DOCUMENT RESUME

ED 143 525 SE 023 003

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TITLE Introduction to Probability, Part 1 - Basic Concepts. Student Text. Revised Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 66
NOTE 155p.; For related document, see SE 023 004; Contains numerous light type

EDRS PRICE MF-$0.83 HC-$8.69 Plus Postage.
DESCRIPTORS Arithmetic; *Instructional Materials; Junior High Schools; *Mathematics; Number Concepts; *Probability; Secondary Grades; *Secondary School Mathematics; Statistics; *Textbooks
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This book is designed to introduce the reader to some fundamental ideas about probability. The mathematical theory of probability plays an increasingly important role in science, government, industry, business, and economics. An understanding of the basic concepts of probability is essential for the study of statistical methods that are widely used in the behavioral and social, as well as the biological and physical, sciences. The material in this book is written for students at approximately the seventh or eighth grade level. It is presumed that the student has an adequate background in arithmetic. No previous experience in probability is assumed. Parts of the text are in programmed form. Each chapter includes problems and exercises. Chapters include: (1) Introduction; (2) Finding Probabilities; (3) Counting Outcomes; (4) Estimating Probabilities; (5) The Probability of A or B; (6) The Probability of A and B; and (7) Conditional Probability. Discussion of problems and exercises are at the end of the text. (RH)

SCHOOL
MATHEMATICS
STUDY GROUP

INTRODUCTION TO PROBABILITY

Part 1 — Basic Concepts

Student Text

(Revised Edition)

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PREFACE

For Students and Teachers

Why Probability?

This book is designed to introduce the reader to some fundamental ideas about probability. The mathematical theory of probability plays an increasingly important role in science, government, industry, business, and economics. An understanding of the basic concepts of probability is essential for the study of statistical methods that are widely used in the behavioral and social, as well as the biological and physical, sciences.

Probability is a mathematical subject about which most people have intuitive ideas. It turns out that one's intuition cannot always be trusted, however. In any event, the problems presented are easily understood, and their solutions are obtainable with a minimum of advanced mathematical experience.

Required Mathematical Background.

The material in Part I is written for students at approximately the seventh or eighth grade level. It is presumed that the student has an adequate background in arithmetic. Wroking with problems in probability will provide excellent practice in handling fractions and decimals. No previous experience with probability is assumed. Thos students who have had an opportunity to study SMSG: PROBABILITY FOR INTERMEDIATE GRADES will, obviously, have a headstart on the ideas of this volume. For such students, Chapters 1 to 6 may be covered very rapidly.

Some familiarity with the language and notation of sets is pre-supposed. Students with no previous experience with this language and notation may need a brief introduction. In particular, Chapter 5 uses extensively the notions of $A \cup B$ and $A \cap B$.

Students who are more advanced will be able to proceed through Part I at a fairly rapid rate. They will find more challenging ideas in Part II.

In Part II there are various sections that require some experience with elementary ideas from algebra.

Outline of Content, Part I

Chapters 1 to 6 form a minimal set of topics. These topics are truly basic to any understanding of probability. Chapters 1, 2, and 3 introduce the notation and vocabulary that will be used. Here, too, ideas are presented which will be explored in more depth later in the text. A brief Chapter 4 points out the fact that many probabilistic judgments rely on experience or on experimental data.

In Chapter 5 we are concerned with the slightly more complicated problem of the probability of the union of two events. That is, if one event occurs with a certain probability and a second occurs with a certain probability, then what is the probability that one or the other occurs? It turns out that the answer to this question hinges on the probability that both occur.

For the special cases considered in Chapter 6, it develops that the probability that both events occur is easily found. It follows that, for these cases, one may also find the probability that one or the other occurs.

The treatment in Chapter 6 is somewhat informal. A more precise treatment of the probability of $A \cap B$ is given in Chapter 7. The material of this chapter is a bit more difficult. For some individuals and for some classes Chapters 1 to 6 might provide a reasonable introduction to probability. However, the notion of conditional probability (Chapter 7) is essential for much of Part II. Moreover, this topic is sufficiently interesting and important in itself that it should be included if at all possible.

For a discussion of the content of Part II, see the Preface to that volume.

Experiments

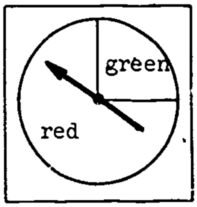
Experiments are used as an important part of the course. An attempt has been made to design experiments which may be carried out by a student working independently. If the text is used by a class of students, it is wise to combine the results obtained by several students (or groups of students). The experiments provide good experience in organizing data in an efficient and systematic way.

Moreover, if several students work together, data from many trials of an experiment can be obtained without undue expenditure of time. In some instances, the number of trials suggested in the text is insufficient to give any good indication of trends. Even doing an experiment a few times is valuable, of course, since it gives the student a better idea about a situation than a verbal description can provide. However, a large number of trials is very likely to yield, in addition, results which indicate long-run tendencies.

Primarily, the experiments are designed to: (a) provide the student with data for further examination and study; (b) lead the student to obtain the "feel" of probabilistic situations; and (c) provide opportunities for the student to "guess" and, perhaps, to formulate generalizations.

The equipment needed for these experiments is rudimentary. Coins, a deck of cards, dice, containers, and colored marbles (or disks) are traditional items for experiments in probability.

Extensive use has been made of spinners, which appear well suited for illustrating ideas in a simple and easily visualized way. Spinners can be made, purchased, or adapted from those that come with many children's games. For example, suppose a spinner with 10 equal regions, numbered 1 through 10, is available. It could be used, if necessary, where one which is $\frac{3}{4}$ red and $\frac{1}{4}$ green is called for. Simply call the 1 and 2 regions "green"; the 3, 4, 5, 6, 7, and 8 regions "red"; and ignore spins which result in 9 or 10.



How To Use the Program

Parts of the volume are in "programmed" form. This allows the text to be used for individual study or to be used at different rates by different class members. It is also possible to use the book as a regular text, supplementing the material by class discussion and using the exercises as out-of-class work. Unless the book is to be expended, each student should have available separate paper for recording responses and doing scratch work.

There are two types of programmed items in the text.

In some places there are items which contain blanks. The blank is to be filled in or the response recorded and then checked with the correct response, which is given at the left of the item.

For example:

11	1. The sum of 5 and 6 is ______.
$\frac{5}{6}$	2. $\frac{1}{3} + \frac{1}{2} = \frac{\square}{6}$.
$\neq$	3. $3 + 5 \underset{(=,\ \neq)}{\underline{\qquad}} 9$.

In Item 2, a box has been used to show that the numerator of the fraction is to be supplied. However, the entire fraction is recorded in the response column. This is because we want to show the complete answer, and not just a bit of it. In Item 3, the response is cued by the symbols written under the blank. Cueing of this sort is used to prevent a misunderstanding of what response is wanted.

It is helpful to cover the response column with a piece of paper or a card while the response is made. In any case, the response should be written before it is compared with the one provided. A wrong response should be corrected before proceeding.

There are also multiple-choice items, like Item 4. For each of these, indicate the letter of the response chosen. The choice should be verified by reading the discussion below the item.

4. In which of the following is there an error in addition?

[A] $5 + 7 + 13 = 25$.

[B] $\frac{2}{3} + \frac{5}{3} + \frac{8}{3} = 5$.

[C] $3.2 + 1 + .7 = 4.0$.

[A] and [B] contain no errors, but in [C] the sum is 4.9, which is apparent if the sum is written $3.2 + 1.0 + 0.7$. Therefore, you should have chosen [C] .

7

Each chapter contains sets of exercises. The exercises provide practice in applying the ideas that have been developed. Some of them are designed to produce deeper insight. Still others offer hints as to the material which will follow. The more difficult exercises are marked †. Section 7-5, in particular, contains three interesting and challenging problems. Section 3-4, while not difficult, is optional.

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Part I

PART I

Chapter 1

INTRODUCTION

1-1. Uncertainty

Some events are certain. If you go swimming, you are certain to get wet. If you select a boy from your class, you are certain to find that he is more than 5 years old.

Some events are not certain. We use words like "probable", "likely", "unlikely", in talking of them. For example, a weatherman makes a forecast of the future weather. His forecast, "Rain", is actually the statement, "It will probably rain." Similarly, you may predict that "The Green Shirts will win the pennant," but what you mean is; "I th[illegible] it is likely that the Green Shirts will win the pennant."

We often must make decisions about what to do in situations where we cannot be certain of what will happen. Very often these decisions have to be made by "weighing the pros and cons" and finally choosing one of two or more alternatives. The phrase "weighing the pros and cons" is used at this point for a special reason. Ordinarily when we weigh something we measure it -- we assign a numerical value to a characteristic of it which we call weight. When we "weigh the pros and cons" we are trying in our minds to give a numerical measure to the circumstances which are "for" one alternative and compare it with a numerical measure of those "against" the alternative. If we can assign <u>numerical</u> values to the pros and cons, we feel happier about our decision.

We are in a supermarket, have collected our groceries in a basket, and push the basket toward the cash registers. Which line do we pick? We try to make a numerical decision -- we count people, estimate the number of packages in their baskets, and then choose a line.

Here is another mathematical decision in the supermarket. The manager watching over the store sees the lines at the cash registers getting longer and longer. A voice over the intercom says -- "Clerk A to Gate 7, Clerk B to Gate 8." This manager may know from experience how long the lines should be before another man is sent to the cash registers. Notice -- he counts the customers. His decision is based on numbers.

Here is a set of paired statements. Which statement of each pair tells more?

1. (a) I think Bill is a better batter than Tom.
   (b) I think Bill is a better batter than Tom. Bill's batting average this year is .300 and Tom's is .190.
2. (a) I think homeroom 207 will beat homeroom 112 in today's game.
   (b) I think homeroom 207 will beat homeroom 112 in today's game. Homeroom 207 has won five of its seven games while 112 has won three out of seven.
3. (a) Weather forecast: rain tomorrow.
   (b) Weather forecast: 80% chance of rain tomorrow.
4. (a) Dr. A: "Try this remedy for your sunburn. It may help you."
   (b) Dr. B: "Try this remedy for your sunburn. It has helped 6 out of 7 patients who have tried it."

You have probably noticed that in each pair of statements the second would be more helpful, because it gives you more definite information. In each case, the additional information involves numerical measures of some sort. You should realize, however, that the numerical information given does not make the conclusion certain. Tom may have played all year with a sore arm, the better team does not always win, it may not rain, and the remedy may not work for you.

In probability we shall study systematic methods of weighing pros and cons. Although we cannot change an uncertain future to a certain one, we can sometimes compare likelihoods of various occurrences.

One of the objects of this course is to learn how to assign appropriate numerical measures to uncertain events. These measures will be called probabilities.

The study of probability has many practical uses. For example, federal and state governments use probability in setting up budget requirements; military experts use it in making decisions on defense tactics; scientists use it in research and study. Engineers use probability in designing and manufacturing reliable machines, planes, and satellites; business firms use it to help make difficult management decisions; it is the main tool of the insurance industry in deciding on premium rates and on size of benefits.

We will use as illustrations several examples of games of chance, employing such familiar objects as coins, dice, and playing cards. The examples have been chosen since they are fairly simple to understand. We all have some intuition about the "chance" of throwing a head when a coin is tossed. The practical situations indicated in the preceding paragraph are too complicated for the present, although we will mention some special problems from these fields of application.

It is of interest that, historically, the mathematical theory of probability arose from the consideration of gambling games.

1-2. Fair and Unfair Games

Suppose that we decide to play a game for two players in which the outcome depends upon chance, not skill. We agree that the game is fair if each player has an equal chance of winning. For the time being, we will assume that you have an intuitive idea of what we mean by "equal chance".

How may one decide whether a game is fair? Sometimes careful thinking about the rules will enable us to decide. Another possibility is to play the game many times, keeping track of the results. This may give us a "feel" as to whether the particular game is fair. We would expect that if a game is grossly unfair, several trials might indicate that fact. If a game is almost fair, it may take lengthy experimentation to discover that fact. For some of the games described in Section 1-3, you may wish to conduct, say, 20 trials to give you a clue as to whether they are fair.

1-3. Exercises

(Answers on page 125 .)

Here are some games to think about. For each game a rule is given which tells whether you or your opponent wins. If neither wins, the game counts as a tie. Decide whether each game is fair.

1. Here is a list of games played by two players with a die having six faces, numbered 1, 2, 3, 4, 5, 6 .
   (a) The die is tossed. You win if a 1 is thrown. Your opponent wins if a 3 is thrown.

(b) You win if an odd number is thrown and he wins if an even number is thrown.

(c) You win if 3 is thrown and he wins if a number greater than 3 is thrown.

2. These games are played by tossing a die with "1" on one face, "2" on two other faces, and "3" on the three remaining faces.

(a) You win if 3 is thrown. Your opponent wins if 1 is thrown.

(b) You win if 3 is thrown and he wins if any number less than 3 is thrown.

(c) You win if 2 is thrown and he wins otherwise.

3. Here are some games played with two ordinary dice, one red and one green. Both dice are thrown.

(a) You win if 1 is thrown on each die. Your opponent wins if 5 is thrown on each die.

(b) You win if there is an even number on the red die and he wins otherwise.

(c) You win if 6 shows on the red die and he wins if 4 shows on the green die.

(d) You win if 1 is on each die. He wins if one die has 1 and the other 2 .

(e) You win if the number on the red die is greater than the number on the green one. He wins otherwise.

Let us summarize. We have used the idea that a game played by two people (or teams) is fair if winning is as likely as losing. "Winning" means that particular events occur; "losing" means that <u>other</u> events occur. You cannot <u>both</u> win and lose.

Consider throwing an ordinary die.

is	1. Throwing a specified number ________ (is, is not) just as likely as throwing any other specified number. (Example: 6 is just as likely as 3 .)
are	2. Throwing an odd number and throwing an even number ________ (are, are not) equally likely events.
less	3. Throwing a specified number is ________ (more, less) likely than not throwing it.

Someone proposes that you play a game in which you win if a 4 is thrown with an ordinary die, and he wins if a number greater than 4 is thrown.

You say, "This game would be unfair, but let us change it to a fair game."

> 4. Which of the following are fair games?
>   [A] You win if the throw is either 4 or 5 . He wins if the number thrown is greater than 4 .
>   [B] You win if the throw is 4 or less. He wins if the throw is a number greater than 4 .
>   [C] You win if the throw is either 3 or 4 . He wins if the throw is a number greater than 4 .

If you chose [A] , you were probably thinking that throwing 4 or 5 is just as likely as throwing 5 or 6 . That is true, but suppose you throw 5 ? You would seem both to win and lose, which is impossible. The "rules" are contradictory. In game [B] , you are more likely to win than your opponent. (You might try it by playing it 20 times.) Hence this game is not fair. Game [C] <u>is</u> fair, because throwing 3 or 4 is just as likely as throwing 5 or 6 .

1-4. An Experiment: Throwing a Die

Throw a die and record which face is up. Repeat until you have made 100 trials of this experiment. For convenience in counting, record the numbers in blocks of five, with five blocks to a row. For example, the first row might look like this:

2 4 1 3 1   6 4 2 3 5   2 6 5 6 3   2 2 5 2 2   2 1 3 5 5

When you have recorded all 100 numbers, look at them carefully. Here are some questions to think about. They are discussed on page 119 .

1. How many 1's are there in the first row? How many 2's? For the first row, record the frequency of each number (that is, the number of times it occurs).

	Number on die face					
	1	2	3	4	5	6
Frequency, first row						
Frequency, second row						
Frequency, third row						
Frequency, fourth row						
Total						

2. Repeat for the second, third, and fourth rows.

3. Out of 100 throws of a die, what fraction do you expect will show 1 ? Show 2 ? Now record the frequency of 1, 2, 3, 4, 5, 6 from all of the 100 throws.

4. You may find one digit repeated several times in succession. Find the longest such run of a single digit. How many triples of like digits? How many pairs?

5. How many times do you find two successive digits in numerical order (e.g., 3 followed by 4)? Do you find any instances of three or more digits in order?

6. Examine your 100 throws by considering them as twenty groups of five. How many groups show three or more digits the same?

Save your results of this experiment. We will refer to them later in the course.

7. Here is the record of 25 throws of a die. Comment.

2 3 3 3 3 3 3 3 2 2 1 2 1 3 3 3 3 2 1 3 2 3 2 3 1

You have just done an experiment with a die. If you think about an honest die -- one which is not "loaded" -- you see no reason to favor one face over another. It seems reasonable to suppose, then, that approximately $\frac{1}{6}$ of the 100 throws, or about 17 throws, will result in a 5 , for example.

Suppose someone suggests a game which you think is fair. You play it 100 times. You win 45 times and lose 55 times. You would probably feel, quite rightly, that this is reasonable enough. You do not expect to win <u>exactly</u> 50 times.

1. After 100 plays of a game using a die you have lost 95 times. Which of the following conclusions is the most reasonable? [A] The game is fair. You have had a run of poor luck. [B] The game is fair. If it is played 100 more times, you will win most of them. [C] The game is not fair. The evidence of 95 losses out of 100 plays is convincing.

[B] is the only really incorrect choice. If the game <u>is</u> fair you should only expect to win about 50 of the next 100 plays. [A] is a possible conclusion, but as you study more about probability you will discover that such a run of bad luck in a fair game is extremely "improbable". We cannot <u>prove</u> that the game is unfair by conducting 100 trials but [C] is the most reasonable conclusion.

Chapter 2

FINDING PROBABILITIES

2-1. Experiments: Drawing a Marble (See p. 120 for discussion.)

1. Take 3 marbles, one red, one green, one yellow, all the same size. Place them in a jar or a box that you can't see through. Mix them. Draw one marble and record its color. Return the marble to the container. Repeat the experiment until you have drawn 12 times in all. Approximately how many times did you expect to draw each color?

2. Again use 3 marbles, but this time use 2 that are white and 1 that is black. Repeat the same process as in the first experiment, recording again the color of each draw. Did you expect approximately the same number of whites as blacks?

2-2. Assigning Probabilities; Some Preliminary Ideas

In the first experiment in Section 2-1, there is no reason to suppose that on any draw one color is more likely than another.

3	1. For each draw, then, we have _____ possible outcomes -- red, green, yellow.
are	2. These outcomes __________ (are, are not) equally likely.

As suggested in Chapter 1, we want to use numbers to express such ideas as "more likely", "equally likely".

Thus we are going to assign to each outcome a nonnegative number -- the probability of the outcome. We shall use the symbol

$$P(\text{red})$$

to stand for the probability of getting red on a draw. Notice that P(red) is the symbol for a number.

probability green	3. Similarly, P(green) is a number. It is the ________ of getting ________ on a draw.
P(yellow)	4. The probability of getting yellow on a draw will be written ________.

P(red), P(green), P(yellow) are all <u>numbers</u>. So far we have not said what these numbers are. However, if they are to fit with our ideas about what is likely, we can see that just any numbers will not do.

We want the probability of an outcome to be a measure of its likelihood, just as the weight of a block is a measure of its heaviness. It seems reasonable to require, therefore, that: <u>equally likely outcomes have equal probabilities</u>.

Since red, yellow, green are equally likely, we agree that P(red), P(yellow), P(green) should all be assigned the <u>same</u> number.

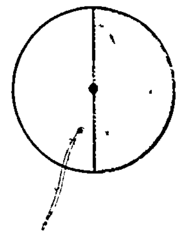
This does not tell us what P(red), P(yellow), P(green) are. Think about what numbers you would choose. Then read on.

It turns out (for reasons which you will see later) that we assign to each of the outcomes red, yellow, green the probability $\frac{1}{3}$. Thus

$$P(\text{red}) = \frac{1}{3}$$
$$P(\text{yellow}) = \frac{1}{3}$$
$$P(\text{green}) = \frac{1}{3}$$

3	5. That $P(\text{red}) = \frac{1}{3}$ may seem reasonable to you if you think of "red" as being one out of _____ equally likely outcomes.
$\frac{1}{3}$	6. That $P(\text{red}) = \frac{1}{3}$ is consistent with the fact that in the experiment we expected red about _____ of the time.
	As you go on you will see another reason that we decide that $P(\text{red}) = \frac{1}{3}$.
$\frac{1}{3}$, $\frac{1}{3}$	7. Of course the same ideas help justify the fact that P(green) = _____ and P(yellow) = _____.

Here is a spinner, with half its area red and half white.



$\frac{1}{2}$	8. By the same reasoning as above, P(red) = P(white) = _____.

Let us next consider finding probabilities of events connected with throwing an honest die.

six	9. A die is a cube and so it has _____ faces.
six	10. When we throw the die, there are _____ possible outcomes.
likely	11. We suppose that the die is honest. Hence, that it will show 4 , for instance, is one of six equally ________ possible outcomes.
$\frac{1}{6}$	12. The probability of each of the six outcomes is the number _____.
$\frac{1}{6}$	13. For example, P(4) = _____.

The simplest possible experiment in probability has to do with tossing a coin. Suppose we have a coin which is perfectly balanced, not weighted in any way. (Such a perfectly balanced coin is called an "honest" coin.) We toss it and let it fall freely.

tail	14. When the coin comes to rest, either a head or a _____ will show.

For an honest coin, the two outcomes are equally likely, and hence have equal probabilities.

Suppose you toss an honest coin five times and it shows a head each time. What is the probability that the coin will show a tail on the next toss? Some people believe that the probability changes from one toss to another. Not so! The probability that the coin will show a head remains $\frac{1}{2}$ for each toss. It is not true that if an honest coin shows a head on the first toss it is more likely to show a tail on the second toss.

$\frac{1}{6}$ 15. If an honest die has been thrown 25 times and has not shown a 6, the probability that it will show a 6 on the next throw is _____.

$\frac{1}{2}$, $\frac{1}{2}$ 16. If the 2-color spinner in Item 8 has shown red on ten spins, the probability of its showing red on the next spin is _____, and the probability of its showing white is _____. (We assume that the spinner is honest.)

Later you will learn more about situations in which you repeat the same action (like tossing a coin) many times.

Now let us think about the second experiment in Section 2-1. In this experiment, we have a box containing 3 marbles, all the same size, of which 2 are white and 1 is black. One marble is picked without looking into the box.

3 17. The likelihood of picking any one of the three marbles is the same as that of picking any other. We can think again of _________ (how many) equally likely possible outcomes.

$\frac{1}{3}$ 18. Choosing a black marble is one possible outcome, and just as before it is reasonable that

P(black) = _____.

19. What is the probability of drawing a white marble?

[A] $\frac{1}{2}$ [B] $\frac{2}{3}$ [C] You can't be sure.

Drawing white, our experiment suggests, is twice as likely as drawing black. So it is reasonable that if $P(\text{black}) = \frac{1}{3}$ then $P(\text{white})$ is $\frac{2}{3}$. Hence [B] is correct.

Notice that we can think about P(white) in another way.

$\frac{1}{3}$	20. We get white whenever we draw <u>either</u> of the white marbles. The probability of drawing one specific white marble is ____.
$\frac{1}{3}$	21. The probability of drawing the other is also ____.
$\frac{1}{3} + \frac{1}{3}$, or $\frac{2}{3}$	22. It seems reasonable that the probability of drawing one <u>or</u> the other is _______, or ____.

For any ordinary die, let us find the probability of throwing an even number.

4 6	23. We throw an even number whenever we throw 2, ____, or ____.
$\frac{1}{6}$	24. As we have seen (Item 12) $P(2) = P(4) = P(6) =$ ____.

Look back at your results in Experiment 1-5. Did you throw an even number about half the time? This is what you would expect, since you would expect to throw 2, 4, 6 each about $\frac{1}{6}$ of the time.

$\frac{1}{2}$	25. $P(2) + P(4) + P(6) =$ ____.
are	26. Note that on a throw of a die "odd" and "even" __________ equally likely. (are, are not)
$\frac{1}{2}$	27. Each has probability ____.
$\frac{2}{6}$, or $\frac{1}{3}$	28. For a die, the probability of throwing a number <u>less</u> than 3 is ____.

2	29. This is clear because the only numbers less than 3 are 1 and ___.
P(1) + P(2)	30. By the reasoning above: P(less than 3) = P( ) + P( ) .

2-3. Events that are Certain to Occur

Again, let us consider a box with 3 marbles, but this time let us suppose that all are white. If you draw once, what is the probability of drawing a white marble? Think about this, and then read on.

Drawing a white marble, in this case, is certain. If you use the ideas illustrated in Section 2-2, you recognize that it is reasonable to say: An event which is certain has probability 1 .

If you draw a marble from a box containing only white marbles you get a white marble all (100%) of the time. In this case we say:

$$P(\text{white}) = 1$$

1	1. When you throw a single die, the probability of getting less than 7 is ___.

We have said that the probability of an event is a measure of its likelihood. You have used the idea of measurement in other situations -- in working with lengths, areas, volumes, etc.

Just as in the case of other kinds of measurement, we need to choose some sort of unit of measure. We do this when we decide, once and for all: if an event is certain to occur then its probability is 1 .

6	2. Recall that when a die is thrown there are ___ possible outcomes. All are equally likely. Hence $P(1) = P(2) = P(3) = P(4) = P(5) = P(6)$. Suppose we had not already decided what number P(1) , for example, should be.

We could reason:

P(number less than 7) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6)

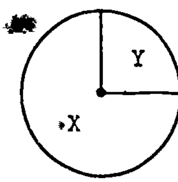
(Look back at Items 22, 25, 30, of Section 2-2, if you don't understand.)

3. Thus by adding 6 equal numbers we get 1 ,

1 since P(number less than 7) = ____.

$\frac{1}{6}$ 4. We see: P(1), P(2), etc., must all be ____, because P(number less than 7) is 1 .

Here is another spinner. Its entire area is white. The X region has an area 3 times that of the Y region.



3 5. P(X) is ____ times P(Y) .

1, certain 6. P(white) = ____, since you are ____ to spin white.

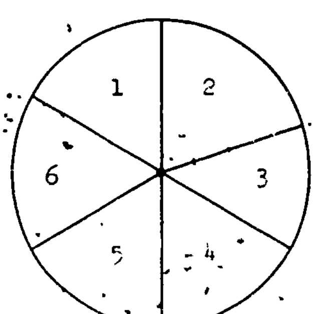
7. It seems reasonable to say that:

$\frac{1}{4}$ P(Y) = ____,

$\frac{3}{4}$ P(X) = ____.

1 8. Then P(X) + P(Y) = ____, which is consistent with Item 6.

Here is still another spinner. Regions 1, 4, 5, 6 all have equal areas, and each of these regions is $\frac{1}{6}$ of the circle.



	We are told that $P(2) = \frac{1}{15}$.
$\frac{1}{5}$	9. We expect, then, that region 2 covers _____ of the circle.
$\frac{2}{15}$	10. What is $P(3)$? _____

If you had trouble with Item 10, complete Items 11 to 14. If not, you may omit them.

$\frac{13}{15}$	11. All together, regions 1, 2, 4, 5, 6 cover _____ of the circle. $[\frac{1}{6} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = ?]$
$\frac{2}{15}$	12. There remains _____ of the circle for region 3.
$\frac{2}{15}$	13. Hence $P(3) =$ _____.
1	14. We might also think: The probabilities of all the outcomes must add up to _____. Hence $\frac{13}{15} + P(3) = 1$, $P(3) = 1$ _____.
$1 - \frac{13}{15}$	

Thinking again of the box with 3 white marbles, we ask: What is the probability of drawing a black marble?

15. The probability is:

[A] 1

[B] $\frac{1}{2}$

[C] 0

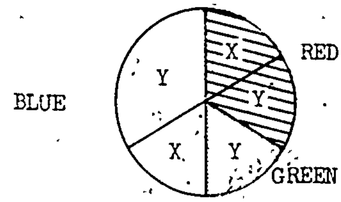
You can't draw a black marble from this box. It seems reasonable to say, therefore, that the probability of drawing a black marble is 0, so [C] is correct. You can think: If you draw a marble from this box, replace it, draw again -- repeating many times -- you will get a black marble 0 times.

Consideration of Item 15 leads us to make another agreement: an event which cannot occur has probability 0 .

0	16. The probability that your birthday is on February 31 is ____.
0	17. If you throw an ordinary die, the probability of getting 7 is ____.

2-4. Outcomes of an Experiment

Let us suppose we have a spinner marked as shown:



The red X , red Y , green Y , and blue X regions all have the same area. The blue Y region has area twice that of the blue X .

blue Y	1. If you spin this spinner, you are certain to get one of the following: red X , red Y , green Y , blue X , or ________.
blue Y	2. Not all are equally likely. In fact, the most likely of them is ________.

Suppose you are playing a game with the spinner above. You win if red is spun, lose if green is spun, and tie if blue is spun.

3. Is this a fair game?

[A] Yes [B] No

In this game you have an advantage, so [B] is correct. Spinning red means spinning red X or red Y . Spinning red X is as likely as green Y , since the red X and green Y regions have the same area. Spinning red Y is also as likely as green Y . You would expect to win about twice as many times as you would lose if you played this game many times.

In analyzing complicated situations it is useful to have some special terms.

Let us consider the spinner above. Spinning once is an example of an experiment.

Before we perform the experiment, we can give a set of possible outcomes, or simple events, that can result: {red X , red Y , green Y , blue X , blue Y} .

You may ask, "What if the spinner stops on a line?" True, this might be listed as one possible outcome. We will always suppose, however, that if the spinner stops on a line we spin again until this does not happen.

When you perform the experiment, using the rules set up for it, you are certain to get exactly one outcome out of the set of possible outcomes.

	Here are other examples.
	4. Experiment: Toss a coin.
tail	Possible outcomes: head, _____.
	5. Experiment: Throw a die.
5, 6	Possible outcomes: 1, 2, 3, 4, ___, ___.

Let us suppose, now, that we have decided to use, as a set of possible outcomes for the spinner, {red X , red Y , green Y , blue X , blue Y} . Suppose we are interested, as in the game above, in the likelihood of spinning red. Red is not one of our outcomes listed. We have red whenever we have either outcome red X or outcome red Y .

We shall speak of "red" in this case as an event. The event "red" occurs whenever outcome red X or outcome red Y occurs. We thus identify the event "red" with the set of outcomes {red X , red Y} .

blue Y, 2	6. Similarly, the event "blue" is the set {blue X , _____} , which contains _____ members.
{red X , blue X}	7. The event X is {_____ , _____} .

event

> 8. Recall that an outcome is sometimes called a <u>simple</u> ______. All other events are built up out of simple events.

Look back once again at the spinner at the beginning of this section.

$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{3}$

> The probabilities of the outcomes are:
>
> 9. P(red X) = ______.
> 10. P(red Y) = ______.
> 11. P(green Y) = ______.
> 12. P(blue X) = ______.
> 13. P(blue Y) = ______.

Notice that in this example the outcomes are <u>not</u> all equally likely.

$\frac{1}{6} + \frac{1}{6}$, or $\frac{1}{3}$

$\frac{1}{6} + \frac{1}{6} + \frac{1}{3}$, or $\frac{2}{3}$

> Let us find the probabilities of some events for the same spinner.
>
> 14. P(red) = ______. Note that event red is {red X , red Y} .
> 15. P(Y) = ______. Note that event Y is {red Y , green Y , blue Y} .

As we see: <u>An event is a set of outcomes</u>. <u>The probability of an event is the sum of the probabilities of the outcomes in it</u>.

When we spin this spinner, we are <u>certain</u> to get a letter between W and Z .

1

> 16. This is consistent with the fact that if we add the probabilities of all the outcomes in the set {red X , red Y , green Y , blue X , blue Y} we get ______.

You might ask: Suppose we are only interested in the color that we spin. Can we think of the set {red , green , blue} as a set of outcomes of the experiment "spin this spinner"? We can, because on a spin we are sure to get exactly <u>one</u> of them. On each spin, you get exactly <u>one</u> color.

17. With the same spinner can you use {red, blue, Y} as a set of outcomes? [A] Yes [B] No

This set cannot be used. If you spin red Y, you have both red and Y. You should have answered [B].

18. Can you use {X,Y} as a set of outcomes for the experiment of spinning the spinner? [A] Yes [B] No

Since every spin gives either X or Y but not both, this is a possible set of outcomes. [A] is correct.

In our examples, the sets of possible outcomes have had only a few numbers. We will soon see some more complicated situations in which there are many possible outcomes. For some experiments, the set of outcomes is infinite. In this text we will consider only situations in which we can use finite sets of outcomes.

2-5. Assigning Probabilities

As we have seen, we can analyze an experiment in the following way. We choose a set of outcomes for the experiment.

one

1. Remember, the set of outcomes must be such that the experiment is sure to result in exactly _____ of them.

Each event is a set of outcomes.

The probability of an outcome is a real number between 0 and 1. We have observed the following properties of the probabilities we have assigned.

1. Equally likely outcomes have equal probabilities.

2. If an event is certain, its probability is 1.

Knowing the probability of each member of the set of outcomes, we can find the probability of an event.

The probability of an event is the sum of the probabilities of the outcomes in it.

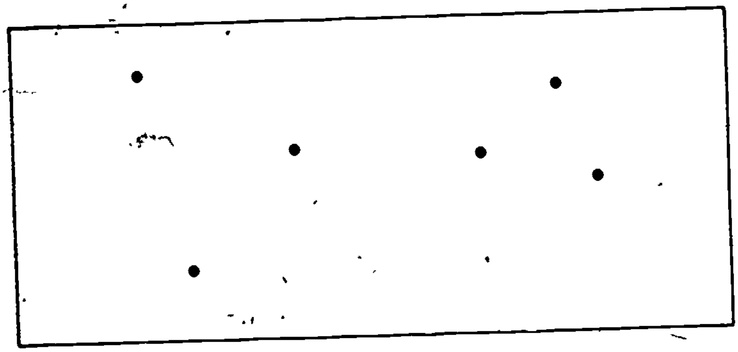
certain	2. The set of all outcomes is the event "something happens". This event is ____.
1	3. Its probability is ____.
1	4. Consequently, the sum of the probabilities of all the members of the set of outcomes is ____.

Here is an example of the way these ideas can be used.

	From a regular deck of playing cards, one card is drawn. What is the probability of drawing an ace? (Assume the cards are well-shuffled, so that any card is just as likely to be drawn as any other.)
52	5. If we regard drawing each card as a different outcome, then the set of outcomes has ____ members.
equal	6. All the outcomes are equally likely. Hence their probabilities are ______.
1	7. The sum of the probabilities of the outcomes is ____.
$\frac{1}{52}$	8. Hence each outcome has probability ____.
4	9. The event "ace" contains ______ (how many) outcomes.
$\frac{4}{52}$, or $\frac{1}{13}$	10. Hence P(ace) = ______.

There is no single simple rule for deciding on the set of outcomes and assigning probabilities to them. Practice and experience will help improve your skill.

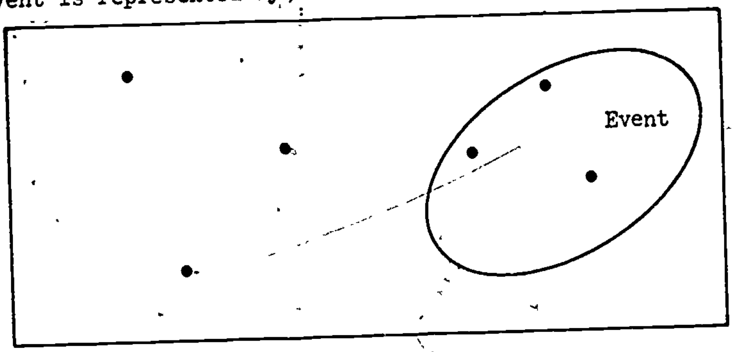
It is easy, and sometimes very helpful, to think in terms of diagrams. We can think of an experiment as represented by a set of dots. Each dot stands for an outcome.



Outcomes of an Experiment

Each dot represents a member of the set of outcomes.

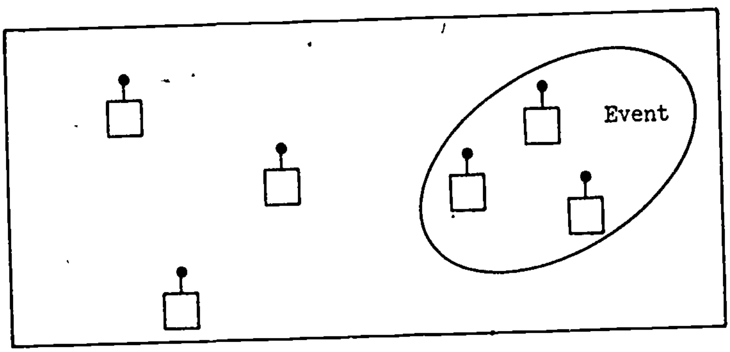
An event is represented by a set of dots.



An Event

An event is a set of outcomes, as shown.

We assigned a probability -- a measure of likelihood -- to each outcome.

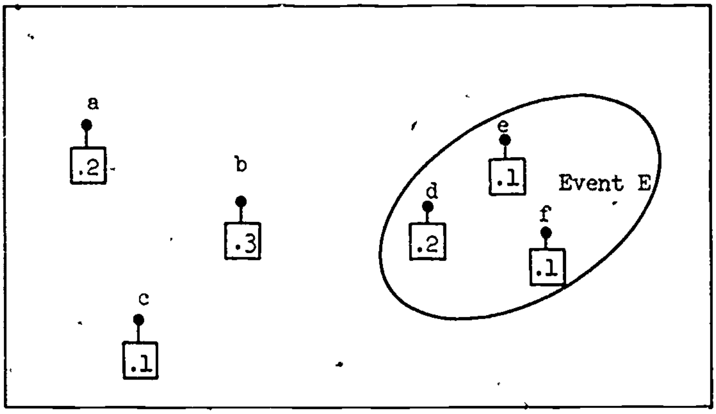


Assigning Probabilities

When we assign probabilities to the outcomes, we are -- so to speak -- attaching a weight to each outcome.

sum

11. The probability of an event is the _____ of the probabilities of the outcomes in the event.



In this diagram event E is the set {d,e,f}. The probability of each outcome is shown.

.1 + .2 + .1 , or .4

12. P(E) = __________.

What is the probability that event E will <u>not</u> occur?

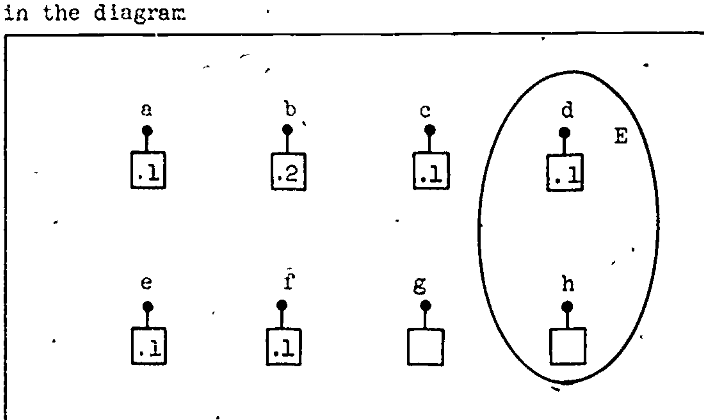
{a,b,c}

13. The event, in this case, is {a, , }.

.2 + .3 + .1 , or 1 - .4, or .6

14. Its probability is __________.

Suppose that for a certain experiment the probabilities of some of the 8 outcomes are known. They are shown in the diagram.



.2

15. If $P(E) = .3$, then we may be sure that $P(h) =$ _____.

.1

16. In this case, $P(g) =$ _____. [Hint: What is the sum of the probabilities of the outcomes?]

For the diagram used in Items 15 and 16, $\{a,b,f\}$ is a subset of the set of outcomes, and so it is an event.

$.1 + .2 + .1$, or $.4$

17. The probability of this event is __________.

18. The empty set -- sometimes written $\emptyset$ -- is a subset of the set of outcomes. Hence it can be regarded as an event. The probability of this event is:

    [A] 0

    [B] You can't tell

The event cannot occur. Hence its probability is 0. We can think: There are <u>no</u> outcomes in the event; hence the sum of the weights is 0.

[A] is the correct response.

2-6. <u>Exercises</u>

(Answers on page 126.)

1. Two black marbles and one white marble are in a box. Without looking inside the box, you are to take out one marble. Find the probability that the marble will be black.

2. For the box in Problem 1, find the probability that the marble drawn will be white.

3. Suppose you toss an honest coin 10 times.
   (a) Are you likely to get a head each time?
   (b) What is the probability that the coin will show a tail on the tenth toss?
   (c) Does the outcome of the first 9 tosses have any effect on the outcome of the tenth toss?

4. There are 25 students in a class, of whom 10 are girls and 15 are boys. The teacher has written the name of each pupil on a separate card. If the cards are shuffled and one is drawn, what is the probability that the name written on the card is:

   (a) the name of a boy?

   (b) your name (assuming you are in the class)?

5. Suppose a box contains 48 marbles. Eight are black and 40 are white. Find the probability that a marble picked without looking in the box will be white.

6. Using the box in Problem 5, consider this event: Nine marbles are taken out simultaneously and all are black.

   (a) Is this event possible?

   (b) What is the probability of this event?

7. A whole number from 1 to 30 (including 1 and 30) is selected at random; that is, the selection is made so that one number is just as likely to be chosen as any other. What is the probability that the number selected will be a prime number?

8. Three hats are in a dark closet. Two belong to Mr. Smith and the other to his friend. Mr. Smith reaches in the closet and draws two hats. What is the probability that he will pick his friend's hat and one of his own?

9. Suppose you have five cards: the ten, jack, queen, king, and ace of hearts. You draw them, one at a time, at random.

   (a) What is the probability that the first card you draw is the ace?

   (b) Assume that you draw the jack on the first draw, and put it aside. What is the probability that the second card you draw is the ace?

   (c) Are your answers for (a) and (b) the same? Why?

   (d) After drawing the jack, and putting it aside, assume that the second card you draw is the ten. Put that aside also. What is the probability that the third card you draw is the ace?

   (e) What is true of the probabilities in (a), (b), and (c).

Chapter 3

COUNTING OUTCOMES

3-1. Experiments: Tossing One and Two Coins

1. Toss a coin. Repeat until you have tossed 10 times. Record the number of heads and tails. How many times did you have heads? Record your result on each throw.

2. Toss 2 coins, a penny and a nickel. Again use 10 trials. How many times did you have 0 heads? Two heads? One head and 1 tail?

3. Would you expect the number of heads in 10 tosses of a single coin to be greater than the number of "two-head" throws in 10 tosses of 2 coins? Explain why.

These experiments are discussed on page 121 .

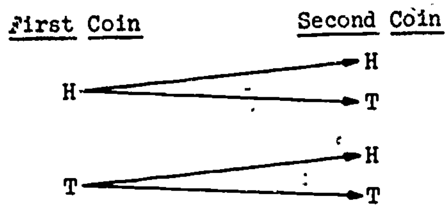
3-2. Listing Outcomes: Tree Diagrams

In determining probability, we often have had occasion to list all of the possible outcomes of an experiment.

For example, if we toss a single coin, there are exactly two possible outcomes: heads, which we might designate by H, and tails, which might be designated by T . If we toss two coins, we have the four possible outcomes shown in the following table:

First Coin	Second Coin
H	H
H	T
T	H
T	T

As the number of outcomes increases, keeping track of the possible outcomes is more difficult. One useful way of listing them is by means of a "tree" diagram, as pictured below:



2 1. How many possibilities are there for 1 coin? ______

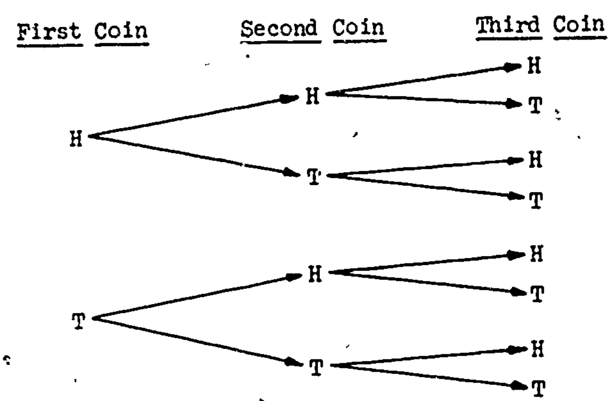
2 2. If 2 coins are tossed, then for each possibility of the first coin there are ______ possibilities for the second coin.

tails 3. If the first coin falls heads, the second coin might fall either heads or __________.

TH, TT 4. The outcomes, which can be seen by reading from left to right along the branches, are: HH, HT, ______, ______.

4 5. The number of outcomes is found by counting the ends of the branches on the right. Thus, for tossing 2 coins there are ______ possible outcomes.

If a third coin is added, the number of possibilities is doubled again, as is seen in this diagram:



4	6. For 2 coins, the total number of possible results is ______.
2	7. For each possible result of 2 coins there are ____ branches for the third coin.
8	8. There are _____ possibilities for 3 coins.
2	9. Thus the number of possibilities for 3 coins is ______ times the number of possibilities for 2 coins,
2	10. and we expect the number of possibilities for 4 coins to be _____ times the number of possibilities for 3 coins.
multiplied	11. Each time 1 more coin is added, the number of possible outcomes is ______________ by 2.

Refer to the tree diagram for tossing 3 coins to help you answer the following:

8	12. If 3 coins are tossed, how many possible outcomes are there? ________
THH	13. Of these, which outcomes have exactly 2 heads? HHT, HTH, and _______. These outcomes are equally likely.
$\frac{3}{8}$	14. If E is the event "exactly 2 heads", the probability of E is _______. (Hint: There are 8 possible outcomes, of which 3 are in the event.)
4	15. How many outcomes of tossing 3 coins have <u>at least 2</u> heads? _________
$\frac{4}{8}$, or $\frac{1}{2}$	16. If F is the event "at least 2 heads", then $P(F) =$ __________.
	If 3 coins are tossed, the probability of:
$\frac{3}{8}$	17. exactly one head is _________;
$\frac{7}{8}$	18. at least one head is _________;
$\frac{1}{8}$	19. no heads is __________;
$\frac{1}{8}$	20. three heads is __________.

We have found that, in coin-tossing, each addition of a coin to the experiment multiplies the number of outcomes by 2. Hence we can write the following table:

Number of Coins	Number of Outcomes
1	2
2	$2 \cdot 2 = 2^2 = 4$
3	$2 \cdot 2 \cdot 2 = 2^3 = 8$
4	$2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$
$\vdots$	$\vdots$
10	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10} = 1024$
$\vdots$	$\vdots$
n	$\underbrace{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}_{n \text{ factors}} = 2^n$, where n is the number of coins

If T is the total number of possibilities, and n is the number of coins, then we can write

$$T = 2^n .$$

You tossed a coin 10 tim (Experiment 1, Section 3-1). The authors also tossed a coin 10 times. What is the probability that your record and ours match toss for toss?

Our result -- T H T H T T T H H H -- was one possible outcome of tossing 10 coins.

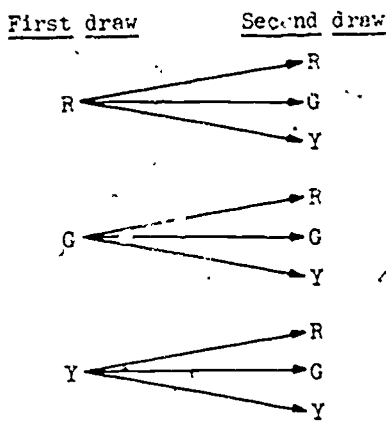
2^{10}, or 1024	21. In all, there are ________ equally likely outcomes for the experiment of tossing a coin 10 times.
$\frac{1}{2^{10}}$, or $\frac{1}{1024}$	22. The probability that you get exactly our result is the probability of one of these outcomes. Thus the probability is ________ . This is less than .001 .

3-3. More About Tree Diagrams

We will think next of an experiment in which we have 3 possibilities for the first step.

Suppose that we have a box containing 3 marbles: 1 red, 1 green, and 1 yellow. If one marble is picked at random, there are 3 possibilities. We shall call them R, G, and Y, for red, green, and yellow.

If the marble is returned to the box, and again a marble is selected at random, we have 3 possibilities for the second draw also. The outcomes of the succession of 2 draws can be described in terms of "color on first draw and color on second draw". They are shown in this tree diagram:



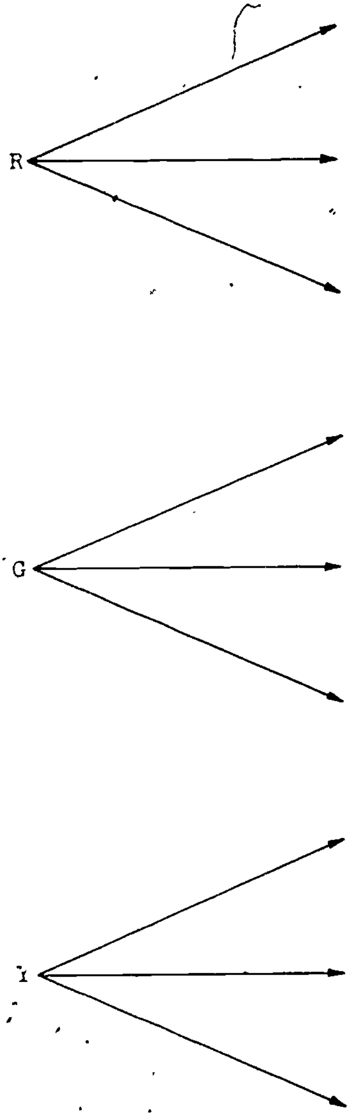
3 1. On the first draw there are ______ possible outcomes.

3 2. For each possibility on the first draw there are ______ possibilities on the second draw.

9 3. The total number of possible outcomes for the 2 draws is ______.

Exercise: [illegible] tree diagram for [illegible] draws of the marbles, assuming that [illegible] returned to the box before the next draw is made.

First draw   Second draw   Third draw



Compare your diagram with the one on page 127 and make any necessary corrections.

Use the tree diagram for picking a marble 3 times to help you answer the following:

27	4. On 3 draws, how many possible outcomes are there? ________
6	5. In how many outcomes is the red marble picked exactly twice? ________
$\frac{6}{27}$, or $\frac{2}{9}$	6. If E is the event "red exactly twice", then $P(E) =$ ________.
7	7. In how many outcomes is the green marble picked at least twice? ________
$\frac{7}{27}$	8. If F is the event "green at least twice", then $P(F) =$ ________.
	On 3 draws, the probability of:
$\frac{3}{27}$, or $\frac{1}{9}$	9. event A, "all 3 the same color", is ________;
$\frac{18}{27}$, or $\frac{2}{3}$	10. event B, "two of one color, one of another", is ________;
$\frac{6}{27}$, or $\frac{2}{9}$	11. event C, "no two the same color", is ________.
1	12. $P(A) + P(B) + P(C) =$ ________.

Now we will think about the number of outcomes when we throw dice.

6,	13. A die is a cube and has 6 faces. If one die is thrown the number of possible outcomes is ________.
6	14. If we throw two dice, the number of possible outcomes in Item 13 is multiplied by ________.
36	15. For two dice the number of possible outcomes is ______.
6^3 216	16. For three dice, the number of outcomes is $6^{\square}$, or ________.
6^n	17. If you had n dice, the number of outcomes would be $\square^n$.

Suppose we have an object such that on a single toss there are s possible outcomes.

	18. If we toss two such objects, or if we toss the one object twice, the number of possible outcomes is multiplied by ________.
s	
s^2	19. For two objects, or for two tosses, we have $s \cdot s$, or $s^{\square}$ possible outcomes.
s	20. Each added object of the same sort, or each additional toss of the single object, multiplies the number of possible outcomes by ________.
$T = s^n$	21. If T is the total number of possible outcomes, and s is the number of possible outcomes for one object, and n is the number of objects (or of trials of a single object), then T = ________.

Let us think of throwing two dice, one red and one green. A convenient way to indicate the outcome " 5 on the red and 2 on the green" is "(5,2)". In a similar way, we can express each outcome as a pair of numbers in which the first is the number on the red die and the second is the number on the green die.

Exercise: Complete the following table showing all of the possible outcomes of rolling the two dice.

Green

Red	1	2	3	4	5	6
1	(1,1)	(1,2)	____	____	____	____
2	(2,1)	____	____	____	____	____
3	____	____	____	____	(3,5)	____
4	____	____	(4,3)	____	____	____
5	____	____	____	____	____	(5,6)
6	____	(6,2)	____	____	____	____

Compare your completed table with that shown on page 128 before you continue.

	Suppose that 2 dice are thrown. Refer to the table to help you with the following:	
36	22. The number of possible outcomes is ________.	
equally	23. All of the outcomes are ________ likely.	
(4,2),(5,1)	24. If E is the event "the sum of the numbers is 6", then the outcomes in E are (1,5), (2,4), (3,3), ( ), ( ).	
5	25. The number of these outcomes is ________.	
$\frac{5}{36}$	26. Hence, $P(E) =$ ________.	
$\frac{4}{36}$	27. If F is the event "the sum of the numbers is 9", then $P(F) = \frac{\square}{36}$.	

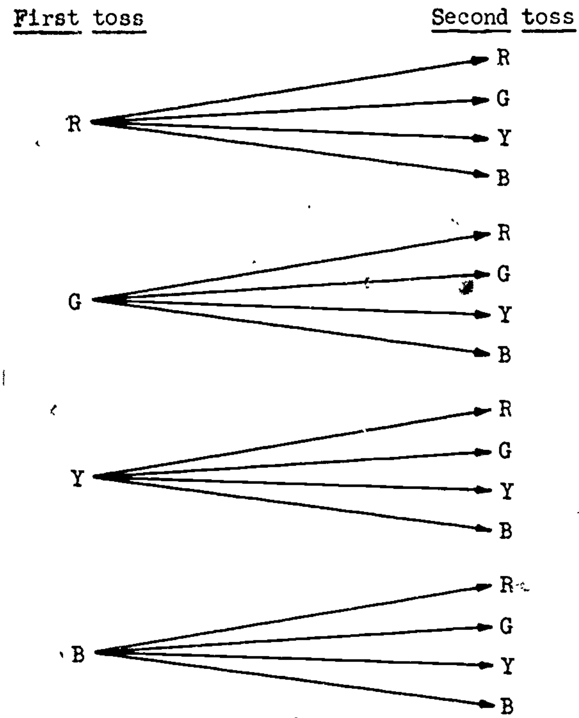
3-4. Tossing a Tetrahedron (optional)

(If you would like to examine another situation very similar to those above, complete this section. If not, omit it.)

On page 42 is a pattern for making a regular tetrahedron. Cut the figure out of stiff paper (or cardboard), color each triangle as indicated, fold on the dotted lines, and fasten the tabs with glue, paste, or scotch tape.

When the tetrahedron is tossed into the air and allowed to fall freely, we describe the outcome by the color of the face on which the tetrahedron rests.

Suppose we make a tree diagram to show the number of possible outcomes for 2 tosses in succession. If we designate red, green, yellow, and blue by R, G, Y, and B, respectively, the diagram looks like this:



4 1. On the first toss there are ________ possible outcomes.

4 2. For each possibility on the first toss, there are ________ possibilities on the second toss.

16 3. The total number of possible outcomes of two tosses is ________.

Make a tree diagram showing the possible outcomes for 3 tosses of the tetrahedron. Compare it with that indicated on page 42 and make any necessary corrections. Use it when you need help in answering the following questions.

4 4. For a single toss of the tetrahedron there are ________ possible outcomes.

4 5. On the second toss there are ________ possibilities for each of the 4 possibilities of the first toss.

16 6. For 2 tosses, the number of possible outcomes is ________.

4	7. Suppose we wish to find the number of outcomes for 3 tosses. The number of outcomes for 2 tosses must be multiplied by ________.
64	8. For 3 tosses, the number of outcomes is ______.
256	9. For 4 tosses, the number of outcomes is ______.
4^n	10. For n tosses, the number of outcomes is $4^{\square}$.
1	11. On 3 tosses, how many outcomes have 3 blues? _____.

Assume now that the outcomes of the toss of the tetrahedron are all equally likely.

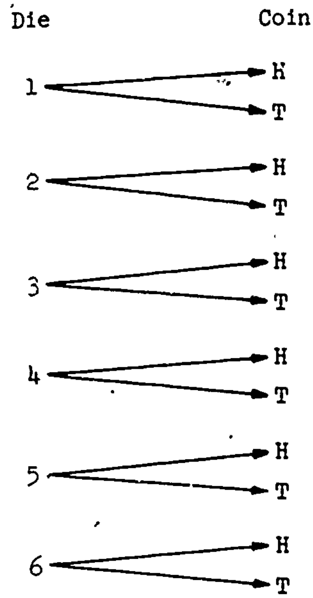
$\frac{1}{64}$	12. The probability of getting 3 blues on 3 tosses is ________.
	If the tetrahedron is tossed 3 times, the probability of:
$\frac{3}{64}$	13. two green and 1 red is ________;
$\frac{9}{64}$	14. two green and 1 not green is ________;
$\frac{36}{64}$, or $\frac{9}{16}$	15. two of one color and one of another is ________.

3-5. Further Example

A die and a coin. Suppose we toss a die and a coin. Again a tree diagram can be used to count outcomes.

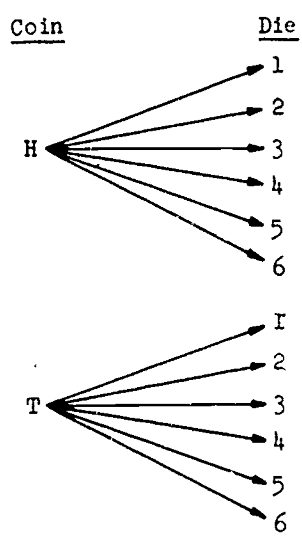
6	1. For the die, we have ______ possibilities.
2	2. For each possibility of the die, there are ______ possibilities for the coin.

Draw the tree. Compare with the diagram below.



12	3. For tossing a die and a coin, there are ______ possible outcomes, all equally likely.

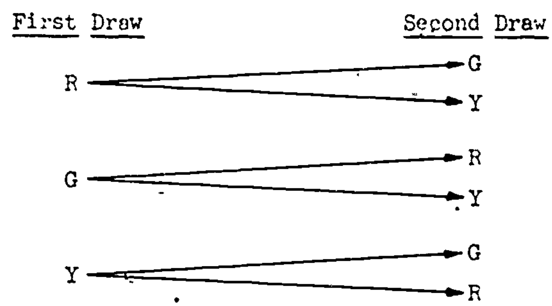
Of course, you might have drawn a different tree diagram in this case. Your diagram might have been:



4. Again, this diagram shows _______ possibilities for the die and the coin.

Drawing without replacement. Let us return to the box with one red marble, one green marble, and one yellow marble. Again we pick a marble at random. This time we do not return it to the box. We select another marble from those remaining in the box.

The possible outcomes of this succession of two draws are shown in the diagram:



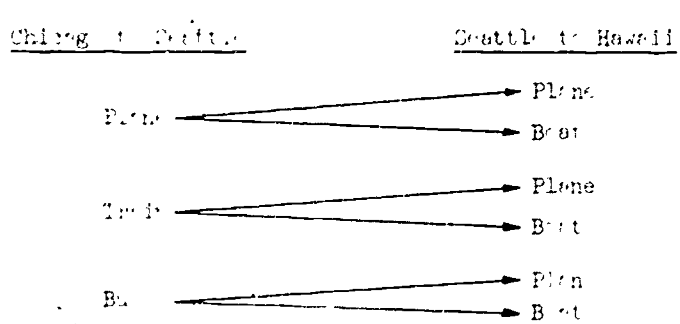
6

5. Note that there are _______ possible outcomes.
6. But if we had replaced the first marble drawn before drawing the second, then there would be _______ possible outcomes.

9

Problem: The Jones family is planning a trip from Chicago to Hawaii, via Seattle. They can go by plane, train, or bus to Seattle, and by plane or boat to Hawaii. In how many different ways can they choose to travel?

Though this is not a problem in which probability is used, a tree diagram can be used to explain it. Draw the diagram and then check with the one below.



7. There are ______ ways in all.

3-6. Exercises

(Answers on page 128 .)

1. If 3 honest coins are tossed, what is the probability that 3 heads will show? Use the tree diagram which shows the 8 possibilities for 3 coins.

2. If 3 honest coins are tossed, what is the probability that 2 heads and 1 tail will show?

3. There are 35 bricks, of which 5 are gold. What is the chance that if you pick a brick at random you will pick a gold one?

4. A bowl contains 10 marbles, of which 5 are white, 3 are black, and 2 are red. We will assume that they are identical in size, hence that each marble is equally likely to be picked if you reach into the bowl and take one marble without looking.

   (a) What is the probability that you will pick a white marble in one draw?

   (b) Assuming you pick a white marble the first time and do not replace it, what is the probability that you will pick a black marble the second time?

   (c) Assuming you pick a white marble the first time and a black marble the second time and do not replace them, what is the probability that you will pick a red marble the third time?

5. The letters A, B, C, D, E, and F are printed on the faces of a cube (one on each face). We describe the outcome of a roll of the cube by the letter on the top face.

   (a) If one cube is rolled, how many possible outcomes are there?

   (b) If 2 cubes are rolled at the same time, how many possible outcomes are there?

   (c) If one cube is rolled, what is the probability of B ?

   (d) What is the probability of two E's when two cubes are rolled at the same time?

6. A regular tetrahedron has one face marked A, one marked B, one marked C, and one marked D. It is tossed in the air and allowed to fall freely. We describe the outcome by the letter on the bottom face.

(a) How many possible outcomes are there?

(b) Find the probability of A .

(c) How many possible outcomes are there if two such tetrahedrons are tossed? What is the probability of one A and one B ?

(d) How many possible outcomes are there if three such tetrahedrons are tossed? What is the probability of two C's and one D ?

7. Add a fourth and a fifth line to the following table showing the pattern involved in a count of the number of outcomes in tossing coins.

Number of coins						
1	1 (1H)	1 (1T)				
2	1 (2H)	2 (1H,1T)	1 (2T)			
3	1 (3H)	3 (2H,1T)	3 (1H,2T)	1 (3T)		
4	____	____	____	____	____	
5	____	____	____	____	____	____

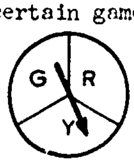
8. Use the table in Problem 7 to find the probability of getting 2 heads and 2 tails if 4 coins are tossed.

9. Give the probabilities of each of the 6 possible outcomes when 5 coins are tossed. Is the sum of the probability equal to 1 ?

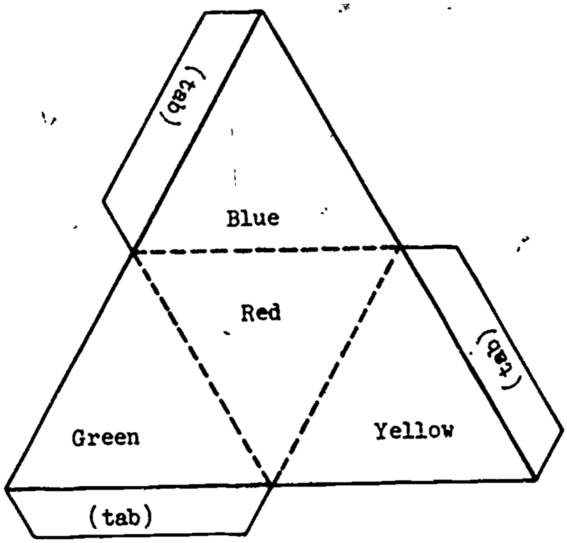
10. If 5 coins are tossed, what combinations of heads and tails are most likely to occur? Why? (Hint: See Problem 7.)

11. When 6 coins are tossed, what is the probability that one and only one coin will show heads?

12. A certain game is played with a spinner -- as shown -- and a die. You spin once and throw the die once. What is the probability that you will spin red and then get 6 on the die?

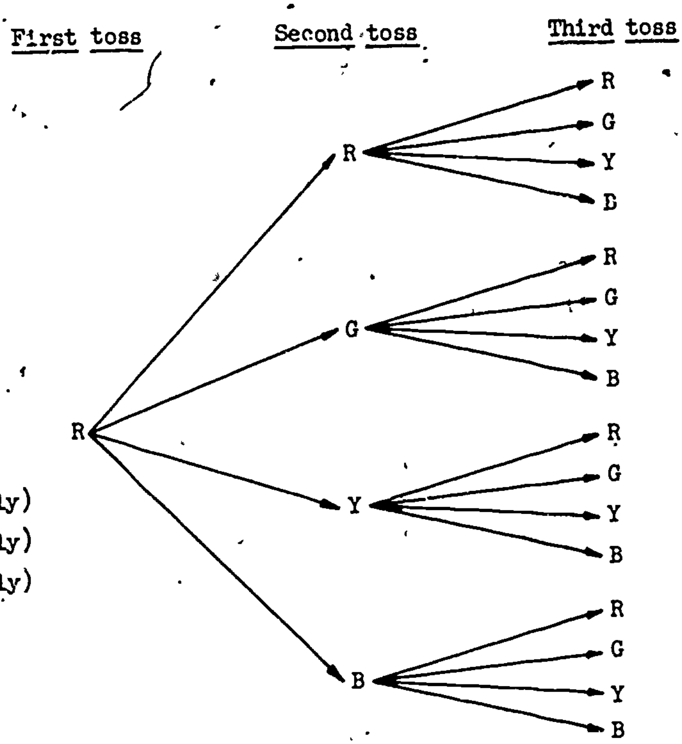


Pattern for regular tetrahedron



Tree diagram for 3 tosses of a regular tetrahedron

(NOTE: That part of the table is shown here which gives the outcomes for red on the first toss. The other parts differ only in the first letter.)



G (similarly)
Y (similarly)
B (similarly)

Chapter 4

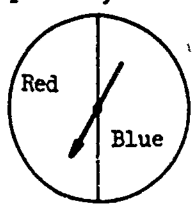
ESTIMATING PROBABILITIES

4-1. Estimation

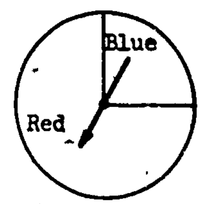
In many of the situations we have studied as examples (using coins, dice, spinners), it is possible to discuss the probability of certain events simply by thinking about the problem. We reason: <u>If</u> we have honest equipment, we can reach certain conclusions about it.

Consider two spinners, I and II .

I



II



> If the pointer is balanced and if it is honestly spun, we are willing to assert
>
> $\frac{1}{2}$ 1. for Spinner I, P(red) = _______,
>
> $\frac{1}{4}$ 2. for Spinner II, P(blue) = _______.

Notice that we assign these probabilities without actually spinning the pointer. We reason that, for Spinner I, the red and blue regions are equally likely <u>if</u> the spinner is "honest". Our reasoning is correct. Whether or not this reasoning applies to a particular spinner can only be decided by actually experimenting many times. Suppose, after many trials, our results show approximately the same number of reds as blues. Then we are somewhat confident that the spinner is fair and that our reasoning is correct.

You may very well raise some questions regarding the last paragraph. For example:

(a) How many experiments should be made?

(b) What precisely is meant by "approximately the same number of reds as blues"?

(c) How confident would we be?

These questions are all related and can only be answered somewhat generally here -- our degree of confidence increases as we conduct more experiments and as the fraction of red comes closer to $\frac{1}{2}$.

Suppose that a friend tells you that he has a spinner colored red and blue. You are not able to see the spinner. Your friend spins the pointer and tells you the result of each spin. After thirty spins, the record looks like this:

R R B B R   B R R B R   R R R B R

B B R B R   R R B R R   R R B B R.

19 3. There were ________ (how many) spins resulting in the outcome

11 red, and ________ (how many) spins resulting in the outcome blue.

$\frac{19}{30}$ 4. What fraction of the spins yielded red? ________

$\frac{19}{30}$ 5. If the spinner is spun once again what is your guess as to the probability of obtaining red? ________

Of course you cannot be <u>sure</u> that about $\frac{2}{3}$ of the spinner dial is colored red, but it is a reasonable guess based on the evidence at hand.

The problem that we have been discussing -- knowing the result of a certain number of trials without knowing the exact design of the spinner -- is illustrative of many real life situations. We may cite many examples where decisions are made on the basis of estimated probabilities. These estimates, in turn, are based on past experience ("trials"). Here are two such examples.

(a) In a baseball game, Robinson comes in to pitch. The opposing manager then orders Jones to bat for Smith. His decision is based on the fact that Jones has had better success than Smith against Robinson in previous games. Regardless of the result, the manager may well claim that he is "playing percentage baseball'.

(b) A doctor decides not to operate on Mr. A. His decision is based on the fact that, in medical experience, a large percentage of patients with Mr. A's symptoms have been cured without recourse to expensive (and, perhaps, dangerous) surgery.

In a particular situation, the confidence that is placed in a decision based on the results of previous trials depends both on the nature of the results and on the number of trials.

more	6. For the spinner of Items 3 to 5, we are led to believe (based on 30 trials) that red is ________ likely (more, less) than blue.
red	7. If we were told that we would win a prize if we picked the correct color on the next spin we would choose ________.
	Suppose that we have a record of 3000 spins of the spinner and that record shows 1900 red.
more	8. As a result of examining this record of 3000 trials, we are ________ (more, less) confident (than after 30 trials) that the spinner favors red.
	Suppose the record of 3000 spins shows 1512 red.
$\frac{1512}{3000} = .504$	9. The estimated probability of red is now $\frac{\square}{3000}$.
	If, on the basis of this record, we wish to pick the correct color of the next spin we would still choose red. However, we no longer would be very confident that the spinner favors red.

4-2. Examples

Example 1. Of many thousands of manufactured articles of a certain type, the company selected a sample of size 100 at random. These were carefully tested and it was found that 98 of the articles met the desired standard. What is the probability that an article made by this process is up to standard?

$\frac{98}{100}$	1. $\frac{\square}{100}$ of the samples tried were satisfactory.
98	2. One could expect about _______ percent of all of the articles manufactured by this process to be up to standard.
$\frac{98}{100}$, or $\frac{49}{50}$	3. The probability that a given article made by this process will be satisfactory could be said to be _______.

We could get more information by testing <u>every</u> article produced. Usually this is not feasible because of the time and expense involved. Sometimes it is not even possible. (An electric fuse may be designed to "blow" when subjected to a 20 ampere current. If the manufacturer tested every fuse produced, then ...?)

<u>Example 2</u>. A random sample of 500 patients with a certain disease were treated with a new drug. Of these, 380 were helped.

$\frac{380}{500}$, or $\frac{76}{100}$	4. What is the estimated probability that a given person with this disease will be helped by the new drug? _______
0.76	5. It is sometimes convenient to express probabilities in decimal form. For this example, P(patient is helped) = 0._______ .
3040 (.76 × 4000)	6. If 4000 patients were treated with the new drug, about how many would you expect to be helped? _______

<u>Example 3</u>. In baseball a player's batting average is computed by dividing his number of hits by his official times at bat. The average thus reflects the player's previous performance.

72	7. After 240 times at bat, a player has a batting average of .300 . He has made _____ hits.

8. What is the estimated probability that he will get a hit on his next time at bat? ________

9. If this player comes to bat 180 more times during the season, we estimate that he will get about ________ more hits.
(how many)

4-3. An Experiment

Obtain a new rivet. (A rivet may be purchased at a hardware store. We used a size $9\frac{1}{2}$ copper rivet. A tack or a flat-headed screw will serve equally well.) Think about tossing the rivet 50 times.

(a) Just by looking at the rivet guess how many times it will fall "up", like this:  , and how many times it will fall "down", like this:  .

Toss the rivet onto a flat surface and record the result. Perform 50 trials of this experiment.

(b) On 50 tosses, how many times did you get "up"?

(c) Does this result fit with your guess?

(d) What is the probability of getting "up" on the toss of a rivet?

See the discussion of this experiment on page 122 .

4-4. Exercises

(Answers on page 130 .)

1. A teacher has taught eighth grade mathematics to 1600 students during the past 10 years. In this period he has given A's to 152 students.
   (a) Based on these data, what is the probability that a student selected at random will receive an A in this teacher's class?
   (b) If this teacher will teach 2000 students in eighth grade mathematics during the next 12 years, about how many A's do you expect the teacher to give?

2. The batting average of a baseball player is 0.333 . What is the probability that this man will make a hit the next time he is at bat?

3. The record of a weather station shows that in the past 120 days its weather prediction of rain or no rain has been correct 89 times. Use this information to state the probability that its prediction for tomorrow will be correct.

4. Car insurance rates are usually higher for male drivers under the age of 25. Explain how data on accidents justify this.

5. Life insurance and life annuity rates are based on tables of mortality. A table of mortality includes statistical data on 100,000 people who were alive at age 10. The following are ten lines from the Actuaries Table of Mortality.

Age	Number living	Number dying during next year	Age	Number living	Number dying during next year
10	100,000	676	40	78,653	815
12	98,650	672	50	69,517	1,108
13	97,978	671	60	55,973	1,698
14	97,307	671	70	35,837	2,327
21	92,588	623	99	1	1

According to the table, 676 of the 100,000 will not be alive at 11 years of age. At age 13, 97,978 of the original 100,000 are alive, but 671 of these persons die within one year.

(a) How many of the original 100,000 were alive at the age of 50 ?

(b) How many were alive at the age of 100 ?

In Problems [illegible] through [illegible], use the Actuaries Table of Mortality given in Problem 5. Find the answers correct to the nearest 0.01 .

6. (a) What is the probability that a person who is 1[illegible] years of age will be alive at the age of 21 ?

(b) What is the probability that a person who is 13 years of age will be alive at the age of [illegible] ?

7. (a) What [illegible] years ago?

(b) D[illegible] table of mortality would be very useful if it actually were constructed [illegible] selecting 100,000 people at age 10 and keeping [illegible] year ? Why ?

(c) [illegible] table be constructed ?

8. (a) What is the probability that a boy who is 10 years of age will live to the age of 99 ?

   (b) What is the probability that a man who is 40 years of age will live to the age of 50 ?

9. One kind of life insurance policy guarantees to pay $1000 to a widow if her husband dies within the next ten years. Would such a policy be most expensive for a man aged 40 , 50 , or 60 ? Why?

10. Consider the following events:

    Event A. It rains on Friday the thirteenth.

    Event B. The sun shines all day on Friday the thirteenth.

    The following table shows the weather on 20 Fridays the thirteenth. Using the data from the table below, find the probabilities for the events A and B . Based on the information in the table, which is more likely to occur over a great number of Fridays the thirteenth, A or B ? Note that it is possible that neither event occurs.

<u>Weather on 20 Fridays the Thirteenth</u>

1.	Heavy rain	11.	Cloudy, no rain
2.	Light rain	12.	Partly cloudy
3.	Sunny	13.	Cloudy with some showers
4.	Sunny	14.	Showers
5.	Sunny	15.	Sunny
6.	Scattered showers	16.	Sunny
7.	Showers	17.	Hot and sunny
8.	Sunny	18.	Sunny
9.	Sunny	19.	Cloudy with some showers
10.	Sunny	20.	Sunny

11. Look up your record of 100 throws of a die. Based on this sample, what is your estimated probability of obtaining each of the faces?

12. Again referring to the 100 throws: considering them as twenty sets of five throws each, what is your estimate of the probability of "three of a kind" in five throws? of "four of a kind"? "five of a kind"?

Chapter 5

THE PROBABILITY OF A OR B

5-1. Union and Intersection of Two Sets

In this section we will use an example to illustrate some ideas about sets which should be familiar to you.

Suppose that a class consists of four boys -- Arthur, Bob, Carl, and Dan -- and three girls -- Elsie, Flora, and Grace. Three members of the class -- Bob, Dan, and Flora -- are in the band.

Before the band concert the teacher says, "Some of you are needed in the gym to help get ready for the concert. Go if you are a boy of if you are in the band." How many members of the class go to the gym? Think about the answer; then read on.

4 + 3
7

1. If you write the names of all the boys and then write the names of all the band members you write 4 + _____, or _____, names in all.

Arthur Bob Carl Dan	There are 4 boys.
Bob Dan Flora	There are 3 band members.

Everyone whose name is on this list goes to the gym. However, on this list some names appear twice, because some members of the class are both boys and band members.

2

2. In fact, _________ (how many) names are listed twice. If we cross each of them out once, then our list will have no duplicates:

Arthur Bob Carl Dan ~~Bob~~ ~~Dan~~ Flora	4 + 3 - 2 members of the class go to the gym.

Notice that this example illustrates the use of "or" to mean "either one, or the other, or both". This is the usual meaning of "or" in mathematics.

Suppose A is the set of boys in the class:

A = {Arthur, Bob, Carl, Dan} .

Suppose B is the set of band members:

B = {Bob, Dan, Flora} .

Then the set of class members who go to the gym can be written as $A \cup B$.

Dan, Flora

3. $A \cup B$ = {Arthur, Bob, Carl, _____, ______} .

4. $A \cup B$ is the set of class members who are boys _____ who are in the band.

or

"$A \cup B$" is often read "A union B" . It may also be read "A or B" .

Notice that the names crossed out in Item 2 above are the names of class members who belong both to A and to B . These class members are members of the set $A \cap B$ -- the intersection of A and B . Its members are in <u>both</u> A <u>and</u> B .

You may read "$A \cap B$" as "A intersect B" or as "A and B" , as you prefer.

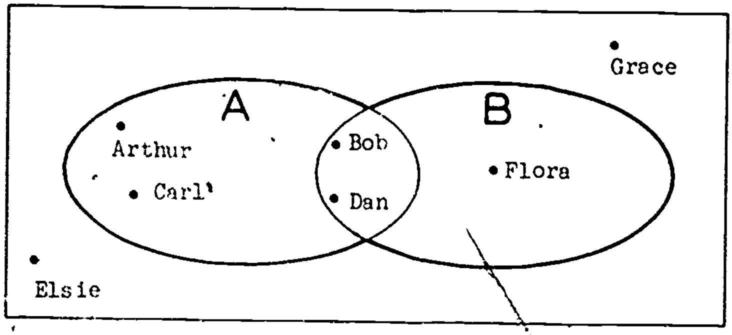
{Bob, Dan}
2

5. $A \cap B$ = {______________} . The number of elements in $A \cap B$ is _____.

6. If we add the number of elements (4) in A to the number of elements (3) in B , and then subtract the number of elements (2) in $A \cap B$, we have 4 + 3 - 2 , or _____. This is the number of elements in $A \cup B$.

5

It is easy to picture the sets in this example by means of a diagram.



In this drawing, set A , {Arthur, Bob, Carl, Dan} , and set B , {Bob, Dan, Flora} , are represented by regions  , 

Set $A \cup B$ is represented by the entire region 

Set $A \cap B$ is represented by the region  common to the A and B regions.

(A diagram of this sort is called a Venn diagram.)

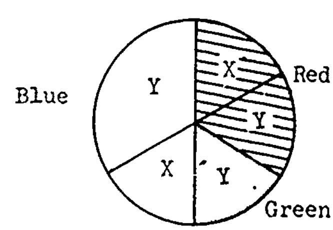
5-2. Event $A \cup B$; Event $A \cap B$

As we have seen, an event is a set of outcomes.

sum

> 1. The probability of an event A is the _____ of the probabilities of the outcomes in A .

Consider again the spinner shown in Section



Suppose you play, with this spinner, a game where you get 1 point for red or Y . (We will always use "or", in such cases, in the sense "either red, or Y , or both".)

5/6

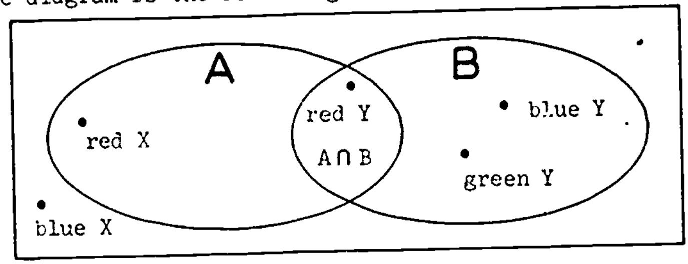
> It is easy to see that your probability of getting 1 point is _____.

Thinking a little more about this easy situation helps us understand an important general idea. We have an experiment for which we may choose, as set of outcomes,

{red X , red Y , green Y , blue X , blue Y} .

	Let A stand for the event "red".
red Y	3. A = {red X , ______}.
	Let B stand for the event "Y" .
red Y	4. B = {______, blue Y , green Y} .
or	5. Then $A \cup B$ is the event "red _____ Y" .
blue Y	6. $A \cup B$ = {red X , red Y , ______ , green Y} .
and	7. Also, $A \cap B$ is the event "red _____ Y" .
{red Y}	8. $A \cap B$ = ______.

An appropriate diagram is the following.



Let us think of throwing two dice, one red and one green. As we have seen, it is sometimes convenient to indicate the outcomes by ordered pairs. Thus "5 on the red and 2 on the green" is indicated (5,2) . There are 36 equally likely outcomes. (See Section 3-3 if you have forgotten.)

(4,2),(5,1)	9. If A is the event "the sum of the numbers is 6", then A = {(1,5),(2,4),(3,3),( ),( )} .
6	10. If B is the event "the sum of the numbers is 7", then F has ______ (how many) members. (List the outcome in F if you weren't sure.)
No	11. On a single throw of two dice, can you get a sum of both 6 and 7 ? ______

0 ; empty

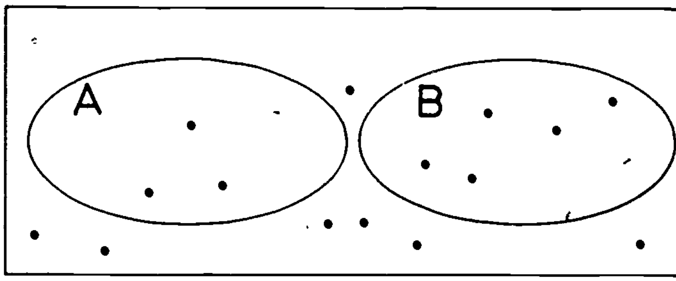
12. $A \cap B$ contains ________ (how many) members. It is the ________ set.

We could write $A \cap B = \emptyset$.

In this example, A and B cannot both occur on a throw.

Two events A and B are said to be <u>mutually exclusive</u> (or <u>disjoint</u>) if the occurrence of either excludes that of the other -- they cannot both occur. In other words, A and B are mutually exclusive if $A \cap B$ is the empty set. For example, on a throw of two dice, a sum of 6 and a sum of 7 are mutually exclusive.

If events A and B are mutually exclusive, then an appropriate diagram is:



If A and B are mutually exclusive, the A and B regions in the diagram do not overlap.

the empty set, or $\emptyset$

13. A and B are mutually exclusive if $A \cap B$ is ________.

are

14. If a single card is drawn from a deck of cards, drawing an ace and drawing a jack ________ (are, are not) mutually exclusive events.

are not

15. Drawing an ace and drawing a spade ________ (are, are not) mutually exclusive events. (Both can occur on a single draw -- you can draw the ace of spades.)

are

16. Two coins are tossed. Consider the events "one head and one tail", "two heads". These events ________ (are, are not) mutually exclusive.

17. Suppose events A, B are mutually exclusive. What is $P(A \cap B)$?

[A] 0 [B] You can't tell

$A \cap B$ is the empty set, since both events cannot occur at once. Hence, $P(A \cap B) = 0$. [A] is the correct response.

Again consider throwing two dice. Let E be the event "the sum is even". Let F be the event "the sum is divisible by 4" .

is

18. The throw (3,5) -- 3 on the first die and 5 on the second is in E (since $3 + 5 = 8$) , and ________ (is, is not) also in F .

is

is not

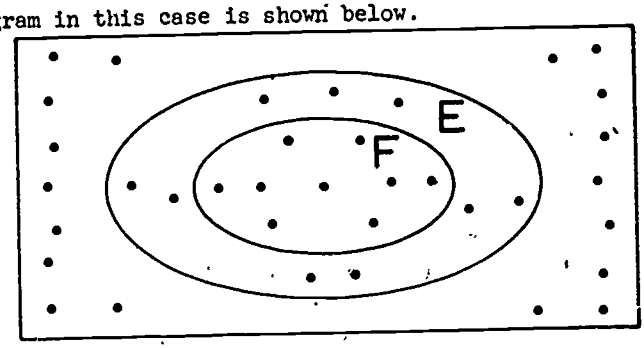
19. The throw (1,5) ________ (is, is not) in E and ________ (is, is not) in F .

Every member of F is also a member of E , since every number divisible by 4 is also divisible by 2 .

subset

20. Hence F is a ________ of E .

A suitable diagram in this case is shown below.



36

21. In this diagram, there are ________ (how many) dots, because for the throw of two dice there are 36 outcomes.

is

22. There are 9 outcomes for the event "sum divisible by 4" . For example (2,6) ________ (is, is not) in F .

is not

23. There are 9 outcomes which are in E but not in F . For example, (2,4) is in E , but (2,4) ______ in F .
(is, is not)

24. If F is a subset of E , then $E \cap F =$

[A] E

[B] F

[C] You can't tell.

If F is a subset of E , then every member of F is in E . In this case $E \cap F = F$, so [B] is the correct response. If you had trouble, look again at Items 18 to 23. In the example discussed there, F is a subset of E .

E

25. If F is a subset of E , then $E \cup F =$ ______.

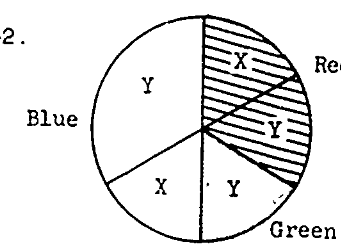
5-3. Probability of $A \cup B$

We have found the probabilities of certain events by counting outcomes. Sometimes, however, we have a situation for which it is difficult or impossible to do this. Sometimes we know only the probabilities of certain events. From this knowledge it is sometimes possible to find out about other probabilities. In this section, you will learn about a formula for $P(A \cup B)$.

Consider again the spinner used in Section 5-2.

A is the event "red".

B is the event "Y" .



$\frac{1}{3}$

$\frac{2}{3}$

$\frac{5}{6}$

$\frac{1}{6}$

Let us write the probabilities for these events.

1. $P(A) =$ ______.

2. $P(B) =$ ______.

3. $P(A \cup B) =$ ______.

4. $P(A \cap B) =$ ______.

It is easy to verify that these probabilities satisfy the following equation:

$$P(A \cup B) = P(A) + P(B) - P(A \cup B)$$

Look at the situation described in Section 5-1. In it, there were 7 students. In the symbols used there, A is the set of boys:

$$A = \{\text{Arthur}, \text{Bob}, \text{Carl}, \text{Dan}\} .$$

B is the set of band members:

$$B = \{\text{Bob}, \text{Dan}, \text{Fora}\} .$$

The set of students who go to the gym is $A \cup B$.

Consider a student whose name is selected at random from the list of students in the class.

The probability that the student is a boy is $P(A)$.

$\frac{4}{7}$ 5. $P(A) =$ ______.

$\frac{3}{7}$ 6. $P(B) =$ ______.

$\frac{2}{7}$ 7. $P(A \cap B) =$ ______.

$\frac{5}{7}$ 8. $P(A \cup B) =$ ______.

Notice that $\frac{4}{7} + \frac{3}{7} - \frac{2}{7} = \frac{5}{7}$.

Once again, it is true that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) .$$

For this very simple example, the probabilities can be found easily by counting. Let us look at a situation in which we cannot count outcomes. In these cases the equation is very useful.

Suppose, for example, that a principal says:

> "$\frac{3}{5}$ of the students in my school are boys. $\frac{1}{10}$ of them are boys who play in the band. In all, $\frac{3}{20}$ of the students in the school play in the band."

do not	9. We ________ know how many students attend the school. (do, do not)
do not	10. Hence, we ________ know the number of possible outcomes of the experiment "select a student at random". (Remember, "select at random" here means "select in such a way that all students are equally likely to be selected".) (do, do not)

However, we can consider probabilities of events associated with this experiment.

Suppose we ask: What is the probability, if a name is chosen at random, that it is the name of either a boy or a band member?

$P(A \cap B)$	11. If A is the event "name ~~of a boy~~", and B is the event "name of a band member", then we are looking for $P(\quad)$.
$\frac{3}{5}$ $\frac{3}{20}$ $\frac{1}{10}$	12. We know: $P(A) =$ ____; $P(B) =$ ____; $P(A \cap B) =$ ____.
and	13. $P(A \cap B)$ is the probability that the name belongs to a student who is a boy ____ who is in the band.

14. In this case, can we use the formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

to find $P(A \cup B)$?

[A] Yes [B] You can't be sure

Though we don't know the numbers of outcomes involved, we can use the formula. $P(A)$ is the sum of the probabilities of the outcomes in A . $P(B)$ is the sum of the probabilities of the outcomes in B . If we add $P(A)$ and $P(B)$, we have added these, but some probabilities are added twice. In fact, each outcome in $A \cap B$ is entered twice. If we subtract $P(A \cap B)$, we have left exactly the sum of the probabilities of the outcomes in $P(A \cup B)$. Hence, we can use the formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) .$$

[A] is the correct response.

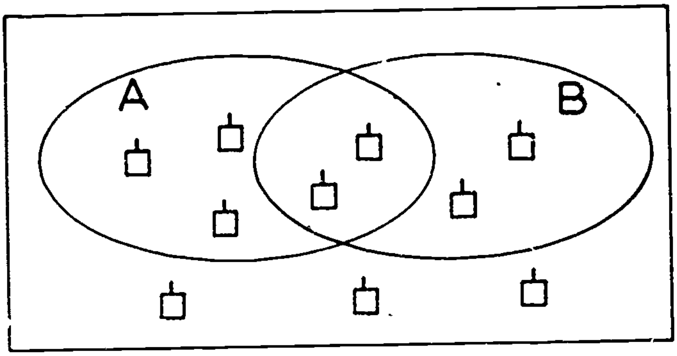
In the preceding example, we asked the probability that, if a name of a student is chosen at random, it is the name of either a boy or a band member.
15. Using the results in Item 12, we have $P(A \cup B) =$ _____. [Hint: Use Item 14.]

$\frac{13}{20}$

We worked through the preceding example very carefully. The reasoning used there can be applied to any situation. It is always true that if A and B are any events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) .$$

The reasoning is suggested by the diagram below. Event A is a set of outcomes; event B another set of outcomes. (The weights hanging on the dots remind us that a probability has been assigned to each outcome. We do not need to know what numbers belong on the weights to follow the reasoning.)



16. Recall that P(A) is the sum of the probabilites of the outcomes in A . Similarly P( ) is the sum of the probabilities in B .

P(B)

If we add the probabilities of the outcomes in A and the probabilities of the outcomes in B, we have added those in $A \cap B$ twice. By subtracting $P(A \cap B)$ from $P(A) + P(B)$, we get exactly the sum of the probabilities in $A \cup B$. But this sum is $P(A \cup B)$. Note that our reasoning is exactly like that used in Section 5-1, Items 1, 2.

Experiment:

Throw a die.

Let A be the event "a number greater than 3".
Let B be the event "an even number".
In this case we can compute directly, by counting equally likely outcomes:

$\frac{1}{2}$	17. $P(A) =$ _____.
$\frac{1}{2}$	18. $P(B) =$ _____.
$\frac{2}{6}$, or $\frac{1}{3}$	19. $P(A \cap B) =$ _____. (Note: $A \cap B = \{4,6\}$)
$\frac{2}{3}$	20. $P(A \cup B) =$ _____.
or	21. $P(A \cup B)$ is the probability of throwing an even number _____ a number greater than 3.
$\{2,4,5,6\}$	22. $A \cup B = \{$ _____ $\}$.
$P(A) + P(B) - P(A \cap B)$	23. We can verify easily that again: $P(A \cup B) = P(A) +$ _____.

5-4. Probability of $(A \cup B)$ for Mutually Exclusive Events

$P(A \cap B)$	1. In Section 5-3, we saw: If A, B are any events, then $P(A \cup B) = P(A) + P(B) - P($ _____ $)$.
$P(A \cap B)$.	2. It is easy to use this result to find $P(A \cup B)$ if we know $P(A)$, $P(B)$, and $P($ _____ $)$.

In some cases, we know $P(A \cap B)$ at once.

Again, we throw 2 dice. Let A be the event "the sum is 6". Let B be the event "the sum is 7". We will find the probability of throwing 6 <u>or</u> 7.

$\frac{5}{36}$	3. $P(A) =$ ______.
$\frac{6}{36}$, or $\frac{1}{6}$	4. $P(B) =$ ______. (See Items 9, 10, Section 5-2, if you weren't sure.)
exclusive	5. In this case events A and B are mutually ______.
0	6. $P(A \cap B) =$ ______.
$\frac{5}{36} + \frac{6}{36} = \frac{11}{36}$	7. Hence, $P(A \cup B) =$ ______ $+ \frac{6}{36} =$ ______.

Of course, you might have found $P(A \cup B)$ by simply counting outcomes.

This example illustrates a general rule. If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B).$$

empty	8. If $A \cap B$ is the ______ set, then $P(A \cup B) = P(A) + P(B)$.

are	9. Drawing a jack and drawing an ace from a regular deck of cards ______ (are, are not) mutually exclusive events.
$\frac{4}{52}$, or $\frac{1}{13}$	10. P(jack) = ______. (There are 52 equally likely outcomes, and 4 of them are in the event "jack".)
$\frac{4}{52}$, or $\frac{1}{13}$	11. P(ace) = ______.
$\frac{2}{13}$	12. P(ace or jack) = ______.

1[illegible]. Suppose you wish to find the probability of getting at least one head when two coins are tossed. Examine the following reasoning and decide whether it is correct. "The probability of getting head on the first coin is $\frac{1}{2}$. The probability of getting head on the second coin is $\frac{1}{2}$. The probability of getting head on one or the other is $\frac{1}{2} + \frac{1}{2}$." This reasoning is

[A] Correct

[B] Incorrect

If this reasoning were correct, we would conclude that the probability of getting at least one head is 1 . This is clearly incorrect, so [B] is the response you should have chosen. The error in the reasoning comes from the fact that head on the first coin and head on the second coin are not mutually exclusive events.

$$P(\text{head on first } \underline{\text{or}} \text{ head on second}) = P(\text{head on first}) + P(\text{head on second}) - P(\text{heads on both}) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}.$$

The probability of throwing the sum 6 with two dice is $\frac{5}{36}$. If you throw two dice, what is the probability of not throwing 6 ? Think about this before you go on.

You could [illegible]:

$\frac{5}{36}$	1. [illegible] event "sum 6" . $P(A) =$ ______ . [illegible] event "sum different from 6" .
1	2. $P(A \cup [illegible]) =$ ______ . [illegible] certain that either the sum [illegible] .
mutually	3. [illegible] ______ [illegible], so that [illegible] $\cap$ [illegible] ______ .
0	
$\frac{31}{36}$	[illegible] $\cup$ [illegible] . [illegible], $1 = \frac{5}{36} + P(C)$. $P(C) = 1 -$ — ______ .

You may have arrived at this result by a different kind of reasoning. You may have thought:

36 | 18. When you throw 2 dice, there are ______ equally likely outcomes. Each outcome has probability ______.

$\frac{1}{36}$

5 | 19. ______ outcomes are in the event "sum 6". The probability of this event is ______.

$\frac{5}{36}$

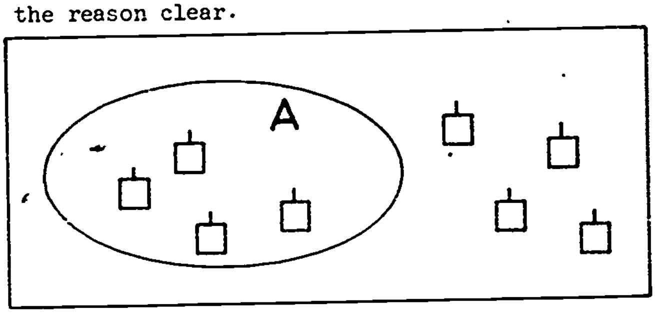
31 | 20. The remaining outcomes -- ______ in all -- are in the event "sum not 6", so the probability of this event is ______.

$\frac{31}{36}$

This example illustrates a general result which is very useful. If $P(A)$ is the probability of event A, then the probability that A does <u>not</u> occur is

$$1 - P(A) .$$

A diagram makes the reason clear.



outcomes | 21. Event A' is a set of ________.

The outcomes not in A (outside A in the diagram) are those outcomes which result when event A does <u>not</u> occur. (This set is sometimes called the <u>complement</u> of A. Sometimes we call this set event <u>not-A</u>.)

1 | 22. Recall that the sum of the probabilities of <u>all</u> the possible outcomes is ______.

P(A) | 23. Recall, too, that the sum of the probabilities of the outcomes in A is <u>P( )</u>.

$1 - P(A)$	24. Then it is clear that the sum of the probabilities of the outcomes outside A is ______ .
does not	25. $1 - P(A)$ is the probability that A ______ (does, does not) occur.

Notice that the event A and the event not-A are always mutually exclusive. No outcome is in both.

.7	26. If a batter's probability of getting a hit is .3 , then his probability of not getting a hit is ______ .
.55	27. If the probability that a student passes a test is .45 . then the probability he fails is ______ .
.983	28. If the probability that a certain manufactured article is defective is .017 , then the probability that it is not defective is ______ .

Ten coins are tossed. What is the probability that at least one shows head?

2^{10}	29. You recall (Section 3-2) that there are ______ outcomes for this experiment.
$\frac{1}{2^{10}}$, or $\frac{1}{1024}$.	30. Only one -- all tails -- fails to show at least one head. Thus the probability that there are no heads is ______ .
$1 - \frac{1}{1024}$, or $\frac{1023}{1024}$	31. The probability of at least one head is consequently $1 -$ ______ , or ______ .

We have seen in the chapter that it is easy to compute $P(A \cup B)$ if you know $P(A)$, $P(B)$, and $P(A \cap B)$. Suppose we know $P(A)$ and $P(B)$. If A and B are mutually exclusive, then $P(A \cap B) = 0$. What if A and B are not mutually exclusive? Can we still find $P(A \cap B)$ from $P(A)$ and $P(B)$? The answer is - sometimes. We will learn more about $P(A \cap B)$ in Chapters 6 and 7.

5-5. Exercises

(Answers on p. 131 .)

1. Which of the following pairs of events are mutually exclusive?
   (a) In tossing a coin: throwing heads; throwing tails.
   (b) In throwing a die: throwing an odd number; throwing a 3 .
   (c) In throwing a die: throwing a 6 ; throwing a 3 .

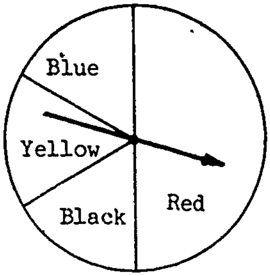
2. During the past 10 years, a teacher of eighth-grade mathematics has taught 1600 students. In this period, he has given A as a final grade to 132 students, and B as a final grade to 508 students. Based on these data, what is the probability that a student selected at random will receive either an A or a B ?

3. Consider a game in which you get 1 point for throwing a number which is even or greater than 3 .
   (a) Are the events "number is even" and "number is greater than 3" mutually exclusive?
   (b) What is the probability that on any toss of the die you get a point?

4. In a bag there are 4 red, 3 white, and 2 blue marbles. One marble is picked at random.
   (a) What is the probability of picking a red marble?
   (b) What is the probability of picking a white marble?
   (c) What is the probability of picking either a red or a white marble?
   (d) What is the probability that the marble picked is neither red nor white?

5. In a neighborhood pet show there are 10 dogs, 8 cats, 3 canaries, and 6 rabbits. Each pet is owned by a different person. A prize is to be awarded by drawing at random the name of an owner from the set of entry blanks.
   (a) What is the probability that the winner will own either a dog or a cat?
   (b) What is the probability that the winner will not be the owner of one of the four-legged pets in the show?

6. The dial of a spinner is $\frac{1}{2}$ red, $\frac{1}{6}$ blue, $\frac{1}{6}$ yellow, and $\frac{1}{6}$ black. The pointer is spun once.



(a) What is the probability that it stops on either red or blue?

(b) What is the probability that it stops on either yellow or blue?

7. The gum machine has just been filled with 100 balls of gum of assorted colors; there are 25 red, 15 black, and 20 each of yellow, green, and white. The balls are mixed throughly so that the chance of getting any one ball is as good as any other. If you buy one ball from the machine, what is the probability that you get:

(a) a red?

(b) a yellow?

(c) either a black or a green?

8. There are 3 boys and 2 girls in a group. Two of them are chosen at random to buy refreshments for a party.

(a) In how many ways can the choice be made?

(b) How many of the pairs consist of two boys?

(c) How many pairs consist of two girls?

(d) How many pairs consist of one boy and one girl?

(e) What is the probability that two boys are selected?

(f) What is the probability that a boy and a girl are picked?

(g) What is the probability that at least one boy is selected?

9. Eight girls are to help with refreshments at a party. Seven of them are chosen at random to bake cookies. The remaining girl, plus four others chosen at random from the cookie-bakers, are to make punch.

(a) How many girls will bake cookies <u>only</u>?

(b) How many will make punch <u>only</u>?

(c) How many will do both?

(d) If Mary is one of the eight girls, what is the probability that she both bakes cookies and makes punch?

10. At a certain boys' camp there are 22 boys. All except 6 of them swim at least once a day. Nine boys swim in the morning, 11 swim in the afternoon, and no one swims at night.

(a) If X is the set of boys who swim in the morning and Y is the set of boys who swim in the afternoon, $X \cup Y$ is the set of those boys who swim every day. How many members does $X \cup Y$ contain?

(b) $X \cap Y$ is the set of boys who swim twice a day. How many members has $X \cap Y$?

(c) Then how many boys swim only in the morning? How many only in the afternoon?

(d) If the name of one camper is picked at random, what is the probability that the name drawn is that of a boy who swims at least once a day.

Chapter 6

$P(A \cap B)$, THE PROBABILITY OF A AND B

6-1. Experiments

1. In a previous experiment we used a box containing a red, a green, and a yellow marble. We shall use the same equipment for the following experiment.

   Draw a marble. Replace it. Draw a second marble. Record the colors of the two marbles drawn. (For example, if the first is red and the second is yellow, record RY.)

   Repeat the steps above until you have recorded 20 pairs of colors.

   How many times was the second color red? green? yellow? What did you expect?

2. Suppose, in the experiment above, that you had not replaced the marble before the second draw. You are still to record the color of the two marbles drawn. Would you still expect to get the same sort of results as before? Decide on your answer first; then try the experiment until you have recorded 20 pairs, and see what your results are.

   Were your results consistent with your guess? Is there any reason to believe that one color is more likely than another?

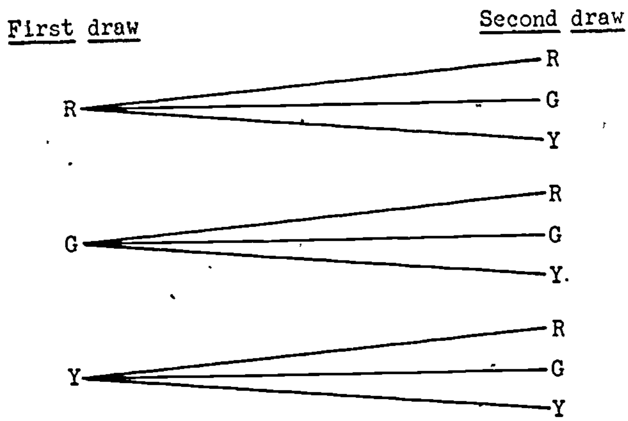
3. For the first of the experiments above, make a guess as to the probability of drawing red on both draws. What is the probability of drawing red on both draws in the second experiment?

Read the discussion of both experiments on page 122 .

6-2. Probability of $A \cap B$: Some Examples

In the first experiment, we had three marbles in a jar: one red, one green, and one yellow. We drew one marble, put it back, and drew a second marble.

Now we are interested in "color of first marble and color of second marble". The following tree diagram shows all the possible outcomes:



YR, YG, YY

1. The possible outcomes are RR, RG, RY, GR, GG, GY, ___, ___, and ___. They are equally likely.

RG, RY

2. The outcomes in the event E : "red on the first draw", are RR, ___, ___.

$\frac{3}{9}$; $\frac{1}{3}$

3. $P(E) = \frac{\square}{9}$, or ___.

Of course, there is another, and easier, way to find P(E) . If we want to find the probability of red on the first draw, we can forget all about the second draw.

4. We can think of E as the event "drawing a red" in the simpler experiment of merely drawing one marble from the box. For this experiment, the possible outcomes are R, ___, and ___.

G; Y

$\frac{1}{3}$

5. P(E) = ___.

We are not surprised to get the same number for P(E) in Item 5 as we got in Item 3.

Similarly, if F is the event "yellow on the second draw", we can compute P(F) in either of two ways.

9

6. We can think of all _____ (how many) outcomes shown in the tree diagram.

7. We can think: We get "yellow on the second draw" in

3 ______ out of 9 outcomes.
(how many)

$\frac{1}{3}$ 8. Hence $P(F) = \frac{3}{9}$, or ___.

We can also think: The possible outcomes of the second draw are R, G, and Y and all are equally likely.

9. Hence, at once,

$\frac{1}{3}$ $P(F) =$ ___.

Look back at Item 1, where we listed the possible outcomes for drawing a marble, replacing it, and drawing another. Suppose we wish to find the probability of the event "red on the first draw and yellow on the second draw".

10. The only outcome which is in both the events "red on the first draw" and "yellow on the second draw", is

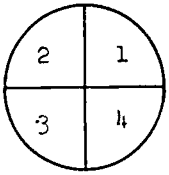
RY _____.

$\frac{1}{9}$ 11. P(red on first and yellow on second) = _____.

= 12. Notice that in this case $P(E \cap F)$ ______ $P(E) \cdot P(F)$.
(=, ≠)

Another similar example will help you to see a pattern emerging.

Let us toss a penny, and then spin a pointer on a dial divided into four equal spaces marked 1, 2, 3, and 4.



Suppose that A is the event "heads on the penny".
Let B be the event "spinner stops on 4".

$\frac{1}{2}$ 13. Then $P(A) =$ ____.

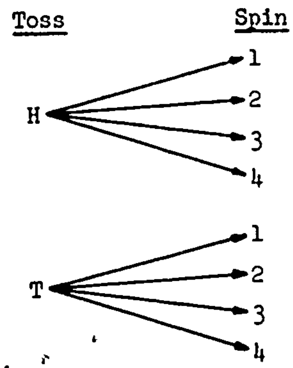
$\frac{1}{4}$ 14. $P(B) =$ _____.

$\frac{1}{8}$ 15. The probability that both A and B occur is _____.

= 16. Once again, we observe: $P(A \cap B)$ ______ $P(A) \cdot P(B)$.
(=, ≠)

If you had trouble with Items 13 - 16, or if you were not sure, continue with Item 17. If you did not have trouble, you may omit Items 17 - 24.

A tree diagram showing the possible outcomes is:



17. From the tree diagram we see that the number of possible outcomes is ____.

8

18. A is the event "heads on the penny". One outcome in A is "H1"; list the other outcomes in A.
____, ____, ____.

H2, H3, H4

19. $P(A) =$ ____.

$\frac{4}{8}$, or $\frac{1}{2}$

20. B is the event "spinner stops on 4". Then B includes two outcomes, H4 and ____.

T4

21. $P(B) =$ ____.

$\frac{2}{8}$, or $\frac{1}{4}$

22. Since $A = \{H1, H2, H3, H4\}$ (Item 18)
$B = \{H4, T4\}$,
we see: The only outcome in $A \cap B$ is ____.

H4

23. The probability that both A and B occur is ____.

$\frac{1}{8}$

24. We can use the results of Items 19, 21, 22 to check:
$P(A \cap B) \underset{(=,\ \neq)}{\underline{\qquad}} P(A) \cdot P(B)$.

=

As we have seen, it sometimes happens that

$$P(A \cap B) = P(A) \cdot P(B).$$

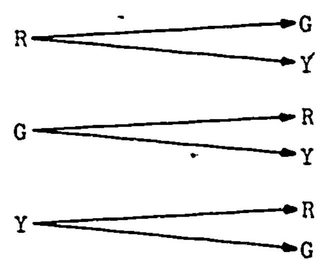
But does this result always hold?

Let us return to the second experiment in Section 6-1. In it, we drew a marble but we did not replace it before drawing a second marble.

In this case, what did you expect? Before you go on, think! Is the probability of the event "red on the first and yellow on the second" the same as in the first experiment? Is the set of equally likely outcomes the same for both experiments?

Read on, to verify your reasoning.

This time, the tree diagram showing the possible outcomes is:



GR, GY, YR	25. The possible outcomes are RG, RY, ___, ___, ___, and YG. All are equally likely.
RY	26. The only outcome in the event "red on the first draw and yellow on the second draw" is ____.
$\frac{1}{6}$	27. The probability of red on the first draw and yellow on the second is ____. That is, $P(S \cap T) = \frac{1}{6}$.
RG; RY	28. If S is the event "red on the first draw", the outcomes in S are ___ and ___.
$\frac{2}{6}$, or $\frac{1}{3}$	29. $P(S) =$ ____.
RY; GY	30. The outcomes in the event T : "yellow on the second draw", are ____ and ____.
$\frac{2}{6}$, or $\frac{1}{3}$	31. $P(T) =$ ____.
$\frac{1}{9}$	32. $P(S) \cdot P(T) =$ ____
$\neq$	33. $P(S \cap T) \underset{(=, \neq)}{\underline{\qquad}} P(S) \cdot P(T)$. [Recall that $P(S \cap T) = \frac{1}{6}$.]

This time the probability of red on the first draw and yellow on the second--$P(S \cap T)$--is a number different from the product $P(S) \cdot P(T)$.

6-3. Independent Events; Some Preliminary Ideas

In each experiment in Section 6-1 we were concerned with drawing two marbles from a box.

On the first experiment we replaced the first marble before drawing the second. In this case, it seems natural to say that the two draws are <u>independent</u>. If we know, for example, that the result of the first draw was red, this knowledge does not affect the probabilities we assign to the possible second draws.

In the second experiment, in which the first marble was not put back, we recognize that knowing what happened on the first draw has a bearing on the probabilities assigned to the second draw. In this case the two draws are <u>not</u> independent.

We will find that a precise definition of independent events is a little more complicated than this simple example would suggest. However, in certain experiments that involve two actions -- like throwing a die and then spinning a spinner, or drawing one marble and then another -- it usually is easy to recognize independence. In such cases, we feel intuitively that two events are independent when the occurrence of one does not affect the probability of the other.

In each of the following experiments, compute $P(A)$, $P(B)$, and $P(A \cap B)$. Decide whether A and B are independent.

Experiment: Throw a die twice.
Event A is 5 on the first throw.
Event B is 3 on the second throw.

$\frac{1}{6}$ 1. $P(A) =$ ____

$\frac{1}{6}$ 2. $P(B) =$ ____

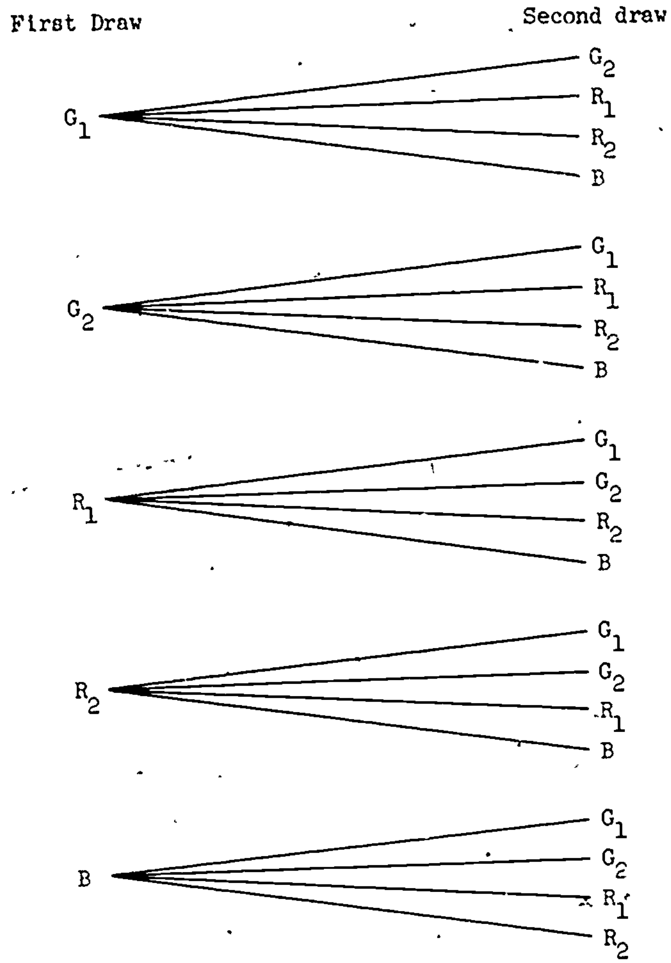
$\frac{1}{36}$ 3. $P(A \cap B) =$ ____

are 4. A and B ____ (are, are not) independent.

	Experiment: Toss a penny and a die.
	Event A is head on the penny.
	Event B is 3 on the die.
$\frac{1}{2}$	5. $P(A) =$ _____
$\frac{1}{6}$	6. $P(B) =$ _____
$\frac{1}{12}$	7. $P(A \cap B) =$ _____
are	8. A and B _____ (are, are not) independent.

	Experiment: We have a jar containing 5 marbles, of which 2 are red, 2 are green, and 1 is blue. We draw a marble and then draw a second marble without replacing the first.
	Event A is green marble on the first draw.
	Event B is green marble on the second draw.
$\frac{2}{5}$	9. $P(A) =$ _____
$\frac{2}{5}$	10. $P(B) =$ _____
$\frac{2}{20}$, or $\frac{1}{10}$	11. $P(A \cap B) =$ _____
are not	12. A and B _____ (are, are not) independent.

If you had trouble with Items 9 to 12, examine the tree on page 76 and complete Items 13 to 18. If not, omit them. In making the tree, we think of the two green marbles as G_1, G_2. Likewise the two red marbles are called R_1, R_2.



20	13. There are _____ outcomes, all equally likely.
8; $\frac{8}{20}$, or $\frac{2}{5}$	14. Of these, _____ are in the event A, "green on the first draw". Hence $P(A) =$ _____.
8; $\frac{8}{20}$, or $\frac{2}{5}$	15. There are _____ outcomes in the event B, "green on the second draw". Hence $P(B) =$ _____.
2; $\frac{2}{20}$, or $\frac{1}{10}$	16. There are _____ outcomes in $A \cap B$. Hence $P(A \cap B) =$ _____.

	Experiment: We use the same jar of marbles as in Items 9 to 12. (2 red, 2 green, 1 blue.) We draw a marble, replace it, and then draw a second marble.
	Event A is red on the first draw.
	Event B is red on the second draw.
$\frac{2}{5}$	17. $P(A) =$ ____
$\frac{2}{5}$	18. $P(B) =$ ____
$\frac{4}{25}$	19. $P(A \cap B) =$ ____
are	20. A and B ________ (are, are not) independent.

21. Two dice are thrown. The event "even on the first die" and the event "odd on the second die" are
   [A] mutually exclusive.
   [B] independent.
   [C] both.
   [D] neither.

If we know that "even on the first die" has occurred, we do not change our idea about the probability of "odd on the second die". Hence the events are independent. Both can occur (as in the throw $(4,3)$). Hence the events A, B are not mutually exclusive. You should have answered [B].

We have observed a number of examples in which the formula

$$P(A \cap B) = P(A) \cdot P(B)$$

is true. We have observed, too, that in all the examples where the formula holds, we feel intuitively that knowing whether A has occurred does not influence our thinking about the probability of B. We have called events A, B for which the formula holds independent events, noting that our examples fit our every day usage of "independent". In some situations it is not evident whether two events are independent. Independence will be discussed further in Chapter 7.

[illegible] E, F are mutually exclusive. $P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{4}$. Are E and F independent?

A [illegible]

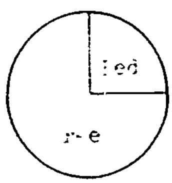
B [illegible]

Since E and F are mutually exclusive, we know that if one occurs the other does not. Hence if we know E has occurred, the probability that F has occurred is 0. [illegible] $P(E \cap F) = 0$, but $P(E) \cdot P(F) \neq 0$. The events E, F are not independent. You should have responded 'B'.

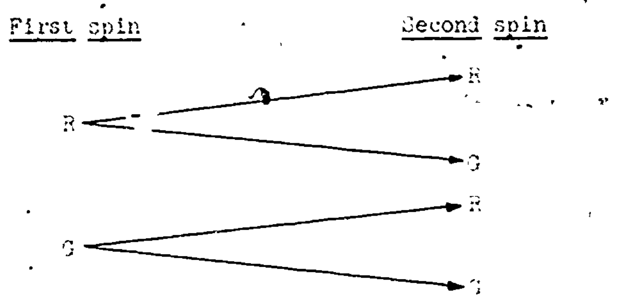
In general, if E and F have non-zero probabilities, and if E and F are mutually exclusive, then they are not independent.

6-4. Experiment [illegible]

[illegible] $\frac{1}{4}$ of the area red and $\frac{3}{4}$ of the [illegible] Continue to [illegible] pair.



[illegible]



GR, GG

1. The possible outcomes of spinning the spinner twice are

RR, RG, ___, ___.

are not

2. We must be sure to observe, however, that these outcomes ___ equally likely.
(are, are not)

Suppose the experiment "spin twice" is repeated many times with this spinner.

3

3. We expect to get green ___ times as often as red on the first spin.

$\frac{3}{4}$

$\frac{1}{4}$

4. ... ,

P(green on first spin) = ___,

and P(red on first spin) = ___.

... ... :

... ... equally often,

3

5. ... ___ times as often ...,

3

6. ... ___ times as often ... R.

. ... In other words, ...

P(RR)

P(GR) or P ...

$\frac{1}{4}$

P ... = P ...

... = ... $\times$ ___ . (Item ...)

... = ... $\times$ ___ (Item ...)

. ... = ___ ... ... is

... red on first spin.

Since this experiment involves spinning twice, it seems reasonable to see whether our formula for independent events applies.

are	9. We reason that "red on first spin" and "red on second spin" ______ (are, are not) independent events.
$\frac{1}{4}$	10. P(red on first spin) = ____.
$\frac{1}{4}$	11. P(red on second spin) = ____.
$\frac{1}{4} \cdot \frac{1}{4}$; $\frac{1}{16}$	12. P(red on both spins) = P(RR) = ___ · ___, or ___.

	13. Similarly, we find
$\frac{1}{4} \cdot \frac{3}{4}$, or $\frac{3}{16}$	P(RG) = ___ · ___, or ___.
$\frac{3}{4} \cdot \frac{1}{4}$, or $\frac{3}{16}$	P(GR) = ___ · ___, or ___.
$\frac{3}{4} \cdot \frac{3}{4}$, or $\frac{9}{16}$	P(GG) = ___ · ___, or ___.

If you check, you will find that these values for the probabilities fit with Items 7 and 8.

	14. Moreover, if the probabilities have these values then
1	P(RR) + P(RG) + P(GR) + P(GG) = ____.

Once again we observe that the formula

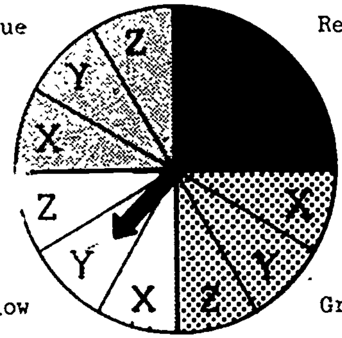
$$P(A \cap B) = P(A) \cdot P(B)$$

applies where A, B are independent events.

In each of our earlier examples in this chapter, we could begin by forming a set of equally likely outcomes. However, as just illustrated, we need not always begin with equally likely outcomes.

6-5. Another Example of Independent Events.

Examine the spinner shown here. One-fourth of it is red, one-fourth is green, one-fourth is blue, and one-fourth is yellow. The region for each color is divided into three smaller regions of equal area. Of the three, one section is labeled X, one Y, and one Z.



If we want to find P(red) on one spin we can, if we like, ignore the letters.

yellow

$\frac{1}{4}$

1. We can think of the total set of equally likely outcomes as {red,green,blue,_____} .
2. We see at once: P(red) = _____.

However, if we want to find P(X) on one spin we can ignore the colors. We need only note that the three outcomes X, Y, Z are equally likely.

$\frac{1}{3}$

3. $P(X) = \frac{1}{\square}$.

We could have found P(red) and also P(X) by considering the set of 12 outcomes {red X, red Y, red Z, green X, green Y, green Z, blue X, blue Y, blue Z, yellow X, yellow Y, yellow Z} . All of the outcomes in this set are equally likely.

3

$\frac{1}{4}$

4

$\frac{1}{3}$

4. There are _____ (how many) outcomes where the color is red.
5. $P(\text{red}) = \frac{3}{12} =$ _____ .
6. There are _____ outcomes where the letter is X .
7. P(X) = _____.

What is the probability of getting a red X on one spin?

1 — 8. We might think: The event $red \cap X$ has ______ (how many) outcome(s).

$\frac{1}{12}$ — 9. $P(red \cap X)$ = ______.

P(X) — 10. Observe that $P(red \cap X) = P(red) \cdot$ ______.

In our previous examples of independent events, we had two actions (like throwing a die and then spinning a spinner, or tossing a coin twice). Here we have only one spin. However, it still seems reasonable to regard X and red as independent events.

Here is why.

Suppose we spin the spinner.

$\frac{1}{3}$ — 11. Our probability of getting X on the spin is ______,

$\frac{1}{3}$ — because the X regions cover ______ of the spinner.

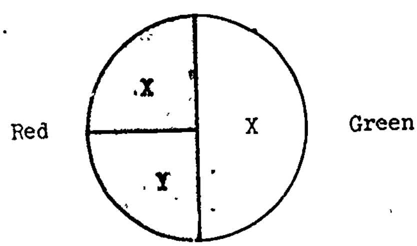
Now suppose we spin, without looking at the spinner. Suppose someone looks and tells us that we got red.

$\frac{1}{3}$; $\frac{1}{3}$ — 12. We continue to feel that our probability of X is ______, because the red X region covers ______ of the red region.

Knowing what color was spun does not change our idea of the probability of X. Hence the situation again illustrates independent events. Again we can use the formula:

$$P(A \cap B) = P(A) \cdot P(B).$$

But now consider this spinner, where the red Y region is $\frac{1}{4}$ the area. Suppose we spin it.



$\frac{1}{4}$

13. The probability of spinning Y is ____.

0

14. Suppose we spin, without looking, and someone tells us that we have spun green. With this knowledge, we say: We judge the probability that we have spun Y to be ____.

are not

15. We feel, in this case, that spinning green and spinning Y $\underset{\text{(are, are not)}}{\underline{\hspace{6em}}}$ independent events.

$\frac{1}{2}$

$\frac{1}{4}$

0

16. Notice that:

P(green) = ____;

P(Y) = ____;

P(green Y) = ____.

$\neq$

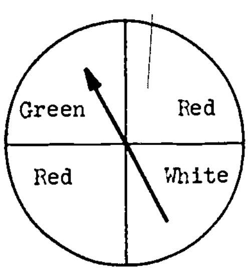
17. P(green Y) $\underset{(=,\ \neq)}{\underline{\hspace{3em}}}$ P(green)P(Y) .

In the next chapter we will learn more about this kind of situation.

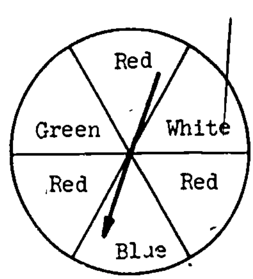
6-6. Exercises

(Answers on p.133 .)

1. You toss a coin twice in succession. Let A be the event that a tail shows on the first toss of the coin. Let B be the event that a head shows on the second toss.
   (a) Are events A and B independent? Explain.
   (b) Find the probability that the coin will show tails on the first toss and heads on the second.

2. 

The four sections are equal.



The six sections are equal.

Both pointers are made to spin.
   (a) What is the probability that both will stop on red?
   (b) What is the probability that both will stop on green?
   (c) What is the probability that the pointer for A stops on white and the pointer for B stops on blue?

3. If you have a bag containing five black marbles and four white marbles, what is the probability of drawing two white marbles from the bag if one is drawn and then replaced before the second drawing?

4. In problem 3, what is the probability of drawing two white marbles if the first one is not replaced before the second drawing?

5. Assume that each time a child is born the probability of a boy is $\frac{1}{2}$, and of a girl is $\frac{1}{2}$.

   (a) If a family with two children is selected at random, what is the probability the children are a boy and a girl?

   (b) What is the probability that the older is a boy and the younger a girl?

   (c) What is the probability that the older is a girl and the younger a boy?

   (d) If this family have a third child, what is the probability that it will not be a girl?

6. On a baseball team, player A has a batting average of .320 and player B's batting average is .280. Both players come to bat in the seventh inning. Assume that "hit for A" and "hit for B" are independent events.

   (a) What is the probability that both A and B get hits in the seventh inning?

   (b) What is the probability that either A, or B, or both, get hits in the inning?

7. John and Jim were born the same year, and each married at age 21. Use the Actuaries Table of Mortality, p. 48 of Chapter 4 to find the following probabilities:

   (a) The probability that John is alive at age 70.

   (b) The probability that both John and Jim are alive at age 70.

   (c) The probability that at least one of them is alive at age 70.

8. In each of 2 laundry bags you have some socks, not sorted into pairs. In one bag there are 5 black socks and 4 blue socks--9 socks in all. The other bag contains 15 socks, of which 7 are black and 8 are blue. If you pick one sock from each bag without looking, what is the probability that:

   (a) both are black?

   (b) both are blue?

   (c) one is black and one is blue?

9. There are 5 socks, unsorted, in a bureau drawer. Of these, 3 are blue and 2 are green. If you reach into the drawer in the dark and take out 2 socks, what is the probability that:

(a) both are green?

(b) both are blue?

(c) one is green and one is blue?

10. A certain problem is to be solved by 2 men, A and B. The probability that A will solve the problem is $\frac{2}{3}$, and the probability that B will solve it is $\frac{5}{12}$. Assume A and B work separately, so that the events involved are independent.

(a) What is the probability that the problem will not be solved?

(b) What is the probability that it will be solved by A and not by B?

(c) What is the probability that it will be solved by B and not by A?

(d) What is the probability that it will be solved by both men?

(e) What is the probability that it will be solved?

The following problems review ideas from earlier chapters.

11. When 6 coins are tossed, what is the probability that at least 1 head will be obtained?

12. There are 5 sticks. One is an inch long, one is 2 inches long, and so on up to 5 inches. A person picks up 3 of these sticks at random. What is the probability that he can form a triangle with them? Remember that the sum of the lengths of any two sides of a triangle must be greater than the length of the third side.

13. If two dice are thrown, what is the probability that the sum of the faces is either odd, or less than 5, or both?

14. Four cards consist of the ace and king of hearts and the ace and king of spades. One card is picked at random.

(a) What is the probability that the card is either an ace or a spade?

(b) What is the probability that it is either an ace or a king?

7

15. Of the 15 boys in homeroom 107 of Smith Junior High School, 11 signed up for noontime intramural baseball and 8 signed up for noontime intramural basketball. Two signed up for cafeteria work and cannot participate in noontime games. Every boy in the room has signed up either for cafeteria work or for one or both of the sports. If Bob is a member of homeroom 107, what is the probability that he

(a) signed up for baseball?

(b) signed up for basketball?

(c) signed up for either baseball or basketball?

(d) signed up for either baseball or cafeteria duty?

16. A spinner has 2 colors, green and yellow. Green is twice as likely as yellow. Find P(green), P(yellow).

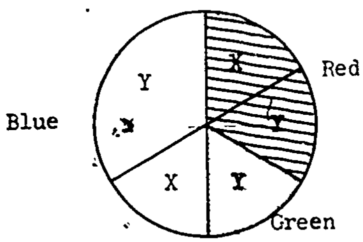
17. A certain experiment has 3 outcomes, A, B, C. A is twice as likely as B, and C is three times as likely as B. Find P(A), P(B), P(C).

Chapter 7

CONDITIONAL PROBABILITY

7-1. An Experiment

Here is a spinner which we have seen before. The red X, red Y, green Y, and blue X regions all have the same area. The blue Y region has twice the area of the blue X region.



We wish to conduct 100 trials of the experiment "spin the spinner" and to record the results -- both by color and by letter. You may either

(a) build such a spinner

OR

(b) read below.

Note that for our spinner: $P(\text{red } X) = \frac{1}{6}$

$$P(\text{red } Y) = \frac{1}{6}$$

$$P(\text{green } Y) = \frac{1}{6}$$

$$P(\text{blue } X) = \frac{1}{6}$$

$$P(\text{blue } Y) = \frac{2}{6} = \frac{1}{3}$$

Is there any experiment that you have already performed in which the probability $\frac{1}{6}$ occurs?

Look back to the experiment of Section 1-5.

For a throw of a die there are six outcomes, each with probability $\frac{1}{6}$. Suppose we regard "5 or 6" as a single outcome. Then

$$P(1) = \frac{1}{6}$$

$$P(2) = \frac{1}{6}$$

$$P(3) = \frac{1}{6}$$

$$P(4) = \frac{1}{6}$$

$$P(5 \text{ or } 6) = \frac{2}{6} = \frac{1}{3}$$

Thus you can use your record of throwing a die 100 times to <u>simulate</u> 100 spins of a spinner. (To simulate is to imitate, to make conform to the same laws.)

You simply match each number thrown with a spinner outcome, as follows:

$$\begin{aligned} 1 &\longrightarrow \text{red } X \\ 2 &\longrightarrow \text{red } Y \\ 3 &\longrightarrow \text{green } Y \\ 4 &\longrightarrow \text{blue } X \\ 5,6 &\longrightarrow \text{blue } Y \end{aligned}$$

Thus if the first 5 numbers in your die experiment were 4 3 5 5 3, you would record:

blue X, green Y, blue Y, blue Y, green Y

Before analyzing your record, it is interesting to see what results might be expected.

<u>Exercises</u> (Answers on page 136 .)

By thinking about the spinner (or the die), you should be able to determine the following probabilities, and answer the questions.

1. P(bl) = ____.
2. P(X) ____.
3. P(blue X) = ____.
   Note: P(blue X) is the same as P(blue <u>and</u> X) .
4. Is it true that P(blue X) = P(blue) · P(X) ?
5. Are the events "blue" and "X" independent?
6. P(green) = ____.

7. Are "green" and "X" independent events?

8. Suppose we know that the spinner has stopped in the red region. With this knowledge, what is the probability that the spinner shows X ?

9. Are "red" and "X" independent events?

Questions on the Experiment

Using your record (either for the spinner or for the 100 throws of a die), answer each of the following. (Our results are on page 123 .)

1. What fraction of the spins are red X ? red Y ? green Y ? blue X ? blue Y ?

2. What fraction of the spins are X ?

3. What fraction of the spins are red ? blue ?

4. Do your results yield (approximately)

$$P(\text{blue X}) = P(\text{blue}) \cdot P(X) \ ?$$

5. Examine only those spins that are blue. What fraction of these are X ?

6. Is the answer to 5 approximately the same as the answer to 2 ?

7. Examine only those spins that are red. What fraction of these are X ?

8. Is the answer to 7 approximately the same as the answer to 2 ?

7-2. Introduction

Sometimes during the course of an experiment we receive partial information which causes us to reconsider our judgment about the probability of some event.

Suppose a friend selects a playing card at random from a regular 52-card deck.

$\frac{13}{52}$, $\frac{1}{4}$

1. The probability that the card selected is a spade is $\frac{\square}{52}$ or $\frac{1}{\square}$

52

$\frac{1}{52}$

13

$\frac{13}{52}$, or $\frac{1}{4}$

2. To obtain P(spade) = $\frac{1}{4}$ we reason as follows:

   Every one of the 52 cards is equally likely. Our set of outcomes has ________ (how many) elements, each having probability ________. Since there are 13 spades, the event "select a spade" consists of ________ (how many) elements.

   Hence P(spade) = ________.

Once again, a friend selects a card at random. This time, however, before we judge the probability of a spade being selected, the friend tells us that he has selected a black card. Stop and think. With this information, is "spade" more likely than before?

$\frac{13}{26}$

3. Reasoning as before, we conclude that, since the card is black, the probability of a spade, given that the card is black, is $\frac{\square}{26}$, or $\frac{1}{2}$.

   We shall write P(spade, given black) in place of "the probability of a spade, given that the card is black".

26, 13

4. We know that our set of possible outcomes now contains only ________ (how many) elements. ________ (how many) are in the event spade?

has

5. The information that the card is black ________ (has, has not) caused us to change our judgment of the probability of the event spade.

greater

6. Notice that P(spade, given black) is ________ (less, greater) than P(spade).

We could express our results of Items 1-4 as follows:

The probability of selecting a spade at random from a 52-card deck is $\frac{1}{4}$.

The probability of selecting a spade at random from a 52-card deck is $\frac{1}{2}$, <u>given that the selected card is black</u>, because we know that the card is one of a certain 26.

	As before, a friend selects a card. This time he tells us that the card selected is red.
0	7. P(spade, given red) = _______.
less	8. Notice that, in this case, P(spade, given red) is ____________ (less, greater) than P(spade).

	Suppose our obliging friend, after selecting a card at random, tells us that he has chosen an ace.
$\frac{1}{4}$	9. P(spade, given ace) = _______.
does not	10. In this case, the additional in rmation ____________ (does, does not) result in a change in our judgment of the probability that the card is a spade.

Here is another example.

{1,2,3,4,5,6}	11. The set of possible outcomes for the throw of a die is {1, ____________}.
$\frac{1}{6}$	12. The probability of getting a 4 is _______.
{2,4,6}	13. A die is thrown. Suppose someone whispers to us that an even number is showing. The set of possible outcomes is now _______.
$\frac{1}{3}$	14. The probability now that we have a 4 is _______. Quite a difference!

Now for a slightly more complicated case. A class consists of 15 boys and 10 girls. Four of the boys and three of the girls are left-handed. One member of the class is selected at random.

$\frac{15}{25}$, or $\frac{3}{5}$	15. P(boy) = _______.
$\frac{7}{25}$	16. P(left-handed) = _______.
$\frac{4}{25}$	17. P(boy and left-handed) = _______.

	Now!
$\frac{4}{15}$	18. P(left-handed, given that a boy is selected) = ________.
$\frac{4}{7}$	19. P(boy, given that a left-hander is selected) = ________.
	If you answered Items 18 and 19 correctly, skip Items 20-25.
15	20. There are ________ boys.
4	21. Of the boys, ________ (how many) are left-handed.
$\frac{4}{15}$	22. P(left-handed, given that a boy is selected) = ________.
7	23. There are ________ (how many) left-handers in the class?
4	24. Of the left-handers, ________ (how many) are boys?
$\frac{4}{7}$	25. P(boy, given that a left-hander is selected) = ________.

In our examples of this section we have encountered situations in which we were asked to determine the probabilities of events after some information was received. That is, we determined probabilities subject to some restriction (or condition) on the set of outcomes. Probabilities such as P(boy, given that a left-hander was selected) are known as conditional probabilities.

26. Some coins are tossed. Let E be the event "exactly two heads show". Then $P(E)$

[A] is $\frac{1}{4}$ [C] is 0

[B] is $\frac{3}{8}$ [D] cannot be determined

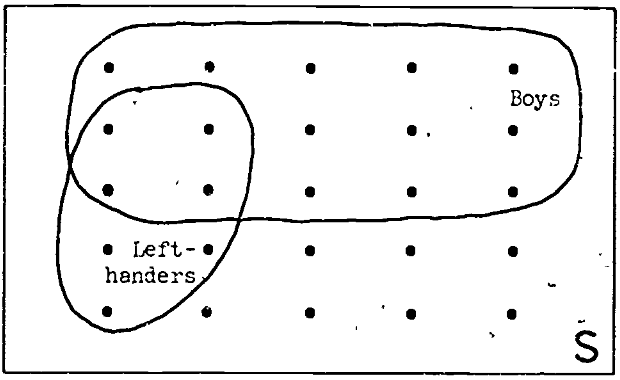
If two coins are tossed, [A] is correct; for three coins [B]; and if only one coin is tossed, $P(E) = 0$. [D] is the correct response. We cannot determine $P(E)$ until we know the set of outcomes of which E is a subset.

Item 26 serves to remind us that all probabilities are in a sense "conditional". For a particular case, if S is the set of all possible outcomes and if E is an event, we have written $P(E)$. We could have written $P(E, \text{given } S)$. If the set S is understood, then there is no confusion about what is meant by $P(E)$.

Review Items 11-14. Before the die is thrown the event "4" is one of six equally likely outcomes and hence $P(4) = \frac{1}{6}$. After we learn that the die shows an even number we are led to consider a reduced set of outcomes $\{2,4,6\}$. With reference to this new set of three equally likely outcomes, $P(4$, given even$) = \frac{1}{3}$.

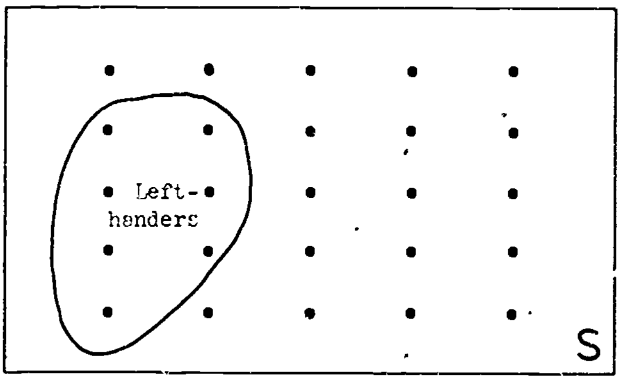
In the example about the left-handed students (Items 15-19), we begin by selecting at random one student from a class of twenty-five. Our set of possible outcomes has twenty-five equally likely elements. Since seven are left-handed, we have $P(\text{left-handed}) = \frac{7}{25}$. When we learn that a boy has been selected, our attention is focused on a reduced set of outcomes which has 15 equally likely outcomes. In this reduced set there are only 4 members of the event "left-handed", hence $P(\text{left-handed, given that a boy is selected}) = \frac{4}{15}$.

A diagram for this situation is the following.



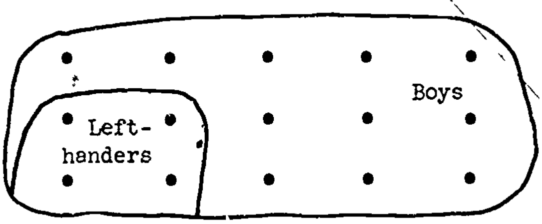
25 25	27. There are ________ students, so our full set of outcomes S is represented by ________ dots.

When we think of $P(\text{left-hander})$, we think



$\frac{7}{25}$ 28. The event "left-hander" is a subset of the original set S of possible outcomes. P(left-hander) = ____.

When we think of P(left-hander, given that a boy is selected) we use the reduced set of outcomes "boys". We look only at part of the diagram:



29. We think about the event "boy and left-handed" as a subset of the set "boys".

$\frac{4}{15}$ P(left-handed, given that a boy is selected) = ____.

In a general case, we are interested in an event E of some known set of possible outcomes S. If some information leads us to consider only some reduced set of outcomes, F, we then wish to find P(E, given F). The question arises: Suppose we know

$$P(E), \quad P(F), \quad P(E \cap F) \qquad \text{(Note: We are given } S.)$$

Can we determine P(E, given F) ?

For all our examples, the answer was "yes". It seems reasonable that a general method (a formula) might be developed. Perhaps we can use our example of left-handed students to help us guess at a formula.

$\frac{15}{25}$ 30. P(boy) = $\frac{\square}{25}$. (Item 15)

$\frac{7}{25}$ 31. P(left-handed) = ____. (Item 16)

$\frac{4}{25}$ 32. P(boy and left-handed) = ____. (Item 17)

$\frac{4}{15}$

33. P(left-handed, given boy) = _____. (Item 18)

How may we (arithmetically) use $\frac{15}{25}$, $\frac{7}{25}$, $\frac{4}{25}$ in some fashion to obtain $\frac{4}{15}$? You might try adding, multiplying, etc.

Did you happen to notice that

$\frac{4}{15}$

34. $$\frac{\frac{4}{25}}{\frac{15}{25}} = \frac{\square}{\square} \;?$$

P(boy)

35. For this case,

$$P(\text{left-handed, given boy}) = \frac{P(\text{boy and left-handed})}{\boxed{P(\qquad)}}.$$

Notice that in Item 34 we do not use P(left-handed) = $\frac{7}{25}$ at all!

Does a similar method work for P(boy, given left-handed) ?

$\frac{4}{7}$

36. From Item 19, P(boy, given left-handed) = _______.

Can you use $\frac{15}{25}$, $\frac{7}{25}$, $\frac{4}{25}$ to obtain $\frac{4}{7}$?

$\frac{4}{7}$

37. Of course!

$$\frac{\frac{4}{25}}{\frac{7}{25}} = \frac{\square}{\square}$$

Notice that, in Item 37, we did not use P(boy) = $\frac{15}{25}$ at all!

P(left-handed)

38. We h[illegible] seen (Item 37)

$$P(\text{boy, given left-handed}) = \frac{P(\text{boy and left-handed})}{\boxed{P(\qquad)}}.$$

$\frac{P(E \cap F)}{P(F)}$

39. Would you like to guess at a general formula now? Items 35 and 38 should lead you to guess:

If E and F are events, then

$$P(E, \text{given } F) = \frac{P(\qquad)}{P(\quad)}.$$

In Section 7-3 we will develop the formula of Item 39 on a more general basis.

Before going on, it is convenient to introduce a notation. Instead of P(E, given F) we will write:

$$P(E|F) .$$

This is read "the conditional probability of E, given F".

E is the event in which we are interested. F is the reduced set of outcomes. Since F is a subset of S, F is also an event.

given	40. $P(A|B)$ is "the conditional probability of A, ____ B".
conditional; X	41. $P(X|Y)$ is the ________ probability of ___, given Y.
$P(R|T)$	42. _____ is the conditional probability of R, given T.

Referring to Items 15-19, let B be the event "a boy is selected", L be the event "a left-hander is selected".

left-hander; boy	43. $P(L|B)$ is the conditional probability that a ____________ is selected, given that a _____ is selected.
$\frac{4}{15}$	44. $P(L|B) = \underset{\text{(number)}}{\underline{\qquad\qquad}}$. (Item 18)
$P(B|L)$	45. $\underline{P(\;|\;)}$ is the conditional probability that a boy is selected, given that a left-hander is selected.
$\frac{4}{7}$	46. $P(B|L) = \underset{\text{(number)}}{\underline{\qquad\qquad}}$. (Item 19)
	Notice that $P(B|L) \neq P(L|B)$.

47. If $P(E) = \frac{1}{2}$, then $P(E|E) =$

[A] 1 [C] $\frac{1}{2}$

[B] 0 [D] You can't tell

$P(E|E)$ is the probability that E occurs given that E occurs. For example, we toss a coin. Let E be the event "head". Someone tells us we got "head". In this case we are sure. $P(E|E) = 1$, so [A] is correct.

Exercise (Answer on page 136.)

Suppose S has 10 members, all equally likely. Suppose E has 4 members, F has 3 members, and $E \cap F$ has 2 members. Draw an appropriate diagram. Find $P(E)$, $P(F)$, $P(E \cap F)$, $P(E|F)$, $P(F|E)$.

Experiment

In your class, determine the number of right- and left-handed boys and girls. Pretend that one student is to be selected at random. Let B, G, L, R have the obvious meanings. Find

(a) $P(B)$	(i) $P(B\|L)$
(b) $P(G)$	(j) $P(G\|L)$
(c) $P(L)$	(k) $P(B\|R)$
(d) $P(R)$	(l) $P(G\|R)$
(e) $P(B \cap L)$	(m) $P(L\|B)$
(f) $P(B \cap R)$	(n) $P(L\|G)$
(g) $P(G \cap L)$	(o) $P(R\|B)$
(h) $P(G \cap R)$	(p) $P(R\|G)$

If it is not convenient to determine right- or left-handedness, take L to be the set of students sitting in the left-hand row of seats in the classroom. R would then be the set of all other students. For a discussion of this experiment, see page 124.

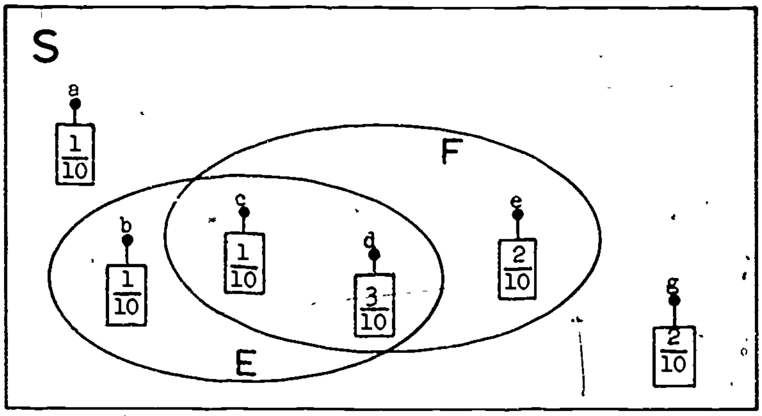
7-3. A Formula for Conditional Probability

We saw that, for our example of Section 7-2,

$$P(B|L) = \frac{P(B \cap L)}{P(L)} .$$

We shall try to discover whether this formula holds for all cases. We begin with another example.

Suppose that a set of outcomes, S, has six elements a, b, c, d, e, f. The probabilities of these outcomes are shown in the following diagram. Event $E = \{b,c,d\}$. Event $F = \{c,d,e\}$. We are interested in $P(E|F)$.



The sum of the probabilities of the elements of S is, of course, 1. That is, $P(S) = 1$. Outcome d, for instance, accounts for $\frac{3}{10}$ of the probability of S.

$\frac{5}{10}$

1. $P(E) = \frac{1}{10} + \frac{1}{10} + \frac{3}{10} = \frac{\square}{10}$.

Event E accounts for $\frac{5}{10}$ of the probability of S.

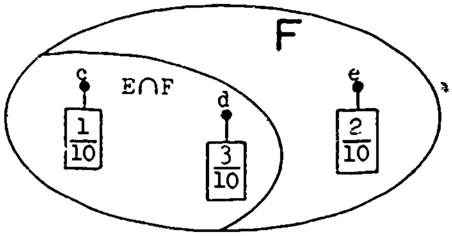
$\frac{6}{10}$

2. $P(F) = \frac{\square}{10}$.

$\frac{4}{10}$

3. $P(E \cap F) = \frac{\square}{10}$, since $E \cap F = \{c,d\}$.

Since we wish to compute $P(E|F)$, our attention is directed toward F, the reduced set of outcomes. If we are given that F has occurred, then we are concerned only with outcomes c, d, e.



$\frac{6}{10}$ 4. From Item 2, $P(F) = \frac{\square}{10}$ (given S!)

$\frac{6}{10}$ 5. The total of the probabilities attached to outcome c, d, e is $\underset{\text{(fraction)}}{\underline{\qquad}}$.

6. Outcome d, which accounts for $\frac{3}{10}$ of the probability of S, accounts for $\frac{3}{6}$ (or $\frac{1}{2}$) of the probability

F of ______.

Notice that $\dfrac{\frac{3}{10}}{\frac{6}{10}} = \frac{3}{6} = \frac{1}{2}$.

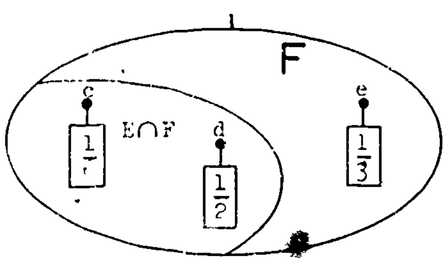
$\dfrac{\frac{1}{10}}{\frac{6}{10}} = \frac{1}{6}$ 7. Outcome c accounts for $\underset{\text{(fraction)}}{\underline{\qquad}}$ of the probability of F.

$\dfrac{\frac{2}{10}}{\frac{6}{10}} = \frac{1}{3}$ 8. Similarly, outcome e accounts for $\underset{\text{(fraction)}}{\underline{\qquad}}$ of the probability of F.

We may think of $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{3}$ as the probabilities of c, d, e relative to F.

1 9. Adding, we have $\frac{1}{2} + \frac{1}{6} + \frac{1}{3}$ ______.

Let us draw a new diagram, attaching numbers to c, d, e that indicate their probabilities (weights) relative to F.



10. Complete the following table.

Outcome	(I) Probability, given S	(II) Probability, given F
c	$\frac{1}{10}$	____
d	____	$\frac{1}{2}$
e	____	____

Compare your table with the one given below.

Outcome	(I) Probability, given S	(II) Probability, given F
c	$\frac{1}{10}$	$\frac{1}{6}$
d	$\frac{3}{10}$	$\frac{1}{2}$
e	$\frac{2}{10}$	$\frac{1}{3}$

Notice that if we multiply the entries in column II (Item 10) for c, d, and e by $\frac{6}{10}$ we obtain the corresponding entry of column I.

$\frac{1}{10}$ 11. For c : $\frac{1}{6} \cdot \frac{6}{10} =$ ________.

$\frac{3}{10}$ 12. For d : $\frac{1}{2} \cdot \frac{6}{10} =$ ________.

$\frac{1}{3} \cdot \frac{6}{10}$ 13. For e : ____ $\cdot \frac{6}{10} = \frac{2}{10}$.

Why is $\frac{6}{10}$ important?

We are interested in F.

$\frac{6}{10}$ 14. P(F) = ________ (given S!)

	Of course, we usually start (as in this example) with the entries in column I.
$\frac{6}{10}$	15. For outcomes c, d, and e the entries in column II are obtained from those in column I by dividing by ____.
$\frac{1}{6}$	16. For c : $\dfrac{\frac{1}{10}}{\frac{6}{10}}$ = ____.
$\frac{1}{2}$	17. For d : $\dfrac{\frac{3}{10}}{\frac{6}{10}}$ = ____.
$\dfrac{\frac{2}{10}}{\frac{6}{10}} = \frac{1}{3}$	18. For e : $\dfrac{\square}{\frac{6}{10}}$ = ____.
P(F)	19. $\frac{6}{10}$ = P( ) . (given S!)

We began with the problem of computing $P(E|F)$. Thus far we have examined the reduced set of outcomes, F. We have attached new probabilities to the outcomes of F. These new probabilities are obtained by dividing the original probabilities (given S) by P(F).

At last we are ready for $P(E|F)$.

	$E = \{b,c,d\}$, $F = \{c,d,e,f\}$
$E \cap F$	20. Since we are confining ourselves to F, we are interested in those elements of E which are also in F. These elements are those in the set E $\underset{(\cup,\cap)}{\underline{\quad\quad}}$ F.
$\{c,d\}$.	21. $E \cap F$ = {c, } .
$\frac{4}{10}$ $(= \frac{2}{5})$	22. Considering $E \cap F$ as an event in S ; $P(E \cap F) = \frac{1}{10} + \frac{3}{10} = \frac{\square}{10}$.
$\frac{4}{6}$ $(= \frac{2}{3})$	23. Considering $\{c,d\}$ as an event in F , $P(E\|F) = \frac{1}{6} + \frac{1}{2} = \frac{\square}{6}$.

It is important to notice (recalling Items 19, 22, and 23)

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{4}{10}}{\frac{6}{10}} = \frac{2}{3}.$$

We have gone through this example quite carefully and we have reached the conclusion

(1) $$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

The argument may be repeated for any set of outcomes S and for any events E, F -- provided $P(F) \neq 0$. Thus, (1) is a <u>formula</u> for finding $P(E|F)$ if we know $P(E \cap F)$ and $P(F)$.

0 — 24. If $P(F) = 0$, then formula (1) involves division by ____, which is never permissible.

In the remainder of the text, we shall assume: if we refer to a reduced set of outcomes, its probability is <u>not</u> 0.

Let us apply Formula (1) to two of our examples of Section 7-2. (You should also review Items 30-39 of Section 7-2.)

A die is thrown.

Let E be the event "2", F the event "an even number shows".

$\frac{1}{2}$ — 25. $P(F) =$ ____.

$\frac{1}{6}$ — 26. $P(E \cap F) =$ ____.

$\frac{1}{3}$ — 27. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{6}}{\frac{1}{2}} =$ ____. (See Item 14, Section 7-2.)

Notice that $P(E) = \frac{1}{6}$, $P(E|F) = \frac{1}{3}$.

$\neq$ — 28. In this case $P(E)$ ____ $P(E|F)$.
(=, ≠)

$\frac{1}{52}$

$\frac{4}{52}$

$\frac{1}{4}$

=

A card is selected at random from a regular 52-card deck. Let E be the event "spade", F the event "ace".

29. $P(E \cap F) =$ ____.

30. $P(F) = \frac{\square}{52}$.

31. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{52}}{\frac{4}{52}} =$ ____. (See Item 9, Section 7-2.)

Notice that $P(E) = \frac{1}{4}$, $P(E|F) = \frac{1}{4}$.

32. In this case $P(E) \underset{(=, \neq)}{\underline{\qquad}} P(E|F)$.

The following exercises are provided in order to give you practice in using formula (1).

Exercises (Answers on page 136.)

1. If $P(E) = .2$, $P(F) = .5$, $P(E \cap F) = .1$, find
   (a) $P(E|F)$ (b) $P(F|E)$
2. If $P(A) = .8$, $P(B) = .2$, $P(A \cap B) = .16$, find
   (a) $P(A|B)$ (b) $P(B|A)$
3. If $P(X) = \frac{1}{3}$, $P(Y) = \frac{1}{4}$, $P(X \cap Y) = \frac{1}{6}$, find
   (a) $P(X|Y)$ (b) $P(Y|X)$
4. If $P(A) = .3$, $P(B) = .2$, $P(A \cap B) = 0$, find
   (a) $P(A|B)$ (b) $P(B|A)$
5. If $P(E) = .7$, $P(F) = .5$, $P(E|F) = .6$, find
   (a) $P(E \cap F)$ (b) $P(E \cup F)$ (Hint: Use result of 5(a).)
6. If $P(E) = .4$, $P(F) = .4$, $P(E \cup F) = .6$, find
   (a) $P(E \cap F)$ (b) $P(E|F)$ (c) $P(F|E)$
7. Suppose E and F are mutually exclusive events; what may we say about $P(E|F)$? $P(F|E)$? (Hint: See Exercise 4.)
8. Suppose E is a subset of F; what may we say about $P(E|F)$? $P(F|E)$?

7-4. Review and Exercises

A, given B	1. $P(A\|B)$ means "the conditional probability of ________".
conditional B	2. $P(B\|A)$ means "the ________ probability of ____ given A".
$P(X\|Y)$	3. The conditional probability of X given Y is written as $P(\quad)$.

=	4. If S is the set of all possible outcomes, then $P(E) \underset{(=,\ \neq)}{\underline{\qquad}} P(E\|S)$.
$P(F) = 0$	5. The conditional probability of E given F is not defined if $P(\quad) = 0$.
$\frac{P(E\cap F)}{P(F)}$	6. $P(E\|F) = \frac{P(\quad)}{P(F)}$.
0 (since $P(E\cap F) = 0$)	7. If E and F are mutually exclusive, then $P(E\|F) =$ ________.
$P(E\cap F)$, $P(F)$	8. $P(E\|F)$ is the ratio of $P(\quad)$ to $P(\quad)$.

9. If $P(E) \neq 0$, $P(F) \neq 0$, then

[I] $P(E|F) = P(F|E)$

[II] $P(E|F) \neq P(F|E)$

[A] I is always true.

[B] II is always true.

[C] Either I or II may be true.

For most of our examples and exercises [II] has been true. Exercise 6, Section 7-2, shows that [I] may also be true. [C] is correct.

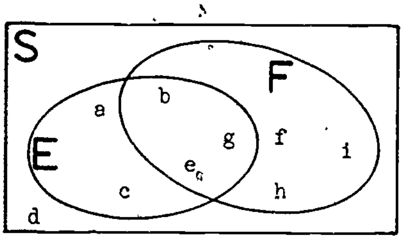
10. If $P(E) \neq 0$, $P(F) \neq 0$, then [I] $P(E\|F) < P(E)$ [II] $P(E\|F) > P(E)$ [III] $P(E\|F) = P(E)$ [A] III is always false. [B] Any of the three may occur. [C] II is never true. [D] I is always true.

Review Items 6, 8, 10 of Section 7-2. [B] is correct. Those special cases where $P(E|F) = P(E)$ are discussed in Section 7-6.

Exercises (Answers on page 138.)

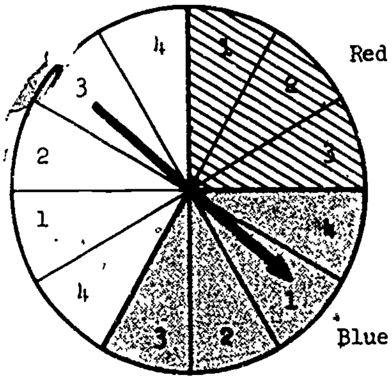
1. A letter is selected at random from the word "about". What is the probability of selecting:
   (a) a vowel?
   (b) an "o" given that a vowel was selected?
   (c) a "b" given that a vowel was selected?
   (d) a "t" given that a consonant was selected?

2. A red and a green die are tossed. What is the probability that:
   (a) the sum is under 5 ?
   (b) the sum is under 5 given that the red die shows a 2 ?
   (c) the sum is under 5 given that the red die is 5 ?
   (d) the sum is under 5 given that the green die shows an odd number?

3. 

   $S = \{a,b,c,d,e,f,g,h,i\}$. All outcomes are equally likely. Find
   (a) $P(E)$,
   (b) $P(F)$,
   (c) $P(E|F)$,
   (d) $P(F|E)$.

4. Consider the spinner shown here. Each of the 12 small regions has the same area.



Find

(a) P(Red)
(b) P(1)
(c) P(Red ∩ 1)
(d) P(Blue ∪ 2)
(e) P(Red|1)
(f) P(1|Red)
(g) P(2)
(h) P(2|White)
(i) P(White|2)
(j) P(4|Red)
(k) P(Red|Odd)
(ℓ) P(White|4)

5. There are 4 balls in a box: 2 black and 2 white. Balls are drawn out in succession and kept out. Any ball in the box has the same chance of being drawn as any other. B_1 means black on first draw, B_2 means black on second, B_2B_3 means black on second <u>and</u> third draws, etc. Find the following probabilities:

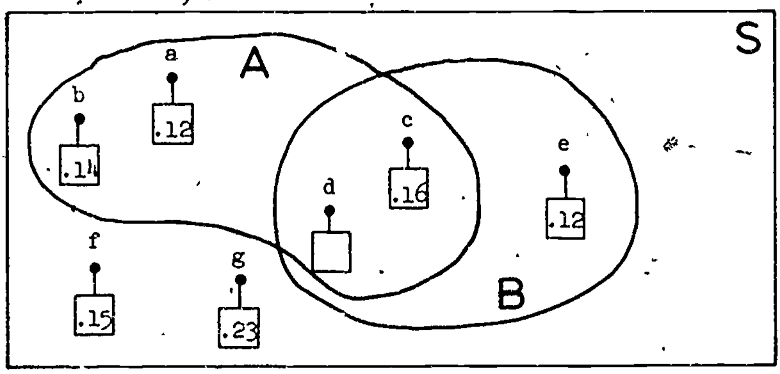
(a) $P(B_2|B_1)$

(b) $P(B_1|B_2B_3)$

(c) $P(B_1|W_2)$

(d) $P(W_1|B_2B_3)$

6. The diagram shows a set, S, of outcomes and the probability of each outcome. Events A, B are indicated.



(a) $P(d)$

(b) $P(A)$

(c) $P(B)$

(d) $P(A \cap B)$

(e) $P(A \cup B)$

(f) $P(A|B)$

(g) $P(B|A)$

7. An analysis of the success of weather forecasts for a 200-day period in a certain city shows

	Wet Day	Dry Day
Forecast: Rain	30	20
Forecast: No Rain	30	120

The table is to be read as follows: there were 50 days for which rain was predicted. Wet weather (rain) occurred on 30 of these days, etc.

Let R be the event "rain predicted",
N be the event "no rain predicted",
W be the event "wet" (i.e., it did rain),
D be the event "dry" (i.e., it did not rain).

(a) A picnic is planned for a certain day. The weather prediction is "rain". What is the probability that it actually is a wet day the day of the picnic?

(b) One day it was actually raining. What is the probability that the forecast was "rain"?

(c) Find $P(D|N)$ and interpret the answer.

(d) Find $P(N|D)$ and interpret the answer.

†8. A poll-taker obtained the following results when he asked teen-agers and adults their reactions to a certain TV program:

25% of those interviewed were teen-agers
60% of the teen-agers enjoyed the program
20% of the adults enjoyed the program.
(By adult we mean someone over age 19.)

If we choose, at random, one person who was interviewed, what is the probability that the person is:

(a) an adult?

(b) an adult who liked the program?

(c) a teen-ager who liked the program?

(d) someone who liked the program?

(e) someone who liked the program, given that the person is a teen-ager?

(f) an adult, given that the person liked the program?

†9. The method and reasoning of Exercise 8 have serious applications in many fields. Here is an example from the field of medicine.

If a person has a certain disease, then a blood test will reveal that fact with probability .90 . Unfortunately this test, like many others, yields "false positives". If a person is healthy the test will falsely indicate the presence of the disease with probability .05 . Suppose further that only 2% of the population has the disease. A person, chosen at random from the population, is given the test. The test shows "positive". What is the probability that the person is, in fact, healthy?

†7-5. Special Exercises

In this section we present three problems in which you can use your knowledge of conditional probability to discover the correct solutions. The problems are of special interest because the answers seem to be contrary to one's intuition. Try to guess the answers before working the problems. See the discussion on page 143 .

1. A new family is moving into the neighborhood. You learn that the family consists of a man, wife, and two children.
   (a) Lacking any further information, what is the probability that both children are boys?
   (b) A friend tells you that he met one of the children and that it was a boy. With this new information, what is the probability that both children are boys? (Clearly, the outcome "two girls" has been ruled out.)
   (c) Your friend supplies the additional detail that the boy he met is the older child. Does this knowledge change your judgment of the probability of "two boys"?

2. Consider a "deck" of four cards - ace of spades, ace of hearts, king of spades, king of hearts. Two cards are chosen at random.
   (a) What is the probability that the two cards are both aces?
   (b) If you know that at least one ace has been selected, then P(two aces|at least one ace) = ?
   (c) Does it make any difference if you know that one of the two cards selected is the ace of spades?

3. This one is a bit harder. Three identical bags contain, respectively, two white, one white and one black, two black marbles. A bag is selected at random. (The other bags are set aside.)

(a) What is the probability that the bag selected contains two black marbles?

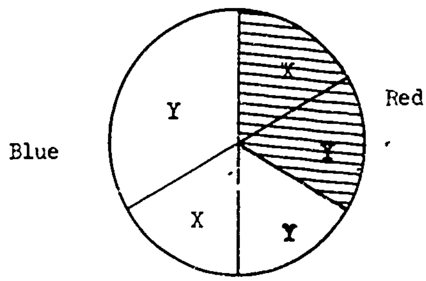
(b) One marble is drawn from the bag and it is black. What is the probability that the marble that remains in the bag is also black?

7-6. $P(E|F) = P(E)$

In several of our examples (Item 9, Section 7-2; Item 32, Section 7-3; Exercise 2, Section 7-3) we have encountered situations in which

$$P(E|F) = P(E) \quad .$$

Let us see another example. Refer to the spinner of Section 7-1.



For this spinner
1. $P(X) =$ _____.
2. $P(X|\text{blue}) =$ _____.
3. So, $P(X|\text{blue})$ _____ $(=, \neq)$ $P(X)$.

Before the spinner is spun we judge $P(X) = \frac{1}{3}$. After the spin, if we learn that the spinner has stopped in the blue region, we still judge that the probability of X is $\frac{1}{3}$. (In symbols, $P(X|\text{blue}) = \frac{1}{3} = P(X)$.) To repeat, our judgment of the probability of X is not affected by the knowledge that the spinner has stopped in the blue region. This is the same type of statement that was made in Chapter 6 -- event "X" is independent of event "blue".

In general, $P(E|F) = P(E)$ means that the probability of E is the same whether we refer to the original set of outcomes or to the reduced set F.

Hence, $P(E|F) = P(E)$ is a mathematical statement of "E is independent of F".

We have learned (Chapter 6) that if E is independent of F, then $P(E \cap F) = P(E) \cdot P(F)$. Let us see if we may obtain this same result by starting with $P(E|F) = P(E)$.

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{3}$	4. For the spinner we have been discussing $P(\text{blue}) =$ ____. $P(X) =$ ____. $P(X\|\text{blue}) =$ ____.
$\frac{P(X \cap \text{blue})}{P(\text{blue})}$	5. From our formula for conditional probability, we know $P(X\|\text{blue}) = \frac{\boxed{}}{P(\text{blue})}$.
$\frac{P(X \cap \text{blue})}{P(\text{blue})}$	6. Since $P(X\|\text{blue}) = P(X)$ we may write $P(X) = \frac{P(X \cap \text{blue})}{\boxed{}}$.
$\frac{1}{3} = \frac{P(X \cap \text{blue})}{\frac{1}{2}}$	7. Using the numerical values of Item 4, ____ $= \frac{P(X \cap \text{blue})}{\boxed{}}$.
$\frac{1}{3} \cdot \frac{1}{2}$	8. Therefore $P(X \cap \text{blue}) = \frac{1}{3} \cdot$ ____ $= \frac{1}{6}$.
$\frac{1}{6}$	9. Look at the spinner. What fraction of the spinner is <u>both</u> X and blue? ____

For the general case, if

$$P(E|F) = P(E) ,$$

then

$$P(E) = \frac{P(E \cap F)}{P(F)}$$

and

$$P(E \cap F) = P(E) \cdot P(F) .$$

We have seen that if $P(E|F) = P(E)$, then E is independent of F and that $P(E \cap F) = P(E) \cdot P(F)$. Is it also true that F is independent of E? We would hope so, since this would agree with our previous ideas about independent events.

Let us assume that $P(E|F) = P(E)$.

$P(E) \cdot P(F)$	10. Since $P(E\|F) = P(E)$, we know $P(E \cap F) =$ ________.
$\frac{P(E \cap F)}{P(E)}$	11. Now, by our formula for conditional probability, $P(F\|E) = \frac{P(E \cap F)}{\boxed{}}$. (Again, you recall that $E \cap F = F \cap E$.)
$P(F)$	12. Substituting $P(E \cap F) = P(E) \cdot P(F)$ from Item 10, we have $P(F\|E) = \frac{P(E) \cdot P(F)}{P(E)} =$ ______.
E	13. $P(F\|E) = P(F)$ means that event F is independent of event ____.

To summarize:

if $P(E|F) = P(E)$, then

(a) $P(F|E) = P(F)$

(b) E and F are independent events

(c) $P(E \cap F) = P(E) \cdot P(F)$.

Here are some familiar examples.

	A card is to be selected at random from a regular 52-card deck.
$\frac{1}{4}$	14. P(spade) = ____.
$\frac{1}{13}$	15. P(ace) = ____.
$\frac{1}{4}$	16. P(spade|ace) = ____.
$\frac{1}{52}\ (=\frac{1}{4} \cdot \frac{1}{13})$	17. P(spade <u>and</u> ace) = ____.
independent	18. The event "a spade is selected" and the event "an ace is selected" are ________ events.

A die is thrown.

$\frac{1}{6}$	19. $P(2) =$ ___.
$\frac{1}{2}$	20. $P(\text{less than } 4) =$ ____. (There are three outcomes.)
$\frac{1}{3}$	21. $P(2 \mid \text{less than } 4) =$ ____.
are not	22. "2" and "less than 4" ________ (are, are not) independent events.
1	23. $P(\text{less than } 4 \mid 2) =$ ____.

24. Two dice are thrown, one green, one red. Let E be the event "green die shows an even number". Let F be the event "the sum of the faces is 9".

[A] E and F are independent events.

[B] E and F are not independent events.

Here is a case where the answer is not intuitively obvious. Of the 36 possible outcomes, 18 are in E, 4 in F. Furthermore, exactly 2 are in $E \cap F$. Hence

$$P(E \mid F) = \frac{\frac{2}{36}}{\frac{4}{36}} = \frac{1}{2} = P(E) .$$

[A] is the correct response.

Exercises (Answers on page 146 .)

1. Two dice are thrown, one red and one green. Let A be the event "sum is 10" . B is the event "sum is 7" . C is the event "red die shows 6" . Find
   (a) $P(A \mid C)$
   (b) $P(B \mid C)$
   (c) $P(A \mid B)$
   (d) $P(C \mid A)$
   (e) $P(C \mid B)$
   (f) $P(B \mid A)$
   (g) which pair(s) of events are independent
   (h) which pair(s) of events are mutually exclusive

2. Using the notation of Exercise 1, find
   (a) $P(A \cup B)$ (b) $P(B \cup C)$ (c) $P(A \cup C)$

In Chapter 6 you worked many exercises in which it was "reasonable" to assume that the events considered are independent. For the exercises that follow, it is interesting to guess whether the events are independent. After guessing, compute the appropriate conditional probability and state whether the events are, in fact, independent.

3. Two dice are thrown. Event A: both dice show even numbers. Event B: the sum is 8 .

4. The Smiths have three children. Assume that "boy" and "girl" are equally likely for each child. Event E: the family includes children of both sexes. Event F: there is at most one girl.

5. The Robinsons have four children. Events E and F are as described in Exercise 4.

6. A number is selected at random from $\{1,2,3, \ldots ,12\}$.
   $C = \{2,3,4,5,6,7\}$, $D = \{6,7,11,12\}$.

7. Three points P, Q, and R are placed at random in a line. A is the event "Q is to the right of P", and B is the event "R is to the right of P".

8. Able Batter has made 140 hits out of the last 400 times at bat. On 160 of these times at there were one or more men on base; he hit safely on 50 of these 160 times at bat. Able Batter now comes to the plate; H is the event "He gets a hit", and R is the event "There is a runner on base".

APPENDIX

Odds

We often hear such expressions as: "the odds are 1 to 1 that a coin will fall heads", or "it's 5 to 1 against a die showing 6".

These two examples should give you a hint as to the relationship between the "odds in favor of an event" and the "probability of an event".

The odds in favor of an event E are simply a to b, where a and b are any two numbers in the ratio of P(E) to P(not-E). (Not-E, you recall, is the event "E does not occur".)

For example, if the probability of E is $\frac{2}{3}$, so that the probability of not-E is $\frac{1}{3}$, then the odds in favor of E are $\frac{2}{3}$ to $\frac{1}{3}$ or, more simply, 2 to 1. (They are also 10 to 5; etc.)

Evidently, if the odds in favor of an event E are a to b, then the odds against E -- that is, in favor of not-E -- are b to a. In the preceding illustration, the odds against E are 1 to 2.

Example 1. Two coins are tossed. A is the event "two heads". What are the odds in favor of A? What are the odds against A?

$\frac{1}{4}$	1. P(A) = ______.
$\frac{3}{4}$	2. P(not-A) = 1 - P(A) = ______.
$\frac{1}{4}$ 1	3. The odds in favor of A are ______ to $\frac{3}{4}$, or ______ to 3.
3	4. The odds against A are ______ to 1.
	The odds are 1 to 3 in favor of two heads on the toss of two coins, and 3 to 1 against.

We know that if two fractions have the same denominator, then the ratio of these fractions is the same as the ratio of their numerators. Therefore if, for any event E, we are given P(E) and P(not-E) as fractions with the same denominator, then we can immediately state the odds in favor of E. We simply read off the numerators.

For example, the ratio of $\frac{2}{3}$ to $\frac{1}{3}$ is the same as the ratio of 2 to 1 (the odds in favor of E in the illustration at the beginning). The ratio of $\frac{1}{4}$ to $\frac{3}{4}$ is the same as the ratio of 1 to 3 (the odds in favor of A in Example 1).

Example 2. A die is tossed. B is the event "a number greater than 2". What are the odds in favor of B ? What are the odds against B ?

$\frac{2}{6}$	5. $P(B) = \frac{4}{6}$, $P(\text{not-}B) = \frac{\_\_\_}{6}$.
	6. The odds in favor of B are $\frac{4}{6}$ to $\frac{2}{6}$
4	or ______ to 2 ,
2	or ______ to 1 .
1	7. The odds against B are ______ to 2 .

Example 3. Swat King's batting average is .325 . What are the odds in favor of C , a hit his next time at bat? The odds against C ?

.325	8. P(C) = ______ .
.675	9. P(not-C) = ______ .
675	10. The odds in favor of C are 325 to ______ ,
13	or ______ to 27 .
27	11. The odds against the hit are ______ to 13 ,
2	or approximately ______ (an integer) to 1 .

If we know that $P(E) = p$, then the odds in favor of E are in the ratio p to q , where $q = 1 - p$. Suppose we know the odds in favor of E ; can we find P(E) ?

	Suppose, for a certain spinner, the odds in favor of red are 3 to 2 .
3 to 2	12. Then the ratio of p to q is ___ to ___ .
$\frac{3}{5}$	Hence P(red) = ______ .

If you had difficulty, reread the discussion following Item 4 and complete Items 14 to 17.

1 | 14. We wish to determine two fractions, p and q, where $p + q =$ ______ .

3 to 2 | 15. Moreover, we wish these fractions to be in the ratio ____ to ____ .

From the earlier discussion, we see that it is simple to use 3 and 2 as the numerators of our fractions. Our task then is to find a denominator (the same denominator for both fractions).

$\frac{5}{d}$ | 16. $p + q = \frac{3}{d} + \frac{2}{d} = \frac{3 + 2}{d} = \frac{\square}{d}$.

5 | 17. But $p + q = 1$, hence we wish $\frac{5}{d} = 1$. Therefore, $d =$ ____ .

$\frac{3}{5}$, $\frac{2}{5}$ | 18. Finally, $p = \frac{3}{\square}$, $q = \frac{2}{\square}$.

In general, if the odds in favor of E are a to b, then $P(E) = \frac{a}{a + b}$.

Exercises

1. Two coins are tossed. What are the odds in favor of throwing both a head and a tail? What are the odds against?

2. A die is thrown. What are the odds in favor of throwing a prime number? Against?

3. A die is thrown. What are the odds in favor of throwing a perfect square? Against?

4. Jimmy's batting average is .200 . What are the odds in favor of a hit his next time at bat? Against?

5. The team standing of homeroom 205 is .750 . What are the odds in favor of the team's winning their next game? Of losing?

6. Another common expression is that a certain event is a "50-50 bet". How may this be interpreted in terms of probability? (NOTE: Some people even say that "the odds are even"!)

7. Bill tells John that "the chances are 10 to 1" that there will be a mathematics quiz tomorrow. Interpret this as a probability statement.

8. A man is willing to give odds of 4 to 3 that the Dodgers will win the World Series. If he thinks this is a fair bet, what is his judgment of probability that the Dodgers will win the series?

9. If the odds against an event are 7 to 5, what is the probability of the event?

Answers to Exercises

1. In favor: 1 to 1. Against: 1 to 1.
2. In favor: 1 to 1. Against: 1 to 1.
3. In favor: 1 to 2. Against: 2 to 1.
4. In favor: 1 to 4. Against: 4 to 1.
5. In favor: 3 to 1. Against: 1 to 3.
6. "50-50" or 50:50 is the same ratio as 1 to 1. Hence, $P(\text{event}) = \frac{1}{2}$.
7. By "chances" Bill means "odds". $P(\text{quiz}) = \frac{10}{11}$. (Notice that if Bill means that the probability of a test is .90, he should say that the chances are 9 to 1.)
8. $\frac{4}{7}$.
9. $\frac{5}{12}$.

Experiment 1-5.

Here are the results obtained by the authors.

1. and 2.

43553	53344	14166	53213	46451
54563	41353	35335	65536	64112
43253	62454	53263	33423	21531
24131	64235	26563	22522	21355

	Number on die face					
	1	2	3	4	5	6
Frequency, first row	4	1	6	6	5	3
Frequency, second row	3	1	7	3	7	4
Frequency, third row	2	5	8	4	4	2
Frequency, fourth row	3	8	4	2	5	3
Total	12	15	25	15	21	12

3. When we throw a die, there is no reason to expect one face to appear more than another. Hence we expect that out of many throws each number will occur about $\frac{1}{6}$ of the time. This means that in 100 throws we expect each number to occur about 17 times. But only about -- it would be quite surprising if our results were nearly on the nose.

4. In this example, there are two triples of like digits: the 333 in the third row and the 222 in the fourth. There are eleven additional pairs: a 11, a 22, three 33's, a 44, three 55's, and two 66's.

5. There is one triple of successive digits in our example: the 456 in the second row. There are thirteen additional pairs: two 12's, two 23's, four 34's, three 45's, and two 56's. You may ask: are successive digits just as probable as like digits?

6. Our trials yielded no group of five-of-a-kind. We have one four-of-a-kind (22522) and two threes-of-a-kind.

7. This really is surprising! It could happen with an ordinary die, but it is certainly very unlikely. This record makes you suspect that someone is using a die that doesn't have any 4's, 5's, or 6's.

We will now tell you a secret. This row of numbers was made up in the following way. Look at the first row in our table of one hundred throws. Here it is:

4 3 5 5 3	5 3 3 4 4	1 4 1 6 6	5 3 2 1 3	4 6 4 5 1
↓ ↓ ↓ ↓ ↓	↓ ↓ ↓ ↓ ↓	↓ ↓ ↓ ↓ ↓	↓ ↓ ↓ ↓ ↓	↓ ↓ ↓ ↓ ↓
2 3 3 3 3	3 3 3 2 2	1 2 1 3 3	3 3 2 1 3	2 3 2 3 1

What did we do?

2 — When we saw a 4, we changed it to a ____.

3 — Each 5 and 6 we changed to a ____.

But 1, 2, 3 we did not change at all.

We were pretending that the die had one face with 1, two faces with 2 (the real 2, and the 4), and three faces with 3 (the real 3, the 5, and the 6). Incidentally, of these 25 throws we have: four 1's, seven 2's, and fourteen 3's. Not surprising!

Do you remember the die in Problem 2 of Exercise 1-2? It has a 1 on one face, a 2 on each of two other faces, and 3's on the remaining faces. This is the kind of result you'd find with such a die. In fact, if you didn't have one (you probably didn't), you could do an experiment about it anyway. You could simply use an ordinary die, and call the 4 "two" and the 5 and 6 "three", just as we did above.

Experiment 2-1.

1. Our record was:

G R Y R G R   R R G Y R Y

We got 6 reds, 3 greens, 3 yellows.

We had expected each color to appear about the same number of times. Twelve trials isn't very many. You may want to try more. If you and several of your classmates put your results together, you will probably find that, in all, about $\frac{1}{3}$ of the marbles drawn are red, $\frac{1}{3}$ are green, and $\frac{1}{3}$ are yellow.

2. Our record:

WBBWWW BWWWWB

We got 8 whites and 4 blacks. We had expected <u>about</u> twice as many whites as blacks -- but not necessarily exactly twice as many.

<u>Experiment 3-1</u>.

1. Here is our record:

THTHT TTHHH

We got 5 heads and 5 tails! Did you? It is more likely that you got 6 of one and 4 of the other, and slightly less likely that you got 7 and 3. You may have had some other result, but it is far less likely.

2. Our record:

Penny	Nickel
T	T
H	T
T	H
H	T
T	H
T	H
T	T
H	T
T	T
T	H

We got no throws with 2 heads, seven with 1 head and 1 tail, three with 2 tails. Compare our results with yours.

3. We expect fewer throws with 2 heads in our second experiment than throws with 1 head in our first. One way to see why is to think: When I get heads on both coins, I have to get heads on the penny <u>and</u> on the dime. But I'd expect that about half of the times that I get heads on the penny I would get tails, rather than heads, on the dime. (You may have had another way of thinking about this.)

Experiment 4-3.

(a) Tossing a rivet provides an example of a situation in which we have no real way of determining the probability of "up" by inspection. Certainly you would expect a broad-head tack to fall "up" more often than you would a long screw with a small head. For a given rivet, whatever guess you make is not likely to be very accurate. We guessed 20 "up" for our rivet.

(b) We actually obtained 9 "up" in 50 trials.

(c) Our guess was not very good.

(d) For our rivet:

$$P(\text{"up"}) = \frac{9}{50} .$$

Experiment 6-1.

1. (a) Our record of pairs of draws:

RY RR GR GY YY GR YR RY RR GR
GY GG YG RY GR RY YY GG GR RG

The second color was red eight times, green four times, and yellow eight times. We expected about $\frac{1}{3}$ of each. There is no reason to believe that one color is more likely than another.

2. (a) Our results were:

GR RY RG GG RY GY YG GR RR GY
YY GY GG YG YR GR RG RY GY YG

The second color was red five times, green seven times, and yellow eight times. Again, our results were consistent with the fact that one color is just as likely as another. It is probable that yours were, too.

3. As you read the next section, keep in mind your guess about the probability of red on both draws in the first experiment. You will find the answer there.

You don't need to guess in the case of the second experiment. The probability of drawing both red in this case is 0 . It can't be done!

Experiment 6-4.

1. We obtained:

GG GG RG GG RG   GG GG GR GR GG   GG RG GG GG GR

GR RG   RG RG GR GG GR   GG GG GR GG RR   GG RG RG GG RR

We have: 15 GG
7 GR
8 RG
2 RR.

We have R on the first spin 10 times, and R on the second spin 9 times. We would expect the number of reds on the first spin and the number of reds on the second spin to be approximately equal.

2. On both first and second spins we expected more greens than reds, and this is what we obtained. In fact, we would expect green about 3 times as often as red for each spin.

3. We had 8 RG's and 2 RR's. This seems reasonable. You would expect to get green on the second spin 3 times as often as red.

Experiment 7-1.

1. Our totals (see our results of Experiment 1-5 on page 119):

red X : 12
red Y : 15
green Y : 25
blue X : 15
blue Y : 33 ("5" showed 21 times, "6" showed 12 times)

2. X shows 27 times out of 100.

3. For us, red shows 27 times out of 100 which is a bit less than $\frac{1}{3}$ of the times. Blue shows 48 times -- very close to $\frac{1}{2}$.

4. For our results $\frac{15}{100}$ is the fraction of blue X, $\frac{48}{100}$ is the fraction of blue, $\frac{21}{100}$ is the fraction of X. $\frac{48}{100} \cdot \frac{27}{100}$ is about $\frac{12}{100}$, which differs somewhat from $\frac{15}{100}$.

5. We have 48 blue spins. Of these 15 are X. Thus, the desired fraction is $\frac{15}{48}$, very nearly $\frac{1}{3}$.

6. $\frac{15}{48} \sim .31$ does not differ greatly from $\frac{27}{100} \sim .27$.

7. We have 27 red spins. $\frac{12}{27}$ of these are X.

8. $\frac{12}{27} \sim .44$ which does differ considerably from .27.

Experiment 7-2.

For your particular class, we cannot tell what probabilities you found. We can, however, help you check your work. Whatever values you found for the probabilities (a) - (p), the following relations should hold (check your results).

(i) $P(B) + P(G) = 1$
(ii) $P(L) + P(R) = 1$
(iii) $P(B|L) + P(B|R) = P(B)$
(iv) $P(G|L) + P(G|R) = P(G)$
(v) $P(L|G) + P(L|B) = P(L)$
(vi) $P(R|G) + P(R|B) = P(R)$

You should be able to see why these relations must be true.

Here is another list of true relations. The reasons why these must be true is the subject of Section 7-3.

(vii) $P(B\cap L) = P(L) \cdot P(B|L)$
(viii) $P(G\cap R) = R(R) \cdot P(G|R)$
(ix) $P(L|B) = \frac{P(B\cap L)}{P(B)}$
(x) $P(R|G) = \frac{P(R\cap G)}{P(G)}$

If your results do not satisfy all these relations, use the information of the example of Items 14-18. Here $P(B) = \frac{15}{25}$, $P(B\cap L) = \frac{4}{25}$, $P(B\cap R) = \frac{11}{25}$, $P(B|L) = \frac{4}{7}$, etc.

ANSWERS TO EXERCISES

Section 1-3.

1. (a) Fair. There is exactly one face marked 1 and one marked 3. All other results yield a tie.
   (b) Fair. There are three odd-numbered faces (1, 3, 5) and three even-numbered faces (2, 4, 6).
   (c) Not fair. You win only if 3 is thrown, he wins whenever 4, 5 or 6 is thrown. He has more "chances" to win than you do.

2. (a) Not fair. You have an advantage -- there are three faces marked "3", but only one face marked "1".
   (b) Fair. Do you see why?
   (c) Not fair. Only two faces give you a win. while four faces favor him.

3. (a) Fair. Think about this game. Most throws will result in a tie, but throwing two 1's is as likely as throwing two 5's.
   (b) Fair. This game also takes a bit of thought. As soon as you realize that it does not matter what happens to the green die you will see that this game is essentially the same as the game described in 1(b) above.
   (c) We tried to trick you here. The rule does not enable us to decide who should win if the dice fall with a green 4 and a red 6 at the same time. Strictly speaking, the "game" is not defined. Notice that if we agree to a tie if this situation occurs, then we have a true game and it is fair.
   (d) Not fair. You win only on the throw red 1, green 1. He wins on two throws: red 1, green 2 and red 2, green 1.
   (e) Not fair. You might list all the possibilities for the two dice (there are 36 of them). You will notice that you win on only 15 of them while he wins on 21.

Section 2-6.

1. $\frac{2}{3}$

2. $\frac{1}{3}$

3. (a) Not likely, although it is possible. The probability, as you will learn later, is $\frac{1}{512}$.

   (b) $\frac{1}{2}$

   (c) No

4. (a) $\frac{3}{5}$

   (b) $\frac{1}{25}$

5. $\frac{5}{6}$

6. (a) No

   (b) 0

7. $\frac{1}{3}$ (There are 10 primes between 1 and 30. Remember, 1 is not a prime.)

8. $\frac{2}{3}$ (You can use a set of 3 outcomes, all equally likely. For simplicity, suppose Mr. Smith has a brown hat and a black hat. The outcomes are: both Mr. Smith's hats; Mr. Smith's black hat and his friend's hat; Mr. Smith's brown hat and his friend's hat.

9. (a) $\frac{1}{5}$

   (b) $\frac{1}{4}$

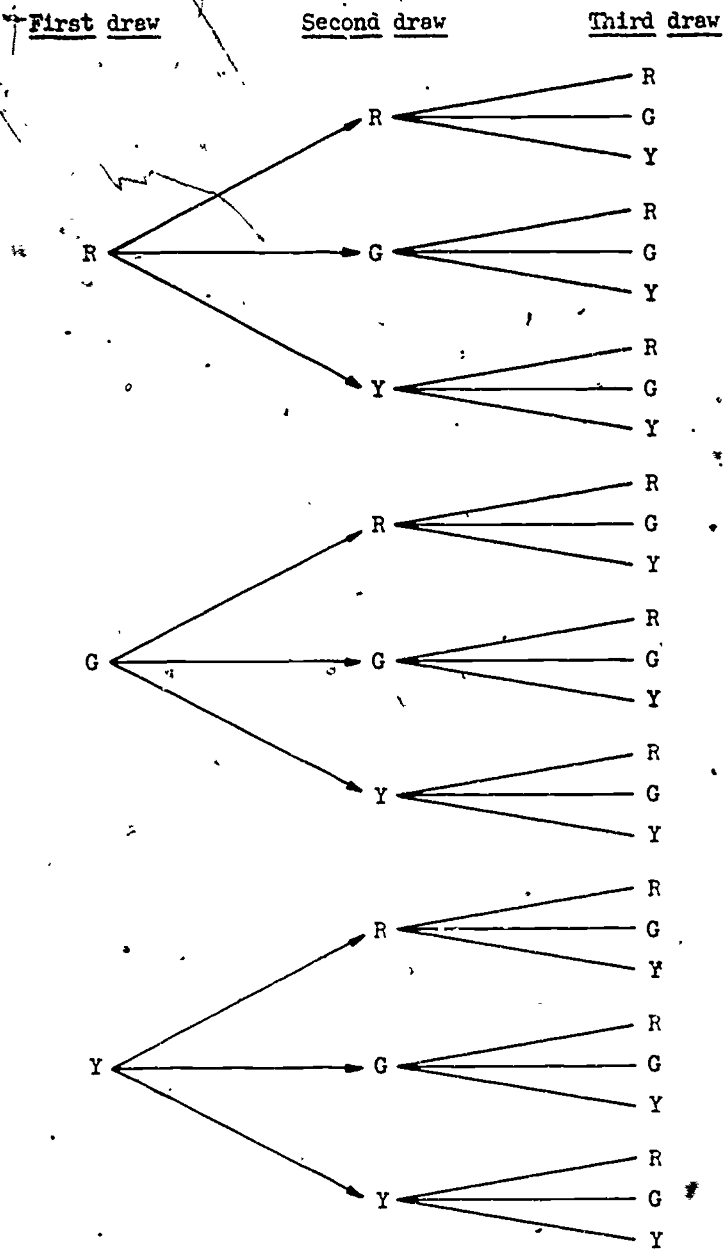
   (c) No; in (a) drawing the ace is one outcome out of five, while in (b) it is one out of four.

   (d) $\frac{1}{3}$

   (e) They are increasing.

Section 3-3.

Here is the tree diagram for 3 draws of a marble from a box containing 3 marbles, 1 red, 1 yellow, 1 green, assuming each marble picked is returned to the box before the next draw is made.



Section 3-4.

Table showing the set of outcomes for throwing two dice.

Green / Red	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Section 3-6.

1. $\frac{1}{8}$. There are 8 equally likely outcomes, only one of which is 3 heads.

2. $\frac{3}{8}$. Again, there are 8 equally likely outcomes. This time, the event contains three outcomes -- HHT , HTH , THH .

3. $\frac{5}{35}$, or $\frac{1}{7}$.

4. (a) $\frac{5}{10}$, or $\frac{1}{2}$.

   (b) $\frac{3}{9}$, or $\frac{1}{3}$. Since you did not replace the white marble, there are 9 marbles -- 4 white, 3 black, 2 red.

   (c) $\frac{2}{8}$, or $\frac{1}{4}$.

5. (a) 6 (c) $\frac{1}{6}$

   (b) 36 (d) $\frac{1}{36}$

   Did you notice that your reasoning here is exactly like that for an ordinary die? Having letters instead of numbers on the faces of the cubes doesn't change the probabilities.

6. (a) 4 (c) 16 ; $\frac{1}{8}$

   (b) $\frac{1}{4}$ (d) 64 ; $\frac{3}{64}$

   If you had trouble, look back at the tree in Sec. 3-4 .

7. Notice that the table has been constructed by counting ends of branches on tree diagrams. You may wish to construct further diagrams.

4-coins:	1	4	6	4	1	
	(4H)	(3H,1T)	(2H,2T)	(1H,3T)	(4T)	
5-coins:	1	5	10	10	5	1
	(5H)	(4H,1T)	(3H,2T)	(2H,3T)	(1H,4T)	(5T)

8. $\frac{6}{16}$, or $\frac{3}{8}$. There are 16 outcomes in all for 4 coins.

9.

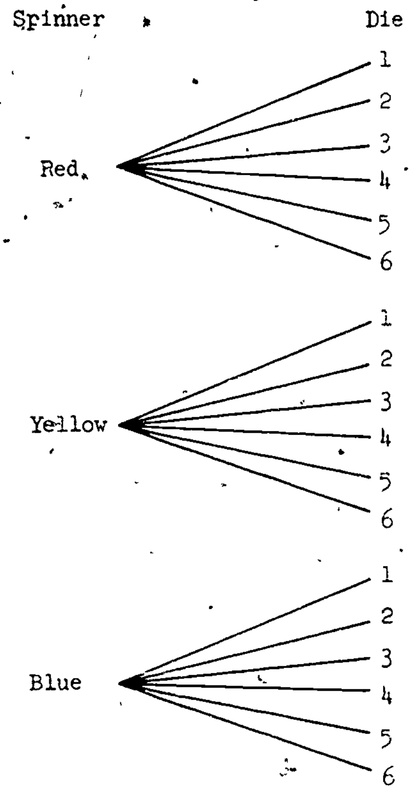
outcome	probability
5 heads	$\frac{1}{32}$
4 heads, 1 tail	$\frac{5}{32}$
3 heads, 2 tails	$\frac{10}{32}$, or $\frac{5}{16}$
2 heads, 3 tails	$\frac{10}{32}$, or $\frac{5}{16}$
1 head, 4 tails	$\frac{5}{32}$
5 tails	$\frac{1}{32}$

The sum is 1 .

10. 3 heads and 2 tails, 2 heads and 3 tails. Each of these events has probability $\frac{10}{32}$, or $\frac{5}{16}$.

11. $\frac{6}{64}$, or $\frac{3}{32}$.

12. $\frac{1}{18}$. The tree diagram for this problem is:



There are 18 possible outcomes.

Section 4-4.

1. (a) $\frac{152}{1600}$, or $\frac{19}{200}$, or 0.095

   (b) $\frac{19}{200} \cdot 2000 = 190$

2. 0.333 (Notice that we are ignoring such things as the skill of the opposing pitcher to be faced the "next time at bat".)

3. $\frac{89}{120}$

4. Insurance companies keep careful records on accidents with related information on age and sex of drivers. These record show that male drivers under 25 years of age are more apt to be involved in accidents than are male drivers over 25 years of age or female drivers of any age. Account would have to be taken of the percent of an age group by sexes that drive cars, the number of drivers who had accidents, and the number of drivers that were not involved in accidents in a given period of time.

5. (a) 69,517

   (b) None

6. (a) $P = \frac{92,588}{97,978}$, or 0.95 .

   (b) $P = \frac{35,837}{97,978}$, or 0.37 .

7. (b) Not for setting insurance rates. Some of the data would be obsolete.

   (c) Take the results of studying many people at various ages for shorter lengths of time.

8. (a) $\frac{1}{100,000}$, or 0.00001 .

   (b) $P = \frac{69,517}{78,653}$, or 0.88 .

9. The man at age 60 . His chance of living another 10 years is less than the others, so the insurance company would have to charge him more for the extra risk the company would take.

10. $P(A) = \frac{7}{20}$, $P(B) = \frac{11}{20}$. Sunshine seems more likely. Note that on cloudy days with no rain neither event occurs.

11. Our results (see discussion of Experiment 1-5):

    $P(1) = .12$; $P(2) = .15$; $P(3) = .25$; $P(4) = .15$; $P(5) = .21$; $P(6) = .12$.

12. Our results: P(three of a kind) $= \frac{2}{20} = .10$; P(four of a kind) $= \frac{1}{20} = .05$; P(five of a kind) $= 0$

Section 5-3.

1. (a), (c), are pairs of mutually exclusive events. (b) is not.

2. $\frac{640}{1600} = \frac{2}{5}$. The events are mutually exclusive, so we use

$$P(A \cup B) = P(A) + P(B)$$
$$= \frac{132}{1600} + \frac{508}{1600} = \frac{640}{1600} .$$

3. (a) No. If A is the event "number is even" and B is the event "number is greater than 3", then $A \cap B = \{4,6\}$.

(b) $\frac{2}{3}$, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$.

Alternatively: $A \cup B = \{2,4,5,6\}$; $A \cup B$ has 4 outcomes, and all possible outcomes are equally likely. $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$.

4. (a) $\frac{4}{9}$ (c) $\frac{7}{9}$

(b) $\frac{3}{9}$, or $\frac{1}{3}$ (d) $\frac{2}{9}$

5. (a) $\frac{18}{27}$, or $\frac{2}{3}$ (b) $\frac{3}{27}$, or $\frac{1}{9}$

6. (a) $\frac{4}{6}$, or $\frac{2}{3}$ (b) $\frac{2}{6}$, or $\frac{1}{3}$

7. (a) $\frac{25}{100}$, or $\frac{1}{4}$

(b) $\frac{20}{100}$, or $\frac{1}{5}$

(c) $\frac{35}{100}$, or $\frac{7}{20}$

8. (a) 10. Hint: Call the boys A, B, C and the girls D and E. The possible pairs are AB, AC, AD, AE, BC, BD, BE, CD, CE, DE. (Note that AB and BA would not count as different pairs.)

(b) 3 : AB, AC, and BC. (e) $\frac{3}{10}$

(c) 1 (f) $\frac{6}{10}$, or $\frac{3}{5}$

(d) 6 (g) $\frac{9}{10}$

9. (a) 3 (c) 4

(b) 1 (d) $\frac{1}{2}$

10. (a) 16

(b) 4 $(16 = 9 + 11 - 4)$

(c) 5 swim only in the morning; 7 swim only in the afternoon.

(d) $\frac{16}{22}$, or $\frac{8}{11}$

Section 6-6.

1. (a) Yes. Regardless of the result of the first throw, head and tail are equally likely on the second throw.

   (b) $\frac{1}{4}$

2. (a) $P(\text{both on red}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

   (b) $P(\text{both on green}) = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}$.

   (c) $P(A \text{ on white and } B \text{ on blue}) = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}$.

3. $\frac{4}{9} \cdot \frac{4}{9} = \frac{16}{81}$.

4. $\frac{1}{6}$. Note that in this case the events "white on the first draw" and "white on the second draw" are <u>not</u> independent. An appropriate tree shows 9 possibilities (all equally likely) for the first draw, each with 8 equally likely branches. Thus, there are 72 equally likely outcomes in all, of which 12 are in the event "two white marbles".

5. (a) $\frac{1}{2}$ (c) $\frac{1}{4}$

   (b) $\frac{1}{4}$ (d) $\frac{1}{2}$

   Note that under the assumption each birth is independent of the others. The tree here is exactly like that for two coins.

6. (a) .090. The events are independent; hence

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) \\ &= (.320)(.280) \\ &= .0896 \\ &\sim .09 . \end{aligned}$$

   (b) .510.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= .320 + .280 - .090 \\ &= .510 . \end{aligned}$$

7. (a) $\frac{35,837}{92,588}$, or about .387.

   (b) about .150 : $P(A \cap B) = P(A) \cdot P(B) = (.387)^2 = .150$.

   (c) about .624 : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= .387 + .387 - .150 = .624 .$$

8. (a) P(black from first bag) $= \frac{5}{9}$.

P(black from second bag) $= \frac{7}{15}$.

These are independent events; hence

$$P(\text{both black}) = \frac{5}{9} \cdot \frac{7}{15} = \frac{35}{135} \quad (= \frac{7}{27}) .$$

(b) P(blue from first bag) $= \frac{4}{9}$.

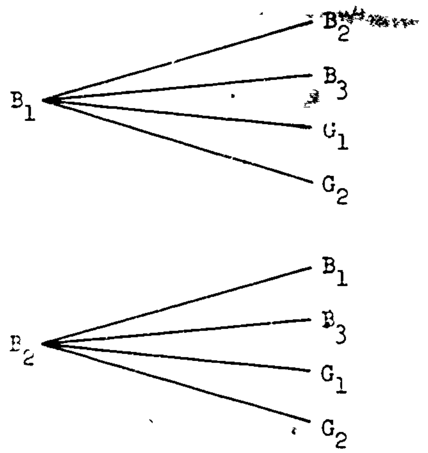
P(blue from second bag) $= \frac{8}{15}$.

P(both blue) $= \frac{4}{9} \cdot \frac{8}{15} = \frac{32}{135}$.

(c) P(one of each color) $= 1 - (\frac{35}{135} + \frac{32}{135}) = \frac{68}{135}$.

9. (a) P(both green) $= \frac{1}{10}$. Make a tree. Using B_1, B_2, B_3 for the blue socks and G_1, G_2 for the green, we have:

First draw Second draw



etc.

The tree shows that we have 20 equally likely outcomes, of which 2 -- G_1G_2 and G_2G_1 -- are in the event "2 greens".

(b) P(both blue) $= \frac{6}{20}$, or $\frac{3}{10}$.

(c) P(one blue and one green) $= 1 - (\frac{1}{10} + \frac{3}{10}) = \frac{6}{10}$, or $\frac{3}{5}$.

Notice that we cannot use the idea of independent events for this problem. In later chapters you will find easier ways of doing problems of this sort.

10. Event A: Mr. A solves the problem.

Event B: Mr. B solves the problem.

Event not-A: A does not solve the problem. $P(\text{not-A}) = 1 - \frac{2}{3} = \frac{1}{3}$.

Event not-B: B does not solve the problem. $P(\text{not-B}) = 1 - \frac{5}{12} = \frac{7}{12}$.

(a) $P(\text{not-A} \cap \text{not-B}) = \frac{1}{3} \cdot \frac{7}{12} = \frac{7}{36}$.

(b) $P(A \cap \text{not-B}) = \frac{2}{3} \cdot \frac{7}{12} = \frac{14}{36}$, or $\frac{7}{18}$.

(c) $P(\text{not-A} \cap B) = \frac{1}{3} \cdot \frac{5}{12} = \frac{5}{36}$.

(d) $P(A \cap B) = \frac{2}{3} \cdot \frac{5}{12} = \frac{10}{36}$.

(e) $P(A \cup B) = 1 - P(\text{not-A} \cap \text{not-B}) = 1 - \frac{7}{36} = \frac{29}{36}$.

11. There are $2^6 = 64$ ways that the 6 coins may have heads and tails appear. There is only one way that will not have at least one head (6 tails). Therefore,

$$P(\text{at least one head}) = 1 - \frac{1}{64} = \frac{63}{64}.$$

12. The list of possible draws of 3 sticks (the order in which they are picked does not matter):

5, 4, 3	5, 3, 2	4, 3, 2	3, 2, 1
5, 4, 2	5, 3, 1	4, 3, 1	
5, 4, 1	5, 2, 1	4, 2, 1	

The possible draws for a triangle are 5, 4, 3 and 5, 4, 2 and 4, 3, 2 . Why?

$$P(\text{triangle}) = \frac{3}{10}.$$

13. $\frac{11}{18}$. (There are 36 equally likely outcomes.

$P(\text{sum is odd}) = \frac{1}{2}$;

$P(\text{sum is less than } 5) = \frac{1}{6}$;

$P(\text{sum is odd or less than } 5) = \frac{1}{2} + \frac{1}{6} - \frac{1}{18} = \frac{11}{18}$.)

14. (a) $\frac{3}{4}$ (b) 1

15. (a) $\frac{11}{15}$ (c) $\frac{13}{15}$

(b) $\frac{8}{15}$ (d) $\frac{13}{15}$

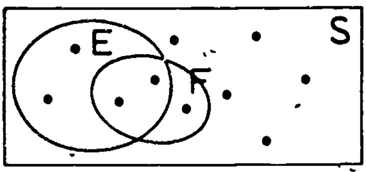
16. P(green) $= \frac{2}{3}$, P(yellow) $= \frac{1}{3}$. (Remember, the sum of the probabilities of the outcomes must be 1 .)

17. $P(A) = \frac{2}{6} = \frac{1}{3}$; $P(B) = \frac{1}{6}$; $P(C) = \frac{3}{6} = \frac{1}{2}$.

Section 7-1.

1. P(blue) $= \frac{1}{2}$.
2. $P(X) = \frac{1}{3}$.
3. P(blue X) $= \frac{1}{6}$.
4. Yes. $\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$.
5. Yes.
6. P(green) $= \frac{1}{6}$.
7. No. P(green X) = 0 which is not equal to P(green) $\cdot$ P(X) .
8. X covers $\frac{1}{2}$ of the red region. If the spinner stops on red, then the probability that it also stops on X is $\frac{1}{2}$.
9. No. P(red X) $= \frac{1}{6}$, while P(red) $\cdot$ P(X) $= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

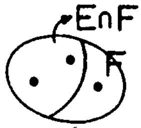
Section 7-2.



All outcomes are equally likely in S.

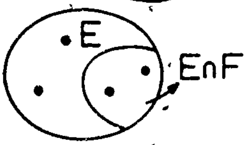
$P(E) = \frac{4}{10} = \frac{2}{5}$; $P(F) = \frac{3}{10}$;

$P(E \cap F) = \frac{2}{10} = \frac{1}{5}$.



All outcomes are equally likely in F.

$P(E|F) = \frac{2}{3}$.



All outcomes are equally likely in E.

$P(F|E) = \frac{2}{4} = \frac{1}{2}$.

Section 7-3.

1. (a) $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{.1}{.5} = .2$

   (b) $P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{.1}{.2} = .5$.

2. (a) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{.16}{.2} = .8$.

(b) $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{.16}{.8} = .2$.

Notice in this case: $P(A|B) = P(A)$,

$P(B|A) = P(B)$,

and $P(A\cap B) = P(A) \cdot P(B)$;

that is, recalling Chapter 6, events A and B are independent.

3. (a) $P(X|Y) = \frac{\frac{1}{6}}{\frac{1}{4}} = \frac{2}{3}$.

(b) $P(Y|X) = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$.

4. (a) $P(A|B) = \frac{0}{.2} = 0$.

(b) $P(B|A) = \frac{0}{.3} = 0$.

5. (a) Since $P(E|F) = \frac{P(E\cap F)}{P(F)}$, we have $.6 = \frac{P(E\cap F)}{.5}$.

Therefore, $P(E\cap F) = (.6)(.5) = .3$.
Notice that we did not need to know $P(E)$.

(b) $P(E\cup F) = P(E) + P(F) - P(E\cap F)$

$= .7 + .5 - .3 = .9$

6. (a) $P(E\cup F) = P(E) + P(F) - P(E\cap F)$

$.6 = .4 + .4 - P(E\cap F)$

$P(E\cap F) = .2$.

(b) $P(E|F) = \frac{.2}{.4} = \frac{1}{2}$ (or .5) .

(c) $P(F|E) = \frac{.2}{.4} = \frac{1}{2}$.

Notice that $P(E|F) = P(F|E)$. Do you see why it turns out this way? Look at $P(E)$, $P(F)$.

7. If E and F are mutually exclusive, then $E\cap F = 0$ and $P(E\cap F) = 0$.
Hence,

$$P(E|F) = \frac{0}{P(F)} = 0$$

$$P(F|G) = \frac{0}{P(G)} = 0 .$$

8. If E is a subset of F, then $E\cap F = E$. Therefore,

$$P(E|F) = \frac{P(E)}{P(F)} .$$

Section 7-4.

1. (a) There are 3 vowels and 5 letters. (All letters are equally likely.)

$$P(\text{a vowel}) = \frac{3}{5}.$$

(b) There are 3 vowels in the reduced set of outcomes.

$$P(o|\text{a vowel}) = \frac{1}{3}.$$

(c) b is not a vowel.

$$P(b|\text{a vowel}) = 0.$$

(d) There are 2 consonants.

$$P(t|\text{a consonant}) = \frac{1}{2}.$$

2. (a) Possible favorable outcomes in the form (red,green) are (3,1), (1,3), (2,2), (2,1), (1,2), and (1,1).

$$P(\text{sum under } 5) = \frac{6}{36} = \frac{1}{6}.$$

(b) Of the favorable outcomes in (a), only (2,1) and (2,2) are now possible since all outcomes with a red 2 are (2,1), (2,2), (2,3), (2,4), (2,5), and (2,6).

$$P(\text{sum under } 5|\text{red die is } 2) = \frac{1}{3}.$$

(c) Since the red die is 5 and the green die is at least 1, the sum cannot be 5 or less than 5. Thus,

$$P(\text{sum under } 5|\text{red die is } 5) = 0.$$

(d) In (a), of the six outcomes whose sum is under 5, four have the green die an odd number. The reduced set of outcomes with the green die odd has 18 elements.

$$P(\text{sum is } 5|\text{green die odd}) = \frac{2}{9}.$$

3. (a) $S = \{a,b,c,d,e,f,g,h,i\}$ (9 elements);
$E = \{a,b,c,e,g\}$ (5 elements);

$P(E) = \frac{5}{9}.$

(b) $F = \{b,e,f,g,h,i\}$ (6 elements);

$P(F) = \frac{6}{9} = \frac{2}{3}.$

(c) $E \cap F = \{b,e,g\}$;

$P(E|F) = \frac{3}{6} = \frac{1}{2}.$

(d) $P(F|E) = \frac{3}{5}.$

4. (a) $P(\text{Red}) = \frac{3}{12} = \frac{1}{4}$.

(b) $P(1) = \frac{3}{12} = \frac{1}{4}$.

(c) $P(\text{Red} \cap 1) = \frac{1}{12}$.

(d) $P(\text{Blue} \cup 2) = \frac{4}{12} + \frac{3}{12} - \frac{1}{12} = \frac{6}{12} = \frac{1}{2}$.

(e) $P(\text{Red}|1) = \frac{P(\text{Red} \cap 1)}{P(1)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$.

(f) $P(1|\text{Red}) = \frac{P(\text{Red} \cap 1)}{P(\text{Red})} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$.

(g) $P(2) = \frac{3}{12} = \frac{1}{4}$.

(h) $P(2|\text{White}) = \frac{P(2 \cap \text{White})}{P(\text{White})} = \frac{\frac{1}{12}}{\frac{5}{12}} = \frac{1}{5}$.

(i) $P(\text{White}|2) = \frac{P(\text{White} \cap 2)}{P(2)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$.

(j) $P(4|\text{Red}) = \frac{P(4 \cap \text{Red})}{P(\text{Red})} = \frac{0}{\frac{1}{4}} = 0$.

(k) $P(\text{Red}|\text{Odd}) = \frac{P(\text{Red} \cap \text{Odd})}{P(\text{Odd})} = \frac{\frac{2}{12}}{\frac{6}{12}} = \frac{1}{3}$.

(l) $P(\text{White}|4) = \frac{P(\text{White} \cap 4)}{P(4)} = \frac{\frac{2}{12}}{\frac{3}{12}} = \frac{2}{3}$.

Notice that, in (e), (f), (h), (i), (j), (k), (l) we didn't <u>need</u> to use the formula. We could have merely counted - since the areas are equally likely. For (k), there are 6 "odd" outcomes and of these 2 are "red".

5. (a) If we know that a black is removed on the first draw, then 1 black and 2 whites are left.

$$P(B_2|B_1) = \frac{1}{3} .$$

(b) It would be impossible to get a black on the first dra. if we got a black on the second and third draws.

$$P(B_1|B_2B_3) = 0 .$$

(c) If you know that a white is to be drawn on the second draw, then only 1 white is available for the first draw.

$$P(B_1|W_2) = \frac{2}{3} .$$

(d) $P(W_1|B_2B_3) = 1$, since black on second and third draws leaves only white for first draw.

Not that it was easier to find these answers without using the formula.

6. (a) We must have $P(S) = 1$. Adding the given probabilities, we obtain .92 . Therefore the probability of outcome d is .08 .

(b) $P(A) = .12 + .14 + .16 + .08 = .50$.

(c) $P(B) = .16 + .08 + .12 = .36$.

(d) $P(A \cap B) = .16 + .08 = .24$.

(e) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= .50 + .36 - .24$

$= .62$

(f) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.16 + .08}{.36} = \frac{.24}{.36} = \frac{2}{3} \sim .67$.

(g) $P(B|A) = \frac{.16 + .08}{.50} = \frac{.24}{.50} = .48$.

7. (a) We have $P(R) = \frac{50}{200}$. (Rain was predicted for 50 of the 200 days.)

$P(W \cap R) = \frac{30}{200}$. (On 30 days rain was predicted and it did rain.)

Hence :

$$P(\text{wet, given "rain forecast"}) = P(W|R) = \frac{\frac{30}{200}}{\frac{50}{200}} = \frac{3}{5} = .6 .$$

That is, if the forecast is "rain", the probability of rain is .6 .

(b) $P(W) = \frac{60}{200}$.

$$P(\text{rain forecast, given wet day}) = P(R|W) = \frac{\frac{30}{200}}{\frac{60}{200}} = \frac{1}{2} = .5 .$$

That is, if it is a wet day, there is a probability of .5 that the weatherman predicted it.

(c) $P(N) = \frac{150}{200}$, $P(D \cap N) = \frac{120}{200}$.

$$P(D|N) = \frac{\frac{120}{200}}{\frac{150}{200}} = \frac{12}{15} = \frac{4}{5} = .8 .$$

That is, if the forecast is "no rain", the probability of a dry day is .8 .

(d) $P(D) = \frac{140}{200}$.

$$P(N|D) = \frac{\frac{120}{200}}{\frac{140}{200}} = \frac{12}{14} \approx .86 .$$

That is, if it is a dry day, the probability that the weatherman predicted it is .86 .

18. Let T be "a teen-ager was chosen"

A be "an adult was chosen"

L be "the person chosen liked the program"

D be "the person chosen did not like the program" .

(a) $P(A) = .75$ (since $P(T) = .25$)

(b) $P(A \cap L) = .15$ (20% of 75% is 15%)

(c) $P(T \cap L) = .15$ (60% of 25% is again 15%)

(d) To find $P(L)$ we reason: anyone who liked the program is either a member of $A \cap L$ or of $T \cap L$. Since these sets are disjoint (no one is both A and T),

$$P(L) = P(A \cap L) + P(T \cap L) = .15 + .15 = .30$$

(e) $P(L|T) = \dfrac{P(L \cap T)}{P(T)} = \dfrac{.15}{.25} = .60$ (Not surprising!) .

(f) $P(A|L) = \dfrac{P(A \cap L)}{P(L)} = \dfrac{.15}{.30} = .50$.

19. Let T be the event "test shows positive"

N be the event "test shows negative"

H be the event "healthy"

D be the event "has the disease" .

We wish $P(H|T)$.

Now $P(H) = .98$

$P(H \cap T) = .049$ (5% of 98% is 4.9%)

$P(D \cap T) = .018$ (90% of 2% is 1.8%)

$$P(T) = P(H \cap T) + P(D \cap T) = .049 + .018 = .067$$

Therefore

$$P(H|T) = \frac{.049}{.067} \sim .73 .$$

Notice the importance of this result. If people are tested "at random", almost $\frac{3}{4}$ of those with a "positive" test are, in fact, healthy.

The numbers used for this exercise do not reflect any real test for a real disease. Some medical tests yield fewer false positives, etc., but the basic problem for medical diagnosis is quite important.

†Section 7-5.

1. (a) Making the assumption that boys and girls are equally likely and also assuming that the sex of the younger is independent of the sex of the older, we have the set of equally likely outcomes.

$$S = \{BB, BG, GB, GG\} .$$

(By "GB" we mean that the older child is a girl, the younger a boy.) Since each of these four outcomes is equally likely, $P(BB) = \frac{1}{4}$.

(b) Our set of equally likely outcomes is now reduced to

$$S_1 = \{BB, BG, GB\} .$$

$P(BB) = \frac{1}{3}$.

Alternate solution to (b): Referring to S :

$$P(BB \text{ and at least one boy}) = \frac{1}{4} .$$

$$P(\text{at least one boy}) = \frac{3}{4} .$$

So $$P(BB|\text{at least one boy}) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} .$$

(c) Our set of equally likely outcomes is now

$$S_2 = \{BB, BG\} .$$

$P(BB) = \frac{1}{2}$.

Alternate solution to (c):

$$P(BB \text{ and older is a boy}) = \frac{1}{4} .$$

$$P(\text{older is a boy}) = \frac{1}{2} .$$

So $$P(BB|\text{older is a boy}) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} .$$

2. (a) The set of possible two card selections is
$\{A_sA_h, A_sK_s, A_sK_h, A_hK_s, A_hK_h, K_sK_h\}$, where we use A_s to mean the ace of spades, etc. Again, assuming that each selection is equally likely, we have $P(A_sA_h) = \frac{1}{6}$.

(b) Knowing that at least one ace has been selected eliminates K_sK_h as a possible outcome. Our reduced set of outcomes now has five equally likely members. Hence $P(A_sA_h \mid \text{at least one ace}) = \frac{1}{5}$.

Alternate solution to (b):

$$P(A_sA_h \cap \text{at least one ace}) = \frac{1}{6}.$$

$$P(\text{at least one ace}) = \frac{5}{6}.$$

$$P(A_sA_h \mid \text{at least one ace}) = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}.$$

(c) If we know that the ace of spades is selected, our reduced set of outcomes is

$$\{A_sA_h, A_sK_s, A_sK_h\} \text{ and } P(A_sA_h \mid A_s) = \frac{1}{3}$$

Alternate solution to (c):

$$P(A_sA_h \cap A_s) = \frac{1}{6}, \text{ since } A_sA_h \cap A_s = A_sA_h.$$

$$P(A_s) = \frac{3}{6}.$$

3. (a) Since each bag is equally likely, $P(2 \text{ black}) = \frac{1}{3}$.

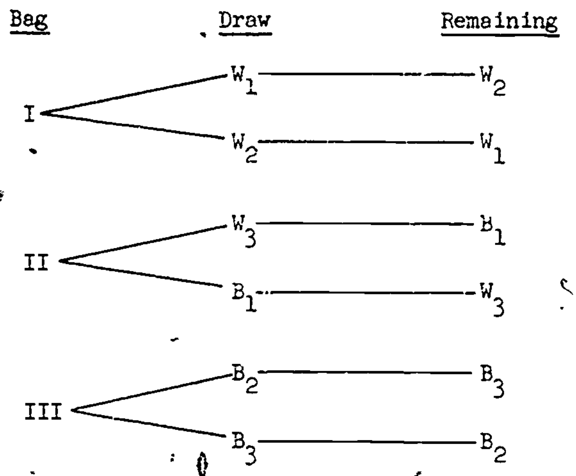
(b) Did you guess $\frac{1}{2}$? This seems reasonable since we know that the bag with two white balls was not selected. Let's analyze a bit more carefully. Think of the bags being I, II, III. Bag I contains two white marbles, W_1, W_2; bag II contains W_3, B_1; bag III contains B_2, B_3. We may think of the problem in three stages.

(1) Which bag is selected?

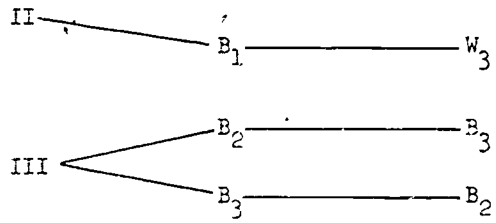
(2) Which color marble is drawn?

(3) Which color marble remains?

A tree diagram may help.



At the beginning, then, there are six equally likely outcomes -- two of which involve bag III (two black marbles). Since a black marble <u>was</u> drawn, we consider only the branches:



Therefore.

$$P(\text{two black}\mid\text{first black}) = \frac{2}{3} \qquad (\underline{\text{not}}\ \tfrac{1}{2}\ !!)$$

Section 7-6.

1. $A = \{(6,4), (4,6), (5,5)\}$
$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$
$C = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

By $(6,4)$, for example, we mean (red 6, green 4). $A \cap B = \emptyset$, $A \cap C = \{(6,4)\}$, $B \cap C = \{(6,1)\}$, so that $P(A \cap B) = 0$, $P(A \cap C) = \frac{1}{36}$, $P(B \cap C) = \frac{1}{36}$.

(a) $P(A|C) = \dfrac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$

(b) $P(B|C) = \dfrac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$

(c) $P(A|B) = 0$

(d) $P(C|A) = \dfrac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$

(e) $P(C|B) = \dfrac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$

(f) $P(B|A) = 0$

(g) $P(A) = \frac{3}{36} = \frac{1}{12}$, $P(B) = \frac{6}{36} = \frac{1}{6}$

$P(C) = \frac{6}{36} = \frac{1}{6}$. Comparing with (a), (b), (c), we see that only B and C are independent.

(h) A and B are mutually exclusive.

2. (a) $P(A \cup B) = P(A) + P(B) = \frac{3}{36} + \frac{6}{36} = \frac{1}{4}$ (A and B are mutually exclusive.)

(b) $P(B \cup C) = P(B) + P(C) - P(B \cap C)$
$= P(B) + P(C) - P(B) \cdot P(C)$ (B and C are independent.)
$= \frac{1}{6} + \frac{1}{6} - (\frac{1}{6} \cdot \frac{1}{6})$
$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}$

(c) $P(A \cup C) = P(A) + P(C) - P(A \cap C)$
$= \frac{3}{36} + \frac{6}{36} - \frac{1}{36}$
$= \frac{8}{36}$, or $\frac{4}{9}$.

Notice that $P(A \cap C) \neq P(A) \cdot P(C)$

$$\frac{1}{36} \neq \frac{3}{36} \cdot \frac{6}{36}.$$

3. There are three ways for each die to show even. Hence, there are nine ways for them both to show even: $P(A) = \frac{9}{36} = \frac{1}{4}$.

B is the event $\{(2,6), (3,5), (4,4), (5,3), (6,2)\}$.

$P(A|B) = \frac{3}{5}$, since three outcomes of B are also in A .

$P(A|B) \neq P(A)$, so the events are <u>not</u> independent.

4. There are eight possible outcomes: bbb, bbg, bgb, ..., etc. (where we list the sex of the eldest child first).

E = {bbg, bgb, gbb, ggb, gbg, bgg}

$P(E) = \frac{6}{8} = \frac{3}{4}$

F = {bbb, bbg, bgb, gbb}

$P(E|F) = \frac{3}{4}$, since three of the elements of F are in E .

$P(E|F) = P(E)$, so the events <u>are</u> independent.

5. After working Exercise 4, it is intuitive to guess that E and F are independent. Let's see.

There are sixteen outcomes: bbbb, bbbg, bbgb, ..., etc. E contains fourteen of these outcomes (all except bbbb and gggg).

$P(E) = \frac{14}{16} = \frac{7}{8}$

F = {bbbb, bbbg, bbgb, bgbb, gbbb}

$P(E|F) = \frac{4}{5}$, so the events are <u>not</u> independent.

6. $P(C) = \frac{6}{12} = \frac{1}{2}$.

$P(C|D) = \frac{1}{2}$, so C and D <u>are</u> independent.

7. Surely, it seem obvious that the placement of P does not influence the placement of Q relative to P . Is it really obvious? The set of possible orders of placement (reading from left to right) is

{PQR, PRQ, QPR, QRP, RPQ, RQP}

A = {PQR, PRQ, RPQ} , $P(A) = \frac{1}{2}$ (obvious!)

B = {PQR, PRQ, QPR} , $P(A|B) = \frac{2}{3}$ (obvious??)

A and B are <u>not</u> independent events.

8. $P(H) = \frac{150}{400} = \frac{3}{8}$

$P(H|R) = \frac{50}{160} = \frac{5}{16}$

H and R are <u>not</u> independent.