equations are called linear equations.

Example: Solve $\frac{2}{3}x+\frac{3}{4}=\frac{1}{7}$

Solution: Suppose the equation has a solution x. Then

$$\frac{2}{3}x + \frac{3}{4} = \frac{1}{7}$$

$$\frac{2}{3}x + \frac{3}{4} + (-\frac{3}{4}) = \frac{1}{7} + (-\frac{3}{4})$$

$$\frac{2}{3}x = \frac{1}{7} + \frac{-3}{4}$$

$$\frac{2}{3}x = \frac{-17}{28}$$

$$\frac{3}{2} \cdot (\frac{2}{3}x) = \frac{3}{2} \cdot (\frac{-17}{28})$$

$$x = \frac{-51}{56},$$

so that if the equation has a solution it must be $x = {}^{-51}/_{56}$. By substitution we verify that $-51/_{56}$ is a solution of the equation.

Summary: Let us look back at what we have done. We have supposed that there is a number system which is an extension of the system of integers and which has the subtraction property and the division property. We defined the rational numbers to be the solutions of the equations $ax = b$, where $a \neq 0$ and a and b are integers. We then discovered the rules for operating with these rational numbers. These were the familiar rules, but they were not just stated or discovered by intuition. They were obtained as logical consequences of the properties of a number system and the subtraction and division properties.

We have not said that the system of rational numbers is the only system having the required properties. There are others. Our discussion shows, however, that any such system must include the rational numbers, so the system of rational numbers is the smallest system of this kind.

For reference we list below the properties, definitions, and theorems formulated in this section.

Division property: For all a and b, $a \neq 0$, there is a unique x such that $a \cdot x = b$.

Definition: Multiplicative Inverse: The multiplicative inverse of a is the number 1/a such that $(1/a)a = a(1/a) = 1$.

Theorem 9: For all a and b, $a \neq 0$, $b/a = b\,(1/a)$.

Theorem 10: If $cb \neq 0$, then $\frac{ca}{cb} = \frac{a}{b}$

Theorem 11: If $bd \neq 0$, then $(\frac{a}{b})(\frac{c}{d}) = \frac{ac}{bd}$

Theorem 12: If $b \neq 0$, then $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

Theorem 13: If $bd \neq 0$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

Theorem 14: For all a and b, $ab \neq 0$, $1/(\frac{a}{b}) = \frac{b}{a}$

Theorem 15: For all a,b,cd, $bcd \neq 0$, $(\frac{a}{b})/(\frac{c}{d}) = \frac{ad}{bc}$

Theorem 16: For all a and b, $b \neq 0$, $-(\frac{a}{b}) = \frac{-a}{b}$

Exercises 6e.

Perform the indicated operations and express the result in its simplest form.

1. $\frac{2}{5} - \frac{3}{10} + \frac{3}{4}$

2. $\frac{x}{3} + \frac{3x}{7} - \frac{x}{14}$

3. $\frac{4}{a} - \frac{2}{3a} + \frac{7}{6a^2}$

4. $\frac{3}{5} \cdot 8$

5. $\frac{8b^2}{3c^2} \cdot 24cb$

6. $\frac{2x-y}{3x} + \frac{3y-2x}{2y}$

7. $\frac{x-1}{2x^2-18} - \frac{x+2}{3x^2-9x}$

8. $\frac{x+y}{3m-9n} \cdot \frac{12}{x+y}$

9. $\frac{x-1}{x^2+1} \Big/ \frac{(x-1)^2}{x^2-1}$

10. $\frac{1}{a^2-4} + \frac{a}{1-a^2} + \frac{2a}{a^2-1}$

11. $a - b - \frac{a^2+b^2}{a+b}$

12. $\frac{2x-3}{x^2-1} \Big/ \frac{4x^2-9}{2x^2+x-3}$

13. $\frac{1}{m+2} \cdot \frac{1}{2-m}$

14. $\frac{c^2-2c-15}{c^2-9} \cdot \frac{c^2-6c+9}{3a-ac}$

15. $x - \frac{x^2+3yx}{x-2y} + 3y$

16. $\frac{3x^2-2xy-y^2}{x^2-y^2} \Big/ (3x^2+4yx+y^2)$

17. $a+6 + \frac{5a+1}{12a^2+5a-2} - \frac{a}{3a+2}$

18. $\dfrac{x^4-y^4}{(x-y)^2} \cdot \dfrac{y^2}{x^2+y^2} \Big/ \dfrac{xy+y^2}{x-y}$

19. $\left[\dfrac{2x}{x-1} + \dfrac{x^2}{x^2-1}\right] \Big/ \dfrac{x^3}{1-x}$

20. $\dfrac{1 - \dfrac{1}{1-\dfrac{1}{a}}}{1 - a}$

21. $\dfrac{\dfrac{1}{2} - \dfrac{4}{x^3}}{\dfrac{1}{x^2} + \dfrac{1}{4} \quad \dfrac{1}{2x}}$

22. $\dfrac{\dfrac{a}{1-a} + \dfrac{1+a}{a}}{\dfrac{1-a}{a} + \dfrac{a}{1+a}}$

23. $\left[\dfrac{3x}{x-3} - \dfrac{3x+2}{x^2-6x+9}\right] \left[\dfrac{x+2}{x+3} - \dfrac{x}{x^2+6x+9}\right]$

24. $\dfrac{\dfrac{x}{y} + \dfrac{y^2}{x^2}}{\dfrac{y}{x^2} - \dfrac{1}{x} + \dfrac{1}{y}}$

Exercises 6f.

Solve the following equations. Check by substitution.

1. $5x - 3/2 = 7/5$
2. $\dfrac{3x}{2} - \dfrac{5}{4} = 5/2$
3. $2x + \dfrac{1}{x} - \dfrac{1}{2} = \dfrac{1}{x} + \dfrac{3}{4}$
4. $\dfrac{x+2}{x-3} + \dfrac{2}{3} = \dfrac{5}{4}$
5. $\dfrac{3y + 6}{2y} = \dfrac{1}{2}$
6. $2x^2 = 3x + 27$
7. $\dfrac{3}{x} = \dfrac{4}{3} - \dfrac{x-3}{2x + 2}$
8. $\dfrac{2}{9x + 18} - \dfrac{x + 2}{8} = 0$

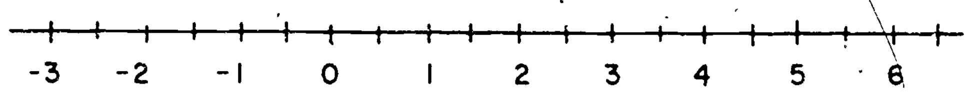
Exercises 6g.

Prove the following.

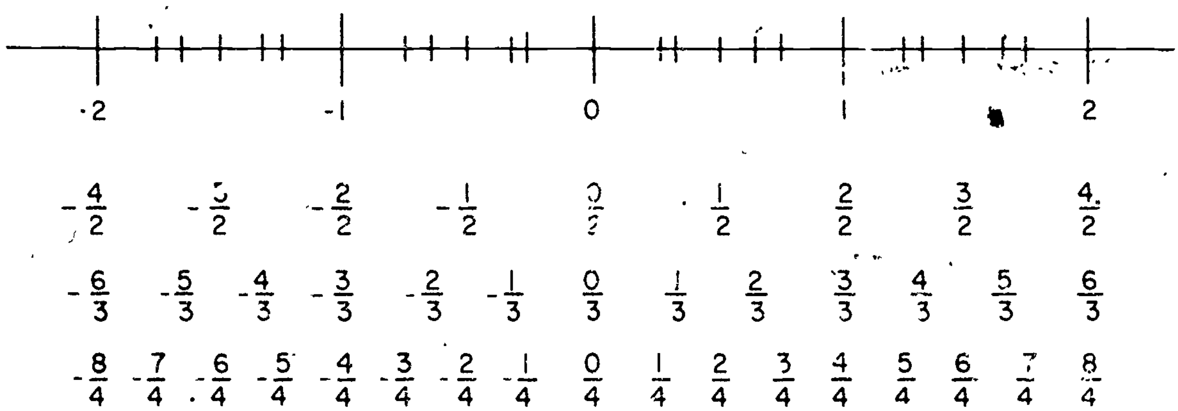
1. $1/(\frac{a}{b}) = \frac{b}{a}$ (Theorem 14)
2. $-(\frac{a}{b}) = \frac{-a}{b}$ (Theorem 16)
3. $(\frac{a}{b}) - (\frac{c}{d}) = \dfrac{ad - bc}{bd}$

7. Order Relations

The rational numbers can be represented by points on a straight line. To do this we choose two distinct points on the line arbitrarily, and label the point on the left with 0 and the one on the right with 1. Using the interval between these points as a unit of measure, and beginning at the point labeled with 1, we locate points equally spaced along the line to the right, and label these points with 2,3,4, and so on. Similarly we locate points equally spaced to the left, beginning at the point labeled with 0, and label these points with -1, -2, -3, and so on.



Starting with the line on which the integers are represented we can label other points by dividing the intervals into halves, thirds, fourths, and so on as indicated in the following diagram.



When every rational number is represented by a point on the line following this scheme, the line is called a <u>number line</u>.

The representation of rational numbers by points on a number line is the basis of the definition of the <u>order relation</u> for rational numbers.

The number a is less than the number b, symbolically $a < b$, if the point on the number line which represents a is to the left of the point which represents b.

For example, $1 < 3$, $-5 < -2$, $-3 < 1$, $\frac{2}{3} < \frac{3}{4}$, $-\frac{3}{2} < -\frac{2}{3}$. The statement "b is greater than a", which is written symbolically $b > a$, is synonomous with "a is less than b".

Exercises 7a.

Use one of the symbols <, >, or = to form a true statement.

1. 6 ____ −3
2. −2 ____ −5
3. −7 ____ 0
4. 8 ____ 0
5. 52.8 ____ −32.9
6. 8.25 ____ 8.2
7. −0.1 ____ −0.01
8. $\frac{7}{5}$ ____ $\frac{4}{5}$
9. −3 + 10 ____ 7
10. $\frac{4}{45}$ ____ $\frac{5}{45}$
11. .682(2.98 + 67.4) ____ 682(2.98)+682(67.4)
12. $(-1)\cdot(-1)\cdot(-1)$ ____ $-3\cdot 2\cdot \frac{7}{10}$
13. $\frac{2}{3} + \frac{1}{2}$ ____ $\frac{2}{3} \times \frac{1}{2}$
14. −45 ____ −30
15. −45 ____ −30 + (−20)
16. −45 + 15 ____ −30
17. 23 ____ 19
18. 23 ____ 19 + 4
19. 15 ____ −1
20. 15 ____ −1 + 16

A statement involving the order relation is called an **inequality**. Inequalities are as important as equations in mathematics. We are going to formulate the fundamental properties of the order relation and show how these are used in operating with inequalities.

If a and b are two numbers then either the points representing them coincide or one is to the left of the other. Thus either $a = b$, $a < b$ or $b < a$, and only one of these statements can hold. This simple but fundamental property is called the trichotomy property.

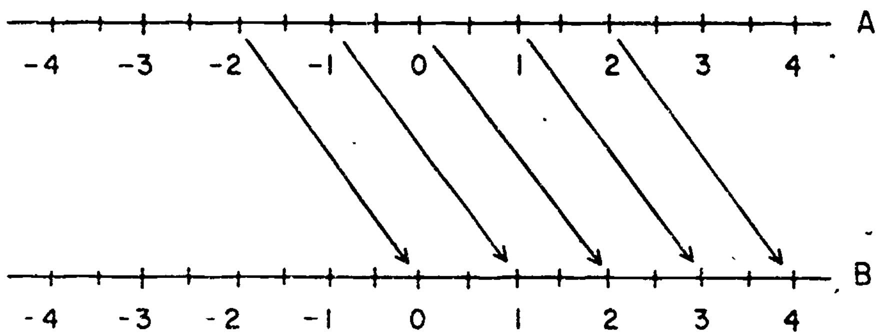
Trichotomy property of order: If a and b are any two numbers then exactly one of the following holds: $a = b$, $a < b$, $b < a$.

The second fundamental property of order, the transitive property is also geometrically evident.

<u>Transitive property of order</u>: <u>If</u> $a < b$ <u>and</u> $b < d$, <u>then</u> $a < c$.

For example, since $-5 < 0$ it follows that if $0 < x$ then $-5 < x$.

Suppose that we add 2 to every rational number. Geometrically this has the effect of moving every point on the number line 2 units to the right, as shown in the following diagram.



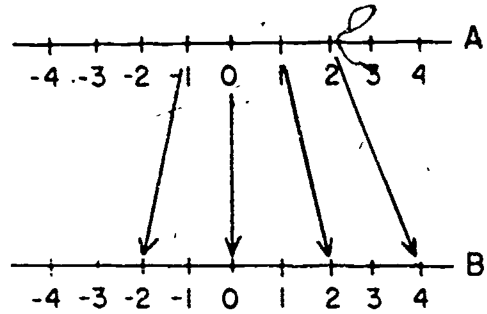
From this diagram it is evident that if $a < b$, then $a + 2 < b + 2$. The student should draw similar diagrams showing the geometric effect of addition of $\frac{2}{3}$, -3, etc. These diagrams will serve to illustrate the following fundamental property.

<u>Addition property of order</u>: <u>If</u> $a < b$, <u>then</u> $a + c < b + c$.

For example, since $2 < 3$, it follows that $2 + x < 3 + x$ for all x.

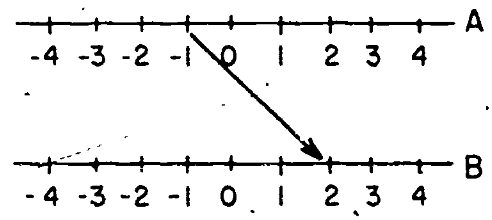
Exercises 7b.

Consider the set of rational numbers on the number line, A. Multiplying each element of the set by $^+2$ makes each point of line A correspond to a point of line B [mapping]. Notice that the order is maintained.



$2 > 1$ and $2(2) < 1\ (2)$

Now, multiply each rational number by -2. Show the mapping from A on to B. What has happened? Is $2(-2) > 1(-2)$?



Fill in one of the symbols, (>, <), to make a true sentence:

1. If $3 > 2$, then $3(5)$____$2(5)$
2. If $1 < 10$, then $1(-3)$____$10(-3)$
3. If $-2 > -3$, then $-2(15)$____$-3(15)$
4. If $-3 < 3$, then $-3(2)$____$3(2)$
5. If $2 > -1$, then $2(-10)$____$-1(-10)$
6. If $5 > 0$, then $5(-2)$____$0(-2)$
7. If $a > b$ and $c > 0$, then ac____bc
8. If $a > b$ and $c < 0$, then ac____bc
9. If $a < b$ and $c > 0$, then ac____bc
10. If $a < b$ and $c < 0$, then ac____bc.

The preceding exercises illustrate the fourth and last fundamental property of order.

Multiplication property of order:

<u>If</u> $a < b$ <u>and</u> $c > 0$, <u>then</u> $ac < bc$.

<u>If</u> $a < b$ <u>and</u> $c < 0$, <u>then</u> $ac > bc$.

For example, from $1 < 2$ it follows that: $1 \cdot 3 < 2 \cdot 3$, $1 \cdot \frac{5}{7} < 3 \cdot \frac{5}{7}$ and generally $1 \cdot x < 2 \cdot x$ if $x > 0$; but $1 \cdot (-3) > 2 \cdot (-3)$, $1 \cdot (-\frac{5}{7}) > 3 \cdot (-\frac{5}{7})$, and generally $1 \cdot x > 2 \cdot x$ if $x < 0$.

We say that a number x is <u>positive</u> if $x > 0$, and <u>negative</u> if $x < 0$. The addition and multiplication properties of order are often stated in words, as follows: "If the same number is added to both sides of an inequality the direction of the inequality is not changed"; "If both sides of an inequality are multiplied by the same <u>positive</u> number the direction of the inequality is not changed, but if both sides of the inequality are multiplied by the same negative number the direction of the inequality is reversed."

The order relation has many other properties, but all of these <u>follow logically</u> from the properties we have stated and the properties of the rational number system we stated in the preceding section. With the properties of a number system, the subtraction and division properties, and the four fundamental properties of order as <u>postulates</u>, every other rule concerning inequalities can be proved as a <u>theorem</u>. We present some examples of theorems we will use.

<u>Theorem</u>: <u>If</u> $a < b$ <u>then</u> $a - c < b - c$.

<u>Proof</u>:

$a < b$	
$a + (-c) < b + (-c)$	Addition property of order
$a - c < b - c$	Theorem on subtraction.

<u>Theorem</u>: <u>If</u> $c > 0$ <u>then</u> $\frac{1}{c} > 0$

<u>If</u> $c < 0$ <u>then</u> $\frac{1}{c} < 0$

<u>Proof</u>: We will prove the second statement of the theorem. Suppose $c < 0$. According to the Trichotomy property exactly one of the statements $\frac{1}{c} = 0$, $\frac{1}{c} > 0$, $\frac{1}{c} < 0$ holds. We will show that the first two are impossible, so that we must have $\frac{1}{c} < 0$.

If $\frac{1}{c} = 0$, then

$$c \cdot \frac{1}{c} = c \cdot 0 \quad \text{Substitution}$$

$$1 = c \cdot 0 \quad \text{Definition of multiplicative inverse}$$

$$1 = 0 \quad \text{Theorem 2}$$

which is a contradiction. Hence $\frac{1}{2} = 0$ cannot hold.

If $\frac{1}{c} > 0$ then since $c < 0$

$$c \cdot \frac{1}{c} < c \cdot 0 \quad \text{Multiplication property of order}$$

$$1 < c \cdot 0 \quad \text{Definition of multiplicative inverse}$$

$$1 < 0 \quad \text{Theorem 2}$$

which is again a contradiction, so that $\frac{1}{c} > 0$ cannot hold. Since $\frac{1}{c} = 0$ and $\frac{1}{c} > 0$ do not hold if $c < 0$ we must have $\frac{1}{c} < 0$.

The proof of the first statement of the theorem is similar.

__Theorem__: __If__ $a < b$ __and__ $c > 0$, __then__ $\frac{a}{c} < \frac{b}{c}$.

__If__ $a < b$ __and__ $c < 0$, __then__ $\frac{a}{c} > \frac{b}{c}$.

__Proof__: We prove the second statement; the proof of the first is similar. If $c < 0$, then

$$\frac{1}{c} < 0 \quad \text{Theorem}$$

$$\frac{1}{c} \cdot a > \frac{1}{c} \cdot b \quad \text{Multiplication property of order}$$

$$\frac{a}{c} > \frac{b}{c} \quad \text{Theorem on division.}$$

It is often convenient to use the symbols $\leq$ and $\geq$ which are read "less than or equal to" and "greater than or equal to", respectively. $a \leq b$ means that either $a = b$ or $a < b$; $b \geq a$ is synonomous. It is easy to verify that, except for the Trichotomy property, the preceding properties and theorems remain true if $a < b$ is replaced by $a \leq b$.

Every statement about inequalities can be proved using the fundamental properties of order, the preceding theorems, and statements about specific numbers, such as $0 < 1$, $-3 < -2$. In the following examples we do not cite the properties and theorems which justify each step but we organize our solution so that the reasons for each step can be seen easily.

Example: Show that if x satisfies the inequality $4x-3 < 2x+5$ then $x < 4$.

Solution:

$$4x - 3 < 2x + 5$$

$$4x - 3 + 3 < 2x + 5 + 3$$

$$4x < 2x + 8$$

$$4x - 2x < 2x + 8 - 2x$$

$$2x < 8$$

$$\frac{1}{2} \cdot 2x < \frac{1}{2} \cdot 8$$

$$x < 4$$

Example: Show that if $\frac{x+3}{2x} > 1$ then $0 < x < 3$, that is, $0 > x$ and $x < 3$.

Solution: We have

$$\frac{x+3}{2x} > 1$$

$$2\left(\frac{x+3}{2x}\right) > 2 \cdot 1$$

$$\frac{x+3}{x} > 2$$

Now we have to consider two cases. If $x > 0$ then

$$x\left(\frac{x+3}{x}\right) > x \cdot 2$$

$$x + 3 > 2x$$

$$x + 3 - x > 2x - x$$

$$3 > x.$$

If $x < 0$ then

$$x\left(\frac{x+3}{x}\right) < x \cdot 2$$

$$x + 3 < 2x$$

$$x + 3 - x < 2x - x$$

$$3 < x$$

But $x > 3$ contradicts $x < 0$, so $x < 0$ cannot hold. Thus either $x = 0$ or $x > 0$. Since division by 0 is not defined, the possibility $x = 0$ is excluded, and we have $x > 0$. Thus, for all x satisfying the given inequality we have $0 < x$ and $x < 3$.

We will generally use the notation $a < b < c$ as an abbreviation for the statement "$a < b$ and $b < c$". Similarly, $a \leq b < c$ means "$a \leq b$ and $b < c$," and so forth.

Exercises 7c.

For each of the following, show that if x satisfies the inequality on the left, it must satisfy the inequalities on the right.

1. If $2x + 3 > 5x - 9$ then $x < 4$
2. If $x + 5 < 4x - 1$ then $x > 2$
3. If $(x - 5)(x - 7) < 0$ then $5 < x < 7$
4. If $(x - 3)^2 \leq 4$ then $1 \leq x \leq 5$
5. If $x/x-5 > 2/x-5$ then $x > 5$ or $x < 2$
6. If $(2x + 5/x \leq 6$ then $x < 0$ or $x \geq 5/4$
7. If $6x + 13 < 28/x$ then $x \neq 0$ and $-7/2 < x < \frac{4}{3}$
8. If $x(x-1)(x+2) < 0$ then $x < -2$ or $0 < x < 1$

Exercises 7d.

Prove the following theorems

1. If $a > 1$, then $a^2 > a$
2. If $a < 1$ and $a > 0$, then $a^2 < a$
3. If $a < 1$ and $a < 0$, then $a < a^2$
4. If $a > 0$ and $b > 0$ and $a < b$, then $a^2 < b^2$.
5. If $a \neq 0$, then $a^2 > 0$.
6. For all a, $a^2 \geq 0$.

7. If $a^2 + b^2 = 0$ then $a = 0$ and $b = 0$.

8. For all a and b, $ab \leq \frac{a^2 + b^2}{2}$.

[Hint: Consider $(a-b)^2$]

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8. Sets of Numbers and Graphs

A <u>set of numbers</u> is a collection (or class) of numbers specified by some condition. The numbers in the set - that is the numbers which satisfy the condition which specifies the set - are called <u>elements</u> or <u>members</u> of the set.

<u>Examples</u> The following are examples of statements which specify sets of numbers:

(a) The set of numbers x such that $x^2 = 4$ - the elements of this set are 2 and -2.

(b) The set of natural numbers x such that $x \leq 3$ - the elements of this set are 1,2 and 3.

(c) The set of rational numbers x such that $1 < x < 3$ - the elements of this set cannot be listed, but we can say for each number whether or not it is a member of the set. 2 and $\frac{47}{32}$ are members of the set; -5, $\frac{7}{11}$ are not members.

(d) The set of natural numbers - again we cannot list all the members of the set. 1,2,3 and so forth, are in the set; 0, -1, $3/2$ are not in the set.

(e) The set of numbers in the list 1, 2/7, -3.

It is convenient to have a notation for the phrase "the set of numbers - such that __________". We use the symbol $\{\_\_ : \_\_\_\_\_\}$. With this notation we may write the preceding examples as follows.

<u>Examples</u>:

(a) $\{x : x^2 = 4\}$

(b) $\{x : x$ is a natural number and $x \leq 3\}$

(c) $\{x : x$ is a rational number and $1 < x < 3\}$

(d) $\{x : x$ is a natural number$\}$

(e) $\{x : x = 1, x = 2/7, \text{ or } x = -3\}$ (In this case we write more briefly $\{1, 2/7, -3\}$; we will use a similar notation whenever a set is specified by listing its members.)

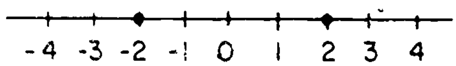
We will often denote sets by capital letters. In particular, we denote the set of natural numbers, the set of integers, and the set of rational numbers by N, I, and R, respectively. We introduce the curious notion of a set which has no members, which we call the <u>empty set</u>, and which we denote by the symbol $\emptyset$; you will see in a moment why this is a useful notion.

Every statement about a number x specifies a set, namely, the set of numbers for which the statement is true. If the statement is not true for any number, the set specified is the empty set, $\emptyset$. For example, $\{x : x > 0 \text{ and } x < 0\}$ is the empty set.

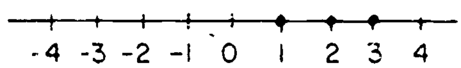
The <u>graph</u> of a set of numbers is the collection of points on the number line which represent the members of the set. We will sketch the graphs of the sets we have discussed as examples.

<u>Examples</u>:

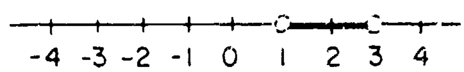
(a) $\{x : x^2 = 4\}$



(b) $\{x : x$ is in N and $x \leq 3\}$

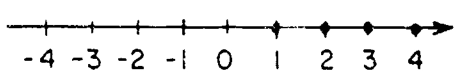


(c) $\{x : x$ is in R and $1 < x < 3\}$



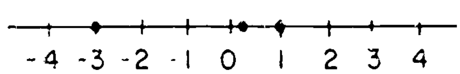
(We draw circles around the points 1 and 3 to emphasize that they are not in the set.)

(d) $\{x : x$ is in N$\}$



(We draw an arrow on the right to indicate that the graph continues to the right.)

(e) $\{1, 2/7, -3\}$



We say that two sets A and B are <u>equal</u>, and write $A = B$, if they have exactly the same elements. For example: $\{x : x^2 = 4\} = \{2, -2\}$. If every member of the set A is also a member of the set B we say that A <u>is contained in</u> B, or B contains A, and write $A \subseteq B$. For example: $\{1, 3\} \subseteq \{1,2,3\}$; $I \subseteq R$.

If every member of A is a member of B and every member of B is a member of A, then A and B have exactly the same members; that is, if $A \subseteq B$ and $B \subseteq A$ then $A = B$. We usually show that two sets are equal by using this fact.

The set of all numbers which satisfy an equation or inequality is called the <u>solution set</u> of the equation or inequality. For example: $\{x : 2x = 3\} = \{3/2\}$ is the solution set of the equation $2x = 3$; $\{x : 2x < 3\} = \{x : x < 2/3\}$ is the solution set of the inequality $2x < 3$. If the solution set of an equation contains all the numbers in a number system, the equation is an identity in the system. For example, the solution set of $x + 2 = 2 + x$, $\{x : x + 2 = 2 + x\}$, is R, so that this equation is an identity in R.

<u>Example</u> (f): Find the solution set of $2x - 1 < x$, and sketch its graph.

<u>Solution</u>: If x is in $\{x : 2x - 1 < x\}$ then

$$2x - 1 < x$$
$$2x - 1 - x + 1 < x - x + 1$$
$$x < 1,$$

so that x is in $\{x : x < 1\}$. Thus $\{x : 2x - 1 < x\} \subseteq \{x : x < 1\}$.

Suppose now that x is in $\{x : x < 1\}$.

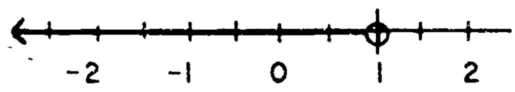
Then

$$x < 1$$
$$x + x - 1 < 1 + x - 1$$
$$2x - 1 < x$$

so that x is in $\{x : 2x - 1 < x\}$. Thus $\{x : x < 1\} \subseteq \{x : 2x - 1 < x\}$

Since the solution set of $2x - 1 < x$ is contained in and contains $\{x : x < 1\}$ it is equal to $\{x : x < 1\}$.

The graph of the solution set $\{x : x < 1\}$ is sketched below.



Graph sketching is helpful in the discussion of statements involving the absolute value of numbers.

<u>Definition of absolute value</u>: <u>The absolute value of</u> a, <u>denoted by</u> $|a|$, <u>is given by</u>

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

The notion of absolute value has a simple and important geometric interpretation. If we regard the rational numbers as points on the number line, then $|a|$ is the distance of a from the origin 0. The distance between any two points a and b is $|a - b|$. For example: the distance between 3 and 5 is $|5 - 3| = |2| = 2 = |-2| = |3 - 5|$; the distance between -1 and -4 is $|-1 - (-4)| = |-1 + 4| = |3| = 3 = |-3| = |-4 - (-1)|$; the distance between -1 and 3 is $|-1 -3| = |-4| = 4 = |4| = |3 - (-1)|$.

<u>Example</u> (g): Sketch the graph of $\{x : |x - 3| < 5\}$

<u>Solution</u>: We have to consider two cases $x \geq 3$ and $x < 3$ since if $x \geq 3$ then $x - 3 \geq 0$ and $|x - 3| = x - 3$

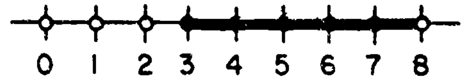
and

if $x < 3$ then $x - 3 < 0$ and $|x - 3| = -(x - 3) = 3-x$.

We suppose first that $x \geq 3$, that is, we find the part of the graph of $\{x : |x - 3| < 5\}$ which is to the right of 3 (or is 3 itself). We have

$$|x - 3| < 5$$
$$x - 3 < 5$$
$$x < 8,$$

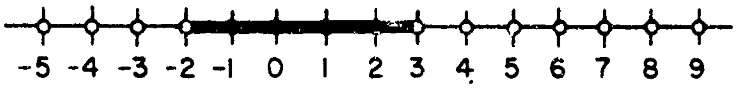
so that the graph to the right of 3 is the graph of $\{x : 3 \leq x < 8\}$, which we sketch below.



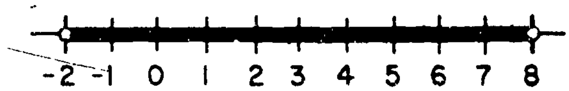
Now suppose $x < 3$. Then

$$|x - 3| < 5$$
$$-(x - 3) < 5$$
$$3 - x < 5$$
$$3 - 5 < x$$
$$-2 < x,$$

so the part of the graph of $\{x : |x - 3| < 5\}$ to the left of 3 is the graph of $\{x : -2 < x < 3\}$. The sketch of this graph is



Thus, the graph of $\{x : |x - 3| < 5\}$ is



<u>Exercises</u> 8a

Find the solution set of each of the following and sketch its graph.

1. $\{x : x \text{ is in } N \text{ and } 2x^2 < 50\}$
2. $\{x : x \text{ is in } N \text{ and } 2x - 4 < 10\}$
3. $\{x : x \text{ is in } I \text{ and } 6 < x < 8\}$
4. $\{x : x \text{ is in } R \text{ and } (2x - 5)(3x - 6) = 0\}$
5. $\{x : x \text{ is in } N \text{ and } (x-4)(x + 3) < 0\}$
6. $\{x : x \text{ is in } R \text{ and } 5x - 4 > 7x + 9\}$
7. $\{x : x \text{ is in } R \text{ and } \frac{2x - 5}{2} > \frac{5x + 4}{5}\}$
8. $\{x : x \text{ is in } R \text{ and } (x + 4)^2 \geq 36\}$
9. $\{x : x^2 < 9 \text{ on } x^2 > 25\}$
10. $\{x : \frac{x - 3}{x} < 1\}$
11. $\{x : \frac{x}{2} - 1 > 3 - x\}$
12. $\{x : \frac{x}{x} \geq \frac{x}{x}\}$

13. $|2x + 3| > 1$
14. $|x + 4| < 2$
15. $|x| > -2$
16. $|2x - 5| \geq 3$

<u>Exercises</u> 8b

Prove the following theorems.

1. For all a and b, $|a - b| = |b - a|$
2. For all a and b, $|ab| = |a|\,|b|$
3. For all a and b, except b = 0, $|\frac{a}{b}| = \frac{|a|}{|b|}$

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9. <u>Real Numbers</u>

In the system of rational numbers every linear equation,

$$ax + b = c, \qquad a \neq 0$$

has a solution. The next simplest kind of equation is the quadratic equation

$$ax^2 + bx + c = 0. \qquad a \neq 0$$

Here again we run into trouble. Even the most simple quadratic equation may not have a solution if the only numbers we have are rational numbers. The equation

$$x^2 - 2 = 0$$

does not have a rational solution because there is no rational number whose square is 2.

This last statement seems difficult to prove. How does one prove that an equation does not have a solution in a certain number system except by testing every number in the system? Although it required genius to discover the proof, the proof is very easy to understand. All that is required is a precise definition of even integer and odd integer. An integer p is even if and only if p = 2m where m is an integer, and odd if and only if p = 2m + 1. Now you can easily show that the square of an even integer is even and the square of an odd integer is odd. From this it follows that if the square of an integer is even the integer must also be even. Why? It also follows that <u>every rational number can be written as</u> p/q <u>where</u> p <u>and</u> q <u>are integers not both</u>

of which are even. Why?

Now suppose that the equation

$$x^2 = 2$$

has a solution in the system of rational numbers. Then x is a quotient of two integers. Let p and q be any two integers such that $x = p/q$ and substitute in equation $x^2 = 2$. Then

$$\left(\frac{p}{q}\right)^2 = 2$$

$$\frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$

which states that p^2, and therefore p is even. Since p is even, $p = 2r$ and substituting in the equation $p^2 = 2q^2$

$$(2r)^2 = 2q^2$$

$$4r^2 = 2q^2$$

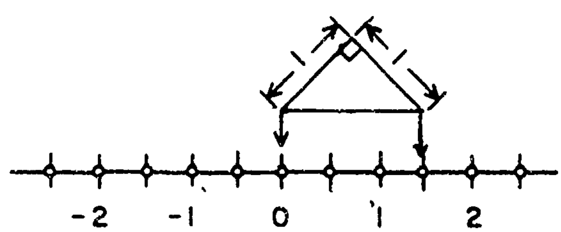
$$2r^2 = q^2$$

so that q^2, and therefore q is even. Thus if the equation $ax^2 + bx + c = 0$, $a \neq 0$ has a rational solution it must be a number x such that if $x = p/q$ where p and q are integers then p and q are both even. But there is no such number.

Since there are quadratic equations with no solution in the system of rational numbers a logical step in view of our previous discussion would be to look for the simplest possible extension of the system of rational numbers in which every quadratic equation has a solution. This is a possible approach, but if we followed it we would have to face the possible need of more extensions when we consider cubic equations, quartic equations, etc. In fact, this is not the way the number system developed historically. The extension of the number system which was used in algebra after the rational numbers was the system of real numbers.

The real number system can be described as a system of numbers which is an extension of the system of rational numbers, which has the subtraction, division, and fundamental order properties and which also has the property that to every real number there corresponds a point on a line in such a way that two different points correspond to two different numbers, and every point corresponds to some number. The property which distinguishes the real number system from the rational number system is this last statement that every point corresponds to some real number. How did a number system having this geometric property come to be considered?

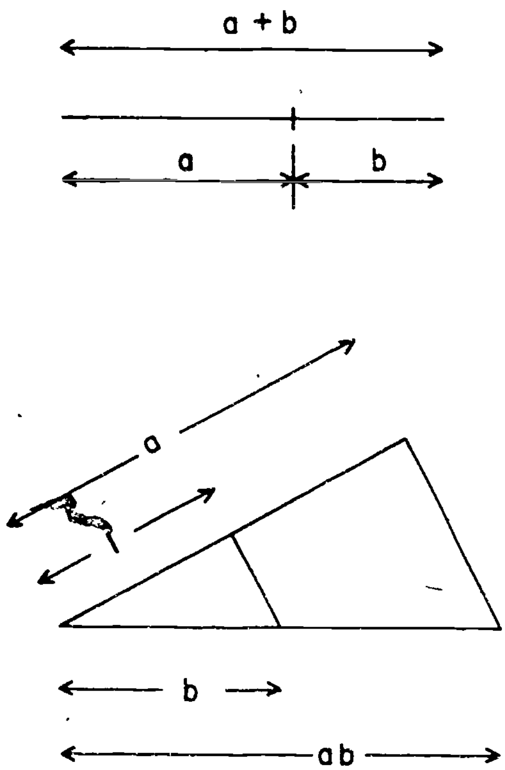
We know that there is no rational number whose square is 2. We can however with straight edge and compass construct a right triangle with legs of length 1. If there were a number whose square is two then by the Pythagorean Theorem, the hypotenuse of this triangle would have that length. Transfering this hypotenuse to the number line we mark a point, as in the following diagram.



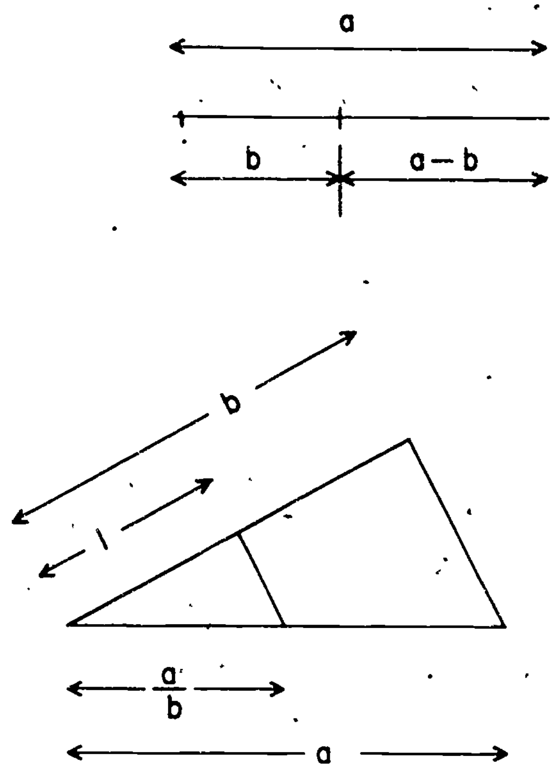
Thus there is a point on the line which does not represent a rational number.

Historically, this led to the idea of assigning a number to every point on the line. The numbers represented by points which did not represent rational numbers were called irrational numbers. The set of rational and irrational numbers was called the set of real numbers.

The operations of addition, multiplication, subtraction and division of positive real numbers can be defined geometrically as indicated in the following diagrams.



Multiplication



Division

Operations with negative real numbers can be performed in terms of operations with positive real numbers, using the same rules as for the rational numbers. In this way we obtain a system of numbers with the geometric property we stated.

The picture of the system of real numbers which this construction provides seems clear enough, except for one question. How do we obtain usable, that is, rational, approximations to irrational numbers? The process of physical measurement which seems to be required is unsatisfactory because of its non-mathematical character and because of its unavoidable lack of precision. To answer this question let us first see how the approximation process could be carried out geometrically in a systematic way.

Suppose that we wish to approximate a real number r between 2 and 3. We write

$$2 \leq r < 3$$

and subdivide the interval from 2 to 3 into ten equal parts. Now suppose r lies in the third subinterval. Then

$$2.2 = 2 + \frac{2}{10} \leq r \leq 2 + \frac{3}{10} = 2.3$$

Subdivide the interval from 2.2 to 2.3 into 10 equal parts and suppose that r lies in the fifth subinterval. Then

$$2.24 = 2 + \frac{2}{10} + \frac{4}{10^2} \leq r < 2 + \frac{2}{10} + \frac{5}{10^2} = 2.25$$

If we continue this process indefinitely we obtain an infinite decimal $a.\ a_1\ a_2 \ldots a_n \ldots$, where $a_1 = 2$ is the first place digit, $a_2 = 4$ is the second place digit, and a_n is the digit in the n-th place, with the property that, no matter how large n is,

$$(1) \quad a + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} \leq r < a + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} + \frac{1}{10^n}.$$

This infinite decimal represents the number r in the sense that by breaking off the decimal far enough out we obtain a rational approximation to r with an error as small as we wish.

The idea of an infinite decimal may seem difficult to you. How can we know infinitely many places of a decimal? Let us make the proposition more general. An infinite decimal may be regarded as an infinite <u>sequence</u> of numbers, where by a sequence of numbers we mean a set of numbers given in a definite order. How can we know any infinite sequence of numbers? We cannot specify the sequence by writing down all of the numbers in it since there are infinitely many of them. But the sequence is <u>specified</u> and therefore <u>known</u> if there is a definite rule which enables us to determine as many of the numbers in the sequence as we wish. The geometrical procedure described above is one kind of definite rule which enables us to determine as many places of the infinite decimal of a real number as we wish, and which therefore specifies the infinite decimal.

The rule which specifies an infinite decimal may be more or less simple. The infinite decimal of a rational number is always periodic, that is, after a certain point the same block of digits is repeated indefinitely. For example, the infinite decimal representation of 2/7 is .285714285714285714 ... where the block of digits 285714 is repeated indefinitely. In this case one can easily tell what the digit in any given place is. A more difficult kind of rule specifies the infinite decimal .12345678910111213 ... which is obtained by writing all the natural numbers in succession. This is the infinite decimal of an irrational number. Here we can again determine what digit is in any given place, but with more difficulty than in the case of a rational number. A still more difficult rule is that which determines the infinite decimal of $\sqrt{2}$, which we will describe below.

The value of the idea of the infinite decimal representation of a real number is that the decimal can be obtained in a purely arithmetic way, without using the picture furnished by the number line. For, suppose that we have already determined the decimal of r to three places $a.\,a_1\,a_2\,a_3$. Then we have only to compare r with the numbers $a.\,a_1\,a_2\,a_3\,1$, $a.\,a_1\,a_2\,a_3\,2$, ..., $a.\,a_1\,a_2\,a_3\,9$. If the first of these which is greater than r has 7 in the fourth place for example, then the fourth place digit is 6. If all are smaller than r then the fourth place digit is 9. If r is specified by some algebraic property then we can use this property and algebraic theorems to make the comparisons.

As an example, we shall find the first few places of the decimal of $\sqrt{2}$. $\sqrt{2}$ is defined by the equation $(\sqrt{2})^2 = 2$. We will use the theorem that if a and b are non-negative numbers and $a^2 < b^2$ then $a < b$. This is easy to prove, but we will put the proof off until later. We compare 2 with the squares of integers and find

$$1^2 = 1 < 2 < 4 = 2^2$$

Next we compute $(1.1)^2$, $(1.2)^2$ ... $(1.5)^2$ and find

$$(1.4)^2 = 1.96 < 2 < 2.25 = (1.5)^2.$$

Continuing we find

$$(1.41)^2 = 1.9881 < 2 < 2.0264 = (1.42)^2$$

$$(1.414)^2 = 1.999396 < 2 < 2.002225 = (1.415)^2$$

$$(1.4142)^2 = 1.99996164 < 2 < 2.00024449 = (1.4143)^2$$

so that

$$(1.4142)^2 < (\sqrt{2})^2 < (1.4143)^2,$$

and

$$1.4142 < \sqrt{2} < 1.4143.$$

The decimal of $\sqrt{2}$ correct to four places is therefore 1.4142. It is important to realize that, although it would be extremely laborious we could use this procedure to determine as many places of the decimal of $\sqrt{2}$ as we desired. We could, in particular, calculate the digit in the one millionth place of $\sqrt{2}$. No one has ever calculated this digit and it is highly unlikely that anyone ever will. But this is irrelevant. What is important is that if we wished we could calculate that digit.

ANSWERS

Exercises 2a, page 2.

1. 12
2. 6
3. 5
4. 3
5. None
6. All
7. 10
8. 7
9. None
10. All
11. 1, 0
12. 0
13. 2
14. None
15. None
16. 0
17. All
18. All
19. All
20. None
21. All

Exercises 2b, page 3.

1. a. yes
   b. no; $2 - 3$ = no natural number
   c. yes
   d. no; $3x = 2$ has no solution in the set.

2. a. yes
   b. yes

3. a. no; Sum of any two odd numbers is even.
   b. yes; $(2n + 1)(2k + 1) = 4kn + 2n + 2k + 1$
   $= 2(2k + n + k) + 1$
   $= 2k + 1$, odd

4. a. no; $7 + 4$
   b. no; 2×5

Exercises 2c, page 5,6.

1. a. Distributive law
   b. False
   c. Addition is commutative.
   d. False
   e. Multiplication is commutative.
   f. Multiplication is associative.
   g. False
   h. Addition is commutative
   i. Addition is commutative
   j. Addition is associative
   k. Multiplication is commutative
   l. Distributive principle
   m. Addition is commutative

2. One must distinguish here between number and decimal representation of that number. At this stage, number is of primary concern, so that the answer intended was

$$7 + 5 + 3 = (7 + 5) + 3 = 12 + 3 = 15.$$

However, the associative property and the commutative property for addition gives

$$(7 + 5) + 3 = 7 + (5 + 3)$$
$$=(3 + 5) + 7$$

Exercises 2d, page 7.

1. $2 + 2 = 2 + (1 + 1)$ Definition
   $= (2 + 1) + 1$ Addition is associative
   $= 3 + 1$ Definition
   $= 4$ Definition

2. The distributive law:

$$(20 + 3)(32) = 20\cdot 32 + 3\cdot 32$$
$$= 640 + 96$$
$$= 736$$

<u>Exercises 2d</u>, (cont'd.)

3. $23 \cdot 8 = (2 \cdot 10 + 3)8$
$= (2 \cdot 10)8 + 3 \cdot 8$
$= 2(10 \cdot 8) + 3 \cdot 8$
$= 2(8 \cdot 10) + 3 \cdot 8$
$= (2 \cdot 8)(10) + 3 \cdot 8$
$= (16)(10) + 24$
$= (10 + 6)(10) + [(2 \cdot 10) + 4]$
$= [10 \cdot 10 + 6 \cdot 10) + (2 \cdot 10)] + 4$
$= \{10 \cdot 10 + [(6 \cdot 10) + (2 \cdot 10)]\} + 4$
$= \{10 \cdot 10 + (6 + 2)(10)\} + 4$
$= [100 + (8)(10)] + 4$
$= 184.$

4. (a) $13 + 25 = [1(10) + 3(1)] + [2(10) + 5(1)]$
$13 + 25 = [1(10) + 2(10)] + [3(1) + 5(1)]$
$= 3(10) + 8(1)$
$= 38$

(b) $38 + 44 = [3(10) + 8(1)] + [4(10) + 4(1)]$
$= [3(10) + 4(10)] + [8(1) + 4(1)]$
$= 7(10) + 12(1)$
$= 7(10) + 1(10) + 2(1)$
$= [7(10) + 1(10)] + 2(1)$
$= 8(10) + 2(1)$
$= 82$

(c) $16 \times 13 = [1(10) + 6(1)] \cdot [1(10) = 3(1)]$
$= [1(10) + 6(1)] \cdot 10 + [1(10) + 6(1)] \cdot 3$
$= [1(100) + 6(10) + 0(1)] + [3(10) + 18(1)]$
$= [1(100) + 6(10) + 0(1)] + [3(10) + (1 \cdot 10 + 8) \cdot 1]$
$= [1(100) + 6(10) + 0(1)] + [+4(10 + 8(1)]$
$= 1(100) + 10(10) + 8(1)$
$= 2(100) + 0(10) + 8(1)$
$= 208$

(d) - (h) These solutions may be obtained in a manner similar to that illustrated above.

Exercises 2e, page 11.

	Answer	Cancellation law
1.	0	Addition
2.	9	Multiplication
3.	10	Addition
4.	No solution in set of whole numbers	
5.	0	Multiplication
6.	No solution in set of whole numbers	
7.	1	Addition
8.	16	Multiplication
9.	5	Addition and Multiplication
10.	No solution in set of whole numbers	
11.	42	Addition and Multiplication
12.	No solution in set of whole numbers	

Exercises 3, page 15.

1. $2(5a + b) + 3(b + 2a)$

$= (10a + 2b) + (3b + 6a)$	Distributive law
$= 10a + [2b + (3b + 6a)]$	Associative law of addition
$= 10a + [(2b + 3b) + 6a]$	Associative law of addition
$= 10a + [5b + 6a]$	Distributive law, and addition tables
$= 10a + [6a + 5b]$	Commutative law of addition
$= (10a + 6a) + 5b$	Associative law of addition
$= 16a + 5b$	Distributive law

2. $(3x)(2y) = (3)(2)(x)(y)$

$= (3\cdot 2)(xy)$

$= 6xy$

3. $(x + 3)(2x + 3)$

$= (x + 3)2x + (x + 3)3$

$= (2x^2 + 6x) + (3x + 9)$

$= 2x^2 + [6x + (3x + 9)]$

$= 2x^2 + [(6x + 3x) + 9]$

$= 2x^2 + 9x + 9$

4. $a + 3(a + 4) = a + 3a + 12$
$= a(1 + 3) + 12$
$= 4a + 12$

5. $x^2y + 5 + 3x^2y + 2 = 4x^2y + 7$

6. $2a(b + 2) = 2ab + 4a$

7. $9a^2 + 8b^2$

8. $10x^2 + 31x + 36$

9. $(x + a)x + (x + a)b = x^2 + ax + bx + ab$

Note: The teacher should point out that this can be written in the form

$$x^2 + (a + b)x + ab$$

and that in this form it is most usable.

10. $(a + b)x + (a + b)y = ax + bx + ay + by$

11. $(ax + b)cx + (ax + b)d = acx^2 + bcx + adx + bd$

Note: The teacher should point out that this can be written in the form

$$acx^2 + (bc + ad)x + bd$$

and that in this form it is most usable.

<u>Exercises 4a,</u> page 17.

1. -5
2. None
3. -9
4. -9
5. None
6. None
7. -3
8. None

Exercises 4b, page 20, 21.

1. 24
2. 24
3. -4
4. -12
5. -2
6. 188
7. $6 \cdot a$
8. -16a
9. 2x + 4
10. 2

Exercises 4c, page 22.

1. -10
2. 10
3. 80
4. -15
5. 6
6. 0
7. 4x - 5
8. -6
9. 0
10. 2(a + b)

Exercises 4d, page 25,26.

1. -a + b
2. x - 5y
3. 5x - 5y
4. 4x - 3y
5. 4x
6. ab
7. 12x + 12y - 18z
8. $6x^2 - 14x$
9. $10x^2 - 4x^2y$
10. $x^2 - xy - 2y^2$
11. $6x^2 - 7xy - 20y^2$
12. $9x^2 - 3x^3 - 9x^4$
13. $30a^3b^3$
14. $y^3 - 9y^2 + 24y - 20$
15. $4c^3 + 19c^2 - 29c + 6$
16. $3x^3 - 12xy^2$

Exercises 4d, (cont'd.)

17. $4x^2 - 20xy + 25y^2$

18. $8x^2 - 10x - 3$

19. $2b^2 - 8$

20. $9a^2 + 18ab + 9b^2$

21. $4a^2 - 25b^2$

22. $6 + 2y + 3x + xy$

23. $x^3 + 9x^2 + 26x + 24$

24. $(a + b)^2 - 4c^2$

25. $9x^2 - 6x(y - z) + (y - z)^2$

26. $(2a + b)^2 - 3(2a + b) - 10$

27. $9 + 3(r + s + a + b) + (r + s)(a + b)$

28. $x^2 - (y - m + n)^2$

29. $(2x - y)^2 - z^2$

30. $b^4 - (2b + 1)^2$

31. $4r^2 + s^2 + t^2 - 4rs - 4rt + 2st$

32. $x^3 - 8y^3$

33. 0

34. $x^3 + 125$

35. $x^2 + 2xy + y^2 - 9$

36. $2ab + 2b^2$

Exercises 4e, page 26.

1.

$(x + y) - y = (x + y) + -y$	Theorem: $a - b = a + -b$
$= x + (y + -y)$	Associative law
$= x + 0$	$y + -y = 0$ meaning of additive inverse
$= x$	0 the additive identity

2.

$(x - y) + y = (x + -y) + y$	Theorem: $a - b = a + -b$
$= x + (-y + y)$	Associative law
$= x + 0$	$y + -y = 0$
$= x$	Additive identity

3.

$x - (y + z) = x + -(y + z)$	Theorem: $a - b = a + -b$
$= x + -y + -z$	$-(a + b) = -a + -b$
$=(x + -y) + -z$	Convention
$=(x - y) - z$	Theorem: $(a + -b) = a - b$

4. To prove $x - (y - z) = (x - y) + z$

$x - (y - z)$	Theorem: $b + -a = b - a$
$= x + -(y + -z)$	
$= x + -1(y + -z)$	Theorem: $-1(a) = -a$
$= x + (-1 \cdot y + -1 \cdot -z)$	Distributive property for multiplication over addition
$= (x + -1 \cdot y) + -1 \cdot -z$	Associative property for addition
$= (x + -y) + -1 \cdot -z$	Theorem: $-1(a) = -a$
$= (x + -y) + z$	Theorem: $-1(-a) = a$
$= (x - y) + z$	Theorem: $b + -a = b - a$

$\therefore x - (y - z) = (x - y) + z$ as a consequence of the above properties and theorems.

5. To prove $(x + y) - z = x + (y - z)$

$(x + y) - z$	Theorem: $b + -a = b - a$
$= (x + y) + -z$	
$= x + (y + -z)$	Associative property for addition
$= x + (y - z)$	Theorem: $b + -a = b - a$

$\therefore (x + y) - z = x + (y - z)$ as a consequence of the above property and theorem.

Exercises 4e. (cont'd.)

6. Same as number 4.

7.	$(x)(y)(-z) = (x)[(y)\cdot(-z)]$	Associative property
	$= (x)[(-z)\cdot(y)]$	Commutative property
	$= (x)[-(zy)]$	Theorem: $(-a)(b) = -(ab)$
	$= (x)\cdot[-(yz)]$	Commutative property
	$= [-(yz)](x)$	Commutative property
	$= -[(yz)(x)]$	Theorem: $(-a)(b) = -(ab)$
	$= -(xyz)$	Commutative property
8.	$(-x)(y)(-z) = (-x)[(y)(-z)]$	Associative property
	$= (-x)[(-z)(y)]$	Commutative property
	$= (-x)[-(zy)]$	Theorem: $(-a)(b) = -(ab)$
	$= (-x)[-(yz)]$	Commutative property
	$= xyz$	Theorem: $(-a)(-b) = ab$
9.	$a(b-c) = a(b + -c)$	Theorem: $b + -a = b - a$
	$= ab + (a)(-c)$	Distributive property
	$= ab + (-c)(a)$	Commutative property
	$= ab - ca$	Theorem: $(-a)(b) = -(ab)$
	$= ab - ac$	Commutative property
10.	$(a - b)^2 = (a - b)(a - b)$	Definition: $a^2 = a\cdot a$
	$= (a + -b)(a + -b)$	Theorem: $a + -b = a - b$
	$= (a + -b)a + (a + -b)(-b)$	Distributive property
	$= a^2 + (-b)(a) + a(-b) + (-b)^2$	Distributive property
	$= a^2 + -ab + -ab + b^2$	Theorem: $(-a)(b) = (a)(-b) = -ab$ and $(-a)(-b) = ab$
	$= a^2 + (-ab)(1 + 1) + b^2$	Distributive property
	$= a^2 - ab(2) + b^2$	Fact of arithmetic; Theorem: $a + (-b) = a - b$
	$= a - 2ab + b^2$	Commutative property

11. $(a + -b)(a^2 + ba + b^2)$ — Commutative property; Theorem: $a - b = a + -b$

$= (a + -b)a^2 + (a + -b)ba + (a + -b)b^2$ — Distributive property

$= a^3 + -ba^2 + ba^2 + -b^2a + ab^2 + -b^3$ — Distributive property

$= a^3 + -b^3$ — Commutative property and additive inverse

$= a^3 - b^3$ — Theorem: $a + -b = a - b$

12. $(a + b)(a^2 + -ba + b^2)$

$= (a + b)a^2 + (a + b)(-ba) + (a + b)b^2$

$= a^3 + ba^2 + (a)(-ba) + b(-ba) + ab^2 + b^3$

$= a^3 + ba^2 + (a)(-ab) + -b^2a + ab^2 + b^3$

$= a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3$

$= a^3 + b^3$

13. $b + (-a) = (-a) + b$

$= (-1)a + -(-b)$ — Theorem: $-(-a) = a$

$= (-1)a + (-1)(-b)$

$= (-1)(a + -b)$

$= -(a - b)$ — Theorem: $a + -b = a - b$

<u>Exercises 5a</u>, page 29-32.

1. $a(x + y)$
2. $5y^3(x^2 + 6y^2)$
3. $3b(x - 2by)$
4. $2c(7d + 3e - f)$
5. $(y - 5)(y + 5)$
6. $(7 - x)(7 + x)$
7. $(3a - 4b)(3a + 4b)$
8. $4(x - 4)(x + 4)$
9. $(4 - 5a)(4 + 5a)$
10. $(x + 2)^2$
11. $(2x - 3y)^2$
12. $(x + 7)(x + 2)$
13. $(y - 3)(y + 5)$
14. $(7 + w)(2 - w)$
15. $(y + 3)(y^2 - 3y + 9)$
16. $(x - 4)(x^2 + 4x + 16)$
17. $(c + 1)(c^2 - c + 1)(c^6 - c^3 + 1)$
18. $(3a - 1)(9a^2 + 3a + 1)$
19. $a(c - 4)(c^2 + 4c + 16)$
20. $(x + y)(a + b)$
21. $(x - y)(a - b)$
22. $(a + 1)(a - 1)(a - 1)$
23. $(3x - y - 3u)(3x - y + 3u)$
24. $(3 - a - 3b)(3 + a + 3b)$
25. $2a^2b^2(3 - b)(9 + 3b + b^2)$
26. $(x + y)(x - y)(x^2 - xy + y^2)(x^2 + xy + y^2)$
27. $(x + 9)(x - 4)$

Exercises 5a. (cont'd.)

28. $m(x - 6)^2$
29. $(a^2 + 4)(a + 2)(a - 2)$
30. $(3 - 2x)(3 + 2x)(2 - x)(2 + x)$
31. $5b(4ab + 1)(4ab - 1)$
32. $(a^4 - 5c^5)(a^8 + 5a^4c^5 + 25c^{10})$
33. $(m + n)(x + y)$
34. $(b + c)(x - y)$
35. $(t + u)(t^2 - tu + u^2 - 5)$
36. $(x - y)(x + y - 1)$
37. $(2a - 1)(4a^2 + 2a + 1)$
38. $(6 + y)(4 - y)$
39. $(3r - s - 1)(3r + s + 1)$
40. $(w - 8)(w - 3)$
41. $(3c - 1)(3c + 1)$
42. $(a^2 + 1)(a + 2)(a - 2)$
43. $(y^2 - 5)(y + 2)$
44. $(a - 2)(a^2 + 2a + 4)(a + 1)(a^2 - a + 1)$
45. $(a - b)(x - y)(x + y)$
46. $(8 - a)^2$
47. $(c^2 + 5)(c^2 - 5)$
48. $(c + d)(c^2 + cd + d^2)$
49. $y(3r + 1)(9r^2 - 3r + 1)$
50. $a(b + y)(b - y)$
51. $(9x^2 - 4yz)(9x^2 + 4yz)$
52. $-3(y - 7)(y + 2)$
53. $(3a + 2b)(9a^2 - 6ab + 4b^2)$
54. $(x - 4)(x + 6y)$
55. $c(x^2 + 2)(x + 2)(x - 2)$
56. $(6m - a)(m - 12a)$
57. $3t(r + 4t)(r^2 - 4rt + 16t^2)$
58. $(w - 5b)(2w + 3)$
59. $8b(2a - b)(4a^2 + 2ab + b^2)$
60. $4x^2y(2x + 1)(x - 3)$
61. $(3a + 5)(a - 3)$
62. $(b + x)(x + y)(x - y)$

Exercises 5a. (cont'd.)

63. $(7z - 1)^2$ or $(1 - 7z)^2$
64. $(3 - 8x)(6 + x)$
65. $(v - 7m)(a - 2b + 2c)$
66. $(5k - 2w)(k + 6w)$
67. $(3x^2 - 5y^2)(2x - y)(2x + y)$
68. $(x - 3y)^3$
69. $(a^2 + 1)(2a - 5)(2a + 5)$
70. $(c - d - a + 2b)[(c - d)^2 + (c - d)(a - 2b) + (a - 2b)^2]$
71. $5(7x - 3y)(x^3 - 3y)$
72. $(x + y + a + 3b)[(x + y)^2 - (x + y)(a + 3b) + (a + 3b)^2]$
73. $(r - 5 - 3s + t)(r - 5 + 3s - t)$
74. $(x - 1)^2(x + 1)^2$
75. $(2x + y + 3)^2(2x + y - 3)^2$

Exercises 5b, page 32, 33.

1. $y = 4$ and $y = 3$
2. $x = -9$ and $y = 2$
3. $x = 5$ and $x = -2$
4. $x = 4$ and $x = -8$
5. $x = 1$
6. $x = 0$ and $x = 1$
7. $x = 0$ and $x = -1$
8. $x = 2$ and $x = -2$
9. $x = 3$ and $x = -3$

Exercises 5c, page 33.

1. $x^3 + x^2(2a + b) + x(2ab + a^2) + a^2b$

(a) $2a + b = -3$ $\qquad 2(-2) + 1 = -3$

$2ab + a^2 = 0$ $\qquad 2(-2)(1) + (-2)^2 = 0$

$a^2b = 4$ $\qquad (-2)^2(1) = 4$

If $a = \pm 1$; $b = 4$

If $a = \pm 2$; $b = 1$

$(x + a)^2(x + b) = (x - 2)^2(x + 1)$

Exercises 5c, (cont'd.)

(b) $2a + b = 5$ $\qquad 2(1) + 3 = 5$

$2ab + a^2 = 7$ $\qquad 2(3)(1) + (1)^2 = 7$

$a^2b = 3$ $\qquad (1)^2(3) = 3$

$\therefore a = \pm 1; \quad b = 3$

$(x + a)^2(x + b) = (x + 1)^2(x + 3)$

(c) $2a + b = 3$ $\qquad 2(1) + 1 = 3$

$2ab + a^2 = 3$ $\qquad 2(1)(1) + (1)^2 = 3$

$a^2b = 1$ $\qquad (1)^2(1) = 1$

$\therefore a = \pm 1; \quad b = 1$

$(x + a)^2(x + b) = (x + 1)^3$

2. $(x - 1)(x - 2)(x - 3) = 0$

$x^3 - 6x^2 + 11x - 6 = 0$.

Exercises 6a, page 35.

1. By the definition of division $\frac{a}{a}$ is a solution of the equation $ax = a$. However, we know Unit 1 is also a solution of $ax = a$. Therefore $\frac{a}{a}$ and 1 are each solutions of an equation which has a unique solution according to the division property. $\therefore \frac{a}{a} = 1$.

2. Both $\frac{a}{1}$ and a are solutions of the equation $1 \cdot x = a$ which has a unique solution according to the division property. $\therefore \frac{a}{1} = a$.

Exercises 6a, (cont'd.)

3. Again we involve the definition of division and the division property to show that each of the numbers $\frac{0}{a}$ and 0 satisfy the equation $ax = 0$. $\therefore\ \frac{0}{a} = 0$.

4. Each of the numbers $\frac{a}{-1}$ and $-a$ is a solution of the equation $(-1)\cdot x = a$. $\therefore\ \frac{a}{-1} = -a$.

5. Each of the numbers 2 and $\frac{2a}{a}$ is a solution of the equation $ax = 2a$ provided $a \neq 0$. $\therefore\ 2 = \frac{2a}{a}$.

Exercises 6b, page 37.

1. $\frac{1}{7}$
2. $\frac{a}{b}$
3. $\frac{3n}{4my}$
4. $\frac{3}{5}$
5. $\frac{1}{a - b}$
6. $\frac{a - 2}{2a + 5}$
7. $\frac{a + 1}{3a - 2}$
8. $\frac{3x + y}{x + 3y}$
9. $-\frac{a + b}{2a + b}$
10. $\frac{x + y}{m - n}$

Exercises 6c, page 39, 40.

1. $\frac{14}{15}$
2. 1
3. $\frac{15x}{4y}$
4. $\frac{b^2}{3a^2}$
5. $\frac{(x + y)(x - y)}{(x + 2y)(2x + y)}$
6. $\frac{x}{x^2 + xy + y^2}$
7. 1
8. $\frac{22}{15}$
9. $\frac{3}{2}$
10. $\frac{3x^2 + 2y^2}{4xy}$
11. $\frac{5a}{b}$
12. $\frac{x^2}{x - y}$
13. $\frac{5(4 - x)}{3x - 12}$
14. 0
15. $\frac{2(a - 1)}{(a^2 + 1)(a + 1)}$

Exercises 6d, page 42.

1. $\frac{3}{2}$

2. $\frac{25}{2}$

3. $\frac{14}{3}$

4. $\frac{y^2}{xz^2}$

5. $\frac{5 - x}{4}$

6. $-\frac{57}{5}$

7. $\frac{7}{8}$

8. $\frac{1}{bc}$

9. $\frac{1}{2}$

10. $\frac{3y^2 - 2}{5y}$

11. $\frac{20x}{(x + 5)(x - 5)}$

12. $-\frac{2(2x + 3)}{x(x + 2)}$

13. $\frac{12m}{m - 3}$

14. $\frac{-2(2x + 3)}{(x + 1)(x - 1)}$

15. $\frac{2x - 3}{x(2x - 1)(2x + 1)}$

Exercises 6e, page 44, 45.

1. $\frac{17}{20}$

2. $\frac{29x}{42}$

3. $\frac{20a + 7}{6a^2}$

4. $\frac{21}{40}$

5. $\frac{64b^3}{c}$

6. $-\frac{6x^2 + 13xy - 2y^2}{6xy}$

7. $\frac{x^2 - 13x - 12}{6x(x - 3)(x + 3)}$

8. $\frac{4}{m - 3n}$

9. $\frac{x + 1}{x^2 + 1}$

10. $\frac{a^3 + a^2 - 4a - 1}{(a + 1)(a - 1)(a + 2)(a - 2)}$

11. $-\frac{2b^2}{a + b}$

12. $\frac{1}{x + 1}$

13. $\frac{1}{4 - m^2}$

14. $\frac{5 - c}{a}$

15. $\frac{-2y(x + 3y)}{x - 2y}$

16. $\frac{1}{(x + y)^2}$

17. $\frac{12a^3 + 73a^2 + 34a - 11}{(4a - 1)(3a + 2)}$

18. y

19. $\frac{-3x - 2}{x^2(x + 1)}$

Exercises 6e, (cont'd.)

20. $-\frac{1}{(1-a)^2}$

21. $\frac{2(x-2)}{x}$

22. $\frac{1+a}{1-a}$

23. $\frac{(3x^2-12x-2)(x^2+4x+6)}{(x-3)^2\cdot(x+3)^2}$

24. $x + y$

Exercises 6f, page 45.

1. $x = \frac{29}{50}$

2. $x = \frac{5}{2}$

3. $x = \frac{5}{8}$

4. $x = -9$

5. $y = -\frac{3}{2}$

6. $x = \frac{9}{2}$ and $x = -3$

7. $x = -\frac{9}{5}$ and $x = 2$

8. $x = -\frac{2}{3}$ and $x = -\frac{10}{3}$

Exercises 6g, page 45.

1. $\frac{1}{(\frac{a}{b})} = \frac{b}{a}$

$(\frac{b}{a})\cdot(\frac{a}{b}) = 1$	Division property
$\frac{ba}{ab} = 1$	Theorem: $(\frac{a}{b})(\frac{c}{d}) = \frac{ac}{bd}$
$\frac{ab}{ab} = 1$	Commutative property
$(\frac{a}{a})(\frac{b}{b}) = 1$	Theorem: $(\frac{a}{b})(\frac{c}{d}) = \frac{ac}{bd}$
$a(\frac{1}{a})b(\frac{1}{b}) = 1$	Theorem: $\frac{b}{a} = b(\frac{1}{a})$
$1\cdot 1 = 1$	Multiplication inverse

2.

$\frac{a}{b} + -\frac{a}{b} = 0$	Definition: Additive inverse
$\frac{a + -a}{b} = \frac{a}{b} + \frac{-a}{b} = 0$	Theorem: $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$
$-\frac{a}{b} = \frac{-a}{b}$	Substitution

Exercises 6g, (cont'd.)

3. $\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + -\frac{c}{d}$ — Subtraction property

$= \frac{a}{b} + \frac{-c}{d}$ — Theorem: $-(\frac{a}{b}) = \frac{-a}{b}$

$= \frac{ad + (-c)(b)}{bd}$ — Theorem: $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

$= \frac{ad + -bc}{bd}$ — Commutative property and Theorem: $(-a)(b) = -ab$

$= \frac{ad - bc}{bd}$ — Theorem: $a + -b = a - b$

Exercises 7a, page 47.

1. $6 > -3$ [or $-3 < 6$]
2. $-2 > -5$
3. $-7 < 0$
4. $8 > 0$
5. $52.8 > -32.9$
6. $8.25 > 8.2$
7. $-0.1 < -0.01$
8. $\frac{7}{5} > \frac{4}{5}$
9. $-3 + 10 = 7$
10. $\frac{4}{45} < \frac{5}{45}$
11. $682(2.98 + 67.4)$
    $= 682(2.98) + 682(67.4)$
12. $(-1)\cdot(-1)\cdot(-1)\cdot > -3\cdot 2\cdot\frac{7}{10}$
13. $\frac{2}{3} + \frac{1}{2} > \frac{2}{3} \times \frac{1}{2}$
14. $-45 < -30$
15. $-45 > -30 + (-20)$
16. $-45 + 15 = -30$
17. $23 > 19$
18. $23 = 19 + 4$
19. $15 > -1$
20. $15 = -1 + 16$

Exercises 7b, page 49.

1. >
2. >
3. >
4. <
5. <
6. <
7. >
8. >
9. <
10. >

Exercises 7c, page 53.

1. $2x + 12 > 5x$

$12 > 3x$

$x < 4$

2. $5 < 3x - 1$

$3x > 6$

$x > 2$

3. $(x - 5)(x - 7) < 0$ means either $(x - 5) < 0$ or $(x - 7) < 0$ but not both

$\therefore$ if $x - 5 < 0$

$x < 5$

$x - 7 > 0$

$x > 7$

but it is impossible for x to be less than 5 and greater than 7.

if $x - 5 > 0$

$x - 7 < 0$

$x < 7$

$x > 5$

$5 < x < 7$

4. $x^2 - 6x + 9 \leq 4$

$x^2 - 6x + 5 \leq 0$

$(x - 5)(x - 1) \leq 0$

$(x - 5)(x - 1) = 0$

$x = 5; \quad x = 1$

$(x - 5)(x - 1) < 0$ means $x - 5 < 0$ and $x - 1 > 0$

$x < 5 \qquad x > 1$

$\therefore 1 \leq x \leq 5$

5. If $x - 5 > 0$

$x > 2$

If $x - 5 < 0$, $x < 5$ and $x < 2$ but in order to insure that both of these conditions are satisfied we must take $x < 5$

Exercises 7c, (cont'd.)

6. If $x > 0$, $2x + 5 \leq 6x$

$5 \leq 4x$

$x \geq \frac{5}{4}$

If $x = 0$ we have an undefined operation

If $x < 0$, $2x + 5 \geq 6x$

$x \leq \frac{5}{4}$

In order to satisfy both of these conditions at the same time we must take $x < 0$

$\therefore$ $x < 0$ or $x \geq \frac{5}{4}$.

7. If $x > 0$, $6x^2 + 13x < 28$

$6x^2 + 13x - 28 < 0$

$(2x + 7)(3x - 4) < 0$

and $2x + 7 > 0$

$2x > -7$

$x > -\frac{7}{2}$

while $3x - 4 < 0$

$3x < 4$

$x < \frac{4}{3}$

$0 < x < \frac{4}{3}$

If $x = 0$ we have an undefined operation.

If $x < 0$ $6x^2 + 13x > 28$

$(2x + 7)(3x - 4) > 0$

then $3x - 4 < 0$

$3x < 4$

$x < \frac{4}{3}$

Exercises 7c, (cont'd.)

7. (cont'd.) while $2x + 7 > 0$

$$2x > -7$$

$$x > -\frac{7}{2}$$

$$-\frac{7}{2} < x < 0$$

$\therefore$ $x \neq 0$ and $-\frac{7}{2} < x < \frac{4}{3}$

8. $x - 1 < x < x + 2$ for all x

$\therefore$ only $x - 1 < 0$ or all factors must be < 0

If $x - 1 < 0$ and $x > 0$

then $x < 1$

$x + 2 < 3$

$\therefore$ $0 < x < 1$

If $x + 2 < 0$ it follows that $x < 0$ and $x - 1 < 0$

so $x < -2$

$\therefore$ $x < -2$ or $0 < x < 1$

Exercises 7d, pages 53, 54.

1. $a > 1$

$a \cdot a > 1 \cdot a$ Multiplication property of order

$a^2 > a$

2. $a < 1$ and $a > 0$

$a \cdot a < 1 \cdot a$ Multiplication property of order

$a^2 < a$

3. $a < 0$

$a < 1$

$a \cdot a > a$ Multiplication property of order

$a^2 > a$ or $a < a^2$.

Exercises 7d, (cont'd.)

4. $a < b$

$a \cdot a < a \cdot b$ Multiplication property of order

$a \cdot b < b \cdot b$ Multiplication property of order

$a^2 < b^2$ Transitive property

5. Case I $a > 0$

$a \cdot a > 0 \cdot a$ Multiplication property of order

$a^2 > 0$

Case II $a < 0$

$a \cdot a > 0 \cdot a$ Multiplication property of order

$a^2 > 0$

6. See prob. 5 for $a > 0$ or $a < 0$

$a = 0$

$a \cdot a = 0 \cdot a$

$a^2 = 0$

$\therefore a^2 \geq 0$

7. $a^2 + b^2 = 0$

Case I $a = 0$

$0 \cdot 0 + b^2 = 0$

$b^2 = 0$

$b \cdot b = 0$

$b = 0$

Case II $a \neq 0$

$a^2 > 0$

$-a^2 < 0$

$b^2 = 0 - a^2$

$b^2 = -a^2 < 0$

Exercises 7d, (cont'd.)

7\. (cont'd.) but $b^2 > 0$ for all b

$\therefore$ a must $= 0$, $b = 0$

8\. $(a - b)^2 > 0$

$a^2 - 2ab + b^2 > 0$

$a^2 - 2ab + 2ab + b^2 > 2ab$ addition property for inequalities

$a^2 + b^2 > 2ab$

$ab < \frac{a^2 + b^2}{2}$ Multiplication property for inequalities

Exercises 8a, pages 58, 59.

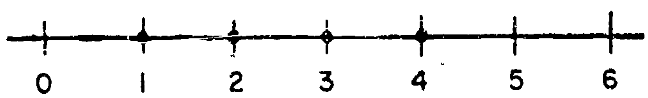
1\. $x^2 < 25$

$x^2 - 25 < 0$

$(x - 5)(x + 5) < 0$

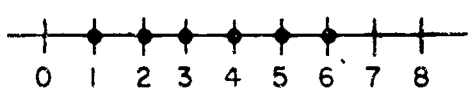
$x - 5 < 0$ and $x + 5 > 0$

$-5 < x < 5$

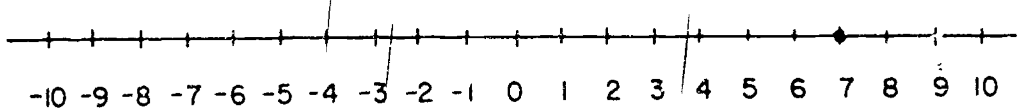


2\. $2x < 14$

$x < 7$

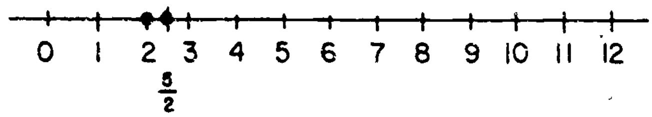


3\. $6 < x < 8$

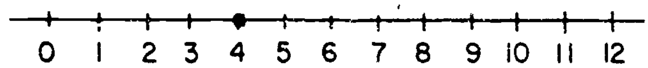


Exercises 8a, (cont'd.)

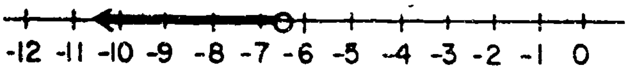
4. $x = \frac{5}{2}$ or $x = 2$



5. $x = 4$ or $x = -3$



6. $x < -\frac{13}{2}$

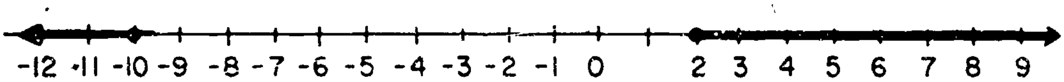


7. $\emptyset$

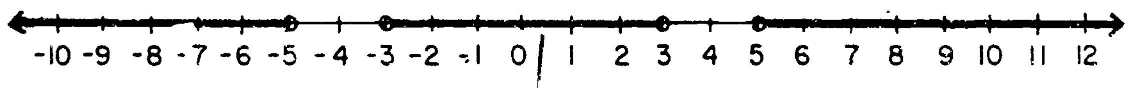
8. $(x + 10)(x - 2) \geq 0$

$x - 2 \geq 0$ or $x + 10 \leq 0$

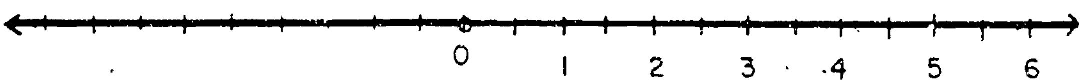
$x > 2$ $x \leq -10$



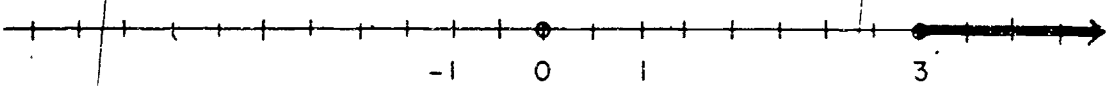
9. $\{x:(-3 < x < 3) \text{ or } (x < -5 \text{ or } x > 5)\}$



10. $\{x: \frac{x - 3}{x} < 1\} = \{x: x \text{ is in } R \text{ and } x \neq 0\}$



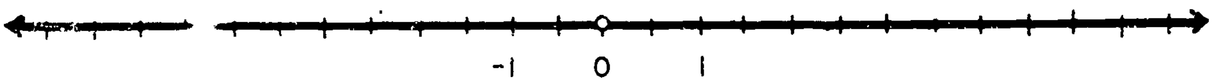
11. $\{x:x > \frac{8}{3}\}$



Exercises 8a, (cont'd.)

12. $\{x:(x = x \text{ or } x > x) \text{ and } (x \neq 0)\}$

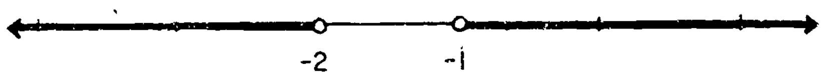
or $\{x: x \text{ is in } R \text{ and } x \neq 0\}$



(The number line except the point 0)

13. $(2x + 4)(2x + 2) > 0$

$2x + 2 > 0$	or	$2x + 4 < 0$
$2x > -2$		$2x < -4$
$x > -1$		$x < -2$



14. $(x + 4)^2 - 2^2 < 0$

If $x + 4 > 0$	$x + 4 < 0$
$x + 4 < 2$	$x + 4 > 2$

$(x + 4 + 2)(x + 4 - 2) < 0$

$(x + 6)(x + 2) < 0$

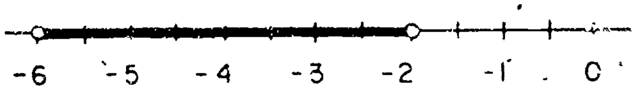
$x + 2 < 0$

$x < -2$

$x + 6 > 0$

$x > -6$

$-6 < x < -2$



15. $|x| < -2$

$\emptyset$

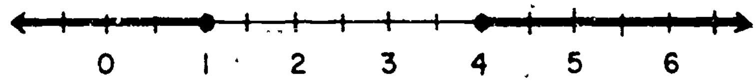
Exercises 8a, (cont'd.)

16. $(2x - 8)(2x - 2) \geq 0$

$(x - 4)(x - 1) \geq 0$

$x - 4 \geq 0$ $\qquad$ $x - 1 \leq 0$

$x \geq 4$ $\qquad$ $x \leq 1$



Exercises 8b, page 59.

1. If $a - b > 0$

$|a - b| = a - b$

and $b - a < 0$

$|b - a| = -(b - a)$

$= a - b$

$\therefore |a - b| = |b - a|$

If $a - b = 0$

$a = b$

and $b - a = 0$

$|a - b| = 0 = |b - a|$

If $a - b < 0$

$|a - b| = -(a - b)$

$= b - a$

and $b - a > 0$

$|b - a| = b - a$

$\therefore |a - b| = |b - a|$

2. If $a > 0; \ b > 0$

$|a| = a \quad |b| = b$

$|a||b| = a \cdot b = |ab|$

If $a = 0 \quad b = 0$

$|a| = 0 \quad |b| = 0$

$|a| \cdot |b| = 0 \cdot 0 = a \cdot b = |ab|$

If $a > 0 \quad b < 0$

$|a| = a \quad |b| = -b$

$|a||b| = a \cdot (-b) = -ab$

$-ab > 0$

$ab < 0$

$|ab| = -ab$

$\therefore |a||b| = |ab|$

If $a < 0$ and $b > 0$ the proof is similar to that just given.

Exercises 8b, (cont'd.)

3. If $a \geq 0$ and $b > 0$

$|a| = a; \qquad |b| = b$

$\frac{|a|}{|b|} = \frac{a}{b} = \left|\frac{a}{b}\right|$

If $a < 0$ and $b > 0$

$|a| = -a \qquad |b| = b$

$\frac{|a|}{|b|} = \frac{-a}{b}$

$-\frac{a}{b} > 0$

$\frac{a}{b} < 0$

$\left|\frac{a}{b}\right| = -\frac{a}{b}$

$\therefore \frac{|a|}{|b|} = \left|\frac{a}{b}\right|$

If $a < 0$ and $b < 0$

$|a| = -a \qquad |b| = -b$

$\frac{|a|}{|b|} = \frac{-a}{-b} = \frac{a}{b} = \left|\frac{a}{b}\right|$