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ABSTRACT

This nineteenth unit in the SMSG secondary school mathematics series is the teacher's commentary for Unit 17. First, a time allotment for each of the chapters in Units 17 and 18 is given. Then, for each of the chapters in Unit 17, the goals for that chapter are discussed, the mathematics is explained, some teaching suggestions are given, answers to exercises are provided, and sample test questions are included. (DT)

School Mathematics Study Group

Intermediate Mathematics

Unit 19

Intermediate Mathematics

Teacher's Commentary, Part I

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Suggested Time Schedule

Following is a suggested schedule for teaching the chapters of this text. This schedule is based on information obtained from the centers where Intermediate Mathematics was taught experimentally during the 1959-60 academic year. Strict adherence to this schedule is not recommended. Teachers should feel free to modify it in accordance with the background and ability level of each class.

		Number Of Weeks
Chapter 1	Number Systems	4-5
Chapter 2	Introduction to Coordinate Geometry	2
Chapter 3	Functions	2
Chapter 4	Quadratic Functions and Quadratic Equations	2
Chapter 5	Complex Numbers	3
Chapter 6	Coordinate Geometry - Straight Lines and Conic Sections	2
Chapter 7	Systems of Equations in Two Variables	2
Chapter 8	Systems of Equations in Three Variables	2
Chapter 9	Logarithms and Exponents	4
Chapter 10	Introduction to Trigonometry	3-4
Chapter 11	Vectors	3
Chapter 12	Polar Form of Complex Numbers	2
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Chapter 15 Algebraic Structures (Supplementary)

Commentary for Teachers

Chapter 1

NUMBER SYSTEMS

1-0. General Observations.

Chapter 1 is devoted to a careful study of the number systems of elementary algebra. (Chapter 15 is an introduction to some of the corresponding systems of higher algebra.)

Scientists often speak of mathematics as a "language," and their point of view is certainly justified by the way they use mathematics. However, there is an implication here that they are the "poets" while mathematicians are the "grammarians." This implication is not very generous, for there is little similarity in the aims of grammarians and mathematicians. If we may say that the grammarian analyzes statements, breaking them down for purposes of classification, we may say on the other hand that the mathematicians' aim is to show the relationships between statements and in particular their logical dependence on each other. Mathematics is concerned with inferences--the processes of drawing conclusions from given statements. Thus mathematicians are concerned with collections of statements and the "structure" of such collections, rather than the "structure" of individual utterances.

Students are introduced to geometry organized as a mathematical, or deductive, system, so they have some familiarity with the mathematical approach. In Chapter 1 we organize our knowledge of the familiar number systems along similar lines. Some of the advantages of this organization are indicated on pages 1 and 2 of the text and need not be repeated here.

The basis of this organization is the Euclidean method of postulates, theorems, and proofs. There is nothing 'modern' about this method; what is 'modern' is the appreciation of its wide range of applicability and the recognition of its power. Studying geometry, a student sees how tidy the subject becomes, and how tightly knit our knowledge of it is when it is organized this way. He sees that by setting down relatively few assertions as postulates and by using some common sense and ingenuity we can derive from these postulates the other assertions which go to make up the body of geometrical knowledge. However, what he is not as likely to see in a first course in geometry is that Euclid's ideas on organizing the subject have a significance far beyond the fact that they enable us to detect a logical structure in a single body of knowledge. Their value is enhanced tremendously when we use them as a basis for the "comparative anatomy" of mathematical systems.

The comparative anatomy of different geometries is something beyond the experience of most students, for they have seen only one geometrical system. On the other hand, they are already familiar with several different number systems. Thus, once these systems are organized along Euclid's lines, a study of their comparative anatomy is possible and appropriate. Moreover, the problems encountered in organizing our knowledge of the simpler numerical systems (the natural numbers, the integers, and the rationals) are much less involved than the logical problems encountered in casting geometry in deductive form. For this reason algebra may well be a better vehicle for presenting Euclid's ideas than geometry!

Why should we study comparative anatomy anyway? We have several reasons. We want to consolidate and reinforce our knowledge of the familiar number systems. We want to bring out as clearly as possible their similarities and differences. And we also want to study two number systems not so familiar and considerably more complicated than the more familiar ones.

(They are the real number system and the complex number system.) The main thing all these systems have in common is their logical structure. The 'numbers' involved in them are quite different. Shifting our attention from the numbers themselves to the structure of the systems, we find that, to a considerable extent, in the various systems we are saying much the same things but saying them about different things. It is this similarity which makes the real and complex number systems fit more comfortably into our thinking about numbers.

We therefore start by "backing up" to the very beginning of our knowledge about numbers and have a good deal to say about counting and other ideas which the students first met as far back as first grade. Many of the things we discuss are often called "obvious"--meaning apparently that they are thoroughly familiar. What we do here with such things is generally anything but obvious. Two statements may be obvious; for example,

$$a(b + c) = ab + ac \quad \text{and} \quad a \cdot 0 = 0 \;;$$

but that one of them follows from the other, or that we can't prove one of them without using the other; these statements are certainly not obvious in the same sense that the formulas themselves are "obvious".

It will be noticed that there is not a single picture in Chapter 1, not even a "number-line". This is no accident; it is quite intentional. One of our objectives is to present the real number system in such a way that there may be no reservations about using it as a basis for geometry--as is done, for example, in the SMSG Tenth Grade Course. By emphasizing that our discussion of the real numbers is based entirely on arithmetical considerations, we demonstrate that there is no possibility of logical circularity when it is used in geometry. There have been many presentations of the real number system using the "number-line" for purposes of illustration of many of its important properties. In some of these presentations it may not be absolutely clear to the students that such illustrations are not central to the presentation, that they are merely asides or aids. We are confident that no such misinterpretations can occur reading Chapter 1.

From time to time in this Commentary we suggest places where such illustrations may contribute to class discussion and help to clarify matters. The fact that they are irrelevant for our proofs in Chapter 1 is what is important.

There are many ways a class can study Chapter 1. We do not think that any class should spend more than 5 weeks on it and that most classes should spend 4 or fewer.

Students who have been through the SMSG Ninth Grade Course and have a thorough understanding of the real number system and the solution of inequalities may find it possible to skip from Section 1-1 directly to Section 1-8, with only a brief review of the basic properties of the rational number system listed on pages 63, 64.

Students without this preparation may pursue a variety of routes. The chapter is written in such a way that a class with relatively weak preparation and slight interest in fine points can skip through the first 10 sections (1-1 to 1-10), reading perhaps only the summaries of each number system (at the ends of Sections 1-3, 1-5, 1-7, 1-9), and then settle down to the review of radicals, factoring and the manipulation of rational expressions in Sections 1-10, 1-11, 1-12. This is one extreme, which should take approximately 4 weeks.

The other extreme is to take the whole thing ca[illegible]lly and in detail. A class with strong preparation and [illegible] great [illegible] of interest might be able to cover everything in 5 we[illegible]

Between these two extremes is a whole spectrum of possibilities. By the time a class hits the second or third set of exercises in Section 1-2, the teacher should have fair ideas of what his students can take and how much he can get them to absorb. Depending on reactions by this time, or at any later time, adjustments can be made in plans. For instance, instead of taking each proof to bits, after the first couple of sections a class could just read the definitions and theorems, illustrate them with numerical examples, omit all problems calling for proofs, and go on to the next number system. Even this way, the class ought to get a better-than-average picture of number systems and their organization and be prepared for the work in later chapters, which makes few demands on the specific proof-techniques presented in Chapter 1. If later, a class finds it needs a better foundation--for instance, in inequalities--it is always possible to go back and get it. There is much to be said for the point of view that a demonstrated need for some skill or piece of knowledge adds to one's appreciation of it and to his motivation in studying it.

We do feel, however, that many students--and most of all, teachers--will have sufficient interest, curiosity, and drive to want to cover everything in this chapter. If we had not thought so, we wouldn't have written it.

OUTLINE OF CHAPTER ONE

Section	Pages	Topic
1-1	1 - 4	Introduction
1-2 1-3	4 - 12 13 - 23	N: Natural Number System $[E_{1-6}, A_{1-3}, M_{1-4}, D, O_{1-5}, O_6(N)]$ Summary: pp. 20 - 23.
1-4 1-5	24 - 31 31 42	I: System of Inte[illegible] [E_{1-} [illegible] 5, [illegible]-4, D, O_{1-5}, $O_6(I)$] Summary: pp. 40 - 42.
1-6 1-7	43 - 52 53 - 65	Q: Rational Number System $[E_{1-6}, A_{1-5}, M_{1-5}, D, O_{1-5}, O_6(Q)]$ Summary: pp. 63-65.
1-8	66 - 72	Rational Decimal Representations
1-9 1-10	72 - 77 77 - 86	R: Real Number System $[E_{1-6}, A_{1-5}, M_{1-5}, D, O_{1-6}, O_7(R)]$ Summary: pp. 75 - 76.
1-11	86 - 95	Polynomials and Review of Factoring
1-12	95 - 102	Rational Expressions
1-13	103 - 113	Additional Exercises for Sections 1-1 through [illegible]
1-14	113 - 117	Miscellaneous Exercises.

1-1. Introduction.

The first page-and-a-half is devoted to telling the student, in language we think he can understand, some of the things we have said in this commentary under the heading "General Observations."

The remainder of Section 1-1 is devoted to a discussion of statements having the form "If A , then B ." We give examples of such statements, discuss their converses, and say how we shall use the expression "only if." The last point is quite important for "only if" is used repeatedly throughout this book and in nearly every piece of mathematical writing in the English language. Its meaning seems to be something students must be told; whether mathematical use of "only if" conforms to that in conversational English is a disputed question.

We resist the temptation to go into any real questions of logic--although an excellent case may be made for presenting a good deal of symbolic logic at this level--for we have so many other things we must get on to. We don't even try to say what "if ... then ..." means, although we do rephrase it in several ways. We rely on the student's past experience with statements having this form and take it that he has some "understanding" of them. We believe that this understanding will be strengthened by making him think about such statements in the exercises.

The Exercises are quite "formal" in the sense that the meanings of the statements to be reworded are irrelevant as far as the problem of rewording them is concerned. Thus, for example, the converse of

> Hottentots are polytheists only if existentialism is nominalistic

is obtained by deleting the word "only"; and this fact has nothing whatever to do with the meaning of the entire statement (if it has any), or any of the words in it (except for "only" and "if"--of course), or whether either the original statement or its converse is "true."

Answers to Exercises

Exercises 1-1a:

1. (a) If y is greater than x, then $x + 3 = y$.
   (b) If a natural number is a multiple of 2, then it is even.
   (c) $x = 1$ if $x^2 = 1$.
   (d) If x is less than z, then x is less than y.
   (e) If "If B, then A" is the converse of "If A, then B", then "If A, then B" is the converse of "If B, then A".

2. (a) If $x + z = y + z$, then $x = y$, and, if $x = y$, then $x + z = y + z$.
   (b) If $y - 1 = x$, then $x + 1 = y$, and, if $x + 1 = y$, then $y - 1 = x$.
   (c) If $x = 3$, then $2x + 1 = 7$, and, if $2x + 1 = 7$, then $x = 3$.
   (d) If x or y is zero, then $(x + y)^2 = x^2 + y^2$, and, if $x^2 + y^2 = (x + y)^2$, then x or y is zero.
   (e) If "If B, then A" is true, then the converse of "If A, then B" is true, and, if the converse of "If A, then B" is true, then "If B, then A" is true.

[pages 2-3]

1-2. The System of Natural Numbers.

In this section we collect a list of properties of the natural number system. Our attitude is that, to a greater or lesser extent, these properties are ones which the student has encountered before. For some classes a considerable amount of review will be required here.

Some students may never have seen these properties stated as general propositions before, although they will have used particular instances in arithmetical work. For them it will be necessary to have some class discussion of "symbols", the meaning of "arbitrary", and--in general--a review of the differences between algebra and arithmetic. (In arithmetic we deal with given numbers and are concerned with calculations; in algebra we deal with arbitrary symbols and are concerned with derivations--with proving formulas and solving equations.)

For other students, a large part of the section will itself be review. For them, the bigg t task will be putting the names to the various properties and getting them all clearly in mind to prepare for what is coming in later sections.

The first page (bottom of 4 and top of 5) is extremely informal and carefree. We merely set the stage and introduce some of the actors. There is no real plot, except for the word "closed." The words "subtraction" and "division" appear here, but they are really out of place from a logical point of view. They are mentioned merely to emphasize that "closure" is an important notion; that one makes a significant assertion when he says such-and-such a system is closed under such-and-such an operation. Subtraction will be introduced formally on page 24, division on page 48; until then they play no official part whatever in our considerations, except that we have to learn to live without them in the meantime.

On pages 5 and 6 we refer to the "relation" equality and the "operations" addition and multiplication. These words require careful interpretation if their use is to be understood. The relations we consider (equality, order) and the algebraic operations we work with (the so-called rational operations) are binary. This means two symbols are involved: $a = b$, $a \neq b$, $a < b$, $a > b$; $a + b$, $a \cdot b$, $a - b$, $\frac{a}{b}$. The grammatical result of compounding a pair of symbols with the sign for a relation between them is to form a statement. Thus $a = b$, $a \neq b$, $a < b$ and $a > b$ are each statements; they assert something about numbers (or more precisely about letters representing numbers). On the other hand, the grammatical status of the result of compounding a pair of symbols with the sign for an operation between them is that of a noun. Thus $a + b$, $a - b$, $a \cdot b$, $\frac{a}{b}$ are "expressions" for certain numbers (or more precisely become so if the letters are replaced by numerals). A mathematical expression is just a more or less complicated noun. It is never a statement as a formula always is.

In Exercise 1-2b, Part 2, the student is asked to supply some proofs. Generally speaking, throughout this book we offer a model--usually called an Example--in the text before we ask the student to undertake any activity on his own. In this case we deliberately failed to offer such a model. Our reasons for this are as follows.

There are many ways to write out proofs using the basic properties presented on pages 6 and 7. Such ways range from the the completely detailed--giving a reason every time we do any thing at all--to the completely sketchy--e.g., dispatching Exercise 1-2b, 2(a) with the words "commutativity, distributivity". In the "Answer" section of the Commentary the proofs will be given in considerable detail, but for class work some middle course seems the most desirable.

One question that will undoubtedly come up is whether students may use their familiar "substitution principle (or axiom)": equals may be substituted for equals. The answer is an emphatic "YES" . Since it is so easy to show that this principle is a theorem in our systems, we suggest that it be proved in class to pr[illegible] example of a very detailed proof. This principle is summarized by the following three theorems:

Theorem 1: If $a = b$, $a = c$, and $b = d$, then $c = d$.

Theorem 2: If $a + b = c$ and $a = d$, then $d + b = c$.

Theorem 3: If $ab = c$ and $a = d$, then $db = c$.

Proof of Theorem 1:

1.	$a = c$	[Hyp.	4.	$c = b$	[E_4 (Trans); 2),3)
2.	$c = a$	[E_3 (Symm); 1)	5.	$b = d$	[Hyp.
3.	$a = b$	[Hyp.	6.	$c = d$	[E_4 (Trans.); 4),5)

Q.E.D.

Proof of Theorem 2:

1.	$a + b = c$	[Hyp.	4.	$a + b = d + b$	[E_5;3)
2.	$c = a + b$	[E_3;1)	5.	$c = d + b$	[E_4; 2),4)
3.	$a = d$	[Hyp.	6.	$d + b = c$	[E_3;5)

Q.E.D.

Proof of Theorem 3: Same as proof of Theorem 2, writing a dot for each plus and citing E_6 in place of E_5 .

These three theorems are almost obvic

If the class wants to see more proofs like these the teacher can work Exercise 1-2b,3 in class. (These theorems are useful variants of the substitution principle.) If they had no trouble following the previous arguments, Exercise 1-2b,3--or only one of its parts--may be assigned as homework.

The important thing, however, is that the students should feel free to use the substitution axioms whenever they wish in all later proofs without filling in the details. (This is the essence of Hardy's remarks.)

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*"There is a certain ambiguity in this phrase which the reader will do well to notice. When one says 'such and such a theorem is almost obvious' one may mean one or other of two things. One may mean 'it is difficult to doubt the truth of the theorem', 'the theorem is such as common sense instinctively accepts', as it accepts, for example, the truth of the propositions '2 + 2 = 4' or 'the base-angles of an isosceles triangles are equal'. That a theorem is 'obvious' in this sense does not prove that it is true, since the most confident of the intuitive judgments of common sense are often found to be mistaken; and even if the theorem is true, the fact that it is also 'obvious' is no reason for not proving it, if a proof can be found. The object of mathematics is to prove that certain premises imply certain conclusions; and the fact that the conclusions may be as 'obvious' as the premises never detracts from the necessity, and often not even from the interest of the proof.

"But sometimes (as for example here) we mean by 'this is almost obvious' something quite different from this. We mean 'a moment's reflection should not only convince the reader of the truth of what is stated, but should also suggest to him the general lines of a rigorous proof'. And often, when a statement is 'obvious' in this sense, one may well omit the proof, not because the proof is unnecessary, but because it is a waste of time to state in detail what the reader can easily supply for himself.

"The substance of these remarks was suggested to me many years ago by Prof. Littlewood."

This footnote appears on Page 130 of the American Reprint of the Eighth Edition (1943) of G. H. Hardy's A Course of Pure Mathematics, Cambridge University Press and The Macmillan Company. It is quoted with the permission of The Macmillan Company, New York.

As an example of their use, the teacher may work Exercise 1-2b, 2(a) in class as a model for the other proofs requested in Exercise 1-2b,2 .

$$(x + y)z = xz + yz$$

Proof:

1. $z(x + y) = zx + zy$ [$\underline{D}$
2. $(x + y)z = xz + yz$ [Subst. ($\underline{M}_2$)

Our first "public" proof, on Page 9, is quite sketchy--considering the detail we could have supplied--but it is not so sketchy the omitted details cannot be inserted easily. Our real interest is in the consequences of the A,M,D,C properties. Their significance would be less apparent if we always buried them among a mass of detail which can well be omitted if it is "almost obvious" anyway. We expect the student to react to our proofs by feeling that we could supply all the details if someone insisted. We think he should write his proofs the same way; and we hope he feels that he could also supply such details, although we shall not insist that he do.

Answers to Exercises

Exercises 1-2a:

1. Addition and multiplication.
2. Addition and multiplication.
3. Multiplication.
4. None.
5. Multiplication and division, assuming that $\frac{1}{0}$ is not defined.
6. Addition and multiplication.

Exercises 1-2b:

1. (a) $\underline{A_2}$ (Commutativity) (d) $\underline{D}$ (Distributivity)

   (b) $\underline{D}$ (Distributivity) (e) $\underline{D}$ (Distributivity)

   (c) $\underline{M_3}$ (Associativity) (f) $\underline{A_3}$ (Associativity)

   Note that parts e and f involve use of different names for a number ($7 \cdot 5 = 35$ and $2 + 3 = 5$), which is the relation of equality rather than a property of the natural numbers.

2. (a) $(x + y)z = z(x + y)$ [Comm.

   $= zx + zy$ [Dist.

   $= xz + yz$ [Comm.

   Note: This proves the "right hand" distributive property. It may be used to make proofs shorter in later work.

   (b) $x + xy = 1 \cdot x + xy$ [Mult. Iden.

   $= x \cdot 1 + xy$ [Comm.

   $= x(1 + y)$ [Dist.

   (c) $x[y + (w + z)] = x[(y + w) + z]$ [Assoc.

   $= x(y + w) + xz$ [Dist.

   An alternate method would be to use $\underline{D}$ three times.

   (d) $x + (y + z) = (z + y) + xz$ [Hypothesis

   $= (y + z) + xz$ [Comm.

   $= xz + (y + z)$ [Comm.

   therefore $x = xz$ [Cancellation

3. $a = b$ and $c = d$ [Hypothesis

$a + c = b + c$ and $b + c = b + d$ [$\underline{E}_5$ (Addition)

$a + c = b + d$ [$\underline{E}_4$ (Transitive)

$a = b$ and $c = d$ [Hypothesis

$ac = bc$ and $bc = bd$ [$\underline{E}_6$ (Multiplication)

$ac = bd$ [$\underline{E}_4$ (Transitive)

Exercises 1-2c:

1. (a) $5p(3 + r) = 5p \cdot 3 + 5p \cdot r$ [Dist.

$= 3(5p) + 5(p \cdot r)$ [Comm., Assoc.

$= (3 \cdot 5)p + 5pr$ [Assoc., Def.

$= 15p + 5pr$

(b) $(2x + 3)(x + 4) = (2x + 3) \cdot x + (2x + 3) \cdot 4$ [Dist.

$= 2x \cdot x + 3x + 4 \cdot 2x + 4 \cdot 3$ [Dist.,Comm.

$= 2x^2 + 3x + 8x + 12$ [Assoc., Def.

$= 2x^2 + (3 + 8)x + 12$ [Dist.

$= 2x^2 + 11x + 12$

(c) $(y + 1)(y + 1) = (y + 1) \cdot y + (y + 1) \cdot 1$ [Dist.

$= y \cdot y + 1 \cdot y + y \cdot 1 + 1 \cdot 1$ [Dist.

$= y^2 + y + y + 1$ [Def., Mult. Iden.

$= y^2 + 2y + 1$ [Def.

(d) $2m(m + n + 3) = 2m \cdot m + 2m \cdot n + 2m \cdot 3$

$= 2(m \cdot m) + 2mn + 3(2m)$

$= 2m^2 + 2mn + (3 \cdot 2)m$

$= 2m^2 + 2mn + 6m$

[page 10]

(e) $(x + 1)(x + y + 2)$

$= (x + 1) \cdot x + (x + 1) \cdot y + (x + 1) \cdot 2$ [Dist.

$= x \cdot x + 1 \cdot x + x \cdot y + 1 \cdot y + x \cdot 2 + 1 \cdot 2$ [Dist.

$= x^2 + x + xy + y + 2x + 2$ [Mult. Iden., Comm., Def.

$= x^2 + xy + x + 2x + 2 + y$ [Comm.

$= x^2 + xy + 1 \cdot x + 2 \cdot x + y + 2$ [Mult. Iden.

$= x^2 + xy + (1 + 2)x + y + 2$ [Dist.

$= x^2 + xy + 3x + y + 2$

2. (a) $(a + b + c) + d = [(a + b) + c] + d$ [Def.

$= (a + b) + (c + d)$ [Assoc.

(b) $(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d$ [Dist.

$= ac + bc + ad + bd$ [Dist.

$= ac + ad + bc + bd$ [Comm.

(c) $(px + q)(rx + t)$

$= (px + q) \cdot rx + (px + q)t$ [Dist.

$= px(rx) + q(rx) + (px)t + qt$ [Dist.

$= rx(px) + q(rx) + t(px) + tq$ [Comm.

$= rx(xp) + q(rx) + (tp)x + qt$ [Comm., Assoc.

$= (rx \cdot x)p + (qr)x + (pt)x + qt$ [Assoc., Comm.

$= p(rx^2) + (qr + pt)x + qt$ [Comm., Dist., Def., Assoc.

$= prx^2 + (qr + pt)x + qt$ [Def.

$= prx^2 + (pt + qr)x + qt$ [Comm.

(d) $a(b + c + d) = a[(b + c) + d]$ [Def.

$= a(b + c) + ad$ [Dist.

$= (ab + ac) + ad$ [Dist.

$= ab + ac + ad$ [Def.

(e) $a(bcd) = a[(bc)d]$ [Def.

$= [a(bc)]d$ [Assoc.

$= [(ab)c]d$ [Assoc.

$= (ab)(cd)$ [Assoc.

3. (a) $4x + 2xy = x \cdot 4 + x \cdot 2y$ [Comm., Assoc.

$= x(4 + 2y)$ [Dist.

(b) $2(4u + 1) + 3(4u + 1) = (2 + 3)(4u + 1)$ [Dist.

$= 5(4u + 1)$

(c) $m(p + q) + m(p + q) = (m + m)(p + q)$ [Dist.

$= (m \cdot 1 + m \cdot 1)(p + q)$ [Mult. Iden.

$= m(1 + 1)(p + q)$ [Dist.

$= 2m(p + q)$ [Comm.

An alternate sequence would be:

$$m(p + q) + m(p + q) = 2[m(p + q)]$$
$$= 2m(p + q)$$

using the agreement that $x + x = 2x$.

(d) $(2x + 1)(x + 1) + (1 + 2x)(1 + x)$

$= (2x + 1)(x + 1) + (2x + 1)(x + 1)$ [Comm.

$= 2[(2x + 1)(x + 1)]$ [Def.

$= 2(2x + 1)(x + 1)$ [Def.

4. An even natural number has the general form $2a$, where a may represent any natural number. Since

$$(2a)^2 = (2a)(2a)$$
$$= 4a^2$$
$$= 2(2a^2) ,$$

and $2a^2$ is a natural number by $\underline{M}_1$ (Closure), then $(2a)^2$ is even.

5. An odd natural number has the general form $(2a + 1)$, where a may represent any natural number. Since

$$\begin{aligned}(2a + 1)^2 &= (2a + 1)(2a + 1)\\ &= 4a^2 + 4a + 1\\ &= 2(2a^2 + 2a) + 1 ,\end{aligned}$$

and $(2a^2 + 2a)$ is a natural number by $\underline{M_1}$ (Closure) and $\underline{A_1}$ (Closure), then $(2a + 1)^2$ is odd.

6. Representing the product as $2a(2b + 1) = 4ab + 2a = 2(2ab + a)$, then the product is even since $(2ab + a)$ is a natural number.

7. Any two digit number ending in 5 can be represented in the decimal system as $10a + 5$ where a represents the first digit on the left. Since

$$\begin{aligned}(10a + 5)^2 &= 100a^2 + 100a + 25\\ &= 100a(a + 1) + 5^2 ,\end{aligned}$$

then $a(a + 1)$ provides the first digit or digits on the left, with the 100 factor fixing their position, and the 5^2 provides the last two digits.

<u>Exercises</u> <u>1-2d</u>:

1. $x + 2 = 3 + 2$

   $x = 3$ [Cancellation - Add.

2. No solution in N , since there is no natural number z such that $1 = z + 3$.

3. $y \cdot 3 = 2 \cdot 3$ [Comm.

   $y = 2$ [Cancellation - Mult.

[pages 10, 12]

4. $2u + 5 = 2 + 5$

$2u = 2$	[Cancellation - Add.
$u \cdot 2 = 1 \cdot 2$	[Comm., Mult. Iden.
$u = 1$	[Cancellation - Mult.

5. $2u + 1 = 3 + 1$

$2u = 3$

No solution in N, since there is no natural number u such that $2u = 3$.

6.

$3p + 4 = p + 4$	[Comm.
$3p = p$	[Cancellation - Add.

No solution in N, since there is no natural number p such that $3p = p$.

7.

$2w + 1 = 3w + 4$	[Comm.
$= 3w + (3 + 1)$	
$= (3w + 3) + 1$	[Assoc.
$2w = 3w + 3$	[Cancellation - Add.

No solution in N.

8.

$3m + 1 = 2m + (3 + 1)$	
$= (2m + 3) + 1$	[Assoc.
$3m = 2m + 3$	[Cancellation - Add.
$m + 2m = 3 + 2m$	[Comm.
$m = 3$	[Cancellation - Add.

1-3. Order in the Natural Number System.

In Section 1-3 we begin our study of inequalities. (This study continues in Sections 1-5, 1-7, 1-9.) We observe that the counting process arranges the natural numbers in a definite order and we [illegible] our order relation on the notion of precedence provided by [illegible] counting process. We [illegible] the order relation to the addition operation in N using the criterion: For a, b in N

$$a < b \quad \text{if and only if} \quad \begin{cases} \text{there is a } c \text{ such} \\ \text{that } a + c = b. \end{cases}$$

This enables us to deduce $\underline{O}_2, \underline{O}_3, \underline{O}_4$ [illegible] $\underline{A}, \underline{M}, \underline{D}, \underline{C}$. We cannot deduce [illegible] from $\underline{E}, \underline{A}, \underline{M}, \underline{D}, \underline{C}$. (We fail [illegible] mention this in the text. That we [illegible] prove it follows from [illegible] fact that there exist systems satisfying all of the $\underline{E}, \underline{A}, \underline{M}, \underline{D}, \underline{C}$ properties of N, I, and ever [illegible] satisfying none of the $\underline{O}$ properties. Examples of such systems are the so-called "modular systems".)

We begin our study of the solution of inequalities on pages 15-16 where we discuss the cancellation properties and their converses and consider the problem of "checks". At the bottom of page 18 and the top of page 19, we solve our first inequalities.

We close Section 1-3 by completing the list of "basic properties" for N and summarize the results of Sections 1-2, 1-3.

We state the Archimedian and Well Order properties, because they are needed for a complete list of "basic properties" as we explained that expression on page 11. We shall have no occasion to use either of them in our work. The Archimedian property holds for all the number systems in Chapter 1. Because of this we state it as a property for N, although in N we could drop it or prove it from the Well Order property. (The Well Order property holds for none of the other systems.)

The importance of property $\underline{O}_5$ (Archimedes) for all of our systems rests on the fact that any number system without this property is inadequate for problems of "measurement." $\underline{O}_5$ asserts that any "length" b can be "measured" by any given "length" a. Thus we know that a yardstick (of length a = 3 feet) can be used to measure a mile (b = 5280 feet) by putting it down some number n of times, where na > b. If our numbers did not have this property, there would be some length N (a mile or perhaps a light-year, or some greater distance) which could not be determined as some multiple of a yard. Since modular systems are not even ordered, they are certainly not "Archimedian ordered"; thus they are useless for questions of measurement. Although they may not help surveyors or astronomers, they are useful to people with other kinds of problems.)

On a more elementary level, the Archimedian property is what makes the long division algorithm work. We divide by subtracting multiples of the divisor. Our quotient is the largest multiple of the divisor not exceeding the dividend. If no multiple of the "divisor" exceeded the dividend, this subtraction procedure would fail in a dismal way to produce any quotient.

How do we prove $\underline{O}_5$ from $\underline{O}_6$ in N? The well order principle implies that no element of N is less than 1. Therefore, if $a \neq 1$ and a is in N, we have $a > 1$ (why?). But then $\underline{O}_4$ gives $ab > b$. Thus b itself is such an n as $\underline{O}_5$ demands. But if $a = 1$, then $a(b + 1) > b$ and $b + 1$ fulfills the requirement on n.

The well order property, $\underline{O}_6(N)$, provides the logical justification for the method of "mathematical induction." The principle of mathematical induction may be expressed as follows:

> If $S_1, S_2, \ldots, S_n, \ldots$ is a sequence of statements and if the following statements are true:
>
> (i) S_1
>
> (ii) If S_{m-1}, then S_m
>
> (the latter being true for each natural number m greater than 1), then every one of the given statements
>
> $$S_1, S_2, \ldots, S_n, \ldots$$
>
> is true.

We may deduce this principle from the well order property as follows. Suppose, contrary to our desired conclusion, that at least one of the statements

$$S_1, S_2, \ldots, S_n, \ldots$$

is not true. We are to force a contradiction from this supposition in conjunction with our hypotheses. Consider the set of natural numbers n having the property that S_n is false. Our supposition tells us this set contains one or more members. From the well order property we conclude that this set has a minimal member, say m. Then S_k is true for k less than m, but S_m itself is false. Surely m cannot be 1, for hypothesis (i) states that S_1 is true. Since $m > 1$, we know that S_{m-1} is true. Our second hypothesis, (ii), tells us that S_m must be true since both

$$S_{m-1}$$

and

$$\text{If } S_{m-1}, \text{ then } S_m$$

are true. This is our contradition, for by the definition of m, we know that S_m is false. Thus we must reject our supposition and conclude that every one of the statements

$$S_1, S_2, \ldots, S_n, \ldots$$

is true. Q.E.D.

We have shown that the principle of mathematical induction is a consequence of the well order property. Conversely, the well order property is a consequence of the principle of mathematical induction. (cf. Birkhoff and Mac Lane, A Survey of Modern Algebra.) These two principles are therefore logically equivalent; neither tells us any more about the natural numbers than the other does. Peano, in his postulational development of the natural number system, takes the principle of mathematical induction as one of his postulates. Here we have chosen instead the well order property, since it seems to be easier to grasp on first reading than the principle of mathematical induction. We do not explicitly use either of these principles in this book (except for the proof of Theorem 15-7b), although detailed proofs for many of the results in chapters 13 and 14 would require one or the other of them. (The statements made in Section 1-2 on page 8 and at the top of page 9 also conceal "inductions.")

Properties $\underline{O}_5$, $\underline{O}_6(N)$ are presented in the text for two reasons: (i) they complete the list of basic properties and hence distinguish N from the other systems I, Q, R, C considered in this book, (ii) they provide a basis for the students' future study of mathematics. We do not believe the teacher need emphasize them beyond pointing out these facts. Certainly their mastery at this stage is unnecessary; it is their existence which is significant.

Answers to Exercises

Exercises 1-3a:

1. { 1, 2, 3, 4 }.
2. 2 + 4 = 6.
3. (a) $2 < 6$.
   (b) $3 < 5$.
   (c) $a < 3a$.
   (d) $1 + a < 2 + a$.
   (e) $b < c$.
   (f) $a < e$.
4. (a) $x \leq 4$.
   (b) $5 < x < 7$.
   (c) $4 \leq y$.
   (d) $m \neq n$.
   (e) $3 \leq x \leq 5$.
5. $\underline{O}_3$. Statement: If there is a d in N such that $a + d = b$, then there is an e in N such that $(a + c) + e = b + c$.
   Proof: Since $a + d = b$, $(a + d) + c = b + c$. Now
   $(a + d) + c = a + (d + c) = a + (c + d) = (a + c) + d$
   $\therefore (a + c) + d = b + c$
   Thus $a + c < b + c$ because we have found a number $e = d$ which added to $a + c$ gives $b + c$.

$\underline{O_4}(N)$: If there is a d in N such that $a + d = b$, then there is an e in N such that $ac + e = bc$. Since $a + d = b$, then $(a + d)c = bc$ and $ac + dc = bc$. Then $e = dc$, where dc is in N by the closure property for multiplication.

6. $x < z$ from transitivity of equals and the order definition.

Exercises 1-3b:

1. (a) If $a + b = c$, then $(a + b) + b = c + b$ and by the definition for order, $a + b < c + b$.

   (b) If $a(b + c) = d$, then $ab + ac = d$. Since ac is N, then $ab < d$.

   (c) If $a < b$ and $c < d$, then there are natural numbers e and f such that $a + e = b$ and $c + f = d$. Then $(a + e) + (c + f) = b + d$, and $(a + c) + (e + f) = b + d$ by use of Commutivity and Associativity for addition. Since $(e + f)$ in N, then $a + c < b + d$. An alternate proof can be made using Transitivity and the $\underline{O_3}$ property.

   (d) Trichotomy provides three cases; (i) $a < b$, (ii) $b < a$, (iii) $a = b$. If $a < b$, then $ac < bc$ by $\underline{O_4}(N)$ which contradicts the hypothesis $ac = bc$. Similarly, case (ii) contradicts the hypothesis, so $a = b$.

   (e) If $a + c < b + c$, then there is a natural number d such that $(a + c) + d = b + c$. By use of Associativity and $\underline{C_1}$, $a + d = b$ and $a < b$.

   (f) Trichotomy provides three cases. If $a = b$ or $b < a$, then $ac = bc$ or $bc < ac$ for c in N. Both of these conclusions contradict the hypotheses, so $a < b$.

2. (a) $m = 1$.
   (b) $p = 1$.
   (c) $x = 1$.
   (d) $x = 1$.
   (e) $y = 1, 2, 3,$ or 4.
   (f) $x = 2, 3, 4,$ or 5.

3. Since $a < b < c$ if and only if $a < b$ and $b < c$, use $\underline{OC_1}$.

Exercises 1-3c:

1. $1 < 2$ and $3 \cdot 1 > 2$.
2. 10.
3. (a) $\underline{M}_2$. (b) $\underline{E}_2$. (c) $\underline{A}_3$. (d) $\underline{D}$. (e) $\underline{O}_2$.
   (f) $\underline{M}_3$. (g) $\underline{O}_4$. (h) $\underline{E}_1$. (i) $\underline{A}_2$. (j) $\underline{E}_5$.
   (k) $\underline{E}_5$. (l) $\underline{O}_1$. (m) $\underline{E}_6$. (n) $\underline{O}_4$ (N).
   (o) $\underline{O}_5$.
4. Suppose p is a solution. Then $p + 2 = 2$ and from the definition for $a < b$ it follows that $p < 2$. But this contradicts $\underline{O}_1$ (Trichotomy).

1-4. The System of Integers.

In Section 1-4 we discuss the integers. Just as in Section 1-2 where we first discuss the elements of N, we do not say what these numbers are. We take that as "known." We feel that for us to attempt any description of the numbers in our various systems, or to discuss their "existence," would becloud the discussion completely. Throughout this chapter, we see our aim as being the organization, in a logical pattern, of properties of the various sets of numbers with which the student is familiar and some (at least) of whose properties he already knows. It is not our job to convince the student that "his" numbers have the basic properties - he has to grant that. What we do do - and it is a pretty ambitious project by itself - is try to convince him that the rest of the properties follow as theorems. Some of these "derived" properties he already knows, of course, but whether or not he does is irrelevant because we prove them.

This attitude is not new to the student. It is precisely the attitude in geometry, where the most "basic" words (point, line, plane) are not defined; the only attributes the geometer requires of whatever they name is that the postulates be satisfied. One may put the same interpretation on our sets of basic properties: that our lists of basic properties are all we need know about the various sets of numbers they describe. This is a possible attitude, and it is the one we adopt here; but it would be misleading to imply that no other alternative is available.

It is possible to begin with the natural numbers as given, and from them to construct, successively, each of the other number systems. Such programs are carried out in Landau's Foundations of Analysis, Thurston's The Number System, Kershner and Wilcox's Anatomy of Mathematics, and other books. This program has been summarized by the great nineteenth-century German mathematician Leopold Kronecker in the epigram "God made the integers, the rest is man's work." Since Kronecker's time, man has presumed to "make" even the natural numbers. How this may be done is described, for example, in Elements of Algebra, by Howard Levi.

We "create" no numbers at all here. We merely "observe" them and arrange their properties in a logical hierarchy: "basic" properties, and theorems.

Thus on page 24, we simply state that I has properties $\underline{A_4}$ (Additive Identity) and $\underline{A_5}$ (Subtraction).

Having $\underline{A_5}$, we define the difference of two integers on page 24, and on page 25 introduce the abbreviation $-a$ for $0-a$. With these items at our disposal we undertake our first serious sequence of proofs. We show how all the "usual" properties of additive inverses follow from $\underline{A_4}$, $\underline{A_5}$ and the properties we inherited from N.

The proofs we give for $a \cdot 0 = 0$ and all the usual "laws of signs" are by no means the only ones we could have selected. Each teacher has his own favorite way of proving these things and we feel we cannot be too emphatic in encouraging him to present alternate proofs to his class. Two proofs are always better than one. More than just twice as good in fact, for they help to tie the result in question to diverse complexes of ideas, thus revealing logical relationships otherwise unnoticed.

The proofs we present in Section 1-4 are all variations of the idea that whenever two expressions satisfy an equation having only one solution, they are equal. This is the idea of "uniqueness," or "unicity." This notion is one of the most important in all of mathematics: two numbers have one and only one sum, product, etc. Such and such a differential equation is satisfied by precisely one function meeting certain initial or boundary conditions. Each non-negative real number has one and only one non-negative square root. Each positive real number has one and only one common logarithm. Indeed <u>any</u> function assigns one and only one element of its range to each element of its domain. (See Chapter 3.)

Answers to Exercises

Exercises 1-4a:

1. (a) -2. (d) -m.
   (b) -(-5) = 5. (e) -(-p) = p
   (c) -0 or 0. (f) -(b - a) or (a - b)
2. (a) Definition 1-4b.
   (b) A_4 (Additive Identity).
   (c) Definition 1-4a.
   (d) Theorem 1-4a.
   (e) Corollary 1-4a.
3. A counter example, $5 - 3 \neq 3 - 5$, proves that subtraction is not commutative for a, b in I. Perhaps question could be raised whether subtraction is ever commutative to emphasize properties of zero, i.e., $a - b = b - a$ if $a = b$.
4. A counter example, $(5 - 3) - 2 \neq 5 - (3 - 2)$, proves that subtraction is not associative for a, b, c in I. An extension of this would be to determine conditions under which subtraction is associative in I.
5. If $-x = -y$, then $x = y$.

   Proof: If $-x = -y$, then $-x + y = -y + y$ by E_4 and $-x + y = 0$. Hence, $y = -(-x)$ or $y = x$.

   If $x = y$, then $-x = -y$.

   Proof: If $x = y$, then $x+(-x) = y+(-x)$ and $0 = y+(-x)$. From this, $-x = -y$.

6. If $a + 0 = a$ for a in I, then $0 + 0 = 0$. Also, if $a + x = 0$, then $x = -a$. Using $0 + 0 = 0$ as a form of $a + x = 0$, then $0 = -0$.
7. Suppose a in I, $a \neq 0$. If $a = -a$, then $a + (-a) = 0$ by Definition 1-4b, or $(-a) + a = 0$. Since $a = -a$ by hypothesis, $-a + (-a) = 0$ and $-a = -(-a)$. But this contradicts Theorem 1-4a, $a = -(-a)$, so $a \neq -a$ for $a \neq 0$ and a in I. From number 6 above, $0 = 0$, so 0 is the only integer which is its own additive inverse.
8. Suppose a in N and $-a$ in N. Since $a+(-a) = 0$ and the system of natural numbers is closed under addition, then zero is a natural number. But zero is not a natural number, so the additive inverse of a natural number cannot be a natural number.

Exercises 1-4b:

1. (a) $1 + (-2) = 1 - [-(-2)]$ [Th. 1-4a,c
   $= 1 - 2$ [Th. 1-4a
   $= -1$ [Def. 1-4a

   (b) $12 - (-4) = 12 + [-(-4)]$ [Th. 1-4c
   $= 12 + 4$ [Th. 1-4a
   $= 16$

   (c) $(-8) - (-7) = (-8) + [-(-7)]$ [Th. 1-4c
   $= (-8) + 7$ [Th. 1-4a
   $= 7 + (-8)$ [Comm.
   $= 7 - 8$ [Th. 1-4c
   $= -1$ [Def. 1-4a

   (d) $(-5) + 7 = 7 + (-5)$ [Comm.
   $= 7-5$ [Th. 1-4c
   $= 2$ [Def. 1-4a

   (e) $(-4)\cdot(5) = -(4\cdot5)$ [Th. 1-4f
   $= -20$

   (f) $(-2)\cdot(-7) = (2\cdot7)$ [Th. 1-4g
   $= 14$

[page 30]

(g) $3(a + 2) - 4(a + 2) = 3(a + 2) + [-4(a + 2)]$ [Th. 1-4c
$= [3 + (-4)] (a + 2)$ [Dist.
$= (3 - 4) (a + 2)$ [Th. 1-4c
$= (-1) (a + 2)$ [Def. 1-4a
$= -(a + 2)$ [Th. 1-4e, Comm.

(h) $-5 (6)(-3) = [-5(6)] (-3)$ [Def.
$= [-(5\cdot 6)] (-3)$ [Th. 1-4f
$= (30) (3)$ [Th. 1-4g
$= 90$

(i) $4(5a)(0) = [4(5a)] (0)$ [Def.
$= 4[(5a)(0)]$ [Assoc.
$= 4 (0)$ [Th. 1-4b
$= 0$ [Th. 1-4b

(j) $-(2a-3) + 4(3-2a) = (3-2a) + 4(3-2a)$ [Def. 1-4b
$= (3-2a)\cdot 1 + (3-2a)\cdot 4$ [Mult. Iden. and Comm.
$= (3-2a) (1+4)$ [Dist.
$= 5(3-2a)$ [Comm.

2. (a) $-(x - y) = -1(x - y)$ [Th. 1-4e, and Comm.
$= -1[x + (-y)]$ [Th. 1-4c
$= (-1)x + (-1)(-y)$ [Dist.
$= -x + y$ [Th.1-4e,Comm.,Th.1-4g,and Mult.Iden.
$= y - x$ [Th. 1-4c, Comm.

(b) $(-x) + (-y) = -1(x) + (-1)(y)$ [Th. 1-4e, Comm.
$= -1(x + y)$ [Dist.
$= (x + y) (-1)$ [Comm.
$= -(x + y)$ [Th. 1-4e

(c) $(-x)y = [(x)(-1)]y$ [Th. 1-4e
$= (x) [(-1)y]$ [Assoc.
$= (x) [y(-1)]$ [Comm.
$= [(x) y] (-1)$ [Assoc.
$= - xy$ [Th. 1-4e

(d) $(-x)(-y) = [x(-1)](-y)$ [Th. 1-4e
$= x[(-1)(-y)]$ [Assoc.
$= x[(-y)(-1)]$ [Comm.
$= xy$ [Th. 1-4e, Th. 1-4a

[page 30]

3.	$a(b - c) = a[b + (-c)]$	[Th. 1-4c
	$= ab + a(-c)$	[Dist.
	$= ab - ac$	[Th. 1-4c, Comm., Th. 1-4f
4. (a)	$5x - 3 = 12$	
	$[5x + (-3)] + 3 = 12 + 3$	[Th. 1-4c and E_5
	$5x + [(-3) + 3] = 15$	[Assoc.
	$5x + 0 = 15$	[Add. Inverse
	$5x = 5 \cdot 3$	[Add. Iden.
	$x \cdot 5 = 3 \cdot 5$	[Comm.
	$x = 3$	[C_2
(b)	$3y + 4 = 2y - 18$	
	$(3y+4) + (-4) = [2y+(-18)] + (-4)$	[Th. 1-4c and E_4
	$3y + [4+(-4)] = 2y+[(-18)+(-4)]$	[Assoc.
	$3y + 0 = 2y + (-22)$	[Add. Inv. and Th. 1-4d
	$3y + (-2y) = [2y+(-22)] + (-2y)$	[E_5
	$3y + (-2)y = [(-22)+2y] + (-2y)$	[Th. 1-4f and Comm.
	$y[3+(-2)] = -22 + [2y+(-2y)]$	[Comm., Dist., and Assoc.
	$y = -22$	[Th. 1-4c, Add. Inv., and Add. Iden.

(c)
$3m - 2(7-2m) = 21$
$3m + \{-2[7 + (-2m)]\} = 21$
$3m + [-2(7) + (-2)(-2m)] = 21$
$3m + [-14 + 4m] = 21$
$3m + [4m + (-14)] = 21$
$[3m + 4m] + (-14) = 21$
$7m + [(-14) + 14] = 21 + 14$
$7m = 35$
$m \cdot 7 = 5 \cdot 7$
$m = 5.$

(d)
$2(6z + 2) + 3 = 12 - 3(2z - 1)$
$(1 z + 4) + 3 = 12 + \{-3[2z + (-1)]\}$
$1z + (4+3) = 12 + [-3(2z) + (-3)(-1)]$
$1z + 7 = 12 + (-6z + 3)$
$1z + 7 = 12 + [3 + (-6z)]$
$1z + 7 = (12 + 3) + (-6z)$
$12z + [7 + (-7)] = (-6z) + [15 + (-7)]$

(d) (continued)

$12z = (-6z) + 8$

$12z + 6z = [8 + (-6z)] + 6z$

$z(12 + 6) = 8 + [(-6z) + 6z]$

$18z = 8 + 0$

No solution in I since there is no z in I such that $18z = 8$.

(e) $x + (-1) = x + (-2)$

$[x + (-1)] + 1 = [x + (-2)] + 1$

$x + [(-1) + 1] = x + [(-2) + 1]$

$x = x + (-1)$

$x = x - 1$

No solution in I since there is no x in I such $x = x - 1$

(f) $100 (p + 4) + 11p = 111p + 400$

$100p + 400 + 11P = 111p + 400$

$100p + 11p + 400 = 111p + 400$

$111p = 111p$

All values of p in I will satisfy this equation.

*5. $ac = bc$ if $-(ac) = -(bc)$

<u>Proof</u>: $-(ac) = -(bc)$

$(ac)(-1) = bc(-1)$ [Th. 1-4e

$ac = bc$ [$\underline{C}_2$

$ac = bc$ only if $-(ac) = -(bc)$

<u>Proof</u>: $ac = bc$

$(ac)(-1) = (bc)(-1)$ [$\underline{E}_6$ (Mult.)

$-(ac = -(bc)$ [Th. 1-4e, Mult. Iden.

There are eight possible cases for a, b, c to be natural numbers or not. Of these, only four will have $ac = bc$ and $c \neq 0$.

<u>Case (i)</u>. $0 < a$, $0 < b$, and $0 < c$. If $ac = bc$, then $a = b$ by $\underline{C}_2(N)$.

<u>Case (ii)</u>. $a < 0$, $b < 0$, and $c < 0$. Then $0 < ac$ and $0 < bc$. If $ac = bc$, then $a = b$ by $\underline{C}_2(N)$.

Case (iii). $0 < a$, $0 < b$, and $c < 0$. Then $ac < 0$ and $bc < 0$, so $0 < -ac$ and $0 < -bc$ and $a = b$ by $\underline{C}_2(N)$ and preliminary Theorem proved.

Case (iv). $a < 0$, $b < 0$, and $0 < c$. Then $ac < 0$ and $bc < 0$, so same argument as Case (iii).

1-5. Order of the Integers.

In this section we continue our study of inequalities. First we must extend the definition of the relation $<$ from the "subsystem" N to the entire system I.

Theorem 1-5a, page 32, permits us to do this (Definition 1-5a) without changing the wording we had in N. However anticipating the next extension to all of Q (the rational number-system), we introduce the terms "positive," "negative" and reword our first definition (1-5a), calling it a "second" definition (1-5c). It is the latter form which we shall carry over to Q in Section 1-7. This procedure has a slight pedagogical disadvantage, perhaps in that it seems to provide two definitions for the same thing. If this should bother any of the students point out that the first (Def. 1-5a) is really the official one here, the one we actually use in Section 1-5; and that the other (Def. 1-5c) may be interpreted as a "criterion" for $a < b$ which we shall need later.

Next we have three theorems on the products of positive and/or negative numbers. In proving them we use the "law of signs" theorems (Theorems 1-4f, 1-4g) in our arguments. It should be clear, however, that Theorems 1-5c, 1-5d are not the same as Theorems 1-4f, 1-4g as the former are concerned with order and the notions "positive," "negative" while the latter have nothing whatever to do with any of these ideas. Theorems 1-4f and 1-4g are valid results in any "ring" (i.e., any system with the E,A,M,D Properties of I) whether or not it has any kind of "order relation." On the other hand Theorems 1-5b, 1-5c, 1-5d are theorems about an order relation.

Using these theorems, properties $\underline{O}_1$, $\underline{O}_2$, $\underline{O}_3$, $\underline{O}_4$ (I) follow easily. Note here, that this time we can prove $\underline{O}_1$, although we couldn't prove it in N. (See Commentary on Section 1-3.) The reason we can prove it now is to be found in the fact that I contains the subsystem N which has the following three properties:

(i) For any a in I, if $a \neq 0$, then either a is in N, or $-a$ is in N

(ii) For a, b in N, $a + b$ is in N

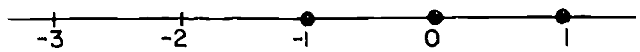
(iii) For a, b in N, ab is in N.

(Property (i) is Corollary 1-4a; (ii) and (iii) are, respectively, $\underline{A}_1$, $\underline{M}_1$ for N)

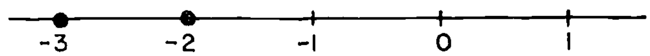
Any ring containing a subset having the properties (i), (ii), (iii) written above for N also has "our" four order properties $\underline{O}_1, \underline{O}_2, \underline{O}_3, \underline{O}_4$. We <u>might</u> have adopted (i), (ii), (iii) as our basic order properties instead of $\underline{O}_1, \underline{O}_2, \underline{O}_3, \underline{O}_4$ which would then be theorems. This is done, for instance, by Birkhoff and Maclane in their <u>Survey of Modern Algebra</u> and by Stabler in his <u>Introduction to Mathematical Thought</u>. We feel the saving to be gained by having 3 basic properties instead of 4 is more than offset by the fact that we are able to throw the entire theory of order back on the counting process, which is - after all - a notion more familiar to students than the idea of the "existence of a set of positive elements."

On pages 35 and 36, we formulate the cancellation properties for order in I. We give none of the proofs because we believe that these results will not startle any of the students after what they've been through already. Any student who is really interested in seeing some proofs for the assertions on these pages can construct his own. If he can't, then he ought to go back and study the proofs for the cancellation properties in N. He can't possibly lose if he tries: he stands to profit either way.

We close Section 1-5 with a discussion of "absolute value" and, on page 38, we solve an inequality involving absolute values. Students should be encouraged to draw number-lines in connection with Example 1-5, page 38, and Exercise 1-5b, 2. They help to keep the cases straight. For instance, in Example 1-5 in case (i) we find $-1 \leq x \leq 1$:



(The "dots" represent elements of the solution set)
In case (ii) we find $-3 \leq x < -1$:



Since the pair of cases (i) $0 \leq x + 1$, (ii) $x + 1 < 0$ represent <u>alternatives</u>, the solution set of the inequality is the "union" of these two sets:

(This device will prove even more useful in Section 1-7.)

<u>Answers to Exercises</u>

<u>Exercises 1-5a</u>:

1. (a) $-2 < 1$.
   (b) $-8 < -7$.
   (c) $-2 < 0$.
   (d) $-x < x$.
   (e) $x - y < y - x$.
   (f) $2x < -3x$.
2. (a) If $x < y$ and $y < z$, then there are a, b in N such that $x + a = y$ and $y + b = z$.
   Since $y = z - b$, then $x + a = z - b$ or $x + (a + b) = z$, and $x < z$ since $(a + b)$ is in N.
   (b) If $x < y$, then there is an a in N such that $x + a = y$. Then $(x + a) + z = y + z$ or $(x + z) + a = y + z$, and $x + z < y + z$.
   (c) If $x < 0$ and $y < 0$, then $0 < -y$ and by $\underline{O_4}$ (Multiplication), $x(-y) < 0(-y)$.
   Hence, $-xy < 0$ or $0 < xy$.
   (d) If $y < 0$ and $0 < x$, then by $\underline{O_2}$, $y < x$ or $x > y$.
   (e) If $x < y$, then $x+(-z) < y+(-z)$ by $\underline{O_4}$.
   Hence, $x - z < y - z$.

(f) If $x < y$, then $x+(-x) < y+(-x)$ and $0 < y - x$ or $y - x > 0$.

(g) If $-x < 0$, then $-x + x < 0 + x$ or $0 < x$.
If $0 < x$, then $0 + (-x) < x + (-x)$ or $-x < 0$.

(h) If $x < 0$, then $x + (-x) < 0 + (-x)$ or $0 < -x$.
If $0 < -x$, then $0 + x < -x + x$ or $x < 0$.

(i) If $xy < yw$, then $xy + [-(xy)] < yw + [-(xy)]$ or $0 < yw + [(-x)y]$. By use of Commutivity and Distribution, $0 < y\,[w + (-x)]$ or $0 < y(w - x)$.

(j) If $x < y$ and $w > z$, then there are a, b in N such that $x + a = y$ and $z + b = w$.
Hence,
$$(x+a) + (-w) = y + (-w)$$
$$[x+(-w)] + a = y + [-(z+b)]$$
$$(x-w) + a = y + [(-z) + (-b)]$$
$$= (y-z) + (-b)$$
$$(x-w) + a + b = (y-z) + (-b) + b$$
$$(x-w) + (a+b) = y - z$$
By the definition of order in I, $x-w < y-z$.

Exercises 1-5b:

1. (a) $m < 3$, so the solution set is $\{1,2\}$.
   (b) $m < 3$, so $\{\ldots-2, -1, 0, 1, 2\}$.
   (c) $z < 5$, so the solution set is $\{1, 2, 3, 4\}$.
   (d) $z < 5$, so $\{\ldots-2, -1, 0, 1, 2, 3, 4\}$.
   (e) $x < 3/2$, so the solution set is $\{1\}$.
   (f) $x < 3/2$, so $\{\ldots, -1, 0, 1\}$.
   (g) $1 < p < 2$, so the solution set is the empty set.
   (h) $2 \leq y \leq 4$, so the solution set is $\{2, 3, 4\}$.
2. (a) 3, -3.
   (b) $-4 < c < 4$, so the solution set is $\{-3,-2,-1,0,1,2,3\}$.
   (c) No solution, since $0 < |a|$ for a in I.
   (d) $\{-2, 1\}$.
   (e) $\{2\}$; There is no solution in I when $4y-1 < 0$.
   (f) $-10 \leq x \leq 4$, so the solution set is $\{-10,-9,-8,\ldots,2,3,4\}$.
   (g) $2 < x < 8$, so the solution set is $\{3,4,5,6,7\}$.

[pages 35, 39]

(h) $-3 \leq x \leq 9$, so the solution set is $\{-3,-2,-1,\ldots,7,8,9\}$.

(i) $-9 < x < -3$, so the solution set is $\{-8,-7,-6,-5,-4\}$.

3\. (a) If $0 < x$, then $0 \cdot x < x \cdot x$ by $\underline{O_4}$ and $0 < x^2$.

(b) If $1 < x$, then since $0 < 1$ it follows by Transitivity that $0 < x$.

Then by $\underline{O_4}$, $1 \cdot x < x \cdot x$ and $x < x^2$.

(c) If $1 \leq x$, then $0 < x$ and $-x < 0$. From part b, $x < x^2$ if $1 < x$. From $-x < 0$, $0 < x$, and $x < x^2$ it follows by Transitivity that $-x < x^2$.

(d) If $x < -1$, then $1 < -x$ by $\underline{O_4}$. Since $x < -1 < 0$, then $x < 0$ and by $\underline{O_4}$, $-x(x) < 1(x)$ or $-x^2 < x$.

4\. Case (iii): Use proof similar to that for case (ii).

Case (iv): $|a| = -a$ and $|b| = -b$. Since $0 < ab$, then $|ab| = ab = (-a)(-b) = |a| \cdot |b|$.

5\. Case (i): $0 < x$ and $0 < y$ so $|x| = x$ and $|y| = y$. Since $0 < x + y$, then $|x+y| = x+y = |x| + |y|$.

Case (ii): $x < 0$ and $0 < y$, so $|x| = -x$ and $|y| = y$. Also, $x + y$ has two cases; $0 \leq x+y$ or $x + y < 0$. If $0 \leq x + y$, then $|x+y| = x + y$. Since $-|x| < |x|$, then $-|x| + |y| < |x| + |y|$ by $\underline{O_3}$. Also, since $|x| = -x$ and $|y| = y$ in this case, then $x = -|x|$ and $-|x| + |y| = x + y$. But $-|x| + |y| < |x| + |y|$ from previous use of $\underline{O_3}$, so $x + y < |x| + |y|$ and $|x+y| < |x| + |y|$ since $|x+y| = x+y$. If $x = y < 0$, then $|x+y| = (-x) + (-y)$. Since $-|y| < |y|$, then $-|y| + |x| < |y| + |x|$ by $\underline{O_3}$, or $-y + (-x) < |y| + |x|$. But $(-y) + (-x) = |x+y|$, so $|x+y| < |x| + |y|$.

Case (iii): $0 < x$ and $y < 0$. Proof is similar to that for case (ii)

Case (iv): $x < 0$ and $y < 0$, so $|x| = -x$ and $|y| = -y$. Since $x + y < 0$ in this case, then $|x+y| = -(x+y) = (-x) + (-y) = |x| + |y|$.

Combining the four cases into a single statement,

$$|x+y| \leq |x| + |y|$$

[page 39]

1-6. The Rational Number System.

When we come to the rational number system we shift our attitude somewhat concerning the extent of understanding we expect of the student regarding the numbers under discussion. In the cases of the natural numbers and the integers we said nothing about how you decide whether a pair of numbers (such as 3 and 13) are equal, how you add them, or multiply them. All these things we took for granted. Unless he has had a good deal of experience working with integers written to various bases (binary, octal, duodecimal, hexadecimal, etc.) the student knows only one name for each number (its decimal name) and so is not likely to be worried about the multiplicity of "aliases" for the numbers he knows.

With the rational numbers, however, the question is quite different, for even when we use only one scheme for naming the integers (i.e., decimal) we still have a multiplicity of names for each rational number:

$$3 = \frac{6}{2} = \frac{9}{3} = \frac{12}{4} = \ldots \quad .$$

We therefore presume to define what we shall mean in saying two rational numbers, $\frac{a}{b}$, $\frac{c}{d}$, are equal. (Definition 1-6a.) We motivate this definition by seeing how it must be stated if it is to agree with the equality relation we already have in I. This we do on pages 43, 44. On pages 45, 46 we treat addition and multiplication for rationals in a similar fashion. The moral of the whole discussion is that it is our E,A,M,D properties which force us to frame the definitions as we do. Relying on these old friends, E,A,M,D, our intent is to dispel any residue of mystery or dogma the student may have been left with after his first exposure to manipulation with fractions.

Since we go so far as to define equality, sum and product in Q, we prove that with this relation and these operations Q actually possess the properties E,A,M,D,C. This program is carried out on pages 46, 47, 48, 49. On page 50 we get to the new property M_5 which Q shares with neither of its predecessors N, I. This property "rounds out" the lists A,M and puts them

[pages 43-50]

on a more nearly "equal" basis than they have previously enjoyed. For instance in N we had

$$\underline{A}_1, \underline{A}_2, \underline{A}_3$$
$$\underline{M}_1, \underline{M}_2, \underline{M}_3, \underline{M}_4.$$

In I we had

$$\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4, \underline{A}_5$$
$$\underline{M}_1, \underline{M}_2, \underline{M}_3, \underline{M}_4,$$

where the A-list takes a temporary lead. In Q, these lists are matched $\underline{A}_1$-$\underline{A}_5$, $\underline{M}_1$-$\underline{M}_5$, the only discrepancy being the exclusion of zero as a divisor in $\underline{M}_5$. As we have emphasized, however, this discrepancy is unavoidable because of the presence of property $\underline{D}$. (See the parenthetical remarks following Theorem 1-4e, page 28.)

We close Section 1-6 by talking about solving equations and give the usual sermon on that cardinal sin: division by zero.

Answers to Exercises

Exercises 1-6a:

1. (a) 3/5
   (b) 5/2
   (c) -4
   (d) $\frac{c - b}{a}$
   (e) $\frac{2a - b + d}{a - c}$
2. (a) 6
   (b) 6
   (c) 5
3. If a, b, c, d in I with $bd \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ by the definition for equality of fractions. Since ad and bc are integers, then by $\underline{E}_3$ (Symmetry) for integers, $bc = ad$ and by Commutativity, $cb = da$. Applying the definition for equality of fractions, $\frac{c}{d} = \frac{a}{b}$.

Exercises 1-6b:

1. (a) $\frac{13}{15}$
   (b) $\frac{xz + 2y}{2z}$

[pages 45, 50-52]

(c) $\frac{a + 2b}{b}$

(d) $\frac{xy + x(y+z)}{(y+z)y}$ or $\frac{2xy + xz}{y(y+z)}$

(e) $\frac{11p - 2}{12}$

(f) $\frac{3}{c} + \frac{4}{-c} = \frac{3}{c} + \frac{-4}{c} = \frac{-1}{c}$

2. (a) $\frac{4}{15}$

(b) $\frac{xy}{4z}$

(c) $\frac{2a}{b}$

(d) $\frac{x^2}{(y+z)y}$ or $\frac{x^2}{y(y+z)}$

(e) 0.

3. If b, d, and f are non-zero integers and $\frac{a}{b} = \frac{c}{d}$, then ad = bc. Since

$$\frac{a}{b} + \frac{e}{f} = \frac{af+be}{bf} = \frac{(af+be)(df)}{bf(df)} = \frac{adf^2+bdef}{bdf^2} = \frac{bcf^2+bdef}{bdf^2}$$

and

$$\frac{c}{d} + \frac{e}{f} = \frac{cf+de}{df} = \frac{(cf+de)(bf)}{df(bf)} = \frac{bcf^2+bdef}{bdf^2},$$

then

$$\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}.$$

4. If b, d, and f are non-zero integers and $\frac{a}{b} = \frac{c}{d}$, then ad = bc. Since

$$\frac{a}{b} \cdot \frac{e}{f} = \frac{ae}{bf} = \frac{ae(df)}{bf(df)} = \frac{adef}{bdf^2} = \frac{bcef}{bdf^2}$$

and

$$\frac{c}{d} \cdot \frac{e}{f} = \frac{ce}{df} = \frac{ce(bf)}{df(bf)} = \frac{bcef}{bdf^2},$$

then

$$\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}.$$

5. $(\frac{a}{b}+\frac{c}{d}) + \frac{e}{f} = \frac{ad+bc}{bd} + \frac{e}{f} = \frac{(ad+bc)f+(bd)e}{(bd)\cdot f} = \frac{adf+bcf+bde}{bdf}.$

$\frac{a}{b} + (\frac{c}{d} + \frac{e}{f}) = \frac{a}{b} + \frac{cf+de}{df} = \frac{a(df)+b(cf+de)}{b(df)} = \frac{adf+bcf+bde}{bdf}.$

[pages 48-49]

6. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \cdot \frac{a}{b}$.

7. $\frac{-a}{a} = -\frac{a}{a} = -(a \cdot \frac{1}{a}) = -1.$

8. $\frac{na}{n} = \frac{an}{n} = a(\frac{n}{n}) = a(n \cdot \frac{1}{n}) = a \cdot 1 = a.$

9. If $b = 0$ and $a \neq 0$, then $b \cdot \frac{1}{a} = 0 \cdot \frac{1}{a}$ by $\underline{E}_6$.

Then $\frac{b}{a} = 0.$

If $\frac{b}{a} = 0$ and $a \neq 0$, then $(b \cdot \frac{1}{a}) \cdot a = 0 \cdot a$ by $\underline{E}_6$

and $b(\frac{1}{a} \cdot a) = 0$ by Association.

Then $b \cdot 1 = 0$ or $b = 0.$

Exercises 1-6c:

Note: There are many different ways in which these solutions can be made and more or less detail can be included.

1. (a) $\frac{2x}{3} = 4$

$2x = 12$ [$\underline{E}_6$

$x = 6$ [$\underline{M}_5$

(b) $3m + \frac{2}{5} = \frac{1}{3}$

$45m + 6 = 5$ [$\underline{E}_6$ and Dist.

$45m = -1$ [$\underline{E}_5$

$m = -\frac{1}{45}$ [$\underline{M}_5$

(c) $\frac{5y - 1}{4} - 1 = \frac{3}{5}$

$25y - 5 - 20 = 12$ [$\underline{E}_6$ and Dist.

$25y = 37$ [$\underline{E}_5$

$y = \frac{37}{25}$ [$\underline{M}_5$

(d) $\frac{2(1-w)}{w-1} + \frac{3w}{5} = 2$

$-2 + \frac{3w}{5} = 2$ [Th. $-(a-b) = b - a$

$-10 + 3w = 10$ [$\underline{E}_6$ and Dist.

$3w = 20$ [$\underline{E}_5$

$w = \frac{20}{3}$ [$\underline{M}_5$

[pages 49, 52]

(e) No solution; The equation can be transformed to $0 \cdot x = 11$

2. If r, s, and t are rational numbers, then $r = s$ if and only if $r + t = s + t$.

Let $r = \frac{a}{b}$, $s = \frac{c}{d}$, and $t = \frac{e}{f}$, where b, d, and f are non-zero integers.

Proof for "if": If $\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$, then

$\frac{af+be}{bf} = \frac{cf+de}{df}$, [Def. 1-6b

and

$(af+be)(df) = bf(cf+de)$. [Def. 1-6a

$adf^2 + bdef = bcf^2 + bdef$ [Dist.

$adf^2 = bcf^2$ [C_1 for I

$ad = bc$ [C_1 for I

$\therefore \frac{a}{b} = \frac{c}{d}$ [Def. 1-6a

Proof for "only if": If $\frac{a}{b} = \frac{c}{d}$, then

$\frac{a}{b} + \frac{e}{f} = \frac{af + be}{bf}$ [Def. 1-6b

$= \frac{(af+be)df}{bf(df)}$ [Th. $\frac{ac}{bc} = \frac{a}{b}$

$= \frac{adf^2+bdef}{bdf^2}$ [Dist.

Similarly

$\frac{c}{d} + \frac{e}{f} = \frac{(cf+de)(bf)}{df(bf)}$

$= \frac{bcf^2+bdef}{bdf^2}$

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ so that

$$\frac{a}{b} + \frac{e}{f} = \frac{adf^2+bdef}{bdf^2} = \frac{bcf^2+bdef}{bdf^2} = \frac{c}{d} + \frac{e}{f}.$$

3. If r, s, and t are rational numbers, then $r = s$ if and only if $rt = st$. Let $r = \frac{a}{b}$, $s = \frac{c}{d}$, and $t = \frac{e}{f}$, where b, d, and f are non-zero integers.

Proof for "if": If $\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}$, then $\frac{ae}{bf} = \frac{ce}{df}$ and $ae(df) = bf(ce)$. By Commutativity and Associativity,

$$ad(ef) = bc(ef).$$

Using $\underline{E}_6$ for integers,

$$ad = bc,$$

so by the definition for equality of fractions,

$$\frac{a}{b} = \frac{c}{d}.$$

Proof for "only if": If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ and $ad(ef) = bc(ef)$.

By Associativity and Commutativity,

$$ae(df) = bf(ce).$$

Then

$$\frac{ae}{bf} = \frac{ce}{df},$$

and

$$\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}.$$

1-7. Order of the Rationals.

In Section 1-7 we begin by doing to < what we did to =, +, · in Section 1-6: we examine what the E,A,M,D,O properties have to tell us about how we must define < in Q so that it will agree with the order relation we have already in I. This we do on pages 53, 54.

On page 55 we announce that $\underline{O}_1, \underline{O}_2, \underline{O}_3, \underline{O}_4$ all hold in Q and that this fact follows from the definition we have given for < (Definition 1-7b). At the bottom of page 55 and the top of page 56, we quote the cancellation properties in "if and only if" form.

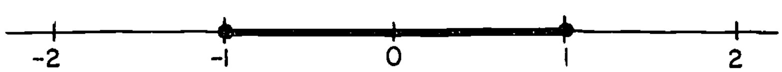
[pages 52-56]

Our attitude by now is that these properties are all "almost obvious" consequences of the definitions and of the "old" properties in I. Proofs are chains of words, and since we now have the same vocabulary in our "new" system Q, we can go back and reread the old words with their new meanings. This way of getting proofs is what shows most spectacularly, perhaps, the power of the study of "structure" as a means of "unifying" these various number systems.

On pages 56, 57 we present Archimedes' property and the special order property $\underline{O}_6$ (Q) of the rational number system. All of the properties of Q which have been mentioned go over intact to the real number system R. (We shall not actually prove this assertion, we shall merely proclaim it.) Thus every single consequence of all these properties will also hold in R. (The big difference between Q and R is the fact that R has one additional order property, $\underline{O}_7$ (R) (page 76), which it does not share with Q. See Commentary on Section 1-10.)

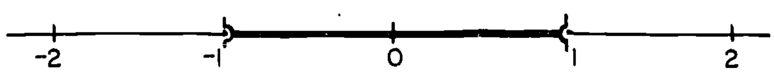
On pages 58, 59 we discuss the solution of inequalities in Q, pointing out how the "density" of Q and the "discretion" of I make a big difference in the way we can specify solution sets.

At page 60, we return to "absolute value" and look again at inequalities involving them. The theorems we give here make it possible to "clean up" our methods for handling such problems. Drawing number-lines to illustrate the solution sets for these problems is highly recommended. Thus



depicts the solution set of the inequality in Example 1-7c, page 60. This picture may be compared with that for the problem

$$|x| < 1 \text{ if and only if } -1 < x < 1$$



which differs from its predecessor by lacking just 2 "points", the "end-points" of the interval.

Answers to Exercises

Exercises 1-7a:

1. (a) $\frac{2}{3} < \frac{5}{7}$ (d) $\frac{3}{y} < \frac{7}{x}$

   (b) $\frac{12}{17} < \frac{5}{7}$ (e) $2x + 1 < \frac{3x + 5}{3}$

   (c) $\frac{x}{5} < \frac{y}{4}$

2. $-\frac{37}{61} < -\frac{12}{20} < \frac{47}{59} < \frac{4}{5} < 2 < \frac{27}{13}$, since $-740 < -732$, $-708 < 940$, $235 < 236$, $4 < 10$, and $26 < 27$.

3. Proof for "if"; If $0 < a$, then $0 < \frac{1}{a}$.

   If $0 < a$, then $0 \cdot a < a \cdot 1$. Using this result with the definition for order in Q, $\frac{0}{a} < \frac{1}{a}$ or $0 < \frac{1}{a}$.

   Proof for "only if": If $0 < \frac{1}{a}$, then $0 < a$.

   If $0 < \frac{1}{a}$, then $\frac{0}{a} < \frac{1}{a}$ and $0 \cdot a < a \cdot 1$ by the order definition in Q. Hence, $0 < a$.

4. Proof for "if": If $ad > bc$, then $\frac{a}{b} > \frac{c}{d}$.

   Since $ad > bc$ means $bc < ad$, then $cb < da$ and $\frac{c}{d} < \frac{a}{b}$, or $\frac{a}{b} > \frac{c}{d}$.

   Proof for "only if": If $\frac{a}{b} > \frac{c}{d}$, then $ad > bc$.

   Since $\frac{a}{b} > \frac{c}{d}$ means $\frac{c}{d} < \frac{a}{b}$, then $cb < ad$, or $bc < da$.

   From the definition for order in Q,

   $\frac{b}{d} < \frac{a}{c}$, or $\frac{a}{c} > \frac{b}{d}$.

5. Since $\frac{a}{b} = \frac{af}{bf}$, if $\frac{a}{b} < \frac{c}{d}$, then $\frac{af}{bf} < \frac{c}{d}$, and

   $af(d) < bf(c)$.

[page 56]

Similarly, $\frac{c}{d} = \frac{cb}{db}$ and if $\frac{c}{d} < \frac{e}{f}$, then $\frac{cb}{db} < \frac{e}{f}$, so $cb(f) < db(e)$.

By Commutativity and Associativity, since $af(d) < bf(c)$,

$$afd < bfc \quad \text{and} \quad bfc < dbe.$$

Then by Transitivity,

$$afd < dbe,$$

or $$a(fd) < b(ed).$$

Hence, $$\frac{a}{b} < \frac{ed}{fd},$$

or $$\frac{a}{b} < \frac{e}{f}.$$

Exercises 1-7b:

1. If O_5 (Archimedes) is to hold for Q, then for positive rational numbers r and s, with $r < s$, there must be a positive integer n such that $nr > s$. Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$, when a, b, c, d are integers with $0 < b$ and $0 < d$. Since $r < s$, then $ad < bc$. But ad and bc are positive integers, so O_5 holds and there is a positive integer n such that $n(ad) > bc$, or $n > \frac{bc}{ad}$. This determines n for $O_5(Q)$. Note that $a \neq 0$, since $r = \frac{a}{b}$ must be positive.

2. If $a < b$, then $a + 2a < b + 2a$, or $3a < 2a + b$. Hence, $a < \frac{2a+b}{3}$. Similarly, if $a < b$, then $a + 2b < b + 2b$, or $a + 2b < 3b$ and $\frac{a+2b}{3} < b$. Also, if $a < b$, then $a + (a+b) < b + (a+b)$, or $2a + b < a + 2b$. Then $\frac{1}{3}(2a+b) < \frac{1}{3}(a+2b)$, or $\frac{2a+b}{3} < \frac{a+2b}{3}$.

   Summarizing, if $a < b$, then $a < \frac{2a+b}{3}$, $\frac{2a+b}{3} < \frac{a+2b}{3}$, and $\frac{a+2b}{3} < b$, so by Transitivity,

$$a < \frac{2a+b}{3} < \frac{a+2b}{3} < b.$$

[pages 56-58]

3. In a similar manner, if $a < b$, then $a + 3a < b + 3a$ and $a < \frac{3a+b}{4}$. Also, if $a < b$, then $a + (b+2a) < b + (b+2a)$, and $3a + b < 2a + 2b$. Hence, $\frac{3a+b}{4} < \frac{2a+2b}{4}$. Using $a + (a+2b) < b + (a+2b)$, then $\frac{2a+2b}{4} < \frac{a+3b}{4}$. Finally, $a + 3b < b + 3b$, so $\frac{a+3b}{4} < b$.
By Transitivity, $a < \frac{3a+b}{4} < \frac{2a+2b}{4} < \frac{a+3b}{4} < b$.

Exercises 1-7c:

1. $\frac{3}{2} < x < \frac{7}{2}$
2. $-\frac{7}{3} < y < -1$
3. $-2 < a < \frac{2}{3}$
4. $1 < m < 3$
5. $-\frac{63}{2} < x < 1$
6. $0 \leq c < \frac{1}{2}$
7. $\frac{5}{3} < x \leq \frac{7}{3}$
8. $-\frac{1}{24} < x < \frac{3}{8}$
9. $-\frac{1}{2} \leq x \leq \frac{7}{2}$
10. $-\frac{1}{7} < x \leq \frac{11}{7}$

Exercises 1-7d:

1. (a) $-5 < x < 3$
   (b) $0 \leq x \leq 1$
   (c) $x < -2$ or $4 < x$
   (d) $x \leq \frac{7}{3}$ or $1 \leq x$
   (e) No solution: $0 \leq |a|$ for all a.

*(f) $2 \leq |x+1| \leq 3$ means the same as $2 \leq |x+1|$ and $|x+1| \leq 3$.
From Theorem 1-7b,
$|x+1|$ if and only if $x + 1 \leq -2$ or $2 \leq x + 1$,
if and only if $x \leq -3$ or $1 \leq x$.
From Theorem 1-7a,
$|x+1| \leq 3$ if and only if $-3 \leq x + 1 \leq 3$,
if and only if $-4 \leq x \leq 2$.
Combining these results,
$2 \leq |x+1| \leq 3$ if and only if $x \leq -3$ and $-4 \leq x \leq 2$.

[pages 58-59, 63]

This has the same meaning as $2 < |x+1| \leq 3$
if and only if $x \leq -3$ and
$-4 \leq x \leq 2$, or $1 \leq x$ and
$-4 \leq x \leq 2$
if and only if $-4 \leq x \leq -3$ or
$1 \leq x \leq 2$.

Another approach would be to use the definition of absolute value as a "side condition."

$0 \leq x + 1$ and $2 \leq x + 1 \leq 3$, or $x + 1 < 0$ and $2 \leq -(x+1) \leq 3$,
$-1 \leq x$ and $1 \leq x \leq 2$, or $x < -1$ and $-3 \leq x + 1 \leq -2$,
$-1 \leq x$ and $1 \leq x \leq 2$, or $x < -1$ and $-4 \leq x \leq -3$.

Since $-1 \leq x$ is included in $1 \leq x \leq 2$, and $x < -1$, is included in $-4 \leq x \leq -3$, then the previous statement becomes,

$$1 \leq x \leq 2 \quad \text{or} \quad -4 \leq x \leq -3.$$

This solution is incomplete since it gives, "if $2 < |x+1| \leq 3$, then $1 \leq x \leq 2$ or $-4 \leq x \leq -3$," but does not guarantee that the converse of this statement is true. The first solution of the other hand is complete, since the theorems used involve the "if and only " phrase.

1-8. Decimal Representation of Rational Numbers.

Mohammed ibn Musa al-Khowarizmi, one of the greatest Arab mathematicians around 800 A.D., wrote a book called Al-Jabr wal-Muqabalak. This book is credited with having much to do with the spread of the arabic decimal system in the Arab world and, later, in Europe. Our word "algebra" had its source in the title of this book, and our word "algorithm," or "algorism" comes from the author's name. An algorithm is any step-by-step procedure for calculation. The term has recently been less widely used than it had been earlier, but is coming back into use concurrently with the development and wide use of automatic computing machines.

Answers to Exercises

Exercises 1-8a:

1. (a) $0.\overline{7}$ (d) $4.\overline{285714}$
   (b) $0.\overline{18}$ (e) $1.\overline{2352941176470588}$
   (c) 0.6875 or $0.6875\overline{0}$

2. (a) $\frac{5}{9}$ (d) $\frac{6409}{4950}$
   (b) $\frac{16}{99}$ (e) $\frac{38651}{11000}$
   (c) $\frac{1423}{3300}$

4. Let $a = 1.\overline{9}$. Then $10a = 19.\overline{9}$ and $9a = 18$, so $a = 2$ or $a = 2.\overline{0}$

5. $\frac{1}{10^n}$ means a decimal expression having $(n - 1)$ zeros followed by a "1" .

1-9. Infinite Decimal Expressions and Real Numbers.

In this section we unveil the whole collection of infinite decimal expressions and talk about their role as names for real numbers. We don't say very much, for there is not much we can say without getting too involved.

We exhibit Liouville's example of an irrational number at the bottom of page 72 to convince the student that there are such things; that the number system R we'd like to talk about really is different from Q, the one we have talked about.

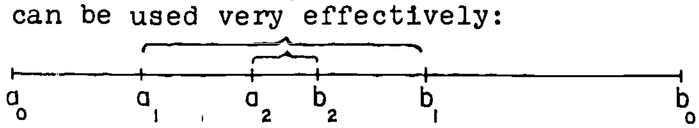
Using decimal expressions to introduce real numbers involves a number of bothersome details which we try to "sweep under the carpet." (An account of this approach may be found in the first few pages of Dienes' Taylor's Series, a book on functions of a complex variable recently reprinted by Dover Publications.)

We go so far as to frame definitions for $=$ and $<$ in the part of R not in Q. We restrict ourselves to this subset to avoid discussing decimals in which the digit 9 repeats and which are all rational. For similar reasons we omit entirely any definition of 'sum' or 'product'.

The details are less involved if real numbers are introduced as "Dedekind Sections" (Hardy, A Course of Pure Mathematics, Chapter 1) or as "Cantor Sequences" but it is difficult to motivate these approaches without a considerable amount of background, for the ideas involved are rather sophisticated. These methods are outlined and compared in Friedrich Waismann's Introduction to Mathematical Thinking, Harper Torchbook 511 (paperback). We believe students have a fairly good idea of what decimals are, so we say about as much as we think they can take in this context and omit all the rest.

We announce that R has all the Q properties, and take this announcement as license to invoke any of the Q properties we wish when working with real numbers throughout the rest of this book.

As our version of the special order property distinguishing R from Q we state, $\underline{O}_7(R)$, the "principle of nested intervals." If any mention is made of $\underline{O}_7(R)$ in class, a picture of the number-line can be used very effectively:



As the "length" of the n^{th} brace is less than $\frac{1}{10^n}$, the intervals get small very fast. Putting it this way, the class may find it remarkable anyone could suspect that two different numbers could both survive this "squeeze" process. If they grant that much, they are saying that at most one real number lies in all these intervals. The rest of the assertion is that one actually does.

Answers to Exercises

Exercises 1-9a:

1. $2.153\ldots < 2.1545 < 2.15\overline{456} < 2.1547 < 2.1547\ldots$

2. (a) $\underline{O}_1$ (Trichotomy) (g) $\underline{E}_1$ (Dichotomy)
   (b) $\underline{O}_2$ (Transitivity) (h) $\underline{A}_2$ (Commutativity)
   (c) $\underline{M}_2$ (Commutativity) (i) $\underline{M}_3$ (Associativity)
   (d) $\underline{C}_1$ (Addition) (j) $\underline{E}_4$ (Transitivity)
   (e) $\underline{O}_4$ (Multiplication) (k) $\underline{A}_5$ (Subtraction)
   (f) $\underline{O}_6$ (Density) (l) $\underline{D}$ (Distributivity)

1-10. The Equation $x^n = a$.

In Section 1-10 we examine one of the ways in which Q and R differ. All such differences depend ultimately on property $\underline{O}_7$(R) which R possesses but which Q does not. That Q does not possess this property is a consequence of one of our calculations in Section 1-10. We do not stop in the text to draw this conclusion, however, for to do so would cause us to digress from our discussion. We have written the text in such a way that it is possible to skip over this point in class, in the event the teacher considers it inappropriate to dwell on the matter. We take the liberty, however, of filling in some of the details in this Commentary for the benefit of teachers who would prefer to see a more thorough treatment.

As we say in the text, Q and R share all their $\underline{E}$,$\underline{A}$,$\underline{M}$,$\underline{D}$ properties. Thus R is the first number system in our chain of extensions of N which is not closed under a new algebraic operation. The differences between N and I can be attributed to the latter's closure under subtraction; the differences between I and Q to the latter's closure under division (0 excepted -- of course). However the lists of $\underline{A}$ and $\underline{M}$ properties are completed when we get to Q. R contributes nothing more in this direction.

The new property, $\underline{O}_7(R)$, however, does produce a very significant enlargement of the set of numbers involved in the new system. Some enlargement of the rational number system is necessary on geometrical grounds if we are to have a number system capable of measuring the diagonal of a square with unit edge, for example, or any of the other lengths one can construct with straight-edge and compasses. However the demand for all such "constructible" lengths is still not enough: the edge of a cube whose volume is twice that of the unit cube is not constructible, nor is the circumference of the unit circle. (These last assertions are proved in texts on "Theory of Equations.")

The real number system is a system "closed" under limiting processes. It is on these grounds that R is adequate for geometry as well as "analysis" (calculus and its extensions). We outline in the text the proof that R contains $\sqrt{2}$, and thus lets us measure the diagonal of the unit square. We announce that R contains $\sqrt[3]{2}$, which implies that, using R, we can "duplicate a cube," even if this is impossible with only straight-edge and compasses. We announce also that R contains a solution for <u>each</u> of the equations $x^n = a$, a non-negative. We do not emphasize, however, that R contains many numbers which are not "algebraic" (i.e., which do not satisfy equations like that in Theorem 1-10a with <u>rational</u> $a_0, a_1, \ldots, a_{n-1}$). Some of these "transcendental" numbers (i.e. non-algebraic numbers) are π, e, and even Liouville's number (0.1010010001..., in Section 1-9). In short, we certainly do not do justice to R or to the many problems it can handle. We merely indicate only a few of its more elementary facets.

With Theorem 1-10a and its Corollary we show the inadequacy of Q when it comes to solving equations more complicated than $bx = a$. We turn then to $x^2 = 2$ and argue that R contains a solution — indeed, two solutions.

We first present an algorithm for determining successive digits of the number we seek. (We can't call it $\sqrt{2}$ yet because we don't even know there <u>is</u> such a number until we prove there is. All we can say with any certainty is that the number we want is not in Q. Until we find it, we know nothing at all about it beyond this.) There are, of course, many other algorithms available, some of which will give us more than just a single digit in each step of the calculation. The one chosen seems to require the least amount of justification to carry out and produces for us two sequences of rationals, the a's and the b's, "closing in" on $\sqrt{2}$. The first part of the argument requires only the recognition that some real number is produced by our algorithm. (Whether it is the one we want is considered in the second part of the argument.) That this algorithm must keep going, that it cannot stop after some number of steps follows from the fact that none of the a's or b's can have their squares equal to 2. The a's and b's are all rational numbers since they have terminating decimal representations.

In the second part of the argument, we "show" that our number (by this time we are calling it "c") has its square equal to 2. We assert that

$$a_n < c < b_n \text{ , all } n,$$

implies

$$a_n^2 < c^2 < b_n^2 \text{ , all } n.$$

This follows from $\underline{O}_5$ for (writing " $\Rightarrow$ " for "implies")

$$0 < a_n \text{ and } a_n < c \Rightarrow a_n^2 < a_n c \text{ and } a_n c < c^2$$

$$\Rightarrow a_n^2 < c^2 .$$

Similarly

$$0 < c \text{ and } c < b_n \Rightarrow c^2 < b_n c \text{ and } cb_n < b_n^2$$

$$\Rightarrow c^2 < b_n^2.$$

Finally we assert $0 < b_n^2 - a_n^2 < \frac{1}{10^n}$, all n.

Here is a proof:

From the manner of their construction, we can say of the a's and b's that

$$b_0 - a_0 = \frac{1}{10}$$

$$b_1 - a_1 = \frac{1}{10^2}$$

[pages 80-81]

. . . .

$$b_n - a_n = \frac{1}{10^{n+1}} .$$

But we also know that $b_0 = 1.5$ and all the other b's and all the a's are less than b_0:

$$b_n \leq b_0 \text{ for } n \geq 0,$$

$$a_n < b_0 \text{ for } n \geq 0.$$

Hence

$$b_n + a_n < 2b_0 = 3, \text{ for all } n.$$

But $\quad b_n + a_n < 3$, all n,

and $\quad b_n - a_n < \frac{1}{10^{n+1}}$, all n,

give

$$b_n^2 - a_n^2 < \frac{3}{10^{n+1}} < \frac{10}{10^{n+1}} = \frac{1}{10^n}, \text{ all } n.$$

This calculation proves that the word "real" cannot be replaced by "rational" in property $\underline{O}_7(R)$. Notice that here all of our a's and b's (and hence their squares) are rational numbers. Our number c, however, is not rational.

The rest of Section 1-10 deals with radicals. (Students detecting the trickery in Exercises 1-10a, 46-52, and interested in learning the pattern behind their construction, may be directed to the references under "Pell's Equation" given in SMSG's "Study Guide to Number Theory.")

Answers to Exercises

Exercises 1-10a:

1. $2\sqrt{2}$
2. $5\sqrt{3}$
3. $7\sqrt{2}$
4. $3\sqrt{3}$
5. $4\sqrt{3}$
6. $6\sqrt{2}$
7. $8\sqrt{3}$
8. $10\sqrt{7}$
9. $9\sqrt{3}$
10. $12\sqrt{2}$
11. $\frac{\sqrt{3}}{2}$
12. $\frac{\sqrt{5}}{3}$
13. $\frac{2\sqrt{3}}{5}$
14. $\frac{2}{5}$
15. $\frac{3\sqrt{3}}{2}$
16. $\frac{\sqrt{15}}{5}$
17. $\frac{\sqrt{6}}{3}$

[pages 81-85]

18. $\frac{2\sqrt{5}}{5}$

19. $\frac{\sqrt{55}}{11}$

20. $\frac{3\sqrt{6}}{4}$

21. 6

22. 10

23. $2\sqrt{15}$

24. 2

25. 4

26. $\frac{\sqrt{30}}{6}$

27. $\frac{1}{27}\sqrt{5}$

28. $\frac{3\sqrt{5}}{5}$

29. $\sqrt{2}$

30. $\frac{\sqrt{15}}{3}$

31. $4\sqrt{2}$

32. $\sqrt{2}$

33. $-\sqrt{6}$

34. $7\sqrt{15}$

35. $31\sqrt{5}$

36. -1

37. $3 + 2\sqrt{2}$

38. 5

39. $\sqrt{2}$

40. $28 + 17\sqrt{6}$

41. $-1 + \sqrt{2}$

42. $2 + 2\sqrt{3}$

43. $-5 - 2\sqrt{6}$

44. $7 + 5\sqrt{2}$

45. $\frac{-68}{167} + \frac{42}{167}\sqrt{3}$

46. $2 - \sqrt{3}$; $7 - 4\sqrt{3}$

47. $3 - 2\sqrt{2}$; $17 - 12\sqrt{2}$

48. $5 - 2\sqrt{6}$; $49 - 20\sqrt{6}$

49. $7 - 4\sqrt{3}$; $97 - 56\sqrt{3}$

50. $8 - 3\sqrt{7}$; $127 - 48\sqrt{7}$

51. $9 - 4\sqrt{5}$; $161 - 72\sqrt{5}$

52. $17 - 12\sqrt{2}$; $577 - 408\sqrt{2}$

1-11. Polynomials and their Factors. (Review)

Answers to Exercises

Exercises 1-11a:

1. $5(x - y)$
2. $-2(3a + 8)$
3. $3(2p - q + 5r)$
4. $5(2y - x + 4w - 2z)$
5. $6b(2a + 1 - 9c)$
6. $(x + y)(a + b)$
7. $(a - b)(x - y)$
8. $u(x + y)$
9. $(x - y)(b + 1)$
10. $2(a + b)$
11. $(x + y)(a + b)$
12. $(x - y)(b + c)$
13. $(x - y)(b - c)$
14. $(a + 1)(3a^2 - 4)$
15. $(2m - 3n)(2m + 3n)$
16. $(a - 2)(a + 2)(a^2 + 4)$
17. $7(c - 3)(c + 3)$
18. $(x + a - b)(x - a + b)$
19. $(a + b + c + d)(a + b - c - d)$
20. $(x - y)(x - y + 2)$

Exercises 1-11b:

1. $(x + 5)(x + 3)$
2. $(w - 8)(w - 3)$
3. $(3a + 5)(a - 3)$
4. $(4x + 3)(x - 2)$
5. $(y - 5)^2$
6. $(3a - 2)(a + 2)$
7. $w(x - 6)^2$
8. $d(y - 6)(y - 5)$
9. $(5x - 3y)^2$
10. $a(9w^2 + 5w - 36)$
11. $(2u + 3v)^2$
12. $(7z + 1)^2$
13. $c(x - 4)(x + 2)$
14. $2(1 - 4a)(1 + a)$
15. $(3 + 4c)(3 - 2c)$
16. $3(7 - y)(2 + y)$
17. $(4 + 3x)(5 - 2x)$
18. $(2ab + 1)^2$
19. $(a + b + c)(a + b - c)$
20. $[(a + b) - 1]^2 = (a + b - 1)^2$

Exercises 1-11c:

1. $(c + d)(c^2 - cd + d^2)$
2. $(w - 4)(w^2 + 4w + 16)$
3. $(x + 1)(x^2 - x + 1)$
4. $(m - 2u)(m^2 + 2mu + 4u^2)$
5. $(3r + y)(9r^2 - 3ry + y^2)$
6. $(2a + x)(4a^2 - 2ax + x^2)$
7. $(r - 4s)(r^2 + 4rs + 16s^2)$
8. $(4 - 3x)(16 + 12x + 9x^2)$
9. $a(c - 4)(c^2 + 4c + 16)$
10. $b(a - 5b)(a^2 + 5ab + 25b^2)$
11. $y(3r + 1)(9r^2 - 3r + 1)$
12. $4(x - 2)(x^2 + 2x + 4)$
13. $16(2 + y)(4 - 2y + y^2)$
14. $(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)$
15. $(m^2 + u^2)(m^4 - m^2u^2 + u^4)$

Exercises 1-11d: (Miscellaneous Exercises)

1. $(6m - 5)(2m + 3)$
2. $(a - 1)^2(a + 1)$
3. $xy(4y - 13)$
4. $(d + h + f)(d + h - f)$
5. $(2a + 3b)(m + x)$
6. $3a^3(2 + 3b - 4ab^2)$
7. $(3 - 5z)(x - y)$
8. $(a - b)(a + b + 4)$
9. $(m - 2u)(m^2 + 2mu + 4u^2)$
10. $(3y + 4z)(2y - 3z)$
11. $(r - s)(k + w)$
12. $4x^2y^2(2x + 1)(x - 3)$
13. $2(2r - 3)(4r^2 + 6r + 9)$
14. $(c + d + h)(c + d - h)$
15. $(x^2 + 1)^2$
16. $a(y - 5)^2$
17. $(10 + t^2)(10 - t^2)$
18. $(r - s)(m + p)$
19. $(a + b + c)(a + b - c)$
20. $y(3r + 1)(9r^2 - 3r + 1)$
21. $(7z - 1)^2$
22. $(5c + x)(x - 1)(x + 1)$
23. $(2r - 5)(4r^2 + 10r + 25)$
24. $(2x + y - z)(2x - y + z)$
25. $a(a - 2)(a + 2)(a^2 + 4)$
26. $(2 - x^2)(2 + x^2)(4 + x^4)$
27. $(2x + 2y - 1)(2x - 2y + 1)$
28. $[x-5(x+y)][x+4(x+y)] = -(4x+5y)(5x+4y)$
29. $(x + y + 7)(x + y - 4)$
30. $(r - 5 + 3s - t)(r - 5 - 3s + t)$

1-12. Rational Expressions. (Review)

Answers to Exercises

Exercises 1-12a:

1. (a) $\frac{1}{u}$
   (b) $\frac{a}{b}$
   (c) $\frac{3n}{4mp}$
   (d) $\frac{x - 3}{3(x - 1)}$
   (e) $\frac{2 - m}{2 + m}$
   (f) $x + w$
   (g) $a - 2$
   (h) $\frac{1 + z}{y - z}$
   (i) $x - y$
   (j) $\frac{1}{c - d}$
   (k) $\frac{x - 5}{x + 5}$
   (l) $\frac{y - 2}{2(y + 2)}$
   (m) $\frac{p - 3}{2p - 5}$
   (n) $\frac{c - 2}{c - 3}$
   (o) $\frac{x^2 + x + 1}{x(x + 5)}$
   (p) $\frac{a}{a^2 - a + 1}$
   (q) $\frac{x - 3}{1 - x}$ or $-\frac{x - 3}{x - 1}$
   (r) $-\frac{a + b}{a + 2b}$
   (s) $x - y + z$
   (t) $\frac{x - y - 2}{3}$

2. (a) $\frac{64b^3}{c}$
   (b) $\frac{4}{m - 3u}$
   (c) $-\frac{1}{6}$
   (d) $-\frac{x - 5}{a}$ or $\frac{5 - x}{a}$
   (e) $\frac{x^2(x - y)}{y^2(x + y)}$
   (f) $\frac{x + 1}{x^2 + 1}$
   (g) $\frac{1}{m + 1}$
   (h) $\frac{1}{(a + b)^2}$
   (i) $\frac{p + 9}{p + 1}$
   (j) $(c - 3d)^2$
   (k) 2
   (l) $\frac{y}{x(y - x)}$
   (m) $\frac{a + 1}{a^2 + 1}$
   (n) $a + b$

(o) $\frac{(x-2)(4x-1)(3x+2)}{2(2x-1)(3x^2-4x+4)}$

(p) $\frac{ac}{8b}$

(q) $\frac{a(a-b)}{c(a+b)}$

(r) $\frac{2x(x-y)}{3y(x+y)}$

(s) $\frac{2x}{2-x}$ or $-\frac{2x}{x-2}$

Exercises 1-12b:

1. $\frac{17x-8}{6}$
2. $\frac{a-4}{5}$
3. $\frac{n+5}{mn^2}$
4. $\frac{-2y^2+16y+5}{4y}$
5. $\frac{19x-210}{42}$
6. $\frac{7a+3}{a(a^2-1)}$
7. $\frac{-(2x+1)}{x^2-1}$ or $\frac{2x+1}{1-x^2}$
8. $\frac{p+1}{p}$
9. $\frac{2(x^2+y^2)}{(3x-2y)(2x+3y)}$
10. $\frac{m^2+18}{m^2-9}$
11. $\frac{a^3+9a^2-5a+1}{(a+3)(3a-1)}$
12. $\frac{6c^2-3c}{c^2-9}$
13. $\frac{13}{2(x+2y)}$
14. $\frac{2x+3y}{xy(x+y)}$
15. $\frac{5a^2-4ab+3b^2}{(a+b)(a-b)^2}$
16. $\frac{b^2-4b}{(b+2)(b^2-1)}$
17. $\frac{2x^2}{(x-3)(x^2-4)}$
18. $\frac{-6m+7}{(m+4)(m-1)}$
19. $\frac{-x^4-x^3+2x^2+3x}{(x+2)^2(x+1)^2}$
20. $\frac{4y^2-7y+15}{(y^2-2y+5)(y-1)}$

Exercises 1-12c:

1. $\frac{3a^2-2}{5a}$
2. $\frac{3xy^2}{10}$
3. $\frac{6a^2-5ab-2b^2}{6ab}$
4. $\frac{4}{x-3y}$
5. $\frac{20x}{x^2-25}$
6. $-\frac{1}{6}$
7. $\frac{-1}{m^2-4}$ or $\frac{1}{4-m^2}$
8. $\frac{2x(x-y^2)}{(x-y)^2}$

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9. $\frac{1}{(m + 1)(m + n)}$

10. $\frac{x^2 - x + 12}{6x(x^2 - 9)}$

11. $5 - a$

12. $\frac{x^2(x - y)}{y^2(x + y)}$

13. $\frac{-x + 4}{(2x - 3)(x + 5)(x + 1)}$

14. $\frac{w + 23}{(w + 2)(w - 1)(w + 5)}$

15. $2 - x$

16. $\frac{-2b^2}{a + b}$

17. $\frac{12m}{m - 3}$

18. $\frac{-2y(x + 3y)}{x - 2y}$

19. $\frac{1 + a}{1 - a}$

1-13. Additional Exercises for Sections 1-1 Through 1-7.

Answers to Exercises

Exercises 1-1a':

1. (a) If $x < y$, then $2x + 1 = y$.
   (b) If the sum of two integers is even, then they are each odd numbers.
   (c) $xy = 0$ if $x = 0$. (Note change of "only if" to "if.")
   (d) If $a = b$, then $a + c = b + c$.
   (e) If the sum of two numbers is an even number, then it is a multiple of 10.
   (f) $\sqrt{a^2 + b^2} = a + b$ only if $(a + b)^2 = a^2 + b^2$. (Note change of "if" to "only if.")
   (g) $3x + 2 = 8$ if $x = 2$.
   (h) If $c = 0$, then $a(b + c) = ab$.
   (i) If xy is negative, then $2xy + 3 = 1$.
   (j) $(a - b) - c = a - (b - c)$ only if $c = 0$.

2. (a) If $3x - 2 = 0$, then $x = 4$; and if $x = 4$, then $3x - 2 = 0$.
   (b) If $y = z$, then $y + x = z + x$; and if $y + x = z + x$, then $y = z$.
   (c) If $m < n$, then $m - a < n - a$; and if $m - a < n - a$, then $m < n$.
   (d) If $abc = 0$, then $c = 0$; and if $c = 0$, then $abc = 0$.
   (e) If $r + s = 0$, then $r = -s$; and if $r = -s$, then $r + s = 0$.
   (f) If $p(r + s) = ps$, then $r = 0$; and if $r = 0$, then $p(r + s) = ps$.
   (g) If x is negative, then $-x$ is positive; and if $-x$ is positive, then x is negative.
   (h) If $a = b$, then $(a - b)(a + b) = 0$; and if $(a - b)(a + b) = 0$, then $a = b$.
   (i) If $x + (y \cdot z) = (x + y) \cdot (x + z)$, then $x = 0$; and if $x = 0$, then $x + (y \cdot z) = (x + y) \cdot (x + z)$.

Note: The A and B phrases may be reversed in these statements.

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Exercises 1-2b':

1. (a) $\underline{C_1}$ (Addition)
   (b) $\underline{D}$ (Distributive)
   (c) $\underline{M_3}$ (Associativity)
   (d) $\underline{M_2}$ (Commutativity)
   (e) $\underline{A_3}$ (Associativity)
   (f) $\underline{A_3}$ (Associativity)
   (g) $\underline{M_4}$ (Identity)
   (h) $\underline{C_2}$ (Multiplication)
   (i) $\underline{A_2}$ (Commutativity)
   (j) $\underline{D}$ (Distributivity)

2. (a) $x(y + z) = xy + xz$ [Dist.
   $= zx + yx.$ [Comm.

   (b) $(x + y) + z = z + (x + y)$ [Comm.
   $= (z + x) + y$ [Assoc.
   $= y + (z + x).$ [Comm.

   (c) $(x + y)(u + v) = (x + y)(v + u)$ [Comm.
   $= (v + u)(x + y)$ [Comm.
   $= (v + u)x + (v + u)y$ [Dist.
   $= x(v + u) + y(v + u)$ [Comm.
   $= y(v + u) + x(v + u)$ [Comm.

   (d) $xy + y = xy + 1 \cdot y$ [Mult. Iden.
   $= yx + y \cdot 1$ [Comm.
   $= y(x + 1)$ [Dist.
   $= y(1 + x)$ [Comm.

   (e) $2[x + (y + c)] = 2x + 2(y + 3)$ [Dist.
   $= 2x + 2y + 2 \cdot 3$ [Dist.
   $= 2y + 2x + 2 \cdot 3$ [Comm.
   $= 2y + 2(x + 3)$ [Def. and Dist.

Note: Each of these proofs can be done in several different ways.

Exercises 1-2c':

1. (a) $(x+1)(3x+2) = (x+1) \cdot 3x + (x+1) \cdot 2$ [Dist.
   $= 3x(x+1) + 2(x+1)$ [Comm.
   $= 3x^2 + 3x + 2x + 2$ [Dist.
   $= (3x^2 + 3x + 2x) + 2$ [Def.
   $= [(3x^2 + 3x) + 2x] + 2$ [Def.
   $= [3x^2 + (3x + 2x)] + 2$ [Assoc.
   $= [3x^2 + 5x] + 2$ [Comm. and Dist.
   $= 3x^2 + 5x + 2$ [Def.

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(b) $2x^2 + 5x + 2$; same properties as for (a).

(c) $2x(x+y+3) = 2x[(x+y) + 3]$ [Def.

$= 2x(x+y) + (2x)(3)$ [Dist.

$= (2x)x + (2x)y + 3(2x)$ [Dist. and Comm.

$= 2(x^2) + 2(xy) + (3 \cdot 2)x$ [Assoc. and Def.

$= 2x^2 + 2xy + 6x$ [Def.

(d) $6x^2 + 3xy + 12x$; same properties as for (c).

(e) $(x + 2)^2 = (x + 2)^x + (x + 2)2$ [Def. and Dist.

$= x(x + 2) + 2(x + 2)$ [Comm.

$= x^2 + x \cdot 2 + 2x + 4$ [Dist.

$= x^2 + 4x + 4$ [Comm., Dist.

(f) $2x^2 + 4x + 2$; same properties as for (e).

(g) $15(2x)(3y) = (15 \cdot 2)x(3y)$ [Assoc.

$= (30x \cdot 3)y$ [Assoc.

$= (3 \cdot 30x)y$ [Comm.

$= (3 \cdot 30)xy$ [Assoc.

$= 90xy$ [Def.

(h) $6xyw$; similar to part (g).

(i) $(x+1)(x+y+2) = (x+1)\ [(x+y) + 2]$ [Def.

$= (x+1)(x+y) + (x+1) \cdot 2$ [Dist.

$= (x+1)x + (x+1)y + (x+1) \cdot 2$ [Dist.

$= x(x+1) + y(x+1) + 2(x+1)$ [Comm.

$= x^2 + x + yx + y + 2x + 2$ [Dist.

$= x^2 + xy + x + y + 2x + 2$ [Comm.

$= x^2 + xy + y + x + 2x + 2$ [Comm.

$= x^2 + xy + y + x \cdot 1 + x \cdot 2 + 2$ [Mult. Iden. and Comm.

$= x^2 + xy + y + 3x + 2$ [Dist. and Comm.

$= x^2 + xy + 3x + y + 2.$ [Comm.

(j) $(x+y+z)^2 = (x+y+x)[(x+y)+z]$ [Def.

$= (x+y+z)(x+y) + (x+y+z)z$ [Dist.

$= (x+y+z)x + (x+y+z)y + (x+y+z)z$ [Dist.

$= x[(x+y)+z] + y[(x+y)+z] + z[(x+y)+z]$ [Comm. and Def.

$= x(x+y) + xz + y(x+y) + yz + z(x+y) + z^2$ [Dist.

$= x^2+xy+xz+yx+y^2+yz+zx+zy+z^2$ [Dist.

$= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.$ [Comm., Dist, and Mult. Iden.

2\. (a) $6 \cdot x + 3xy = 2(3x) + (3x)y$ [Assoc. and Def.

$= (3x) \cdot 2 + (3x)y$ [Comm.

$= 3x(2 + y)$ [Dist.

(b) $4yz + 2z = (4y)z + (2z) \cdot 1$ [Def. and Mult. Iden.

$= z(4y) + (2z) \cdot 1$ [Comm.

$= (4z)y + (2z) \cdot 1$ [Assoc. and Comm.

$= 2z(2y) + (2z) \cdot 1$ [Assoc. and Comm.

$= 2z(2y + 1)$ [Dist.

(c) $7(m + n)$; Comm. and Dist.

(d) $7(3x + 1)$; Comm. and Dist.

(e) $a(x + y) + a(x + y) = (x + y)a + (x + y)a$ [Comm.

$= (x + y)(a + a)$ [Dist.

$= (x + y)(2a)$ [Def.

$= 2a(x + y).$ [Comm.

(f) $2x(a + 2b)$; similar to (e).

(g) $y + 3xy = y \cdot 1 + (3x)y$ [Def., Mult. Iden.

$= y \cdot 1 + y(3x)$ [Comm.

$= y(1 + 3x).$ [Dist.

(h) $p(5q + 1)$; similar to (g).

(i) $ab + ac + ad = (ab + ac) + ad$ [Def.

$= a(b + c) + ad$ [Dist.

$= a[(b + c) + d]$ [Dist.

$= a(b + c + d).$ [Def.

(j) $ab + ac + bd + cd = [(ab + ac) + bd] + cd$ [Def.

$= [a(b + c) + bd] + cd$ [Dist.

$= a(b + c) + (bd + cd)$ [Assoc.

$= a(b + c) + d(b + c)$ [Comm. and Dist.

$= (b + c)(a + d).$ [Comm. and Dist.

3\. (a) $(x + y) + (w + z) = [(x + y) + w] + z$ [Assoc.

$= (x + y + w) + z.$ [Def.

(b) $xy + xz + yw + wz = [(xy+xz) + yw] + xz$ [Def.
$= [x(y+z) + yw] + wz$ [Dist.
$= x(y + z) + (wy + wz)$ [Comm. and Assoc.
$= x(y + z) + w(y + z)$ [Dist.
$= (y + z)(x + w).$ [Comm. and Dist.

(c) $(xy)(uv) = [(xy) \cdot u] \cdot v$ [Assoc.
$= (xyu) \cdot v$ [Def.
$= xyuv.$ [Def.

(d) $(a+b)(x+y+z) = (a+b) \cdot x + (a+b) \cdot y + (a+b) \cdot z$ [Dist.
$= x(a+b) + y(a+b) + z(a+b).$ [Comm.

(e) $x^2 + 2xy + y^2 = x^2 + xy(1 + 1) + y^2$ [Comm.
$= x^2 + xy + xy + y^2$ [Dist. and Mult. Iden.
$= x \cdot x + x \cdot y + y \cdot x + y \cdot y$ [Def. and Comm.
$= x(x + y) + y(x + y)$ [Dist.
$= (x + y)(x + y)$ [Comm. and Dist.
$= (x + y)^2.$ [Def.

<u>Exercises</u> <u>1-2d'</u>:

1. 1; C_1 (Addition)
2. 7; C_1
3. 8; C_2 (Multiplication)
4. 3; C_2
5. 4; C_1 and C_2
6. $8z + 3 = 24 + 3$ [Comm.
   $z \cdot 8 = 3 \cdot 8$ [C_1 and Comm.
   $z = 3$ [C_2.
7. $2a + 5 = a + 3 + 5$
   $2a = a + 3$ [C_1.
   $a(1 + 1) = 3 + a$ [Comm.
   $a + a = 3 + a$ [Dist.
   $a = 3$ [C_1.

8. $3p = p + 14$ [C_1.
$p(2 + 1) = p + 14$ [Comm.
$p \cdot 2 + p = 14 + p$ [Dist., Mult. Iden., and Comm.
$p \cdot 2 = 7 \cdot 2$ [C_1.
$p = 7$. [C_2.

9. $4w + 5 = 5w + 1 + 5$ [Comm.
$4w = 5w + 1$ [C_1.
No solution over N.

10. $3x + 9 + 6 = 5x + 6$ [Comm.
$3x + 9 = 5x$ [C_1.
$9 + 3x = x(2 + 3)$ [Comm.
$9 + 3x = x \cdot 2 + 3 \cdot x$ [Dist. and Comm.
$9 = 2x$ [C_1 and Comm.
No solution over N.

Exercises 1-3a':

1. {1, 2, 3}.
2. {1}.
3. $1 + x = 3$; or $1 + 2 = 3$.
4. $1 + x = 5$; or $1 + 4 = 5$.
5. (a) $3 < 4$.
   (b) $7 < 12$.
   (c) $x < 2x$.
   (d) $a < a + 2$.
   (e) $3x + 1 < 2x + 4$.
   (f) $5m + 1 < 4m + 3$.
   (g) $y < x$.
6. (a) $x \not< y$.
   (b) $x \not> y$.
   (c) $x < 9 < y$.
   (d) $x < 5 \leq y$.
   (e) $x \leq 2$.
   (f) $1 < x < 3$.
   (g) $2 \leq 3 < 5$.
   (h) $4 \geq x$.
   (i) $2 \leq x \leq 5$.
   (j) $x \leq y \leq z$.

Exercises 1-3b':

1. (a) {1, 2}.
   (b) {1, 2, 3}.
   (c) {1, 2, 3, 4, 5, 6}.
   (d) {1, 2}.
   (e) {1, 2}.
   (f) {1, 2, 3}.
   (g) {1}.
   (h) {1}.
   (i) $2 < x < 3$, so no solution over N.
   (j) $3 > x > 2$, so no solution over N.

2. (a) If $x < y$, then $x + a = y$ where a is a natural number. Then $(x + a) + z = y + z$ by $\underline{O_3}$ or $x + (a + z) = y + z$. Since $(a + z)$ is in N, then by the definition for order in N, $x < y + z$.
   (b) If $x(y + z) = wz$, then $xy + xz = wz$, or $xz + xy = wz$. Since xy is in N, then $xz < wz$. Hence, $x < w$ by $\underline{OC_2}$.
   (c) If $x(y + z + w) = a$, then $x[y + (z + w)] = a$ by definition. Then $xy + x(z + w) = a$, or $x(z + w) + xy = a$. Since xy is in N, then $x(z + w) < a$.
   (d) If $x > y$ and $w > z$, then $y + a = x$ and $z + b = w$ where a, b in N. Then $(y + a) + (z + b) = x + w$, or by Association and Commutation, $(y + z) + (a + b) = x + w$. Since $a + b$ is in N, then $y + z < x + w$, or $x + w > y + z$.
   (e) If $a < y$, then $a + c = y$, where c is in N. Since $x = a + b$, then $(a + c) + (a + b) = y + x$, or $2a + (b + c) = x + y$. But $(b + c)$ is in N, so $2a < x + y$.

Exercises 1-4a':

1. (a) -4
   (b) 5
   (c) x
   (d) x - 1
   (e) y
   (f) -x

2. (a) $\underline{A_5}$ (Subtraction) Note: $\underline{A_4}$ and $\underline{A_5}$ are properties; not Theorems or definitions as required in the exercise instructions.
   (b) $\underline{A_4}$ (Add. Iden.)
   (c) Def. 1-4b.
   (d) Def. 1-4a.
   (e) Def. 1-4b.
   (f) $\underline{EC_2}$
   (g) Def. 1-4a.
   (h) Corollary 1-4a.

3. Let $x = (a + b) - c$ and $y = b - c$. Then $x + c = a + b$ and $y + c = b$, so $x + c = a + (y + c)$. By Associativity and $\underline{C_1}$, $x = a + y$. Hence, $(a + b) - c = a + (b - c)$.

4. Let $x = b - c$ and $y = a - b$. Then $x + c = b$ and $y + b = a$, so $(x + c) + (y + b) = b + a$, or $(y + c) + x = a$ and $y + c = a - x$. Hence, $(a - b) + c = a - (b - c)$.

5. Let $x = a - (b + c)$ and $y = a - b$. Then $(b + c) + x = a$ and $b + y = a$, so $(b + c) + x = b + y$. By Commutivity, Associativity, and $\underline{C_1}$, $c + x = y$, or $x = y - c$. Hence, $a - (b + c) = (a - b) - c$.

<u>Exercises 1-4b'</u>:

1. (a) $(x + y)(-1) = (-1)(x + y)$ [Comm.
   $= -(x + y)$ [Th. 1-4f, Mult. Iden.
   (b) $3x$; Th. 1-4g.
   (c) $6 - (-2) = 6 + 2 = 8$; Th. 1-4c.
   (d) -12; Th. 1-4f.
   (e) 0; Th. 1-4b.
   (f) $(-x) + (-2) = -(x + 2)$ [Th. 1-4d
   (g) $-2(3)(4) = -6(4)$ [Th. 1-4f
   $= 24$ [Th. 1-4f
   (h) $-8 + 12 = 12 + (-8)$ [Comm.
   $= 12 - 8 = 4$ [Th. 1-4c

(i) $-4 - (-7) = -4 + [-(-7)] = -4 + 7$ [Th. 1-4c
$= 7 + (-4)$ [Comm.
$= 7 - 4 = 3$ [Th. 1-4c

(j) $(-5) - (-9) = (-5) + [-(-9)] = (-5) + 9$ [Th. 1-4c
$= 9 + (-5)$ [Comm.
$= 9 - 5 = 4$ [Th. 1-4c

2. (a) $4x - 2 = 8$ if and only if $4x = 8 + 2 = 10$ [Def.
No solution over I, since there is no x in I such that $4x = 10$.

(b) $6m+1 = 13$ if and only if $6m+1+(-1) = 13+(-1)$ [$\underline{EC}_1$
if and only if $6m = 12$ [Add. Inv., Add. Iden., Th. 1-4c
if and only if $m \cdot 6 = 2 \cdot 6$ [Comm.
if and only if $m = 2$ [$\underline{EC}_2$

(c) $5y-3 = 2y+6$ if and only if $5y+(-3)+3 = 2y+6+3$ [$\underline{EC}_1$ and Th. 1-4c
if and only if $5y = 2y + 9$ [Add. Inv., Add. Iden.
if and only if $5y+(-2) = 2y+9+(-2y)$ [$\underline{EC}_1$
if and only if $5y - 2y = 9$ [Th. 1-4c, Comm. Add. Inv., Add. Iden.
if and only if $3y = 3 \cdot 3$ [Th. 1-4c, Comm., Dist.
if and only if $y = 3$ [Comm. and $\underline{EC}_2$

(d) $3p + 7 = p + 9$
$3p + 7 + (-p) = p + 9 + (-p)$ [$\underline{EC}_1$
$2p + 7 = 9$ [Comm., Th. 1-4c, Add. Inv., Add. Iden.
$2p = 2$ [Add. Inv., Th. 1-4c, Add. Iden.
$p = 1$ [$\underline{EC}_2$, Mult. Iden., Comm.

(e) $4x - 2x - 2 = 6$ [Dist. Th. 1-4c
$2x = 6 + 2 = 8$ [$\underline{EC}_1$, Th. 1-4c, Dist.
$x = 4$

(f)	$7y + 6y + 9 = 17$	[Dist.,
	$13y = 17 + (-9) = 8$	[Dist., Comm., $\underline{EC_1}$, Th. 1-4c
	No solution over I.	
(g)	$4a + 31 = 6a + 21$	[Dist., Comm.
	$31 = 2a + 21$	[$\underline{EC_1}$, Add. Inv., Comm., Add. Iden.
	$10 = 2a$	[$\underline{EC_1}$, Add. Inv., Add. Iden.
	$a = 5$	[$\underline{EC_2}$ and $\underline{E_2}$ (Symmetry)
(h)	$5 - 6x - 8 = 3x + 6 - 18$	[Dist., Th. 1-4c
	$-6x - 3 = 3x - 12$	[Th. 1-4c, Comm.
	$9 = 9x$	[$\underline{EC_1}$, Add. Inv., Add. Iden., Dist., Th. 1-4c
	$x = 1$	[$\underline{E_2}$, $\underline{EC_2}$
(i)	$3y - 3 + 2 = 6 - 2y - 6$	[Dist., Th. 1-4c
	$3y - 1 = -2y$	[Comm., Th. 1-4c, Add. Inv., Add. Iden.
	$5y = 1$	[Add. Iden., Add. Inv., $\underline{EC_1}$, Th. 1-4c
	No solution over I.	
(j)	$13 - 3w + 4 = 1 - 2 + 6w$	[Dist., Th. 1-4c,g, $\underline{EC_1}$, Add. Inv., Mult. Iden. Add. Iden.
	$17 - 3w = -1 + 6w$	[Th. 1-4c
	$18 = 9w$	[$\underline{EC_1}$, Comm., Dist., Add. Iden., Add. Inv.
	$w = 2$	[$\underline{EC_2}$, $\underline{E_2}$

3. (a)

$a - (b - c) = a + [- b + (-c)]$	[Th. 1-4c
$= a + [(-b) + -(-c)]$	[Th. 1-4d
$= a + [(-b) + c]$	[Th. 1-4a
$= [a + (-b)] + c$	[Assoc.
$= a - b + c$	[Def. and Th. 1-4c.

(b)

$a(b - c) = a[b + (-c)]$	[Th. 1-4c
$= ab + a(-c)$	[Dist.
$= ab + [-(ac)]$	[Th. 1-4f and Comm.
$= ab - ac$	[Th. 1-4c

(c) $(a-b)(a+b) = (a-b)a + (a-b)b$ [Dist.

$= a[a + (-b)] + b[a + (-b)]$ [Comm., Th. 1-4c

$= a \cdot a + a(-b) + b \cdot a + b(-b)$ [Dist.

$= a^2 + [-(ab)] + ab + [-(b^2)]$ [Def., Th. 1-4f Comm.

$= a^2 + 0 - b^2$ [Add. Inv., Th. 1-4c

$= a^2 - b^2.$ [Add. Iden.

(d) $(a-b)^2 = [a + (-b)][a + (-b)]$ [Th. 1-4c, Def.

$= [a + (-b)]a + [a + (-b)](-b)$ [Dist.

$= a \cdot a + a(-b) + (-b)a + (-b)(-b)$ [Comm., Dist.

$= a^2 + [-(ab)] + [-(ab)] + b^2$ [Def., Th. 1-4f, Th. 1-4g

$= a^2 + 2[-(ab)] + b^2$ [Mult. Iden., Dist., Comm.

$= a^2 + [-2(ab)] + b^2$ [Th. 1-4f

$= a^2 - 2ab + b^2$ [Th. 1-4c, Def.

(e) $(a-b)(a^2+ab+b^2) = (a-b)a^2+(a-b)ab+(a-b)b^2$ [Dist.

$= a^2 \cdot a+a^2(-b)+ab(a)+ab(-b)+b^2 \cdot a+b^2(-b)$

[Comm., Dist.

$= a^3+(-a^2b)+a^2b+(-ab^2)+ab^2+(-b^2)$

[Def., Th. 1-4f, Comm., Assoc.

$= a^3 + 0 + 0 + (-b^3)$ [Add. Inv.

$= a^3 - b^3$ [Add. Iden., Th. 1-4c

Exercises 1-5a':

1. (a) $-6 < 4$.
   (b) $-3 < -2$.
   (c) $-5 < 2$.
   (d) $x < -x$.
   (e) $1 \leq y$.
   (f) $3w \leq 2w$.
   (g) $z \leq -3z$.
   (h) $y - 1 < y + 1$.
   (i) $-2x \leq 2x$.
   (j) $2p - 1 < 2p + 1$.

2. (a) Since $y > 0$, y is a natural number. Also $x+y = x+y$. Therefore $x < x + y$ by Definition 1-5a.

   (b) If $x < y$, then $x + (-y) < y + (-y)$ by $\underline{O}_3$. Then $x - y < 0$ by Theorem 1-4c and $\underline{A}_4$.

[pages 108-109]

(c) Since $x < y$, $x + a = y$ where $a > 0$. Then $[x + (-y)] + a = 0$ or $(x - y) + a = 0$. Therefore, $x - y < 0$ by Definition 1-5a, because a is in N.

(d) If $x < y$, then $x + a = y$ where a is in N. Then $(x + a) + (-w) = y + (-w)$ and $(x - w) + a = y - w$, or $x - w < y - w$. If $z < w$, then $-w < -z$ and $-w + y < -z + y$, or $y - w < y - z$. Since $x - w < y - w$ and $y - w < y - z$, then by Transitivity, $x - w < y - z$.

(e) If $0 < x < y$, then $0 < x$ and $-x < 0$. Since $-x < 0$ and $0 < x$, then $-x < x$ and $-x + y < x + y$, or $y - x < x + y$.

Exercises 1-5b':

1. (a) $x < 7$, so {1, 2, 3, 4, 5, 6}.
   (b) No solution over N.
   (c) {1, 2}.
   (d) {1}.
   (e) {1, 2, 3}.
   (f) {..., -1, 0, 1, 2, 3, 4}
   (g) No solution over I.
   (h) {-2, -1, 0, 1, 2}.
   (i) No solution over I.
   (j) {-1, 0, 1, 2, ...}.

2. (a) {-3, 3}.
   (b) No solution, since $0 \leq |a|$ for a in N.
   (c) {-4, -3, -2}, i.e., integers z such that $-5 < z < -1$.
   (d) {0, 1, 2, 3, ..., 10}.
   (e) {-4, 3}.
   (f) No solution, since $0 \leq |a|$ for a in N.
   (g) {-1, 0, 1, 2}, i.e., integers x such that $-1 \leq x \leq 2$.
   (h) {-1, 5}.

(i) $\{-6, -5, -4, -3, -2, -1, 0\}$; $-2|y + 3| \geq -6.$

$$|y + 3| \leq 3$$

$$[0 \leq y + 3 \text{ and } y + 3 \leq 3] \text{ or } [y + 3 < 0 \text{ and } -(y + 3) \leq 3]$$

$$[-3 \leq y \text{ and } y \leq 0] \text{ or } [y < -3 \text{ and } -3 \leq y + 3]$$

$$[-3 \leq y \leq 0] \text{ or } [y < -3 \text{ and } -6 \leq y]$$

$$[-3 \leq y \leq 0] \text{ or } [-6 \leq y < -3]$$

Combining these results; $-6 \leq y \leq 0$.

(j) $\{-2, -1, 0, 1, 2, 3\}$

Exercises 1-6a':

1. (a) $\frac{7}{3}$
   (b) $\frac{2}{5}$
   (c) $-\frac{1}{3}$
   (d) $\frac{9}{8}$
   (e) $\frac{3}{2}$
   (f) $\frac{b - a}{3}$
   (g) $\frac{3b - 2}{a}$
   (h) $\frac{5a + 3}{2}$
   (i) $\frac{a - c}{b}$
   (j) $\frac{d - bc}{a - b}$

2. (a) 16
   (b) 12
   (c) 35
   (d) 16
   (e) $\frac{20}{3}$
   (f) 5

Exercises 1-6b':

1. (a) $\frac{17}{12}$
   (b) $\frac{3x + 2y}{6}$
   (c) $\frac{7a + 5b}{7b}$
   (d) $\frac{5x + 3y}{xy}$
   (e) $\frac{3a + 2}{3}$
   (f) $\frac{a + c}{b}$
   (g) $\frac{ac - bc + 2d}{d(a - b)}$
   (h) $\frac{8a + 2}{15}$
   (i) $\frac{ab + a - 2b + 2}{b}$
   (j) 0

2. (a) $\frac{15}{28}$
(b) $\frac{xy}{6}$
(c) $\frac{5a}{7b}$
(d) $\frac{2a}{3}$
(e) $\frac{ac}{b^2}$
(f) $\frac{zc}{d(a - b)}$
(g) $\frac{a^2 - 1}{15}$
(h) $\frac{a^2 + a - 2}{3}$
(i) 0
(j) $-\frac{a^2}{(a - b)^2}$

Exercises 1-6c':

1. (a) $\frac{5x}{3} = 4$ if and only if $3(\frac{5x}{3}) = 3(4)$, [EC_2
if and only if $5x = 12$, [Mult. Inv.
if and only if $x = \frac{12}{5}$. [M_5

Note: Only the major reason involved in the statements are given. For example, in forming the second statement use is made of a number of other reasons such as Associativity, $\frac{a}{b} = a \cdot \frac{1}{b}$, etc.

(b) $\frac{2y}{7} + 1 = 6$ if and only if $\frac{2y}{7} + 1 + (-1) = 6 + (-1)$, [$EC_1$
if and only if $\frac{2y}{7} = 5$, [Add. Inv., Th. 1-4c
if and only if $7 \cdot \frac{2y}{7} = 7 \cdot 5$ [EC_2
if and only if $2y = 35$, [Mult. Inv.
if and only if $y = \frac{35}{2}$ [M_5

(c) $\frac{1}{8}$
(d) $\frac{14}{15}$
(e) $\frac{1}{8}$
(f) $\frac{23}{20}$

(g) $\frac{83}{9}$

(h) $\frac{1}{61}$

(i) Since $(1 - x) = -(x - 1)$, then the equation is the same as $1 - 2(-1) = \frac{x + 2}{3}$, so $x = 7$.

(j) $\frac{11}{10}$

2. (a) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$ by $\underline{E}_5$. Adding, $\frac{a + b}{b} = \frac{c + d}{d}$.

(b) If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. Using Commutativity, $da = cb$, and by definition for equal rational numbers, $\frac{d}{c} = \frac{b}{a}$. Hence, $\frac{b}{a} = \frac{d}{c}$ by $\underline{E}_2$.

(c) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$ from (b) above. By $\underline{E}_5$, $\frac{b}{a} + \frac{a}{a} = \frac{d}{c} + \frac{c}{c}$, so $\frac{b + a}{a} = \frac{d + c}{c}$. Using Commutativity, $\frac{a + b}{a} = \frac{c + d}{c}$.

(d) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + (\frac{-b}{b}) = \frac{c}{d} + (\frac{-d}{d})$. Adding, $\frac{a + (-b)}{b} = \frac{c + (-d)}{d}$, or $\frac{a - b}{b} = \frac{c - d}{d}$.

Exercises 1-7a':

1. (a) $\frac{11}{7} < \frac{13}{8}$.

(b) $\frac{2}{31} < \frac{4}{61}$.

(c) $\frac{2x}{3} < \frac{9y}{13}$.

(d) $\frac{m - 1}{9} \leq \frac{m + 1}{7}$ for $-8 \leq m$;

$\frac{m + 1}{7} < \frac{m - 1}{9}$ for $m < -8$.

(e) $\frac{28}{17b} < \frac{5}{3a}$.

2. $\frac{-26}{13} < \frac{-53}{27} < \frac{-15}{16} < \frac{1}{3} < \frac{12}{35} < \frac{13}{7} < \frac{25}{13}$.

3. If $a < 0$, then $\frac{1}{a} < 0$.

   Proof: If $a < 0$, then $0 < -a$ by O_5. Since $0 \cdot 1 < -a \cdot 1$ by $\underline{M}_4$, then $0 < 1(-a)$ and by Def. 1-7a, $0 < \frac{1}{-a}$. But $\frac{1}{-a} = -\frac{1}{a}$ for $a \neq 0$, so $0 < -\frac{1}{a}$. Hence, $\frac{1}{a} < 0$ by $\underline{O}_5$.

   If $\frac{1}{a} < 0$, then $a < 0$.

   Proof: If $\frac{1}{a} < 0$, then $0 < -\frac{1}{a}$ and $0 < \frac{1}{-a}$. By Def. 1-7a, $0 < 1(-a)$ or $0 < -a$. Hence, $a < 0$ by $\underline{O}_5$.

4. If $\frac{a}{b} > \frac{c}{d}$ and $\frac{c}{d} > \frac{e}{f}$, then $\frac{c}{d} < \frac{a}{b}$ and $\frac{e}{f} < \frac{c}{d}$. By Transitivity, $\frac{e}{f} < \frac{a}{b}$. Hence, $\frac{a}{b} > \frac{e}{f}$.

Exercises 1-7b':

1. If average method is used; $\frac{5}{7} < \frac{11}{14} < \frac{6}{7}$,

$$\frac{5}{7} < \frac{21}{28} < \frac{11}{14} < \frac{23}{28} < \frac{6}{7},$$

$$\frac{5}{7} < \frac{41}{56} < \frac{21}{28} < \frac{43}{56} < \frac{11}{14} < \frac{45}{56} < \frac{23}{28} < \frac{6}{7}.$$

   An alternate method would be to form $\frac{5}{7} = \frac{30}{42}$ and $\frac{6}{7} = \frac{36}{42}$. Then

$$\frac{5}{7} = \frac{30}{42} < \frac{31}{42} < \frac{32}{42} < \frac{33}{42} < \frac{34}{42} < \frac{35}{42} < \frac{36}{42} = \frac{6}{7}.$$

2. $\left|\frac{5}{6} + \frac{3}{4}\right| = \frac{19}{12}$, $\left|\frac{5}{6} - \frac{3}{4}\right| = \frac{1}{12}$, $\left|\frac{5}{6} \cdot \frac{3}{4}\right| = \frac{15}{24}$, $\left|\frac{3}{4}\right| - \left|\frac{5}{6}\right| = -\frac{1}{12}$.

   Hence, $\left|\frac{3}{4}\right| - \left|\frac{5}{6}\right| < \left|\frac{5}{6} - \frac{3}{4}\right| < \left|\frac{5}{6} \cdot \frac{3}{4}\right| < \left|\frac{5}{6} + \frac{3}{4}\right|$.

3. The proof is exactly the same as that used for Theorem 1-5e except that a is in Q rather than a in I.

4. The proof is exactly the same as that used for Theorem 1-5f and part 4 of Exercise 1-5b except that a in Q rather than a in I.

Exercises 1-7c':

1. $6 < 3x + 2 < 10$ if and only if $4 < 3x < 8$, [OC_1
$\frac{4}{3} < x < \frac{8}{3}$. [$OC_2$

2. $-\frac{11}{2} < y < -\frac{3}{2}$.

3. $-2 < \frac{2w+3}{5} < 2$ if and only if $-10 < 2w + 3 < 10$, [OC_2
$-13 < 2w < 7$, [OC_1
$\frac{-13}{2} < w < \frac{7}{2}$. [$OC_2$

4. $-1 < 3 - x < 1$ if and only if $-4 < -x < -2$ [OC_1
$2 < x < 4$. [OC_2

5. $\frac{-25}{3} < y < \frac{1}{3}$.

6. $0 \leq m < \frac{1}{2}$.

7. $\frac{1}{2} < a \leq \frac{7}{2}$.

8. $-\frac{1}{18} < p < \frac{5}{8}$.

9. $-1 \leq \frac{4 - 3x}{-2} \leq 1$ if and only if $-2 \leq 4 - 3x \leq 2$, [OC_2
$-6 \leq -3x \leq -2$, [OC_1
$\frac{2}{3} \leq x \leq 2$. [$OC_2$

10. $\frac{2}{5} \leq x \leq \frac{6}{5}$.

Exercises 1-7d':

1\. (a) $|x + 3| < 5$ if and only if $-5 < x + 3 < 5$, [Th. 1-7a
$-8 < x < 2$. [OC_1

(b) $-8 < x < 6$.

(c) $0 \leq y \leq \frac{4}{3}$.

(d) $0 \leq y \leq 5$.

(e) $|4 - m| > 6$ if and only if $4-m < -6$ or $6 < 4-m$, [Th. 1-7
$-m < -10$ or $2 < -m$, [OC_1
$10 < m$ or $m < -2$. [O_4

(f) $p < -2$ or $6 < p$.

(g) $x \leq 0$ or $6 \leq x$.

(h) $x \leq \frac{2}{3}$ or $2 \leq x$.

(i) $1 \leq |x + 2| \leq 3$ if and only if

[$x + 2 \leq -1$ or $1 \leq x + 2$] and [$-3 \leq x + 2 \leq 3$],

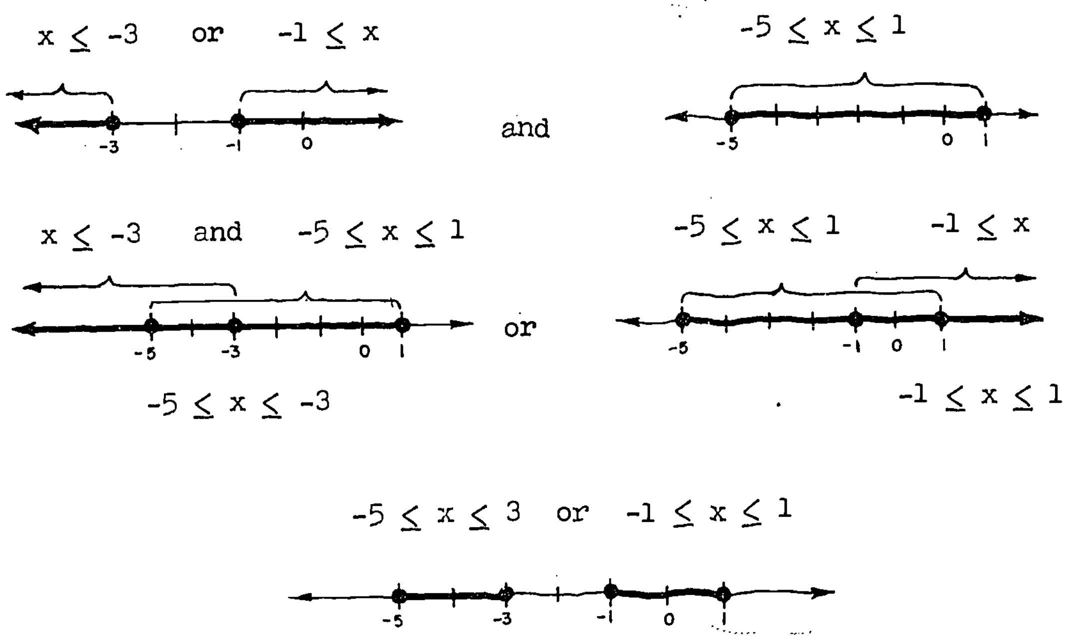
[$x \leq -3$ or $-1 \leq x$] and [$-5 \leq x \leq 1$].

Combining these cases,

[$x \leq -3$ and $-5 \leq x \leq 1$] or [$-1 \leq x$ and $-5 \leq x \leq 1$],

[$-5 \leq x \leq -3$] or [$-1 \leq x \leq 1$].

The combining of the two cases is actually the use of the Distributive Property of propositional logic: $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, where $a = (x \leq -3)$, $b = (-1 \leq x)$, and $c = (-5 \leq x \leq 1)$; using $\wedge$ for the conjunction "and" and $\vee$ for the inclusive "or." This point will necessarily have to be made by appealing to student's acceptance of the same meaning for the two cases. Number line diagrams may be helpful to make this point.



(j) $1 \le x \le 2$.

2. Theorem 1-17a for $a = 0$. $|x| \le 0$ if and only if $-0 \le x \le 0$, i.e., $x = 0$.

<u>Proof</u> <u>for</u> <u>"if"</u>: If $x = 0$, then $|x| \le 0$.

If $x = 0$, then $|x| = 0$ and $|x| \le 0$.

<u>Proof</u> <u>for</u> <u>"only if"</u>: If $|x| \le 0$, then $x = 0$.

Since $0 \le |x|$ for x in Q, then $|x| \le 0$ only for $|x| = 0$. Hence, $x = 0$ from the definition for order.

1-14. Miscellaneous Exercises.

Answers to Exercises

1. (a) I, Q, R.
   (b) Q, R.
   (c) I, Q, R.
   (d) I, Q, R.
   (e) I, Q, R.
   (f) I, Q, R.
   (g) N, I, Q, R
   (h) R
   (i) Q, R.
   (j) N, I, Q, R.

2. (a) False
   (b) False
   (c) False
   (d) True
   (e) True
   (f) False
   (g) True
   (h) True
   (i) False

3. (a) $\underline{M}_3$ (Associativity)
   (b) $\underline{E}_4$ (Transitivity)
   (c) $\underline{O}_4$ (Multiplication)
   (d) $\underline{M}_4$ (Mult. Identity)
   (e) $\underline{E}_3$ (Symmetry)
   (f) $\underline{M}_2$ (Commutativity)
   (g) $\underline{D}$ (Distributive); ($\underline{M}_4$ is also involved).
   (h) $\underline{A}_4$ (Add. Identity)
   (i) $\underline{O}_3$, or $\underline{OC}_1$; also $\underline{A}_5$ and $\underline{A}_4$
   (j) $\underline{A}_2$ (Commutativity); also possibly $\underline{EC}_2$ and Dist.

4. (a) True; $\underline{D}$ (Distributivity)
   (b) False
   (c) True; $\underline{A}_2$ (Commutivity)
   (d) False
   (e) True; $\underline{M}_2$ (Commutativity)
   (f) True; $\underline{M}_3$ (Associativity)
   (g) False
   (h) True; $\underline{A}_2$ (Commutativity)
   (i) True; $\underline{A}_2$ (Commutativity)
   (j) True; $\underline{M}_2$ (Commutativity)
   (k) True; $\underline{D}$ (Distributivity)
   (l) True; $\underline{A}_2$ (Commutativity)

5. Commutativity for addition.

6. (a) $-3 < 6$.
   (b) $-5 < -2$.
   (c) $-7 < 0$.
   (d) $8.2535 < 8.2536$
   (e) $-0.1 < -0.001$
   (f) $\frac{4}{5} < \frac{7}{5}$
   (g) $-\frac{3}{23} < \frac{4}{23}$
   (h) $\frac{6}{13} < \frac{5}{7}$
   (i) $-\frac{4}{15} < -\frac{5}{19}$
   (j) $\frac{12}{17} < \frac{5}{7}$
   (k) $a < a^2$
   (l) $a^2 < a$
   (m) $a < -a$
   (n) $a^2 < a$
   (o) $a < a^2$

7. (a) $\{1\}$ $(x < 2)$.
   (b) $\{1,2,3,4\}$ $(y \leq 4)$.
   (c) $\{1,2,3\}$ $(p < 4)$.
   (d) No solution in N $(m < \frac{6}{7})$.
   (e) No solution in I $(\frac{6}{5} < x < 2)$.
   (f) $\{1,2\}$ $(c < \frac{8}{3})$.
   (g) $\{2,3,4,\ldots\}$ $(\frac{21}{11} < x)$.
   (h) $\{1\}$ $(y < \frac{33}{25})$.
   (i) $-\frac{10}{3} < x \leq -\frac{1}{3}$.
   (j) $-\frac{5}{2} \leq w \leq \frac{5}{2}$
   (k) $\frac{8}{3} < y \leq 22$.
   (l) $\{\ldots -8, -7\}$ $(d < -6)$.
   (m) $\frac{x-3}{x} \leq 1$ iff $1 - \frac{3}{x} \leq 1$ iff $-\frac{3}{x} \leq 0$.
   Hence, $0 < x$.

8. (a) $\{-3, -2, -1, 0, 1, 2, 3\}$ $(-4 < x < 4)$.
   (b) $y < -4$ or $4 < y$.
   (c) No solution; $0 \leq |c|$ for c in R.
   (d) $\{1, 2, 3\}$ $(p < 4)$.
   (e) $\{\ldots -4, -3, -2, 2, 3, \ldots\}$ $(n < -\frac{5}{3}$ or $\frac{5}{3} < n)$.
   (f) $\{-6, 6\}$.
   (g) No solution; $|3m - 1| < -1$.

(h) Case (i) $0 \leq \frac{x+1}{2}$ and $\frac{x+1}{2} + x = 6$,

$-1 \leq x$ and $x = \frac{11}{3}$,

$x = \frac{11}{3}$.

Case (ii) $\frac{x+1}{2} < 0$ and $-\frac{x+1}{2} + x = 6$,

$x < -1$ and $x = 13$.

No solution, since no x in N satisfies $\frac{x+1}{2} < 0$.

8. (i) No solution; Case (i) $-\frac{3}{4} \leq y$ and $y = -\frac{5}{2}$,

Case (ii) $y < -\frac{3}{4}$ and $y = -\frac{1}{6}$.

(j) Case (i) $0 < \frac{2c-1}{3}$ and $4 + \frac{2c-1}{3} < 6$,

$\frac{1}{2} \leq c$ and $c < \frac{7}{2}$.

$\frac{1}{2} \leq c < \frac{7}{2}$.

Case (ii) $\frac{2c-1}{3} < 0$ and $4 - \frac{2c-1}{3} < 6$,

$c < \frac{1}{2}$ and $-\frac{5}{2} < c$,

$-\frac{5}{2} < c < \frac{1}{2}$.

Combining the two cases,

$-\frac{1}{2} \leq c < \frac{7}{2}$ or $-\frac{5}{2} < c < \frac{1}{2}$.

Hence, $-\frac{5}{2} < c < \frac{7}{2}$.

(k) No solution, since $0 \leq |a|$ for a in R.

(l) Case (i) $0 \leq x$ and $x = 8$. Case (ii) $x < 0$ and $x = \frac{8}{3}$. Only Case (i) gives a solution, $x = 8$.

(m) $-3 < z < -2$ or $1 < z < 2$. This can be done by use of Theorems 1-7a, b; see Teachers Commentary for 1-13, Exercise 1-7d', part 1 (i) for a discussion of a similar problem.

(n) If $|x| \leq 3$, then $-3 \leq x \leq 3$ by Theorem 1-7a.
If $|x| \geq 5$, then $x \leq -5$ or $5 \leq x$ by Theorem 1-7b.
Since $|x| \leq 3$ or $|x| \geq 5$, then

$$[-3 \leq x \leq 3] \text{ or } [x \leq -5 \text{ or } 5 \leq x].$$

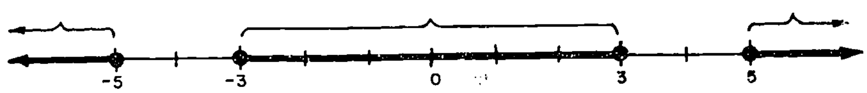
Combining these gives

$$-3 \leq x \leq 3 \text{ or } x \leq -5 \text{ or } 5 \leq x.$$

This is the use of the general associative property of propositional logic, e.g.,

$$a \vee (b \vee c) = (a \vee b) \vee c = a \vee b \vee c$$

where in this case $a = (-3 \leq x \leq 3)$, $b = (x \leq -5)$, and $c = (5 \leq x)$.
A number line diagram is useful to picture these statements.



9. (a) $x - (y+z) = x + [-(y+z)]$ [Th. 1-4c
$= x + [(-y) = (-z)]$ [Th. 1-4d
$= [x + (-y)] + (-z)$ [Assoc.
$= (x - y) - z$ [Th. 1-4c

(b) $(x-y) + (x-z) = x + (-y) + w + (-z)$ [Th. 1-4c and Def.
$= x + w + (-y) + (-z)$ [Comm.
$= (x+w) + [-(y+z)]$ [Def. and Th. 1-4d
$= (x+w) - (y+z)$ [Th. 1-4c

(c) Use $\underline{OC_2}$; $0 < x$ if and only if
$x(-1) < 0(-1)$,
or $-x < 0$.

(d) Use $\underline{OC_2}$; $x < 0$ if and only if
$0(-1) < x(-1)$, or
$0 < -x$.

10. (a) $x(y + z) = x(z + y)$ [Comm.

$= (z + y)x$ [Comm.

(b) $(x + y) + z = x + (y + z)$ [Assoc.

$= x + (z + y)$ [Comm.

$= (x + z) + y$ [Assoc.

(c) $(x + y)z = z(x + y)$ [Comm.

$= zx + zy$ [Dist.

$= xz + yz$ [Comm.

(d) $x + (y + z) = (x + y) + z$ [Assoc.

$= (y + x) + z$ [Comm.

(e) $(x+y)(w+z) = (x+y)w + (x+y)z$ [Dist.

$= w(x+y) + z(x+y)$ [Comm.

$= wx + wy + zx + zy$ [Dist.

$= xw + yw + xz + yz$ [Comm.

$= xw + xz + yw + yz$ [Comm.

Illustrative Test Questions

A. Multiple choice items.

1. Which of the following statements is a correct logical inference from the statement: Every element of set A is an element of set B.?
   (a) If P is an element of B, then it is an element of A.
   (b) P is an element of A if and only if it is an element of B.
   (c) P is an element of B if and only if it is an element of A.
   (d) P is an element of A only if it is an element of B.
   (e) If P is not an element of A, then it is not an element of B.
2. Which one of the following is NOT equal to $\frac{4}{9}$?
   (a) $\frac{1}{3} + \frac{1}{9}$; (c) $\left(\frac{2}{3}\right)^2$; (e) $\frac{7}{9} - \frac{1}{3}$.
   (b) $\frac{1}{3} \div \frac{3}{4}$; (d) $2 \div \frac{2}{9}$.
3. Which of the following is an equation with integral coefficients that has the same solution as
   $\frac{3}{14}x = \frac{4}{21}$?
   (a) $6x = 12$; (c) $8x = 6$; (e) $9x = 8$.
   (b) $7x = 12$; (d) $8x = 9$;
4. Which of the following properties of zero is the basis for excluding "division by zero" ?
   (a) $a + 0 = a$ for every integer a.
   (b) $a + (-a) = 0$ for every integer a.
   (c) $a \cdot 0 = 0$ for every integer a.
   (d) 0 is its own additive inverse.
   (e) 0 is the additive identity.
5. If $a < 0$ and $0 < b$, then $2\sqrt{b} + \sqrt{a^2 b}$ equals
   (a) $(2 + a)\sqrt{b}$;

(b) $(2 - a)\sqrt{b}$; (d) $\sqrt{(4 + a^2)b}$;

(c) $(2 + a)b$; (e) $\sqrt{(4 - a^2)b}$.

6. If p, q, r and s are arbitrary natural numbers, four of the following expressions have the same value. Which one has a <u>different</u> value?
   (a) $pq + rs$; (d) $qp + sr$;
   (b) $sr + pq$; (e) $rs + qp$.
   (c) $pr + sq$;
7. Which one of the following is <u>NOT</u> true if p, q, and r are integers such that $p < q < r < 0$?
   (a) $p + r < q + r$; (d) $0 < pr$;
   (b) $pq < rq$; (e) $qr < pr$.
   (c) $0 < r - p$;
8. What is the smallest positive integer n such that $n\left(\frac{1}{3} - \frac{1}{5}\right) > \frac{11}{10}$?
   (a) 6; (b) 8; (c) 9; (d) 10; (e) 15.
9. Which one of the following sets of numbers is <u>NOT</u> closed under the "operation" of squaring?
   (a) {2, 4, 8, 16, 32, ...}.
   (b) {-1, 0, 1}.
   (c) {-2, 4, -6, 8, -10, ...}.
   (d) {-1, 2, -3, 4, -5, ...}.
   (e) {1, 3, 5, 7, 9, ...}.
10. If a, b, c, and d are arbitrary integers such that $0 < a < b < c < d$, which of the following is a correct conclusion?
   (a) $\frac{a}{b} < \frac{c}{d}$; (c) $\frac{a}{d} < \frac{b}{c}$;
   (b) $\frac{a}{c} < \frac{b}{d}$; (d) $\frac{a}{b + c} < \frac{b}{b + d}$.

B. Short answer items.

1. Express each of the following rational numbers as the quotient of two integers:

   (a) $1 + \frac{1}{3} + \frac{3}{2}$; (b) $\dfrac{\frac{3}{2}}{\frac{1}{3}}$

2. Find the integer k such that the rational number $\frac{k-3}{k}$ is equal to the integer 2.

3. For what integer k will the pair of rational numbers $\frac{k+1}{k-1}, \frac{k}{k-3}$ be equal?

4. Express the reciprocal of $6 - 2\sqrt{5}$ in the form $A + B\sqrt{5}$ where A and B are rational numbers in simplest form.

5. Identify the following numbers as being rational or irrational,

   $\sqrt[3]{-8}$, $\sqrt{3}$, $(-2)^4$, $0.\overline{9}$, $3.\overline{14}$, 2.121121112 . . .,
   $\sqrt{\frac{16}{3}}$, $(\sqrt{2})^4$, $\sqrt{(-2)^2}$, $\sqrt{\frac{9}{4}}$

6. Which of the following properties of the system of integers are NOT also properties of the system of natural numbers?

   (a) The existence of a multiplicative identity.
   (b) The existence of an additive identity.
   (c) Closure under multiplication.
   (d) The existence of a solution of $a + x = b$ for all a and b in the system.
   (e) If $a < b$ there exists an element c in the system such that $a + c = b$.
   (f) If $a \leq b$, then $a + c \leq b + c$.

7. State whether or not each of the following sets of numbers is closed under the operation specified. If it is not closed, give a counter-example to illustrate this.

   (a) Natural numbers under division.

(b) Odd integers under subtraction;
(c) Irrational numbers under squaring;
(d) Rational numbers under multiplication;
(e) Rational numbers of the form $\frac{a}{b}$, where a and b are natural numbers and $a < b$, under multiplication.

8. Any odd integer can be expressed in the form $2n + 1$, where n is an integer. Using this definition, prove that
   (a) the set of odd integers is closed under multiplication,
   (b) the set of odd integers is not closed under addition.

9. (a) Show that the subset of integers consisting of all integral multiples of 3 is closed under subtraction and multiplication.
   (b) Give a counter-example to show it is not closed under division.

10. Find the subset of the set
    $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
    which satisfy the conditions

$$|x - 7| \leq 6 \quad \text{and} \quad |x + 1| < 5.$$

<u>Answers to the Illustrative Test Questions</u>

A. 1. (d). 2. (d). 3. (e). 4. (c).
5. (b). Note that since $\sqrt{a^2} = |a|$, then for $a < 0$, $\sqrt{a^2} = -a$. 6. (c) 7. (b). 8. (c). 9. (d). 10. (c).

B. 1. (a) $\frac{17}{6}$; (b) $\frac{9}{2}$. 2. -3. 3. -3. 4. $\frac{3}{8} + \frac{1}{8}\sqrt{5}$.
5. Rational; $\sqrt[3]{-8}$, $(-2)^4$, $0.\overline{9}$, $3.\overline{14}$, $(\sqrt{2})^4$, $\sqrt{(-2)^2}$, $\sqrt{\frac{9}{4}}$. Irrational; $2.121121112\ldots$, $\sqrt{\frac{16}{3}}$, $\sqrt{3}$.
6. (b), (d). 7. (a) Not closed; $\frac{3}{2}$. (b) Not closed; $3 - 1 = 2$. (c) Not closed; $(\sqrt{2})^2 = 2$. (d) Closed.

8. (a) Let $2a + 1$ and $2b + 1$ represent odd numbers, where a,b in I. Then
$(2a+1)(2b+1) = 4ab + 2(a+b) + 1 = 2[2ab+(a+b)] + 1.$
Since $(2ab + a + b)$ is in I,
then $2[2ab+(a+b)] + 1 = (2a+1)(2b+1)$ is odd.

(b) $(2a+1) + (2b+1) = 2(a+b) + 2 = 2[(a+b) = 1].$
Since $(a + b = 1)$ in I,
Then $2[(a+b) + 1] = (2a+1) + (2b+1)$ is even.

9. (a) Let $3a$ and $3b$ represent integral multiples of 3; a, b in I. Then $3a - 3b = 3(a - b)$.
Since $a - b$ in I, then $3(a - b) = 3a - 3b$ is an integral multiple of 3.
Since $3a(3b) = 3(3ab)$, where $3ab$ in I,
Then $3(3ab) = 3a(3b)$ is an integral multiple of 3.

(b) $\frac{6}{3} = 2$.

10. $\{1, 2. 3\}$.

Commentary for Teachers

Chapter 2

AN INTRODUCTION TO COORDINATE GEOMETRY IN THE PLANE

2-0. Introduction.

Chapter 2 is only an introduction to analytic or coordinate geometry in the plane. It is important that the teacher keep this in mind and that he make this clear to his students. The really important idea of this chapter is that we have two ways of looking at the same situation, a geometric view and an algebraic one. This comes by setting up a one-to-one correspondence between points in the plane and ordered pairs of real numbers. We then look at whichever formulation happens to be more convenient for our purpose in a particular problem. As the text says on the first page, the method gives us a way of solving geometric problems algebraically. The advantage of the new method is that we have quite a formidable array of algebraic rules and techniques already available and ready to apply to a geometric problem, once it has been translated from its geometrical form into the language of algebra.

The new coordinate geometry is then useful primarily in solving two kinds of problems.

(1) Given an algebraic relation, find the set of points whose coordinates satisfy the relation. In most of the problems we will meet in this course this relation is given by an algebraic equation (occasionally by an inequality).

(2) Given a geometric condition find an algebraic relation which must be satisfied by the coordinates of all points satisfying the geometric condition. If we keep in mind that these two problems are the central ones, then the details which are important and which must be studied carefully, can be kept in the proper perspective.

In addition to solving these problems, the analytic method is a powerful tool in constructing proofs of geometric theorems.

Its great advantage here is that in general it offers a straightforward method for proving the theorem, which relies on the machinery of algebra rather than ingenuity, as is so often the case with proofs in synthetic geometry.

The subsequent work in analytic geometry in Chapters 6, 7 and 8 develops systematically the relationship between geometric and algebraic concepts. In particular, the correspondence between lines in the plane and linear equations in two unknowns, parabolas (and more generally conic sections) and quadratic equations in two unknowns, planes and linear equations in three unknowns is developed and exploited in these chapters.

A notion that needs emphasis appears in several exercises and proofs in this section, i.e. the absolute value of a number. Repetition in all possible ways of the idea that the absolute value of a is a only if a is non-negative and that $|a| = -a$ if a is negative may be helpful.

While some of the material of this chapter is discussed in grades 9 and 10, this is the beginning of a systematic development of analytic geometry. It is important, therefore, that the fundamental idea of the one-to-one correspondence between points in the plane and pairs of real numbers be made clear. The one-to-one property of the correspondence is useful for most purposes; however it is worth noting that other coordinate systems are possible. At a later stage polar coordinates are introduced. This system has definite advantages in representing certain types of curves and in describing certain kinds of geometric conditions. In this system the one-to-one property is sacrificed for other desirable properties. In solid analytic geometry besides the obvious extension of rectangular or cartesian coordinates, spherical and cylindrical coordinates are useful. In many branches of physics still more general curvilinear coordinates have been found useful. In fact the study of some branches of physics would be difficult if not impossible without the framework of these more general coordinate systems. (See Bell's THE DEVELOPMENT OF MATHEMATICS, p.521 ff.)

[pages 119-120]

Exercises 2-1. Answers.

1. -
2. A(2,1); B(1,6); C(-2,6); D(-5,1); E(-5,-6); F(-1,-1); G(1,-4); H(5,-5).
3. (3,3)
4. (a) (-4,-4).

   (b) (-4,4), (4,-4).

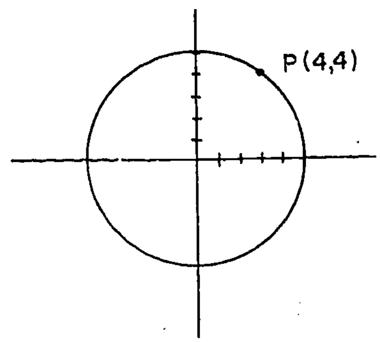
   (c) $(4\sqrt{2},0)(0,4\sqrt{2})$, $(-4\sqrt{2},0)$, $(0,-4\sqrt{2})$.

   From inspecting the graph, the students may approximate the coordinates. To 3 decimal places they are $(\pm 5.656,0)$, $(0,\pm 5.656)$.

   (d) (1) The points are located outside the circle.

   (2) The points are located inside the circle.

   (3) The points are those located outside the circle and those on the circle.



*5. (a) (-x,-y)

   (b) $(-x_1,y_1)$, $(x_1,-y_1)$

6. (a) y = 2, 8, -4, 0. The number pairs will be (2,2), (8,8), (-4,-4) and (0,0).

   (b) y = x.

7. (a) (-6,6), (-6,-6), (6,-6).

   (b) (6,0), (0,6), (-6,0), (0,-6).

   (c) $\sqrt{288}$ or $12\sqrt{2}$ units (by the Pythagorean theorem.)

8. AC = 6.

   BC = 6

   AB = $6\sqrt{2}$ (by the Pythagorean theorem.)

   Area = 18

9. $A'(-6,-8)$

$d(A,A') = d(A,O) + d(O,A')$

$d(A,O) = \sqrt{6^2 + 8^2} = 10$, Pythagorean theorem.

Since $d(A,O) = d(O,A')$

Then $d(A,A') = 10 + 10 = 20$

10. (a) $A'(0,5)$

(b) $B'(6,0)$

(c) $M(6,5)$

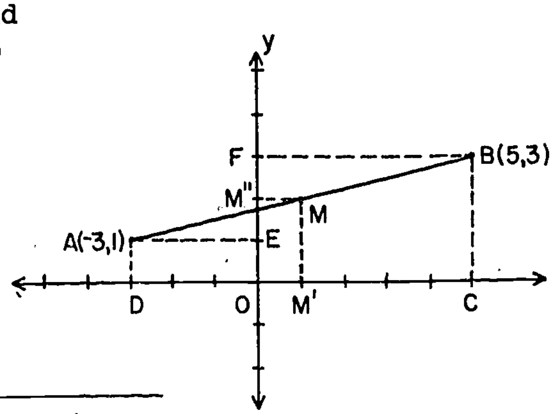
*11. (a) $D(-3,0)$

(b) $C(5,0)$

(c) $DC = 8$

(d) $M'(1,0)$. $DM' = M'C = \frac{1}{2} DC = 4$ because of segments intercepted by parallel lines theorem in geometry $DC = 8$, $OC = 5$, then $OM' = 1$. Hence the coordinates of M' are $(1,0)$.

(e) $M(1,2)$. Project A and B on the y-axis at E and F respectively. $EF = 2$. Since $\overline{MM''}$ bisects $\overline{EF}$, the coordinates of M'' are $(0,2)$. Hence the coordinates of M are $(1,2)$.



2-2. The Distance Between Two Points.

In section 2-2 the use of subscripts on the coordinates to denote particular points sometimes causes difficulty. The convenient tradition that $P(x,y)$ represents a variable point, while $P_0(x_0,y_0)$, $P_1(x_1,y_1)$, etc. represent particular points, which may have a certain kind of generality in a particular problem, needs a good deal of explaining and emphasizing. For instance, if we say let $P(x,y)$ be any point on a circle with radius r and center at $C(h,k)$, then the equation for the circle is $(x - h)^2 + (y - k)^2 = r^2$. If we wish to state that a particular point $P_o(x_o,y_o)$ is on the circle, this is clearly indicated by writing $(x_o - h)^2 + (y_o - k)^2 = r^2$. $P(x,y)$ is the variable point. For appropriate values of x and y it can stand for any point on the circle. $P_o(x_o,y_o)$ on the other hand is a fixed point. We still may mean any point on the circle, but the subscripts tell us that in this particular discussion P_o is fixed (whatever point on the circle it may be) and does not change.

The real point of Section 2-2 of course is to establish the distance formula, one of the fundamental tools of coordinate geometry. Many teachers find that the proof goes more easily if they first illustrate the idea of the proof with specific line segments in the first quadrant, say the segment $P_1(1,2)P_2(4,6)$. Then take general coordinates, drawing the picture in quadrant I. This is usually convincing enough so that drawing a diagram with P_1 and P_2 in various positions in other quadrants offers no great difficulty. It may trouble some students that $d(P_1,R) = |x_2 - x_1|$ regardless of the signs of x_1 and x_2. This is a rather important point and will be used in many other problems. Repeated numerical examples should help to get the point across. Another idea which needs emphasizing is that the distance $d(P_1,P_2)$ is always non-negative. We have defined the distance <u>between</u> two points, which is not a directed distance. Hence $d(P_1,P_2) = d(P_2,P_1)$.

After the midpoint formula, an excellent exercise for a good class (or an especially good student) would be to try to get the class to guess at a formula for the coordinates of a point which divides a line segment in a given ratio; that is $\frac{d(P_1,P)}{d(P,P_2)} = \frac{r_1}{r_2}$. It can be pointed out that the midpoint divides the segment in the ration $\frac{1}{1}$. Then ask what the coordinates would be if the point was 1/3 of the way between P_1 and P_2, etc. The general formula is

$$x = \frac{r_2x_1 + r_1x_2}{r_1 + r_2}, \quad y = \frac{r_2y_1 + r_1y_2}{r_1 + r_2}.$$

This ability to generalize is one of the marks of a potential mathematician. The practice of making conjectures, testing them, and then proving them if they seem to be true, should be encouraged whenever possible. While the midpoint formula is a useful bit of information, it is not a central result in analytic geometry and it is not used nearly as frequently and with as significant results as is the distance formula. Both of these results are tools -- important ones -- but still tools to be used in solving problems and in proving other results.

Exercises 2-2. Answers.

1. (a) $5\sqrt{5}$ (b) $2\sqrt{26}$
2. $C(1,3)$
3. $AB = 5$, $BC = 2\sqrt{85}$, $AC = 17$ and the perimeter $= 22 + 2\sqrt{85}$.
4. 39
5. $M(\frac{3}{2}, -\frac{13}{2})$

   Length of $P_1M = \frac{3}{2}\sqrt{2}$

   Length of $P_1P_2 = 3\sqrt{2}$.
6. $d(P,R) = 13$

   $d(Q,S) = 17$
7. $d(A,B) = 5$

   $d(A,C) = \sqrt{50} = 5\sqrt{2}$.

   $d(B,C) = 5$

   Since $d(A,B) = d(B,C)$ the triangle is isosceles.
8. The length of the radius is 13.

   No, since the distance between the points $(0,0)$ and $(4,-3)$ is 5.
9. $B(4,-6)$
10. Assign to C the coordinates (x,y).

    Then $d(A,C) = d(B,C)$ fulfills the gometric condition

$$\sqrt{(x + 1)^2 + y^2} = \sqrt{(x + 1)^2 + (y - 5)^2}$$
$$y^2 = y^2 - 10y + 25$$
$$y = 2.5$$

[pages 124-129]

10. cont. Hence, d(A,C) = d(B,C) if C is (x,2.5) for any real number x. However if x is -1, A, B, C do not form a triangle.

11. $d(P,O) = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2}$
$= \sqrt{x_1^2 + y_1^2}$

12. M(3,1); N(4,-2); P(2,0)
$d(M,N) = \sqrt{10}$; $d(N,P) = 2\sqrt{2}$; $d(M,P) = \sqrt{2}$
perimeter $\triangle$ MNP = $3\sqrt{2} + \sqrt{10}$
Here the students may recall from geometry that the length of the line segment joining the midpoints of two sides of the triangle is equal to $\frac{1}{2}$ the length of the other side. So, the perimeter of $\triangle$ ABC is twice that of $\triangle$ MNP. Otherwise, the student will use the distance formula to find the length of each side of $\triangle$ ABC in order to find its perimeter in order to compare the perimeters.
$d(A,B) = 4\sqrt{2}$
$d(A,C) = 2\sqrt{10}$
$d(B,C) = 2\sqrt{2}$
Hence, the perimeter of ABC = $2(3\sqrt{2} + \sqrt{10})$

13. In this set of exercises the student is expected to use the distance formula to show that d(A,B) + d(B,C) = d(A,C).
$d(A,B) = \sqrt{61}$
$d(B,C) = 2\sqrt{61}$
$d(A,C) = 3\sqrt{61}$
In case he remembers the slope concept from the 9th grade, he may find the slope of each segment and then compare them, as,
slope (A,B) = $\frac{6}{5}$
slope (B,C) = $\frac{6}{5}$
slope (A,C) = $\frac{6}{5}$
Or, after section 2-3, where slope has been discussed, you might suggest that this concept now be used to solve this problem.

14. $(\frac{3x}{2}, \frac{3y}{2})$

$d(P_1, P_2) = \sqrt{x_1^2 + y_1^2}$

15. (b) $d(N,O) = 2\sqrt{5}$ $d(P,M) = 2\sqrt{5}$

$d(O,P) = 2\sqrt{5}$ $d(M,N) = 2\sqrt{5}$

perimeter $= 8\sqrt{5}$

(c) Since $d(N,O) = d(P,M)$ and $d(O,P) = d(N,M)$, the quadrilateral MNOP is a parallelogram. Furthermore, since $d(N,O) = d(P,M) = d(O,P) = d(N,M)$, the quadrilateral is a rhombus.

*16. Four different solutions are possible.

They are $(a + c,b)$, $(a + c,b + c)$, $(a,b + c)$;

$(a + c,b)$, $(a + c,b - c)$, $(a,b - c)$;

$(a - c,b)$, $(a - c,b + c)$, $(a,b + c)$;

$(a - c,b)$, $(a - c,b - c)$, $(a,b - c)$.

The midpoints are

$(\frac{2a + c}{2}, b)$, $(a + c, \frac{2b + c}{2})$, $(\frac{2a + c}{2}, b + c)$, $(a, \frac{2b + c}{2})$;

$(\frac{2a + c}{2}, b)$, $(a + c, \frac{2b - c}{2})$, $(\frac{2a + c}{2}, b - c)$, $(a, \frac{2b - c}{2})$;

$(\frac{2a - c}{2}, b)$, $(a - c, \frac{2b + c}{2})$, $(\frac{2a - c}{2}, b + c)$, $(a, \frac{2b + c}{2})$;

$(\frac{2a - c}{2}, b)$, $(a - c, \frac{2b - c}{2})$, $(\frac{2a - c}{2}, b - c)$, $(a, \frac{2b - c}{2})$.

*17. $d(A,B) = 2\sqrt{2}$

$d(B,C) = \sqrt{2}$

$d(A,C) = 3\sqrt{2}$

Since $d(A,B) + d(B,C) = d(A,C)$, the points A, B, and C are collinear.

Here is an opportunity to suggest that the students investigate for collinearity such sets of points:

(a) $A(1, 2 + h)$; $B(3, 6 + h)$; $C(5, 10 + h)$.

(b) $A(1, -2 + h)$; $B(5, -10 + h)$; $C(4, -8 + h)$.

Others may be given by the students.

2-3. <u>The Slope of a Line</u>.

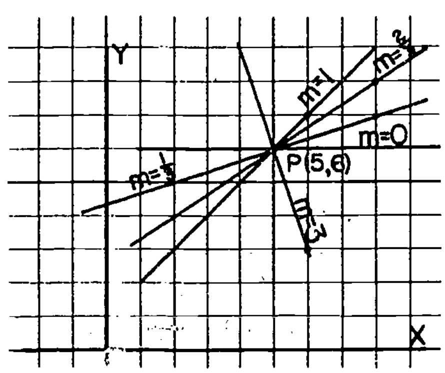
The formula for the slope will be review for students who have been through either the SMSG First Course in Algebra or the SMSG Geometry Course. It is introduced prematurely here in order to have the idea available for use in Chapters 3, 4 and 5 before a systematic study of the first degree equation is attempted in Chapter 6. The descriptive terminology "rise" and "run" was purposely omitted. If the teacher feels this is natural, helpful, or meaningful, its use is hallowed by tradition. The point that the slope is independent of the points P_1 and P_2 on the line should be stressed; that is,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} .$$

The theorems on slope and parallel and perpendicular lines are extremely useful. The proofs may be omitted in favor of informal arguments with numerical examples. However the proofs in the text are novel and well worth the good student's time and effort.

<u>Exercises 2-3</u>. <u>Answers</u>.

1. (a) $m = -\frac{3}{4}$
   (b) $m = 2$
   (c) $m = \frac{1}{2}$
   (d) Slope is undefined.
   (e) $m = \frac{3}{5}$
   (f) $m = 0$
2. (a)



2. (b) The line having slope -3, since the magnitude of the steepness of a line is measured by the absolute value of its slope.

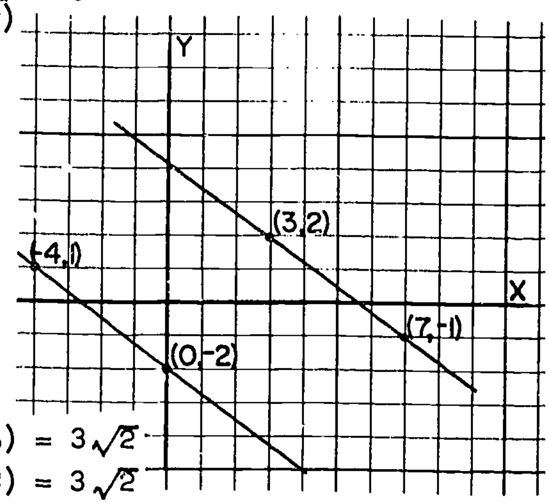
(c) They become steeper.

(d) They are perpendicular, since $\frac{1}{3}(-3) = -1$.

3. (b) Slope of line through (3,2) and (7,1) is $-\frac{3}{4}$.

Slope of line through (-4,1) and (0,-2) is $-\frac{3}{4}$.

(c) They are parallel.



4. (a) m(A,B) = 1   d(A,B) = $3\sqrt{2}$

m(B,C) = -1   d(B,C) = $3\sqrt{2}$

m(A,C) is undefined   d(A,C) = 6

(b) Δ ABC is a right triangle with right angle B, since the product of m(A,B) and m(B,C) equals -1.

(c) Midpoint of $\overline{AB}$ is (4.5,3.5); of $\overline{BC}$ is (4.5,6.5); of $\overline{AC}$ is (3,5).

(d) $m(M_2, M_3) = 1$

$m(M_1, M_3) = -1$

$m(M_1, M_2)$ is undefined.

5. $m(A,B) = \frac{6}{5}$

$m(B,C) = \frac{6}{5}$

$m(A,C) = \frac{6}{5}$

Hence the points A, B, and C lie on a straight line.

6. $m(A,B) = -\frac{2}{1 - b}$ $[b \neq 1]$

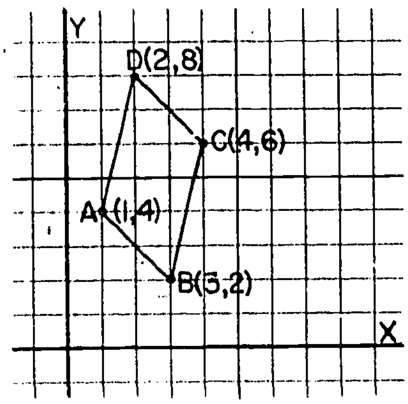
$m(B,C) = \frac{2}{3}$

If $-\frac{2}{1 - b} = \frac{2}{3}$, then $b = 4$

7. (a) $\frac{2 - 0}{p - 1} = \frac{3 - 1}{2 + 2}$; p

(b) $\frac{2}{p - 1} \cdot \frac{1}{2} = -1$; p

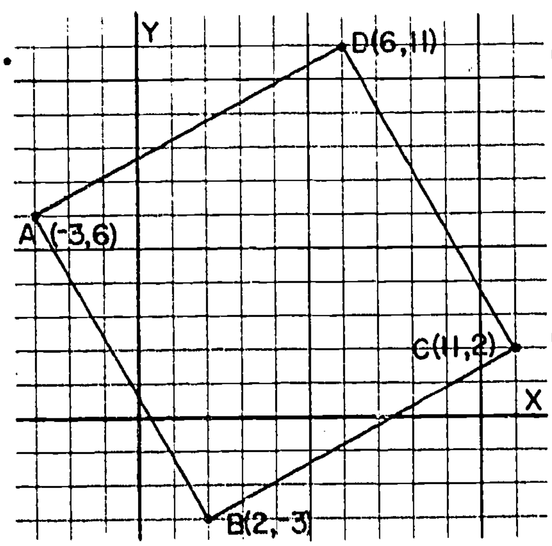
8.



(a) $m(A,B) = -1$ $m(A,D) = 4$
$m(D,C) = -1$ $m(B,C) = 4$
Since the opposite sides have the equal slopes, they are parallel and ABCD is a parallelogram.

(b) No, since the slope of any one side is not the negative reciprocal of an adjoining side.

9.



(a) $m(A,B) = -\frac{9}{5}$ $m(B,C) = \frac{5}{9}$
$m(C,D) = -\frac{9}{5}$ $m(A,D) = \frac{5}{9}$
Hence ABCD is a parallelogram. $d(A,B) = \sqrt{106}$, $d(B,C) = \sqrt{106}$. Hence the parallelogram is a rhombus.

(b) ABCD is also a square, since $m(A,B) \times m(B,C) = \frac{-9}{5} \cdot \frac{5}{9} = -1$.

10. $m(A,C) = 1$

$m(B,D) = -1$

Since $1 \cdot -1 = -1$, $\overline{AC} \perp \overline{BD}$.

11. (a) $\frac{p}{q}$

(b) $-\frac{q}{p}$

12. (a) $m(P_1,P_2) = \frac{a - b}{b - a} = -1$

(b) Slope of a line $\perp$ P_1P_2 is 1.

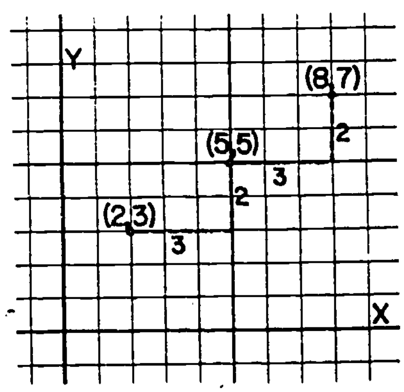
13. C is the vertex of the right angle because the slope of the line segments AC and BC are negative reciprocals, i.e. $m(A,C) = \frac{2}{21}$, $m(B,C) = -\frac{21}{2}$ and $\frac{2}{21}\left(\frac{-21}{2}\right) = -1$.

14. (a) (5,5), (8,7).

(b) Yes, Some students may find that the coordinates can be found by the following:

$x = 2 + 3n$

$y = 3 + 2n$



*15. $m(A,C) = \frac{b - (b + c)}{a - (a + c)} = 1$

$m(B,D) = \frac{(b + c) - b}{a - (a + c)} = -1$

Since $1 \times (-1) = -1$, $\overline{AC} \perp \overline{BD}$.

*16. $m(A,B) = \frac{(c + a) - (b + c)}{b - a} = -1$

$m(B,C) = \frac{(a + b) - (c + a)}{c - b} = -1$

$m(A,C) = \frac{(a + b) - (b + c)}{c - a} = -1$

Since the slopes of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are equal, the points A, B, and C are collinear.

*17. (a) $m(A,B) = 0$ which means that $\overline{AB}$ is parallel to the x-axis $m(B,C)$ is undefined which means that $\overline{BC}$ is parallel to the y-axis.

Hence $\overline{AB} \perp \overline{BC}$ and ΔABC is a right triangle.

(b) The coordinates are given by

$x = \frac{a + (a + c)}{2} = a + \frac{1}{2}c$

$y = \frac{b + (b + d)}{2} = b + \frac{1}{2}d$

$m(a + \frac{1}{2}c, b + \frac{1}{2}d)$

2-4. Sketching Graphs of Equations and Inequalities.

In Section 2-4 it helps to remember that this is an introduction to sketching graphs. Many of the curves treated here will be studied in detail later when more machinery can be brought to bear. In fact some curves can not be satisfactorily discussed until the derivative concept is available in a calculus course. So the aims here are:

(1) To give the students experience in plotting points on curves, and after they have a good deal of practice at this,

(2) To help them see that obtaining information about the inter[illegible]ymmetry of the curve can be more helpful in [illegible]ses than plotting great numbers of points.

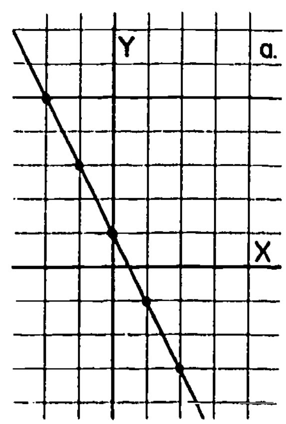
A discussion of asymptotes and "extent" (that is, values of x for which y is not real and vice versa) is omitted in this section in order to avoid confusing the central issues with useful techniques which occur for rather special curves. Asymptotes will be treated in Chapter 6 when the hyperbola is discussed. Extent will also be touched on in the discussion of the conic sections in that chapter.

Exercises 2-4. Answers.

1. Sample number pairs,

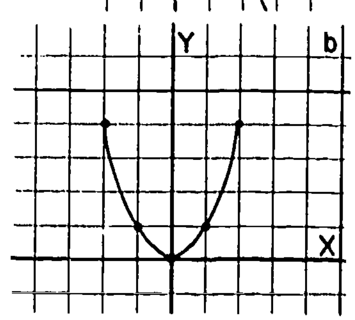
(a)

x	-2	-1	0	1	2
y	5	3	1	-1	-3



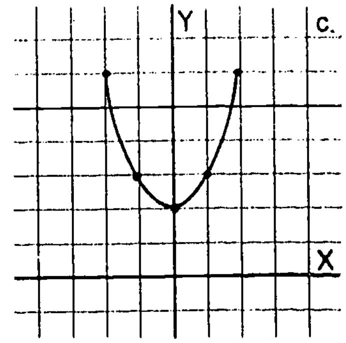
(b)

x	-2	-1	0	1	2
y	4	1	0	1	4



(c)

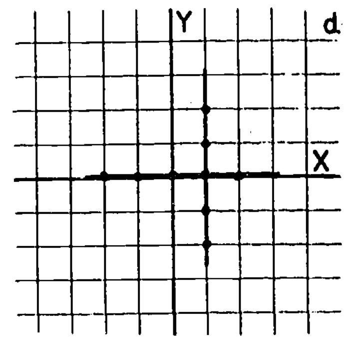
x	-2	-1	0	1	2
y	6	3	2	3	6



(d)

x	-2	-1	0	1	2
y		0	0	0, -1, -2, 1, 2 ...	0

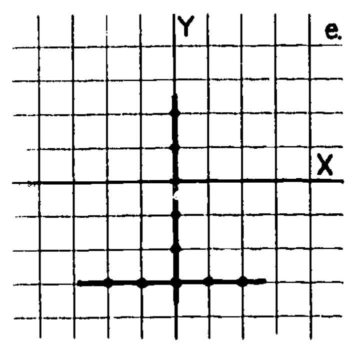
{(x,y): x - 1 = 0 or y = 0}



(e)

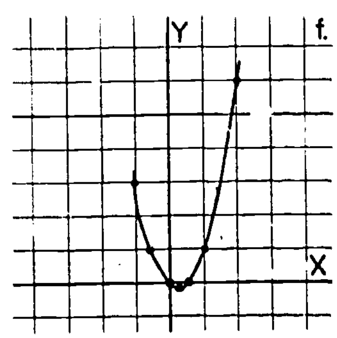
x	-2	-1	0	1	2
y	-3	-3	-3, -2, -1, 0, 1, 2, ...	[illegible]	-3

{(x,y): x = 0 or y = -3}



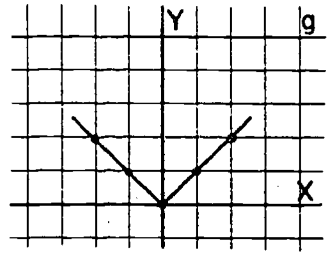
(f)

x	-1	-.5	0	.25	.5	1	2
y	3	1	0	-.125	0	1	6



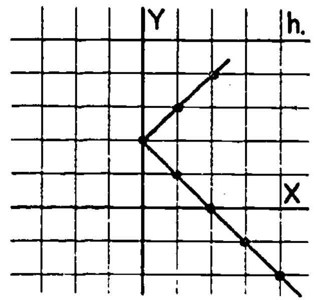
(g)

x	-2	-1	0	1	2
y	2	1	0	1	2



(h)

x	4	3	2	1	0	1	2
y	-2	-1	0	1	2	3	4



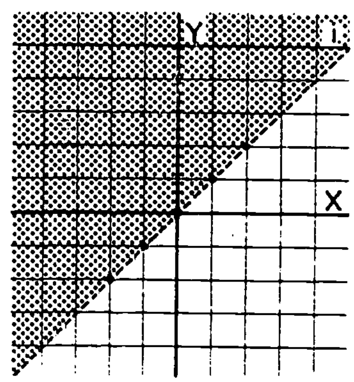
(i) Sample pairs for $y = x$ are:

x	-2	-1	0	1	2
y	-2	-1	0	1	2

Plot these number pairs. But, the number pairs that we need are $\{(x,y): y > x\}$. Some of them are

x	-2	[illegible]	2
y	-1	[illegible]	2.5

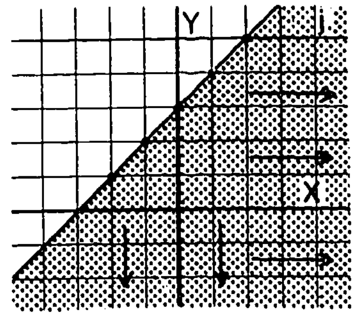
and are plotted in the shaded portion. None of these belong to $\{(x,y): y = x\}$, thus the line $y = x$ is not included.



(j) Some number pairs for $y = x + 3$ are:

x	-2	-1	0	1	2
y	1	2	3	4	5

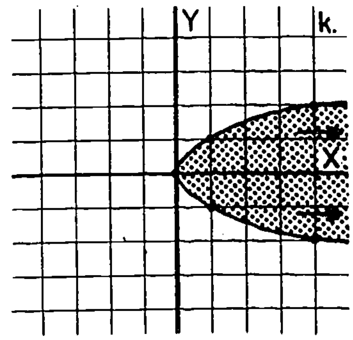
The graph is $\{(x,y): y = x + 3\} \cup \{(x,y): y < x + 3\}$ shown by the line <u>and</u> the region.



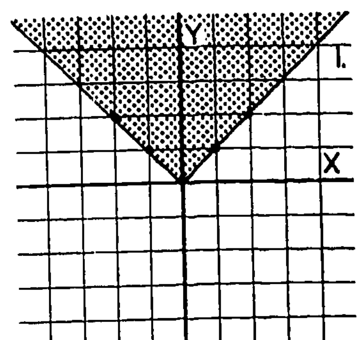
(k) Some number pairs for $x = y^2$ are,

x	4	1	0	1	4
y	-2	-1	0	1	2

The graph is $\{(x,y): x > y^2\}$, the shaded region.

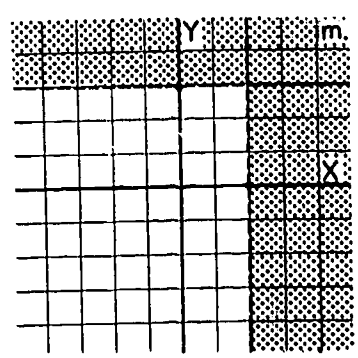


(l) Some number pairs for $y = |x|$ may be found in (g). The shaded region is the graph of $\{(x,y): y > |x|\}$.

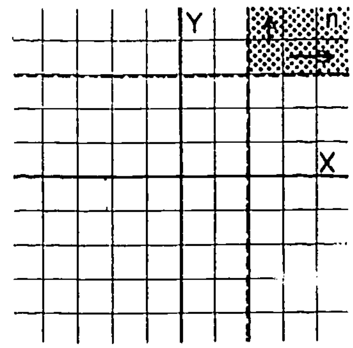


(m) The graph of $\{(x,y): x > 2 \text{ or } y > 3\}$ is the shaded region shown. This is $\{(x,y): x > 2\} \cup \{(x,y): y > 3\}$, i.e. all the points whose first coordinate is greater than 2 and all the points whose second coordinate is greater than 3.

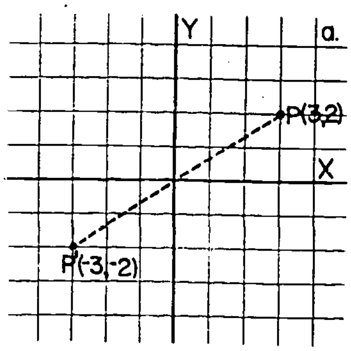
(m)



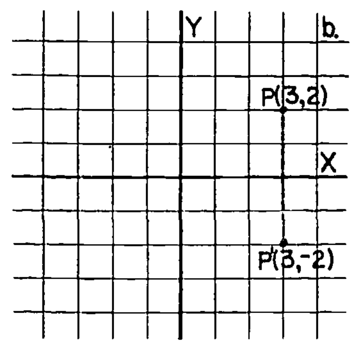
(n) The graph of $\{(x,y): x > 2 \text{ and } y > 3\}$ is the shaded region shown. This is $\{(v,y): x > 2\} \cap \{(x,y): y > 3\}$, i.e. all the points whose first coordinate is greater than 2 and whose second coordinate is greater than 3. Note the distinction between the problems m and n.



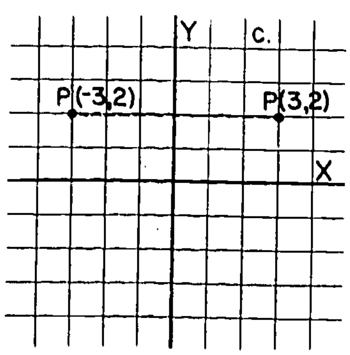
2. (a)



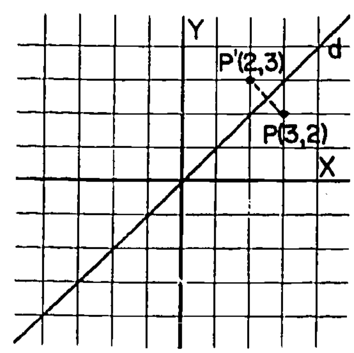
(b)



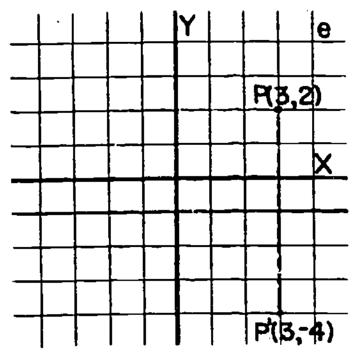
(c)



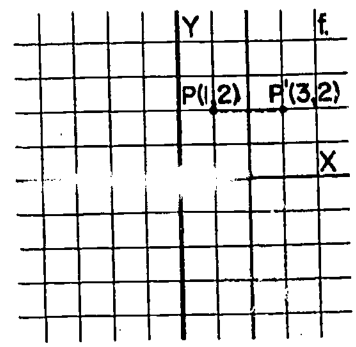
(d)



2. (e)



(f)



3.

	x-intercept(s)	y-intercept(s)
(a)	3	-6
(b)	±1	±1
(c)	0	0
(d)	1	±1
(e)	±2	-4
(f)	None	None
(g)	9	There is no real number whose square is a negative number.
(h)	0	0
(i)	There is no real number whose absolute is a negative number.	15
(j)	-3	-9

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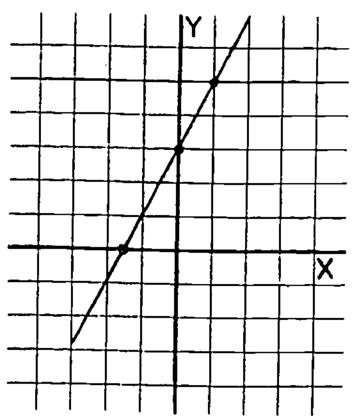
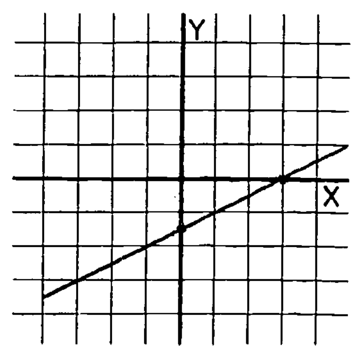
4. Symmetry with respect to:

	x-axis	y-axis	origin
(a)	Yes	Yes	Yes
(b)	No	Yes	No
(c)	No	No	No
(d)	No	No	Yes
(e)	No	No	No
(f)	Yes	Yes	Yes
(g)	Yes	No	No
(h)	Yes	Yes	Yes
(i)	No	Yes	No
(j)	No	Yes	No
(k)	No	No	Yes
(l)	No	Yes	No
(m)	No	No	Yes

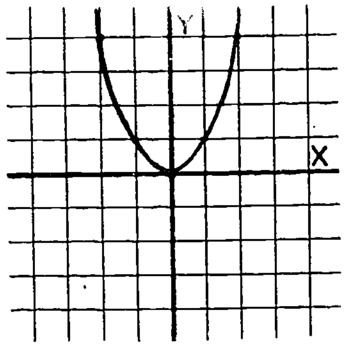
5. (a)



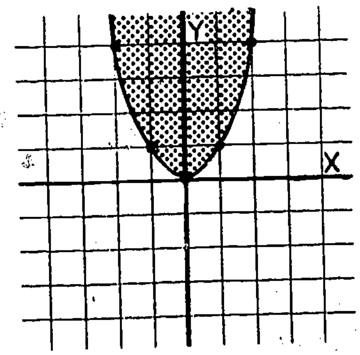
(b)



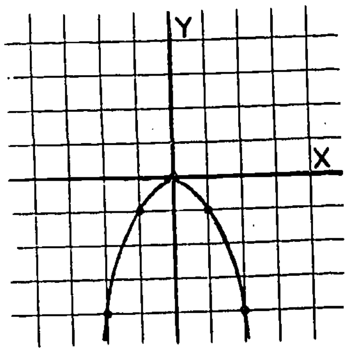
5. (c)



(d)



(e)



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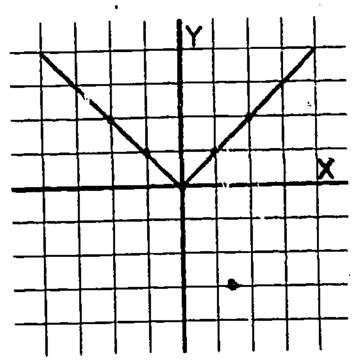
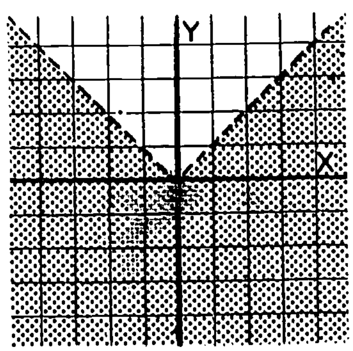
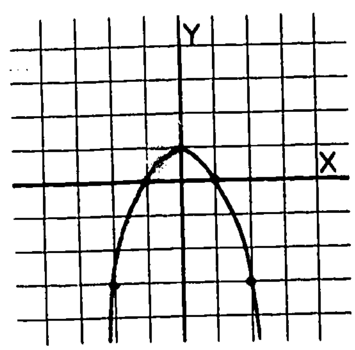


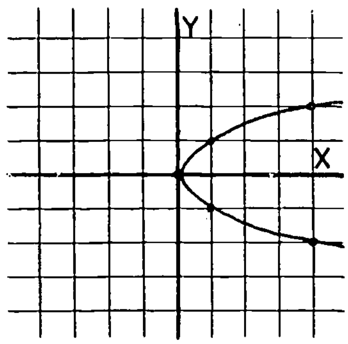
(g)



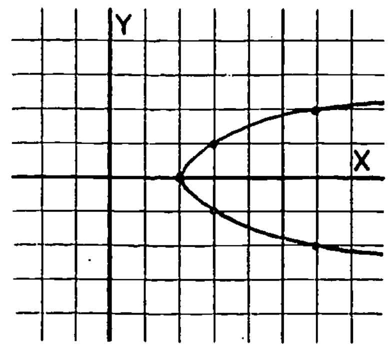
(h)



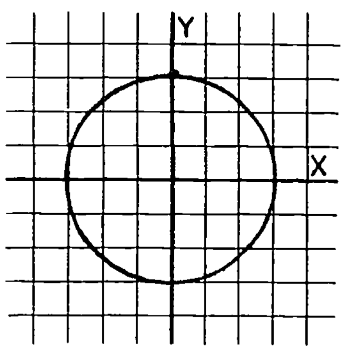
(i)



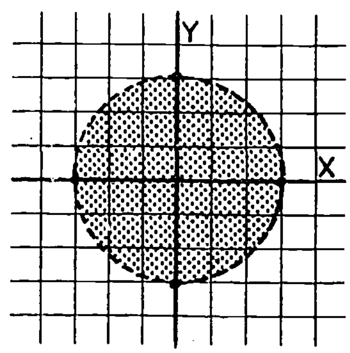
(j)



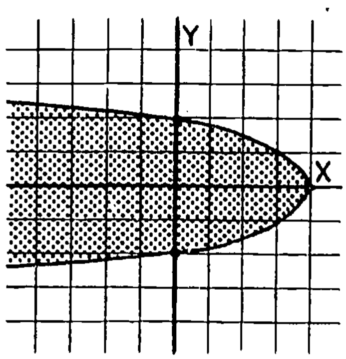
(k)



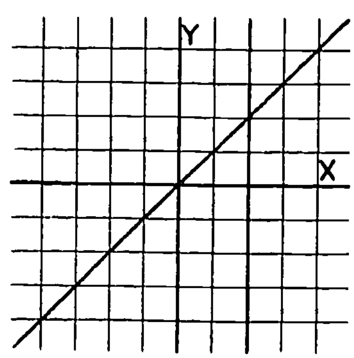
(l)



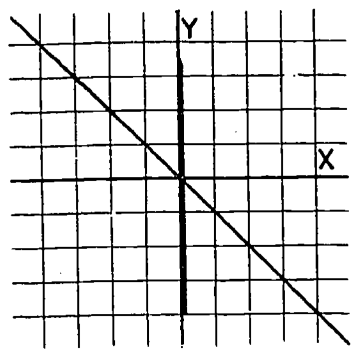
(m)



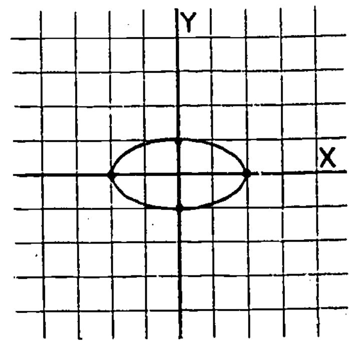
(n)



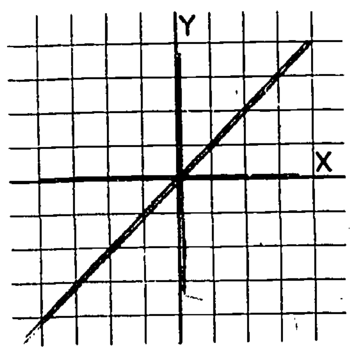
5. (o)



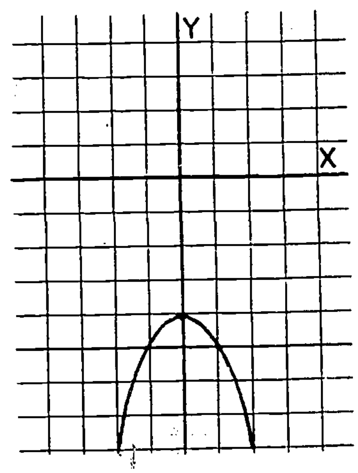
(p)



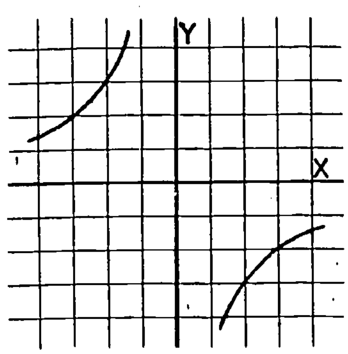
(q)



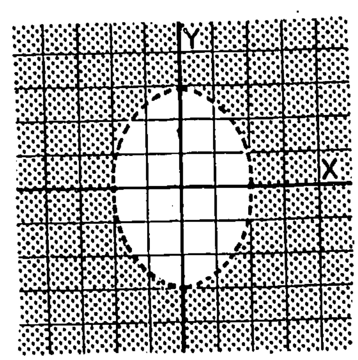
(r)



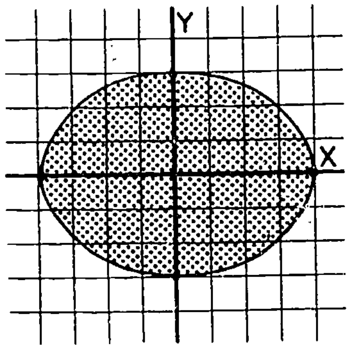
*(s)



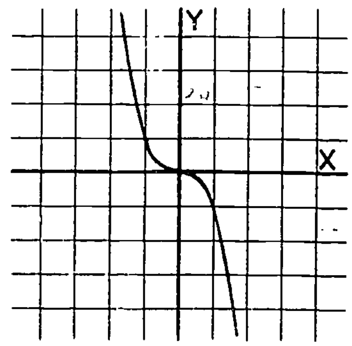
*(t)



5. *(u)



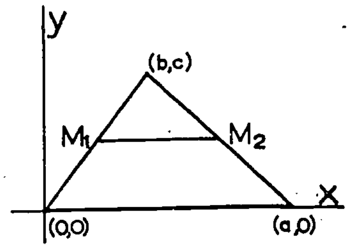
*(v)



*2-5. Analytic Proofs of Geometric Theorems.

In section *2-5 there are two principal difficulties which you are likely to encounter. One is that in assigning coordinates in the figure, it is easy for the student to assume properties equivalent to the ones he is trying to prove. The other is that care must be taken to avoid taking a figure or a position of a figure which results in proving a special case of the proposition rather than the proposition itself. There are no magic prescriptions for eliminating these difficulties, but the student needs to be constantly reminded of the need for avoiding both of these pitfalls.

*Exercises 2-5. Answers.



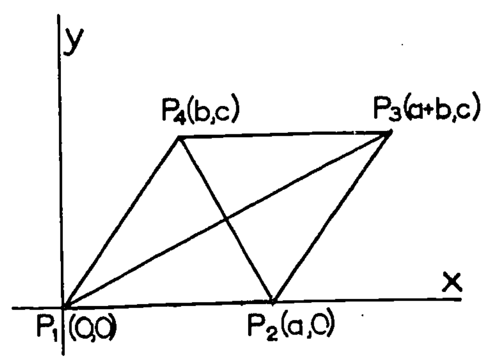
1. Given: A line connecting the midpoints of two sides of a triangle.

   Prove: The line is parallel to the third side of the triangle and equal to half of it.

1. $M_1(\frac{b}{2}, \frac{c}{2})$.	1. Midpoint formula.
2. $M_2(\frac{a+b}{2}, \frac{c}{2})$.	2. Midpoint formula.
3. Slope $M_1M_2 = \dfrac{\frac{c}{2} - \frac{c}{2}}{\frac{b}{2} - \frac{a+b}{2}} = 0$	3. Definition of slope.
4. $\overline{M_1M_2} \parallel \overline{OA}$.	4. Slope of x-axis is 0 and parallel lines have same slope.
5. $d(M_1, M_2) = \left\lvert\frac{a+b}{2} - \frac{b}{2}\right\rvert = \frac{a}{2}$.	5. Distance formula.
6. Length of base of triangle = a.	6. Distance formula.
7. Hence, $M_1M_2 = \frac{1}{2}(OA)$.	7. By steps 5 and 6.
8. $\therefore$ theorem is proved.	8. Steps 1 - 5.

2. Given: A parallelogram with diagonals perpendicular to each other.

   Prove: Parallelogram is a rhombus.



1. Slope $P_1P_3 = \frac{c}{a+b}$	1. Definition of slope.
2. Slope $P_2P_4 = \frac{c}{b-a}$	2. Definition of slope.

2\. 3. Slope $P_1P_3 = \frac{a - b}{c}$ — 3. Given $m_{d_1} = -\frac{1}{m_{d_2}} = \frac{a - b}{c}$. (Definition of perpendicular lines.)

4\. $\frac{a - b}{c} = \frac{c}{a + b}$ — 4. Steps 1 and 3.

5\. $a^2 = b^2 + c^2$ — 5. Pythagorean Theorem.

6\. $d(P_1P_4) = \sqrt{b^2 + c^2}$ — 6. Distance formula.

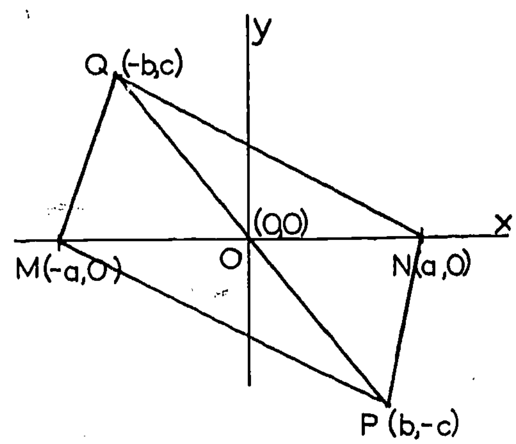
7\. $d(P_1P_4) = |a|$ — 7. Substitution, steps 5 and 6.

8\. $d(P_1P_2) = |a|$ — 8. Distance formula.

9\. $\therefore P_1P_2P_3P_4$ is a rhombus — 9. A parallelogram with adjacent sides equal is a rhombus.

3\. Given: A quadrilateral with diagonals MN and QP bisecting each other at O.

Prove: MPNQ is a parallelogram.



Proof: Choose the x and y axes so that the diagonal MN lies along the x-axis. Then the midpoint O of this diagonal will have (0, 0) as coordinates. If we let N have coordinates (a,0), then the coordinates of M must be (-a,0). We can name the coordinates of either P or Q arbitrarily, but when those for one of these points have been labeled those for the other are determined. If we label P as (b,-c), then Q has the coordinates (-b,c), since P is reflected in the origin.

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If MPNQ is a parallelogram, then QM = NP, and MP = QN.

But $QM = \sqrt{(-b + a)^2 + c^2}$,

and $NP = \sqrt{(b - a)^2 + a^2}$.

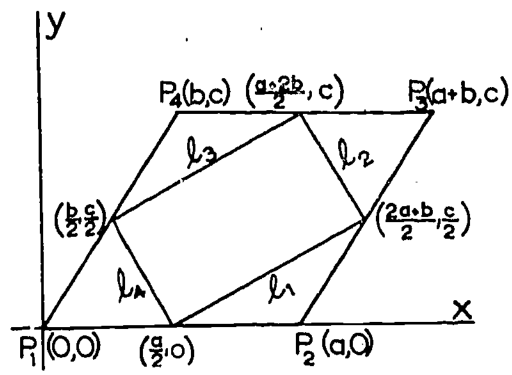
Hence QM = NP

In a similar manner it can be shown that MP = QN.

$\therefore$ MPNQ is a parallelogram.

4. Given: Rhombus $P_1P_2P_3P_4$ with midpoints of the sides.

Prove: Figure formed by joining the midpoints of the sides is a rectangle.



Proof:

1. Slope of $\ell_1 = \dfrac{\frac{c}{2}}{\frac{a+b}{2}} = \dfrac{c}{a+b}$	1. Slope formula.
2. Slope of $\ell_2 = \dfrac{c - \frac{c}{2}}{\frac{a+2b}{2} - \frac{2a+b}{2}} = \dfrac{c}{b-a}$	2. Slope formula.
3. Slope of $\ell_3 = \dfrac{c - \frac{c}{2}}{\frac{a+2b}{2} - \frac{b}{2}} = \dfrac{c}{a+b}$	3. Slope formula .
4. Slope of $\ell_4 = \dfrac{\frac{c}{2}}{\frac{b}{2} - \frac{a}{2}} = \dfrac{c}{b-a}$	4. Slope formula.
5. $\therefore$ $\ell_1 \ell_2 \ell_3 \ell_4$ forms a parallelogram.	5. A quadrilateral with opposite sides parallel is a parallelogram.
6. $a^2 = b^2 + c^2$	6. $d(P_1,P_2) = d(P_1,P_4)$ since $P_1P_2P_3P_4$ is given a rhombus.

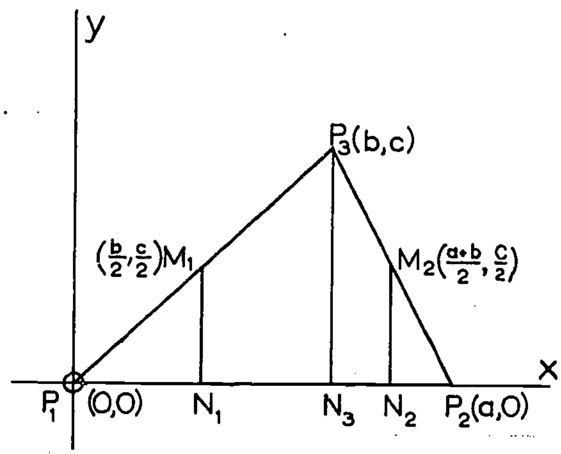
4. 7. (Slope of l_1)×(Slope of l_2) $= \frac{c}{a+b} \cdot \frac{c}{b-a} = -1$ — 7. By step 6.

8. $l_1 \perp l_2$ — 8. By Theorem 6-2.

9. ∴ $l_1 l_2 l_3 l_4$ is a rectangle. — 9. A parallelogram with a right angle is a rectangle.

5. Given: M_1 and M_2 are the midpoints of two sides of a triangle, and perpendiculars are drawn to the third side from M_1 and M_2.

Prove: The sum of the lengths of these perpendiculars equals the length of the altitude drawn from P_3 to P_1P_2.

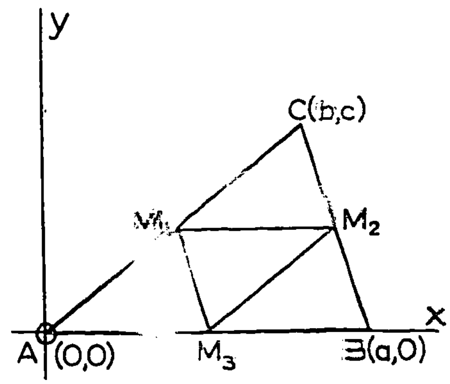


Proof:

1. Coordinates of M_1 are $(\frac{b}{2}, \frac{c}{2})$. Coordinates of M_2 are $(\frac{a+b}{2}, \frac{c}{2})$.	1. Midpoint formula.
2. $d(M_1,N_1) = d(M_2N_2) = \frac{\lvert c \rvert}{2} = \frac{1}{2} d(P_3N_3)$.	2. Distance formula.
3. ∴ Theorem is proved.	3. Steps 1 and 2.

6. Given: M_1, M_2, M_3, the mid-points of the sides of the triangle ABC

Prove: The four triangles formed are congruent.



Proof:

1. $d(A,M_3) = d(M_3,B) = \frac{1}{2} d(A,B) = \frac{|a|}{2}$. — 1. Distance formula and M_3 is a midpoint.

2. $d(B,M_2) = d(M_2,C) = \frac{1}{2} d(B,C) = \frac{\sqrt{c^2 + (b-a)^2}}{2}$ — 2. Distance formula.

3. $d(A,M_1) = d(M_1,C) = \frac{1}{2} d(A,C) = \frac{\sqrt{b^2 + c^2}}{2}$ — 3. Distance formula.

4. $M_1M_2 = AM_3 = M_3B$ — 4. No.1, Example 2-5.

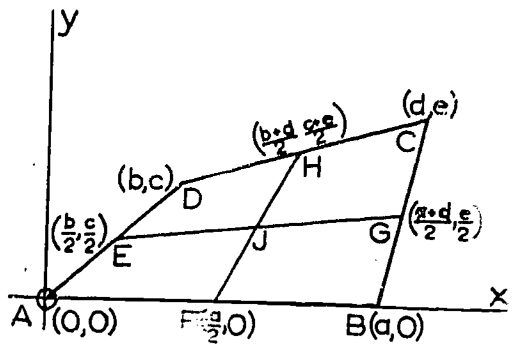
   $M_1M_3 = CM_2 = M_2B$

   $M_3M_2 = CM_1 = M_1A$

5. ∴ the four triangles are congruent. — 5. S.S.S. = S.S.S.

7. Given: The lines connecting the midpoints of the opposite sides of a quadrilateral.

Prove: These lines bisect each other.



Proof:

1. F is $(\frac{a}{2}, 0)$ — 1. Midpoint formula.

   G is $(\frac{a + d}{2}, \frac{e}{2})$

   H is $(\frac{b + d}{2}, \frac{c + e}{2})$

   E is $(\frac{b}{2}, \frac{c}{2})$

2. The midpoint of EG is $(\frac{a + b + d}{4}, \frac{c + e}{4})$. — 2. Midpoint formula.

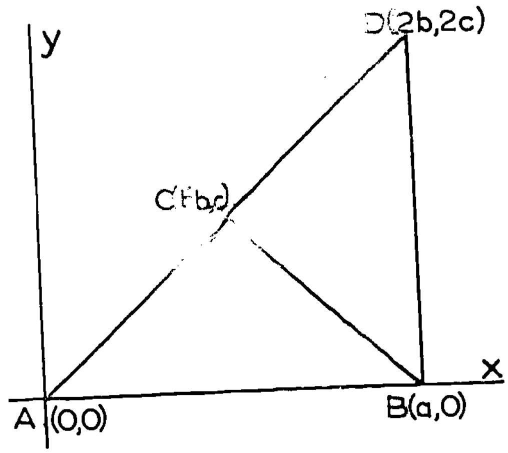
   The midpoint of FH is $(\frac{a + b + d}{4}, \frac{c + e}{4})$.

3. ∴ EG and FH bisect each other — 3. Because the coordinates of the midpoints are the same.

8. Given: d(A,C) = d(C,B) = d(C,D).

Prove: BD is perpendicular to AB.

Select the coordinates of A, B, and C as indicated in the drawing.

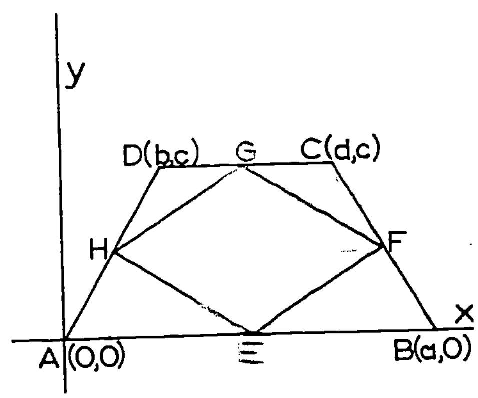


Proof:

1. Coordinates of D are (2b,2c). — 1. Midpoint formula.
2. d(A,C) = d(C,B)
   $b^2 + c^2 = (b-a)^2 + c^2$
   $0 = -2ab + a^2$ — 2. Hypothesis and distance formula.
3. $0 = -2b + a$ [$a \neq 0$] and $a = 2b$ — 3. a cannot be zero since B is distinct from A.
4. DB is vertical. — 4. B and D have the same abscissas.
5. AB is horizontal and ∴ perpendicular to BD.

9. Given: Isosceles trapezoid ABCD with lines connecting the midpoints E,F,G,H.

Prove: EFGH is a rhombus.



Proof:

1. $E(\frac{a}{2}, 0)$, $F(\frac{a+d}{2}, \frac{c}{2})$, $G(\frac{b+d}{2}, c)$, $H(\frac{b}{2}, \frac{c}{2})$ — 1. Midpoint formula.

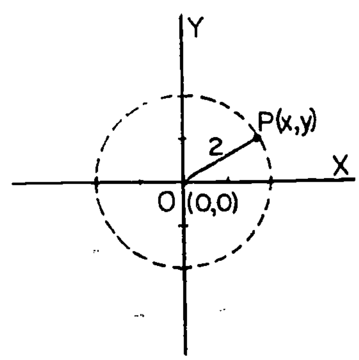
9. 2.	$d(A,D) = d(B,C)$ $b^2 + c^2 = a^2 - 2ad + d^2 + c^2$ $b^2 = (a - d)^2$ $b = a - d$ $d = a - b$	2. Hypothesis and distance formula.
3.	Slope EF = slope $HG = \frac{c}{d}$ Slope FG = slope $EH = \frac{c}{b-a}$	3. Slope formula.
4.	$\therefore EF \parallel HG$ and $FG \parallel EH$	4. Lines having equal slopes are parallel
5.	$\therefore$ EFGH is a parallelogram	5. If opposite sides of a quadrilateral are $\parallel$, it is a parallelogram.
6.	$d(E,F) = \sqrt{(\frac{d}{2})^2 + (\frac{c}{2})^2}$ $d(E,H) = \sqrt{(\frac{a-b}{2})^2 + (-\frac{c}{2})^2}$	6. Distance formula.
7.	$d(E,H) = \sqrt{(\frac{d}{2})^2 + (\frac{-c}{2})^2}$	7. Substituting in step 6 for d in step 2.
8.	$\therefore$ $d(E,F) = d(E,H)$ and EFGH is a rhombus.	8. A parallelogram having two adjacent sides congruent is a rhombus.

2-6. Sets Satisfying Geometric Conditions.

In this section the problem really is in translating a geometric condition which is usually stated in words into an algebraic relation stated in mathematical symbols. The best advice for the student is, as in the case of "word problems", to read the problem through once. Then read it again. Begin by letting $P(x,y)$ be any point which satisfies the geometric condition. Then write down the algebraic condition which must be satisfied by the coordinates x and y of any point in the set. Simplify the resulting expression, if possible. Illustrations of the technique by many examples will help the student to get the idea, but no amount of watching the teacher can take the place of attempts on the part of the student to set up the algebraic equation for himself. The only way to gain confidence in solving such problems is to attempt enough of them, proceeding from rather simple ones like finding the set of points at a given distance from a fixed point, with which most students can succeed, to more difficult ones which require the use of several of the formulas of sections 2-2 and 2-3.

The teacher will have noticed that these are the problems which are usually called "locus" problems in geometry. The word "locus" was not mentioned in this section since it may have unpleasant connotations for some students from experiences in geometry and more important because it really contributes nothing to the point the section is trying to make -- the formulation of an algebraic statement of geometric conditions.

Another omission from the traditional treatment of this topic is the discussion showing that the coordinates of every point in the set satisfy the equation and conversely that every point whose coordinates satisfy the equation belongs to the set. This discussion was deliberately omitted at this point since the difficulty never arises in the easy examples which the student meets at this time. The proper place for this discussion is in Chapters 4 and 7 where equivalent equations and operations leading to equivalent equations are treated. If the teacher wants to caution students about squaring both sides of an equation and assuming that the solution set of the resulting equation is the same as that of the original equation, it is entirely appropriate. It seemed that to make a big issue of this point at this time was inappropriate, almost useless, and certainly ineffective.



<u>Exercises 2-6. Answers.</u>

1. $d(O,P) = \sqrt{x^2 + y^2}$

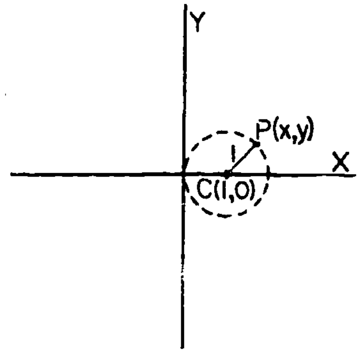
$2 = \sqrt{x^2 + y^2}$

$4 = x^2 + y^2$, the equation.

2. $d(C,P) = \sqrt{(x - 1)^2 + y^2}$

$1 = \sqrt{(x - 1)^2 + y^2}$

$1 = (x - 1)^2 + y^2$



3. $d(C,P) = \sqrt{x^2 + (y - 2)^2}$

$3 = \sqrt{x^2 + (y - 2)^2}$

$9 = x^2 + (y - 2)^2$

4. $d(C,P) = \sqrt{(x - 2)^2 + (y - 3)^2}$

$5 = \sqrt{(x - 2)^2 + (y - 3)^2}$

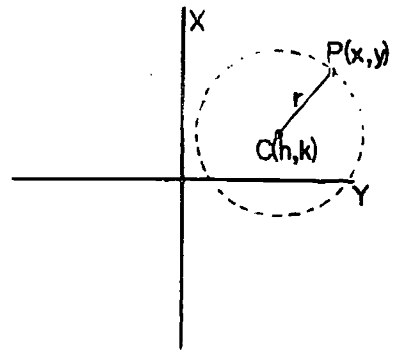
$25 = (x - 2)^2 + (y - 3)^2$

5. $d(C,P) = \sqrt{(x + 1)^2 + (y - 3)^2} = k$

$(x + 1)^2 + (y - 3)^2 = k^2$

6. $d(C,P) = \sqrt{(x - h)^2 + (y - k)^2} = r$

$(x - h)^2 + (y - k)^2 = r^2$



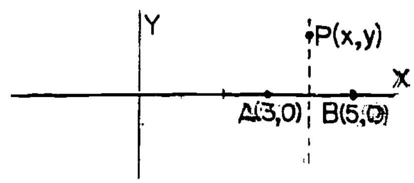
The set is a circle with radius r, center at $C(h,k)$.

7. $d(A,P) = d(B,P)$

$\sqrt{(x - 3)^2 + y^2} = \sqrt{(x - 5)^2 + y^2}$

$(x - 3)^2 + y^2 = (x - 5)^2 + y^2$

$x = 4$



8. $d(A,P) = d(B,P)$

$\sqrt{(x + 2)^2 + (y + 5)^2} = \sqrt{(x - 3)^2 + (y - 2)^2}$

$5x - 7y = -8$

*9. $d(P_1,P) = d(P_2,P)$

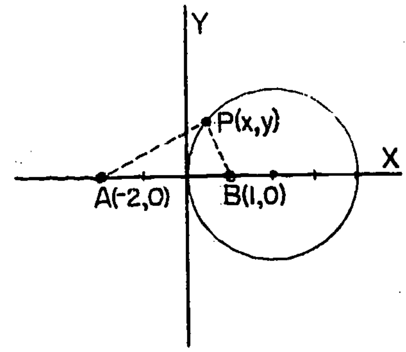
$\sqrt{(x - x_1)^2 - (y - y_1)^2} = \sqrt{(x - x_2)^2 + (y - y_2)^2}$

$2x(x_2 + x_1) + 2y(y_2 - y_1) + (x_1^2 + x_2^2 + y_1^2 + y_2^2) = 0$

10. $d(P,A) = 2d(P,B)$

$\sqrt{(x + 2)^2 + y^2} = 2\sqrt{(x - 1)^2 + y^2}$

$x^2 - 4x + y^2 = 0$



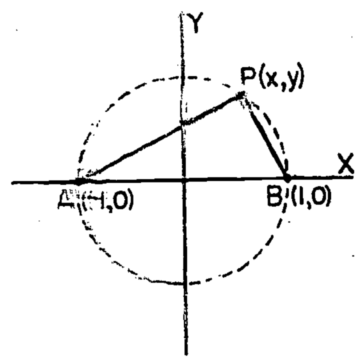
11. $m(P,A) \cdot m(P,B) = -1$, the condition for $\overline{PA} \perp \overline{PB}$

$m(P,A) = \frac{y}{x + 1}$; $m(P,B) = \frac{y}{x - 1}$

$\frac{y}{x + 1} \cdot \frac{y}{x - 1} = -1$

$x^2 + y^2 = 1, \; y \neq 0.$

This set consists of the set of all points except A and B on the circle with center (0,0) and the length of the radius equal to 1.



*12. Using the midpoint formula,

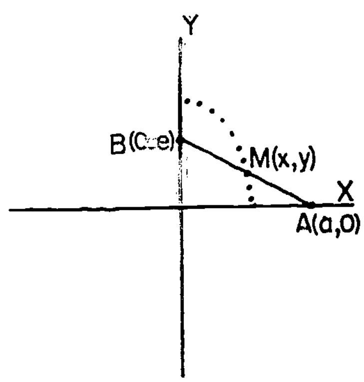
$P(x,y) = P(\frac{a}{2}, \frac{b}{2})$, hence

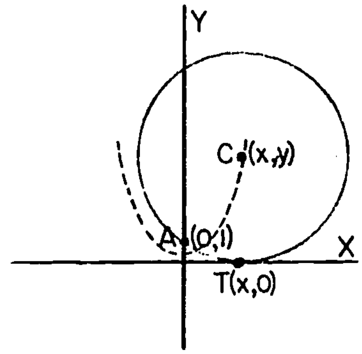
$a = 2x$, $b = 2y$. Using the distance formula,

$d(A,B) = \sqrt{a^2 + b^2} = 2$

$\sqrt{(2x)^2 + (2y)^2} = 2$, by substitution.

$x^2 + y^2 = 1$





13. d(C,T) = d(C,A)

$$y^2 = \sqrt{x^2 + (y - 1)^2}$$

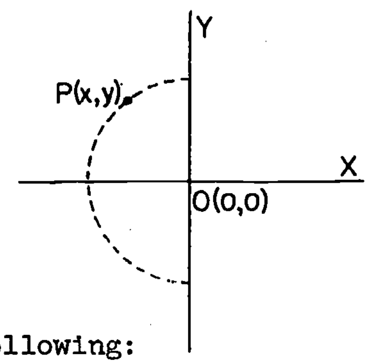
$$y = \frac{1}{2} x^2 + \frac{1}{2}$$

(The set of points is a parabola.)

14. This problem can be considered as that of finding the set of all points 0 distance from the origin. Hence

$$d(P,O) = \sqrt{x^2 + y^2} = 0$$

$$x^2 + y^2 = 0$$



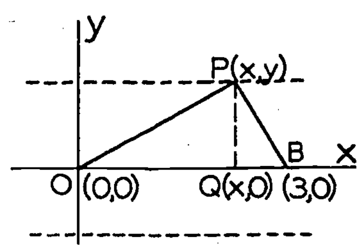
15. $d(P,O) = \sqrt{x^2 + y^2} = 2$

$$x^2 + y^2 = 4$$

But, this includes all values for x and y. To exclude the "right" part of the circle ,the description may be written as any one of the following:

(a) $\{ (x,y):x = -\sqrt{4 - y^2}\}$

(b) $\{ (x,y)\ x < 0:x^2 + y^2 = 4 \}$



16. $d(P,A) = \sqrt{y^2} = |y|$

d(O,B) = 3

Area $= \frac{1}{2}$ d(P,A)·d(O,B)

$2 = \frac{1}{2} |y|\cdot 3$

$\frac{4}{3} = |y|$

$y = \pm \frac{4}{3}$, which means that the set of points (x,y) is the graph of two lines parallel to the x-axis. This may be described as,

$\{ (x,y):y = \frac{4}{3} \} \cup \{ (x,y):y = -\frac{4}{3} \}$.

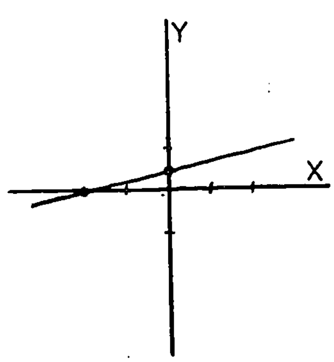
[pages 159-160]

2-7. Supplementary Exercises for Chapter 2 - Answers.

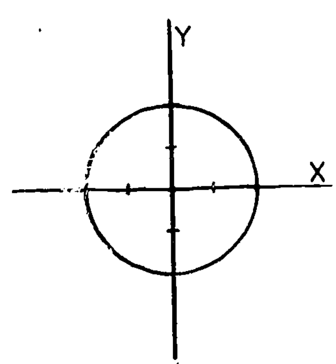
1.

	Symmetric with respect to,			Intercepts	
	x-axis	y-axis	origin	x	y
(a)	No	No	No	-2	$\frac{2}{5}$
(b)	Yes	Yes	Yes	± 2	± 2
(c)	No	Yes	No	$\pm \frac{\sqrt{6}}{2}$	-3
(d)	Yes	Yes	Yes	± 3	$\pm\sqrt{6}$
(e)	Yes	Yes	Yes	$\pm 2\sqrt{3}$	None
(f)	No	No	No	6 and 1	-6
(g)	No	No	No	None	-2
(h)	No	No	No	0	0 and -1
(i)	Yes	Yes	Yes	None	± 4
(j)	No	No	No	$\frac{3}{2}$	3
(k)	Yes	No	No	0	0
(l)	No	Yes	No	None	4
(m)	Yes	Yes	Yes	± 3	± 4
(n)	Yes	No	No	-7 and 1	$\pm\sqrt{7}$
(o)	No	No	No	1 and 2	-4
(p)	No	No	Yes	0	0

2. (a)



(b)



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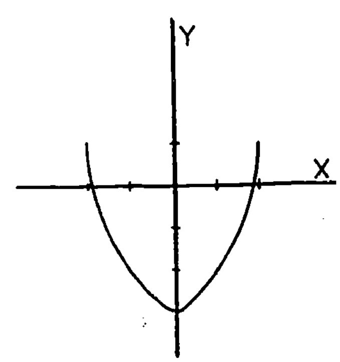

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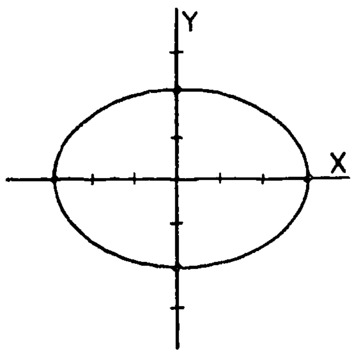

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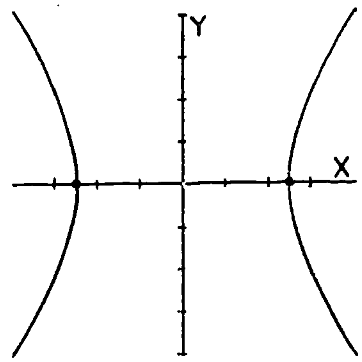

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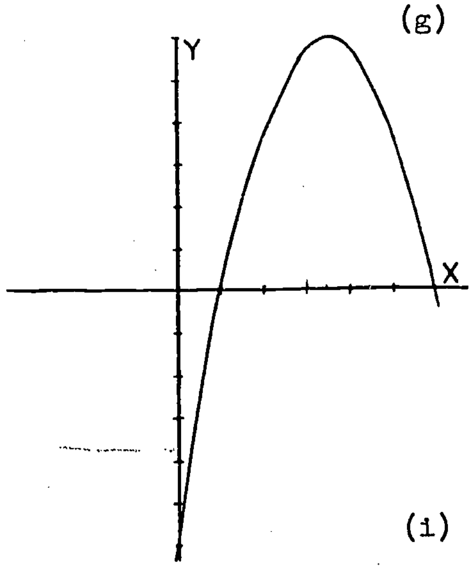

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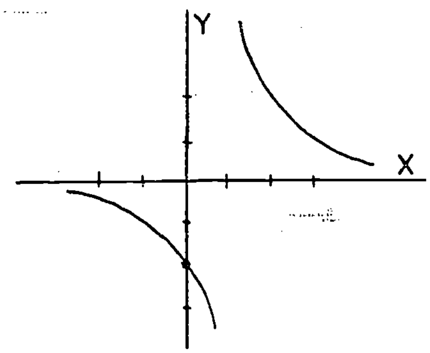

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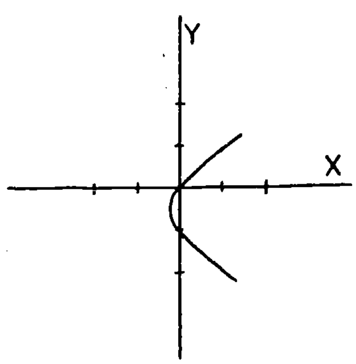

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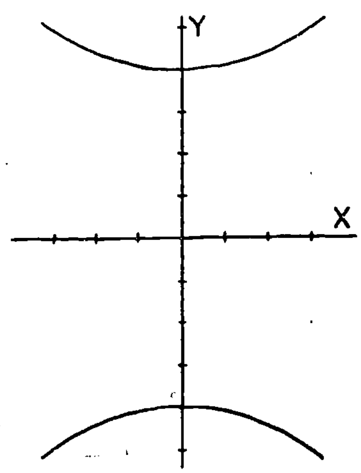
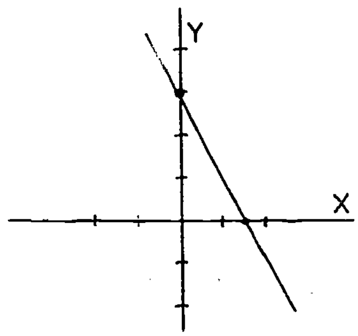
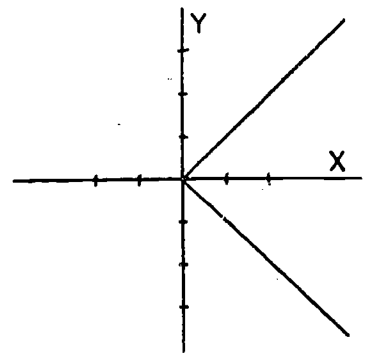


2. (j)



(k)



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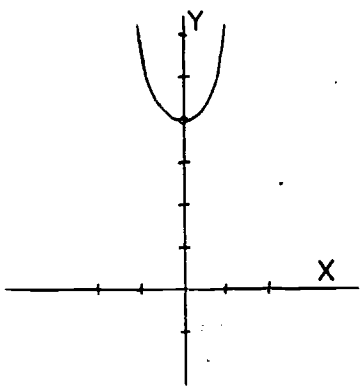
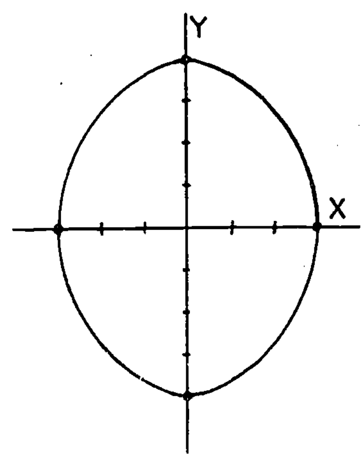
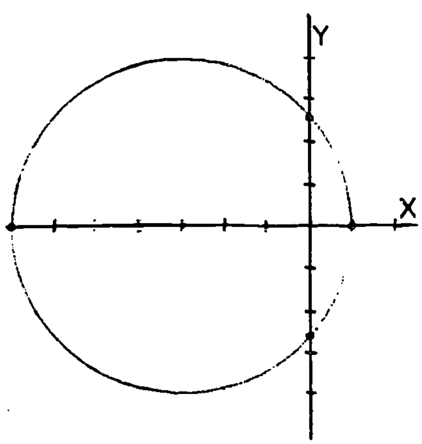


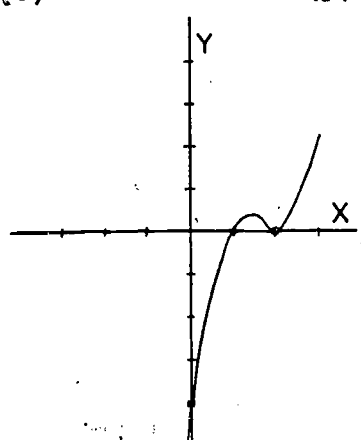
(m)



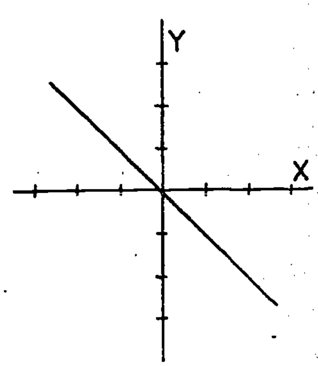
(n)



(o)

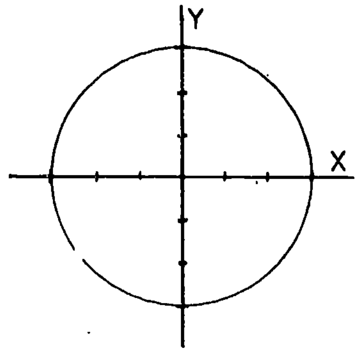


(p)

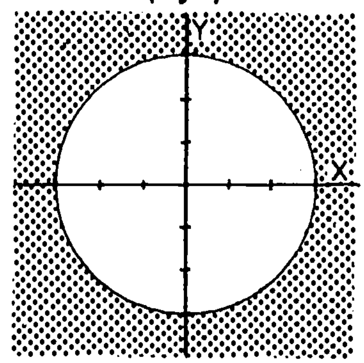


3. (a) Each point has coordinates of the form (a,y).
   (b) Each point has coordinates of the form (x,b).

4. (a)



(b)



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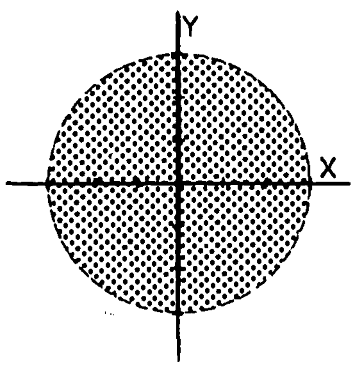

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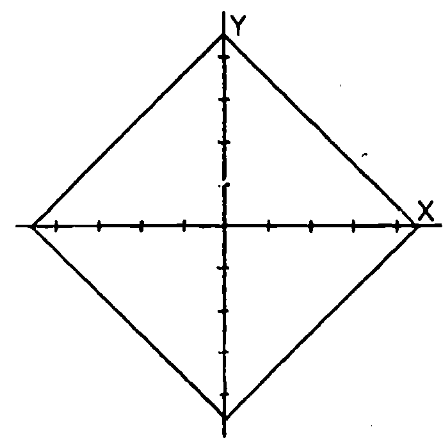

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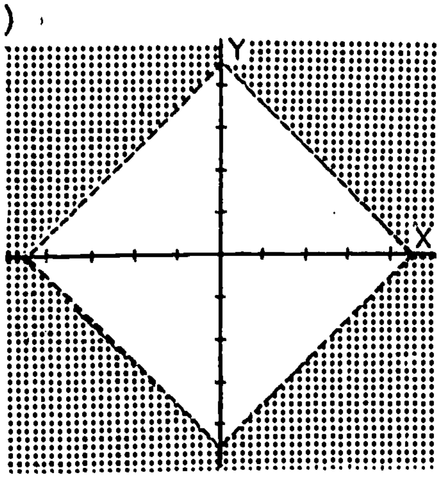

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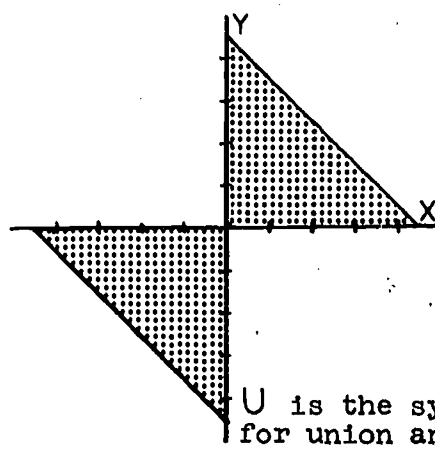
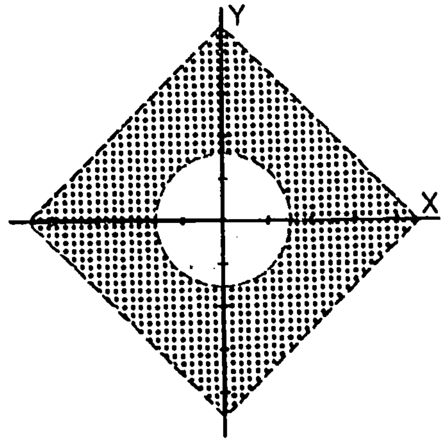


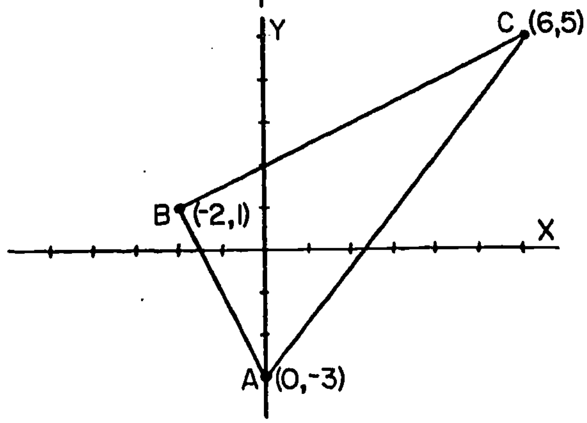
∪ is the symbol for union and means all the points in either set.

4. (g)



$\cap$ is the symbol for intersection, and means all the points that are in both sets.

5. (a)



(b) $m(A,B) = -2$; $m(C,B) = \frac{1}{2}$

Since $-2(\frac{1}{2}) = -1$, $\overline{AB} \perp \overline{CB}$ and $\triangle ABC$ is a right triangle.

(c) $m(A,C) = \frac{4}{3}$

(d) $d(A,B) = 2\sqrt{5}$; $d(B,C) = 4\sqrt{5}$

Area $= \frac{1}{2}\, 2\sqrt{5} \cdot 4\sqrt{5}$

$= 20$

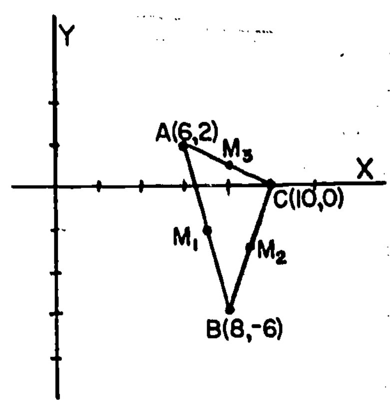
6. (a) The coordinates of

$M_1(A,B)$ are $(7,-2)$

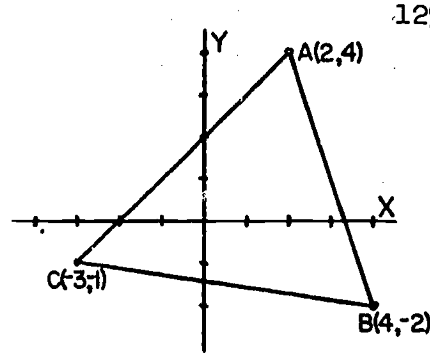
$M_2(B,C)$ are $(9,-3)$

$M_3(A,C)$ are $(8,1)$

$d(M_1,M_3) = \sqrt{10}$



(b) $m(M_1, M_2) = -\frac{1}{2}$



7. Δ ABC is isosceles since

$d(C,B) = 5\sqrt{2}$

$d(C,A) = 5\sqrt{2}$

$d(A,B) = 2\sqrt{10}$

8. $m(C,A) = \frac{a}{a - 3}$ [$a \neq 3$]

$m(C,B) = \frac{a - 5}{a}$ [$a \neq 0$]

(a) If $\overleftrightarrow{CA} \perp \overleftrightarrow{CB}$,

then $\frac{a}{a - 3} \cdot \frac{a - 5}{a} = -1$ and $a = 4$

(b) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

$$\frac{5}{-3} = \frac{3a - (a - 3)}{2a - (a - 1)}$$

$$a = -\frac{14}{11}$$

9. $d(P,A) = d(P,B)$

$$\sqrt{x^2 + y^2} = \sqrt{(x - 6)^2 + (y - 3)^2}$$

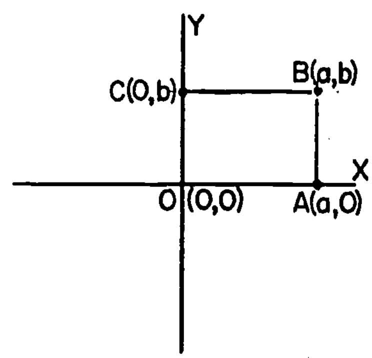
$$2y = -4x + 15$$

10. $d(P,A) = 3$

$$\sqrt{(x - 2)^2 + y^2} = 3$$

The required set is, $\{ (x,y): y > 0 \text{ and } x^2 - 4x + y^2 = 5 \}$

11. Plot the vertices as shown in the figure. Use the distance formula to show $d(A,C) = d(O,B)$.



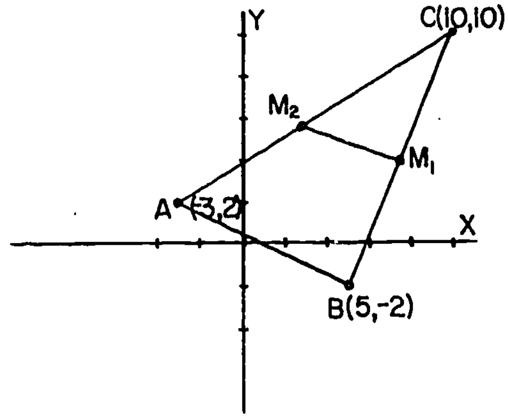
$d(A,C) = \sqrt{a^2 + b^2}$

$d(O,B) = \sqrt{a^2 + b^2}$

12. $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$ formula for coordinates of the midpoint

$\frac{1}{2} = \frac{-4 + x_2}{2}$, $\frac{3}{2} = \frac{8 + y_2}{2}$

$x_2 = 5$ $\qquad y_2 = -5$



13. The coordinates of

$M_1(B,C)$ are $(\frac{15}{2}, 4)$

$M_2(A,C)$ are $(\frac{7}{2}, 6)$

$m(M_1,M_2) = -\frac{1}{2}.$

$m(A,B) = -\frac{1}{2}$

$\therefore \overline{M_1M_2} \parallel \overline{AB}$

$d(M_1M_2) = 2\sqrt{5}$

$d(A,B) = 4\sqrt{5}$

$\therefore d(M_1M_2) = \frac{1}{2} d(A,B)$

14. $d(P,A) = d(P,B)$

$$\sqrt{(1-3)^2 + (y-2)^2} = \sqrt{(1-7)^2 + (y-6)^2}$$

$$y = 8$$

$\therefore P(1,y) = P(1,8)$

15. (a) $\{(x,y): y = 7 \text{ or } y = -7\}$

(b) $\{(x,y): x = 7 \text{ or } x = -7\}$

(c) $\{(x,y): x^2 + y^2 = 7^2\}$

(d) $\{(x,y): |x| = 7 \text{ and } |y| = 7\}$

16. The equation of the perpendicular bisector is derived by using the distance formula and a related theorem from plane geometry. It is

$$x + y = 9.$$

The coordinates of point C satisfy this equation, hence C is on the perpendicular bisector of $\overline{AB}$.

17. Each point is on the circle. The reflection of (a,b) in the x-axis is (a,-b); of (a,b) in the origin is (-a,-b); of (a,b) in the y-axis is (-a,b).

18. (a) $\{(x,y): (x > 0 \text{ and } y > 0) \text{ and } (x^2 + y^2 = 9)\}$

(b) $\{(x,y): (x < 0 \text{ and } y > 0) \text{ and } (x^2 + y^2 = 9)\}$

(c) $\{(x,y): x < 0 \text{ and } x^2 + y^2 = 9 \text{ and } y \neq 0\}$

(d) $\{(x,y): [(x > 0 \text{ and } y > 0) \text{ or } (x < 0 \text{ and } y < 0)] \text{ and } (x^2 + y^2 = 9)\}$

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Another description is $\{(x,y):(x > 0 \text{ and } y > 0) \text{ and } (x^2 + y^2 = 9)\} \cup \{(x,y):(x < 0 \text{ and } y < 0) \text{ and } (x^2 + y^2 = 9)\}$.

19. $\{(x,y):(x > 0 \text{ and } y > 0) \text{ and } (2x + y < 3)\}$.

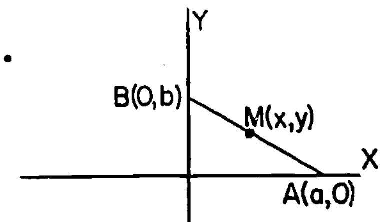
20. Since the vertices of such triangles will be the set of points, namely, the perpendicular bisector of $\overline{AB}$, the equation is derived from

$$d(P,A) = d(P,B).$$

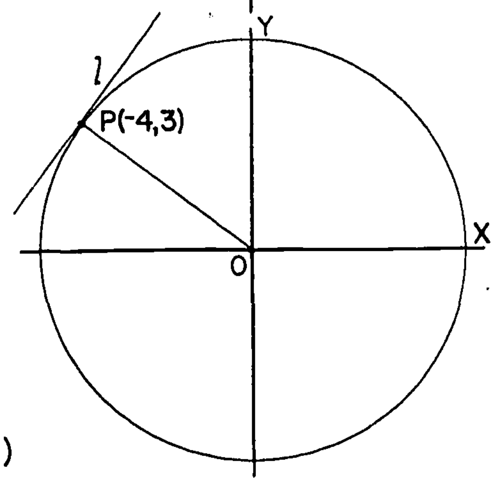
The equation is

$$14x - 12y = -17.$$

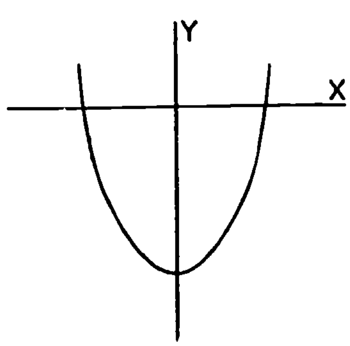
21. Area $= \frac{1}{2}ab = k$.
The coordinates of M give,
$x = \frac{a}{2}$, $y = \frac{b}{2}$
or $2x = a$, $2y = b$
$\therefore \frac{1}{2}(2x)(2y) = k$ by substitution
$2xy = k$



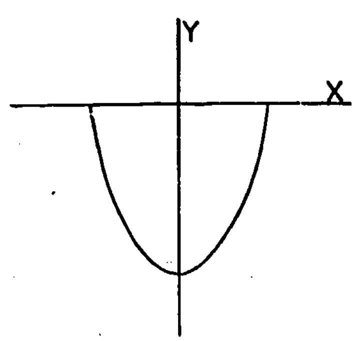
22. The slope of l is the negative reciprocal of $\overline{PO}$, the radius.
$m(P,O) = -\frac{3}{4}$.
$\therefore$ slope of l is $+\frac{4}{3}$



23. (a)



(b)



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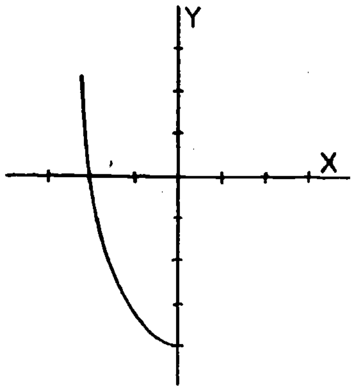
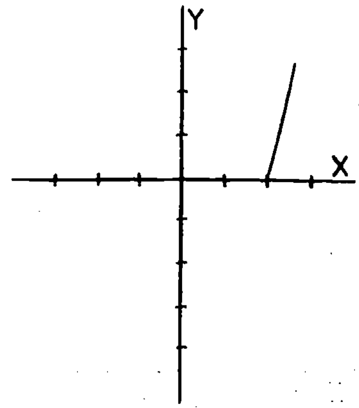
23. (c)

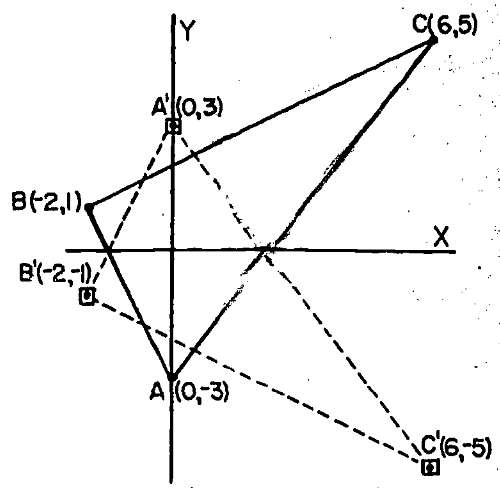


(d)



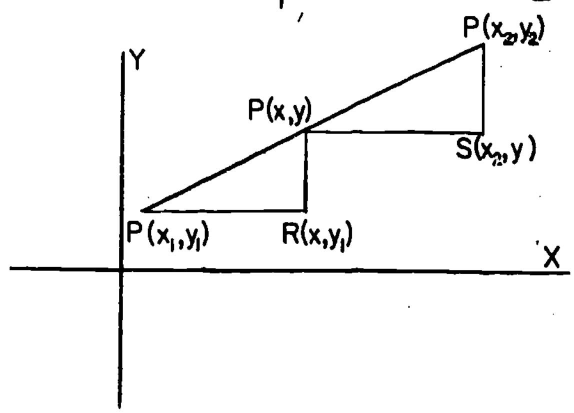
24. $m(A,B) = -2$; $d(A,B) = 2\sqrt{5}$
$m(B,C) = \frac{1}{2}$; $d(B,C) = 4\sqrt{5}$
$\therefore \overline{AB} \perp \overline{BC}$ $d(A,C) = 10$
Area $\Delta ABC = 20$
Perimeter $\Delta ABC = 10 + 6\sqrt{5}$

$m(A',B') = 2$; $d(A',B') = 2\sqrt{5}$
$m(B',C') = -\frac{1}{2}$; $d(B',C') = 4\sqrt{5}$
$\therefore \overline{A'B'} \perp \overline{B'C'}$ $d(A',C') = 10$
Area $\Delta A'B'C' = 20$
Perimeter $\Delta A'B'C' = 10 + 6\sqrt{5}$



Challenge Problems Answers

1. $d(P_1,R) = x - x_1$
$d(P,S) = x_2 - x$
$d(R,P) = y - y_1$
$d(S,P_2) = y_2 - y$
$\Delta P_1RP \sim \Delta PSP_2$
$\therefore \frac{P_1P}{PP_2} = \frac{P_1R}{PS} = \frac{RP}{SP_2}$.



1. cont.

Let $P_1P = r_1$ and $PP_2 = r_2$

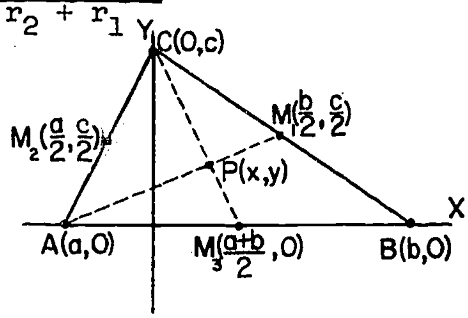
$$\frac{P_1P}{PP_2} = \frac{r_1}{r_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$$r_2(x - x_1) = r_1(x_2 - x) \quad , \quad r_2(y - y_1) = r_1(y_2 - y)$$

$$r_2x + r_1x = r_1x_2 + r_2x_1 \qquad \text{and,}$$

$$x(r_2 + r_1) = r_1x_2 + r_2x_1$$

$$x = \frac{r_1x_2 + r_2x_1}{r_2 + r_1} \quad , \quad y = \frac{r_1y_2 + r_2y_1}{r_2 + r_1}$$



There is no loss of generality in taking one side on the x-axis and the third vertex on the y-axis. Let the centroid be P(x,y), P_1 be M_1, and P_2 be A.

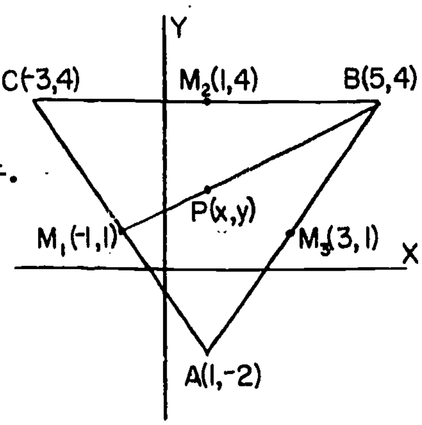
$$x = \frac{1(a) + 2(\frac{b}{2})}{3} = \frac{a + b}{3}$$

$$y = \frac{1(0) + 2(\frac{c}{2})}{3} = \frac{c}{3}$$

The same results are found when P_1 is M_2, P_2 is B; when P_1 is M_3 and P_2 is C.

2. P(x,y) divides P_1 and P_2 in the ratio $r_1:r_2$ if,

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2} , \quad y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2} .$$



Assigning P_1 to M_1 and P_2 to B,

$$x = \frac{1(5) + 2(-1)}{3} = 1$$

$$y = \frac{1(4) + 2(1)}{3} = 2$$

Assigning P_1 to M_3 and P_2 to C,

$$x = \frac{1(-3) + 2(3)}{3} = 1$$

$$y = \frac{1(4) + 2(1)}{3} = 2$$

An alternate solution is given by,

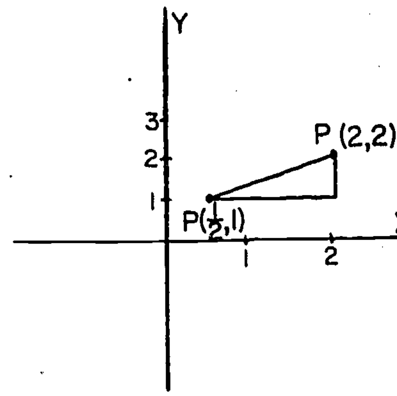
equation of $\overline{M_1B}$ is $y - 1 = \frac{3}{6}(x + 1)$

equation of $\overline{M_2C}$ is $y - 1 = \frac{-3}{6}(x - 3)$

solving for x and y, $x = 1$ and $y = 2$.

Hence the coordinates of P are $(1,2)$.

3. Length of abscissa x = 2 times length of ordinate y. In (x_1,y_1), the length of x_1 is actually $2x_1$.



Use "old" distance formula,

$$d(P_1P_2) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2},$$

replacing x_1 by $2x_1$, the "new" distance formula is,

$$d(P_1,P_2) = \sqrt{(2x_2-2x_1)^2 + (y_2-y_1)^2}$$

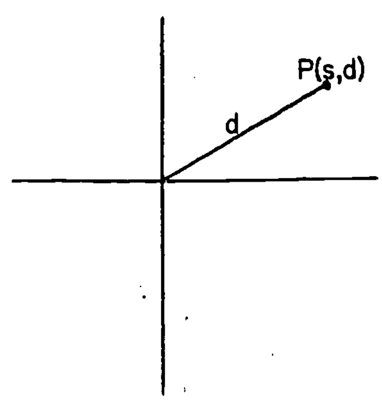
$$= \sqrt{4(x_2-x_1)^2 + (y_2-y_1)^2}$$

Check this by using the coordinates used in the above drawing.

$$d(P_1,P_2) = \sqrt{4(2 - \tfrac{1}{2})^2 + (2 - 1)^2}$$

$$= \sqrt{10}$$

4. This system is really a disguised form of polar coordinates. Instead of the usual polar coordinates (r,θ) we are using $(\tan\theta, r)$ with special conventions about the signs of $s = \tan\theta$ and $d = r$. For instance (c) in this problem, $d = ks$ is essentially the polar equation $r = k\tan\theta$.



[pages 163-164]

Use these transformation equations,

Rectangular coordinates

$x = \frac{\pm d}{\sqrt{1 + s^2}}$,

{+ if d and s have same sign
{- if d and s have opposite signs

$y = \frac{ds}{1 + s^2}$

"New" coordinates

$d = \sqrt{x^2 + y^2}, \quad y > 0$

$d = -\sqrt{x^2 + y^2}, \quad y < 0$

$s = \frac{y}{x}$

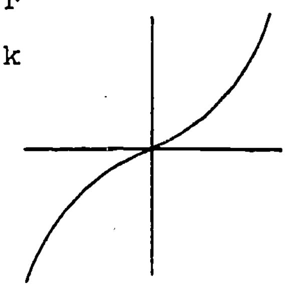
Rectangular	"New"
(a) $x^2 + y^2 = r^2$	$d^2 = r^2$
(b) $\frac{y}{x} = k$	$s = k$

(c) Graph of $d = ks$

The equation in rectangular coordinates is,

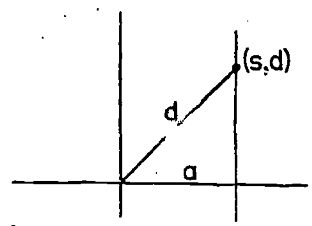
$+\sqrt{x^2 + y^2} = k\,\frac{y}{x}$ if $y > 0$

$-\sqrt{x^2 + y^2} = k\,\frac{y}{x}$ if $y < 0$



(d) $x = a$

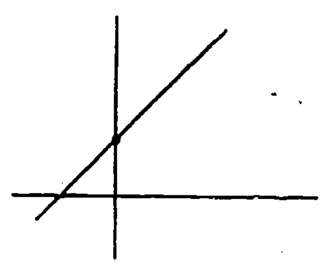
$\frac{\pm d}{\sqrt{1 + s^2}} = a$ by the transformation equations



(e) $Ax + By + C = 0$ general linear equation

$\frac{\pm d}{\sqrt{1 + s^2}} A + \frac{sd}{\sqrt{1 + s^2}} B + C = 0$ by the transformation equations

$\pm Ad + Bsd + C\sqrt{1 + s^2} = 0$



[pages 163-164]

2-8. Illustrative Test Questions for Chapter 2.

The following is a set of illustrative test items for Chapter 2. It is not intended that this be used as a Chapter test, but rather as a model for making test questions.

1. The coordinates of the projections of point P on the axes are (-2,0) and (0,3); the corresponding coordinates for point Q are (4,0) and (0,5). What are the coordinates of the midpoint of $\overline{PQ}$?
2. Which of the following terms apply to the triangle whose vertices are (2,4), (4,-2) and (-3,-1)?
   (a) acute
   (b) right
   (c) obtuse
   (d) non-isosceles
   (e) isosceles and non-equilateral
   (f) equilateral
3. Given the points A(1,2), B(-11,4) and C(13,-6). Find the distance between the midpoint of $\overline{AB}$ and the midpoint of $\overline{AC}$.
4. Given the points A(1,2), B(-11,4), and C(13,-6). Find the slope of the line through the midpoint of $\overline{AB}$ and the midpoint of $\overline{AC}$.
5. Find the value of k for which the line through the point A(k,2k - 1) and B(2k + 1,k) will have a slope of $\frac{2}{3}$.
6. Given the points A(2,-3), B(-1,2), and P(a - 1, a - 3). Find the values of a for which $\overline{PA}$ will be perpendicular to $\overline{PB}$.
7. Given the three points A(-4,-2), B(0,2) and C(3,y). Find the value of y for which these points are the vertices of a right triangle if,
   (a) the right angle is at B.
   (b) the right angle is at A.
   Show that the right angle cannot be at C for any real value of y.
8. Given the points $P_1(3,5)$, $P_2(r,1)$, $P_3(-1,2)$ and $P_4(-4,s)$. Find r and s such that,
   (a) $\overline{P_1P_2}$ is parallel to $\overline{P_3P_4}$.
   (b) $\overline{P_1P_3}$ is perpendicular to $\overline{P_3P_4}$.

9. If a line l_1 has a slope -3 and y-intercept 5 while line l_2 has slope 2 and y-intercept 8, in which quadrant do the graphs of l_1 and l_2 intersect?
10. Find the point P(x,y) that is symmetric to the point P'(3,2) when reflected in the line $y = x$.
11. Determine in which quadrants the graph of each of the following equations has points. Indicate for each graph whether it is symmetric with respect to the x-axis; with respect to the y-axis; with respect to the origin.
    (a) $y = -2x + 3$
    (b) $y = |x|$
    (c) $y = 2x^2 + 3$
    (d) $x^2 - y^2 = 9$
    (e) $y = x^3 - 3x^2 + 3x - 1$
12. Write the equation of the set of points equidistant from the two points A(-5,3) and B(5,3).
13. Describe the graph and write the equation of the set of points whose distance from the point A(0,3) is 5 and for which $y > 0$.
14. Write the equation, sketch, and describe the set of points P(x,y) such that $\overline{PA} \perp \overline{PB}$. The coordinates are A(-2,3) and B(2,-3).
15. The points whose coordinates are (5,2) and (-1,4) are symmetric with respect to a line.
    (a) Give the coordinates of the point that is on the line of symmetry and on the line connecting the two given points.
    (b) What is the slope of the line of symmetry?
16. What are the coordinates of the point on the line $y = x$, if its projection on one axis has coordinates (0,-7)?
17. Given the points A(0,1), B(4,-3), C(6,0).
    (a) Find the coordinates of M_1, the midpoint of $\overline{AB}$.
    (b) Find the coordinates of M_2, the midpoint of $\overline{BC}$.
    (c) Find the length of $\overline{M_1M_2}$.
    (d) Find the slope of $\overline{M_1M_2}$

17. (e) Find the slope of $\overline{AC}$.

18. Given the points A(3,5), B(p,-1), C(-1,2), D(2,-3).
(a) Find p such that $\overline{AC} \parallel \overline{DB}$.
(b) Find p such that $\overline{AC} \perp \overline{DB}$.

19. Give the equation of the set of points equidistant from the points whose coordinates are (5,7) and (7,5).

20. Find the equation of the set of points equidistant from the points whose coordinates are (3,25) and (-5,25).

21. Find the equation of the set of points which is equidistant from the two points (-2,5) and (3,-2).

22. Find the equation of the set of points which is 5 units from the point (-4,-5).

23. What is the slope of the line whose equation is,
(a) $y = x$?
(b) $x = 0$?
(c) $y = 0$?

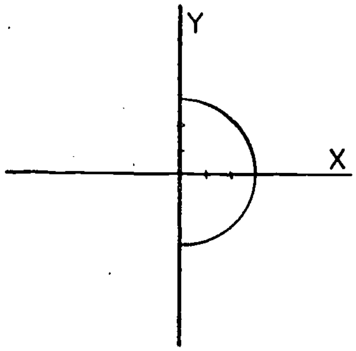
24. A line L with slope $-\frac{2}{3}$ which intersects the x-axis at (2,0) will pass through which quadrants?

25. Give the coordinates of the x-intercept(s) and/or the y-intercept(s) of each of the following equations:
(a) $y = 2x$.
(b) $y = x^2$.
(c) $x^2 + y^2 = 9$.
(d) $y = |x|$.
(e) $x^2 = 4 + y^2$.
(f) $x = y^2 + 7$.
(g) $3y + 6x = 9$.
(h) $y = x^2 - 3$.
(i) $x = 2$.
(j) $y = -2$.
(k) $y = 0$.

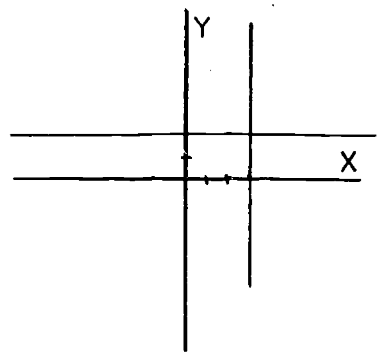
26. Give the line(s) of symmetry for the equations labeled (b), (c), (d), (f), (h) in Problem 25 above.

27. Select a sentence from the list given below the figures which will completely describe each of the graphs. [The dotted --- portions of the graph are excluded.]

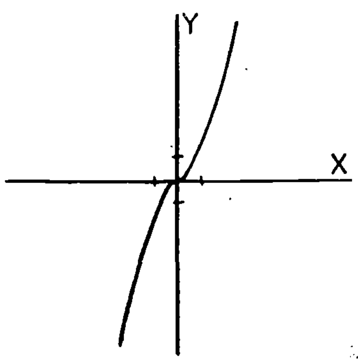
(1)



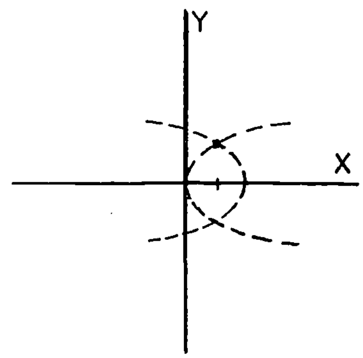
(2)



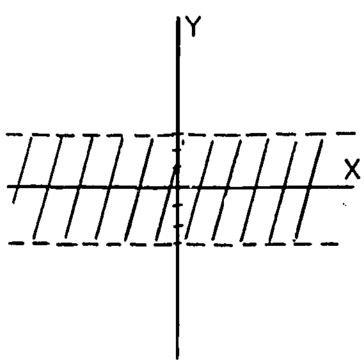
(3)



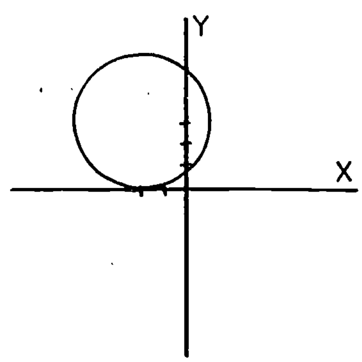
(4)



(5)



(6)



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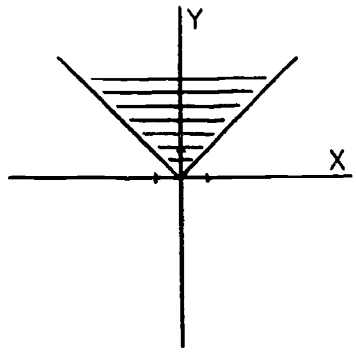

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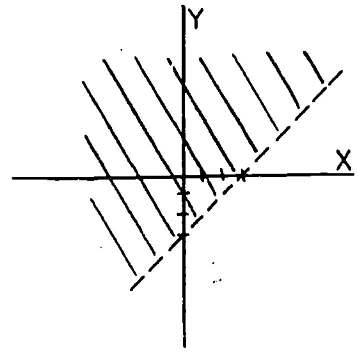

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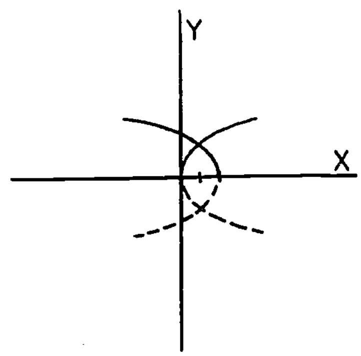


(a) $\{(x,y): x = 3 \text{ or } y = 2\}$
(b) $\{(x,y): x = 3 \text{ and } y = 2\}$
(c) $\{(x,y): y \geq |x|\}$
(d) $\{(x,y): y \geq x^2\}$
(e) $\{(x,y): x - y < 3\}$
(f) $\{(x,y): x + y < 3\}$
(g) $\{(x,y): y = x^3\}$
(h) $\{(x,y): y = -x^3\}$
(i) $\{(x,y): x^2 + y^2 = 9\}$
(j) $\{(x,y): x^2 + y^2 \text{ and } x \geq 0 \text{ and } y \geq 0\}$
(k) $\{(x,y): x^2 + y^2 = 9 \text{ and } x \geq 0\}$
(l) $\{(x,y): -3 < y < 3\}$
(m) $\{(x,y): (x + 2)^2 + (y - 3)^2 = 9\}$
(n) $\{(x,y): -3 > y > 3\}$
(o) $\{(x,y): x = y^2 \text{ and } x = -y^2 + 2 \text{ and } y > 0\}$
(p) $\{(x,y): x > y^2 \text{ and } x > -y^2 + 2 \text{ and } y > 0\}$
(q) $\{(x,y): (x = y^2 \text{ or } x = -y^2 + 2) \text{ and } y > 0\}$

2-9. Illustrative Test Questions for Chapter 2. Answers.

1. (1,4)

2. (a) acute.

   (e) isosceles and non-equilateral.

3. 13

4. $\frac{-5}{12}$

5. $\frac{1}{5}$

6. {0,4}

7. (a) When y = -1, the right angle is at B.

   (b) When y = -9, the right angle is at A.

   For the right angle to be at C,

$$\frac{y - 2}{3} \cdot \frac{y + 2}{7} = -1$$

$$y^2 - 4 = -21$$ which has no real number solution.

8. (a) s = 6

   (b) r = 6

9. 2nd quadrant

10. P(2,3)

11. (a) I, II, and IV. Not symmetric with respect to either axis or the origin.

    (b) I and II. Symmetric with respect to y-axis only.

    (c) I and II. Symmetric with respect to y-axis only.

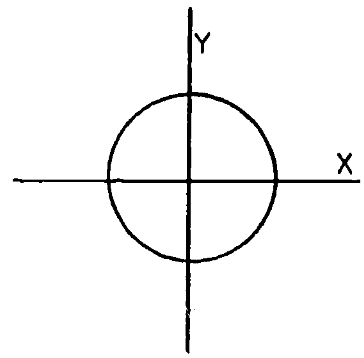
    (d) I, II, III and IV. Symmetric with respect to both axes also origin.

    (e) I, III and IV. Not symmetric with respect to either axis or the origin.

12. x = 0 (the y-axis)

13. $x^2 + (y - 3)^2 = 25$ [$y > 0$] is the "upper" half of a circle whose center is (0,3) and radius 5. The description may be given in set notation as,

    $\{(x,y): y > 0 \text{ and } x^2 + (y - 3)^2 = 25\}$



14. $m(P,A) = \frac{y - 3}{x + 2}$

$m(P,B) = \frac{y + 3}{x - 2}$

$\overline{PA} \perp \overline{PB}$ $\quad \frac{y - 3}{x + 2} \cdot \frac{y + 3}{x - 2} = -1$

$x^2 + y^2 = 13$

The graph is a circle with center at O(0,0) and radius $\sqrt{13}$.

15. (a) (2,3)
    (b) slope is 3
16. (-7,-7)
17. (a) $M_1(2,-1)$
    (b) $M_2(5,\frac{-3}{2})$
    (c) $\frac{\sqrt{37}}{2}$
    (d) slope $\overline{M_1M_2} = -\frac{1}{6}$
    (e) slope $\overline{AC} = -\frac{1}{6}$
18. (a) $p = \frac{14}{3}$
    (b) $p = -\frac{1}{2}$
19. $\sqrt{(x - 5)^2 + (y - 7)^2} = \sqrt{(x - 7)^2 + (y - 5)^2}$

$y = x$

20. $x = -1$
21. $5x - 7y = -8$
22. $\sqrt{(x + 4)^2 + (y + 5)^2} = 5$

$(x + 4)^2 + (y + 5)^2 = 25$

23. (a) slope is 1.
    (b) slope is undefined.
    (c) slope is 0.
24. Quadrants I, II and IV.

25.

	x-intercept(s)	y-intercept(s)
(a)	0	0
(b)	0	0
(c)	± 3	± 3
(d)	0	0
(e)	± 2	None
(f)	$\pm\sqrt{7}$	None
(g)	$\frac{3}{2}$	3
(h)	$\pm\sqrt{3}$	-3
(i)	2	None
(j)	None	-2
(k)	Every real number	0

26. (b) y-axis
(c) x and y axes
(d) y-axis
(f) x-axis
(h) y-axis

27. (1) (k)
(2) (a)
(3) (g)
(4) (o)
(5) (l)
(6) (m)
(7) (c)
(8) (e)
(9) (q)

Commentary for Teachers

Chapter 3

THE FUNCTION CONCEPT AND THE LINEAR FUNCTION

3-0. General Introduction.

This chapter is about functions, but the student should not feel that when he is finished with the chapter that he has finished with the concept. He will meet the idea throughout the rest of his mathematical studies. Some of the functions he will meet are the traditional ones - the linear function, the quadratic function, the trigonometric functions, the logarithmic and exponential function. He will also meet a more abstract kind of function that may pair other objects besides numbers. In geometry, for instance, it is instructive to regard congruence, similarity, angle measure, and the like in terms of functions. These are functions which involve sets of points as well as numbers. Also, in the twelfth grade algebra course, the study of groups, rings and fields is pursued in terms of functions which involve quite abstract objects. The student should emerge from this chapter with some understanding of how widely applicable the function idea is. He should be able to recognize a function when he sees one, and he should be able to see one wherever he looks.

3-1. Informal Background of the Function Concept.

Once the student grasps the function idea he should be able to find functions in all sorts of unlikely places.

He will probably get the pairing idea first, and the classroom has many examples: students and their first names, students and seats, heights of objects above the floor, etc. Some points to establish are:

(1) that the pairings need not be one-to-one (It is all right if several students are named Joe).

(2) Every member of the domain set must have exactly one object assigned to it. (Each student has exactly one first name.)

(3) Every member of the range set must be assigned to at least one member of the domain set. (If no one in class is named "Algernon," then "Algernon" is not in the range.)

There are usually <u>three</u> sets involved in a function, its domain, its range and a third set which includes the range. For instance, assigning the first name of a student to a student involves as domain the set of all students, as range the set of all names which are names of students. It also suggests as a third set, the set of <u>all</u> first names. This latter set has no special name but in more advanced studies of functions such sets are mentioned.

<u>Suggestions for section 3-1.</u>

1. These exercises are better suited for class discussion than for written work. The fact that some of them have many interpretations and many answers is an asset in a discussion, it is a liability in a writing situation.

2. For some of the problems the domain or the range cannot be explicitly given. For instance in problem 5, no one knows the set of ages of living people. In every case, however, a description phrase can be given which is correct. The teacher should not accept an answer like "whole numbers less than 8,000,000" as the range sought in problem 6. This set includes the range, but a more accurate answer should be required.

3. The teacher should also ask for relations suggested by these phrases which are not functions. For instance in problem 1, assigning triangles to areas does not define a function.

4. It should be possible to have the students volunteer examples of their own, from science, politics and everyday living. The teacher could even challenge his class to specify a

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topic which does not suggest functions. The brighter students can be counted on both to produce far fetched topics and ingenious solutions.

Answers to Exercises 3-1

1. Areas of triangles

   Domain: Set of all triangles.

   Range: Set of all positive real numbers.

   Rule: To each triangle is assigned the number $\frac{1}{2}bh$, where b is the length of its base and h is the length of its altitude.

2. Multiplication table for positive integers

   Domain: Set of all positive integers.

   Range: Set of all positive integers.

   Rule: Assign to each pair of integers (x,y) the integer xy.

3. Election returns

   Domain: Set of all offices.

   Range: Set of all elected candidates.

   Rule: Assign to each office the single candidate elected to that office.

4. Peoples' first names

   Domain: Set of all people.

   Range: Set of all first names of living people.

   Rule: Assign to each person his first name.

5. Peoples' ages

   Domain: Set of all living people.

   Range: Set of all positive integers which are ages in years of living people.

   Rule: Assign to each person his age.

   Note: (1) The domain of this function is continally changing.

   (2) We are not able to specify its range precisely. It is roughly the set of

positive integers x such that $1 \leq x \leq 110$. There may be a few larger numbers in the set, too. Also babies' ages are not generally measured in years at all but in months, days and even hours.

(3) It is recommended that the teacher not try to gloss over these difficulties. It is a sad fact that every day language is full of unclear phrases. However, he can promise the class that when functions are used in mathematical contexts such difficulties do not arise.

6. Population of cities

   Domain: Set of all cities.

   Range: Set of all positive integers that are population of cities.

   Rule: Assign to each city a positive integer that is the number of its inhabitants.

7. A dictionary

   Domain: Set of all words.

   Range: Set of all meanings.

   Rule: Assign to each word its meaning.

8. Relative nearness to sun of planets

   Domain: Earth, Jupiter, Mars, Mercury, Neptune, Pluto, Saturn, Venus, Uranus.

   Range: $\{ 1, 2, 3 .... 9 \}$

   Rule: Assign to each planet its rank.

9. Batting averages

   Domain: Set of all baseball players.

   Range: Set of all 3-place decimals which are actual batting averages.

   Rule: Assign to each player his average.

10. Absolute values

   Domain: Set of all real numbers.

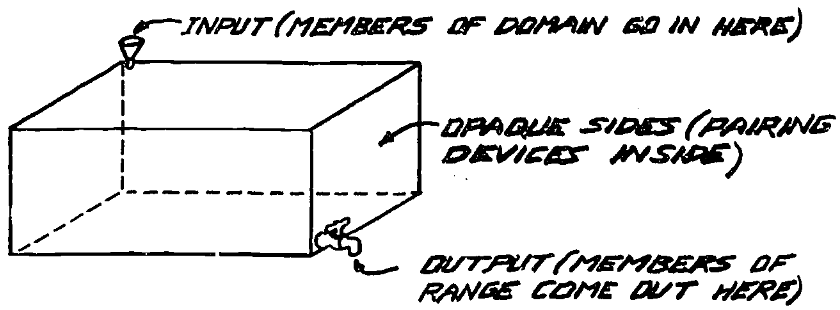
   Range: Set of all non-negative real numbers.

Rule: To each non-negative real number x assign x.
To each negative number x assign -x.

Note: The second part of the answer to this problem looks peculiar. We wished to assign positive numbers and in this part we assigned -x, which looks negative. Actually we assigned -x only in the case that x was negative, and in this case -x is positive.

3-2. Formal Definition of Function.

After the student has learned to discuss functions informally, he can be led to try to give a formal definition. He will almost certainly get stuck on the "rule" part of the definition. This is no disgrace. In fact, mathematicians have come to recognize that it is hard to define "rule" and they avoid this problem. They think of a function as being like a box with an input funnel and an output end.



It is a device which feeds something out of the output end after something is fed into the input end. The rule of the function corresponds to the machinery inside the box. It is not necessary to know exactly how the machinery inside the box converts the input to the output. All that is required is that the machine always yields the same output element for any given input element. In advanced treatments of function the idea of a rule disappears altogether. The pairings are taken as the primary thing and no questions are asked as to how they are arrived at.

It will be helpful to give some examples of functions in which the "rule" is just an arbitrary pairing. For instance if A is the set {1, 2, 3} and B is the set {4, 5}, assigning 5 to 1, 5 to 2 and 4 to 3 is a function whose domain is A and whose range is B. The teacher should judge for himself how much of this kind of arbitrary pairing can be used.

Our text treats two classes of functions - those whose domain and range are sets of numbers, and those with more abstract domain and range. In the ninth grade text, only the former kind are discussed and therefore this is probably the student's first exposure to the second kind. The teacher should be willing to adopt pretty much the same attitude toward both these. When the familiar linear and quadratic functions come up the approach to functions introduced here should be maintained.

Suggestions for section 3-2. The problems are suitable for written work.

Answers to Exercises 3-2

1. (a) Domain: Set of all real numbers.
   Range: Set of all real numbers.
   (b) Domain: Set of all real numbers.
   Range: Set of all real numbers.
   (c) Domain: Set of all real numbers.
   Range: Set of all non-negative real numbers.
   (d) Domain: Set of all real numbers.
   Range: Set of all non-negative real numbers.
   (e) Domain: Set of all real numbers.
   Range: Set of all non-negative real numbers.
   (f) Domain: Set of all real numbers.
   Range: { 4 }
   (g) Domain: Set of all integers.
   Range: { 0, 1 }
   (h) Domain: Set of all points in the plane.
   Range: Set of all points in the plane.

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(i) Domain: Set of all rectangles.
Range: Set of all positive real numbers.

(j) Domain: Set of all pairs of points in the plane.
Range: Set of all non-negative real numbers.

2. (a) $f(1) = 4$, $f(2) = 5$, $f(3) = 4$

(b) $f(1) = 2$, $f(2) = 1$, $f(3) = 3$

(c) $f(4) = 4$, $f(5) = 5$

(d) Any function whose domain is B has a range containing at most two elements. A is therefore ruled out; it has three elements.

3-3. Notation for Functions.

In traditional discussions of functions the language is often sloppy. The teaching of functions can be made less difficult if a precise language is established and maintained. Consider, for instance, f, $f(x) = x^2$, $y = f(x)$. The first f should be used to denote a function, the second, $f(x)$, should be used to denote that member of its range that the function f assigns to the number x of its domain, $f(x) = x^2$ should be used to mean that the function f assigns x^2 to x. The equation $y = f(x)$ suggests a function but does not always define one. This point is discussed at length in the next section. The trickiest part of the function notation involves the technique of substitution. It should cause no difficulty to use the symbol $f(3)$ to denote the object assigned to 3 by the function f. Things get a little more complicated when letters appear and substitutions are made. For instance if $f(x) = x^2$ then $f(2x) = (2x)^2$ or $4x^2$. It probably is best not to discuss substitution as a separate topic, but to try to establish the rules by use of examples.

The symbol $f(x)$ does not necessarily denote an algebraic expression. If f is the function which assigns to each state of the U.S.A. its capitol then $f(\text{New York}) = \text{Albany}$.

Suggestions for Section 3-3.

1. The fast learners will catch on to this material quickly

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and enjoy it. Some of the slow learners may feel hopelessly lost at the start. The teacher should try to get across the recognition that this notation for functions is only a new way of saying something, what is said is not new.

2. The text uses only the letters "f" and "g" to denote functions. In classroom discussion it is a good idea to use other letters as well.

Answers to Exercises 3-3

1. (a) The range of f is the set of positive integers $\{3, 6, 9 \ldots\}$
   (b) $f(4) = 12$
   (c) $f(6) = 18$
   (d) $f(a) = 3a$
   (e) $f(3a) = 9a$
   (f) $f(2 + x) = 6 + 3x$
   (g) Yes, because $f(3x) = 3(3x) = 9x$ and $3f(x) = 3(3x) = 9x$
   (h) No, because
   $f(3h + 4) = 3(3x + 4) = 9x + 12$
   $3f(x) + 4 = 3(3x) + 4 = 9x + 4$
2. (a) The range of f is $\{0, 1\}$
   (b) $f(2) = 0$
   (c) $f(3) = 1$
   (d) $f(104) = 0$
   (e) No, because $f(3) + f(5) = 1 + 1 = 2$ and $f(3 + 5) = f(8) = 0$
   (f) Yes, because $f(3) + f(4) = 1 + 0 = 1$ and $f(3 + 4) = f(7) = 1$
   (g) Yes, because $f(2) + f(4) = 0 + 0 = 0$ and $f(2 + 4) = f(6) = 0$
   (h) Yes, because $f(3) \cdot f(4) = 1 \cdot 0 = 0$ and $f(3 \cdot 4) = f(12) = 0$
   (i) Yes, because $f(2) \cdot f(4) = 0 \cdot 0 = 0$ and $f(2 \cdot 4) = f(8) = 0$

(j) Yes. If $x + 2$ is even, so is x and $f(x + 2) = f(x) = 0$. If $x + 2$ is odd, so is x and $f(x + 2) = f(x) = 1$

(k) No. If x is even, then $x + 1$ is odd so $f(x) = 0$ and $f(x + 1) = 1$. If x is odd, then $x + 1$ is even, so $f(x) = 1$ and $f(x + 1) = 0$

(l) No. If x is odd, then $2x$ is even so $f(x) = 1$ and $f(2x) = 0$. (For even values of x, $f(x) = f(2x)$, but since this doesn't hold for all values of x, the answer is "no".)

3. $\{ x : 0 \leq x < 2 \}$
4. $\{ x : x \geq 0 \}$

3-4. Functions Defined by Equations.

The notion of the solution set of an equation has already been defined and discussed. This section is concerned with the possibility of using such solution sets to define functions. A solution set of an equation in x and y consists of pairs (a,b). If each first member a occurs only once, then assigning b to a defines a function whose domain is the set of all first members whose range is the set of all second members.

For some equations this condition is fulfilled, for others it is not. For instance the equation $y = x^2$ defines a function - for each x there is only one y. The equation $y^2 = x^2$ does not define a function since for every x other than 0 there are two values of y.

The traditional study of functions tended to concentrate exclusively on functions defined by equations. It should be made clear here that

(a) Not all equations define functions

(b) Not all functions are defined by equations.

Suggestions for section 3-4.

Some points to be made are:

1. Not all functions are defined by equations

2. Not all equations define functions
3. To find the domain and range of a function defined by an equation, it helps to obey these rules:
   (a) don't divide by zero
   (b) don't take the square root of a negative number (or any even root)
   (c) The square of a real number must be non-negative (or more generally any even power of a real number must be non-negative)
   (d) Negative numbers have real roots of odd index and odd powers of negative numbers are negative.

Answers to Exercises 3-4

1. $f(x) = 2x + 6$
   (a) The domain of f is the set of all real numbers.
   (b) The range of f is the set of all real numbers.
   (c) $f(2) = 10$
   (d) $f(x) = 100, \quad x = 47$
   (e) $f(x) = 0 \quad , \quad x = -3$

2. (a) $y = 3x$
   Domain: Set of all real numbers.
   Range: Set of all real numbers.
   (b) $y = \frac{3}{x}$
   Domain: Set of all real numbers except zero.
   Range: Set of all real numbers except zero.
   (c) $y = \sqrt{x}$
   Domain: Set of all non-negative real numbers.
   Range Set of all non-negative real numbers.
   (d) $y = x^3$
   Domain: Set of all real numbers.
   Range: Set of all real numbers.
   (e) $y = \sqrt[3]{x}$
   Domain: Set of all real numbers.
   Range: Set of all real numbers.

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3. $y = x^2$
   (a) Domain: Set of all real numbers.
   (b) Range: Set of non-negative real numbers.
   (c) Yes, $x = \sqrt{6}$. Also $-\sqrt{6}$.
   (d) No. The square of a real number cannot be negative.
4. $y = x^3$
   (a) Domain: Set of all real numbers.
   (b) Range: Set of all real numbers.
   (c) Yes, $x = \sqrt[3]{6}$.
   (d) Yes, $x = \sqrt[3]{-6}$.
5. $y = x^n$, n is a positive integer.
   (a) Domain: Set of all real numbers.
   (b) Range: Set of all real numbers if n is odd.
       : Set of all non-negative real numbers if n is even.
   (c) Yes, $x = \sqrt[n]{6}$.
   (d) No, if n is even.
       Yes, if n is odd, $y = \sqrt[n]{-6}$.
6. $y = \frac{1}{x}$
   (a) Range: Set of all real numbers except zero.
   (b) Domain: Set of all real numbers except zero.
   (c) Yes. If $f(x) = 6$, $x = \frac{1}{6}$.
   (d) Yes. If $f(x) = -6$, $x = -\frac{1}{6}$.
7. $y = \frac{1}{x^2}$
   (a) Range: Set of all positive real numbers.
   (b) Domain: Set of all real numbers except zero.
   (c) Yes. $x = \frac{1}{\sqrt{6}}$.
   (d) No. y has to be positive.
8. $y = \frac{1}{x^n}$
   (a) Range: If n is even, all positive real numbers.
       If n is odd, all real numbers except zero.
   (b) Domain: Set of all real numbers except zero.

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(c) Yes. $x = \sqrt[n]{\frac{1}{6}}$.

(d) Yes, if n is odd, $x = \sqrt[n]{-\frac{1}{6}}$.
No, if n is even.

3-5. The Graph of a Function.

The word "graph" is already part of the student's vocabulary. It occurs in this text as a technical word with a very precise meaning and it is probably best to use it only in this way. Recommended usages are to talk about "the graph of a function," "the graph of an equation," "Plot, draw or sketch the graph of a function." While it is probably harmless to talk about "graphing an equation" or "graphing a function" these usages are are not recommended. In other words "graph" should be used as a noun and not as a verb.

The graph of a function is a set of ordered pairs. If these pai are pairs of numbers then they can be regarded as points and the graph becomes a geometric figure. There is no geometric figure attached to the functions which do not pair numbers with numbers. Nevertheless, these functions have graphs too. The graph of such a function f is the set of all pairs (a, f(a)) where a is the domain of f and f(a) is what f assigns to a. Thus if f were the function which assigned to each state of the U.S.A. its capitol then (New York, Albany) would be in its graph. This pair is clearly not a point on a geometric object of any kind. The student will not have much occasion to use the term "graph" for non-geometric objects in this course. In his later work such usage will be more frequent.

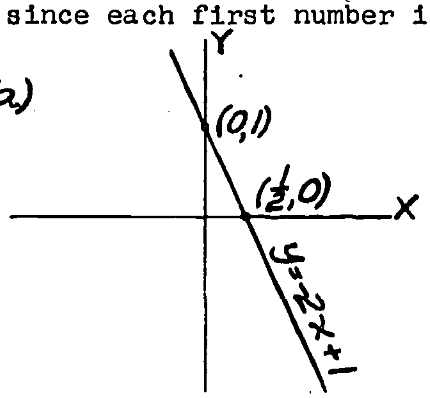
Suggestions for section 3-5.

1. Problems 1 and 2 are suitable for classroom discussion.
2. Problems 3 and 4 should be handled lightly as far as geometry and calculation are concerned. The main thing is that for each x there is exactly one y.

Answers to Exercises 3-5

1. No, (1,2) and (1,3) cannot occur in the graph of same function for to a single value of domain must be assigned only one value of the range.
2. Yes, (2,1) and (3,1) can occur in the graph of same function since each first number is paired with a single second member.

3. (a)



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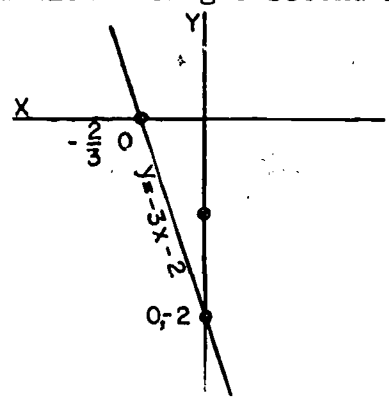

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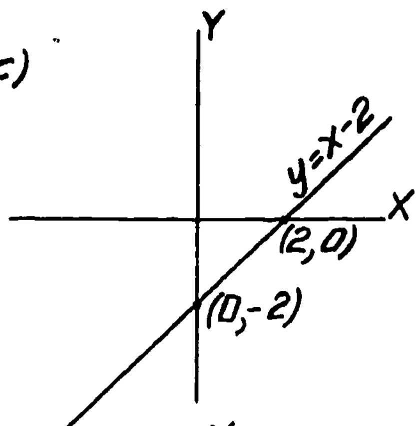

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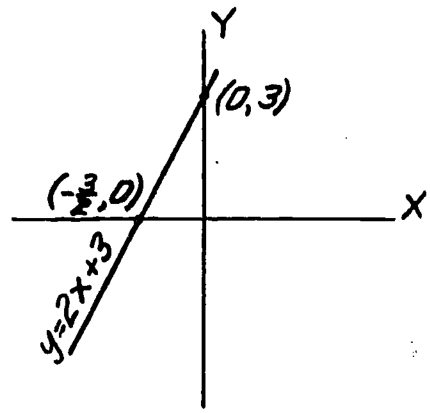
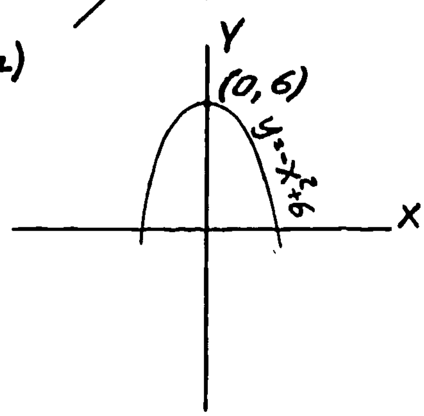
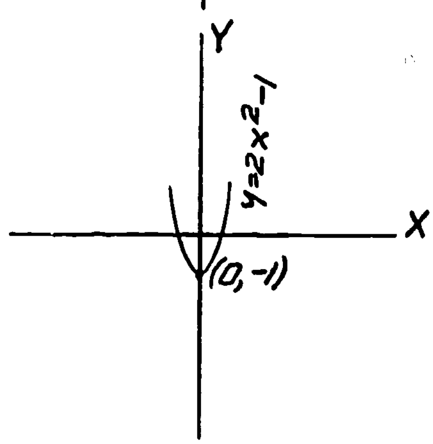
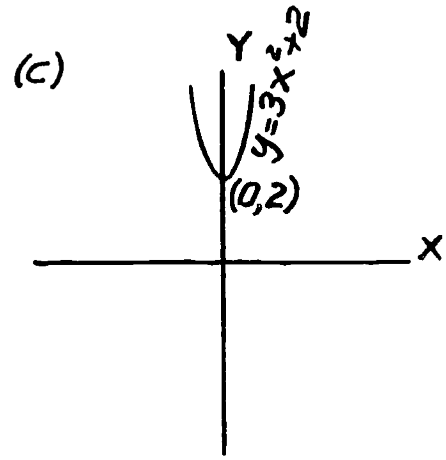


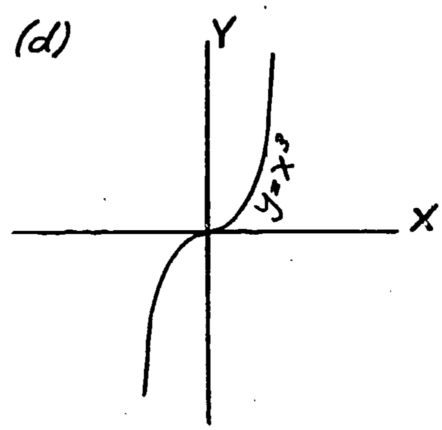
4. (a)



(b)

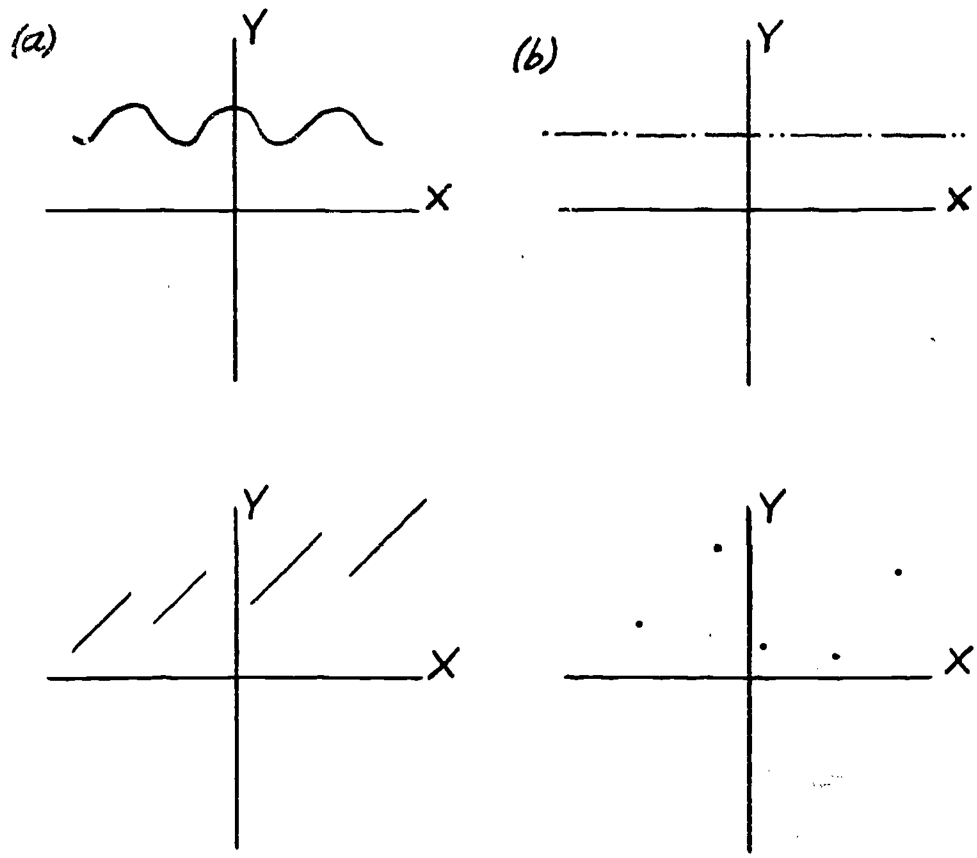






3-6. Functions Defined Geometrically.

This section reverses the relation studied in section 3-5. In section 3-5 we start with a function and plot its graph. In this section we start with a set of points and seek a function of which it is the graph. Two facts are noteworthy. In the first place, not all sets of points are graphs of functions. The second fact is that a set of points can be the graph of a function and fail to be the graph of an algebraic equation. A set of points is the graph of a function if and only if each vertical line intersects it in at most one point. This permits some strange looking set of points to define functions.



<u>Suggestions</u> <u>for</u> <u>section</u> <u>3-6</u>.

1. All these problems except problem 8 are suitable for classroom discussion. The "vertical line" test is the important thing.
2. Problem 8 calls for a proof. It cannot be answered completely by a graph or by some numerical examples. Nevertheless, a graph and numerical examples can show whether the student is thinking along correct lines.

[pages 176-179]

Answers to Exercises 3-6

1. No, because some vertical lines intersect the circle in more than one point.
2. Yes, if the diameter is parallel to x-axis.
3. Yes, if diameter is not parallel to the x-axis.
4. No, for some vertical lines cross the triangle in more than one point.
5. Yes, any line not parallel to y-axis is the graph of a function.
6. Yes, because no line parallel to the y-axis is the graph of a function.
7. Use the vertical line test on each of these.
   (a) Yes.
   (b) No.
   (c) No.
   (d) Yes.
   (e) Yes.
   (f) No.
   (g) No.
   (h) Yes.
   (i) No.
8. If n is odd for each real number x, there is one and only one number y for which $y^n = x$. Therefore $y^n = x$ defines a function for odd n.
   If n is even for each negative number x, there is no number y and for each positive number x. There are two numbers y such that $y^n = x$. Therefore $y^n = x$ does not define a function if n is even.

3-7. Functions Defined by Physical Processes.

This may be a good place to point out that science and common sense are not the same thing. Common sense can see that a falling body has a speed at each instant but it needs deep insight to find an algebraic equation which will relate speed and time. This is the task of the physicists. Some students

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will know a few such equations of physics but the teacher should not require or expect it.

Suggestions for section 3-7.

1. Most of the answers involve a constant k. This is to allow for variations in the physical conditions of the problem, choice of units, and other relevant factors not mentioned explicitly.
2. One of the objectives of these problems is to obtain mathematical statements which express what the student knows. He should not be expected to know the correct formula but should be required to devise one which is qualitatively (if not quantitatively) reasonable. This means that if he knows that the temperature diminishes with time, he should devise a formula in which this actually happens.

Answers to Exercises 3-7

1.	Domain:	Pressures
	Range:	Volumes
	Rule :	Assign to each pressure its corresponding volume.
	Algebraic Formula:	$V = \frac{k}{P}$, where P denotes pressure, V denotes volume, and k is a number which depends on the properties of the gas and the container.
2.	Domain:	Set of lengths of pendulums.
	Range:	Set of time intervals.
	Rule:	Assign to each length the time it takes a pendulum of that length to complete a swing.
	Algebraic Formula:	(approximate) $t = k\sqrt{L}$, where L denotes length, t denotes time, and k is a suitable constant.
3.	Domain:	Set of all positions in space of the body.

Range: Magnitude of the gravitational attractions.

Rule: Assign to each position of the body, the gravitational attraction of the earth on it in that position.

Algebraic Rule: $F = \frac{k}{r^2}$, where r denotes the distance of the position of the body from the center of the earth, F denotes the attraction in question and where k is a suitable constant.

4. Domain: Set of all weights of the objects.

Range: Set of displacements of the beam.

Rule: Assign to each weight the displacement it produces.

Algebraic Formula: $d = kw$, where w denotes the weight, and d denotes the displacement. (This is a reasonable guess, which is not entirely accurate.)

5. Domain: Set of distances of the observer.

Range: Set of apparent brightnesses.

Rule: Assign to each distance the corresponding apparent brightness.

Algebraic Formula: $b = \frac{k}{r^2}$, where r denotes the distance, b denotes the apparent brightness and k is a suitable constant.

6. Domain: Set of all distances of fulcrum.

Range: Set of all forces exerted.

Rule: Assign to each distance the corresponding force.

Algebraic Rule: $F = \frac{k}{L}$, where L denotes the distance, F denotes the corresponding force and k is a suitable constant.

7. Domain: Set of all times of flow.

Range: Set of all volumes of the water.

Rule: Assign to each time the corresponding volume.

Algebraic Formula: $V = kt$, where t denotes time of flow, V denotes volume and k is a suitable constant.

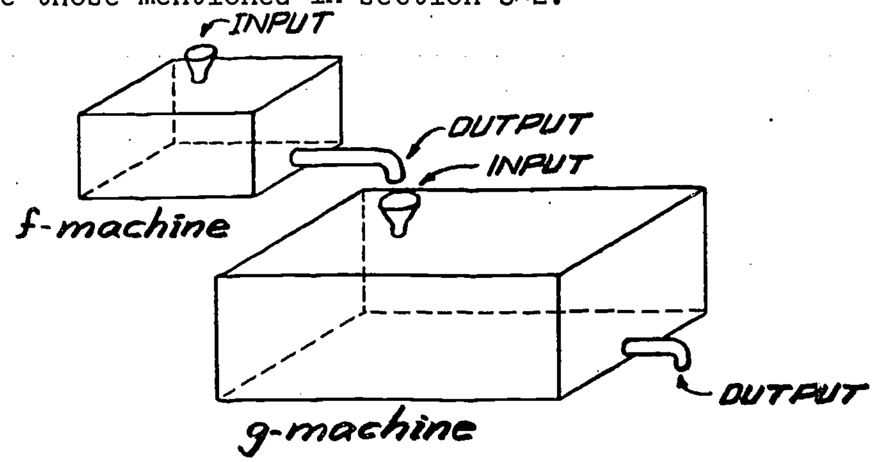
8. Domain: Set of all cooling times.
Range: Set of all temperatures of coffee.
Rule: Assign to each cooling time the corresponding temperature.
Algebraic Formula: $T = T_r + \frac{a}{t + b}$, where T denotes temperature, T_r denotes room temperature, t denotes time and where a and b are suitable constants. (This is a reasonable guess. Chapter 9, on exponentials gives a better one.)

9. Domain: Set of all altitudes.
Range: Set of all boiling points of water.
Rule: Assign to each altitude the corresponding boiling point.
Algebraic Formula: $B = 212 - kh$, where B denotes the boiling point, h denotes the altitude and where k is a suitable constant. (This is a reasonable guess. It is far from accurate.)

10. Domain: Speeds of an automobile.
Range: Times needed to stop.
Rule: Assign to each speed its corresponding time.
Algebraic Formula: $T = kv^2$, where T denotes stopping time, v denotes speed of automobile and where k is a suitable constant. (This is a reasonable guess. It shows that if the speed of an automobile is doubled, its stopping time is quadrupled.)

3-8. Function Defined by Composition; Inverses.

Composition of functions is easily illustrated by using boxes like those mentioned in section 3-2.



If the output of the f-machine is fed into the input of the g-machine, then the whole device illustrates the composition of g with f. Notice that it is assumed that the output of the f-machine can be fed into the input of the g-machine. This only requires that the range of f be included in the domain of g. Our formal definition goes a little farther and asks that the range of f be the same as the domain of g. This makes for a neater treatment later. The notion of inverse functions is treated in Chapter 9, in the discussion of logarithmic and exponential functions. Some preview here for the student (and also for the teacher) might not be amiss.

Suggestions for section 3-8.

1. In classroom discussion students can be invited to find examples of composition of functions from everyday life. Elections in which voters elect the electors, and the electors elect the office holders is an example.
2. Addition and multiplication have good examples on inverses. Note that $x + 6$ has an inverse, but $x + y$ does not.
3. Algebraic technique for actually finding inverses should not be stressed.

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Answers to Exercises 3-8

1. (a) $f(g(x)) = (x^3)^2 = x^6$.

   (b) $g(f(x)) = (x^2)^3 = x^6$.

   (c) Yes.

2. (a) $f(g(x)) = (x^3 + 1)^2 + 1 = x^6 + 2x^3 +$.

   (b) $g(f(x)) = (x^2)^3 + 1 = x^6 + 1$

   (c) No.

3. $f(x) = 2x + 3$.

   $g(x) = \frac{1}{2}x - \frac{3}{2}$.

   $f(g(x)) = 2(\frac{1}{2}x - \frac{3}{2}) + 3 = x$.

   $g(f(x)) = \frac{1}{2}(2x + 3) - \frac{3}{2} = x$.

   Therefore f and g are inverse functions.

4. $f(x) = 4x + 5$.

   $g(x) = \frac{1}{4x + 5}$

   $f\big(g(x)\big) = 4(\frac{1}{4x + 5}) + 5 = \frac{4}{4x + 5} + 5 = \frac{29 + 20x}{4x + 5} \neq x$.

   $g\big(f(x)\big) = \frac{1}{4(4x + 5) + 5} = \frac{1}{16x + 25} \neq x$.

   Therefore f and g are not inverses.

5. Only a, d, e, and h define functions and none of these has an inverse because each of the graphs is crossed by some horizontal line in more than one point.

6. (a) The line $y = x$ is the perpendicular bisector of the segment whose endpoints are (a,b) and (b,a). To prove this, use the distance formula.

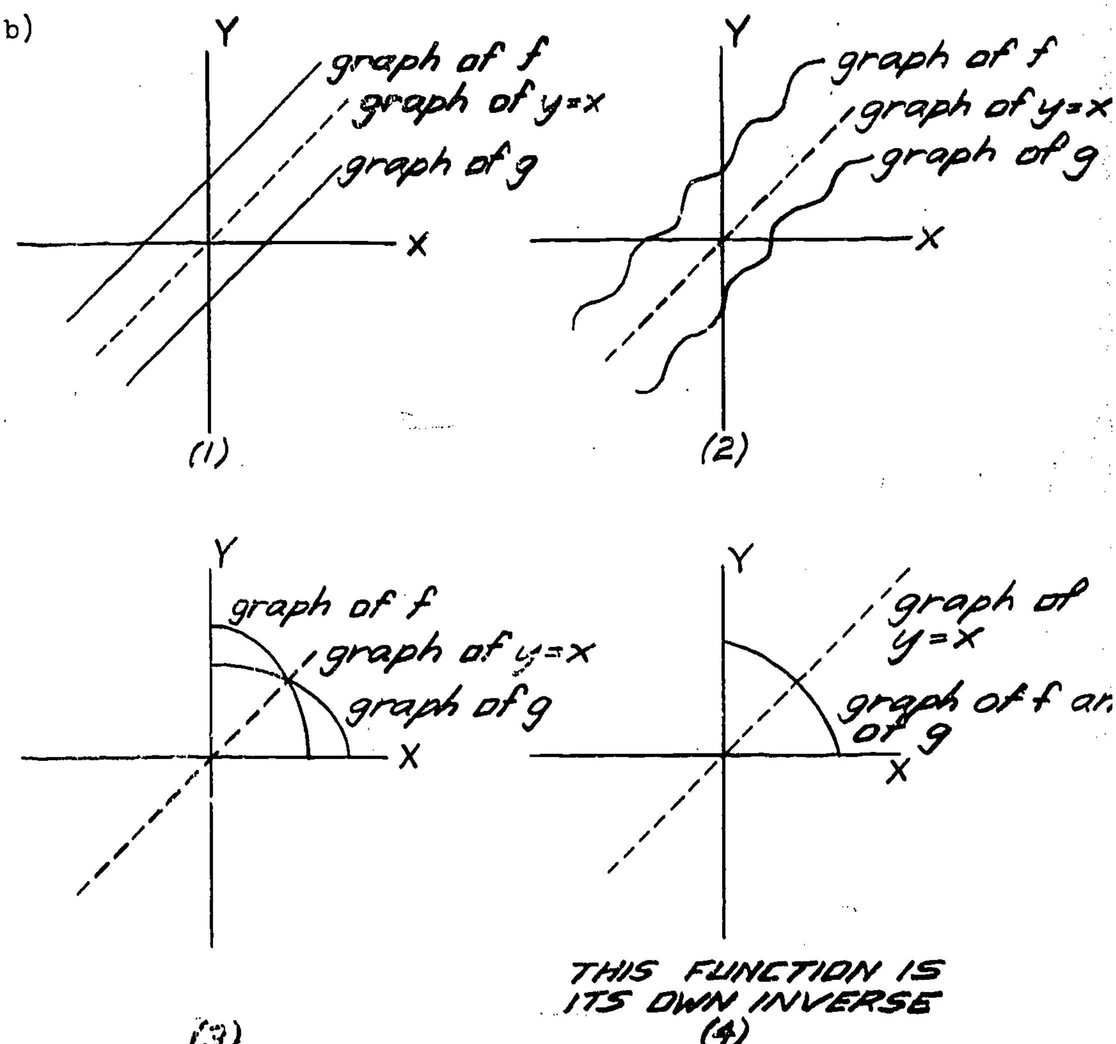
$$(x - a)^2 + (y - b)^2 = (x - b)^2 + (y - a)^2$$

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = x^2 - 2bx + b^2 + y^2 - 2ay + a^2$$

$$x = y.$$

The rule we infer is that if a point P is on the graph of a function f which has an inverse g, then the mirror image of P in the line $y = x$ is on the graph of g. The whole graph of g can be obtained by reflecting the graph of f in the line $y = x$.

6. (b)



3-9. The Linear Function.

The traditional treatment of the linear function emphasizes the graph to the exclusion of almost all its other properties. It is hoped that the teacher will be willing to stand behind the somewhat different viewpoint stressed here; that the linear function is a special kind of pairing of numbers with numbers.

Theorem 3-9a relates to an interesting paradox. Consider all the positive integers $\{1, 2, 3, \ldots, n, \ldots\}$ and all the even positive integers $\{2, 4, 6, \ldots, 2n, \ldots\}$.

There is a natural pairing of the members of the two sets, n with $2n$, which is in fact a one-to-one correspondence. It follows that there are as many even integers as there are integers. This fact is somewhat distressing because it is clear that the second set is a proper subset of the first, and so ought to have fewer members. Theorem 3-9a can be regarded as a generalization of this fact.

Theorem 3-9a also implies that every linear function has an inverse. The corollary which states this fact is an example of what mathematicians call an "existence theorem." It should be compared with Theorem 3-9b which shows how to find the inverse in question and identifies the inverse as a linear function.

Theorem 3-9c and 3-9d are rarely stated in elementary courses because when the treatment is largely geometric they seem almost trivial. The proof of Theorem 3-9d may create some problems. It ends up by representing $f(x)$ as the expression $tx - tq_o + f(q_o)$. It is not easy to see that this really is of the form

"number times x plus number"

on account of all the subscripts, letters and parentheses. It also may occur to a bright student to wonder whether a different linear function could have been obtained if a different value from q_o, say q_1, had been used at the outset. Starting the proof with q_1 would have led to $f(x) = tx - tq_1 + f(q_1)$. The question is whether the expression $-tq_o + f(q_o)$ and $-tq_1 + f(q_1)$ are equal.

This can be settled by appealing to the hypothesis, which

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implies that

$$\frac{f(q_1) - f(q_o)}{q_1 - q_o} = t, \text{ and}$$

then clearing fractions.

Suggestions for section 3-9.

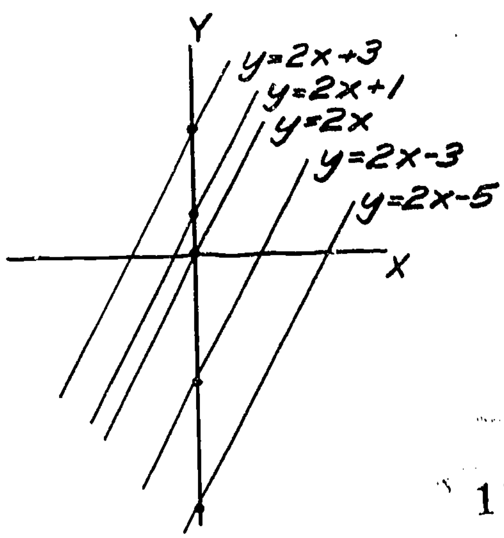
The algebra of finding the inverse of a linear function is easy. If $f(x) = ax + b$, then its inverse g is found by solving $y = ax + b$ for x and substituting x for y in the result. The student should know primarily what he is looking for; the method of finding it is subordinate.

Answers to Exercises 3-9

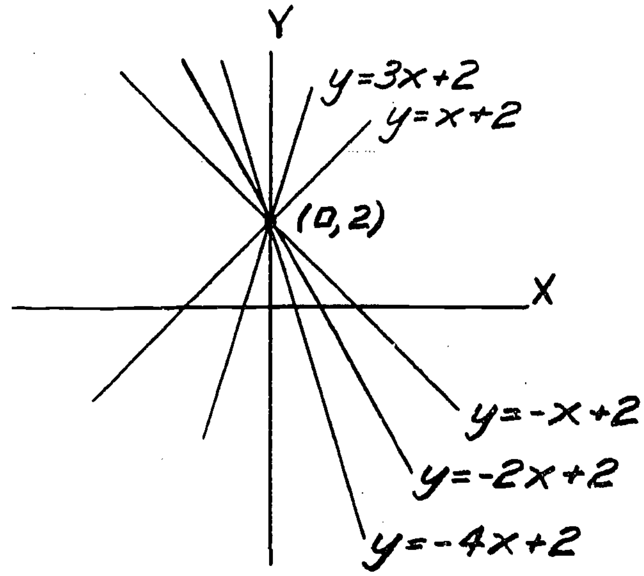
1. (a) Yes.
   (b) Yes.
   (c) Yes.
   (d) No, (constant function).
   (e) Yes.
   (f) Yes.
   (g) No.
   (h) No, (constant function).

2. $y = 5x + 6$
   (a) $f(0) = 6$
   (b) $f(\frac{1}{2}) = \frac{17}{2}$
   (c) $f(11) = 61$
   (d) $x = -\frac{6}{5}$
   (e) $x = -\frac{11}{10}$
   (f) $x = 1$

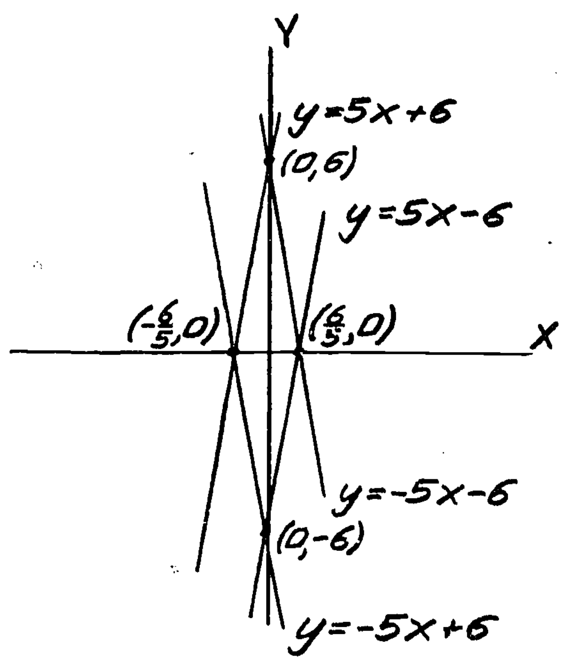
3.



4.


Y
X
y=3x+2
y=x+2
(0,2)
y=-x+2
y=-2x+2
y=-4x+2

5.


Y
X
y=5x+6
(0,6)
y=5x-6
(-6/5,0)
(6/5,0)
y=-5x-6
(0,-6)
y=-5x+6

6. (a) $y = 2x - 1$

$2x = y + 1$

$x = \frac{1}{2}y + \frac{1}{2}$

$y = \frac{1}{2}x + \frac{1}{2}$ (Inverse)

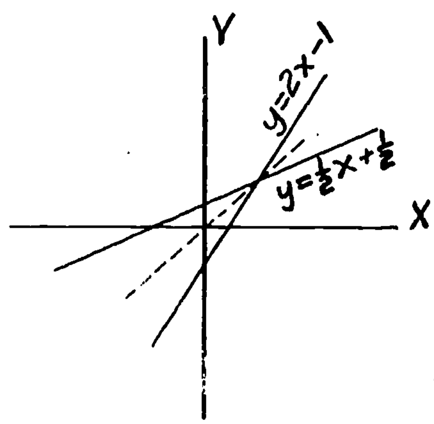
(b) $y = \frac{1}{3}x - \frac{5}{3}$

(c) $y = -\frac{1}{2}x + 3$

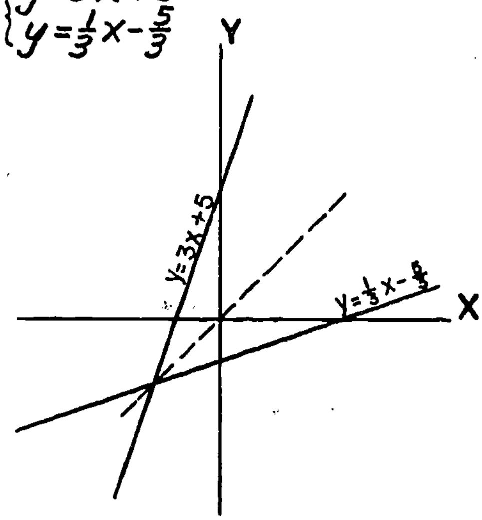
(d) $y = -x - 4$

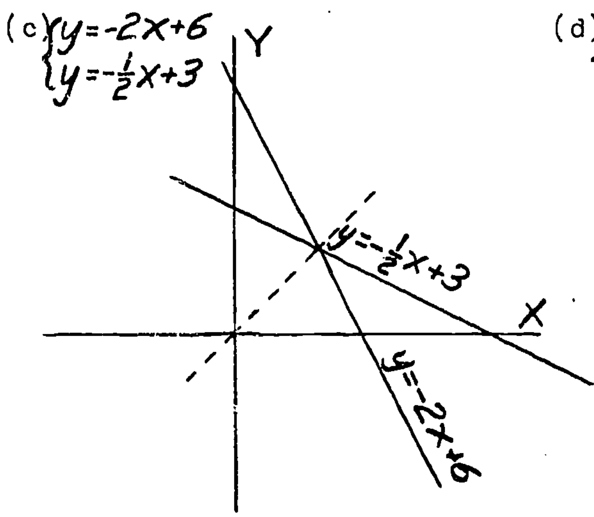
(e) $y = \frac{1}{6}x - \frac{7}{6}$

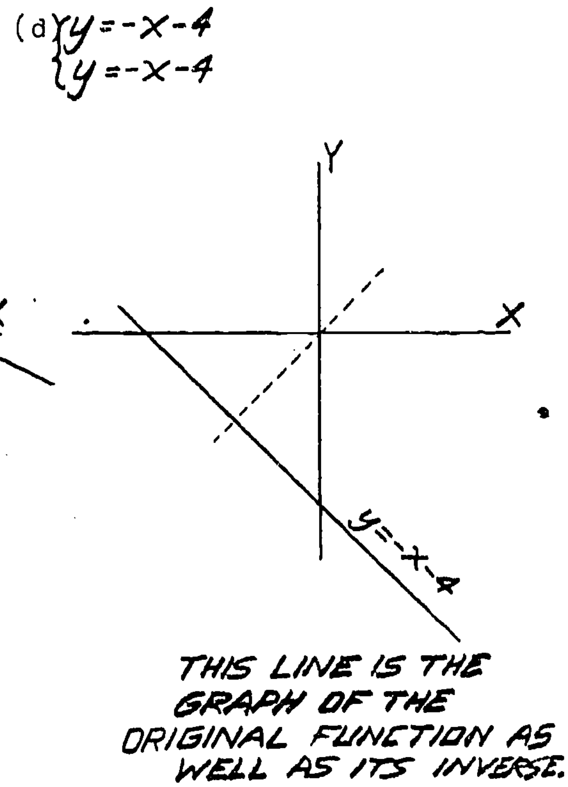
7. (a) $\begin{cases} y = 2x - 1 \\ y = \frac{1}{2}x + \frac{1}{2} \end{cases}$

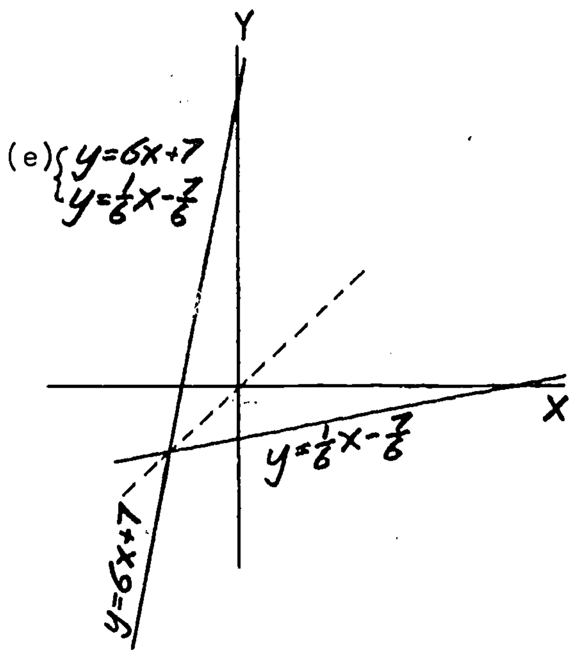


(b) $\begin{cases} y = 3x + 5 \\ y = \frac{1}{3}x - \frac{5}{3} \end{cases}$









8.
(a) $g(x) = \frac{1}{2}x + \frac{7}{2}$

(b) $f(6) = 5$

(c) $g(f(6)) = 6$

(d) $g(6) = \frac{13}{2}$

(e) $f(g(6)) = 6$

9. Prediction: -3 using Theorem 3-9c.

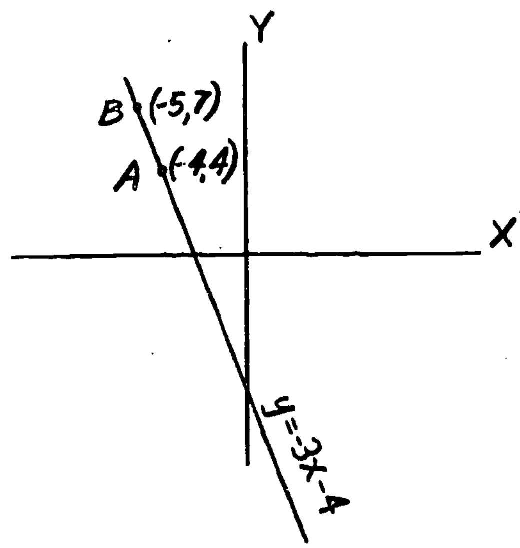
Computation: $\frac{(-3004) + 304}{900} = -\frac{2700}{900} = -3$

10. For example choose points

A(-4,4) and B(-5,7)

$\frac{f(7) - f(4)}{7 - 4} = a$

$\frac{-5 - 4}{7 - 4} = -3.$



11. $\frac{f(3) - f(5)}{3 - 5} = \frac{9 - 25}{3 - 5} = \frac{-16}{-2} = 8$

$\frac{f(4) - f(6)}{4 - 6} = \frac{16 - 36}{4 - 6} = \frac{-20}{-2} = 10$

Therefore

$\frac{f(3) - f(5)}{3 - 5} \neq \frac{f(4) - f(6)}{4 - 6}$

12. (a) We solve

$\frac{x^2 - 49}{x - 7} = 16.$ Since $x \neq 7$ (and hence $x - 7 \neq 0$) we divide both numerator and denominator of the fraction on the left by $x - 7$ to obtain

$$x + 7 = 16$$
Hence $x = 9$

(b) Every real x except 7 (this follows from Theorem 3-9c).

3-10. Linear Functions having Prescribed Values.

In some traditional algebra courses students are given sample pairings and are asked to fill in others, as in the following:

1	2	3	4	5	?	
4	6	8	10	?	2	

While it is true that such problems are often dealt with in an acceptable way, the problems themselves really are not acceptable problems. They involve hidden assumptions without which no definite answer is possible. For instance, there is no way of telling what $f(5)$ is knowing that $f(1) = 4$, $f(2) = 6$, $f(3) = 8$, $f(4) = 10$. The guess that $f(5) = 12$ is reasonable but there is no sound way of justifying this unless further assumptions are made about f.

The state of affairs with the linear function is quite different. If f is known to be a linear function then from information such as $f(1) = 4$, $f(2) = 6$, the value of $f(x)$ for any x whatsoever can be computed. It is with this that Theorem 3-10b is concerned. It asserts that any two pairings of a linear function determines all other pairings. It also asserts that a linear function can be found to assign to given numbers x_1, x_2 given numbers y_1, y_2.

When we study the quadratic function in Chapter 4 we shall see a theorem there which is like Theorem 3-10a. It asserts that any three pairings of a quadratic function determines all its other pairings and that generally a quadratic function can be found to assign given numbers y_1, y_2, y_3 to given numbers x_1, x_2, x_3.

Suggestions for section 3-10.

Use the graphs to illustrate the algebra, not to develop it.

Answers to Exercises 3-10

1. $y = ax + b$

   (a) $1 = a + b$

   $3 = 3a + b$

   $a = 1$

   $b = 0$

   $y = x$

   (b) $y = -x + 4$

   (c) $y = -\frac{1}{4}x + \frac{13}{4}$

   (d) $y = 42x - 294$

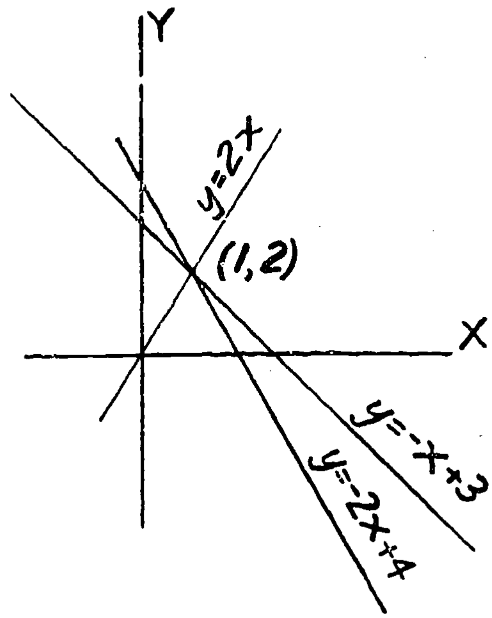
2. Same as problem 1.

   (a) $y = x$

   (b) $y = -x + 4$

   (c) $y = -\frac{1}{4}x + \frac{13}{4}$

   (d) $y = 42x - 294$



3.

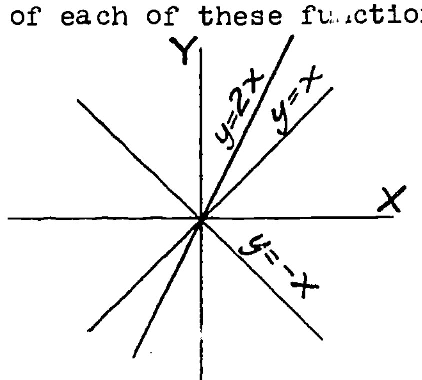
   (a) $f(x) = x + 1$

   $f(x) = 2x$

   (b) $f(x) = a(x - 1) + 2$ for any $a \neq 0$

   (c) The graphs of each of these functions goes through (1,2).

4.

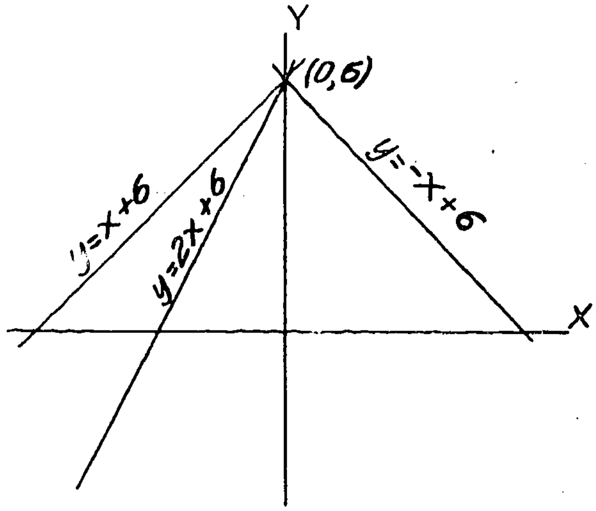


   (a) $f(x) = ax$ for all non-zero real a.

   All graphs have $(0,0)$ in common.

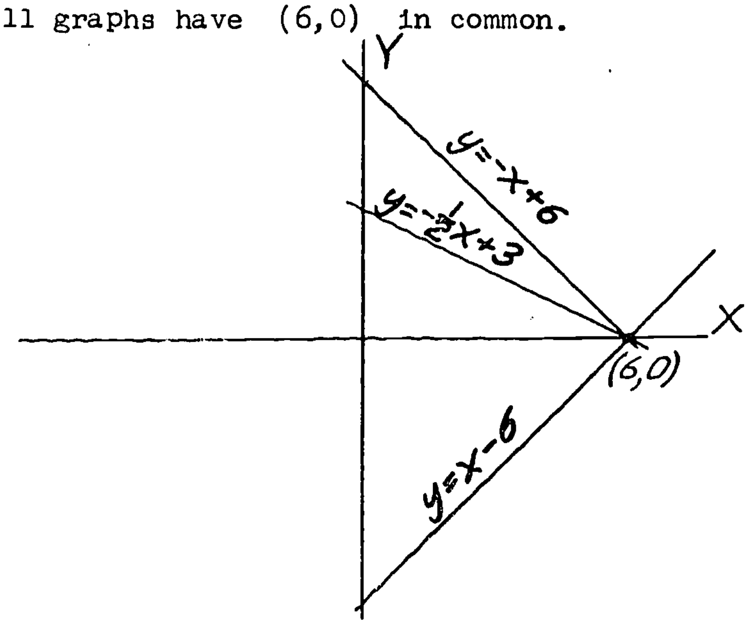
(b) $f(x) = ax + 6$ for all non-zero real a.

All graphs have (0,6) in common.



(c) $f(x) = (-\frac{b}{6})x + b$ for all non-zero real b.

All graphs have (6,0) in common.



Answers to Miscellaneous Problems 3-11

1. (a) Domain: Set of all hexagons.
   Range: Set of all non-negative real numbers.
   Rule: To each hexagon is assigned the number which is sum of lengths of its 6 sides.

   (b) Domain: Set of all circles.
   Range: Set of all positive real numbers.
   Rule: To each circle is assigned the number πd where d is the diameter.

   (c) Domain: {1, 2, 3}
   Range: {4, 9}
   Rule: Assign 9 to 1, 4 to 2 and 9 to 3.

   (d) Domain: Set of times.
   Range: Set of temperatures.
   Rule: Assign to each time its temperature.

   (e) Domain: Set of all real numbers.
   Range: Set of all real numbers greater than or equal to -6.
   Rule: Assign to each x the number $x^2 + 4x - 2$.

   (f) Domain: {1, 2, 3}
   Range: {a}
   Rule: To each number in {1, 2, 3} assign a.

   (g) Domain: Set of all real numbers.
   Range: Set of all real numbers greater than or or equal to 3.
   Rule: To each number x, assign the number $x^2 + 3$.

   (h) Domain: Set of all possible speeds r and elapsed times t.
   Range: Set of all possible distances d.
   Rule: $d = rt$.

   (i) Domain: Set of all integers.
   Range: {1, -1}
   Rule: To each even positive integer assign 1, d to each odd positive integer assign -1.

(j) Domain: Set of all state capitols.

Range: Set of all distances of state capitols from Washington, D. C. Each number in this set is a positive integer.

Rule: To each capitol assign its distance from Washington, D. C.

2.

(a) Domain: Set of all real numbers.

(b) Range: Set of all real numbers greater than or equal to -5.

(c) $f(0) = -5$

(d) $f(-1) = -2$

(e) $f(5) = 70$

(f) $f(a) = 3a^2 - 5$

(g) $f(a - 1) = 3(a - 1)^2 - 5 = 3a^2 - 6a - 2$

(h) $f(\pi) = 3\pi^2 - 5$

3.

(a) Range: $\{1, -1\}$

(b) $f(-3) = -1$

(c) $f(0) = 1$

(d) $f(3) = 1$

(e) $f(2 - 6) = -1$

(f) $f(2) - f(6) = 0$

(g) $f(4) + f(2) = 2$

(h) $f(4 + 2) = 1$

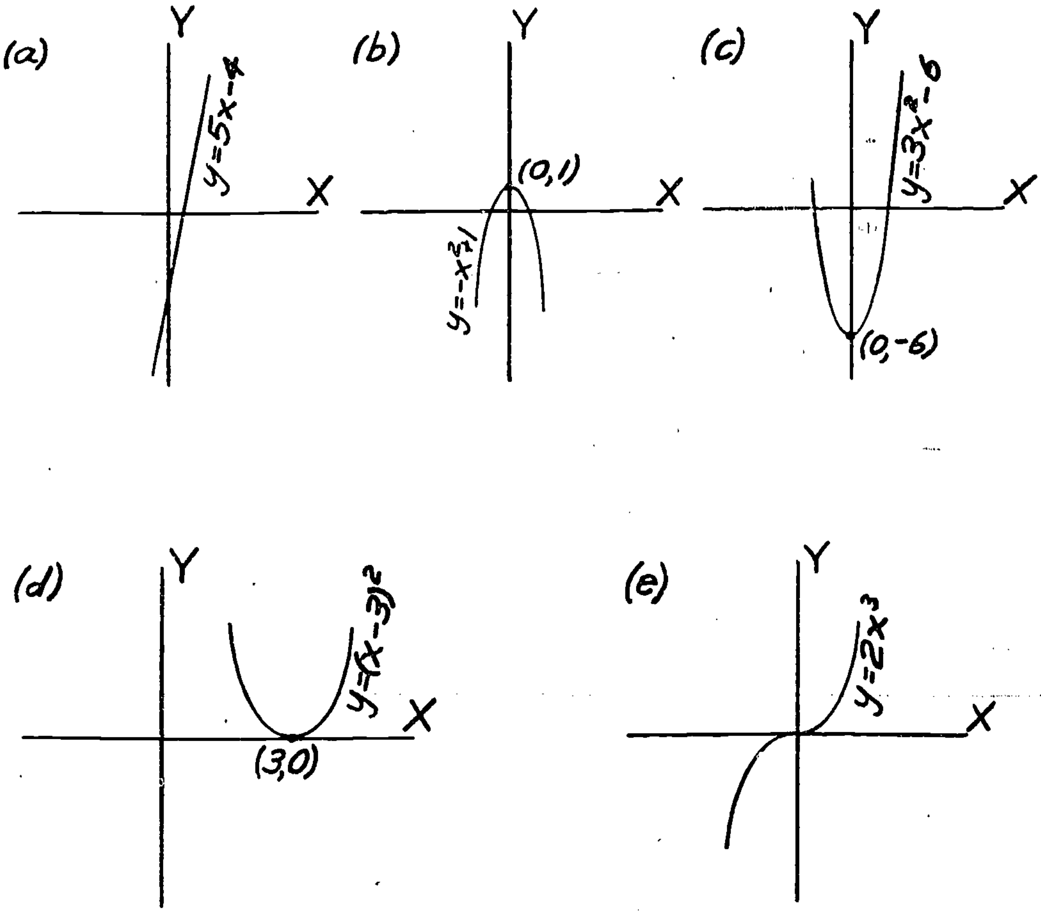
(i) $f(-6) = -1$

(j) $f(-6) + 3 = 2$

(k) $f(3 \cdot 6) = 1$

(l) $3f(6) = 3$

4.



5. $f(x) = x^2 + 3$ and $g(x) = 2x + 5$

   (a) $f(g(x)) = (2x + 5)^2 + 3 = 4x^2 + 20x + 28$

   (b) $g(f(x)) = 2(x^2 + 3) + 5 = 2x^2 + 11$

6.

   (a) $f: y = x + 5$

   $x = y - 5$

   $g: y = x - 5$

   $f(g(x)) = (x - 5) + 5 = x$

   $g(f(x) = (x + 5) - 5 = x$

   (b) $f: y = -2x - 1$

   $+2x = -y - 1$

   $x = -\frac{1}{2}y - \frac{1}{2}$

(continued - next page)

[page 200]

$y = -\frac{1}{2}x - \frac{1}{2}$

$f(g(x)) = -2(-\frac{1}{2}x - \frac{1}{2}) - 1 = x$

$g(f(X)) = -\frac{1}{2}(-2x - 1) - \frac{1}{2} = x$

(c) $f: y = -3x + 7$

$-3x = y - 7$

$x = -\frac{1}{3}y + \frac{7}{3}$

$g: y = -\frac{1}{3}x + \frac{7}{3}$

$f(g(x)) = -3(-\frac{1}{3}x + \frac{7}{3}) + 7 = x$

$g(f(x) = -\frac{1}{3}(-3x + 7) + \frac{7}{3} = x$

(d) $f: y = 5x - 6$

$5x = y + 6$

$x = \frac{1}{5}y + \frac{6}{5}$

$g: y = \frac{1}{5}x + \frac{6}{5}$

$f(g(X)) = 5(\frac{1}{5}x + \frac{6}{5}) - 6 = x$

$g(f(X)) = \frac{1}{5}(5x - 6) + \frac{6}{5} = x$

7. (a) $\frac{f(x) - f(3)}{x - 3} = -1$

$f(x) - 5 = -x + 3$

$f(x) = -x + 8$

(b) $\frac{f(x) - f(1)}{x - 1} = -\frac{1}{4}$

$f(x) = -\frac{1}{4}(x - 1)$

$f(x) = -\frac{1}{4}x + \frac{1}{4}$

(c) $\frac{f(x) - f(-2)}{x + 2} = -1$

$f(x) - 3 = -x - 2$

$f(x) = -x + 1$

(d) $\frac{f(x) - f(0)}{x - 0} = -\frac{3}{5}$

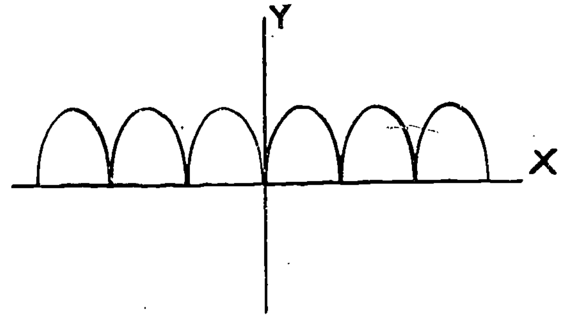
$f(x) - 5 = -\frac{3}{5}x$

$f(x) = -\frac{3}{5}x + 5$

8. (a) Yes.

(b) No. Here is the graph of a function f which is not the constant function for which $f(x + 1) = f(x)$.

[page 200]



9. (a). Yes.
   (b) No, same as graph in 8(b).

10. $f: y = x^3 + 1$

    $g: y = \sqrt[3]{x - 1}$

    $f(g(x)) = (\sqrt[3]{x - 1})^3 + 1$

    $= x - 1 + 1 = x$

    $g(f(x)) = \sqrt[3]{x^3 + 1 - 1} = x$

11. $y = \frac{1}{x + 1}$

    (a) Domain: Set of all real numbers except -1.
        Range: Set of all real numbers except 0.

    (b) $f: y = \frac{1}{x + 1}$

        $g: y = \frac{1}{x} - 1$

        Domain of g is all real numbers except 0.

        Range of g is all real numbers except -1.

12. $f(x + 1) = a(x + 1)^2 + b(x + 1) + c$

    $f(x) = ax^2 + 6x +$

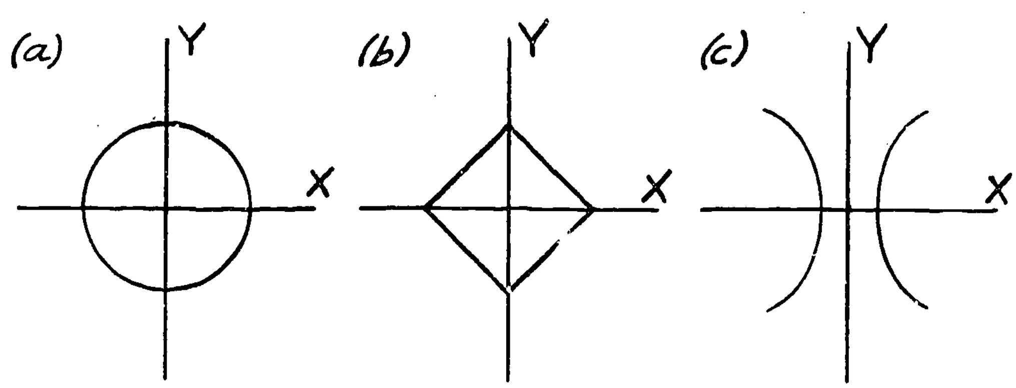
    $f(x + 1) - f(x) = 2ax + a + b$

    Therefore, since $g(x) = f(x + 1) - f(x)$ and since $a \neq 0$, g is a linear function.

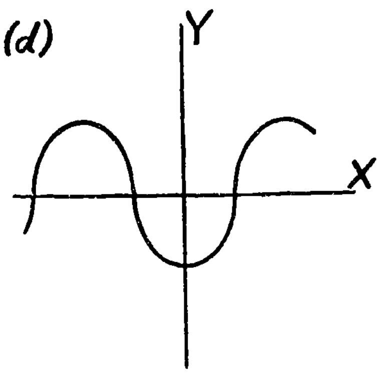
13. (a) $y = x$ defines a function which is its own inverse.
    (b) $y = -x + b$ for any real b
        or
        $y = x$

Illustrative Test Questions 3-12

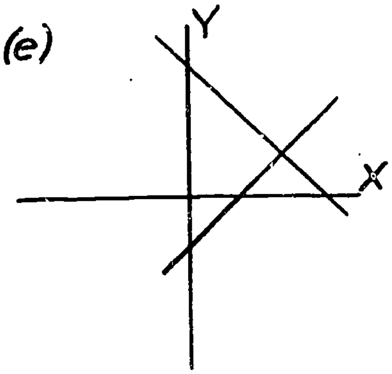
1. If f is the function defined by $f(x) = (x - 1)^2 + 1$ whose domain is the set of all positive real numbers, find
   (a) $f(1)$ and $f(2)$
   (b) the range of f
   (c) the value of x for which $f(x) = 5$.
   (d) Does f have an inverse?
2. Answer (a), (b), (c) of question 1 for the function defined by $f(x) = \frac{2x - 1}{x + 1}$, if the domain is the set of all positive real numbers.
3. Find the linear function such that
   (a) $f(-2) = -1$, $f(4) = 2$
   (b) $f(1) = 2$, $f(2) = 5$
4. For which value of k will the linear function $y = kx + k$ pair -3 with 2?
5. Given that $f(x) = 3x^2 + 5$
   $g(x) = 5x + 3$
   (a) $f(g(x)) = ?$
   (b) $g(f(x)) = ?$
   (c) Does $f(g(x)) = g(f(x))$?
6. Which of following graphs defines a function?



(d)

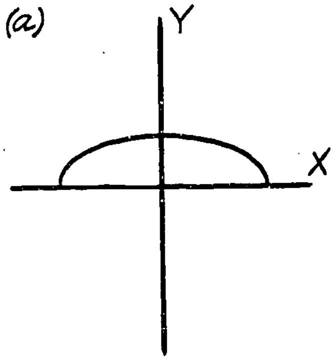


(e)

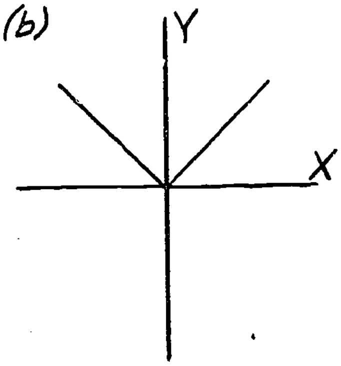


7. Which of following graphs does <u>not</u> define a function?

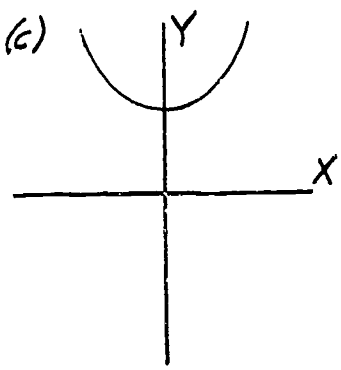
(a)



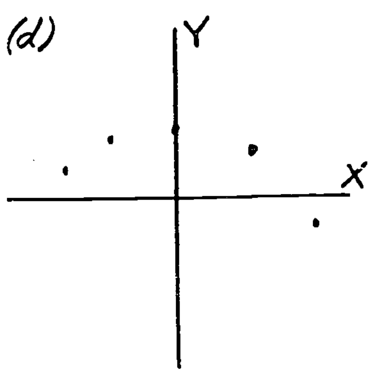
(b)



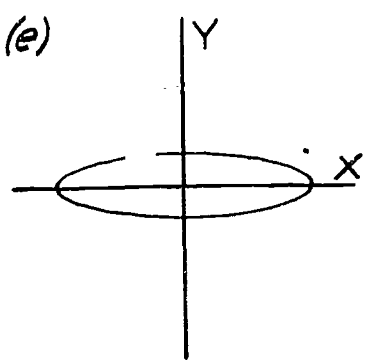
(c)



(d)



(e)



8. Given the following linear functions. Find the inverse of each. For each function plot its graph and the graph of its inverse using a single set of axes.
   (a) $y = 3x + 8$
   (b) $y = -x + 6$
   (c) $y = 2x - 3$
   (d) $y = x - 6$

9. Give an example of each of the following:
   (a) A function whose range is a single real number
   (b) A function whose domain is an infinite set of real numbers and whose range is a finite set of real numbers.
   (c) A non-linear function whose domain and range are the same.

10. Which of the following equations does not define a function?
    (a) $y = |x|$
    (b) $|y| = x$
    (c) $\sqrt{y} = x$
    (d) $y = \sqrt{x}$
    (e) $y = \frac{1}{x^2 + 1}$

11. If the domain of a function f is $\{x: -5 \leq x \leq 2\}$ and if $f(x) = 2x + 4$ what is the range of this function?
    (a) $\{y: -14 \leq y \leq 8\}$
    (b) $\{y: -6 \leq y \leq 8\}$
    (c) $\{y: 10 \leq y \leq 12\}$
    (d) $\{y: -10 \leq y \leq 4\}$
    (e) $\{y: -6 \leq y \leq 12\}$

12. The domain of the function defined by $f(x) = \frac{3}{x - 3}$ is
    (a) All real numbers
    (b) $\{x: -3 < x < 3\}$
    (c) $\{x: x \neq 1\}$
    (d) $\{x: x \neq 3\}$
    (e) None of these.

13. Given that f is a linear function, that g is its inverse, that $f(2) = 4$ and that $f(3) = -2$.
    $g(4) = ?$, $g(-2) = ?$, $g(5) = ?$

14. Plot the graph of each of the following linear functions using a single set of coordinate axes.
    (a) $y = 3x + 2$
    (b) $y = 5x + 2$
    (c) $y = -x + 2$
    (d) $y = x + 2$

15. Let $f(x) = |x| + x$
    (a) What is the domain of f?
    (b) What is the range of f?
    (c) Sketch the graph of f.

16. Which of the following pairings are the pairings of a function?
    (a) (1,2), (2,3), (3,4)
    (b) (1,2), (2,2), (3,2)
    (c) (1,2), (1,3), (1,4)
    (d) (2,1), (3,1), (2,3)

17. Describe the domain of the function defined by each of the following equations.
    (a) $y = \frac{1}{x}$
    (b) $y = \frac{1}{1 - x^2}$
    (c) $y = \frac{1}{x^2 - 2}$
    (d) $y = \sqrt{x}$
    (e) $y = \sqrt{4 - x}$

18. For what value of c is the function defined by $y = cx + c$ its own inverse?

19. (a) Which of the following equations defines a function which has an inverse?
    (b) Which, if any, of these functions is its own inverse?
    1) $y = |x|$
    2) $y = -x - 3$
    3) $y = \frac{x + 1}{x - 1}$
    4) $y = x^2 + 1$
    5) $y = x^3 + 1$

20. (a) Given $f(x) = 2x$ find $g(x)$ such that $f(g(x)) = -x$.

(b) Given $f(x) = \frac{1}{2}x + 1$ fine $g(x)$ such that $f(g(x)) = -x$.

Answers to Illustrative Test Questions 3-12

1. (a) $f(1) = 1$, $f(2) = 2$

(b) $\{y: y \geq 1\}$

(c) 3, (Even though -1 also satisfies $(x - 1)^2 + 1 = 5$, is not in the domain of f).

(d) No.

2. (a) $f(1) = \frac{1}{2}$, $f(2) = 1$

(b) $\{y: y \neq 2\}$ i.e. all real numbers except 2.

(c) -2

(d) Yes.

3.

(a) $\frac{f(x) - f(-2)}{x + 2} = \frac{f(4) - f(-2)}{4 + 2}$

$\frac{f(x) - (-1)}{x + 2} = \frac{2 + 1}{6}$

$f(x) + 1 = \frac{1}{2}(x + 2)$

$f(x) = \frac{1}{2}x$ (answer)

(b) $\frac{f(x) - f(1)}{x - 1} = \frac{f(2) - f(1)}{2 - 1}$

$\frac{f(x) - 2}{x - 1} = \frac{5 - 2}{1}$

$f(x) = 3(x - 1) + 2$

$f(x) = 3x - 3 + 2$

$f(x) = 3x - 1$ (answer)

4. $y = kx \div k$

$-3 = 2k + k$

$3k = -3$

$k = -1$ (answer)

5. (a) $f(g(x)) = 3(5x + 3)^2 + 5 = 75x^2 + 90x + 32$

(b) $g(f(x)) = 5(3x^2 + 5) + 3 = 15x^2 + 28$

(c) No.

6. (d)

7. (e)

8.

(a) $f: y = 3x + 8$

$3x = y - 8$

$x = \frac{1}{3}y - \frac{8}{3}$

$g: y = \frac{1}{3}x - \frac{8}{3}$

(b) $f: y = -x + 6$

$x = -y + 6$

$g: y = -x + 6$

(c) $f: y = 2x - 3$

$-2x = -y - 3$

$g: x = \frac{1}{2}y + \frac{3}{2}$

(d) $f: y = x - 6$

$x = y + 6$

$g: y = x + 6$

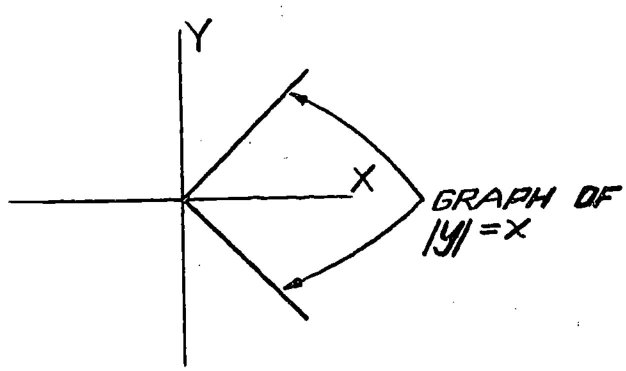
9. (a) Assign 17 to each real number.

(b) Assign 0 to all the rational numbers and 1 to all the irrational numbers.

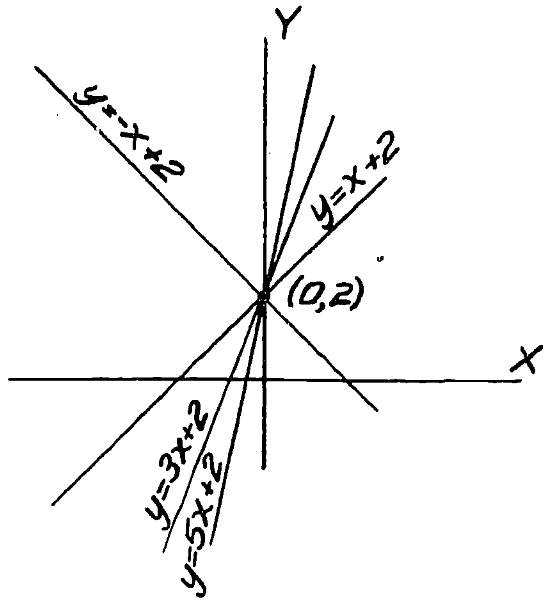
(c) The function defined by $y = x^3$.

10.

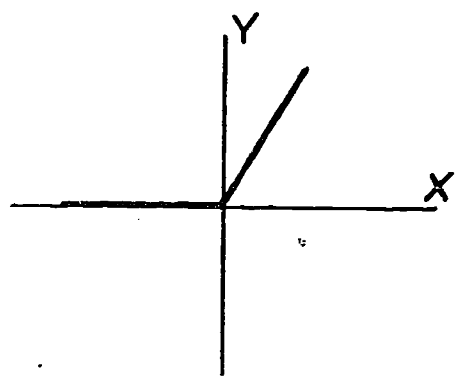
(b)



11. (b) $\{y: -6 \leq y \leq 8\}$.
12. (d) $\{x: x \neq 3\}$.
13. $g(4) = 2$, $g(-2) = 3$, $g(5) = \frac{11}{6}$.
14.



15.
(a) Domain: Set of all real numbers.
(b) Range: Set of all non-negative real numbers.



16. (a) and (b).

17. (a) $\{x: x \neq 0\}$

    (b) $\{x: x \neq \pm 1\}$

    (c) $\{x: x \neq \pm \sqrt{2}\}$

    (d) $\{x: x \geq 0\}$

    (e) $\{x: x \leq 4\}$.

18. -1.

19. (a) 2), 3), 5)

    (b) 2), 3) .

20. (a) $f(g(x)) = 2(g(x)) = -x$

    $g(x) = -\frac{x}{2}$ (answer)

    (b) $f(g(x)) = \frac{1}{2}(g(x)) + 1 = -x$

    $g(x) = -2x - 2$ (answer)

Commentary for Teachers

Chapter 4

QUADRATIC FUNCTIONS AND EQUATIONS

4-0. Introduction.

A reasonable teaching time for this chapter is three weeks, allotting generally one classroom hour per section. Some of this material can be covered at a faster rate, if the class has a good background. Section 4-1 and 4-2 can be combined into a single lesson. Sections 4-3, 4-4, 4-5 and 4-6 are so much alike that they could be covered adequately in as few as two days. Section 4-10 is a fairly solid one and 4-11 is relatively light. A total of two days would be adequate for both of these, most of the two days being given to 4-10. It would be possible to spend a whole term on working the kind of problems which are introduced in Section 4-13. It is recommended that the teacher aim at an introduction to this material rather than to strive for perfection; two teaching days should be an adequate amount of time.

4-1. Quadratic Functions.

Most of the important facts about quadratic functions are clearly illustrated by their graphs. This tempts most teachers to start their discussion of this subject with the graphs and to work with them exclusively. While this practice is pedagogically easy it is mathematically unsound. The graph can't be drawn with any assurance until the function itself is studied. In this chapter the quadratic function is treated primarily as a special kind of pairing of real numbers with real numbers. Its analysis is based on certain properties of real numbers -- that no square of a real number is negative and that every positive real number b is the square of the two real numbers $\sqrt{b}$ and $-\sqrt{b}$. The graph is used as a way of displaying the facts after they are derived.

Suggestions for Section 4-1

Problems 1 - 15 can be covered by oral discussion. Problems 17 - 20 are suitable for written work.

Answers to Exercises 4-1

1. Yes.
2. No.
3. Yes.
4. No.
5. Yes.
6. No.
7. Yes.
8. Yes.
9. No.
10. No.
11. All non-zero real numbers.
12. All real numbers.
13. All real numbers, except $t = 2$.
14. All non-zero real numbers.
15. $t = 2$.
16. $a = 3$, $b = 0$, $c = 0$.
17. $a = 3$, $b = -24$, $c = 48$.
18. $a = 3$, $b = -24$, $c = 53$.
19. $a = 1$, $b = -1$, $c = -6$.
20. $a = 12$, $b = 13$, $c = -14$.

4-2. The Function Defined by $y = x^2$.

The irregularity of figure 4-2a can best be described in geometric language -- that certain lines intersect the graph in more than two points. It will be shown in Chapter 7 that no line cuts the graph of a quadratic function in more than two points.

The teacher should maintain the distinction between properties of the graph which were formally derived and those which were used without proof. Properties of the first kind are (a) that the curve has a single minimum point at (0, 0); (b) that each horizontal line above the x-axis intersects the curve in exactly two points, symmetric with respect to the y-axis; (c) that as x increases indefinitely through positive values y increases indefinitely. Properties of the second kind are (a) that it is unwrinkled; (b) that it is concave upward.

Suggestions for Section 4-2

1. Problem 1 asks the student to check the whole sketching process of making graphs. In this process he plots only a few points and then draws the rest of the graph freehand. Problem 1 can be used to give him assurance that his freehand sketch really produces the desired graph.
2. Problem 3(b) is a tricky one. The drawing looks exactly like the graph of $y = x^2$ for real values of x. This is so because the rational numbers are so densely distributed among the real numbers that the thickness of the pencil makes it impossible to differentiate between the two graphs.

Answers to Exercises 4-2

1. (a) $(\frac{3}{2}, \frac{9}{4})$

   (b) $(\frac{-5}{2}, \frac{25}{4})$

   (c) $(-1\frac{1}{4}, \frac{25}{16})$

   (d) $(\frac{4}{3}, \frac{16}{9})$

   (e) $(-\frac{1}{2}, \frac{1}{4})$

[pages 204-205]

2. Have the student plot the graph of the equation $y = x^2$.

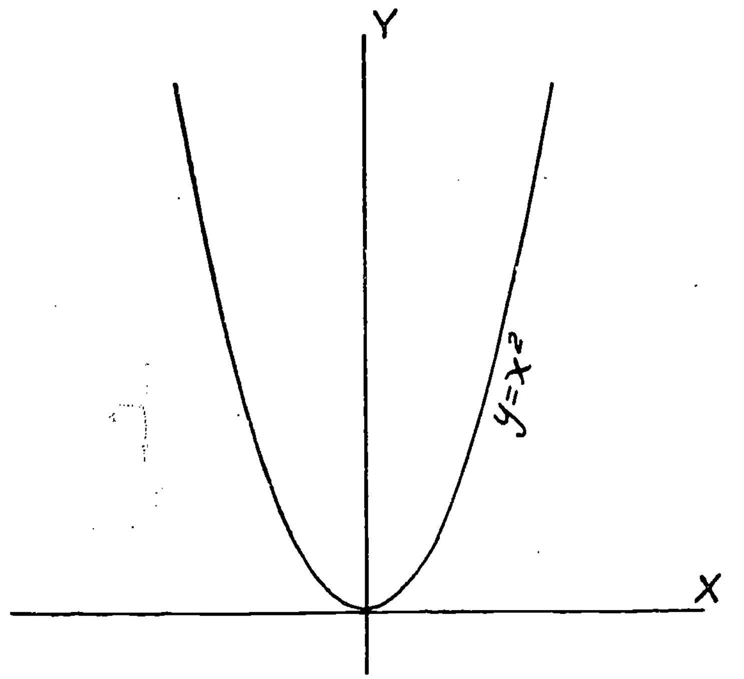
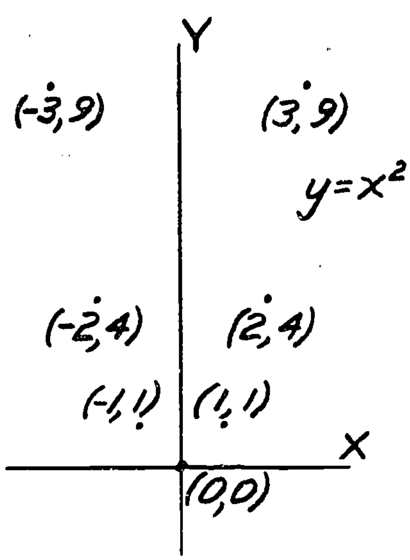


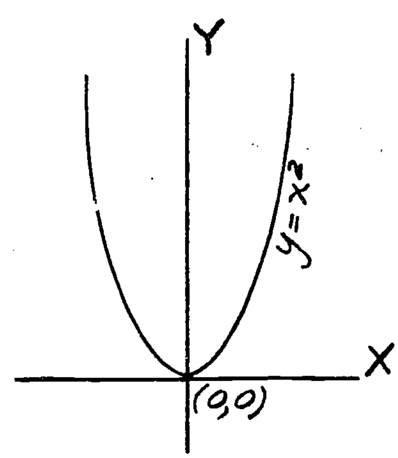
Figure 4-2a

3.

(a)



(b)



4-3. The Function Defined by $y = ax^2$.

In this section we study the graph of $y = ax^2$, where a is any non-zero real number. We show how the graph can be obtained from the graph of $y = x^2$. The sign of a and the absolute value of a are the determining factors. If a is positive the curve opens upward, if a is negative the curve opens downward. If $|a|$ is small the curve is flat, if $|a|$ is large the curve is steep. This can be proved, but it is probably best shown by examples. Figure 4-3a illustrates some of these facts.

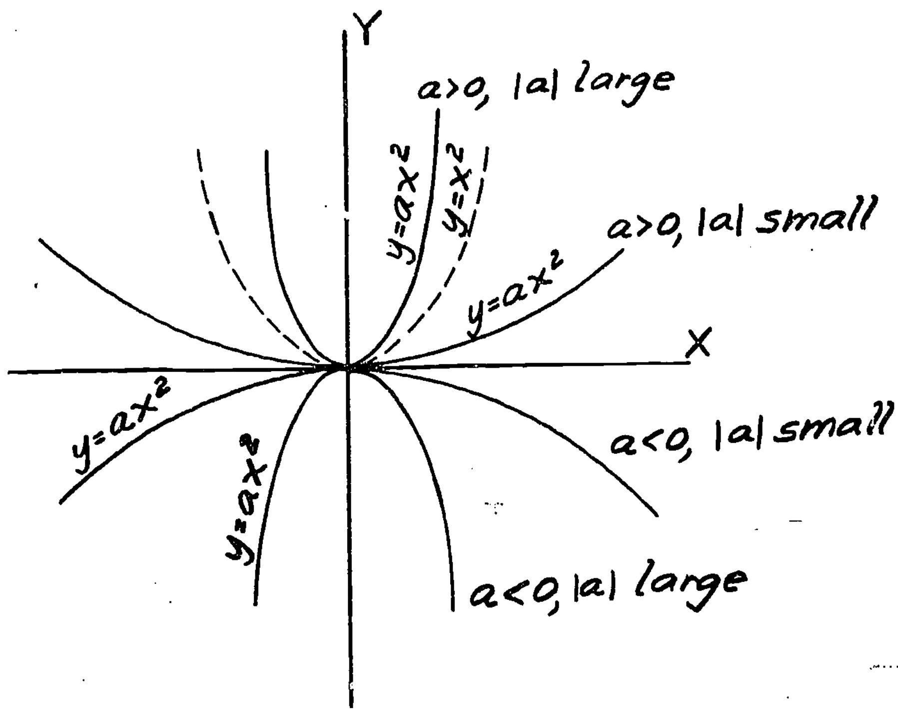


Figure 4-3a

Suggestions for Section 4-3

1. The student should be asked to plot the graphs of Problem 1 in the usual way -- to draw up a short table of values, plot the corresponding points and pass a smooth curve through them. His study of the significance of the coefficient a gives him qualitative information but does not by itself enable him

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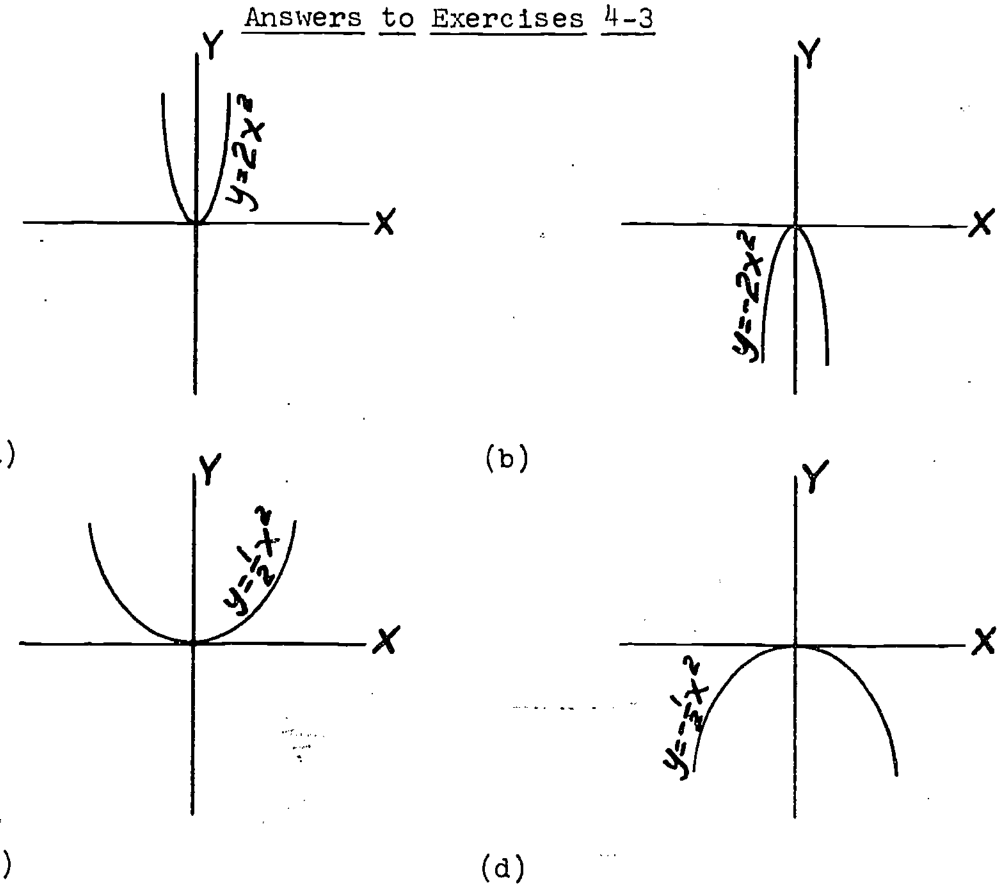
to draw the graph.

2. Problem 3 is intentionally stated in algebraic form. It will help the student to solve this problem if he sees what it means in geometric language. The algebraic question about which of the numbers is the greater means, in geometric language, which of the curves is above the other.

Answers to Exercises 4-3

1.



2. $y = ax^2$.

(a) $a = 1$

(b) $a = 2$

(c) $a = \frac{1}{4}$

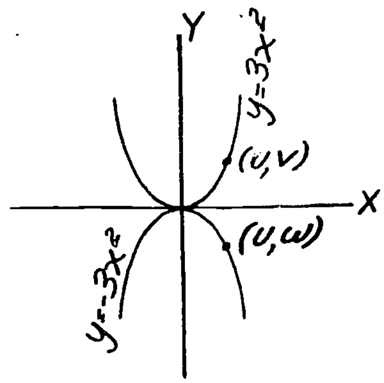
(d) $a = -1$

(e) $a = \frac{1}{4}$

(f) $a = \frac{1}{2}$

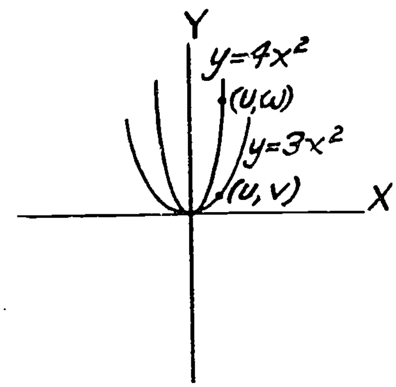
3\.

(a)



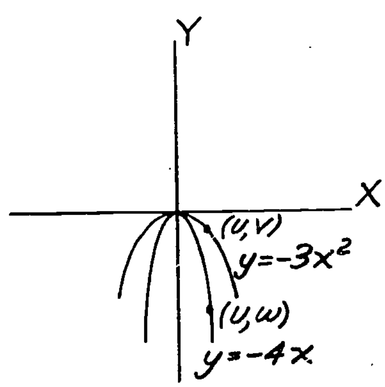
$v > w$

(b)



$v < w$

(c)



$v > w$

4-4. The Function Defined by $y = ax^2 + c$.

In this section we show how the graph of $y = ax^2 + c$ can be obtained from the graph of $y = ax^2$. The graph of $y = ax^2 + c$ is congruent to the graph of $y = ax^2$ and is obtainable from the latter by translating it upward or downward, der[illegible] value of c. The student needs to know how to [illegible] c to locate the graph of $y = ax^2 + c$. The vert[illegible] graph of $y = ax^2 + c$ is at (0, c), and its axis is the line $x = 0$. The curve is $|c|$ units above the graph of $y = ax^2$ if c is positive, and $|c|$ units below the graph of $y = ax^2$ if c is negative. Here, too, examples, rather than formal proofs should be stressed. Figure 4-4a shows some of these facts.

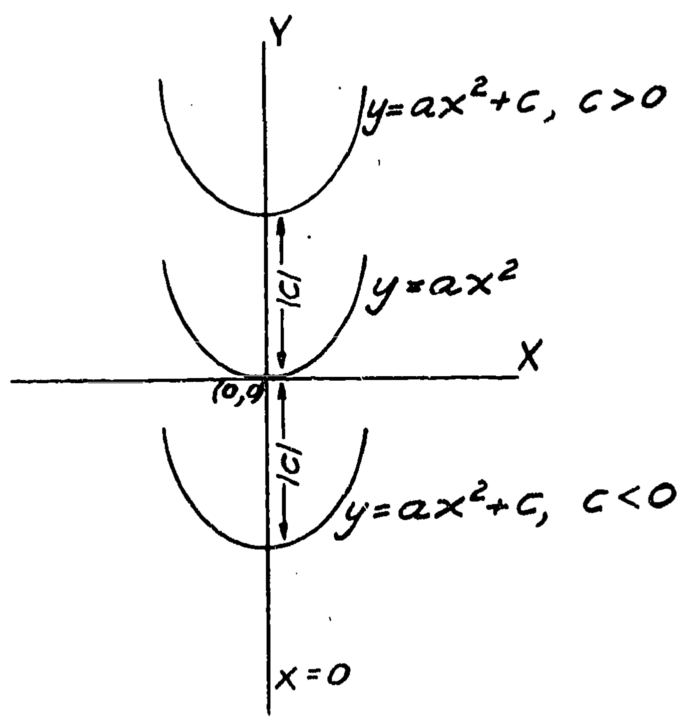


Figure 4-4a

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Suggestions for Section 4-4

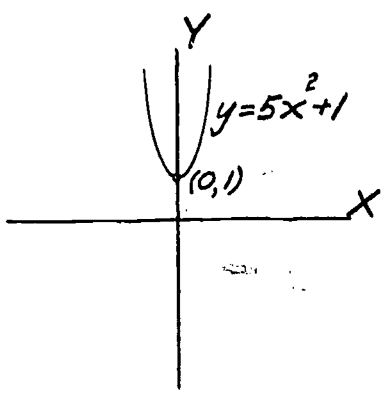
Problem 2 suggests a topic for the brighter students. The table of values for $y = \frac{1}{3}x^2 - 1$ can be obtained from the table of values of $y = \frac{1}{3}x^2$ by the simple arithmetic operation of subtraction; the graph of $y = \frac{1}{3}x^2 - 1$ can be obtained from the graph of $y = \frac{1}{3}x^2$ by the simple geometric operation of sliding. The topic in question is about the relation between sliding and subtraction.

Answers to Exercises 4-4

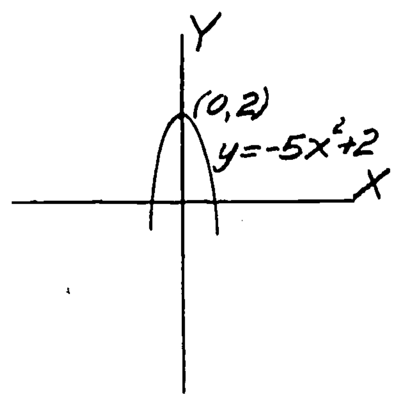
1.

	Vertex	Axis
(a)	(0, 1),	x = 0
(b)	(0, 2),	x = 0
(c)	(0, -1),	x = 0
(d)	$(0, -\frac{1}{3})$,	x = 0
(e)	(0, 6),	x = 0

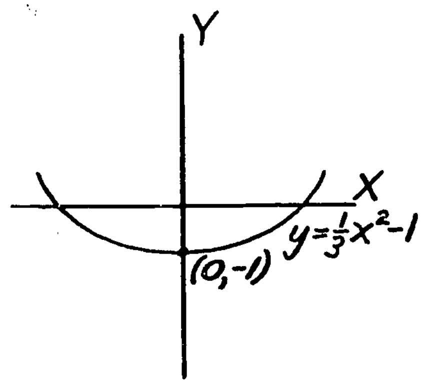
2.



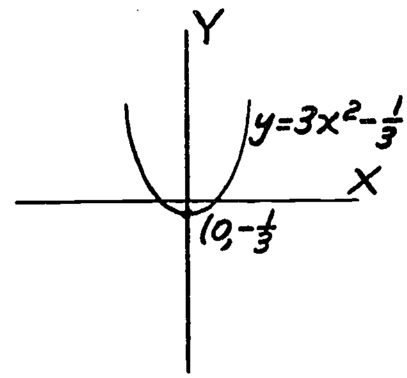
(a)



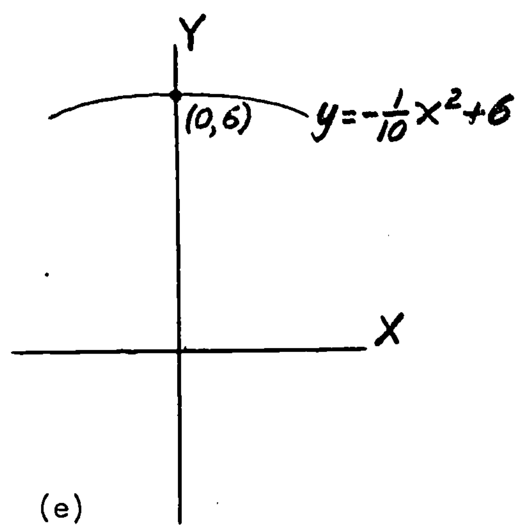
(b)



(c)

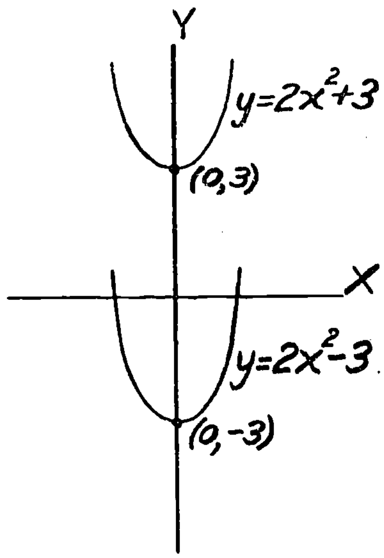


(d)

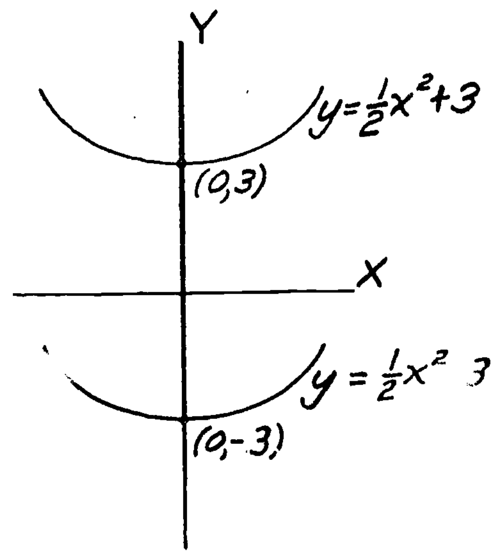


(e)

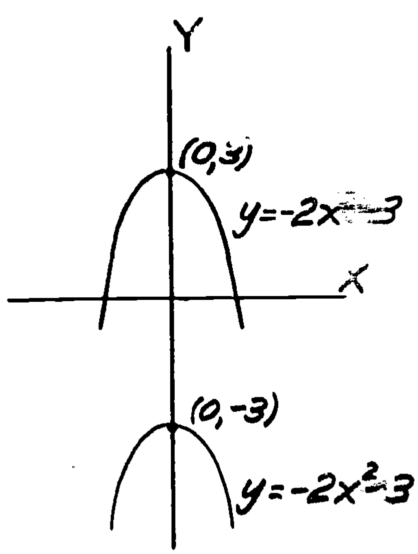
3.


Y
y=2x²+3
(0,3)
X
y=2x²-3
(0,-3)

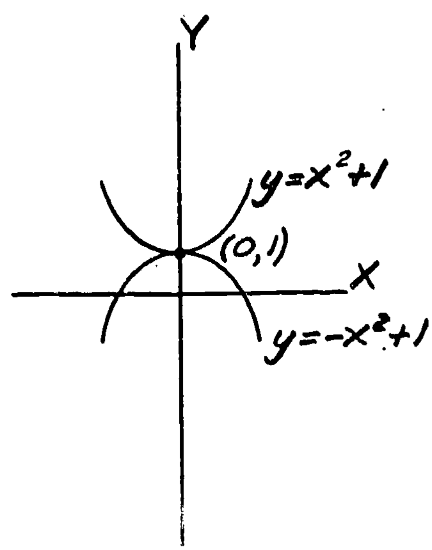
(a)


Y
y=½x²+3
(0,3)
X
y = ½x² 3
(0,-3)

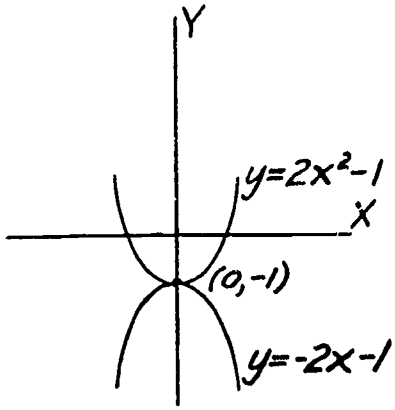
(b)


Y
X
(0,-3)
y=-2x²-3

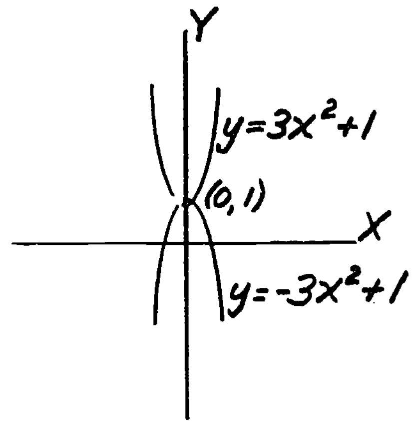
(c)


Y
y=x²+1
(0,1)
X
y=-x²+1

(d)



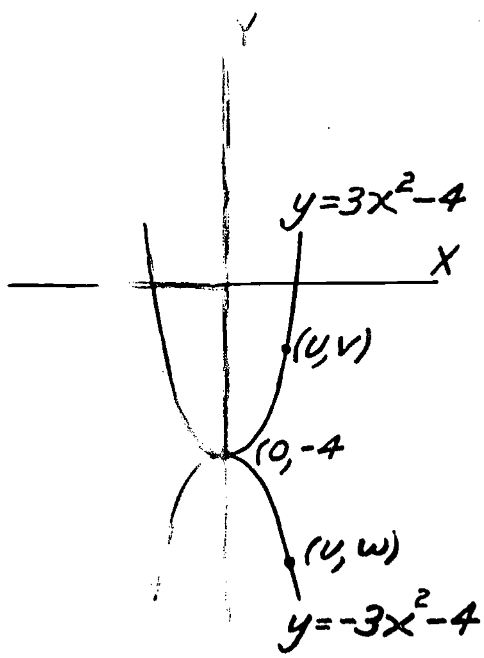
(e)



(f)

4. (a) Minimum: 1
   (b) Maximum: 2
   (c) Minimum: -1
   (d) Minimum: $-\frac{1}{3}$
   (e) Maximum: 6

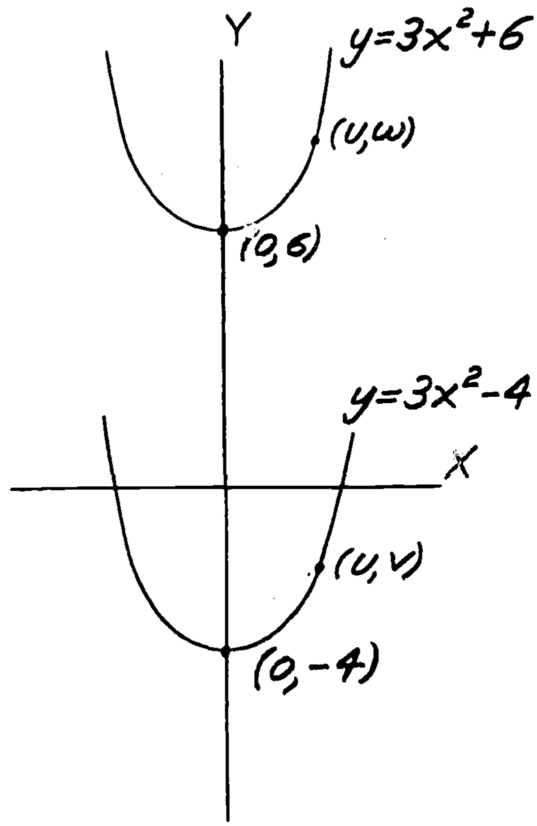
5. (a)



$v > w$

[page 211]

(b)



$v < w$

4-5. The Function Defined by $y = a(x - k)^2$.

In this section we show how the graph of $y = a(x - k)^2$ can be obtained from the graph of $y = ax^2$. The graph of $y = a(x - k)^2$ is congruent to the graph of $y = ax^2$ and is obtainable from the latter by translating it, either to the right or to the left, depending on the value of k. The student needs to know how to use the number k to locate the graph of $y = a(x - k)^2$. The vertex of the graph of $y = a(x - k)^2$ is at $(k, 0)$, and its axis is the line $x = k$. The curve is $|k|$ units to the right of the graph of $y = ax^2$ if k is positive and $|k|$ units to the left of this graph if k is negative. Figure 4-5a shows some of these facts. ($a < 0$ for these graphs.)

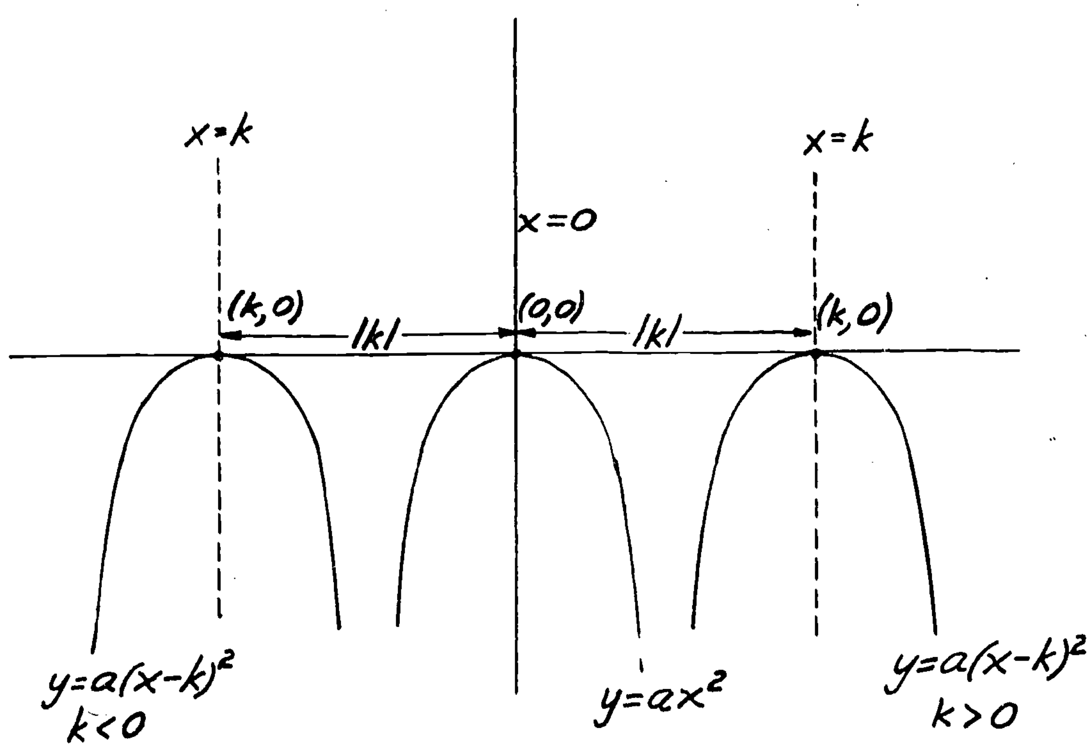


Figure 4-5a

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Suggestions for Section 4-5

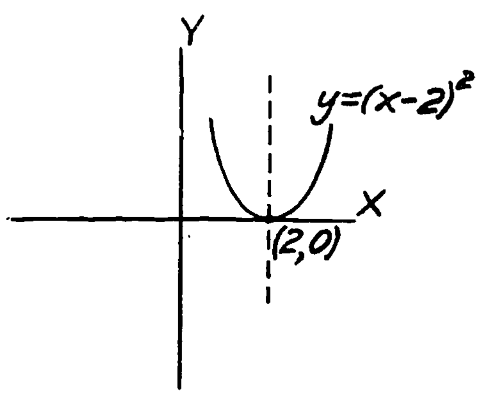
The novelty of these problems is that we work with x instead of with y and slide the curve to the right or to the left instead of up or down.

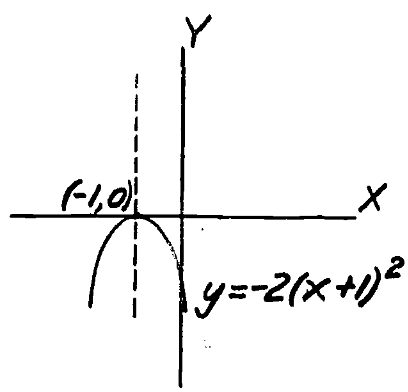
Answers to Exercises 4-5

1.

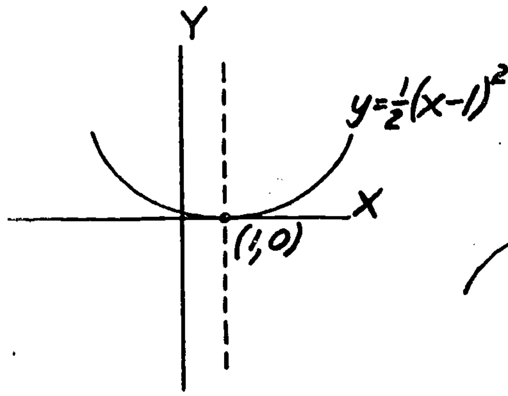
	Vertex	Axis
(a)	(2, 0),	$x = 2$
(b)	(-1, 0),	$x = -1$
(c)	(1, 0),	$x = 1$
(d)	(-2, 0),	$x = -2$
(e)	(2, 0),	$x = 2$
(f)	(1, 0),	$x = 1$

2.

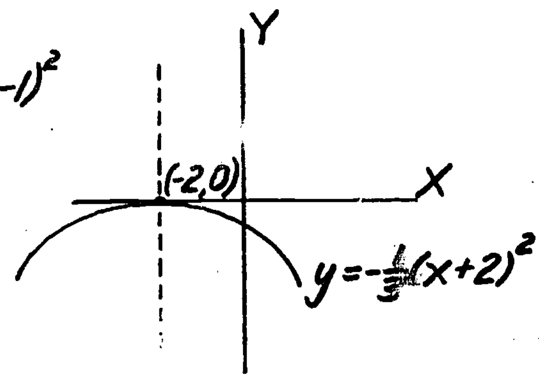




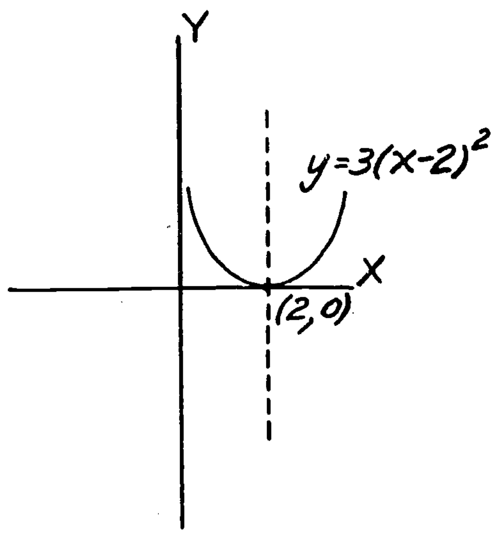
(a) (b)



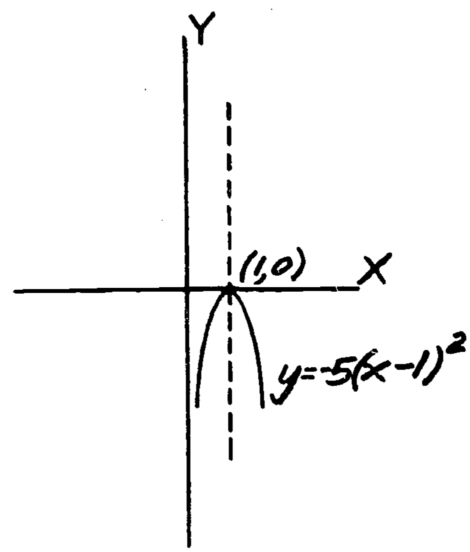
(c)



(d)

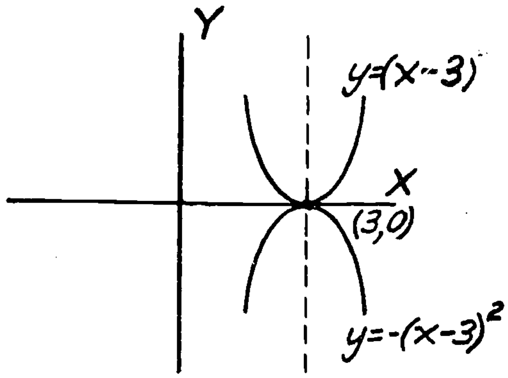


(e)

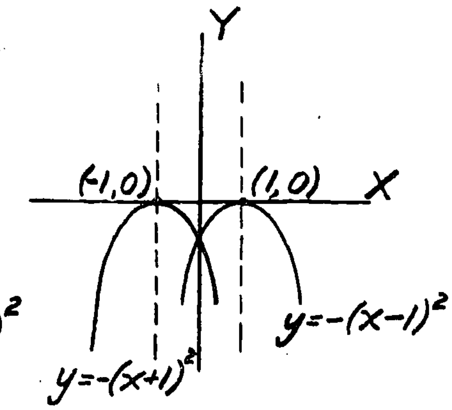


(f)

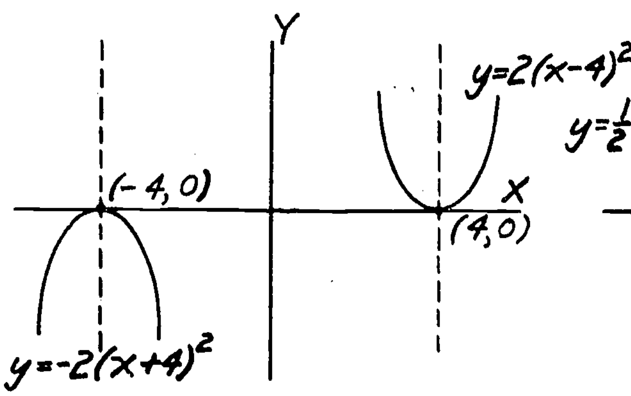
3.


Y
y=(x-3)
X
(3,0)
y=-(x-3)²

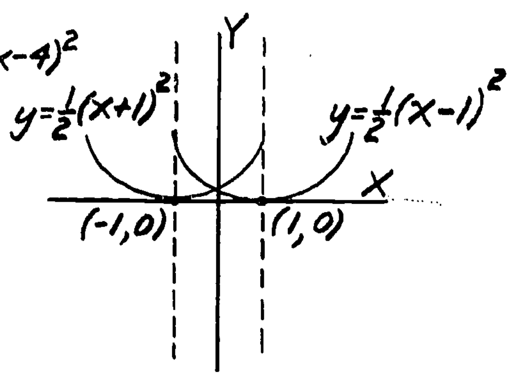
(a)


Y
(-1,0)
(1,0)
X
y=-(x-1)²
y=-(x+1)²

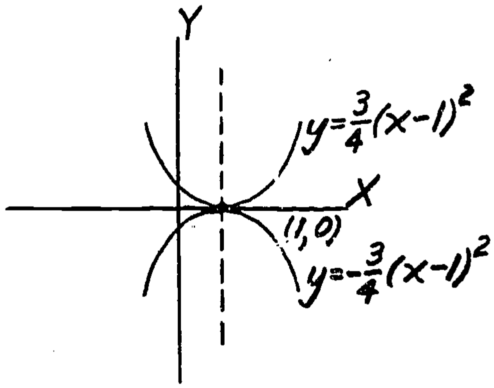
(b)


Y
y=2(x-4)²
(-4,0)
X
(4,0)
y=-2(x+4)²

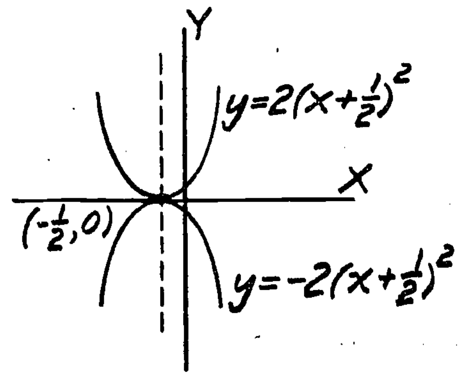
(c)


Y
y=½(x+1)²
y=½(x-1)²
X
(-1,0)
(1,0)

(d)



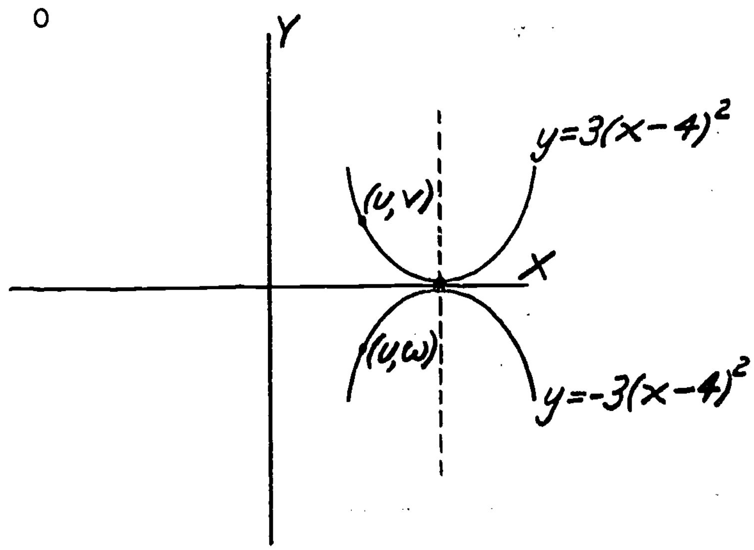
(e)



(f)

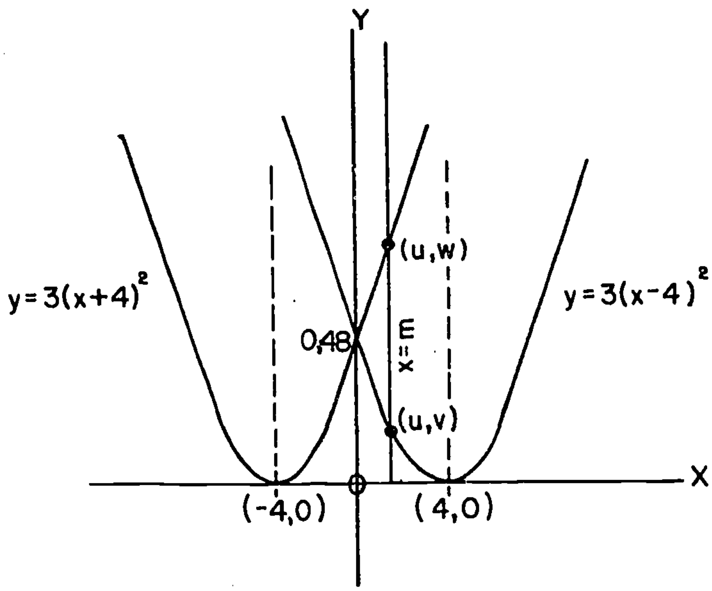
4. (a) Minimum: 0
   (b) Maximum: 0
   (c) Minimum: 0
   (d) Maximum: 0
   (e) Minimum: 0
   (f) Maximum: 0

5.



(a) For $u = 4$, $v = w$
For $u < 4$ or $u > 4$, $v > w$

[page 214]



(b) For u = 0, v = w

For u > 0, v < w

For u < 0, v > w

4-6. The Function Defined by $y = a(x - k)^2 + p$.

In this section we show how the graph of $y = a(x - k)^2 + p$ can be obtained from the graph of $y = ax^2$. The graph of $y = a(x - k)^2 + p$ is congruent to the graph of $y = ax^2$ and is obtainable from this latter graph by translating it up or down depending on the value of p and right or left depending on the value of k. The student needs to know how to use the numbers k and p to locate the graph of $y = a(x - k)^2 + p$. The vertex of the graph of $y = a(x - k)^2 + p$ is (k, p) and the axis is the line $x = k$. The curve is $|p|$ units above the graph of $y = ax^2$ if p is positive and $|p|$ units below this graph if p is negative, $|k|$ units to the right if k is positive, and $|k|$ units to the left if k is negative.

Here, too, examples rather than formal proofs should be stressed. Figure 4-6a shows some of these facts.

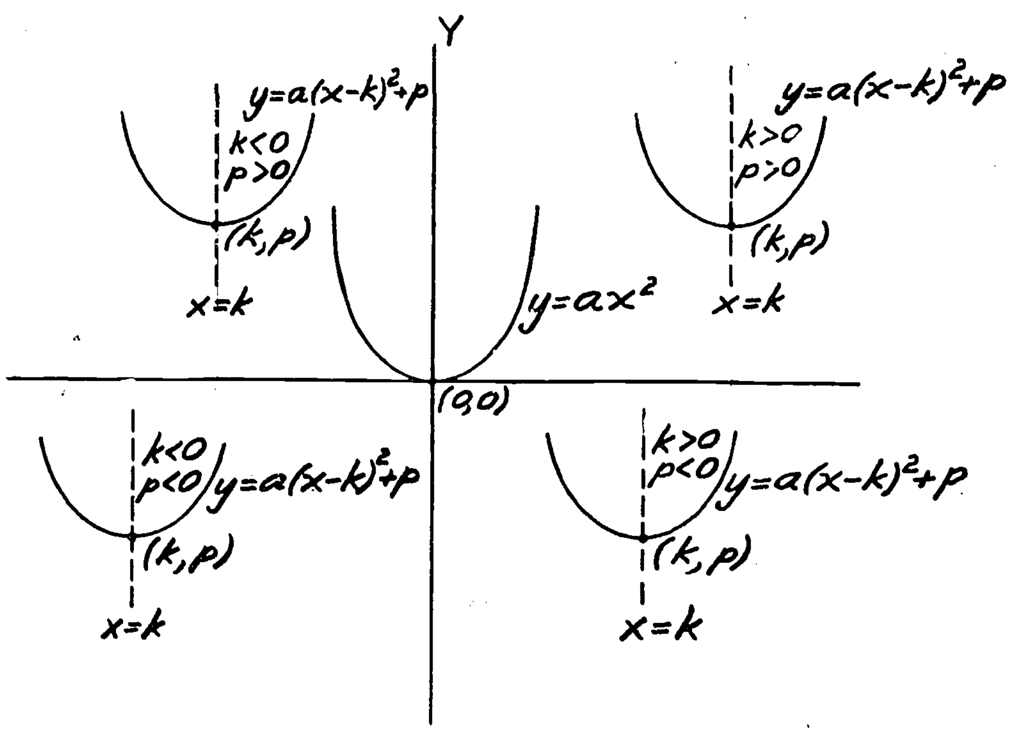


Figure 4-6a

Suggestions for Section 4-6

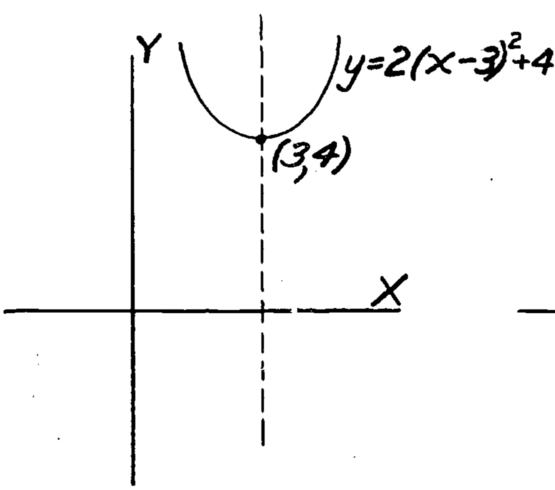
The novelty of these exercises is that the student works simultaneously with x and y and he has to slide both right or left as well as up or down.

Answers to Exercises 4-6

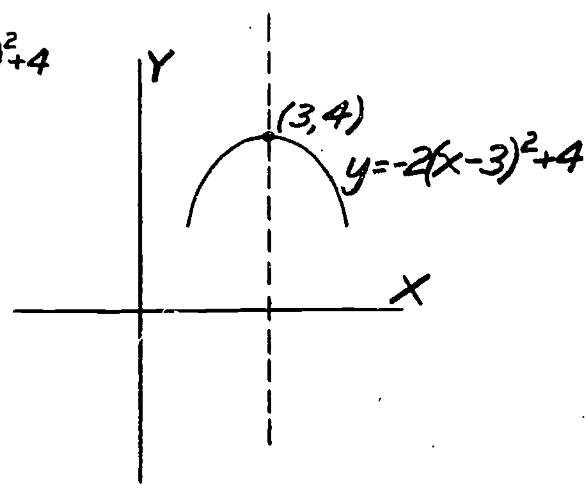
1.

	Vertex	Axis
(a)	(3, 4),	x = 3
(b)	(3, 4),	x = 3
(c)	(-3, 0)	x = -3
(d)	(1, -1),	x = 1
(e)	(-1, 2),	x = -1
(f)	(2, -3),	x = 2

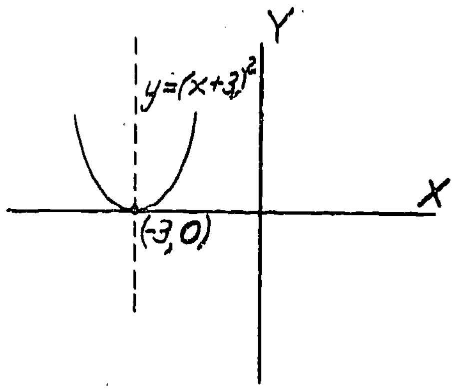
2.



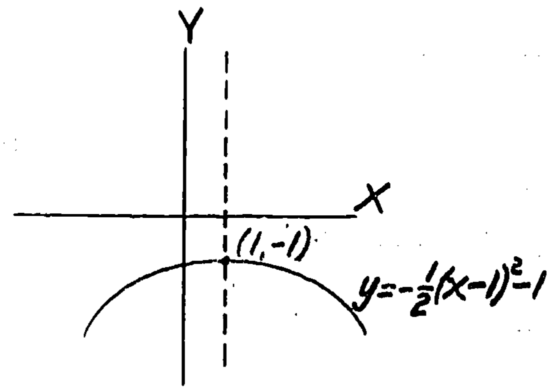
(a)



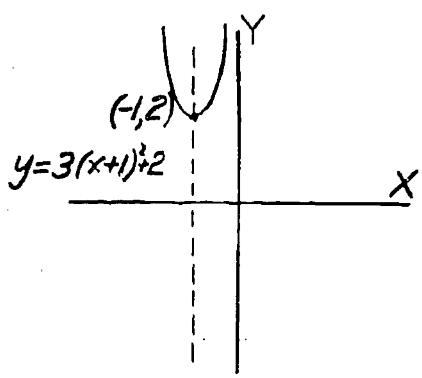
(b)



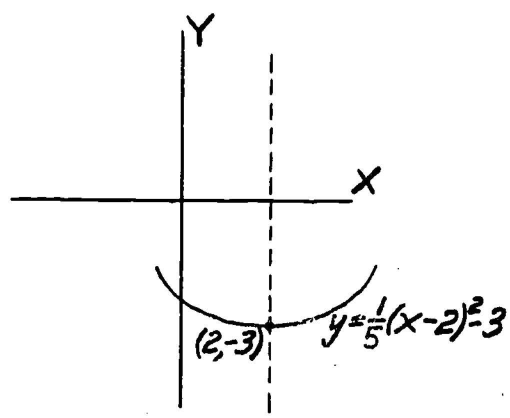
(c)



(d)

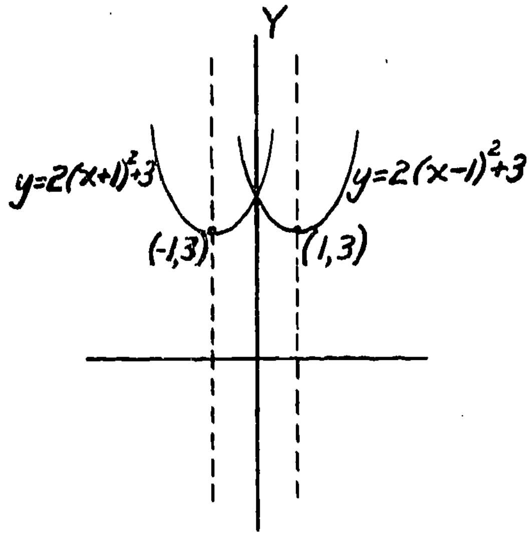


(e)

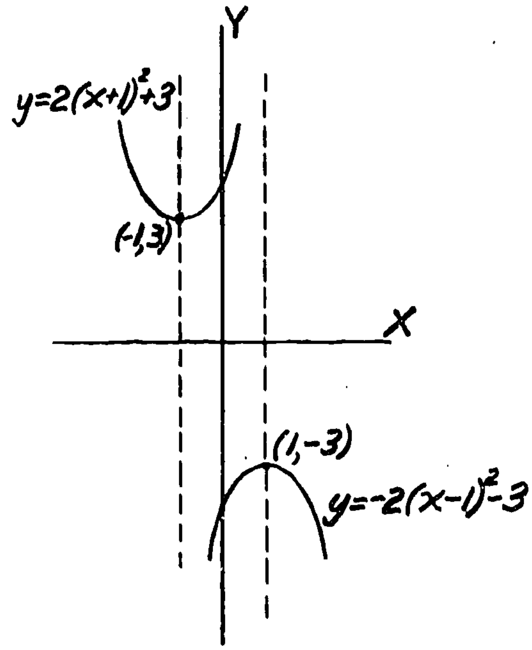


(f)

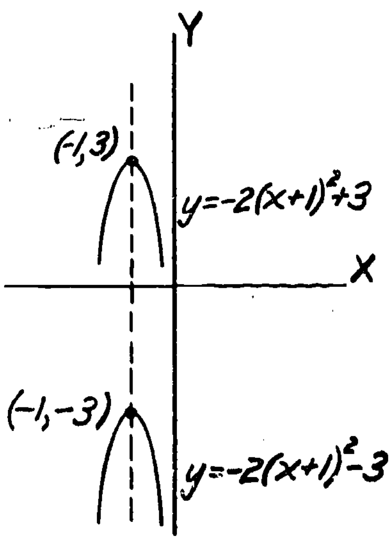
3.


Y
y=2(x+1)²+3
y=2(x-1)²+3
(-1,3)
(1,3)

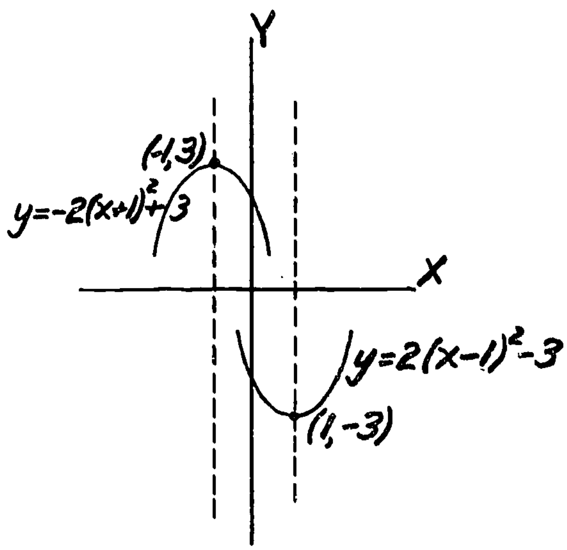
(a)


Y
y=2(x+1)²+3
(-1,3)
X
(1,-3)
y=-2(x-1)²-3

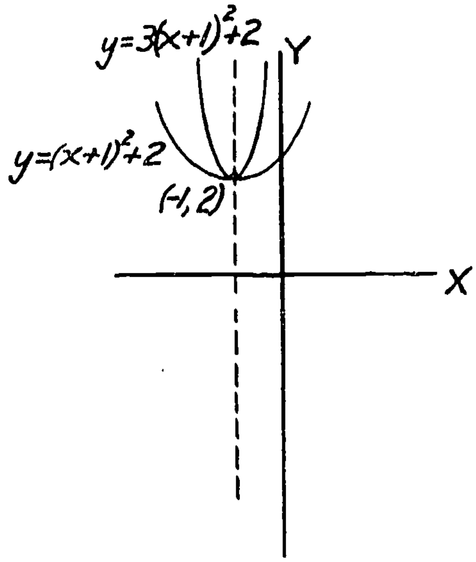
(b)

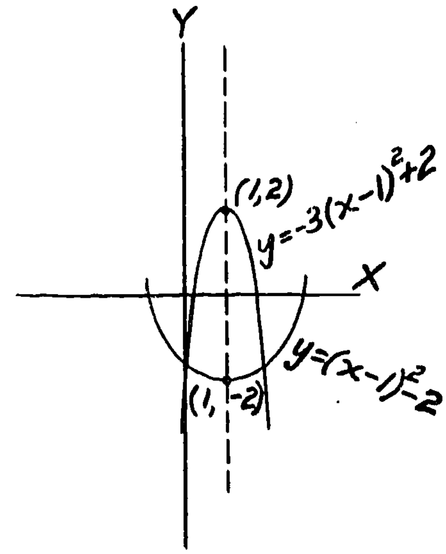

Y
(-1,3)
y=-2(x+1)²+3
X
(-1,-3)
y=-2(x+1)²-3

(c)


Y
(-1,3)
y=-2(x+1)²+3
X
y=2(x-1)²-3
(1,-3)

(d)

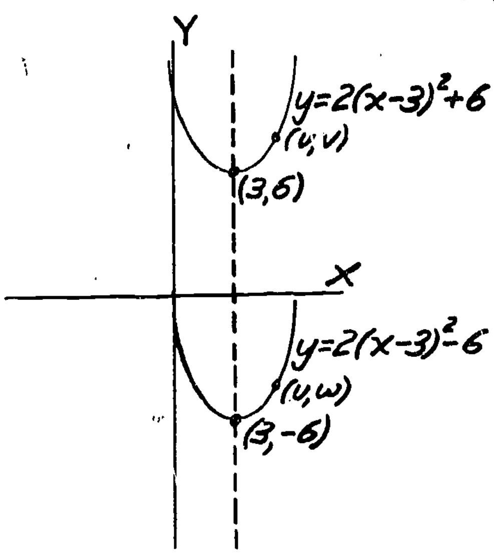


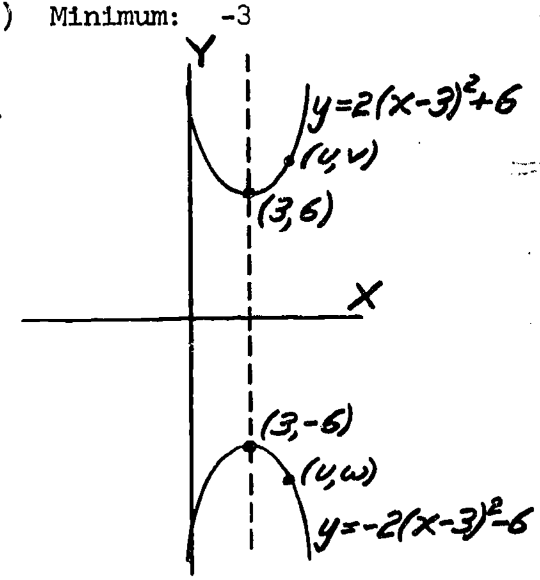


(e) (f)

4. (a) Minimum: 4
(b) Maximum: 4
(c) Minimum: 0
(d) Maximum: -1
(e) Minimum: 2
(f) Minimum: -3

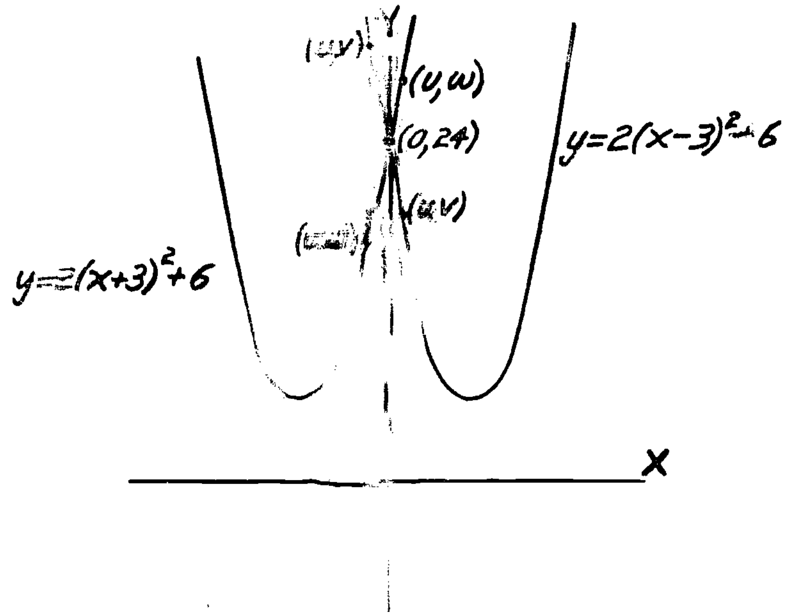
5.





(a) $v > w$

(b) $v > w$



(c) For $u > 0$, $v < w$

For $u = 0$, $v = w$

For $u < 0$, $v > w$

4-7. The Function Defined by $y = ax^2 + bx + c$.

The text shows how to represent $ax^2 + bx + c$ in the form $a(x - k)^2 + p$, by completing the square. It is probably best not to let the students read off the values of k and p from the formula of the text, but rather to have them carry out the steps of completing the square in each individual case. The student has been exposed to many details in progressing from $y = x^2$ to $y = ax^2 + bx + c$. This is a good time to try to give him some unifying perspective so that he doesn't drown in a sea of algebraic tricks. He studied the function defined by $y = x^2$ by referring to properties of the real number system. The information obtained in this way was then applied to the case $y = ax^2 + bx + c$ by a few algebraic and geometric devices. The material about $y = x^2$ is probably deeper than the steps leading to $y = ax^2 + bx + c$, (although it is not easy to justify such a value judgment). However, the student will probably be more interested in these latter steps, because there he has been asked to perform rather than to meditate.

Suggestions for Section 4-7

The emphasis in these problems should not be on the plotting of the graphs, but rather on the algebraic reduction of the problem to the previous case.

Answers to Exercises 4-7

1. (a) $y = x^2 - 4x$
$y = x^2 - 4x + 4 -$
$y = (x - 2)^2 - 4$

(b) $y = -x^2 + 2x$
$y = -(x^2 - 2x + 1)$
$y = -(x - 1)^2 + 1$

(c) $y = x^2 + 3$
$y = (x - 0)^2 + 3$

(d) $y = 3x^2 + 5$
$y = 3(x - 0)^2 + 5$

(e) $y = -x^2 + 6x + 7$
$y = -(x^2 - 6x + 9) + 7 + 9$
$y = -(x - 3)^2 + 16$

(f) $y = x^2 - 144$
$y = (x - 0)^2 - 144$

$y = x^2 + 2x +$ [illegible]
(g) $y = (x^2 + 2x + 1) - 3 - 1$
$y = (x + 1)^2 - 4$

(h) $y = 2x^2 + 8x - 5$
$y = 2(x^2 + 4x + 4) - 5 - 8$
$y = 2(x + 2)^2 - 13$

(i) $y = x^2 + 2x - 24$
$y = x^2 + 2x + 1 - 24 - 1$
$y = (x + 1)^2 - 25$

(j) $y = -5x^2 + 5x + 10$
$y = -5(x^2 - x + \frac{1}{4}) + 10 + \frac{5}{4}$
$y = -5(x - \frac{1}{2})^2 + \frac{45}{4}$

2. and 3.

(a) $y = x^2 + 7x - 8$

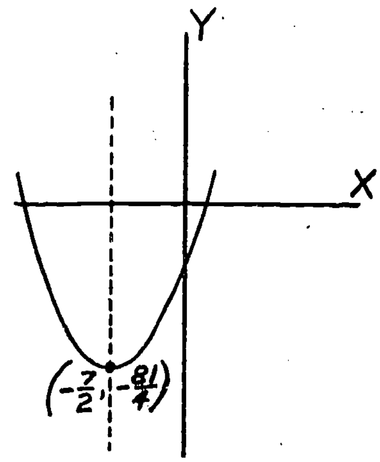
$y = x^2 + 7x + \frac{49}{4} - 8 - \frac{49}{4}$

$y = (x + \frac{7}{2})^2 - \frac{49}{4} - \frac{32}{4}$

$y = (x + \frac{7}{2})^2 - \frac{81}{4}$

Vertex at $(-\frac{7}{2}, -\frac{81}{4})$.

Axis is line $x = -\frac{7}{2}$.



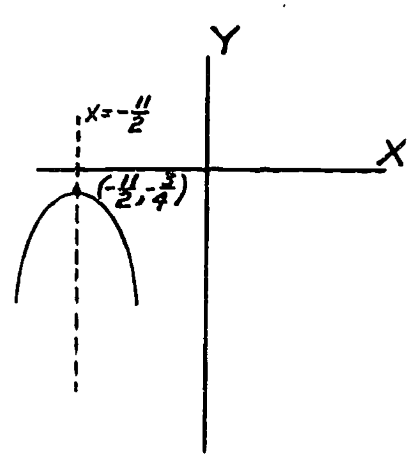
(b) $y = -x^2 - 11x - 31$

$y = -(x^2 + 11x + \frac{121}{4}) + \frac{121}{4} - \frac{124}{4}$

$y = -(x + \frac{11}{2})^2 - \frac{3}{4}$

Vertex at $(-\frac{11}{2}, -\frac{3}{4})$.

Axis is line $x = -\frac{11}{2}$.



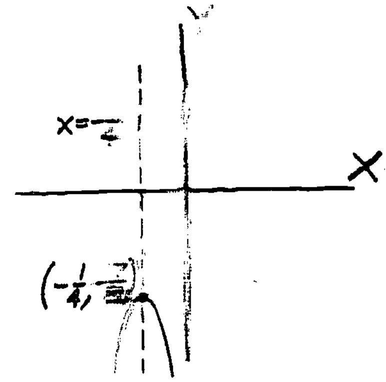
(c) $y = -2x^2 - x - 1$

$y = -2(x^2 + \frac{1}{2}x + \frac{1}{16}) - 1 + \frac{1}{8}$

$y = -2(x + \frac{1}{4})^2 - \frac{7}{8}$

Vertex at $(-\frac{1}{4}, -\frac{7}{8})$.

Axis is line $x = -\frac{1}{4}$.



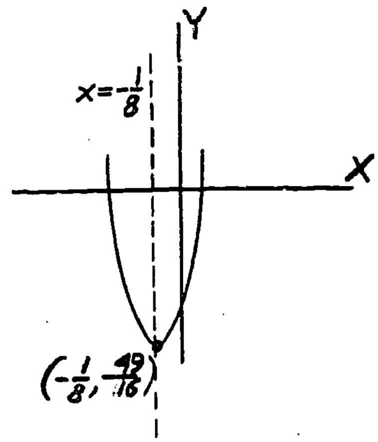
(d) $y = 4x^2 + x - 3$

$y = 4(x^2 + \frac{1}{4}x + \frac{1}{64}) - 3 - \frac{1}{16}$

$y = 4(x + \frac{1}{8})^2 - \frac{49}{16}$

Vertex at $(-\frac{1}{8}, -\frac{49}{16})$.

Axis is line $x = -\frac{1}{8}$.



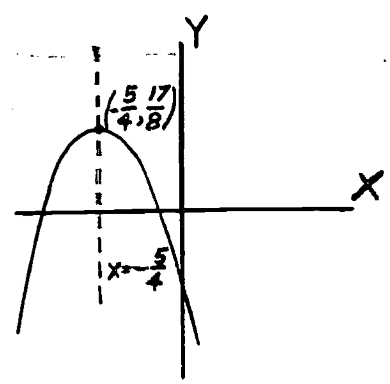
(e) $y = -2x^2 - 5x - 1$

$y = -2(x^2 + \frac{5}{2}x + \frac{25}{16}) - 1 + \frac{25}{8}$

$y = -2(x + \frac{5}{4})^2 + \frac{17}{8}$

Vertex is $(-\frac{5}{4}, \frac{17}{8})$.

Axis is the line $x = -\frac{5}{4}$.



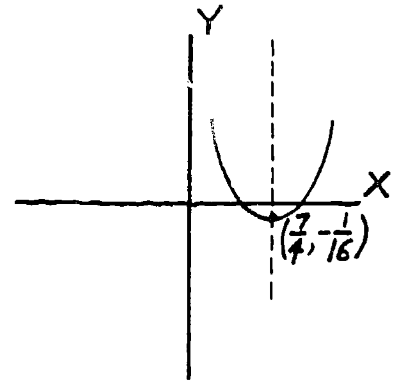
(f) $y = x^2 - \frac{7}{2}x +$ [illegible]

$y = x^2 - \frac{7}{2}x + \frac{49}{16} + 3 - \frac{49}{16}$

$y = (x - \frac{7}{4})^2 - \frac{1}{16}$

Vertex is at point $(\frac{7}{4}, -\frac{1}{16})$.

Axis is the line $x = \frac{7}{4}$.



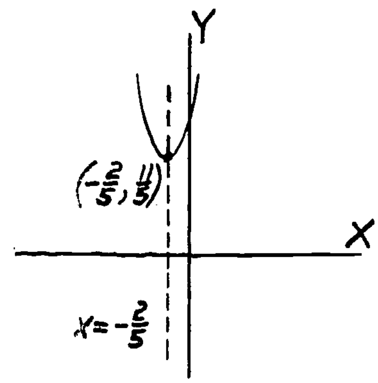
(g) $y = 5x^2 + 4x + 3$

$y = 5(x^2 + \frac{4}{5}x + \frac{4}{25}) + 3 - \frac{4}{5}$

$y = 5(x + \frac{2}{5})^2 + \frac{11}{5}$

Vertex is at point $(-\frac{2}{5}, \frac{11}{5})$.

Axis is the line $x = -\frac{2}{5}$.



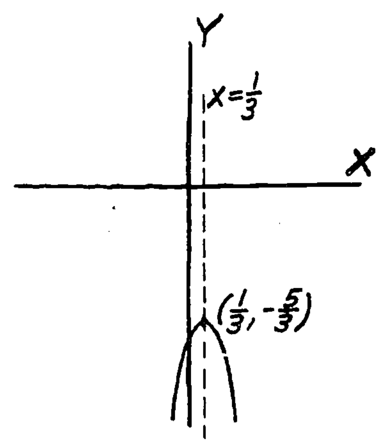
(h) $y = -3x^2 + 2x - 2$

$y = -3(x^2 - \frac{2}{3}x + \frac{1}{9}) - 2 + \frac{1}{3}$

$y = -3(x - \frac{1}{3})^2 - \frac{5}{3}$

Vertex at $(\frac{1}{3}, -\frac{5}{3})$.

Axis is line $x = \frac{1}{3}$.



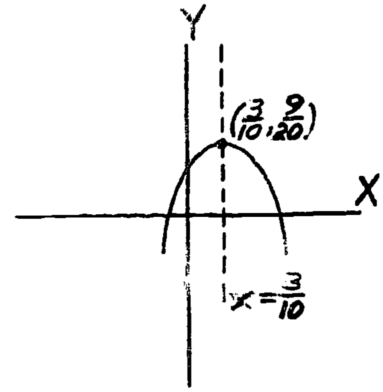
(i) $y = -5x^2 + 3x$

$y = -5(x^2 - \frac{3}{5}x + \frac{9}{100}) + \frac{9}{20}$

$y = -5(x - \frac{3}{10})^2 + \frac{9}{20}$

Vertex is at $(\frac{3}{10}, \frac{9}{20})$.

Axis is line $x = \frac{3}{10}$.

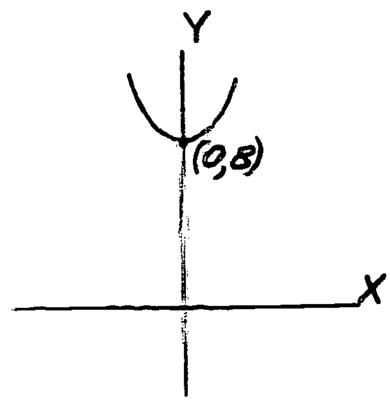


(j) $y = 2x^2 + 8$

$y = 2(x - 0)^2 + 8$

Vertex at (0, 8).

Axis is line $x = 0$.



4-8. Quadratic Functions Having Prescribed Values.

We saw in Section 3-10 that if x_1 and x_2 are distinct numbers and if y_1 and y_2 are distinct numbers then there is one and only one linear function which pairs y_1 with x_1 and y_2 with x_2. This section is about the corresponding state of affairs for quadratic functions -- to discuss how many and what kind of prescribed pairings determine a quadratic function. It is instructive to compare the conclusion about the linear function with those about the quadratic function. It is helpful to consider the constant function in this connection as well.

The following table summarizes the conclusions and suggests their similarities.

	Constant Function	Linear Function	Quadratic Function
Defining Equation	$y = a$	$y = ax + b$	$y = ax^2 + bx + c$
Auxiliary Condition	none	$a \neq 0$	$a \neq 0$
Number of Pairs (x, y) that can be Prescribed	one (x_1, y_1)	two (x_1, y_1) and (x_2, y_2)	three (x_1, y_1), (x_2, y_2) and (x_3, y_3)
Supplementary Condition on the Pairs	none	Not the pairings of a constant function.	Not the pairings of a linear nor a constant function.
Algebraic Statement of Supplementary Conditions	none	$y_1 \neq y_2$	$\frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_3 - y_1}{x_3 - x_1}$
Graph	Any horizontal line	Any line which is neither horizontal nor vertical.	Any parabola with axis parallel to the y-axis.
Conditions that can be imposed on graph	To contain any given point	To contain any two points (not on a horizontal or vertical line).	To contain any three non-collinear points, no two of which have same x-coordinate.

The teacher can let the brighter students try to extend the table to include the functions defined by

$$y = ax^3 + bx^2 + cx + d$$
$$y = ax^4 + bx^3 + cx^2 + dx + e$$
etc.

The proof that there is a quadratic function which makes three prescribed pairings depends on solving three equations in three unknowns. It is postponed until that topic is discussed in Chapter 8. The student can probably handle the calculations here. However, he is not prepared for an important subtlety which is a part of the problem. He probably can find the a, b and c of the quadratic function which makes prescribed assignments. He is not yet able to prove that the coefficient <u>a</u> he discovers by his calculations will be different from zero if and only if

$$\frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_3 - y_1}{x_3 - x_1} .$$

<u>Suggestions for Section 4-8</u>

1. Problems 2 and 3 are basically the same, Problem 2 is stated in geometric language and Problem 3 in algebraic language.
2. The exercises cannot be counted on to get across all the material of this section. The teacher will want to discuss the material apart from its problem solving aspect.

<u>Answers to Exercises 4-8</u>

1.
$$y = ax^2 + bx + c$$
$$0 = 0 + 0 + c$$
$$1 = a + b$$
$$1 = a - b$$
by inspection $a = 1, \quad b = 0, \quad c = 0$

hence $y = x^2$.

2.
$$
\begin{aligned}
y &= ax^2 + bx + c \\
0 &= 0 + 0 + c \\
0 &= 4a + 2b \\
-1 &= a + b \\
2a &= 2 \\
a &= 1 \\
b &= -2 \\
c &= 0 \\
y &= x^2 - 2x
\end{aligned}
$$

3. Answer is $y = x^2 - 2x$. Calculations same as those in Problem 2.

4. We need to determine t so that $(0, 0)$, $(1, 2)$, $(-1, t)$ are not collinear.

   If the points are collinear,
$$
\begin{aligned}
\frac{2 - 0}{1 - 0} &= \frac{t - 0}{-1 - 0} \\
2 &= -t \\
t &= -2.
\end{aligned}
$$

   Hence, for the points to be non-collinear, $t \neq -2$.

4-9. Equivalent Equations, the Equation $ax^2 + bx + c = 0$.

This section gives a definition of equivalent equations. It then shows some ways of transforming an equation to obtain an equivalent equation. The virtue of these transformations is that they can be applied to any equation and always yield an equivalent equation. In Section 4-13 we discuss transformations of equations which need not lead to equivalent equations.

Quadratic equations can occur in various forms, such as $3x^2 + 4x + 5 = 6$ or $x - 3 = x^2 - 4$. In solving a quadratic equation a useful first step is to transform it to the standard form, $ax^2 + bx + c = 0$. All later instructions use this as a point of departure.

Suggestions for Section 4-9

1. Problems 1 - 9 are suited for oral discussion. The remaining problems are suitable for written work.

2. Some of these problems are tricky. The equations in Problem 18 look alike to the eye, but are not equivalent. The equations in Problem 21 look quite different, but are equivalent. The moral is to use the formal test for equivalence rather than a visual one.

Solutions to Exercises 4-9

1. Multiply by $\frac{1}{3}$.
2. Add -6 and multiply by $\frac{1}{2}$.
3. Multiply by $\frac{1}{2}$.
4. Add -3.
5. Multiply by 2 and add 16.
6. Add -11.
7. Add -20.
8. Add $-x - 6$.
9. Multiply by $\frac{1}{5}$.
10. Add $-bx - c$.
11. $x^2 - 8x + 15 = 0$.
12. $x^2 + x - 6 = 0$.
13. $x^2 - 14x + 49 = 0$.
14. $x^2 + 1 = 0$.
15. $4x^2 + 8x - 5 = 0$.
16. Yes.
17. No.
18. No.
19. No.
20. Yes.
21. Yes, because the solution set of each is {50}. The multiplicity of the root 50 is 125 for the first equation and 13 for the second. This is discussed at length in Chapter 5, Section 5-9.

4-10. Solution of $ax^2 + bx + c = 0$ by Completing the Square.

Theorems 4-10a and 4-10b set forth a procedure for solving quadratic equations and also furnish a formula for their roots. The student should master the procedure as well as the forumla because he will need to complete the square over and over again in a mathematic career. He should be able to start with a quadratic equation such as $2x^2 - 6x + 3 = 0$ and (a) complete the square without referring to a formula, (b) decide from his own calculations without referring to formulas whether the equation has a solution or not, (c) be able to find its solutions, if they exist, without referring to formulas. He should also know the quadratic formula and be able to use it. He should know the discriminant test for the existence of real solutions and be able to use it. Some drill on identifying a, b and c is indicated, especially on such equations as $3x + 4 = 2x^2 - 5$.

The requirement of being able to work without the forumlas and yet to know them and be able to use them might require some justification. If the student is expected to be able to derive the formulas himself he can see why it is valuable to know both the procedure and the result.

We comment briefly on equations with exactly one root, such as $x^2 - 2x + 1 = 0$. There is a tradition which calls such a root a "double root," but this language should be avoided here. In Chapter 5 the notion of "multiplicity" of a root is discussed carefully. This is a refinement which goes beyond solution sets, and here we are only concerned with solution sets. Chapter 5 also shows that if complex numbers are allowed as solutions, every quadratic equation has a root. The teacher should explain that the discussion of this chapter applies only to quadratic equations with real coefficients and is concerned only with their real roots.

Suggestions for Section 4-10

1. Unless you have an exceptionally slow class you will not need to cover all the problems of this section.
2. It is recommended that the teacher stress the theory of quadratic equations as well as the technique of solving them.

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This means that the discriminant test should be given at least as much importance as the use of the quadratic formula.

Answers to Exercises 4-10

1. 2 roots.
2. 2 roots.
3. No roots.
4. 2 roots.
5. No roots.
6. 1 root.
7. No roots.
8. 1 root.
9. No root.
10. 2 roots.
11. No roots.
12. 2 roots.
13. 1 root.
14. 2 roots.
15. No roots.
16. 2 roots.
17. No roots.
18. 2 roots.
19. $\{7, -1\}$
20. $\{18, -10\}$
21. $\{9, -5\}$
22. $\{13, -11\}$
23. $\{9, 5\}$
24. $\{-\frac{5}{2} + \frac{\sqrt{21}}{2}, \ -\frac{5}{2} - \frac{\sqrt{21}}{2}\}$
25. $\{-5, 2\}$
26. $\{-\frac{1}{2}, -1\}$
27. $\{\frac{1}{3}, -4\}$
28. $\{\frac{1}{3}, \frac{1}{2}\}$
29. $\{-1 + \sqrt{5}, -1 - \sqrt{5}\}$
30. $\{-1, -\frac{4}{5}\}$
31. $\{\frac{1}{3} + \frac{\sqrt{7}}{3}, \ \frac{1}{3} - \frac{\sqrt{7}}{3}\}$
32. $\{\frac{3}{8} + \frac{\sqrt{41}}{8}, \ \frac{3}{8} - \frac{\sqrt{41}}{8}\}$

33. $\{-\frac{7}{2} + \frac{\sqrt{53}}{2}, \quad -\frac{7}{2} - \frac{\sqrt{53}}{2}\}$

34. $-\frac{1}{2} - \frac{\sqrt{5}}{2}, -\frac{1}{2} + \frac{\sqrt{5}}{2}\}$

35. $\{-\frac{3}{2} + \frac{\sqrt{497}}{14}, \quad -\frac{3}{2} - \frac{\sqrt{497}}{14}\}$

36. $\{\frac{1}{4}, -\frac{1}{3}\}$

37. $\{-\frac{11}{4} + \frac{\sqrt{145}}{4}, \quad -\frac{11}{4} - \frac{\sqrt{145}}{4}\}$.

38. $\{\frac{-b + \sqrt{b^2 - 4ac}}{2c}, \frac{-b - \sqrt{b^2 - 4ac}}{2c}\}$

39. $\{\frac{-a + \sqrt{a^2 - 4bc}}{2b}, \frac{-a - \sqrt{a^2 - 4bc}}{2b}\}$

40. $\{\frac{-c + \sqrt{c^2 - 4ab}}{2b}, \frac{-c - \sqrt{c^2 - 4ab}}{2b}\}$

41. $\{0, \frac{5}{3}\}$

42. $\{\frac{3}{2}, -1\}$

43. $\{\frac{-1}{2} + \frac{\sqrt{5}}{2}, \quad \frac{-1}{2} - \frac{\sqrt{5}}{2}\}$

44. $\{-4 + 3\sqrt{2}, \quad -4 - 3\sqrt{2}\}$

45. $\{\frac{1}{10} - \frac{\sqrt{61}}{10}, \quad \frac{1}{10} + \frac{\sqrt{61}}{10}\}$

46. $\{\frac{5}{4} + \frac{\sqrt{73}}{4}, \quad \frac{5}{4} - \frac{\sqrt{73}}{4}\}$

47. $\{-\frac{3}{2}, \quad 1\}$

48. $\{-\frac{1}{5}\}$

49. $\{\frac{1}{4} + \frac{\sqrt{10}}{4}, \quad \frac{1}{4} - \frac{\sqrt{10}}{4}\}$

50. $\{\frac{-5 + \sqrt{109}}{6}, \quad \frac{-5 - \sqrt{109}}{6}$ 234

4-11. Solutions of Quadratic Equations by Factoring.

In traditional algebra courses a great deal of time is spent on factoring and on solving quadratic equations by factoring. We do not linger over this topic here. Factoring an algebraic expression rarely accomplishes anything of value, and the quadratic equation which can be solved by factoring is rarely met outside the classroom in which drill problems on the subject are provided. The connection between factors and roots is discussed in Chapter 5, especially in Theorem 5-8. The mathematically valuable aspect of the subject considered there and here is more the information that roots give about factors than with the information that factors give about roots.

Suggestions for Section 4-11

Factoring is primarily a mental process. Many of the problems of this section can be handled in oral discussion.

Solutions to Exercises 4-11

1. $\{2, 3\}$
2. $\{4\}$
3. $\{4, -4\}$
4. $\{9, -6\}$
5. $\{\frac{3}{2}, 1\}$
6. $\{-\frac{3}{2}, 1\}$
7. $\{-\frac{5}{4}, \frac{5}{4}\}$
8. $\{0, \frac{1}{3}\}$
9. $\{-13, 5\}$
10. $\{\frac{3}{5}, -\frac{7}{2}\}$
11. $\{-\frac{3}{5}, \frac{2}{3}\}$
12. $\{\frac{7}{2}, \frac{5}{3}\}$
13. $\{\frac{2}{5}, -\frac{1}{3}\}$
14. $\{\frac{3}{2}, -4\}$
15. $\{\frac{4}{3}, -\frac{1}{2}\}$
16. $\{-\frac{3}{5}, 2\}$
17. $\{0, -\frac{5}{3}\}$
18. $\{\frac{2}{5}, \frac{1}{3}\}$
19. $\{\frac{4}{3}, -\frac{4}{3}\}$

20. $\{-\frac{5}{7}, 1\}$

21. $\{-\frac{7}{3}, \frac{3}{7}\}$

22. $\{\frac{1}{7}, -\frac{2}{3}\}$

23. $\{0, -\frac{1}{2}\}$

24. $\{\frac{1}{6}, \frac{1}{3}\}$

25. $\{\frac{1}{8}\}$

26. $\{6a, -4a\}$

27. $\{-b, 4b\}$

28. $\{a, b\}$

29. $\{-a, b\}$

30. $\{\frac{a}{t}, \frac{b}{t}\}$

4-12. Some Properties of the Roots of a Quadratic Equation.

This section reverses the emphasis of the previous sections. Instead of starting with an equation and seeking its solution we start with a set and seek an equation of which it is the solution set.

Several points should be emphasized; (a) If r and s are any real numbers, there is one and only one quadratic equation $x^2 + px + q = 0$ whose solution set is $\{r, s\}$, (b) the numbers p and q can be expressed simply in terms of r and s, (c) there are infinitely many quadratic equations whose solution set is $\{r, s\}$, (d) a, b, c are not determined by r and s but the quotients $\frac{b}{a}$ and $\frac{c}{a}$ are so determined.

Suggestions for Section 4-12

All the problems in this section are suitable for a written assignment.

Answers to Exercises 4-12

1. $x^2 - 11x + 30 = 0.$
2. $x^2 + 4x - 21 = 0.$
3. $x^2 - 8x + 16 = 0.$
4. $x^2 - 36 = 0.$
5. $x^2 = 0.$

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6. $3x^2 - 11x - 4 = 0.$
7. $5x^2 - 2x = 0.$
8. $12x^2 + x - 6 = 0.$
9. $x^2 - 8x + 11 = 0.$
10. $rx^2 - (r^2 + 1)x + r = 0.$

	Sum	Product (If roots exist)
11.	13	40
12.	-5	-50
13.	3	$\frac{5}{2}$
14.	3	0
15.	$\frac{11}{7}$	$-\frac{8}{7}$
16.	p + q	pq

17. Sum: 4, Product: 1 $\quad x^2 - 4x + 1 = 0.$

18. $9x^2 + 24x + 11 = 0.$

19. (a) $\{\frac{9}{20}\}$ (b) $\{2\sqrt{15}, -2\sqrt{15}\}$ (c) $\{\frac{9}{8}\}$

20. $$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

If $x_1 = 2x_2$

then

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} = 2[-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}]$$

$$-\frac{b}{2a} + \frac{2b}{2a} = \frac{2\sqrt{b^2 - 4ac}}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\frac{b}{2a} = \frac{3\sqrt{b^2 - 4ac}}{2a}$$

$$b = 3\sqrt{b^2 - 4ac}$$

$$b^2 = 9b^2 - 36ac$$
$$-8b^2 = -36ac$$
$$2b^2 = 9ac$$

4-13. Equations Transformable to Quadratic Equations.

This section could have been entitled "Solution of Equations by Ingenious Devices." There are in fact three specific types of equations which are discussed here, but the interest of the section goes beyond showing the student how to handle these three types. He should learn that a little bit of algebraic "get up and go" can occasionally make a new problem solvable.

The central topic is the transformation of given equations to quadratic equations. The actual transformations include multiplying both members of an equation by an expression containing the unknown, and squaring both members of an equation. These procedures can: (a) introduce new roots, (b) lose roots, (c) yield an equivalent equation, and there is no simple way of telling from the procedure just which of these outcomes has actually been attained. The student should therefore be urged to check his answers in the original equation and to seek some kind of assurance that he has all the roots. The student should see some simple examples of gaining roots and of losing roots.

Example on losing roots

To solve $x^2 = 2x$ divide both members by x. This transformation yields the equation $x = 2$ and the root 0 has been lost.

Example on gaining a solution

To solve $\sqrt{2x - 5} = x - 4$, square both sides. This transformation yields the equation

$$2x - 5 = x^2 - 8x + 16$$

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or

$$x^2 - 10x + 21 = 0.$$

This last equation has solution set {3, 7}. The number 7 is a solution of the original equation but the number 3 does not satisfy the original equation because

$$\sqrt{2 \cdot 3 - 5} = 1 \text{ and } 3 - 4 = -1.$$

This is a good place to compare general procedures and ingeneous devices. Linear and quadratic equations can be solved by straightforward means which do not require inspiration or luck. The problems of this section cannot be handled by a uniform method. Some temperaments prefer the first kind and some prefer the second. Mathematics includes both kinds of problems.

Some students will be curious about equations of higher degree than two. The equation of third degree

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0$$

and the equation of fourth degree

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad a \neq 0$$

are solvable by formulas involving extraction of roots somewhat like the quadratic formula. It has been proved that there are no such formulas for equations of higher degree. All that is available of a general nature for these cases are methods for solving them approximately. Some individual cases yield to ingenious devices, but no uniform methods are known or can be discovered.

Suggestions for Section 4-13

The teacher with a slow class may want to spend three days on this section, devoting one day to each type. With a better class it is preferable to deal with all the types in the same lesson. The student should get the feeling that he needs a little initiative in tackling these problems.

Answers to Exercises 4-13a

1. $\{-4, 1\}$
2. $\{3, -3\}$
3. $\{-1\}$
4. $\{1\}$
5. $\{0, 2\}$
6. $\{-2, 1\}$
7. $\{-7, 5\}$
8. $\{0, 17\}$
9. $\{2\}$ 5 is a root of the auxiliary quadratic but doesn't satisfy the original equation.
10. Empty set.

Answers to Exercises 4-13b

In some of the problems in this section the auxiliary quadratic equation has roots which do not satisfy the original equation. These extraneous roots are not listed with the answers.

1. $\{0, 10\}$
2. $\{3, -1\}$
3. $\{7\}$
4. $\{5\}$
5. $\{1\}$
6. $\{\frac{1}{2}, \frac{1}{18}\}$
7. $\{2\}$
8. $\{\frac{1}{3}, -1\}$
9. $\{3, 1\}$
10. Empty set.

Answers to Exercises 4-13c

1. $\{1, -1, \sqrt{3}, -\sqrt{3}\}$
2. $\{2, -2, \sqrt{2}, -\sqrt{2}\}$
3. $\{2, -2, 3, -3\}$
4. $\{2, -2, 5, -5\}$
5. $\{1, -1, 2, 4\}$
6. $\{-1, -2, -\frac{3}{2} + \frac{\sqrt{5}}{2}, -\frac{3}{2} - \frac{\sqrt{5}}{2}\}$
7. $\{\frac{17}{9}, \frac{9}{4}\}$
8. $\{2\}$

Answers to Exercises 4-13d

1. $\{1\}$
2. $\{1, 7\}$
3. $\{\frac{5}{3}, -1\}$
4. $\{-3, 3\}$
5. $\{+5, -5, -i, i\}$
6. $\{-1, 2, -3, 4\}$
7. No solution.
8. $\{\sqrt{6}, -\sqrt{6}\}$
9. $\{2\}$
10. $\{27\}$
11. $\{\frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}, \frac{-9 + \sqrt{141}}{6}, \frac{-9 - \sqrt{141}}{6}\}$
12. $\{-\frac{3}{2}, -\frac{1}{3}\}$
13. $\{-1\}$
14. $\{\frac{1}{2}\}$
15. $\{0, +4, -4\}$
16. $\{1, -6, -\frac{5}{2} - \frac{\sqrt{21}}{2}, -\frac{5}{2} + \frac{\sqrt{21}}{2}\}$
17. $\{-\frac{5}{2} + \frac{\sqrt{5}}{2}, -\frac{5}{2} - \frac{\sqrt{5}}{2}\}$
18. $\{-3 + \sqrt{2}, -3 - \sqrt{2}\}$
19. $\{2, -\frac{43}{2}\}$
20. $\{-\frac{a}{2}, -b\}$

4-14. Quadratic Inequalities.

Plotting the solution set of a quadratic inequality on the number line can yield one of four possible figures:

(a) the whole line

(b) the empty set

(c) all the points of a segment except its end points

(d) all the points not on a segment.

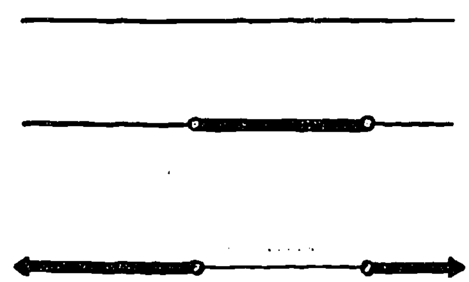

Figure 4-14a

These facts become a little less mysterious if the graph of the associated quadratic function is drawn. Cases (a) and (b) are illustrated in Figure 4-14b. The whole line is the solution set of $ax^2 + bx + c > 0$ and of $a'x^2 + b'x + c' < 0$. The empty set is the solution of $ax^2 + bx + c < 0$ and of $a'x^2 + b'x + c' > 0$.

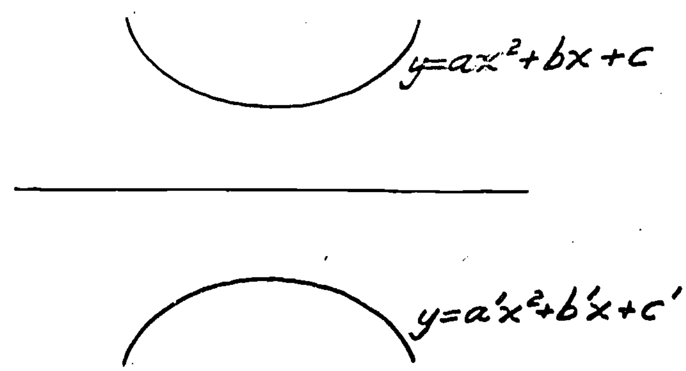

Figure 4-14b

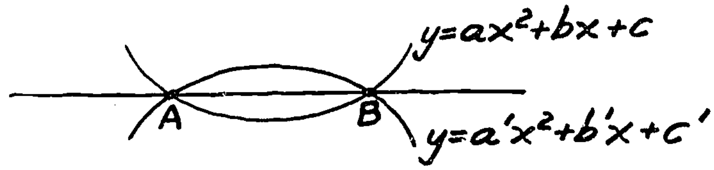

Figure 4-14c

Cases (c) and (d) are illustrated in Figure 4-14c. All the points of $\overline{AB}$ except A and B constitute the solution set of $ax^2 + bx + c < 0$ and of $a'x^2 + b'x + c' > 0$. All the points of the line not on $\overline{AB}$ constitute the solution set of $ax^2 + bx + c > 0$ and of $a'x^2 + b'x + c' < 0$.

The solution set of quadratic inequalities can also be found by factoring but the method is tricky and not generally applicable. For instance, to solve

$$3x^2 + x - 2 > 0$$

notice that

$$3x^2 + x - 2 = (3x - 2)(x + 1).$$

The inequality is equivalent to

$$(3x - 2)(x + 1) > 0.$$

For this to hold either

$$3x - 2 > 0 \text{ and } x + 1 > 0 \text{ or}$$
$$3x - 2 < 0 \text{ and } x + 1 < 0.$$

The first pair are satisfied if $x > \frac{2}{3}$ and the second pair are satisfied if $x < -1$. Therefore, the solution set is the set shown in the Figure 4-14d.

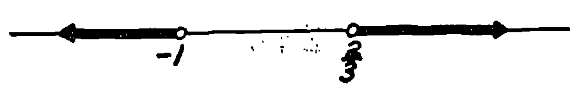

Figure "-14d
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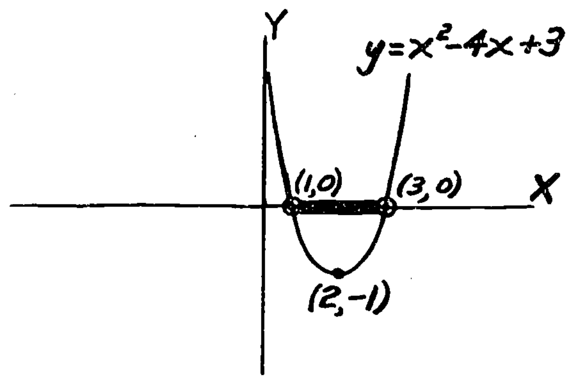
This method is not recommended for general consumption.

Suggestions for Section 4-14

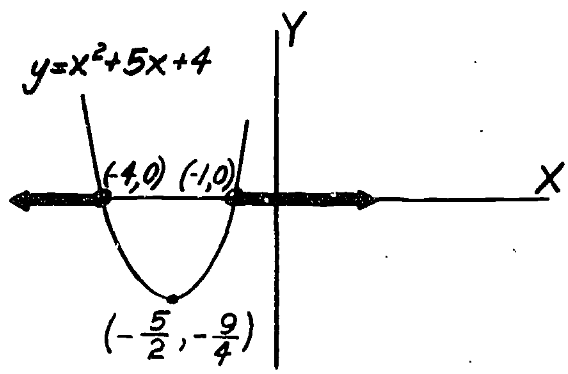
Notice that Problem 9 and those that follow are not in standard form. The student should put them into standard form.

Answers to Exercises 4-14

1. $x^2 - 4x + 3 < 0$
   $y = x^2 - 4x + 4 - 4 + 3$
   $y = (x - 2)^2 - 1$
   $\{x: 1 < x < 3\}$

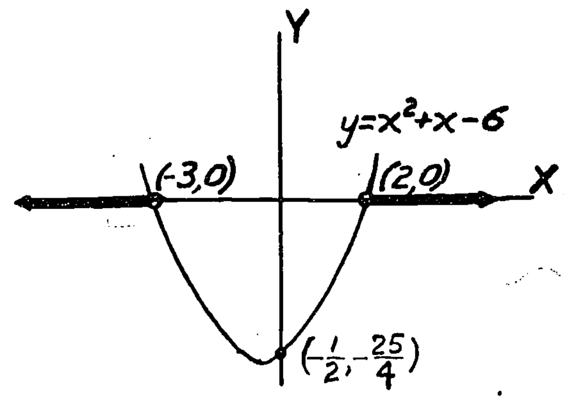


2. $x^2 + 5x + 4 > 0$
   $\{x: x < -4 \text{ or } x > -1\}$



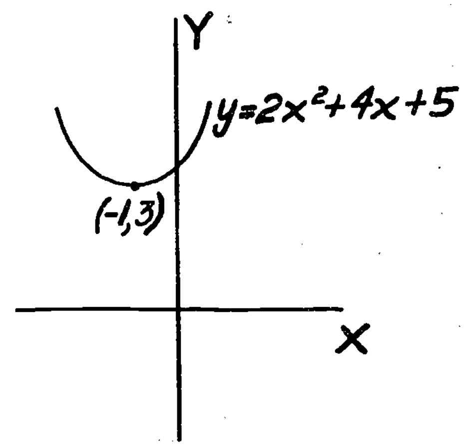
3. $x^2 + x - 6 > 0$

$\{x: x < -3 \text{ or } x > 2\}$



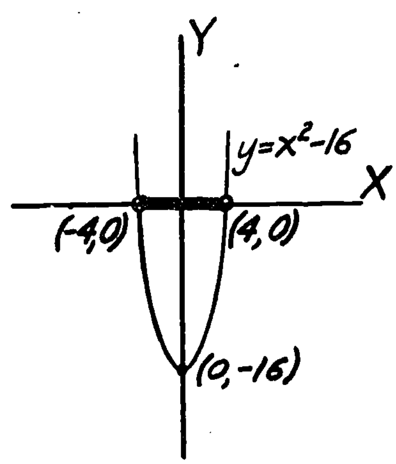
4. $2x^2 + 4x + 5 < 0$

Empty Set



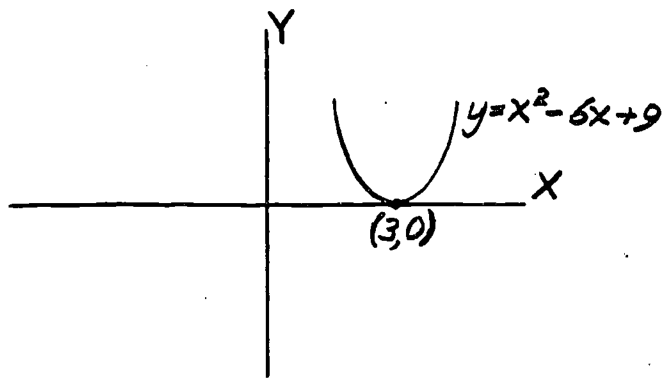
5. $x^2 - 16 < 0$

$\{x: -4 < x < 4\}$



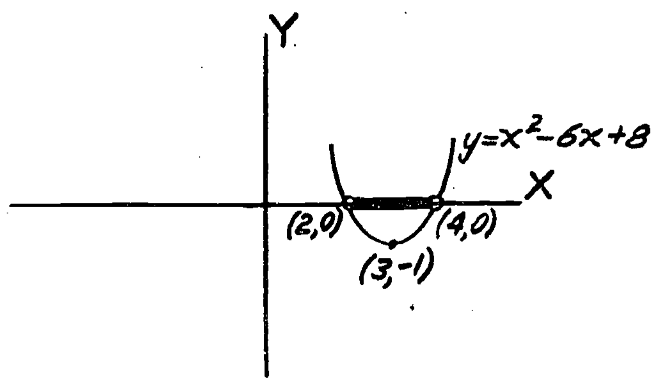
6. $x^2 - 6x + 9 < 0$

Empty Set.



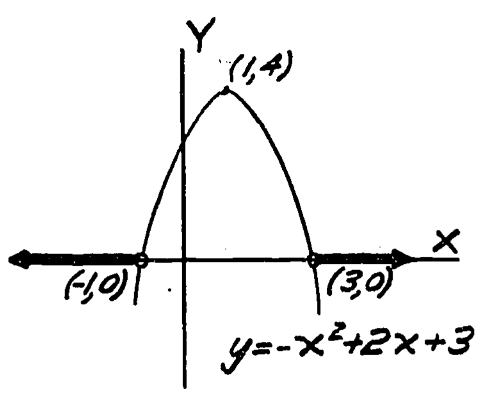
7. $x^2 - 6x + 8 < 0$

$\{x: 2 < x < 4\}$



8. $-x^2 + 2x + 3 < 0$

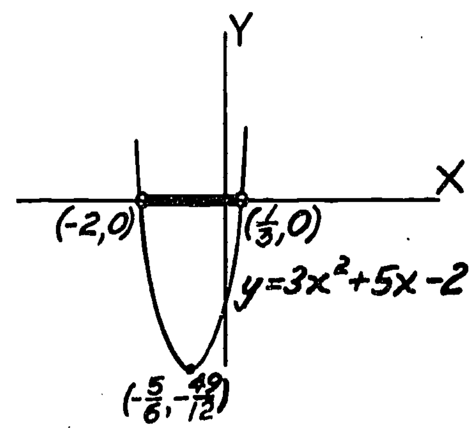
$\{x: x < -1 \text{ or } x > 3\}$



9. $5x < 2 - 3x^2$

$3x^2 + 5x - 2 < 0$

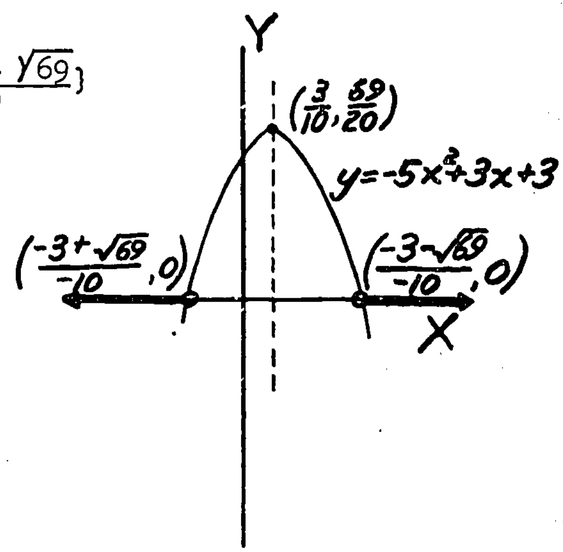
$\{x: -2 < x < \frac{1}{3}\}$



10. $3(x + 1) < 5x^2$

$-5x^2 + 3x + 3 < 0$

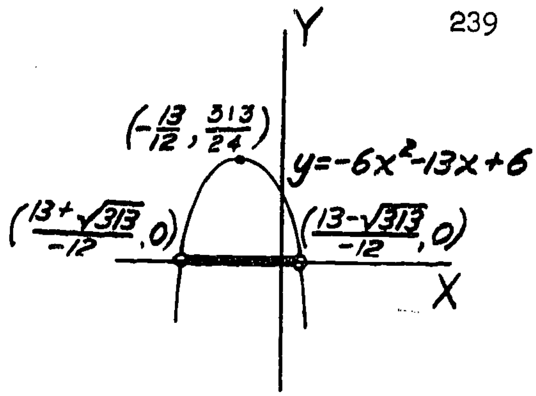
$\{x: x < \frac{-3 + \sqrt{69}}{-10} \quad \text{or} \quad x > \frac{-3 - \sqrt{69}}{-10}\}$



11. $6(-x^2 + 1) > 13x$

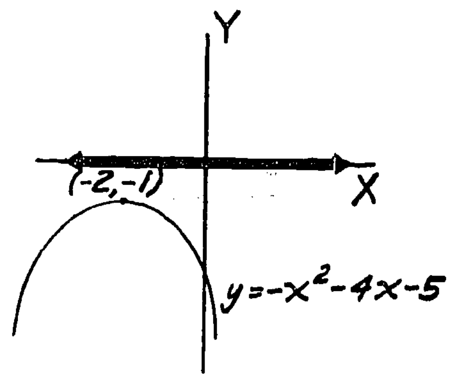
$-6x^2 - 13x + 6 > 0$

$\{x: \frac{13 + \sqrt{313}}{-12} < x < \frac{13 - \sqrt{313}}{-12}\}$



12. $-x^2 - 4x - 5 < 0$

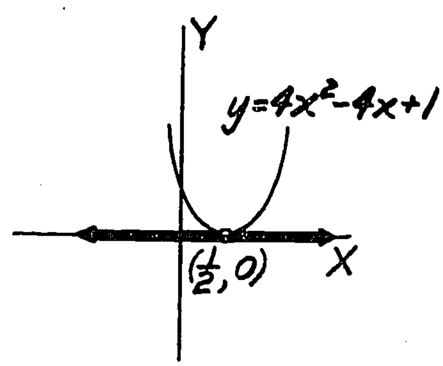
x is set of all real numbers



13. $4x^2 + 1 > 4x$

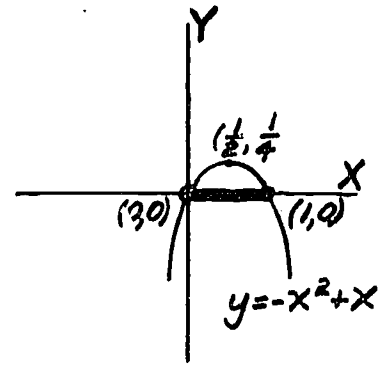
$4x^2 - 4x + 1 > 0$

$\{x: x < \frac{1}{2} \text{ or } x > \frac{1}{2}\}$



14. $-x^2 + x > 0$

$\{x: 0 < x < 1\}$

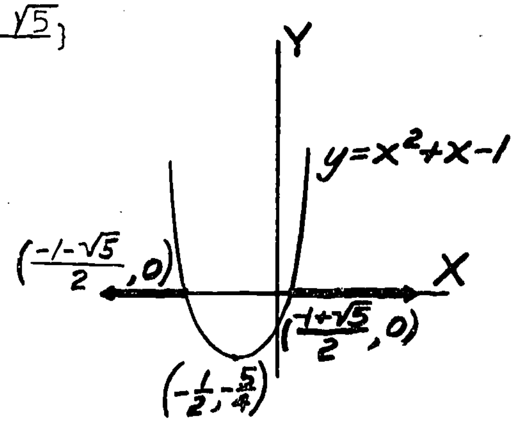


15. $x^2 + x - 1 > 0$

$x^2 + x + \frac{1}{4} - 1 - \frac{1}{4} > 0$

$(x + \frac{1}{2})^2 - \frac{5}{4} > 0$

$\{x: x < \frac{-1 - \sqrt{5}}{2} \text{ or } x > \frac{-1 + \sqrt{5}}{2}\}$



16. (a) $x^2 + hx + 9 = 0$

$a = 1, b = h, c = 9$

$b^2 - 4ac = h^2 - 36$

For no roots $h^2 - 36 < 0$. The solution set is $\{h: -6 < h < 6\}$.

For one root $h^2 - 36 = 0$. The solution set is $\{-6, 6\}$.

For two roots $h^2 - 36 > 0$. The solution set is $\{h: h < -6 \text{ or } h > 6\}$

(b) $x^2 + hx + 9h = 0$

$a = 1,\ b = h,\ c = 9h$

$b^2 - 4ac = h^2 - 36h$

For no roots $h^2 - 36h < 0$. The solution set is $\{h: 0 < h < 36\}$.

For one root $h^2 - 36h = 0$. The solution set is $\{0, 36\}$.

For two roots $h^2 - 36h > 0$. The solution set is $\{h: h < 0 \text{ or } h > 36\}$.

4-15. Applications.

Mathematics is often thought of as a means to an end -- the end being to solve problems which arise in other areas of inquiry. This is an undesirable point of view. While it is true that mathematics can be so applied, it is a valuable body of knowledge in its own right. Moreover, the people who are most successful in applying mathematics to these external problems are those who have the best grasp of pure mathematics.

At the eleventh grade the applications of mathematics appear in "word problems." Solving word problems can be a rewarding experience if

(a) the problem is realistic and its solution is of some interest to someone,

(b) its algebraic formulation is reasonably obvious,

(c) the algebraic techniques that the student recently learned contribute significantly to solving the problem.

Unfortunately most interesting problems require for their solution some technical scientific information as well as algebraic skill. It is unfair to expect a student to tackle such a problem.

Some teachers keep a record of interesting problems that they run across in textbooks, magazines and in discussion with other teachers. Our Example 4-15 (b) is an example of a reasonably interesting problem. Our Example 4-15 (a) is not.

Suggestions for Section 4-15

Some students in your class may know of word problems of general interest, or may even have themselves met situations in which mathematics could be usefully applied. Problems which come from living students are more interesting than those which come from dead physicists.

Answers to Exercises 4-15

1. Width is 3 feet.
   Length is 7 feet.
2. 3 inches is width of frame.
3. 20, 21.
4. (a) 2 seconds.
   (b) 10 seconds.
   (c) During 8th second: 320 ft.
       During 10th second: 384 ft.
5. 12, 10.
6. Let x = number of oranges in bag

   $\frac{100}{x}$ = price of each orange

   $\frac{100}{x} + \frac{5}{6}$ = new price of each orange

   $x - 4$ = new number of oranges in bag

   $$\frac{(600 + 5x)(x - 4)}{6x} = 100$$

   from which
   (a) 24 oranges per bag
   (b) \$ .50 per dozen
7. Let x = speed of 1st train
   $x + 5$ = speed of 2nd train

   $$\frac{140}{x} + \frac{200}{x + 5} = 9$$

   from which
   $x = 35$, $x = -\frac{20}{9}$ (not acceptable).
   $x + 5 = 40$.

8. Solve equation

$\sqrt{2x - 3} = 1 + \sqrt{2 + x}$

for x

$x = 14.$

9. Width is 6 inches.
Length is 8 inches.

10. $x + \frac{1}{x} = 1$

$x^2 - x + 1 = 0$

discriminant is -3
hence no real solution.

11. Yes 13 width
20 length.

12. 7, 8 and -8, -7.

13. $(100 + x)^2 + x^2 = (500)^2$

$x = 300$

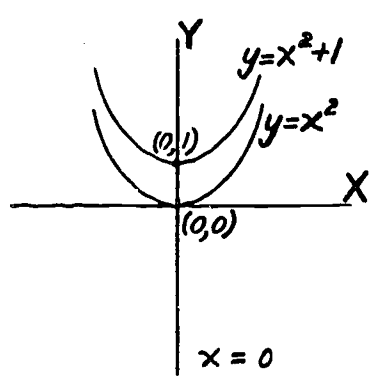
$x + 100 = 400$

14. 7, 28.

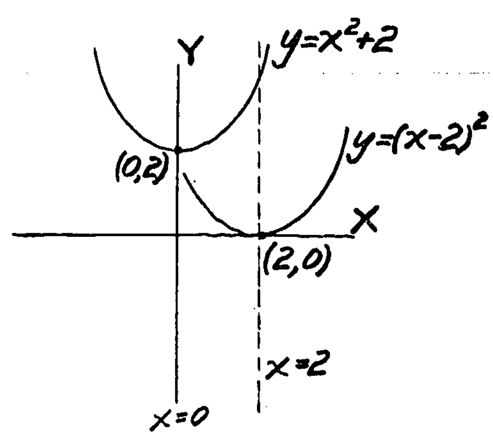
15. Width 25 yd.
Length 25 yd.

Answers to Exercises 4-16

1.



2.



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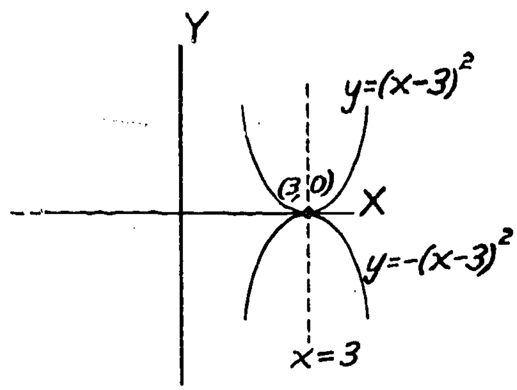
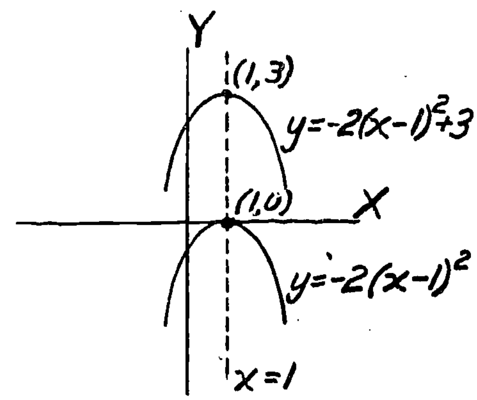
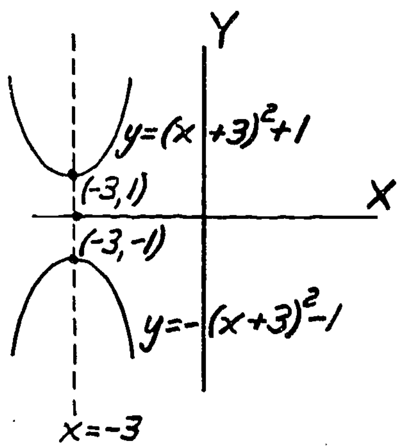
3.



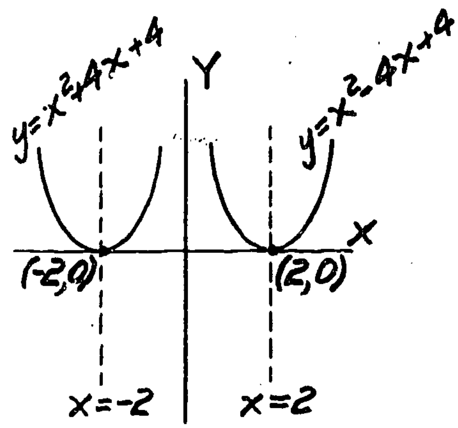
4.



5.



6.



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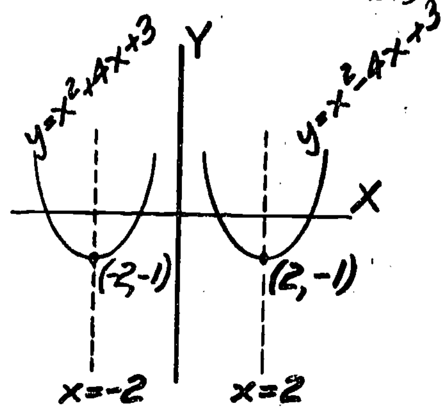
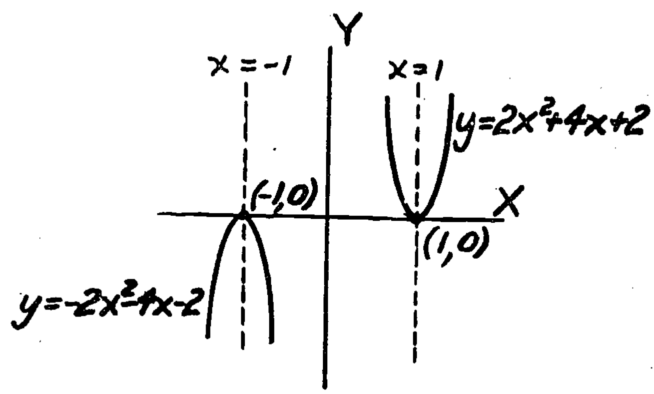
7. $y = x^2 - 4x + 3$
$y = x^2 - 4x + 4 - 4 + 3$
$y = (x - 2)^2 - 1$

$y = x^2 + 4x + 4 - 4 + 3$
$y = (x + 2)^2 - 1$

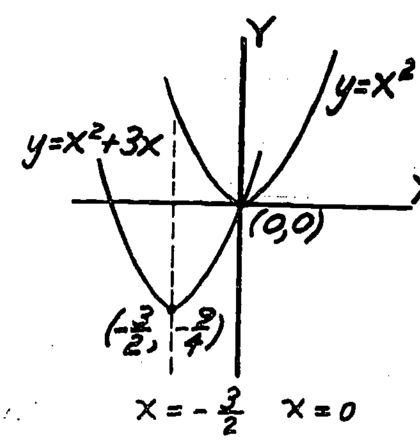


8. $y = -2(x^2 + 2x + 1)$
$y = -2(x + 1)^2$

$y = 2(x^2 - 2x + 1)$
$y = 2(x - 1)^2$

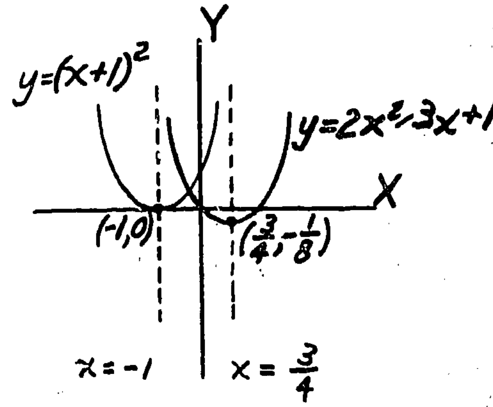


9. $y = x^2 + 3x + \frac{9}{4} - \frac{9}{4}$
$y = (x + \frac{3}{2})^2 - \frac{9}{4}$



10. $y = 2(x^2 - \frac{3}{2}x + \frac{9}{16})^2 - \frac{9}{8} + 1$

$y = 2(x - \frac{3}{4})^2 - \frac{1}{8}$.



	Number of real roots	Sum	Product	Discriminant
11.	0	—	—	-3
12.	1	-3	$\frac{9}{4}$	0
13.	2	-2	-2	12
14.	0	—	—	-31
15.	2	$\frac{3}{5}$	$-\frac{4}{5}$	89
16.	2	7	-18	121
17.	2	$\frac{13}{10}$	$-\frac{3}{10}$	289
18.	0	—	—	-16
19.	2	-8	6	40
20.	2	-7	0	49

Note: We are presently confined to real roots. If there are no real roots, it makes no sense to talk about their sum or product.

21. $\{-2, +\frac{3}{2}\}$

22. $\{-3, +\frac{5}{12}\}$

23. $\{-\frac{9}{2}, -3\}$

24. $\{-3 + \sqrt{5}, \ -3 - \sqrt{5}\}$

25. Empty set.

26. $\{-\frac{1}{2}, -1\}$

27. $\{-\frac{5}{2} + \frac{\sqrt{21}}{2}, \ -\frac{5}{2} - \frac{\sqrt{21}}{2}\}$

28. $\{\frac{3}{8} - \frac{\sqrt{41}}{8}, \ \frac{3}{8} + \frac{\sqrt{41}}{8}\}$

29. $\{-\frac{11}{4} + \frac{\sqrt{145}}{4}, \ -\frac{11}{4} - \frac{\sqrt{145}}{4}\}$

30. $\{-2\}$

31. $\{-1 + \sqrt{10}, -1 - \sqrt{10}\}$

32. $\{\frac{5}{2} + \frac{\sqrt{37}}{2}, \quad \frac{5}{2} - \frac{\sqrt{37}}{2}\}$

33. Empty set. Discriminant = -3(

34. $\{-1 + \frac{3}{2}\sqrt{2}, \quad -1 - \frac{3}{2}\sqrt{2}\}$

35. $\{\frac{1}{12} + \frac{\sqrt{73}}{12}, \quad \frac{1}{12} - \frac{\sqrt{73}}{12}\}$

36. $\{0, 6\}$

37. $\{\frac{15}{4}, \quad -\frac{1}{6}\}$

38. $\{2 + \frac{1}{2}\sqrt{2}, \quad 2 - \frac{1}{2}\sqrt{2}\}$

39. Empty set. Discriminant = -60

40. $\{1 + \sqrt{7}, \quad 1 - \sqrt{7}\}$

41. $b^2 - 4ac = 0$

$900 - 36k = 0$

$36k = 900$

$k = 25$

42. $36 + 16k = 0$

$16k = -36$

$k = -\frac{9}{4}$

43. $64 - 8k = 0$

$-8k = -64$

$k = 8$

44. $9x^2 - 8kx + 4 = 0$

$64k^2 - 144 = 0$

$k = \pm\frac{3}{2}$

45. $k^2 - 4k = 0$

$k(k - 4) = 0$

$k = 0$ (not acceptable)

$k = 4$

46. $(x - 3)(x + 2) = 0$

$x^2 - x - 6 = 0$

47. $(x + 5)(x - 5) = 0$

$x^2 - 25 = 0$

48. $[x - (2 + \sqrt{2})][x - (2 - \sqrt{2})] = 0$

$x^2 - (2 - \sqrt{2})x - (2 + \sqrt{2})x + 2 = 0$

$x^2 - 4x + 2 = 0$

49. $(x - \frac{1}{4})(x - \frac{3}{2}) = 0$

$x^2 - \frac{7}{4}x + \frac{3}{8} = 0$

$8x^2 - 14x + 3 = 0$

50. $(x - \frac{1}{3})(x - 3) = 0$

$x^2 - \frac{10}{3}x + 1 = 0$

$3x^2 - 10x + 3 = 0$

51. $x - \sqrt{5x + 9} - 1 = 0$

$x - 1 = \sqrt{5x + 9}$

$x^2 - 2x + 1 = 5x + 9$

$x^2 - 7x - 8 = 0$

$(x + 1)(x - 8) = 0$

~~$x = -1$~~

$x = 8$ (checks)

Check

$-1 - \sqrt{5(-1) + 9} - 1 \neq 0$

$8 - \sqrt{5 \cdot 8 + 9} - 1 = 0$

52. $\sqrt{x^2 + 3} + \dfrac{4}{\sqrt{x^2 + 3}} = 4$

$x^2 + 3 + 4 = 4\sqrt{x^2 + 3}$

$x^2 + 7 = 4\sqrt{x^2 + 3}$

$x^4 + 14x^2 + 49 = 16(x^2 + 3)$

$x^4 - 2x^2 + 1 = 0$

$(x^2 - 1)(x^2 - 1) = 0$

$x^2 = 1$

$x = +1$

$x = -1$

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53. $\sqrt{3x - 5} + \sqrt{2x + 3} + 1 = 0$

$\sqrt{3x - 5} = -1 - \sqrt{2x + 3}$

$3x - 5 = 1 + 2\sqrt{2x + 3} + 2x + 3$

$x - 9 = 2\sqrt{2x + 3}$

$x^2 - 18x + 81 = 4(2x + 3)$

$x^2 - 26x + 69 = 0$

$(x - 3)(x - 23) = 0$

Empty set.

Note: It was evident from the beginning that the sum of one and two non negative numbers could not be zero.

54. $2x^4 - 17x^2 - 9 = 0$

$(2x^2 + 1)(x^2 - 9) = 0$

$\{3, -3\}$

55. $\frac{2x^2}{x + 1} + 1 = \frac{2}{x + 1}$

$2x^2 + x + 1 = 2$

$2x^2 + x - 1 = 0$

$(2x - 1)(x + 1) = 0$

$\{\frac{1}{2}\}$

56. Let $t = x^2 - 3x + 1$

$t^2 - 4t - 5 = 0$

$(t - 5)(t + 1) = 0$

$t = 5$

$t = -1$

$t = x^2 - 3x + 1 = 5$

$x^2 - 3x - 4 = 0$

$t = x^2 - 3x + 1 = -1$

$x^2 - 3x + 2 = 0$

$\{-1, 1, 2, 4\}$

57. $(x + 7)(x - 7) - 2(x - 1) = 12x$

$x^2 - 49 - 2x + 2 = 12x$

$x^2 - 14x - 47 = 0$

$\{7 - 4\sqrt{6}, 7 + 4\sqrt{6}\}$

58. $(a + b)^2x^2 - (a + b)^2x - ab = 0$

$$x = \frac{(a + b)^2 \pm \sqrt{(a + b)^4 - 4(a + b)^2ab}}{2(a + b)^2}$$

$x = \frac{a}{a + b}$ or $x = \frac{b}{a + b}$.

59. $\sqrt{x + 4} + \sqrt{x - 1} = \sqrt{x - 4}$

$x + 4 + 2\sqrt{(x + 4)(x - 1)} + x - 1 + x - 4$

$x + 7 = -2\sqrt{(x + 4)(x - 1)}$

$x^2 + 14x + 49 = 4(x + 4)(x - 1)$

$x^2 + 14x + 49 = 4x^2 + 12x - 16$

$-3x^2 + 2x + 65 = 0$

$3x^2 - 2x - 65 = 0$

$(3x + 13)(x - 5) = 0$

$x = -\frac{13}{3}$

$x = 5$

Empty set. Why should this result have been anticipated?

60. $3x^4 - 4x^2 - 7 = 0$

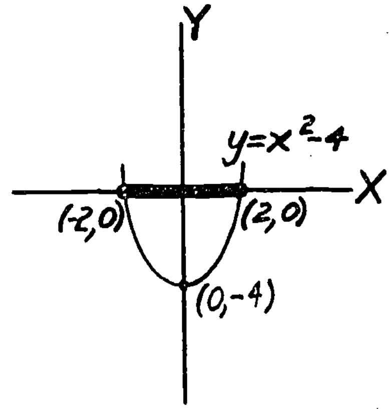
$(3x^2 - 7)(x^2 + 1) = 0$

$\{\frac{\sqrt{21}}{3}, -\frac{\sqrt{21}}{3}\}$

61. $y = x^2 - 4$

$x^2 - 4 < 0$

$\{x: -2 < x < 2\}$



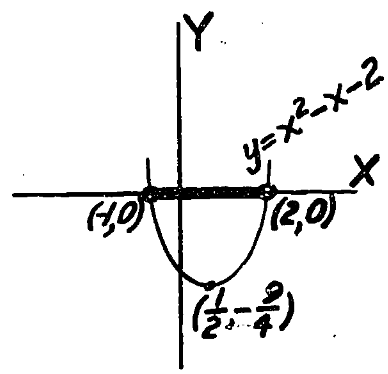
62. $y = x^2 - x - 2$

$y = (x^2 - x + \frac{1}{4}) - 2 - \frac{1}{4}$

$y = (x - \frac{1}{2})^2 - \frac{9}{4}$

$x^2 - x - 2 < 0$

$\{x: -1 < x < 2\}$



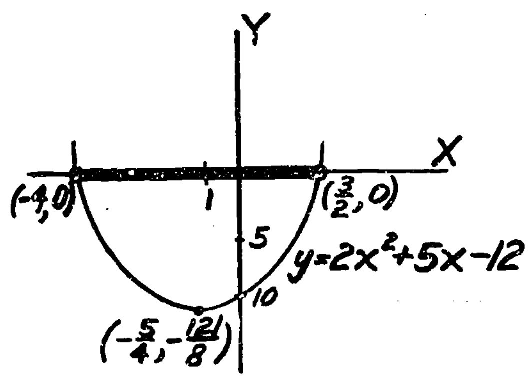
63. $2x^2 + 5x - 12 > 0$

$y = 2x^2 + 5x - 12$

$y = 2(x^2 + \frac{5}{2}x + \frac{25}{16}) - 12 - \frac{25}{8}$

$y = 2(x + \frac{5}{4})^2 - \frac{121}{8}$

$\{x: x < -4 \text{ or } x > \frac{3}{2}\}$



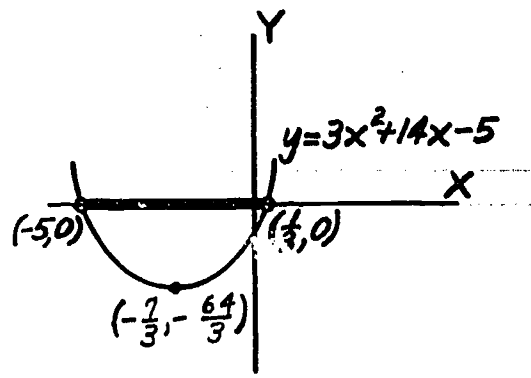
64. $3x^2 + 14x - 5 < 0$

$y = 3x^2 + 14x - 5$

$y = 3(x^2 + \frac{14}{3}x + \frac{49}{9}) - 5 - \frac{49}{3}$

$y = 3(x + \frac{7}{3})^2 - \frac{64}{3}$

$\{x\colon -5 < x < \frac{1}{3}\}$

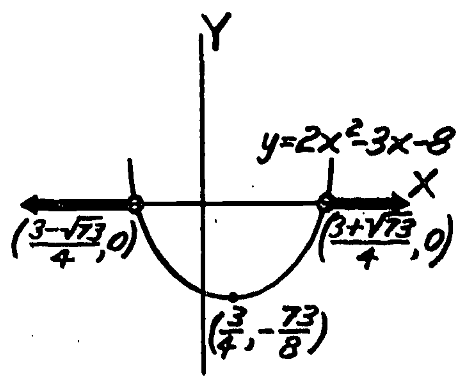


65. $2x^2 - 3x - 8 > 0$

$y = 2(x^2 - \frac{3}{2}x + \frac{9}{16}) - 8 - \frac{9}{8}$

$y = 2(x - \frac{3}{4})^2 - \frac{73}{8}$

$\{x\colon x < \frac{3 - \sqrt{73}}{4} \text{ or } x > \frac{3 + \sqrt{73}}{4}\}$



66. Let x = the number

$3x^2 - 9x = 120$

$x^2 - 3x - 40 = 0$

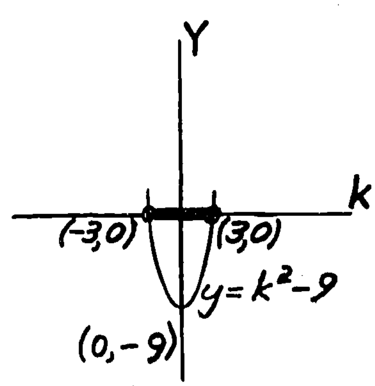
$\{8, -5\}$

67. Let x = width
$2x + 6$ = length
$x^2 + (2x + 6)^2 = 39^2$
$x^2 + 4x^2 + 24x + 36 = 1521$
$5x^2 + 24x - 1485 = 0$
Width: 15
Length: 36

68. $\sqrt{x + 72} = \sqrt{x} + 4$
$x + 72 = x + 8\sqrt{x} + 16$
$8\sqrt{x} = 56$
$\sqrt{x} = 7$
$x = 49$

69. $y = x(50 - x)$
$y = -x^2 + 50x$
$y = -(x^2 - 50x + 625) + 625$
$y = -(x - 25)^2 + 625$
Maximum at $x = 25$
25

70. $b^2 - 4ac < 0$
$4k^2 - 36 < 0$
$k^2 - 9 < 0$
Plot $y = k^2 - 9$
$\{k: -3 < k < 3\}$



71. $y = 3x^2 - 6x + 5$
$y = 3(x^2 - 2x + 1) + 5 - 3$
$y = 3(x - 1)^2 + 2$
$\{y: y \geq 2\}$

72. $kx^2 - 8x + 3 = 0$

(a)
$$64 - 12k = 0$$
$$12k = 64$$
$$3k = 16$$
$$k = \frac{16}{3}$$

(b)
$$3 + x_1 = \frac{8}{k}$$
$$3x_1 = \frac{3}{k}$$
$$x_1 = \frac{1}{k}$$
$$3 + \frac{1}{k} = \frac{8}{k}$$
$$3k + 1 = 8$$
$$3k = 7$$
$$k = \frac{7}{3}$$

OR
$$9k - 24 + 3 = 0$$
$$\therefore k = \frac{7}{3}$$

73. $kx^2 - 2x + 3 = 0$
$$4 - 12k > 0$$
$$1 - 3k > 0$$
$$3k < 1$$
$$k < \frac{1}{3} \text{ and } k \neq 0$$

74. Use the formula for the sum and the product of the roots of a quadratic equation.
$$r + s = -(r - 1)$$
$$rs = 2s$$
From the last equation either $s = 0$ or $r = 2$. If $s = 0$ then, from the first equation $r = -(r - 1)$, so $r = \frac{1}{2}$.
If $r = 2$ then $2 + s = -(2 - 1)$, so $s = -3$.
$$r = \frac{1}{2},\ s = 0$$
$$r = 2,\ s = -3$$
answers

75. $\frac{CB}{AC} = \frac{AC}{AB}$
$$\frac{20 - x}{x} = \frac{x}{20}$$
$$x^2 = 20(20 - x)$$
$$x^2 = 400 - 20x$$
$$x^2 + 20x - 400 = 0$$
$$x = \frac{-20 \pm \sqrt{2000}}{2}$$

$x = -10 \pm 10\sqrt{5}$

$x = 12.36$

$20 - x = 7.64$

4-17. Illustrative Test Questions.

1. Sketch the graphs of each of the following using a single set of axes.
   (a) $y = -2x^2$
   (b) $y = -\frac{1}{2}x^2$
   (c) $y = \frac{1}{2}x^2$
   (d) $y = 2x^2$

2. Find any maximum or minimum values of the functions whose values are given by
   (a) $f(x) = x^2 + 3$
   (b) $f(x) = 5 - 2x^2$
   (c) $f(x) = 2(3x^2 - 4)$

3. Find the minimum value of the function whose values are given by $y = 2x^2 - 12x + 14$.

4. Sketch the graphs of the following equations, using a single set of axes. Specify the coordinates of the vertex and the equation of the axis of each.
   (a) $y = -2(x - 6)^2$
   (b) $y = 8 - 2(x - 6)^2$

*5. A ball is thrown upward from a vertical cliff. Its distance s in feet above ground is given by $s = 200 + 100t - 16t^2$, where t denotes time (in seconds). Find
   (a) the time at which the ball is 336 above the ground.
   (b) the time at which the ball reaches a maximum height.
   (c) the maximum height.

6. What is the solution set of $(x - 1)(x + 2) = 4$?

7. Which of the following equations are equivalent to the equation $2x - 1 = 0$?

   (a) $x = \frac{1}{2}$

   (b) $4x^2 - 1 = 0$

   (c) $4x^2 - 4x + 1 = 0$

   (d) $12x^2 - 4x - 1 = 0$

8. Find the value of k for which the equation $x^2 - 3kx + k = 2$ has 4 as a root.

9. Show that the expression

$$a\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)^2 + b\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + c$$

   equals zero (a) by calculation (b) by an argument using no calculation.

10. Sketch the graph of $y = 3 + x - 2x^2$. Specify the coordinates of the vertex and the equation of its axis.

11. Find a quadratic equation with integral coefficients whose roots are 3 and $-\frac{2}{5}$.

12. Find a quadratic equation whose roots are the negative reciprocals of the roots of $6x^2 + 11x - 10 = 0$.

13. Find the sum and product of the roots of the equation $3x^2 + 5x - 7 = 0$.

14. Find the solution set of the equation $\frac{2x^2}{x + 1} + 1 = \frac{2}{x + 1}$.

15. Solve the inequality

$$2x^2 < x + 15.$$

16. Find the solution set of the equation

$$|x + 1|^2 + |x + 1| = 6.$$

17. Solve the equation $\sqrt{x + 5} + \sqrt{x - 4} = 9$.

18. Solve the equation $(x^2 - 5x)^2 + 2(x^2 - 5x) - 24 = 0$.

19. Solve the equation $x^4 - 10x^2 + 9 = 0$.

20. Determine k so that the equation $9x^2 - 8kx = -4$ has only one root.

21. Which of the following quadratic equations has the roots r and s?

(a) $x^2 = rs$

(b) $x^2 - r^2s^2 = 0$

(c) $x^2 - rsx + r^2s^2 = 0$

(d) $x^2 - (r + s)x + rs = 0$

(e) $x^2 - rsx + rs = 0$

22. Which of the following is the axis of the graph of the equation $y = 2x^2 + 8x - 3$?

(a) $x = -4$

(b) $x = -3$

(c) $x = -2$

(d) $x = 2$

(e) $x = 4$

23. Which of the following equations has a graph which lies entirely below the x-axis?

(a) $y = 2x^2 - 4x - 5$

(b) $y = -x^2 + 4x$

(c) $y = x^2 - 10$

(d) $y = -x^2 + 5$

(e) $y = -2x^2 + 4x - 4$

24. For each of the following quadratic equations find the set of all values of k for which the equation has two roots.

(a) $3x^2 + kx + 3 = 0$

(b) $x^2 + 6x - k = 0$

(c) $kx^2 - 4x + 2 = 0$

(d) $3x^2 + 6x + k = 0$

25. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the coordinates of 3 non-collinear points, how many quadratic functions can be found whose graphs pass through these three points?

(a) 0

(b) 1

(c) 2

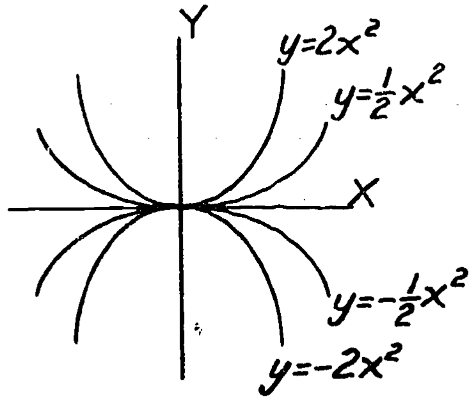
(d) Infinitely many

(e) It cannot be determined from the information given.

26. Let x_1, x_2, x_3 be distinct real numbers and y_1, y_2, y_3 be any real numbers. Prove that there is a quadratic function which pairs y_1 to x_1, y_2 to x_2 and y_3 to x_3 if and only if $(y_1 - y_2)(x_2 - x_3) - (x_1 - x_2)(y_2 - y_3) \neq 0$.

Answers to Illustrative Test Questions

1.

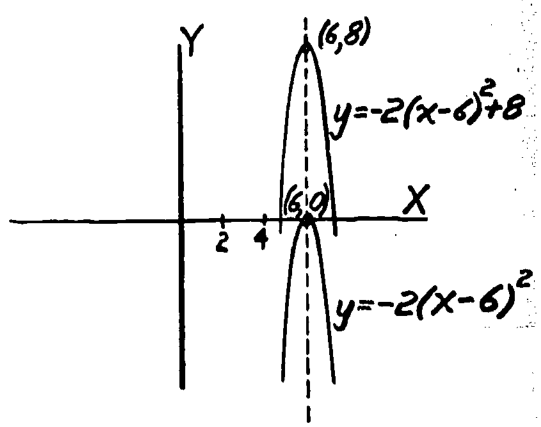


2. (a) $f(x) = x^2 + 3$; minimum, 3
   (b) $f(x) = -2x^2 + 5$; maximum, 5
   (c) $f(x) = 6x^2 - 8$; minimum, -8

3. $y = 2x^2 - 12x + 14$
   $y = 2(x^2 - 6x + 9) + 14 - 18$
   $y = 2(x - 3)^2 - 4$
   minimum, -4

4. $y = -2(x - 6)^2$
   Vertex is (6, 0).
   Equation of axis is $x = 6$.

   $y = -2(x - 6)^2 + 8$
   Vertex is (6, 8).
   Equation of axis is $x = 6$.



5. (a) $336 = 200 + 100t - 16t^2$

$16t^2 - 100t + 136 = 0$

$4t^2 - 25t + 34 = 0$

$(4t - 17)(t - 2) = 0$

2 seconds, $\frac{17}{4}$ seconds (answer).

(b) $S = -16t^2 + 100t + 200$

$S = -16(t^2 - \frac{25}{4}t + \frac{625}{64}) + \frac{625}{4} + 200$

$S = -16(t - \frac{25}{8})^2 + \frac{1425}{4}$

$\frac{25}{8}$ seconds (answer).

(c) the maximum height is $356\frac{1}{4}$ ft.

6. $\{-3, 2\}$

7. (a), (c)

8. $\frac{14}{11}$

9. (a) The given expression equals

$$\frac{a}{4a^2}(b^2 - 2b\sqrt{b^2 - 4ac} + b^2 - 4ac) - 2ab^2 + \frac{2ab\sqrt{b^2 - 4ac}}{4a^2} + \frac{4a^2c}{4a^2}$$

which reduces to zero.

(b) Follows from quadratic formula.

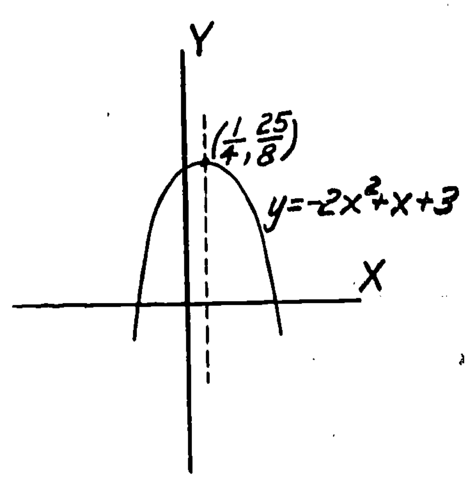
10. $y = -2x^2 + x + 3$

$y = -2(x^2 - \frac{1}{2}x + \frac{1}{16}) + 3 + \frac{1}{8}$

$y = -2(x - \frac{1}{4})^2 + \frac{25}{8}$

Vertex is $(+\frac{1}{4}, \frac{25}{8})$.

Equation of axis is $x = +\frac{1}{4}$.



11. $(x - 3)(x + \frac{2}{5}) = 0$

$5x^2 - 13x - 6 = 0$

12. $6x^2 + 11x - 10 = 0$

$(3x - 2)(2x + 5) = 0$

$x = \frac{2}{3} \quad x = -\frac{5}{2}$

$(x + \frac{3}{2})(x - \frac{2}{5}) = 0$

$(2x + 3)(5x - 2) = 0$

$10x^2 + 11x - 6 = 0$

13. Sum of roots $= -\frac{5}{3}$.

Products of roots $= -\frac{7}{3}$.

14. $\{\frac{1}{2}\}$

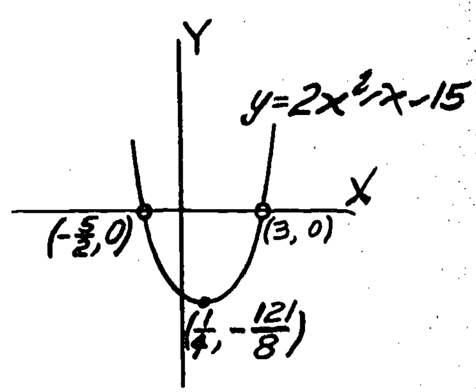
15. $2x^2 - x - 15 < 0$

$y = 2x^2 - x - 15$

$y = 2(x^2 - \frac{1}{2}x + \frac{1}{16}) - 15 - \frac{1}{8}$

$y = 2(x - \frac{1}{4})^2 - \frac{121}{8}$

$\{x: -\frac{5}{2} < x < 3\}$



16. Let $|x + 1| = z$

$z^2 + z - 6 = 0$

$(z + 3)(z - 2) = 0$

$z = -3, \quad z = 2$

$|x + 1| = -3$ Impossible

$|x + 1| = 2$

or $\{-3, 1\}$

17. $\sqrt{x + 5} = 9 - \sqrt{x - 4}$

$x + 5 = 81 - 18\sqrt{x - 4} + x - 4$

$-72 = -18\sqrt{x - 4}$

$4 = \sqrt{x - 4}$

$16 = x - 4$

$x = 20$

18. $(x^2 - 5x)^2 + 2(x^2 - 5x) - 24 = 0$

Let $z = x^2 - 5x$

$z^2 + 2z - 24 = 0$

$(z + 6)(z - 4) = 0$

$z = -6, \quad z = 4$

$x^2 - 5x = -6, \quad x^2 - 5x = 4$

$\{2, 3, \frac{5 + \sqrt{41}}{2}, \frac{5 - \sqrt{41}}{2}\}$

19. $x^4 - 10x^2 + 9 = 0$

$(x^2 - 9)(x^2 - 1) = 0$

$\{3, -3, 1, -1\}$

20. $9x^2 - 8kx + 4 = 0$

$64k^2 - 144 = 0$

$k^2 = \frac{144}{64} = \frac{9}{4}$

$\frac{3}{2}$ or $-\frac{3}{2}$ (answer).

21. (d)

22. (c)

23. (e)

24. $b^2 - 4a \geq 0$

(a) $3x^2 + kx + 3 = 0$

$k^2 - 36 > 0$

$\{k: k < -6 \text{ or } k > 6\}$

(b) $x^2 + 6x - k = 0$

$36 + 4k > 0$

$4k > 36$

$k > -9$

(c) $kx^2 - 4x + 2 = 0$

$16 - 8k > 0$

$k \leq 2$

(d) $3x^2 + 6x + k = 0$

$36 - 12k > 0$

$k < 3$

25. (b)

26. Proof:

We see from the slope formula that the given points are not collinear if and only if

$$\frac{y_1 - y_2}{x_1 - x_2} \neq \frac{y_2 - y_3}{x_2 - x_3} .$$

Therefore the points are not collinear if and only if

$$(y_1 - y_2)(x_2 - x_3) - (x_1 - x_2)(y_2 - y_3) \neq 0.$$

Chapter 5

COMMENTARY FOR TEACHERS

THE COMPLEX NUMBER SYSTEM

5-0. Introduction.

The complex number system is one of the supreme achievements of the human intellect. Compelling reasons for extending the real number system are easy to find. In the context of the real number system the theory of quadratic equations is most unsatisfactory, for some quadratic equations with real coefficients have real solutions, while others have no real solutions. The desire to remedy this situation is surely reasonable and modest. What is remarkable is the fact that this modest aim, once attained, yields a system so rich that no further extensions are necessary to capture the roots of any algebraic equation of whatever degree. However, the solution of algebraic equations is only one of the achievements of the complex number system. It is surely lamentable that we are unable, at this level of the students' development, to indicate the profusion of important and beautiful results to be found in the theory of functions of a complex variable. We can only state -- with all the enthusiasm we can muster -- that this field of mathematics (and others closely related to it) is probably the most intensively cultivated at the present time, and that its applications in the sciences and engineering seem to grow daily.

The extension of the real number system to the complex number system can be regarded as the solution of a problem -- the problem of constructing a number system with certain properties. The solution of any problem generally proceeds in three stages (the solution of an equation is typical): 1. statement of the problem; 2. identification of a possible solution, assuming that a solution exists; 3. verification that the possible solution actually is a solution. Accordingly in Section 5-1 we state the properties that the system is required to have; in Sections 5-2, 5-3, 5-4 we identify the system by finding its elements and the rules for operating with them, assuming that such a system exists; and in Section *5-11 we verify that the system constructed with these elements and rules of operation has the required properties.

In the complex number system, classical algebra -- the theory of equations -- finds its proper setting. The role of the complex number system in the theory of equations is discussed in Sections 5-5 and 5-8.

The connection between the complex number system and geometry is of great importance for geometry and analysis as well as for algebra. This connection is introduced in Sections 5-6 and 5-7 and further explored in Chapter 12.

5-1. Comments on the Introduction to Complex Numbers.

In Section 5-1 we review the inadequacy of the real number system with respect to the solution of quadratic equations and announce our intention to attempt a remedy by extending the real number system. We state that we will find a system in which every quadratic equation with real coefficients has a solution if we seek one in which the equation $x^2 + 1 = 0$ has a solution. This is so, of course, because every quadratic equation

$$ax^2 + bx + c = 0$$

with negative discriminant can be transformed into the equivalent equation

$$\left(\frac{x + \frac{b}{2a}}{\sqrt{\frac{4ac - b^2}{4a^2}}} \right)^2 = -1 .$$

This is not discussed in the text until Section 5-6, but a brief informal class discussion might be appropriate at this time.

The Properties C-1, C-2, and C-3 which we require our new number system to possess are just explicit statements of the simple and natural requirements that the system have all the algebraic properties of the real number system, include the real number system, and contain a solution of the equation $x^2 + 1 = 0$. In Section 5-2 we shall impose a fourth requirement -- also simple, but not so natural (see 5-2).

It should be observed that in past extensions of the number system the extended system was required to have many of the <u>order</u> properties of the original system, but this is not done here. It is not done because it cannot be done. If the complex number system had the order properties of the real number system the theorem that the square of every number is non-negative would have to hold, but this contradicts $i^2 = -1$.

Problems 1 and 4 of Exercises 5-2 can be assigned after Section 5-1, if desired: Problem 1 reviews the reasons for previous extensions of the number system; Problem 4 is intended to stimulate discussion of the fact, mentioned above, that the order properties of a number system may not be preserved when the system is extended.

5-2. Complex Numbers.

In the preceding section we stated a problem which we tacitly assumed had a unique solution. It does not -- as we will see later. An additional condition is needed to make the problem definite, that is, to insure that it has a unique solution.

To expose this difficulty let us consider it in a more familiar setting. Suppose that our number system is the system of rational numbers and that we wish to extend it to a system in which the equation $x^2 = 2$ has a solution. Explicitly, we seek a system which has Properties C-1; Properties C-2 with the word "real" replaced by "rational" wherever it occurs; and which has the third property -- corresponding to C-3 -- that it contains a number $\sqrt{2}$, such that $(\sqrt{2})^2 = 2$. Let us call these Properties S-1, S-2 and S-3, respectively.

We know that the system of real numbers has these properties, but looking ahead, so does the system of complex numbers. Our problem does not have a unique solution; it has at least two solutions, and possibly more.

It would seem foolish to extend the system of rational numbers to the system of complex numbers just to achieve Properties S-1, S-2 and S-3; the system of complex numbers is too large -- it contains a number system (the real numbers) which already has all the properties we require. Pursuing this objection, the system of real numbers might be larger than we require. It seems natural to add to our conditions the requirement S-4 that the system be as small as possible. With this condition added our problem has a unique solution S: The elements of S are those real numbers which can be written in the form $a + b\sqrt{2}$, where a and b, are rational numbers; and the operations in S are addition and multiplication of real numbers.

It is obvious that S has Properties S-2 and S-3. That it has all the Properties S-1 except (i), (iv) and (vii) follows immediately from the fact that the system of real numbers has these properties, and from S-2. It can be verified by calculation that the sum, product, opposite and reciprocal of real numbers which can be expressed in the form $a + b\sqrt{2}$, a and b rational, can also be expressed in this form, so that S has properties S-1(i), (iv) and (vii). Thus, S is a solution of our problem. Notice that in this argument the only statements whose proofs were not immediate are those asserting that the sum, product, additive inverse and multiplicative inverse of numbers in S are in S.

It is easy to see that S is the smallest system which solves our problem. Consider any other set of real numbers which, with addition and multiplication of real numbers as operations, forms a system S' which is a solution of the problem. Then S' contains all rational numbers and $\sqrt{2}$, and is closed with respect to addition and multiplication. Hence it must contain all real numbers which can be expressed in the form $a + b\sqrt{2}$, a and b rational -- that is, it must contain S.

We summarize the salient features of this discussion: The properties we have required do not determine a unique number system; The natural additional condition to impose to determine a unique system is that the system be the smallest possible one having the given properties; This additional condition is logically equivalent to the condition that every number in the system be expressible in a certain form; The essential part of the proof of the equivalence of the two conditions is the proof that the sum, product, additive and multiplicative inverses of numbers which can be expressed in the stated form can also be expressed in that form.

The problem of extending the system of real numbers to the system of complex numbers is entirely analogous to the problem we have just discussed. Each of the summary statements we have just made holds also for the extension from the real numbers to the complex numbers.

We could have presented a discussion analogous to that given here in the text. Such a discussion, however, would have been an extensive and sophisticated preliminary to a program whose first objective is the introduction of complex numbers and the rules for calculating with them. Instead we have adopted a middle course.

In Section 5-2 we add Property C-4 to our requirements instead of the more natural condition that the system be the smallest possible system having Properties C-1, C-2, and C-3. The connection between these two conditions is suggested through brief discussion. However, in the discussion of addition, multiplication, additive inverse and multiplicative inverse in sections 5-3, 5-4 and 5-5 we make no essential use of Property C-4; we use it only as a guide. Thus, at the end of Section 5-5 one can look back and see that Property C-4 is not <u>necessary</u>, but that to find a system having Properties C-1, C-2 and C-3 it is <u>sufficient</u> to consider the system with Property C-4. Better students should be encouraged to do this, and all students should be aware of the need to check at each stage the compatibility of Property C-4 with the other properties of the system.

We have still to present an example of a system larger than the system of complex numbers which has properties C-1, C-2, and C-3. The simplest example is the following. Let H contain the complex numbers, an element j which is not a complex number, and all expressions of the form

$$\frac{a_0 j^n + a_1 j^{n-1} + \cdots + a_{n-1} j + a_n}{b_0 j^m + b_1 j^{m-1} + \cdots + b_{m-1} j + b_m}$$

where n and m are non-negative integers, $a_0, a_1, \cdots a_n$ and $b_0, b_1, \cdots, b_m$ are any complex numbers, and $a_0 \neq 0$, $b_0 \neq 0$. Thus H is the set of all quotients of polynomials in j with complex coefficients. Addition and multiplication are defined according to the usual rules for operating with polynomials. Then H has all the desired properties.

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Problem 2 of Exercises 5-2 is intended to point out that in previous extensions of the number system the system sought was the smallest one having the desired properties. Problem *5 provides an opportunity for the student to carry through for himself the discussion presented above.

Exercises 5-2. Answers.

1. (a) The system of integers has an additive identity element, and each integer has an additive inverse.

   (b) In the rational number system each element except zero has a multiplicative inverse.

   (c) In the real number system every non-negative number has two real even roots, and every negative number has one real odd root.

   (d) The complex number system contains an element i which has the property $i^2 = -1$.

2. (a) System of integers.

   (b) Rational number system.

   (c) Rational number system.

3. (a) $1 + 0i$ (e) $3 + 0i$

   (b) $0 + 0i$ (f) $0 + 2i$

   (c) $-1 + 0i$ (g) $-1 + 0i$

   (d) $0 + (1)i$

4. (a) The natural number system has the Well Order property. Every subset has a least element.

   (b) The real number system has an order relation. No order relation has been defined for the complex number system.

5. (a) If $\sqrt{3}$ were in S we could write

$$\sqrt{3} = a + b\sqrt{2}$$

where a and b are rational. If we square both sides of this equation we get

$$3 = a^2 + 2ab\sqrt{2} + 2b^2$$

or

$$\frac{3 - a^2 - 2b^2}{2ab} = \sqrt{2}.$$

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5. Since a and b are rational, the left side of the last equation is rational, and the equations says that $\sqrt{2}$ is rational. Since we know this is false, the assumption that $\sqrt{3}$ belongs to S must be false.

(b) $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$, and if a, b, c, d are rational so are a + c and b + d, since the rational numbers are closed with respect to addition.

$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2}$
and if a, b, c, d are rational so are ac + 2bd and bc + ad, since the rational numbers are closed with respect to addition and multiplication.

(c) The additive inverse of $a + b\sqrt{2}$ in the real number system is $-(a + b\sqrt{2})$. But

$$-(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2}$$

and if a and b are rational so are -a and -b.

The additive identity in S is $0 = 0 + 0\sqrt{2}$. If $a + b\sqrt{2}$ is not zero it has a multiplicative inverse $\frac{1}{a + b\sqrt{2}}$ in the real number system. But

$$\frac{1}{a + b\sqrt{2}} = \left(\frac{a}{a^2 - 2b^2}\right) + \left(\frac{-b}{a^2 - 2b^2}\right)\sqrt{2}$$

and if a and b are rational so are $\frac{a}{a^2 - 2b^2}$ and $\frac{-b}{a^2 - 2b^2}$, since the rational number system is closed with respect to addition, multiplication, subtraction and division.

(d) Property (i) of C-1 was established in part (b) of this problem.

Property (ii) is established by observing that addition in S is addition of real numbers and addition of real numbers is associative and commutative. To be more explicit, addition is commutative since $x + y = y + x$ if x and y are any real numbers, and hence, in particular if $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$. Properties (v) and (viii) are established in the same way.

Property (iii) is established by observing that $0 = 0 + 0\sqrt{2}$ is in S, and $x + 0 = x$ for any real number. Thus, in particular, if $x = a + b\sqrt{2}$, $x + 0 = x$, and 0 is an additive identity in S. 0 is the only additive identity in S since any other additive identity c in S would be a real number which satisfied $x + c = x$ for all x in S. But, taking $x = 0$, this becomes $0 + c = 0$ or $c = 0$. Property (vii) is established in a similar way.

Since 0 is the additive identity in S, an additive inverse of a number $a + b\sqrt{2}$ in S is a solution x in S of the equation $x + (a + b\sqrt{2}) = 0$. There is one and only one real number $-(a + b\sqrt{2})$ which satisfies this equation. We showed in part (c) that $-(a + b\sqrt{2})$ is in S, and since this is the only real number which satisfies the equation, it is the only number in S which satisfies the equation. This establishes property (iv). Property (vii) is established in a similar way.

(e) S has the stated properties. Let S' be another part of the real number system with the stated properties, and let a and b be any rational numbers. Then a, b and $\sqrt{2}$ are in S'. Since S' is closed with respect to multiplication it contains $b\sqrt{2}$ and since it is closed with respect to addition it contains $a + b\sqrt{2}$. Thus every number in S is in S', and S' contains the system S.

5-3. Addition, Multiplication and Subtraction.

In this section we begin the discussion of operations with complex numbers. It should be emphasized that our objective is to perform operations with complex numbers in terms of operations with real numbers. The discussion of addition and multiplication is straightforward, but that of subtraction deserves some comment.

Subtraction is, as usual, defined as the inverse of addition. We show that the equation

$$z_1 + z = z_2$$

has at least one solution $z = z_2 + (-z_1)$. Notice, however, that

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in order to define $z_2 - z_1$ as the solution of this equation, and to assert

$$z_2 - z_1 = z_2 + (-z_1),$$

it is essential to show that the equation has at most one solution -- a unique solution. The teacher may find it desirable to present the proof of uniqueness to the class.

The additive inverse $-z$ of z is defined by the equation

$$z + (-z) = 0.$$

According to Property C-4, $-(a + bi) = x + yi$ where x and y are real. Substituting in the equation defining $-(a + bi)$ we obtain two real equations in x and y which have the solution $x = -a$ and $y = -b$. We therefore conclude that

$$-(a + bi) = -a + (-b)i.$$

Notice however that here we have been using Property C-4 only as a guide. To prove the last equation it is only necessary to verify that

$$[a + bi] + [(-a) + (-b)i] = 0,$$

and this is done without using C-4.

Exercises 5-3 provide practice in addition, multiplication and subtraction of complex numbers.

Exercises 5-3. Answers.

1. (a) $4 + 9i$
   (b) $4 + 0i$
   (c) $3 + 7i$
   (d) $(4 + \pi) + \pi i$
   (e) $(\sqrt{2} + 1) + 5i$
   (f) $-1 + 7i$
   (g) $8 + (1)i$
   (h) $0 + 7i$
   (i) $15 + (1)i$
   (j) $(3 + \sqrt{2}) + (9 + \sqrt{3})i$.
2. (a) $5i$
   (b) yi
   (c) $\sqrt{3}\, i$
   (d) $5i$

   Yes, any real number might have been added to the answer given here.
3. (a) $-13 + 26i$
   (b) $24 + (-10)i$

3. (c) $5 + 5i$
   (d) $-5 + 3i$
   (e) $2 + 2\sqrt{2}\, i$
   (f) $(8 - \sqrt{6}) + (8\sqrt{3} + \sqrt{2})i$
   (g) $-7 + 24i$
   (h) $2 + 0i$
   (j) $-18 + 0i$
   (k) $14 + (-84)i$
   (l) $70 + 40i$
   (m) $-106 - 83i$
   (n) $92 - 18i$
   (o) $(cx - dy) + (cy + dx)i$, if c,d,x,y are real numbers
   (p) $(x^2 - xy) + (xy - y^2)i$, if x, y are real numbers.

Note: Part (i) was omitted in text so is omitted here.

4. (a) $-3 + 0i$
   (b) $0 + (-1)i$
   (c) $-1 + (-1)i$
   (d) $-2 + (-3)i$
   (e) $-5 + 4i$
   (f) $4 + 3i$
   (g) $-a + bi$, if a,b are real numbers
   (h) $-x + (-y)i$, if x,y are real numbers.
5. (a) $5 + 8i$
   (b) $-2 + 2i$
   (c) $0 + 10i$
   (d) $-1 + 0i$
   (e) $(\sqrt{3} - 2) + (1 - \sqrt{2})i$
   (f) $1 + i$
   (g) $\pi + (-\pi)i$
   (h) $0 + 6i$
   (i) $1 + (-3)i$
6. (a) $i^3 = i^2 \cdot i = (-1)i = \quad 0 + (-1)i$
   (b) $i^4 = i^2 \cdot i^2 = (-1)(-1) = \quad 1 + 0i$
   (c) $i^9 = (i^2)^4 \cdot i = (-1)^4 i = \quad 0 + (1)i$

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6. (d) $i^{15} = (i^2)^7 i = (-1)^7 i = \quad 0 + (-1)i$

(e) $i^{4n+1} = (i^2)^{2n} \cdot i = [(-1)^2]^n i = 0 + (1)i$

(f) $i^{79} = (i^2)^{39} \cdot i = (-1)^{39} i = \quad 0 + (-1)i$

7. General rule: the values of the powers of i recur in cycles of 4.

To explain the general rule first note that

$$i^1 = i,$$
$$i^2 = -1,$$
$$i^3 = i^2 \cdot i = (-1)i = -i,$$
$$i^4 = (i^2)^2 = (-1)^2 = 1.$$

Making use of the first four powers we have

$$i^5 = i^4 \cdot i = (1)i = i,$$
$$i^6 = i^4 \cdot i^2 = (1)(-1) = -1,$$
$$i^7 = i^4 \cdot i^3 = (1)(-i) = -i,$$
$$i^8 = i^4 i^4 = (1)(1) = 1.$$

In general, if n and m are natural numbers such that $n = 4m$, we have

$$i^n = i^{4m} = (i^4)^m = 1^m = 1.$$

Thus, $i^{4m+1} = i^{4m} \cdot i = (1)i = i,$

$$i^{4m+2} = i^{4m} \cdot i^2 = (1)(-1) = -1,$$
$$i^{4m+3} = i^{4m} \cdot i^3 = (1)(-i) = -i,$$
$$i^{4m+4} = i^{4m} \cdot i^4 = (1)(1) = 1.$$

These possibilities are all there are, for if n is a natural number and we divide it by 4, the only non-negative remainders less than 4 which we can get are 0,1,2,3.

8. (a) $1 + (-1)i$

(b) $0 + (-1)i$

(c) $0 + 107i$

(d) $-7 + 84i$

(e) $-1 + (-1)i$

(f) $11 + 20i$

(g) $2abc + [-a^3 - b^3 - c^3 - (b + c)(c + a)(a + b)]i$

(h) $-1 + 0i$

(i) $-10 + 0i.$

9. $$2\left(\frac{3 + \sqrt{7}\,i}{4}\right)^2 - 3\left(\frac{3 + \sqrt{7}\,i}{4}\right) + 2$$
$$= \frac{2 + 6\sqrt{7}\,i}{8} - \frac{9 + 3\sqrt{7}\,i}{4} + 2$$
$$= \frac{2 + 6\sqrt{7}\,i - 18 - 6\sqrt{7}\,i}{8} + 2$$
$$= -2 + 2 = 0$$

5-4. Standard Form of Complex Numbers.

Section 5-4 is devoted to proving Theorem 5-4 and to defining some important terms. Theorem 5-4 asserts that each complex number z may be written in the form $a + bi$ (a and b real) in only one way. (C-4 asserts that z may be written in this form in at least one way.) This theorem justifies the definite article in the expression "the standard form" used to describe this way of writing complex numbers. (One advantage of Theorem 5-4 is that it shows us we can have only one answer for exercises like those in Section 5-3 where the student is asked to express certain complex numbers in what we now call "standard form".) The double-barrelled way in which Theorem 5-4 is stated gives the teacher an opportunity to refresh the students' minds on the distinction between "if" and "only if", a distinction which cannot be overemphasized. However, the statement containing "only if" is the only part that requires a proof.

Any tendency to regard Theorem 5-4 as obvious can be overcome by emphasizing that the requirement in the hypothesis that a,b,c,d be real is essential; without this requirement the conclusion is false. Example 5-4a demonstrates this.

It is worth observing that the proof of Theorem 5-4 can be based on the following special case of the theorem: If a and b are real, then $a + bi = 0$ ($= 0 + 0i$) if and only if $a = 0$ and $b = 0$. Let us suppose this has been proved and show how the general case follows from it. Let a,b,c,d be real. Then

$$a + bi = c + di$$

if and only if

$$(a - c) + (b - d)i = 0.$$

The equation in the last line holds if and only if $a - c = 0$ and

$b - d = 0$. This proves Theorem 5-4.

A word (or two) about the terms defined in Section 5-4 may be in order. "Standard form" should cause no trouble; though one must emphasize that the a and b appearing in the standard form are <u>real</u> <u>numbers</u>. (Throughout the rest of the chapter we sometimes say "$a + bi$, in standard form" and sometimes "$a + bi$, where a and b are real numbers"; these expressions have identical meanings.) "Real part" is straightforward and should cause no trouble. Mathematicians have used the expression "imaginary part" as defined in the text for many years: The imaginary part of a complex number is a <u>real</u> number. This terminology may be unfortunate, but it is standard. Writers of many elementary books have departed from the mathematicians' usage, saying that bi is the "imaginary part" of $a + bi$. Students reading other books will notice that they are not all in agreement. (This experience is a valuable part of anyone's education.) A student who goes on in mathematics has to learn sooner or later that in advanced work b is called the imaginary part of $a + bi$. Since it seems a shame to teach him something he must later unlearn, we stick to the mathematicians' standard terminology: The imaginary part of a complex number is a real number.

Observe that 0 is both real and <u>pure</u> imaginary, but that it is not imaginary. This may be momentarily disconcerting; but it should be so only momentarily. One has only to remember that everyday connotations and relations of [illegible] and phrases are irrelevant to their technical use: A technical term means only what its definition says it means.

Problems 1 - 5 of Exercises 5-4 are practice problems. Problem 6 refers to the special case $c = d = 0$ of Theorem 5-4 discussed above, and emphasizes again the necessity of the condition that a and b be real. Problem 7 generalizes Theorem 5-4: Theorem 5-4 is the special case obtained by setting $z_1 = 1$, $z_2 = i$.

Exercises 5-4. Answers.

1.

	Real part	Imaginary part
(a)	0	2
(b)	0	0
(c)	0	1
(d)	5	-1
(e)	2x	3
(f)	a	-2
(g)	1	$-2\sqrt{2}$
(h)	-2	$-2\sqrt{3}$
(i)	-3	1
(j)	2	0
(k)	0	3
(l)	1	2

2. (a) -3
   (b) 0
   (c) -5
   (d) -5.

   There is only one way in each case.

3. (a) $x = 3, y = -6$
   (b) $x = 3, y = 0$
   (c) $x = 0, y = -4$
   (d) $x = \frac{1}{2}, y = -3$
   (e) $x = -\frac{4}{3}, y = 2$
   (f) $x = 4, y = 2$
   (g) $x = 2, y = 6$
   (h) $x = 0, y = 0$
   (i) $x = \pm 1, y = 0$
   (j) $x = 0, y = -1$.

4. (a) $8 + 3i$
   (b) $-2 + 0i$
   (c) $6 + 12i$
   (d) $4 + 8i$
   (e) $11 + (-16)i$
   (f) $10 + (-11)i$
   (g) $18 + 14i$
   (h) $(a^2 + 2ab + b^2 + c^2) + 0i$
   (i) $(x^3 - 3xy^2) + (3x^2y - y^3)i$.

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5. Let $z^2 = (x + yi)^2 = 8 + 6i$.

Then $(x^2 - y^2) + 2xyi = 8 + 6i$.

Since x and y are real we must have

$$\text{(i)}\quad x^2 - y^2 = 8,$$

$$\text{(ii)}\quad 2xy = 6.$$

Squaring both members of the last two equations, we obtain

$$\text{(iii)}\quad x^4 - 2x^2y^2 + y^4 = 64,$$

$$\text{(iv)}\quad 4x^2y^2 = 36.$$

Adding the last two equations, we get

$$\text{(v)}\quad (x^2 + y^2)^2 = 100.$$

Since $x^2 + y^2$ must be positive, it follows that

$$\text{(vi)}\quad x^2 + y^2 = +10.$$

Adding (i) and (vi), we get

$$2x^2 = 18, \qquad 2y^2 = 2.$$

Hence

$$x = \pm 3, \qquad y = \pm 1.$$

From (ii) x and y have the same sign so $\begin{cases} x = 3 \\ y = 1 \end{cases}$ and $\begin{cases} x = -3 \\ y = -1 \end{cases}$

Note. In a sense the problem appears to be that of finding the square root of the complex number $8 + 6i$; however, we have not defined the symbol $\sqrt{\ }$ for complex numbers.

6. Let $a = x + yi$ and $b = u + vi$ where $x, y, u,$ and v are all real.

(a) If $a = 0$ and $b = 0$, then $a + bi$ and $a - bi$ are both zero.

(b) Suppose $a + bi = 0$, then

$$x + yi + (u + vi)i = 0$$

or

$$(x - v) + (y + u)i = 0.$$

By Theorem 5-4, we have

$$x - v = 0 \quad \text{and} \quad y + u = 0$$

or,

$$\text{(1)}\quad x = v \text{ and } y = -u$$

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i.e. if $a = v - ui$ and $b = u + vi$, $a + bi = 0$ with neither a or b zero.

Since $a - bi = 0$ also, we have $x + v + (y - u)i = 0$, or

(2) $x = -v$ and $y = u$.

Both (1) and (2) can be satisfied only if $x = u = 0$ and $y = v = 0$. In this case $a = 0$ and $b = 0$.

7. Let $z = x + yi$ and $z_1 = x_1 + iy_1$, $y_1 \neq 0$. Then

$$z = a + bz_1$$

if and only if

$$x = a + bx_1, \quad y = by_1$$

that is, if and only if

$$b = \frac{y}{y_1}, \quad a = x - \frac{yx_1}{y_1}.$$

5-5. Division.

The discussion of division in this section parallels that of subtraction in Section 5-3. The comments made about subtraction hold also, with obvious modifications, for division. Once again it should be emphasized that our objective is to express calculations with complex numbers in terms of calculations with real numbers.

The central problem of this section is to express the multiplicative inverse $\frac{1}{z}$ of $z = a + bi$ in terms of a and b. Since $\frac{1}{z}$ is defined by the equation

$$\frac{1}{z} \cdot z = 1,$$

and since, by the Property C-4, $\frac{1}{z} = x + yi$, x and y real, the problem reduces to solving the equation

$$(x + yi)(a + bi) = 1$$

for real values of x and y. This equation can be transformed into the equation

$$(ax - by) + (bx + ay)i = 1.$$

Now, if x and y are real then the expressions in parentheses are real; here we are using Property C-2. Hence, by the theorem on standard form, the equation above is satisfied if and only if

$$ax - by = 1,$$

and $$bx + ay = 0.$$

The problem has thus been reduced to that of solving a pair of linear equations with real coefficients for the real unknowns x and y. The solution of this system proceeds in the familiar way, and we conclude

$$\frac{1}{a + bi} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i .$$

To *find* this result we used Property C-4. However, to *establish* the result we have only to verify that

$$\left(\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i\right) (a + bi) = 1,$$

and this verification makes no use of Property C-4.

Now looking back over the discussion in Sections 5-3, 5-4 and 5-5 we see that, as promised in Section 5-2, we have proved that the sum, product, and additive and multiplicative inverses of numbers given in the form $a + bi$ can again be expressed in this form. Thus, if we had required that the system we sought be the smallest possible system having Properties C-1, C-2, and C-3, we could have established Property C-4 as a theorem.

Of equal importance is the fact that we have achieved our objective of expressing all operations with complex numbers in terms of operations with real numbers.

Exercises 5-5 either provide practice in operations with complex fractions, or require the proof of statements made in the text without proof.

Exercises 5-5. Answers.

1. (a) $1 + 0i$

   (b) $\frac{1}{5} + 0i$

   (c) $0 + (-1)i$

   (d) $0 + (1)i$

   (e) $\frac{1}{2} + (-\frac{1}{2})i$

1. (f) $\frac{2}{13} + (-\frac{3}{13})i$

   (g) None

   (h) $\frac{4}{25} + \frac{3}{25}i$.

2. Zero does not have a multiplicative inverse.

3. 1, -1

4. i, -i

5. (a) $\frac{2}{5} + (-\frac{1}{5})i$

   (b) $0 + (-\frac{3}{2})i$

   (c) $-\frac{5}{29} + (-\frac{2}{29})i$

   (d) $\frac{5}{2} + (-\frac{13}{2})i$

   (e) $\frac{1}{5} + \frac{3}{5}i$

   (f) $\frac{23}{29} + (-\frac{14}{29})i$

   (g) $-\frac{3}{25} + \frac{46}{25}i$

   (h) $\frac{21}{65} + \frac{12}{65}i$

   (i) $-\frac{25}{34} + (-\frac{15}{34})i$

   (j) $-\frac{1}{3} + \frac{2\sqrt{2}}{3}i$

   (k) $\frac{\sqrt{2} + \sqrt{6}}{3} + \frac{\sqrt{3} - 2}{3}i$

   (l) $\frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$

   (m) $\frac{2a^2 - 2b^2}{4a^2 + b^2} + \frac{5ab}{4a^2 + b^2}i$

   (n) $\frac{m^2 - n^2}{m^2 + n^2} + \frac{2mn}{m^2 + n^2}i$

   (o) $\frac{3x^2 - 2y^2}{x^2 + y^2} + \frac{5xy}{x^2 + y^2}i$

6. Let z_3 and z_4 be two solutions of the equation $z_1 z = z_2$, so that

$$z_1 z_3 = z_2, \text{ and } z_1 z_4 = z_2.$$

Multiply both members of each of the last two equations by $\frac{1}{z_1}$.

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Then

$$\frac{1}{z_1} z_2 = \frac{1}{z_1}(z_1 z_3) = (\frac{1}{z_1} z_1) z_3 = 1 \cdot z_3 = z_3;$$

$$\frac{1}{z_1} z_2 = \frac{1}{z_1}(z_1 z_4) = (\frac{1}{z_1} z_1) z_4 = 1 \cdot z_4 = z_4.$$

Therefore $z_3 = z_4$.

Alternate solution for 6:

Suppose u and z are solutions of the equation. Then

$$z_1 u = z_2$$

$$z_1 z = z_2$$

and $$z_1(u - z) = 0.$$

By (5-5f) this can happen only if one of the factors is zero. $z_1 \neq 0$, $\therefore$ $u - z = 0$ or $u = z$.

7. Let $z = a + bi$. (Note that $a^2 + b^2 \neq 0$.)

Then $\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$.

Thus, the real part of $\frac{1}{z} = \frac{a}{a^2 + b^2} = \frac{1}{2}$,

and

$$2a = a^2 + b^2.$$

(a) If $b = 0$, then $a = 2$ (since a and b cannot both be zero); and $z = a + bi = 2 + 0i$.

(b) If $b = \frac{1}{2}$, then $2a = a^2 + \frac{1}{4}$,

$$4a^2 - 8a + 1 = 0;$$

and $$a = 1 + \frac{\sqrt{3}}{2}, \text{ or}$$

$$a = 1 - \frac{\sqrt{3}}{2}.$$

So there are two possible numbers z:

$z_1 = (1 + \frac{\sqrt{3}}{2}) + \frac{1}{2}i$; $z_2 = (1 - \frac{\sqrt{3}}{2}) + \frac{1}{2}i$.

(c) If $b = 1$, then $2a = a^2 + 1$,

$$a^2 - 2a + 1 = 0,$$

$$a = 1.$$

Hence $$z = 1 + i.$$

8. The "if" part of the proof follows immediately from the fact

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that $0 \cdot z = 0$. To prove the "only if" part of the statement suppose that $z_1 z_2 = 0$. If $z_1 \neq 0$, then C contains a number $\frac{1}{z_1}$. Multiplying by $\frac{1}{z_1}$, we get $z_2 = 0 \cdot \frac{1}{z_1} = 0$.
Thus if $z_1 z_2 = 0$, and $z_1 \neq 0$, we have $z_2 = 0$. Similarly if $z_1 z_2 = 0$ and $z_2 \neq 0$, then we have $z_1 = 0$. Therefore $z_1 z_2 = 0$ implies that either $z_1 = 0$ or $z_2 = 0$, or possibly both (since $0 \cdot 0 = 0$).

9. Let w_o be the unique solution of $z_2 w = z_1$; then $w_o = \frac{z_1}{z_2}$ and $z_2 w_o = z_1$. Similarly let w_1 be the unique solution of $z_4 w = z_3$; then $w_1 = \frac{z_3}{z_4}$ and $z_4 w_1 = z_3$. Also let w_2 be the unique solution of $(z_2 z_4)w = z_1 z_3$; then $w_2 = \frac{z_1 z_3}{z_2 z_4}$. We must show that $w_o w_1 = w_2$. From $z_2 w_o = z_1$ and $z_4 w_1 = z_3$ we get $(z_2 w_o)(z_4 w_1) = z_1 z_3$ or $(z_2 z_4)(w_o w_1) = z_1 z_3$. Thus $w_o w_1$ satisfies $(z_2 z_4)w = z_1 z_3$. But w_2 is the <u>only</u> solution of this equation. Hence $w_o w_1 = w_2$.

10. Let $w_o = \frac{z_1}{z_2}$ and $w_1 = \frac{z_3}{z_4}$, and let w_3 be the unique solution of $(z_2 z_4)w = z_1 z_4 + z_2 z_3$. To show $w_o + w_1 = w_3$. From $z_2 w_o = z_1$ and $z_4 w_1 = z_3$ we get $z_4(z_2 w_o) = z_1 z_4$ and $z_2(z_4 w_1) = z_2 z_3$; so, adding, $z_2 z_4(w_o + w_1) = z_1 z_4 + z_2 z_3$. Thus $w_o + w_1$ satisfies the equation whose only root is w_3. Hence $w_o + w_1 = w_3$.

11. (a) $\frac{6}{5} + 0i$

(b) $-\frac{7}{50} + (-\frac{12}{25})i$

(c) $8 + 0i$

(d) $-\frac{1}{2} + 0i$

(e) $\frac{2a^4 - 12a^2b^2 + 2b^4}{(a^2 + b^2)^2} + 0i$.

*12. Whether or not a and b are real numbers, provided that $a^2 + b^2 \neq 0$, we can multiply the factors $a + bi$, $\frac{a - bi}{a^2 + b^2}$ and get $(a + bi)\frac{a - bi}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} + \frac{ab - ab}{a^2 + b^2} i = 1$

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12. (for there is nothing in the proof of Theorem 5-3b which requires a and b to be real). Thus $\frac{a - bi}{a^2 + b^2}$ is an inverse of a + bi, if $a^2 + b^2 \neq 0$. But we know already that no complex number can have more than one inverse, for if it did Property C-1-(vii) (as stated in the text) would be false.

5-6. Quadratic Equations.

Section 5-6 extends the theory of quadratic equations with real coefficients by treating the case of a negative discriminant. Since the quadratic formula involves the expression $\sqrt{b^2 - 4ac}$ and we are interested in the case $b^2 - 4ac < 0$, we are obliged to precede our discussion of the formula by a definition of $\sqrt{r}$ for r real and negative. Hence we begin with the examples $z^2 = -1$ and $z^2 = r$, $r < 0$, and lead up to the extended definition of $\sqrt{r}$ (Definition 5-6). With the definition of $\sqrt{r}$ available, we summarize our results on the special quadratics (those having no first degree term) in Theorem 5-6a, a result we need in the proof of Theorem 5-6b. Theorem 5-6b is proved by the usual process of completing the square, and then using Theorem 5-6a to solve

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} .$$

Since $\sqrt{4a^2} = 2|a|$, the square roots of the right member are

$$\frac{\sqrt{b^2 - 4ac}}{2|a|}, \qquad \frac{-\sqrt{b^2 - 4ac}}{2|a|} .$$

One of these is $\frac{\sqrt{b^2 - 4ac}}{2a}$, the other is $\frac{-\sqrt{b^2 - 4ac}}{2a}$ (which is which depends on whether $a > 0$ or $a < 0$). Theorem 5-6b solves the problem of finding the solutions of the general quadratic equation with real coefficients. We find that every quadratic with real coefficients is one of three types: (1) It has one root -- which is real -- if its discriminant is zero; (2) It has two (different) real roots if its discriminant is positive; (3) It has two (different) non-real complex roots if its discriminant is negative.

Exercises 5-6, Problems 1-3 provide practice in calculating with the square root symbol. It should be emphasized that when a variable appears in the radicand it is in general necessary to distinguish several cases. One reason for this is that the statement $\sqrt{r}\sqrt{s} = \sqrt{rs}$ which holds for $r \geq 0$, $s \geq 0$ is not true in general. Problem 5 requires a proof of the extension of this statement to the case in which r and s are not both negative; Problem 4 is intended to show why the statement is not true when r and s are both negative.

Problems 6 - 17 provide practice in the solution of quadratic equations. Problem 16 deserves particular comment. Although we have established the "quadratic formula" only for the case of real coefficients, it continues to hold when the coefficients are complex provided the discriminant is real; in this case the formula can be established exactly as it was for the case of real coefficients. Thus the quadratic equation

$$z^2 + \beta z + \alpha = 0$$

with complex coefficients β, α can be solved by means of the quadratic formula if

$$\beta^2 - 4\alpha = r,$$

where r is a real number. We can construct quadratic equations for which this is true by choosing the complex number β and the real number r arbitrarily, and determining α from

$$\alpha = \frac{\beta^2 - r}{4}.$$

The equation of Problem 16 is determined by choosing $\beta = -i$, $r = -9$. In Chapter 12 we shall discuss quadratic equations with complex coefficients without the restriction that the discriminant be real.

Problems 18 - 24 provide an opportunity for the student to investigate by himself questions which will be discussed in detail in Section 5-9 and Chapter 12. We mention in particular Problems 19 and 20, which state important results of algebra;

these will be stated more generally in Section 5-9. The approach suggested in the hint for Problem 22 could be used for the solution of quadratic equations with complex coefficients in general, but the method is too cumbersome to be useful. Some students might be interested in pursuing this point, however.

Exercises 5-6. Answers.

1. (a) $0 + 7i$
   (b) $0 + (-13\sqrt{5})i$
   (c) $0 + (4\sqrt{2})i$
   (d) $-2\sqrt{5} + 0i$
   (e) $0 + (-4\sqrt{3})i$
   (f) $0 + \frac{1}{2}i$
   (g) $\frac{3\sqrt{2}}{4} + 0i$
   (h) $\frac{\sqrt{6}}{3} + 0i$
2. (a) $0 + 2i$
   (b) $2 + 0i$
   (c) $0 + 2i$
   (d) $|c| + 0i$
   (e) $|c| + 0i$
   (f) $0 + |c|i$
   (g) $0 + |c|i$
3. (a) $0 + (a + b)i$
   (b) $-2a^2\sqrt{b} + 0i$
   (c) $-(a + \sqrt{ab}) + 0i$
   (d) $\frac{5\sqrt{a}}{3} + 0i$
   (e) $0 + (-a\sqrt{2})i$
   (f) $-a^2 + 0i$
   (g) $0 + 2(a + b)i$
4. Proof that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ if a and b are non-negative real numbers:
   By the definition of the square root of a non-negative number we know that

$$(\sqrt{a})^2 = a,$$
$$(\sqrt{b})^2 = b.$$

Thus $$(\sqrt{a}\sqrt{b})^2 = ab,$$

and we know that $(\sqrt{a}\sqrt{b})$ is a square root of ab. Since $\sqrt{a}$ and $\sqrt{b}$ are both non-negative by definition, it follows that $(\sqrt{a}\sqrt{b})$ is non-negative. Hence $(\sqrt{a}\sqrt{b})$ must be the square root of ab; that is,

$$\sqrt{ab} = \sqrt{a}\sqrt{b}.$$

Now if a and b are negative, then

$$\sqrt{a} = i\sqrt{-a},$$
$$\sqrt{b} = i\sqrt{-b};$$

and $(\sqrt{a}\sqrt{b}) = (i\sqrt{-a})(i\sqrt{-b}) = -\sqrt{ab}$.

Again $(\sqrt{a}\cdot\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab$

but as we have just seen

$\sqrt{a}\cdot\sqrt{b} = -\sqrt{ab}$, a negative number

which cannot be the square root of ab.

5. $r < 0$ and $s > 0$, then
$\sqrt{r}\sqrt{s} = i\sqrt{-r}\sqrt{s} = i\sqrt{-rs}$;
also $\sqrt{rs} = i\sqrt{(-r)(s)} = i\sqrt{-rs}$.

6. $0 + i,\ 0 + (-1)i$

7. $\frac{-1 + \sqrt{5}}{2} + 0i,\ \frac{-1 - \sqrt{5}}{2} + 0i$

8. $-1 + (1)i,\ -1 + (-1)i$

9. $\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{1}{2} + (-\frac{\sqrt{3}}{2})i$

10. $-\frac{1}{3} + \frac{\sqrt{11}}{3}i,\ -\frac{1}{3} + (-\frac{\sqrt{11}}{3})i$

11. $-2 + 2i,\ -2 + (-2)i$

12. $2 + 2i,\ 2 + (-2)i$

13. $-\frac{1}{4} + \frac{\sqrt{7}}{4}i,\ -\frac{1}{4} + (-\frac{\sqrt{7}}{4})i$

14. If $a \geq -\frac{1}{2}$: $(2 + 2\sqrt{1 + 2a}) + 0i,\ (2 - 2\sqrt{1 + 2a}) + 0i$
If $a < -\frac{1}{2}$; $2 + 2\sqrt{-(1 + 2a)}\,i,\ 2 + [-2\sqrt{-(1 + 2a)}]i$

15. $-\frac{1}{2m} + \frac{\sqrt{3}}{2m}i,\ -\frac{1}{2m} + (-\frac{\sqrt{3}}{2m})i$

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16. $0 + 2i, \quad 0 + (-1)i$

17. If $\frac{c}{a} \geq 0$: $\quad 0 + \sqrt{\frac{c}{a}}\, i \,, \quad 0 + (-\sqrt{\frac{c}{a}})i$

If $\frac{c}{a} < 0$: $\quad \sqrt{-\frac{c}{a}} + 0i \,, \quad -\sqrt{-\frac{c}{a}} + 0i$

18. $z^3 - 8 = (z - 2)(z^2 + 2z + 4)$

$z^3 - 8 = 0$ if and only if $z - 2 = 0$ or $z^2 + 2z + 4 = 0$

The solutions are $2, \; -1 + \sqrt{3}\, i, \; -1 + (-\sqrt{3})i$.

19. Using Theorem 5-6b we obtain the following solutions for the given equation:

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$z_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Thus,

$$z_1 + z_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - b}{2a} = -\frac{b}{a}$$

and

$$z_1 z_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (b^2 - 4ac)}{4a^2},$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}.$$

20. $az^2 + bz + c = a(z^2 + \frac{b}{a} z + \frac{c}{a})$

By making use of the results of Problem 19, the right side can be written as

$$a[z^2 - (z_1 + z_2)z + z_1 z_2].$$

Hence,

$$az^2 + bz + c = a(z - z_1)(z - z_2),$$

or alternately, multiplying out the right side of the last equation, the left may be obtained directly.

21. (a) $z^2 - 2z + 2 = 0$

(b) $z^2 - (2 + 2i)z - 1 + 2i = 0$

(c) $z^2 = 0$

(d) $z^2 - [(a_1 + a_2) + (b_1 + b_2)i]z$

$\quad + [(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i] = 0$

*22. Let $z = x + yi$, where x and y are real.

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Then $z^2 = x^2 - y^2 + 2xyi$.

But $z^2 = i$, so

$$\text{(i)} \quad x^2 - y^2 = 0,$$
$$\text{(ii)} \quad 2xy = 1.$$

Squaring both sides of (i) and (ii) and adding, we have

$$\text{(iii)} \quad (x^2 + y^2)^2 = 1.$$

Since $x^2 + y^2 > 0$, taking square roots of both members of (iii) we have

$$\text{(iv)} \quad x^2 + y^2 = 1.$$

Adding (i) and (iv) we obtain

$$2x^2 = 1.$$

From whence

$$x = \pm \frac{\sqrt{2}}{2}.$$

From (ii) the corresponding values of y are

$$y = \pm \frac{\sqrt{2}}{2}.$$ (Note that from (ii) x and y have the same sign.)

Therefore $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad -\frac{\sqrt{2}}{2} + (-\frac{\sqrt{2}}{2})i.$

*23. Employing the method displayed in the solution of Problem *22, we obtain

$$z = \frac{\sqrt{2}}{2} + (-\frac{\sqrt{2}}{2})i, \quad -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$$

*24. Extending the idea of Problem 20, we have

$$[z - (1 + 2i)]\,[z - (1 - i)]\,[z - (1 + i)] = 0,$$

or, multiplying out the left member, we obtain

$$z^3 - (3 + 2i)z^2 + (4 + 4i)z - (2 + 4i) = 0.$$

There is no quadratic equation having all three solutions, for the formula in Problem 20 shows that no quadratic equation may have more than two solutions: If $az^2 + bz + c = a(z - z_1)(z - z_2) = 0$, $a \neq 0$, then either $z - z_1 = 0$ or $z - z_2 = 0$; i.e., $z = z_1$ or $z = z_2$. Moreover no quadratic expression such as $az^2 + bz + c$ can be written as a product of three first degree factors, say $(z - z_1)(z - z_2)(z - z_3)$, times a constant: For any such product produces a z^3 term and no quadratic can have such a term.

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5-7. Graphical Representation -- Absolute Value.

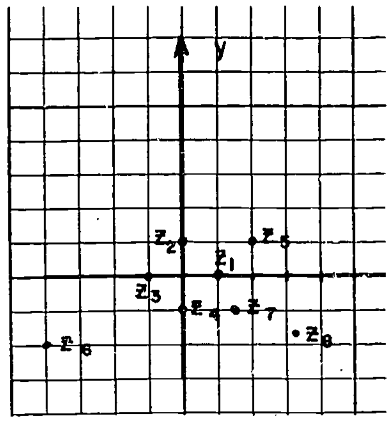
The representation of complex numbers by points in the plane had a great effect historically on the acceptance of the complex number system by mathematicians. This geometric representation overcame the feeling that the complex number system was not concrete; the employment of the complex number system in the solution of geometric problems, which it permitted, promoted an appreciation of the usefulness of the system. The discussion in Section 5-7, and its continuation in Section 5-8 and Chapter 12, may be expected to have a similar effect upon students.

The discussion in the text calls for little comment. We mention only that the notion of absolute value is a purely algebraic one, even though its definition is geometrically motivated; all of the properties of absolute value can be established algebraically. In particular, the relations $|z_1 z_2| = |z_1||z_2|$, $|z_1 + z_2| \leq |z_1| + |z_2|$ can be established algebraically. It is remarkable that although the geometric interpretation of the first relation is obscure and that of the second very clear, the algebraic proof of the first is relatively simple while that of the second is quite difficult. Because of this difficulty the algebraic proof is not presented in the text. The interested teacher can find such a proof in almost any text on the theory of functions of a complex variable. (See, for example, R.V. Churchill, Introduction to Complex Variables and Applications.)

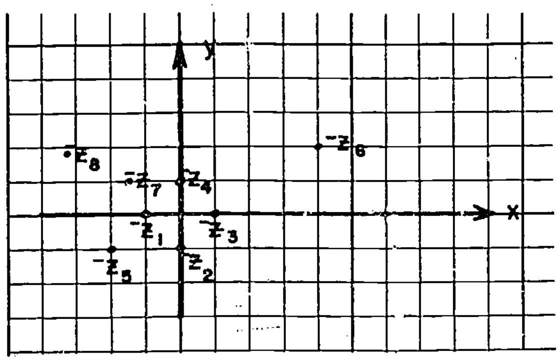
Exercises 5-7, Problems 1 - 4 provide practice in the graphical representation of complex numbers and the graphical interpretation of addition and subtraction. Problems 5 - 7 involve the calculation of absolute values. Problems 8 - 10 require the proof of statements made in the text without proof. Problems 11 - 12 refer to the geometric interpretation of operations with complex numbers in special cases.

Exercises 5-7. Answers.

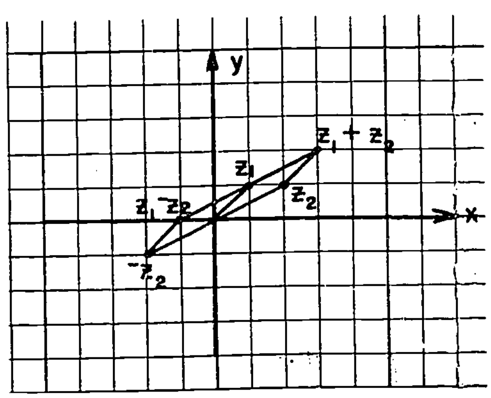
1.



2.



3. (a)



$z_1 + z_2 = 3 + 2i$

$z_1 - z_2 = -1 + 0 \cdot i$

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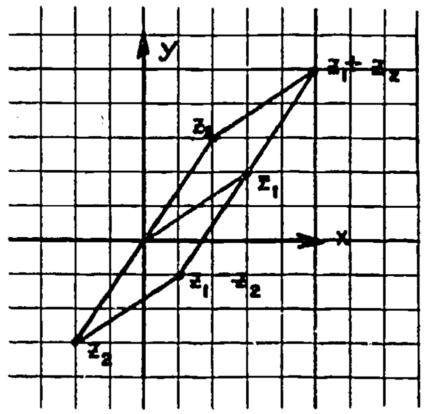
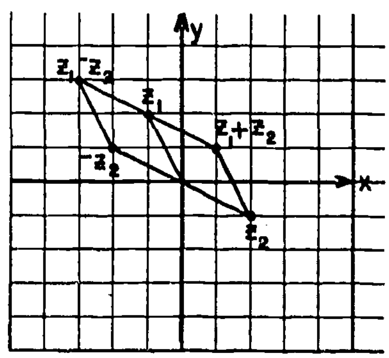
3. (b)



$z_1 + z_2 = 5 + 5i$

$z_1 - z_2 = 1 - i$

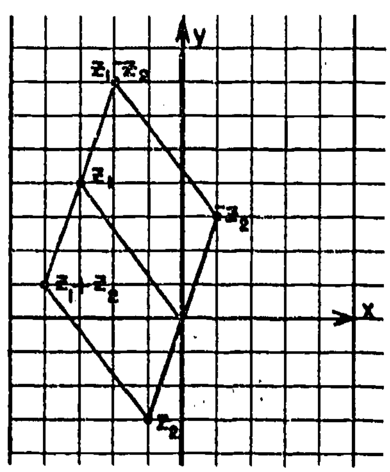
(c)



$z_1 + z_2 = 1 + i$

$z_1 - z_2 = -3 + 3i$

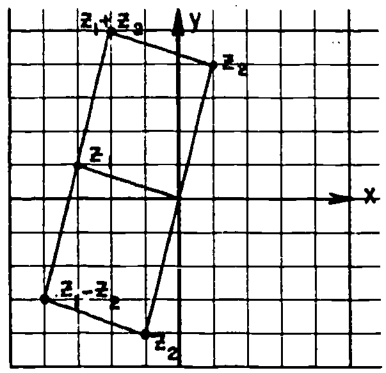
(d)



$z_1 + z_2 = -4 + i$

$z_1 - z_2 = -2 + 7i$

3. (e)



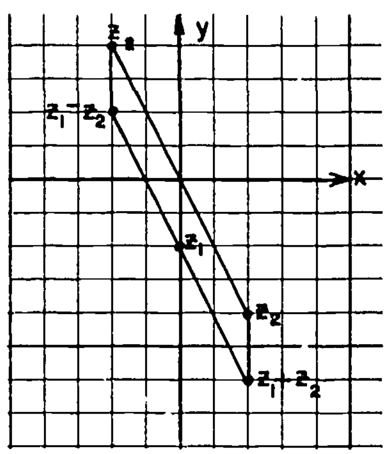
$z_1 + z_2 = -2 + 5i$

$z_1 - z_2 = -4 - 3i$

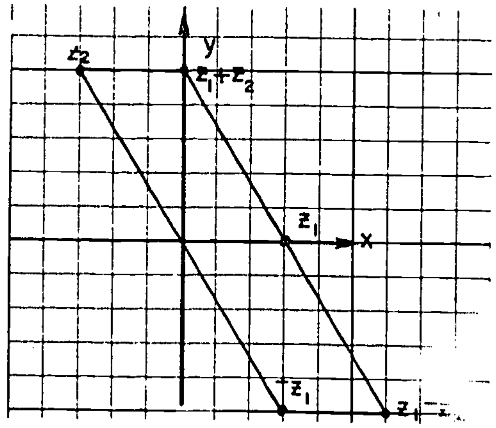
(f)

$z_1 + z_2 = 2 - 6i$

$z_1 - z_2 = -2 + 2i$



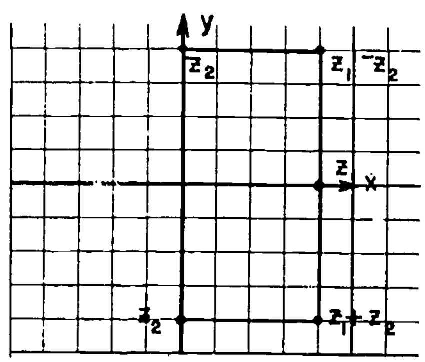
3. (g)



$z_1 + z_2 = 5i$

$z_1 - z_2 = 6 - 5i$

(h)



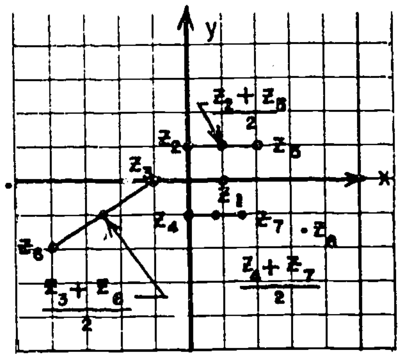
$z_1 + z_2 = 4 - 4i$

$z_1 - z_2 = 4 + 4i$

4. $\frac{z_2 + z_5}{2} = \frac{i + (2 + i)}{2} = 1 + i.$

$\frac{z_3 + z_6}{2} = \frac{-1 + (-4 - 2i)}{2} = -\frac{5}{2} + (-1)i.$

$\frac{z_4 + z_7}{2} = \frac{-i + (\sqrt{2} - i)}{2} = \frac{\sqrt{2}}{2} + (-1)i.$



5. (a) 5
   (b) 2
   (c) 0
   (d) $\sqrt{2}$
   (e) $\sqrt{\pi^2 + 2}$

6. Let $z = x + yi$

   then $\frac{z}{|z|} = \frac{x + yi}{\sqrt{x^2 + y^2}}$,

   and $\left|\frac{z}{|z|}\right| = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{\frac{x^2 + y^2}{x^2 + y^2}} = 1.$

7. (a) The single point (1,0).
   (b) Let $z = x + yi$, x and y real.

   Then $x + yi = \sqrt{x^2 + y^2}$.

   Hence $y = 0$, and $x = \sqrt{x^2}$.

   Therefore, the set of points is the non-negative x-axis.
   (c) Since z cannot be zero, the given equation may be transformed into the equation $|z| = 1$, and this is the equation of the unit circle.

8. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

Then $|z_1 z_2| = \left|(x_1+y_1 i)(x_2+y_2 i)\right| = \left|(x_1x_2-y_1y_2) + (x_1y_2+x_2y_1)i\right|$

$$= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_2x_2y_1 + x_2^2y_1^2}$$

$$= \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}$$

$$= \sqrt{x_1^2 + y_1^2}\ \sqrt{x_2^2 + y_2^2}$$

$$= |z_1| \cdot |z_2|.$$

9. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

Then $\dfrac{z_1}{z_2} = \dfrac{(x_1 + y_1 i)(x_2 - y_2 i)}{x_2^2 + y_2^2} = \dfrac{(x_1x_2 + y_1y_2) + (x_2y_1 - x_1y_2)i}{x_2^2 + y_2^2}$

and $\left|\dfrac{z_1}{z_2}\right| = \dfrac{\sqrt{x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_2^2y_1^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2}}{x_2^2 + y_2^2}$

$$= \frac{\sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}}{x_2^2 + y_2^2}$$

$$= \frac{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}}{x_2^2 + y_2^2}$$

$$= \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}}$$

$$= \frac{|z_1|}{|z_2|}.$$

10. Using the fact that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side, we have

$|z_1 - z_2| + |z_2| \geq |z_1|$ and $|z_1 - z_2| + |z_1| \geq |z_2|$

or

$|z_1 - z_2| \geq |z_1| - |z_2|$ and $|z_1 - z_2| \geq |z_2| - |z_1|$.

From this we conclude

$$|z_1 - z_2| \geq \left||z_1| - |z_2|\right| .$$

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11. If 0, $z_1 = a + bi$, and $z_2 = c + di$ are collinear, then the slope of the segment joining 0 and z_1 is the same as the slope of the segment joining 0 and z_2. Thus,

$$\text{(i)} \quad \frac{b}{a} = \frac{d}{c} .$$

If $z_3 = z_1 + z_2$,

then $z_3 = (a + c) + (b + d)i$.

The slope of the segment joining 0 and z_3 is

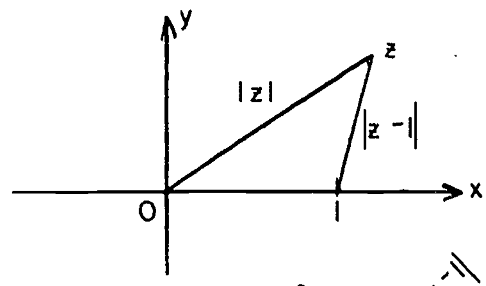
$$\text{(ii)} \quad \frac{b + d}{a + c} .$$

But (ii) is equal to both members of (i); that is

$$\frac{b}{a} = \frac{d}{c} \rightarrow \frac{b}{d} = \frac{a}{c} \rightarrow \frac{b + d}{d} = \frac{a + c}{c} \rightarrow \frac{b + d}{a + c} = \frac{d}{c} .$$

Hence, the slope of the segment joining 0 and z_3 is the same as the slopes of the segments joining 0 and the points z_1 and z_2 respectively, and since all three segments pass through 0, the points 0, z_1, z_2 and z_3 are collinear.

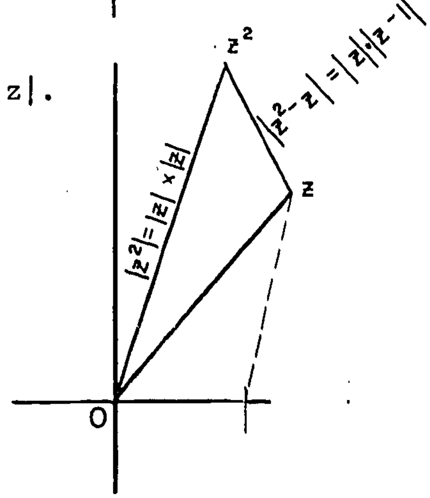
12. The triangle with vertices 0,1,z is shown in the figure at the right. The lengths of the sides of the triangle are 1, $|z|$, $|z - 1|$.



If we multiply each of these lengths by $|z|$, we obtain

$|z|\cdot 1$, $|z|\cdot|z|$, $|z||z - 1| = |z^2 - z|$.

These are the lengths of the sides of a triangle whose vertices are $0, z, z^2$ as the second figure clearly shows.



The two triangles are similar because corresponding sides are proportional.

To obtain a geometric construction for z^2, one must

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choose a unit of length on the x-axis, draw a triangle with vertices 0, 1, z, and then construct a second triangle similar to the first one by making each side of the second triangle $|z|$ times as long as the sides of the first. The vertex of the second triangle which corresponds to z of the first triangle is z^2.

5-8. Complex Conjugate.

The introduction of the notion of complex conjugate has several important consequences. It makes possible: the simplification of computations involving absolute values and multiplicative inverses; the algebraic representation of the geometric operation of reflection in a line; the algebraic formulation and manipulation of statements involving the real and imaginary parts of complex numbers; and the algebraic representation of <u>all</u> geometric relations in terms of complex numbers.

In connection with the last of these features it should be observed that only geometric conditions which are satisfied by a finite number of points can be expressed in terms of the complex variable z alone, since an equation in z has only a finite number of solutions. The solution set of an equation in z alone is, in general, a finite set of points; the solution set of an equation in z and $\bar{z}$ is, in general, a curve.

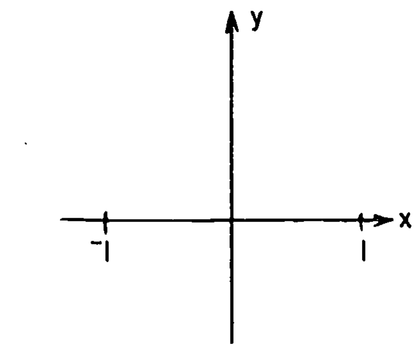
The examples and exercises of Section 5-8 illustrate the statements made above. In particular, Problems 2, 9 and 11 are concerned with computations involving absolute value and multiplicative inverse; Problems 6 and 14 are concerned with reflection in lines; Problems 7, 8 and 10 are concerned with the algebraic formulation of statements about the real and imaginary parts of complex numbers; and Problems 3,4,12,13,15 are concerned with the complex algebraic formulation of geometric conditions. Problem 1 provides practise in computing conjugates, and Problem 5 requires the proof of statements made in the text without proof.

Exercises 5-8. Answers.

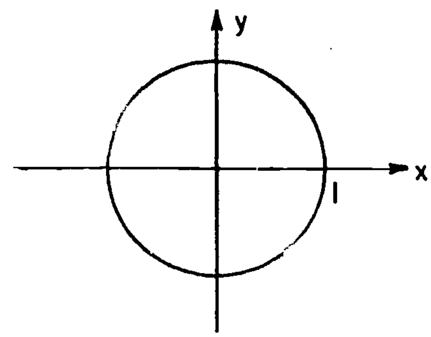
1. (a) $2 + (-3)i$
   (b) $-3 + (-2)i$
   (c) $1 + (1)i$
   (d) $-5 + 0i$
   (e) $0 + 2i$
   (f) $1 + (1)i$
   (g) $0 + \pi i$
   (h) $3 + 0i$
   (i) $-\sqrt{3} + 3i$

2. (a) $\frac{3}{2} + (-\frac{1}{2})i$
   (b) $\frac{1}{10} + \frac{3}{10}i$
   (c) $-\frac{1}{13} + (-\frac{5}{13})i$
   (d) $\frac{7}{29} + \frac{26}{29}i$
   (e) $-\frac{3}{25} + \frac{46}{25}i$
   (f) $\frac{1}{2} + (-\frac{3}{4})i$
   (g) $-\frac{9}{25} + (-\frac{38}{25})i$
   (h) $-3 + (-\frac{3}{2})i$
   (i) $\frac{15}{14} + (-\frac{5\sqrt{5}}{14})i$
   (j) $(\frac{15 + \sqrt{6}}{28}) + (\frac{3\sqrt{3} - 5\sqrt{2}}{28})i$
   (k) $(\frac{3 + \sqrt{35}}{8}) + (\frac{\sqrt{15} - \sqrt{21}}{8})i$
   (l) $\frac{1}{2} + \frac{1}{2}i$
   (m) $(\frac{2a^2 + 3b^2}{4a^2 + 9b^2}) + (\frac{-ab}{4a^2 + 9b^2})i$
   (n) $(\frac{2x^2 - y^2}{4x^2 + y^2}) + (\frac{3xy}{4x^2 + y^2})i$
   (o) $-\frac{2}{13} + (-\frac{3}{13})i$
   (p) $\frac{1}{5} + 0i$

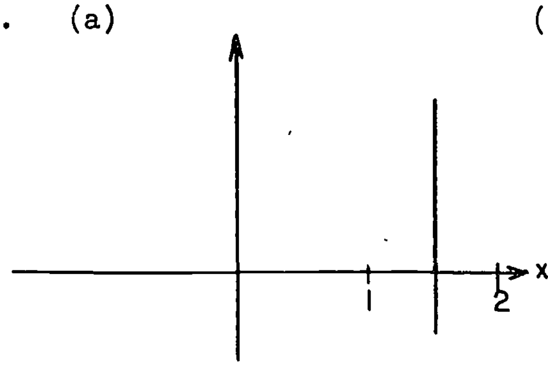
3. (a)



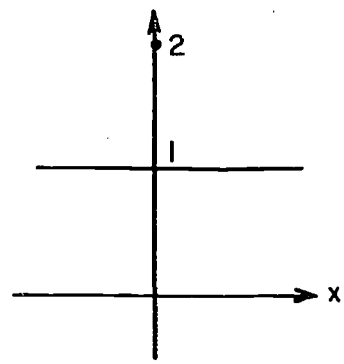
(b)



4. (a)



(b)



(c) There is no complex number z which satisfies the given equation. Hence the set is empty.

5. (a) $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i.$

$$\overline{z_1 + z_2} = (x_1 + x_2) - (y_1 + y_2)i$$
$$= (x_1 - y_1 i) + (x_2 - y_2 i)$$
$$= \overline{z}_1 + \overline{z}_2.$$

(b) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$

$$\overline{z_1 \cdot z_2} = (x_1 x_2 - y_1 y_2) - (x_1 y_2 + x_2 y_1)i$$

But the expression in the right member is equal to the following:

$\overline{z_1}\cdot\overline{z_2} = (x_1 - y_1 i)(x_2 - y_2 i)$

$= (x_1x_2 - y_1y_2) - (x_1y_2 + \ldots)$

Hence

$\overline{z_1 \cdot z_2} = \overline{z_1}\cdot\overline{z_2}$.

(c) $\overline{(-z_2)} = \overline{(-x_2 - y_2 i)} = -x_2 + y_2 i;$

$-(\overline{z_2}) = -(\overline{x_2 + y_2 i}) = -(x_2 - y_2 i) = -x_2 + y_2 i;$

hence $\overline{(-z_2)} = -(\overline{z_2})$.

Since $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, we can now write

$\overline{z_1 - z_2} = \overline{z_1 + (-z_2)} = \overline{z_1} + \overline{(-z_2)} = \overline{z_1} - \overline{z_2}$.

(d) $\overline{\left(\frac{1}{z_2}\right)} = \overline{\left(\frac{x_2 - y_2 i}{x_2^2 + y_2^2}\right)} = \frac{x_2 + y_2 i}{x_2^2 + y_2^2}$;

$\frac{1}{\overline{z}_2} = \frac{1}{x_2 - y_2 i} = \frac{x_2 + y_2 i}{x^2 + y^2}$;

hence $\overline{\left(\frac{1}{z_2}\right)} = \frac{1}{\overline{z}_2}$.

Since $\overline{z_1 \cdot z_2} = \overline{z_1}\cdot\overline{z_2}$, we can now write

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{z_1}\,\overline{\left(\frac{1}{z_2}\right)} = \overline{z_1}\left(\frac{1}{\overline{z_2}}\right) = \frac{\overline{z_1}}{\overline{z_2}} .$$

6. The reflection of any point w in the y-axis is $-\overline{w}$. Hence the reflection of $z^3 - (3 + 2i)z^2 + 5iz - 7$ is

$-[\overline{z^3 - (3 + 2i)z^2 + 5iz - 7}] = -[\overline{(z^3)} - \overline{(3 + 2i)}\,\overline{(z^2)}$

$+ \overline{5i}(\overline{z}) - \overline{7}] = -[(\overline{z})^3 - (3 - 2i)(\overline{z})^2 - 5i(\overline{z}) - 7]$

$= -\overline{z}^3 + (3 - 2i)\overline{z}^2 + 5i\overline{z} + 7$.

7. If $z^2 = \overline{z}^2$ then $0 = z^2 - \overline{z}^2 = (z + \overline{z})(z - \overline{z})$, so either $z + \overline{z} = 0$ or $z - \overline{z} = 0$. In the first case z is pure imaginary, in the second case z is real.

8. A number w is pure imaginary if and only if $w = -\overline{w}$. Thus $z_1\overline{z}_2$ is pure imaginary if and only if

$$z_1\overline{z_2} = -\overline{(z_1\overline{z}_2)}$$

$$z_1\overline{z_2} = -\overline{z_1}\;\overline{\overline{z_2}}$$

$$z_1\overline{z_2} = -\overline{z}_1 z_2.$$

Dividing this last equation by $z_2\overline{z_2}$ we obtain

$$\frac{z_1}{z_2} = -\frac{\overline{z_1}}{\overline{z_2}},$$

and

$$\frac{z_1}{z_2} = -\overline{\left(\frac{z_1}{z_2}\right)},$$

which holds if and only if $\frac{z_1}{z_2}$ is pure imaginary.

9. $$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2}) = (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$
$$= z_1\overline{z_1} + z_2\overline{z_2} - \overline{z_1}z_2 - z_1\overline{z_2}$$
$$= |z_1|^2 + |z_2|^2 - \overline{z_1}z_2 - z_1\overline{z_2}.$$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$
$$= z_1\overline{z_1} + z_2\overline{z_2} + \overline{z_1}z_2 + z_1\overline{z_2}$$
$$= |z_1|^2 + |z_2|^2 + \overline{z_1}z_2 + z_1\overline{z_2}.$$

Thus $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

10. Let $z_1 = x_1 + y_1 i$ and let $z_2 = x_2 + y_2 i$.

$z_1 + z_2$ is real if $y_1 + y_2 = 0$, and

$z_1 z_2$ is real if $x_1 y_2 + x_2 y_1 = 0$.

But if $y_1 + y_2 = 0$, then either $y_1 = y_2 = 0$, or $y_2 \neq 0$ and $y_1 = -y_2$. In the first case z_1 and z_2 are both real, and in the second case we have $x_1(y_2) + x_2(-y_2) = 0$, or $x_1 = x_2$. So in the second case $z_1 = \overline{z}_2$.

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11. It is sufficient to show that $\left| \frac{z_1}{z_2} \right|^2 = \frac{|z_1|^2}{|z_2|^2}$.

But $\left| \frac{z_1}{z_2} \right|^2 = \overline{\left(\frac{z_1}{z_2}\right)} \left(\frac{z_1}{z_2}\right) = \left(\frac{\overline{z_1}}{\overline{z_2}}\right)\left(\frac{z_1}{z_2}\right) = \frac{\overline{z_1}z_1}{\overline{z_2}z_2} = \frac{|z_1|^2}{|z_2|^2}$

12. If $y = 3x + 2$ then since $x = \frac{1}{2}(\overline{z} + z)$, $y = \frac{1}{2}(\overline{z} - z)$ we have

$$\tfrac{1}{2}(\overline{z} - z) = 3\cdot\tfrac{1}{2}(\overline{z} + z) + 2,$$

or simplifying

$$(-3 + i)\overline{z} + (-3 - i)z = 4,$$

which may also be written

$$(-3 + i)\overline{z} + \overline{(-3 + i)}z = 4.$$

13. Let $z = x + yi$ and $K = A + Bi$ where x,y and A,B are real. Substituting in

$$K\overline{z} + \overline{K}\,z = C$$

we get

$$(A + Bi)\overline{(x + yi)} + \overline{(A + Bi)}(x + yi) = C$$
$$(A + Bi)(x - yi) + (A - Bi)(x + yi) = C$$

$$[(Ax + By) + (Bx - Ay)i] + [(Ax + By) + (-Bx + Ay)i] = C$$

$$2(Ax + By) = C.$$

If $B \neq 0$ then

$$y = \frac{C - 2Ax}{B}$$

which is the equation of a straight line. If $B = 0$ then

$$x = \frac{C}{2A}$$

which is the equation of a straight line parallel to the y-axis.

14. The points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are symmetric with respect to the line $y = x$ if and only if $y = x$ is the perpendicular bisector of the segment joining z_1 and z_2. This is equivalent to two conditions: the midpoint of the segment joining z_1 and z_2 is on the line $y = x$; the segment joining z_1 and z_2 is perpendicular to the line $y = x$. The first of these conditions is algebraically

$$\frac{x_1 + x_2}{2} = \frac{y_1 + y_2}{2};$$

the second condition is

$$\frac{y_2 - y_1}{x_2 - x_1} = -1.$$

Thus for symmetry with respect to $y = x$ the following pair of equations must be satisfied:

$$x_1 + x_2 = y_1 + y_2$$

$$y_2 - y_1 = x_1 - x_2.$$

Multiplying the second equation by i and adding the result to the first we obtain

$$x_1 + x_2 + i(y_2 - y_1) = y_1 + y_2 + i(x_1 - x_2)$$

$$(x_1 - iy_1) + (x_2 + iy_2) = (ix_1 + y_1) - (ix_2 - y_2)$$

$$(x_1 - iy_1) + (x_2 + iy_2) = i(x_1 - iy_1) - i(x_2 + iy_2)$$

$$\overline{z_1} + z_2 = i\overline{z_1} - iz_2$$

$$(1 - i)\overline{z_1} + (1 + i)z_2 = 0$$

which was to be proved.

15. Let $z_1 = x_1 + y_1i$, $z_2 = x_2 + y_2i$. (We assume $z_1 \neq 0$, $z_2 \neq 0$ since otherwise the problem has no geometric meaning.) Then

$$z_1\overline{z_2} = (x_1 + y_1i)(\overline{x_2 + y_2i}) = (x_1 + y_1i)(x_2 - y_2i)$$

$$= (x_1x_2 + y_1y_2) + (y_1x_2 - x_1y_2)i,$$

so that if $z_1\overline{z_2}$ is real

$$y_1x_2 - x_1y_2 = 0.$$

If $x_1 = 0$ then since $y_1 \neq 0$ it follows from this equation that $x_2 = 0$, so that both z_1 and z_2 are on the y-axis and the segments joining them to the origin are parallel. The same conclusion is obtained in the same way if $x_2 = 0$. In the general case $x_1 \neq 0$ and $x_2 \neq 0$ so that we may divide our last equation by x_1x_2 to obtain

$$\frac{y_1}{x_1} - \frac{y_2}{x_2} = 0$$

or

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} .$$

Thus the slopes of the segments joining z_1 and z_2 to the origin are equal and the segments are parallel. In every case therefore if $z_1\overline{z_2}$ is real the segments z_1 and z_2 to the origin are parallel.

5-9. Polynomial Equations.

In this section we discuss the ultimate significance of the system of complex numbers for algebra. We state without proof the Fundamental Theorem of Algebra, and consider simple examples in which it applies.

Properly speaking, the Fundamental Theorem of Algebra states that every polynomial equation of positive degree with complex coefficients has at least one complex solution. The theorem we have stated as the Fundamental Theorem is obtained by combining the preceding statement with the Factor Theorem which asserts that if r is a solution of the polynomial equation $P(z) = 0$ then $z - r$ is a factor of $P(z)$. According to the Fundamental Theorem if $P(z)$ is a polynomial of degree $n > 0$ then the equation $P(z) = 0$ has a complex solution r_1. By the Factor Theorem then, $P(z) = (z - r_1)P_1(z)$ where $P_1(z)$ is a polynomial of degree $n - 1$. If $n - 1 > 0$ then applying the same argument to $P_1(z)$ we conclude that $P_1(z) = (z - r_2)P_2(z)$ or $P(z) = (z - r_1)(z - r_2)P_2(z)$. Continuing in this way we obtain the theorem stated in the text.

The teacher may wish to present the preceding discussion and a proof of the Factor Theorem to the class. The following simple proof of the Factor Theorem is based on the factoring identity

$$z^k - r^k = (z - r)(z^{k-1} + z^{k-2}r + z^{k-3}r^2 + \cdots + r^{k-1})$$

Let

$$P(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$$

be a polynomial and let r be a solution of $P(z) = 0$, that is,

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$P(r) = 0$. Then,

$$P(z) = P(z) - P(r)$$
$$= (a_o z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_o) - (a_o r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_o)$$
$$= a_o(z^n - r^n) + a_1(z^{n-1} - r^{n-1}) + \cdots + a_{n-1}(z - r)$$
$$= a_o(z-r)(z^{n-1} + z^{n-2}r + \cdots + r^{n-1}) + a_1(z-r)(z^{n-2} + z^{n-3}r + \cdots + r^{n-2})$$
$$+ \cdots + a_{n-1}(z - r)$$
$$= (z-r)[a_o(z^{n-1} + z^{n-2}r + \cdots + r^{n-1}) + a_1 \ (z^{n-2} + z^{n-3}r + \cdots + r^{n-2})$$
$$+ \cdots + a_{n-1}]$$
$$= (z - r)\,Q(z).$$

Exercises 5-9. Answers.

1. (a) 1, multiplicity 1
   -2, multiplicity 3
   (b) 0, multiplicity 4
   $-\frac{1}{2}$, multiplicity 2
   3, multiplicity 1
   (c) $3 - 2i$, multiplicity 2
   -1, multiplicity 5
2. (a) Since $z^5 + z^4 + 3z^3 = z^3[z - (\frac{-1 - \sqrt{11}i}{2})][z - (\frac{-1 + \sqrt{11}i}{2})]$, we have the following zeros:
   0, multiplicity 3
   $\frac{-1 - \sqrt{11}i}{2}$, multiplicity 1
   $\frac{-1 + \sqrt{11}i}{2}$, multiplicity 1
   (b) Since $z^4 + 2z^2 + 1 = (z + i)^2(z - i)^2$, we have the following zeros:
   $-i$, multiplicity 2
   i, multiplicity 2
   (c) Since $z^3 + 3z^2 + 3z + 1 = (z + 1)^3$, we have
   -1, multiplicity 3
3. (a) Example 1: $(z - 1)(z - 2) = 0$.
   Example 2: $a(z - 1)(z - 2) = 0$, where a is real,

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non-zero, and not equal to 1.

(b) Example 1: $(z - 1)(z - 2) = 0$. This equation is of degree 2 and each zero is of multiplicity 1.

Example 2: $(z - 1)(z - 2)^2 = 0$. This equation is of degree 3 and has zeros which are of multiplicity 1 and 2 respectively.

4. An equation of degree 4 can have either <u>one</u>, <u>two</u>, <u>three</u>, or <u>four</u> solutions. The number which it has depends on the multiplicity of the zeros of the polynomial associated with the equation. The following examples are illustrative.

One solution: $(z - 1)^4 = 0$.

The polynomial $(z - 1)^4$ has the zero 1 of multiplicity four. Hence the solution of the equation is the single value $z = 1$.

Two solutions: $(z - 1)(z - 2)^3 = 0$.

The zeros of the polynomial $(z - 1)(z - 2)^3$ are 1 (multiplicity one) and 2 (multiplicity three). The solutions of the equation are $z = 1,2$. Another example is $(z^2 + 1)^2 = 0$; its solutions are $z = i, -i$. Each solution is a zero of multiplicity two of the polynomial $(z^2 + 1)^2$. Note that here we have two pairs of conjugate complex numbers.

Three solutions: $(z - 1)(z - 2)(z - 3)^2 = 0$.

The zeros of the polynomial are 1 (multiplicity one), 2 (multiplicity one), and 3 (multiplicity two). The solutions of the equation are $z = 1,2,3$.

Four solutions: $(z - 1)(z - 2)(z - 3)(z - 4) = 0$.

The zeros of the polynomial are 1,2,3,4; each is of multiplicity one. The solutions of the equation are $z = 1,2,3,4$.

5. $z^3 + 1 = (z + 1)(z^2 - z + 1) = 0$
Hence $z = -1$ is one solution.
To obtain the remaining solution, put

$$z^2 - z + 1 = 0.$$

Then
$$z = \frac{1 \pm \sqrt{3}i}{2}.$$

The solutions of the given equation are

$$-1, \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2}.$$

6. (a) Since $z = 4$ is one solution, $(z - 4)$ is a factor of the polynomial in the left member of the given equation. Dividing the polynomial in the left member by $(z - 4)$, we find that the given equation can be rewritten in the form

$$(z - 4)(3z^2 - 8z + 4) = 0.$$

Factoring again, we have

$$(z - 4)(3z - 2)(z - 2) = 0.$$

The solutions of the equation are $4, \frac{2}{3}, 2$.

(b) $2, 1 + i, 1 - i$

7. (a) $-1, -2, \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2}.$

(b) $4, 1, -1 + \sqrt{2}i, -1 - \sqrt{2}i.$

8. (a) $(z - 1)(z + 2i)$ or $z^2 + (2i - 1)z - 2i$.
The polynomial is of degree 2.

(b) In order for the polynomial to have real coefficients it must have the conjugate of $-2i$ as a zero because it has $-2i$ as a zero. Hence the polynomial must be of degree 3; the required polynomial is

$(z - 1)(z + 2i)(z - 2i)$ or $z^3 - z^2 + 4z - 4$.

(c) The polynomial of lowest possible degree must contain the square of a polynomial of degree 2 which has both $-2i$ and $2i$ for its zeros. Thus, the required polynomial is of degree 5; it is

$$(z - 1)[(z + 2i)(z - 2i)]^2 \text{ or } z^5 - z^4 + 8z^3 - 8z^2 + 16z - 16$$

9. Since $3 + \sqrt{2}i$ is a solution of the equation so is $3 - \sqrt{2}i$. Thus

$$\{[z - (3 + \sqrt{2}i)][z - (3 - \sqrt{2}i)]\} = (z^2 - [illegible] + 11$$

is a factor of the polynomial in the left member of the given equation. By long division it can be found that the other factor is $(z^2 -$ [illegible] ence the solutions of the equation are

$$3 + \sqrt{2}i, \; [illegible] \sqrt{2}i, \; 3, \; -3.$$

10. $- \sqrt{5}\, i, \; 1 + \sqrt{5}\, i, \; [illegible], \; -\sqrt{2}.$

11. a) $(z - r_1)(z - r_2)(z - r_3) =$

$$z^3 + [(-r_1)+(-r_2)+(-r_3)]z^2 + [(-r_1)(-r_2)+(-r_1)(-r_3)+(-r_2)(-r_3)]z + [(-r_1)(-r_2)(-r_3)] =$$

$$z^3 - (r_1 + r_2 + r_3)z^2 + (r_1r_2 + r_1r_3 + r_2r_3)z - (r_1r_2r_3).$$

(b) $(z - r_1)(z - r_2)(z - r_3)(z - r_4) =$

$$z^4 - (r_1+r_2+r_3+r_4)z^3+(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)z^2$$
$$- (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)z + (r_1r_2r_3r_4).$$

(c) $(z - r_1)(z - r_2)\cdots(z - r_7) =$

$$z^7 - (r_1 + r_2 + \cdots + r_7)z^6 + (r_1r_2 + r_1r_3 + \cdots + r_6r_7)z^5$$
$$- (r_1r_2r_3 + r_1r_2r_4 + \cdots + r_5r_6r_7)z^4 + \cdots +$$
$$+ (-1)^7(r_1r_2 \cdots r_7).$$

5-10. Answers to Miscellaneous Exercises.

1. $-(2 - 3i) = -2 + 3i$

$\overline{(2 - 3i)} = 2 + 3i$

$|2 - 3i| = \sqrt{4 + 9} = \sqrt{13}$

$|\overline{2 - 3i}| = |2 - 3i| = \sqrt{13}$

$\frac{1}{2 - 3i} = \frac{\overline{2 - 3i}}{|2 - 3i|^2} = \frac{2 + 3i}{13} = \frac{2}{13} + \frac{3}{13} i$

$|2 - 3i|^2 = (\sqrt{13})^2 = 13$

$|(2 - 3i)^2| = |2 - 3i|^2 = 13$

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$$\frac{4 + 5i}{2 - 3i} = (4 + 5i)\left(\frac{1}{2 - 3i}\right) = (4 + 5i)\left(\frac{2}{13} + \frac{3}{13} i\right)$$
$$= -\frac{7}{13} + \frac{22}{13} i.$$

2. $[z - (c + di)][z - (c - di)] = 0$

$z^2 - [(c + di) + (c - di)]z + (c + di)(c - di) = 0$

$z^2 - 2cz + (c^2 + d^2) = 0.$

3. It is closed with respect to multiplication, but not with respect to addition since [illegible] 1 is not in the set.

4.
$$y^2 \geq 0$$
$$x^2 + y^2 \geq x^2$$
$$\sqrt{x^2 + y^2} \geq \sqrt{x^2}$$
$$|x + iy| \geq |x| \geq x$$
$$|z| \geq x$$

5. (a) circle of radius 3 with center at $(2,0)$.
(b) set of points exterior to circle of radius 3 with center at $(-2,0)$.
(c) set of points interior to circle of radius 4 with center at $(0,2)$.
(d) set of points interior to, or on, circle of radius 5 with center at z_0.

6.
$$|x + yi - (2 + 3i)| = 5$$
$$|(x - 2) + (y - 3)i| = 5$$
$$\sqrt{(x - 2)^2 + (y - 3)^2} = 5$$
$$(x - 2)^2 + (y - 3)^2 = 25$$
$$x^2 + y^2 - 4x - 6y - 12 = 0.$$

The set of points satisfying the given equation is the circle of radius 5 with center at $(2,3)$.

7. (a) the distance from the origin of z_1 is less than that of z_2.
(b) z is on the circle of radius 5 with center at the origin.

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7. (c) z_1 and z_2 are symmetric with respect to the origin.

(d) z_1 and z_2 are symmetric with respect to the y-axis.

(e) z_1 and z_2 are symmetric with respect to the x-axis.

8. If $z = x + yi$ the stated conditions become

$$x = y , \quad \sqrt{x^2 + y^2} = 1$$

The solutions of this pair of equations are $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ and $x = -\frac{1}{\sqrt{2}}$, $y = -\frac{1}{\sqrt{2}}$. The solutions of the problem are therefore $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i, -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$.

9. If the coefficients are real and $3 + 2i$ is a solution then $3 - 2i$ must also be a solution. If the equation is quadratic it can have no more than these two solutions. Thus the equation must be

$$a[z - (3 + 2i)][z - (3 - 2i)] = 0$$

or

$$az^2 - 6az + 13a = 0$$

where $a \neq 0$ is any real number.

10. We show generally that if $z = x + yi$ is any complex number (not zero) the quadrilateral with vertices z, iz, i^2z, i^3z is a square. The midpoints of the diagonals of this quadrilateral are

$$\frac{z + i^2z}{2} = \frac{z - z}{2} = 0$$

$$\frac{iz + i^3z}{2} = \frac{iz - iz}{2} = 0$$

so that the diagonals bisect each other at the origin. Thus the quadrilateral is a parallelogram. The slope of the segment joining the origin to $z = x + yi$ is $\frac{y}{x}$; the slope of the segment joining the origin to $iz = i(x + yi) = -y + xi$ is $-\frac{x}{y}$. Since these slopes are negative reciprocals the diagonals are perpendicular. Thus , the parallelogram is a rhombus. Finally, each diagonal is equal to $2|z|$ and hence the rhombus, having equal diagonals, is a square.

11. If z_0 is a solution then $\overline{z}_0$ is also a solution, since the coefficients are real. By the Fundamental Theorem

$$az^2 + bz + c = a(z - z_0)(z - \overline{z}_0)$$
$$= a[z^2 - (z_0 + \overline{z}_0)z + z_0\overline{z}_0]$$
$$= az^2 - a(z_0 + \overline{z}_0)z + az_0\overline{z}_0 .$$

Equating coefficients we obtain

$$b = -a(z_0 + \overline{z}_0), \quad c = az_0\overline{z}_0$$

or

$$z_0 + \overline{z}_0 = -\frac{b}{a}, \quad z_0\overline{z}_0 = \frac{c}{a} .$$

The curve $z + \overline{z} = -\frac{b}{a}$ is the straight line $x = -\frac{b}{2a}$. The curve $z\overline{z} = \frac{c}{a}$ is the circle $x^2 + y^2 = \frac{c}{a}$. Since z lies on both curves it is one of the points of intersection of these two curves (the other is $\overline{z}_0$). Thus to construct the roots of the quadratic equation $az^2 + bz + c = 0$ $(b^2-4ac < 0)$ draw a circle of radius $\sqrt{\frac{c}{a}}$ about the origin and draw the straight line parallel to the y-axis through $(-\frac{b}{2a}, 0)$. The solutions of the equation are the points of intersection of these curves.

12. $a(z^2 - z + 4) = 0$ a real $a \neq 0$

13. If $z = x + yi$ then $z^2 = x^2 - y^2 + 2xyi$ so that the real part of z^2 is 0 if and only if $x^2 - y^2 = 0$. Since $x^2 - y^2 = (x + y)(x - y)$, $x^2 - y^2 = 0$ if and only if $x + y = 0$ or $x - y = 0$. Thus the set of points satisfying the given condition is the set of points on the lines of slope 1 and -1 through the origin.

We have $(\frac{1}{z})^2 = (\frac{\overline{z}}{|z|^2})^2 = \frac{\overline{z}^2}{|z|^4}$. If the real part of z^2 is zero then the real part of $\overline{z}^2$ is zero, since z^2 and $\overline{z}^2 = \overline{z^2}$ are conjugates. Since $(\frac{1}{z})^2 = \frac{1}{|z|^4}\overline{z}^2$, and $\frac{1}{|z|^4}$ is real, it follows that the real part of $(\frac{1}{z})^2$ is zero.

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14. The discriminant of the equation is

$$(1 + r)^2 - 4\cdot 2\cdot r = r^2 - 6r + 1.$$

The equation has only one real root when the discriminant is 0, that is, when r is one of the zeros -- $3 - 2\sqrt{2}$, $3 + 2\sqrt{2}$ -- of the discriminant. The equation has complex roots when the discriminant is negative. For very large values of r the discriminant is positive, so that it will be negative if and only if r is between its zeros -- $3 - 2\sqrt{2} < r < 3 + 2\sqrt{2}$.

15. If $a = 1$, $b = i$, then $a + bi = 1 + i\cdot i = 0$, $\overline{a + bi} = \overline{0} = 0$; $a - bi = 1 - i\cdot i = 2$. Thus $\overline{a + bi} \neq a - bi$ in this case.

16. The set of points equidistant from z_1 and z_2 is the set of points z which satisfy the equation

$$|z - z_1| = |z - z_2|.$$

Squaring this equation we have

$$|z - z_1|^2 = |z - z_2|^2$$

from which we get

$$(z - z_1)\overline{(z - z_1)} = (z - z_2)\overline{(z - z_2)}$$

$$(z - z_1)(\overline{z} - \overline{z}_1) = (z - z_2)(\overline{z} - \overline{z}_2)$$

$$z\overline{z} - \overline{z}_1 z - z_1\overline{z} + z_1\overline{z}_1 = z\overline{z} - \overline{z}_2 z - z_2\overline{z} + z_2\overline{z}_2$$

$$(\overline{z}_1 - \overline{z}_2)z + (z_1 - z_2)\overline{z} = z_1\overline{z}_1 - z_2\overline{z}_2 .$$

The last equation is the equation of the perpendicular bisector of the segment.

17. The point z belongs to the set if and only if $|z - \overline{z}_0| < |z - z_0|$, that is, if and only if the distance from z to $\overline{z}_0$ is less than the distance from z to z_0. This will be true if and only if the point z lies on the same side as $\overline{z}_0$ of the perpendicular bisector of the segment joining z_0 and $\overline{z}_0$. This perpendicular bisector is the x-axis. Thus the set is the set of all points z which lie on the same side of the x-axis as $\overline{z}_0$. This can

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also be established by calculation.

18. Let $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c, and d are real.

(1) $\frac{z_1}{z_2} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} \cdot i.$

Hence $\frac{z_1}{z_2}$ is real <u>if and only if</u> (2) $bc - ad = 0$.

It can be shown that $bc - ad = 0$ <u>if and only if</u> z_1 and z_2 are on a straight line through the origin. To establish this we must prove two if-then statements.

(a) If z_1 and z_2 are on a straight line through the origin then $bc - ad = 0$, <u>and</u>

(b) If $bc - ad = 0$ then z_1 and z_2 are on a straight line through the origin.

<u>Proof</u> of (a): If the line is the y-axis then $a = 0$ and $c = 0$ and we have at once $b \cdot 0 - 0 \cdot d = 0$. If the line is not the y-axis then the slope of this line joining the origin to z_1 is equal to the slope of the line joining the origin to z_2, i.e. $\frac{b}{a} = \frac{d}{c}$.

Hence $bc = ad$ or $bc - ad = 0$.

<u>Proof</u> of (b): We have (2) $bc = ad$. If z_1 is on the y-axis then $a = 0$ and by (2) $bc = 0$. But $b \neq 0$ because $z = a + bi \neq 0$ by hypothesis. Hence $c = 0$ and z_2 is also on the y-axis. This proves that z_2 is on the y-axis if z_1 is and the two points are on a straight line through the origin.

If z_1 is not on the y-axis then $a \neq 0$. From this we see that $c \neq 0$ because if $c = 0$ and $a \neq 0$ we must conclude from (2) that $d = 0$ and this would mean that $z_2 = c + di = 0$ in violation of our hypothesis that z_2 is a non-zero complex number. Hence $ac \neq 0$ and we may divide both members of (2) by ac to obtain

$\frac{b}{a} = \frac{d}{c}$ which is precisely the condition that z_1 and z_2 lie on a straight line through the origin.

We summarize our argument:

$\frac{z_1}{z_2}$ is real if and only if $bc - ad = 0$ and $bc - ad = 0$ if and only if z_1 and z_2 lie on a straight line through the origin. Therefore $\frac{z_1}{z_2}$ is real if and only if z_1 and z_2 lie on a straight line through the origin.

19. $z^4 = -1$ or $z^4 + 1 = 0$

$(z^2 + i)(z^2 - i) = 0$

Hence $z^2 = -i$ or $z^2 = i$. The solution set is evident by the union of the solution sets of the equations solved in problems 22 and 23 of Exercise 5-6,

namely $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$.

20. It will be sufficient to show that the law of trichotomy is inconsistent with O_4 for the element i. Certainly $i \neq 0$. Then either $i > 0$ or $i < 0$. In either case by O_4 we have $i^2 > 0$ and we are confronted by the contradiction $-1 > 0$.

21. If x and y are real it is evident that the conjugate of $x + yi$ is $x - yi$. Moreover, it can be shown that if $\overline{x + yi} = x - yi$ then x and y are real.

Let $x = a + bi$ and $y = c + di$ where a, b, c and d are real.

$$x + yi = (a - d) + (b + c)i$$
$$\overline{x + yi} = (a - d) - (b + c)i$$
$$x - yi = (a + d) + (b - c)i$$

Since $\overline{x + yi} = x - yi$ we have

$$(a - d) - (b + c)i = (a + d) + (b - c)i$$

According to Theorem 5-4

$$a - d = a + d \quad \text{and}$$
$$-(b + c) = b - c$$

From these equations we conclude that $d = 0$ and $b = 0$.

$\therefore x = a$ and $y = c$ where a and c are real.

Hence, $\overline{x + yi} = x - yi$ if and only if x and y are real.

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22. The proposition stated is true provided x and y are real. In this event we have

$$|x| + |y| \leq \sqrt{2}|z| \text{ if and only if}$$

$$(|x| + |y|)^2 \leq 2|z|^2.$$

Now $|z|^2 = x^2 + y^2$ and we have

$$|x|^2 + 2|x||y| + |y|^2 \leq 2|x|^2 + 2|y|^2.$$

This reduces to $0 \leq |x|^2 - 2|x||y| + |y|^2$ or $0 \leq (|x| - |y|)^2$ which is true because the square of any real number is non-negative. Q.E.D.

The proposition is not true for all complex values of x and y as the following counter example will show.

Let $x = 8 + 2i$ and $y = -1 + 4i$

then $|x| = \sqrt{68} = 2\sqrt{17}$ and $|y| = \sqrt{17}$.

$$|x| + |y| = 3\sqrt{17}$$

$z = x + iy = (8 + 2i) + i(-1 + 4i) = 4 + i.$

$|z| = \sqrt{17}$. It is false that $\sqrt{2}\sqrt{17} \geq 3\sqrt{17}$, hence in this case $|x| + |y|$ is not equal to or less than $\sqrt{2}\ |z|$.

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*5-11. Construction of the Complex Number System.

Section *5-11 outlines Gauss's construction of the complex number system. As a source of historical information we suggest The Development of Mathematics, by E.T. Bell (McGraw-Hill, 1945, Second Edition): Wessell and Argand, p.177; Gauss, p.179; Cauchy, p.194.

5-12. Sample Test Questions for Chapter 5.

Note: In the questions included in this section a,b,c,d, x,y are real numbers and z is a complex number.

Part I: True-False.

Directions: If a statement is true, mark it T ; if the statement is false, mark it F.

1. The imaginary part of a + bi is bi.
2. The discriminant of the equation $x^2 + 2 = 0$ is -8.
3. Every complex number has an additive inverse.
4. A one-to-one correspondence can be established between points of the xy-plane and the elements of C.
5. The product of a complex number and its conjugate is a complex number.
6. The sum of a complex number and its conjugate is a pure imaginary number.
7. If the coefficients of a quadratic equation are real numbers, then the roots of the equation are real numbers.
8. $|z|$ is a non-negative real number.
9. The sum of z and $-\overline{z}$ is a real number.
10. If z is a complex number, z and $\overline{z}$ correspond to points in the xy-plane which are symmetric with respect to the y-axis.
11. The multiplicative inverse of (x - yi) is $\frac{x - yi}{x^2 + y^2}$.
12. If (a + bi)(x + yi) = 1, then ax - by = 1.
13. $\overline{z_1 + z_2} = \overline{z}_1 - \overline{z}_2$.

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14. $(a + bi)\overline{(a + bi)} = a^2 + b^2$.
15. If $|z| = 1$, then z is its own multiplicative inverse.
16. The set of numbers {1, -1, i, -i} is closed under multiplication.
17. $|z_1| + |z_2| \leq |z_1 + z_2|$.
18. The reflection of $\overline{z}$ in the y-axis is $-z$.

Part II: Multiple Choice.

Directions: Select the response which best completes the statement or answers the question.

19. Which one of the following equations does not have a solution in the real number system?
    A. $x + 5 = 5$
    B. $(x + 5)^2 = 9$
    C. $x^2 + 5 = 0$
    D. $x^2 - 5 = 0$
    E. $\sqrt{x + 5} = 0$
20. What ordered pair of real numbers (x,y) satisfies the equation $x - 4yi = 20i$?
    A. (20,0)
    B. (0,-5)
    C. (0,20)
    D. (0,5)
    E. (0,0)
21. If $z = (5 - 6i) - (3 - 4i)$, then the standard form of z is
    A. $2 - (10)i$
    B. $2 + (2)i$
    C. $2 + (-10)i$
    D. $2 + (-2)i$
    E. $2 + (10)i$
22. The additive inverse of $c - di$ is
    A. $-c + di$
    B. $\frac{1}{c - di}$
    C. $c + di$
    D. 1
    E. 0
23. If the complex number $5 + 5i$ is represented by the point P in an Argand diagram, then the slope of the line segment joining P and the origin is
    A. $\frac{2}{3}$ B. $\frac{3}{2}$ C. 5 D. 1 E. 0

24. Which one of the following expressions does not represent a real number?

A. $i^2 + \sqrt{2}$
B. $3i^4$
C. $\sqrt{(-3)^2}$
D. $6 \div 2i$
E. $(2i)^6 - \sqrt{3}$.

25. The multiplicative inverse of i is

A. i B. $-i$ C. 1 D. -1 E. $-\frac{1}{i}$.

26. Which one of the following equations has non-real solutions?

A. $x - 4 = 6$
B. $4x^2 - 3x + 6 = 0$
C. $6x^2 + 5x - 8 = 0$
D. $2x^2 - 14x + 3 = 0$
E. $x^2 = \sqrt{14}$.

27. The conjugate of -4 written in standard form is

A. $4 + 0i$
B. $-\frac{1}{4} - 0i$
C. $\frac{-4}{16} - \frac{i}{16}$
D. $-4 + 0i$
E. None of these.

28. Which one of the following is not equivalent to each of the other four?

A. $\sqrt{(2)^2}$
B. $\sqrt{(-2)^2}$
C. $\sqrt{-(2)^2}$
D. $\sqrt{-(2i)^2}$
E. $\sqrt{4}$.

29. The product of $(2 + 3i)$ and $(5 - 3i)$ is

A. $19 + 9i$
B. $19 + 21i$
C. $1 + 9i$
D. $1 - 21i$
E. $10 - 9i$.

30. When written in standard form the real part of $(2 - i)^2$ is

A. 1 B. -1 C. 5 D. -3 E. 3.

31. Given $z = -3i$, then $\overline{z}$ in standard form is

A. $3i$ B. $0 + 3i$ C. $|3|i$ D. $0 + (-3)i$ E. $-3i$.

32. The smallest set which contains the absolute value of every complex number is the set of

   A. Natural numbers
   B. Integers
   C. Real numbers
   D. Rational numbers
   E. Complex numbers.

33. The additive inverse of i is

   A. 1 B. -1 C. i D. $-i$ E. 0

34. Which one of the following pairs of complex numbers can be represented by points which are symmetric with respect to the origin in an Argand diagram?

   A. $3 + 2i,\ 3 - 2i$
   B. $3 + 2i,\ 2 + 3i$
   C. $3 + 2i,\ -3 - 2i$
   D. $3 + 2i,\ -3 + 2i$
   E. $3 + 2i,\ -2 - 3i$.

35. In an Argand diagram the set of points defined by the equation $|z|^2 = 5$ is

   A. A point
   B. A straight line
   C. Two perpendicular lines
   D. A circle
   E. Two parallel lines.

36. If z is a complex number such that $\frac{z}{\bar{z}} = -1$ and $z\bar{z} = 1$, then z is

   A. i
   B. $-i$
   C. i or $-i$
   D. 1 or -1
   E. i or $-i$ or 1 or -1.

37. Which of the following ordered pairs of real numbers (x,y) satisfies the equation $3x + 5yi - 8 = 5x - yi + 6i^2$?

   A. $(-4,1)$ B. $(-1,0)$ C. $(0,-1)$ D. $(4,1)$ E. $(-4,-4)$.

38. Which of the following equations has the solutions $2 - i$ and $3i$?

   A. $z^2 - 4z + 5 = 0$
   B. $z^2 - (2+4i)z + (3+6i) = 0$
   C. $z^2 - (2 + 2i)z + (3 + 6i) = 0$
   D. $z^2 - (2 + 2i)z + (-3 + 6i) = 0$
   E. $z^2 - (2 - 2i)z + (6 - 3i) = 0$.

39. The equation $z^3 - 2z^2 + z - 2 = 0$ has i as one of its solutions. The other solutions of the equation are
A. $-i, 2$ B. $-i, -2$ C. $-i, 1$ D. $-i, 2$ E. $0, -i$.

40. Which one of the following complex numbers is the reflection of $2 - 3i$ in the y-axis?

A. $-2 - 3i$
B. $-2 + 3i$
C. $2 + 3i$
D. $-3 - 2i$
E. $3 - 2i$.

41. The solution set of the equation $z^2 + a^4 = 0$, where a is a real number, is

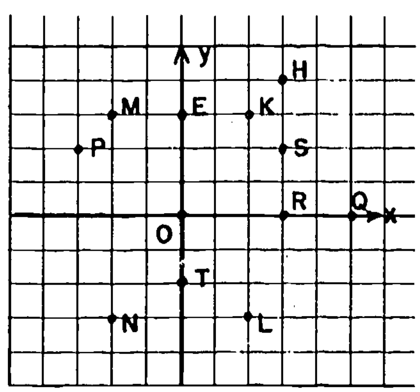
A. $\{a^2, -a^2\}$
B. $\{a, -a, ai, -ai\}$
C. $\{a, -a, i, -i\}$
D. $\{a^2 i, -a^2 i\}$
E. $\{-a^2, a^2 i, -a^2 i\}$.

42. The length of the line segment which joins the points representing $3 + 4i$ and $-4 + 5i$ is
A. $\sqrt{2}$ B. $2\sqrt{2}$ C. $5\sqrt{2}$ D. 8 E. 50.

Part III: Matching.

Directions: In questions 43 - 49 choose the point on the Argand diagram which represents the given number. Write the letter which identifies the point of your choice on an answer sheet. Any choice may be used once, several times or not at all.

43. $2 - 3i$
44. $3 - 0i$
45. $\overline{-2 + 3i}$
46. $|3 + 4i|$
47. $(2 + 3i) + (1 - i)$
48. $(3 + 2i) - (5 + 5i)$
49. z_1 such that $|z_1| = 2$



Part IV: Problems.

50. Express the quotient $\frac{1 - i^3}{3 + i}$ in standard form.

51. If $z = 4 + 2i - i^6$, find the standard form of z.

52. Find the ordered pair of real numbers (x,y) that satisfies the equation $x - 15i = 5yi$.

53. Find the real values of x and y which satisfy the equation $x - y + (x + y)i = 2 + 6i$.

54. Solve the equation $(x + yi)(2 + i) + 3x - 11 = 0$ for real values of x and y.

55. For what real values of k does the equation $z^2 + kz + 1 = 0$ have solutions that are not real?

56. Write a quadratic equation with real coefficients which has $5 + i$ as one of its roots.

57. If $z_1 = -2 + i$ and $z_2 = 1 + 4i$, find $z_1 + z_2$ in standard form and exhibit the sum graphically.

58. Describe the set of points in the plane which satisfy the condition $|z|$ = the real part of z.

59. Solve each of the following equations and express the solutions in standard form:

(a) $3z^2 + z + 1 = 0$

(b) $z^2 + z + c = 0$, c is a positive integer

(c) $pz^2 + q = 0$, $p < 0$, $q > 0$, and p and q are real.

60. Given the following numbers: 2, -12, $4i$, $\frac{2}{3}$, $-\sqrt{16}$, 0, π, $\sqrt{-9}$, $\sqrt[3]{-27}$, $\sqrt{50}$, $i\sqrt{3}$, $-\frac{3}{5}$, $\sqrt{5}$, $2\sqrt{16}$, $\sqrt{23}$, $\sqrt[3]{4}$, 1.74, $\sqrt{-3}$, $3.\overline{37}$, $-i\sqrt{7}$, i^2, $2 + \sqrt{3}$, $2 - \sqrt{-5}$.

(a) Classify the given numbers into two lists, real numbers and imaginary numbers.

(b) Reclassify the real numbers into rational and irrational numbers.

Answers to Sample Test Questions.

Part I: True-False.

1.	F	10.	F
2.	T	11.	F
3.	T	12.	T
4.	T	13.	F
5.	T	14.	T
6.	F	15.	F
7.	F	16.	T
8.	T	17.	F
9.	F	18.	T

Part II: Multiple Choice.

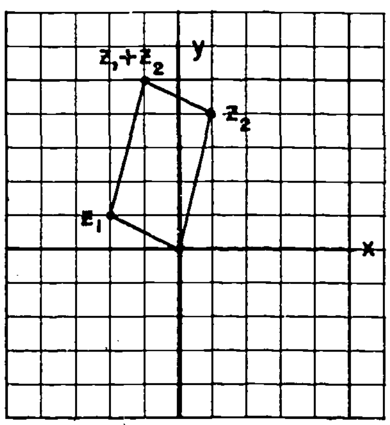
19. C	37. B
20. B	38. C
21. D	39. D
22. A	40. A
23. D	41. D
24. D	42. C
25. B	
26. B	
27. D	
28. C	
29. A	
30. E	
31. B	
32. C	
33. D	
34. C	
35. D	
36. C	

Part III: Matching

43. L
44. R
45. N
46. Q
47. S
48. N
49. T

Part IV: Problems

50. $\frac{2}{5} + \frac{1}{5} i$
51. $5 + 2i$
52. $(0,-3)$
53. $x = 4, y = 2$
54. $x = 2, y = -1$
55. $|k| < 2$
56. $z^2 - 10z + 26 = 0$
57. $z_1 + z_2 = -1 + 5i$



58. Non-negative part of x-axis
59. (a) $z = -\frac{1}{6} + \frac{\sqrt{11}}{6} i, \quad -\frac{1}{6} + (-\frac{\sqrt{11}}{6})i$

(b) $z = -\frac{1}{2} + \frac{\sqrt{4c - 1}}{2} i, \quad -\frac{1}{2} + (-\frac{\sqrt{4c - 1}}{2})i$

(c) $z = \sqrt{-\frac{q}{p}} + 0i, \quad -\sqrt{-\frac{q}{p}} + 0i$

60. The table shows the answers to both parts (a) and (b).

REAL		IMAGINARY
RATIONAL	IRRATIONAL	
2		
-12		
		$4i$
$\frac{2}{3}$		
$-\sqrt{16}$		
0		
	π	
		$\sqrt{-9}$
$\sqrt{-27}$		
	$\sqrt{50}$	
		$i\sqrt{3}$
$-\frac{3}{5}$		
	$\sqrt{5}$	
$2\sqrt{16}$		
	$\sqrt{23}$	
	$\sqrt[3]{4}$	
1.7^4		
		$\sqrt{-3}$
$3.\overline{37}$		
		$-i\sqrt{7}$
i^2		
	$2 + \sqrt{3}$	
		$2 - \sqrt{-5}$

Commentary for Teachers

Chapter 6

EQUATIONS OF THE FIRST AND SECOND DEGREE IN TWO VARIABLES

6-0. Introduction

In Chapter 6 we again take up the study of analytic geometry. Chapter 2 was designed as a very general introduction to the subject. In this chapter we systematically study equations of the first and second degree and their graphs.

Sections 6-1 and 6-2 are devoted to linear equations and the straight lines which are their graphs. While the last four sections are concerned with the conic sections. A few words need to be said about the approach we have adopted toward the conics. The parabola is discussed in rather great detail. While much work has been done in Chapter 4 with parabolas, the emphasis was on the numerical properties of the quadratic function and the graph was used informally as a visual aid. In this section we are not interested in the quadratic function, but the parabola -- that is, the set of points, P, which are equidistant from a fixed point and a fixed line. This definition naturally follows the sections in which we have been considering the straight line -- the set of points P equidistant from two fixed points.

In the following Section 6-3 we generalize the definition of the parabola to ask for the equation of the set of points P whose distance from a fixed point is a constant times its distance from a fixed line. This, the general definition of a conic, has the virtue of unifying the study of second degree equations in a way which the piecemeal definitions which are sometimes used can not. Introducing the definition after, and not before, the parabola, was by design. Generalizing is one of the most characteristic and most powerful devices of the mathematician. It can't be pointed out too often.

The general definition is used to derive the standard forms for the equation of the ellipse and the hyperbola, the special case of the parabola having been discussed in the preceding section. The detailed study of these curves is reserved for the next several sections.

The fact that the general equation of the second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

always has for its graph either a conic, or a so-called degenerate form of one of these curves, is only mentioned. While the exercises indicate how this can be seen, the full story can only be told after the student knows enough trigonometry to be able to use the formulas for rotating the coordinate axes. The fact is that by rotating the axes through an acute angle such that $\cot 2\theta = \frac{A - C}{B}$, the xy term can always be eliminated. Then a translation of the axes which moves the origin to the vertex of a parabola or to the center of an ellipse or hyperbola, will put the equation in one of the standard forms. So, while we have not been able to carry out the program of showing that every second degree equation has for its graph a conic, the main ideas are indicated. (For reference, the translation formulas for moving the origin to the point $P_0(x_0,y_0)$ are

$$\begin{aligned} x' &= x - x_0 \\ y' &= y - y_0 \end{aligned} \quad \text{or} \quad \begin{aligned} x &= x' + x_0 \\ y &= y' + y_0 \end{aligned}$$

The formulas for rotating the axes through the angle θ are

$$\begin{aligned} x' &= x \cos \theta + y \sin \theta \\ y' &= -x \sin \theta + y \cos \theta \end{aligned} \quad \text{or} \quad \begin{aligned} x &= x' \cos \theta - y' \sin \theta \\ y &= x' \sin \theta + y' \cos \theta. \end{aligned}$$

The problem of finding the equation of a conic with the focus and directrix in general positions was not undertaken either. We began by taking the axes of the conics to be one of the coordinate axes and the directrix perpendicular to one of the coordinate axes. We moved by easy stages to the case in which the axes were parallel to the coordinate axes, but the directrix still perpendicular to one of the coordinate axes. For the case in which the directrix is a line in general position, we would have needed the formula for the distance from a point to a line. This formula really follows from the normal form for the equation of a straight line-- $x \cos \theta + y \sin \theta - p = 0$, where p is the distance of the line from the origin and θ is the angle the perpendicular from the origin to the line makes with the positive x-axis. To derive this equation we need trigonometry again. For reference the formula for the distance from the point $P_0(x_0,y_0)$ to the line L:

$$Ax + By + C = 0 \quad \text{is} \quad d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

For the few difficult problems in the text where the directrix is the line $y = x$, we simply expect the student to use an argument by similar triangles to determine the distance from the point to the line.

The last two sections develop more fully the specific details about the equations and graphs of the ellipse and the hyperbola. The standard forms are extended to include cases in which the center is not the origin in the problem sets.

The idea of inverse variation is introduced in connection with the hyperbola. This union is not the happiest one. The tie is simply that if one quantity varies inversely as a second, the graph of this relation is an equilateral hyperbola. What needs to be done with inverse variation is to stress the definition which allows us to translate the words into mathematical equations. One quantity may vary inversely as the square of another, the product of two others, etc. In all of these, the graph is really irrelevant and certainly is not a hyperbola. However, this seemed as good a place to mention inverse variation as any other.

The real purpose of this chapter, then, is to develop the details of the relation between linear and quadratic equations in two variables and their graphs, the straight line and the conic sections. These results are not ends in themselves, but a thorough grounding in these details and in the methods used to derive them, will be helpful for most further work in mathematics. These results do have applications, but their importance derives not from the physical applications, but from their fundamental role in analytic geometry -- one of the most useful tools in all mathematics.

COMMENTS ON SECTIONS

6-1. The Straight Line

The proof given in this section that the equation of the line through $P_1(x_1,y_1)$ with slope m is $y - y_1 = m(x - x_1)$, is deceptively simple. While line is a basic undefined term of geometry, we assume certain properties of this undefined term in order to derive its equation. We assume in this proof that any two distinct points do determine a line and further that the slope of a line is constant. (Another property of a line which we do not use in this proof is the property that for three distinct points P_1,P_2, and P_3 on a line $d(P_1,P_2) + d(P_2,P_3) = d(P_1,P_3)$.) This assumption about the slope really means that we are assuming what we pretend to prove in Chapter 2 -- namely, that the slope is the same regardless of the pair of points picked to determine the line. The proof we gave in that chapter depends on the geometric picture that the line is really "straight"; that is, the slope is constant. The remainder of the section is concerned with developing various useful forms for the equations of non-vertical lines. These forms should be looked on as useful devices for determining the equation of straight lines. We derive more than one so that we can easily write down the equation no matter what information is given to determine the line. For instance, we could always find the equation of a non-vertical line from the slope-intercept form $y = mx + b$. Whether we are given two points on the line, a point and the slope, or the two intercepts, we could use any of these to determine m and b and thus determine the equation of the line. However, in each case a special form makes it easier to write the equation directly. Probably the most useful form is the slope-intercept form $y = mx + b$. It would be more profitable for the student to be able when asked, to derive all the other forms rather than for him simply to memorize them.

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Exercises 6-1 - Answers.

1. (a) $m = \frac{5 - 4}{4 - 2} = \frac{1}{2}$, slope formula.

$y - 4 = \frac{1}{2}(x - 2)$, substituting for m, x_1 and y_1 in $y - y_1 = m(x - x_1)$.

$y = \frac{1}{2}x + 3$

(b) $m = -1$

$y - 4 = -1(x - 2)$

$y = -x + 6$

(c) $m = 5$

$y - 0 = 5(x - 0)$

$y = 5x$

(d) $m = \frac{1}{5}$

$y - 0 = \frac{1}{5}(x - 0)$

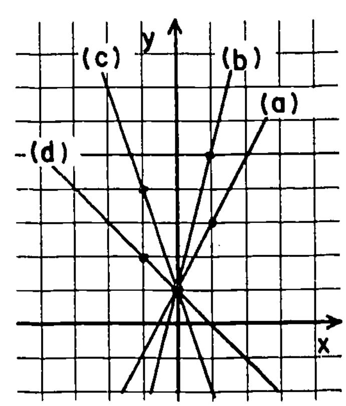
$y = \frac{1}{5}x$

(e) $m = \frac{1}{5}$

$y - 5 = \frac{1}{5}(x + 8)$

$y = \frac{1}{5}x + \frac{33}{5}$

2. (a) Slope is 2 and the coordinates of the y-intercept are (0,1). Plot (0,1), then locate a second point (1,3) by going to the right 1 and up 2. Or, plot (0,1), then plot the point $(-\frac{1}{2},0)$, which is the x-intercept.

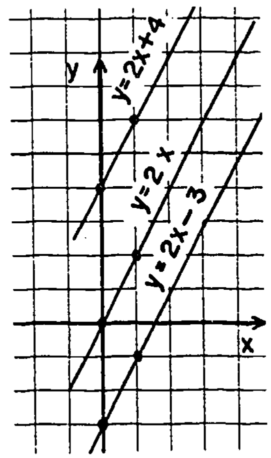


(b) Slope is 4 and the coordinates of the y-intercept are (0,1).

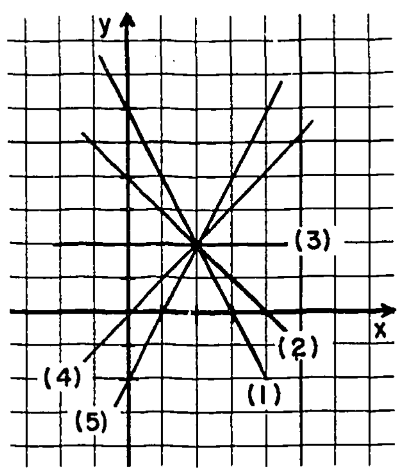
(c) Slope is -3 and the same y-intercept as the preceding graphs. Plot (0,1), then locate a second point (-1,4) or (1,-2) by going to the left 1 then up 3 or by going to the right 1 and down 3.

(d) Slope is -1 and the same y-intercept.

3. Use the slope-intercept concept to draw these graphs. However, the x and y intercepts may be found and plotted.



4. (a)



(b)

	Slope	Equation
(1)	-2	$y = -2x + 6$
(2)	-1	$y = -x + 4$
(3)	0	$y = 2$
(4)	1	$y = x$
(5)	2	$y = 2x - 2$

5. The equation of this vertical line is $x = 3$.

6. The equation $y = mx + 3$ is satisfied by the coordinates $(-1,-2)$, hence,

$$-2 = m(-1) + 3$$
$$m = 5$$

7. In $y = mx + b$, the x-intercept is,

$$0 = mx + b$$

$$-\frac{b}{m} = x, \quad \text{hence,}$$

$$\frac{-b}{-\frac{1}{2}} = 2; \quad b = 1$$

Alternate Solution

The line with slope $-\frac{1}{2}$ passes through $(2,0)$. Hence,

$$y - 0 = -\frac{1}{2}(x - 2)$$
$$y = -\frac{1}{2}x + 1$$

The equation is $y = -\frac{1}{2}x + 1$ or $2y + x = 2$

8. $m = \frac{3 - 0}{-1 - 0} = -3$, Substituting in $y - y_1 = m(x - x_1)$ we have,

$$y - 3 = -3(x + 1)$$
$$y = -3x$$

Alternate Solution

$$y = mx + b$$
$$0 = m(0) + b$$
$$0 = b$$
$$y = mx$$
$$3 = m(-1)$$
$$-3 = m$$
$$y = -3x$$

9. $m = \frac{y_1}{x_1}$, Substituting for m in $y - y_1 = m(x - x_1)$ we have,

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$
$$y = \frac{y_1}{x_1}x$$

10. $y - 0 = m(x - 0)$

$y = mx$

When written in the above form, we may say that m is the constant of proportionality.

11. (a) $P = ks$, $k = 3$, $\therefore P = 3s$

(b) $S = kt^2$

(c) $I = kE$

12. $x = ky$ $\quad x = \frac{8}{15} \cdot 10$

$8 = k \cdot 15$ $\quad x = \frac{16}{3}$

$k = \frac{8}{15}$

13. $V = kT$ $\quad 2500 = 5T$

$1500 = k \cdot 300$ $\quad T = 500^{\circ}$ (absolute)

$5 = k$

14. $y = k(x + 1)$, Since the point $(-\frac{1}{2}, -3)$ is on the line, we write $-3 = k(-\frac{1}{2} + 1)$

$k = -6$

15. Slope of line $y = 2x + 2$ is 2. Slope of a line parallel to this line is 2.

$y - 4 = 2(x - 3)$

$y = 2x - 2$

} the equation of a line $\parallel$ $y = 2x + 2$ and passing through point $(3,4)$

16. Slope of line $y = \frac{1}{4}x + \frac{1}{2}$ is $\frac{1}{4}$, Slope of a line perpendicular to this line is -4, since $m_1 m_2 = -1$

$y - 0 = -4(x - 0)$

$y = -4x$

17. Slope of $5x - 2y = 2$ is $\frac{5}{2}$; slope of a line perpendicular to this one is $-\frac{2}{5}$.

$y - 5 = -\frac{2}{5}(x + 2)$

$y = -\frac{2x}{5} + \frac{21}{5}$

} the required equation

18. The slope of the perpendicular line is $-\frac{4}{3}$.

$$y + 12 = -\frac{4}{3}(x - 8)$$

$$y = -\frac{4}{3}x - \frac{4}{3}$$

19. $5x + 3y - c = 0$

$$y = -\frac{5}{3}x + \frac{c}{3}$$

$$m = -\frac{5}{3}$$

(a) $\frac{c}{3} = -\frac{5}{2}$; $c = -\frac{15}{2}$

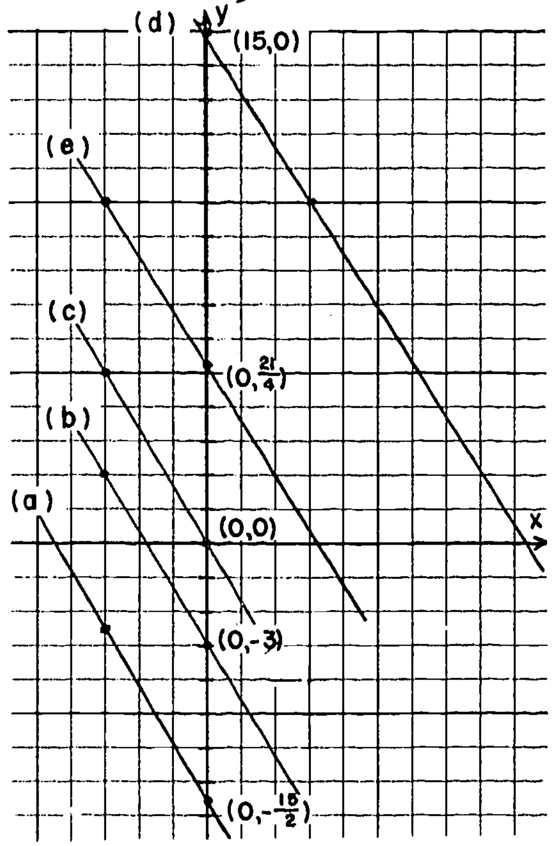
(b) $\frac{c}{3} = -1$; $c = -3$

(c) $\frac{c}{3} = 0$; $c = 0$

(d) $\frac{c}{3} = 5$; $c = 15$

(e) $\frac{c}{3} = \frac{7}{4}$; $c = \frac{21}{4}$

The slope, $-\frac{5}{3}$, may be used to plot the second point, or the x-intercept may be found and used instead.



20. (a) Since the line parallel to the y-axis and passing through the point $(-5,7)$ is a vertical line which contains points $(-5,y)$, its equation is $x = -5$.

(b) When the line through the point $(-5,7)$ is parallel to the y-axis, it contains points $(x,7)$ and has slope 0. its equation is, $y = 7$. This may be derived by using the point-slope, $y - 7 = 0(x + 5)$.

21. Slope of the line whose equation is $2y + x = 5$ is $-\frac{1}{2}$; its point of intersection with the y-axis is $(0, -\frac{5}{2})$; with the x-axis is $(5,0)$; the slope of a line perpendicular to this line is 2.

(a) $y - \frac{5}{2} = 2(x - 0)$
$y = 2x + \frac{5}{2}$
equation of line through $(0, -\frac{5}{2})$ with slope 2.

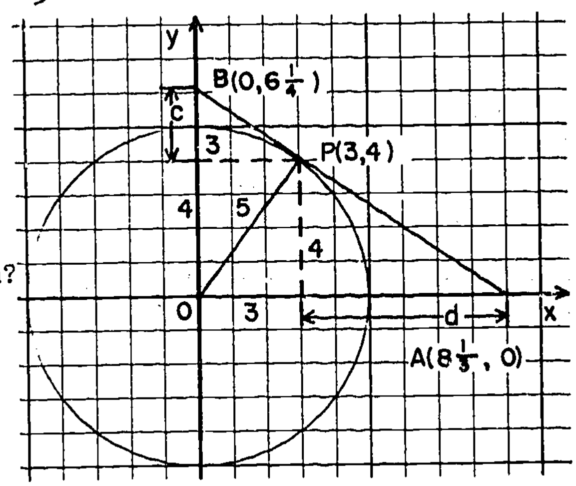
(b) $y - 0 = 2(x - 5)$
$y = 2x - 10$
equation of line through $(5, 0)$ with slope 2.

22. Equation of line $\overleftrightarrow{OP}$ is $y = \frac{4}{3}x$

Equation of tangent to OP is

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y = -\frac{3}{4}x + \frac{25}{4}$$



Alternate solution:

$\frac{3}{4} = \frac{4}{d}$ What geometry theorem?

$d = 5\frac{1}{3}$

$\frac{4}{3} = \frac{3}{c}$

$c = 2\frac{1}{4}$

Coordinates of A are $(8\frac{1}{3}, 0)$; of B are $(0, 6\frac{1}{4})$; slope is $-\frac{3}{4}$. Use the point-slope form to write equation

$$y - 0 = -\frac{3}{4}(x - 8\frac{1}{3})$$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

23. $3c = 16$

$c = 5\frac{1}{3}$

$d(A,B) = 3 + 5\frac{1}{3} = 8\frac{1}{3}$

$d(O,A) = 5$

$(AB)^2 = (OA)^2 + (OB)^2$ Pythagorean Theorem.

$(\frac{25}{3})^2 = 5^2 + (OB)^2$

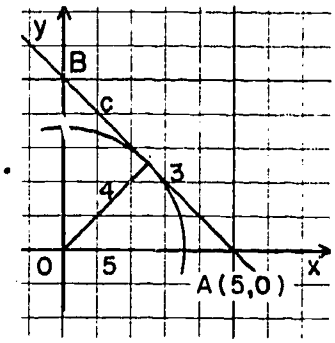
$(OB)^2 = \frac{400}{9}$

$d(O,B) = \frac{20}{3}$

Use slope-intercept, $B(0,\frac{20}{3})$, and slope

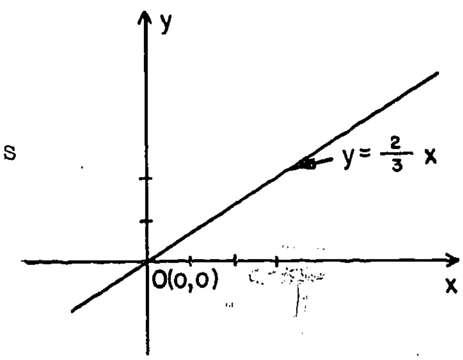
$-\frac{\frac{20}{3}}{5} = -\frac{4}{3}$ to write the equation,

$y = -\frac{4}{3}x + \frac{20}{3}$.



24. The line intersects the x-axis at the point $A(-\frac{c}{2},0)$ and the y-axis at the point $B(0, -\frac{c}{3})$. The mid-point of $\overline{AB}$ is $M(-\frac{c}{4}, -\frac{c}{6})$. Slope of line $\overleftrightarrow{OM}$ is $\frac{2}{3}$, hence, the equation of this line is,

$$y = \frac{2}{3}x.$$



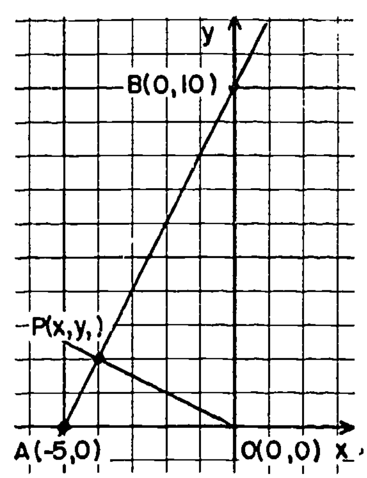
25. Intercepts are: $A(-5,0)$, $B(0,10)$

$m = 2$

$\overline{OP} \perp \overline{AB}$ given

Equation of line $\overleftrightarrow{OP}$ is,

$$y = -\frac{1}{2}x.$$



6-2. The General Linear Equation $Ax + By + C = 0$

The main idea in 6-2 is to show that any linear equation has for its graph a straight line and conversely. This is slightly more general than the preceding section, since vertical lines are included in the general form of the equation of the line as well as all non-vertical lines. Given the general equation, the student should be able to transform the equation into slope-intercept form and read off the slope and the y-intercept. Another good exercise in algebraic manipulation is to ask that the student put the equation in intercept form. Many students find this difficult, but, it is a healthy algebraic exercise.

Exercises 6-2. - Answers

1. Substitute for m and (x_1, y_1) in,

$$y - y_1 = m(x - x_1)$$
$$y + 2 = \frac{3}{5}(x + 1)$$
$$5y + 10 = 3x + 3$$
$$3x + (-5)y + (-7) = 0$$ which is in the form $Ax + By + C = 0$.

2. Substitute for a and b in,

$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\frac{x}{2} + \frac{y}{3} = 1$$

3. Transform each equation into the slope-intercept form,

$$y = mx + b.$$

	slope	y-intercept
(a)	$\frac{3}{2}$	-3
(b)	$\frac{1}{8}$	$\frac{1}{4}$
(c)	$\frac{9}{5}$	$\frac{1}{5}$
(d)	4	-7
(e)	4	$-\frac{7}{2}$
(f)	4	-28

4. (a) $3x + 2y - 6 = 0$

$3x + 2y = 6$

$\frac{3x}{6} + \frac{2y}{6} = 1$

$\frac{x}{2} + \frac{y}{3} = 1$

(b) $\frac{x}{3} + \frac{y}{-4} = 1$

(c) $\frac{x}{2} + \frac{y}{5} = 1$

(d) $\frac{x}{5} + \frac{y}{-\frac{20}{7}} = 1$

(e) $\frac{x}{-\frac{10}{3}} + \frac{y}{2} = 1$

(f) $\frac{x}{-\frac{5}{2}} + \frac{y}{\frac{5}{3}} = 1$

	x-intercept	y-intercept
(a)	2	3
(b)	3	-4
(c)	2	5
(d)	5	$\frac{-20}{7}$
(e)	$\frac{-10}{3}$	2
(f)	$\frac{-5}{2}$	$\frac{5}{3}$

[pages 313-314]

5. (a) Since $\frac{3}{6} = \frac{-2}{-4} = \frac{-2}{-4} = \frac{1}{2}$, the equations represent the same line.

(b) Since $\frac{2}{3} \neq \frac{-2}{-6} \neq \frac{7}{1}$, the equations do not represent the same line.
Since the slopes are different $(m_1 = 1$ and $m_2 = \frac{1}{2})$, the lines are not parallel.

(c) Since $\frac{1}{1} \neq \frac{-1}{1} \neq \frac{-6}{-6}$, the lines are not the same. Since the slopes are different $(m_1 = 1$ and $m_2 = -1)$, the lines are not parallel.

(d) Since $\frac{6}{1} \neq \frac{2}{3} \neq \frac{5}{5}$, the lines are not the same. Since the slopes are not the same, they are not parallel.

(e) Rewrite the equations in the form $Ax + By + C = 0$.
$- x + 6y + 3 = 0$
$- 3 \frac{1}{2}x + 21y + 2 = 0$
Since $\frac{-1}{-3\frac{1}{2}} \neq \frac{6}{12} \neq \frac{3}{2}$, the lines are not the same.
Since the slopes are the same, $\frac{1}{6}$, the lines are parallel.

(f) Rewrite the equations,
$3x + y - 1 = 0$
$6x + 2y - 2 = 0$
Since $\frac{3}{6} = \frac{1}{2} = \frac{-1}{-2} = \frac{1}{2}$, the lines are the same.

(g) Rewrite the equations,
$2x - y + 1 = 0$
$2x - y - 5 = 0$
Since $\frac{2}{2} = \frac{-1}{-1} \neq \frac{1}{-5}$, the lines are not the same.
Since the slopes are the same, 2, the lines are parallel.

6. Slope of the line of $2x - y - 5 = 0$ is 2. Slope of the line parallel to this line is 2. The equation of the line through point $(0,0)$ and a slope 2 is, $y = 2x$.

7. $\frac{x}{5} - \frac{y}{6} = 1$

$6x - 5y = 30$

$y = \frac{6}{5}x - 6$

$m = \frac{6}{5}$

Slope of a line perpendicular to this line is $-\frac{5}{6}$. Equation of the line through $(-2, \frac{1}{2})$ with slope $-\frac{5}{6}$ is,

$y - \frac{1}{2} = -\frac{5}{6}(x + 2)$

$5x + 6y + 7 = 0$

6-3. The Parabola. No comments.

Exercises 6-3 - Answers.

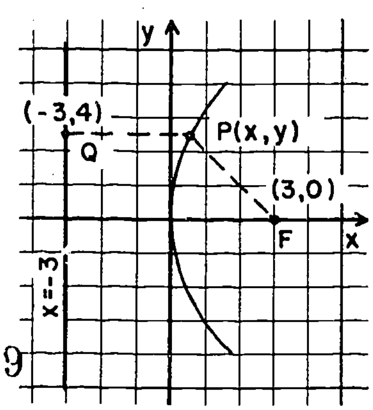
1. (a) Select point $P(x,y)$ any point which is equidistant from the line (directrix) whose equation is $x = -3$ and the point (focus) whose coordinates are $(3,0)$. Let Q be the point of intersection of the perpendicular from P to the line $x = -3$. Then $\overleftrightarrow{PQ}$ is horizontal, Q has coordinates $(-3,y)$. Since P is equidistant from $F(3,0)$ and the line $x = -3$,

$d(P,F) = d(P,Q)$

$$\sqrt{(x-3)^2 + (y-0)^2} = \sqrt{(x+3)^2 + (y-y)^2}$$

$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$$

$$y^2 = 12x$$



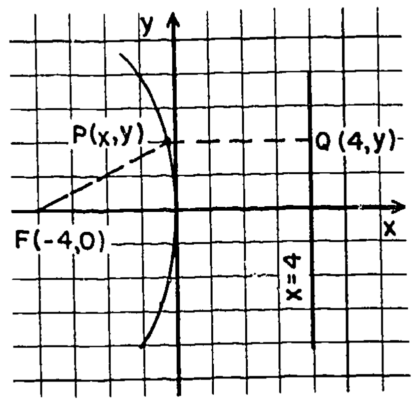
[pages 314, 315-321]

(b) $P = (x,y)$ directrix $x = 4$ $F(-4,0)$

$$\sqrt{(x + 4)^2 + (y - 0)^2} = \sqrt{(x - 4)^2 + (y - y)^2}$$

$$x^2 + 8x + 16 + y^2 = x^2 - 8x + 16$$

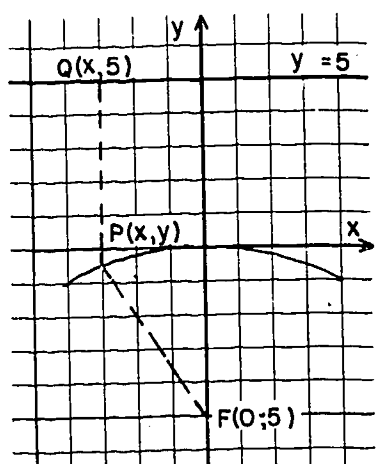
$$y^2 = -16x$$



(c) $$\sqrt{(x - 0)^2 + (y + 5)^2} = \sqrt{(x - x)^2 + (y - 5)^2}$$

$$x^2 + y^2 + 10y + 25 = y^2 - 10y + 25$$

$$x^2 = -20y$$

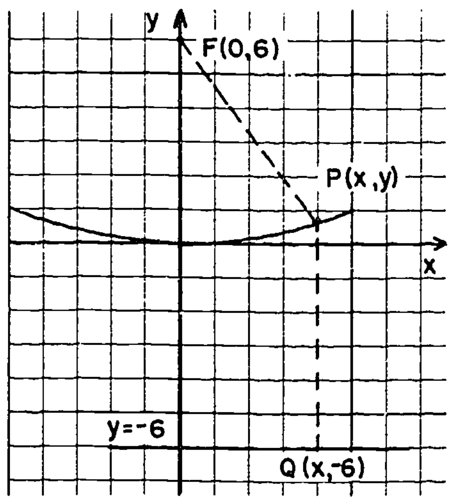


[page 321]

(d) $\sqrt{(x - 0)^2 + (y - 6)^2} = \sqrt{(x - x)^2 + (y + 6)^2}$

$$x^2 + y^2 - 12y + 36 = y^2 + 12y + 36$$

$$x^2 = 24y$$



2. (a) From the Equation 6-3a,

$x^2 = 4cy$

The given equation,

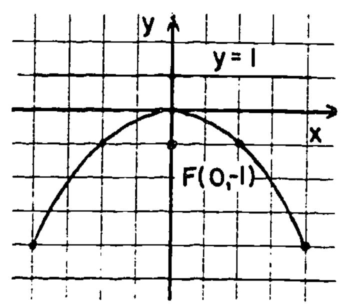
$x^2 = -4y$

$x^2 = 4(-1)y$

$\therefore c = -1$

$F = (0,c) = (0,-1)$

directrix the line $y = -c = 1$

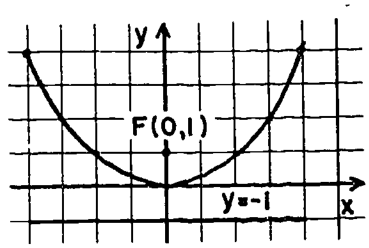


[pages 321-322]

(b) $x^2 = 4y$

$c = 1$

$F = (0,1) \quad y = -1$



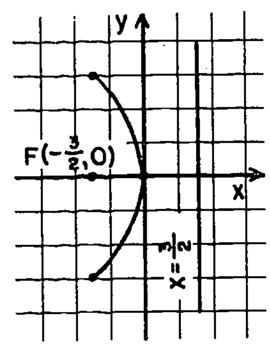
(c) From the Equation 6-3b

$y^2 = 4cx$

For $y^2 = 6x, \quad c = -\frac{3}{2}$

Hence, the focus $F(-\frac{3}{2},0)$

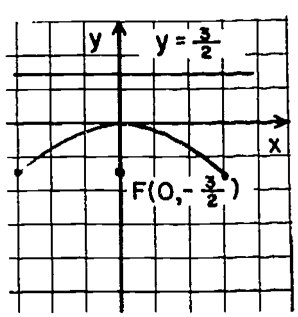
and the directrix $x = \frac{3}{2}$.



(d) $x^2 = -6y$

$c = -\frac{3}{2}$

$F = (0,-\frac{3}{2}), \quad y = \frac{3}{2}$

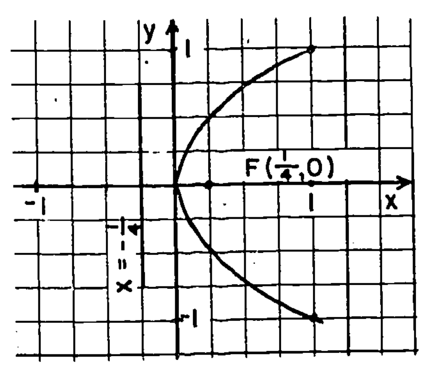


(e) $x = y^2$

$y^2 = x$

$c = \frac{1}{4}$

$F = (\frac{1}{4},0) \quad x = -\frac{1}{4}$



(f) $x^2 + y = 0$

$x^2 = -y$

$c = -\frac{1}{4}$

$F(0,-\frac{1}{4}),\quad y = \frac{1}{4}$

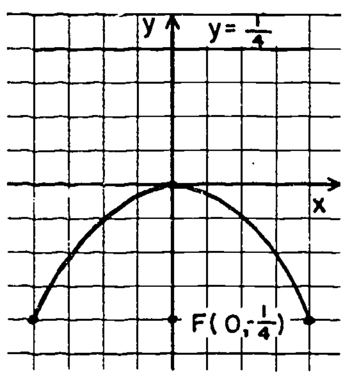
(g) $2x^2 - 4y = 0$

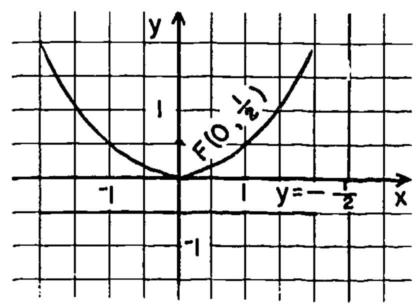
$2x^2 = 4y$

$x^2 = 2y$

$c = \frac{1}{2}$

$F(0,\frac{1}{2}),\quad y = -\frac{1}{2}$





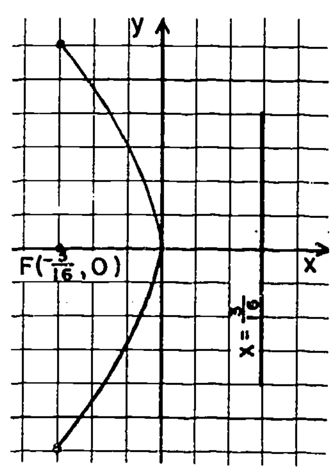
(h) $3x + 4y^2 = 0$

$4y^2 = -3x$

$y^2 = -\frac{3}{4}x$

$c = -\frac{3}{16}$

$F = (-\frac{3}{16},0),\quad x = \frac{3}{16}$



[page 322]

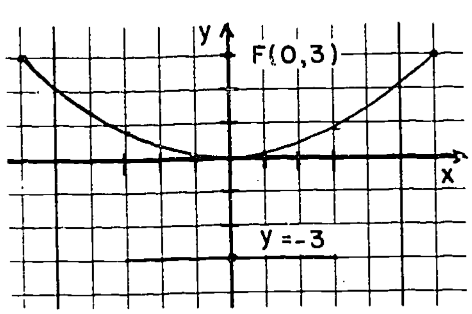
3. Cross section of an automobile headlight reflector, cross section of a radar antenna, trajectories of missiles, and cables of a suspension bridge are a few examples which may be mentioned. A chain or rope suspended from two supports appear to be parabolas; however, these are catenaries whose equation is

$$y = \frac{e^{ax} + e^{-ax}}{2}$$

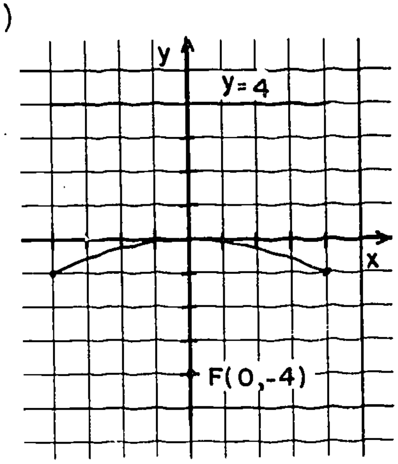
4.

	Equation of		Coordinates of	
	axis	directrix	vertex	focus
(a)	$x = 0$	$y = -3$	$(0,0)$	$(0,3)$
(b)	$x = 0$	$y = 4$	$(0,0)$	$(0,-4)$
(c)	$y = 0$	$x = -5$	$(0,0)$	$(5,0)$
(d)	$y = 0$	$x = \frac{1}{8}$	$(0,0)$	$(-\frac{1}{8},0)$
(e)	$y = 0$	$x = \frac{1}{4}$	$(0,0)$	$(-\frac{1}{4},0)$
(f)	$x = 0$	$y = -\frac{1}{4}$	$(0,0)$	$(0,\frac{1}{4})$

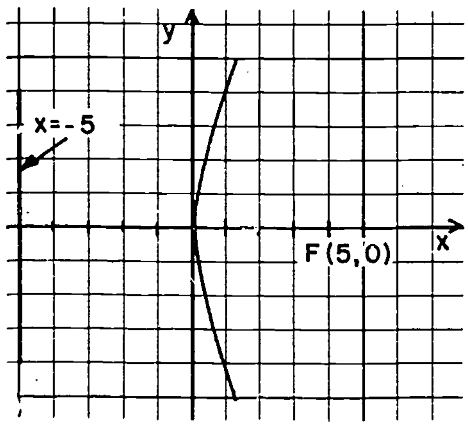
(a)



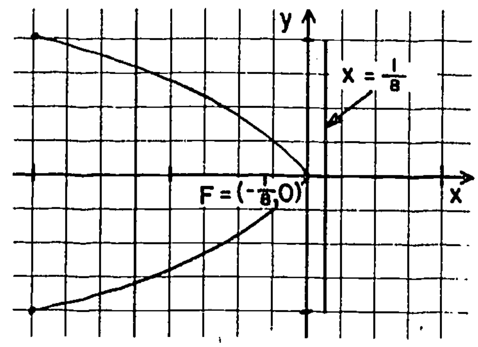
(b)



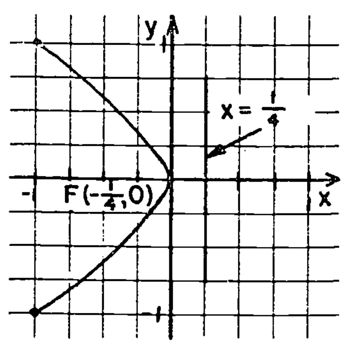
(c)



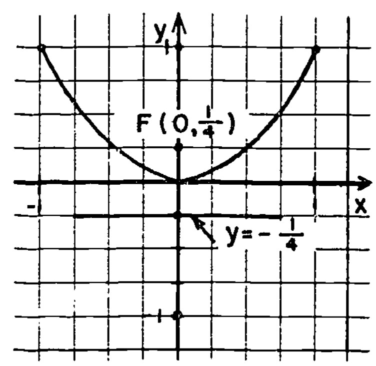
(d)



(e)



(f)



After completing #1, #2, and #4 in this set of exercises, the student should be expected to have a general notion of how the graph will appear before he starts to sketch it.

5. $P(x,y)$ satisfies $y^2 = 8x$

adding $-4x$ to each member, $-4x + y^2 = -4x + 8x$

$-4x + y^2 = 4x$

adding $x^2 + 4$ to each member, $x^2 - 4x + 4 + y^2 = x^2 + 4x + 4$

which can be written as, $(x - 2)^2 + y^2 = (x + 2)^2$

or as, $(x - 2)^2 + (y - 0)^2 = (x + 2)^2 + (y - y)^2$

taking the square root $\sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x+2)^2 + (y-y)^2}$

Since the left member gives the distance from the point (2,0) and the right member gives the distance from the point (-2,y), the proof is complete.

6. (a) π

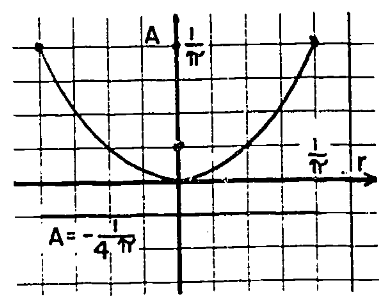
(b) $A = \pi r^2$ or $r^2 = \frac{1}{\pi} A$

(c) $r^2 = \frac{1}{\pi} A$

$c = \frac{1}{4\pi}$

Focus at $(0, \frac{1}{4\pi})$.

Directrix: $A = -\frac{1}{4\pi}$.



(d) $A = \pi r^2$

$A = \pi (\frac{d}{2})^2$

$A = \pi \frac{d^2}{4}$

$4A = \pi d^2$

$A = 63$

$4(63) = \pi d^2$

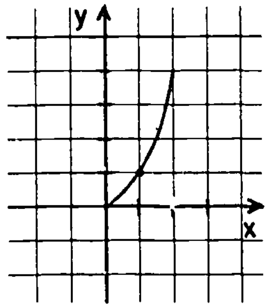
$252 = \pi d^2$

$\frac{252}{\pi} = d^2$

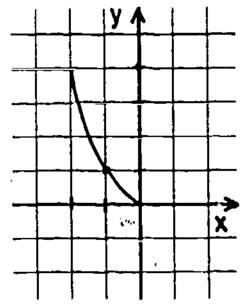
$\sqrt{\frac{252}{\pi}} = d$

$\frac{1}{\pi}\sqrt{252\pi} = d$

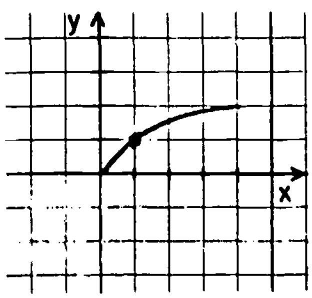
7. (a) $x = +\sqrt{y}$



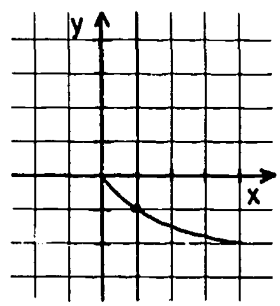
(b) $x = -\sqrt{y}$



(c) $y = +\sqrt{x}$



(d) $y = -\sqrt{x}$

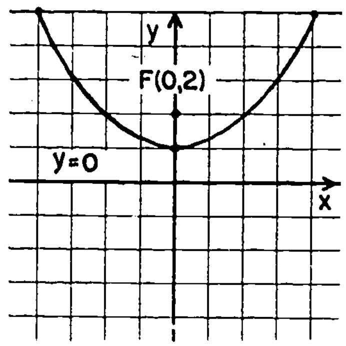


No! Notice that every solution of $x = +\sqrt{y}$ is a solution of $x^2 = y$. Notice also that the solution of $x = -\sqrt{y}$ is a solution of $x^2 = y$. Now, every solution of $x^2 = y$ is a solution of <u>either</u> $x = -\sqrt{y}$ <u>or</u> $x = +\sqrt{y}$, but not of both (except (0,0)). We can say that the graph of $x^2 = y$ is the <u>union</u> of the graph of $x = -\sqrt{y}$ and the graph of $x = -\sqrt{y}$. In other words, we might say that the solution set of $x = +\sqrt{y}$ is a subset of the solution set of $x^2 = y$. A similar discussion can be given for the other parts of the problem.

8. (a) $\sqrt{(x - 0)^2 + (y - 2)^2} = \sqrt{(x - x)^2 + (y - 0)^2}$

$$x^2 + y^2 - 4y + 4 = y^2$$

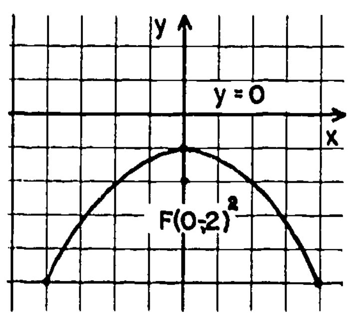
$$x^2 = 4y - 4$$



(b) $\sqrt{(x - 0)^2 + (y + 2)^2} = \sqrt{(x - x)^2 + (y - 0)^2}$

$$x^2 + y^2 + 4y + 4 = y^2$$

$$x^2 = -4y - 4$$

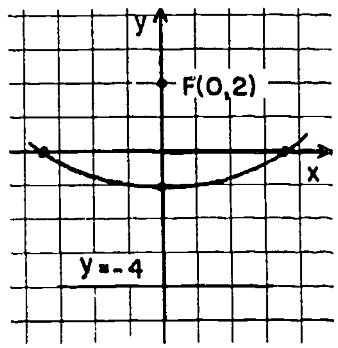


(c) $\sqrt{(x - x)^2 + (y + 4)^2} = \sqrt{(x - 0)^2 + (y - 2)^2}$

$$y^2 + 8y + 16 = x^2 + y^2 - 4y + 4$$

$$12y + 12 = x^2$$

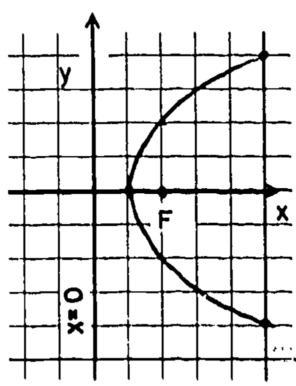
$$x^2 = 12y + 12$$



(d) $\sqrt{(x - 0)^2 + (y - y)^2} = \sqrt{(x - 2)^2 + (y - 0)^2}$

$$x^2 = x^2 - 4x + 4 + y^2$$

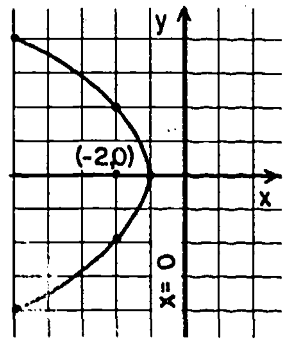
$$y^2 = 4x - 4$$



(e) $\sqrt{(x + 2)^2 + (y - 0)^2} = \sqrt{(x - 0)^2 + (y - y)^2}$

$$x^2 + 4x + 4 + y^2 = x^2$$

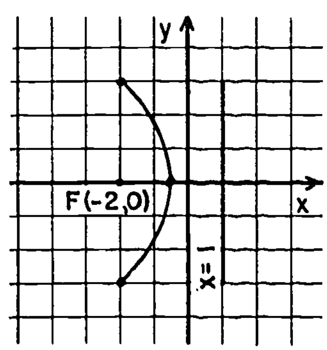
$$y^2 = -4x - 4$$



(f) $\sqrt{(x + 2)^2 + (y - 0)^2} = \sqrt{(x - 1) + (y - y)^2}$

$$x^2 + 4x + 4 + y^2 = x^2 - 2x + 1$$

$$y^2 = -6x - 3$$

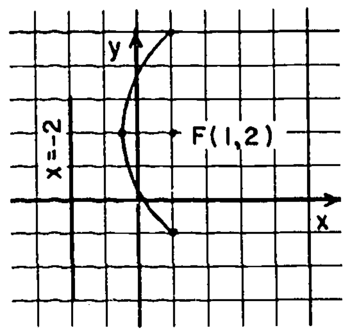


(g) $\sqrt{(x - 1)^2 + (y - 2)^2} = \sqrt{(x + 2)^2 + (y - y)^2}$

$$x^2 - 2x + 1 + (y - 2)^2 = x^2 + 4x + 4 + 0$$

$$(y - 2)^2 = 6x - 3$$

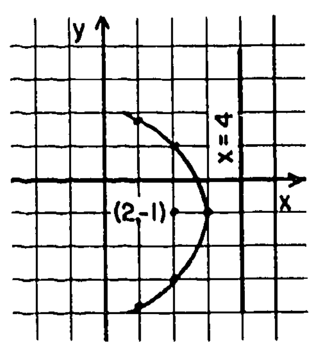
or $\quad y^2 - 4y = 6x - 1$



(h) $\sqrt{(x - 2)^2 + (y + 1)^2} = \sqrt{(x - 4)^2 + (y - y)^2}$

$$x^2 - 4x + 4 + (y + 1)^2 = x^2 - 8x + 16$$

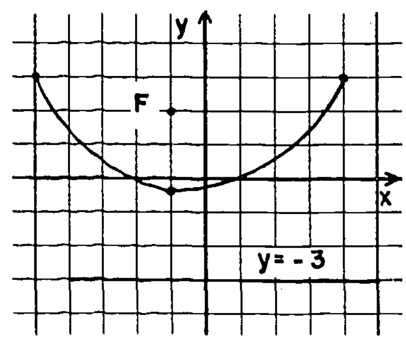
$$(y + 1)^2 = - 4x + 12$$



(i) $\sqrt{(x + 1)^2 + (y - 2)^2} = \sqrt{(x - x)^2 + (y + 3)^2}$

$$(x + 1)^2 + y^2 - 4y + 4 = y^2 + 6y + 9$$

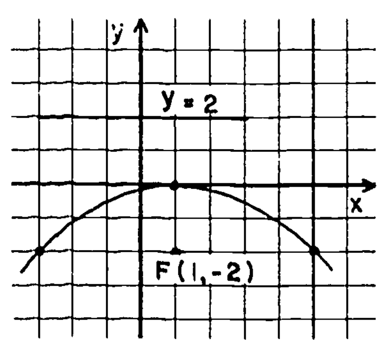
$$(x + 1)^2 = 10y + 5$$



(j) $\sqrt{(x - 1)^2 + (y + 2)^2} = \sqrt{(x - x)^2 + (y - 2)^2}$

$$(x - 1)^2 + y^2 + 4y + 4 = y^2 - 4y + 4$$

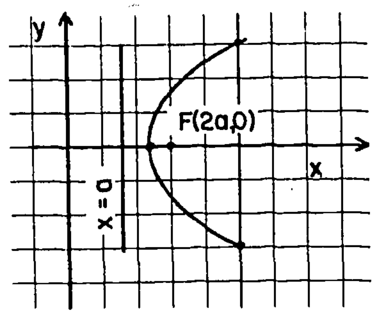
$$(x - 1)^2 = - 8y$$



(k) $\sqrt{(x - a)^2 + (y - y)^2} = \sqrt{(x - 2a)^2 + (y - 0)^2}$

$$x^2 - 2ax + a^2 = x^2 - 4ax + 4a^2 + y^2$$

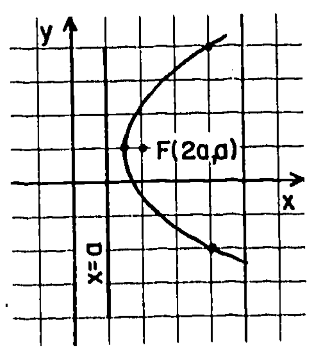
$$y^2 = 2ax - 3a^2$$



(ℓ) $\sqrt{(x - a)^2 + (y - y)^2} = \sqrt{(x - 2a)^2 + (y - a)^2}$

$$x^2 - 2ax + a^2 = x^2 - 4ax + 4a^2 + (y - a)^2$$

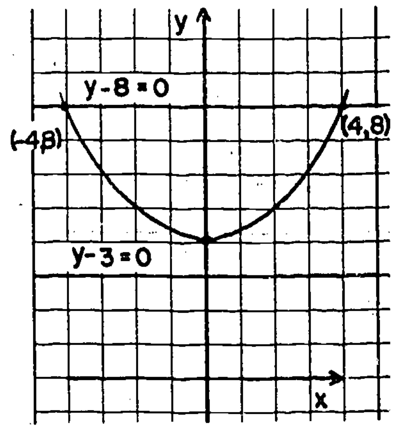
$$(y - a)^2 = 2ax - 3a^2$$



9. (a) $x^2 = 4y - 16$

(b) (4,8) and (-4,8)

(c) The line whose equation is $y - 3 = 0$ does not intersect the curve. The intersection may be described as the empty set, ($\emptyset$).



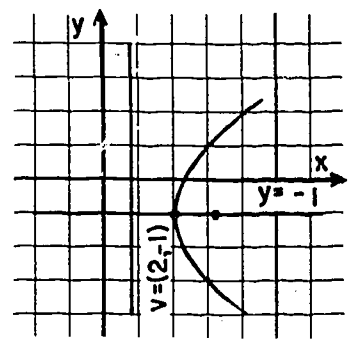
10. (a) $y^2 + 2y - 5x + 11 = 0$

$y^2 + 2y + 1 = 5x - 11 + 1$

$(y + 1)^2 = 5x - 10$

$(y + 1)^2 = 5(x - 2)$

V = (2,-1) axis y = - 1

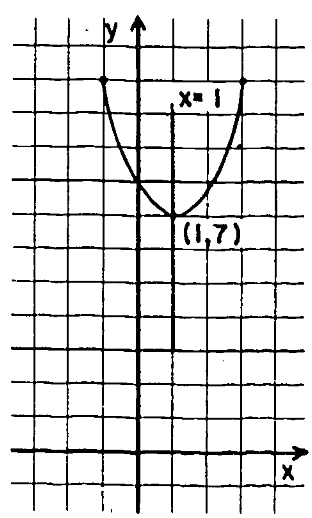


(b) $x^2 - 2x - y + 8 = 0$

$x^2 - 2x + 1 = y - 8 + 1$

$(x - 1)^2 = y - 7$

V = (1,7) axis x = 1



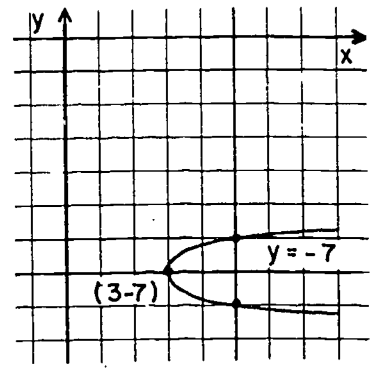
(c) $2y^2 + 28y - x + 101 = 0$

$$y^2 + 14y + 49 = \frac{1}{2}x - \frac{101}{2} + \frac{98}{2}$$

$$(y + 7)^2 = \frac{1}{2}x - \frac{3}{2}$$

$$(y + 7)^2 = \frac{1}{2}(x - 3)$$

$V = (3,-7)$ axis $y = -7$

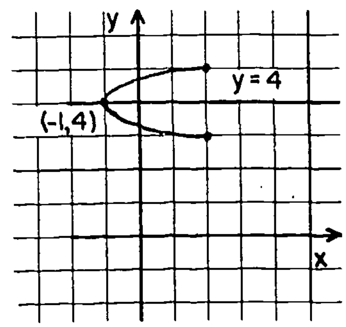


(d) $3y^2 - 24y - x + 47 = 0$

$$y^2 - 8y + 16 = \frac{1}{3}x - \frac{47}{3} + \frac{48}{3}$$

$$(y - 4)^2 = \frac{1}{3}(x + 1)$$

$V = (-1,4)$ axis $y = 4$

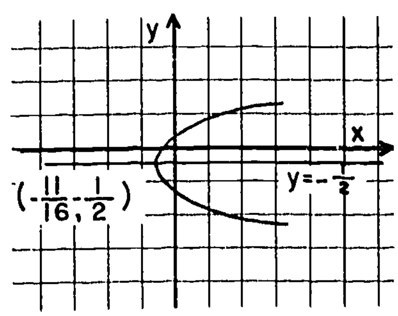


(e) $140y^2 + 140y - 80x - 20 = 0$

$$y^2 + y + \frac{1}{4} = \frac{4}{7}x + \frac{1}{7} + \frac{1}{4}$$

$$(y + \frac{1}{2})^2 = \frac{16}{28}(x + \frac{11}{16})$$

$V = (-\frac{11}{16}, -\frac{1}{2})$ axis $y = -\frac{1}{2}$



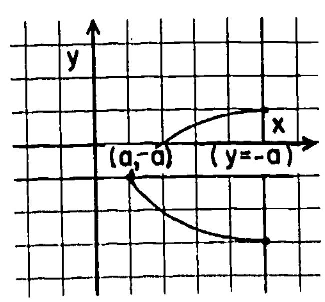
(f) $4a^2y^2 + 8a^3y - x + 4a^4 + a = 0 \quad a > 0$

$$y^2 + 2ay + a^2 = \frac{1}{4a^2}x - a^2 - \frac{1}{4a} + a^2$$

$$(y + a)^2 = \frac{1}{4a^2}x - \frac{1}{4a}$$

$$(y + a)^2 = \frac{1}{4a}(x - a)$$

$V = (a, -a)$ axis $y = -a$



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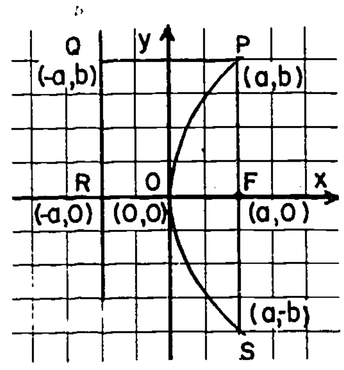
11. Latus Rectum is $\overline{PS}$

Show that $d(P,S) = 2 \cdot d(F,R)$

Select the coordinate axes such that the x-axis passes through the vertex and the focus; the y-axis passes through the vertex. Note the coordinates of the points P,F,S, and Q on the drawing.



$d(O,F) = d(O,R) = |a|$ by Definition 6-3a

$\therefore d(F,R) = 2\ |a|$

$d(P,F) = d(P,Q)$ by Definition 6-3a.

$d(P,Q) = 2\ |a| = d(F,R)$

$\therefore d(P,F) = d(F,R)$

But $d(P,S) = 2 \cdot d(P,F)$ by symmetric property of the parabola.

$d(P,S) = 2 \cdot d(F,R)$ substitution.

This proof appears to be made for a particular parabola with vertex at the origin and axis the x-axis. It is important to emphasize to the student that the coordinate axes may always be conveniently chosen. Some inquisitive student may want to tackle the problem when the axes are not so conveniently chosen, such as those in 6-3c.

12. (a) $y^2 = x$ $c = \frac{1}{4}$ $F = (\frac{1}{4}, 0)$ directrix $x = -\frac{1}{4}$

latus rectum $2\left|(-\frac{1}{4}) - \frac{1}{4}\right| = 1$

Note that the absolute value is used here since distance is always positive

(b) $x^2 = y$ $c = \frac{1}{4}$ $F = (0, \frac{1}{4})$ directrix $y = -\frac{1}{4}$

latus rectum $2(\frac{1}{2}) = 1$

(c) 4

(d) 12

(e) 6

(f) 3

13. $$d(P,S) = 4$$

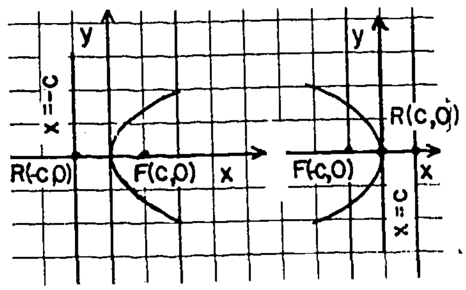
Since $$d(P,S) = 2\cdot d(F,R)$$

$$4 = 2\cdot 2\ c$$

$$1 = c$$

$\therefore$ the equation is, $y^2 = 4x, \quad c > 0$

The equation is, $y^2 = -\ 4x, \quad c < 0$



14. Refer to 6-3c in the text and,

$$F(h + c,\ k) = F(2,\ -3)$$

$$V(h,k) \qquad = V(1,\ -3)$$

$\therefore$ the equation is,

$$(y + 3)^2 = 4\cdot 1(x - 1)$$

$$(y + 3)^2 = 4(x - 1)$$

The equation may be found by using the Definition 6-3a.

15. Since the vertex of the parabola is at (0,0) and it is symmetric with respect to the x-axis, the equation is, $y^2 = 4cx$. Since the parabola passes through the point whose coordinates are $(-3,-2)$, they must satisfy $y^2 = 4cx$;

hence, $$(-2)^2 = 4c(-3)$$

$$c = -\frac{1}{3}$$

The focus $(-\frac{1}{3},0)$; the equation $y^2 = -\frac{4}{3}x$.

16. Notice that this problem is the same as Problem 15, except that the axis is the y-axis. The equation is, $x^2 = 4\cdot \frac{9}{20}y$ or $x^2 = \frac{9}{5}y$.

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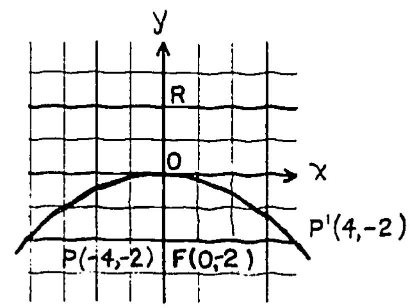
17. With the vertex at the origin and the endpoints of the latus rectum being at (4,8) and (4,-8), we know that the parabola is symmetric with respect to the x-axis. Hence, the problem is essentially the same as Problem 15. The equation is, $y^2 = 16x$.

18. $d(P,P') = 2 \cdot d(F,R) = 8$

$d(F,R) = 4$

Since F(0,-2), then R(0,2) and the vertex of the parabola is V(0,0). The axis is the y-axis. Hence, the equation $x^2 = -8y$



19. (a) $a = 2$

(b) $a = \frac{y_o}{(x_o)^2}$ Any point (x_o, y_o), $x_o \neq 0$.

20.
$$y = x^2 + x + 5$$
$$y - \frac{19}{4} = (x + \tfrac{1}{2})^2$$
$V(-\frac{1}{2}, \frac{19}{4})$, $F(-\frac{1}{2}, 5)$, directrix $y = \frac{9}{2}$ [6-3c]

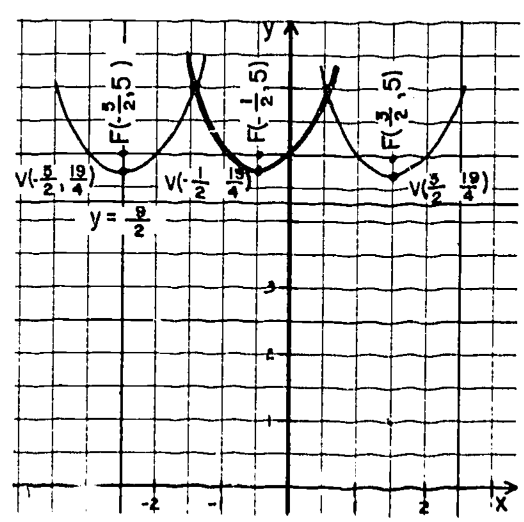
(a) Replacing x by x - 2, the equation becomes,

$y = (x-2)^2 + (x-2) + 5$

$y - \frac{19}{4} = (x - \frac{3}{2})^2$

$V(\frac{3}{2}, \frac{19}{4})$, $F(\frac{3}{2}, 5)$,

directrix $y = \frac{9}{2}$.



(c) Replacing x by x + 2 in $y = x^2 + x + 5$.
The equation is, $y - \frac{19}{4} = (x + \frac{5}{2})^2$

$V(-\frac{5}{2}, \frac{19}{4})$, $F(-\frac{5}{2}, 5)$, directrix $y = \frac{9}{2}$

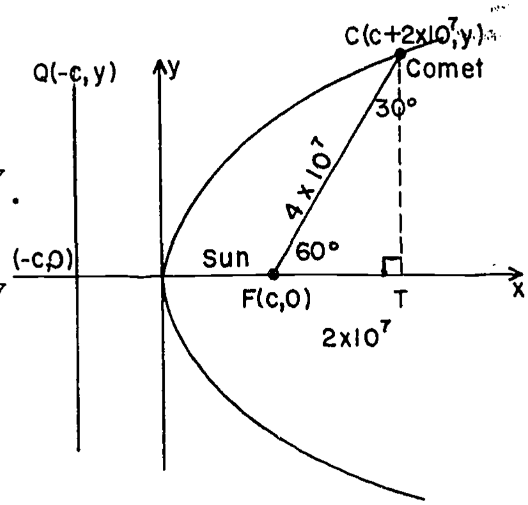
(d) Replacing x by x-2 moves the graph of the parabola two units to the right and replacing x by x + 2 moves the curve two units to the left.

21. Choose the axes as shown.

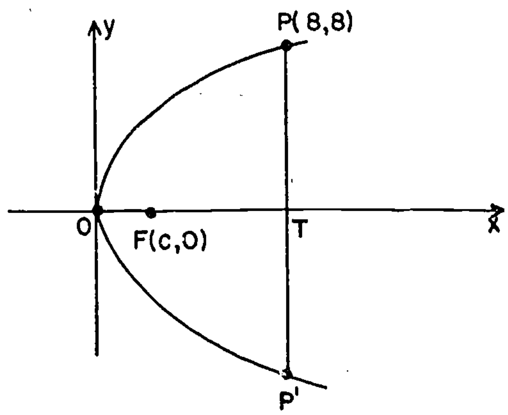
$d(F,T) = \frac{1}{2}d(F,C)$ by property of right triangle.

$\therefore d(F,T) = \frac{1}{2}(4\times10^7) = 2\times10^7$.

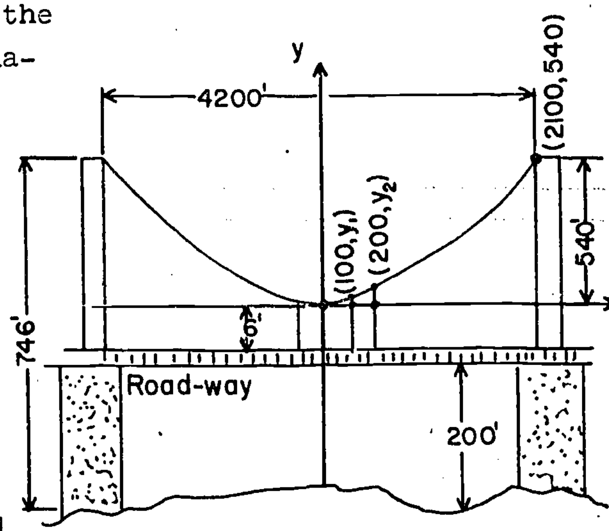
$$\begin{aligned} d(C,F) &= d(C,Q) \\ 4\times10^7 &= |-c| + c + 2\times10^7 \\ 4\times10^7 &= 2c + 2\times10^7 \\ 10^7 &= c \end{aligned}$$



22. $d(P,P') = 16$, $d(O,T) = 8$
$d(P,T) = 8$, $P(8,8)$
The equation of this parabola is $y^2 = 4cx$. Since the parabola passes through (8,8), then $8^2 = 4c(8)$
$c = 2$
$\therefore d(O,F) = 2$



23. By convenient choice of the coordinate axis, the equation of the parabola is,



$x^2 = 4cy$

$(2100)^2 = 4c(540)$

$c = \frac{(2100)^2}{4(540)}$

$\therefore \quad x^2 = \frac{(2100)^2}{540} y$

(equation of parabola)

The length of the first support is 6'

The length of the second support is $(6 + y_1)'$.

The second support will meet the parabola at $(100, y_1)$

$$\therefore \quad (100)^2 = \frac{(2100)^2}{540} y_1 , \quad y_1 = 100^2 \cdot \frac{540}{21^2 \cdot 100^2} = \frac{540}{441}$$

So, the length of the second support is,

$$6 + \frac{540}{441} \approx 7.2$$

To find the length of the second support, proceed in a similar manner, but using the coordinates of the point $(200, y_2)$.

Continue in this manner until the lengths of 22 supports are obtained. (The length of the first and the last are known; hence, there are only 20 left to compute.)

The measures are:

$$y_0 = 0^2 . \frac{540}{(21)^2} + 6 = 6'$$

$$y_1 = 1^2 . \frac{540}{(21)^2} + 6 \approx 7.2'$$

$$y_2 = 2^2 . \frac{540}{(21)^2} + 6 \approx 10.9'$$

$$y_3 = 3^2 . \frac{540}{(21)^2} + 6$$

$$y_4 = 4^2 \cdot \frac{540}{(21)^2} + 6$$

$$\vdots$$

$$y_{20} = (20)^2 \cdot \frac{540}{(21)^2} + 6$$

$$y_{21} = (21)^2 \cdot \frac{540}{(21)^2} + 6$$

By leaving in the above form, the students will observe the pattern. Some may want to find the approximate answers; however, this is not necessary in this problem.

6-4. The General Definition of the Conic

The emphasis in this section is on deriving equations of conics from the definition. We would like for the student to be able to do this in many cases for himself. The importance of the value of e in determining the kind of conic should be stressed. The actual graphing of the resulting equations, and finding coordinates of the vertex, focus, etc. will be covered more thoroughly in the next two sections.

Exercises 6-4 - Answers

1. Since $e < 1$, $(e = \frac{1}{3})$, the conic is an ellipse.

$$\sqrt{(x - 2)^2 + (y - 0)^2} = \frac{1}{3}\sqrt{(x - x)^2 + (y - 3)^2}$$

$$(x - 2)^2 + y^2 = \frac{1}{9}(y - 3)^2$$

$$9x^2 - 36x + 36 + 9y^2 = y^2 - 6y + 9$$

$$9x^2 + 8y^2 - 36x + 6y + 27 = 0$$

2. (a) Hyperbola, since $e = \frac{3}{2}$ $e > 1$

(b) $\sqrt{(x - 0)^2 + (y - 0)^2} = \frac{3}{2}\sqrt{(x - x)^2 + (y + 2)^2}$

$x^2 + y^2 = \frac{9}{4}(y^2 + 4y + 4)$

$4x^2 + 4y^2 = 9y^2 + 36y + 36$

$4x^2 - 5y^2 - 36y - 36 = 0$

3. (a) Parabola, since $e = 1$

(b) $\sqrt{(x + 2)^2 + (y - 3)^2} = \sqrt{(x - 4)^2 + (y - y)^2}$

$x^2 + 4x + 4 + (y - 3)^2 = x^2 - 8x + 16$

$(y - 3)^2 = -12x + 12$

4. (a) Hyperbola, since $e = \sqrt{2}$, $e > 1$,

(b) $-9x^2 + 9y^2 + 78x + 73 = 0$

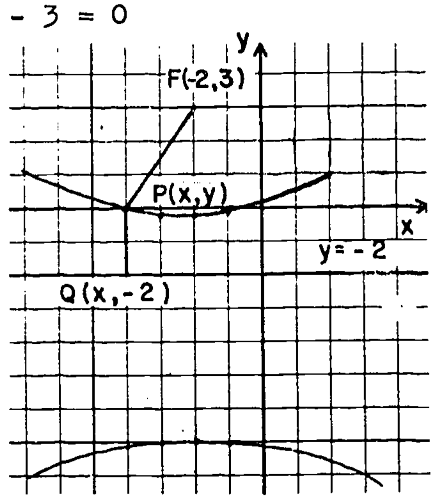
5. (a) Hyperbola, since $e = 2\sqrt{5}$, $e > 1$.

(b) $-19x^2 + y^2 + 22x - 6y + 5 = 0$

6. (a) $\sqrt{(x+2)^2 + (y-3)^2} = 2\sqrt{(x-x)^2 + (y+2)^2}$

$x^2 + 4x + 4 + y^2 - 6y + 9 = 4(y^2+4y+4)$

$x^2 - 3y^2 + 4x - 22y - 3 = 0$



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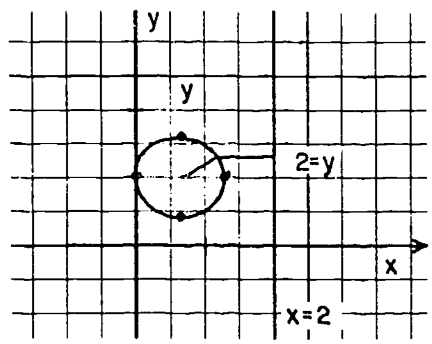
(b) $\sqrt{(x - 1)^2 + (y - 1)^2} = \frac{1}{2}\sqrt{(x - 2)^2 + (y - y)^2}$

$$(x - 1)^2 + (y - 1)^2 = \frac{1}{4}(x - 2)^2$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 = \frac{1}{4}(x^2 - 4x + 4)$$

$$4x^2 - 8x + 4 + 4y^2 - 8y + 4 = x^2 - 4x + 4$$

$$3x^2 + 4y^2 - 4x - 8y + 4 = 0$$



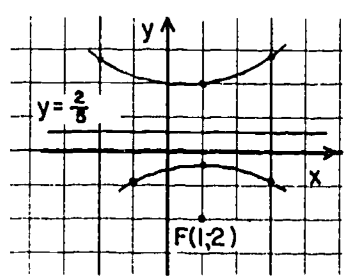
(c) $\sqrt{(x - 1)^2 + (y + 2)^2} = \frac{5}{2}\sqrt{(x - x)^2 + (y - \frac{2}{5})^2}$

$$(x - 1)^2 + (y + 2)^2 = \frac{25}{4}(y - \frac{2}{5})^2$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = \frac{25}{4}(y^2 - \frac{4}{5}y + \frac{4}{25})$$

$$4x^2 - 8x + 4 + 4y^2 + 16y + 16 = 25y^2 - 20y + 4$$

$$4x^2 - 21y^2 - 8x + 36y + 16 = 0$$



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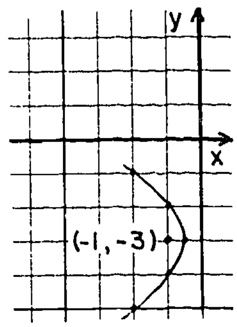
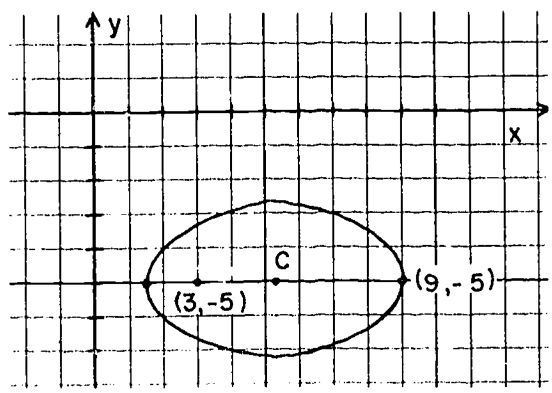
(d) $\sqrt{(x + 1)^2 + (y + 3)^2} = \sqrt{(x - 0)^2 + (y - y)^2}$

$$x^2 + 2x + 1 + (y + 3)^2 = x^2$$

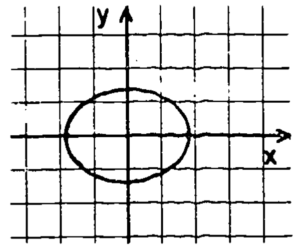
$$(y + 3)^2 = -2x - 1$$



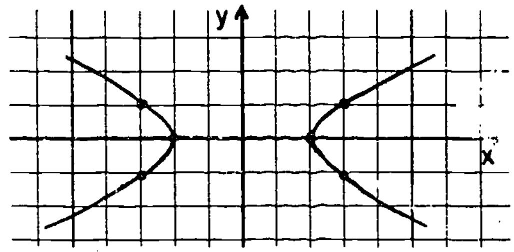
(e) $5x^2 + 9y^2 - 54x + 90y + 306 = 0$



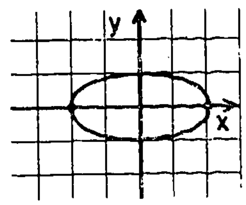
7. (a) ellipse



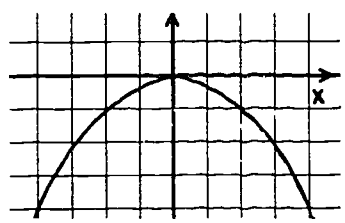
(b) hyperbola



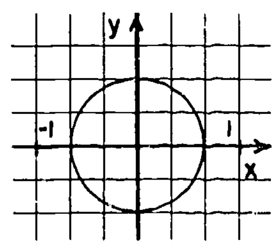
(c) ellipse



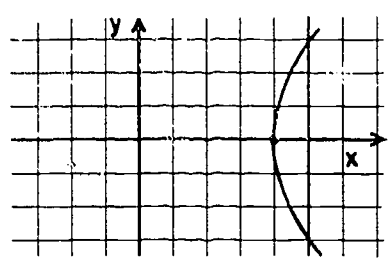
(d) parabola



(e) circle



(f) parabola



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8. (i) $Ax^2 + Cy^2 + F = 0$

$Ax^2 + Cy^2 = -F$

(ii) $Ax^2 + Cy^2 = M$, replacing $-F$ by M.

(a) When $A \cdot C > 0$

Case 1: $M = 0$ Now, when $M = 0$ and A and C have the same signs, the graph of (ii) consists of a <u>single point</u> (0,0).

Case 2: $M \neq 0$ Hence, (ii) can be written as,

$$\frac{x^2}{\frac{M}{A}} + \frac{y^2}{\frac{M}{C}} = 1$$

Now, when A and C have the same sign, then M must have the same sign as A and C; (if M has the opposite sign, there will be no real pairs (x,y) which satisfy the equation and consequently no graph), hence, $\frac{M}{A}$ and $\frac{M}{C}$ will have the same sign, and the graph will be an ellipse.

(b) When $A \cdot C < 0$

Case 1: $M = 0$

When $M = 0$ and A and C have opposite signs (ii) is factorable, and the graph consists of two straight lines. This is called a degenerate conic.

Case 2: $M \neq 0$ Hence, (ii) can be written as,

$$\frac{x^2}{\frac{M}{A}} + \frac{y^2}{\frac{M}{C}} = 1$$

Now, when A and C have opposite signs, $\frac{M}{A}$ and $\frac{M}{C}$ will have opposite signs, and the graph of the equation is a hyperbola.

(c) When $A \cdot C = 0$

Case 1: $M = 0$

When <u>either</u> A or C is 0, the graph is either the x-axis or the y-axis. This is another instance of the degenerate conic.

Case 2: $M \neq 0$

Suppose we choose $A = 0$, then (ii) becomes,

$$Cy^2 = M$$

$$y^2 = \frac{M}{C}$$

$$y^2 = \sqrt{\frac{M}{C}} \quad \text{or} \quad y = -\sqrt{\frac{M}{C}}$$

Hence, the graph of the equation is two lines parallel to the x-axis, provided $\frac{M}{C} > 0$. If we choose $C = 0$, then the graph is two lines parallel to the y-axis, provided $\frac{M}{A} > 0$. This is another instance of the degenerate conic.

9. (i) $Ax^2 + Cy^2 + Dx + Ey + F = 0$

Completing the square on x and on y,

$$A(x + \frac{D}{2A})^2 + C(y + \frac{E}{2C})^2 = \frac{D^2C + E^2A - 4ACF}{4AC}$$

Let $M = \frac{D^2C + E^2A - 4ACF}{4AC}$, $h = \frac{-D}{2A}$, $k = \frac{-E}{2C}$ [h, k, and M real numbers], then

(ii) $A(x - h)^2 + C(y - k)^2 = M$, where h, k, and M are real numbers depending on the coefficients of (i).

(a) When $A \cdot C > 0$

Case 1: $M = 0$

With $M = 0$ and A and C with the same sign, the graph of (ii) is the <u>single point</u> (h, k).

Case 2: $M \neq 0$

Then (ii) can be written,

$$\frac{(x - h)^2}{\frac{M}{A}} + \frac{(y - k)^2}{\frac{M}{C}} = 1$$

When A and C have the same sign, then M must have the same sign as A and C; hence, $\frac{M}{A}$ and $\frac{M}{C}$ will have the same sign, and the graph will be an ellipse.

(b) When $A \cdot C < 0$

Case 1: $M = 0$

With A and C having opposite signs and $M = 0$, (ii) is factorable.

$\left[\sqrt{A}\,(x - h) \pm \sqrt{B}\,(y - k) \right] = 0$, hence, the graph consists of two straight lines. This is called a degenerate conic.

Case 2: $M \neq 0$

Then (ii) can be written in the form,

$$\frac{(x - h)^2}{\frac{M}{A}} + \frac{(y - k)^2}{\frac{M}{C}} = 1$$

When A and C have opposite signs, then $\frac{M}{A}$ and $\frac{M}{C}$ have opposite signs, and the graph will be a hyperbola.

(c) When $A \cdot C = 0$ [either A or C is 0]

Case 1: $M = 0$

When A is 0, then (ii) is, $c(y - k)^2 = 0$
$y = k$, hence, the graph is a line k units from the x-axis and parallel to it. When C is 0, then the graph is a line h units from the y-axis and parallel to it.
This is another instance of a degenerate conic.

Case 2: $M \neq 0$

Suppose we choose $A = 0$, then (i) becomes

$Cy^2 + Dx + Ey + F = 0$ completing the square on y

$$y^2 + \frac{E}{C}y + \frac{D}{C}x = -\frac{F}{C}$$

$$\left(y + \frac{E}{2C}\right)^2 + \frac{D}{C}x = \frac{-4CF + E^2}{4C^2}$$

Let $k = -\frac{E}{2C}$, $p = \frac{D}{C}$, $M = \frac{-4CF + E^2}{4C^2}$

Hence, the equation, $(y - k)^2 + px = m$

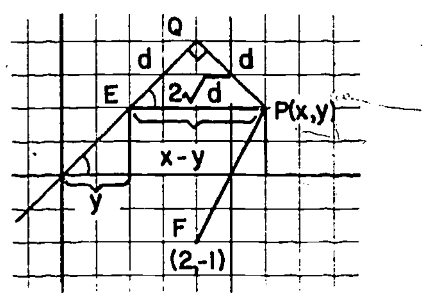
The graph of this equation is a parabola.

In the case where $A = C = 0$, (i) becomes,

$Dx + Ey + F = 0$, and its graph is a straight line.

Note to Teacher: After completing these two exercises, one might wish to use these to identify the equations given in Problem 7. Others may be found in the Supplementary Set.

*10. Ellipse, since $e < 1$, $(e = \frac{2}{3})$.



$$\sqrt{2}d = x - y$$

$$d = \frac{x - y}{\sqrt{2}}$$

$$\sqrt{(x - 2)^2 + (y + 1)^2} = \frac{2}{3}\left|\frac{x - y}{\sqrt{2}}\right|$$

$$(x - 2)^2 + (y + 1)^2 = \frac{4}{9}\,\frac{(x - y)^2}{2}$$

$$x^2 - 4x + 4 + y^2 + 2y + 1 = \frac{4}{9}\left(\frac{x^2 - 2xy + y^2}{2}\right)$$

$$x^2 - 4x + 4 + y^2 + 2y + 1 = \frac{2}{9}(x^2 - 2xy + y^2)$$

$$9x^2 - 36x + 36 + 9y^2 + 18y + 9 = 2x^2 - 4xy + 2y^2$$

$$7x^2 + 4xy + 7y^2 - 36x + 18y + 45 = 0$$

6.5. The Circle and the Ellipse

The somewhat backwards introduction of the circle before the ellipse was an expedient for reviewing the results about the circle which we derived in Chapter 2. The way in which the circle is a limiting form of an ellipse is difficult to explain from the focus-directrix definition and it is only after we have the property that the ellipse is the set of points the sum of whose distances from two fixed points is constant, that this can be satisfactorily explained. The fact is that as e approaches zero, the two foci converge to the center and the directrices recede to infinity. The general definition $d(F,P) = e \cdot d(P,Q)$ then degenerates into the statement $r = 0 \cdot \infty$. This statement is really meaningful and and in a sense true, but it certainly would be confusing to try to explain to a high school student.

To aid in solving the problems which ask for the coordinates of the vertex, center, focus, equations of the directrix and the axes, it may be helpful to stress the relations between a, b, c, and e. These are $\boxed{a^2 = b^2 + c^2}$, $\boxed{b = a\sqrt{1 - e^2}}$, and $\boxed{c = ae}$

Also a word about our use of 2a for the major axis is in order. We developed the equation for the ellipse with the focus at the point $F(ae,0)$ and the directrix, the line $x = \frac{a}{e}$. The motivation for using the letter a for the abscissa of the vertex was that we had used this notation for the x-intercept for the straight line and we might as well call the x-intercept of the ellipse by the same letter. However, once the notation is set that the major axis has lenght 2a, it seemed inadvisable to change this if the focus was on the y-axis or at some other point in the plane. Consequently although our standard form may be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,$$

depending on the position of the focus, 2a is always the length of the major axis. With this agreement a is always greater than either b or c. The major axis with length 2a is always greater in length than the minor axis of length 2b. Hence, if we see

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the equation for an ellipse in standard form the greater of the numbers under the squared terms is a^2.

Exercises 6-5. - Answers

1. (a) $(x - h)^2 + (y - k)^2 = r^2$ by 6-5a

$h = 0,\ k = 0,\ r = 3$

$\therefore\ (x - 0)^2 + (y - 0)^2 = 3^2$ by replacing h, k and r.

$x^2 + y^2 = 9$

(b) $(x - 0)^2 + (y - 2)^2 = 9$

$x^2 + y^2 - 4y - 5 = 0$

(c) $(x - 2)^2 + y^2 = 9$

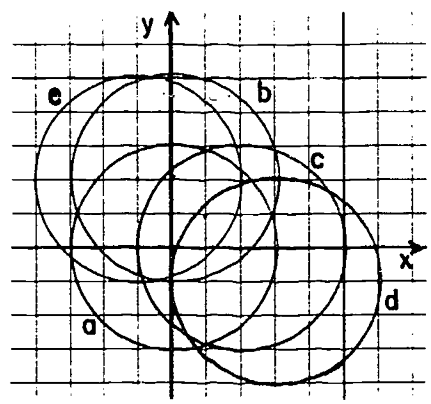
$x^2 + y^2 - 4x - 5 = 0$

(d) $(x - 3)^2 + (y + 1)^2 = 9$

$x^2 - 6x + y^2 + 2y + 1 = 0$

(e) $(x + 1)^2 + (y - 2)^2 = 9$

$x^2 + 2x + y^2 - 4y - 4 = 0$



2. Solutions of this kind of problem depend solely on transforming the equation of the circle to the form

$$(x - h)^2 + (y - k)^2 = r^2.$$

This is done merely by completing the square in both x and y.

	Center	Rad
(a)	(0,0)	5
(b)	(2,-3)	3
(c)	$(-\frac{1}{2},6)$	$\sqrt{5}$
(d)	(0,-5)	$\sqrt{\frac{7}{3}}$ or $\frac{\sqrt{21}}{3}$
(e)	(-4,0)	2
(f)	(5,0)	$\sqrt{7}$
(g)	(3,4)	4

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(h)	(1,-2)	$\sqrt{7}$
(i)	(2,-3)	$\sqrt{14}$
(j)	(1,6)	5
(k)	$(\frac{1}{2}, -\frac{7}{2})$	$\frac{\sqrt{62}}{2}$
(ℓ)	$(-\frac{3}{2}, 2)$	3

Sample solutions for,

(d) $3x^2 + 3(y + 5)^2 = 7$

$(x + 0)^2 + (y + 5)^2 = \frac{7}{3}$

$h = 0 \quad k = -5 \quad r^2 = \frac{7}{3}$

Center $(h,k) = (0,-5)$ radius $= \frac{\sqrt{21}}{3}$

(j) $3x^2 + 3y^2 - 6x - 36y + 36 = 6$

$x^2 + y^2 - 2x - 12y + 12 = 0$

$(x^2 - 2x \quad) + (y^2 - 12y \quad) = -12$

$(x^2 - 2x + 1) + (y^2 - 12y + 36) = -12 + 1 + 36$

$(x - 1)^2 + (y - 6)^2 = 25$

$h = 1, \quad k = 6, \quad r^2 = 25$

Center (1,6) Radius 5.

3.

				Coordinates of			Length of		Equation of
	a	b	c	Vertices	Foci	e	Major axis	Minor axis	Directrix
a	4	3	$\sqrt{7}$	$(\pm 4, 0)$	$(\pm\sqrt{7}, 0)$	$\frac{\sqrt{7}}{4}$	8	6	$x = \frac{16\sqrt{7}}{7}$
b	4	2	$2\sqrt{3}$	$(0, \pm 4)$	$(0, \pm 2\sqrt{3})$	$\frac{\sqrt{3}}{2}$	8	4	$y = \frac{8\sqrt{3}}{3}$
c	5	3	4	$(\pm 5, 0)$	$(\pm 4, 0)$	$\frac{4}{5}$	10	6	$x = \frac{25}{4}$
d	5	2	$\sqrt{21}$	$(0, \pm 5)$	$(0, \pm\sqrt{21})$	$\frac{\sqrt{21}}{5}$	10	4	$y = \frac{25\sqrt{21}}{21}$
e	3	2	$\sqrt{5}$	$(\pm 3, 0)$	$(\pm\sqrt{5}, 0)$	$\frac{\sqrt{5}}{3}$	6	4	$x = \frac{9\sqrt{5}}{5}$
f	$5\sqrt{2}$	5	5	$(0, \pm 5\sqrt{2})$	$(0, \pm 5)$	$\frac{\sqrt{2}}{2}$	$10\sqrt{2}$	10	$y = 10$
g	6	1	$\sqrt{35}$	$(0, \pm 6)$	$(0, \pm\sqrt{35})$	$\frac{\sqrt{35}}{6}$	12	2	$y = \frac{36\sqrt{35}}{35}$
h	$2\sqrt{2}$	$\sqrt{6}$	$\sqrt{2}$	$(\pm 2\sqrt{2}, 0)$	$(\pm\sqrt{2}, 0)$	$\frac{1}{2}$	$4\sqrt{2}$	$2\sqrt{6}$	$x = 4\sqrt{2}$
i	$2\sqrt{3}$	$\sqrt{2}$	$\sqrt{10}$	$(\pm 2\sqrt{3}, 0)$	$(\pm\sqrt{10}, 0)$	$\frac{\sqrt{30}}{6}$	$4\sqrt{3}$	$2\sqrt{2}$	$x = \frac{6\sqrt{10}}{5}$
j	$2\sqrt{3}$	$\sqrt{2}$	$\sqrt{10}$	$(0, \pm 2\sqrt{3})$	$(0, \pm\sqrt{10})$	$\frac{\sqrt{30}}{6}$	$4\sqrt{3}$	$2\sqrt{2}$	$y = \frac{6\sqrt{10}}{5}$
k	$\frac{\sqrt{6}}{2}$	$\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{6}}{6}$	$(\pm\frac{\sqrt{6}}{2}, 0)$	$(\pm\frac{\sqrt{6}}{6}, 0)$	$\frac{1}{3}$	$\sqrt{6}$	$\frac{4\sqrt{3}}{3}$	$x = \frac{3\sqrt{6}}{2}$

In order to solve the ellipse completely, first transform to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The larger denominator will always be a^2 and its numerator will indicate which axis is parallel to the major axis of the ellipse. Since $a^2 = b^2 + c^2$, c is readily found, $e = \frac{c}{a}$ to give e, and length of the major axis is 2a and length of the minor axis is 2b.

Example solution for,

(g)

$$y^2 = 36(1 - x^2)$$

$$y^2 = 36 - 36x^2$$

$$36x^2 + y^2 = 36$$

$$\frac{x^2}{1} + \frac{y^2}{36} = 1$$

$$a = \sqrt{36} = 6$$

$$b = \sqrt{1} = 1$$

Major axis || y-axis

Center at (0,0)

Vertices $(0,\pm 6)$ and $(\pm 1,0)$

$c^2 = a^2 - b^2 = 36 - 1$

$c = \sqrt{35}$

Foci $(0,\pm \sqrt{35})$

$e = \frac{c}{a}$

$e = \frac{\sqrt{35}}{6}$

Directrix $y = \frac{c}{e^2}$

$$y = \frac{\sqrt{35}}{\frac{35}{36}}$$

$$y = \frac{36\sqrt{35}}{35}$$

4. (a) $a = 5 \quad e = .2$

$c = ae = 1$

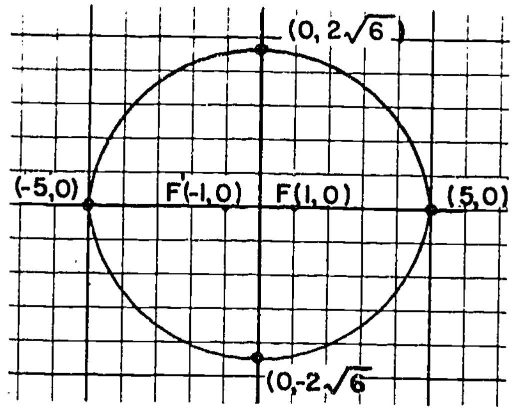
$b^2 = a^2 - c^2 = 25 - 1 = 24$

$$\frac{x^2}{25} + \frac{y^2}{24} = 1$$

$F(1,0)$; $F'(-1,0)$

Directrix $x = \frac{c}{e^2} = \frac{1}{\frac{4}{100}}$ or $x = 25$

$b = \sqrt{24} = 2\sqrt{6}$



(b) $a = 5 \quad e = .4$

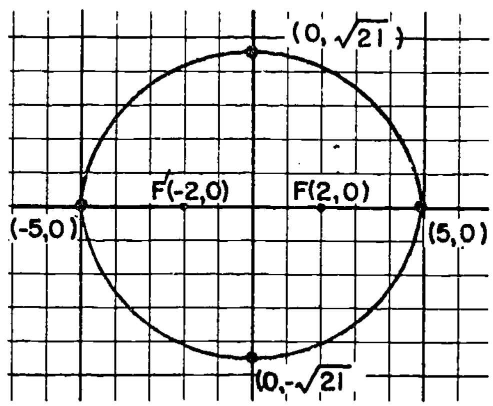
$c = ae = 5 \times .4 = 2$

$b^2 = a^2 - c^2 = 25 - 4 = 21$

$$\frac{x^2}{25} + \frac{y^2}{21} = 1$$

$F(2,0)$; $F'(-2,0)$

Directrix; $x = \frac{c}{e^2} = \frac{2}{\frac{16}{100}} = \frac{200}{16} = \frac{25}{2}$



(c) $a = 5 \quad e = .6$

$c = ae = 5 \times .6 = 3$

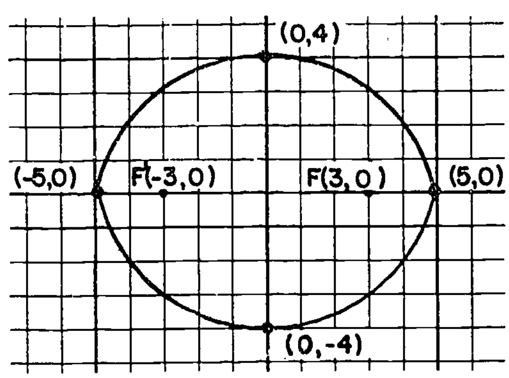
$b^2 = a^2 - c^2 = 25 - 9 = 16$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$F(3,0)$; $F'(-3,0)$

Directrix $x = \frac{c}{e^2} = \frac{3}{\frac{36}{100}} = \frac{300}{36}$

$x = \frac{25}{3}$



(d) $a = 5 \quad e = .8$

$c = ae = 5 \times .8 = 4$

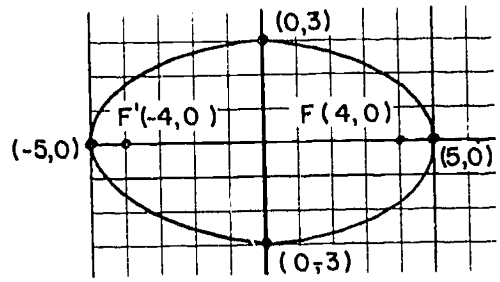
$b^2 = a^2 - c^2 = 25 - 16 = 9$

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

$F(4,0)$; $F'(-4,0)$

Directrix $x = \frac{c}{e^2} = \frac{4}{\frac{64}{100}} = \frac{400}{64}$

$x = \frac{25}{4}$



5. (a) $c = 2, \quad a = 5$

$c = ae$ or $e = \frac{2}{5}$

$b^2 = a^2 - c^2 = 25 - 4 = 21$

$\frac{x^2}{25} + \frac{y^2}{21} = 1$

(b) $b = 2, \quad a = 4$

$\frac{x^2}{16} + \frac{y^2}{4} = 1$

(c) $a = 7 \quad e = \frac{2}{3}$

$c = ae = 7 \cdot \frac{2}{3} = \frac{14}{3}$

$b^2 = a^2 - c^2$

$b^2 = 49 - \frac{196}{9} = \frac{441 - 196}{9} = \frac{245}{9}$

$\frac{x^2}{49} + \frac{y^2}{\frac{245}{9}} = 1$

(d) $b = \sqrt{3} \quad e = \frac{1}{2}$

$c = ae = \frac{1}{2}a$

$b^2 = a^2 - c^2 = a^2 - \frac{1}{4}a^2$

$b^2 = \frac{3}{4}a^2$

$3 = \frac{3}{4}a^2$

$a^2 = 4$

$a = 2$

$\frac{x^2}{4} + \frac{y^2}{3} = 1$

(e) $c = 6 \quad e = \frac{3}{4}$

$c = ae$

$6 = a \cdot \frac{3}{4}$

$a = 8$

$b^2 = a^2 - c^2$

$b^2 = 64 - 36 = 28$

$\frac{x^2}{64} + \frac{y^2}{28} = 1$

(f) $c = ae = 8; \quad x = \frac{a}{e} = 10$

$a = \frac{8}{e}; \quad a = 10e$

$\therefore \quad 10e = \frac{8}{e}$

$e = \frac{2\sqrt{5}}{5}$

$a = 10 \cdot \frac{2\sqrt{5}}{5} = 4\sqrt{5}$

$b = a\sqrt{1 - e^2} = 4$

$\frac{x^2}{80} + \frac{y^2}{16} = 1$

(g) $a = 3$; $x = \frac{a}{e} = 6$; $e = \frac{1}{2}$

$b = \frac{3\sqrt{3}}{2}$

$\frac{x^2}{9} + \frac{4y^2}{27} = 1$

6. (a) As $d(F,F')$ approaches 0, ae approaches 0. Since the sum of the distances from $F(ae,0)$ and $F'(ae,0)$ to any point on the ellipse is constant and equal to 2a, then <u>a</u> is constant. So, as ae approaches 0 and with <u>a</u> a constant, then e must approach 0. Now, when e is very close to 0, then $b = a\sqrt{1 - e^2}$ is very close to a. So, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes $x^2 + y^2 = a^2$, a circle with center, $C(0,0)$ and radius a.

(b) As $d(F,F')$ approaches 2a, ae approaches a. Hence,

e approaches 1. Consequently, $b = a\sqrt{1 - e^2}$ approaches 0. Since, $\frac{y^2}{b^2}$ has no meaning when $b = 0$, we look at $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in this form,

$$b^2x^2 + a^2y^2 = a^2b^2$$
$$0x^2 + a^2y^2 = a^2 \cdot 0$$
$$y^2 = 0$$
$$y = 0$$

Hence, the ellipse approaches a line segment of length 2a, its major axis.

7. Given,

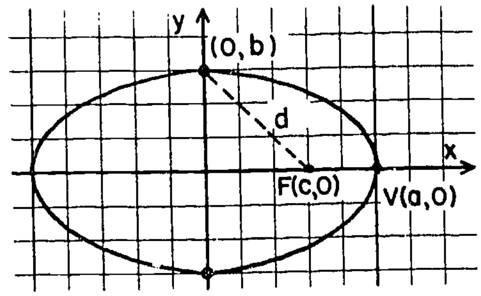
Ellipse, center: (0,0)

Length of $\frac{1}{2}$ minor axis is b

Length of $\frac{1}{2}$ major axis is a

To Prove: $d(P,F) = a$

Proof:



$$d(P,F) = \sqrt{(b-0)^2 + (0-c)^2}$$

$$= \sqrt{b^2 + c^2}$$

For the ellipse

$$b = a\sqrt{1 - e^2}$$

$$b^2 = a^2 - a^2e^2$$

since $c = ae$

$$b^2 = a^2 - c^2$$

or $a^2 = b^2 + c^2$

or $a = \sqrt{b^2 + c^2}$

Therefore $d(P,F) = a$

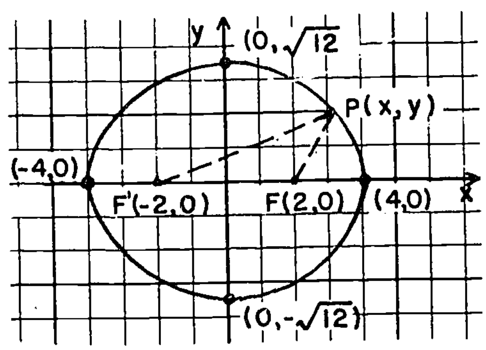
8. Given $\frac{x^2}{16} + \frac{y^2}{12} = 1$

Then, $y^2 = 12(1 - \frac{x^2}{16}) = \frac{192 - 12x^2}{16}$

By the distance formula,

(i) $d(P,F) = \sqrt{(x-2)^2 + y^2}$

(ii) $d(P,F') = \sqrt{(x+2)^2 + y^2}$



Since $P(x,y)$ must be on the ellipse whose equation is $\frac{x^2}{16} + \frac{y^2}{12} = 1$, replace y^2 in (i) and (ii) with $\frac{192 - 12x^2}{16}$,

$d(P,F) = \sqrt{(x-2)^2 + \frac{192 - 12x^2}{16}} = \sqrt{(4 - \frac{x}{2})^2} = \frac{8-x}{2}$, since

$|x| < 4, \sqrt{(4 - \frac{x}{2})^2} = 4 - \frac{x}{2}$ and not $\frac{x}{2} - 4$.

$d(P,F') = \sqrt{(x+2)^2 + \frac{192 - 12x^2}{16}} = \sqrt{(\frac{x}{2} + 4)^2} = \frac{x}{2} + 4$

$\therefore \; d(P,F) + d(P,F') = \frac{8-x}{2} + \frac{x+8}{2} = \frac{16}{2} = 8$ Q.E.D.

*9. Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $F(c,o)$; $F'(-c,0)$

Show that $d(P,F) + d(P,F') = 2a$

(i) $d(P,F) = \sqrt{(x-c)^2 + y^2}$

(ii) $d(P,F') = \sqrt{(x+c)^2 + y^2}$

From the equation of the ellipse, $y^2 = b^2(1 - \frac{x^2}{a}) = \frac{b^2}{a^2}(a^2 - x^2)$

Replacing $\frac{b^2}{a^2}(a^2 - x^2)$ for y^2 in (i) and (ii)

$$d(P,F) = \sqrt{(x-c)^2 + \frac{b^2}{a^2}(a^2 - x^2)}$$

$$d(P,F') = \sqrt{(x+c)^2 + \frac{b^2}{a^2}(a^2 - x^2)}$$

but $b = a\sqrt{1 - e^2}$

Hence,

$$d(P,F) = \sqrt{(x-c)^2 + \frac{a^2(1-e^2)(a^2-x^2)}{a^2}}$$

$$d(P,F') = \sqrt{(x+c)^2 + \frac{a^2(1-e^2)(a^2-x^2)}{a^2}}$$

$$d(P,F) = \sqrt{x^2 - 2cx + c^2 + a^2 - x^2 - a^2e^2 + e^2x^2}$$

$$d(P,F') = \sqrt{x^2 + 2cx + c^2 + a^2 - x^2 - a^2e^2 + e^2x^2}$$

But $c = ae$

$$d(P,F) = \sqrt{x^2 - 2aex + a^2e^2 + a^2 - x^2 - a^2e^2 + e^2x^2}$$

$$d(P,F') = \sqrt{x^2 + 2aex + a^2e^2 + a^2 - x^2 - a^2e^2 + e^2x^2}$$

$$d(P,F) = \sqrt{a^2 - 2aex + e^2x^2}$$

$$d(P,F') = \sqrt{a^2 + 2aex + e^2x^2}$$

Then $d(P,F) = (a - ex)$

$d(P,F') = a + ex$

$\therefore\ d(P,F) + d(P,F') = a - ex + a + ex = 2a.$ Q.E.D.

10. $ae = 1$ and $\frac{a}{e} = 4$

$\therefore\ e = \frac{1}{2}$

Use 6 - 4

$d(P,F) = e \cdot d(P,Q)$

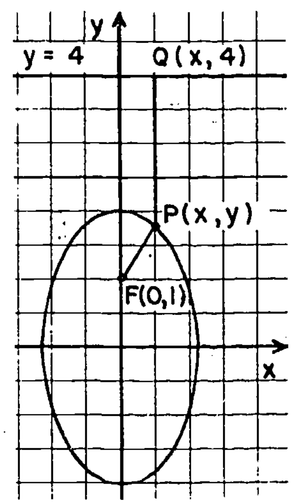
$$\sqrt{(x - 0)^2 + (y - 1)^2} = \frac{1}{2}\sqrt{(x - x)^2 + (y - 4)^2}$$

$$x^2 + y^2 - 2y + 1 = \frac{1}{4}(y^2 - 8y + 16)$$

$$4x^2 + 4y^2 - 8y + 4 = y^2 - 8y + 16$$

$$4x^2 + 3y^2 = 12$$

$$\frac{x^2}{3} + \frac{y^2}{4} = 1$$



11. Use a similar procedure to that used in Problem 9. The equation for these conditions is $\frac{x^2}{4} + \frac{y^2}{3} = 1$. This equation may obtained from that in Problem 9 by merely interchanging the coordinate axes.

12. Use 6 - 4. Let $a^2(1 - e^2) = b^2$

$d(P,F) = e \cdot d(P,Q)$

$$\sqrt{(x - 0)^2 + (y - ae)^2} = e\sqrt{(x - x)^2 + (y - \frac{a}{e})^2}$$

$$x^2 + y^2 - 2aey + a^2e^2 = e^2\ (y^2 - 2\frac{a}{e}y + \frac{a^2}{e^2})$$

$$x^2 + y^2 - 2aey + a^2e^2 = e^2y^2 - 2aey + a^2$$

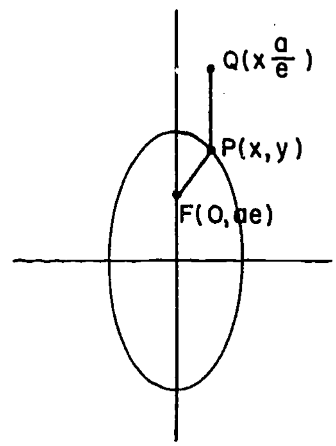
$$x^2 + (1 - e^2)y^2 = (1 - e^2)a^2$$

$$\frac{x^2}{(1 - e^2)a^2} + \frac{(1 - e^2)y^2}{(1 - e^2)a^2} = 1$$

since $1 - e^2 = \frac{b^2}{a^2}$

$$\frac{x^2}{\frac{b^2}{a^2} \cdot a^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



2a is still the major axis (the axis containing the focus) and 2b is called the minor axis.

*13. $d(P,F) = e \cdot d(P,Q)$

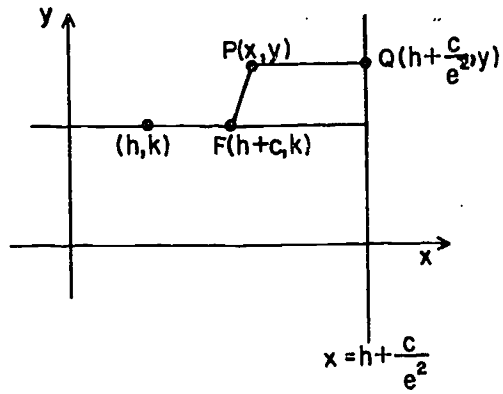
$$\sqrt{[x - (h+c)]^2 + (y-k)^2} = e \cdot \sqrt{[x - (h+\frac{c}{e^2})]^2 + (y-y)^2}$$

$$[x - (h+c)]^2 + (y-k)^2 = e^2[x-(h+\frac{c}{e^2})]^2$$

$$(x-h)^2(1-e^2) + (y-k)^2 = \frac{c^2}{e^2}(1-e^2)$$

Let $\frac{c^2}{e^2} = a^2$ and $b^2 = a^2(1-e^2)$

$$\therefore \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



A follow-up of this problem for the better student would be to suggest finding the equation of the ellipse if the coordinates of the focus are $(h,k+c)$ and the equation of the directrix is $y = k + \frac{c}{e^2}$.

14. (a) Find the coordinates of the midpoint of segment VV'.

$M(VV') = (1,2)$, the center of the ellipse.

$\frac{1}{2}$ major axis $= a = 5-1 = 4$

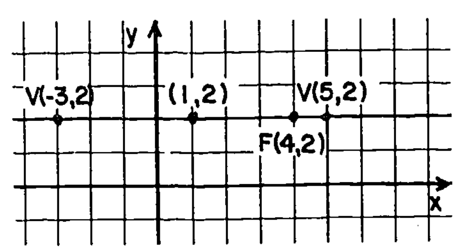
Length of segment from center to focus $= c = 4 - 1 = 3$

$b^2 = a^2 - c^2$

$\therefore b^2 = 4^2 - 3^2 = 7$

Hence, the equation,

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{7} = 1$$ Problem 12.



(b) Coordinates of center (4,1)

$c = 3 - 1 = 2$

$c = ae$

$2 = a\cdot\frac{1}{3}$; $a = 6$

$b^2 = 32$

$$\frac{(x - 4)^2}{32} + \frac{(y - 1)^2}{36} = 1$$

(c) Coordinates of center (-5,2)

$a = 3 - 2 = 1$

$c = ae = 1\cdot\frac{2}{3} = \frac{2}{3}$

$b^2 = \frac{5}{9}$

$$\frac{(x + 5)^2}{\frac{5}{9}} + \frac{(y - 2)^2}{1} = 1 \quad \text{or} \quad \frac{9(x+5)^2}{5} + \frac{(y - 2)^2}{1} = 1$$

(d) Coordinates of center (-2,-1)

$a = 5$ and $b = 3$

$$\frac{(x + 2)^2}{9} + \frac{(y + 1)^2}{25} = 1$$

(e) Coordinates of center $(-3,-\frac{1}{2})$

$c = 6$; $b = \frac{11}{2}$; $a^2 = \frac{265}{4}$

$$\frac{(x + 3)^2}{\frac{265}{4}} + \frac{(y + \frac{1}{2})^2}{\frac{121}{4}} = 1 \quad \text{or} \quad \frac{4(x + 3)^2}{265} + \frac{4(y + \frac{1}{2})}{121} = 1$$

(f) Coordinates of center (-5,-3)

$a = 7$; $c = 5$; $b^2 = 24$

$$\frac{(x + 5)^2}{49} + \frac{(y + 3)^2}{24} = 1$$

*(g) $a = 3 \quad \frac{c}{e^2} = 7 \quad c = 3e \quad h = 0 \quad k = 2$

$x = h + \frac{c}{e^2}$

$\frac{3e}{e^2} = 7$

$e = \frac{3}{7}$

$c = 3e = \frac{9}{7}$

$b^2 = a^2 - c^2 = 9 - \frac{81}{49}$

$b = \frac{6\sqrt{10}}{7}$

Equation $= \frac{x^2}{9} + \frac{(y-2)^2}{\frac{360}{49}} = 1$

*(h) Given $h = 3$ assume $k = 0$ [k can be any real number < 4].

Then $c = 4 \quad \frac{c}{e^2} = 5$

$5 = \frac{4}{e^2}$

$e^2 = \frac{4}{5}$

$e = \frac{2\sqrt{5}}{5}$

$c = ae$

$4 = a \cdot \frac{2\sqrt{5}}{5}$

$a = \frac{20}{2\sqrt{5}} = 2\sqrt{5}$

$b^2 = a^2 - c^2$

$b^2 = 20 - 16 = 4$

$b = 2$

$\frac{(x-3)^2}{20} + \frac{y^2}{4} = 1$ This gives a family of ellipses with every assignment of k.

[page 340]

*(i) I. Assume axis || the x-axis and $h = 0$, $k = 2$ then

$c = 5 \quad e = \frac{1}{5}$

$c = ae$

$5 = a \cdot \frac{1}{5}$

$a = 25$

$b^2 = a^2 - c^2 = 625 - 25$

$b^2 = 600$

$$\frac{x^2}{625} + \frac{(y - 2)^2}{600} = 1$$

This gives a family of ellipses according to the value of h.

II. Assume axis || the y-axis and $k = 0, h = -5$

$c = 2 \quad e = \frac{1}{5}$

$c = ae$

$2 = a \cdot \frac{1}{5}$

$a = 10$

$b^2 = a^2 - c^2 = 100 - 4 = 96$

$$\frac{(x + 5)^2}{100} + \frac{y^2}{96} = 1$$

This gives a family of ellipses according to the value of k.

15. (a) Coordinates of the center (h,k) are $(3,5)$; $a = 5$; $b = 3$;

$c^2 = a^2 - b^2 = 25 - 9 = 16$

$c = 4$

$c = ae$; $4 = 5e$; $e = \frac{4}{5}$, <u>eccentricity</u>

$x = h + \frac{c}{e^2} = 3 + \frac{4}{(\frac{4}{5})^2} = \frac{37}{4}$, <u>directrix</u>

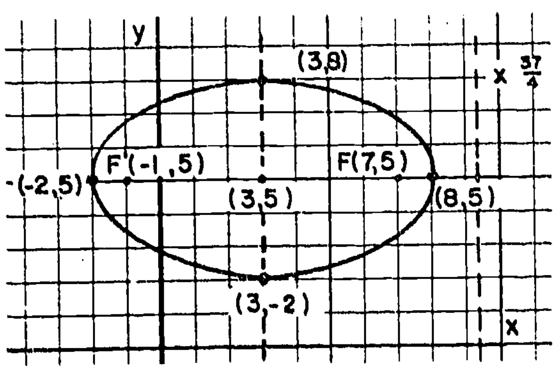
$F(h + c,k) = (3 + 4,5) = (7,5)$, <u>coordinates of the focus</u>

$V_x(h + a,k)$ and $(h - a,k) = (8,5)$ and $(-2,5)$ <u>coordinates of vertices</u>.

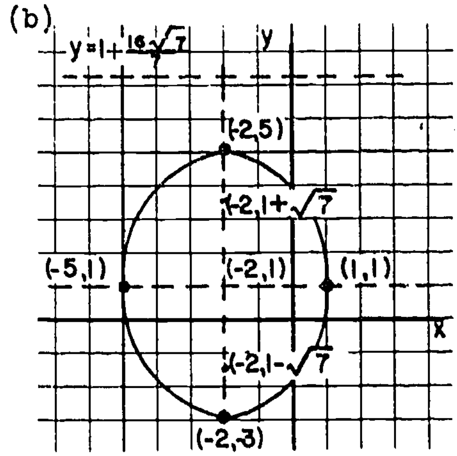
For problems beginning with (c), change each to the form, $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. The results are in the table.

	Coordinates of Vertices	Coordinates of Focus	Equation of Directrix	Eccentricity
(a)	(8,5)(-2,5)	(7,5)	$x = \frac{37}{4}$	5
(b)	(-2,5)(-2,-3)	$(-2, 1+\sqrt{7})$	$y = 1 + \frac{16\sqrt{7}}{7}$	$\frac{\sqrt{7}}{4}$
(c)	(1,0)(-7,0)	$(-3+2\sqrt{3}, 0)$	$x = -3 + \frac{8\sqrt{3}}{3}$	$\frac{\sqrt{3}}{2}$
(d)	(3,0)(3,-8)	$(3, -4+\sqrt{7})$	$y = -4 + \frac{16\sqrt{7}}{7}$	$\frac{\sqrt{7}}{4}$
(e)	(2,2)(-4,2)	$(-1+\sqrt{5}, 2)$	$x = -1 + \frac{9\sqrt{5}}{5}$	$\frac{\sqrt{5}}{3}$

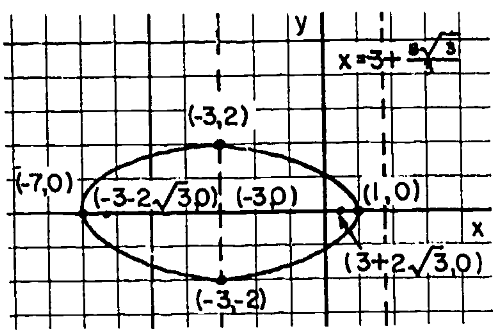
(a)



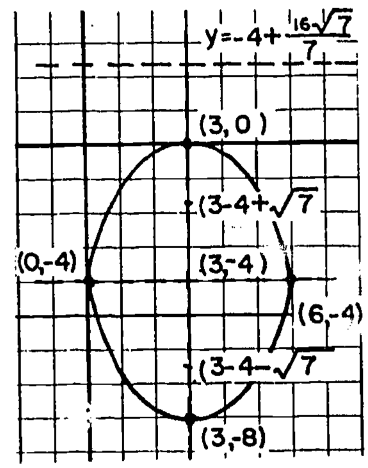
(b)



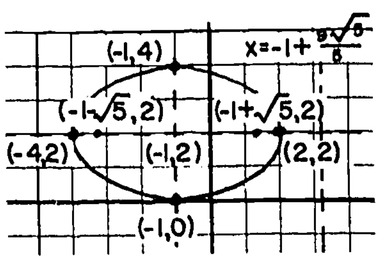
(c)



(d)



(e)



16. Foci: (0,4000), (0,-4000)

Center: (0,0)

$c = 4000, \quad b = 4200.$

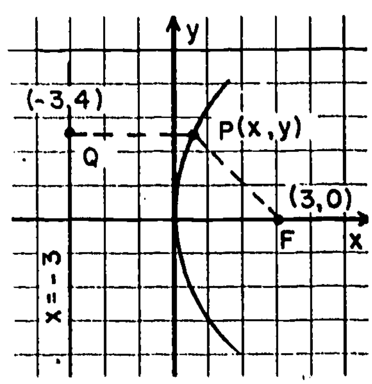
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$a^2 = (4200)^2 + (4000)^2 = 3364 \cdot 100^2$$

$$\frac{x^2}{1764} + \frac{y^2}{3364} = 10000$$

When $x = 0$, $y = 100\sqrt{3364} \approx 5800$

$\therefore$ satellite will be 1800 miles above North Pole.



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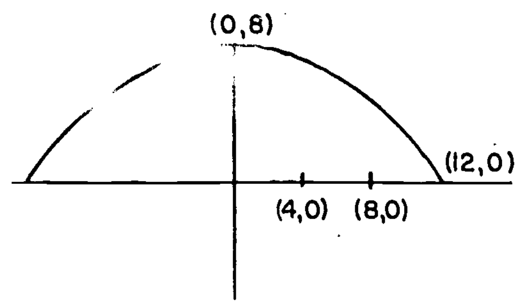
17. Equation of the curve is

$$\frac{x^2}{144} + \frac{y^2}{64} = 1$$

at $x = 8$, $\frac{y^2}{64} = 1 - \frac{8^2}{144}$; $y = \frac{8}{3}\sqrt{5}$

at $x = 4$, $\frac{y^2}{64} = 1 - \frac{4^2}{144}$; $y = \frac{16}{3}\sqrt{5}$

at $x = 0$, $y = 8$



*18. Let $(\overline{x}, \overline{y})$ be coordinates of one of the vertices of the square. Since ellipse is symmetric with respect to both axes and the origin, then $2\overline{x} = 2\overline{y}$ = length of side of the square. Hence, $\overline{x}^2 = \overline{y}^2$ and from the equation of the ellipse $5\overline{x}^2 = 80$ or $\overline{x} = \pm 4 = \overline{y}$. Required coordinates are $(\pm 4, \pm 4)$.

6-6. The Hyperbola.

In this section we develop the details of the graph of the hyperbola in much the same way that we did in the previous section for the ellipse. The new complication here is asymptotes. Students should be encouraged to use the asymptotes to get a quick accurate sketch of the curve. The proof that the diagonals of the rectangle mentioned in this section are actually asymptotes is hinted at but not carried out in detail. The proof involves limits and is best discussed informally at this point. The relation between the constants in this case are

$$\boxed{a^2 + b^2 = c^2} \quad \boxed{c = ae} \quad \text{and} \quad \boxed{b = a\sqrt{e^2 - 1}}$$

Again the same sort of discussion about the notation for the constants a and b is relevant. The notation was devised in the beginning so that a is the x-intercept. However, if we allow the focus to be on the y-axis, we still let 2a be the length of the transverse axis (the segment joining the vertices) in spite of the fact that this forces us to write the equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ and a is no longer the x-intercept (in fact the curve doesn't have one). The equation for the more general position for the hyperbola [illegible] to the exercises.

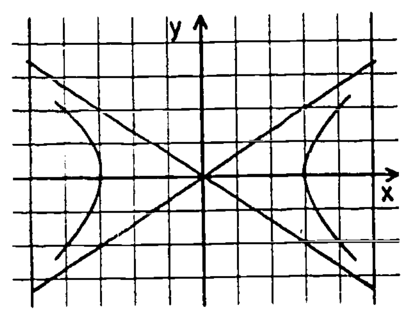
When the chapter is completed, it is to be hoped that the student can quickly identify a conic by looking at its equation (provided the xy term is missing) and that he can also draw a good sketch of any of these curves quickly.

<u>Exercises</u> <u>6-6</u>. <u>Answers</u>

1. (a) $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$

   (b) (3,0) (-3,0)

   (c)

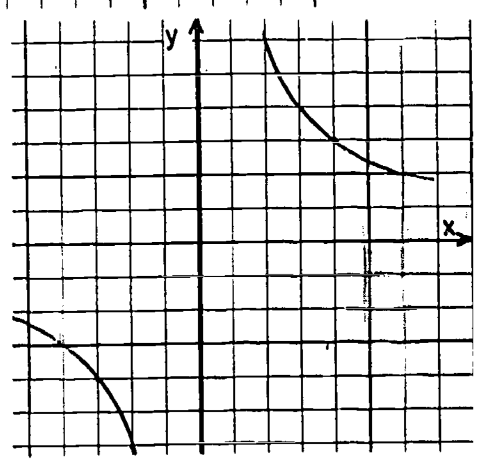


2. $3xy = 36$

   (a) $x = 0$ $y = 0$

   (b) $(2\sqrt{3}, 2\sqrt{3})(-2\sqrt{3}, -2\sqrt{3})$

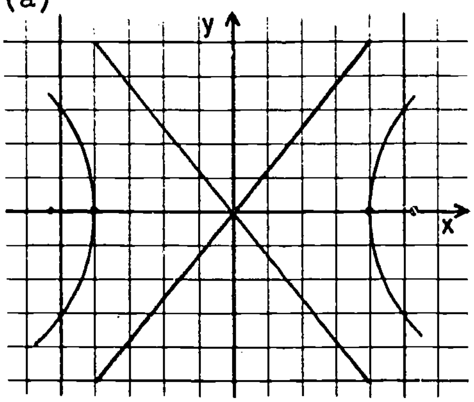
   (c)



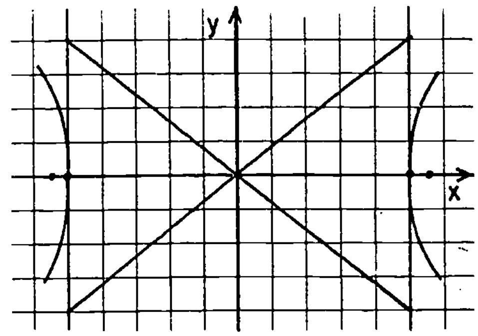
3.

	Coordinates Of		Equation Of		Eccentricity
	Vertices	Focus	Directrix	Asymptotes	
(a)	$(4,0);(-4,0)$	$(\sqrt{41},0)$	$x = \frac{16\sqrt{41}}{41}$	$y = \frac{5}{4}x, y = -\frac{5}{4}x$	$\frac{\sqrt{41}}{4}$
(b)	$(5,0);(-5,0)$	$(\sqrt{41},0)$	$x = \frac{25\sqrt{41}}{41}$	$y = \frac{4}{5}x, y = -\frac{5}{4}x$	$\frac{\sqrt{41}}{5}$
(c)	$(6,0);(-6,0)$	$(6\sqrt{2},0)$	$x = 3\sqrt{2}$	$y = x, y = -x$	$\sqrt{2}$
(d)	$(0,6);(0,-6)$	$(0,6\sqrt{2})$	$y = 3\sqrt{2}$	$x = y, x = -y$	$\sqrt{2}$
(e)	$(0,3);(0,-3)$	$(0,\sqrt{21})$	$y = \frac{3\sqrt{21}}{7}$	$z = \frac{\sqrt{3}}{2}x$ $z = -\frac{\sqrt{3}}{2}y$	$\frac{\sqrt{21}}{3}$
(f)	$(2\sqrt{3},0);$ $(-2\sqrt{3},0)$	$(\sqrt{21},0)$	$x = \frac{4\sqrt{21}}{7}$	$y = \frac{\sqrt{3}x}{2}$, $y = \frac{-\sqrt{3}x}{2}$	$\frac{\sqrt{7}}{2}$

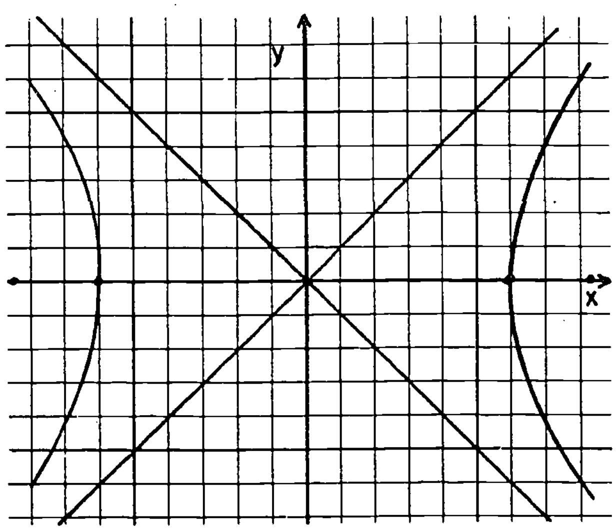
(a)



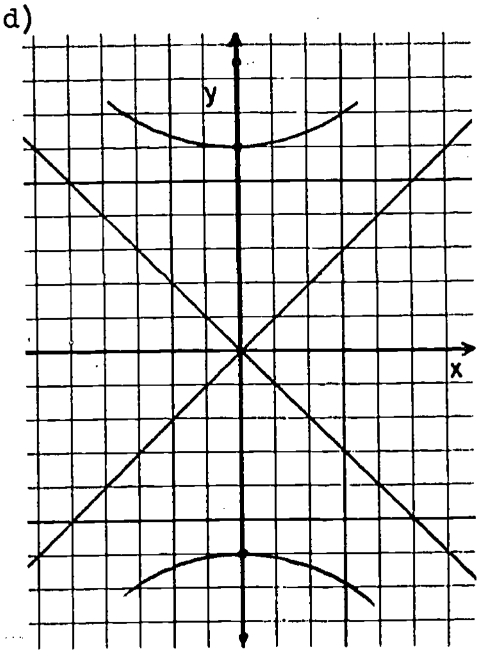
(b)



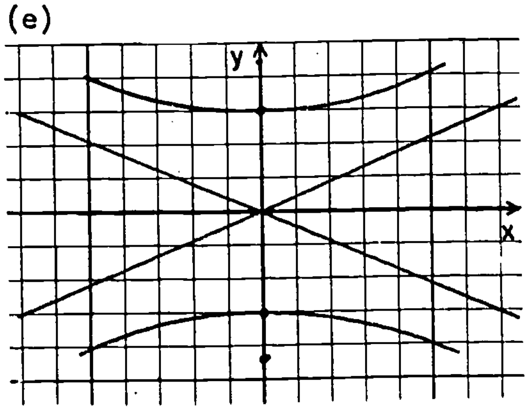
(c)



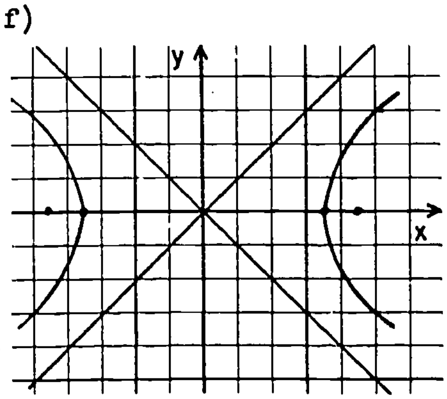
(d)



(e)



(f)



4. Given the hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$

Where $a = 1$, $b = 2$; $b^2 = a^2(e^2 - 1)$

$$4 = 1\,(e^2 - 1)$$

$$5 = e^2$$

$F(\sqrt{5},0)$ and $F'(-\sqrt{5},0)$ $e = \sqrt{5}$

$$|d(F',P) - d(F,P)| = \left|\sqrt{(x+\sqrt{5})^2 + y^2} - \sqrt{(x-\sqrt{5})^2 + y^2}\right|$$

(Since $y^2 = 4(x^2 - 1)$)

$$= \left|\sqrt{x^2 + 2\sqrt{5}x + 5 + 4x^2 - 4} - \sqrt{x^2 - 2\sqrt{5}x + 5 + 4x^2 - 4}\right|$$

$$= \left|\sqrt{5x^2 + 2\sqrt{5}x + 1} - \sqrt{5x^2 - 2\sqrt{5}x + 1}\right|$$

Note:

(Since x in Quadrant I is greater than 1

$\sqrt{5x^2 - 2\sqrt{5}x + 1} = |\sqrt{5}x - 1| = \sqrt{5}x - 1$)

$$= |(\sqrt{5}x + 1) - (\sqrt{5}x - 1)|$$

$$= |2|$$

$$= 2$$

5. $\left|d(F',P) - d(F,P)\right| = \left|\sqrt{(x+ae)^2 + y^2} - \sqrt{(x-ae)^2 + y^2}\right|$

Since $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$

$$= \left|\sqrt{(x+ae)^2 + \frac{b^2}{a^2}x^2 - b^2} - \sqrt{(x-ae)^2 + \frac{b^2}{a^2}x^2 - b^2}\right|$$

$$= \left|\sqrt{\frac{a^2+b^2}{a^2}x^2 + 2aex + a^2e^2 - b^2} - \sqrt{\frac{a^2+b^2}{a^2}x^2 - 2aex + a^2e^2 - b^2}\right|$$

$$= \left|\sqrt{e^2x^2 + 2aex + a^2} - \sqrt{e^2x^2 - 2aex + a^2}\right|$$

$$= \left|\sqrt{(ex+a)^2} - \sqrt{(ex-a)^2}\right|$$

$\left(\sqrt{(ex+a)^2} = ex - a \quad \text{Since } |x| > a \text{ and } e > 1\right)$

$$= \left|(ex+a) - (ex-a)\right|$$

$$= |2a|$$

$$= 2a$$

6. (a) $\left|d(P,F) - d(P,F')\right| = 2a$

$$2a = 6$$

$$a = 3$$

$$a^2 = 9$$

$c = 4$

$c = ae$

$4 = 3e$

$\frac{4}{3} = e$

$b^2 = a^2(e^2 - 1)$

$b^2 = 9\left(\frac{16}{9} - 1\right)$

$= 9\left(\frac{7}{9}\right)$

$b^2 = 7$

$\therefore \frac{x^2}{9} - \frac{y^2}{7} = 1$

(b) $\frac{y^2}{9} - \frac{x^2}{7} = 1$

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7. $d(F,P) = 2d(P,Q)$

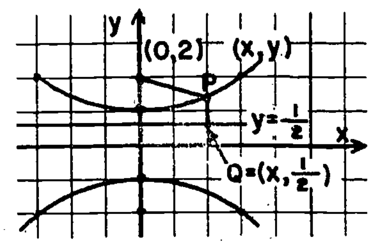
$$\sqrt{(x)^2 + (y - 2)^2} = 2\sqrt{(x - x)^2 + (y - \tfrac{1}{2})^2}$$

$$x^2 + y^2 - 4y + 4 = 4(y^2 - y + \tfrac{1}{4})$$

$$x^2 + y^2 - 4y + 4 = 4y^2 - 4y + 1$$

$$x^2 - 3y^2 = -3$$

$$\frac{y^2}{1} - \frac{x^2}{3} = 1$$



8. $d(F,P) = ed(P,Q)$

$$\sqrt{(x - 0)^2 + (y - ae)^2} = e\sqrt{(x - x)^2 + (y - \tfrac{a}{e})^2}$$

$$x^2 + y^2 - 2aey + a^2e^2 = e^2(y^2 - 2\tfrac{a}{e}y + \tfrac{a^2}{e^2})$$

$$x^2 + y^2 - 2aey + a^2e^2 = e^2y^2 - 2aey + a^2$$

$$x^2 - e^2y^2 + y^2 = -a^2e^2 + a^2$$

$$a^2e^2 - a^2 = y^2(e^2 - 1) - x^2$$

$$a^2(e^2 - 1) = y^2(e^2 - 1) - x^2$$

$$1 = \frac{y^2}{a^2} - \frac{x^2}{a^2(e^2 - 1)}$$

let $b^2 = a^2(e^2 - 1)$

$$1 = \frac{y^2}{a^2} - \frac{x^2}{b^2}$$

9. (a) $F(\pm 8,0)$; $c = 8$; vertices $(\pm 5,0)$; $a = 5$

$c = ae$ $\quad$ $b^2 = a^2(e^2 - 1)$

$8 = 5e$ $\quad$ $b^2 = 25(\frac{64}{25} - 1)$

$\frac{8}{5} = e$ $\quad$ $b^2 = 25(\frac{39}{25})$

$b^2 = 39$

$$\frac{x^2}{25} - \frac{y^2}{39} = 1$$

(b) vertices $(\pm 3,0)$ $a = 3$;

distance between foci equal to 8 $\quad c = 4$

$c = ae$ $\quad$ $b^2 = a^2(e^2 - 1)$

$4 = 3e$ $\quad$ $b^2 = 9\ (\frac{16}{9} - 1)$

$\frac{4}{3} = e$ $\quad$ $b^2 = 9(\frac{7}{9})$

$b^2 = 7$

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

(c) vertices $(\pm 3,0)$ $\quad e = 2$

$a = 3$

$b^2 = a^2(e^2 - 1)$

$b^2 = 9(4 - 1)$

$b^2 = 27$ $\qquad$ $\frac{x^2}{9} - \frac{y^2}{27} = 1$

(d) directrix $x = 2$ vertex $(4,0)$

$\frac{c}{e^2} = 2$ $\quad$ $a = 4$

$c = ae$

$\frac{4}{e} = 2$ $\quad$ $c = 4e$ $\quad$ $b^2 = a^2(e^2 - 1)$

$c = 4 \cdot 2$ $\quad$ $b^2 = 16(4 - 1)$

$2 = e$ $\quad$ $c = 8$ $\quad$ $b^2 = 48$

$$\frac{x^2}{16} - \frac{y^2}{48} = 1$$

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(e) $F(7,0) \quad e = \frac{4}{3}$

$c = 7$

$c = ae$

$7 = \frac{4}{3}a$

$a = \frac{21}{4}$

$b^2 = a^2(e^2 - 1)$

$b^2 = \frac{441}{16}(\frac{16}{9} - \frac{9}{9})$

$b^2 = \frac{441}{16}(\frac{7}{9})$

$b^2 = \frac{343}{16}$

$$\frac{x^2}{\frac{441}{16}} - \frac{y^2}{\frac{343}{16}} = 1$$

(f) $y = \pm 3x$ vertex $(2,0)$

$y = \pm\frac{b}{a}x$ $a = 2$

$\frac{b}{a} = 3$

$\frac{b}{2} = 3$

$b = 6$

$$\frac{x^2}{4} - \frac{y^2}{36} = 1$$

(g) $3x + 2y = 0$ and $3x - 2y = 0$ $F = (0,3)$

Since the asymptotes are given by $y = \pm \frac{b}{a}x$,

we have $\frac{b}{a} = \pm \frac{3}{2}$

$b = \pm \frac{3}{2}a$

$c^2 = a^2 + b^2$

$9 = a^2 + \frac{9}{4}a^2$

Hence $a = \frac{6}{\sqrt{13}}$ and $b = \frac{9}{\sqrt{12}}$ and the required equation is

$$\frac{y^2}{\frac{81}{13}} - \frac{x^2}{\frac{36}{13}} = 1$$

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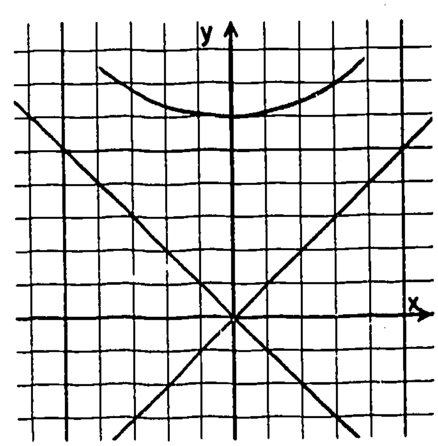
10.

	Equation of Asymptotes	Coordinates of Vertices
(a)	$y = \pm \frac{3}{2} x$	$(\pm 2,0)$
(b)	$y = \pm\ 2x$	$(\pm 1,0)$
(c)	$y = \pm\ \frac{2}{3}x$	$(\pm 3,0)$
(d)	$y = \pm \frac{1}{2} x$	$(\pm 2,0)$
(e)	$y = \pm\ 3x$	$(\pm 1,0)$
(f)	coordinate axes, $x = 0$, $y = 0$	$(2,2)$, $(-2,-2)$
(g)	$y = \pm\ 3x$	$(0,\pm 3)$
(h)	$x = 0$, $y = 0$	$(1,1)$, $(-1,-1)$
(i)	$x = 0$, $y = 0$	$(-2,2)$, $(2,-2)$
(j)	$y = \frac{5}{2}x$, $y = -\frac{5}{2}x + 10$	$(4,5)$, $(0,5)$

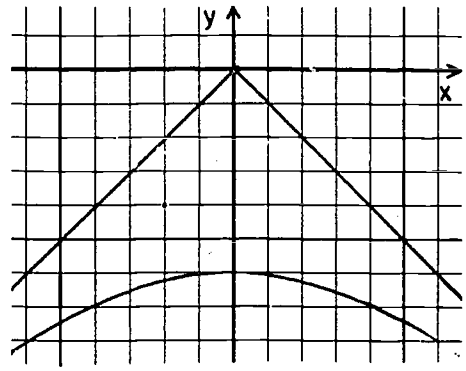
11. (a) $y = \sqrt{36 + x^2}$

y can not be less than 0.



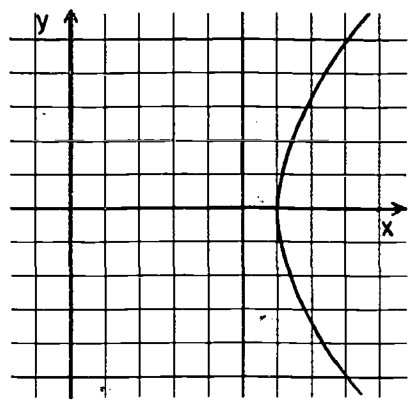
(b) $y = -\sqrt{36 - x}$

y cannot be greater than 0.



(c) $x = \sqrt{36 + y^2}$

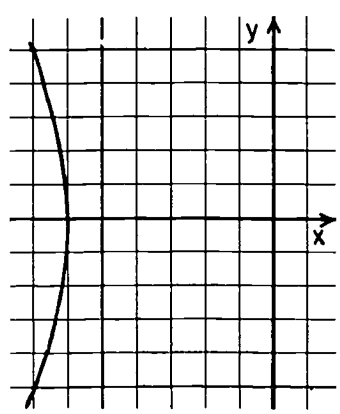
x cannot be less than 0.



(d) $x = -\sqrt{36 + y^2}$

x cannot be greater than 0.

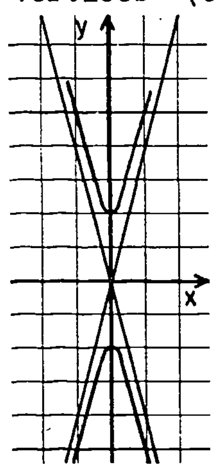
See Problem 7, Exercises 6-3 Answers



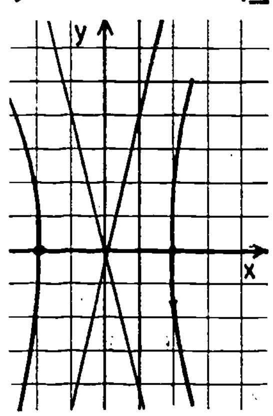
12. Since asymptotes given by $3x - 5y = 0$ and $3x + 5y = 0$, then they are also given by $(3x - 5y)(3x + 5y) = 0$, or $9x^2 - 25y^2 = 0$. Reversing the procedure used to find the asymptote equations gives the hyperbola equation as $9x^2 - 25y^2 = k$, where k is the right member constant. But the hyperbola passes through the point of $(2,3)$ so by use of the locus property $9(2)^2 - 25(3)^2 = k$ and k is found to be -189. Hence, the required equation is $9x^2 - 25y^2 = -189$.

13. $9x^2 - 25y^2 = 200$.

14. $16x^2 - y^2 = -4$. [Sketch; hyperbola with center $(0,0)$, asymptotes $y = \pm 4x$, vertices $(0, \pm 2)$].



15. $16x^2 - y^2 = 64$. [Sketch; hyperbola with center $(0,0)$, asymptotes $y = \pm 4x$, vertices $(\pm 2,0)$].



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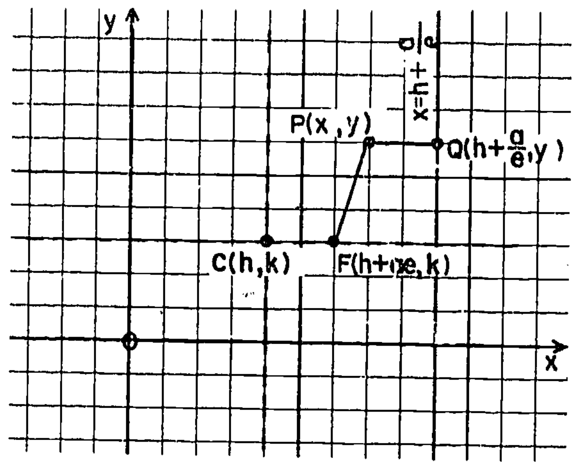
*16.



$$d(F,P) = ed(P,Q)$$

$$\sqrt{\left(x - (h + ae)\right)^2 + (y - k)^2} = e\sqrt{\left(x - (h + \frac{a}{e})\right)^2 + (y - y)^2}$$

$$\left((x - h) - ae\right)^2 + (y - k)^2 = e^2 \left((x - h) + \frac{a}{e}\right)^2$$

$$(x-h)^2 - 2ae(x-h) + a^2e^2 + (y-k)^2 = e^2 \left((x-h)^2 - 2\frac{a}{e}(x-h) + \frac{a^2}{e^2}\right)$$

$$(x-h)^2 - 2ae(x-h) + a^2e^2 + (y-k)^2 = e^2(x-h)^2 - 2ae(x-h) + a^2$$

$$a^2e^2 - a^2 = e^2(x-h)^2 - (x-h)^2 - (y-k)^2$$

$$a^2(e^2-1) = (x-h)^2(e^2 - 1) - (y-k)^2$$

$$1 = \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{a^2(e^2-1)}$$

$$b^2 = a^2(e^2 - 1)$$

$$1 = \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2}$$

17. (a) $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$

$\frac{(x - \frac{3}{2})^2}{\frac{9}{4}} - \frac{(y - 1)^2}{4} = 1$

(b) $\frac{(x - 1)^2}{4} - \frac{(y + 3)^2}{5} = 1$

(c) $\frac{(x - \frac{3}{2})^2}{\frac{196}{25}} - \frac{(y - 7)^2}{\frac{441}{100}} = 1$

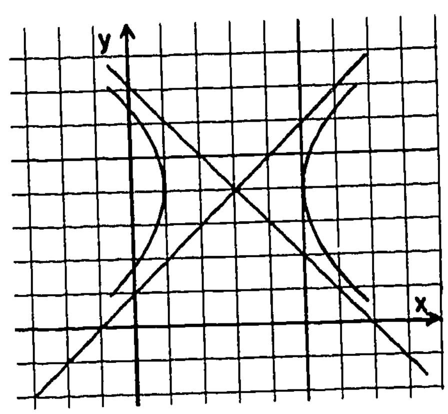
(d) $\frac{(x + \frac{3}{2})^2}{\frac{49}{4}} - \frac{(y - 7)^2}{\frac{245}{16}} = 1$

18. $h = 3 \quad k = 4 \quad ae = 2\sqrt{2} \quad \frac{a}{e} = \sqrt{2}$

$ae = \sqrt{2}e^2 = 2\sqrt{2} \quad \therefore \quad e = \sqrt{2}$ and $a = 2$

$b = a\sqrt{e^2 - 1} = 2\sqrt{2 - 1} = 2 \cdot 1 = 2$

$\frac{(x - 3)^2}{4} - \frac{(y - 4)^2}{4} = 1$



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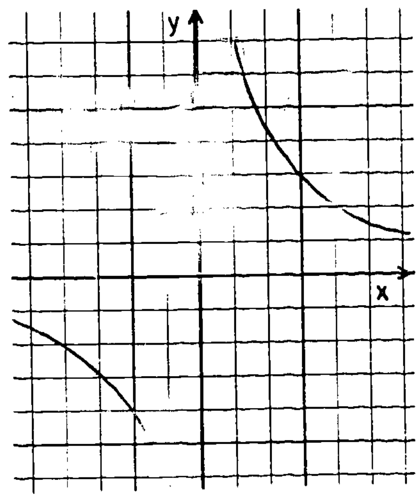
19.

	Center	Vertices	Focus	Directrix	Asymptotes	E
(a)	$(-2,5)$	$(1,5)(-5,5)$	$(-2+\sqrt{13},5)$	$x=-2+\frac{9}{\sqrt{13}}$	$3(y-5)=\pm 2(x+2)$	[illegible]
(b)	$(1,4)$	$(1,0)(1,8)$	$(1,4+\sqrt{17})$	$y=4+\frac{16}{\sqrt{17}}$	$y-4=\pm 4(x-1)$	[illegible]
(c)	$(0,0)$	$(0,4)(0,-4)$	$(0,\sqrt{41})$	$y=\frac{16}{\sqrt{41}}$	$y=\pm\frac{4}{5}x$	[illegible]
(d)	$(3,-2)$	$(3+\sqrt{6},-2)(3-\sqrt{6},-2)$	$(3+\sqrt{15},-2)$	$x=3+\frac{2\sqrt{15}}{5}$	$\sqrt{6}(y+2)=\pm 3(x-3)$	[illegible]
(e)	$(2,3)$	$(3,3)(1,3)$	$(2+\sqrt{2},3)$	$x=2+\frac{\sqrt{2}}{2}$	$y-3=\pm(x-2)$	[illegible]
(f)	$(1,-3)$	$(4,-3)(-2,-3)$	$(1+3\sqrt{2},-3)$	$x=1+\frac{3}{2}\sqrt{2}$	$y+3=\pm(x-1)$	[illegible]
(g)	$(4,-3)$	$(0,-3)(8,-3)$	$(9,-3)$	$x=\frac{36}{5}$	$4(y+3)=\pm 3(x-4)$	[illegible]
(h)	$(2,-\frac{3}{2})$	$(6,-\frac{3}{2})(-2,-\frac{3}{2})$	$(2+2\sqrt{5},-\frac{3}{2})$	$x=2+\frac{8\sqrt{5}}{5}$	$y=\pm\frac{1}{2}(x-2)-\frac{3}{2}$	[illegible]
(i)	$(-\frac{1}{2},1)$	$(\frac{5}{2},1)(-\frac{7}{2},1)$	$(-\frac{1}{2}+\sqrt{13},1)$	$x=-\frac{1}{2}+\frac{9}{13}\sqrt{13}$	$3(y-1)=\pm 2(x+\frac{1}{2})$	[illegible]
(j)	Degenerate conic consisting of two lines $2x-5y=-13$ and $2x+5y=-3$					

20. (a) $xy = k$; $y = $ [illegible] [illegible] $= \frac{k}{2}$ so $k = 8$

(b) $xy = 8$ $5y = $ [illegible] [illegible] $= \frac{8}{5}$



21. $IR = k$

$20 \cdot 15 = k$ $I(50) = 300$

$\therefore K = 300$ $I = 6$

22. $3 = \frac{K}{25}$ $y = \frac{75}{4}$

$K = 75$

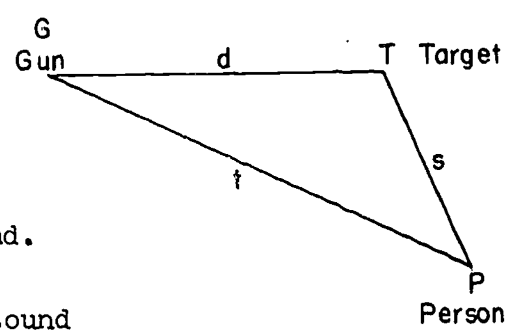
23. $MS^2 = K$ $M(4,200)^2 = 800,000,000$

$50(4000)^2 = K$ $M = \frac{800,000,000}{17,640,000}$

$50(16,000,000) = K$ $M = 45.35$

$800,000,000 = K$

*24. Suppose that the distance from the gun to the target is $\underline{d}$ feet. The speed of the bullet is $\underline{a}$ feet per second and the speed of sound is $\underline{b}$ feet per second. P is the point where the report of the gun and the sound of the bullet hitting the target are heard simultaneously.



Then $\frac{d}{a} + \frac{s}{b} = \frac{t}{b}$

$$\frac{d}{a} = \frac{t}{b} - \frac{s}{b} = \frac{t - s}{b}$$

Since $\frac{d}{a}$ is constant, the[illegible] $\frac{t-s}{b}$ [illegible] constant. Hence, t [illegible] is constant. The person wi[illegible] [illegible] on one branch of an hyperbola whose foci are the [illegible] nd the gun.

Exercises 6-7 Supplementary Exe[illegible] - Answers

1. (a) $3y - 2x - 15 = 0$ [illegible] $\frac{2}{3}$ y intercept + 5
$3y = 2x + 15$ x intercept = $-\frac{15}{2}$
$y = \frac{2}{3}x + 5$
$3 \cdot 0 - 2x - 15 = 0$
$-2x + 15 = 0$
$x = -\frac{15}{2}$

x-intercept = value of x when $y = 0 = -\frac{\text{y-intercept}}{m}$

	m	y-intercept	x-intercept
(b)	$-\frac{3}{2}$	10	$\frac{20}{3}$
(c)	$+\frac{3}{2}$	4	$-\frac{8}{3}$
(d)	not defined	every real number	0
(e)	$\frac{1}{3}$	-2	6
(f)	0	5	none
(g)	$-\frac{6}{5}$	3	$\frac{5}{2}$
(h)	$\frac{8}{21}$	$\frac{4}{3}$	$-\frac{7}{2}$

2. $y = mx + b$ $(6,0)$ and $(0, -\frac{5}{2})$

$0 = m \cdot 6 + b$

$-\frac{5}{2} = 0 \cdot m + b$

$0 = 6m - \frac{5}{2}$

$m = \frac{5}{12}$

$y = \frac{5}{12}x - \frac{5}{2}$

3. $y = mx + b$ $(-1,-5)$ and $(\frac{2}{3}, -5)$

Since ordinate is -5 for both points, line is $y = -5$.

4. (a) $y = 3x + 2$

(b) $y = -x + 4$

(c) $y = \frac{3}{2}x - 3$

5.

	(a) slope	(b) y-intercept	(c) direction
1.	+ 1	+ 6	rise
2.	+ 3	0	rise
3.	$+\frac{2}{3}$	- 4	rise
4.	$+\frac{3}{4}$	0	rise
5.	$-\frac{5}{6}$	+ 2	sink

(d) Steepest line is # 2

(e) Slope of 1 is + 1   Intercept of 3 is - 4

$y = x - 4$

6. $y = 3x + b$

$3 = 3(-2) + b$

$b = 9$

$y = 3x + 9$

[pages 353-354]

7. $y = -\frac{2}{5}x + b$

$-2 = -\frac{2}{5} \cdot 3 + b$

$b = -2 + \frac{6}{5} = -\frac{4}{5}$

$y = -\frac{2}{5}x - \frac{4}{5}$

8. A(5,10) B(10,-7) C(-5,-5)

AB $10 = 5m + b$ $\qquad y = -\frac{17}{5}x + 27$

$\underline{-7 = 10m + b}$

$17 = -5m$

$m = -\frac{17}{5}$

$10 = 5(-\frac{17}{5}) + b$

$b = 27$

AC $10 = 5m + b$ $\qquad y = \frac{3}{2}x + \frac{5}{2}$

$\underline{-5 = -5m + b}$

$5 = 2b$

$b = \frac{5}{2}$

$10 = 5m + \frac{5}{2}$

$5m = \frac{15}{2}$

$m = \frac{3}{2}$

BC $-7 = 10m + b$ $\qquad y = -\frac{2}{15}x - \frac{17}{3}$

$\underline{-5 = -5m + b}$

$-2 = 15m$

$m = -\frac{2}{15}$

$-7 = 10(-\frac{2}{15}) + b$

$b = -\frac{17}{3}$

9. (a) $2x - 3y = 5$ or $y = \frac{2}{3}x - \frac{5}{3}$

$3x + 2y - 4 = 0$ $\quad y = -\frac{3}{2}x + 2$

Since $m_1 \cdot m_2 = \frac{2}{3}(-\frac{3}{2}) = -1$ these are perpendicular.

(b) Neither $\perp$ nor $||$.

(c) Perpendicular

(d) Parallel

(e) Parallel

(f) Neither

(g) Neither

(h) Neither

10. Given

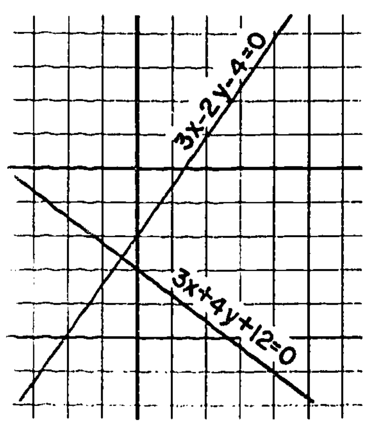
(i) $3x - 2y - 4 = 0$ $\quad y = \frac{3}{2}x - 2$

(ii) $3x + 4y + 12 = 0$

(a)

	y-intercept	slope
(i)	-2	$+\frac{3}{2}$
(ii)	-3	$-\frac{3}{4}$

10(b).



(c) $m_i = +\frac{3}{2}$

$m_{ii} = -\frac{3}{4}$

$m_i m_{ii} = (\frac{3}{2})(-\frac{3}{4}) \neq -1$

Lines not perpendicular.

11. $y =$ [illegible] x

12. $m = \frac{2}{3}$

$-1 = 2(\frac{2}{3}) + b$

$b = -\frac{7}{3}$

$y = \frac{2}{3}x - \frac{7}{3}$

13. $y = mx + 4$

$1 = -5m + 4$

$5m = -1 + 4$

$m = \frac{3}{5}$

$y = \frac{3}{5}x + 4$

14. x-intercept = 3

slope $= \frac{1}{3}$

$\frac{\text{y-intercept}}{-m} = \text{x-intercept}$

$\frac{b}{-\frac{1}{3}} = 3$

$b = -1$

$y = \frac{1}{3}x - 1$

15. $S = 2^{\frac{t_2 - t_1}{10}}$

$S = 2^{\frac{100-20}{10}} = 2^8 = 256$ times as great at 100° or at 20°

16. (a) Given $A = k_1C$; $B = k_2C$

$A + B = k_1C + k_2C$

$A + B = C(k_1 + k_2)$

(b) $A - B = k_1C - k_2C$

$A - B = C(k_1 - k_2)$

[pages 355-356]

(c) $\sqrt{AB} = \sqrt{k_1C \cdot k_2C}$

$\sqrt{AB} = \sqrt{C^2 k_1 k_2}$

$\sqrt{AB} = C\sqrt{k_1 k_2}$

17. (a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$; $16x^2 + 25y^2 = 400$ Ellipse

(b) $\frac{x^2}{16} + \frac{y^2}{25} = 1$; $25x^2 + 16y^2 = 400$ Ellipse

(c) $8x^2 - y^2 = 8$; $\frac{x^2}{1} - \frac{y^2}{8} = 1$ Hyperbola

(d) $8y^2 - x^2 = 8$; $\frac{y^2}{1} - \frac{x^2}{8} = 1$ Hyperbola

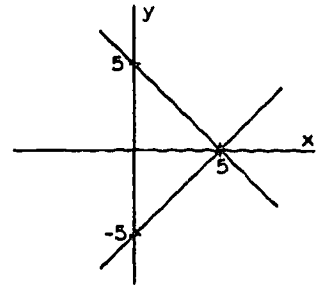
(e) $4y = x^2$ Parabola

(f) $4x = y^2$ Parabola

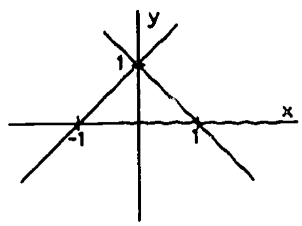
18. (a) circle
(b) ellipse
(c) hyperbola
(d) ellipse
(e) parabola
(f) parabola
(g) circle
(h) ellipse
(i) parabola
(j) hyperbola
(k) ellipse
(l) parabola
(m) hyperbola

19.

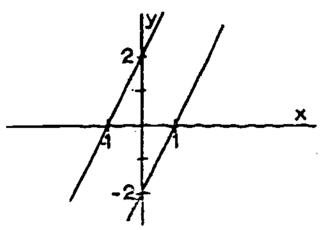
(a)



(b)



(c)



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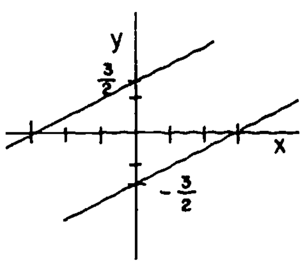

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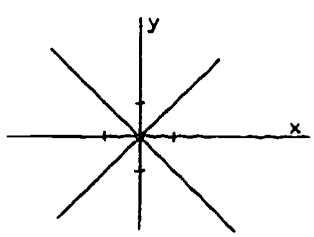
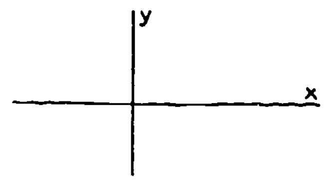
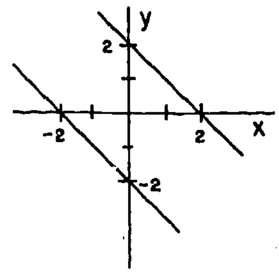


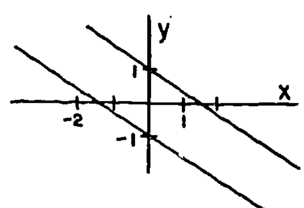
(f)



(g) Simplifying to
$(x + y)^2 - 4 = 0$

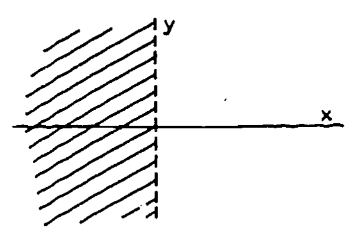


(h)

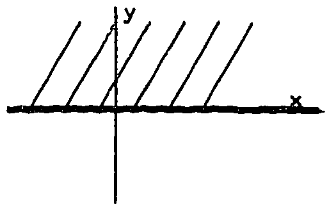


Simplifying to
$(2x + 3y)^2 - 9 = 0$

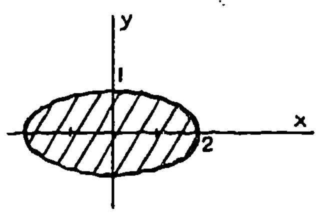
(i)



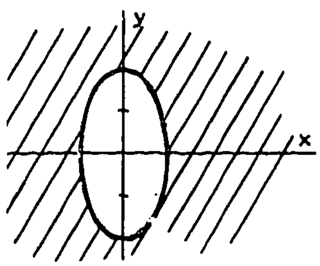
(j)



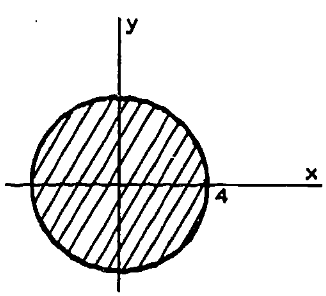
(k)



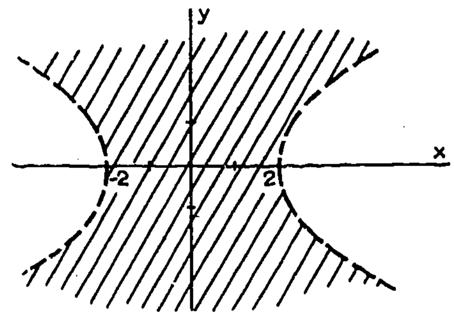
(l)



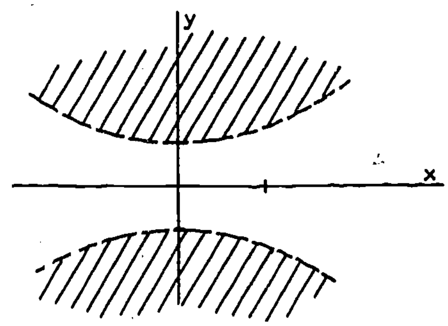
(m)



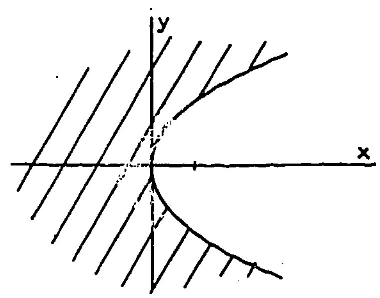
(n)



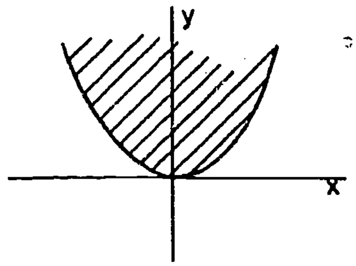
(o)



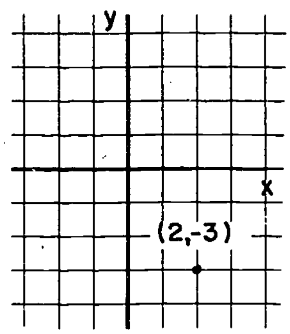
(p)



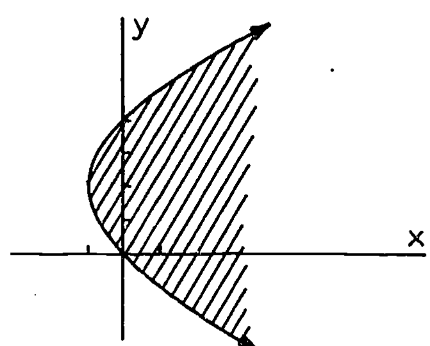
(q)



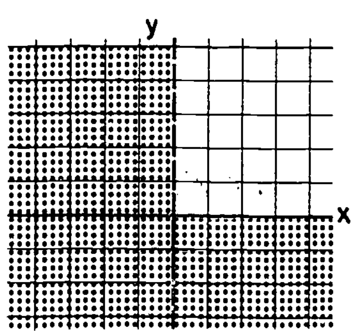
(r) Simplify to $(x - 2)^2 + (y + 3)^2 \leq 0$. The only element of the solution set is (2,-3), so the graph consists of this single point



(s) Simplify to $(y - 2)^2 < 4(x + 1)$. The graph of $(y - 2)^2 = 4(x + 1)$ is a parabola with vertex at (-1,2) opening to the right



(t)



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20. $y^2 = 16x$

$y^2 = 4cx$

$\therefore c = 4$

Latus rectum $= 4c = 16$.

Radius of circle is 8, center at $(4,0)$.

Equation: $(x - 4)^2 + y^2 = 8^2$.

21. Equation is evidently of the form $3x^2 - 5y^2 = K$. Since $(2,3)$ is on this hyperbola we have

$3 \cdot 2^2 - 5 \cdot 3^2 = K = -33$.

$\therefore$ required equation is

$$5y^2 - 3x^2 = 33$$

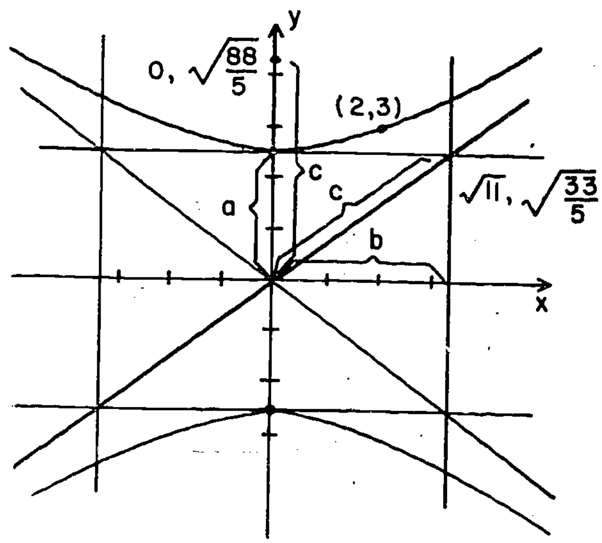
or $\frac{y^2}{\frac{33}{5}} - \frac{x^2}{11} = 1$

$a = \sqrt{\frac{33}{5}} \approx 2.6$

$b = \sqrt{11} \approx 3.3$

$c^2 = \frac{33}{5} + 11 = \frac{88}{5}$

$c = \sqrt{\frac{88}{5}} \approx 4.2$

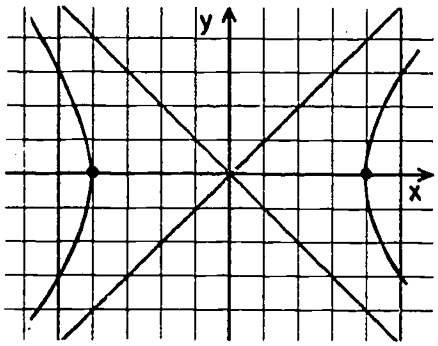


22. $2x^2 - 7y^2 = 18$.

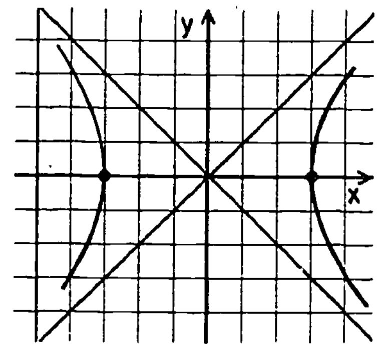
23. $a^2x^2 - b^2y^2 = a^2b^2$

24. Sketches of hyperbolas as indicated. Each has asymptotes $y = \pm x$ and vertices at a. $(\pm 4,0)$. b. $(\pm 3,0)$. c. $(\pm 2,0)$ d. $(\pm 1,0)$ e. $(0,\pm 1)$ f. $(0,\pm 2)$ g. The curve consists of the two lines of the asymptotes. h. $(0,\pm 4)$

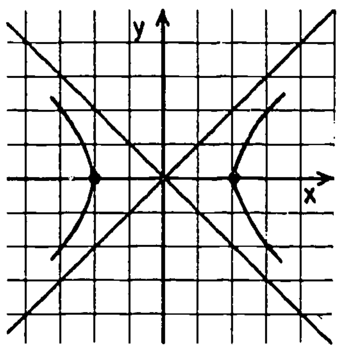
(a)



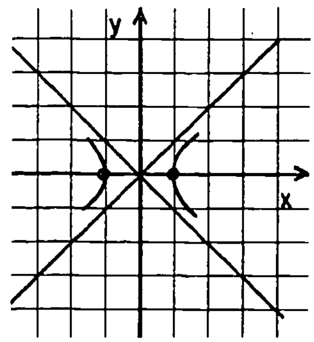
(b)



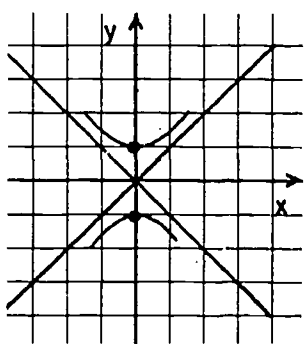
(c)



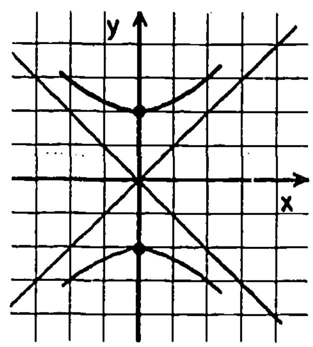
(d)



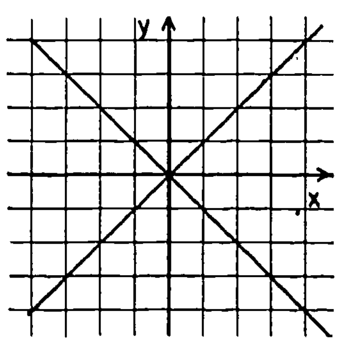
(e)



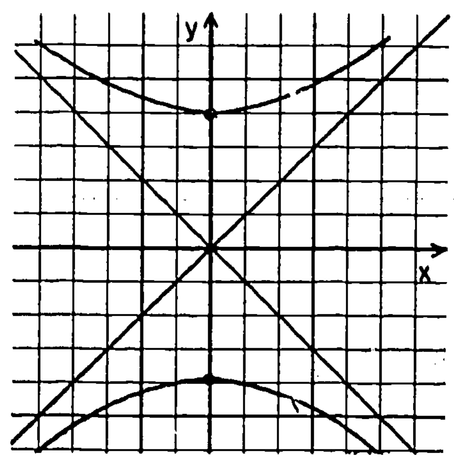
(f)



(g)



(h)



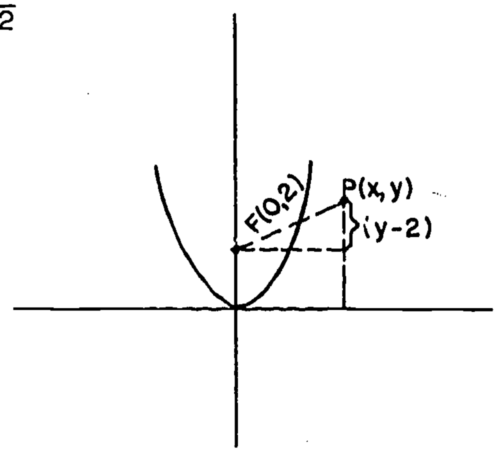
25. The coordinates are $(c, \pm \frac{b^2}{a})$. The development follows the same pattern as for the ellipse. The length of the focal chord is $\frac{2b^2}{a}$. It applies also to the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

26. $\sqrt{x^2 + y^2} = 2\sqrt{x^2 + (y - 2)^2}$

$x^2 + y^2 = 4(x^2 + y^2 - 4y + 4)$

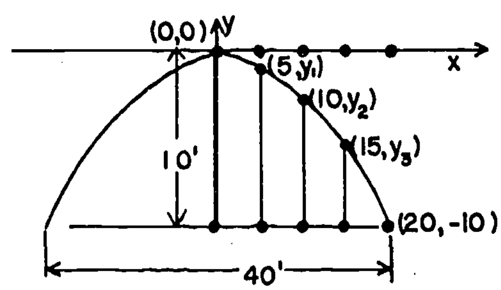
$x^2 + y^2 = 4x^2 + 4y^2 - 16y + 16$

$3x^2 + 3y^2 - 16y + 16 = 0$



27. The equation of the parabola, $x^2 = 4cy$ through $(20,-10)$ gives $c = -10$ Hence, the equation is, $x^2 = -40y$

The height of the arch at each interval is given by $10 - |y_n|$.



At $(0,0)$, $y_0 = 0$. Hence, $10 - 0 = 10'$

At $(5,y_1)$, $(5)^2 = -40y_1$

$$y_1 = -\frac{5^2}{40} = -\frac{5}{8}$$

$$\therefore\ 10 - \left|-\frac{5}{8}\right| = \frac{75}{8} \text{ the height at } (5,y_1)$$

At $(10,y_2)$, $10^2 = -40y_2$

$$y_2 = -\frac{10^2}{40} = -\frac{5}{2}$$

$$\therefore\ 10 - \left|-\frac{5}{2}\right| = \frac{15}{2} \text{ the height at } (10,y_2)$$

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At $(15, y_3)$, the height is $\frac{35}{8}$.

At $(20, y_4)$, the height is 0.

28. $\sqrt{(y - b)^2 + (x - a - c)^2} = x - a + c$

$$(y - b)^2 + (x - a - c)^2 = (x - a + c)^2$$

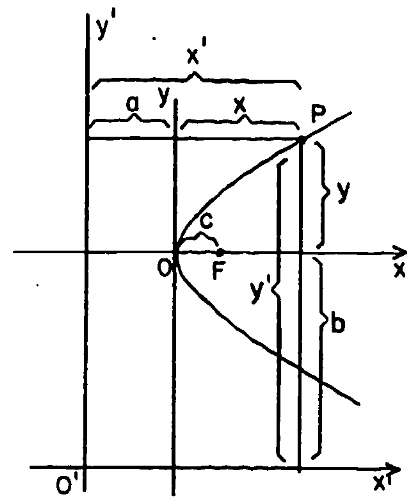
$$(y - b)^2 + x^2 + a^2 + c^2 - 2ax - 2cx + 2ac =$$
$$x^2 + a^2 + c^2 - 2ax + 2cx - 2ac$$

$$(y - b)^2 = 4cx - 4ax$$

$$(y - b)^2 = 4c(x - a)$$

29. Similarly $(x - a)^2 = 4c(y - b)$

30. Identical to #28.



<u>Note on Translation of Axes</u>. We know that the equation of a parabola with vertex at the origin and focus at $(c, 0)$ is $y^2 = 4cx$. From this and a process known as the translation of axes we can derive the equation required in Exercise 28. If the coordinate of O with respect to the translated axes (origin O) are a and b, we have (see drawing) $x = x' - a$ and $y = y' - b$. Substituting these new names for x and y we obtain $(y' - b)^2 = 4c(x' - a)$ as the equation of a parabola whose vertex is at (a, b) and whose focus is at $(a + c, b)$ with respect to the new coordinate system.

This "translation of axes" process can be used to derive many other formulas in this chapter.

Challenge Problems - Answers

1. $y^2 = 4ax$ a positive, open to right.

a negative, open to left.

2. When $x = a$

$y^2 = 4ax$

becomes $y^2 = 4a^2$

$y = \pm 2a$

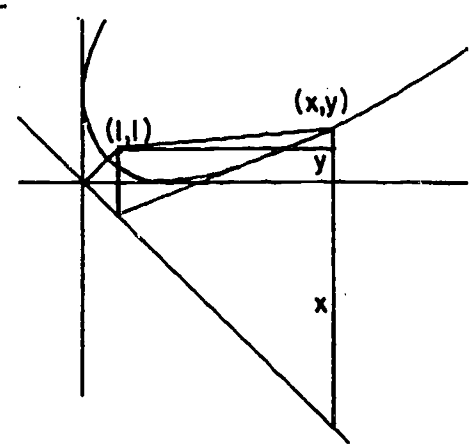
Points are $(a,2a)(a,-2a)$.

3. $\sqrt{(x-1)^2 + (y-1)^2} = \frac{(x+y)}{\sqrt{2}}$

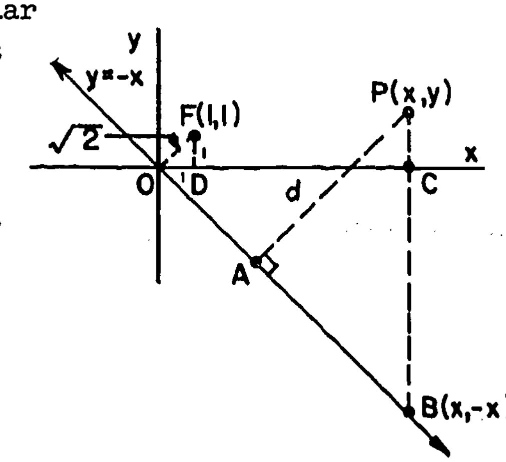
$x^2 - 2x + 1 + y^2 - 2y + 1 = \frac{x^2 + 2xy + y^2}{2}$

$2x^2 - 4x + 2 + 2y^2 - 4y + 2 = x^2 + 2xy + y^2$

$x^2 + y^2 - 2xy - 4x - 4y + 4 = 0$



*3 Let $P(x,y)$ be a point on the parabola, then PC perpendicular to the x-axis determines point B having coordinates $(x,-x)$ since B lies on the line of $y = -x$. These coordinates indicate that Δ OCB is right isosceles, so $\angle B = 45^\circ$. Then Δ ABP is also right isosceles and is similar to right isosceles triangle ODF.



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From proportional sides of these similar triangles, $\frac{d}{\overline{PB}} = \frac{1}{\sqrt{2}}$ and since $\overline{PB} = y - (-x) = x + y$, then $d = \frac{x + y}{\sqrt{2}}$. From the locus definition of the parabola,

$$\sqrt{(x - 1)^2 + (y - 1)^2} = \frac{x + y}{\sqrt{2}}$$

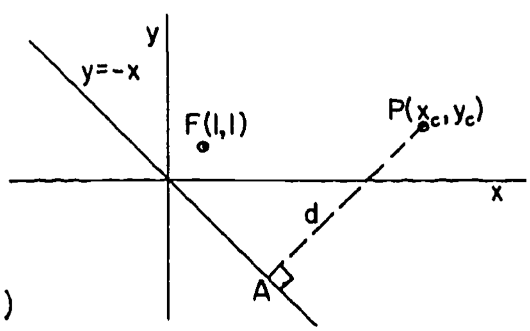
Simplifying yields the equation

$$x^2 - 2xy + y^2 - 4x - 4y + 4 = 0$$

An alternate solution determines $\overline{PA} = d$ from coordinates of A and P by use of the distance formula. The coordinates of A can be found by simultaneous solution of the equations for the lines of PA and $y = -x$. Let $P(x_c, y_c)$ be the coordinates of point P on the parabola. The slope of the line of $y = -x$ is -1, so the slope of PA is 1 and the equation of PA has the form $y = x + b$

Since $P(x_c, y_c)$ lies on this line, then

$$y_c = x_c + b$$
$$b = y_c - x_c$$



The equation of PA is then $y = x + (y_c - x_c)$ when (x,y) are the coordinates of any point on PA. Solving the system $\begin{cases} y = -x \\ y = x + (y_c - x_c) \end{cases}$

determines the coordinates of A as $\left(\frac{x_c - y_c}{2}, \frac{y_c - x_c}{2}\right)$.

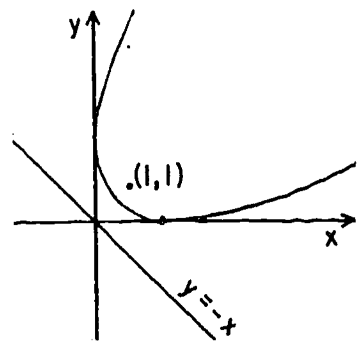
Applying the distance formula,

$$\overline{PA} = d = \sqrt{[x_c - \frac{(x_c - y_c)}{2}]^2 + [y_c - \frac{(y_c - x_c)}{2}]^2}$$

$$= \sqrt{(\frac{x_c + y_c}{2})^2 + (\frac{x_c + y_c}{2})^2}$$

$$= \sqrt{\frac{(x_c + y_c)^2}{2}}$$

$$= \frac{x_c + y_c}{\sqrt{2}}$$

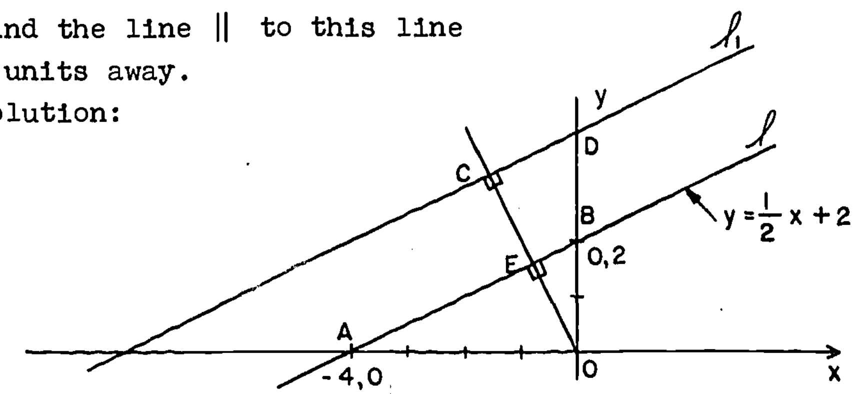


The equation of the parabola then follows as in previous solution.

4. Problem: Given $y = \frac{1}{2}x + 2$.

Find the line || to this line 2 units away.

Solution:



In the drawing ℓ is the graph of the line whose equation is $y = \frac{1}{2}x + 2$ and ℓ_1 is the required line which lies above ℓ (another, ℓ_2, not shown lies below ℓ). OC is perpendicular to ℓ and ℓ_1 and hence EC = 2. In Δ AOB we have $|AO| = 4; |OB| = 2$ and by the Pythagorean Theorem $AB = 2\sqrt{5}$. The area of $\Delta AOB = \frac{1}{2} \cdot 4 \cdot 2 = 4 = \frac{1}{2}|AB| \cdot |OE| = \frac{1}{2} \cdot 2\sqrt{5} \cdot |OE|$.

$\therefore OE = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$. It is clear that $\Delta EOB \sim \Delta COD$

hence $\frac{OE}{OC} = \frac{OB}{OD}$ or $\frac{\frac{4\sqrt{5}}{5}}{2 + \frac{4\sqrt{5}}{5}} = \frac{2}{OD}$.

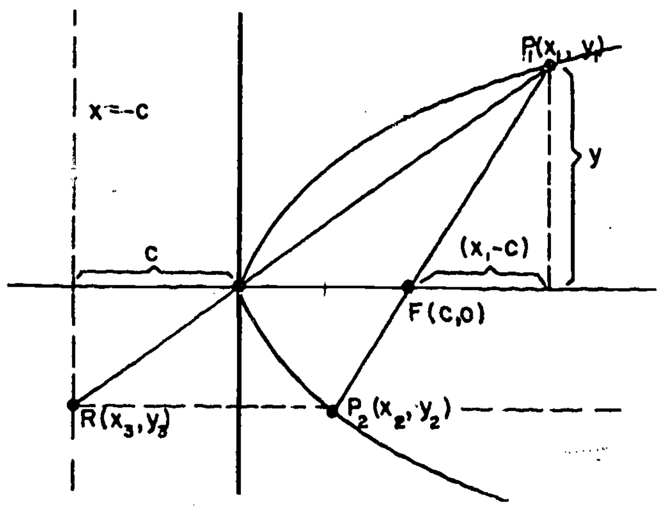
$\therefore$ $OD = 2 + \sqrt{5}.$

We seek the equation of a line whose slope is $\frac{1}{2}$ and whose y-intercept is $2 + \sqrt{5}.$ Our equation is

$$y = \frac{1}{2}x + 2 + \sqrt{5}.$$

Evidently for ℓ_2 the y-intercept is $2 - \sqrt{5}$ and the equation is $y = \frac{1}{2}x + 2 - \sqrt{5}.$ We can write the equations of ℓ_1 or ℓ_2 in the form $y = \frac{1}{2}x + 2 \pm \sqrt{5}.$

5.



Given: $y^2 = 4cx$ and line from $P_1(x_1,y_1)$ through F to $P_2(x_2,y_2)$. Also line from $P_1(x_1,y_1)$ through vertex to R on directrix.

Find: Coordinates of $P_2(x_2, y_2)$ in terms of x_1, y_1, and c.

Prove: $\overline{RP} \parallel$ axis $y = 0$.

$$y^2 = 4cx$$

$$\therefore \quad y_2^2 = 4cx_2$$

$$\text{or} \quad x_2 = \frac{y_2^2}{4c}$$

$$y = mx + b$$

$$m = \frac{y_1}{(x_1 - c)} = \frac{y_2}{(x_2 - c)}$$

$$y_2 = \frac{y_1(x_2 - c)}{(x_1 - c)} \qquad x_2 = \frac{y_2^2}{4c}$$

$$y_2 = \frac{y_1\left(\frac{y_2^2}{4c} - c\right)}{(x_1 - c)}$$

$$y_2 = \frac{y_1\left(y_2^2 - 4c^2\right)}{(x_1 - c)4c}$$

$$y_2(x_1 - c)4c = y_1 y_2^2 - y_1 4c^2$$

$$y_1 y_2^2 - 4c(x_1 - c)y_2 - 4c^2 y_1 = 0$$

A quadratic in y_2

$$y_2 = \frac{4c(x_1 - c) \pm \sqrt{16c^2(x_1 - c)^2 + 16c^2 y_1^2}}{2y_1}$$

$$y_2 = \frac{2c(x_1 - c) \pm 2c\sqrt{(x_1 - c)^2 + y_1^2}}{y_1}$$

Substitute $y_1^2 = 4cx_1$

$$y_2 = \frac{2c(x_1 - c) \pm 2c\sqrt{x_1^2 - 2cx_1 + c^2 + 4cx_1}}{y_1}$$

$$y_2 = \frac{2c(x_1 - c) \pm 2c(x_1 + c)}{y_1}$$

$$y_2 = \frac{2c(x_1 - c + x_1 + c)}{y_1} \quad \text{or} \quad \frac{2c(x_1 - c - x_1 - c)}{y_1}$$

$$y_2 = \frac{4cx_1}{y_1} \qquad \text{or} \quad \frac{-4c^2}{y_1}$$

Examination of these two values indicates that $y_2 = \dfrac{4cx_1}{y_1}$ cannot be possible. The sign of y_2 must be different from that of y_1.

$$y_2 = \frac{-4c^2}{y_1} \quad \text{so} \quad y_2^2 = \frac{16c^4}{y_1^2}$$

Hence, $\quad y_2^2 = 4cx_2$

$$\frac{16c^4}{y_1^2} = 4cx_2$$

or $\quad x_2 = \dfrac{4c^3}{y_1^2} \quad$ x always positive.

Coordinates of $P_2 \left(\dfrac{4c^3}{y_1^2}, \dfrac{-4c^2}{y_1}\right)$

Where line from P_1 passes through vertex to R.

Now consider m as the slope of the line RP.

$$m = \frac{y_1}{x_1} = \frac{y_3}{-c}$$

$$y_3 = -\frac{cy_1}{x_1}$$

$$x_3 = -c$$

Coordinates of $R = \left(-c, -\frac{cy_1}{x_1}\right)$

Prove $RP_2 \parallel$ x-axis

Prove $y_2 = y_3$

$$y_3 = \frac{-cy_1}{x_1}$$

but $$x_1 = \frac{y_1^2}{4c}$$

∴ $$y_3 = -\frac{cy_1}{\frac{y_1^2}{4c}} = -\frac{4c^2}{y_1}$$

Since $$y_2 = \frac{-4c^2}{y_1}$$

Then $y_2 = y_3$ Q.E.D.

Illustrative Test Questions

1. Find an equation of the line parallel to the line whose equation is $y = 2x - 3$ and passing through the point $(5, -2)$.

2. Write an equation of the line perpendicular to the line whose equation is $y = 3 - 2x$ and intersecting it on:

   (a) the y-axis.
   (b) the x-axis.

3. Write an equation of the line with,

   (a) slope -2 and x-intercept 2.

   (b) intercepts $(4,0)$ and $(0, -3)$.

4. (a) Sketch the graphs of,

      (i) $2x + y = 4$

      (ii) $x + 3y = -3$

   (b) What are the coordinates of the point of intersection?

   (c) Give the slope of each.

   (d) Give the x and y intercepts of each.

5. Determine, without sketching graphs, whether each of the following pairs of equations represent lines which are the same, are parallel, are perpendicular, or none of these.

   (a) $y = \frac{2}{3}x + 2$
   $y = -\frac{3}{2}x + 2$

   (b) $5x + 2y = 6$
   $y + 5x = 1$

   (c) $3x - y + 1 = 0$
   $6x - 2y + 1 = 0$

   (d) $-3x + 4y - 3 = 0$
   $-8y + 6x + 6 = 0$

   (e) $2\frac{1}{2}x - 7y - 2 = 0$
   $10x - 28y - 4 = 0$

   (f) $x = -3$
   $x = 0$

(g) $y + 6 = 0$
$x - 3 = 0$

(h) $3x - 5y = 0$
$y = 0$

(i) $x + 2y - 1 = 0$
$x = 3$

6. Find the slope-intercept form of the line through the point $(2,1)$ perpendicular to the line $\frac{2}{3}x + \frac{3}{2}y + 2 = 0$.

7. Given line L_1 with equation $2y = 3x + 2$. What is the y-intercept of line L_2 if the slope of L_2 is twice that of L_1 and if the two lines intersect on the x-axis?

8. Given the line whose equation is $2y = 3x + 2$,

   (a) Find the equation of the line whose slope is twice that of the given line and whose y-intercept is two units above that of the given line.

   (b) How many units apart are the x-intercepts of the two lines?

9. Find the equation when x is replaced by $x + a$ in the equation $y^2 = 4cx$. Discuss the "family" of curves of the equations for $a > 0$ and $a < 0$.

10. The graph of the hyperbola $4x^2 - 9y^2 = 36$ has no points in the vertical strip between what two lines?

11. Find the center and the radius of the circle $x^2 + y^2 + 2x - 4y - 1 = 0$.

12. Given the line whose equation is $3x - 4y + 12 = 0$. Find the equation of,

   (a) The line parallel to the given line with the x-intercept 3 units closer to the origin.

   (b) The line through the origin perpendicular to the given line.

(c) The line with slope $\frac{1}{2}$ that intersects the given line on the y-axis.

(d) The line through the point (1,2) with the same slope as the given line.

13. The graph of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ has no points in the vertical strip between the lines,

(a) $x = 4$ and $x = -4$
(b) $x = 2$ and $x = -2$
(c) $x = 9$ and $x = -9$
(d) $x = 3$ and $x = -3$
(e) $x = 6$ and $x = -6$

14. What are the coordinates of the center of the circle whose equation is,

$2x^2 + 2y^2 + 4x - 4y - 5 = 0$?

(a) (- 1,1)
(b) (1, - 1)
(c) (-2,2)
(d) (2,-2)
(e) none of these

15. What is the axis of the parabola defined by,

$y = 3x^2 + 6x - 4$?

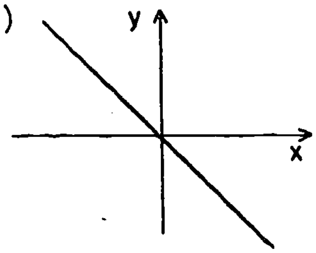
(a) $x = -7$
(b) $x = -4$
(c) $x = -3$
(d) $x = -1$
(e) $x = 1$

16. Which of the following equations has the same graph as $2y - 3x + 2 = 0$?

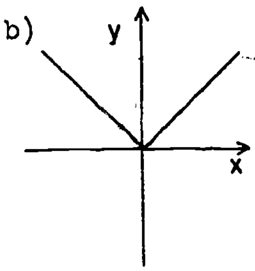
(a) $2y = 3x + 2$
(b) $6x - 4y + 4 = 0$
(c) $y + 1 = \frac{3}{2}x$
(d) $y + 1 = -\frac{3}{2}x$
(e) $3y - 2x + 2 = 0$

17. If y varies inversely as x, which of the following could be the graph of y as a function of x?

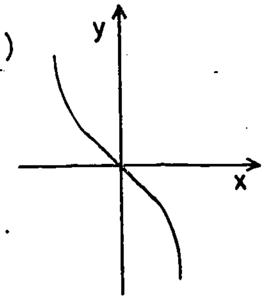
(a)



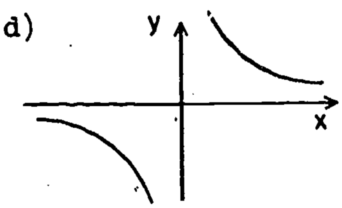
(b)



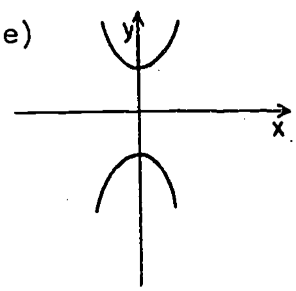
(c)



(d)



(e)



18. An ellipse with center at the origin has its focus at (4,0) and its directrix the line whose equation is $x = 16$. Find,

(a) Its eccentricity.
(b) The length of its major axis.
(c) The length of its minor axis.
(d) Its equation.

19. For each of the following, identify the conic section of which it is the equation,

(a) $4x^2 + 4y^2 - 16 = 0$

(b) $x^2 + 4y^2 - 2x - 3 = 0$

(c) $x = \frac{3}{y}$

(d) $x^2 + 4y = 0$

20. Find the equation of the parabola with vertex at (0,1) and x-intercepts at $x = 2$ and $x = -2$.

21. Given the ellipse whose equation is $x^2 + \frac{y^2}{9} - 2x = 0$, find,

(a) Its center

(b) Its x-intercepts

(c) Its vertices.

Directions: Select the response which best completes the statement or answers the question.

22. The graph of $y^2 = (2x - 1)^2$ is

(a) A circle

(b) An ellipse which is not a circle

(c) A hyperbola

(d) Two intersecting straight lines

(e) Two parallel straight lines

23. The parabola at the right has the focus (2,0) and the line $x = -2$ for its directrix. The equation of the parabola is,

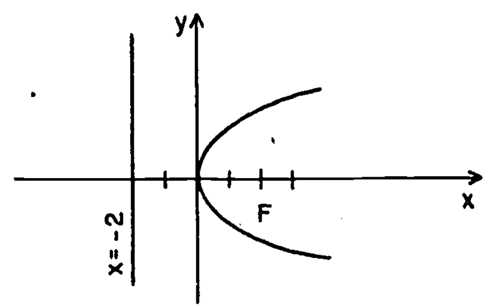
(a) $y^2 = x - 2$

(b) $y^2 = x + 2$

(c) $(x - 2)^2 = y$

(d) $x^2 = 8y$

(e) $y^2 = 8x$



24. The y-intercepts of the graph of the equation $9x^2 + 4y^2 = 36$ are,

(a) 2 and -2.

(b) 3 and -3

(c) 6 and -6

(d) 4 and -4

(e) 9 and -9

25. If the foci of the hyperbola are (-2,2) and (8,2), and one vertex is (6,2), what is the length of the transverse axis?

(a) 4
(b) 6
(c) 8
(d) 10
(e) None of these

26. Write the equation of the ellipse which has foci (0,3) and (0,-3) and vertices (0,6) and (0,-6).

(a) $\frac{x^2}{9} + \frac{y^2}{36} = 1$
(b) $x^2 + 6y^2 = 216$
(c) $\frac{x^2}{27} + \frac{y^2}{36} = 1$
(d) $\frac{x^2}{36} + \frac{y^2}{9} = 1$
(e) $x^2 + 3y^2 = 108$

27. Write the equation of the hyperbola which has foci (3,0) and (-3,0) and vertices (1,0) and (-1,0)

(a) $\frac{x^2}{1} - \frac{y^2}{8} = 1$
(b) $\frac{y^2}{8} - \frac{x^2}{1} = 1$
(c) $\frac{x^2}{1} - \frac{y^2}{10} = 1$
(d) $8y^2 - x^2 = 4$
(e) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

28. Which of the following statements about the conic section which has the equation $\frac{(x-1)^2}{9} + \frac{y^2}{4} = 1$ is true?

(a) It has the point (0,1) for its center.
(b) It is symmetric with respect to the line x = 1.
(c) It has the point (3,0) as a focus.
(d) It has the point (1,2) as a vertex.
(e) It does not cross the y-axis.

29. If x varies inversely as y, and $x = 8$ when $y = 2$, what is the value of x when $y = 5$?

(a) 80 (d) $\frac{4}{5}$

(b) 20 (e) $\frac{1}{20}$

(c) $\frac{16}{5}$

30. Which equation expresses the following fact? "The central angle C of a regular polygon varies inversely as the number of sides n." (Assume k is a constant.)

(a) $C = \frac{n}{k}$ (d) $C = kn$

(b) $Cn = k$ (e) $n = kC$

(c) $\frac{C}{n} = k$

31. The graph of $y^2 - x^2 = 0$ is

(a) an ellipse (d) a circle

(b) a hyperbola (e) two parallel lines

(c) two perpendicular lines

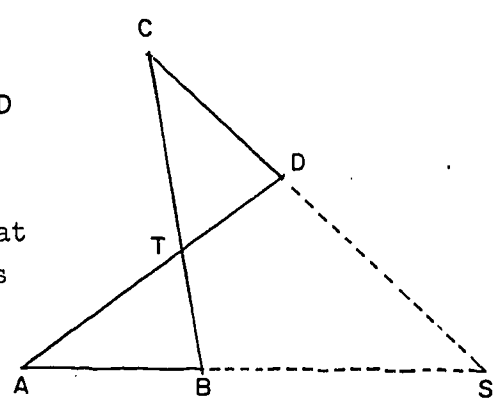
32. The graph of $9x^2 + 4y^2 = 0$ is

(a) a circle (d) a parabola

(b) an ellipse (e) two straight lines

(c) a point

33. The figure ABCD is a "twisted parallelogram" in which $AB = CD$, $CB = AD$ and $AB < BC$. If we hold A and B fixed and allow the figure to rotate so that C and D describe circles with centers B and A respectively, the point T will move on



(a) a straight line (d) an ellipse

(b) a hyperbola (e) a circle

(c) a parabola

34. Refer again to the drawing for Problem 33. Let S be the intersection of AB and CD extended. If we hold B and C fixed and allow the figure to rotate so that D and A describe circles with centers C and B respectively, the point S will move on

(a) a parabola
(b) a hyperbola
(c) a straight line
(d) a circle
(e) an ellipse

35. The line segment AB is 10 units long. If the mid-point M of AB moves on the circle $x^2 + y^2 = 25$ while A moves on the x-axis, then B moves on

(a) an ellipse
(b) a line parallel to the x-axis
(c) a hyperbola
(d) a parabola
(e) the y-axis

*36. Which of the following statements is false?

(a) If AB and A'B' are the focal chords of any two parabolas whose vertices are respectively, V and V' then Δ VAB $\sim$ Δ V'A'B'.

(b) If A and B are any two points on an ellipse then the perpendicular bisector of AB will pass through the center of the ellipse.

(c) If a line ℓ intersects the hyperbola $xy = 12$ and its asymptotes in the four points A, B, C, D, then AB = CD.

(d) The hyperbolas whose equations are $xy = 18$ and $x^2 - y^2 = 36$ are congruent.

(e) If P moves in such a way that the ratio of its distances to the fixed points A and B is a constant not equal to 1, then P moves on a circle.

37. If a hyperbola has the lines $y = \pm \frac{1}{2}x$ as asymptotes and passes thru the point (5,3) then its conjugate axis is

(a) $2\sqrt{11}$

(b) $\sqrt{11}$

(c) $\frac{\sqrt{11}}{2}$

(d) $\sqrt{\frac{11}{2}}$

(e) $\sqrt{22}$

38. The graph of $x^2 - 25 = -y(y + 2x)$ is

(a) a hyperbola

(b) an ellipse

(c) two parallel lines

(d) a circle

(e) two perpendicular lines

Part II Matching

Directions: Match each equation on the left with a locus on the right, a locus may be used once, more than once, or not at all.

39. $9x^2 + 4y^2 = 36$	(a) Circle
40. $xy = 20$	(b) Point
41. $x^2 + 4y^2 - 4x - 8y - 8 = 0$	(c) Two straight lines.
42. $y^2 = x - 2$	(d) Ellipse
43. $y^2 - 9 = 0$	(e) Hyperbola
44. $y^2 + 9 = 0$	(f) Parabola
	(g) No locus in the real number plane.

Part III: Problems

45. Write an equation for the locus of points equidistant from the line $x = -4$ and the point $(-2,2)$.

46. Write an equation for the locus of points the sum of whose distances from $(0,3)$ and $(0,-3)$ is 10.

47. Write an equation for the conic section whose foci are $F(2,0)$ and $F'(-2,0)$ and every point on the curve satisfies the condition $d(PF) - d(PF') = 2$.

48. Rewrite the equation $x^2 + 2y^2 - 8x + 8y + 16 = 0$ in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

49. Sketch the graphs of the equations in the following system and label the intersection points of the two curves with the letters P_1, P_2, P_3, ... Use as many letters as there are intersection points. You need not find the coordinates of the intersection points.

$$\begin{cases} x = y^2 + 1 \\ x^2 + y^2 = 25 \end{cases}$$

6-8 Illustrative Test Questions - Answers

1. $y = 2x + b$

$-2 = 2\cdot5 + b$

$b = -12$

$y = 2x - 12$

2. $y = \frac{1}{2}x + b$

(a) $y = \frac{1}{2}x + 3$

(b) $0 = 3 - 2x$

$x = \frac{3}{2}$

$0 = \frac{1}{2} \cdot \frac{3}{2} + b$

$b = -\frac{3}{4}$

$y = \frac{1}{2}x - \frac{3}{4}$

3. (a) $0 = -2\cdot2 + b$

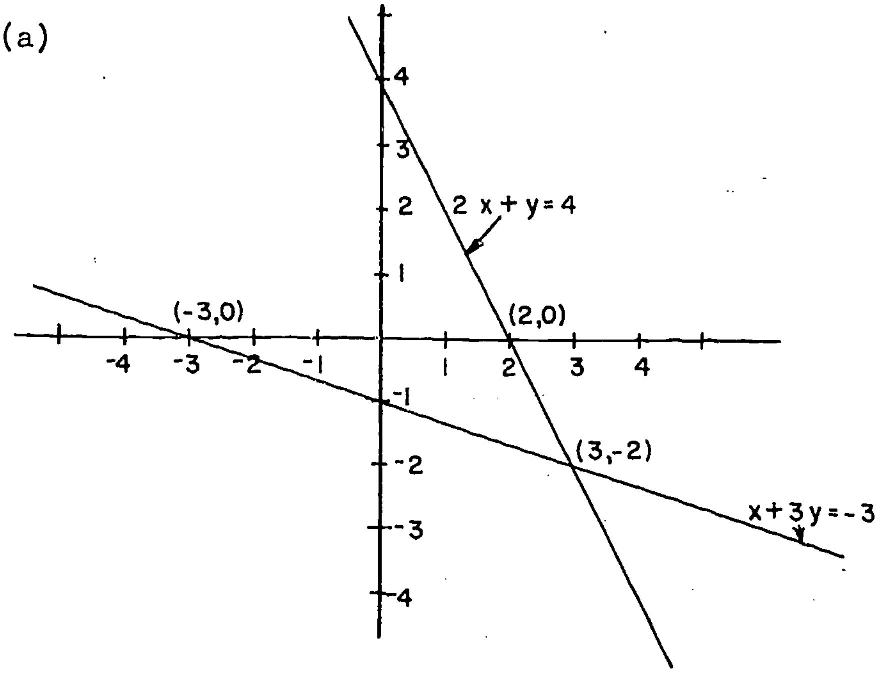
$b = 4$

$y = -2x + 4$

(b) $m = \frac{3}{4}$ and $b = -3$

Hence $y = \frac{3}{4}x - 3$

4. (a)



(b) $2x + y = 4$

$\underline{x + 3y = -3}$

$6x + 3y = 12$

$\underline{x + 3y = -3}$

$5x = 15$

$x = 3$

$3 + 3y = -3$

$3y = -6$

$y = -2$

Intersection (3,-2)

(c) (i) $m = -2$ and (ii) $m = -\frac{1}{3}$

(d) (i) (2,0) and (ii) (-3,0)

(0,4) (0,-1)

5. (a) Perpendicular
   (b) None
   (c) Parallel
   (d) Same
   (e) Parallel
   (f) Parallel
   (g) Perpendicular
   (h) None
   (i) None

6. $\frac{2}{3}x + \frac{3}{2}y + 2 = 0$

   $\frac{3}{2}y = -\frac{2}{3}x - 2$

   $y = -\frac{4}{9}x - \frac{4}{3}$

   $1 = \frac{9}{4}\cdot 2 + b$

   $b = 1 - \frac{9}{2} = -\frac{7}{2}$

   $y = \frac{9}{4}x - \frac{7}{2}$

7. L_1 x-intercept is $(-\frac{2}{3}, 0)$

   L_2 $y = 3x + b$

   $0 = 3(-\frac{2}{3}) + b$

   $b = 2$

   $y = 3x + 2$

8. (a) $y = 3x + 3$
   (b) distance is $|-1 - (-\frac{2}{3})| = \frac{1}{3}$

9. $y^2 = 4c(x + a)$

$y^2 = 4cx + 4ca$

The curves are parabolas whose vertices are on the x-axis, open to the right, and symmetric about the x-axis. When $a > 0$ the vertices are to the right of the y-axis by the amount $4ca$. When $a < 0$ the vertices are to the left by the amount $4ca$.

10. $a = 3$ so there are no points such that $-3 < x < +3$.

11. $(x^2 + 2x + 1) + (y^2 - 4y + 4) - 1 = 0 + 1 + 4$

$(x + 1)^2 + (y - 2)^2 = 6$

Center $(-1,2)$ radius $\sqrt{6}$.

12. (a) original x-intercept is -4

new x-intercept is -1

$y = \frac{3}{4}x + \frac{3}{4}$

(b) $y = -\frac{4}{3}x$

(c) $y = \frac{1}{2}x + 3$

(d) $2 = \frac{3}{4}\cdot 1 + b$

$b = \frac{5}{4}$

$y = \frac{3}{4}x + \frac{5}{4}$

13. (d) Same as 10

14. (a) $(-1,1)$

15. (d) $x = -1$

16. (c) $y + 1 = \frac{3}{2}x$

17. (d) equilateral hyperbola

18. (a) $c = 4 \quad x \doteq \frac{c}{e^2}$

$16 = \frac{4}{e^2} \quad e^2 = \frac{1}{4} \quad e = \frac{1}{2}$

(b) $c = ae$

$4 = \frac{1}{2}a$

$a = 8$

Major axis $= 2a = 2 \cdot 8 = 16$

(c) $b^2 = a^2 - c^2$

$b^2 = 64 - 16 = 48$

$b = 4\sqrt{3}$

Minor axis $= 2b = 8\sqrt{3}$

(d) $\frac{x^2}{64} + \frac{y^2}{48} = 1$

19. (a) circle
(b) ellipse
(c) hyperbola
(d) parabola

20. $(y-1) = ax^2$

$0 - 1 = a \cdot 4$

$a = -\frac{1}{4}$

$(y - 1) = -\frac{1}{4}x^2$

$y = -\frac{1}{4}x^2 + 1$

21. $x^2 + \frac{y^2}{9} - 2x = 0$

$(x^2 - 2x + 1) + \frac{y^2}{9} = 1$

$\frac{(x-1)^2}{1} + \frac{y^2}{9} = 1$

(a) center is at (1,0)
(b) x-intercepts are 0 and 2.
(c) $a = 1 \quad b = 3$

$V_x = (0,0)$ and $(2,0)$

$V_y = (1, \pm 3)$

22. d
23. e
24. b
25. b
26. c
27. a
28. b
29. c
30. b
31. c
32. c
33. d
34. b
35. e
36. b
37. a
38. c
39. d
40. e
41. d
42. f
43. c
44. g

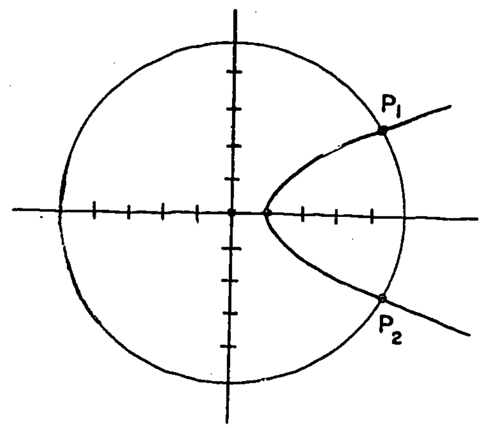
III. 45. $(y - 2)^2 = 4(x + 3)$

46. $\frac{x^2}{16} + \frac{y^2}{25} = 1$

47. $\frac{x^2}{1} - \frac{y^2}{3} = 1$

48. $\frac{(x - 4)^2}{8} + \frac{(y + 2)^2}{4} = 1$

49. Two points of intersection.



Chapter 7

SYSTEMS OF EQUATIONS IN TWO VARIABLES

7-0. Introduction.

In this Chapter we take advantage of the opportunity to discuss several ideas which have been mentioned indirectly and used without any explanation at various points in the text -- these ideas are the ideas of solution set of an equation (inequality) and equivalent equations and systems of equations.

We try to define the terms and to collect the information we have about operations which yield equivalent equations. The important thing to stress here is the meaning of the "solution set of an equation" and later on "the solution set of a system of equations". In connection with this latter idea, the solution set is derived by substituting an equivalent equation in a system to obtain an equivalent system of equations.

The purpose of Section 7-1 and 7-2 is simply to establish the framework for talking about solutions of systems of equations and inequalities. We are content with merely stating and illustrating the definitions. The exercises for the sections are designed simply to clarify these definitions. Ways of determining the solution sets will be treated in succeeding sections. In the last part of Section 7-2 we state principles which allow us to replace a system of equations with an equivalent system. The point is that in solving systems we want to be sure without checking, if possible, that when we arrive at a solution, it really is a solution. This is certain if we deal only with equivalent systems. This gets cumbersome at times, but the point is important -- to be sure that you have the solution, the intermediate systems you work with must be equivalent to the original system.

The method of solving systems by using the principle of linear combination (7-2b) to obtain equivalent systems is actually the method which is sometimes called "elimination by addition and subtraction". The discussion of Case III in Section 7-3 is actually a justification of the fact that this method yields an equivalent system of linear equations and leads inevitably to the solution set, if this set consists of a single ordered pair.

The geometric interpretation of the linear combination principle shows how a system consisting of two intersecting lines can be converted into an equivalent system consisting of two lines with equations $x = s$ and $y = t$. We have really said that the family of lines passing through the intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by the equation $k_1(a_1x + b_1y + c_1) + k_2(a_2x + b_2y + c_2) = 0$, where k_1 and k_2 are two parameters. If k_1 is not zero we can write the equation in the form

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \text{ with } k = \frac{k_2}{k_1}.$$

Similarly if k_2 is not zero we could have obtained the form

$$k(a_1x + b_1y + c_1) + (a_2x + b_2y + c_2) = 0 \text{ with } k = \frac{k_1}{k_2}.$$

So there is really only one parameter in the equation for the family of lines.

We have taken the view that students have seen a thorough discussion of systems of linear equations in the first course in algebra. The methods of solving such systems by the usual elimination by addition and subtraction and elimination by substitution must be familiar to them. What we have tried to do in this Chapter is to look at the problem from a slightly more sophisticated point of view and stress the importance of dealing with equivalent systems at all times. The idea of the linear combination of two expressions is an important one and will be met in many other phases of the student's mathematical life.

As for solving other systems, we can settle completely the problem of solving systems consisting of one linear and one quadratic equation. The method we have used in the examples is of course the familiar method usually called substitution. We have refrained from using this word in the text preferring to concentrate on using the definition of the solution set of a system. The solutions are derived in such a way that there is no doubt that the number pairs we get are actually solutions. Of course what we really are doing is to solve the linear equation for one variable and substitute this expression in the quadratic equation. The resulting quadratic equation in one variable is then solved and the two solutions are again substituted in the linear equation, yielding two, one, or no real number pairs as the solution set. The reason for not using this straight forward substitution, but rather talking about number pairs $(a,f(a))$ or $(g(b),b)$ is that these pairs are first of all members of the solution set of the linear equation: then necessary and sufficient conditions that they also be members of the solution set of the quadratic equation are obtained. When these conditions are met we know that the pairs are then members of the solution set of the system. There is no confusion between our solution set and points (x,y) which may well be on one or the other of the two curves, but not both.

While we are able to completely settle the problem of solving systems of two linear equations and systems of one linear and one quadratic, when we consider systems of quadratics, we are unable to finish the job. No mention was made of systems which only contained variable terms which are squared, symmetric systems, homogeneous systems, etc. While the methods used on linear systems also suffice for systems containing only variable terms involving squares and similarly the method of linear combination together with the Principles 7-2a and 7-2b suffice to reduce many other systems to manageable form, the fact is that, short of solving equations of the fourth degree, we cannot devise methods which will solve every system of two quadratic equations.

We therefore resort to the graphs of the equations. From these we are able to get approximate solutions and to find out how many elements there are in the solution set. This method gives us a good chance to review and use the analytic geometry of Chapter 6 as well.

This then completes the program for Chapter 7. We seek to derive solution sets of equations, inequalities, and systems of equations. We re-examine the solution of systems of linear equations and systems of one linear and one quadratic equation from a more sophisticated point of view and we solve some systems of two quadratic equations. A chance to employ our new skills in analytic geometry is offered in considering systems of inequalities. Much time or little time can be devoted to these problems depending on the interests of the teacher and his students and the limitation of time.

7-1. Solution Sets of Systems of Equations and Inequalities.

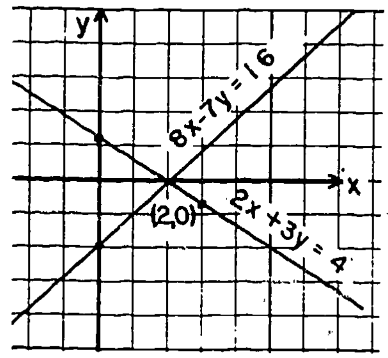
The terms inconsistent, consistent, and dependent are introduced in this Section solely as terms describing the solution set of a system. The student at this point is not expected to solve the system to find out about the solution set. He is to be encouraged to see what we mean by a point belonging to the solution set. He may also be able simply to look at the pair of equations and see that no pair (x,y) could satisfy both at once. While some of the problems ask him to draw the graphs of the component equations, the explicit relation between the various kind of systems and their graphs will be stated in Section 7-3.

[pages 361-366]

Exercises 7-1. - Answers

1. Yes. Is $\begin{cases} 2(2) + 3(0) = 4 \\ 8(2) - 7(0) = 16 \end{cases}$ true?

$\begin{cases} 4 = 4 \\ 16 = 16. \end{cases}$ Yes.



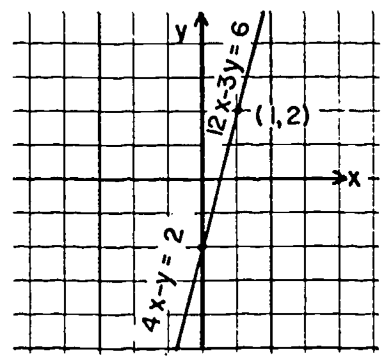
The graphs intersect at the point (2,0). This may be described in the following way,

$\{(x,y):2x + 3y = 4\} \cap \{(x,y):8x - 7y = 16\} = \{(2,0)\}.$

2. Yes. Is $\begin{cases} 4(1) - 2 = 2 \\ 12(1) - 3(2) = 6 \end{cases}$ true?

$\begin{cases} 4 - 2 = 2 \\ 12 - 6 = 6 \end{cases}$

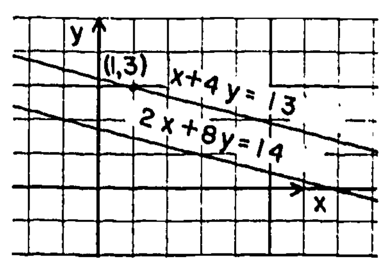
$\begin{cases} 2 = 2 \\ 6 = 6. \end{cases}$ Yes.



The graph of each is the same line. Any solution of one is a solution of the other. There is an infinite number of these number pairs in the solution set of the system.

3. No. Is $\begin{cases} 1 + 4(3) = 13 \\ 2(1) + 8(3) = 14 \end{cases}$ true?

$\begin{cases} 1 + 12 = 13 \\ 2 + 24 = 14. \end{cases}$ No.



The graphs of the two equations are parallel lines. The solution set of this system is the empty set.

4. (a) If b is any real number except b = 44 the solution set is empty.

(b) b = 44.

(c) b = 44.

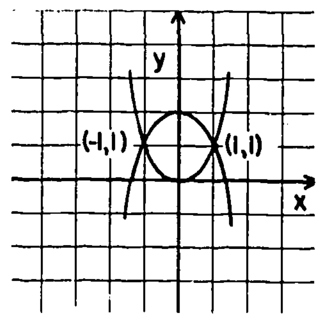
5. The solution set can be empty, (no solution) or contain at least one member (at least one common solution) or be the same as the solution set of one of the component equations (an infinite number of number pairs).

6. The following systems are consistent, but not dependent: b, c, e, f, g, h, i, k, ℓ. The following systems are also dependent: c, k, ℓ.

7. Yes.

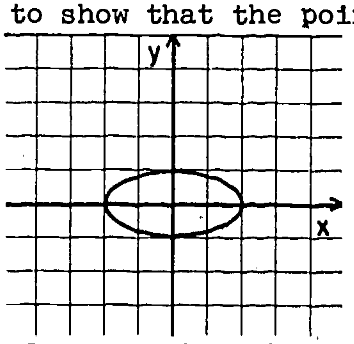
Yes. (-1,1)

Two.



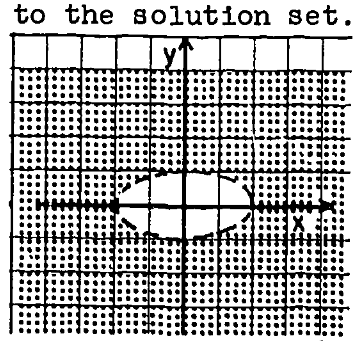
8. Dotted lines are used to show that points belonging to these lines are not included in the solution set; solid lines are used to show that the points belong to the solution set.

(a)



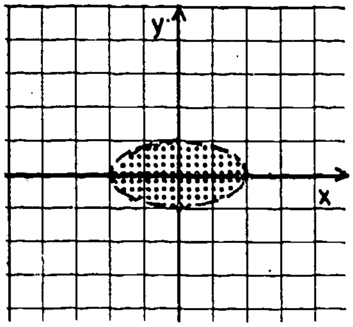
The solution set is the set of ordered pairs which are coordinates of points on the ellipse.

(b)

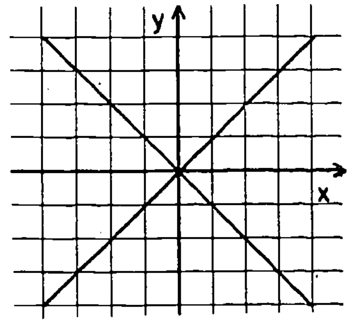


The solution set is the set of ordered pairs which are coordinates of points in the shaded region.

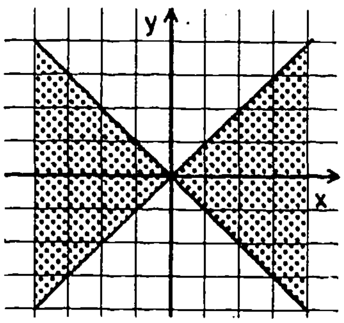
(c)



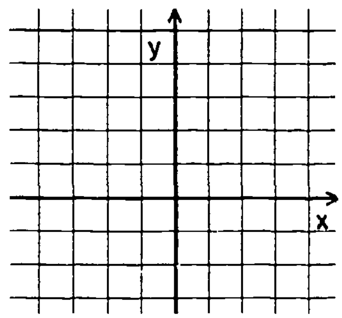
(d)



(e)

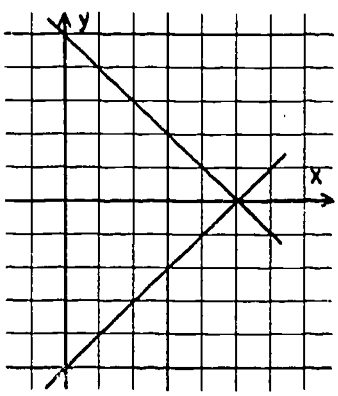


(f)

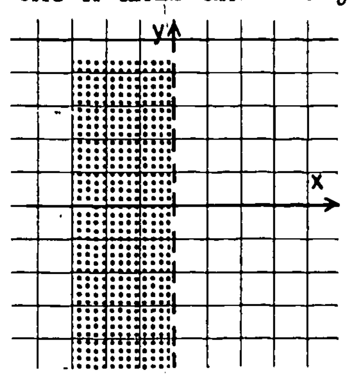


The solution set is the set of ordered pairs of the x-axis and the y-axis.

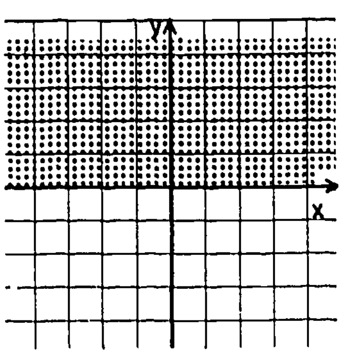
(g)



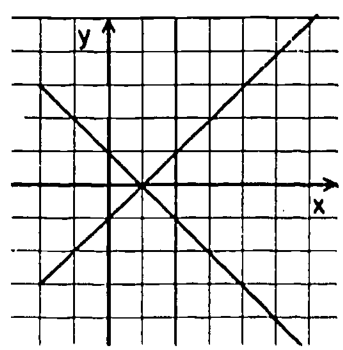
(h)



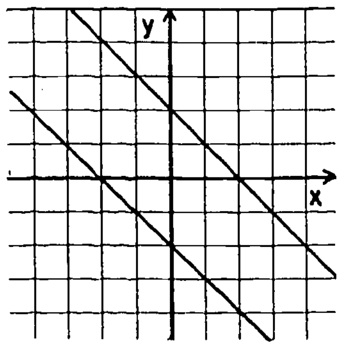
(i)



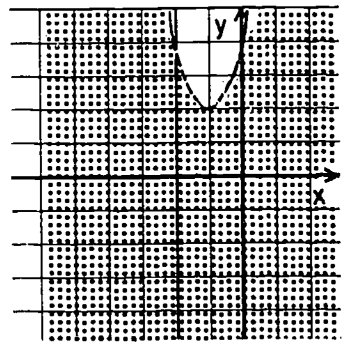
(j)



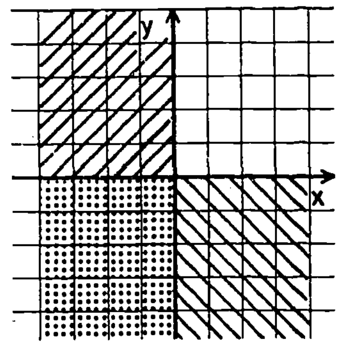
(k)



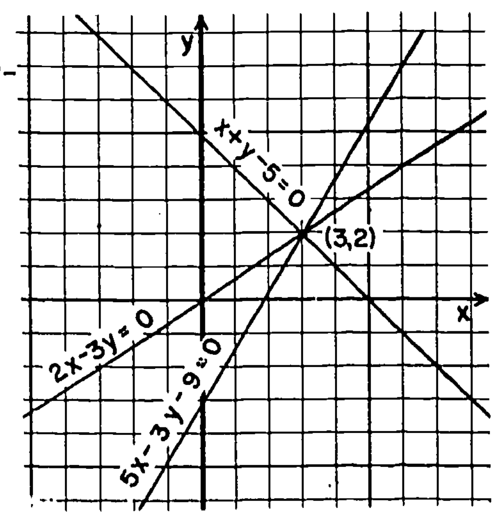
(l)



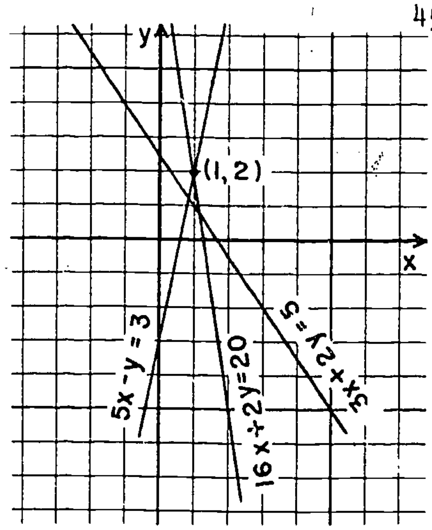
(m) The solution set consists of the ordered pairs which are coordinates of the points in the <u>entire</u> shaded region <u>and</u> the x-axis. Suggest to the students that they consider the problem with the union symbol U be replaced with the intersection symbol ∩. The solution set of the new problem is the ordered pairs which are coordinates of the points in the "dotted" region.



9. Yes. When the graphs are sketched, the lines intersect at the point whose coordinates are (3,2).



10. No. The graphs of two of the equations, $5x - y = 3$ and $16x + 2y = 20$ pass through the point whose coordinates are $(1,2)$; however, the graph of the equation $3x + 2y = 5$ does not.



11. The graphs of the components of the system must intersect at a common point.

*12. One would probably expect the solution set of $m(ax + by - c) + n(dx + ey - f) = 0$ to also contain the single element in the solution set of the given system. In Problem 9, it was found that $(3,2)$ was an element of the solution set $\begin{cases} 2x - 3y = 0 \\ x + y - 5 = 0 \\ 5x - 3y - 9 = 0. \end{cases}$

For any $m \neq 0$ and $n \neq 0$ such as $m = 3$, $n = 2$, $3(2x - 3y) + 2(x + y - 5) = 0$ has $(3,2)$ as an element of its solution set.

Is $3(2 \cdot 3 - 3 \cdot 2) + 2(3 + 2 - 5) = 0$ true?

$3(0) + 2(0)$

$0 = 0$ yes.

In Problem 12, if (x_1,y_1) is any solution of the given system, then $dx_1 + ey_1 - f = 0$ and $ax_1 + by_1 - c = 0$. Hence (x_1,y_1) satisfies $m(ax + by - c) + n(dx + ey - f) = 0$ because, $m(ax_1 + by_1 - c) + n(dx_1 + ey_1 - f) = m \cdot 0 + n \cdot 0 = 0$

13. $\left[\frac{1 + i\sqrt{15}}{2}, \frac{-7 + i\sqrt{15}}{2} ; \frac{1 - i\sqrt{15}}{2}, \frac{-7 - i\sqrt{15}}{2} \right]$

14. $\{(3 + 3i, 3 - 3i)\}$.

7-2. Equivalent Equations and Equivalent Systems of Equations.

The important ideas here are equivalent equations, systems of equations, and the two principles which give equivalent systems. These roles stated as principles allow us to proceed from one system to an equivalent one until we reach one for which the solution set is obvious. The idea of linear combination is worth some time. Here we use it to justify the elimination method of solving systems of linear equations. However, it is a very useful mathematical idea, and is used in many different branches of mathematics.

Exercises 7-2. - Answers

1. (a) No, since the solution sets $x = 2$ and $x = \frac{3}{2}$ are not the same, since the solution set of $x = 2$ is $\{2\}$ and that of $x = \frac{3}{2}$ is $\{\frac{3}{2}\}$.

   (b) No, since $\{(\frac{12}{7}, \frac{12}{7})\}$ is the solution set for both equations.

   (c) Yes, since $\{3\}$ is the solution set for both equations.

   (d) Yes, since every ordered pair of numbers which belong to the solution set of $8x - 10 = 2y$ belong to the solution set of $4x - y = 5$.

   (e) No, since the solution sets are not the same.

   (f) No, since the solution set of $x = -\sqrt{y + 3}$ contains only negative values of x while the solution set of $x^2 = y + 3$ contains positive and negative values of x.

   (g) No, since the solution set of $x = \sqrt{y - 6}$ contains only positive values of x while the solution set of $x^2 = y - 6$ contains positive and negative values of x.

   (h) No, since the solution set of $y = |x - 2|$ has only positive values of y and $y = x - 2$ has positive and negative values of y.

[pages 370-376]

(i) No, since the solution sets of the equations are not the same. Notice that the only ordered pairs which belong to both solution sets are (0,0), (1,1), (-1,1).

(j) No, since the solution sets are not the same.
$xy + x^2$ has the solution set {(0,a), (a,-a)}
$x = -y$ has the solution set {(a,-a)}

(k) No, since the solution set of the first equation is {6,-2} while that of the second equation is {6}.

2. Yes, since {(3,5)} satisfies both systems.

3. Yes, since {(6,2)} satisfies both systems.

4. (a) Yes, since {(8,2)} is the solution set of both systems.

(b) Yes, since {(3,-3)} is the solution set of both systems.

(c) Yes, since {(3,-2)} is the solution set of both systems.

(d) Yes, since {(-4,3)} is the solution set of both systems.

(e) Yes, since {(1,3)} is the solution set of both systems.

(f) Yes, since the solution sets are the same.

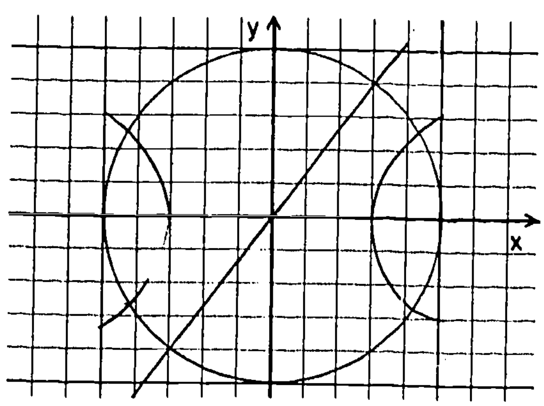
(g) Yes, since $\{(1,-\frac{1}{2})\}$ is the solution set of both systems.

(h) No, since the first system has the {(-10,2)} as its solution set and the second system has {(-10,2),(-10,2), (10,-2),(10,2)} as its solution set.

(i) No, since the solution sets are not the same.

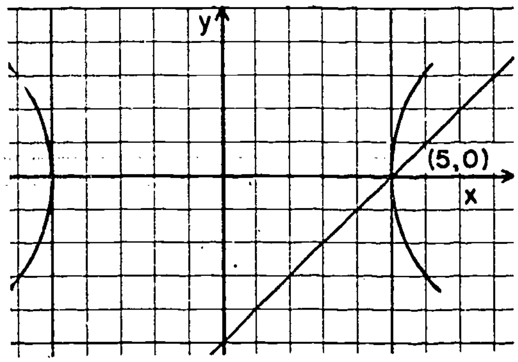
(j) No, since the solution sets are not the same.

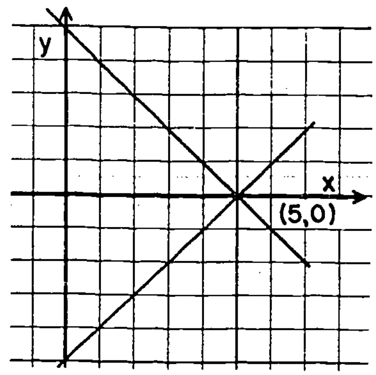
5. No.



[pages 376-378]

6.

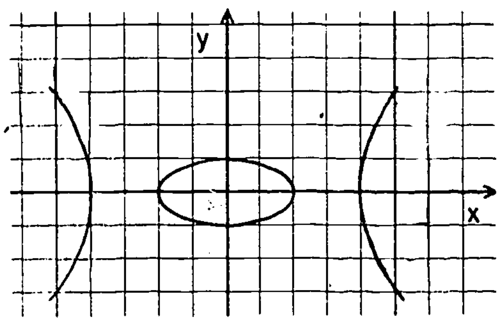


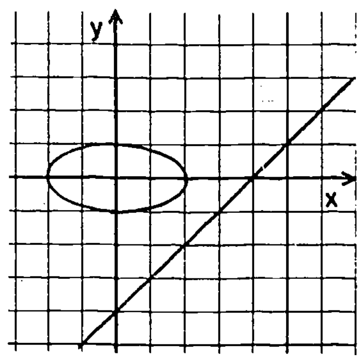


Yes. Another system could be $\begin{cases} 5y = x - 5 \\ 3x + 5y = 15. \end{cases}$

Yes.

7.





Since the graphs of the equations in system (i) do not intersect, the solution set when restricted to real number pairs is the empty set. This is also true for the solution set of system (ii). The empty set is equivalent to the empty set. Hence, the two systems are equivalent. Note: when not restricted to the real numbers, the solution set of each system contains complex number pairs which are not the same for both systems. So systems (i) and (ii) are not equivalent.

8. (a) $a = 1$, $b = -1$; $a = 2$, $b = -2$; a equal to the opposite of b.

(b) $a = 3$, $b = -7$.

(c) $a = 3$, $b = 2$.

(d) $a = 2$, $b = 9$.

[page 378]

9. (a) $a = 1$, $b = 1$

   (b) $a = 5$, $b = 1$

   (c) $a = 11$, $b = 5$

   (d) $a = -11$, $b = 5$

10. (a) $a = -2$, $b = 1$ will eliminate x.

   (b) $a = 3$, $b = 1$ will eliminate x.

   (c) $a = 5$, $b = 2$ will eliminate y.

   (d) $a = 1$, $b = -2$ will eliminate y.

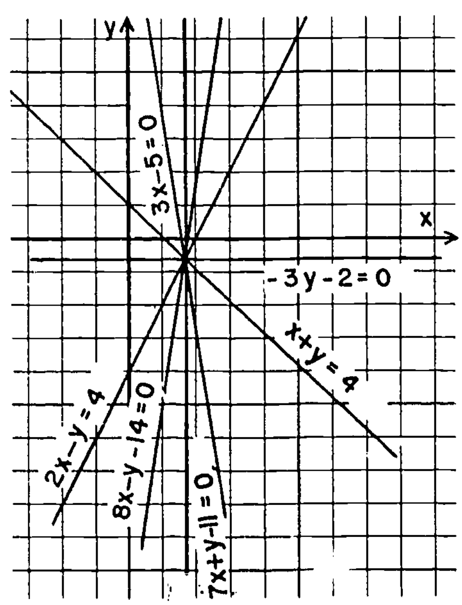
11. (a) $\begin{cases} x + y = 1 \\ a(x + y - 1) + b(2x - y - 4) = 0 \end{cases}$ $\begin{cases} 2x - y = 4 \\ a(x + y - 1) + b(2x - y - 4) = 0 \end{cases}$

   (b) if $a = 2$, $b = 3$ $a = 3$, $b = 2$

   $\begin{cases} x + y = 1 \\ 8x - y - 14 = 0 \end{cases}$ $\begin{cases} 2x - y = 4 \\ 7x + y - 11 = 0 \end{cases}$

   (c) if $a = 1$, $b = 1$ if $a = -2$, $b = 1$

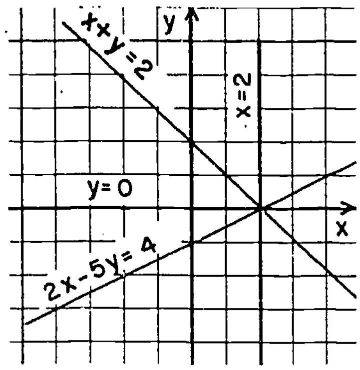
   $\begin{cases} x + y = 1 \\ 3x - 5 = 0 \end{cases}$ $\begin{cases} 2x - y = 4 \\ -3y - 2 = 0 \end{cases}$



[page 379]

12. (a) $a = 5, \quad b = 1; \qquad a = 1, \quad b = -1$

(b)



(c) $\{(2,0)\}$

13. (a) $\begin{cases} 2x - y - 4 = 0 \\ x - 2y + 7 = 0, \end{cases} \qquad \begin{cases} 1(2x - y - 4) + (-2)(x - 2y + 7) = 0 \\ 2x - y - 4 = 0, \end{cases}$

$\begin{cases} y - 6 = 0 \\ 2x - y - 4 = 0, \end{cases} \qquad \begin{cases} y - 6 = 0 \\ 1(2x - y - 4) + 1(y - 6) = 0, \end{cases}$

$\begin{cases} y - 6 = 0 \\ x - 5 = 0, \end{cases} \qquad \begin{cases} y = 6 \\ x = 5. \end{cases}$

(b) $(3,-2)$

(c) $(25,7.5)$

(d) inconsistent. Solution set is the empty set.

(e) $(-2,5)$

(f) $(3,\frac{5}{2})$

(g) dependent. Solution set is every real ordered pair $(a, \frac{2}{3}a - 5)$

(h) $(-5,0)$

(i) $(-\frac{3}{2},-2)$

(j) $(\frac{25}{11},-\frac{2}{11})$

14. If System (1) is equivalent to System (2) then (x_1,y_1) will satisfy both systems and if System (2) is equivalent to System (3) then (x_1,y_1) will satisfy both systems. Therefore, System (1) and System (3) are satisfied by (x_1,y_1). Since System (1) and System (3) have the same solution set they are equivalent systems.

15. We must prove that any solution of (i) $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ is a solution of (ii) $\begin{cases} af(x,y) + bg(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ and any solution of (ii) is a solution of (i).

If (x_1,y_1) satisfies (i) then $f(x_1,y_1) = 0$ and $g(x_1,y_1) = 0$. Hence, by 7-2b $af(x_1,y_1) + bg(x_1,y_1) = 0$ which means that (x_1,y_1) satisfies the component equations of (ii).

If (x_1,y_1) satisfies (ii) we have $af(x_1,y_1) + bg(x_1,y_1) = 0$ and $g(x_1,y_1) = 0$. Since $a \neq 0$ it follows that $f(x_1,y_1) = 0$ which means that (x_1,y_1) satisfies the component equations of (i).

16. Suppose (x_1,y_1) is a pair of values satisfying

(i) $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ so that $f(x_1,y_1) = 0$, $g(x_1,y_1) = 0$.

Then it will follow that $af(x_1,y_1) + bg(x_1,y_1) = 0$ and $cf(x_1,y_1) + dg(x_1,y_1) = 0$, so that each solution of (i) is a solution of

(ii) $\begin{cases} af(x,y) + bg(x,y) = 0 \\ cf(x,y) + dg(x,y) = 0. \end{cases}$

Conversely, suppose that $ad - bc \neq 0$ and that (x_1,y_1) is a pair satisfying (ii). Then

$$\begin{cases} af(x_1,y_1) + bg(x_1,y_1) = 0 \\ cf(x_1,y_1) + dg(x_1,y_1) = 0. \end{cases}$$

Thus $d\left(af(x_1,y_1) + bg(x_1,y_1)\right) - b\left(cf(x_1,y_1) + dg(x_1,y_1)\right) = 0$ which reduces to $(ad - bc) \cdot f(x_1,y_1) = 0$. Since $ad - bc \neq 0$ it follows that $f(x_1,y_1) = 0$.

Similarly, we may show that $g(x_1,y_1) = 0$. Hence, each solution of (ii) is a solution of (i) and we conclude that (i) and (ii) are equivalent.

7-3. Systems of Linear Equations.

This is the time to review the methods students have seen for solving systems of linear equations. Much facility in solving such systems is necessary in many applications of mathematics. While the text concentrates on the geometric interpretation of the various possible solution sets, it is important also to review the algebraic methods of solution.

In the discussion of Case III, the teacher may want to introduce determinants simply as a way of remembering the solution.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1, \quad \begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix} = -c_1b_2 + c_2b_1$$

$$\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix} = -a_1c_2 + a_2c_1.$$

Then, the solution set can be written,

$$\left(\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \right).$$

We did not think it advisable to try to define determinants and develop their properties at this point. However, they are a very useful device for remembering this general solution.

Exercises 7-3. - Answers

1. (a) Since if $a_1b_2 - a_2b_1 \neq 0$, by Case III the lines intersect and if $a_1 = 5$, $b_1 = 4$, $a_2 = 2$, and $b_2 = -7$ we have

$$5(-7) - 2(4) \neq 0$$
$$-35 - 8 \neq 0$$
$$43 \neq 0$$

and the graphs of the component equations of the system $\begin{cases} 5x + 4y + 7 = 0 \\ 2x - 7y + 5 = 0 \end{cases}$ intersect in one point, and therefore the system is consistent.

(b) If $a_1b_2 - a_2b_2 = 0$ the lines are parallel

$3(2) - 2(3) = 0$ and the graphs of the system $\begin{cases} 3x + 3y + 1 = 0 \\ 2x + 2y + 1 = 0 \end{cases}$ are parallel.

If the graphs are parallel the system is inconsistent.

(c) The system $\begin{cases} 3x = 1 - 2y \\ \frac{9}{2}x - 6y = 3 \end{cases}$ is equivalent to the system $\begin{cases} 3x + 2y - 1 = 0 \\ 9x - 12y - 6 = 0 \end{cases}$ and

$a_1b_2 - a_2b_1 = 3(-12) - 9(2) = -54 \neq 0$ and therefore the lines intersect and the system is consistent.

(d) parallel and inconsistent

(e) intersect and consistent

(f) intersect and consistent

(g) parallel and inconsistent

(h) parallel and inconsistent

(i) same line and dependent

(j) parallel and inconsistent

2. (a) $\begin{cases} x + 3y - 9 = 0 \\ x - 3y + 3 = 0 \end{cases}$ since $a_1b_2 - a_2b_1 = -3 - 3 = -6 \neq 0$

the system is consistent and the solution set

$$\left\{\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)\right\}$$ by Case III.

$$\left\{\left(\frac{9 - 27}{-3 - 3}, \frac{-9 - 3}{-3 - 3}\right)\right\}$$

$\{(3,2)\}$

It is expected that at this point, the students will use the results discussed in Case III. However, the solution set may be found by the method used in 7-2 by using the principle of linear combination and equivalent systems.

(b) $\begin{cases} 4x + y - 5 = 0 \\ 2x - 3y - 13 = 0 \end{cases}$ consistent $\{(2,-3)\}$

(c) $\{(4,\frac{1}{3})\}$

(d) $\{(-2,-1)\}$

(e) $\{(1,-2)\}$

(f) $\{(-\frac{2}{3}, -\frac{2}{3})\}$

(g) $\{(0,4)\}$

(h) $\{(8,-12)\}$

(i) Since by Case II $a_1b_2 = a_2b_1$ i.e. $1(4) = 4(1)$, the system is inconsistent, and the solution set is the empty set.

(j) $\{(160,-110)\}$

(k) $\{(3,2)\}$

(l) Since the corresponding coefficients are proportional, $k = 4$ (or $\frac{1}{4}$), by Case I, the system is dependent.

(m) $\{(0,-\frac{5}{3})\}$

(n) $\{\frac{148}{17}, \frac{112}{17}\}$

(o) $\{(\frac{c + d}{2b},\frac{d - c}{2a})\}$

(p) $\{(\frac{b^2 + a^2}{2a + b},\frac{2b - a}{2a + b})\}$

*3. We consider four cases, (I) $b_1 \neq 0$ and $b_2 \neq 0$; (II) $b_1 \neq 0$ and $b_2 = 0$; (III) $b_1 = 0$ and $b_2 \neq 0$; (IV) $b_1 = 0$, $b_2 = 0$.

Case I

Since $a_1b_2 - a_2b_1 = 0$, we have $a_1 = \frac{b_1}{b_2} a_2$. Define $k = \frac{b_1}{b_2} \neq 0$, then $a_1 = ka_2$, $b_1 = kb_2$ and since $b_1c_2 - b_2c_1 = 0$ we also have $c_1 = kc_2$.

Case II

Since $a_1b_2 - a_2b_1 = 0$ and $b_2 = 0$ we have $a_2b_1 = 0$. But $b_1 \neq 0$, hence $a_2 = 0$. But this is impossible since $a_2^2 + b_2^2 > 0$. Thus Case (II) is impossible.

Case III

The same argument as used in Case (II) shows this case is also impossible.

Case IV

Since $b_1 = b_2 = 0$ we must have $a_1 \neq 0$ and $a_2 \neq 0$. Defining $k = \frac{a_1}{a_2} \neq 0$ and using $a_1b_2 - a_2b_1 = 0$ and $a_2c_1 - a_1c_2 = 0$ we see that $a_1 = ka_2$, $b_1 = kb_2$, and $c_1 = kc_2$.

4. $4\frac{1}{2}$ miles per hour in still water

   $1\frac{1}{2}$ miles per hour (rate of stream)

5. Width, 60 rods

   Length, 80 rods

6. 52° and 33°

7. A's rate $6\frac{7}{8}$ miles per hour

   B's rate $3\frac{1}{8}$ miles per hour

8. 12 pounds of 1st. alloy

   2 pounds of 2nd. alloy

9. (a) $7\frac{1}{2}$ and $4\frac{1}{2}$

   (b) $\frac{1}{10}$ and $\frac{1}{14}$

10. $s_o = 10,650$

    $v_o = -50$ ft./sec.

    (Note: The negative velocity should be interpreted as a body which is falling at the rate of 50 ft./sec.).

11. Every line which passes through the point of intersection of the lines whose equations are $4x + y = 2$ and $2x - 3y = 8$ will have the equation $a(4x + y - 2) + b(2x - 3y - 8) = 0$ so if the equation must also pass through the origin $(0,0)$ we have $a\Big(4(0) + (0) - 2\Big) + b\Big(2(0) - 3(0) - 8\Big) = 0$

$$-2a - 8b = 0$$

$$a = -4b$$

$$-4b(4x + y - 2) + b(2x - 3y - 8) = 0$$

$$-16x - 4y + 8 + 2x - 3y - 8 = 0$$

$$-14x - 7y = 0$$

$$-7y = 14x$$

$$y = -2x$$

12. Every line which passes through the point of intersection will have the equation $a(x + 4y - 2) + b(2x + 3y + 1) = 0$ so if it must also pass through the point $(5,4)$ we have

$$a(5 + 16 - 2) + b(10 + 12 + 1) = 0$$
$$19a + 23b = 0$$
$$b = \frac{-19a}{23}$$
$$a(x + 4y - 2) - \frac{19}{23}a(2x + 3y + 1) = 0$$
$$23x + 92y - 46 - 38x - 57y - 19 = 0$$
$$-15x + 35y - 65 = 0$$
$$3x - 7y + 13 = 0$$

7-4. Systems of One Linear and One Quadratic Equation.

The method used in this Section is really substitution. We use the letter a to represent the first element of an ordered pair which satisfies the linear equation. The second element will then be a linear expression such as $2a + 3$ as in Example 7-4a. After the values of a have been found, we obtain the corresponding second elements by substituting in the linear expression ($2a + 3$ in this example). Thus, for each first element a, we obtain exactly one second element to complete an ordered pair which will satisfy the linear equation. This eliminates the possibility of getting "extraneous solutions". These "solutions", may be encountered if we substitute in the second degree equation. Such substitution will assign two second elements for each first element and some y the resulting "solutions" may not check. The procedure we want students to avoid is illustrated by the following discussion of the system

$$\begin{cases} x^2 + y^2 = 25 \\ y = x + 1. \end{cases}$$

The solution set for the linear equation is $(a, a + 1)$. By the usual method, we obtain $a = 3$ or -4. If we substitute -4 in the quadratic equation we obtain

$$y^2 = 9 \rightarrow y = 3 \quad \text{or} \quad y = -3.$$

Thus we have the "solution", $(-4,3)$ and $(-4,-3)$. Clearly the first of these is not a member of our solution set because it does not satisfy the linear equation. Hence, the solution $(-4,3)$ is "extraneous" and is in fact a solution of the system

$$\begin{cases} x^2 + y^2 = 25 \\ y^2 = (x + 1)^2 \end{cases}$$

which is not equivalent to the original system.

Exercises 7-4. - Answers

1. (a) $\begin{cases} x^2 + y^2 = 50 \\ x - y = 0. \end{cases}$ If (x,y) belongs to the solution set of the system then it must be of the form (a,a) for some real number a, in order to belong to the solution set of the second equation. The pair will satisfy the first equation if and only if

$$a^2 + a^2 = 50$$
$$2a^2 = 50$$
$$a^2 = 25$$
$$a = \pm 5.$$

Hence, the solution set of the system is $\{(5,5), (-5,-5)\}$.

(b) $\begin{cases} x^2 - 4x + 3 = 0 \\ x - y + 1 = 0. \end{cases}$ The elements of the solution set of the second equation must have the form $(a, a + 1)$. The pair will satisfy the first equation also if and only if

$$a^2 - 4a + 3 = 0$$
$$(a - 3)(a - 1) = 0$$
$$a = 3 \quad a = 1.$$

Hence, the solution set of the system is $\{(3,4),(1,2)\}$.

(c) $\{(-3,2)\}$

(d) $\begin{cases} x^2 - y^2 = 0 \\ x + y = 0. \end{cases}$ The elements of the solution set of the second equation must have the form $(a,-a)$. The pair will satisfy the first equation if and only if

$$a^2 - a^2 = 0.$$

Since this equation is satisfied by every real number a, its solution set is the set of all real numbers. Therefore, the solution set of the original equation is the set of all pairs $(a,-a)$ where a is any real number.

(e) $\{(-\frac{3}{2},-4),(2,3)\}$

(f) $\begin{cases} y = 2x^2 \\ y + 1 = 2x. \end{cases}$ The elements of the second equation must have the form $(a,2a - 1)$. The pair will satisfy the first equation also, if and only if

$$2a - 1 = 2a^2$$
$$2a^2 - 2a + 1 = 0$$
$$a = \frac{2 \pm \sqrt{-4}}{4}.$$

But this equation is not satisfied by any real number a. Hence the solution set of the system is the empty set. (Note: When the solution set is not restricted to real number a, it is not the empty set, and the solution set contains complex number pairs.)

(g) $\{(-12 + \sqrt{41},-2 + \sqrt{41}),(-12 - \sqrt{41},-2 - \sqrt{41})\}$

(h) $\{(4,6),(-4,-6)\}$

(i) $\{(-2,6),(-12,1)\}$

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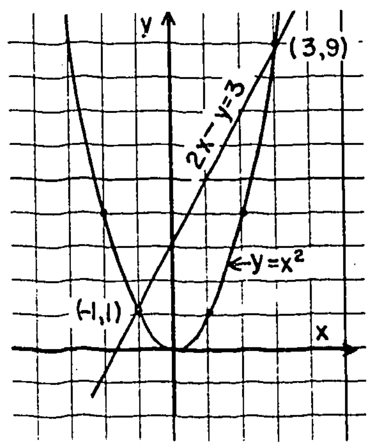
(j) The solution set is the set of all pairs (a,a) where a is any real number.

(k) $\{(-2,3), (\frac{38}{11}, \frac{63}{11})\}$

(l) The solution set is the set of all pairs (a,a + 1) where a is any real number.

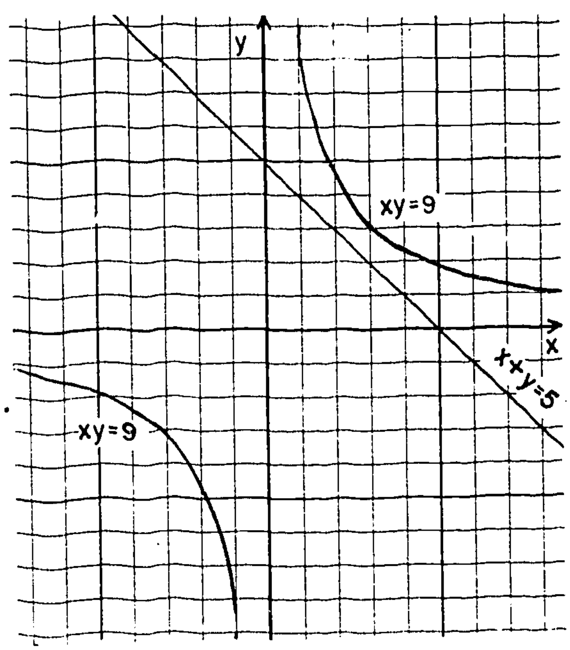
(m) $\{(-2,3)\}$

2. (a) $\begin{cases} y = x^2 \\ 2x - y = -3 \end{cases}$ has the solution set $\{(3,9),(-1,1)\}$



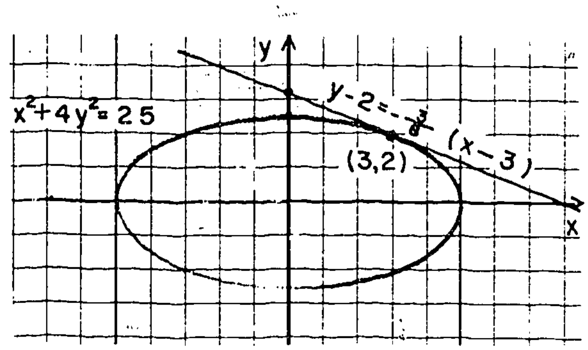
(b) $\begin{cases} xy = 9 \\ x + y = 5 \end{cases}$ has the empty set for the solution set when the members are restricted to real numbers. Otherwise, the solution set is

$\{(\frac{5 + i\sqrt{11}}{2}, \frac{5 - i\sqrt{11}}{2}),$
$(\frac{5 - i\sqrt{11}}{2}, \frac{5 + i\sqrt{11}}{2})\}.$



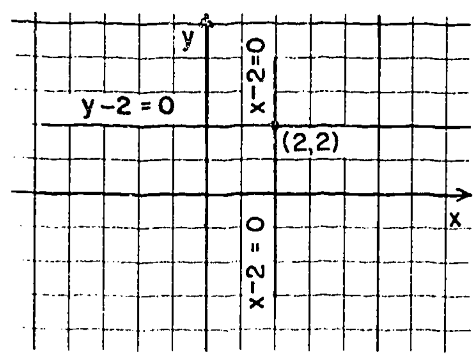
(c) $\begin{cases} x^2 + 4y^2 = 25 \\ y - 2 = -\frac{3}{8}(x - 3) \end{cases}$

has the solution set $\{(3,2)\}$.



(d) $\begin{cases} xy - 2x + 2y + 4 = 0 \\ x - 2 = 0 \end{cases}$

has the solution set $\{(2,b)\}$ where b is any real number.



3. (a) The line intersects the circle in two points.

(b) The line intersects the pair of lines in two points.

(c) The line intersects the parabola in one point.

(d) The graph of the first equation $x^2 - y^2 = 0$ is the graph of $x + y = 0$ and $x - y = 0$. Hence, the graph of the second equation is the same as the graph of $x + y = 0$ of the first equation.

(e) The line intersects the hyperbola in two points.

(f) The line and the parabola do not intersect.

(g) The line and the parabola intersect in two points.

(h) The line intersects the circle in two points.

(i) The line intersects the hyperbola in two points.

(j) The line is the same as one of the pair of intersecting lines.

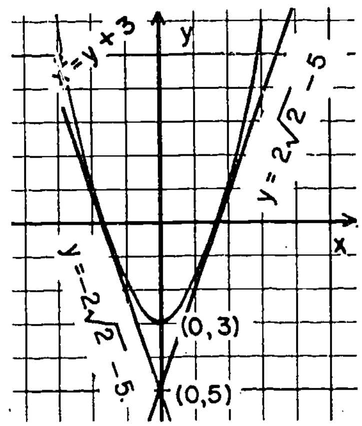
(k) The line intersects the hyperbola in two points.

(l) The two graphs have infinite many points in common.

(m) The line intersects the parabola in one point.

4. An equation of a line through the point whose coordinates are $(0,-5)$ is,

$$y = mx - 5.$$



The solution set of this equation is $(a, ma - 5)$ for some real $\underline{a}$ and $\underline{m}$. The coordinates $(a, ma - 5)$ will satisfy $x^2 = y + 3$, if and only if,

$$a^2 = (ma - 5) + 3$$

$$a^2 - ma + 2 = 0.$$

The condition for the line $y = mx - 5$ to be tangent is that the roots of $a^2 - ma + 2 = 0$, a quadratic equation for $\underline{a}$ must be equal. Hence, the discriminant must be 0. Then,

$$m^2 - 8 = 0$$

$$m^2 = 8$$

$$m = \pm 2\sqrt{2}.$$

There will be <u>two</u> lines passing through $(0,-5)$ and tangent to $x^2 = y + 3$, namely,

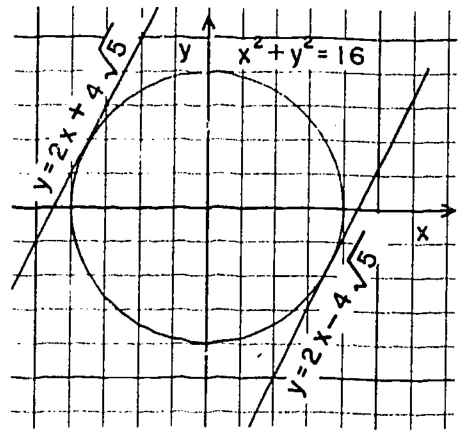
$$y = 2\sqrt{2}\,x - 5$$

$$y = -2\sqrt{2}\,x - 5$$

5. An equation of a line whose slope is 2 is $y = 2x + b$ the solution set of this equation is $(a, 2a + b)$ for some real $\underline{a}$ and $\underline{m}.$ The coordinates of $(a, 2a + b)$ will satisfy $x^2 + y^2 = 16$, if and only if

$$a^2 + (2a + b)^2 = 16$$
$$a^2 + 4a^2 + 4ab + b^2 = 16$$
$$5a^2 + 4ab + b^2 - 16 = 0.$$

The coordinates for the line $y = 2x + b$ to be tangent is that the roots of $5a^2 + 4ab + b^2 - 16 = 0$ must be equal. Hence, the discriminant must be 0. Then



$$16b^2 - 20(b^2 - 16) = 0$$
$$b^2 = 80$$
$$b = \pm 4\sqrt{5}.$$

There will be two lines with slope 2 tangent to $x^2 + y^2 = 16$, namely,

$$y = 2x + 4\sqrt{5}$$
$$y = 2x - 4\sqrt{5}.$$

6. If the line is tangent to the circle, then the system $\begin{cases} y = mx + k \\ x^2 + y^2 = r^2 \end{cases}$ must have a single element in its solution set. The elements of the solution set of $y = mx + k$ must have the form $(a, ma + k)$. The pair will satisfy $x^2 + y^2 = r^2$, if and only if,

$$a^2 + (ma + k)^2 = r^2$$
$$a^2 + m^2a^2 + 2mka + k^2 - r^2 = 0.$$

(i) $(1 + m^2)a^2 + 2mka + (k^2 - r^2) = 0.$

The condition for the line whose equation is $y = mx + k$ be tangent to the circle whose equation is $x^2 + y^2 = r^2$ is that the roots of the quadratic equation for a must be equal. Hence, the discriminant of the quadratic equation must be zero.

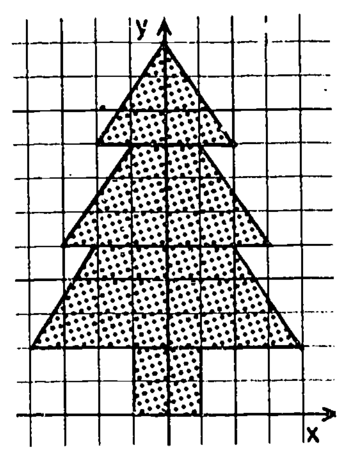
$$(2mk)^2 - 4(1 + m^2)(k^2 - r^2) = 0$$

$$4m^2k^2 - 4(k^2 + m^2k^2 - r^2 - m^2r^2) = 0$$

$$k = \pm r\sqrt{1 + m^2}.$$

7-5. Other Systems.

The time spent on this Section is probably best spent in using graphical methods for finding the solution sets of both systems of equations and inequalities. This affords a wonderful review of Chapter 6 and is usually quite enjoyable for both students and teacher. Here the teacher might suggest that the students write systems of equations whose solution sets will give a Christman tree, a jack-O'-lantern, a boat, and many other interesting figures. Another suggestion is to sketch a solution set (as show here), then ask the student to set up the equations (inequalities) of the system.



Exercises [illegible] - Answers

1. $\begin{cases} 4x^2 + y^2 - 100 = 0 \\ 4x - y^2 + 20 = 0 \end{cases}$ is equivalent to $\begin{cases} x^2 + x - 20 = 0 \\ 4x - y^2 + 20 = 0. \end{cases}$

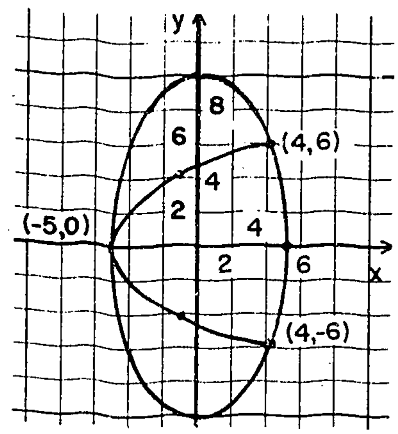
This system can be written as, $\begin{cases} (x + 5)(x - 4) = 0 \\ 4x - y^2 + 20 = 0. \end{cases}$

The solution set of the first equation are of the form $(-5, b)$ or $(4, b)$ for some real b. These are elements of the solution set of $4x - y^2 + 20 = 0$, if and only if $4(-5) - b^2 + 20 = 0$

$b = 0$

or $4(4) - b^2 + 20 = 0$

$b = \pm 6.$



Hence, the solution set of the system is $\{(-5,0), (4,6), (4,-6)\}$.

2. $\begin{cases} x^2 + 4y^2 = 100 \\ 2y^2 = 50 - \frac{1}{2}x^2 \end{cases}$ is equivalent to $\begin{cases} x^2 + 4y^2 = 100 \\ x^2 + 4y^2 = 100 \end{cases}$

hence, any ordered pair having the form $\left(a, \frac{\sqrt{100 - a^2}}{2}\right)$ will satisfy both equations for any real number a. Two ellipses which coincide.

3. $\begin{cases} x^2 + y^2 = 20 \\ \frac{1}{2}x^2 = 3 - \frac{1}{2}y^2 \end{cases}$ is equivalent to $\begin{cases} x^2 + y^2 = 20 \\ x^2 + y^2 = 6 \end{cases}$

by linear combination $a = 1$ $b = -1$

$$x^2 + y^2 - 20 - x^2 - y^2 + 6 = 0$$

$$14 = 0$$

hence, the solution set is the empty set. Two circles which are concentric.

4. $\begin{cases} x^2 + y^2 - 25 = 0 \\ y^2 - x^2 - 2x - 1 = 0 \end{cases}$ if $a = 1$ $b = -1$ is equivalent to

$\begin{cases} x^2 + x - 12 = 0 \\ x^2 + y^2 - 25 = 0 \end{cases}$ is equivalent to $\begin{cases} (x + 4)(x - 3) = 0 \\ x^2 + y^2 - 25 = 0 \end{cases}$

hence, the solution set $\{(-4,3),(-4,-3),(3,4),(3,-4)\}$ geometrically this is two intersecting straight lines, intersecting the circle at four distinct points.

5. $\begin{cases} x^2 - 5xy + 4y^2 = 0 \\ xy = 1. \end{cases}$ Since no ordered pair with first element zero satisfies the second equation, this system is equivalent to the system $\begin{cases} x^2 - 5xy + 4y^2 = 0 \\ y = \frac{1}{x}. \end{cases}$

The solution set of $y = \frac{1}{x}$ is of the form $(a,\frac{1}{a})$, therefore, it is a member of the solution set of the system, if and only if

$$a^2 - 5a(\tfrac{1}{a}) + \frac{4}{a^2} = 0$$

$$a^4 - 5a^2 + 4 = 0$$

$$(a^2 - 4)(a^2 - 1) = 0$$

$$(a - 2)(a + 2)(a - 1)(a + 1) = 0$$

hence, the pair $(a,\frac{1}{a})$ belongs to the solution set, if and only if $a = 2$, $a = -2$, $a = 1$, or $a = -1$. Hence, the solution set is $\{(2,\frac{1}{2}),(-2,-\frac{1}{2}),(1,1),(-1,-1)\}$.

6. (a) $\{(\sqrt{10},\sqrt{6}),(\sqrt{10},-\sqrt{6}),(-\sqrt{10},\sqrt{6}),(-\sqrt{10},-\sqrt{6})\}$

(b) $\{(\sqrt{37},5),(-\sqrt{37},5),(4,2),(-4,2)\}$

(c) $\{(4,\sqrt{5}),(4,-\sqrt{5}),(-4,\sqrt{5}),(-4,-\sqrt{5})\}$

(d) $\{(1,-2),(-1,2),(\frac{3\sqrt{2}}{2},\frac{\sqrt{2}}{2}),(-\frac{3\sqrt{2}}{2},-\frac{\sqrt{2}}{2})\}$

(e) Empty set if considering only real roots.
$\{(2,3i),(2,-3i),(-2,3i),(-2,-3i)\}$

(f) $\{(1,0),(-1,0)\}$

(g) $\{(\frac{1}{2},-1),(-\frac{1}{2},1)(\sqrt{2},\frac{\sqrt{2}}{3})(-\sqrt{2},\frac{-\sqrt{2}}{3})\}$

(h) $\{(4,1),(-4,-1),(2,2),(-2,-2)\}$

(i) $\{(5,3),(5,-3),(-5,3),(-5,-3)\}$

(j) $\{(2,-2),(-2,2),(\frac{2\sqrt{3}}{3},\frac{4\sqrt{3}}{3}),(\frac{2\sqrt{3}}{3},-\frac{4\sqrt{3}}{3})\}$

(k) $\{(2,-2)(3,-3)\}$

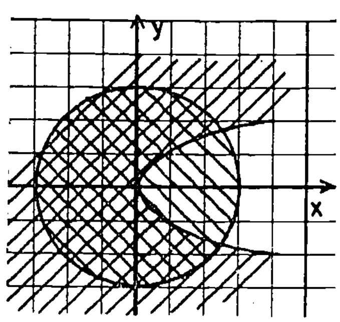
(l) $\{(1,2),(-4,2),(4,-2),(-1,-2)\}$

(m) $\{(6,0),(-6,0),(0,6),(0,-6)\}$

(n) $\{(1,1),(-1,-1),(i,0),(-i,0)\}$

7. The solution set of each system is the <u>intersection</u> of the solution set of each of the component equations.

(a)



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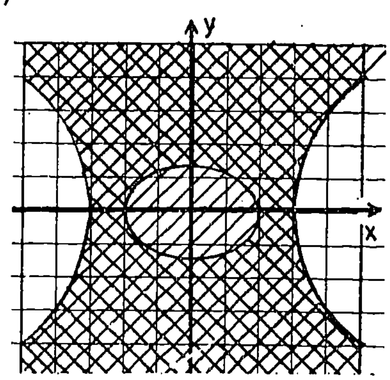

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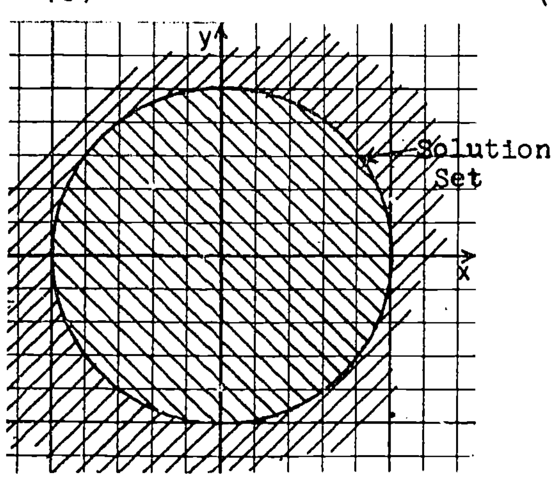

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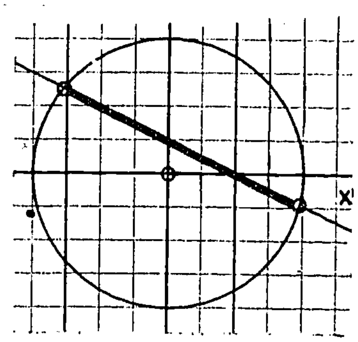

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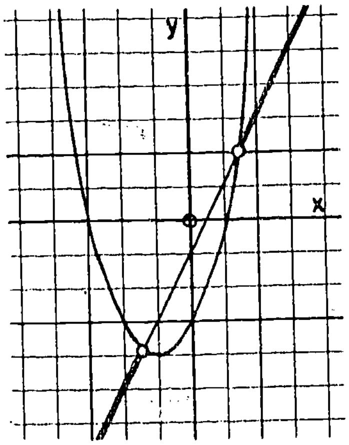

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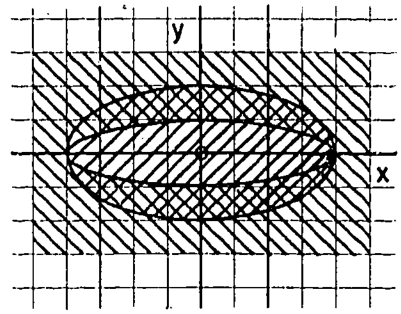

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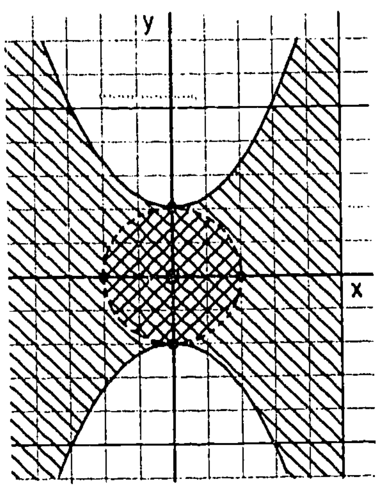

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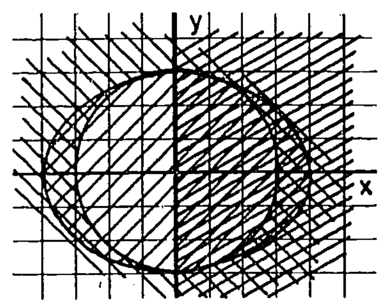
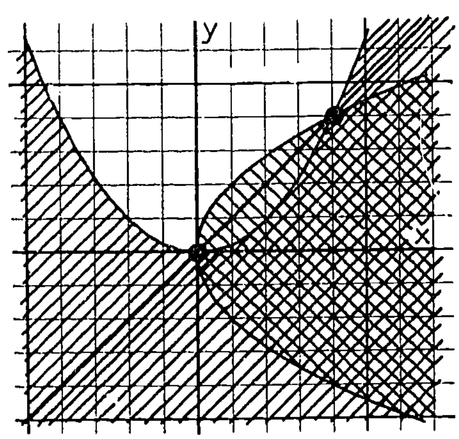
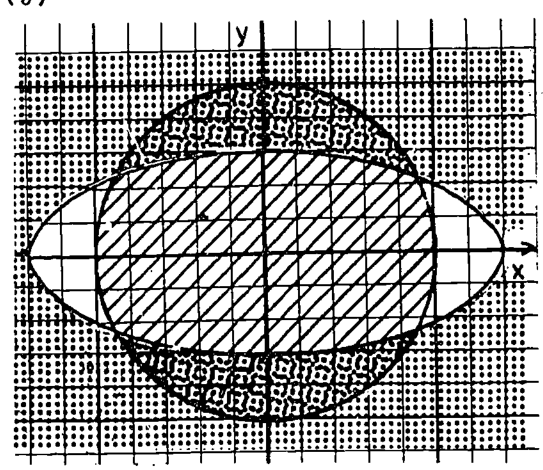


(i)

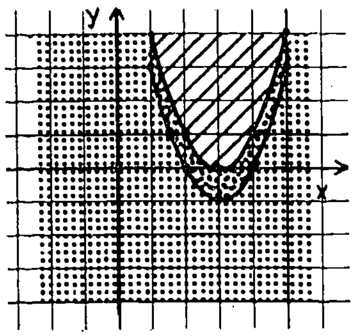


(j)



[pages 406-407]

(k)



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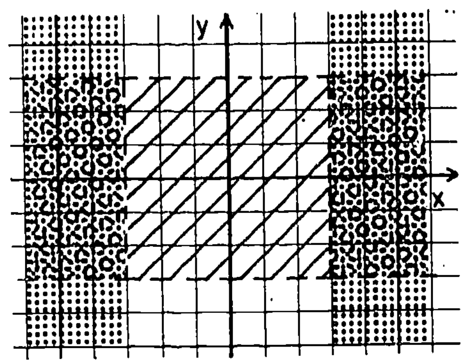

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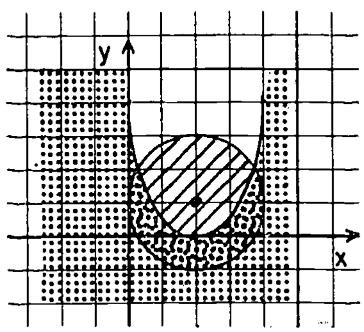
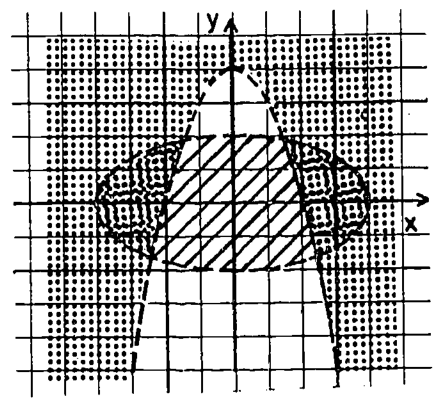


(n)



7-6. Supplementary Exercises - Answers

1. (a) $[(4,-1)]$

(b) $\{(\frac{41}{19},-\frac{9}{19})\}$

(c) $\{(\frac{22}{13},\frac{25}{13})\}$

(d) $\{(\frac{1}{2},8)\}$

(e) Empty set

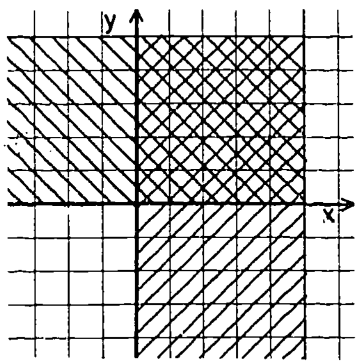
(f) $\{(\frac{ad + 3b}{3(2 + a)},\frac{2d - 3b}{3(2 + a)})\}$

(g) $\{(\frac{2v + 3rw}{sr + 1},\frac{2vs - 3w}{sr + 1})\}$

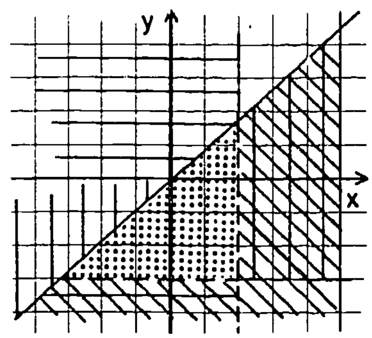
(h) $\{(\frac{ed - fb}{ad - cb},\frac{af - ce}{ad - cb})\}$

2. The solution set of each of the systems are the number pairs which are coordinates of the points in the dotted region. This is the intersection of the solution sets of the component inequalities.

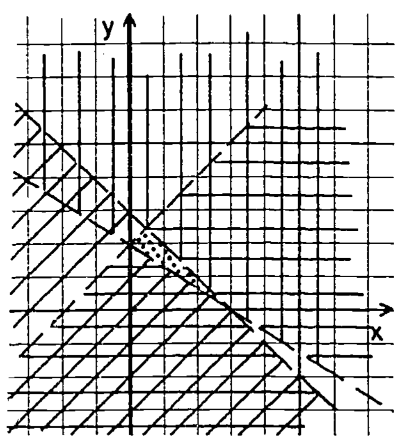
(a)



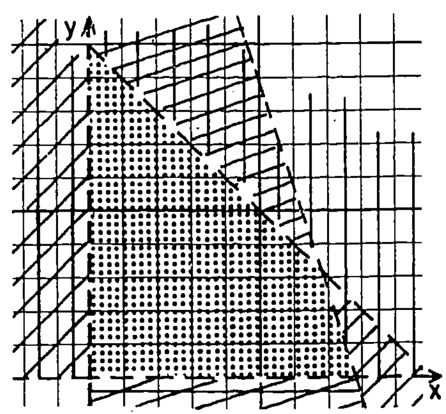
(b)



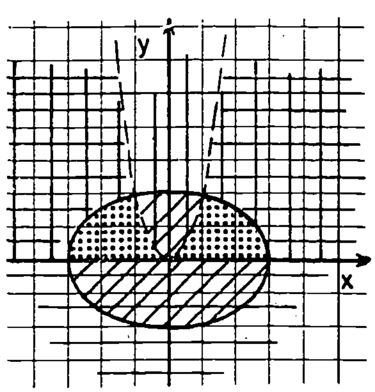
(c)



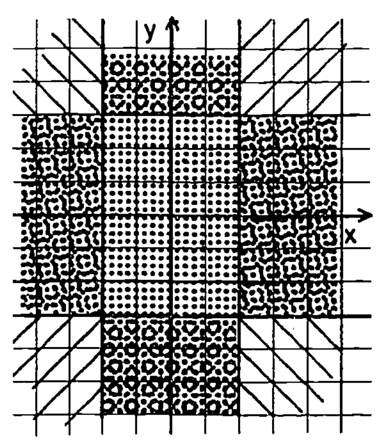
(d)



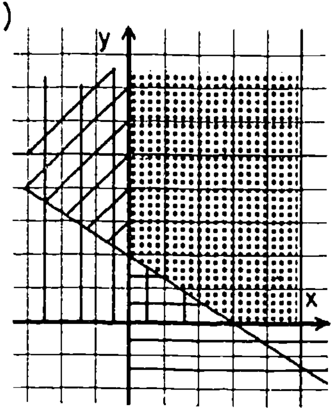
(e)



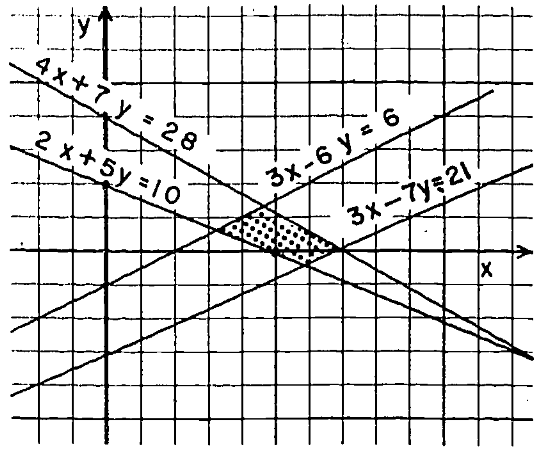
(f)



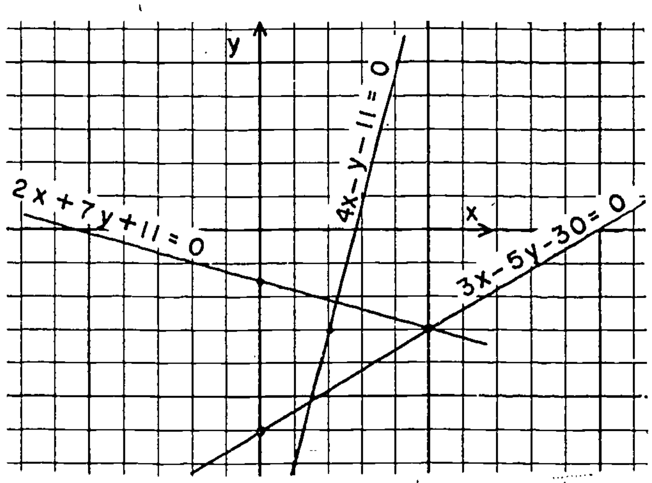
(g)



(h)



(i)



Solution set is null set.

(j) Null set. Complex roots: $\{(2i\sqrt{6}, -i\sqrt{6}), (-2i\sqrt{6}, i\sqrt{6})\}$

(k) Null set. Complex roots: $\{(\frac{3i\sqrt{15}}{5}, \frac{4\sqrt{10}}{5}), (\frac{3i\sqrt{15}}{5}, \frac{-4i\sqrt{10}}{5}),$
$(\frac{-3i\sqrt{15}}{5}, \frac{4\sqrt{10}}{5}), \quad (\frac{-3i\sqrt{15}}{5}, \frac{-4\sqrt{10}}{5})\}$

(l) $\{(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{3}), (\frac{\sqrt{30}}{3}, -\frac{\sqrt{30}}{3}), (-\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{3}), (-\frac{\sqrt{30}}{3}, -\frac{\sqrt{30}}{3})\}$

(m) $\{(5,-5), (0,-10)\}$

(n) $\{(2,3), (-2,-3), (1,-1), (-1,1)\}$

(o) $\{(0,-5), (+2\sqrt{6},1), (-2\sqrt{6},1)\}$

(p) $\{(2,\frac{4}{3}), (-2,-\frac{4}{3}), (\sqrt{2},\sqrt{2}), (-\sqrt{2},-\sqrt{2})\}$

(q) $\{(1,-4),(-1,4),(\frac{1}{\sqrt{2}},-3\sqrt{2}),(-\frac{1}{\sqrt{2}},3\sqrt{2})\}$

(r) $\{(0,-6),(\sqrt{11},5),(-\sqrt{11},5)\}$

(s) $\{(\frac{2}{3},5),(-\frac{2}{3},-5),(\frac{2}{3},-5),(-\frac{2}{3},5)\}$

(t) $\{(1,4),(4,1)\}$

(u) $\{(-1,1)\}$

(v) $\{(\frac{12}{5}\sqrt{5},\ \frac{12}{5}\sqrt{5}),(-\frac{12}{5}\sqrt{5},\frac{12}{5}\sqrt{5}),(\frac{12}{5}\sqrt{5},-\frac{12}{5}\sqrt{5}),$
$(-\frac{12}{5}\sqrt{5},-\frac{12}{5}\sqrt{5})\}$

(w) $\{(\sqrt{\frac{12}{31}},\frac{3}{2}\sqrt{\frac{12}{31}}),(-\sqrt{\frac{12}{31}},-\frac{3}{2}\sqrt{\frac{12}{31}})\}$

(x) $\{(1,-4),(\frac{19}{5},\frac{8}{5})\}$

(y) $\{(4,2),(\frac{22}{7},-\frac{4}{7})\}$

*3. The solution set of $y = mx + k$ is in the form, $(a, ma + k)$ for real $\underline{a}$, $\underline{m}$, and $\underline{k}$. The pair belongs to the solution set of the system, $\begin{cases} y = x^2 \\ y = mx + k \end{cases}$

if and only if,

$$\begin{cases} ma + k = a^2 \\ a^2 - ma - k = 0. \end{cases}$$

For the line whose equation is $y = mx + k$ to be tangent to the conic whose equation is $y = x^2$, the discriminant of the equation $a^2 - ma - k = 0$ for $\underline{a}$ must be 0. Hence,

$$m^2 + 4k = 0$$
$$m^2 = -4k$$
$$m = \pm 2\sqrt{-k}, \quad \text{for real } m,\ k \leq 0.$$

$$\therefore\ y = \pm 2\sqrt{-k}\,x + k,\quad k \leq 0,\quad \text{is tangent to } y = x^2.$$

7-7. **Illustrative Test Questions**

1. Find all the ordered pairs (x,y) such that the square of the first is one more than three times the second while the square of the second minus the square of the first is one less than five times the second.

2. Find the solution set of the system $\begin{cases} x^2 + 2y = 19 \\ 2x - y = 1. \end{cases}$

3. The ordered pair $(6,0)$ belongs to the solution set of the statements;

   (a) $x^2 + y^2 = 25.$
   (b) $x^2 + y^2 < 25.$
   (c) $x^2 + y^2 > 25.$
   (d) $x = y - 6.$
   (e) $x + y = -6.$

4. Which of the following systems has no real number pairs in its solution set?

   (a) $\begin{cases} x^2 - y^2 = 16 \\ 9x^2 - 25y^2 = 0 \end{cases}$
   (b) $\begin{cases} x = y \\ x^2 + y^2 = 25 \end{cases}$
   (c) $\begin{cases} 4x^2 + 4y^2 = 4 \\ 9x^2 + 4y^2 = 36 \end{cases}$
   (d) $\begin{cases} x = 0 \\ x^2 - y^2 = 1 \end{cases}$
   (e) $\begin{cases} x^2 - y^2 = 36 \\ x^2 + y^2 = 4 \end{cases}$

5. A choice of a and b which will eliminate y^2 from the equation $a(x^2 - 2y^2 + 2x - 3) + b(3x^2 + y^2 - x - 2) = 0$ is:

   (a) $a = 3, \quad b = 2.$
   (b) $a = -1, \quad b = -2.$
   (c) $a = 1, \quad b = -2.$
   (d) $a = 7, \quad b = 2.$
   (e) $a = 0, \quad b = -2.$

6. Which of the following systems are equivalent to the system
$$\begin{cases} 5x + 4y - 3 = 0 \\ x + 2y = 0, \end{cases}$$

(a) $\begin{cases} -6y = 3 \\ x + 2y = 0 \end{cases}$

(b) $\begin{cases} 5x + 4y - 3 = 0 \\ (5x + 4y - 3) + 2(x + 2y) = 0 \end{cases}$

(c) $\begin{cases} y = -\frac{1}{2} \\ x = 1 \end{cases}$

(d) $\begin{cases} x(5x + 4y - 3) - 2x(x + 2y) = 0 \\ x + 2y = 0 \end{cases}$

7. Determine whether or not the following systems are consistent:

(a) $\begin{cases} x = y^2 \\ x + 1 = 0 \end{cases}$

(b) $\begin{cases} y = x^2 \\ y = x^2 + 6 \end{cases}$

(c) $\begin{cases} 4x^2 + 2y^2 = 0 \\ 2x^2 + y^2 = 0 \end{cases}$

(d) $\begin{cases} x^2 + 4y^2 = 4 \\ x^2 - y^2 = 9 \end{cases}$

(e) $\begin{cases} x^2 - y^2 = 1 \\ x = 2y^2 \end{cases}$

8. Show graphically the region in the xy-plane which satisfies <u>both</u> of the following inequalities:
$$\frac{x^2}{36} + \frac{y^2}{16} \leq 1, \quad \text{and} \quad x^2 + y^2 \geq 9.$$

9. Sketch the graphs of the equations in the following system and label the intersection points of the two curves with the letters P_1, P_2, P_3, ... Use as many letters as there are intersection points. You need not find the coordinates of the intersection points.
$$\begin{cases} x = y^2 + 1 \\ x^2 + y^2 = 25. \end{cases}$$

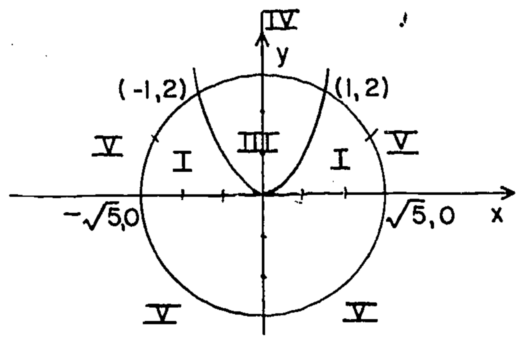
10. For what values of k will the system of equations

$$\begin{cases} 3x + 4y = 12 \\ 4x + ky = 16 \end{cases}$$ be dependent.

(a) None
(b) 3
(c) $\frac{16}{3}$
(d) $\frac{4}{3}$
(e) All values.

11. The solution of the system of inequalities, $\begin{cases} x^2 + y^2 < 5 \\ 2x^2 < y \end{cases}$ is represented by which of the numbered regions in the accompanying figure?

(a) I
(b) II
(c) III
(d) IV
(e) V



12. Which of the following systems is equivalent to any system of equations in two variables whose solution set is the single ordered pair (p,q)?

(a) $\begin{cases} x + y = p \\ x - y = q \end{cases}$

(b) $\begin{cases} x = p \\ y = q \end{cases}$

(c) $\begin{cases} x = q \\ y = p \end{cases}$

(d) $\begin{cases} x - y = p \\ x + y = q \end{cases}$

(e) $\begin{cases} x^2 = p^2 \\ y^2 = q^2 \end{cases}$

13. An ordered pair of numbers with first element equal to p is in the solution set of the system $\begin{cases} x - y = 0 \\ x + 2y^2 - 3 = 0 \end{cases}$

if and only if,

(a) $\frac{3 - p}{2} = p^2$ (d) $\sqrt{3 - p} = 2p$

(b) $\sqrt{\frac{3 - p}{2}} = p$ (e) $3p^2 - 3 = 0$

(c) $6 - 2p = p^2$

14. For what value of k will the lines whose equations are

$$x + 2y = 3 \quad \text{and} \quad (k + 1)x + (3k - 1)y = 4$$

be parallel?

(a) $\frac{3}{5}$ (d) 2

(b) 1 (e) 3

(c) $\frac{5}{3}$

15. If a system of equations consists of a linear equation and a quadratic equation, how many pairs (x,y) does the solution set contain?

(a) 0 or 1 or 2 (d) 1 or 2 or 3

(b) 1 or 2 (e) 0 or 1 or 2 or infinitely many.

(c) 2

16. Which of the following describes the graph of the solution set of the system $\begin{cases} xy = 0 \\ x = 0? \end{cases}$

(a) The point $(0,0)$

(b) The x-axis

(c) The y-axis

(d) The x-axis and the y-axis.

(e) There is no graph because the solution set is the empty set.

17. What are all ordered pairs (x,y) in the solution set of the system $\begin{cases} x^2 + 4y^2 = 16 \\ y^2 = 4x + 4 \end{cases}$

(a) $(-16,-2i\sqrt{15}),(-16,2i\sqrt{15}),(16,2i\sqrt{15}),(16,-2i\sqrt{15})$ $(0,2),(0,-2)$.

(b) $(-16,2i\sqrt{15}),(-16,-2i\sqrt{15}),(16,2i\sqrt{15}),(16,-2i\sqrt{15})$.

(c) $(-16,2i\sqrt{15}),(-16,-2i\sqrt{15}),(0,2),(0,2)$.

(d) $(0,2),(0,-2)$.

(e) $(0,2),(0,-2),(2,0),(-2,0)$

18. How many ordered pairs of real numbers are there in the solution set of the system $\begin{cases} x^2 + y^2 = 7 \\ x^2 - y^2 = 1 \end{cases}$?

(a) 0

(b) 1

(c) 2

(d) 4

(e) Infinitely many.

7-8. Illustrative Test Questions - Answers

1. {(1,0),(-1,0),(5,8),(-5,8)}

2. {(-7

3. c

4. (c), (d), (e)

5. (b)

6. (a), (b), (c)

7. (a) inconsistent "over the reals"; consistent "over the complex".

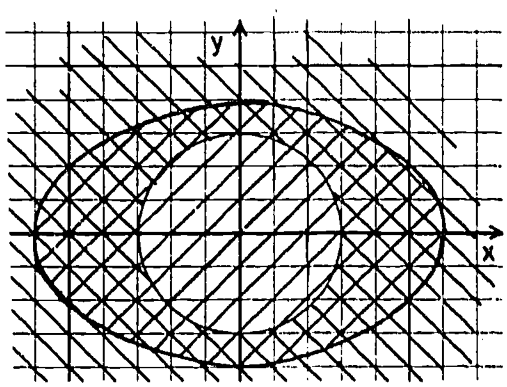
   (b) inconsistent

   (c) dependent

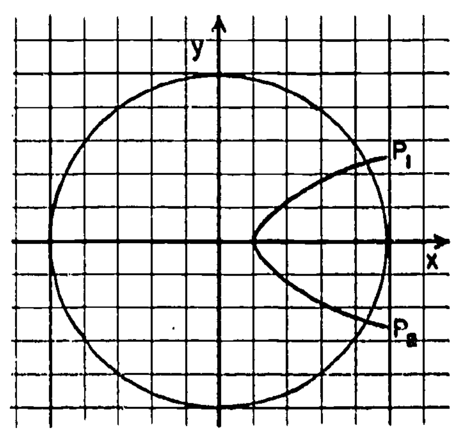
   (d) inconsistent "over the reals"; consistent over the complex".

   (e) consistent.

8.



9.



10. (c)
11. (c)
12. (b)
13. (a)
14. (e)
15. (e)
16. (c)
17. (c)
18. (d)

Chapter 8

SYSTEMS OF FIRST DEGREE EQUATIONS IN THREE VARIABLES

8-0. Introduction

Systems of first degree equations arise in many branches of mathemat[illegible]d science as well as in many modern theories of econo[illegible] business problems, particularly those concerned with in[illegible] and production questions. The geometrical significance of the subject is emphasized by our presentation of the geometry along with the algebra of the problem.

With the current use of computing machines to solve engineering and scientific problems, this subject has become one of the most important branches of applied mathematics. Every day, industrial and research organizations must solve systems of first degree equations, some of them with hundreds of equations and hundreds of variables. A thorough understanding of the simpler cases is therefore a necessity for any one hoping to take almost any kind of mathematical job in industry or scientific research.

The central problem studied in this chapter is an algebraic one: under what circumstances do two or more equations in three variables with real coefficients have common solutions, and if there are common solutions, how many are there and how are they related to one another? Because we restrict our attention to first degree equations with real coefficients having only three variables, we are able to translate the problem into geometric language. This translation makes it possible to cast our results in the form of statements about planes in three dimensional space in such a way that statements about common solutions of the equations become statements about configurations of planes and their intersections. The insights gained in this way are perhaps most strikingly illustrated by the diagrams in Figure 8-9b where the many types of intersection and parallelism of planes are used to describe the types of solution sets that may be expected when a system of three first degree equations in three variables is studied. In this presentation, Section 8-9 gives cases where there are solutions, and also gives cases where the solution set is empty. Although it is not essential that the student understand all these cases, some students will enjoy the opportunity to see a classification of this kind. Thus, those students who learn to handle the ideas presented in discussing the correspondence between the geometry and algebra of these (and other) systems of equations will benefit from the ability to visualize the geometry. If they go on to the study of mathematics at college, they will find that the development of this ability gives them real advantages.

Degrees of Freedom

One of the basic ideas that will throw light on the approach to the problems studied in this chapter is the concept of degrees of freedom. An understanding of this concept will improve the teacher's intuitions about all the problems discussed here. (For a more sophisticated treatment of this question and of linear dependence, see Birkhoff-MacLane, Survey of Modern Algebra, p. 166ff.)

A point in space has 3 coordinates (x, y, z). It is said to have 3 degrees of freedom since each of the variables may be assigned arbitrary values. As each of x, y, and z assumes all possible real values, the point (x, y, z) assumes all possible positions in space. If, however, the values of the variables are constrained to satisfy a single equation (here an equation of first degree - the equation of a plane), the number of variables that may be assigned arbitrary values is reduced to 2. The point now has only 2 degrees of freedom and is constrained to remain in the plane whose equation is given. We say that the number triples in the solution set of the equation can be described in terms of 2 parameters. (This case is treated in the first part of Section 8-4, but the word parameter is not used in the text.)

If 2 first degree equations are given, there are 3 possibilities:

1. In the most interesting case there are points (x,y,z) whose coordinates satisfy the 2 equations; the 2 equations impose 2 independent conditions on the point, and only one variable may be assigned arbitrarily. The point (x,y,z) now has only one degree of freedom. The number triples in the solution set of the system of 2 equations can be described in terms of a single parameter. The point is constrained to remain on a line--the line of intersection of the 2 given planes.

   Section 8-8 develops methods for describing the line of intersection in different ways, depending on which variable is assigned arbitrarily (which variable serves as parameter).

2. In the second case, the 2 equations are inconsistent. They represent parallel planes (discussed below and in starred Section 8-7), and no number triple can satisfy both equations. Here there is no point that is common to both planes.

3. In the third case, the 2 equations represent the same plane, and we have actually only 1 condition on the coordinates of points in this plane. Again, the number triples in the solution set are described in terms of 2 parameters. (This case is also discussed below and in Section 8-7.)

If a third first degree equation, consistent with the first two and independent of them, is given, an additional condition is imposed on the number triple (x,y,z). In this case, no variable may be chosen arbitrarily. The coordinates (x,y,z) are completely determined, and the point (x,y,z) is the single point of intersection of the three planes. This is one of the cases studied in Section 8-9. The cases in which the systems are dependent reduce to one of the two cases studied above: a line of intersection (one degree of freedom)--one parameter, or a plane of intersection (two degrees of freedom)--two parameters. If the system is inconsistent (and this can happen in a variety of ways, as illustrated in Figure 8-9b), then there is again no point that is common to the three planes.

Throughout Chapter 8 the manipulations that enable us to find the solution set for a system of equations are justified by the fact that the given system is consistently replaced by an equivalent system. The new system is equivalent to the old because the new equations are derived by taking linear combinations of the expressions defining the given equations. Hence, the planes defined by the new equations pass through the intersection of the given planes; when we have described the solution set of the new equations, we have also described the solution set of the given equations.

General Comments: An Outline of our Procedure--Suggestions to the Teacher.

We give here an outline of our procedure in this chapter and an indication, in certain parts, of teaching techniques that may make the presentation easier or of aspects of the problem that are not developed in the text but may be useful for the teacher to know.

The Purpose of the First Three Sections (Sections 8-1, 8-2, 8-3).

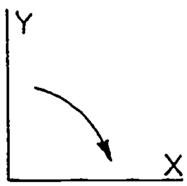
The first three sections are included in order to establish our basic geometric-algebraic correspondence; namely, the theorem that the equation of a plane is always of first degree, and that a first degree equation always represents a plane. There are several points that should be made here:

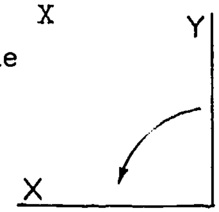
8-1. Comments. The Coordinate System (Section 8-1)

The coordinate system used is a "right-handed" one. This is an arbitrary choice that is made throughout this book because of the widespread use of this system in physics, vector analysis, and other courses in mathematics and its applications.

What do we mean by a "right-handed" or a "left-handed" system?

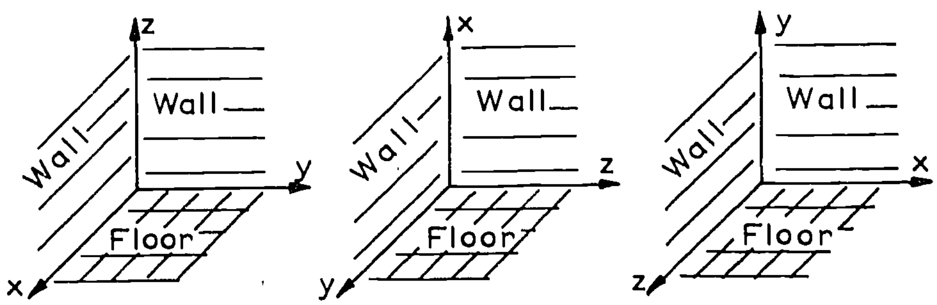
The difference between these systems may be expressed somewhat picturesquely perhaps, as follows: In a right-handed coordinate system, a person impaled on the positive Z-axis and looking toward the XY plane views it just as he always did in plane geometry--the positive X axis positive to the <u>right</u> of the positive Y axis. In a left-handed system our observer on the positive Z axis, on looking toward the XY plane sees the positive X axis extending to the <u>left</u> of the positive Y axis.



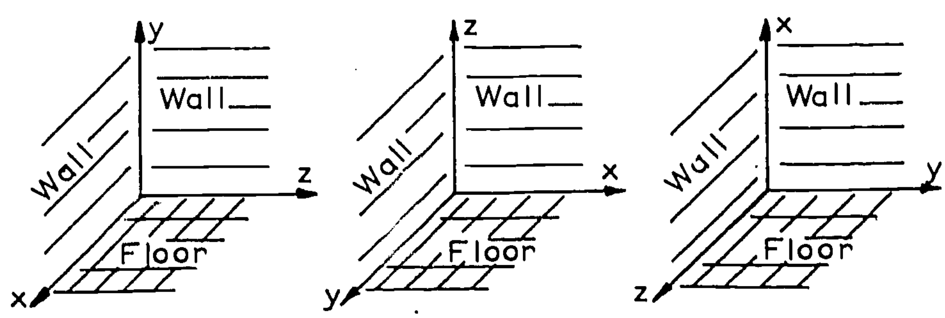


The following sketches illustrate a variety of the views an observer may have of each type of system.

Right Handed System



Left Handed System



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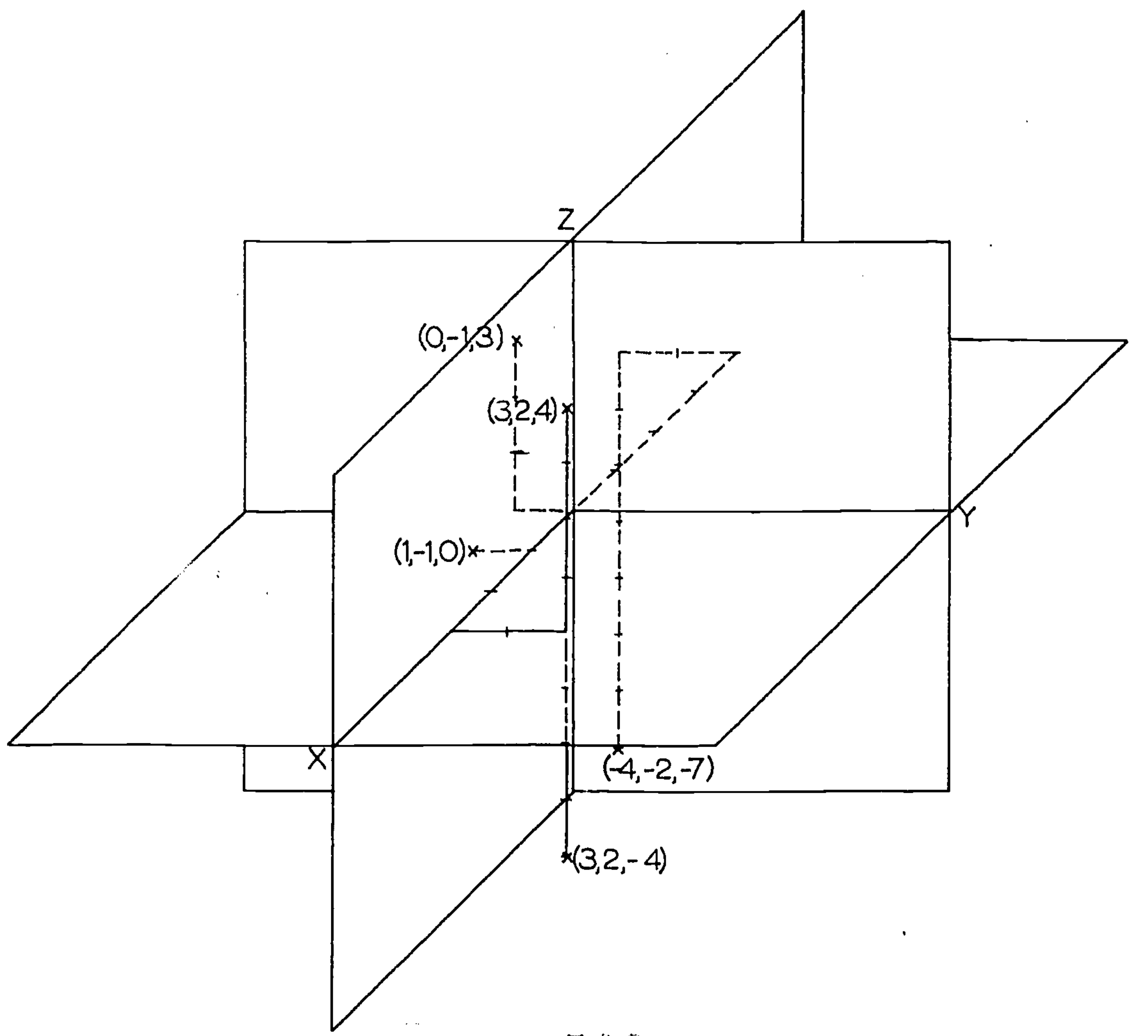
Exercises 8-1. - Answers.

1. See graph.
2. Point not plotted.
3. See graph.
4. Point not plotted.
5. See graph.
6. Point not plotted.
7. See graph.
8. Point not plotted.
9. See graph.
10. Point not plotted.



11. (a) On the yz plane

    (b) On a plane || to the yz plane cutting the x axis at 2.

    (c) On a plane || to the yz plane cutting the x axis at -3.

12. (a) On the xz plane

    (b) On a plane || to the xz plane and cutting the y axis at 3.

13. (a) On a plane || to the xy plane and cutting the z axis at 2.

    (b) On a plane || to the xy plane and cutting the z axis at -2.

14. A plane || to the z axis and cutting the x and y axis at 4.

8-2. Comments. The Distance Formula in Space.

In teaching the distance formula in space, many teachers have found that a box or other model constructed with pieces of hardware, cloth, window screen, or ordinary cardboard is very helpful. Even the corner of a room can be used to assist the student to visualize this, and other parts of geometry in space. The industrial arts teacher can be very helpful in providing large drawings for display purposes; and it may be possible to secure film strips that will show figures in three dimensions.

Exercises 8-2. - Answers.

1. $4\sqrt{2}$
2. 13
3. 7
4. 5
5. 12
6. 29
7. 3
8. $\sqrt{129}$
9. $\sqrt{41}$
10. $\sqrt{14}$

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8 [illegible]mments. The Corr[illegible]ondence Between Planes and First [illegible]egree Equations in [illegible] Variables.

In pr[illegible]ving the theorem establishing the correspondence between planes in space and first degree equations in 3 variables (Section 8-3), we did not have available the customary techniques of solid analytic geometry for deriving the equation of a plane. Thus, instead of viewing the plane as the locus of points on lines perpendicular to a given line through a point on that line, we have adopted a different definition. We have viewed the plane as the locus of points equidistant from two given points. This definition enables us to derive the equation of the plane with no analytic machinery beyond the distance formula. Since our definition embodies a property that characterizes a plane, the equation we derive represents precisely the plane with all the properties studied in geometry. In particular, a pair of distinct planes are either parallel or they intersect in a line.

If the teacher is pressed for time, it is suggested that the proof in Section 8-3 be omitted. The student should then accept without proof the theorem that every plane in three dimensions can be represented by an equation of the form

$$Ax + By + Cz + D = 0$$

where A, B, C, D, are real constants, and A, B, C, are not all zero; and the converse theorem, that every equation of this form represents a plane.

Exercises 8-3. - Answers.

1. (a) $10x - 10y - 8z - 10 = 0$

   (b) $2x - 6y - 12z + 6 = 0$

   (c) $-20x + 4y - 8z = 0$

(d) $4x + 4y - 16z = 32$

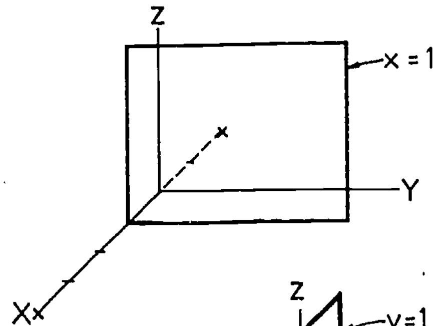
(e) $6x + 8y - 6z = -14$

(f) $4x - 8y + 12z = 0$

2. (a) (4,0,0), (-2,0,0).
Plane has equation

$$x = 1$$

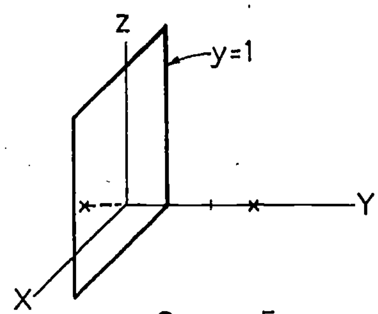
Plane is parallel to YZ plane and cuts the X axis at 1.



(b) (0,3,0), (0, -1,0).
Plane has equation

$$y = 1 .$$

Plane is parallel to XZ plane and cuts the Y axis at 1.

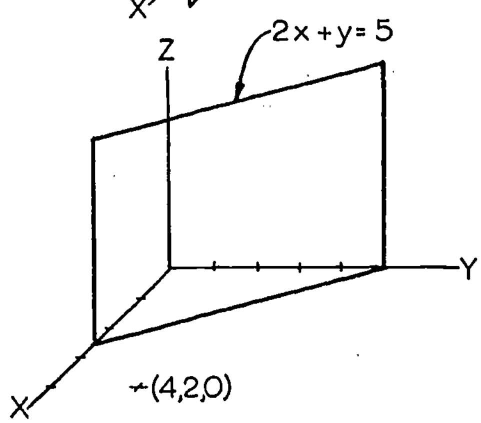


(c) (0,0,0), (4,2,0).
Plane has equation

$$8x + 4y = 20$$

or $\quad 2x + y = 5$

Plane is parallel to the Z axis, and cuts the X axis at $\frac{5}{2}$ and the Y axis at 5.

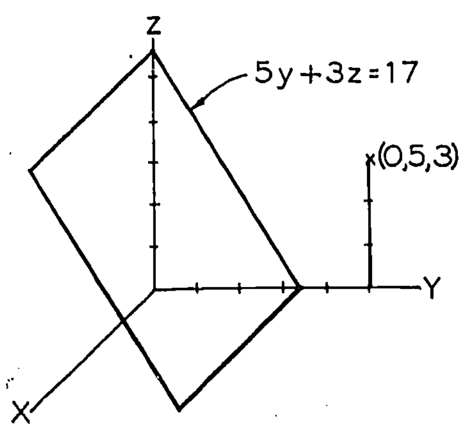


(d) (0,0,0), (0,5,3).
Plane has equation

$$10y + 6z = 34$$

or $\quad 5y + 3z = 17$

Plane is parallel to the X axis, and cuts the Y axis at $\frac{17}{5}$ and the Z axis at $\frac{17}{3}$.



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3. Let $P(x,y,z)$ be any point on the plane that is the locus of points equidistant from $R(A,B,C)$ and $S(-A,-B,-C)$. Since these points are symmetric with respect to the origin, we would expect the required plane to pass through the origin.

$$PR = PS$$

$$(x - A)^2 + (y - B)^2 + (z - C)^2 = (x + A)^2 + (y + B)^2 + (z + C)^2$$

$$-2Ax - 2By - 2Cz + A^2 + B^2 + C^2 = 2Ax + 2By + 2Cz + A^2 + B^2 + C^2$$

$$4Ax + 4By + 4Cz = 0$$

$$Ax + By + Cz = 0$$

Since $(0,0,0)$ is a point in the solution set of this equation, the plane passes through the origin.

8-4, 8-5. Comments. The Graph of a First Degree Equation in Three Variables.

The correspondence between a plane and a first degree equation is introduced to throw light on the algebraic problem. The ability to draw the graph of an equation will enable the student to gain insight into some of the special situations that may occur when, in the later sections, we study the solution sets of systems of two or three equations. In Sections 8-4 and 8-5 we try to develop this ability to draw graphs, first for the special planes that are parallel to an axis but not parallel to a coordinate plane (one variable has a zero coefficient, e.g., $x + y = 4$); second for planes parallel to a coordinate plane (two variables have zero coefficients--the equation gives a constant value for one coordinate, e.g., $x = 3$); and last for planes that have equations with no coefficients equal to zero. In all these cases, we consider the trace of the plane in each of the coordinate planes.

(The trace is the intersection of the given plane with a coordinate plane.) The trace, like every line, is described by two first, degree equations, but one of these is the equation of a coordinate plane, i.e., $x = 0$ or $y = 0$ or $z = 0$.

Some Generalizations About Planes and Their Equations--Suggestions for Constructing Problems.

We are now in a position to make certain general observations:

1. Any plane has infinitely many equations.

   If a given plane is represented by the equation

$$Ax + By + Cz + D = 0,$$

   it is also represented by

$$k(Ax + By + Cz + D) = 0,$$

   where k is any non-zero constant.

   The proof of this may be either algebraic or geometric:

   (a) every number triple satisfying either equation satisfies the other; or

   (b) the traces of the two planes are identical.

   The converse is also true. Equations in which the coefficients are proportional represent coincident planes.

   For if two planes have equations

$$A_1x + B_1y + C_1z + D_1 = 0$$

$$A_2x + B_2y + C_2z + D_2 = 0$$

   and

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2} = k$$

   then the first equation is k times the second, and the equations represent the same plane.

   These results are useful throughout our algebraic study. For example, if we have a system of two equations in which one equation is a multiple of the other, we know that the second equation contributes no information not already given by the first. Thus, a point whose coordinates satisfy the two

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equations still has the two degrees of freedom that characterize the point whose coordinates satisfy a single equation. This is the third case described under Degrees of Freedom for two equations.

2. If two planes have equations that can be reduced to the form

$$Ax + By + Cz = D$$

$$\text{where } D \neq F$$

$$Ax + By + Cz = F$$

the planes are parallel, and there is no common solution. Again, the proof may be algebraic or geometric:

(a) If (x_0, y_0, z_0) is any number triple, it cannot satisfy both these equations since

$$Ax_0 + By_0 + Cz_0$$

cannot be equal to both D and F if $D \neq F$; or

(b) the traces of the planes are parallel lines.

3. These two cases are summarized in the following rule:

If corresponding coefficients of two first degree equations are proportional, then their graphs

(a) are the same plane if the constant terms have the same ratio as the coefficients,

(b) are parallel planes if their constant terms are not in the same ratio as the coefficients.

This information gives us a way to recognize at a glance two equations that are inconsistent or dependent. It also gives the teacher the ability to make up problems with great ease. He need merely put down any left member of first degree and any constant term for the first equation. For the first case, double the first equation, triple it, or transpose some terms.

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For example:

Start with the equation $3x + 5y - z = 7$

Double the equation and transpose the y term

$$6x - 2z = 14 - 10y.$$

The resulting equations represent the same plane. For the second case, copy down the same left member for the equation but make its constant term different:

$$3x + 5y - z = 12.$$

This equation represents a plane parallel to the first. A more sophisticated version of this procedure involves doubling, trebling, or changing the sign of the left member while taking care to do something else with the constant.

4. Conversely, two planes meet in a line if and only if their corresponding coefficients are not proportional.

Again, examples of this sort can be invented in the time it takes to write them down: take any first degree equation as the first, and change the coefficients for the second one somehow, so that they are not proportional to the first ones. Having accomplished this much one is safe. Any constant term whatever will do. New coefficients not porportional to the first ones can be obtained in many ways. For example: keep one of them the same and change some other one; or add one to each of them; or change some of the signs, but not all, etc. We give a collection of such equations:

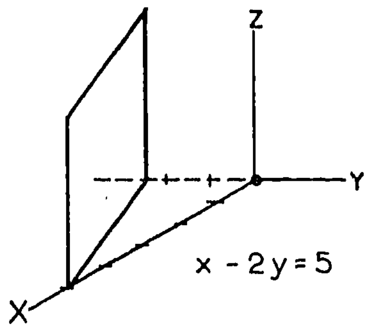
$$3x + 5y - z = 7$$
$$3x + 5y + z = 7$$
$$4x - 5y + 2z = 5, \text{ etc.}$$

The four results stated above may be made the basis of a preliminary examination of a system of equations. If we can tell by inspection that two of the given equations are inconsistent, we know immediately that the system has no solution. If we can tell by inspection that one of the given equations is dependent on the others, we know that the number of degrees of freedom is larger than would be in the case if all the equations were independent.

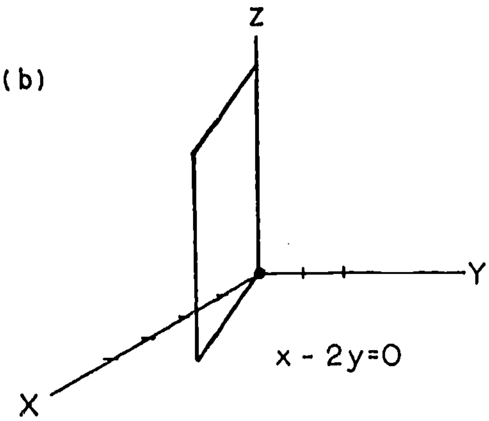
[pages 417-421]

Exercises 8-4. Answers.

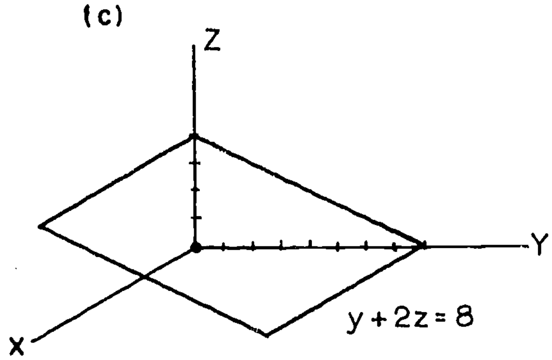
1.(a)



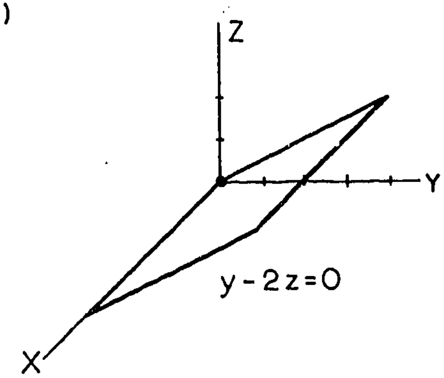
(b)

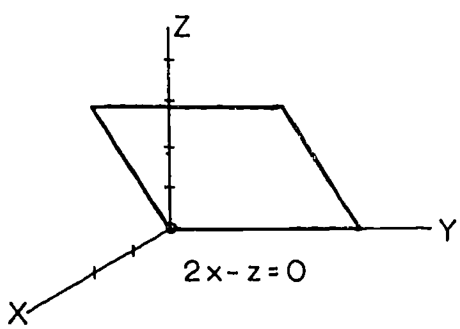


(c)



(d)





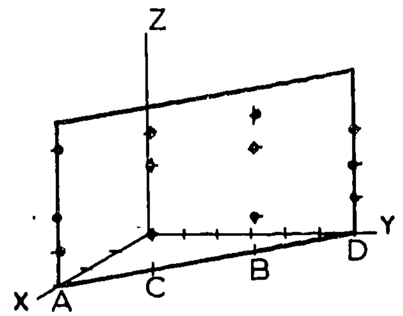
2. 2x + y = 6

A(3,0,0). Also on the graph are
(3,0,1), (3,0,2), (3,0,4).

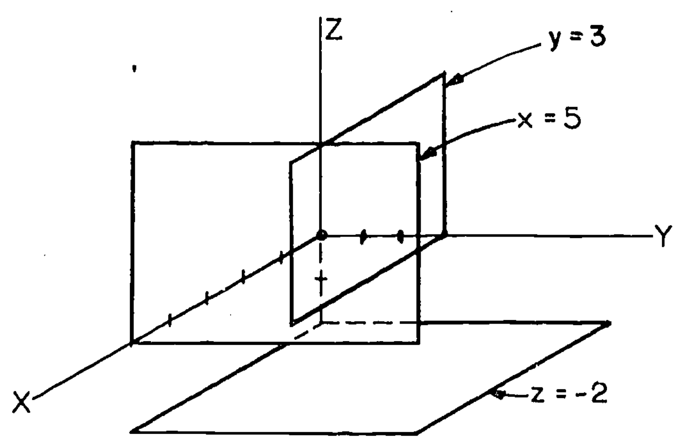
B(1,4,0). Also on the graph are
(1,4,1), (1,4,3), (1,4,4).

C(2,2,0). Also on the graph are
(2,2,1), (2,2,3), (2,2,4).

D(0,6,0). Also on the graph are
(0,6,1), (0,6,2), (0,6,3).

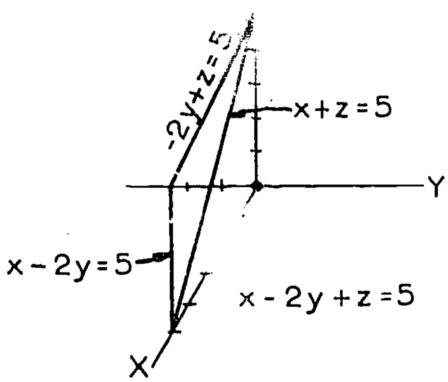


3.

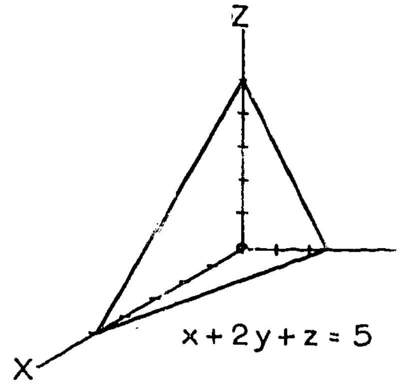


Exercises 8-5. - Answers.

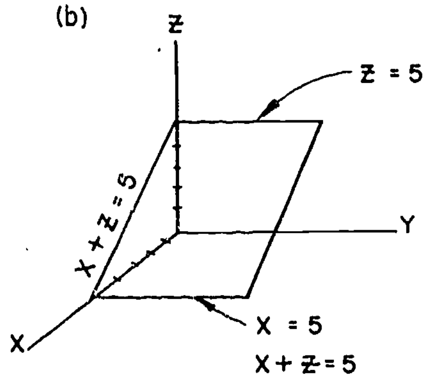
1. (a)



(d)



(b)



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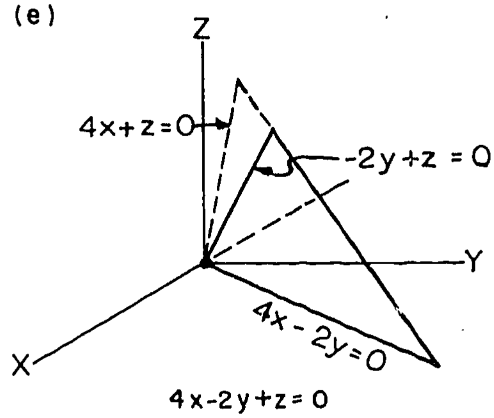

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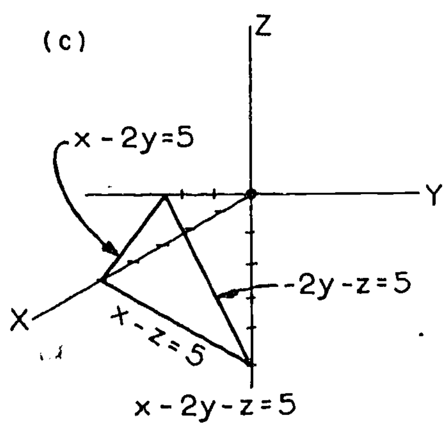

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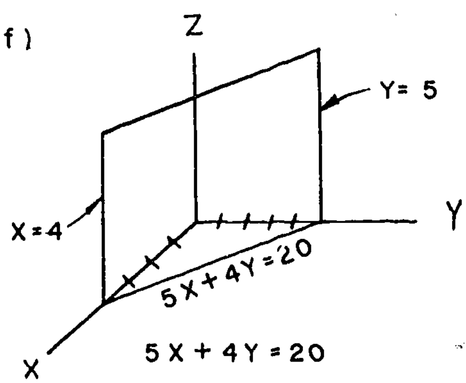
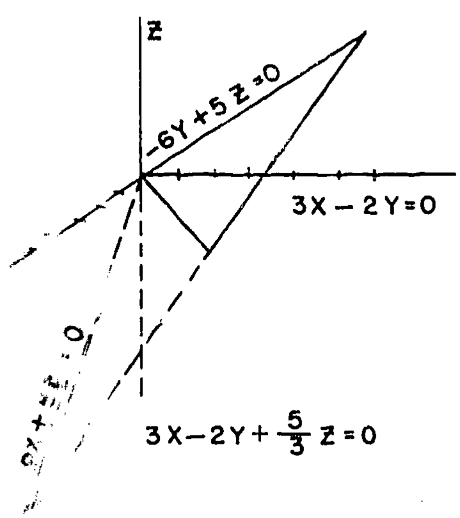
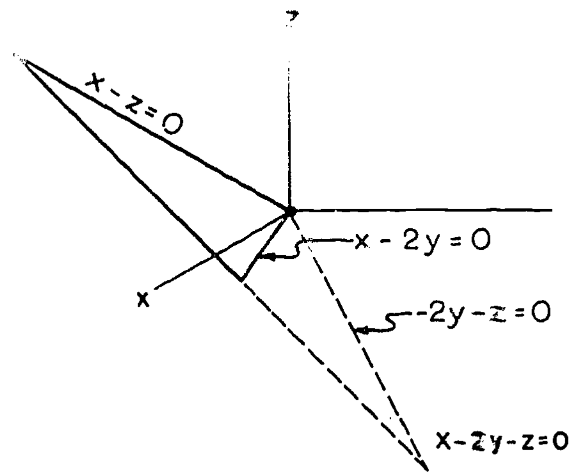


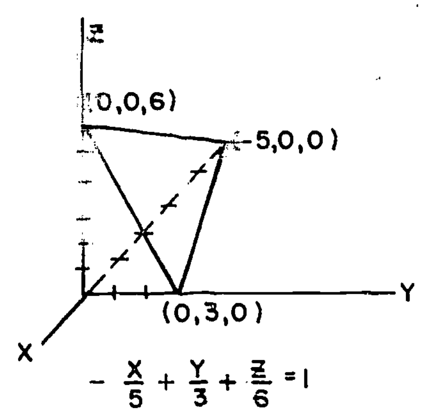
(g)


Z
-6Y + 5Z = 0
Y
3X - 2Y = 0
3X - 2Y + $\frac{5}{3}$ Z = 0

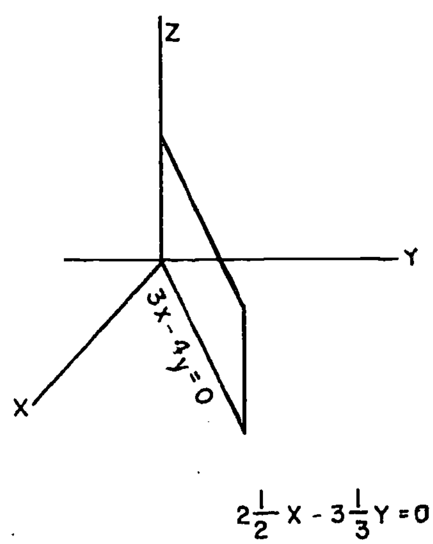
(i)


X - Z = 0
x - 2y = 0
X
-2y - z = 0
X - 2y - z = 0

(h)


Z
(0,0,6)
(-5,0,0)
(0,3,0)
Y
X
$-\frac{X}{5} + \frac{Y}{3} + \frac{Z}{6} = 1$

(j)


Z
Y
X
3X - 4y = 0
$2\frac{1}{2}X - 3\frac{1}{3}Y = 0$

8-7. Comment.

The geometric information presented in Section 8-4 and 8-5 is sufficient for the minimal purposes of the chapter. Section 8-7 presents additional geometric information that will throw light on the later work and will be of interest to those students who enjoy three-dimensional studies. If the teacher finds time to include this material, and if some students find the graphing of space figures excessively difficult, it may be helpful if such assignments are made to groups of 2 or 3 students. If Section 8-7 is omitted, the teacher should be sure to teach the material presented in Examples 2 and 3 of the section. This is covered by the discussion summarized in point (3) above. (See page 491.)

Parametric Representation of the Line of Intersection of Two Intersecting Planes.

(a) The line intersects all the coordinate planes.

Once we have disposed of systems of 2 equations in 3 variables in which the planes are coincident or parallel, we must undertake the more formidable problem of representing the line of intersection of planes that do intersect. Actually this line is represented by the two equations of the given intersecting planes; but since we know from our discussion above that the point describing the line has a single degree of freedom, we seek a representation of the line in which the three coordinates of the point are described in terms of a single parameter. We seek to describe the coordinates of any point on the line as functions of a single variable--this is the variable to which we can assign arbitrary values in finding as many points as we want in the solution set. Our manipulation of the given equations is aimed at expressing all three variables in terms of one of them so that the variable that is arbitrary is clearly indicated. In the non special case, when the line of intersection cuts all three coordinate planes, any one of the three variables may be chosen arbitrarily, so that there are three different parametric representations of the line, one in which x is arbitrary, one in which y is arbitrary, and one in which z is arbitrary. To derive each of these, we find the equations of a pair of planes through the line, each equation having in it one variable with a zero coefficient. This is achieved by eliminating each variable in turn from the given equations, and combining the resulting three equations, two by two. For example, an equation containing only x and y (the coefficient of z is zero) is combined with an equation containing only x and z. Since x and y are in the first equation, y can be expressed in terms of x. Since x and z are in the second equation, z can be expressed in terms of x. In this case, x serves as parameter. (Note that geometrically the planes corresponding to equations that have a single zero coefficient are parallel to an axis, e.g., Equation 8-8d is $x + 3z + 1 = 0$; this plane is parallel to the y-axis.)

(b) The line is perpendicular to one of the coordinate planes. If the line we seek to describe is perpendicular to one of the coordinate planes, the situation is special. For example, if the line is perpendicular to the XY-plane, the plane that passes through it and is parallel to the x-axis is also parallel to the XZ-plane. Similarly, the plane that passes through the given line, parallel to the y-axis, is also parallel to the YZ plane. Indeed any plane that passes through the given line is parallel to the z-axis. Thus the coordinates of a point on the line have a very special parametric representation, namely,

$x = a$ (y and z have zero coefficients)
$y = b$ (x and z have zero coefficients)
z is arbitrary

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(c) <u>The line is parallel to one of the coordinate planes</u>. In a similar way, we find that if the line we seek to describe is parallel to one coordinate plane, but intersects the other two, the situation is special. Here, one variable will be constant, but either of the others can be expressed in terms of the third. This case is discussed in Example 2 of Section 8-8.

<u>The Method of Elimination</u>

The justification for the familiar procedure used in eliminating one variable from the given equations is the theorem that is illustrated for a special case in starred Section 8-10. This argument is the same as that given in Chapter 7 for equations in two variables. It establishes the fact that the solution set for any given system

$$\begin{cases} f_1 = 0, \\ f_2 = 0; \end{cases}$$

is the same as for a new system

$$\begin{cases} f_1 = 0, \\ a_1f_1 + a_2f_2 = 0. \quad (a_1 \text{ and } a_2 \text{ not both zero}) \end{cases}$$

or for a second system

Eq. (1) $\quad a_1f_1 + a_2f_2 = 0,$

Eq. (2) $\quad b_1f_1 + b_2f_2 = 0. \quad (b_1 \text{ and } b_2 \text{ not both zero})$

The two new systems are thus <u>equivalent</u> to the given system. The expressions in the left members of Equations (1) and (2) are <u>linear combinations</u> of f_1 and f_2. Equations (1) and (2) represent planes through the line of intersection of the planes of the given system if these intersect; they represent planes parallel to the given planes if these are parallel; they represent the same plane if the original equations did. It is the case for intersecting planes that is studied for a special pair of equations in Section 8-10. (The teacher is urged to study this section, even if it is not covered in class, to gain some familiarity with these ideas.)

Exercises 8-[illegible] - Answers.

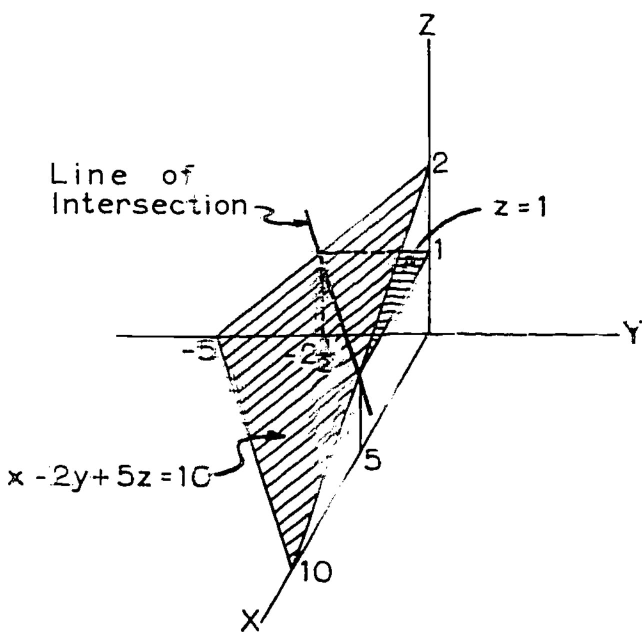
1. $x - 2y + 5z = 10$

   Intercepts: (10,0,0)

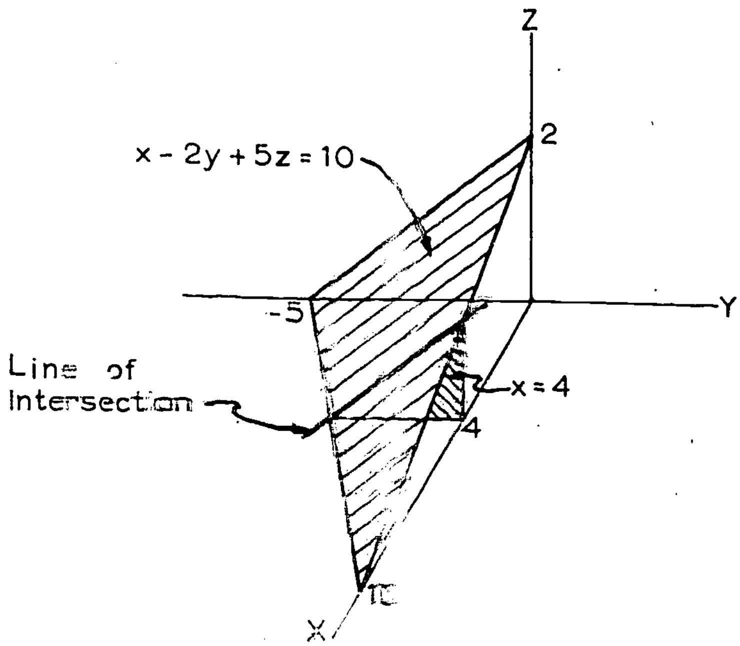
   (0,-5,0)

   (0,0,2)

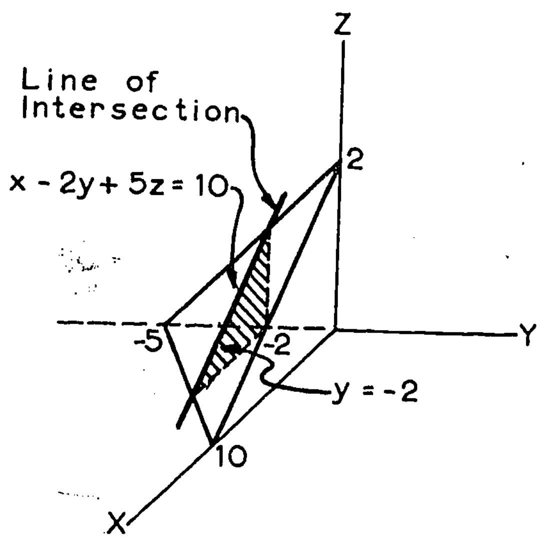
   Z = 1 Parallel to XY plane.



2. $x - 2y + 5z = 10$
$x = 4$

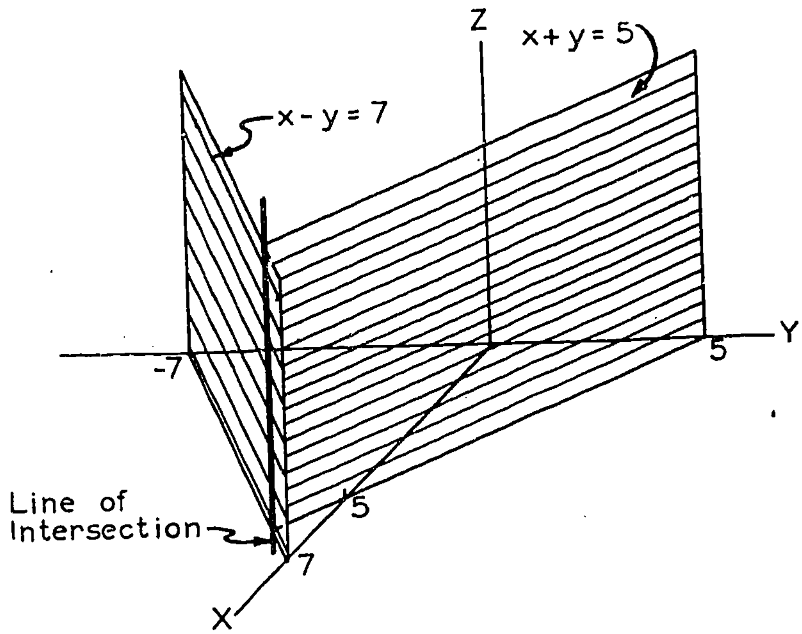


3. $x - 2y + 5z = 10$
$y = -2$

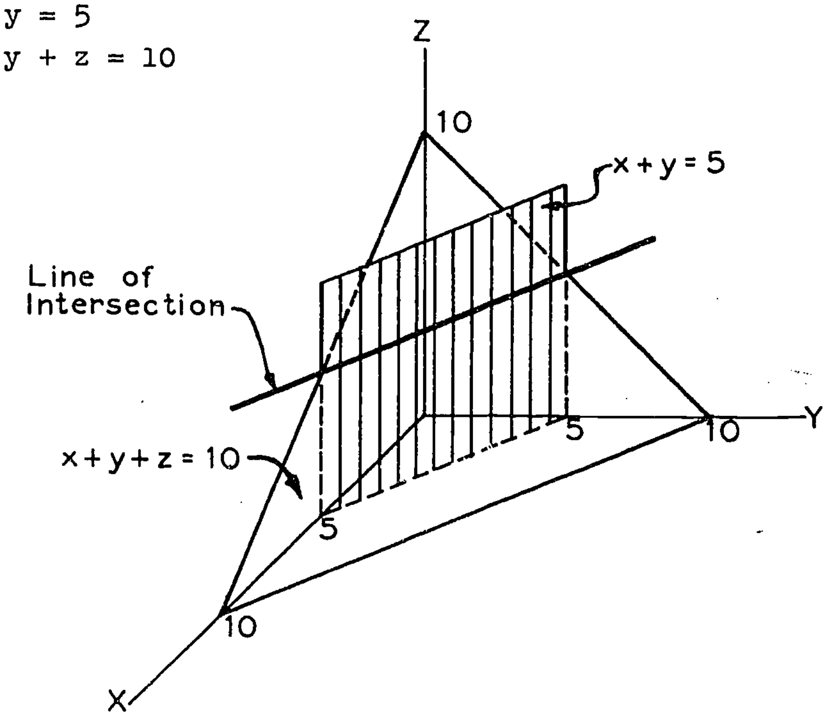


[page 448]

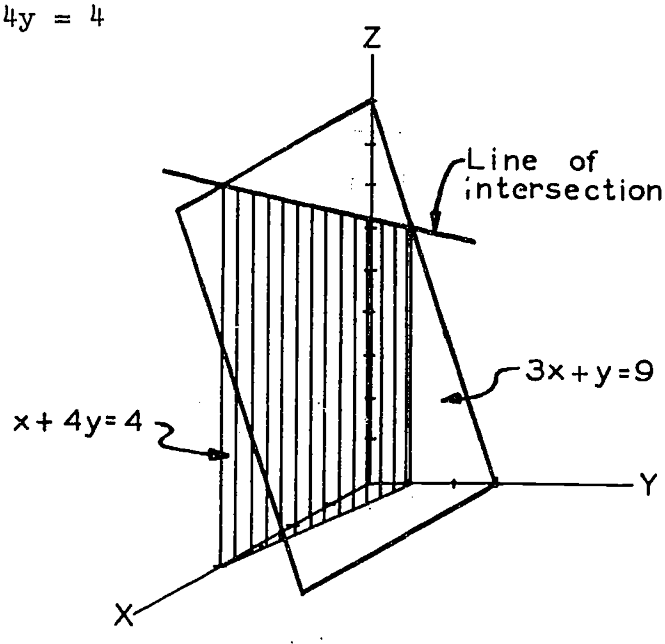
4. $x + y = 5$
   $x - y = 7$



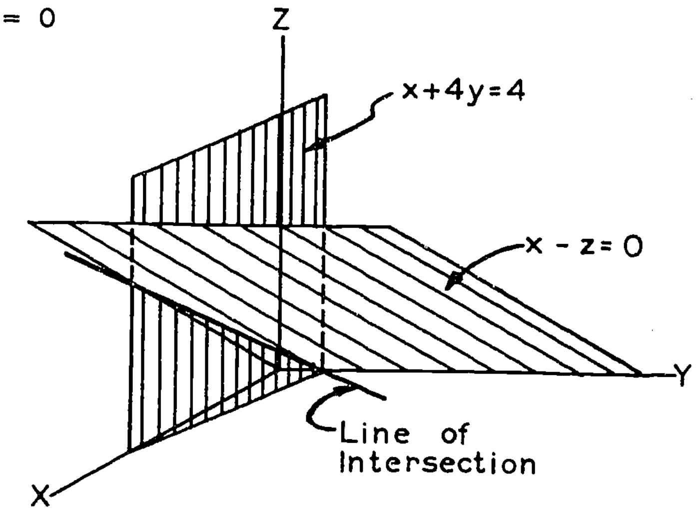
5. $x + y = 5$
$x + y + z = 10$



6. $3y + z = 9$
$x + 4y = 4$



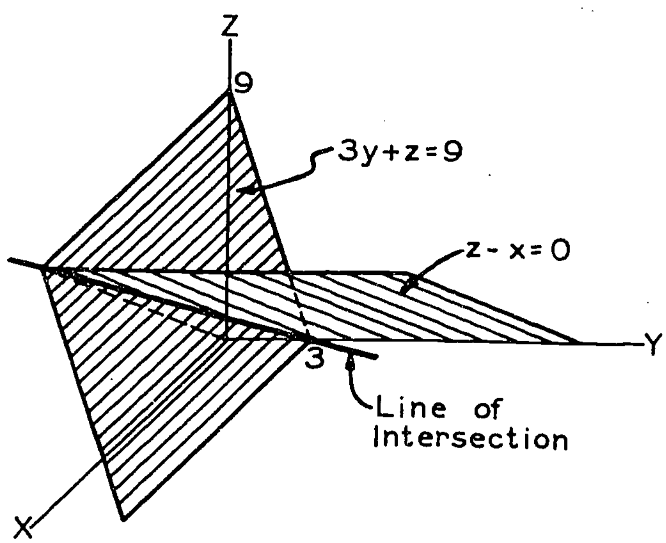
7. x + 4y = 4
   x - z = 0



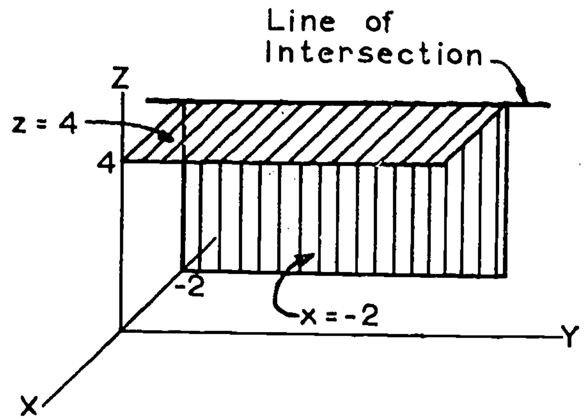
8. 3x + y - z = 2
   2z = 6x + 2y - 4

Same plane

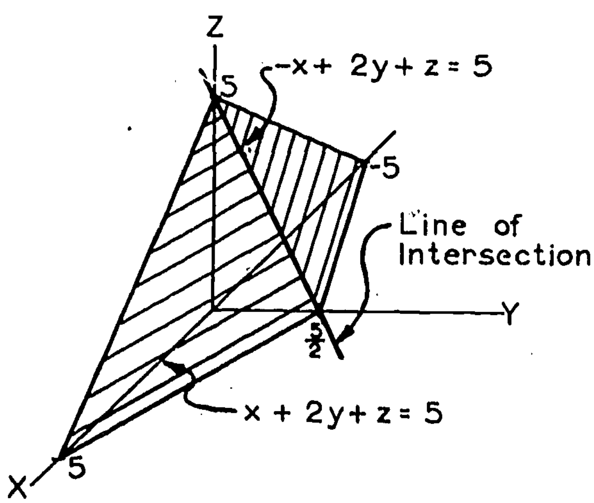
9. z - x = 0
   3y + z = 9.



10. $x = -2$
$z = 4$

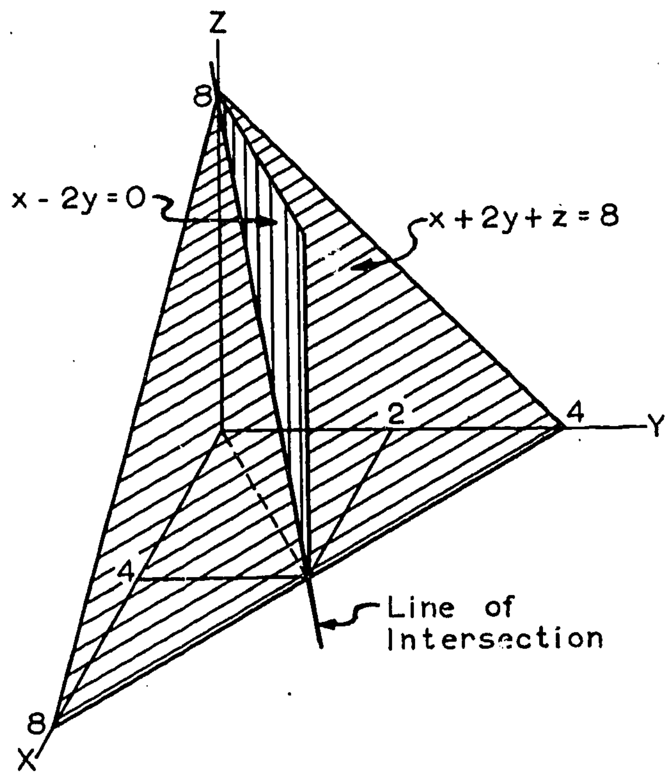


11. $x + 2y + z = 5$
$-x + 2y + z = 5$



12. $x + 2y + z = 8$

$x - 2y = 0$



8-8. Comments.

In Section 8-8 the equations we obtain represent planes through the line of intersection we seek to describe. Therefore when we use the new equations to write in parametric form the coordinates of a point on the line of intersection, we have actually described a point on the line of intersection of the given planes as well as on the line of intersection of the planes represented by the new equations.

Exercises 8-8. - Answers.

1. $x - 3y - z = 11$

$x - 5y + z = 1$

$2x - 8y = 12$

$-2y + 2z = -10$

$\therefore \quad x = 4y + 6$

y arbitrary

$z = y - 5$

x	-2	6	14	22
y	-2	0	2	4
z	-7	-5	-3	-1

Check by substituting in given equations.

$4y + 6 - 3y - y + 5 \stackrel{?}{=} 11$

$4y + 6 - 5y + y - 5 \stackrel{?}{=} 1$

2. $x + 2y - z = 8,$

$x + y + z = 0.$

$y - 2z = 8$

$2x + 3y = 8$

$\therefore \quad x = \frac{1}{2}(-3y + 8)$

y arbitrary

$z = \frac{1}{2}(y - 8)$

x	4	1	-2	-5
y	0	2	4	6
z	-4	-3	-2	-1

Check by substituting in the given equations.

$-3y + 8 + 4y - y + 8 \stackrel{?}{=} 16$

$-3y + 8 + 2y + y - 8 \stackrel{?}{=} 0$

3. $x + y - z = 5$
$x + 2y + z = 0$
$y + 2z = -5$
$2x + 3y = 5$
$x = \frac{1}{2}(-3y + 5)$
y arbitrary
$z = \frac{1}{2}(-y - 5)$

x	$\frac{5}{2}$	1	-2	-5
y	0	1	3	5
z	$-\frac{5}{2}$	-3	-4	-5

Check by substituting in the given equations.
$-3y + 5 + 2y + y + 5 \stackrel{?}{=} 10$
$-3y + 5 + 4y - y - 5 \stackrel{?}{=} 0$

4. $2x + 4y - 5z = 7$
$4x + 8y - 5z = 14$
$2x + 4y = 7$
$z = 0$
y arbitrary
$x = -2y + \frac{7}{2}$

x	$\frac{7}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$
y	0	1	2	3
z	0	0	0	0

Check by substituting in the given equations.
$-4y + 7 + 4y - 5(0) \stackrel{?}{=} 7$
$-8y + 14 + 8y - 5(0) \stackrel{?}{=} 14$

5. $-2x + y + 3z = 0,$
$-4x + 2y + 6z = 0.$

Since the second equation is twice the first, the two planes coincide. Thus there is no line of intersection.

6. $2x + 6z - 18y = 6$

$x - 3z - y = -3$

$2x - 10y = 0$

$6z - 8y = 6$

$\therefore \quad x = 5y$

y arbitrary

$z = \frac{1}{3}(4y + 3)$

x	-15	0	15	30
y	-3	0	3	6
z	-3	1	5	9

Check by substituting in the given equations.

$10y + 8y + 6 - 18y \stackrel{?}{=} 6$

$5y - 4y - 3 - y \stackrel{?}{=} -3$

7. $3x - 4y + 2z = 6$

$6x - 8y + 4z = 14.$

If we divide both members of the second equation by 2, we obtain

$$3x - 4y + 2z = 7$$

We can see by inspection that no number triple can be in the solution set of both these equations. Therefore the corresponding planes have no point in common. The planes are parallel.

8. $-5x + 4y + 8z = 0$

$-3x + 5y + 15z = 0$

$13x + 20z = 0$

$13y + 51z = 0$

$x = -\frac{20}{13}z$

$y = -\frac{51}{13}z$

z arbitrary

x	20	0	-20	-40
y	51	0	-51	-102
z	-13	0	13	26

Check by substituting in the given equations

$\frac{100}{13}z - \frac{204}{13}z + 8z \stackrel{?}{=} 0$

$\frac{60}{13}z - \frac{255}{13}z + 15z \stackrel{?}{=} 0$

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9. $4x - 7y + 6z = 13,$

$5x + 6y - z = 7.$

$-59y + 34z = 37$

$34x + 29y = 55$

$x = \frac{1}{34}(-29y + 55)$

y arbitrary

$z = \frac{1}{34}(59y + 37)$

x	$\frac{55}{34}$	$\frac{13}{17}$	$\frac{42}{17}$	$-\frac{3}{34}$
y	0	1	-1	2
z	$\frac{37}{34}$	$\frac{48}{17}$	$-\frac{11}{17}$	$\frac{155}{34}$

10. $-10x + 4y - 5z = 20$

$2x - \frac{4}{5}y + z = 4$

If we multiply both members of equation 2 by -5, we obtain

$$-10x + 4y - z = -20$$

We can see by inspection that no number triple can be in the solution set of both these equations. Therefore, the corresponding planes have no point in common. The planes are parallel.

8-9. Comments. Application of the Method of Elimination

The same idea dominates Section 8-9. Consider an example studied there. We discussed the system (Example 1)

$$\begin{aligned} x + 2y - 3z &= 9 \\ 2x - y + 2z &= -8 \\ -x + 3y - 4z &= 15 \end{aligned}$$

and converted into the equivalent system

$$\begin{aligned} x + 2y - 3z &= 9 \\ -5y + 8z &= -26 \\ z &= -2 \end{aligned}$$

by repeated application of exactly the same technique used in Section 8-8: selecting two of our equations and playing them off against one another to get rid of variables one at a time.

Even the final stage of the discussion of Example 1 is an instance of the same process. We arrive eventually at the system

$$x = -1$$
$$y = 2$$
$$z = -2$$

by subtracting appropriate multiples of the last equation from the first two, eliminating z, and then using the second equation to get y out of the first. This leaves us with the last system given above which is equivalent to the original system. The last system is so extraordinarily simple that we can read off its solution set at a glance.

A Systematic Method for Studying Three Equations in Three Variables

The problem discussed in Section 8-9 is the most complicated case we consider with three variables--the case in which there are as many equations as variables. Figure 8-9b illustrates the eight essentially different configurations formed by three planes in space. These pictures are included only for their interest. It is not important at this point that the student understand all the details.

With three planes there are four different types of solution sets (there were only three in the case of two planes):

1. The empty set
2. A single point
3. A line
4. A plane

The main business of Section 8-9 is the presentation of a systematic algebraic method for determining everything there is to know about systems of first degree equations: whether there are any solutions and how to find all of them. This method, "elimination", is applicable to systems having any number of equations and any number of variables. It is spelled out in detail only for three equations in three variables, since this case is probably the smallest one complicated enough to be of any real interest.

Restricting ourselves to this case, we give examples to illustrate the application of the method not only to the type of system in which the solution set consists of a single number triple, but also to several types in which the systems are inconsistent or dependent.

This method (sometimes called triangulation) is attributed to Gauss (1777-1855), the greatest mathematician since Newton. It gives the student the basic point of view he will need if he goes on to work in a large computing center. Its popularity reflects the fact that it gives an orderly procedure for handling systems of linear equations which, for many important cases, involves substantially fewer arithmetic operations than other methods. Using this method, we have the system essentially solved by the time we discover whether or not the solution is unique.

Relation of Method of Elimination to Cramer's Rule.

Those familiar with Cramer's rule (this is discussed in many of the older texts on College Algebra; it is usually not included in the newer texts) may be interested in its relation to the method of Gauss that we have presented. Observe first that, if the given equations are

$$A_1x + B_1y + C_1z = D_1$$

$$A_2x + B_2y + C_2z = D_2$$

$$A_3x + B_3y + C_3z = D_3$$

then our "triangulation" method replaces the given system by

$$\begin{cases} A_1x + B_1y + C_1z = D_1 \\ \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} y + \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} z = \begin{vmatrix} A_1 & D_1 \\ A_2 & D_2 \end{vmatrix} \\ \begin{vmatrix} A_1 & B_1 \\ A_3 & B_3 \end{vmatrix} y + \begin{vmatrix} A_1 & C_1 \\ A_3 & C_3 \end{vmatrix} z = \begin{vmatrix} A_1 & D_1 \\ A_3 & D_3 \end{vmatrix} \end{cases}$$

[pages 426-435]

and then by

$$A_1x \quad + B_1y \quad + C_1z \quad = D$$

$$\begin{vmatrix} A_1B_1 \\ A_2B_2 \end{vmatrix} y + \begin{vmatrix} A_1C_1 \\ A_2C_2 \end{vmatrix} z = \begin{vmatrix} A_1D_1 \\ A_2D_2 \end{vmatrix}$$

$$\begin{vmatrix} \begin{vmatrix} A_1B_1 \\ A_2B_2 \end{vmatrix} & \begin{vmatrix} A_1C_1 \\ A_2C_2 \end{vmatrix} \\ \begin{vmatrix} A_1B_1 \\ A_3B_3 \end{vmatrix} & \begin{vmatrix} A_1C_1 \\ A_3C_3 \end{vmatrix} \end{vmatrix} z = \begin{vmatrix} \begin{vmatrix} A_1B_1 \\ A_2B_2 \end{vmatrix} & \begin{vmatrix} A_1D_1 \\ A_2D_2 \end{vmatrix} \\ \begin{vmatrix} A_1B_1 \\ A_3B_3 \end{vmatrix} & \begin{vmatrix} A_1D_1 \\ A_3D_3 \end{vmatrix} \end{vmatrix}$$

The last equation can be shown to be equivalent to

$$A_1 \begin{vmatrix} A_1B_1C_1 \\ A_2B_2C_2 \\ A_3B_3C_3 \end{vmatrix} z = A_1 \begin{vmatrix} A_1B_1D_1 \\ A_2B_2D_2 \\ A_3B_3D_3 \end{vmatrix}$$

This is one of the equations derived by applying Cramer's rule. But for practical computing, the "elimination" or "triangulation" method has the great advantage that the nature of the solution becomes clear at this point; if it is unique, we find the solution with a minimum of additional computation. The mastery of this method should be a principal objective in teaching the chapter.

Exercises 8-9. - Answers

1. (3, 4, 5)
2. (2, 3, 3)
3. (2, -1, 1)

4. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations.

$$x = 3z + 5$$
$$y = 2z + 4$$
$$z \text{ arbitrary}$$

5. (1, -1, 1)

6. (1, -1, 2)

7. The three planes coincide. The solution set is an infinite set of triples corresponding to all the points in the plane.

8. (-1, -2, 3)

9. The system is inconsistent. The solution set is empty.

10. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = 2z$$
$$y = \frac{1}{2}z$$
$$z \quad \text{arbitrary}$$

11. (4, 6, 3)

12. (1, 2, -1)

13. The system is inconsistent. The solution set is empty.

14. The system is inconsistent. The solution set is empty.

15. $(\frac{1}{3}, \frac{1}{2}, 1)$

16. The three planes have a line in common. (The second equation represents the same plane as the first equation.) The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = \frac{9}{2} - y$$
$$y \text{ arbitrary}$$
$$z = -\frac{3}{2}$$

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17. $(\frac{3}{4}, \frac{1}{2}, 3)$

18. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = \frac{1}{5}(-7z + 17)$$

$$y = \frac{1}{5}(z - 1)$$

$$z \quad \text{arbitrary}$$

19. $(\frac{1}{3}, -\frac{2}{5}, \frac{1}{2})$

20. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = -\frac{1}{7}(6z - 5)$$

$$y = +\frac{1}{7}(16z - 11)$$

$$z \quad \text{arbitrary}$$

21. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = \frac{1}{7}(-z + 2)$$

$$y = \frac{1}{7}(-5z + 17)$$

$$z \quad \text{arbitrary}$$

22. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = -7z - 10$$

$$y = -5z - 6$$

$$z \quad \text{arbitrary}$$

23.

Food	Vitamin Content A	B	C
I	1	3	4
II	2	3	5
III	3	0	3
Requirements	11	9	20

If we buy x units of I, y of II, z of III, we want

$$x + 2y + 3z = 11, \quad 3x + 3y = 9, \quad 4x + 5y + 3z = 20; \qquad \text{or} \qquad x + 2y + 3z = 11, \quad x + y = 3, \quad 4x + 5y + 3z = 20.$$

Eliminate x:

$$x + 2y + 3z = 11, \quad -y - 3z = -8, \quad -3y - 9z = -24; \qquad \text{or} \qquad x + 2y + 3z = 11, \quad y + 3z = 8, \quad y + 3z = 8.$$

Answer for (a): No--Our conditions are dependent.
for (b): Consider the system,

$$x + 2y + 3z = 11,$$
$$y + 3z = 8,$$
$$6x + y + z = 10.$$

Eliminate x:

$$x + 2y + 3z = 11,$$
$$y + 3z = 8,$$
$$11y + 17z = 56.$$

Eliminate y:

$$x + 2y + 3z = 11,$$
$$y + 3z = 8,$$
$$16z = 32.$$

Thus $z = 2$, $y = 8 - 3z = 2$, $x = 11 - 2y - 3z = 1$

Answer for (b): Yes. 1 unit of I and 2 each of II,III.

24. $x + y - 5 = 0,$

$-x + 3z - 2 = 0,$

$x + 2y + z - 1 = 0,$

$y + z + 4 = 0.$

Suppose we apply the standard procedure given in Section 8-9, if only to see what happens. We eliminate x by subtracting appropriate multiples of the first equation from others:

$$x + y - 5 = 0,$$
$$y + 3z - 7 = 0,$$
$$y + z + 4 = 0,$$
$$y + z + 4 = 0.$$

We have found that, in our original system, the first, third and fourth equations are dependent; indeed

$$x + y - 5 = 1(x + 2y + z - 1) - 1(y + z + 4).$$

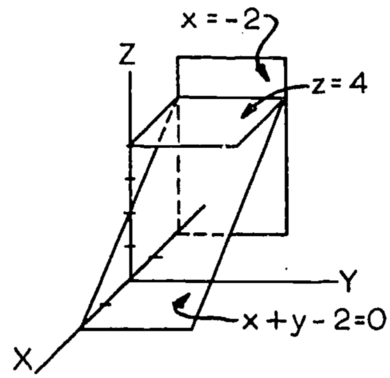
These three therefore all meet in a line. Since we were given the fact that the system has only one solution triple, this line must pierce the second plane in a single point. Hence any one of the four equations except for the second may be omitted, the line being determined by any pair of the three planes containing it.

Exercises 8-10. - Answers.

1. (a) $a(x + 2) + b(z - 4) = 0$

$a = 1,\ b = 1;$

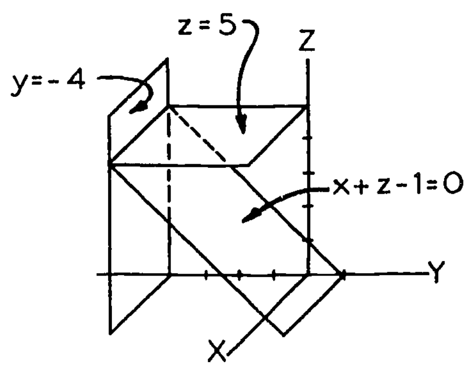
$x + z - 2 = 0$



(b) $a(y + 4) + b(z - 5) = 0$

$a = 1,\ b = 1;$

$y + z - 1 = 0$



2. (a) $a(x + 2y - 3z) + b(x - y + z - 1) = 0$

Substituting (1, 2, 1) for (x, y, z):

$a(1 + 4 - 3) + b(1 - 2 + 1 - 1) = 0$

$2a - b = 0 \quad ; \quad b = 2a$

Take $a = 1,\ b = 2;$

$(x + 2y - 3z) + 2(x - y + z - 1) = 0$

or $3x - z = 2$

(b) $a(2y - 3z - 2) + b(x + y + z) = 0$

Substituting $(3, -1, 0)$ for (x, y, z):

$a(-2 + 0 - 2) + b(3 - 1 + 0) = 0$

$-4a + 2b = 0$

$2a = b$

Take $a = 1$, $b = 2$

$(2y - 3z - 2) + 2(x + y + z) = 0$

$2x + 4y - z - 2 = 0$

(c) $a(x + z) + b(2x - y + z - 8) = 0$

Substituting $(0,0,0)$ for (x, y, z):

$-8b = 0$; $b = 0$.

The equation is $x + z = 0$.

This shows that the plane represented by the first equation is the only plane through the given line of intersection that also passes through the origin.

(d) $a(2x - y + z - 3) + b(x - 3y + 4) = 0$

Substituting $(2,2,1)$ for (x,y,z)

$a(4 - 2 + 1 - 3) + b(2 - 6 + 4) = 0$

$0 = 0$

For all values of a and b the plane

$a(2x - y + z - 3) + b(x - 3y + 4) = 0$

passes through the point $(2,2,1)$. This is because the given point lies on the line of intersection of the given planes.

*3. Since the second equation can be written

$$3(2x - y + 3z) = 5,$$

it is clear that any triple in the solution set of the first equation (and therefore reducing the parenthesis to 1) will not be in the solution set of the second equation. Similarly for any triple in the solution set of the second equation. Thus the planes have no point in common and are parallel.

The equation 8-10e

$$a(2x - y + 3z - 1) + b(6x - 3y + 9z - 5) = 0$$

can be written

$$(a + 3b)(2x - y + 3z) + (-a - 5b) = 0$$

If this plane passes through a point on the first plane, we know that

$$2x - y + 3z = 1$$

Therefore $a + 3b - a - 5b = 0$

$$-2b = 0$$

$$b = 0$$

Thus, any plane represented by (8-10e) that passes through a point in the first plane must coincide with the first plane. Similarly, if a plane represented by (8-10e) passes through a point of the second plane, we have

$$(a + 3b)\left(\frac{5}{3}\right) + (-a - 5b) = 0$$

$$2a = 0$$

$$a = 0$$

Therefore the plane coincides with the second plane.

We conclude that if $a \neq 0$ and $b \neq 0$, any plane represented by (8-10e) has no point in common with either of the given planes. It is therefore parallel to these planes.

*4. $a(x + y - 3) + b(z - 4) = 0$

Substituting $(1,-1,1)$ for x,y,z

$a(-3) + b(-3) = 0;\ a = -b.$

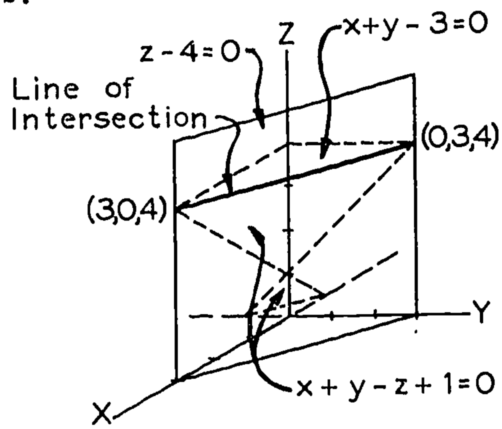
Take $a = 1,\ b = -1$

$x + y - z + 1 = 0$

The trace of this plane in the XZ plane is $x - z + 1 = 0$. This line intersects the XZ trace of

$x + y - 3 = 0$

which is $x - 3 = 0$. The point of intersection is $(3,0,4)$. Similarly the trace of the plane, $x + y - z + 1 = 0$, in the YZ plane intersects the trace of $x + y - 3 = 0$ in the YZ plane in the point $(0,3,4)$. These points are both in the plane $z = 4$. Thus the line joining these 2 points is the line of intersection of the three planes.



Exercises 8-11. Miscellaneous Exercises - Answers.

1. The number is 364.
2. $3x + 4y + 5z = a,$

   $4x + 5y + 6z = b,$

   $5x + 6y + 7z = c.$

   Eliminate x: $3x + 4y + 5z = a,$

   $y + z = 4a - 3b,$

   $2y + 2z = 5a - 3c.$

   Condition: $5a - 3c = 2(4a - 3b)$

   or $a + c = 2b$

3. The number is 456 or 654.

4. \$6500, \$1300, \$2200.

5. a = 7; the line is given by $\begin{array}{l} x = -2y + 7 \\ y \quad \text{arbitrary} \\ z = y - 1 \end{array}$

6. 5 cu. yds., 6 cu. yds., 8 cu. yds.

7. 12 dimes, 8 nickels, 20 pennies.

8. 75 units, 80 units, 50 units.

9. 12 days, 8 days, 6 days.

10. 8 hours, 4 hours, 8 hours.

11. $AS = AR = 4\frac{1}{2}$; $BS = BT = 5\frac{1}{2}$; $CT = CR = 2\frac{1}{2}$

12. $y = -x^2 + 2x + 4$

13. $y = 3x^2 + 2x - 1$

14. 160 elementary school pupils
    80 high school pupils
    80 adults

15. $K = -8$; $A = 50$, $B = 0$

16. Rewrite the given equation

$$w_1T + w_2Q + w_3E = (w_1 + w_2 + w_3)A$$

as

$$w_1(T - A) + w_2(Q - A) + w_3(E - A) = 0.$$

Using the table of scores we construct the following table:

	T - A	Q - A	E - A
Frank	-4	-4	4
Joyce	-2	18	-6
Eunice	3	-17	5

from which we write our system

$$w_1 + w_2 - w_3 = 0,$$
$$w_1 - 9w_2 + 3w_3 = 0,$$
$$3w_1 - 17w_2 + 5w_3 = 0.$$

Eliminate x:

$$[w_1] \quad w_1 + w_2 - w_3 = 0,$$
$$-10w_2 + 4w_3 = 0,$$
$$-20w_2 + 8w_3 = 0.$$

Each of the last two equations reduces to

$$5w_2 - 2w_3 = 0$$

So $w_2 = \frac{2}{5}w_3$ and $w_1 = w_3 - w_2 = \frac{3}{5}w_3$.

(Equivalently, $w_1 : w_2 : w_3 = 3:2:5$)

For $w_1 + w_2 + w_3 = 1$, we can write

$\frac{3}{5}w_3 + \frac{2}{5}w_3 + w_3 = 1$

or

$10w_3 = 5$ so $w_3 = 0.5$, $w_2 = 0.2$, $w_1 = 0.3$.

17. Let a = number of air mail stamps purchased

f = number of 4 cent stamps purchased

s = number of one cent stamps purchased

$$.07a + .04f + .01s = 10$$
$$a = 2f$$

observe that there are only 2 equations in 3 unknowns. However, s must be an integer less than 18 (the price of one 4 cent stamp and 2 air mail stamps) since only the change is spent for 1 cent stamps.

Substituting $a = 2f$ in

$$7a + 4f + s = 1000$$

we have $18f + s = 1000$

$$s = 1000 - 18f$$

$f = 55$ is the largest integral value that leaves s positive. Therefore

$$f = 55, \ s = 10, \ a = 110$$

Note that the problem can be solved simply by observing that we are to buy the largest number possible of 18 cent units consisting of 2 air mail and 1 four cent stamp), and spend the change on 1 cent stamps.

18. Actual score is 81. Reported score is 63. Par is 60.

*19. If A, B, C, D are the coefficients of our desired plane,

$$Ax + By + Cz + D = 0,$$

we obtain three equations for the four "unknowns" A, B, C, D by demanding that the coordinates of the three given points shall satisfy this equation:

$$A\cdot(-1) + D = 0,$$
$$A\cdot 1 + B\cdot(-1) + D = 0,$$
$$A\cdot(-1) + B\cdot 3 + C\cdot 2 + D = 0;$$

or

$$-A + D = 0,$$
$$A - B + D = 0,$$
$$-A + 3B + 2C + D = 0.$$

Eliminating A from the second and third:

$$-A + D = 0$$
$$-B + 2D = 0,$$
$$3B + 2C = 0.$$

Eliminating B from the third:

$$-A + D = 0,$$
$$-B + 2D = 0,$$
$$2C + 6D = 0.$$

Hence $A = D,\ B = 2D,\ C = -3D,$

an answer being $x + 2y - 3z + 1 = 0$ $(D = 1)$

Since any other choice of D will give an equation with coefficients proportional to these, only one plane is determined.

20. A. (a) Ag + (b) $HNO_3 \longrightarrow$ (c) Ag NO_3 + (d) NO + (e) H_2O

Ag: a = c; H: b = 2e; N: b = c + d; O: 3b = 3c + d + e

$3Ag + 4HNO_3 \longrightarrow 3AgNO_3 + NO + 2H_2O$

B. (a) $AuCl_3$ + (b) KI $\longrightarrow$ (c) AuCl + (d) KCl + (e) I_2

Au: a = c; Cl: 3a = c + d; K: b = d; I: b = 2e

$AuCl_3 + 2KI \longrightarrow AuCl + 2KCl + I_2$

C. (a) HNO_3 + (b) HI $\longrightarrow$ (c) NO + (d) I_2 + (e) H_2O

H: a + b = 2e; N: a = c; O: 3a = c + e; I: b = 2d

$2HNO_3 + 6HI \longrightarrow 2NO + 3I_2 + 4H_2O$

D. (a) MnO_2 + (b) HCl $\longrightarrow$ (c) $MnCl_2$ + (d) Cl_2 + (e) H_2O

Mn: a = c; O: 2a = e; H: b = 2e; Cl: b = 2c + 2d

$MnO_2 + 4HCl \longrightarrow MnCl_2 + Cl_2 + 2H_2O$

E. (a) $Cr(OH)_3$ + (b) NaOH + (c) $H_2O_2 \longrightarrow$ (d) Na_2CrO_4 + (e) H_2O

Cr: a = d; O: 3a + b + 2c = 4d + e; H: 3a + b + 2c = 2e;

Na: b = 2d

$2Cr(OH)_3 + 4NaOH + 3H_2O_2 \longrightarrow 2Na_2CrO_4 + 8H_2O$

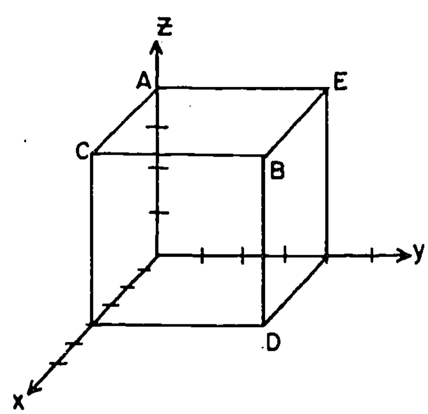
Illustrative Test Questions

Part I: Multiple Choice.

Directions: Select the response which best completes the the statement or answers the question.

1. The set of points in space equidistant from two given points is
   (a) a cylinder.
   (b) a plane.
   (c) a straight line.
   (d) the midpoint of the line segment which joins the two points.
   (e) two parallel straight lines.

2. The point whose coordinates are (4,0,4) is
   (a) A.
   (b) B.
   (c) C.
   (d) D.
   (e) E.



3. Which of the following is an ordered triple of real numbers that corresponds to a point in the xz-plane
   (a) (0,2,0).
   (b) (0,3,-2).
   (c) (3,2,0).
   (d) (2,3,2)
   (e) (-2,0,3)

4. The distance between the points (2,3,4) and (4,3,2) is

   (a) 0.
   (b) 4.
   (c) $2\sqrt{2}$.
   (d) 8.
   (e) $3\sqrt{2}$.

5. Which one of the following points is 5 units from the origin?

   (a) (-4,3,0).
   (b) (1,2,0).
   (c) $(\sqrt{2}, 1, \sqrt{2})$.
   (d) $(\sqrt{2}, \sqrt{3}, 0)$.
   (e) (5,3,4).

6. The equation $ax + by + cz + d = 0$, where a, b, c, d are real constants, represents a plane if and only if

   (a) all four constants are different from zero.
   (b) $d \neq 0$.
   (c) a, b, c are all different from zero.
   (d) at least one of the constants a, b, c, is different from zero.
   (e) at least one of the constants a, b, c, d, is different from zero.

7. Which of the following statements about the plane whose equation is $x + y + z = 0$ is <u>not</u> true?

   (a) It is the perpendicular bisector of the line segment joining (1,1,1) and (-1,-1,-1).
   (b) It passes through the origin.
   (c) It contains the point (0,1,-1)
   (d) It intersects the xy-plane in the line $x + y = 0$.
   (e) It intersects the z-axis in the point (1,-1,0)

8. The set of points in space defined by the equation $y = 5$ is

   (a) a plane parallel to the y-axis.

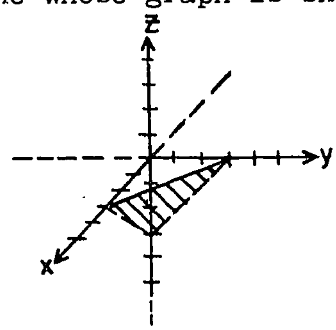
   (b) a plane perpendicular to the y-axis.

   (c) a plane containing the y-axis.

   (d) a line intersecting the y-axis.

   (e) a point on the y-axis.

9. What is the equation of the plane whose graph is sketched at the right?



   (a) $x + y = 3$.

   (b) $x + y - z = 3$.

   (c) $-x - y + z = 3$.

   (d) $x - y + z = 3$.

   (e) $x - y - z = 3$.

10. Which one of the following points lies in the plane whose equation is $x - 2y = 6$?

   (a) (0,-3,9).

   (b) (2,2,7).

   (c) (0,6,0).

   (d) (0,3,-6).

   (e) (12,-3,6).

11. The solution set of the equation $px + qy + rz = 0$ contains the element

   (a) (p,q,-r).

   (b) (r,-p,q).

   (c) (0,0,r).

   (d) (0,r,q).

   (e) (r,0,-p).

12. Which of the following number triples is in the solution set of the system $\begin{cases} x - 2y + z = 4 \\ z = 2 \end{cases}$?

I. (2,0,2).

II. (0,-2,0).

III. (4,1,2).

(a) I. only (c) III. only (e) I., II. and III

(b) II. only (d) I. and III. only

13. How many number triples are in the solution set of three equations which represent three coincident planes?

(a) 0 (d) 3

(b) 1 (e) Infinitely many

(c) 2

14. The trace of the graph of the equation $x - 2y + z = 5$ in the xy-plane is

(a) $-2y + z = 5$. (d) $x - 2y = 0$.

(b) $x - 2y = 5$. (e) $x + z = 0$.

(c) $x + z = 5$.

15. The trace of the graph of the equation $ax + by + cz = d$ in the xz-plane is given by

(a) $by = d$. (d) $x + z = \frac{d}{a + c}$.

(b) $\begin{cases} ax + by + cz = d. \\ y = 0 \end{cases}$ (e) none of the above.

(c) $ax + cz = 0$.

16. Which of the following represents a straight line in a three dimensional coordinate system?

(a) $\begin{cases} x = -3z + 1 \\ y = 2z + 3 \end{cases}$ (c) $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$ (e) $x = 3$

(b) $\begin{cases} x + y = 6 \\ x + y = 7 \end{cases}$ (d) $x = y$

17. In each of the following systems the three equations represent three planes. In which system do the three planes intersect in a line?

(a) $\begin{cases} z = 0 \\ y = 0 \\ z + y = 2 \end{cases}$

(b) $\begin{cases} x = 2 \\ y = 4 \\ 2x - y = 0 \end{cases}$

(c) $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$

(d) $\begin{cases} x - 2y = 0 \\ x - 2y = 4 \\ z = 5 \end{cases}$

(e) $\begin{cases} x + y + z = 4 \\ 2x + 2y + 2z = 8 \\ 3x + 3y + 3z = 12 \end{cases}$

18. Which statement is true of the solution set of the following system of equations?

$$\begin{cases} 3x - y + 2z = 6 \\ 6x - 2y + 4z = 7 \end{cases}$$

(a) The solution set has an infinite number of elements.

(b) The graph of the solution set is a straight line.

(c) The solution set is empty.

(d) The solution set contains exactly one element.

(e) None of the above statements is true.

19. Which one of the following systems of equations represents a pair of parallel planes?

(a) $\begin{cases} 2x + 3y + 4z = 0 \\ x + 2y + 3z = 0 \end{cases}$

(b) $\begin{cases} 2x + 3y - 4z = 1 \\ 2x + 3y + 4z = 1 \end{cases}$

(c) $\begin{cases} 2x - 3y - 4z = 3 \\ 2x - 6y - 8z = 6 \end{cases}$

(d) $\begin{cases} 2x + 3y - 4z = 2 \\ 4x + 6y - 8z = 2 \end{cases}$

(e) $\begin{cases} x = 4 \\ y = 3 \end{cases}$

20. The solution set of the system $\begin{cases} x + y + z = 2 \\ 3x - 3y + 3z = 9 \\ x + y - z = 6 \end{cases}$

(a) is empty.

(b) contains a single number triple.

(c) contains an infinite set of number triples which correspond to points of a straight line.

(d) contains an infinite set of number triples which correspond to points of a plane.

(e) contains exactly three number triples.

21. What is the solution set of the following system? $\begin{cases} x + y + z = 4 \\ x + y = 2 \\ y = -3 \end{cases}$

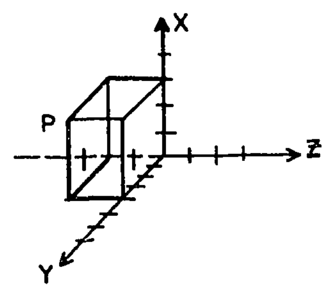
(a) (-1,3,2). (d) (5,-3,-4).

(b) (1,-3,6). (e) (-5,-3,12).

(c) (5,-3,2).

Part II: Problems.

22. If the x, y, and z axes are chosen as shown in the figure, what triple of real numbers (x,y,z) are the coordinates of P?



23. Find the distance between the points (3,4,2) and (-3,4,0).

24. Find an equation for the locus of points equidistant from the points (2,4,-1) and (0,5,6).

25. Make a free-hand drawing of the graphs of the following equations in a three dimensional coordinate system.

(a) $x + y = 2$. (b) $3x + y + 2z = 6$.

26. If the planes whose equations are given in the following system intersect in a line, express two of the variables of the solution set in terms of the third variable. If the planes do not intersect in a line, describe their position with respect to each other.

$$\begin{cases} x + 3y - 2z = 6 \\ x - 2y + z = 4 \end{cases}$$

27. Find the solution set of the following system of equations.

$$\begin{cases} 2x - 4y + 3z = 17 \\ x + 2y - z = 0 \\ 4x - y - z = 6 \end{cases}$$

28. Find a three digit number such that the sum of the digits is 19; the sum of the hundreds digit and the units digit is one more than the tens digit, and the hundreds digit is four more than the units digit.

29. Three tractors, A, B, and C, working together can plow a field in 8 days. Tractors A and B can do the work in 14 days. Tractor A can plow the entire field in half the time that it takes Tractor C. Write a system of equations which could be solved to find the number of days it would take each tractor to do the work alone. (You need not solve the system).

30. Give the coordinates of the point which is symmetric to the point (1,-2,3) with respect to

(a) the origin.
(b) the x-axis
(c) the y-axis
(d) the z-axis
(e) the yz-plane
(f) the zx-plane
(g) the xy-plane

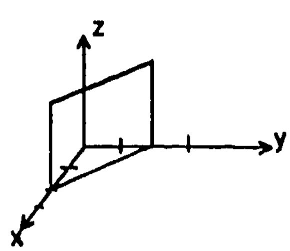
Answers to Illustrative Test Questions

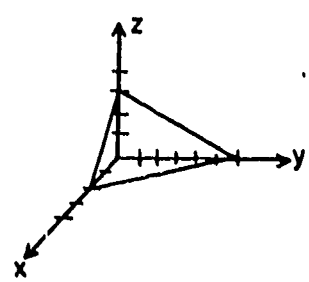
Part I Multiple Choice:

1.	B	12.	D
2.	C	13.	E
3.	E	14.	B
4.	C	15.	B
5.	A	16.	A
6.	D	17.	B
7.	E	18.	C
8.	B	19.	D
9.	B	20.	B
10.	A	21.	C
11.	E		

Part II Problems:

22. (3,4,-2)
23. $2\sqrt{10}$
24. $2x - y - 7z + 15 = 0$
25. (a) (b)





26. $\begin{cases} x = \dfrac{z + 24}{5} \\ \\ y = \dfrac{3z + 2}{5} \end{cases}$ or $\begin{cases} y = 3x - 14 \\ z = 5x - 24 \end{cases}$ or $\begin{cases} x = \dfrac{y + 14}{3} \\ z = \dfrac{5y - 2}{3} \end{cases}$

27. (3,1,5)

28. 793.

29. $\begin{cases} \dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C} = \dfrac{1}{8} \\ \dfrac{1}{A} + \dfrac{1}{B} = \dfrac{1}{14} \\ \dfrac{1}{A} = \dfrac{2}{C} \end{cases}$

30. (a) (-1,2,-3)
(b) (1,2,-3)
(c) (-1,-2,-3)
(d) (-1,2,3)
(e) (-1,-2,3)
(f) (1,2,3)
(g) (1,-2,-3)