DOCUMENT RESUME

ED 135 617 SE 021 987

AUTHOR	Allen, Frank B.; And Others
TITLE	First Course in Algebra, Student's Text, Part I, Unit 9.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	61
NOTE	259p.; For related documents, see SE 021 988-SE 002 002 and ED 130 870-877; Contains occasional light and broken type
EDRS PRICE	MF-$0.83 HC-$14.05 Plus Postage.
DESCRIPTORS	*Algebra; *Curriculum; Elementary Secondary Education; Instruction; *Instructional Materials; Mathematics Education; Number Systems; *Secondary School Mathematics; *Textbooks
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

Unit 9 in the SMSG's secondary school mathematics series is a student text covering the following topics in Algebra I: sets and the number line, numerals and variables, sentences and properties of operations, open sentences and English sentences, the real numbers, properties of addition, properties of multiplication, properties of order, and subtraction and division for real numbers. (DT)

STUDENT'S TEXT

UNIT NO. 9

FIRST COURSE IN ALGEBRA

PART I

SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

School Mathematics Study Group

First Course in Algebra

Unit 9

First Course in Algebra

Student's Text, Part I

Prepared under the supervision of
the Panel on Sample Textbooks
of the School Mathematics Study Group:

Frank B. Allen	Lyons Township High School
Edwin C. Douglas	Taft School
Donald E. Richmond	Williams College
Charles E. Rickart	Yale University
Henry Swain	New Trier Township High School
Robert J. Walker	Cornell University

New Haven and London, Yale University Press

Printed in the United States of America.

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

Below are listed the names of all those who participated in any of the writing sessions at which the following SMSG texts were prepared: First Course in Algebra, Geometry, Intermediate Mathematics, Elementary Functions, and Introduction to Matrix Algebra.

H.W. Alexander, Earlham College
F.B. Allen, Lyons Township High School, La Grange, Illinois
Alexander Beck, Olney High School, Philadelphia, Pennsylvania
E.F. Beckenbach, University of California at Los Angeles
E.G. Begle, School Mathematics Study Group, Yale University
Paul Berg, Stanford University
Emil Berger, Monroe High School, St. Paul, Minnesota
Arthur Bernhart, University of Oklahoma
R.H. Bing, University of Wisconsin
A.L. Blakers, University of Western Australia
A.A. Blank, New York University
Shirley Boselly, Franklin High School, Seattle, Washington
K.E. Brown, Department of Health, Education, and Welfare, Washington, D.C.
J.M. Calloway, Carleton College
Hope Chipman, University High School, Ann Arbor, Michigan
R.R. Christian, University of British Columbia
R.J. Clark, St. Paul's School, Concord, New Hampshire
P.H. Daus, University of California at Los Angeles
R.B. Davis, Syracuse University
Charles DePrima, California Institute of Technology
Mary Dolciani, Hunter College
Edwin C. Douglas, The Taft School, Watertown, Connecticut
Floyd Downs, East High School, Denver, Colorado
E.A. Dudley, North Haven High School, North Haven, Connecticut
Lincoln Durst, The Rice Institute
Florence Elder, West Hempstead High School, West Hempstead, New York
W.E. Ferguson, Newton High School, Newtonville, Massachusetts
N.J. Fine, University of Pennsylvania
Joyce D. Fontaine, North Haven High School, North Haven, Connecticut
F.L. Friedman, Massachusetts Institute of Technology
Esther Gassett, Claremore High School, Claremore, Oklahoma
R.K. Getoor, University of Washington
V.H. Haag, Franklin and Marshall College
R.R. Hartman, Edina-Morningside Senior High School, Edina, Minnesota
M.H. Heins, University of Illinois
Edwin Hewitt, University of Washington
Martha Hildebrandt, Proviso Township High School, Maywood, Illinois
R.C. Jurgensen, Culver Military Academy, Culver, Indiana
Joseph Lehner, Michigan State University
Marguerite Lehr, Bryn Mawr College
Kenneth Leisenring, University of Michigan
Howard Levi, Columbia University
Eunice Lewis, Laboratory High School, University of Oklahoma
M.A. Linton, William Penn Charter School, Philadelphia, Pennsylvania
A.E. Livingston, University of Washington
L.H. Loomis, Harvard University
R.V. Lynch, Phillips Exeter Academy, Exeter, New Hampshire
W.K. McNabb, Hockaday School, Dallas, Texas
K.G. Michaels, North Haven High School, North Haven, Connecticut
E.E. Moise, University of Michigan
E.P. Northrop, University of Chicago
O.J. Peterson, Kansas State Teachers College, Emporia, Kansas
B.J. Pettis, University of North Carolina
R.S. Pieters, Phillips Academy, Andover, Massachusetts
H.O. Pollak, Bell Telephone Laboratories
Walter Prenowitz, Brooklyn College
G.B. Price, University of Kansas
A.L. Putnam, University of Chicago
Persis O. Redgrave, Norwich Free Academy, Norwich, Connecticut
Mina Rees, Hunter College
D.E. Richmond, Williams College
C.E. Rickart, Yale University
Harry Ruderman, Hunter College High School, New York City
J.T. Schwartz, New York University
O.E. Stanaitis, St. Olaf College
Robert Starkey, Cubberley High Schools, Palo Alto, California
Phillip Stucky, Roosevelt High School, Seattle, Washington
Henry Swain, New Trier Township High School, Winnetka, Illinois
Henry Syer, Kent School, Kent, Connecticut
G.B. Thomas, Massachusetts Institute of Technology
A.W. Tucker, Princeton University
H.E. Vaughan, University of Illinois
John Wagner, University of Texas
R.J. Walker, Cornell University
A.D. Wallace, Tulane University
E.L. Walters, William Penn Senior High School, York, Pennsylvania
Warren White, North High School, Sheboygan, Wisconsin
D.V. Widder, Harvard University
William Wooton, Pierce Junior College, Woodland Hills, California
J.H. Zant, Oklahoma State University

CONTENTS

PREFACE

To The Student:

This textbook is written for you to read. It is not just a list of problems.

Your mathematical growth and, as a consequence, the satisfaction and enjoyment which you derive from the study of algebra will depend largely on careful reading of the book. For this reason, you will find it important to develop effective habits of reading mathematics.

Reading mathematics is not the same as reading a novel. You will find that there are times when you will understand only part of a paragraph the first time you read it, a little more the second time, and will feel sure of yourself only after the third reading. Sometimes working out details or examples with paper and pencil will be necessary.

You have a new and enriching experience ahead of you. Make the most of it. Go to it.

Chapter 1

SETS AND THE NUMBER LINE

1-1. Sets and Subsets

Can you give a description of the following:

Alabama, Arkansas, Alaska, Arizona?

How would you describe these?

Monday, Tuesday, Wednesday, Thursday, Friday?

Include:

Saturday, Sunday

in the preceding group and then describe all seven. Give a description of the collection of numbers:

1, 2, 3, 4, 5;

of the collection of numbers:

2, 3, 5, 7, 8.

You may wonder if you drifted into the wrong class. What do these questions have to do with mathematics? Each of the above collections is an example of a set. Your answers to the questions should have suggested that each set was a particular collection of members or elements with some common characteristic. This characteristic may be only the characteristic of being listed together.

The concept of a set will be one of the simplest of those you will learn in mathematics. A set is merely a collection of elements (usually numbers in our work).

Now we need some symbols to indicate that we are forming or describing sets. If the members of the set can be listed, we may include the members within braces, as for the set of the first five odd numbers:

{1, 3, 5, 7, 9};

or the set of all even numbers between 1 and 9;

{2, 4, 6, 8};

or the set of states whose names begin with C:

{California, Colorado, Connecticut}.

Can you list all the elements of the set of all odd numbers? Or the names of all citizens of the United States? You see that in these cases we may prefer or even be forced to give a description of the set without attempting to list all its elements.

It is convenient to use a capital letter to name a set, such as

$$A = \{1, 3, 5, 7\}$$

or

A is the set of all odd numbers between 0 and 8.

A child learning to count recites the first few elements of a set N which we call the set of <u>counting numbers</u>:[*]

$$N = \{1, 2, 3, 4, \ldots\}.$$

We write this set with enough elements to show the pattern and then use three dots to mean "and so forth." When we take the set N and include the number 0, we call the new set the set W of <u>whole numbers</u>:

$$W = \{0, 1, 2, 3, 4, \ldots\}.$$

An interesting question now arises. How shall we describe a set such as the set of all even whole numbers which are greater than 8 and at the same time less than 10? Does this set contain <u>any</u> elements? You may not be inclined to call this a set, because there is no way to list its elements. In mathematics we say that the set which contains no elements is described as the <u>empty</u>, or <u>null</u>, set. We shall use the symbol $\emptyset$ to denote the empty set.

Warning! The set $\{0\}$ is <u>not</u> empty; it contains the element 0. On the other hand, we never write the symbol for the empty set $\emptyset$ with braces.

Perhaps you can think of more examples of the null set, such as the set of all whole numbers between $\frac{4}{3}$ and $\frac{5}{4}$.

Notice that when we talk in terms of sets, we are concerned more with collections of elements than with the individual elements themselves. Certain sets may contain elements which also

---

[*] We sometimes refer to counting numbers as "natural numbers."

belong to other sets. For example, let us consider the sets

$$R = \{0, 1, 2, 3, 4\} \text{ and } S = \{0, 2, 4, 6\}.$$

Form the set T of all numbers which belong to both R and S. Thus,

$$T = \{0, 2, 4\}.$$

We see that every element of T is also an element of R. We say that T is a <u>subset</u> of R.

> If every element of a set A belongs to a set B, then A is a <u>subset</u> of B.

Is T a subset of S?

One result is that every set is a subset of itself! Check for yourself that $\{0, \frac{1}{2}, 3, 4\}$ is a subset of itself. We shall also agree that the null set $\emptyset$ is a subset of every set.

Problem Set 1-1a

1. List the elements of the set of
   (a) All odd whole numbers from 1 to 12 inclusive.
   (b) All numbers from 0 to 50, inclusive, which are squares of whole numbers.
   (c) All two-digit whole numbers, each of whose units digit is twice its tens digit.
   (d) All whole numbers from 0 to 10, inclusive, which are the square roots of whole numbers.
   (e) All cities in the U. S. with population exceeding five million.
   (f) All numbers less than 10 which are squares of whole numbers.
   (g) Squares of all those whole numbers which are less than 10.
   (h) All whole numbers less than 5 and at the same time greater than 10.

2. Given the following sets:

   P, the set of whole numbers greater than 0 and less than 7;

   Q, the set of counting numbers less than $\frac{13}{2}$;

   R, the set of numbers represented by the symbols on the faces of a die;

   S, the set {1, 2, 3, 4, 5, 6}.

   (a) List the elements of each of the sets P, Q, R.

   (b) Give a description of set S.

   (c) From the answers to (a) and (b) decide how many possible descriptions a set may have.

3. Find U, the set of all whole numbers from 1 to 4, inclusive. Then find T, the set of squares of all members of U. Now find V, the set of all numbers belonging to both U and T. (Did you include 2 in V? But 2 is not a member of T, so that it cannot belong to both U and T.) Does every member of V belong to U? Is V a subset of U? Is V a subset of T? Is U a subset of T?

4. Returning to problem 3, let K be the set of all numbers each of which belongs either to U or to T or to both. (Did you include 2 in K? You are right, because 2 belongs to U and hence belongs to either U or to T. The numbers 1 and 4 belong to both U and T but we include them only once in K.) Is K a subset of U? Is U a subset of K? Is T a subset of K? Is U a subset of U?

*5. Consider the four sets

$$\emptyset$$
$$A = \{0\}$$
$$B = \{0, 1\}$$
$$C = \{0, 1, 2\}$$

How many different subsets can be formed from the elements of each of these four sets? Can you tell, without writing out the subsets, the number of subsets in the set

$$D = \{0, 1, 2, 3\}?$$

What is the rule you discovered for doing this?

A set may have any number of elements. How many elements are in the set of all odd numbers between 0 and 100? Could you count the number of elements? Do you need to count them to determine how many elements there are?

How many elements are in the set of whole numbers which are multiples of 5? (A multiple of 5 is a whole number times 5.) Can you count the elements of this set (that is, with the counting coming to an end)?

Consider a set whose elements can be counted, even though the job of counting would entail an enormous amount of time and effort. Such a set is the set of all living human beings at a given instant. On the other hand, there are sets whose elements cannot possibly be counted because there is no end to the number of elements.

We shall say that a set is <u>finite</u> if the elements of the set can be counted, with the counting coming to an end, or if the set is the null set. Otherwise, we call it an <u>infinite</u> set. We say that an infinite set has <u>infinitely many</u> elements.

Sometimes a finite set may have so many members that we prefer to abbreviate its listing. For example, we might write the set E of all even numbers between 2 and 50 as

$$E = \{4, 6, 8, \ldots, 48\}.$$

Problem Set 1-1b

1. How many elements has each of the following sets?

   (a) The set of all whole numbers from 10 to 27, inclusive.

   (b) The set of all odd numbers between 0 and 50.

   (c) The set of all multiples of 3.

   (d) The set of all multiples of 3 from 0 to 99, inclusive.

   (e) The set of all multiples of 10 from 10 to 1,000 inclusive.

2. Classify the following sets (finite or infinite):
   - (a) All natural numbers.
   - (b) All squares of all counting numbers.
   - (c) All citizens of the United States.
   - (d) All natural numbers less than one billion.
   - (e) All natural numbers greater than one billion.

3. Given the sets S = {0, 5, 7, 9} and T = {0, 2, 4, 6, 8, 10}.
   - (a) Find K, the set of all numbers belonging to both S and T. Is K a subset of S? Of T? Are S, T, K finite?
   - (b) Find M, the set of all numbers each [illegible] belongs to S or to T or to both. (We never include the same number more than once in a set.) Is M a subset of S? Is T a subset of M? Is M finite?
   - (c) Find R, the subset of M, which contains all the odd numbers in M. Of which others of our sets is this a subset?
   - (d) Find A, the subset of R, which contains all the members of M which are multiples of 11. Did you find that A has no members? What do we call this set?
   - (e) Are sets A and K the same? If not, how do they differ?
   - (f) From your experience with the last few problems, could you draw the conclusion that subsets of finite sets are also finite?

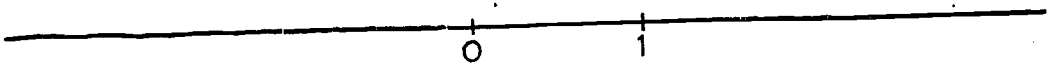
4. Referring to the definition of a multiple of 5 given above in the text, define an even whole number in terms of multiples. Is 0 an even number?

1-2. The Number Line.

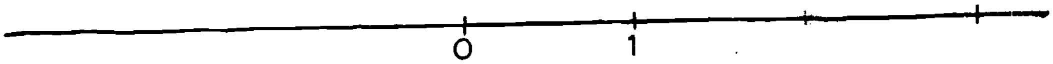
In your study of arithmetic you began by using numbers to count, and later you thought of them in terms of symbols apart from counting. You have also worked with points on scales or lines, such as on a ruler or a thermometer. Suppose we now associate numbers with points on a line.

First, we draw a line, regarding it as a set of points. How many points? Certainly, there are infinitely many points. Now let us associate some of these points with numbers in the following way:

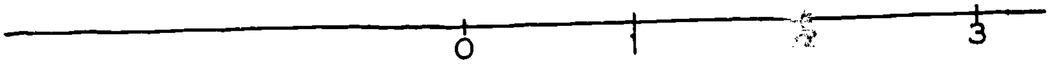
Choose two distinct points on the line and label the point on the left with the symbol 0 and on the right with 1.



Using the interval between these points as a unit of measure, and beginning at the point associated with 1, locate points equally spaced along the line to the right. We think of this process as continuing without end, even though we cannot show the process beyond the right margin of the page.



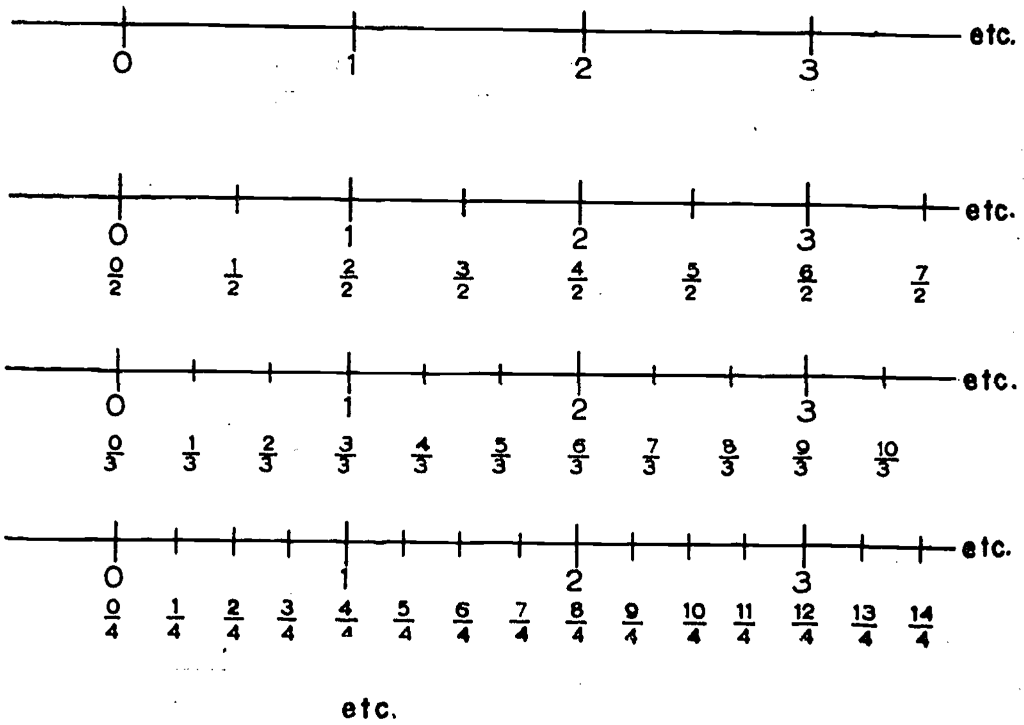
Now label these points from left to right, labeling each point with the successive whole numbers. Our line now looks like this:



We notice that each whole number is followed on the right by its <u>successor</u>. What is the successor of 105? Of 1000000005? Choose as large a counting number as you can imagine. Does it have a successor? Give a rule for finding the successor of a counting number. What does this imply? For one thing, since each counting number has a successor, there cannot be a largest counting number, and the set of counting numbers is infinite.

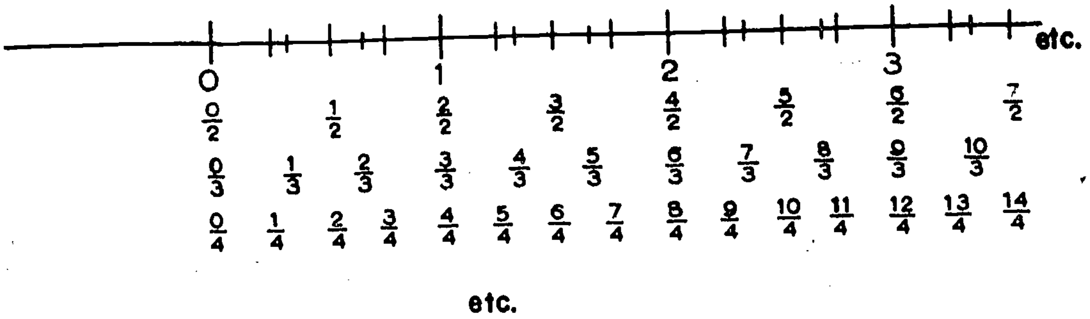
Let us be certain to understand that every whole number is now attached to a point of the extended line and every point so far located on this line is associated with a whole number. This correspondence between sets of numbers and sets of points on a line is an idea we shall use many times in this course.

Starting with the line on which points are labeled with whole numbers, we can label other points by dividing the intervals into halves, thirds, fourths, etc., to obtain



where we visualize an unending set of lines with the intervals divided into successively greater numbers of parts.

Now put all these together on one line. We have a labeling of points corresponding to some of the elements of a set of numbers which are called rational numbers.



We shall call a line on which we label points with numbers, as we did above, a <u>number line</u>. The number associated with a point is called the <u>coordinate</u> of the point.

At this time let us review what is meant by a "fraction." Notice that the coordinate of the point on the number line corresponding to 2 has many different names:

$$\frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \text{ etc.}$$

Each of these symbols is a <u>fraction</u> and each is a different name for the same number 2. The number named by "$2\frac{1}{2}$" can also be represented by fractions: $\frac{5}{2}$, $\frac{10}{4}$, $\frac{15}{6}$, etc. Also, the number named by ".6" can be represented by: $\frac{3}{5}$, $\frac{6}{10}$, $\frac{9}{15}$, etc. In general, we shall mean by a <u>fraction</u> a symbol which indicates the quotient of two numbers.

A number which can be represented by a fraction indicating the <u>quotient of two whole numbers</u>, excluding division by zero, is called a <u>rational number</u>. Thus, 2, $2\frac{1}{2}$, .6, $\frac{3}{2}$, $\frac{1001}{37}$ are examples of rational numbers. Later we shall study rational numbers which correspond to points to the left of 0 on the number line. We shall also see later that not all fractions represent rational numbers, but by definition every rational number can be represented by a fraction. Represent the following rational numbers by fractions: $3\frac{1}{3}$, 7, 3.5, 0.

The important points to remember are:

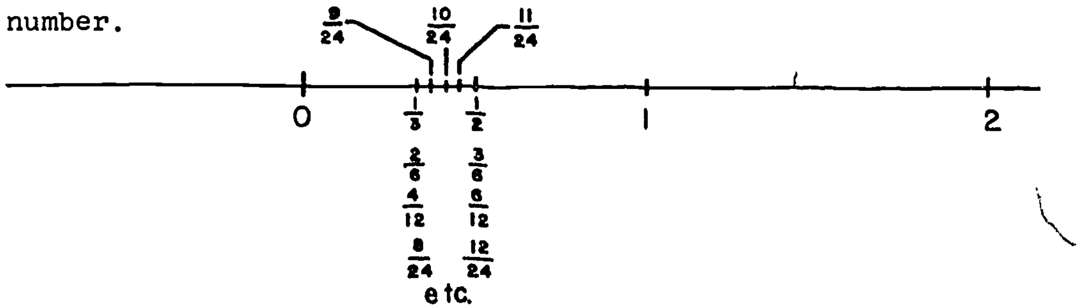
(1) A rational number can be represented by a fraction. (<u>Must</u> it be represented by a fraction?)

(2) The set of all whole numbers is a subset of the set of all rational numbers; that is, all whole numbers are rational numbers. (Are all rational numbers whole numbers?)

(3) There are many possible names for the same number.

(4) We have a process for locating the point on the number line corresponding to any given rational number; that is, for each element of the set of rational numbers there is a corresponding point on the number line.

You may think we will use up the supply of points in this process of assigning numbers to points. Are we sure that between any two distinct points, no matter how close, there is another point? We can answer this question for points corresponding to rational numbers as follows: Choose two such points, for example the points with coordinates $\frac{1}{3}$ and $\frac{1}{2}$. We know that $\frac{1}{3}$ has the names: $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{8}{24}$, etc., and $\frac{1}{2}$ has the names: $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, $\frac{12}{24}$, etc. Then a number between $\frac{1}{3}$ and $\frac{1}{2}$ is a number between $\frac{8}{24}$ and $\frac{12}{24}$. We can choose any one of $\frac{9}{24}$, $\frac{10}{24}$, $\frac{11}{24}$ as such a number.



Thus, the points with coordinates $\frac{9}{24}$, $\frac{10}{24}$, $\frac{11}{24}$ are between the points with coordinates $\frac{1}{3}$ and $\frac{1}{2}$.

This process of finding a point between points by finding a number between numbers could be carried out for any two points, no matter how close. This serves to show us that there are not only a great many points between any two given points, but infinitely many points.

Now we are quite sure that every rational number corresponds to a point on the number line. Do you think that every point on the number line (to the right of 0) corresponds to a rational number? In other words, do you think we can label every point to the right [illegible] with a rational number?

The answer to this question, amazingly, is "No." Later this year we shall prove this fact to you. And we shall soon associate numbers with points to the left of 0. Meanwhile, we assume that <u>every</u> point to the right of 0 has a number coordinate, although some of these numbers are <u>not</u> rational.

To summarize the above statements: There are infinitely many points on the number line. There are also infinitely many points with rational numbers as coordinates. Indeed, there are infinitely many points between each pair of points on the number line. Although we have seen this only for points with rational coordinates, it is also true for all points.

In Chapters 1 through 4 we shall be concerned with the set of numbers consisting of 0 and all numbers corresponding to points on the right of 0. In these chapters when we speak of "numbers of arithmetic" we shall mean numbers of this set.

Problem Set 1-2a

1. How many numbers are there between 2 and 3? Between $\frac{2}{500}$ and $\frac{3}{500}$? List two numbers between 2 and 3; between $\frac{2}{500}$ and $\frac{3}{500}$.
2. On the number line indicate with heavy dots the points whose coordinates are
   (a) $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$.
   (b) Rational numbers represented by fractions with denominators 5, beginning with $\frac{1}{5}$ and ending with $\frac{20}{5}$.
   (c) 0, 0.5, 0.7, 1.1, 1.5, 1.8, 2, 2.7, 3.5, 4, 4.1.

(d) Which of the symbols shown in parts (a) and ([illegible]) are not fractions? Which of the symbols in (a) [illegible] ([illegible]) represent rational numbers?

3. In the problem 2(b), the coordinate of the point associated with $\frac{20}{5}$ could have had another name, the common name of a natural number. What would this be? Can you write four other names of the number that could serve as the coordinate of this point?

4. Write six names for the coordinate of the point associated with $\frac{3}{4}$.

5. On the number line we see that some points lie to the right of others, some to the left of others, some between others. How is the point with coordinate 3.5 located with respect to the point with coordinate 2? Which is the greater of 3.5 and 2? How is the point with coordinate 1.5 located with respect to the point with coordinate 2? Which is the greater of 1.5 and 2?

6. Between which successive whole numbers will you find $\frac{22}{7}$? Is $\frac{22}{7}$ greater than 3.1? Does the point with coordinate $\frac{22}{7}$ lie to the left of the point with coordinate 3.2? Between what two numbers expressed in tenths does $\frac{22}{7}$ lie?

*7. What can you say about a set S of whole numbers if it has the two characteristics:

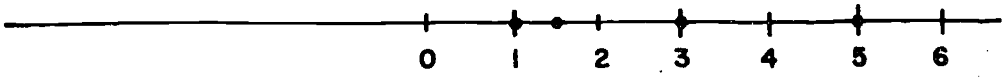
(1) 2 is an element of S;

(2) whenever a number belongs to S its successor also belongs to S?

Let us return to the idea of a set of numbers and visualize a set on the number line. For example, each element of the set

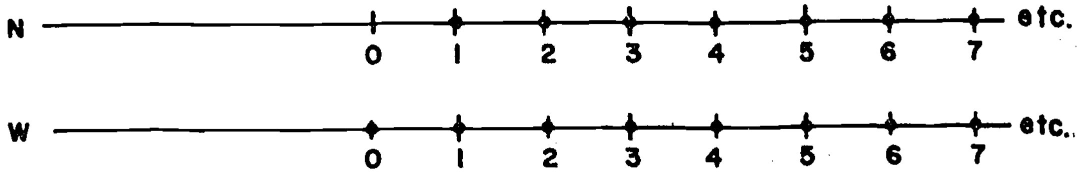
$$A = \{1, \tfrac{3}{2}, 3, 5\}$$

is a number associated with a point on the number line. We call this set of associated points the graph of the set A. Let us indicate the points of the graph by marking them specially with heavy dots:



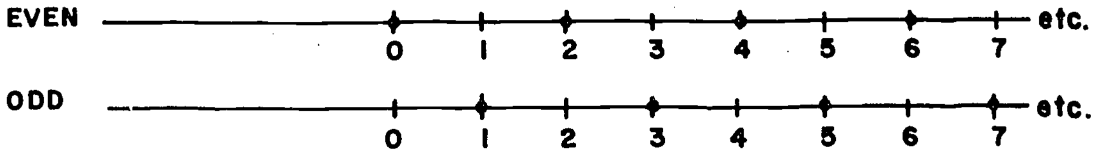
Thus, the graph of a set of numbers is the corresponding set of points on the number line whose coordinates are the numbers of the set, and only those points.

In passing, we note that the graphs of the set N of counting numbers and the set W of whole numbers are:



From these diagrams we see immediately that N is a subset of W.

The graphs of the sets of even and odd whole numbers are:



Problem Set 1-2b

1. Given the sets S = {0, 3, 4, 7} and T = {0, 2, 4, 6, 8, 10}.
   (a) Find the set K, the set of all numbers belonging to both S and T, and the set M, the set of all numbers each of which belongs to S or T or to both.
   (b) Draw four identical number lines, one below the other. On successive lines show the graphs of the sets S, T, K, and M.
   (c) What schemes do you see for obtaining the graphs of K

and M from the graphs of S and T?

2. Consider the sets $A = \{0, 5, 7, 9\}$ and $B = \{1, \frac{5}{2}, 8, 10\}$.

   (a) Draw two identical number lines, one below the other, and on these lines show the graphs of A and B.

   (b) If set C is the set of numbers which are members of both A and B, what do you infer about set C from the graphs of A and B? What is the name of this set?

*3. Every finite set except the null set has the property that it can be paired off, one-to-one, with a finite set of natural numbers. For example, the set of all letters of the English alphabet can be paired off, one-to-one, with the set of the first 26 natural numbers.

$$\begin{array}{ccccc} 1, & 2, & 3, & 4, \ldots, & 26 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ a, & b, & c, & d, \ldots, & z \end{array}$$

An infinite set, however, cannot be paired off, one-to-one, with a finite set. Furthermore, it has the surprising property that all its members can be paired off, one-to-one, with the elements of some <u>proper</u> subset of itself. (A proper subset of a set is one which does not contain all the elements of the set.) How can the set of whole numbers (which is infinite) be paired off, one-to-one, with the set of all multiples of 3 (which is a proper subset of the set of whole numbers?)

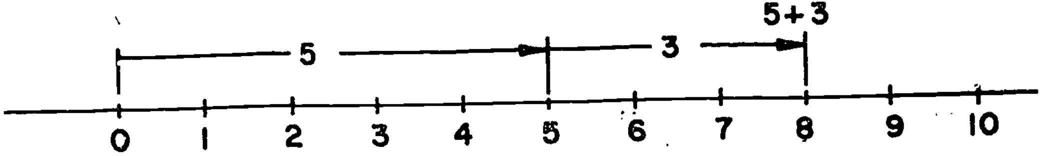
1-3. Addition and Multiplication on the Number Line.

We have seen how to portray sets of numbers on a number line. Now let us use the number line to visualize addition and multiplication of numbers.

At the beginning we recall that

$$5 + 3$$

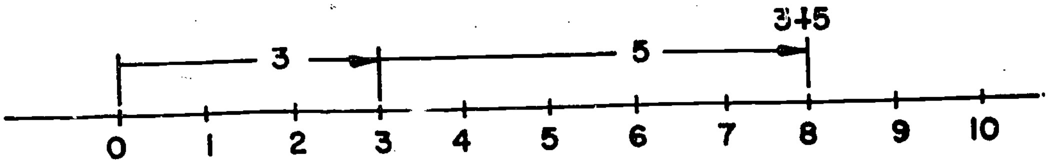
is a symbol for a number obtained by adding 3 to 5. This can be interpreted as moving from 0 to 5 on the number line and then moving from this point three units to the right, thereby locating a point with coordinate 5 + 3.



As a different example let us find

$$3 + 5$$

on the number line. We move from 0 to 3 and then move 5 units to the right, thereby locating the point with coordinate 3 + 5.



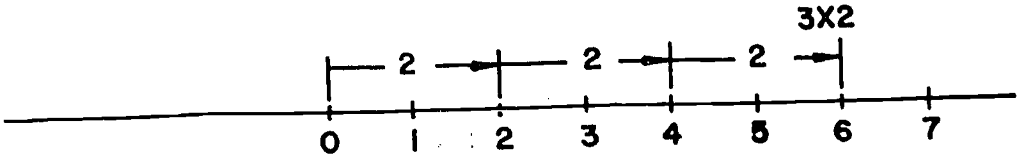
These two addition processes are different, as we see by the diagrams, although they terminate at the same point. That is, "5 + 3" and "3 + 5" are different symbols for the same number 8.

We may wonder whether addition on the number line will always be possible. Will the sum of any two rational numbers on the number line be a rational number? It is suggested that you think about this question carefully.

The procedure of multiplying two counting numbers on the number line is similar to that of addition if we recall that, for example,

$$3 \times 2$$

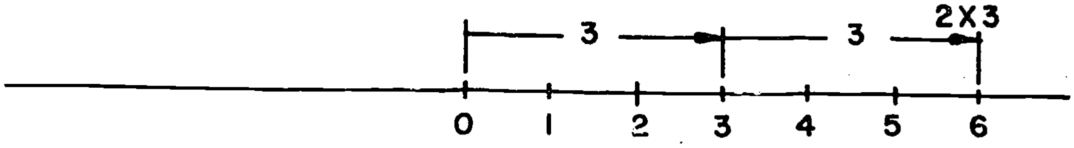
is a symbol for the number obtained by adding three 2's. On the number line this is interpreted as using the segment from 0 to 2 as a measure and moving to the right from 0 a distance of three such segments; we thereby locate the point with coordinate 3 × 2.



As a different example, consider

$$2 \times 3$$

which is a symbol for the number obtained by adding two 3's. On the number line we use the segment from 0 to 3 and move to the right from 0 a distance of two such segments. The terminal point has coordinate 2×3.



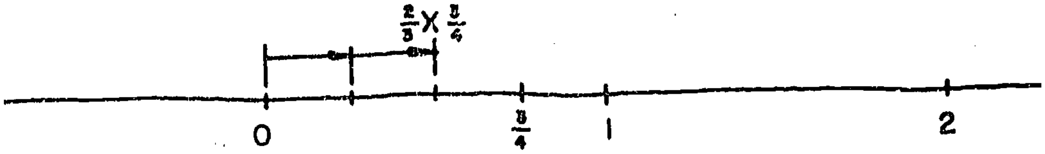
From the diagrams we observe that these two multiplications on the number line are different, but again they terminate at the same point.

Problem Set 1-3

1. Perform the following operations on the number line:
   - (a) $4 + 6$
   - (b) 3×4
   - (c) $0 + 0.8$
   - (d) 5×2
   - (e) $\frac{4}{3} + 3\frac{1}{3}$
   - (f) 4×1
2. Describe a procedure for subtracting numbers on the number line. Apply your procedure to

   (a) $6 - 2$, (b) $7 - 3$, *(c) $1.8 - \frac{9}{5}$

*3. We may visualize the multiplication of rational numbers on the number line as follows: Consider " $\frac{2}{3} \times \frac{3}{4}$ ", which is a symbol representing two-thirds of $\frac{3}{4}$. On the number line we divide the segment from 0 to $\frac{3}{4}$ into three equal parts and locate the point corresponding to the right end of the second part.



Perform the following operations on the number line:

(a) $\frac{1}{2} \times 4$, (b) $\frac{3}{4} \times \frac{2}{3}$.

4. (a) If we add any two odd numbers, will their sum be an odd number? Give an example.

   (b) If we multiply any two odd numbers, will their product be an odd number? Give an example.

5. Consider the set T = {1, 2, 3, 4}. If we select any element of this set and add to it any element of the set (including adding an element to itself), what is the set S of all possible sums? Is S a subset of T?

6. Consider the set Q = {0, 1}. Choose any element of Q and multiply it by any element of Q, including the same element. What is the set P of all possible products? Is P a subset of Q?

*7. Consider the set Q = {1, 3, 5, 7, 9 ...}. As in problem 6, list all the products of any element of Q times any element of Q. This set is P = {1, 3, 5, 7, 9, ...}. P is a subset of Q, and we describe this property of the odd numbers by saying that the set Q is "closed under multiplication." In general, we say that if a set K is given, and a certain operation is applied to all pairs of elements of K, and if the resulting set P is a subset of K, then set K is <u>closed under</u> the operation.

(a) Is the set Q above closed under addition?
(b) Is the set of even numbers closed under addition? Under the operation of taking the average?
(c) Is the set of whole numbers closed under addition? Under multiplication?
(d) Is the set of rational numbers closed under addition? Under multiplication?

*8. (a) Describe a set that is closed under the operation "twice the product."
(b) Describe a set that is closed under the operation "twice the product and add one."

9. Summarize the important ideas in this chapter.

Chapter 2

NUMERALS AND VARIABLES

2-1. Numerals and Numerical Phrases.

Most of your life you have been reading, writing, talking about, and working with numbers. You have also used many different names for the same number. Some numbers have one or more common names which are the ones most often used in referring to the numbers. Thus the common name for the number five is "5" and for one gross is "144". Common names for the rational number one-half are "$\frac{1}{2}$" and "0.5". A problem in arithmetic can be regarded as a problem of finding a common name for a number which is given in some other way; for example, 17 times 23 is found by arithmetic to be the number 391.

The names of numbers, as distinguished from the numbers themselves, are called numerals. Two numerals, for example, which represent the same number are the indicated sum "$4 + 2$" and the indicated product "2×3". The number represented in each case is 6 and we say that "the number $4 + 2$ is 6", "the number 2×3 is 6", and "the number $4 + 2$ is the same as the number 2×3". These statements can be written more briefly as "$4 + 2 = 6$", "$2 \times 3 = 6$", and "$4 + 2 = 2 \times 3$". This use of the equal sign illustrates its general use with numerals: An equal sign standing between two numerals indicates that the numerals represent the same number.

We shall need sometimes to enclose a numeral in parentheses in order to make clear that it really is a numeral. Hence it is convenient to regard the symbol obtained by enclosing a numeral for a given number in parentheses as another numeral for the same number. Thus "$(4 + 2)$" is another numeral for 6 and we might write "$(4 + 2) = 6$". In order to save writing, the symbol for multiplication "$\times$" is often replaced by a dot "$\cdot$". Hence "2×3" can be written as "$2 \cdot 3$". Also to save writing, we agree that two numerals placed side-by-side is an indicated product. For example "$2(7 - 4)$" is taken to mean the same as "$2 \times (7 - 4)$". Notice, however, that "23" is already established as the common name for the number twenty-three and so cannot be interpreted as the indicated product "2×3". A similar exception is "$2\frac{1}{4}$" which means "$2 + \frac{1}{4}$" rather than "$2 \times \frac{1}{4}$".

We may, however, write "2(3)" or "(2)(3)" in place of "2 × 3". Similarly, "2 × $\frac{1}{4}$" might be replaced by "2$\cdot\frac{1}{4}$" or "2($\frac{1}{4}$)".

Consider the expression "2 × 3 + 7". Is this a numeral? Perhaps you will argue that it is since 2 × 3 = 6 and hence

$$2 \times 3 + 7 = 6 + 7 = 13.$$

On the other hand, someone else might decide that since 3 + 7 = 10,

$$2 \times 3 + 7 = 2 \times 10 = 20.$$

Let us examine the expression more carefully. How do we read it? What numerals are involved in it? Obviously "2", "3", and "7" are numerals, but what about "2 × 3" and "3 + 7"? It is true that "2 × 3", as an indicated product, and "3 + 7", as an indicated sum, are both numerals. However, within the expression "2 × 3 + 7", if "2 × 3" is an indicated product, then "3 + 7" cannot be an indicated sum, or, if "3 + 7" is an indicated sum, then "2 × 3" cannot be an indicated product. Therefore, without additional information to decide between these alternatives, the expression "2 × 3 + 7" is really not a numeral since it does not represent a definite number. Another expression in which the same problem arises is "10 - 5 × 2". In order to avoid the confusion in expressions of this kind, we shall agree to give multiplication preference over addition and subtraction unless otherwise indicated. In other words, "2 × 3 + 7" will be read with "2 × 3" as an indicated product, so that 2 × 3 +7 = 13. Similarly, "10 - 5 × 2" will be read with "5 × 2" as an indicated product so that 10 - 5 × 2 = 0.

Parentheses can also be used to indicate how we intend for an expression to be read. We have only to enclose within parentheses those parts of the expression which are to be taken as a numeral. Thus, in the case of "2 × 3 + 7", we can write "(2 × 3) + 7" if we want "2 × 3" to be a numeral and "2 × (3 + 7)" if we want "3 + 7" to be a numeral. In other words, the operations indicated within parentheses are taken first. You should always feel free to insert whatever parentheses are needed to remove all doubt as to how an expression is to be read.

Another case in which we need to agree on how an expression should be read is illustrated by the following example:

$$\frac{5(6 - 2)}{13 - 3}$$

It is understood that the expressions above and below the fraction bar are to be taken as numerals. Therefore the expression [illegible] an indicated quotient of the numbers 5(6 - 2) and 1[illegible]

Problem Set 2-1a

1. Write six other numerals for the number 8. How many numerals are there for 8?

2. In each of the following, check whether the numerals name the same number.

   (a) $2 + 4 \times 5$ and 22
   (b) $(2 + 4) \times 5$ and 30
   (c) $3 \times 3 - 1$ and 6
   (d) $2 \times 5 + 1$ and $(2 \times 5) + 1$
   (e) $4 + 3 \times 2$ and $(4 + 3) \times 2$
   (f) $(3 + 2) + 5$ and $3 + (2 + 5)$
   (g) $14 - 4 \times 3$ and 2
   (h) $(4 + \frac{2}{3}) + \frac{1}{3}$ and $4 + (\frac{2}{3} + \frac{1}{3})$

3. Write a common name for each numeral.

   (a) $2 \times 5 + 7$
   (b) $2(5 + 7)$
   (c) $(4 + 15)(2 + 5)$
   (d) $2 + 3(5 + 1)$
   (e) $(8 - 3)6 + 4$
   (f) $4 + 15(2) + 5$
   (g) $\frac{1}{2}(5 + 7 \times 3)$
   (h) $4(5) + \frac{9}{3}$
   (i) $\frac{5(6 - 2)}{10 - 3}$
   (j) $\frac{(7 - 2)(3 + 1)}{15}$
   (k) $\frac{6}{8 - 5}$
   (l) $\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}$

Expressions such as "4 + 2", "2 × 3", "2(3 + 7), "$(1 - \frac{1}{2})(16 + 4) - 5$" are examples of numerical phrases. Each of these is a numeral formed from simpler numerals. A numerical phrase is any numeral given by an expression which involves other numerals along with the signs for operations. It needs to be emphasized that a numerical phrase must actually name a number, that is, it must be a numeral. Therefore, a meaningless expression such as "(3 +) × (2 +)-" is not a numerical phrase. Even the expression "2 × 3 + 7" is not a numerical phrase without the agreement giving preference to multiplication.

Numerical phrases may be combined to form numerical *sentences*; i.e., sentences which make statements about numbers. For example,

$$2(3 + 7) = (2 + 3)(4 + 0)$$

is a sentence which states that the number represented by "$2(3 + 7)$" is the same as the number represented by "$(2 + 3)(4 + 0)$". It is read "$2(3 + 7)$ is equal to $(2 + 3)(4 + 0)$", and you can easily verify that it is a true sentence.

Consider the sentence,

$$(3 + 1)(5 - 2) = 10.$$

This sentence asserts that the number $(3 + 1)(5 - 2)$ is 10. Does this bother you? Perhaps you are wondering how the author could have made such a ridiculous mistake in arithmetic, because anyone can see that $(3 + 1)(5 - 2)$ is 12 and not 10 ! However, "$(3 + 1)(5 - 2) = 10$" is still a perfectly good sentence in spite of the fact that it is false.

The important fact about a sentence involving numerals is that it is either true or false, but not both. Much of the work in algebra is concerned with the problem of deciding whether or not certain sentences involving numerals are true.

Problem Set 2-1b

1. Tell which of the following sentences are true and which are false.

   (a) $(3 + 7)4 = 3 + 7(4)$

   (b) $4(5) + 4(8) = 4(13)$

   (c) $2(5 + \frac{1}{2}) = 2(5) + 2(\frac{1}{2})$

   (d) $23 - 5(2) = 36$

   (e) $\frac{7 + 9}{2} = 7 + \frac{9}{2}$

   (f) $\frac{16}{2} + 4 - 3 = \frac{16}{2} + (4 - 3)$

   (g) $5(7 + 3) = 10(\frac{20}{5} + 1)$

   (h) $\frac{12}{3(2)} = \frac{12}{3}(2)$

   (i) $3(8 + 2) = 6 \times 5$

   (j) $12 + (2 \times 3) = 12(9)$

   (k) $3 + 7(9 + 2) = (3 + 7)(9 + 2)$

2. Write a common name for each numeral.

   (a) $8 + 3(5 - 2) - (9 - 5)$

   (b) $3 \cdot 2(5) + 7 \cdot 4$

   (c) $(\frac{2}{3})(\frac{3}{5}) + 2(6 - 3)$

(d) $\frac{3(2) + 18}{\text{[illegible]}}$

(e) $\frac{5(7 + 9)}{5}$

(f) $0.6(2(3) + 4)(3) - \frac{12}{4}$

3. You are explaining the use of parentheses to a friend who does not know about them. Insert parentheses in each of the following expressions in such a way that the expression will still be a numeral for the same number.

(a) $\frac{1}{2} \times 6 + 3$	(c) $2 \times 3 + 4 \times 3$
(b) $2 \cdot 5 + 6 \cdot 2$	(d) $3 \times 8 - 4$

4. Insert parentheses in each of the following expressions so that the resulting sentence is true.

(a) $10 - 7 - 3 = 6$	(j) $3 \times 5 - 2 \times 4 = 7$
(b) $3 \cdot 5 + 7 = 36$	(k) $3 \times 5 - 2 \times 4 = 52$
(c) $3 \cdot 5 + 7 = 22$	(l) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 51$
(d) $3 \cdot 5 - 4 = 3$	(m) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 3$
(e) $3 \cdot 5 - 4 = 11$	(n) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 18$
(f) $3 \times 5 + 2 \times 4 = 23$	(o) $3 + 4 \cdot 6 + 1 = 49$
(g) $3 \times 5 + 2 \times 4 = 84$	(p) $3 + 4 \cdot 6 + 1 = 31$
(h) $3 \times 5 + 2 \times 4 = 68$	(q) $3 + 4 \cdot 6 + 1 = 43$
(i) $3 \times 5 - 2 \times 4 = 36$	(r) $3 + 4 \cdot 6 + 1 = 28$

2-2. Some Properties of Addition and Multiplication.

At the end of Chapter 1, you recalled addition and its representation on the number line. We are now going to consider some of the properties* of addition. First of all, addition is a binary operation, in the sense that it is always performed on two numbers. This may not seem very reasonable at first sight, since you have often added long strings of figures. Try an experiment on yourself. Try to add the numbers 7, 8, 3 simultaneously. No

---

*Property, in the most familiar sense of the word, is something you have. A property of addition is something addition has; i.e., a characteristic of addition. A similar common usage of the word would be "sweetness is a property of sugar".

matter how you attempt it, you are forced to choose two of the numbers, add them, and then add the third to this sum. Thus, when we write $7 + 8 + 3$, we really mean $(7 + 8) + 3$ or $7 + (8 + 3)$. We use parentheses here, as we have in the past, to single out certain groups of numbers to be operated on first. Thus, $(7 + 8) + 3$ implies that we add 7 and 8 and then add 3 to that sum, so that we think "$15 + 3$". Similarly, $7 + (8 + 3)$ implies that the sum of 8 and 3 is added to 7, giving $7 + 11$. Let us now go one step further and observe that $15 + 3 = 18$, and $7 + 11 = 18$. We have thus found that

$$(7 + 8) + 3 = 7 + (8 + 3)$$

is a true sentence.

Check whether or not

$$(5 + \frac{3}{2}) + \frac{1}{2} = 5 + (\frac{3}{2} + \frac{1}{2})$$

is a true sentence.

Check similarly

$$(1.2 + 1.8) + 2.6 = 1.2 + (1.8 + 2.6),$$

and

$$(\frac{1}{3} + \frac{1}{2}) + \frac{2}{3} = \frac{1}{3} + (\frac{1}{2} + \frac{2}{3}).$$

It is apparent that these sentences have a common pattern, and they all turned out to be true. We conclude that every sentence having this pattern is true. This is a property of addition of numbers; we hope that you will try to formulate it for yourself. Compare your effort with a statement such as this: If you add a second number to a first number and then a third number to their sum, the outcome is the same if you add the second number and the third number and then add their sum to the first number.

This property of addition is called the associative property of addition. It is one of the basic facts about the number system—one of the facts on which all of mathematics depends. Incidentally, it is often handy for cutting down the work in adding. In the second example above, for instance, "$\frac{3}{2} + \frac{1}{2}$" is another name for 2, so that "$5 + (\frac{3}{2} + \frac{1}{2})$" produces a simpler addition than "$(5 + \frac{3}{2}) + \frac{1}{2}$". Similarly, in the third example, $1.2 + 1.8 = 3$; which of the two expressions "$(1.2 + 1.8) + 2.6$"

and "$1.2 + (1.8 + 2.6)$" leads to a simpler second addition?

Now let us look at the fourth example. Neither "$(\frac{1}{3} + \frac{1}{2}) + \frac{2}{3}$" nor "$\frac{1}{3} + (\frac{1}{2} + \frac{2}{3})$" gives a particularly simple first sum to help us with the second sum. If we could only add $\frac{2}{3}$ to $\frac{1}{3}$ first, this would give 1, and adding $\frac{1}{2}$ to 1 is easy! What we would like is to take the first indicated sum in "$(\frac{1}{3} + \frac{1}{2}) + \frac{2}{3}$", and write it instead as "$(\frac{1}{2} + \frac{1}{3})$", in order to put "$\frac{1}{3}$" next to "$\frac{2}{3}$". To do this we need to know that

$$\frac{1}{3} + \frac{1}{2} = \frac{1}{2} + \frac{1}{3}$$

is a true sentence.

Although we can perfectly well do the arithmetic to check this, let us first consider some simpler examples. Is the sentence

$$3 + 5 = 5 + 3$$

true? Perhaps you think: "If I earn \$3 today and \$5 tomorrow, I shall earn the same amount as if I earn \$5 today and \$3 tomorrow." Perhaps John thinks: "Walking three miles before lunch and five miles after lunch covers the same distance as walking 5 miles before lunch and 3 miles after lunch."

Now recall the number line. What did we find out, in Chapter 1, about moving from 0 to 5 and then moving three units to the right, and how did this compare with moving from 0 to 3 and then moving 5 units to the right? What does this say about $5 + 3$ and $3 + 5$? Similarly, on the number line, decide whether the following are true sentences:

$$0 + 6 = 6 + 0,$$

$$2\frac{1}{2} + 5 = 5 + 2\frac{1}{2}.$$

This property of addition is probably very familiar to you. It is called the <u>commutative property of addition</u>. Try to formulate it for yourself, and compare your statement with the following: If two numbers are added in different orders, the results are the same.

The associative property of addition tells us that an indicated sum may be written with or without parentheses as grouping symbols, as we wish. The commutative property, in turn, tells us that the additions, which are always of two numbers at a time, may be performed in any order. For instance, if we consider

$$32 + 16 + 18 + 4,$$

the associative property tells us that we do not have to use parentheses to group this indicated sum, because any way we group it gives the same result. We may, if we wish, just add 16 to 32, then 18 to their sum, and then 4 to <u>that</u> sum. The commutative property tells us that we may choose any other order. For purposes of mental arithmetic, it is easier to choose pairs whose sums are multiples of 10 and consider them first. We may think of "32 + 16 + 18 + 4" as "(32 + 18) + (16 + 4)", where the indicated sums mean that we first add 32 and 18 (because this gives the "easy" sum 50), then 16 and 4, and finally the two partial sums 50 and 20. In our thinking, we first used the commutative property to interchange the 16 and the 18 in the original indicated sum.

Problem Set 2-2a

1. Consider various ways to do the following computations mentally, and find the one that seems easiest (if there is one). Then perform the indicated additions in the easiest way.

(a) $6 + (8 + 4)$

(b) $\frac{2}{5} + \frac{2}{3} + 1 + \frac{1}{3} + \frac{8}{5}$

(c) $5\frac{4}{7} + 6 + 14\frac{3}{7}$

(d) $\frac{3}{4} + \frac{1}{3}$

(e) $2\frac{1}{5} + 3\frac{2}{3} + 6 + 7\frac{4}{5}$

(f) $(2\frac{1}{3} + 1) + \frac{6}{5}$

(g) $(1.8 + 2.1) + (1.5 + .5) + 1.2$

(h) $(8 + 7) + 4 + (3 + 6)$

2. From the tip of a mouse's nose to the back of his head is 32 millimeters; from the back of his head to the base of his tail 71 millimeters; from the base of his tail to the tip of his tail 76 millimeters. What is the length of the mouse from the tip of his nose to the tip of his tail? Is he the same length from the tip of his tail to the tip of his nose? Why do you think we include this exercise here?

We shall now look at the corresponding properties of multiplication. Consider this sentence,

$$(7 \times 6) \times \frac{1}{3} = 7 \times (6 \times \frac{1}{3}).$$

Check whether or not this is a true sentence; be sure to carry out the multiplications as indicated. Similarly check the truth of the sentences

$$(4 \times 1.5) \times 3 = 4 \times (1.5 \times 3)$$

and

$$(\frac{3}{4} \times 7) \times 4 = \frac{3}{4} \times (7 \times 4).$$

Once again, we find that these sentences are true, and that they fit a common pattern. We conclude that all sentences of this pattern are true, and call this property of multiplication the <u>associative property of multiplication</u>. Recall your effort towards stating in words the associative property of addition, and make a similar statement for the associative property of multiplication.

In the examples above, the indicated multiplications were not always equally difficult. In the first sentence, "$(7 \times 6) \times \frac{1}{3}$", which becomes "$42 \times \frac{1}{3}$", is more work to carry out than "$7 \times (6 \times \frac{1}{3})$" which becomes just "$7 \times 2$". Which phrase in the second sentence is easier to handle? Thus the associative property of multiplication, just as the associative property of addition, can be used to simplify mental arithmetic.

In the third sentence, neither form led to a particularly simple second product. The easiest thing to do would be to take $\frac{3}{4} \times 4$ first, even though $\frac{3}{4}$ and 4 are not adjacent in either phrase, and then to multiply by 7. Is this legal? We could be sure that it is if we knew that

$$\frac{3}{4} \times 7 = 7 \times \frac{3}{4}$$

were a true sentence. This is a possible interchange we might like to make before applying the associative property. (What would be another?)

As in the previous section, make up some simple problems about walking or earning money which verify the truth of a sentence such as

$$2 \times 5 = 5 \times 2.$$

You have also had the experience in Chapter 1 of seeing on the number line that

$$2 \times 3 = 3 \times 2$$

is a true sentence. You also know, from arithmetic, that you may perform long multiplication in either order, and you have probably used this to check your work:

$$\begin{array}{r} 256 \\ \underline{63} \\ 768 \\ \underline{1536} \\ 16128 \end{array} \qquad\qquad \begin{array}{r} 63 \\ \underline{256} \\ 378 \\ 315 \\ \underline{126} \\ 16128 \end{array}$$

All these, in various situations, are instances of the <u>commutative property of multiplication</u>: If two numbers are to be multiplied, they may be multiplied in either order with the same result.

As in the case of addition, the associative and commutative properties of multiplication tell us that we may, in an indicated product, think of the numbers grouped as we choose, and may also rearrange such a product at will. Thus in finding $9 \times 2 \times 3 \times 50$, it is convenient to handle 2×50 first, and then to multiply 9×3, or 27, by 100.

<u>Problem Set 2-2b</u>

1. Consider various ways to do the following computations mentally, and find the one that seems easiest (if there is one). Then perform the indicated operations in the easiest way.

(a) $4 \times 7 \times 25$

(b) $\frac{1}{5} \times (26 \times 5)$

(c) $73 + 62 + 27$

(d) $(3 \times 4) \times (7 \times 25)$

(e) 12×14

(f) $\frac{1}{2} \times \frac{1}{3} \times \frac{5}{6}$

(g) $6 \times 8 \times 125$

(h) $(1.25) \times 5.5 \times 8$

(i) $(2 \times 5) \times 1.97$

(j) $\frac{5}{4} \times 6 \times \frac{4}{3} \times \frac{1}{5}$

2. Is it easier to compute

$$\begin{array}{r} 957 \\ \times\ \underline{222} \end{array} \quad \text{or} \quad \begin{array}{r} 222 \\ \times\ \underline{957} \end{array}\ ?$$

How about

$$\begin{array}{r} 3.89 \\ \times\ \underline{141} \end{array} \quad \text{or} \quad \begin{array}{r} 141 \\ \times\ \underline{3.89} \end{array}\ ?$$

2-3 The Distributive Property.

John collected money in his homeroom. On Tuesday, 7 people gave him 15¢ each, and on Wednesday, 3 people gave him 15¢ each. How much money did he collect? He figured,

$$\begin{gathered} 15(7) + 15(3) = \\ 105 + 45 = \\ 150. \end{gathered}$$

So he collected $1.50.

But now we shall ask him to keep different records. Since everyone gave him the same amount, it is also possible to keep an account only of the number of people who have paid him, and then to multiply the total number by 15. Then his figuring looks like this:

$$\begin{gathered} 15(7 + 3) = \\ 15(10) = \\ 150. \end{gathered}$$

The final result is the same in both methods of keeping accounts; therefore

$$15(7) + 15(3) = 15(7 + 3)$$

is a true sentence. Since the above true sentence means that $15(7) + 15(3)$ and $15(7 + 3)$ are names for the same number, we might also have written

$$15(7 + 3) = 15(7) + 15(3).$$

Half the money John collected is to be used for one gift, and one third of it for another. How much is spent? Again, the computation can be performed in two ways:

$150(\frac{1}{2}) + 150(\frac{1}{3}) =$	$150(\frac{1}{2} + \frac{1}{3}) =$
$75 + 50 =$	$150(\frac{3}{6} + \frac{2}{6}) =$
$125.$	$150(\frac{5}{6}) =$
	$125.$

As before, we have found a true sentence,

$$150(\tfrac{1}{2}) + 150(\tfrac{1}{3}) = 150(\tfrac{1}{2} + \tfrac{1}{3}).$$

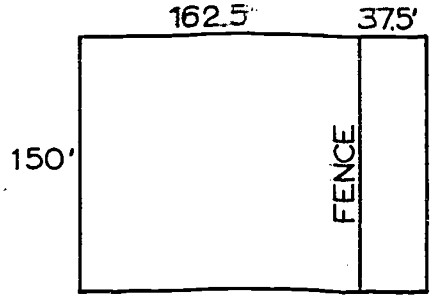
Another way of writing the same true sentence would be

$$150(\tfrac{1}{2} + \tfrac{1}{3}) = 150(\tfrac{1}{2}) + 150(\tfrac{1}{3}).$$

Let us try another example. Mr. Jones owned a city lot, 150 feet deep, with a front of 162.5 feet. Adjacent to his lot, and separated from it by a fence, is another lot with the same depth, but with a front of only 37.5 feet. What are the areas, in square feet, of each of these two lots, and what is their sum? Mr. Jones buys the second lot and removes the fence. Now what is the area of the lot?

The number of square feet in the new lot is

$$150(162.5 + 37.5) =$$
$$150(200) =$$
$$30000.$$



The total area of the two separate lots is

$$150(162.5) + 150(37.5) =$$
$$24375 + 5625 =$$
$$30000.$$

Thus,

$$150(162.5 + 37.5) = 150(162.5) + 150(37.5)$$

is a true sentence.

Let us look more closely at two of our true sentences. We wrote

$$15(7) + 15(3) = 15(7 + 3).$$

15(7) represents one number, which we have chosen to write as an indicated product; so does 15(3). Thus 15(7) + 15(3) is an indicated sum of two numbers. On the other hand, 7 + 3 represents a number which we have chosen to write as an indicated sum, and so 15(7 + 3) is an indicated product. Thus the sentence

$$15(7) + 15(3) = 15(7 + 3)$$

asserts that the indicated sum and the indicated product are names for the same number. The true sentence

$$150(\frac{1}{2} + \frac{1}{3}) = 150(\frac{1}{2}) + 150(\frac{1}{3})$$

makes a similar statement.

Problem Set 2-3a

1. Test the truth of the following sentences:
   (a) $2 \cdot 4 + 5 \cdot 4 = 7 \cdot 4$
   (b) $15 \cdot 2 = 7 \cdot 2 + 8 \cdot 2$
   (c) $25(40 + 3) = 25(40) + 25(3)$
   (d) $3(2) + 6(3) = 9(6)$
   (e) $13(19 + 1) = 13(19) + 13(1)$
   (f) $2(\frac{1}{2} + \frac{1}{4}) = 2\frac{1}{2} + 2\frac{1}{4}$
   (g) $12(\frac{2}{3} + \frac{3}{4}) = 12(\frac{2}{3}) + 12(\frac{3}{4})$
   (h) $3(2.5) + 3(1.5) = 3(2.5 + 1.5)$

It appears that we have found a pattern by which true sentences may be formed. Try to say to yourself in various ways what this pattern is. After you have made an effort, compare your result with the following: The product of a number times the sum of two others is the same as the product of the first and second plus the product of the first and third.. This property is called the distributive property for multiplication over addition, or just, as we shall frequently say, the distributive property.

As in the case of the other properties we have studied, the distributive property has much to do with arithmetic, both with devices for mental facility and with the very foundations of the subject. In our first example, the comparison in arithmetic labor between "15(7) + 15(3)" and "15(7 + 3)" favors the indicated product, because 7 + 3, or 10, leads to an easy multiplication. In the next example, however, the comparison between "$150(\frac{1}{2}) + 150(\frac{1}{3})$" and "$150(\frac{1}{2} + \frac{1}{3})$" favors the indicated sum, because it is more work to add the fractions $\frac{1}{2}$ and $\frac{1}{3}$ than it is to add 75 and 50. Which form was easier in the third example? In the sentences of Problem Set 2-3a?

More important than these niceties of mental manipulation is the role of the distributive property in much of our arithmetic technique such as, for example, in long multiplication. How do we perform

$$\begin{array}{r} 62 \\ \times\ 23 \\ \hline \end{array} \ ?$$

We write

$$\begin{array}{r} 62 \\ \underline{\times\ 23} \\ 186 \\ \underline{124} \\ 1426 \end{array}$$

This really means that we take 62(20 + 3) as 62(20) + 62(3), or 1240 + 186. (The "0" at the end of "1240" is understood in our long multiplication form.) Thus the distributive property is the foundation of this standard technique.

Suppose we wish to consider several ways of computing the indicated product

$$(\frac{1}{3} + \frac{1}{4})12.$$

This phrase does not quite fit the pattern of the distributive property as we have discussed it so far. You can probably guess on the basis of your previous experience, that

$$(\frac{1}{3} + \frac{1}{4})12 = (\frac{1}{3})12 + (\frac{1}{4})12$$

is a true sentence. Let us, however, see how the properties as we have discovered them thus far permit us to conclude the truth of this sentence.

First we know that

$$(\frac{1}{3} + \frac{1}{4})12 = 12(\frac{1}{3} + \frac{1}{4})$$

is a true sentence (by what property of multiplication?). Then we may apply the distributive property as we have discovered it thus far to write

$$12(\frac{1}{3} + \frac{1}{4}) = 12(\frac{1}{3}) + 12(\frac{1}{4}),$$

and apply the commutative property twice more to write

$$12(\frac{1}{3}) + 12(\frac{1}{4}) = (\frac{1}{3})12 + (\frac{1}{4})12.$$

The last step, which would be unnecessary if we were just trying to compute "$(\frac{1}{3} + \frac{1}{4})12$" in a simple fashion, finally leads to the desired sentence, namely

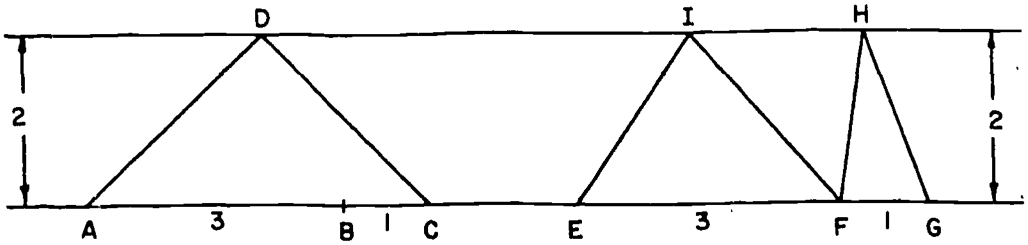
$$(\frac{1}{3} + \frac{1}{4})12 = (\frac{1}{3})12 + (\frac{1}{4})12.$$

This sentence, once again, seems to have a simple form, and in fact suggests an alternate pattern for the distributive property which is obtained from our previous pattern by several applications of the commutative property of multiplication. In your own words, state this alternate pattern. What pattern is suggested by the sentence

$$(\frac{1}{3})12 + (\frac{1}{4})12 = (\frac{1}{3} + \frac{1}{4})12\ ?$$

Problem Set 2-3b

1. Follow the pattern of any convenient form of the distributive property in completing each of the following as a true sentence:
   (a) $12(3 + 4) = 12(\ ) + 12(\ )$
   (b) $3(\ ) + \quad (7) = 3(5 + 7)$
   (c) $(2.5 + 4.5) \quad = (2.5)4 + (4.5)4$
   (d) $\qquad 24(\frac{1}{8} + \frac{1}{6})$
   (e) $7(\ ) + 6(\ ) = 13(\ )$
   (f) $(3 + 11)2 =$
2. Consider in two different ways and in each case decide which, if any, is the easier computation. Then perform the indicated operations in the easier way.
   (a) $27(\frac{7}{8}) + 27(\frac{1}{8})$
   (b) $(\frac{1}{3})12 + (\frac{1}{4})12$
   (c) $(.36 + .14).6$
   (d) $12(5 + 5)$
   (e) $(2.3 + 4.6) + 7.7$
   (f) $6(\frac{3}{2} + \frac{2}{3})$
   (g) $.71(.8) + .2(.71)$
   (h) $3(2 + 7 + 6 + 5)$
3. The figure below shows a number of triangles



What relation holds between the area of triangle ACD and the areas of the two triangles EFI and FGH? Use the formula for the area of a triangle in terms of lengths of base and altitude.

4. Write a common name for

$$(\frac{1}{2} + \frac{2}{3})11 + (\frac{1}{2} + \frac{2}{3})7.$$

(Hint) Think of $\frac{1}{2} + \frac{1}{3}$ as one numeral, and don't start working until you have thought of a way of doing this exercise which isn't much work.

5. Write the common names for

(a) $8(\frac{3}{5} + \frac{2}{3}) + (\frac{2}{3} + \frac{3}{5})7$

(b) $7(\frac{1}{2} + \frac{1}{3} + \frac{3}{4}) + 5(\frac{5}{6} + \frac{3}{4})$

*(c) $5(\frac{1}{2} + \frac{1}{3} + \frac{1}{5}) + 7(\frac{1}{2} + \frac{1}{3})$

6. Write common names for

(a) $3 \times (17 + 4) \times \frac{1}{3}$

(b) $\frac{3(17 + 4)}{3}$

(c) $\frac{1}{5}(3 + 8)10$

(d) $3(7) + 3(11)$

(e) $3(7) + 3(2)$

(f) $3(7) + 6$

(g) $7(4) + 42$

(h) $3(\frac{1}{5}) + \frac{2}{5}$

(i) $3(17) + 12$

(j) $6(19) + 19$

(k) $(10 + 2) \times 4 \times \frac{1}{2}$

(l) $\frac{(10 + 2) \times 4}{2}$

2-4. Variables.

Consider, for a moment, a simple exercise in mental arithmetic:

"Take 6, add 2, multiply by 7, and divide by 4".

Following these instructions, you will no doubt think of the succession of numbers

6, 8, 56, 14

and observe that 14 is the answer to the exercise. Pretend now that your best friend is absent from class and that you have promised to give him a detailed report on the day's work. With your friend in mind, you write down the instructions for each step of the exercise as follows:

$$6$$
$$6 + 2$$
$$7(6 + 2)$$
$$\frac{7(6 + 2)}{4}$$

Does this method of writing the exercise give more information or less information? It clearly has the advantage of showing exactly what operations are involved in each step, but it does have the disadvantage of not ending up with an answer to the exercise. On the other hand, the phrase "$\frac{7(6 + 2)}{4}$" is a numeral for the answer 14, a fact which is readily shown by doing the indicated arithmetic.

This is an imaginary situation in which you were led to record for your friend the form of the exercise, rather than just the answer. It illustrates a point of view which is basic to mathematics. There will be many places in this course where it is the pattern or form of a problem which is of primary importance rather than the answer. As a matter of fact, we are rarely interested only in the answer to a problem.

Try the exercise with the following instructions:

> "Take 7, multiply by 3, add 3,
> multiply by 2, and divide by 12."

What is the phrase which shows all of the operations? Is it a numeral for the same number you obtained mentally?

Let us now do one of these exercises with the added feature that you are permitted to choose at the start any one of the numbers from the set

$$S = \{1, 2, 3, \ldots, 1000\}.$$

The instructions this time are:

> "Take any number from S, multiply by 3,
> add 12, divide by 3, and subtract 4."

This exercise might be thought of as actually consisting of 1000 different exercises in arithmetic, one for each choice of a starting number from the set S. Consider the exercise obtained by starting with the number 17. The arithmetic method and the pattern method lead to the following steps:

Arithmetic	Pattern
17	17
51	$3(17)$
63	$3(17) + 12$
21	$\frac{3(17) + 12}{3}$
17	$\frac{3(17) + 12}{3} - 4$

Notice that, by the distributive property for numbers and since $12 = 3(4)$,

$$3(17) + 12 = 3(17) + 3(4) = 3(17 + 4),$$

so that

$$\frac{3(17) + 12}{3} = \frac{3(17 + 4)}{3} = 17 + 4.$$

Therefore

$$\frac{3(17) + 12}{3} - 4 = 17 + 4 - 4 = 17.$$

In other words, the final phrase obtained in the "pattern" is a numeral for 17. Try some more choices from the set S. Will you always end up with the number you chose at the start? One method of answering this question would be to work out each of the 1000 different exercises! Perhaps you have already guessed, from working the pattern for several cases, that it may not be necessary to do all of the 1000 exercises to answer the above question.

Let us examine the pattern of the exercise more closely. Observe first that the pattern really does not depend on the number chosen from the set S. In fact, if we refer to the number chosen by the word "number", the steps in the exercise become:

$$\text{number}$$
$$3(\text{number})$$
$$3(\text{number}) + 12$$
$$\frac{3(\text{number}) + 12}{3}$$
$$\frac{3(\text{number}) + 12}{3} - 4$$

In order to save writing, denote the chosen number by "n". Then the steps become:

$$n$$

$$3n$$

$$3n + 12$$

$$\frac{3n + 12}{3}$$

$$\frac{3n + 12}{3} - 4.$$

Note that "n" is used here as a numeral for the chosen number and that the phrase in each of the other steps is also a numeral. (Thus, if n happens to be 17, then the indicated product "3n" is a numeral for 51.) In particular, the phrase "$\frac{3n + 12}{3} - 4$" is a numeral for the "answer" to the exercise. Moreover, by the distributive property for numbers,

$$3n + 12 = 3n + 3(4) = 3(n + 4).$$

Hence

$$\frac{3n + 12}{3} = \frac{3(n + 4)}{3} = n + 4$$

Therefore,

$$\frac{3n + 12}{3} - 4 = n + 4 - 4 = n.$$

Since "n" can represent *any* particular element of the set S, we conclude that the end result in this exercise is indeed always the same as the number chosen at the start.

The above discussion illustrates the great power of methods based on pattern or form rather than on simple arithmetic. The method, in a sense, enables us to replace 1000 different arithmetic problems by a single problem!

Would the discussion of the exercise be changed in any essential way if we had decided to denote the chosen number from S by some letter other than n, say m or x?

A letter used, as "n" was used in the above exercise, to denote one of a given set of numbers, is called a *variable*. In a given computation involving a variable, the variable is a numeral which represents a definite though unspecified number from a given set of admissible numbers. The admissible numbers for the variable "n" in the above exercise are the whole numbers from 1 to 1000. A number which a given variable can represent is called a *value* of the variable. The set of values of a variable is sometimes called its domain. The domain of the variable "n" in the above exercise is the set S = {1, 2, 3, . . ., 1000}. Unless the domain of a variable is specifically stated, we shall assume it to be the set of all numbers. For the time being, we are considering only the numbers of arithmetic.

Problem Set 2-4.

1. If the sum of a certain number t and 3 is doubled, which of the following would be a correct form:

   $$2t + 3 \quad \text{or} \quad 2(t + 3) \; ?$$

2. If 5 is added to twice a certain number n and the sum is divided by 3, which is the correct form:

   $$\frac{2n + 5}{3} \quad \text{or} \quad \frac{2n}{3} + 5 \; ?$$

3. If one-fourth of a certain number x is added to one third of four times the same number, which is the correct form:

   $$\frac{1}{3}(4x) + \frac{1}{4}(x) \quad \text{or} \quad \frac{4}{3}(x) + \frac{1}{4}(x) \; ?$$

4. If the number of gallons of milk purchased is y, which is the correct form for the number of quart bottles that will contain it:

   $$4y \quad \text{or} \quad \frac{y}{4} \; ?$$

5. If a is the number of feet in the length of a certain rectangle and b is the number of feet in the width of the same rectangle, is either form the correct form for the perimeter:

   $$a + b \quad \text{or} \quad ab \; ?$$

6. If a is 2, b is 3, c is $\frac{3}{4}$, m is 1 and n is 0, then what is the value of (that is, what number is represented by):

   (a) $6b + ac$ (f) $n(c + ac)$

   (b) $(a + b)(a + m)$ (g) $\frac{2a + 3b}{m}$

   (c) $6(b + ac)$ (h) $m(b - 4c)$

   (d) $\frac{2b + c}{b}$ (i) $\frac{5(3a + 4b)}{2(a + b)}$

   (e) $nc + ac$ (j) $a + 2(b + m)$

7. Find common names for

   (a) $\frac{9}{5}C + 32$, when C is 85

(b) $\frac{h(a + b)}{2}$, when h is 4, a is 3, and b is 5

(c) $P(1 + rt)$, when P is 500, r is 0.04, t is 3

(d) $\frac{r\ell - a}{r - 1}$, when a is 4, r is 2, ℓ is 48

(e) ℓwh, when ℓ is 18, w is 10, h is 6

8. If a is 3, b is 2, and c is 4, then what is the value of

(a) $\frac{(3a + 4b) - 2c}{3}$?

(b) $\frac{(6a - 4b) + 5c}{5}$?

(c) $\frac{(\frac{7a}{2} + \frac{3b}{2}) - \frac{5c}{2}}{2}$?

(d) $\frac{(1.5a + 3.7b) - 2.1c}{7}$?

9. Summarize the new ideas, including definitions, which have been introduced in this chapter.

Chapter 3

SENTENCES AND PROPERTIES OF OPERATIONS

3-1. Sentences, True and False

When we make assertions about numbers we write <u>sentences</u>, such as

$$(3 + 1)(5 + 2) = 10$$

Remember that a sentence is either true or false, but not both. This particular sentence is false.

Some sentences, such as the one above, involve the verb "$=$", meaning "is" or "is equal to". There are other verb forms that we shall use in mathematical sentences. For example, the symbol "$\neq$" will mean "is not" or "is not equal to". Then

$$8 + 4 \neq 28 \cdot 2$$

is a true sentence, and

$$8 + 4 \neq \frac{24}{2}$$

is a false sentence.

Problem Set 3-1

Which of the following sentences are true? Which are false?

1. $4 + 8 = 10 + 5$
2. $8 + 3 = 10 + 1$
3. $4 + 8 = 8 + 4$
4. $5 + 7 \neq 6 + 6$
5. $\frac{1}{2} + \frac{5}{8} = 1 + \frac{1}{8}$
6. $\frac{85}{1} \neq 85$
7. $13 + 0 \neq 15 + 0$
8. $12(5) \neq 5(12)$
9. $7(6 \times 3) = (7 \times 6) \times 3$
10. $8(\frac{1}{2} - \frac{1}{4}) = 8(\frac{1}{2}) - 8(\frac{1}{4})$
11. $65 \times 1 = 65$
12. $13 \times 0 = 13$
13. $\frac{2}{3}(7) \neq 2(\frac{7}{3})$
14. $4(\frac{3}{5}) = \frac{12}{5}$
15. $8(\frac{3}{5}) = \frac{24}{40}$

3-2. Open Sentences

We have no trouble deciding whether or not a numerical sentence is true, because such a sentence involves specific numbers. Consider the sentence

$$x + 3 = 7.$$

Is this sentence true? You will protest that you don't know what number "x" represents; without this information you cannot decide. In the same way you cannot decide whether the sentence, "He is a doctor," is true until "he" is identified. In this sense, the variable "x" is used in much the same way as a pronoun in ordinary language.

Consider the sentence

$$3n + 12 = 3(n + 4),$$

with which we worked in Section 2-4 when the idea of a variable was first introduced. Again we cannot decide whether this sentence is true on the basis of the sentence alone, but here we have a different situation. As before, we could decide if we knew what number "n" represents. But in this case we can decide without knowing the value of n. We can recall a general property of numbers to show that this sentence is true <u>no matter what number "n" represents</u>. (What did we call this general property of numbers?)

We say that sentences such as

$$x + 3 = 7$$

and

$$3n + 12 = 3(n + 4),$$

which contain variables, are <u>open</u> sentences. The word "open" is suggested by the fact that we do not know whether they are true without more information. An <u>open sentence</u> is a sentence involving one or more variables, and the question of whether it is true is left open until we have enough additional information to decide. In the same way, a phrase involving one or more variables is called an <u>open phrase</u>.

Problem Set 3-2a

In each of the following open sentences, determine whether the sentence is true if the variables have the suggested values:

1. $7 + x = 12$; let x be 5
2. $7 + x \neq 12$: let x be 5
3. $y + 9 \neq 11$; let y be 6
4. $t + 9 = 11$: let t be 6
5 $\frac{5x + 1}{7} \neq 3$; let x be 3; let x be 4
6. $2y + 5x = 23$; let x be 4 and y be 3; let x be 3 and y be 4
7. $2a - 5 \neq (2a + 4) - b$; let a be 9 and b be 9; let a be 3 and b be 9
8. $5m + x = (2m + 3) + x$; let x be 4

If we are given an open sentence, and the domain of the variable is specified, how shall we determine the values, if any, of the variable that will make it a true sentence? We could guess various numbers until we hit on a "truth" number, but after the first guess, a bit of thinking could guide us.

Let us experiment with the open sentence "$2x - 11 = 6$". As a first guess, try a number x large enough so that $2x$ is greater than 11; let x be 9. Then

$$\begin{aligned} 2x - 11 &= 2(9) - 11 \\ &= 7. \end{aligned}$$

Thus, the numeral on the left represents 7, which is different from 6. Apparently 9 was too large; so we try 8 for x. Then

$$\begin{aligned} 2x - 11 &= 2(8) - 11 \\ &= 5. \end{aligned}$$

Here the numeral on the left represents 5, which is also different from 6. Since 8 was too small, we try a number between 8 and 9, say $8\frac{1}{2}$. Then

$$\begin{aligned} 2x - 11 &= 2(8\tfrac{1}{2}) - 11 \\ &= 6. \end{aligned}$$

"$6 = 6$" is a true sentence; so we find that the open sentence "$2x - 11 = 6$" is true if x is $8\frac{1}{2}$. Do you think there are other

[sec. 3-2]

values of x making it true? Do you think every open sentence has a value of the variable which makes it true? which makes it false?

Problem Set 3-2b

Determine what numbers, if any, will make each of the following open sentences true:

1. $12 - y = 8$
2. $4y + 5 = 45$
3. $3x - 2 = 10$
4. $3x - 2 = 15$
5. $4x + 3x = 14$
6. $4x - 3x = 14$
7. $s + 3 = s + 2$
8. $t + 2t \neq 27 + 3t$
9. $t + 3 = 3 + t$
10. $(x + 1)2 \neq 2x + 2$

If a variable occurs in an open sentence in the form "a·a" meaning "a multiplied by a", it is convenient to write "a·a" as "a^2", read "a squared".

Problem Set 3-2c

1. Try to find values of the variables which make the following open sentences true:
   (a) $x^2 = 9$
   (b) $4 - x^2 = 0$
   (c) $x^2 = x$
   (d) $x^2 - 1 = 3$
   (e) $x + 2 = 9$
   (f) $(x - 1)^2 = 4$
   (g) $4 + x^2 = 0$
   (h) $x^2 + 7 = 7$
2. What is a value of x for which
   $$x^2 = \frac{9}{16}$$
   is a true sentence?
3. A number of interest to us later is a value of x for which "$x^2 = 2$" is a true sentence. We call this number the square root of 2, and write it $\sqrt{2}$. Later you will find that $\sqrt{2}$ is the coordinate of a point on the number line. Approximately where on the number line would it lie?

3-3. Truth Sets of Open Sentences

Let the domain of the variable in the sentence

$$3 + x = 7$$

be the set of all numbers of arithmetic. If we specify that x has a particular value, then the resulting sentence is true or is false. For instance,

If x is	the sentence	is
0	$3 + 0 = 7$	false
1	$3 + 1 = 7$	false
$\frac{1}{2}$	$3 + \frac{1}{2} = 7$	false
2	$3 + 2 = 7$	false
4	$3 + 4 = 7$	true
6	$3 + 6 = 7$	false

In this way the sentence "$3 + x = 7$" can be thought of as a sorter: it sorts the domain of the variable into two subsets. Just as you might sort a deck of cards into two subsets, black and red, the domain of the variable is sorted into a subset of all those numbers which make the sentence true and another subset of all those numbers which make the sentence false. Here we see that 4 belongs to the first subset, while 0, 1, $\frac{1}{2}$, 2, 6 belong to the second subset.

The truth set of an open sentence in one variable is the set of all those numbers from the domain of the variable which make the sentence true. If we do not specify otherwise we shall continue to assume that the domain of the variable is the set of all numbers of arithmetic. (Recall that the numbers of arithmetic consist of 0 and all numbers which are coordinates of points to the right of 0.)

Problem Set 3-3a

1. Test whether the number belongs to the truth set of the given open sentence:
   (a) $7 + x = 12$; 5
   (b) $y + 9 \neq 11$; 6
   (c) $\frac{5x + 1}{7} \neq 3$; 3

(d) $2x + 1 = 2(x + 1)$; 3

(e) $x + \frac{1}{x} = 2$; 3

(f) $3m = m + 2m$; 5

(g) $n^2 + 2n \neq n(n + 2)$; 3

2. With each of the following open sentences is given a set which contains all the numbers belonging to its truth set, with possibly some more. You are to find the truth set.

   (a) $3(x + 5) = 17$; $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$

   (b) $x^2 - (4x - 3) = 0$; $\{1, 2, 3, 4\}$

   (c) $x^2 + \frac{1}{6} - \frac{7}{6}x = 0$; $\{1, 2, 6, \frac{1}{6}\}$

   (d) $x + \frac{1}{x} = 2$; $\{1, 2, 3\}$

   (e) $x(x + 1) = 3x$; $\{0, 1, 2\}$

   (f) $\frac{5x + 1}{7} = 3$; $\{0, 2, 4\}$

   (g) $x + 1 = 5x - 1$; $\{1, \frac{1}{2}, 2\}$

   (h) $x + 2 = x + 7$; $\{0, 2, 3\}$

3. Write an open sentence whose truth set is the null set $\emptyset$.

Many formulas used in science and business are in the forms of open sentences in several variables. For example, the formula

$$V = \frac{1}{3}Bh$$

is used to find the volume of a cone. The variable h represents the number of units in the height of the cone; B represents the number of square units in the base; V represents the number of cubit units in the volume. When values are specified for all but one of the variables in such a formula, the resulting open sentence contains one remaining variable. Then the truth set of this sentence gives information about the number represented by this variable.

Continuing the example, let us consider a particular cone whose volume is 66 cubic feet and the area of whose base is 33 square feet. From this information we determine that V is 66 and B is 33, and we write the corresponding open sentence in one variable h,

$$66 = \frac{1}{3}(33)h.$$

The truth set of this sentence is $\{6\}$.

Applying this information to the cone, we find that the height of the cone is 6 feet.

Problem Set 3-3b

1. The formula used to change a temperature F measured in degrees Fahrenheit to the corresponding temperature C in degrees Centigrade is
$$C = \frac{5}{9}(F - 32).$$
Find the value of C when F is 86.

2. The formula used to compute simple interest is
$$i = prt,$$
where i is the number of dollars of interest, p is the number of dollars of principal, r is the interest rate, and t is the number of years. Find the value of t when i is 120, r is 0.04, and p is 1000.

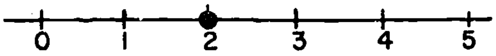
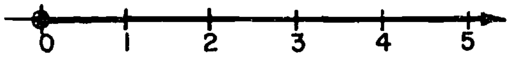
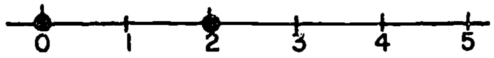
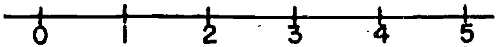
3. A formula used in physics to relate pressure and volume of a given amount of a gas at constant temperature is
$$pv = PV,$$
where V is the number of cubic units of volume at P units of pressure and v is the number of cubic units of volume at p units of pressure. Find the value of V when v is 600, P is 75, and p is 15.

*4. The formula for the area of a trapezoid is
$$A = \frac{1}{2}(B + b)h,$$
where A is the number of square units in the area, B is the number of units in the one base, b is the number of units in the other base, and h is the number of units in the height. Find the value of B when A is 20, b is 4, and h is 4.

3-4. Graphs of Truth Sets

The graph of a set S of numbers, it will be recalled, is the set of all points on the number line corresponding to the numbers of S, and only those points.

Thus, the graph of the truth set of an open sentence containing one variable is the set of all points on the number line whose coordinates are the values of the variable which make the open sentence true. Let us draw the graphs of a few open sentences.

	Sentence	Truth Set	Graph
(a)	$x = 2$	$\{2\}$	
(b)	$x \neq 3$	All numbers of arithmetic except 3	
(c)	$3 + x = (7 + x) - 4$	All numbers of arithmetic	
(d)	$y(y + 1) = 3y$	$\{0, 2\}$	
(e)	$3y = 7$	$\{\frac{7}{3}\}$	
(f)	$2x + 1 = 2(x + 1)$	$\emptyset$	 (Graph contains no points)

You will notice in (b) that we indicate that a point is included in the graph if it is marked with a heavy dot, but not included if it is circled. The heavy lines indicate all the points that are covered. The arrow at the right end of the number line in (b) and (c) indicates that all of the points to the right are on the graph.

Problem Set 3-4

State the truth set of each open sentence and draw its graph:

1. $x + 7 = 10$
2. $2x = x + 3$
3. $x + x \neq 2x$
4. $x + 3 = 3 + x$
5. $(x)(0) = x$
6. $3 + x \neq 6$
7. $2x + 3 = 8$
8. $5 \neq 3n + 1$
9. $y \cdot (1) \neq y$
10. $x^2 = 2x$

3-5. Sentences Involving Inequalities

If we consider any two different numbers, then one is less than the other. Is this always true? This suggests another verb form that we shall use in numerical sentences. We use the symbol "<" to mean "is less than" and ">" to mean "is greater than".

To avoid confusing these symbols, remember that in a true sentence, such as

$$8 < 12$$

or

$$12 > 8,$$

the point of the symbol (the small end) is directed toward the smaller of the two numbers.

Find the two points on the number line which correspond to 8 and 12. Which point is to the left? Will the lesser of two numbers always correspond to the point on the left of the other? Verify your answer by locating on the number line points corresponding to several pairs of numbers, such as $\frac{5}{2}$ and 2.2; $\frac{8}{6}$ and $\frac{8}{5}$.

Just as "$\neq$" means "is not equal to", "$\ngtr$" means "is not greater than". What does "$\nless$" mean?

Problem Set 3-5

Which of the following sentences are true? Which are false? Use the number line to help you decide.

1. $4 + 3 < 3 + 4$
2. $5(2 + 3) > 5(2) + 3$
3. $\frac{1}{2} + \frac{1}{3} \nless 1$
4. $5 + 0 \nless 5$
5. $2 > 2 \times 0$
6. $0.5 + 1.1 = 0.7 + 0.9$
7. $5.2 - 3.9 < 4.6$
8. $2 + 1.3 > 3.3$
9. $2 + 1.3 \ngtr 3.3$
10. $4 + (3 + 2) < (4 + 3) + 2$
11. $\frac{2}{3}(8 + 4) < \frac{2}{3} \times 8 + \frac{2}{3} \times 4$
12. $5 + (\frac{2}{5} + \frac{3}{5}) \neq (4 - 1)2$

3-6. Open Sentences Involving Inequalities

What is the truth set of the open sentence

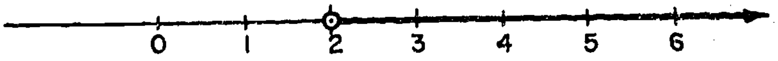
$$x + 2 > 4?$$

We can answer this question as follows: We know that the truth set of

$$x + 2 = 4$$

is {2}. When x is a number greater than 2, then $x + 2$ is a number greater than 4. When x is a number less than 2, then $x + 2$ is a number less than 4. Thus, every number greater than 2 makes the sentence true, and every other number makes it false. That is, the truth set of the sentence "$x + 2 > 4$" is the set of all numbers greater than 2.

The graph of this truth set is the set of all points on the number line whose coordinates are greater than 2. This is the set of all points which lie to the right of the point with coordinate 2:



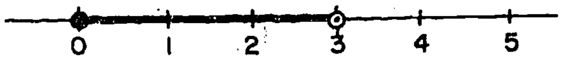
As another example, consider the graph of the truth set of

$$1 + x < 4:$$

Truth set

All numbers of arithmetic from 0 to 3, including 0, not including 3.

Graph

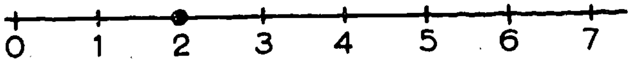


It is customary to call a simple sentence involving "=" an <u>equation</u> and a sentence involving "<" or ">" an <u>inequality</u>.

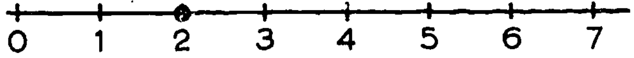
Problem Set 3-6

1. Determine whether the indicated set of points is the graph of the truth set of the given open sentence. If the graph is not the graph of the truth set, explain why.

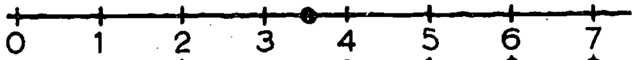
(a) $2 + x = 4$



(b) $3x = 5$

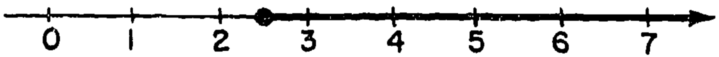


(c) $2y = 7$



(d) $x > 1$

(e) $2x > 5$



2. Draw the graphs of the truth sets of the following open sentences:

(a) $y = 3$

(b) $x \neq 2$

(c) $x > 2$

(d) $3 + y = 4$

(e) $3 + y \neq 4$

(f) $2x = 5$

(g) $2x > 5$

(h) $3 + y > 4$

(i) $3 + y < 4$

(j) $m < 3$

(k) $m > 3$

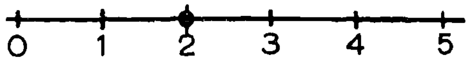
(l) $x(x + 2) = 4x$

(m) $\frac{y}{3} = 4$

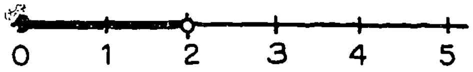
(n) $3a + 2 = 3(a + 2)$

3. Below are some graphs. For each graph, find an open sentence whose truth set is the set whose graph is given.

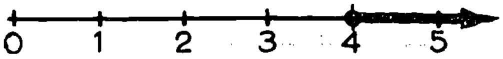
(a)



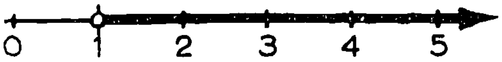
(b)



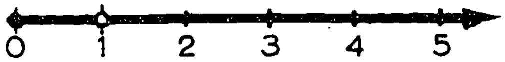
(c)



(d)



(e)



4. If the domain of the variable of each open sentence below is the set {0, 1, 2, 3, 4, 5}, find the truth set of each, and draw its graph.

(a) $4 + x = 7$

(b) $4x + 3 = 6$

(c) $2x > 5$

(d) $x + 3 < 6$

(e) $2x + 6 = 2(x + 3)$

5. If the domain of the variable of each open sentence in problem 4 is the set consisting of 0, 5, and <u>all</u> numbers greater than 0 and less than 5, find the truth set of each and draw its graph.

6. Which of the truth sets in problems 4 and 5 are finite sets?

3-7. Sentences With More Than One Clause

All the sentences discussed so far have been simple---that is, they contained only one verb form. Let us consider a sentence such as

$$4 + 1 = 5 \text{ and } 6 + 2 = 7.$$

Your first impression may be that we have written two sentences. But if you read the sentence from left to right, it will be one compound sentence with the connective and between two clauses. So in mathematics, as well as in English, we encounter sentences (declarative sentences) which are compounded out of simple sentences

Recall that a numerical sentence is either true or false. The compound sentence

$$4 + 1 = 5 \text{ and } 6 + 2 = 7$$

is certainly false, because the word and means "both", and here the second of the two clauses is false. The compound sentence

$$3 \not> 1 + 2 \text{ and } 4 + 7 > 10$$

is true, because both clauses are true sentences.

In general, a compound sentence with the connective and is true if all its clauses are true sentences; otherwise, it is false.

Problem Set 3-7a

Which of the following sentences are true?

1. $4 = 5 - 1$ and $5 = 3 + 2$
2. $5 = \frac{11}{2} - \frac{1}{2}$ and $6 < \frac{2}{3} \times 9$
3. $3 > 3 + 2$ and $4 + 7 < 11$
4. $3 + 2 > 9 \times \frac{1}{3}$ and $4 \times \frac{3}{2} \not> 5$
5. $3.2 + 9.4 \not< 12.6$ and $\frac{7}{8} < \frac{11}{12}$
6. $3.25 + 0.3 \not< 6.25$

Consider next the sentence

$$4 + 1 = 5 \text{ or } 6 + 2 = 7.$$

This is another type of compound sentence, this time with the connective or. Here we must be very careful. Possibly we can get a hint from English sentences. If we say, "The Yankees or the Indians will win the pennant", we mean that exactly one of the two will win; certainly, they cannot both win. But when we say, "My package or your package will arrive within a week," it is possible that both packages may arrive; here we mean that one or more of the packages will arrive, including the possibility that both may arrive. The second of these interpretations of "or" is the one which turns out to be the better suited for our work in mathematics.

Thus we agree that a compound sentence with the connective or is true if one or more of its clauses is a true sentence; otherwise, it is false.

We classify

$$4 + 1 = 5 \quad \text{or} \quad 6 + 2 = 7$$

as a true compound sentence because its first clause is a true sentence; we also classify

$$5 < 4 + 3 \quad \text{or} \quad 2 + 1 \neq 4$$

as a true compound sentence, because one or more of its clauses is true (here both are true).

Is the sentence

$$3 \neq 2 + 1 \quad \text{or} \quad 2 > 4 + 1$$

false? Why?

Problem Set 3-7b

Which of the following sentences are true?

1. $3 = 5 - 1$ or $5 = 3 + 2$
2. $7 = \frac{11}{2} + \frac{3}{2}$ or $2 = \frac{11}{2} - \frac{3}{2}$
3. $4 > 3 + 2$ or $6 < 3 + 1$
4. $2 + 3 > 9 \times \frac{1}{3}$ and $4 \times \frac{3}{2} \neq 6$
5. $6.5 + 2.3 \neq 8.8$ or $\frac{3}{5} < \frac{7}{15}$
6. $5 + 4 < 9$ or $\frac{3}{4} < \frac{9}{12}$.

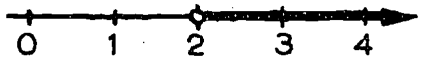
3-8. Graphs of Truth Sets of Compound Open Sentences

Our problems in graphing have so far involved only simple sentences. Graphs of compound open sentences require special handling. Let us consider the open sentence

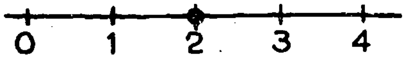
$$x > 2 \quad \text{or} \quad x = 2.$$

The clauses of this sentence and the corresponding graphs of their truth sets are:

$x > 2$



$x = 2$



If a number belongs to the truth set of the sentence "$x > 2$" or to the truth set of the sentence "$x = 2$", it is a number belonging to the truth set of the compound sentence "$x > 2$ or $x = 2$". Therefore, every number greater than or equal to 2 belongs to the truth set. On the other hand, any number less than 2 makes both clauses of the compound sentence false and so fails to belong to its truth set. The graph of the truth set is then

$x > 2$ or $x = 2$

We abbreviate the sentence "$x > 2$ or $x = 2$" to "$x \geq 2$", read "x is greater than or equal to 2". Give a corresponding meaning for "$\leq$"

Let us make a precise statement of the principle involved:

The graph of the truth set of a compound sentence with connective <u>or</u> consists of the set of all points which belong to either one of the graphs of the two clauses of the compound sentence.

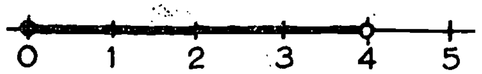
Finally, we consider the problem of finding the graph of an open sentence such as

$$x > 2 \quad \text{and} \quad x < 4.$$

Again we begin with the two clauses and the graphs of their truth sets:

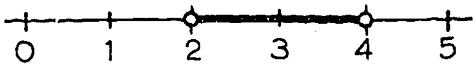
$x > 2$

$x < 4$



Then it follows (using an argument similar to that above) that the graph of the truth set of the compound sentence is

$x > 2$ and $x < 4$



Sometimes we write "$x > 2$ and $x < 4$" as "$2 < x < 4$".

We see that the graph of the truth set of a compound sentence with connective *and* consists of all points which are common to the graphs of the truth sets of the two clauses of the compound sentence.

It has required many words, carefully chosen, to describe the various connections between sentences, truth sets and graphs. We consistently referred to the *graph of the truth set of an open sentence*. In the future, let us shorten this phrase to the *graph of a sentence*. It will be a simpler description, and no confusion will result if we recall what is really meant by the description.

For the same reasons we shall find it convenient to speak of the point 3, or the point $\frac{1}{2}$, when we mean the point with coordinate 3 or the point with coordinate $\frac{1}{2}$. Points and numbers are distinct entities to be sure, but they correspond exactly on the number line. Whenever there is any possibility of confusion we shall remember to give the complete descriptions.

Problem Set 3-8

Construct the graphs of the following open sentences:

1. $x = 2$ or $x = 3$
2. $x = 2$ and $x = 3$
3. $x > 5$ or $x = 5$
4. $x > 2$ and $x < \frac{11}{2}$
5. $x > 5$ and $x = 5$
6. $x + 1 = 4$ and $x + 2 = 5$
7. $x + 1 = 4$ or $x + 3 = 5$
8. $x > 3$ or $x = 3$
9. $x < 3$ or $x = 3$
10. $x \neq 3$ and $x \neq 4$

3-9. Summary of Open Sentences.

We have examined some sentences and have seen that each one can be classified as either true or false, but not both. We have established a set of symbols to indicate relations between numbers:

"$=$" means "is" or "is equal to"
"$\neq$" means "is not" or "is not equal to"
"$<$" means "is less than"
"$>$" means "is greater than"
"$\leq$" means "is less than or is equal to"
"$\geq$" means "is greater than or is equal to"

We have discussed compound sentences which have two clauses. If the clauses are connected by the word or, the sentence is true if at least one clause is true; otherwise it is false. If the clauses are connected by and, the sentence is true if both clauses are true; otherwise it is false.

An open sentence is a sentence containing one or more variables.

The truth set of an open sentence containing one variable is the set of all those numbers which make the sentence true. The open sentence acts as a sorter, to sort the domain of the variable into two subsets: a subset of numbers which make the sentence true, and a subset which make the sentence false.

The graph of a sentence is the graph of the truth set of the sentence.

Problem Set 3-9

State the truth set of each of the following open sentences and construct its graph. Some examples of how you might state the truth sets are:

Open Sentence	Truth Set
$x + 3 = 5$	$\{2\}$
$2x \neq x + 3$	The set of all numbers of arithmetic except 3.
$x + 1 < 5$	The set of all numbers of arithmetic less than 4.
$2x \geq 9$	The set of numbers consisting of $4\frac{1}{2}$ and all numbers greater than $4\frac{1}{2}$.

1. $z + 8 = 14$
2. $2 + v < 15$
3. $2x = 3$
4. $6 > 1 + 3$ and $5 + t = 4$
5. $6 > 1 + 3$ or $2 + t = 1$
6. $x^2 = x$
7. $x + 2 = 3$ or $x + 4 = 6$
8. $\frac{x}{2} > 3$
9. $t + 4 = 5$ or $t + 5 \neq 5$
10. $3a \neq a + 5$
11. $3x^2 = 12$
12. $(3x)^2 = 36$
13. $9 + t < 12$ or $5 + 1 \neq 6$
14. $5x + 3 < 19$
15. $(x - 1)^2 = 4$
16. $3x = (8 + x) - 2$
17. $x^2 + 2 = 3x$
18. $t + 6 \leq 7$ and $t + 6 \geq 7$
19. $3(x + 2) = 3x + 6$
20. $t + 2 \neq 3$ and $8 + 2 < 5$

3-10. Identity Elements

Consider the truth sets of the open sentences

$$5 + x = 5$$
$$3 + y = 3$$
$$2\tfrac{1}{2} + a = 2\tfrac{1}{2}.$$

Do you find that the truth set of each of these sentences is $\{0\}$? For what number n is it true that

$$n + 0 = n?$$

Here we have an interesting property which we shall call the addition property of zero. We can state this property in words: "The sum of any number and 0 is equal to the number."

We can state this property in the language of algebra as follows:

For every number a,

$$a + 0 = a.$$

Since adding 0 to any number gives us identically the same number, 0 is often called the identity element for addition.

Is there an identity element for multiplication? Consider the truth sets of the following open sentences:

$$3x = 3$$
$$\frac{2}{3}n = \frac{2}{3}$$
$$.7 = .7y$$
$$n(5) = 5.$$

You have surely found that the truth set of each of these is {1}.

Thus

$$n(1) = n$$

seems to be a true sentence for all numbers. How could you state in words this property, which we shall call the <u>multiplication property of one</u>?

We can also state this in the language of algebra:

For every number a,

$$a(1) = a.$$

We can see that the identity element for multiplication is 1.

There is another property of zero which will be evident if you answer the following questions:

1. What is the result when any number of arithmetic is multiplied by 0?

2. If the product of two numbers is 0, and one of the numbers is 0, what can you tell about the other number?

The property that becomes apparent is called the <u>multiplication property of zero</u>, and can be stated as follows:

For every number a,

$$a(0) = 0.$$

These properties of zero and 1 are very useful. For instance, we use the multiplication property of 1 in arithmetic in working with rational numbers. Suppose we wish to find a numeral for $\frac{5}{6}$ in the form of a fraction with 18 as its denominator. Of the many names for 1, such as $\frac{2}{2}, \frac{3}{3}, \frac{5}{5}, \ldots$, we choose "$\frac{3}{3}$" because 3 is the number which multiplied by 6 gives 18. We then have

$$\begin{aligned}\frac{5}{6} &= \frac{5}{6}(1) \\ &= \frac{5}{6}\left(\frac{3}{3}\right) \\ &= \frac{5(3)}{6(3)} \\ &= \frac{15}{18}.\end{aligned}$$

Suppose we now wish to add $\frac{7}{9}$ to $\frac{5}{6}$. To do this, we desire other names for $\frac{7}{9}$ and $\frac{5}{6}$, names which are fractions with the same denominator. What denominator should we choose? It must be a multiple of both 6 and 9, but it cannot be 0. Thus 36, or 18, or 54, or many others, are possible choices. For simplicity we pick the smallest, which is 18. (This is called the least common multiple of 6 and 9.) In order, now, to add $\frac{7}{9}$ to $\frac{5}{6}$, we already know that

$$\frac{5}{6} = \frac{15}{18}.$$

Similarly,

$$\begin{aligned} \frac{7}{9} &= \frac{7}{9}(1) \\ &= \frac{7}{9}\left(\frac{2}{2}\right) \\ &= \frac{7(2)}{9(2)} \\ &= \frac{14}{18}. \end{aligned}$$

Then

$$\begin{aligned} \frac{5}{6} + \frac{7}{9} &= \frac{15}{18} + \frac{14}{18} \\ &= \frac{29}{18}. \end{aligned}$$

Example. Find a common name for $\dfrac{\frac{2}{3} + 5}{\frac{3}{7}}$.

$$\begin{aligned} \frac{\frac{2}{3} + 5}{\frac{3}{7}} &= \frac{\frac{2}{3} + 5}{\frac{3}{7}} \quad \frac{21}{21} \text{ (Why did we use } \frac{21}{21}\text{?)} \\ &= \frac{(\frac{2}{3} + 5)21}{(\frac{3}{7})21} \\ &= \frac{\frac{2}{3}(21) + 5(21)}{\frac{3}{7}(21)} \quad \text{(Note use of the distributive property)} \\ &= \frac{14 + 105}{9} \\ &= \frac{119}{9}. \end{aligned}$$

Problem Set 3-10

In problems 1 to 10, show how you use the properties of 0 and 1 to find a common name for each of the following:

1. $\frac{1}{2} + \frac{1}{3}$
2. $\frac{7}{12} + \frac{5}{18}$
3. $\dfrac{7 + \frac{2}{3}}{\frac{5}{6}}$
4. $\dfrac{\frac{1}{2} + \frac{3}{5}}{\frac{3}{20}}$
5. $27 + \left((15\frac{2}{3} + 2\frac{1}{3}) - 18\right)$
6. $\frac{7}{8}\left((3.7 + 0.3) - 4\right)$
7. $163\ (\frac{7}{12} + \frac{1}{6} + \frac{1}{4})$
8. $\left(\frac{2}{3} - \frac{1}{3}(2)\right) + 17$
9. $(6 - 6)(46)(36)\left(17 - \frac{2}{3}(12)\right)$
10. $(\frac{16}{3} - 5)(3)(5280)$

*11. (a) If you know that the product of two numbers is 0, and that one of the numbers is 3, what can you tell about the other number?

(b) If the product of two numbers is 0, what can you tell about at least one of the numbers?

(c) Does the multiplication property of 0 provide answers to these questions? Is another property of 0 implied here?

3-11. Closure

In our work so far we have often combined two numbers by addition or multiplication to obtain a number. We have never doubted that we always do get a number because our experience is that we always do. However, there are some primitive tribes who can count only to three. Suppose you tried to teach such people to add -- what would you tell them when you came to "2 + 2" and "2 + 3"? Obviously, you would have to enlarge their set of numbers until the sum of any two numbers would be a number of the set.

The set of all numbers of arithmetic is such a set. If you add any two of these numbers the sum is always a number of this set. When a certain operation is performed on elements of a given subset of the numbers of arithmetic and the resulting number is always a member of the same subset, then we say that the subset is closed under the operation. We say, therefore, that the set of numbers of arithmetic is closed under addition. Likewise, since the product of any two numbers is always a number, the set of numbers is closed under multiplication. We state these properties in the language of algebra as follows:

Closure Property of Addition: For every number a and every number b, $a + b$ is a number.

Closure Property of Multiplication: For every number a and every number b, ab is a number.

3-12. Associative and Commutative Properties of Addition and Multiplication.

In Chapter 2 we discussed a number of patterns for forming true sentences about numbers, and saw that these patterns were closely connected with many of the techniques of arithmetic. What were some of these patterns? For instance, we found true sentences such as

$$(7 + 8) + 3 = 7 + (8 + 3)$$

and

$$(1.2 + 1.8) + 2.6 = 1.2 + (1.8 + 2.6).$$

We concluded a pattern for true sentences from these examples, which we verbalized as follows: If you add a second number to a first number, and then a third number to their sum, the outcome is the same if you add the second number and the third number, and then add their sum to the first number. What was the name of this property?

The algebraic language with which we have been becoming familiar permits us, as in the case of the properties of 1 and 0 which we have just studied, to give a statement about the above property in this language. We have three (not necessarily different) numbers to deal with at once. Let us call the first "a", the second "b", and the third "c". "Add a second number to a first number" is then interpreted as "a + b"; "add a third number to their sum" is interpreted as "(a + b) + c". (Why did we insert the parentheses?) Write the second half of our verbal statement in the language of algebra. The words "the outcome is the same" now tell us that we have two names for the same number. Our statement becomes

$$(a + b) + c = a + (b + c).$$

For what numbers is this sentence true? We have concluded previously that it is true for <u>all</u> numbers. And so we write, finally,

For every number a, for every number b, for every number c,

$$(a + b) + c = a + (b + c).$$

Some other true sentences were of the form

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{3} + \frac{1}{2}$$

$$3 + 5 = 5 + 3.$$

The property of addition which states that all sentences following this pattern are true we called the commutative property of addition. It was verbalized as follows: If two numbers are added in different orders, the results are the same. In the language of algebra, we say

For every number a and every number b,

$$a + b = b + a.$$

How would you state the associative property of multiplication in the language of algebra?

What property is given by the following statement?

For every number a and every number b,

$$ab = ba.$$

These properties of the operations enable us to write open phrases in "other forms". For example, the open phrase 3d(d) can be written in the form 3(d·d), i.e., $3d^2$, by applying the associative property of multiplication. Thus, two "forms" of an open phrase are two numerals for the same number.

Among the properties with which we have just been concerned are the commutativity of addition and multiplication. Why are we so concerned whether binary operations like addition and multiplication are commutative? Aren't all the operations of arithmetic commutative? Let us try division, for example. Recall that

$$6 \div 3$$

means "6 divided by 3". Now, test whether

$$6 \div 3 = 3 \div 6$$

is a true sentence. This is enough evidence to show that division is <u>not</u> a commutative operation. (By the way, can you find some a and some b such that $a \div b = b \div a$?) Is the division operation associative?

Another very interesting example for the counting numbers is the following: let 2 ** 3 be defined to mean (2)(2)(2); and 3 ** 2 to mean (3)(3). In general, a ** b means a has been used as a factor b times. Is the following sentence true:

$$5 ** 2 = 2 ** 5?$$

Do you conclude that this binary operation on counting numbers is commutative? Is it associative?

You may complain that this second example is artificial. On the contrary, the ** operation defined above is actually used in the language of certain digital computers. You see, a machine is much happier if you give it all its instructions on a line, and so a "linear" notation was devised for this operation. But you see that to the machine the order of the numbers makes a great difference in this operation. Is there any restriction on the types of numbers on which we may operate with **?

Problem Set 3-12

1. If x and y are numbers of arithmetic, the closure property assures us that $3xy$, $2x$ and therefore, $(3xy)(2x)$ are numbers of arithmetic. Then the associative and commutative properties of multiplication enable us to write this in another form:

$$(3xy)(2x) = (3 \cdot 2)(x \cdot x)y$$
$$= 6x^2y$$

Write the indicated products in another form as in the above example:

(a) $(2m)(mn)$

(b) $(5p^2)(3q)$

(c) $n(2n)(3m)$

(d) $(\frac{1}{2}ab)(6c)$

(e) $(10a)(10b)$

(f) $(3x)(12)$

2. If x and y are numbers of arithmetic then the closure property allows us to think of $12x^2y$ as a numeral which represents a single number. The commutative and associative properties of multiplication enable us to write other numerals for the same number. $(4xy)(3x)$, $(2x)(6xy)$, and $(12x^2y)(1)$ are some of the many ways of writing $12x^2y$ as indicated products. Similarly, write three possible indicated products for each of the following.

(a) $8ab^2$

(b) $7xy^2$

(c) $10mn$

(d) x^2y^2

(e) $64a^2bc^2$

(f) $2c$

3. Which of the following sentences are true for every value of the variable? Explain which of the properties aided in your decision.

(a) $m(2 + 5) = (2 + 5)m$

(b) $(m + 1)2 = (2 + 1)m$

(c) $(a + 2y) + b = (a + b) + 2y$

(d) $3x + y = y + 3x$

(e) $(2a + c) + d = 2a(c + d)$

(f) $2c + 6 = 6 + 2c$

(g) $.5b(200) = 200(.5b)$

(h) $(2uv)z = 2u(vz)$

(i) $(\frac{m}{2} + \frac{n}{2}) + \frac{s}{2} = \frac{m}{2} + (\frac{n}{2} + \frac{s}{2})$

(j) $3x + y = x + 3y$

4\. The set A is given by $A = \{0, 1\}$

(a) Is A closed under addition?

(b) Is A closed under multiplication?

5\. (a) Is the set S of all multiples of 6 closed under addition?

(b) Is set S closed under multiplication?

*6. Let us define some binary operations other than addition and multiplication. We shall use the symbol "∘" each time. We might read "a ∘ b" as "a operation b". Since we are giving the symbol various meanings, we must define its meaning each time. For instance, for every a and every b,

if $a \circ b$ means $2a + b$, then $3 \circ 5 = 2(3) + 5$;

if $a \circ b$ means $\frac{a + b}{2}$, then $3 \circ 5 = \frac{3 + 5}{2}$;

if $a \circ b$ means $(a - a)b$, then $3 \circ 5 = (3 - 3)5$;

if $a \circ b$ means $a + \frac{1}{3}b$, then $3 \circ 5 = 3 + (\frac{1}{3})(5)$;

if $a \circ b$ means $(a + 1)(b + 1)$, then $3 \circ 5 = (3 + 1)(5 + 1)$.

For each meaning of $a \circ b$ stated above, write a numeral for each of the following:

(a) $2 \circ 6$

(b) $(\frac{1}{2}) \circ 6$

(c) $6 \circ 2$

(d) $(3 \circ 2) \circ 4$

*7. Are such binary operations on numbers as those defined in Problem 6 commutative? In other words, is it true that for every a and every b, $a \circ b = b \circ a$? Let us examine some cases. For instance, if $a \circ b$ means $2a + b$, we see that

$$3 \circ 4 = 2(3) + 4$$
$$4 \circ 3 = 2(4) + 3$$

But "$2(3) + 4 = 2(4) + 3$" is a false sentence. Hence; we conclude that the operation here indicated by "∘" is not commutative. In each of the following, decide whether or not the operation described is commutative:

(a) For every a and every b, $a \circ b = \frac{a + b}{2}$

(b) For every a and every b, $a \circ b = (a - a)b$

(c) For every a and every b, $a \circ b = a + \frac{1}{3}b$

(d) For every a and every b, $a \circ b = (a + 1)(b + 1)$

What do you conclude about whether all binary operations are commutative?

*8. Is the operation "$\circ$" associative in each of the above cases? For instance, if, for every a and every b, $a \circ b = 2a + b$, is $(4 \circ 2) \circ 5 = 4 \circ (2 \circ 5)$ a true sentence?

$$\begin{aligned}(4 \circ 2) \circ 5 &= 2(2(4) + 2) + 5\\ &= 2(10) + 5\end{aligned}$$

while

$$\begin{aligned}4 \circ (2 \circ 5) &= 2(4) + (2(2) + 5)\\ &= 8 + 9\end{aligned}$$

Since the sentence $2(10) + 5 = 8 + 9$ is false, we conclude that this operation is not associative. Test the operations described in problem 7 (a)-(d) for the associative property.

3-13. The Distributive Property.

Our work with numbers in Chapter 2 has shown us a variety of versions of the distributive property. Thus

$$15(7 + 3) = 15(7) + 15(3)$$

and

$$(\frac{1}{3})12 + (\frac{1}{4})12 = (\frac{1}{3} + \frac{1}{4})12$$

are two true sentences each of which follows one of the patterns which we have recognized. We have seen the importance of this property in relating indicated sums and indicated products. We may now state the distributive property in the language of algebra:

For every number a, every number b, and every number c,

$$a(b + c) = ab + ac.$$

Does this statement agree with our verbalization in Chapter 2? Since we have stated that "$a(b + c)$" and "$ab + ac$" are numerals for the same number, we may equally well write

For every number a, every number b, and every number c

$$ab + ac = a(b + c).$$

We may also apply the commutative property of multiplication to write:

For every number a, every number b, and every number c,

$$(b + c)a = ba + ca$$

and:

For every number a, every number b, and every number c,

$$ba + ca = (b + c)a.$$

Which of these patterns does the first of our above examples follow? the second?

Any one of the four sentences above describes the distributive property. All forms are useful in the study of algebra.

Example 1. Write the indicated product, $x(y + 3)$ as an indicated sum.

$x(y + 3) = xy + x(3)$ by the distributive property

$= xy + 3x$

Example 2. Write $5x + 5y$ as an indicated product.

$5x + 5y = 5(x + y)$ by the distributive property

Example 3. Write the open phrase $3a + 5a$ in simpler form.

$3a + 5a = (3 + 5)a$ by the distributive property

$= 8a$

Example 4. Write the open phrase, $2x + 3y + 4x + 6y$ in simpler form.

$2x + 3y + 4x + 6y = (2x + 4x) + (3y + 6y)$

by the commutative and associative properties of addition.

$= (2 + 4)x + (3 + 6)y$

by the distributive property

$= 6x + 9y$

Problem Set 3-13a

1. Write the indicated products as indicated sums.

(a) $6(r + s)$	(d) $(7 + x)x$
(b) $(b + 3)a$	(e) $6(8 + 5)$
(c) $x(x + z)$	(f) $(a + b)b$

2. Write the indicated sums as indicated products.
   (a) $3x + 3y$
   (b) $am + an$
   (c) $x + bx$ Hint: $x = (1)x$
   (d) $\frac{1}{2}x + \frac{1}{2}y$
   (e) $2a + a^2$ Hint: How is a^2 defined?
   (f) $x^2 + xy$
3. Use the associative, commutative, and distributive properties to write the following open phrases in simpler form, if possible:
   (a) $14x + 3x$
   (b) $\frac{3}{4}x + \frac{3}{2}x$
   (c) $\frac{2}{3}a + 3b + \frac{1}{3}a$
   (d) $7x + 13y + 2x + 3y$
   (e) $4x + 2y + 2 + 3x$
   (f) $1.3x + 3.7y + 6.2 + 7.7x$
   (g) $2a + \frac{1}{3}b + 5$

The distributive property stated by the sentence

For every number a, every number b, and every number c,

$$a(b + c) = ab + ac$$

concerns the three numbers a, b and c. However, the closure property allows us to apply the distributive property in many cases where an open phrase apparently contains more than three numerals. For example, suppose we wish to express the indicated product $2r(s + t)$ as a sum. The open phrase contains the four numerals 2, r, s, and t. The closure property, however, allows us to consider 2r as the name of one number so we can think in terms of three numerals, 2r, s, and t. Thus

$$\begin{aligned} 2r(s+t) &= (2r)(s+t) \\ &= (2r)s + (2r)t \\ &= 2rs + 2rt \end{aligned}$$

Example 1. Write $3u(v + 3z)$ as an indicated sum.

By the closure property we can regard 3u, v, and 3z each as the name of one number. Then by the distributive property

$$\begin{aligned} 3u(v+3z) &= (3u)v + (3u)(3z) \\ &= 3uv + 9uz \quad \text{by the commutative and associative properties of multiplication} \end{aligned}$$

Example 2. Write the indicated sum, $2rs + 2rt$, as an indicated product.

We can do this in three ways:

$$\begin{aligned} (1)\quad 2rs + 2rt &= 2(rs) + 2(rt) \\ &= 2(rs + rt) \\ (2)\quad 2rs + 2rt &= r(2s) + r(2t) \\ &= r(2s + 2t) \\ (3)\quad 2rs + 2rt &= (2r)s + (2r)t \\ &= 2r(s + t) \end{aligned}$$

Although all three ways are correct, the third is usually preferred.

Example 3. Express the indicated product, $3(x + y + z)$, as an indicated sum.

$$3(x + y + z) = 3x + 3y + 3z$$

Problem Set 3-13b

1. Write each of the indicated products as an indicated sum.

(a) $m(6 + 3p)$	(d) $(2x + xy)x$
(b) $2k(k + 1)$	(e) $(e + f + g)h$
(c) $6(2s + 3r + 7q)$	(f) $6pq(p + q)$

2. Which of the following open sentences are true for every value of every variable.

   (a) $2a(a + b) = 2a^2 + ab$

   (b) $4xy + y^2 = (4x + y)y$

(c) $3ab + 6bc = 3b(a + 2c)$

(d) $2a(b + c) = 2ab + c$

(e) $(4x + 3)x = 4x^2 + 3 + x$

(f) $(2y + xy) = (2 + x)y$

3. Write each of the indicated sums as an indicated product.

(a) $3uv + v^2$

(b) $7pq + 7qr$

(c) $3x + 3x^2$ Hint: think of $3x$ as $(3x)(1)$

(d) $2c + 4cd$ Hint: $4cd = (2c)(2d)$

(e) $3x + 6x^2$

(f) $xz^2 + 2xz$

Another important application of the distributive property is illustrated by the following example.

<u>Example 1</u>. Write $(x + 2)(x + 3)$ as an indicated sum without parentheses.

If we write the distributive property with the indicated product beneath it, we can see which names we must regard as separate names of numbers.

$$\begin{array}{rcll}
a(b + c) & = & ab + ac & \\
\downarrow\ \ \searrow\ \ \ \searrow & & \swarrow\downarrow\ \ \ \swarrow\searrow & \\
(\overbrace{x+2})(x + 3) & = & (\overbrace{x+2})x + (\overbrace{x+2})3 & \\
 & = & x^2 + 2x + 3x + 6 & \text{distributive property} \\
 & = & x^2 + (2 + 3)x + 6 & \text{distributive property} \\
 & = & x^2 + 5x + 6 &
\end{array}$$

Could you have used a different form of the distributive property to begin your work?

<u>Example 2</u>. Write $(a + b)(c + d)$ as an indicated sum without parentheses. Supply the reason for each step.

$$\begin{aligned}
(a + b)(c + d) &= (a + b)c + (a + b)d \\
&= ac + bc + ad + bd
\end{aligned}$$

Problem Set 3-13c

Write the indicated products as indicated sums without parentheses.

1. $(x + 4)(x + 2)$
2. $(x + 1)(x + 5)$
3. $(x + a)(x + 3)$
4. $(x + 2)(y + 7)$
5. $(m + n)(m + n)$
6. $(2p + q)(p + 2q)$

3-14. Summary: Properties of Operations on Numbers of Arithmetic.

Numbers can be added and multiplied. We have learned that numbers and their operations have basic properties which we shall list below and always refer to as the properties of numbers.

1. Closure Property of Addition: For every number a and every number b, $a + b$ is a number.

2. Closure Property of Multiplication: For every number a and every number b, ab is a number.

3. Commutative Property of Addition: For every number a and every number b, $a + b = b + a$.

4. Commutative Property of Multiplication: For every number a and every number b, $ab = ba$

5. Associative Property of Addition: For every number a and every number b and every number c, $(a + b) + c = a + (b + c)$

6. Associative Property of Multiplication: For every number a and every number b and every number c, $(ab)c = a(bc)$

7. Distributive Property: For every number a and every number b and every number c, $a(b + c) = ab + ac$

8. Addition Property of 0: For every number a,
   $a + 0 = a$

9. Multiplication Property of 1: For every number a,
   $a(1) = a$

10. **Multiplication Property of 0**: For every number a,

$$a(0) = 0$$

Problem Set 3-14

1. By putting one of the signs, +, ×, -, in each of the blanks, and inserting parentheses to indicate grouping, find the common name for each number which can be obtained from

$$8\_\_\_\_\_\_3\_\_\_\_\_\_2$$

As examples, $8 - (3 + 2) = 3$, and $(8 + 3) \times 2 = 22$

2. State indicated products as indicated sums, and indicated sums as indicated products:

(a) $(1 + x)5$

(b) $\frac{3}{4}a + \frac{2}{4}a$

(c) $33(95) + 15(11)$

(d) $c + 2c$

(e) $(\frac{1}{2} + \frac{1}{3})y$

(f) $x(y + 1)$

(g) $xy + x$

(h) $3ax + 2ay$

(i) $(5 + y)y + 2y$

(j) $(2 + a)b + (2 + a)3$

Hint: (2 + a) is a number, by closure

(k) $b + 3ab$

(l) $2a(a + b + c)$

(m) $(u + 2v)(u + v)$

(n) $(a + 1)^2$

3. Use the properties to write the following in simpler form:

(a) $17x + x$

(b) $2x + y + 3x + y$

(c) $3(x + 1) + 2x + 7$

(d) $1.6a + .7 + .4a + .3b$

(e) $by + 2by$

(f) $9x + 3 + x + 2 + 11x$

*4. Here you are going to see how to test whether a whole number is exactly divisible by 9. Keep a record, as you go, of the properties of addition and multiplication which are used.

Try the following:

$$\begin{aligned}2357 &= 2(1000) + 3(100) + 5(10) + 7(1)\\ &= 2(999 + 1) + 3(99 + 1) + 5(9 + 1) + 7(1)\\ &= 2(999) + 2(1) + 3(99) + 3(1) + 5(9) + 5(1) + 7(1)\\ &= \big(2(999) + 3(99) + 5(9)\big) + \big(2(1) + 3(1) + 5(1) + 7(1)\big)\\ &= \big(2(111) + 3(11) + 5(1)\big)9 + \big(2 + 3 + 5 + 7\big)\\ &= (222 + 33 + 5)9 + (2 + 3 + 5 + 7)\end{aligned}$$

Is 2357 divisible by 9? Try the same procedure with 35874. Can you formulate a general rule to tell when a whole number is divisible by 9?

*5. Look for the pattern in the following calculation:

$$\begin{aligned} 19 \times 13 &= 19(10 + 3) \\ &= 19(10) + 19(3) \quad \text{(what property?)} \\ &= 19(10) + (10 + 9)3 \\ &= 19(10) + \big(10(3) + 9(3)\big) \quad \text{(what property?)} \\ &= \big(19(10) + 10(3)\big) + 9(3) \quad \text{(what property?)} \\ &= (19 + 3)10 + 9(3) \quad \text{(what properties?)} \end{aligned}$$

The final result indicates a method for "multiplying teens" (whole numbers from 11 through 19): Add to the first number the units digit of the second, and multiply by 10; then add to this the product of the units digits of the two numbers. Use the method to find 15 × 14, 13 × 17, 11 × 12.

Review Problems

1. (a) Write a description of the set H if
$$H = \{21, 23, 25, 27, \ldots, 49\}.$$
   (b) Consider the set A of all whole numbers greater than 20. Is H a subset of A? Is A a subset of H?
   (c) Classify sets H and A (finite or infinite).
2. Find the coordinate of a point which lies on the number line between the two points with coordinates $\frac{3}{4}$ and $\frac{5}{6}$. How many points are between these two?
3. Consider the set
$$T = \{0, 3, 6, 9, 12, \ldots\}.$$
Is T closed under the operation of addition? under the operation of "averaging"?
4. Let the domain of the variable t be the set R of all numbers between 3 and 5, inclusive.
   (a) Draw the graph of the set R.
   (b) Decide whether each of the following numbers is an admissible value of t: $\frac{3}{2}$, π, $\frac{17}{4}$, $4\frac{1}{3}$, $\frac{11}{2}$, $\sqrt{2}$.

5. Is each of the following sentences true?

$$(3 + 1)^2 = 3^2 + 2(3)(1) + 1^2$$

$$(5 + 2)^2 = 5^2 + 2(5)(2) + 2^2$$

$$(\frac{1}{3} + \frac{2}{3})^2 = (\frac{1}{3})^2 + 2(\frac{1}{3})(\frac{2}{3}) + (\frac{2}{3})^2$$

Decide what pattern these sentences follow and describe the pattern in words. Then formulate it in the language of algebra as an open sentence with two variables. Use the properties of the numbers of arithmetic as we have discovered them to test whether the resulting sentence is true for all values of the variables.

6. In which of A, B, C, D, E does the sentence have the same truth set as the sentence "$x \leq 5$"?
   (A) $x > 5$ or $x = 5$
   (B) $x < 5$ and $x = 5$
   (C) $x \ngtr 5$
   (D) $x \nless 5$
   (E) $x \nleq 5$

7. Find the truth set for each of the following sentences:
   (a) $n - 5 = 7$
   (b) $2n - 5 = 7$
   (c) $3n - 5 = 7$
   (d) $4n - 5 = 7$
   (e) $6n - 5 = 7$
   (f) $12n - 5 = 7$

8. If m is a number of arithmetic, find the truth set of
   (a) $m + m = 2$
   (b) $m + m = 2m$
   (c) $m \geq 2m$
   (d) $m + 3 < m$

   Which of the above sets is a subset of all the others? Which is a subset of none other than itself? If the domain of m is the set of counting numbers, answer questions (a) through (d) above.

9. Let T be the truth set of

$$x + 3 = 5 \quad \text{or} \quad x + 1 = 4.$$

   (a) Is 3 an element of T?
   (b) Is 2 an element of T?
   (c) Is $\emptyset$ a subset of T?

10. If S is the truth set of

$$x + 1 < 5 \text{ and } x - 1 \geq 2,$$

draw the graph of S.

11. Consider the open sentence

$$2x \leq 1.$$

What is its truth set if the domain of x is the set of

(a) all counting numbers?

(b) all whole numbers?

(c) all numbers of arithmetic?

12. (a) Is the following sentence true?

$$\frac{17}{2}(8 + 1) = \frac{17}{2}(4 + 5)$$

(b) Do you have to perform any multiplication to answer part (a)? Explain.

13. Which of the following sentences are true?

(a) $5(4 + 2) = (4 + 2)5$

(b) $10(\frac{1}{2} + \frac{1}{3}) = \frac{21}{2} + \frac{31}{3}$

(c) $10(\frac{1}{2} + \frac{1}{3}) = 5 + \frac{10}{3}$

(d) $4 + 7 + 1.817 = 5 + 1.817 + 6$

(e) $12 \times 8 + 12 \times 92 = 1200$

(f) $(5 \times \frac{25}{4})16 = 500$

14. Explain how the property of 1 is used in performing the calculation

$$\frac{\frac{3}{5} + \frac{2}{3}}{\frac{3}{4}}$$

15. Explain why

$$3x + y + 2x + 3y = 5x + 4y$$

is true for all values of x and y.

16. (a) Write the indicated products

$$(x + 1)(x + 1)$$

$$(x + 2)(x + 2)$$

as indicated sums without parentheses.

(b) Use the pattern of the results of part (a) to write the indicated sum

$$x^2 + 6x + 9$$

as an indicated product.

Chapter 4

OPEN SENTENCES AND ENGLISH SENTENCES

4-1. Open Phrases and English Phrases

Every day we are building up a new language of symbols which is becoming more and more a complete language. We have used mathematical phrases, such as "$8 + 3y$"; mathematical verb forms, including "$=$" and "$>$"; and mathematical sentences, such as "$7n + 3n = 50$"

We recall that a variable, such as "n", is the name of a definite but unspecified number. The translation of "n" into English will then mean relating an unspecified number to something of interest to us. Thus, the numeral "n" might represent "the number of problems that I worked", "the number of students at the rally", "the number of dimes in Sam's pocket", or "the number of feet in the height of the school flagpole". What are some other possible translations?

Consider the phrase "$5 + n$". Can we invent an English phrase for this? Suppose we use the translations suggested above. If "n" is the number of problems I shall be working today, then the phrase "$5 + n$" represents "the total number of problems including the five worked last night"; or, if I have 5 dimes and "n" represents the number of dimes in Sam's pocket, then "$5 + n$" represents "the total number of dimes, including my five and those in Sam's pocket." Notice that the translation of "$5 + n$" depends on what translation we make of "n".

Which of the apparently limitless number of translations do we pick? We are reminded that the variable appearing in the open phrase, whether "n" or "x", or "w", or "b", is the name of a <u>number.</u> Whether this is the number of dimes, the number of students, the number of inches, etc., depends upon the use we plan to make of the translation. The context itself will frequently suggest or limit translations. Thus it would not make sense to translate a phrase such as "$2,500,000 + y$" in terms of the number of dimes in Sam's pocket, but it would make sense to think of "y" as representing the number giving the population increase in a state which had 2,500,000 persons at the time of the preceding census, or as the number of additional miles traveled by a satellite which had gone 2,500,000 miles at the time of the last report. Similarly, the variable in the phrase ".05 + k" would hardly be translated as the number of cows or students, but possibly as the number giving the increase in the rate of interest which had previously been 5 per cent.

How can we translate the phrase "3x + 25"? In the absence of any special reasons for picking a particular translation, we might let x be the number of cents Tom earns in one hour, mowing the lawn. Then 3x is the number of cents earned in 3 hours. If Tom finished the job in three hours and was paid a bonus of 25 cents, then the phrase "3x + 25" represents the total number of cents in Tom's possession after working three hours. How can this phrase be translated if we let x be the number of students in each algebra class, if algebra classes are of the same size? Or, if x is the number of miles traveled by a car in one hour at a constant speed?

There are many English translations of the symbol "+", indicating the operation of addition of two numbers. A few of them are: "the sum of", "more than", "increased by", "older than", and others. There are also many English translations of the symbols indicating the operation of multiplication of two numbers, including: "times", "product of", and others. What are English translations of the symbol "-" ?

Problem Set 4-1

In Problems 1-6, write English phrases which correspond to the given open phrases. Try to vary the English phrases as much as possible. Tell in each case what the variable represents.

1. 7w (If one bushel of wheat costs w dollars, the phrase is: "the number of dollars in the cost of 7 bushels of wheat.")

2. n + 7

3. $n - 7$
4. $\frac{\text{[illegible]}}{2}$
5. $2r + 5$
6. $a + b$

In each of Problems 7-17, find an open phrase which is a translation of the given English phrase. In each problem, tell explicitly what the variable represents.

7. The number of feet in y yards.
   (If y is the number of yards, then 3y is the number of feet.)
8. The number of inches in f feet.
9. The number of pints in k quarts.
10. The number of miles in k feet.
11. The successor of a whole number.
12. The reciprocal of a number. (Two numbers are reciprocals of each other if their product is 1.)
13. The number of ounces in k pounds and t ounces.
14. The number of cents in d dollars and k quarters.
15. The number of cents in m dollars, k quarters, m dimes and n nickels.
16. The number of inches in the length of a rectangle which is twice as long as it is wide. (Suggestion: Draw a figure to help visualize the situation.)
17. The number of feet in the height of a triangle if its height is 6 feet greater than its base.

In Problems 18-25, write English phrases which are translations of the given open phrases. In each problem identify the variable explicitly.

18. $n + 7n$
19. $(2r - 5) + 7$
20. $3x + (2x + 1) + x$
21. $5000 + 4y$
22. $y + .04y$

*23. $x(x + 3)$ (Hint: This phrase might be interpreted as the expression for the number of square units in an area.)

*24. $\frac{1}{2}y(y + 5)$

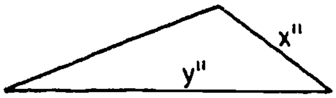
*25. $w(w + 3)(2w + 5)$ (Hint: This phrase might be interpreted as the expression for the number of cubic units in a volume.)

26. Choose a variable for the number of feet in the length of one side of a square. Write an open phrase for the number of feet in the perimeter of the square.

27. Choose a variable for the number of inches in the height of a man's head. Then write an open phrase for the number of inches in the man's height if it is known that his height is $7\frac{1}{2}$ times the height of his head.

28. Write an open phrase for the number of inches in the length of a second side of a triangle, if it is three inches longer than the first side.

29. One side of a triangle is x inches long and a second is y inches long. The length of the third side is one-half the sum of the lengths of the first two sides.



(a) Write an open phrase for the number of inches in the perimeter of the triangle.

(b) Write an open phrase for the number of inches in the length of the third side.

30. In a certain community there are six-fifths as many girls as boys. Write an open phrase for the number of girls in terms of the number of boys.

31. The admission price to a performance of "The Mikado" is $2.00 per person. Write an open phrase for the total number of dollars received in terms of the number of people who bought tickets.

32. If a man can paint a house in d days, write an open phrase for the part of the house he can paint in one day.

33. If a pipe fills $\frac{1}{5}$ of a swimming pool in one hour, write an open phrase for how much of the pool is filled by that pipe in x hours.

34. When a tree grows it increases its radius each year by adding a ring of new wood. If a tree has r rings now, write an open phrase for the number of growth rings in the tree twelve years from now.

*35. A plant grows a certain number of inches per week. It is now 20 inches tall. Write an open phrase giving the number of inches in its height five weeks from now.

*36. Suppose that when a man immerses his arms in hot water, the temperature of his feet will rise one degree per minute, beginning at 10 minutes after his arms are put in the water. Write an open phrase for the rise in temperature of the man's feet at any time (more than ten minutes) after his arms are immersed.

37. Three sons share in an inheritance.
    (a) Write an open phrase for the number of dollars of one son's share which is one-half of the inheritance.
    (b) Write an open phrase for the number of dollars of the second son's share, which is fifty dollars more than one-tenth of the inheritance.
    (c) Write an open phrase for the third son's share.
    (d) Write an open phrase for the sum of the three sons' shares.

38. Choose a variable for the number of feet in the width of a rectangle.
    (a) Write an open phrase for the length of the rectangle if the length is five feet less than twice the width. Draw and label a figure.
    (b) Write an open phrase for the perimeter of the rectangle described in part (a).
    (c) Write an open phrase for the area of the rectangle described in part (a).

4-2. Open Sentences And English Sentences

It is a natural step from translation of phrases to translation of sentences.

Example: $45 +$ [illegible] $= 108$

How shall we write an English sentence for this open sentence? We might say, "A book salesman is paid \$45 a week plus \$3 for each set of books he sells. In one week he was paid \$108." Another translation of this open sentence could be, "A freight shipment consisted of a box weighing 45 pounds and a number of small cartons each weighing 3 pounds. The whole shipment weighed 108 pounds."

What English sentences could you write for this same open sentence?

Problem Set 4-2a

Write your own English sentences for the following open sentences.

1. $n + 7 = 82$
2. $2n = 500$
3. $\frac{b}{2} = 17$
4. $w(w + 4) = 480$
5. $a + (2a + 3a) = (a + 2a) + 3a$
6. $4n + 7n = 44$
7. $4k + 7k = 47$
8. $x + x + x + x = 100$
9. $y + 5 = 5 + y$
10. $\frac{1}{4}a + \frac{1}{2}b = 6$

More often we want to translate English sentences into open sentences. We find such open sentences particularly helpful in word problems when the English sentence is about a quantity which we are interested in finding.

Example 1. "Carl has a board 44 inches long. He wishes to cut it into two pieces so that one piece will be three inches longer than the other. How long should the shorter piece be?"

We may sometimes see more easily what our open sentence should be if we guess a number for the quantity asked for in the problem.

If the shorter piece is 18 inches long, then the longer piece is (18 + 3) inches long. Since the whole board is 44 inches long, we then have the sentence

$$18 + (18 + 3) = 44.$$

Although this sentence is not true, it suggests the pattern which we need for an open sentence. Notice that the question in the problem has pointed out our variable. We can now say:

> If the shorter piece is K inches long, then the longer piece is (K + 3) inches long, and the sentence is

$$K + (K + 3) = 44.$$

We say that this sentence is false when K is 18. There probably is some value of K for which the open sentence is true. If we wanted to find the length of the shorter piece, this could be done by finding the truth set of the above open sentence. Perhaps you feel an urge to find a number which does make the above sentence true. If so, go ahead and try. For the present, however, our objective is practice in writing the open sentences. Later we shall be concerned with finding the truth sets of such sentences and thus answering the questions in the problems.

In this example we tried some particular numbers for the quantities involved to help see a pattern for the open sentence. You may sometimes see the open sentence immediately without having to try particular numbers.

Notice that the English sentences are often about inches or pounds or years or dollars, but the open sentences are always about numbers only.

Notice also that we are very careful in describing our variable to show what it measures, whether it is the number of inches, the number of donkeys, or the number of tons.

Example 2. "Two cars start from the same point at the same time and travel in the same direction at constant speeds of 34 and 45 miles per hour, respectively. In how many hours will they be 35 miles apart?"

If they travel 4 hours, the faster car goes 45(4) miles and the slower car goes 34(4) miles. Since the faster car should then be 35 miles farther from the starting point than the slower car, we have the sentence

$$45(4) - 34(4) = 35,$$

which is false. It suggests, however, the following:

If they travel h hours, then the faster car goes 45h miles and the slower car goes 34h miles, and

$$45h - 34h = 35.$$

Example 3. "A man left $10,500 for his widow, a son and a daughter. The widow received $5,000 and the daughter received twice as much as the son. How much did the son receive?"

If the son received n dollars, then the daughter received 2n dollars, and

$$n + 2n + 5000 = 10,500.$$

Problem Set 4-2b

Write open sentences that would help you solve problems 1-13, being careful to give the meaning of the variable for each. Your work may be shown in the form indicated in Example 3 above. It is not necessary to find the truth sets of the open sentences.

1. Henry and Charles were opposing candidates in a class election. Henry received 30 votes more than Charles, and 516 members of the class voted. How many votes did Charles receive? (Hint: If Charles received c votes, write an open phrase the number of votes Henry received. Then write your open sentence.)
2. A rectangle is 6 times as long as it is wide. Its perimeter is 144 inches. How wide is the rectangle? (Remember to draw a figure.)
3. The largest angle of a triangle is 20° more than twice the smallest, and the third angle is 70°. The sum of the angles of a triangle is 180°. How large is the smallest angle?

4. A bridge has three spans, one of which is 100 feet longer than each of the other two. If the bridge is 2500 ft. long, how long is each of the shorter spans?
5. A class of 43 students was separated into two classes. If there were 5 more students in Mr. Smith's class than in Miss Jones's class, how many students were in each class? (Can you do this one in two ways? If there were y students in Miss Jones's class, find two ways to say how many were in Mr. Smith's class.)
6. The length of a rectangle is 5 inches more than its width. What is the length of the rectangle if its area is 594 square inches?
7. John is three times as old as Dick, Three years ago the sum of their ages was 22 years. How old is each now? (Hint: Find a phrase for the age of each three years ago in terms of Dick's age now.)
8. John has 1.65 in his pocket, all in nickels, dimes, and quarters. He has one more quarter than he has dimes, and the number of nickels he has is one more than twice the number of dimes. How many dimes has he? (Hint: If he has d dimes, write a phrase for the value of all his dimes, a phrase for the value of all his quarters, and a phrase for the value of all his nickels; then write your open sentence.)
9. I bought 23 postage stamps, some of them 4-cent stamps and some 7-cent stamps. If the total cost was 1.19, how many of each kind did I buy?
10. A passenger train travels 20 miles per hour faster than a freight train. At the end of 5 hours the passenger train has traveled 100 miles farther than the freight train. How fast does the freight train travel? (Hint: For each train find a phrase for the number of miles it has traveled.)
11. A store has 39 quarts of milk, some in pint cartons and some in half-pint cartons. There are 6 times as many pint cartons as half pint cartons. How many half-pint cartons are there?

12. Mr. Brown is employed at an initial salary of \$3600, with an annual increase of \$300, while Mr. White starts at the same time at an initial salary of \$4500, with an annual increase of \$200. After how many years will the two men be earning the same salary?

13. A table is three times as long as it is wide. If it were 3 feet shorter and 3 feet wider, it would be a square. How long and how wide is it? (Draw two pictures of the table top.)

14. Write your own problem for each of the following open sentences.

    (a) $5n + 10(n + 2) + 50(2n) = 480$
    (b) $a(3a) = 300$
    (c) $.60x + 1.10(85 - x) = 78.50$
    (d) $a + (a + 3) + (a + 6) = 69$
    (e) $b = 2(18 - b)$
    (f) $30h + 4(5 - h) = 59$
    (g) $\frac{1}{4} + \frac{1}{3} + x = 1$

15. Translate the following into an open sentence. Then translate back from the open sentence to a different problem in English.

    "Three hundred sight-seers went on a trip around the bay in two boats. The capacity of one boat was 80 more passengers than that of the other and both boats were full. How many passengers were in each boat?

4-3. Open Sentences Involving Inequalities

Our sentences need not all be equalities. Problems concerning "greater than" or "less than" have real meaning.

Suppose we say, "Make a problem for the sentence $d + 2 > 5$." The word problem could be, "If I added two dollars to what I now have, I would have more than five dollars. How much do I have now?"

Problem Set 4-3a

Write problems for the following open sentences.

1. $a > 3$
2. $n + 1 < 17$
3. $25q \leq 175$
4. $5(n + 3) < 100$
5. $p + 10,000 > 160,000$
6. $a + 2a + 3a \geq 48$
7. $a + 2a + 3a = 50$

*8. $y + 3 < 10$ and $y + 3 > 5$

9. $\frac{1}{2}b < 9$

*10. $s > 3(12 - s)$

As with equations, it will sometimes help to find an open sentence in problems about inequalities if we try a particular number first.

Example 1. "In six months Mr. Adams earned more than \$7000. How much did he earn per month?"

If he earned \$1100 per month, in 6 months he would earn 6×1100 dollars. The sentence would then be

$$6 \times 1100 > 7000.$$

This, of course, is not true, but it suggests what we should do.

If Mr. Adams earned a dollars per month, in 6 months he would earn $6a$ dollars. Then

$$6a > 7000.$$

Example 2. "The distance an object falls during the first second is 32 feet less than the distance it falls during the second second. During the two seconds it falls 48 feet or less, depending on the air resistance. How far does it fall during the second second?

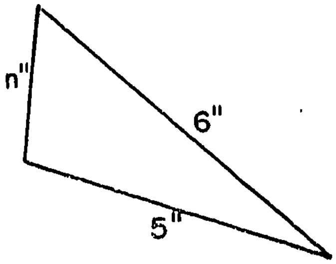
If the object falls 42 feet during the second second, then it falls $(42 - 32)$ feet during the first second. Since the total distance fallen is less than or equal to 48 feet, our sentence is

$$(42 - 32) + 42 \leq 48.$$

This suggests how to write the open sentence. If the object falls d feet during the second second, then it falls $(d - 32)$ feet during the first second, and

$$(d - 32) + d \leq 48.$$

Example 3. "Two sides of a triangle have lengths of 5 inches and 6 inches. What is the length of the third side?"



You may have drawn many triangles in the past and have become aware of the fact that the length of any side of a triangle must be less than the sum of the lengths of the other two sides. Thus if the third side of this triangle is n inches long,

$$n < 5 + 6.$$

At the same time the six inch side must be less in length than the sum of the lengths of the other two; thus

$$6 < n + 5.$$

Since both of tnese conditions must hold, the open sentence for our problem is

$$n < 5 + 6 \text{ and } 6 < n + 5.$$

Problem Set 4-3b

Write open sentences for problems 1-10, being careful to give the meaning of the variable for each.

1. One third of a number added to three-fourths of the same number is equal to or greater than 26. What is the number?
2. Bill is 5 years older than Norman, and the sum of their ages is less than 23. How old is Norman?
3. A square and an equilateral triangle have equal perimeters. A side of the triangle is five inches longer than a side of the square. What is the length of the side of the square? Draw a figure.
4. A boat, traveling downstream, goes 12 miles per hour faster than the rate of the current. Its velocity downstream is less than 30 miles per hour. What is the rate of the current?
5. John said, "It will take me more than 2 hours to mow the lawn and I must not spend more than 4 hours on the job or I won't be able to go swimming." How much time can he expect to

spend on the job?

6. On a half-hour TV show the advertiser insists there must be at least three minutes for commercials and the network insists there must be less than 12 minutes for commercials. Express this in a mathematical sentence. How much time must the program director provide for material other than advertising?
7. A teacher says, "If I had 3 times as many students in my class as I do have, I would have at least 26 more than I now have." How many students does he have in his class?
8. The amount of $205 is to be divided among Tom, Dick and Harry. Dick is to have $15 more than Harry and Tom is to have twice as much as Dick. How must the money be divided?
9. An amount between $205 and $225, inclusive, is to be divided among three brothers, Tom, Dick and Harry. Dick is to have $15 more than Harry, and Tom is to have twice as much as Dick. How much money can Harry expect?

*10. A student has test grades of 75 and 82. What must he score on a third test to have an average of 88 or higher? If 100 is the highest score possible on the third test, how high an average can he achieve? What is the lowest average he can achieve?

11. Using two variables, write an open sentence for each of the following English sentences.
    (a) The enrollment in Scott School is greater than the enrollment in Morris School.
    (b) The enrollment in Scott School is 500 greater than the enrollment in Morris School.

Review Problems

1. Write English phrases which are realistic translations of the following open phrases. In each problem be careful to identify the variable explicitly.
   (a) $x + 15$
   (b) $3p + 2(p - 1)$
   (c) $60 - 10t$
   (d) $9(15r + 25)$
   (e) $\frac{1}{2}b(3b - 4)$

In problems 2, 3, 4, 5, write open phrases which are translations of the word phrases. In each problem be careful to indicate what the variable represents, if it is not already given. In certain problems you may use more than one variable.

2. (a) A number diminished by 3.
   (b) Sam's age seven years from now.
   (c) Mary's age ten years ago.
   (d) Temperature 20 degrees higher than the present temperature.
   (e) Cost of n pencils at 5 cents each
3. (a) The amount of money in my pocket: x dimes, y nickels, and 6 pennies.
   (b) A number increased by twice the number.
   (c) A number increased by twice another number.
   (d) The number of days in w weeks.
   (e) One million more than twice the population of a city in Kansas.
   (f) Annual salary equivalent to x dollars per month.
4. (a) One dollar more than twice Betty's allowance.
   (b) The distance traveled in h hours at a constant speed of 40 m.p.h.
   (c) The real estate tax on property having a valuation of y dollars, the tax rate being $25.00 per $1000 valuation.
   (d) Forty pounds more than Earl's weight.

(e) Area of a rectangle having one side 3 inches longer than another.

5. (a) Cost of x pounds of steak at \$1.59 per pound.
   (b) Catherine's earnings for z hours at 75 cents an hour.
   (c) Cost of g gallons of gasoline at 33.2 cents a gallon.
   (d) Cost of purchases: x melons at 29 cents each and y pounds of hamburger at 59 cents a pound.
   (e) A two-digit number whose tens' digit is t and whose units' digit is u.

6. Write English sentences which are translations of the open sentences.
   (a) $x < 80$
   (b) $y = 3600$
   (c) $z > 100{,}000{,}000$
   (d) $u + v + w = 180$
   (e) $z(z + 18) = 360$

In problems 7, 8, 9, write open sentences corresponding to the word sentences, using one variable in each. In each problem be careful to state what the variable represents, if this is not already indicated.

7. (a) Mary, who is 16 and has two brothers, is 4 years older than her sister.
   (b) Bill bought b bananas at 9 cents each and paid 54 cents.
   (c) If a number is added to twice the number, the sum is less than 39.
   (d) Arthur's allowance is one dollar more than twice Betty's, but is two dollars less than 3 times Betty's.
   (e) The distance from Dodge City to Oklahoma City, 260 miles, was traveled in t hours at an average speed of 40 miles an hour.
   (f) The auto trip from St. Louis to Memphis, 300 miles, was made in t hours at a maximum speed of 50 miles an hour.

8. (a) Pike's Peak is more than 14,000 feet above sea level.
   (b) A book, 1.4 inches thick, has n sheets; each sheet is

0.003 inches thick, and each cover is $\frac{1}{10}$ inches thick and is blue.

(c) Two million is more than twice the population of any city in Colorado.

(d) A square of side x has a larger area than has a rectangle of sides (x - 1) and (x + 1).

(e) The tax on real estate is calculated at $24.00 per $1000 valuation. The tax assessment on property valued at y dollars is $348.00.

(f) If Earl added 40 pounds to his weight, he would still not weigh more than 152 pounds.

9. (a) The sum of a whole number and its successor is 575.

(b) The sum of a whole number and its successor is 576.

(c) The sum of two numbers, the second greater than the first by 1, is 576.

(d) A board 16 feet long is cut in two pieces such that one piece is one foot longer than twice the other.

(e) Catherine earns $2.25 baby-sitting for 3 hours at x cents an hour.

10. A two-digit number is 7 more than 3 times the sum of the digits. Restate this by an open sentence. (Hint: Express the number by means of two variables, as in Problem 5(e).)

11. The sum of two numbers is 42. If the first number is represented by n, write an expression for the second number using the variable n.

12. (a) A number is increased by 17 and the sum is multiplied by 3. Write an open sentence stating that the resulting product equals 192.

(b) If 17 is added to a number and the sum is multiplied by 3, the resulting product is less than 192. Restate this as an open sentence.

13. One number is 5 times another. The sum of the two numbers is 15 more than 4 times the smaller. Express this by an open sentence.

14. Sue has 16 more books than Sally. Write an open sentence showing that together they have more than 28 books.

15. (a) A farmer can plow a field in 7 hours with one of his tractors. How much of the field can he plow in one hour with that tractor?

    (b) With his other tractor he can plow the field in 5 hours. If he had both tractors going for 2 hours, how much of the field would be plowed?

    (c) How much of the field would then be left unplowed?

    (d) Write an open sentence which indicates that, if both tractors are used for x hours, the field will be completely plowed.

*16. If you fly from New York to Los Angeles, you gain three hours. If the flying time is h hours, when do you have to leave New York in order to arrive in Los Angeles before noon? Write an open sentence for this problem.

17. Mr. Brown is reducing. During each month for the past 8 months he has lost 5 pounds. His weight is now 175 pounds. What was his weight m months ago if $m < 8$? Write an open sentence stating that m months ago his weight was 200 pounds.

Write open sentences for problems 18 to 23. Tell clearly what the variable represents, but do not find the truth set of the open sentence.

18. (a) The sum of a whole number and its successor is 45. What are the numbers?

    (b) The sum of two consecutive odd numbers is 76. What are the numbers?

19. Mr. Barton paid $176 for a freezer which was sold at a discount of 12% of the marked price. What was the marked price?

20. A man's pay check for a week of 48 hours was $166.40. He is paid at the rate of $1\frac{1}{2}$ times his normal rate for all hours worked in excess of 40 hours. What is his hourly pay rate?

*21. A man fires a rifle at a target. Two seconds after he fires it he hears the sound of the bullet striking the target. If the speed of sound is 1100 feet per second and the speed of the bullet is 1700 feet per second, how far away is the target?

22. The students attending Lincoln High School have a habit of cutting across a vacant lot near the school instead of following the side-walk around the corner. The lot is a rectangular lot 200 feet by 300 feet, and the short-cut follows a straight line from one corner of the lot to the opposite corner. How long is the short-cut? (Hint: Use the Pythagorean Theorem.)

23. One end of a 50-foot wire is attached to the top of a vertical telephone pole. The wire is pulled taut and the lower end is attached to a concrete block on the ground. This block is 30 feet from the base of the telephone pole, on level ground. What is the height of the pole?

24. (a) At an auto parking lot, the charge is 35 cents for the first hour, or fraction of an hour, and 20 cents for each succeeding (whole or partial) one-hour period. If t is the number of one-hour periods parked after the initial hour, write an open phrase for the parking fee.

(b) With the same charge for parking as in the preceding problem, if h is the total number of one-hour periods parked, write an open phrase for the parking fee.

*25. Two quarts of alcohol are added to the water in the radiator, and the mixture then contains 20 per cent alcohol; that is, 20 per cent of the mixture is pure alcohol. Write an open sentence for this English sentence. (Hint: Write an open phrase for the number of quarts of alcohol in terms of the number of quarts of water originally in the radiator.)

*26. (a) Two water-pipes are bringing water into a reservoir. One pipe has a capacity of 100 gallons per minute, and the second 40 gallons per minute. If water flows from the first pipe for x minutes and from the second for y minutes, write an open phrase for the total flow in gallons.

(b) In the preceding problem, if the flow from the first pipe is stopped at the end of two hours, write the expression for the total flow in gallons in y minutes, where y is greater than 120.

(c) With the data in part (a), write an open sentence stating that the total flow is 20,000 gallons.

27. (a) Plant A grows two inches each week, and it is now 20 inches tall. Write an open phrase for tne number of inches in its height w weeks from now.

(b) Plant B grows three inches each week, and it is now 12 inches tall. Write an open phrase for the number of inches in its height w weeks from now.

(c) In the course of some weeks, the plants will be equally tall. Express this by means of an open sentence.

*28. A man breathes 20 times per minute at sea level and takes one extra breath per minute for each 1500 feet of ascent.

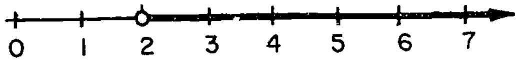
(a) Write an open phrase for the number of breaths he takes each minute h feet above sea level.

(b) At y feet above sea level a man breathes 24 times per minute. With the information obtained in (a), write an open sentence stating this fact.

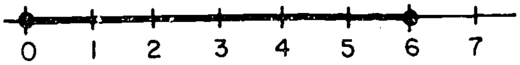
29. Use each of the verb symbols $=$, $\neq$, $<$; $\not<$, $>$, $\not>$, $\leq$, $\geq$, in $x \underline{\hspace{3em}} \frac{5}{2}$, and graph each of the open sentences which you get.

30. Find open sentences whose graphs are the following:

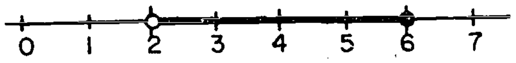
(a)



(b)



(c)



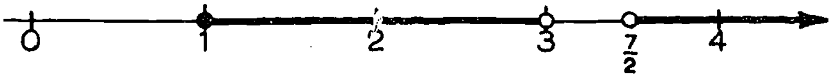
31. Graph the following open sentences:

(a) $x = 2$ or $x > 5$

(b) $x = 2$ and $x > 5$

(c) $x > 2$ or $x < 5$
(d) $x < 2$ and $x < 5$

*32. Find an open sentence whose graph will be:



You will probably need to make a compound sentence using "and" and "or", as well as inequalities.

33. A man, with five dollars in his pocket, stops at a candy store on his way home with the intention of taking his wife two pounds of candy. He finds candy by the pound box selling for \$1.69, \$1.95, \$2.65, and \$3.15. If he leaves the store with two one-pound boxes of candy,

(a) What is the smallest amount of change he could have?
(b) What is the greatest amount of change he could have?
(c) What sets of two boxes can he not afford?

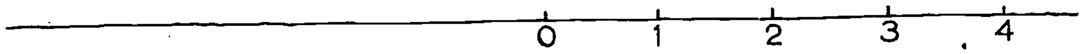
*34. At the end of Chapter 3 (Exercises 3-14, Problem 5) you discovered a "rule for multiplying teens". Using a and b, respectively, to stand for the units digits of the two numbers, you should now be able to write an open sentence which expresses the product p in terms of a and b. When you have written your sentence, use the distributive property to verify the correctness of your choice.

Chapter 5

THE REAL NUMBERS

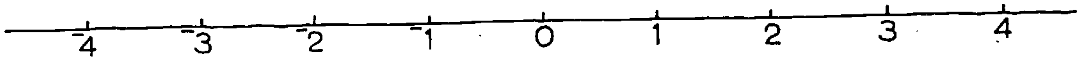
5-1. The Real Number Line

As you worked with the number line, you may have been curious about several things. For one thing, a line extends without end to the left as well as to the right. We have, however, labeled only those points on the right of 0. This raises a question which we shall answer in this section: How shall we label the points on the left?



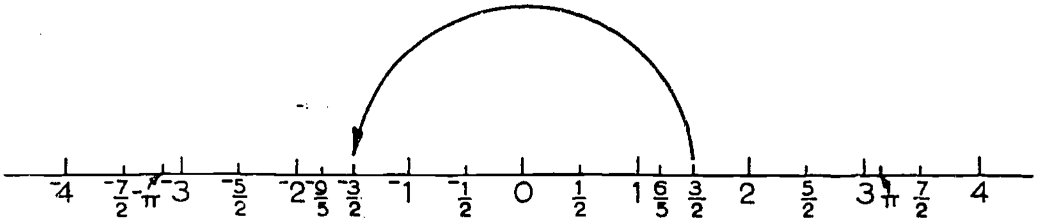
In Chapter 1 you were told that there are rational numbers to be associated with points on the left half of the number line, but meanwhile you have dealt only with rational numbers on the right half. For another thing, you were told that some points on the number line do not correspond to rational numbers. This raises a second question: Where are some of these points on the number line which do not correspond to rational numbers, and what new numbers are associated with them?

Let us return to the first of these questions: How shall we label the points on the left of 0? There is no doubt that the line contains infinitely many points to the left of 0. It is an easy matter to label such points if we follow the pattern we used to the right of 0. As before, we use the interval from 0 to 1 as the unit of measure, and locate points equally spaced along the line *to the left*. The first of these we label



$^-1$, the second $^-2$, etc., where the symbol "$^-1$" is read "<u>negative</u> 1", $^-2$ is read "negative 2", etc. What is the coordinate of the point which is 7 units to the left of 0?

Proceeding as before, we can find additional points to the left of 0 and label them with symbols similar to those used for numbers to the right, with an upper dash to indicate that the number is to the left of 0. Thus, for example, $\frac{^-3}{2}$ is the same distance from 0 on the left as $\frac{3}{2}$ is on the right, etc.



The set of <u>all</u> numbers associated with points on the number line is called the set of <u>real numbers</u>. The numbers to the left of zero are called the <u>negative real numbers</u> and those to the right are called the <u>positive real numbers</u>. In this language, the numbers of arithmetic are the <u>non-negative real numbers</u>.

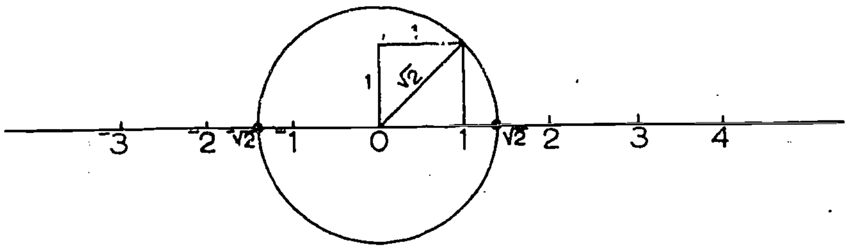
The set of all whole numbers {0, 1, 2, 3, ...} combined with the set {$^-1$, $^-2$, $^-3$, ...} is called the set of <u>integers</u> { ..., $^-3$, $^-2$, $^-1$, 0, 1, 2, 3, ... }. The set of all rational numbers of arithmetic combined with the negative rational numbers is called the set of <u>rational numbers</u>. (Certainly, all rational numbers are real numbers.)

Remember that each rational number is now assigned to a point of the number line, but there remain many points to which rational numbers cannot be assigned. The numbers associated with these points are called the <u>irrational numbers</u>. (Thus, all irrational numbers are also real numbers.) Hence, we can regard the set of real numbers as the combined set of rational and irrational numbers.

For example, all integers, such as $^{-}4$, 0, 2, are rational numbers; find examples of rational numbers which are not integers. Furthermore, all rational numbers such as $\frac{3}{2}$, 0, 6, are real numbers.

The second question remains: Where are some of the points on the number line which do not correspond to rational numbers? It will be proved in a later chapter that, for example, the real number $\sqrt{2}$ is an irrational number. Let us locate the points with coordinates $\sqrt{2}$ and $^{-}\sqrt{2}$, respectively.

First of all, we recall that $\sqrt{2}$ is a number whose square is 2. You may have learned that the length of a diagonal of a square, whose sides have length 1, is a number whose square is 2. (Do you know any facts about right triangles which will help you verify this?) In order to locate a point on the number line for $\sqrt{2}$, all we have to do is construct a square with side of length 1 and transfer the length of one of its diagonals to our number line. This we can do, as in the figure, by drawing



a circle whose center is at the point 0 on the number line and whose radius is the same length as the diagonal of the square. This circle cuts the number line in two points, whose coordinates are the real numbers $\sqrt{2}$ and $^{-}\sqrt{2}$, respectively.

Later you will prove that the number $\sqrt{2}$ is not a rational number. Maybe you believe that $\sqrt{2}$ is 1.4. Test for yourself whether this is true by squaring 1.4. Is $(1.4)^2$ the same number as 2? In the same way, test whether $\sqrt{2}$ is 1.41; 1.414. The square of each of these decimals is closer to 2 than the preceding, but there seems to be no rational number whose square is 2.

There are many more points on the real number line which have coordinates which are not rational numbers. Do you think $\frac{1}{2}\sqrt{2}$ is such a point? $3 + \sqrt{2}$? Why?

Problem Set

1. Draw the graphs of the followi
   - (a) $\{0, 3, {}^-5, {}^-\frac{1}{2}\}$
   - (b) $\{{}^-\frac{2}{3}, \frac{2}{3}, \frac{5}{2}, {}^-\frac{5}{2}\}$
   - (c) $\{{}^-\frac{3}{2}, 5, {}^-7, {}^-\frac{11}{3}\}$
   - (d) $\{{}^-\frac{12}{7}, {}^-\frac{6}{5}, \frac{6}{5}, \frac{7}{12}\}$
   - (e) $\{\sqrt{2}, {}^-\sqrt{2}, 3, {}^-3\}$
   - (f) $\{{}^-1, {}^-(1 + \frac{1}{2}), (1 + \frac{1}{2})\}$
   - (g) $\{{}^-\frac{1}{2}, (\frac{1}{2})^2, {}^-\frac{6}{4}, (3 - 3)\}$

2. Of the two points whose coordinates are given, which is to the right of the other?
   - (a) $3, {}^-4$
   - (b) $5, {}^-4$
   - (c) ${}^-2, {}^-4$
   - (d) ${}^-\sqrt{2}, 1$
   - (e) ${}^-\frac{5}{2}, 0$
   - (f) ${}^-\frac{5}{2}, {}^-\frac{10}{4}$
   - (g) $0, 3$
   - (h) ${}^-4, \sqrt{2}$
   - (i) ${}^-\frac{16}{3}, {}^-\frac{21}{4}$
   - (j) ${}^-\frac{1}{2}, \frac{1}{2}$

3. The number π is the ratio of the circumference of a circle to its diameter. Thus, a circle whose diameter is of length 1 has a circumference of length π. Imagine such a circle resting on the number line at the point 0. If the circle is rolled on the line, without slipping, one complete revolution to the right, it will stop on a point. What is the coordinate of this point? If rolled to the left one revolution it will stop on what point? Can you locate these points approximately

on the real number line? (The real number π, like $\sqrt{2}$, is not a rational number.)

4. (a) Is $^{-}2$ a whole number? An integer? A rational number? A real number?
   (b) Is $^{-}\frac{10}{3}$ a whole number? An integer? A rational number? A real number?
   (c) Is $^{-}\sqrt{2}$ a whole number? An integer? A rational number? A real number?

5. Which of the following sets are the same?
   A is the set of whole numbers, B is the set of positive integers, C is the set of non-negative integers, I is the set of integers, N is the set of counting numbers.

5-2. Order on the Real Number Line

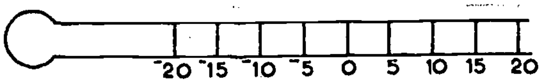
How did we describe order for the positive real numbers? Since, for example, "5 is to the left of 6" on the number line, and since "5 is less than 6", we agreed that these two sentences say the same thing about 5 and 6. We wrote this as the true sentence

$$5 < 6.$$

Thus, for a pair of positive real numbers, "is to the left of" on the number line and "is less than" describe the same order.

What shall we mean by "is less than" for any two real numbers, whether they are positive, negative, or 0? Our answer is simply: "is to the left of" on the real number line.

Let us look for a justification in common experience. All of us are familiar with thermometers and are aware that scales on thermometers use numbers above 0 and numbers below 0, as well as 0 itself. We know that the cooler the weather, the lower on the scale we read the temperature. If we place a thermometer in a horizontal position, we see that it resembles part of our real number line. When we say "is less than" ("is a lower tem-



perature than"), we mean "is to the left of" on the thermometer scale. On this scale, which number is the lesser, $^-5$ or $^-10$?

Thus we extend our former [illegible]ing of "is less than" to the whole set of real numbers. [illegible] ag[illegible]at:

> "is less than" [illegible] real numbers means "is to the left of" on the real number line. If a and b are real numbers, "a is less than b" is written
>
> $$a < b.$$

(Now and in the future a variable is understood to have as its domain the set of real numbers, unless otherwise stated.)

Can you give a meaning for "is greater than" for real numbers? As before, use the symbol ">" for "is greater than". In the same way, explain the meanings of "$\leq$", "$\geq$", "$\nless$", "$\ngtr$" for real numbers.

Problem Set 5-2a

1. For each of the following sentences, determine which are true and which false.

   (a) $3 \leq {}^-1$

   (b) $2 < {}^-\frac{7}{2}$

   (c) $^-4 \nless 3.5$

   (d) $^-\frac{12}{5} < {}^-2.2$

   (e) $^-\frac{3}{5} \leq {}^-\left(\frac{3 + 0}{5}\right)$

   (f) $^-4 \neq 3.5$

   (g) $^-6 > {}^-3$

   (h) $3.5 < {}^-4$

   (i) $^-3 < {}^-2.8$

   (j) $^-\pi \nless {}^-2.8$

2. Consider the following pairs of real numbers and tell which of the sentences below each pair are true and which are false. For example, for the pair

$$^-2 \text{ and } \frac{3 \times 4}{2}:$$

$$^-2 < \frac{3 \times 4}{2} \text{ is true;}$$

$$^-2 = \frac{3 \times 4}{2} \text{ is false;}$$

$$^-2 > \frac{3 \times 4}{2} \text{ is false.}$$

(a) $^-3.14$ ar

$^-3.14 < {}^-3$

$^-3.14 = {}^-3$

$^-3.14 > {}^-3$

2 and $^-2$:

$2 < {}^-2$

$2 = {}^-2$

$2 > {}^-2$

(c) $\frac{5 + 3}{2}$ and 2×2:

$\frac{5 + 3}{2} < 2 \times 2$

$\frac{5 + 3}{2} = 2 \times 2$

$\frac{5 + 3}{2} > 2 \times 2$

(d) $^-0.001$ and $^-\left(\frac{1}{1000}\right)$:

$^-0.001 < {}^-\left(\frac{1}{1000}\right)$

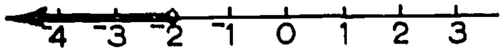
$^-0.001 = {}^-\left(\frac{1}{1000}\right)$

$^-0.001 > {}^-\left(\frac{1}{1000}\right)$

3. Draw the graph of the truth set of each of the following sentences. For example:

$x > {}^-1$,

$x \leq {}^-2$,



(a) $y < 2$

(b) $u \nless 3$

(c) $v \geq \frac{^-3}{2}$

*(d) $r \ngtr {}^-2$

(e) $x = 3$ or $x < {}^-1$

(f) $c < 2$ and $c > {}^-2$

(g) $a \leq {}^-3$ and $a \geq {}^-3$

(h) $d \leq {}^-1$ or $d > 2$

(i) $a < 6$ and $a < {}^-2$

(j) $u > 2$ and $u < {}^-3$

4. For each of the following sets, write an open sentence involving the variable x which has the given set as its truth set:

   (a) A is the set of all real numbers not equal to 3.

   (b) B is the set of all real numbers less than or equal to $^{-}2$.

   (c) C is the set of all real numbers not less than $^{-}\frac{5}{2}$.

5. Choose any positive real number p; choose any negative real number n. Which, if any, of the following sentences are true?

$$n < p, \quad p < n, \quad n \leq p, \quad n \neq p.$$

6. Let the domain of the variable p be the set of integers. Then find the truth set of

   (a) $^{-}2 < p$ and $p < 3$.

   (b) $p \leq {}^{-}2$ and $^{-}4 < p$.

   (c) $p = 2$ or $p = {}^{-}5$.

7. (a) During a cold day the temperature rises 10 degrees from $^{-}5^{\circ}$. What is the final temperature?

   (b) On another day the temperature rises 5 degrees from $^{-}10^{\circ}$. How high does it go?

   (c) During a January thaw the temperature rises from $^{-}15^{\circ}$ to 35°. How much did it rise?

8. In the blanks below use one of $=$, $<$, $>$ to make a true sentence, if possible, in each case.

   (a) $\frac{3}{5}$ _____ $\frac{6}{10}$

   (b) $\frac{3}{5}$ _____ $\frac{3}{6}$

   (c) $\frac{9}{12}$ _____ $\frac{8}{12}$

   (d) $^{-}\frac{173}{29}$ _____ $^{-}\frac{183}{29}$

   (e) $^{-}\frac{3}{5}$ _____ $\frac{3}{6}$

   (f) $^{-}\frac{3}{5}$ _____ $^{-}\frac{3}{6}$

There are certain simple but highly important facts about the order of the real numbers on the real number line. If we choose any two different real numbers, we are <u>sure</u> that the first is less than the second or the second is less than the first, but not both. Stated in the language of algebra, this property of order for real numbers becomes the <u>comparison property</u>:

> If a is a real number and b is a real number, then exactly one of the following is true:
>
> $a < b, \; a = b, \; b < a.$

Problem Set 5-2b

1. For each of the following pairs of numbers verify that the comparison property is true by determining which one of the three possibilities actually holds between the numbers:

   (a) $^-2$ and $^-1.6$

   (b) 0 and $^-2$

   (c) $\frac{2 \times 3 \times 4}{5}$ and $^-(\frac{2 \times 3 \times 4}{5})$

   (d) $^-16$ and $\frac{^-32}{2}$

   (e) 12 and $(5 + 2)(\frac{1}{7} \times \frac{36}{3})$

   (f) $^-2$ and 2

2. Make up true sentences, using $<$, involving the following pairs:

   (a) $2, \; ^-3$

   (b) $\frac{4}{2}, \; \frac{^-5}{2}$

   (c) $\frac{^-4}{5}, \; \frac{^-6}{5}$

   (d) $\frac{4}{5}, \; \frac{11}{10}$

   (e) $\frac{^-4}{50}, \; \frac{11}{100}$

   (f) $\frac{103}{13}, \; \frac{205}{26}$

   (g) $\frac{13}{15}, \; \frac{2}{3}$

   (h) $\frac{12}{119}, \; ^-(\frac{25}{238})$

   (i) $^-\sqrt{2}, \; ^-1.5$

   (j) $\sqrt{2} + \pi, \; 1.5 + 3$

3. The comparison property stated in the text is a statement involving "<". Try to formulate the corresponding property involving ">", and test it with the pairs of numbers in problem 1.

4. Try to state a comparison property involving "$\geq$".

Which is less than the other, $\frac{4}{5}$ or $\frac{5}{6}$? You can find out by applying the multiplication property of 1 to each number to get $\frac{4}{5} = \frac{4}{5} \times \frac{6}{6} = \frac{24}{30}$ and $\frac{5}{6} = \frac{5}{6} \times \frac{5}{5} = \frac{25}{30}$. Then $\frac{4}{5} < \frac{5}{6}$, because $\frac{24}{30}$ is to the left of $\frac{25}{30}$ on the number line.

You should now be able to compare any two rational numbers. How would you decide which is the lesser, $\frac{337}{113}$ or $\frac{167}{55}$? (Describe the process; do not actually carry it out.)

Perhaps you noticed, in comparing $\frac{337}{113}$ and $\frac{167}{55}$, that $\frac{337}{113} < 3$ (i.e., $\frac{337}{113} < \frac{339}{113}$) and $3 < \frac{167}{55}$ (i.e., $\frac{165}{55} < \frac{167}{55}$). Could you now decide about the order of $\frac{337}{113}$ and $\frac{167}{55}$ without writing them as fractions with the same denominator? How could you find out similarly which is lesser, $\frac{40}{27}$ or $\frac{\pi}{2}$? Or suppose that x and y are real numbers and that $x < {}^{-}1$ and ${}^{-}1 < y$. Again using the number line, what can you say about the order of x and y?

The property of order used in these last three examples we call the transitive* property:

If a, b, c are real numbers
and if $a < b$ and $b < c$,
then $a < c$.

Problem Set 5-2c

1. In each of the following groups of three real numbers, determine their order:

For example, $\frac{3}{4}, \frac{3}{2}, {}^{-}\frac{4}{5}$ have the order: ${}^{-}\frac{4}{5} < \frac{3}{4}$, $\frac{3}{4} < \frac{3}{2}$, ${}^{-}\frac{4}{5} < \frac{3}{2}$.

Footnote

*From ~~the~~ Latin, transire, to go across.

(a) $^{-}\frac{1}{5}$, $\frac{3}{2}$, and 12,

(b) π, $^{-}\pi$, and $^{-}\sqrt{2}$,

(c) 1.7, 0, and $^{-}1.7$,

(d) $^{-}\left(\frac{27}{15}\right)$, $^{-}\left(\frac{3}{15}\right)$, and $^{-}\left(\frac{2}{15}\right)$,

(e) $\frac{12 \times (\frac{1}{2} + \frac{1}{3})}{3}$, $^{-}\frac{5}{3}$, and $\frac{6}{3}$,

(f) $\frac{3 \times (27 + 6)}{9}$, $\frac{(2 \times 3) + (7 \times 9)}{6}$, and $\frac{(22 \times 3) \times \frac{1}{3}}{2}$,

(g) 3^2, 4^2, $(3 + 4)^2$,

(h) $^{-}\frac{1}{2}$, $^{-}\frac{1}{3}$, $^{-}\frac{1}{4}$,

(i) $1 + \frac{1}{2}$, $1 + (\frac{1}{2})^2$, $(1 + \frac{1}{2})^2$.

2. State a transitive property for ">", and illustrate this property in two of the exercises in Problem 1.

3. Art and Bob are seated on opposite ends of a see-saw (teeter-totter), and Art's end of the see-saw comes slowly to the ground. Cal gets on and Art gets off, after which Bob's end of the see-saw comes to the ground. Who is heavier, Art or Cal?

4. Is there a transitive property for the relation "="? If so, give an example.

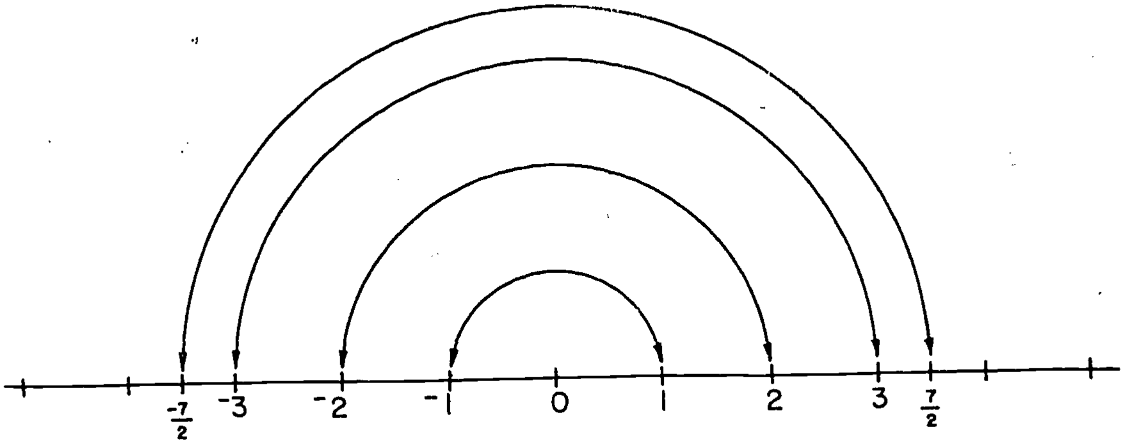
5. State a transitive property for "≥", and give an example.

6. The set of numbers greater than 0 we have called the <u>positive real numbers</u>, and the set of numbers less than 0 the <u>negative real numbers</u>. Describe the
   (a) non-positive real numbers,
   (b) non-negative real numbers.

7. Find the order of each of the following pairs of numbers:
   (a) $^{-}\frac{15}{8}$ and $^{-}\frac{25}{12}$
   (b) $^{-}\frac{17}{35}$ and $^{-}\frac{7}{13}$
   (c) $^{-}\frac{145}{28}$ and $^{-}\frac{104}{21}$
   (d) $^{-}\frac{192}{46}$ and $^{-}\frac{173}{44}$

5-3. Opposites

When we labeled points to the left of 0 on the real number line, we began by marking off successive unit lengths to the left of 0. We can also think, however, of pairing off points at equal distances from 0 and on opposite sides of 0. Thus, $^{-}2$ is at the same distance from 0 as 2. What number is at the same distance from 0 as $\frac{7}{2}$? If you choose any point on the number line, can you find a point at the same distance from 0 and on the opposite side? What about the point 0 itself?



Since the two numbers in such a pair are on opposite sides of 0, it is natural to call them <u>opposites</u>. The opposite of a non-zero real number is the other real number which is at an equal distance from 0 on the real number line. What is the opposite of 0?

Let us consider some typical real numbers. Write them in a column. Then write their opposites in another column; then study the adjacent statements.

2,	$^{-}2$;	$^{-}2$ is the opposite of 2.
$^{-}\frac{1}{2}$,	$\frac{1}{2}$;	$\frac{1}{2}$ is the opposite of $^{-}\frac{1}{2}$.
0,	0;	0 is the opposite of 0.

The statements themselves [illegible] cumbersome to write, and we need a symbol meaning "the [illegible] of". Let us use the [illegible] dash "-" to mean "the opposite of [illegible] this symbol the three statements become the true sentences:

$$^{-}2 = -2$$

$$\frac{1}{2} = -\,^{-}\frac{1}{2}$$

$$0 = -0.$$ (Read these sentences carefully.)

We can learn two things from these sentences. First, it appears that "$^{-}2$" and "-2" are different names for the same number. That is, "negative 2" and "the opposite of 2" represent the same number. Hence, it makes no difference at what height the dash is drawn, since the meaning is the same for the upper and lower dash. This being the case, we do not need both symbols.

Which shall we retain? The upper dash refers only to negative numbers, whereas the lower dash may apply to *any* real number. (Note that the opposite of the positive number 2 is the negative number $^{-}2$, and the opposite of the negative number $^{-}\frac{1}{2}$ is the positive number $\frac{1}{2}$.) Hence it is natural to retain the "opposite of" symbol to mean either "negative" or "opposite of" when the number in question is *positive*. Now the sentences may be written

$-2 = -2$, (read "negative 2 is the opposite of 2")

$\frac{1}{2} = -(-\frac{1}{2})$, (read "$\frac{1}{2}$ is the opposite of negative $\frac{1}{2}$")

$0 = -0$

The second of these sentences can be read also as:

$\frac{1}{2}$ is the opposite of the opposite of $\frac{1}{2}$.

Second, we observe in general that the *opposite of the opposite* of a number *is the number itself*; in symbols:

For every real number y,

$$-(-y) = y.$$

What is the opposite of the opposite of the opposite of a number? What is the opposite of the opposite of a negative number?

When we attach the dash "-" to a variable x we are performing on x the operation of "determining the opposite of x". Do not confuse this with the binary operation of subtraction, which is performed on two numbers, such as $3 - x$, meaning "x subtracted from 3." What kind of number is $-x$ if x is a positive number? If x is a negative number? If x is 0?

We shall read "$-x$" as the "opposite of x". Thus, if x is a number to the right of 0 (positive), then $-x$ is to the left (negative); if x is to the left of 0 (negative), then $-x$ is to the right (positive).

Problem Set 5-3a

1. Form the opposite of each of the following numbers:

   (a) 2.3 (c) $-(-2.3)$ (e) $-(42 \times 0)$

   (b) -2.3 (d) $-(3.6 - 2.4)$ (f) $-(42 + 0)$

2. What kind of number is $-x$ if x is positive? if x is negative? if x is zero?

3. What kind of number is x if $-x$ is a positive number? if $-x$ is a negative number? if $-x$ is 0?

4. (a) Is every real number the opposite of some real number?
   (b) Is the set of all opposites of real numbers the same as the set of all real numbers?
   (c) Is the set of all negative numbers a subset of the set of all opposites of real numbers?
   (d) Is the set of all opposites of real numbers a subset of the set of all negative numbers?
   (e) Is every opposite of a number a negative number?

The ordering of numbers on the real number line specifies that $-\frac{1}{2}$ is less than 2. Is the opposite of $-\frac{1}{2}$ less than the opposite of 2? Make up other similar examples of pairs of numbers. After you have determined the ordering of a pair, then find the ordering of their opposites. You will see that there is a general property for opposites:

For real numbers a and b,
if $a < b$, then $-b < -a$.

Problem Set 5-3b

1. For each of the following pairs, determine which is the greater number:

   (a) 2.97, -2.97
   (b) -12, 2
   (c) -358, -762
   (d) -1, 1
   (e) -370, -121
   (f) 0.12, 0.24
   (g) 0, -0
   (h) -0.1, -0.01
   (i) 0.1, 0.01

2. Write true sentences for the following numbers and their opposites, using the relations "<" or ">".
   Example: For the numbers 2 and 7, $2 < 7$ and $-2 > -7$.

   (a) $\frac{2}{7}, -\frac{1}{6}$
   (b) $\sqrt{2}, -\pi$
   (c) $\pi, \frac{22}{7}$
   (d) $3(\frac{4}{3} + 2), \frac{5}{4}(20 + 8)$
   (e) $-(\frac{8 + 6}{7}), -2$
   (f) $-((3 + 17)0), -((5 + 0)3)$

3. Graph the truth sets of the following open sentences:
   (Hint: In parts (c) and (d) use order property of opposites before graphing.)

   (a) $x > 3$
   (b) $x > -3$
   (c) $-x > 3$
   (d) $-x > -3$

4. Describe the truth set of each open sentence:

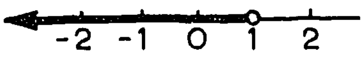
(a) $-x \neq 3$
(b) $-x \neq -3$
(c) $x < 0$
(d) $-x < 0$
(e) $-x \geq 0$
(f) $-x \leq 0$

5. For the following sets, give two open sentences each of whose truth sets is the given set (use opposites when convenient):

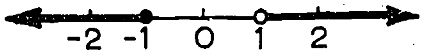
(a) A is the set of all non-negative real numbers.
(b) B is the set of all real numbers not equal to -2.
(c) C is the set of all real numbers not greater than -3.
(d) $\emptyset$.
(e) E is the set of all real numbers.

6. Write an open sentence for each of the following graphs:

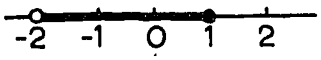
(a)



(c)



(b)



(d)

7. For each of the following numbers write its opposite, and then choose the greater of the number and its opposite:

(a) 3
(b) 0
(c) 17
(d) -7.2
(e) $-\sqrt{2}$
(f) -0.01
(g) $-(-2)$
(h) $(1 - \frac{1}{4})^2$
(i) $1 - (\frac{1}{4})^2$
(j) $-(\frac{1}{2} - \frac{1}{3})$

*8. Let us write "$\succ$" for the phrase "is further from 0 than" on the real number line. Does "$\succ$" have the comparison property enjoyed by "$>$", that is, if a and b are different real numbers, is it true that $a \succ b$ or $b \succ a$ but not both? Does "$\succ$" have a transitive property? For which subset of the set of real numbers do "$\succ$" and "$>$" have the same meaning?

9. Translate the following English sentences into open sentences, describing the variable used:
   - (a) John's score is greater than negative 100. What is his score?
   - (b) I know that I don't have any money, but I am no more than \$200 in debt. What is my financial condition?
   - (c) Paul has paid \$10 of his bill, but still owes more than \$25. What was the amount of Paul's bill?

10. Change the numerals "$-\frac{13}{42}$" and "$-\frac{15}{49}$" to forms with the same denominators. (Hint: First do this for $\frac{13}{42}$ and $\frac{15}{49}$.) What is the order of $-\frac{13}{42}$ and $-\frac{15}{49}$? (Hint: Knowing the order of $\frac{13}{42}$ and $\frac{15}{49}$, what is the order of their opposites?) Now state a general rule for determining the order of two negative rational numbers.

5-4. Absolute Value

We now want to define a new and very useful operation on a single real number: the operation of taking its absolute value.

> The absolute value of a non-zero real number is the greater of that number and its opposite. The absolute value of 0 is 0.

The absolute value of 4 is 4, because the greater of 4 and -4 is 4. The absolute value of $-\frac{3}{2}$ is $\frac{3}{2}$. (Why?) What is the absolute value of -17? Which is always the greater of a non-zero number and its opposite: the positive or the negative number? Explain why the absolute value of any real number is a positive number or 0.

As usual, we agree on a symbol to indicate the operation. We write

$$|n|$$

to mean the absolute value of the number n. For example,

$$|4| = 4, \quad \left|-\frac{3}{2}\right| = \frac{3}{2}, \quad |-\sqrt{2}| = \sqrt{2}, \quad |12| = 12.$$

Note that each of these is non-negative.

If you look at these numbers and their absolute values on the number line, what can you conclude about the distance between a number and 0? You notice that the distance between 4 and 0 is 4; between $-\frac{3}{2}$ and 0 is $\frac{3}{2}$, etc. Notice that the distance between any two points of the number line is a non-negative real number.

> The distance between a real number and 0 on the real number line is the absolute value of that number.

Problem Set 5-4a

1. Find the absolute values of the following numbers:

   (a) -7 (c) $(6 - 4)$ (e) $-(14 + 0)$

   (b) $-(-3)$ (d) 14×0 (f) $-(-(-3))$

2. (a) What kind of number is $\frac{4}{3}$; what kind of number is $\left|\frac{4}{3}\right|$? (Non-negative or negative?)

   If x is a non-negative real number, what kind of number is $|x|$?

   (b) What kind of number is $-\frac{2}{5}$; what kind of number is $\left|-\frac{2}{5}\right|$? (Non-negative or negative?)

   If x is a negative real number, what kind of number is $|x|$?

   (c) Is $|x|$ a non-negative number for every x? Explain.

3. For a negative number x, which is greater, x or $|x|$?

4. Is the set $\{-1, -2, 1, 2\}$ closed under the operation of taking absolute values of its elements?

We note that for a non-negative number, the greater of the number and its opposite is the number itself. That is:

For every real number x
which is 0 or <u>positive</u>,
$|x| = x$.

What can be said of a negative number and its absolute value? Write common numerals for the following pairs for yourself.

$|-5| =$ $\qquad$ $|-3.1| =$

$-(-5) =$ $\qquad$ $-(-3.1) =$

$\left|-\frac{1}{2}\right| =$ $\qquad$ $|-467| =$

$-(-\frac{1}{2}) =$ $\qquad$ $-(-467) =$

(What kind of numbers are -5, $-\frac{1}{2}$, -3.1, -467?) You found that

$$|-5| = -(-5), \qquad |-3.1| = -(-3.1),$$
$$\left|-\frac{1}{2}\right| = -(-\frac{1}{2}), \qquad |-467| = -(-467).$$

Is it now clear that we may say, "The absolute value of a <u>negative</u> number is the <u>opposite</u> of the negative number"? That is:

For every <u>negative</u> real number x,
$|x| = -x$.

Problem Set 5-4b

1. Which of the following sentences are true?

(a) $|-7| < 3$

(b) $|-2| \leq |-3|$

(c) $|4| < |1|$

(d) $2 \not< |-3|$

(e) $|-5| \not< |2|$

(f) $-3 < 17$

(g) $-2 < |-3|$

(h) $|\sqrt{16}| > |-4|$

(i) $|-2|^2 = 4$

2. Write each as a common numeral:

(a) $|2| + |3|$

(b) $|-2| + |3|$

(c) $-(|2| + |3|)$

(d) $-(|-2| + |3|)$

(e) $|-7| - (7 - 5)$

(f) $7 - |-3|$

(g) $|-5| \times 2$

(h) $-(|-5| - 2)$

(i) $|-3| - |2|$

(j) $|-2| + |-3|$

(k) $-(|-3| - 2)$

(l) $-(|-2| + |-3|)$

(m) $3 - |3 - 2|$

(n) $-(|-7| - 6)$

(o) $|-5| \times |-2|$

(p) $-(|-2| \times 5)$

(q) $-(|-5| \times |-2|)$

3. What is the truth set of each open sentence?

(a) $|x| = 1$, (b) $|x| = 3$, (c) $|x| + 1 = 4$, *(d) $5 - |x| = 2$.

4. Which of the following open sentences are true for all real numbers x?

(a) $|x| \geq 0$, (b) $x \leq |x|$, (c) $-x \leq |x|$, (d) $-|x| \leq x$.

(Hint: Give x a positive value; then give x a negative value. Now come to a decision.)

5. Describe the variables used and translate into an open sentence: John has less money than I, and I have less than $20. How much money does John have?

6. Graph the truth sets of the following sentences:

   (a) $|x| < 2$

   (b) $x > -2$ and $x < 2$

   (c) $|x| > 2$

   (d) $x < -2$ or $x > 2$

7. Compare the graphs of the sentences in problems 6(a) and (b). In 6(c) and (d).

8. Show that if x is a negative real number, then x is the opposite of the absolute value of x; that is, if $x < 0$, then $x = -|x|$. (Hint: What is the opposite of the opposite of a number?)

9. Graph the set of integers less than 5 whose absolute values are greater than 2. Is -5 an element of this set? Is 0 an element of this set? Is -10 an element of this set?

10. If R is the set of all real numbers, P the set of all positive real numbers, and I the set of all integers, write three numbers:

    (a) in P but not in I,

    (b) in R but not in P,

    (c) in R but not in P nor in I,

    (d) in P but not in R.

11. Translate into an open sentence: The temperature today remained within 5 degrees of 0.

12. Compare the truth sets of the two sentences

$$|x| = 0, \qquad |x| = -1.$$

5-5. Summary

1. Points to the left of 0 on the number line are labeled with negative numbers, the set of real numbers consists of all numbers of arithmetic and their opposites.

2. Many points on the number line are not assigned rational number coordinates. These points are labeled with irrational numbers. The set of real numbers consists of all rational and irrational numbers.

3. "Is less than" for real numbers means "to the left of" on the number line.

4. Comparison Property. If a is a real number and b is a real number, then exactly one of the following is true: $a < b$, $a = b$, $b < a$.

5. Transitive Property. If a, b, c are real numbers and if $a < b$ and $b < c$, then $a < c$.

6. The opposite of 0 is 0 and the opposite of any other real number is the other number which is at an equal distance from 0 on the real number line.

7. The absolute value of 0 is 0, and the absolute value of any other real number n is the greater of n and the opposite of n.

8. If x is a positive number, then $-x$ is a negative number. If x is a negative number, then $-x$ is a positive number.

9. The absolute value of the real number n is denoted by $|n|$. Also, $|n|$ is a non-negative number which is the distance between 0 and n on the number line.

10. If $n \geq 0$, then $|n| = n$;
    if $n < 0$, then $|n| = -n$.

Review Problems

1. Which of the following sentences are true?

(a) $-2 < -5$

(b) $-(5 - 3) = -(2)$

(c) $-(5 - 3) < -|-2|$

(d) $-\frac{1}{2} \geq -(\frac{6 + 1}{14})$

(e) $-2 < |-5|$

(f) $|-2| < |-5|$

2. Which of the following sentences are false:

(a) $-(-3) + 5 > -(-2) + 6$

(b) $3 \neq -3$ or $3 > -3$

(c) $3 \neq -3$ and $3 > -3$

(d) $\frac{2}{3} < \frac{3}{4}$ and $-\frac{4}{5} < -\frac{5}{6}$

(e) $|\frac{3}{4}| < -(-|-1|)$

(f) $-(|-3| \cdot 2) > -(|-5| - 3)$

3. Draw the graph of each of the following sentences:

(a) $v \geq 1$ and $v < 3$

(b) $|r| = 2$

(c) $x < 4$ or $x > 2$

(d) $|x| = x$

4. Describe the truth set of each sentence.

(a) $y \leq 3$ and $y > 4$

(b) $-|u| < 2$

(c) $-3 < x < 2$

(d) $|x| = -x$

(e) $|y| < -2$

(f) $|v| \geq 0$

5. Consider the open sentence "$|x| < 3$". Draw the graph of its truth set if the domain of x is the set of:

(a) real numbers

(b) integers

(c) non-negative real numbers

(d) negative integers

6. Describe the variables used and translate each of the following into an open sentence:

(a) On Thursday the average temperature was 4° lower than on Friday, and on Friday it was below -10°. What was the average temperature on Thursday?

(b) On Sunday the average temperature remained within 6° of -5°.

7. If R is the set of all real numbers, P the set of all positive real numbers, F the set of all rational numbers, I the set of all integers, which of the following are true statements?
   (a) F is a subset of R.
   (b) Every element of I is an element of F.
   (c) There are elements of I which are not elements of R.
   (d) Every element of I is an element of P.
   (e) There are elements of R which are not elements of F.

8. Draw the graph of the set of integers less than 6 whose absolute values are greater than 3. Is -8 an element of this set?

9. When a certain integer and its successor are added, the result is the successor itself.
   (a) Write an open sentence translating the English sentence above.
   (b) Find the truth set of this sentence.

10. The perimeter of a square is less than 10 inches.
   (a) What do you know about the number of units, s, in the side of this square. Graph this set.
   (b) What do you know about the number of units, A, in the area of this square. Graph this set.

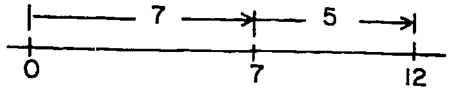
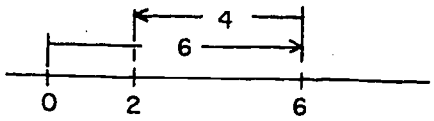
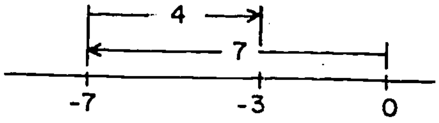
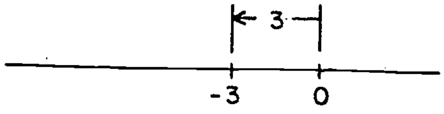
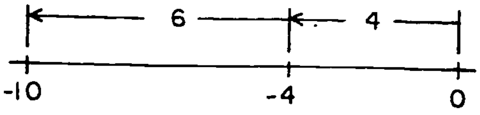
Chapter 6

PROPERTIES OF ADDITION

6-1. Addition of Real Numbers

Ever since first grade you have been adding numbers, the non-negative numbers of arithmetic. Now we are dealing with a larger set of numbers, the real numbers. Our long experience with adding non-negative numbers, both in arithmetic and on the number line, should quickly give us a clue as to how we add real numbers.

Let us consider the profits and losses of an imaginary ice cream vendor during his 10 days in business. We use positive numbers to represent profits and negative numbers for losses. Let us describe his business activity, and then make two columns, one for the arithmetic of computing his net income over two-day periods, and the other for picturing the same on the number line.

Business	Arithmetic	Number Line
Mon.: Profit of \$7 Tue.: Profit of \$5	$7 + 5 = 12$, net income	
Wed.: Profit of \$6 Thu.: Loss of \$4 (Tire trouble)	$6 + (-4) = 2$, net income	
Fri.: Loss of \$7 (another tire) Sat.: Profit of \$4	$(-7) + 4 = -3$	
Sun.: Day of rest Mon.: Loss of \$3 (cold day)	$0 + (-3) = -3$	
Tue.: Loss of \$4 (colder) Wed.: Loss of \$6 (gave up)	$(-4) + (-6) = -10$	

These accounts illustrate almost every possible sum of real numbers: a positive plus a positive, a positive plus a negative, a negative plus a positive, 0 plus a negative, and a negative plus a negative.

One of the purposes of this section will be to learn how to translate into the language of algebra operations which we first describe geometrically. Addition on the number line is such an operation; we shall try to define it in the language of algebra.

If we take $7 + 5$ and picture this addition on the number line, we first go from 0 to 7, and then from 7 we move 5 more units to the <u>right</u>. If we consider $(-7) + 4$, we first go from 0 to (-7), and then from (-7) move 4 units to the right. These examples remind us of something we already know: To add a <u>positive</u> number, we move to the <u>right</u> on the number line. It should now be clear from our other examples above what happens on the number line when we add a <u>negative</u> number. When we added (-4), we moved 4 units to the <u>left</u>; when we added (-6), we moved 6 units to the left. We have one more case to consider: If we add 0, what motion, if any, results?

We have now described the motion in all cases; let us see if we can learn to say <u>algebraically</u> how far we move. Forget for the moment the direction; we just want to know <u>how far</u> we go when we go from a to $a + b$. When b is positive we go to the right. Yes, but how far? We go just b units. When b is negative, we go to the left. How far? We go $(-b)$ units. (Remember $(-b)$ is positive if b is negative.) If b is 0, we don't go at all. What symbol do we know which means "b if b is positive, $-b$ if b is negative, and 0 if b is 0"? "$|b|$", of course. And so we have learned that to find $a + b$ on the number line, we start from a and move the distance $|b|$

to the right, if b is positive;
to the left, if b is negative.

Problem Set 6-1

1. Do the following problems, using positive and negative numbers:
   (a) A football team lost 6 yards on the first play and gained 8 yards the second play. What was the net yardage on the two plays?
   (b) John paid Jim the 60¢ he owed, but John collected the 50¢ Al owed him. What is the net result of John's two transactions?
   (c) If a thermometer registers -15 degrees and the temperature rises 10 degrees, what does the thermometer then register? What if the temperature had risen 30 degrees instead?
   (d) Miss Jones lost 6 pounds during the first week of her dieting, lost 3 pounds the second week, gained 4 pounds the third week, gained 5 pounds the last week. What was her net gain or loss?
2. Perform the indicated operations on real numbers, using the number line to aid you:
   (a) $\left(4 + (-6)\right) + (-4)$
   (b) $4 + \left((-6) + (-4)\right)$
   (c) $-\left(4 + (-6)\right)$
   (d) $3 + \left((-2) + 2\right)$
   (e) $2 + \left(0 + (-2)\right)$
   (f) $((-3) + 0) + (-2.5)$
   (g) $|-2| + (-2)$
   (h) $(-3) + (|-3| + 5)$
3. Tell in your own words what you do to the two given numbers to find their sum:
   (a) $7 + 10$
   (b) $7 + (-10)$
   (c) $10 + (-7)$
   (d) $(-10) + (-7)$
   (e) $10 + 7$
   (f) $(-7) + (-10)$
   (g) $(-7) + 10$
   (h) $(-10) + 7$
   (i) $(-10) + 0$
   (j) $0 + 7$
4. In which parts of Problem 3 did you do the addition just as you added numbers in arithmetic?
5. What could you always say about the sum when both numbers were negative?

6-2. Definition of Addition.

We now want to use what we have just learned about addition on the number line to say first in English and then in the language of algebra, what we mean by $a + b$ for all real numbers a and b. The accounts of the ice cream vendor have introduced us to a number of cases. First of all, we know from previous experience how to add a and b if both are non-negative numbers. So let us consider another example, namely, a negative plus a negative. What is

$$(-4) + (-6)?$$

We have found, on the number line, that

$$(-4) + (-6) = (-10).$$

Our present job is to think a bit more carefully about just how we reached (-10). We begin by moving from 0 to (-4). Where is (-4) on the number line? It is to the left of 0. How far? "Distance between a number and 0" was one of the meanings of the absolute value of a number. Thus the distance between 0 and (-4) is $|-4|$. (Of course we realize that it is easier to write 4 than $|-4|$, but the expression $|-4|$ reminds us that we were thinking of "distance from 0", and this is worth remembering at present.)

(-4) is thus $|-4|$ to the left of 0. When we now consider

$$(-4) + (-6),$$

we move another $|-6|$ to the left. Where are we now? At

$$-(|-4| + |-6|).$$

Thus our thinking about distance from 0 and about distance moved on the number line has led us to recognize that

$$(-4) + (-6) = -(|-4| + |-6|)$$

is a true sentence.

You can reasonably ask at this point what we have accomplished by all this. We have taken a simple expression like $(-4) + (-6)$, and made it look more complicated! Yes, but the expression $-(|-4| + |-6|)$, complicated as it looks, has one great advantage. It contains only operations which we know how to do from previous experience! Both $|-4|$ and $|-6|$ are positive numbers, you see, and we know how to add positive numbers; and $-(|-4| + |-6|)$ is the opposite of a number, and we know how to find that. Thus we have succeeded in expressing the sum of two negative numbers for which sum we previously had just a picture on the number line, in terms of the language of algebra as we have built it up thus far.

Think through $(-2) + (-3)$ for yourself, and see that by the same reasoning you arrive at the true sentence

$$(-2) + (-3) = -(|-2| + |-3|).$$

From these examples we see that the following defines the sum of two negative numbers in terms of operations which we already know how to do:

In English: The sum of two negative numbers is negative; the absolute value of this sum is the sum of the absolute values of the numbers.

In the language of algebra:

If a and b are both negative numbers, then

$$a + b = -(|a| + |b|).$$

Problem Set 6-2a

1. Use the definition above to find a common name for each of the following indicated sums, and then check by using intuition concerning gains and losses, or by using the number line.

   Example: by definition

$$\begin{aligned}(-2) + (-3) &= -(|-2| + |-3|)\\ &= -(2 + 3)\\ &= -5\end{aligned}$$

   Check: A loss of \$2 followed by a loss of \$3 is a net loss of \$5.

   (a) $(-2) + (-7)$
   (b) $(-4.6) + (-1.6)$
   (c) $(-3\frac{1}{3}) + (-2\frac{2}{3})$
   (d) $(-25) + (-73)$
   (e) $5\frac{1}{2} + 2\frac{1}{2}$

2. Find a common name for each of the following by any method you choose:

   (a) $(-6) + (-7)$
   (b) $(-7) + (-6)$
   (c) $-(|-7| + |-6|)$
   (d) $6 + (-4)$

(e) $(-4) + 6$

(f) $|6| - |-4|$

(g) $0 + (-3)$

(h) $-(|-3| - |0|)$

(i) $3 + ((-2) + 2)$

3. Find the truth set of each of the following sentences:
   (a) $(-5) + (x) = -(|-5| + |-3|)$
   (b) $x + (-5) = -(|-5| + |-3|)$
   (c) $(-5) + (x) = -(|-3| + |-5|)$
   (d) $x + (-5) = -(|-3| + |-5|)$
4. Think again of Problem 3 in Problem Set 6-1. When one number is positive and one is negative, how far is their sum from 0?
5. When one number is positive and the other is negative, how do you know whether the sum is positive or negative?

*6. Is the following a true sentence for all non-negative values of x?

$$(-1) + (-x) = -(|-1| + |-x|).$$

So far, we have considered the sum of two non-negative numbers, and the sum of two negative numbers. Next we consider the sum of two numbers, of which one is <u>positive</u> and the other is <u>negative</u>.

Let us look again at a few examples of gains and losses:

Profit of \$7 and loss of \$3; $7 + (-3) = 4$; $|7| - |-3| = |4|$

Profit of \$3 and loss of \$7; $3 + (-7) = -4$; $|-7| - |3| = |-4|$

Loss of \$7 and profit of \$3; $(-7) + 3 = -4$; $|-7| - |3| = |-4|$

Loss of \$3 and profit of \$7; $(-3) + 7 = 4$; $|7| - |-3| = |4|$

Loss of \$3 and profit of \$3; $(-3) + 3 = 0$; $|3| - |-3| = |0|$

Consider these examples on the number line, and also think again about the questions in Problems 4 and 5 in Problem Set 6-2a. From these it appears that the sum of two numbers, of which one is positive (or 0) and the other is negative, is obtained as follows:

The absolute value of the sum is the difference of the absolute values of the numbers.

The sum is positive if the positive number has the greater absolute value.

The sum is negative if the negative number has the greater absolute value.

The sum is 0 if the positive and negative numbers have the same absolute value.

In the language of algebra,

If $a \geq 0$ and $b < 0$, then:

$$a + b = |a| - |b|, \text{ if } |a| \geq |b|$$

and

$$a + b = -(|b| - |a|), \text{ if } |b| > |a|.$$

If $b \geq 0$ and $a < 0$, then:

$$a + b = |b| - |a|, \text{ if } |b| \geq |a|$$

and

$$a + b = -(|a| - |b|), \text{ if } |a| > |b|.$$

Problem Set 6-2b

1. In each of the following, find the sum, first according to the definition, and then by any other method you find convenient.

(a) $(-5) + 3$	(e) $18 + (-14)$
(b) $(-11) + (-5)$	(f) $12 + 7.4$
(c) $(-\frac{8}{3}) + 0$	(g) $(-\frac{2}{3}) + 5$
(d) $2 + (-2)$	(h) $(-35) + (-65)$

2. Is the set of all real numbers closed under the operation of addition?

3. Is the set of all negative real numbers closed under addition? Justify your answer.

4. In the course of a week the variations in mean temperature from the seasonal normal of 71 were -7, 2, -3, 0, 9, 12, -6. What were the mean temperatures each day? What is the sum of the variations?

5. For each of the following open sentences, find a real number which will make the sentence true:
   - (a) $x + 2 = 7$
   - (b) $3 + y = -7$
   - (c) $a + 5 = 0$
   - (d) $b + (-7) = 3$
   - (e) $(-\frac{5}{6}) + x = -\frac{5}{6}$
   - (f) $c + (-3) = -7$
   - (g) $y + \frac{2}{3} = -\frac{5}{6}$
   - (h) $\frac{1}{2}x + (-4) = 6$
   - (i) $\big(y + (-2)\big) + 2 = 3$
   - (j) $(3 + x) + (-3) = -1$

6. Which of the following sentences are true?
   - (a) $(-4) + 0 = 4$
   - (b) $-(|-1.5| - |0|) = -1.5$
   - (c) $(-3) + 5 = 5 + (-3)$
   - (d) $\big(4 + (-6)\big) + 6 = 4 + \big((-6) + 6\big)$
   - (e) $(-5) + \big(-(-5)\big) = -10$
   - (f) $(-7) + \big((-5) + (-3)\big) = \big((-7) + (-5)\big) + 3$
   - (g) $-\big(6 + (-2)\big) = (-6) + (-2)$
   - (h) $(-7) + (-9) = -(7 + 9)$
   - (i) $(-3) + 7 = -\big(3 + (-7)\big)$

7. Translate the following English sentences into open sentences. For example: Bill spent 60¢ on Tuesday and earned 40¢ on Wednesday. He couldn't remember what happened on Monday, but he had 30¢ left on Wednesday night. What amount did he have on Monday?

   If Bill had x cents on Monday, then

   $$x + (-60) + 40 = 30.$$

   This can be written

   $$x + (-20) = 30.$$

   - (a) If you drive 40 miles north and then drive 55 miles south, how far are you from your starting point?
   - (b) The sum of (-9), 28, and a third number is (-52). What is the third number?
   - (c) At 8 A.M. the temperature was -2°. Between 8 A.M. and

noon the temperature increased 15°. Between noon and 4 P.M. the temperature increased 6°. At 8 P.M. the temperature was -9°. What was the temperature change between 4 P.M. and 8 P.M.?

(d) If a 200-pound man lost 4 pounds one week, lost 6 pounds the second week, and at the end of the third week weighed 195 pounds, how much did he gain in the third week?

(e) A stock which was listed at 83 at closing time Monday dropped 5 points on Tuesday. Thursday morning it was listed at 86. What was the change on Wednesday?

6-3. Properties of Addition

We were careful to describe and list the properties of addition when we dealt with the numbers of arithmetic. Now that we have decided how to add real numbers, we want to verify that these properties of addition hold true for the real numbers generally.

We know that our definition of addition includes the usual addition of numbers of arithmetic, but we also want to be able to add as simply as we could before. Can we still add real numbers in any order and group them in any way to suit our convenience? In other words, do the commutative and associative properties of addition still hold true? If we are able to satisfy ourselves that these properties do carry over to the real numbers, then we are assured that the structure of numbers is maintained as we move from the numbers of arithmetic to the real numbers. Similar questions about multiplication will come up later.

Consider the following questions: Are $4 + (-3)$ and $(-3) + 4$ names for the same number? Check this on the number line. Do similarly for $(-1) + 5$ and $5 + (-1)$; and for $(-2) + (-6)$ and $(-6) + (-2)$.

Do the above sentences cover every possible case of addition of real numbers? If not, supply examples of the missing cases.

It appears that the sum of any two real numbers is the same for either order of addition. This is the

<u>Commutative Property of Addition</u>: For any two real numbers a and b,

$$a + b = b + a.$$

Next, compute the following pairs of sums:

$\big(7 + (-9)\big) + 3$, and $7 + \big((-9) + 3\big)$;
$\big(8 + (-5)\big) + 2$, and $8 + \big((-5) + 2\big)$;
$(4 + 5) + (-6)$, and $4 + \big(5 + (-6)\big)$.

What do you observe about the results?

We could list many more examples. Do you think the same results would always hold? We have the

<u>Associative Property of Addition</u>: For any real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Of course, if the associative and commutative properties hold true in several instances it is not a proof that they will hold true in every instance. A complete proof of the properties can be given by applying the precise definition of addition of real numbers to every possible case of the properties. They are long proofs, especially of the associative property, because there are many cases. We shall not take the time to give the proofs, but perhaps you may want to try the proof for the commutative property in some of the cases.

The associative property assures us that in a sum of three real numbers it doesn't matter which adjacent pair we add first; it is customary to drop the parentheses and leave such sums in an unspecified form, such as $4 + (-1) + 3$.

Another property of addition, which is new for real numbers and one that we shall find useful, is obtained from the definition of addition. For example, the definition tells us that $4 + (-4) = 0$; that $(-4) + \big(-(-4)\big) = 0$. In general, the sum of a number and its opposite is 0. We state this as the

Addition Property of Opposites: For every real number a,

$$a + (-a) = 0.$$

One more property that stems directly from the definition is the

Addition Property of 0: For every real number a,

$$a + 0 = a.$$

Make up several examples to illustrate this and the preceding property.

Problem Set 6-3

1. Show how the properties of addition can be used to explain why each of the following sentences is true:

   Example:

   $$5 + \big(3 + (-5)\big) = 3 + 0$$

   The left numeral is

   $$5 + \big(3 + (-5)\big) = \big(5 + (-5)\big) + 3$$ associative and commutative properties of addition.

   $$= 0 + 3$$ addition property of opposites.

   $$= 3 + 0$$ commutative property of addition.

   The right numeral is

   $$3 + 0.$$

   (a) $3 + \big((-3) + 4\big) = 0 + 4$

   (b) $\big(5 + (-3)\big) + 7 = \big((-3) + 5\big) + 7$

   (c) $\big(7 + (-7)\big) + 6 = 6$

   (d) $|-1| + |-3| + (-3) = 1$

   (e) $(-2) + \big(3 + (-4)\big) = \big((-2) + 3\big) + (-4)$

   (f) $(-|-5|) + 6 = 6 + (-5)$

2. Consider various ways to do the following computations mentally, and find the one that seems easiest (if there is one). Then perform the additions in the easiest way.

   (a) $\frac{5}{16} + 28 + (-\frac{5}{16})$

   (b) $.27 + (-18) + 3 + .73$

   (c) $(-5) + 32 + 3 + (-8)$

   (d) $(-\frac{1}{2}) + 7 + (-2) + (-\frac{3}{2}) + 2$

   (e) $\frac{5}{3} + (-3) + 6 + \frac{1}{3} + (-2)$

   (f) $253 + (-67) + (-82) + (-133)$

   (g) $|-\frac{3}{2}| + \frac{5}{2} + (-7) + |-4|$

   (h) $(x + 2) + (-x) + (-3)$

   (i) $w + (w + 2) + (-w) + 1 + (-3)$

3. Using the associative and commutative properties of addition, write a simpler name for one phrase of each of the following sentences, and find the truth set of each:

   (a) $x = x + \big((-x) + 3\big)$ (b) $m + \big(7 + (-m)\big) = m$

   (c) $n + (n + 2) + (-n) + 1 + (-3) = 0$

   (d) $(y + 4) + (-4) = 9 + (-4)$

*4. Use the definition of addition for negative numbers to show that if $a < 0$ and $b < 0$, then $a + b = b + a$.

*5. Use the definition of addition to show that $a + 0 = a$ for all real numbers a.

*6. Use the definition of addition to show that $a + (-a) = 0$ for all real numbers a. (Hint: Separate out the cases $a = 0$ and $a \neq 0$. If $a \neq 0$, one of a and $-a$ is positive, the other negative (Why?)).

6-4. The Addition Property of Equality.

There is another fact about addition to which we must give attention. We know that

$$4 + (-5) = (-1).$$

This means that $4 + (-5)$ and (-1) are two names for one number.

Let us add 3 to that number. Then $\left(4 + (-5)\right) + 3$ and $(-1) + 3$ are again two names for one number. Thus

$$\left(4 + (-5)\right) + 3 = (-1) + 3.$$

Also, for example,

$$\left(4 + (-5)\right) + 5 = (-1) + 5.$$

Similarly, since

$$7 = 15 + (-8),$$
$$7 + (-7) = \left(15 + (-8)\right) + (-7).$$

This suggests the

Addition Property of Equality: For any real numbers a, b, c,

$$\text{if } a = b, \text{ then } a + c = b + c$$

In words, if a and b are two names for one number, then $a + c$ and $b + c$ are two names for one number.

Let us use the previously stated properties of addition, and the above property of equality in some examples.

Example 1. Determine the truth set of the open sentence

$$x + \frac{3}{5} = -2.$$

Can you guess numbers which make this sentence true? If you don't see it easily, could you use properties of addition to help? Let us see. We do not really know whether there is any number making this sentence true. If, however, there is such a number x which makes the sentence true (that is, if the truth set is not empty), then $x + \frac{3}{5}$ and -2 are the same number.

Let us add $-\frac{3}{5}$ to this number; then by the addition property of equality we have

$$\left(x + \frac{3}{5}\right) + \left(-\frac{3}{5}\right) = (-2) + \left(-\frac{3}{5}\right).$$

Why did we add $-\frac{3}{5}$? Because in this case we wish to change the left numeral so it will contain the numeral "x" alone. Watch this happening in the next few lines.

Continuing, we have

$$x + \left(\frac{3}{5} + (-\frac{3}{5})\right) = (-2) + (-\frac{3}{5}). \qquad \text{(Why?)}$$

$$x + 0 = -\frac{13}{5}. \qquad \text{(Why?)}$$

$$x = -\frac{13}{5}. \qquad \text{(Why?)}$$

Thus, we arrive at the new open sentence $x = -\frac{13}{5}$. If a number x makes the original sentence true, it also makes this new sentence true. Of this we are certain because we applied properties which hold true for all real numbers. This tells us that $-\frac{13}{5}$ is the only possible truth value of the original sentence. But it does not guarentee that it is a truth value. Does $-\frac{13}{5}$ make the original sentence true? Yes, because $(-\frac{13}{5}) + \frac{3}{5} = -2$.

Here we have discovered a very important idea about sentences such as the above. We have shown that if there is a number x making the original sentence true, then the only number which x can be is $-\frac{13}{5}$. The minute we check and find that $-\frac{13}{5}$ does make the sentence true, we have found the one and only number which belongs to the truth set.

The sentence in the previous example is an equation. We shall often call the truth set of an equation its solution set, and its members solutions, and we shall write "solve" instead of "determine the truth set of".

Example 2. Solve the equation

$$5 + \frac{3}{2} = x + (-\frac{1}{2})$$

If $5 + \frac{3}{2} = x + (-\frac{1}{2})$ is true for some x,

then $(5 + \frac{3}{2}) + \frac{1}{2} = \left(x + (-\frac{1}{2})\right) + \frac{1}{2}$ is true for the same x;

$5 + 2 = x + 0$ is true for the same x;

$7 = x$ is true for the same x.

If $x = 7$,

the <u>left</u> side is:
$$5 + \frac{3}{2} = \frac{10}{2} + \frac{3}{2}$$
$$= \frac{13}{2}$$

the <u>right</u> side is:
$$7 + (-\frac{1}{2}) = \frac{14}{2} + (-\frac{1}{2})$$
$$= \frac{13}{2}$$

Hence the truth set is $\{7\}$.

Problem Set 6-4

Solve each of the following equations. Write your work in the form shown for Example 2 above.

1. $x + 5 = 13$
2. $(-6) + 7 = (-8) + x$
3. $(-1) + 2 + (-3) = 4 + x + (-5)$
4. $(x + 2) + x = (-3) + x$
5. $(-2) + x + (-3) = x + (-\frac{5}{2})$
6. $|x| + (-3) = |-2| + 5$
7. $(-\frac{3}{8}) + |x| = (-\frac{3}{4}) + (-1)$
8. $x + (-3) = |-4| + (-3)$
9. $(-\frac{4}{3}) + (x + \frac{1}{2}) = x + (x + \frac{1}{2})$

6-5. The Additive Inverse

Two numbers whose sum is 0 are related in a very special way. For example, what number when added to 3 yields the sum 0? What number when added to -4 yields 0? In general, if x and y are real numbers such that

$$x + y = 0,$$

we say that y is an <u>additive inverse</u> of x. Under this definition, is x then also an additive inverse of y?

Now let us think about any number z which is an additive inverse of, say, 3. Of course we know one such number, namely -3 for by the addition property of opposites, $3 + (-3) = 0$. Can there be any other number z such that

$$3 + z = 0?$$

All of our experience with numbers tells us "No, there is no other such number". But how can we be absolutely sure? We can settle this question with the use of our properties of addition, just as we did in Example 1 in the preceding section. If, for some number z,

$$3 + z = 0$$

is a true sentence, then

$$(-3) + (3 + z) = (-3) + 0$$

is also a true sentence, by the addition property of equality. (Why did we add -3?) Then, however,

$$\bigl((-3) + 3\bigr) + z = -3$$

is true for the same z by the associative property of addition and the 0 property of addition. This finally tells us that

$$z = -3$$

must also be true; we have, for this last step, used the addition property of opposites.

What have we done here? We started out by choosing z as <u>any</u> number which is an additive inverse of 3; we found out that z <u>had</u> to equal -3; that is, that -3 is not just <u>an</u> additive inverse of 3, but also the <u>only</u> additive inverse of 3.

Is there anything special about 3? Do you think 5 has more than one additive inverse? How about (-6.3)? We certainly doubt it, and we can show that they do not by the same line of reasoning as the above. Can we, however, check <u>all</u> numbers? What we need is a result for <u>any</u> real number x, a result which is supposed to tell us something like the following: We know that (-x) is <u>one</u> additive inverse of x; we doubt if there is any other, and this is how we prove there is none. Let us parallel the reasoning we used in the special case in which x = 3, and see if we can arrive at the corresponding conclusion.

Suppose z is any additive inverse of x, that is, any number such that

$$x + z = 0.$$

What corresponds to the first step in our previous special case?

We use the addition property of equality to write

$$(-x) + (x + z) = (-x) + 0.$$

We then have that

$$\big((-x) + x\big) + z = -x.$$

What are the two reasons we have used in arriving at this last sentence?

Then

$$0 + z = -x, \qquad \text{(Why?)}$$

and finally

$$z = -x. \qquad \text{(Why?)}.$$

We have succeeded in carrying out our program, not just when $x = 3$ but for any x. Each number x has a unique (meaning "just one") additive inverse, namely $-x$.

You probably have all kinds of qualms and questions at this point, and these are to be expected since this is the first proof which you have seen in this course. What we have done is to use facts which we have previously known about all real numbers in order to argue out a new fact about all real numbers, a new fact which you certainly expected to be true, but which nevertheless took this kind of checking. We shall do a number of proofs in this course, and you will become more and more accustomed to this kind of reasoning as you progress In the meantime, let us make one more comment about the proof just completed. The successive steps we took were of course chosen quite deliberately in order to make the proof succeed. This might give you the impression that the proof was "rigged", that it couldn't come out any other way. Is this fair? Yes, it is, and in fact every proof is "rigged" in the sense that we take only steps to help us towards our goal, and do not take steps which fail to do us any good. When we started from

$$3 + z = 0,$$

we chose to use the addition property of equality to add (-3); we could have added any other number instead, but it wouldn't have helped us. And so we didn't add a different number, but added -3.

Statements of new facts or properties, which can be shown to follow from previously established properties, are frequently (but not always!) called "theorems". Thus the property about additive inverses obtained above can be stated as a theorem:

> Theorem 6-5a. Any real number x has exactly one additive inverse, namely $-x$.

An argument by which a theorem is shown to be a consequence of other properties is called a proof of the theorem.

Problem Set 6-5a

1. For each sentence, find its truth set.

(a) $3 + x = 0$	(f) $\left(-(-\frac{2}{3})\right) + y = 0$
(b) $(-2) + a = 0$	(g) $\left(-(2 + \frac{1}{3})\right) + a = 0$
(c) $3 + 5 + y = 0$	(h) $2 + x + (-5) = 0$
(d) $x + (-\frac{1}{2}) = 0$	(i) $3 + (-x) = 0$
(e) $\lvert -4 \rvert + 3 + (-4) + c = 0$	

2. Were you able to use Theorem 6-5a to save work in solving these equations?

Let us look at another example for this technique of showing a general property of numbers. Of course we cannot prove a general property of numbers until we suspect one; let us find one to suspect. Recall the picture of addition on the number line, or the definition of addition if you prefer, to see that

$$(-3) + (-5) = -(3 + 5).$$

Another way of writing that $(-3) + (-5)$ and $-(3 + 5)$ are names for the same number is that

$$-(3 + 5) = (-3) + (-5).$$

This might lead us to suspect that the opposite of the sum of two numbers is the sum of the opposites. Of course we have checked this only for the numbers 3 and 5, and it is wise to check a few more cases. Is

$$-(2 + 9) = (-2) + (-9)?$$

Is

$$-\big(4 + (-2)\big) = (-4) + \big(-(-2)\big)?$$

(What is another name for $\big(-(-2)\big)$?)

Is

$$-\big((-1) + (-4)\big) = 1 + 4?$$

Our hunch seems to be true, at least in all the examples we have tried. Let us now, instead of checking any more examples by arithmetic, state the general property which we hope to prove as a theorem.

<u>Theorem</u> <u>6-5b</u>. For any real numbers a and b
$-(a + b) = (-a) + (-b)$.

<u>Proof</u>. We need to prove that $(-a) + (-b)$ names the same number as $-(a + b)$. Let us check that $(-a) + (-b)$ <u>acts</u> like the opposite of $(a + b)$. We look at $(a + b) + \big((-a) + (-b)\big)$, for if this expression is 0, $(-a) + (-b)$ will be the opposite of $(a + b)$.

$$\begin{aligned}(a + b) + \big((-a) + (-b)\big) &= a + b + (-a) + (-b)\\ &= \big(a + (-a)\big) + \big(b + (-b)\big) && \text{(Why?)}\\ &= 0 + 0 && \text{(Why?)}\\ &= 0\end{aligned}$$

And so we find that for all real numbers a and b,

$$(-a) + (-b)$$

is an additive inverse of $(a + b)$, and, since there is only one additive inverse, that

$$-(a + b) \text{ and } (-a) + (-b)$$

name the same number.

<u>Problem Set 6-5b</u>

1. Which of the following sentences are true for all real numbers? Hint: Remember that the opposite of the sum of two numbers is the sum of their opposites.

(a) $-(x + y) = (-x) + (-y)$

(b) $-x = -(-x)$

(c) $-(-x) = x$

(d) $-\big(x + (-2)\big) = (-x) + 2$

(e) $-\big(a + (-b)\big) = (-a) + b$

(f) $\big(a + (-b)\big) + (-a) = b$

(g) $-\big(x + (-x)\big) = x + (-x)$

2. In the following proof supply the reason for each step:
For all numbers x, y and z,

$$(-x) + \big(y + (-z)\big) = y + \big(-(x + z)\big).$$

<u>Proof</u>.

$$\begin{aligned}(-x) + \big(y + (-z)\big) &= (-x) + \big((-z) + y\big)\\ &= \big((-x) + (-z)\big) + y\\ &= \big(-(x + z)\big) + y\\ &= y + \big(-(x + z)\big).\end{aligned}$$

3. Is $-\big(3 + 6 + (-4) + 5\big) = (-3) + (-6) + 4 + (-5)$? What do you think is true for the opposite of the sum of more than two numbers?

Tell which of the following sentences are true.

(a) $-\big((-2) + 6 + (-5)\big) = 2 + (-6) + 5$

(b) $-\big(3a + (-b) + (-2)\big) = 3a + b + 2$

(c) $-\big(a + (-b) + (-5c) + .7d\big) = (-a) + b + 5c + (-.7d)$

(d) $-\big(\frac{5}{3}x + 2y + (-2a) + (-3b)\big) = (-\frac{5}{3}x) + 2y + (-2a) + (-3b)$

*4. Give an argument for the conclusion you made for the second question in Problem 3.

*5. Prove the following property of addition:

For any real number a and any real number b
and any real number c,

if $a + c = b + c$, then $a = b$.

6-6. Summary

We have defined addition of real numbers as follows:

The sum of two positive (or 0) numbers is familiar from arithmetic.

The sum of two negative numbers is negative; the absolute value of this sum is the sum of the absolute values of the numbers.

The sum of two numbers, of which one is positive (or 0) and the other is negative, is obtained as follows:

The absolute value of the sum is the difference of the absolute values of the numbers.

The sum is positive if the positive number has the greater absolute value.

The sum is negative if the negative number has the greater absolute value.

The sum is 0 if the positive and negative numbers have the same absolute value.

We have satisfied ourselves that the following properties hold for addition of real numbers:

Commutative Property of Addition: For any two real numbers a and b:

$$a + b = b + a.$$

Associative Property of Addition: For any real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Addition Property of Opposites: For every real number a,

$$a + (-a) = 0.$$

Addition Property of 0: For every real number a,

$$a + 0 = a.$$

Addition Property of Equality: For any real numbers a, b, and c,

$$\text{if } a = b, \text{ then } a + c = b + c.$$

We have used the addition property of equality to determine the truth sets of open sentences.

We have proved that the additive inverse is unique - that is, that each number has exactly one additive inverse, which we call its opposite.

We have discovered and proved the fact that the opposite of the sum of two numbers is the same as the sum of their opposites.

Review Problems

1. Find a common name for each of the following:

   (a) $3(8 + (-6))$ (d) $(-\frac{2}{5}) + \frac{3}{5}$

   (b) $(-3) + 2 \times 3$ (e) $|-6| \cdot |3| + (-3)$

   (c) $2 \times 7 + (-14)$ (f) $6(1 + |-4|)$

2. Which of the following sentences are true?

   (a) $3 + (-8) = (-8) + 3$

   (b) $|-8| + (-8) = 0$

   (c) $6 \times 3 - 3 = 0$

   (d) $(2 + (-3)) + 6 = 2 + ((-3) + 6)$

   (e) $5 - 3 \leq |-5| - |-3|$

   (f) $5(|10 - 7|) < 3 \times 3 \times 2$

   (g) $6 - 6 = 6 + (-6)$

3. Show how the properties of addition can be used to explain why each of the following sentences is true:

   (a) $\frac{2}{3} + (7 + (-\frac{2}{3})) = 7$

   (b) $|-5| + (-.36) + |-.36| = 10 + (2 + (-7))$

4. Find the truth set of each of the following:

   (a) $\frac{5}{9} + 32 = x + \frac{5}{9}$

   (b) $x + 5 + (-x) = 12 + (-x) + (-3)$

   (c) $x + \frac{15}{2} + x = 10 + x + (-\frac{7}{2})$

   (d) $|x| + 3 = 5 + |x|$

5. For what set of numbers is each of the following sentences true?

   (a) $|3| + |a| > |-3|$ (c) $|3| + |a| < |-3|$

   (b) $|3| + |a| = |-3|$

6. Two numbers are added. What do you know about these numbers if
   (a) their sum is negative?
   (b) their sum is 0?
   (c) their sum is positive?

7. A salesman earned a basic salary of \$80 a week. In addition he received a commission of 3% of his total sales. During one week he earned \$116. What was the amount of his sales for the week? Write an open sentence for this problem.

8. A figure has four sides. Three of them are 8 feet, 10 feet, and 5 feet, respectively. How long is the fourth side?
   (a) Write a compound open sentence for this problem.
   (b) Graph the truth set of the open sentence.

9. If a, b, and c are numbers of arithmetic, write each of the indicated sums as an indicated product, and each of the indicated products as an indicated sum:

(a)	$(2b + c)a$	(e)	$x^2y + xy$
(b)	$2a(b + c)$	(f)	$6a^2b + 2ab^2$
(c)	$3a + 3b$	(g)	$ab(ac + 3b)$
(d)	$5x + 10ax$	(h)	$3a(a + 2b + 3c)$

10. Given the set $\{-5, 0, \frac{3}{4}, -.75, 5\}$
    (a) Is this set closed under the operation of taking the opposite of each element of the set?
    (b) Is this set closed under the operation of taking the absolute value of each element?
    (c) If a set is closed under the operation of taking the opposite, is it closed under the operation of taking the absolute value? Why?

11. Given the set $\{-5, 0, \frac{3}{4}, 5, 7\}$
    (a) Is this set closed under the operation of taking the absolute value of each element of the set?
    (b) Is this set closed under the operation of taking the opposite of each element?
    (c) If a set is closed under the operation of taking the absolute value, is it closed under the operation of taking the opposite? Why?

12 Two automobiles start from the same city travelling in the same direction. Write an open phrase for the time it takes the faster car to get m miles ahead of the slower car, if the rates of the cars are 30 m.p.h. and 20 m.p.h.

Chapter 7

PROPERTIES OF MULTIPLICATION

7-1. Multiplication of Real Numbers

Now let us decide how we should multiply two real numbers to obtain another real number. All that we can say at present is that we know how to multiply two non-negative numbers.

Of primary importance here, as in the definition of addition, is that we maintain the "structure" of the number system. We know that if a, b, c are any numbers of arithmetic, then

$$\begin{aligned} ab &= ba, \\ (ab)c &= a(bc), \\ a\cdot 1 &= a, \\ a\cdot 0 &= 0, \\ a(b+c) &= ab + ac. \end{aligned}$$

(What names did we give to these properties of multiplication?) Whatever meaning we give to the product of two real numbers, we must be sure that it agrees with the products which we already have for non-negative real numbers and that the above properties of multiplication still hold for all real numbers.

Consider some possible products:

$(2)(3)$, $(3)(0)$, $(0)(0)$, $(-3)(0)$, $(3)(-2)$, $(-2)(-3)$.

(Do these include examples of every case of multiplication of positive and negative numbers and zero?) Notice that the first three products involve only non-negative numbers and are therefore already determined:

$$(2)(3) = 6, \quad (3)(0) = 0, \quad (0)(0) = (0).$$

Now let us try to see what the remaining three products will have to be in order to preserve the basic properties of multiplication listed above. In the first place, if we want the multiplication property of 0 to hold for all real numbers, then we must have $(-3)(0) = 0$. The other two products can be obtained as follows:

$0 = (3)(0)$	
$0 = (3)\big(2 + (-2)\big),$	by writing $0 = 2 + (-2)$; (Notice how this introduces a negative number into the discussion.)
$0 = (3)(2) + (3)(-2),$	if the distributive property is to hold for real numbers;
$0 = 6 + (3)(-2),$	since $(3)(2) = 6$.

We know from uniqueness of the additive inverse that the only real number which yields 0 when added to 6 is the number -6. Therefore, if the properties of numbers are expected to hold, the only possible value for $(3)(-2)$ which we can accept is -6.

Next, we take a similar course to answer the second question.

$0 = (-2)(0)$	if the multiplication property of 0 is to hold for real numbers;
$0 = (-2)\big(3 + (-3)\big)$,	by writing $0 = 3 + (-3)$;
$0 = (-2)(3) + (-2)(-3),$	if the distributive property is to hold for real numbers;
$0 = (3)(-2) + (-2)(-3),$	if the commutative property is to hold for real numbers;
$0 = (-6) + (-2)(-3),$	by the previous result, which was $(3)(-2) = -6$.

Now we have to come to a point where $(-2)(-3)$ must be the opposite of -6; hence, if we want the properties of multiplication to hold for real numbers, then $(-2)(-3)$ must be 6.

Let us think of these examples now in terms of absolute value. Recall that the product of two positive numbers is a positive number. Then what are the values of $|3||2|$ and $|-2||-3|$? How do these compare, respectively, with $(3)(2)$ and $(-2)(-3)$? Compare $(-3)(4)$ and $-(|-3||4|)$; $(-5)(-3)$ and $|-5||-3|$; $(0)(-2)$ and $|0||-2|$.

This is the hint we needed. If we want the structure of the number system to be the same for real numbers as it was for the numbers of arithmetic, we must define the product of two real numbers a and b as follows:

> If a and b are both negative or both non-negative, then $ab = |a||b|$.
>
> If one of the numbers a and b is non-negative and the other is negative, then $ab = -(|a||b|)$.

It is important to recognize that $|a|$ and $|b|$ are numbers of arithmetic for any real numbers a and b; and we already know the product $|a||b|$. (Why?) Thus, the product $|a||b|$ is a positive number, and we obtain the product ab as either $|a||b|$ or its opposite. Again we have used only the operations with which we are already familiar: multiplying positive numbers or 0, and taking opposites. It will help you to remember the definition by completing the sentences: (Supply the words "positive","negative", or "zero".)

The product of two positive numbers is a __________number.
The product of two negative numbers is a __________number.
The product of a negative and a positive number is a ____________number.
The product of a real number and 0 is __________.

Since the product ab is either $|a||b|$ or its opposite and since $|a||b|$ is non-negative, we can state the following property of multiplication:

For any real numbers a, b,

$$|ab| = |a||b|.$$

Problem Set 7-1

1. Use the definition of multiplication to calculate the following.

   Examples: (a) $(5)(-3) = -(|5||-3|) = -15$
   (b) $(-5)(-3) = (|-5||-3|) = 15$
   (c) $(-5)(3) = -(|-5||3|) = -15$

   (a) $(-7)(-8)$
   (b) $(\frac{2}{3})(-12)$
   (c) $|(-3)(2)|(-2)$
   (d) $(-18)(\frac{3}{5})$
   (e) $(-\frac{3}{4})(-\frac{2}{5})$
   (f) $|-2|\left((-3) + |-3|\right)$

2. Calculate the following:

   (a) $(-\frac{1}{2})(-4)$
   (b) $\left((-\frac{1}{2})(2)\right)(-5)$
   (c) $(-\frac{1}{2})\left((2)(-5)\right)$
   (d) $(-3)(-4) + (-3)(7)$
   (e) $(-3)\left((-4) + 7\right)$
   (f) $(-3)(-4) + 7$
   (g) $|-3|(-4) + 7$
   (h) $|3||-2| + (-6)$
   (i) $(-3)|-2| + (-6)$
   (j) $(-3)\left(|-2| + (-6)\right)$
   (k) $(-0.5)\left(|-1.5| + (-4.2)\right)$

3. Find the values of the following for $x = -2$, $y = 3$, $a = -4$:

   (a) $2x + 7y$
   (b) $3(-x) + \left((-4)y + 7(-a)\right)$
   (c) $x^2 + 2(xa) + a^2$
   (d) $(x + a)^2$
   (e) $x^2 + \left(3|a| + (-4)|y|\right)$
   (f) $|x + 2| + (-5)|(-3) + a|$

4. Which of the following sentences are true?

   (a) $2x + 8 = 12$, for $x = -10$
   (b) $2(-y) + 8 = 28$, for $y = -10$
   (c) $(-3)\left((2)(-x)\right) + 8 \neq 20$, for $x = 2$
   (d) $(-5)\left((-b)(-4) + 30\right) < 0$, for $b = 2$
   (e) $|x + 3| + (-2)\left(|x + (-4)|\right) \geq 1$, for $x = 2$

5. Find the truth sets of the following open sentences and draw their graphs.

Example: Find the truth set of $(3)(-3) + c = 3(-4)$.

If $(3)(-3) + c = 3(-4)$ is true for some c,
then $-9 + c = -12$ is true for the same c;
$(-9 + c) + 9 = -12 + 9$ is true for the same c;
$c = -3$.

If $c = -3$,
then the left member is $(3)(-3) + (-3) = -12$,
and the right member is $3(-4) = -12$.
Hence. the truth set is $\{-3\}$.

(a) $x + (-3)(-4) = 8$
(b) $2(-2) + y = 3(-2)$
(c) $x + 2 = 3(-6) + (-4)(-8)$
(d) $x + (-5)(-6) = (-2)(3)$
(e) $x = (-5)(-6) + |-2|(3)$
(f) $x > (-4)(-2) + (-5)(2)$
(g) $|x| = (-\frac{2}{3})(7) + (-1)(-5)$

6. Given the set $S = \{1, -2, -3, 4\}$, find the set P of all products of pairs of elements of S obtained by multiplying each element of S by each element of S.

7. Given the set R of all real numbers, find the set Q of all products of pairs of elements of R. Is Q the same set as R? Can you conclude that R is closed under multiplication?

8. Given the set N of all negative real numbers, find the set T of all products of pairs of elements of N. Is the set of negative real numbers closed under multiplication?

9. Given the set $V = \{1, -2, -3, 4\}$, find the set K of all positive numbers obtained as products of pairs of elements of V.

10. Prove that the absolute value of the product ab is the product $|a|\cdot|b|$ of the absolute values; that is,

$$|ab| = |a||b|.$$

11. What can you say about two real numbers a and b in each of the following cases?
    (a) ab is positive.
    (b) ab is negative.
    (c) ab is positive and a is positive.
    (d) ab is positive and a is negative.
    (e) ab is negative and a is positive.
    (f) ab is negative and a is negative.

*12. Give the reason for each numbered step in the proof of the following theorem.

<u>Theorem</u>. If a and b are numbers such that both a and ab are positive, then b is also positive.

<u>Proof</u>. We assume that $0 < a$ and $0 < ab$. Then:

1. Exactly one of the following is true
   $b < 0$, $b = 0$, $0 < b$.
2. If $b = 0$, then $ab = 0$.
   Therefore, "$b = 0$" is false, since $ab = 0$ and $0 < ab$ are contradictory.
3. If $b < 0$, then $ab = -(|a||b|)$.
4. If $b < 0$, then ab is negative.
   Therefore, "$b < 0$" is false, since ab is negative and $0 < ab$ are contradictory.
   Therefore, $0 < b$; that is, b is positive, since this is the only remaining possibility.

7-2. Properties of Multiplication

The definition of multiplication for real numbers given in the preceding section was <u>suggested by</u> the structure properties which we wish to preserve for all numbers. On the other hand, we have not actually <u>assumed</u> these properties, since the definition could have been given at the outset without any reference to the properties. However, now that we have stated a definition for multiplication, it becomes important to satisfy ourselves that this definition really leads to the desired properties. In other words, we need to prove that multiplication so defined does have the properties. Since the definition is stated in terms of operations on positive numbers and 0 and of taking opposites, these operations are the only ones available to us in the proofs.

Multiplication property of 1: For any real number a,

$$a \cdot 1 = a .$$

Proof: If a is positive or 0, we know that $a(1) = a$. If a is negative, our definition of multiplication states that

$$\begin{aligned} a \cdot 1 &= -(|a| \cdot 1) \\ &= -|a| \\ &= a. \end{aligned}$$

Try to explain the reason why each step in the above proof is true.

Multiplication property of 0: For any real number a,

$$a \cdot 0 = 0.$$

Write out the proof of this property for yourself.

Commutative property of multiplication: For any real numbers a and b,

$$ab = ba.$$

Proof: If one or both the numbers a, b are zero, then $ab= ba$. (Why?) If a and b are both positive or both negative, then

$$ab= |a||b|, \text{ and } ba = |b||a| .$$

Since $|a|$ and $|b|$ are numbers of arithmetic,

$$|a||b| = |b||a|.$$

Hence,

$$ab = ba$$

for these two cases.

If one of a and b is positive or 0 and the other is negative, then

$$ab = -(|a||b|) \text{ and } ba = -(|b||a|).$$

Since

$$|a||b| = |b||a|,$$

and since if numbers are equal their opposites are equal,

$$-(|a||b|) = -(|b||a|).$$

Hence,

$$ab = ba$$

for this case also.

Here we have given a complete proof of the commutative property for all real numbers. We have based this proof on the precise definition of multiplication of real numbers.

Problem Set 7-2a

1. Apply the properties proved so far to compute the following:

   (a) $(-\frac{3}{4})\left((-\frac{4}{3})(-17)\right)$

   (b) $(-8)\left(5 + (-6)(\frac{2}{3})\right)$

   (c) $(-4)\left((-\frac{3}{2})(4) + (-\frac{6}{5})(-5)\right)$

   (d) $\left((4)(-6) + (-8)(-3)\right)(-\frac{47}{13})$

2. Illustrate the proof of the commutative property by replacing a and b as follows:

   (a) $(-3)(5)$ Example: $(-3)(5) = -(|-3||5|)$
   $= -15$
   $(5)(-3) = -(|5||-3|)$
   $= -15$

   (b) $(3)(-5)$

   (c) $(-\frac{2}{3})(0)$

   (d) $(-3)(-4)$

   (e) $(-7)(\frac{5}{7})$

Associative property of multiplication: For any real numbers a, b, and c,

$$(ab)c = a(bc) .$$

Proof: The property must be shown to be true for one negative, two negatives, or three negatives. This is lengthy, but we shall be able to simplify it by observing that

$$\begin{aligned} |(ab)c| &= |ab||c| \qquad \text{(Why?)} \\ &= |a||b||c| , \end{aligned}$$

and

$$\begin{aligned} |a(bc)| &= |a||bc| \\ &= |a||b||c| . \end{aligned}$$

Thus $|(ab)c| = |a(bc)|$ for all real numbers a, b, c.

This reduces the proof of the associative property of multiplication to the problem of showing that (ab)c and a(bc) are either both positive, both zero, or both negative.

For example, if both (ab)c and a(bc) are negative, then $|(ab)c| = -\big(a(bc)\big)$ and $|a(bc)| = -\big((ab)c\big)$. Thus $-\big(a(bc)\big) = -\big((ab)c\big)$ and hence $a(bc) = (ab)c$.

If one of a, b, c is zero, then (ab)c = 0 and a(bc) = 0. (Why?) Hence, for this case (ab)c = a(bc).

If a, b, and c are all different from zero, we need to consider eight different cases, depending on which numbers are positive and which are negative, as shown in the table below.

If a is	+	+	+	+	-	-	-	-
and b is	+	+	-	-	+	+	-	-
and c is	+	-	+	-	+	-	+	-
then ab is				-				
bc is				+				
(ab)c is				+				
a(bc) is				+				

One column has been filled in for positive a, and negative b and c. In this case, ab is negative and bc is positive. Hence (ab)c is positive and a(bc) is positive. Therefore (ab)c = a(bc) in this case.

The associative property states that, in multiplying three numbers, we may first form the product of any adjacent pair. The effect of associativity along with commutativity is to allow us to write products of numbers without grouping symbols and to perform the multiplication in any groups and any orders.

Problem Set 7-2b

1. Copy the table given above, and complete it. Use it to check the remaining cases and finish the proof.

2. In each of the following verify that the two numerals name the same number.

   (a) $\big((3)(2)\big)(-4)$ and $(3)\big((2)(-4)\big)$

   (b) $\big((3)(-2)\big)(-4)$ and $(3)\big((-2)(-4)\big)$

   (c) $\big((3)(-2)\big)(4)$ and $(3)\big((-2)(4)\big)$

   (d) $\big((-3)(2)\big)(-4)$ and $(-3)\big((2)(-4)\big)$

   (e) $\big((-3)(-2)\big)(-4)$ and $(-3)\big((-2)(-4)\big)$

   (f) $\big((3)(-2)\big)(0)$ and $(3)\big((-2)(0)\big)$

3. Explain how the associative and commutative properties can be used to perform the following multiplications in the easiest manner.

   (a) $(-5)(17)(-20)(3)$

   (b) $(\frac{2}{3})(\frac{7}{5})(-\frac{3}{4})$

   (c) $(\frac{1}{3})(\frac{6}{5})(-21)$

   (d) $(-\frac{2}{3})(\frac{4}{5})(\frac{3}{2})(-\frac{5}{4})$

   (e) $(\frac{1}{5})(-19)(-3)(50)$

   (f) $(-7)(-25)(3)(-4)$

Another property is one which ties together the operations of addition and multiplication.

Distributive property. For any real numbers, a, b, and c,

$$a(b + c) = ab + ac.$$

We shall consider only a few examples:

$(5)\big(2 + (-3)\big) = ?$ and $(5)(2) + (5)(-3) = ?$
$(5)\big((-2) + (-3)\big) = ?$ and $(5)(-2) + (5)(-3) = ?$
$(-5)\big((-2) + (-3)\big) = ?$ and $(-5)(-2) + (-5)(-3) = ?$

The distributive property does hold for all real numbers. It could be proved by applying the definitions of multiplication and addition to all possible cases, but this is even more tedious than the proof of associativity.

Problem Set 7-2c

Use the distributive property, if necessary, to perform the indicated operations with the minimum amount of work:

1. $(-9)(-92) + (-9)(-8)$
2. $(.63)(6) + (-1.63)(6)$
3. $(-\frac{3}{2})\big((-4) + 6\big)$
4. $(-7)(-\frac{3}{4}) + (-7)(\frac{1}{3})$
5. $(-\frac{3}{4})\big((-93) + (-7)\big)$
6. $(-7)(\frac{2}{3}) + (-5)(\frac{2}{3})$

We can use the distributive property to prove another useful property of multiplication.

Theorem 7-2a. For every real number a,

$$(-1)a = -a.$$

To prove this theorem, we must show that $(-1)a$ is the opposite of a, that is, that

$$a + (-1)a = 0.$$

Proof:

$$\begin{aligned} a + (-1)a &= 1(a) + (-1)a && \text{(Why?)} \\ &= \big(1 + (-1)\big) a && \text{(Why?)} \\ &= 0 \cdot a && \text{(Why?)} \\ &= 0. \end{aligned}$$

Here we have shown that $(-1)a$ is an additive inverse of a. Since we also know that $-a$ is an additive inverse of a and that the additive inverse is unique, we have proved that

$$(-1)a = -a.$$

Problem Set 7-2d

Use Theorem 7-2a to prove the following:

1. For any real numbers a and b, $(-a)(b) = -(ab)$.
2. For any real numbers a and b, $(-a)(-b) = ab$.
3. Write the common names for the products:

(a) $(-5)(ab)$

(b) $(-2a)(-5c)$

(c) $(3x)(-7y)$

(d) $(-5c)(\frac{3}{5}d)$

(e) $(\frac{2}{9}bc)(-6a)$

(f) $(-0.5d)(1.2c)$

7-3. Use of the Multiplication Properties

We have seen in the above problems that we may now write

$$\begin{aligned} -5a &\text{ for } (-5)a, \\ -xy &\text{ for } (x)(-y), \\ 6b &\text{ for } (-6)(-b). \end{aligned}$$

In fact,

$$-ab = -(ab) = (-a)(b) = (a)(-b).$$

Now that we can multiply real numbers and have at our disposal the properties of multiplication of real numbers, we have a strong basis for dealing with a variety of situations in algebra.

Problem Set 7-3a

1. Use the distributive property to write the following as indicated sums:

   (a) $3(x + 5)$
   (b) $\left(7 + (-k)\right) a$
   (c) $2(a + b + c)$
   (d) $(-9)(a + b)$
   (e) $\left((-p) + q\right)(-3)$
   (f) $(-1)\left(y + (-z) + 5\right)$
   (g) $(13 + x)y$
   (h) $(-8)\left((-4) + (-m)\right)$
   (i) $(-g)\left(r + 1 + (-s) + (-t)\right)$

2. In doing Problem 1, you probably used the property: For any real numbers a and b ,

   $$(-a)(-b) = ab.$$

   Find the parts of the problem in which you used it.

3. Use the distributive property to write the following phrases as indicated products:

   (a) $5a + 5b$
   (b) $(-9)b + (-9)c$
   (c) $12 + 18$
   (d) $3x + 3y + 3z$
   (e) $km + kp$
   (f) $(a + b)x + (a + b)y$
   (g) $7(\frac{1}{8}) + 3(\frac{1}{8})$
   (h) $(-6)a^2 + (-6)b^2$
   (i) $ca + cb + c$
   (j) $2a + (-2b)$

4. Apply the distributive and other properties to the following:

   Example: $3x + 2x = (3 + 2)x = 5x$

   (a) $12t + 7t$
   (b) $9a + (-15a)$
   (c) $(-5)y + 14y$
   (d) $12z + z$ (Hint: $z = 1 \cdot z$)
   (e) $(-3m) + (-8m)$
   (f) $\frac{1}{2}a + \frac{3}{2}a$
   (g) $(1.6)b + (2.4)b$
   (h) $(-5)x + 2x + 11x$
   (i) $3a + 7y$ (Careful!)
   (j) $4p + 3p + 9p$
   (k) $8r + (-14r) + 6r$
   (l) $6a + (-4a) + 5b + 14b$

In a phrase which has the form of an indicated sum A + B, A and B are called <u>terms</u> of the phrase; in a phrase of the form A + B + C, A, B, and C are called terms, etc. The distributive property is very helpful in simplifying a phrase. Thus we found that

$$5a + 8a = (5 + 8)a = 13a$$

is a possible and a desirable simplification. However, in

$$5x + 8y$$

no such simplification is possible. Why?

We may sometimes be able to apply the distributive property to some, but not all, terms of an expression. Thus

$$6x + (-9)x^2 + 11x^2 + 5y = 6x + \left((-9) + 11\right)x^2 + 5y = 6x + 2x^2 + 5y.$$

We shall have frequent occasion to do this kind of simplification. For convenience we shall call it <u>collecting terms</u> or <u>combining terms</u>. We shall usually do the middle step mentally. Thus

$$15w + (-9)w = 6w.$$

<u>Problem Set 7-3b</u>

1. Collect terms in the following phrases:
   (a) $3x + 10x$
   (b) $(-9)a + (-4a)$
   (c) $11k + (-2)k$
   (d) $(-27b) + 30b$
   (e) $17n + (-16)n$
   (f) $x + 8x$ (Hint: $x = 1x$)
   (g) $(-15a) + a$
   (h) $\frac{7}{8}a + \frac{9}{8}a$
   (i) $5p + 4p + 8p$
   (j) $7x + (-10x) + 3x$
   (k) $12a + 5c + (-2c) + 3c^2$
   (ℓ) $6a + 4b + c$
   (m) $9p + 4q + (-3)p + 7q$
   (n) $4x + (-2)x^2 + (-5x) + 5x^2+1$

2. What other properties of real numbers besides the distributive property did you use in Problem 1, parts (f), (g), (k), and (m)?

3. Find the truth set of each of the following open sentences. (Where possible, collect terms in each phrase.)

(a)	$6x + 9x = 30$	(f)	$(-3a) + 3a + 5 = 5$
(b)	$(-3a) + (-7a) = 40$	(g)	$x + 2x + 3x = 42$
(c)	$x + 5x = 3 + 6x$	(h)	$x + 9 = 20$
(d)	$3y + 8y + 9 = -90$	(i)	$2y = y + 1$
(e)	$14x + (-14)x = 15$	(j)	$12 = 4y + 2y$

7-4. Further Use of the Multiplication Properties

We have seen how the distributive property allows us to collect terms of a phrase. The properties of multiplication are helpful also in certain techniques of algebra related to products involving phrases.

Example 1. "$(3x^2y)(7ax)$" can be more simply written as "$21ax^3y$."

Give the reasons for each of the following steps which show this is true.

$$\begin{aligned}(3x^2y)(7ax) &= 3\cdot x\cdot x\cdot y\cdot 7\cdot a\cdot x\\ &= 3\cdot 7\cdot a\cdot x\cdot x\cdot x\cdot y\\ &= (3\cdot 7)a(x\cdot x\cdot x)y\\ &= 21ax^3y.\end{aligned}$$

(Notice that we write $x\cdot x\cdot x$ as x^3.)

While in practice we do not write down all these steps, we must continue to be aware of how this simplification depends on our basic properties of multiplication, and we should be prepared to explain the intervening steps at any time.

Problem Set 7-4a

Simplify the following expressions and, in Problem 11, write the steps which explain the simplification.

1. $(-3)(8b)$	8. $(\frac{3}{4}abc)(\frac{1}{2}bcd)$
2. $(4c)(-3c)$	9. $(-12pq)(-4pq)$
3. $(9b)(-8)$	10. $(20b^2c^2)(10bd)$
4. $(-6y)(-7z)$	11. $(\frac{1}{3}ab)(9a^2)$
5. $(-3bc)(-6c)$	12. $(-7b)(-4a)c$
6. $(5w^3)(-3w)$	13. $(-2x)(3ax)(-4a)$
7. $(4y^2)(-3ay)$	14. $(6ab)(-2abc)(-a)$

We can combine the method of the preceding exercises with the distributive property to perform multiplications such as the following:

$$(-3a)\big(2a + 3b + (-5)c\big) = (-6a^2) + (-9ab) + 15ac.$$

Furthermore, since we have shown in Section 7-2 that

$$-a = (-1)a,$$

we may again with the help of the distributive property simplify expressions such as

$$-\big(x^2 + (-7x) + (-6)\big) = (-1)\big(x^2 + (-7x) + (-6)\big)$$
$$= (-x^2) + 7x + 6.$$

Problem Set 7-4b

Write in the form indicated in the examples above:

1. $(-3)(c + d)$
2. $2\left(8 + (-3b) + 7b^2\right)$
3. $6x(3y + z)$
4. $(-3)b^2c^2(4b + 7c)$
5. $5x(x + 6)$
6. $10b\left(2b^2 + 7b + (-4)\right)$
7. $-(p + q + r)$
8. $(-7)\left(3a + (-5b)\right)$
9. $6xy(2x + 3xy + 4y)$
10. $-(a^2 + 2ab + b^2)$
11. $(-4c)\left(2a + (-5b) + (-c)\right)$
12. $(-x)\left(x + (-1)\right)$

As you will remember from some of the work we did in Chapter 3, sometimes the distributive property is used several times in one example.

Example 1.
$$\begin{aligned}(x + 3)(x + 2) &= (x + 3)x + (x + 3)2\\ &= x^2 + 3x + 2x + 6\\ &= x^2 + (3 + 2)\ x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

Example 2.
$$\begin{aligned}\left(a + (-7)\right)(a + 3) &= \left(a + (-7)\right)a + \left(a + (-7)\right)3\\ &= a^2 + (-7)a + 3a + (-21)\\ &= a^2 + \left((-7) + 3\right)a + (-21)\\ &= a^2 + (-4)a + (-21)\end{aligned}$$

Example 3.
$$\begin{aligned}(x + y + z)(b + 5) &= (x + y + z)b + (x + y + z)5\\ &= bx + by + bz + 5x + 5y + 5z.\end{aligned}$$

Problem Set 7-4c

1. Perform the following multiplications.
   (a) $(x + 8)(x + 2)$
   (b) $\left(y + (-3)\right)\left(y + (-5)\right)$
   (c) $\left(6a + (-5)\right)\left(a + (-2)\right)$
   (d) $(a + 2)(a + 2)$
   (e) $(x + 6)\left(x + (-6)\right)$
   (f) $(y + 3)\left(y + (-3)\right)$

2. Show that for real numbers a, b, c, d,

$$(a + b)(c + d) = ac + (bc + ad) + bd.$$

(Notice that ac is the product of the first terms, bd is the product of the second terms, and (bc + ad) is the sum of the remaining products.

3. Multiply the following:

(a)	$(a + 3)(a + 1)$	(d)	$\left(y + (-4)\right)\left(y^2 + (-2y) + 1\right)$
(b)	$(2x + 3)(3x + 4)$	(e)	$(m + 3)(m + 3)$
(c)	$(a + c)(b + d)$	(f)	$(2 + z)(7 + z)$

4. Multiply the following:

(a)	$(3a + 2)(a + 1)$	(d)	$\left(2pq + (-8)\right)(3pq + 7)$
(b)	$(x + 5)(4x + 3)$	(e)	$\left(8 + (-3y) + (-y^2)\right)\left(2 + (-y)\right)$
(c)	$(1 + n)(8 + 5n)$	(f)	$\left(5y + (-2x)\right)\left(3y + (-x)\right)$

7-5. Multiplicative Inverse.

We found in section 6-4 that every real number has an additive inverse. In other words, for every real number there is another real number such that the sum of the two numbers is 0. Since a given real number remains unchanged when 0 is added to it (Why?), the number 0 is called the identity element for addition.

Is there a corresponding notion of multiplicative inverse for real numbers? First, we must have an identity element for multiplication. Since a given real number remains unchanged when it is multiplied by 1 (Why?), the number 1 is called the identity element for multiplication. For a given real number is there another real number such that the product of the two numbers is 1?

Consider, for example, the number 6. Is there a real number whose product with 6 is 1? By experiment or from your knowledge of arithmetic, you will probably say that $\frac{1}{6}$ is such a number, because $6 \cdot \frac{1}{6} = 1$. Find a number whose product with -2 is 1. Do the same for $-\frac{1}{3}$ and for $\frac{3}{4}$. Before going any further, let us write down a precise definition of <u>multiplicative inverse</u>.

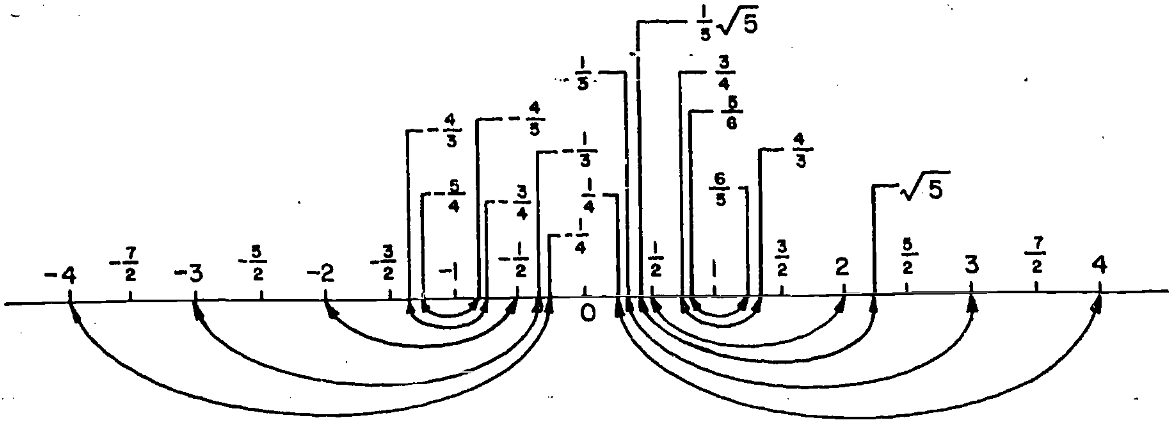
If c and d are real numbers such that

$$cd = 1,$$

then d is called a multiplicative inverse of c.

If d is a multiplicative inverse of c, then is c a multiplicative inverse of d? Why? Does <u>every</u> real number have a multiplicative inverse? What is a multiplicative inverse of 0?

We can observe something of the way these inverses behave by looking at them on the number line. On the diagram below, some numbers and their inverses under multiplication are joined by double arrows. How can you test to see that these pairs of numbers really are multiplicative inverses? Can you visualize the pattern of the double arrows if a great many more pairs of these inverses were similarly marked?



How about the number 0? With what number can 0 be paired? Is there a number b such that $0 \cdot b = 1$? What can you conclude about a multiplicative inverse of 0?

As you look at the double arrows in the above diagram, you may get the impression that the reciprocal of a number must be in a form obtained by interchanging numerator and denominator. What about $\sqrt{5}$? Certainly, $(\frac{1}{5}\sqrt{5})\cdot\sqrt{5} = \frac{1}{5}(\sqrt{5}\cdot\sqrt{5}) = \frac{1}{5}\cdot 5 = 1$, so that $\sqrt{5}$ and $\frac{1}{5}\sqrt{5}$ are reciprocals.

The property toward which we have been working can now be stated. It really is a new property of the real numbers, since it cannot be derived as a consequence of the properties which we have stated up to this point.

> Existence of multiplicative inverses: For every real number c different from 0, there exists a real number d such that $cd = 1$.

That the real numbers actually have this property, if it is not already obvious to you, will become clearer as we do more problems. It is also obvious from experience that each non-zero number has exactly one multiplicative inverse; that is, the multiplicative inverse of a number is unique. We shall assume uniqueness, although it could be proved from the other properties, just as we did for the additive inverse. (See Problem Set 7-8a.)

Problem Set 7-5

1. Find the inverses under multiplication of the following numbers: $3, \frac{1}{2}, -3, -\frac{1}{2}, \frac{3}{4}, 7, \frac{5}{6}, -\frac{3}{7}, -7, \frac{3}{10}, \frac{1}{100}, -\frac{1}{100}, 0.45, -6.8$.
2. Draw a number line and mark off with double arrows the numbers $3, \frac{1}{3}, -3, -\frac{1}{2}, \frac{3}{4}, 7$, and their multiplicative inverses.
3. Draw a number line and mark off with double arrows the numbers $3, \frac{1}{3}, -3, -\frac{1}{2}, \frac{3}{4}, 7$, and their additive inverses. How does the pattern of double arrows differ from the pattern in Problem 2?
4. If b is a multiplicative inverse of a, what values for b do we obtain if a is larger than 1? What values of b do we obtain if a is between 0 and 1? What is a multiplicative inverse of 1?

5. If b is a multiplicative inverse of a, what values for b do we get if a is less than -1? If $a < 0$ and $a > -1$? What is a multiplicative inverse of -1?

6. For inverses under multiplication, what values of the inverse b do you obtain if a is positive? If a is negative?

7-6. Multiplication Property of Equality

In the previous chapter, we stated the addition property of equality. Can we find a corresponding multiplication property? Consider the following statements:

Since $(-2)(3) = -6$, then $\big((-2)(3)\big)(-4) = (-6)(-4)$.

Since $(-5)(-3) = 15$, then $\big((-5)(-3)\big)(\frac{1}{3}) = (15)(\frac{1}{3})$.

Notice that "$(-2)(3)$" and "-6" are different names for the same number, and when we multiply (-4) by this number we obtain "$\big((-2)(3)\big)(-4)$" and "$(-6)(-4)$" as different names for a new number. In general, we have the

Multiplication property of equality. For any real numbers a, b, and c, if $a = b$, then $ac = bc$.

Problem Set 7-6

1. Explain the statement, "Since $(-5)(-3) = 15$, then $\big((-5)(-3)\big)(\frac{1}{3}) = (15)(\frac{1}{3})$," in words in the same manner as above.

2. Which of the following statements are true?
   (a) If $2x = 6$, then $2x(\frac{1}{2}) = 6(\frac{1}{2})$.
   (b) If $\frac{1}{3}a = 9$, then $\frac{1}{3}a(3) = 9(3)$.
   (c) If $\frac{1}{4}n = 12$, then $\frac{1}{4}n(4) = 12(\frac{1}{4})$.
   (d) If $\frac{2}{3}y = 16$, then $\frac{2}{3}y(\frac{3}{2}) = 16(\frac{3}{2})$.
   (e) If $24 = \frac{3}{5}m$, then $24(\frac{3}{5}) = \frac{3}{5}m(\frac{3}{5})$.

3. Find the truth set of each of the following sentences.

Example: Determine the truth set of $\frac{5}{2}x = 60$.

If $\frac{5}{2}x = 60$ is true for some x,

then $\frac{5}{2}x(\frac{2}{5}) = 60(\frac{2}{5})$ is true for the same x;
(Why did we multiply by $\frac{2}{5}$?)

$\left((\frac{5}{2})(\frac{2}{5})\right) x = 24$ is true for the same x;

$x = 24.$

If $x = 24,$

then the left member is $\frac{5}{2}(24) = 60$, and the right member is 60.

So "$\frac{5}{2}(24) = 60$" is a true sentence, and the truth set is $\{24\}$.

(a) $12x = 6$
(b) $7x = 6$
(c) $6x = 6$
(d) $15 = 5y$
(e) $5 = 5y$
(f) $2 = 5y$
(g) $\frac{2}{3}z = 1$
(h) $\frac{2}{3}z = \frac{2}{3}$
(i) $\frac{2}{3}z = \frac{3}{2}$

4. Determine the truth sets of the following open sentences:

(a) $7a = 35$
(b) $\frac{1}{7}x = 5$
(c) $7n = 5$
(d) $\frac{4}{9}c = -2$
(e) $5 = \frac{1}{3}a$
(f) $3 + x = -\frac{3}{2}$
(g) $-12 = 3k$
(h) $0 = \frac{x}{12}$

5. (a) In the formula $V = \frac{1}{3}Bh$, find the value of B if you know that V is 84 and h is 7.

(b) If $P = 5$, $V = 260$, and $v = 100$, use the formula $PV = pv$ to find the value of p.

7-7. Solutions of Equations

In the past, you found possible elements of truth sets of certain sentences, such as the equation

$$3x + 7 = x + 15,$$

and then checked these possibilities. Now we are prepared to solve such equations by a more general procedure. (To "solve" means to find the truth set.)

First, we know that any value of x for which

$$3x + 7 = x + 15$$

is true is also a value of x for which

$$(3x + 7) + \left((-x) + (-7)\right) = (x + 15) + \left((-x) + (-7)\right)$$

is true by the addition property of equality. The numerals in each member of this sentence can be simplified to give

$$\left(3x + (-x)\right) + \left(7 + (-7)\right) = \left(x + (-x)\right) + \left(15 + (-7)\right),$$

$$2x = 8.$$

Here we added the real number $\left((-x) + (-7)\right)$ to each member of the sentence and obtained the new sentence "$2x = 8$." Thus, each number of the truth set of "$3x + 7 = x + 15$" is a number of the truth set of "$2x = 8$," because the addition property of equality holds for all real numbers.

Next, we apply the multiplication property of equality to obtain

$$(\tfrac{1}{2})(2x) = (\tfrac{1}{2})(8),$$

$$x = 4.$$

Thus, each number of the truth set of "$2x = 8$" is a number of the truth set of "$x = 4$."

We can now deduce that every solution of "$3x + 7 = x + 15$" is a solution of "$x = 4$." The solution of the latter equation is obviously 4. But are we sure that 4 is a solution of

"$3x + 7 = x + 15$"? We could easily check that it is, but let us use this example to suggest a general procedure.

The problem is this: We showed that, if x is a solution of

$$3x + 7 = x + 15,$$

then x is a solution of

$$x = 4.$$

What we must now show is that, if x is a solution of

$$x = 4,$$

then x is a solution of

$$3x + 7 = x + 15.$$

These two statements are usually written together as

> x is a solution of "$3x + 7 = x + 15$" if and only if x is a solution of "$x = 4$."

One way to show that the second of these statements is true is to reverse the steps in the proof of the first. Thus, if $x = 4$, we multiply by 2 to obtain

$$2x = 8.$$

(Notice that 2 is the reciprocal of $\frac{1}{2}$.) Then we add $(x + 7)$ to obtain

$$2x + (x + 7) = 8 + (x + 7),$$
$$3x + 7 = x + 15.$$

(Notice that $(x + 7)$ is the opposite of $\left((-x) + (-7)\right)$. Hence, every solution of "$x = 4$" is a solution of "$3x + 7 = x + 15$." That is, the one and only solution is 4.

> We say that "$x = 4$" and "$3x + 7 = x + 15$" are equivalent sentences in the sense that their truth sets are the same.

What have we learned? If to both members of an equation we add a real number or multiply by a non-zero real number, the new sentence obtained is equivalent to the original sentence. This is true because these operations are "reversible." Then if we succeed in obtaining an equivalent sentence whose solution is obvious, we are sure that we have the required truth set without checking. Of course, a check may be desirable to catch mistakes in arithmetic.

As another example, solve the equation

$$5y + 8 = 2y + (-10).$$

This equation is equivalent to

$$(5y + 8) + \big((-2y) + (-8)\big) = \big(2y + (-10)\big) + \big((-2y) + (-8)\big),$$

that is, to

$$\big(5y + (-2y)\big) + \big(8 + (-8)\big) = \big(2y + (-2y)\big) + \big((-10) + (-8)\big)$$

and to

$$3y = -18.$$

In other words, y is a solution of "$5y + 8 = 2y + (-10)$" if and only if y is a solution of "$3y = -18$." The latter sentence is equivalent to

$$(\tfrac{1}{3})(3y) = (\tfrac{1}{3})(-18),$$

that is, to

$$y = -6.$$

Thus y is a solution of "$3y = -18$" if and only if y is a solution of "$y = -6$." Hence, all three sentences are equivalent, and their truth set is $\{-6\}$. Here, we were certain that each step was reversible without actually doing it. When we solve an equation we ask ourselves at each step, "Is this step reversible?" If it is, we obtain an equivalent equation.

Later, we shall learn how to solve other types of sentences by means of applying properties of numbers. To do this we shall learn more about operations which yield equivalent sentences and others which do not.

Problem Set 7-7

1. For each of the pairs of sentences given below, show how the second is obtained from the first. Show if possible how the first can be obtained from the second. Which pairs of sentences are equivalent?

   (a) $x + (-3) = 5$, $x = 8$
   (b) $\frac{1}{2}b = 8$, $b = 16$
   (c) $\left(x + (-2)\right) = 3$, $3\left(x + (-2)\right) = 9$
   (d) $z + (-7) = -z + 3$, $2z = 10$
   (e) $6x = 7$, $(6x)0 = 7 \cdot 0$
   (f) $4y + (-6) = 5y + (-6)$, $0 = y$
   (g) $-3a = -6$, $a = 2$
   (h) $5m + 5 = -m + (-7)$, $12 = -6m$
   (i) $x = 3$, $|x| = |3|$

2. Find the truth set of each of the following equations.

   Example:

   Solve: $(-3) + 4x = (-2x) + (-1)$.

   This sentence is equivalent to

$$\left((-3) + 4x\right) + (2x + 3) = \left((-2x) + (-1)\right) + (2x + 3),$$
$$6x = 2;$$

   and this is equivalent to

$$\frac{1}{6}(6x) = \frac{1}{6}\cdot 2,$$
$$x = \frac{1}{3}.$$

   Hence, the truth set is $\{\frac{1}{3}\}$.

   (a) $2a + 5 = 17$
   (b) $4y + 3 = 3y + 5 + y + (-2)$
   (c) $12x + (-6) = 7x + 24$

(d) $8x + (-3x) + 2 = 7x + 8$ (Collect terms first.)
(e) $6z + 9 + (-4z) = 18 + 2z$
(f) $12n + 5n + (-4) = 3n + (-4) + 2n$
(g) $15 = 6x + (-8) + 17x$
(h) $5y + 8 = 7y + 3 + (-2y) + 5$
(i) $(-6a) + (-4) + 2a = 3 + (-a)$
(j) $0.5 + 1.5x + (-1.5) = 2.5x + 2$
(k) $\frac{1}{2} + (-\frac{1}{2}c) + (-\frac{5}{2}) = 4c + (-2) + (-\frac{7}{2}c)$
(l) $2y + (-6) + 7y = 8 + (-9y) + (-10) + 18y$
(m) $(x + 1)(x + 2) = x(x + 2) + 3$

3. Translate the following into open sentences and find their truth sets, then answer the question in each problem.

(a) The perimeter of a triangle is 44 inches. The second side is three inches more than twice the length of the third side, and the first side is five inches longer than the third side. Find the lengths of the three sides of this triangle.

(b) If an integer and its successor are added, the result is one more than twice that integer. What is the integer?

(c) The sum of two consecutive odd integers is 11. What are the integers?

(d) Mr. Johnson bought 30 ft. of wire and later bought 55 more feet of the same kind of wire. He found that he paid $4.20 more than his neighbor paid for 25 ft. of the same kind of wire at the same price. What was the cost per foot of the wire?

(e) Four times an integer is ten more than twice the successor of that integer. What is the integer?

*(f) In an automobile race, one driver, starting with the first group of cars, drove for 5 hours at a certain speed and was then 120 miles from the finish line. Another driver, who set out with a later heat, had traveled at the same rate as the first driver for 3 hours and was 250 miles from the finish. How fast were these men driving?

(g) Plant A grows two inches each week, and it is now 20 inches tall. Plant B grows three inches each week, and it is now 12 inches tall. How many weeks from now will they be equally tall?

(h) A number is increased by 17 and the sum is multiplied by 3. If the resulting product is 192, what is the number?

(i) One number is 5 times another and their sum is 15 more than 4 times the smaller. What is the smaller number?

(j) Two quarts of alcohol are added to the water in a radiator and the mixture then contains 20 percent alcohol. How many quarts of water are in the radiator?

7-8. Reciprocals

We shall find it convenient to use the shorter name "reciprocal" for the multiplicative inverse, and we represent the reciprocal of a by the symbol "$\frac{1}{a}$". Thus for every a except 0, $a \cdot \frac{1}{a} = 1$.

You probably noticed that for positive integers the symbol we chose for "reciprocal" is the familiar symbol of a fraction. Thus, the reciprocal of 5 is $\frac{1}{5}$. This certainly agrees with your former experience.

The reciprocal of $\frac{2}{3}$, however, is $\frac{1}{\frac{2}{3}}$; of -9 is $\frac{1}{-9}$; of 2.5 is $\frac{1}{2.5}$. Since $\frac{1}{\frac{2}{3}}$ is the reciprocal of $\frac{2}{3}$, and $\frac{2}{3} \times \frac{3}{2} = 1$, it follows that $\frac{1}{\frac{2}{3}}$ and $\frac{3}{2}$ must name the same number; since $\frac{1}{-9}$ is the reciprocal of -9 and since $-9 \times (-\frac{1}{9}) = 1$, $\frac{1}{-9}$ and $-\frac{1}{9}$ must name the same number. Since $(2.5)(0.4) = 1$, 0.4 and $\frac{1}{2.5}$ must name the same number. We shall be in a better position to continue this discussion after we consider division of real numbers in a later chapter.

Problem Set 7-8a

1. If the reciprocal of any non-zero number a is represented by the symbol "$\frac{1}{a}$", represent the reciprocal of:
   - (a) 15
   - (b) -8
   - (c) $\frac{1}{5}$
   - (d) $-\frac{1}{6}$
   - (e) $\frac{5}{3}$
   - (f) 0.3
   - (g) $-\frac{3}{4}$

2. Write the common name for the reciprocal of each number of Problem 1. In which cases is it the same as the reciprocal written in Problem 1?

3. Prove the theorem: For each non-zero real number a there is only one multiplicative inverse of a. (Hint: We know there is a multiplicative inverse of a, namely $\frac{1}{a}$. Assume there is another, say x. Then ax = 1.)

Why did we exclude 0 from our definition of reciprocals? Suppose 0 did have a reciprocal. What could it be? If there were a number b which is the reciprocal of 0, then $0 \cdot b = 1$. What is the truth set of the sentence $0 \cdot b = 1$? You should conclude that 0 simply cannot have a reciprocal. Here we have an opportunity to demonstrate, using a rather simple example, a very powerful type of proof. This proof depends on the fact that a given sentence is either true or it is false, but not both. An assertion that a given sentence is both true and false is called a contradiction. If we reach a contradiction in a chain of correct reasoning, then we are forced to admit that the reasoning is based on a false statement. This is the idea behind the proof of the next theorem.

Theorem 7-8a. The number 0 has no reciprocal.

Proof: The sentence "0 has no reciprocal" is either true or false, but not both. Assuming that it is false, that is, assuming that 0 does have a reciprocal, we have the following chain of reasoning:

1. There is a real number a such that $0 \cdot a = 1$.	Assumption.
2. $0 \cdot a = 0$	The product of 0 with any real number is 0.
3. Therefore $0 = 1$.	

Thus we are led to the assertion that "$0 = 1$" is a true sentence. But "$0 = 1$" is obviously a false sentence, and so we have a contradiction. Conclusion: Step 1 of the argument cannot be true. Therefore it is false that 0 has a reciprocal; that is, 0 has no reciprocal.

A proof of the above type is called indirect or a proof by contradiction or a reductio ad absurdum.

We should like now to see what we can discover and what we can prove about the way reciprocals behave.

In each of the following sets of numbers, find the reciprocals. What conclusion do you draw about reciprocals on examining the two sets?

$$\text{I:} \quad 12, \frac{1}{8}, 150, 0.09, \frac{8}{9}$$

$$\text{II:} \quad -5, -\frac{1}{3}, -700, -2.2, -\frac{5}{3}$$

Observation of reciprocals on the number line strengthens our belief that the following theorem is true.

Theorem 7-8b. The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.

Proof: We know that $a \cdot \frac{1}{a} = 1$; that is, the product of a non-zero number and its reciprocal is the positive number 1. Let us first assume that a is positive. Then exactly one of three possibilities must be true:

$$\frac{1}{a} < 0, \ \frac{1}{a} = 0, \ 0 < \frac{1}{a}.$$

We see that if $\frac{1}{a}$ is negative, then $a \cdot \frac{1}{a}$ is negative, a contradiction of the fact that $a \cdot \frac{1}{a}$ is positive. Also, if $\frac{1}{a} = 0$ when a is positive, then $a \cdot \frac{1}{a} = 0$, again a contradiction. This leaves but one remaining possibility: $\frac{1}{a}$ is positive.

In the same way, we may show that if a is negative, then $\frac{1}{a}$ is negative.

For each of the following numbers, find the reciprocal of the number; then find the reciprocal of that reciprocal. What conclusion is suggested?

$$-12, \ 80, \ \frac{19}{20}, \ -\frac{1}{9}, \ 1.6$$

Theorem 7-8c. The reciprocal of the reciprocal of a non-zero real number a is a itself.

Proof: The reciprocal of the reciprocal of a is $\frac{1}{\frac{1}{a}}$. Since $\frac{1}{\frac{1}{a}}$ and a are both reciprocals of $\frac{1}{a}$ (why?), and since there is exactly one reciprocal of $\frac{1}{a}$, it follows that "$\frac{1}{\frac{1}{a}}$" and "a" must be names of the same number. Hence,

$$\frac{1}{\frac{1}{a}} = a.$$

Problem Set 7-8b

1. Find the reciprocals of the following numbers:

   $\frac{3}{4}$, 0.3, -0.3, 0.33, -0.33, 1, -1, $\sqrt{2}$, $\frac{1}{x^2 + 4}$, $y^2 + 1$.

2. For what real values of a do the following numbers have no reciprocals?

   $a + (-1)$, $a + 1$, $a^2 + (-1)$, $a(a + 1)$, $\frac{a}{a + 1}$, $a^2 + 1$, $\frac{1}{a^2 + 1}$.

3. Consider the sentence

   $$\left(a + (-3)\right)(a + 1) = a + (-3),$$

   which has the truth set {0, 3}. (Verify this fact.) If both members of the sentence are multiplied by the reciprocal of $a + (-3)$, that is by $\frac{1}{a + (-3)}$, and some properties of real numbers are used (which properties?), we obtain the sentence

   $$a + 1 = 1.$$

   For $a = 3$, we have $3 + 1 = 1$, and this is clearly a false sentence. Why doesn't the new sentence have the same truth set as the original sentence?

4. Write a property of opposites which corresponds to Theorem 7-8b. Write a property of opposites which corresponds to Theorem 7-8c.

5. Consider three pairs of numbers: (1) $a = 2$, $b = 3$; (2) $a = 4$, $b = -5$; (3) $a = -4$, $b = -7$. Does the sentence $\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab}$ hold true in all three cases?

6. Is the sentence $\frac{1}{a} > \frac{1}{b}$ true in all three cases of Problem 5? Show the numbers and their reciprocals on the number line.

7. Is it true that if $a > b$ and a, b are positive, then $\frac{1}{b} > \frac{1}{a}$? Try this for some particular values of a and b.

8. Is it true that if $a > b$ and a, b are negative, then $\frac{1}{b} > \frac{1}{a}$? Substitute some particular values of a and b.

9. Could you tell immediately which reciprocal is greater than another if one of the numbers is positive and the other is negative? Illustrate on the number line.

*10. Finish the proof of Theorem 7-8b for the case when a is negative.

In Problem 5 above you showed that for three particular pairs of values of a and b, $\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab}$. In other words, the product of the reciprocals of these two numbers is the reciprocal of their product. How many times would we need to test this sentence for particular numbers in order to be sure it is true for all real numbers except zero? Would 1,000,000 tests be enough? How would we know that the sentence would not be false for the 1000,001st test?

We can often reach <u>probable</u> conclusions by observing what happens in a number of particular cases. We call this <u>inductive reasoning</u>. No matter how many cases we observe, inductive reasoning alone cannot assure us that a statement is <u>always</u> true.

Thus we cannot use inductive reasoning to prove that $\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab}$ is always true. We <u>can</u> prove it for all non-zero real numbers by <u>deductive</u> reasoning as follows. (Remember that in the proof we may use only properties which have already been stated.)

<u>Theorem</u> 7-8d. For any non-zero real numbers a and b,

$$\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab}.$$

Discussion: Since our object is to prove that $\frac{1}{a}\cdot\frac{1}{b}$ is the reciprocal of ab, we recall the definition of reciprocal. Then we concentrate on the product $ab\cdot(\frac{1}{a}\cdot\frac{1}{b})$ and try to show that it is 1.

Proof: $ab\cdot(\frac{1}{a}\cdot\frac{1}{b}) = a(\frac{1}{a})\cdot b(\frac{1}{b})$ by commutative and associative properties

$= 1\cdot 1$ since $x\cdot\frac{1}{x} = 1$

$= 1$

Hence, $\frac{1}{a}\cdot\frac{1}{b}$ is the reciprocal of ab. In other words,

$$\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab} .$$

Notice how closely the proof of Theorem 7-8d parallels the proof that the sum of the opposites of two numbers is the opposite of their sum. Remember how this result was proved: $(a + b) + \left((-a) + (-b)\right) = \left(a + (-a)\right) + \left(b + (-b)\right) = 0$; hence, $(-a) + (-b) = -(a + b)$.

Problem Set 7-8c

1. Do the following multiplications. (In these and in future problem sets we assume that the values of the variables are such that the fractions have meaning.)

   (a) $(\frac{1}{2a})(\frac{1}{3b})$

   (b) $(\frac{1}{x})(\frac{1}{3ax})$

   (c) $(\frac{1}{-3y})(\frac{1}{-7z})$

   (d) $\left(\frac{1}{\frac{1}{3}ab}\right)\left(\frac{1}{9a^2}\right)$

   (e) $\left(\frac{1}{-2m^2n}\right)\left(-\frac{1}{3mn^2}\right)$

   (f) $\left(\frac{1}{x}\right)\left(\frac{1}{\frac{1}{x}}\right)$

2. What is the value of $87 \times (-9) \times 0 \times \frac{2}{3} \times 642$?

3. Is $8\cdot 17 = 0$ a true sentence? Why?

4. If $n\cdot 50 = 0$, what can you say about n?

5. If $p\cdot 0 = 0$, what can you say about p?

6. If $p\cdot q = 0$, what can you say about p or q?

7. If $p\cdot q = 0$, and we know that $p > 10$, what can we say about q?

The idea suggested by the above exercises will be a very useful one, especially in finding truth sets of certain equations. We are able to prove the following theorem now by using the properties of reciprocals.

Theorem 7-8e. For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Because of the "if and only if," we really must prove two theorems: (1) If $a = 0$ or $b = 0$, then $ab = 0$; (2) If $ab = 0$, then $a = 0$ or $b = 0$.

Proof: If $a = 0$ or $b = 0$, then $ab = 0$ by the multiplication property of 0. Thus, we have proved one part of the theorem.

To prove the second part of the theorem, note that either $a = 0$ or $a \neq 0$, but not both. If $a = 0$, the requirement that $a = 0$ or $b = 0$ is satisfied. Why?

If $ab = 0$ and $a \neq 0$, then there is a reciprocal of a and

$$\begin{aligned}
(\tfrac{1}{a})(ab) &= \tfrac{1}{a}\cdot 0, && \text{(Why?)}\\
(\tfrac{1}{a})(ab) &= 0, && \text{(Why?)}\\
(\tfrac{1}{a}\cdot a)b &= 0, && \text{(Why?)}\\
1\cdot b &= 0, && \text{(Why?)}\\
b &= 0.
\end{aligned}$$

Thus, in this case also the requirement that $a = 0$ or $b = 0$ is satisfied; hence, we have proved the second part of the theorem.

Problem Set 7-8d

1. If $\left(x + (-5)\right)\cdot 7 = 0$, what must be true about 7 or $\left(x + (-5)\right)$? Can 7 be equal to 0? What about $x + (-5)$ then?

2. Explain how we know that the only value of y which will make $9 \times y \times 17 \times 3 = 0$ a true sentence is 0.

*3. If a is between p and q, is $\frac{1}{a}$ between $\frac{1}{p}$ and $\frac{1}{q}$? Explain.

4. Theorem 7-8e enables us to determine the truth set of an equation such as

$$\left(x + (-3)\right)\left(x + (-8)\right) = 0$$

without guesswork. With $a = \left(x + (-3)\right)$ and $b = \left(x + (-8)\right)$, the theorem tells us that this sentence is equivalent to the sentence

$$x + (-3) = 0 \quad \text{or} \quad x + (-8) = 0.$$

From this sentence we read off the truth set as $\{3, 8\}$. Find the truth set of each of the following equations:

(a) $\left(x + (-20)\right)\left(x + (-100)\right) = 0$
(b) $(x + 6)(x + 9) = 0$
(c) $x\left(x + (-4)\right) = 0$
(d) $\left(3x + (-5)\right)\left(2x + (-1)\right) = 0$
(e) $\left(x + (-1)\right)\left(x + (-2)\right)\left(x + (-3)\right) = 0$
(f) $2\left(x + (-\frac{1}{2})\right)(x + \frac{3}{4}) = 0$
(g) $\left(3x + (-5)\right)\left(2x + 1\right) = 0$
(h) $9|x + (-6)| = 0$
(i) $x(x + 4) = x^2 + 8$

5. Prove: If a, b, c are real numbers, and if $ac = bc$ and $c \neq 0$, then $a = b$.

7-9. The Two Basic Operations and the Inverse of a Number Under These Operations

In the last two chapters we have focused our attention on addition and multiplication and on the inverses under these two operations. These four concepts are basic to the real number system. Addition and multiplication have a number of properties by themselves, and one property connects addition with multiplication, namely, the distributive property. All our work in algebraic simplification rests on these properties and on the various consequences of them which relate addition, multiplication, opposite, and reciprocal.

We have pointed out that the distributive property connects addition and multiplication. It is instructive to see whether some relationship occurs which connects every combination of addition, multiplication, opposite, and reciprocal in pairs. Let us write down all possible combinations.

1. Addition and multiplication: The distributive property, $a(b + c) = ab + ac$.
2. Addition and opposite: We have proved that $-(a + b) = (-a) + (-b)$.
3. Addition and reciprocal: We find that there <u>is</u> <u>no</u> <u>simple</u> relationship connecting $\frac{1}{a} + \frac{1}{b}$ and $\frac{1}{a + b}$. In fact, there are no real numbers at all for which these two phrases represent the same number. This unfortunate lack of relationship is considerable cause of trouble in algebra for students who unthinkingly assume that these expressions represent the same number.
4. Multiplication and opposite: We have proved that $-(ab) = (-a)(b) = (a)(-b)$.
5. Multiplication and reciprocal: We have proved that $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.
6. Opposite and reciprocal: $\frac{1}{(-a)} = -(\frac{1}{a})$.

   This last relation is a new one and should be proved. The proof may be obtained from (5) above by replacing b by -1. The proof is left to the students. (Hint: What is the reciprocal of -1?)

State (1), (2), (4), (5), (6) in words. Do you see any similarity between addition and opposite, on the one hand, and multiplication and reciprocal, on the other, in these properties? Explain.

Review Problems

1. Write a summary of the important ideas in this chapter, similar to that written at the end of Chapter 6.

2. Change to indicated sums:
   (a) $3a\left(a + (-2)\right)$
   (b) $(x + 1)(x + 6)$
   (c) $(a + b)\left(a + (-b)\right)$
   (d) $\left(m + (-5)\right)^2$
   (e) $\left(x + (-4)\right)(2x + 3)$

3. Write each of the following as an indicated product:
   (a) $2ax + 2ay$
   (b) $ac + (-bc) + c$
   (c) $c(a + b)x + (a + b)y$
   (d) $10x^2 + (-15x) + (-5)$
   (e) $9x^3 + 6x^2 + (-3x)$

4. Find the truth set of each of the following equations:
   (a) $4a + 7 = 2a + 11$
   (b) $8x + (-18) = 3x + 17$
   (c) $7x + 2 + (-5x) = 3 + 2x + (-1)$
   (d) $|-2| + 2x = (-3) + 3x + 5$
   (e) $3x^2 + (-2)x = x^2 + 2 + 2x^2$

5. Collect terms in the following phrases:
   (a) $3a + b + a + (-2b) + 4b$
   (b) $7x + b + (-3x) + (-3b)$
   (c) $6a + (-7a) + 13.2 + (-5)a + (-8.6)$
   (d) $|x| + 3|-x| + (-2)|-x|$

*6. Given the set $S = \{-2, -1, 0, 2\}$
   (a) Find the set P of all products of the elements of S taken 3 at a time. (Hint: First find the set of all products of pairs of elements. Then find the set of the products of each element of this new set with each element of S.)
   (b) Find the set R of all sums of elements of S taken 3 at a time.

7. For each of the following problems, write an open sentence, find its truth set, and answer the question asked in the problem.
   (a) Jim and I plan to buy a basketball. Jim is working, so he agrees to pay \$2 more than I pay. If the basketball costs \$11, how much does Jim pay?
   (b) The sum of two consecutive odd integers is 41. What are the integers?
   (c) The length of a rectangle is 27 yards more than the width. The perimeter is 398 yards. Find the length and the width.
   (d) Mary and Jim added their grades on a test and found the sum to be 170. They subtracted the grades and Mary's grade was 14 points higher than Jim's. What were their grades?
   (e) A man worked 4 days on a job and his son worked half as long. The son's daily wage was $\frac{2}{5}$ that of his father. If they earned a total of \$96, what were their daily wages?
   (f) In a farmer's yard were some pigs and chickens, and no other creatures except the farmer himself. There were, in fact, sixteen more chickens than pigs. Observing this fact, and further observing that there were 74 feet in the yard, not counting his own, the farmer exclaimed happily to himself--for he was a mathematician as well as a farmer, and was given to talking to himself--"Now I can tell how many of each kind of creature there are in my yard." How many were there? (Hint: Pigs have 4 feet, chickens 2 feet.)
   (g) At the target shooting booth at a fair, Montmorency was paid 10¢ for each time he hit the target, and was charged 5 ¢ each time he missed. If he lost 25¢ at the booth and made ten more misses than hits, how many hits did he make?

Chapter 8

PROPERTIES OF ORDER

8-1. The Order Relation for Real Numbers

In Chapter 5 we extended the concept of order from the numbers of arithmetic to all real numbers. This was done by using the number line, and we agreed that:

> "Is less than," for real numbers, means "to the left of" on the real number line.
>
> If a and b are real numbers, then "a is less than b" is written "$a < b$."

We speak of the relation "is less than" for real numbers as an <u>order relation</u>. It is a <u>binary</u> relation since it expresses a relation between <u>two</u> numbers. What are some of the facts which we already know about the order relation? What are some of its general properties?

Two basic properties of the order relation for real numbers were obtained in Section 5-2.

> <u>Comparison property</u>: If a is a real number, then exactly <u>one</u> of the following is true:
>
> $$a < b, \quad a = b, \quad b < a.$$
>
> <u>Transitive property</u>: If a, b, c are real numbers and if $a < b$ and $b < c$, then $a < c$.

Another property of order which was obtained in Section 5-3 connects the order relation with the operation of taking opposites:

> If a and b are real numbers and if $a < b$, then $-b < -a$.

You may wonder at this point why we are so careful to avoid talking about "greater than". As a matter of fact, the relation "is greater than", for which we use the symbol "$>$", is also an order relation. Does this order relation have the comparison property and the transitive property? Since it does, we actually have two different (though very closely connected!) order relations for the real numbers, and we have chosen to concentrate our attention on "less than". We could have decided to concentrate on "greater than"; but if we are going to study an order relation and its properties, we must not confuse the issue by shifting from one order relation to another in the middle of the discussion.

Thus we state the last property mentioned above in terms of "<", but in applying the property we feel free to say, "If $a < b$, then $-a > -b$".

In the next two sections we obtain some properties of the order relation "<" which involve the operations of addition and multiplication. Such properties are essential if we are to make much use of the order relation in algebra.

Problem Set 8-1

1. For each pair of numbers, determine their order.

   (a) $-\frac{3}{2}, -\frac{4}{3}$

   (b) $-|-7|, -|7|$

   (c) $c, 1$ (Consider the comparison property.)

2. Continuing Problem 1(c), what can you say about the order of c and 1 if it is known that $c > 4$? What property of order did you use here?

3. Decide in each case whether the sentence is true.

   (a) $-3 + (-2) < 2 + (-2)$

   (b) $(-3) + (0) < 2 + 0$

   (c) $(-3) + 5 < 2 + 5$

   (d) $(-3) + a < 2 + a$

4. Decide in each case whether the sentence is true.

   (a) $(-3)(5) < (2)(5)$

   (b) $(-3)(0) < (2)(0)$

   (c) $(-3)(-2) < (2)(-2)$

   (d) $(-3)(a) < (2)(a)$ (What is the truth set of this sentence?)

5. Decide in each case whether the sentence is true.
   - (a) $|4|(3 + (-2)) < |-6|(2 + (-3))$
   - (b) $(-\frac{3}{2})|\frac{2}{3}| < |-\frac{3}{2}|(-\frac{2}{3})$
   - (c) $(|-2| + (-2))((-4) + (-6)) < -(-3)|-4|$
   - (d) $\frac{4}{5}(|2| + |-2|) + 3(-|-2| + (-2)) < 0$

6. A given set may be described in many ways. Describe in three ways the truth set of
   - (a) $3 < 3 + x$
   - (b) $3 + x < 3$

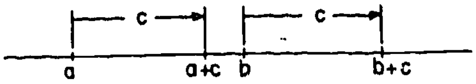
7. Determine the truth set of
   - (a) $3 = 2 + x$
   - (b) $3 = (-2) + x$
   - (c) $-3 = (-4) + x$
   - (d) $\pi = \sqrt{2} + c$

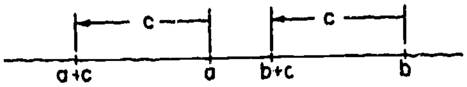
8. Determine the truth set of
   - (a) $y < 3$
   - (b) $|y| < 3$
   - (c) $-|y| < 3$
   - (d) $-y < 3$

8-2. Addition Property of Order

What is the connection between order of numbers and addition of numbers? We shall find a basic property, and from it prove other properties which relate order and addition. As before, we concentrate on the order relation "<"; similar properties can be stated for the order relation ">".

It is helpful to view addition and order on the number line. We remember that adding a positive number means moving to the right; adding a negative number means moving to the left. Let us fix *two* points a and b on the number line, with $a < b$. If we add the same number c to a and to b, we move to the right of a and of b if c is *positive*, to the left if c is *negative*. We could think of two men walking on the number line carrying a ladder between them. At the start the man at a is to the left of the man at b. If they walk c units in <u>either</u> direction, the fixed length of the ladder will insure that the man to the left will stay to the left. In their new positions the man at $a + c$ will still be to the left of the man at $b + c$. Thus,





Here we have found a fundamental property of order which we shall assume for all real numbers.

> <u>Addition Property of Order</u>. If a, b, c are real numbers and if $a < b$, then
> $$a + c < b + c.$$

Illustrate this property for $a = -3$ and $b = -\frac{1}{2}$, with c having, successively, the values -3, $\frac{1}{2}$, 0, -7. Here $-3 < -\frac{1}{2}$. Is "$(-3) + (-3) < (-\frac{1}{2}) + (-3)$" a true sentence? Continue with the other values of c. Phrase the addition property of order in words. Is there a corresponding property of equality?

<u>Problem Set 8-2a</u>

1. By applying the addition properties of order, determine which of the following sentences are true.
   (a) $(-\frac{6}{5}) + 4 < (-\frac{3}{4}) + 4$
   (b) $(-\frac{5}{3})(\frac{6}{5}) + (-5) > (-\frac{5}{2}) + (-5)$
   (c) $(-5.3) + (-2)(-\frac{4}{3}) < (-0.4) + \frac{8}{3}$
   (d) $(\frac{5}{2})(-\frac{3}{4}) + 2 \geq (-\frac{15}{8}) + 2$
2. Formulate an addition property of order for each of the relations "$\leq$", "$>$", "$\geq$".

3. An extension of the order property states that:

   If a, b, c, d are real numbers such that
   $a < b$ and $c < d$, then $a + c < b + d$.

   This can be proved in three steps. Give the reason for each step:

   If $a < b$, then $a + c < b + c$;
   if $c < d$, then $b + c < b + d$;
   hence,

   $$a + c < b + d.$$

4. Find the truth set of each of the following sentences.

   Example: If $(-\frac{3}{2}) + x < (-5) + \frac{3}{2}$ is true for some x,

   then $x < (-5) + \frac{3}{2} + \frac{3}{2}$ is true for the same x.

   $x < -2$ is true for the same x.

   Thus, if x is a number which makes the original sentence true, then $x < -2$. If "$x < -2$" is true for some x, then

   $(-\frac{3}{2}) + x < (-\frac{3}{2}) + (-2)$ is true for the same x,

   $(-\frac{3}{2}) + x < (-\frac{3}{2}) + \left((-5) + 3\right)$,

   $(-\frac{3}{2}) + x < (-5) + \left(3 + (-\frac{3}{2})\right)$,

   $(-\frac{3}{2}) + x < (-5) + \frac{3}{2}$ is true for the same x.

   Hence, the truth set is the set of all real numbers less than -2.

   (a) $3 + x < (-4)$
   (b) $x + (-2) > -3$
   (c) $2x < (-5) + x$
   (d) $3x > \frac{4}{3} + 2x$
   (e) $(-\frac{2}{3}) + 2x \geq \frac{5}{3} + x$
   (f) $(-x) + 4 < (-3) + |-3|$
   (g) $(-5) + (-x) < \frac{2}{3} + |-\frac{4}{3}|$
   (h) $(-2) + 2x < (-3) + 3x + 5$
   (i) $(-\frac{3}{4}) + \frac{5}{4} \geq x + |-\frac{3}{2}|$

5. Graph the truth sets of parts (a), (c), and (h) of Problem 4.

6. In Chapter 5 the following property of order was stated: "If $a < b$, then $-b < -a$." Prove this property, using the addition property of order. (Hint: Add $\left((-a) + (-b)\right)$ to both members of the inequality $a < b$; then use property of additive inverses.)

*7. Show that the property:

"If $0 < y$, then $x < x + y$."

is a special case of the addition property of order. (Hint: In the statement of the addition property of order, let $a = 0$, $b = y$, $c = x$.)

Many results about order can be proved as consequences of the addition property of order. Two of these are of special interest to us, because they give direct translations back and forth between statements about order and statements about equality.

The first of these results will be a special case of the property. Let us consider a few numerical examples of the property with $a = 0$. If $a = 0$, then "$a < b$" becomes "$0 < b$"; that is, b is a positive number. Thus, we may write: If $0 < b$, then $c + 0 < c + b$.

Let $a = 0$, $b = 3$ and $c = 4$; then $4 + 0 < 4 + 3$; that is, since $7 = 4 + 3$, then $4 < 7$.

Let $a = 0$, $b = 5$ and $c = -4$; then $(-4) + 0 < (-4) + 5$; that is, since $1 = (-4) + 5$, then $-4 < 1$.

These two examples can be thought of as saying:

Since $7 = 4 + 3$ and 3 is a positive number, then

$$4 < 7.$$

Since $1 = (-4) + 5$ and 5 is a positive number, then

$$-4 < 1.$$

This result we state as

Theorem 8-2a. If $z = x + y$ and y is a positive number, then $x < z$.

Proof. We may change the addition property of order to read:

If $a < b$, then $c + a < c + b$. (Why?)

Since the property is true for all real numbers a, b, c, we may let $a = 0$, $b = y$, $c = x$. Thus,

if $0 < y$, then $x + 0 < x + y$.

If $z = x + y$, then "$x + 0 < x + y$" means "$x < z$". Hence we have proved that if $z = x + y$ and $0 < y$ (y is positive) then $x < z$.

Theorem 8-2a now gives us a translation from a statement about equality, such as

$$-4 = (-6) + 2,$$

to a statement about order, in this case

$$-6 < -4.$$

Notice that adding 2, a positive number, to (-6) yields a number to the right of -6.

Change the sentence

$$4 = (-2) + 6$$

to a sentence involving order.

The second result of the addition property is a theorem which translates from order to equality, instead of from equality to order, as Theorem 8-2a does. You have seen that if y is positive and x is any number, then x is always less than $x + y$. If $x < z$, then does there exist a positive number y such that $z = x + y$? Consider, for example, the numbers 5 and 7 and note that $5 < 7$. What is the number y such that

$$7 = 5 + y?$$

How did you determine y? Did you find y to be positive? Consider the numbers -3 and -6, noting that $-6 < -3$. What is the truth set of

$$-3 = -6 + y?$$

Is y again positive?

$4 < 9,$	$9 = 4 + (\quad)$
$-3 < 5,$	$5 = (-3) + (\quad)$
$-4 < -1$	$-1 = (-4) + (\quad)$
$-6 < 0$	$0 = (-6) + (\quad)$

What kind of number makes each of the above equations true? In each case you added a positive number to the smaller number to get the greater.

By this time you see that the theorem we have in mind is

> Theorem 8-2b. If x and z are two real numbers such that $x < z$, then there is a positive real number y such that
> $$z = x + y.$$

*Proof. There are really two things to be proved. First, we must find a value of y such that $z = x + y$; second, we must prove that the y we found is positive, if $x < z$.

It is not hard to find a value of y such that $z = x + y$. Your experience with solving equations probably suggests adding $(-x)$ to both members of "$z = x + y$" to obtain $y = z + (-x)$.

Let us try this value of y. Let

$$y = z + (-x).$$

Then

$$x + y = x + \big(z + (-x)\big) \qquad \text{(Why?)}$$
$$= \big(x + (-x)\big) + z \qquad \text{(Why?)}$$
$$= 0 + z$$
$$x + y = z$$

Thus we have found a y, namely $z + (-x)$, such that $z = x + y$.

It remains to show that if $x < z$, then this y is positive. We know there is exactly one true sentence among these: y is negative, y is zero, y is positive. (Why?) If we can show that two of these possibilities are false, the third must be true. Try the first possibility: If it were true that y is negative and $z = x + y$, then the addition property of order would assert that $z < x$. (Let $a = y$, $b = 0$, $c = x$.) But this contradicts the fact that $x < z$; so it cannot be true that y is negative. Try the second possibility: If it were true that y is zero and $z = x + y$, then it would be true that $z = x$. This again contradicts the fact that $z < x$; so it cannot be true that y is zero. Hence, we are left with only one possibility, y is positive, which must be true. This completes the proof.

Theorem 8-2b allows us to translate from a sentence involving order to one involving equality. Thus,

$$-5 < -2$$

can be replaced by

$$-2 = (-5) + 3,$$

which gives the same "order information" about -5 and -2. That is, there is a positive number, 3, which when added to the lesser, -5, yields the greater, -2.

Problem Set 8-2b

1. For each pair of numbers, determine their order and find the positive number b which when added to the smaller gives the larger.

   (a) -15 and -24

   (b) $\frac{63}{4}$ and $-\frac{5}{4}$

   (c) $\frac{6}{5}$ and $\frac{7}{10}$

   (d) $-\frac{1}{2}$ and $\frac{1}{3}$

   (e) -254 and -345

   (f) $-\frac{33}{13}$ and $-\frac{98}{39}$

   (g) 1.47 and -0.21

   (h) $(-\frac{2}{3})(\frac{4}{5})$ and $(\frac{3}{2})(-\frac{5}{4})$

2. Show that the following is a true statement: If a and c are real numbers and if $c < a$, then there is a negative real number b such that $c = a + b$. (Hint: Follow the similar discussion for b positive.)

3. Which of the following sentences are true for all real values of the variables?

(a) If $a + 1 = b$, then $a < b$.
(b) If $a + (-1) = b$, then $a < b$.
(c) If $(a + c) + 2 = (b + c)$, then $a + c < b + c$.
(d) If $(a + c) + (-2) = (b + c)$, then $b + c < a + c$.
(e) If $a < -2$, then there is a positive number d such that $-2 = a + d$.
(f) If $-2 < a$, then there is a positive number d such that $a = (-2) + d$.

4. (a) Use $5 + 8 = 13$ to suggest two true sentences involving "<" relating pairs of the numbers 5, 8, 13.
(b) Since $(-3) + 2 = (-1)$, how many true sentences involving "<" can you write using pairs of these three numbers?
(c) If $5 < 7$, write two true sentences involving "=" relating the numbers 5, 7.

5. Show on the number line that if a and c are real numbers and if b is a negative number such that $c = a + b$, then $c < a$.

6. Which of the following sentences are true for all values of the variables?
(a) If $b < 0$, then $3 + b < b$.
(b) If $b < 0$, then $3 + b < 3$.
(c) If $x < 2$, then $2x < 4$.

7. Verify that each of the following is true.
(a) $|3 + 4| \leq |3| + |4|$
(b) $|(-3) + 4| \leq |-3| + |4|$
(c) $|(-3) + (-4)| \leq |-3| + |-4|$
(d) State a general property relating $|a + b|$, $|a|$ and $|b|$ for any real numbers a and b.

8. What general property can be stated for multiplication similar to the property for addition in Problem 7?

9. Translate the following into open sentences and find their truth sets.
(a) The sum of a number and 5 is less than twice the number. What is the number?

(b) When Joe and Moe were planning to buy a sailboat, they asked a salesman about the cost of a new type of a boat that was being designed. The salesman replied, "It won't cost more than \$380." If Joe and Moe had agreed that Joe was to contribute \$130 more than Moe when the boat was purchased, how much would Moe have to pay?

(c) Three more than six times a number is greater than seven increased by five times the number. What is the number?

(d) A teacher says, "If I had twice as many students in my class as I do have, I would have at least 26 more than I now have." How many students does he have in his class?

*(e) A student has test grades of 82 and 91. What must he score on a third test to have an average of 90 or higher?

*(f) Bill is 5 years older than Norman, and the sum of their ages is less than 23. How old is Norman?

8-3. Multiplication Property of Order

In the preceding section we stated a basic property giving the order of $a + c$ and $b + c$ when $a < b$. Let us now ask about the order of the products ac and bc when $a < b$.

Consider the true sentence

$$5 < 8.$$

If each of these numbers is multiplied by 2, the products are involved in the true sentence

$$(5)(2) < (8)(2).$$

What is your conclusion about a multiplication property of order? Before making a decision, let us try more examples. Just as above, where we took the two numbers 5 and 8 in the true sentence "$5 < 8$" and inserted them in "$(\ )(2) < (\ )(2)$" to make a true sentence, do the same in the following.

1. $7 < 10$ and $(\ )(6) < (\ )(6)$
2. $-9 < 6$ and $(\ )(5) < (\ )(\ )$
3. $2 < 3$ and $(\ )(-4) < (\ )(-4)$
4. $-7 < -2$ and $(\ )(2) < (\ )(2)$
5. $-1 < 8$ and $(\ )(-3) < (\ )(-3)$
6. $-5 < -4$ and $(\ )(-6) < (\ )(-6)$

We are concerned here with the order relation "<", observing the pattern when each of the numbers in the statement "$a < b$" is multiplied by the same number. Did you notice that it makes a difference whether we multiply by a positive number or a negative number?

The above experience suggests that if $a < b$, then

$ac < bc$, *provided* c *is a positive number*;
$bc < ac$, *provided* c *is a negative number*.

Thus we have found another important set of properties of order.

How can you use these properties to tell quickly whether the following sentences are true?

Since $\frac{1}{4} < \frac{2}{7}$, then $\frac{5}{4} < \frac{10}{7}$.

Since $-\frac{5}{6} < -\frac{14}{17}$, then $\frac{14}{51} < \frac{5}{18}$.

Since $\frac{5}{3} < \frac{7}{4}$, then $-\frac{7}{16} < -\frac{5}{12}$.

These properties of order turn out to be consequences of the other properties of order, and we state them together as

Theorem 8-3a. Multiplication Property of Order. If a, b, and c are real numbers and if $a < b$, then
$ac < bc$, if c is positive,
$bc < ac$, if c is negative.

Proof. There are two cases. Let us consider the case of *positive* c. Here we must prove that if $a < b$, then $ac < bc$.

You fill in the reason for each step of the proof.

1. There is a positive number d such that $b = a + d$.
2. Therefore, $bc = (a + d)c$.
3. $bc = ac + dc$
4. The number dc is positive.
5. Hence, $ac < bc$.

The proof of the case for <u>negativ</u> [illegible] to the student in the problems.

We could equally well have discussed the multiplication property of the order relation "is greater than" instead of "is less than".

When we are comparing numbers, the two statements "$a < b$" and "$b > a$" say the same thing about a and b. Thus, when we are concerned primarily with numbers rather than a particular order relation, it may be convenient to shift from one order relation to another and write such sentences as:

Since $3 < 5$, then $3(-2) > 5(-2)$.
Since $-2 > -5$, then $(-2)(8) > (-5)(8)$.
Since $3 > 2$, then $(3)(-7) < (2)(-7)$.

Verify that these sentences are true.

When we are focusing on the numbers involved instead of on an order relation, we can say that

$$\text{if } a < b, \text{ then } \begin{cases} ac < bc & \text{if } c \text{ is positive,} \\ ac > bc & \text{if } c \text{ is negative.} \end{cases}$$

State these properties of orders in your own words.

In our study we shall also need some results such as

<u>Theorem</u> <u>8-3b</u>. If $x \neq 0$, then $x^2 > 0$.

<u>Proof</u>. If $x \neq 0$, then either x is negative or x is positive, but not both. If x is positive, then

$$x > 0,$$
$$(x)(x) > (0)(x), \qquad \text{(Why?)}$$
$$x^2 > 0.$$

If x is negative, then

$$x < 0,$$

$$(x)(x) > (0)(x), \qquad \text{(Why?)}$$

$$x^2 > 0.$$

In either case, the result is the desired one.

Theorem 8-3b states that the square of a non-zero number is positive. What can be said about x^2 for any x?

The properties of order can be used to advantage in finding truth sets of inequalities. For example, let us find the truth set of

$$(-3x) + 2 < 5x + (-6).$$

By the addition property of order we may add $(-2) + (-5x)$ to both members of this inequality to obtain

$$\left((-3x) + 2\right) + \left((-2) + (-5x)\right) < \left(5x + (-6)\right) + \left((-2) + (-5x)\right),$$

which when simplified is

$$-8x < -8.$$

Since $\left((-2) + (-5x)\right)$ is a real number for every value of x, the new sentence has the same truth set as the original. (What must we add to the members of "$-8x < -8$" to obtain the original sentence, that is, to reverse the step?)

Then, by the multiplication property of order.

$$(-8)(-\tfrac{1}{8}) < (-8x)(-\tfrac{1}{8})$$

$$1 < x$$

Here we multiplied by a non-zero real number. Thus, this sentence is equivalent to the former sentence. (What must we multiply the members of "$1 < x$" by to obtain the former sentence?) Obviously, the truth set of "$1 < x$" is the set of all numbers greater than 1, and this is the truth set of the original inequality.

Problem Set 8-3

1. Solve each of the following inequalities, using the form of the following example. (Recall that to "solve" a sentence is to find its truth set.)

   Example: $(-3x) + 4 < -5$.

   This sentence is equivalent to

   $-3x < (-5) + (-4)$, (add (-4) to both members)

   $-3x < -9$,

   which is equival

   $(-\frac{1}{3})(-9) < (-\frac{1}{3})(-3x)$ (multiply both members by $(-\frac{1}{3})$)

   $3 < x$.

   Thus, the truth set consists of all numbers greater than 3.

   (a) $(-2x) + 3 < -5$

   (b) $(-2) + (-4x) > -6$

   (c) $(-4) + 7 < (-2x) + (-5)$

   (d) $5 + (-2x) < 4x + (-3)$

   (e) $(\frac{2}{3}) + (-\frac{5}{6}) < (-\frac{1}{6}) + (-3x)$

   (f) $\frac{1}{2}x + (-2) < (-5) + \frac{5}{2}x$

   (g) $2x < 3 + |-2||-\frac{4}{3}|$

   (h) $(-2) + 5 + (-3x) < 4x + 7 + (-2x)$

   (i) $-(2 + x) < 3 + (-7)$

2. Graph the truth sets of parts (a) and (b) of Problem 1.

3. Translate the following into open sentences and solve.

   (a) Sue has 16 more books than Sally. Together they have more than 28 books. How many books does Sally have?

   (b) If a certain variety of bulbs are planted, less than $\frac{5}{8}$ of them will grow into plants. If, however, the bulbs are given proper care more than $\frac{3}{8}$ of them will grow. If a careful gardener has 15 plants, how many bulbs did he plant?

4. Prove that if $a < b$ and c is a negative number, then $bc < ac$. Hint: There is a negative number e such that $a = b + e$. Therefore, $ac = bc + ec$. What kind of number is ec? Hence, what is the order of ac and bc?

*5. If c is a negative number, then $c < 0$. By taking opposites, $0 < (-c)$. Since $(-c)$ is a positive number, we may prove the theorem of Problem 4 by noting that if $a < b$, then $a(-c) < b(-c)$; i.e., $-(ac) < -(bc)$. Why does the conclusion then follow?

6. If $a < b$ and a and b are both positive real numbers, prove that $\frac{1}{b} < \frac{1}{a}$. Hint: Multiply the inequality $a < b$ by $(\frac{1}{a} \cdot \frac{1}{b})$. Demonstrate the theorem on the number line.

7. Does the relation $\frac{1}{b} < \frac{1}{a}$ hold if $a < b$ and both a and b are negative? Prove it or disprove it.

8. Does the relation $\frac{1}{b} < \frac{1}{a}$ hold if $a < b$ and $a < 0$ and $b > 0$? Prove or disprove.

9. State the addition and multiplication properties of the order ">".

10. Prove: If $0 < a < b$, then $a^2 < b^2$. Hint: use properties of order to obtain $a^2 < ab$ and $ab < b^2$.

8-4. The Fundamental Properties of Real Numbers

In this and the preceding three chapters, we have been dealing with two main problems. The first problem was to extend the order relation and the operations of addition and multiplication from the numbers of arithmetic to all real numbers. Until this was done we really did not have the real number system to work with. The second problem was to discover and state carefully the fundamental properties of the real number system. The two problems, as we have been forced to deal with them, are closely intertwined. In this section we shall separate out the most important problem, the second one, by summarizing the fundamental properties which have been obtained.

Before continuing, we should admit that the decision as to what *is* a fundamental property is not made because of strict mathematical reasons but is to a large extent a matter of convenience and common agreement. We tend to think of the real number system and its many properties as a "structure" built upon a foundation consisting of fundamental properties. This is the way you should begin to think of the real number system. A good question, which can now be answered more precisely than before, is: What *is* the real number system?

The real number system is a set of elements for which binary operations of addition, "+", and multiplication, "$\cdot$", along with an order relation, "<", are given with the following properties.

1. For any real numbers a and b,
   $a + b$ is a real number. (Closure)
2. For any real numbers a and b,
   $a + b = b + a$. (Commutativity)
3. For any real numbers a, b, and c,
   $(a + b) + c = a + (b + c)$. (Associativity)
4. There is a special real number 0
   such that, for any real number a,
   $a + 0 = a$. (Identity element)
5. For every real number a there is
   a real number $-a$ such that
   $a + (-a) = 0$. (Inverses)

6. For any real numbers a and b,
   $a \cdot b$ is a real number. (Closure)
7. For any real numbers a and b,
   $a \cdot b = b \cdot a$. (Commutativity)
8. For any real numbers a, b, and c,
   $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (Associativity)
9. There is a special real number 1
   such that, for any real number a,
   $a \cdot 1 = a$. (Identity element)

10. For any real number a different from 0, there is a real number $\frac{1}{a}$ such that
    $a \cdot (\frac{1}{a}) = 1.$ (Inverses)

11. For any real numbers a, b, and c,
    $a \cdot (b + c) = a \cdot b + a \cdot c.$ (Distributivity)

12. For any real numbers a and b, exactly one of the following is true: $a < b$, $a = b$, $b < a$. (Comparison)
13. For any real numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$. (Transitivity)
14. For any real numbers a, b, and c, if $a < b$, then $a + c < b + c$. (Addition property)
15. For any real numbers a, b, and c,
    if $a < b$ and $0 < c$, then
    $a \cdot c < b \cdot c$,
    if $a < b$ and $c < 0$, then
    $b \cdot c < a \cdot c$. (Multiplication property)

You have probably noticed that there are several familiar and useful properties which we have failed to mention. This is not an oversight. The reasons for omitting them is that they can be proved from the properties listed here. In fact, by adding just one new property, we could obtain a list of properties from which everything about the real numbers could be proved. We shall not consider this additional property since that would take us beyond the limits of this course. You will see it in a later course.

Practically all of the algebra in this course can be based on the above list of properties. It is by means of proofs that we bridge the gap between these basic properties and all of the many ideas and theorems which grow out of them. The chains of reasoning involved in proofs are what hold together the whole structure of mathematics -- or of any logical system.

Thus, if we are going to appreciate fully what mathematics is like, we should begin to examine how ideas are linked in these chains of reasoning -- we should do some proving and not always be satisfied with a plausible explanation. It is true that some of the statements we have proved seem very obvious, and you might wonder, quite justifiably, why we should bother to prove them. As we progress further in mathematics, there will be more ideas which are not at all obvious and which are established only through proofs. During the more elementary stages of our training we need the experience of seeing some simple proofs and developing gradually some feeling for the chain of reasoning on which the whole structure of mathematics depends. This is our reason for looking closely at proofs of some rather obvious statements.

The ability to discover a method for proving a theorem is something which does not develop overnight. It comes with seeing a variety of different proofs, by learning to look for connecting links between something you know and something you want to prove, by thinking about the suggestions which are given to lead you into a proof. On the other hand, the kind of thinking required is not used only in mathematics but is involved in all logical reasoning.

Let us now return to the fundamental properties of real numbers and summarize a few of the other properties which can be proved from those given above. Some of these were proved in the text and some were included in exercises.

16. Any real number x has just one additive inverse, namely $-x$.
17. For any real numbers a and b,
$$-(a + b) = (-a) + (-b).$$
18. For real numbers a, b, and c, if $a + c = b + c$, then $a = b$.
19. For any real number a, $a \cdot 0 = 0$.
20. For any real number a, $(-1)a = -a$.

21. For any real numbers a and b, $(-a)b = -(ab)$ and $(-a)(-b) = ab$.
22. The opposite of the opposite of a real number a is a.
23. Any real number x different from 0 has just one multiplicative inverse, namely $\frac{1}{x}$.
24. The number 0 has no reciprocal.
25. The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.
26. The reciprocal of the reciprocal of a non-zero real number a is a.
27. For any non-zero real numbers a and b,
$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$
28. For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.
29. For real numbers a, b, and c with $c \neq 0$, if $ac = bc$, then $a = b$.
30. For any real numbers a and b, if $a < b$, then $-b < -a$.
31. If a and b are real numbers such that $a < b$, then there is a positive number c such that $b = a + c$.
32. If $x \neq 0$, then $x^2 > 0$.
33. If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.
34. If $0 < a < b$, then $a^2 < b^2$.

You may have noticed that we gave a proof of the multiplication property of order in Section 8-3. In fact, this property (No. 15 in the list) follows from the other 14 fundamental properties. Therefore it could have been omitted from the list without limiting in any way its scope. However, we have included the property in order to emphasize the parallel between the properties of addition and the properties of multiplication.

You may have noticed also that nowhere in the above discussion of fundamental properties is there any mention of absolute values. This important concept can be brought into the framework of the basic properties by the definition:

If $0 \leq a$, then $|a| = a$.
If $a < 0$, then $|a| = -a$.

We close this summary with a mention of some properties of a rather different kind, namely the <u>properties of equality</u>. These are properties of the language of algebra rather than properties of real numbers. Recall that the sentence "a = b", where "a" and "b" are numerals, asserts that "a" and "b" name the same number. The first two properties of equality which we list have not been stated before but have actually been used many times. In the following, a, b, and c are any real numbers.

35. If $a = b$, then $b = a$. (Symmetry)
36. If $a = b$ and $b = c$, then $a = c$. (Transitivity)
37. If $a = b$, then $a + c = b + c$. (Addition property)
38. If $a = b$, then $ac = bc$. (Multiplication property)
39. If $a = b$, then $-a = -b$.
40. If $a = b$, then $|a| = |b|$.

Review Problems

1. For each pair [illegible]ers, determine the order.
   - (a) $-100, -$[illegible]
   - (b) $0.2, -0.1$
   - (c) $|-3|, |-7|$
   - (d) $\frac{6}{7}, \frac{5}{8}$
   - (e) $3 \cdot 4 + (-4), \ 3\left(4 + (-4)\right)$
   - (f) $x^2 + 1, 0$

2. If $p > 0$ and $n < 0$, determine which sentences are true and which are false.
   - (a) If $5 > 3$, then $5n < 3n$.
   - (b) If $a > 0$, then $ap < 0$.
   - (c) If $3x > x$, then $3px > px$.
   - (d) If $(\frac{1}{n})x > 1$, then $x > n$.
   - (e) If $p > n$, then $\frac{1}{p} < \frac{1}{n}$.
   - (f) If $\frac{1}{p} > \frac{1}{x}$ and $\frac{1}{x} > 0$, then $p < x$ and $x > 0$.

3. Which of the following pairs of sentences are equivalent?
   - (a) $3a > 2, (-3)a > (-2)$
   - (b) $3x > 2 + x, 2x > 2$
   - (c) $3y + 5 = y + (-1), 2y = (-6)$
   - (d) $-x < 3, x > (-3)$
   - (e) $-p + 5 < p + (-1), 6 > 2p$
   - (f) $\frac{1}{m} < \frac{1}{2}$ and $m > 0, m < 2$

4. If $p > 0$ and $n < 0$, determine which represent positive numbers, which represent negative numbers.
   - (a) $-n$
   - (b) n^2
   - (c) $-n^2$
   - (d) pn
   - (e) $\left(-p + (-n)\right)^2$
   - (f) $|n|$

5. Solve each of the following inequalities.
   - (a) $-x > 5$
   - (b) $(-1) + 2y < 3y$
   - (c) $(-\frac{1}{2})z < 3$
   - (d) $(-4) + (-x) > 3x + 8$
   - (e) $b + b + 5 + 2b + 12 \leq 381$
   - (f) $x(x + 1) < x$

6. Find the truth set of each of the following sentences.

(a) $\frac{1}{x} < \frac{1}{2}$ and $x > 0$ (d) $(\frac{1}{x})^2 > 0$ and $x \neq 0$

(b) $\frac{1}{x} = \frac{1}{2}$ (e) $0 \leq 2x < 180$

(c) $\frac{1}{x} < \frac{1}{2}$ and $x < 0$ (f) $x^2 + 1 = 0$

7. If the domain of the variable is the set of integers find the truth sets of the following sentences.

(a) $3x + 2x = 10$ (d) $2\left(x + (-3)\right) = 5$

(b) $x + (-1) = 3x + 1$ (e) $3x + 5 < 2x + 3$

(c) $2x + 1 = -3x + (-9)$ (f) $(\frac{1}{2}) + (-x) > (-\frac{1}{2}) + (-2x)$

8. Solve the following equations.

(a) $3x = 5$ (d) $7y + 3 = y + (-3)$

(b) $3 + x = 5$ (e) $3x = 7x + (-2)x$

(c) $2n + n + (-2) = 0$ (f) $3q + (-q) + 5 + q = (-2)$

9. Solve the following equations.

(a) $3(x + 5) = (x + 3) + x$

(b) $x(x + 3) = \left(x + (-4)\right)(x + 3)$

(c) $\frac{1}{2}y + (-\frac{1}{3}) = (-\frac{1}{2})y + (-\frac{1}{3})$

(d) $a^2 = a(a + 1)$

(e) $(x + 2)(x + 3) = x(x + 5) + 6$

(f) $2q^2 + 2q + q^2 = (3q + 5)(q + 1)$

10. The length of a rectangle is known to be greater than or equal to 6 units and less than 7 units. The width is known to be 4 units. Find the area of the rectangle.

11. The length of a rectangle is known to be greater than or equal to 6 units and less than 7 units. The width is known to be greater than or equal to 4 units and less than 5 units. Find the area of the rectangle.

*12. The length of a rectangle is known to be greater than or equal to 6.15 inches and less than 6.25 inches. The width is known to be greater than or equal to 4.15 inches and less than 4.25 inches. Find the area of the rectangle.

13. (a) A certain variety of corn plant yields 240 seeds per plant. Not all the seeds will grow into new plants when planted. Between $\frac{3}{4}$ and $\frac{5}{6}$ of the seeds will produce new plants. Each new plant will also yield 240 seeds. From a single corn plant whose seeds are harvested in 1960, how many seeds can be expected in 1961?

    (b) Suppose instead that a corn plant did not yield exactly 240 seeds, but between 230 and 250 seeds. Under this condition how many seeds can be expected in 1961 from the 240 seeds planted at the beginning of the season?

14. Write open sentences and find the solution to each of the questions which follow.

    (a) A square and an equilateral triangle have equal perimeters. A side of the triangle is 3.5 inches longer than a side of the square. What is the length of the side of the square?

    (b) A boat traveling downstream goes 10 miles per hour faster than the rate of the current. Its velocity downstream is not more than 25 miles per hour. What is the rate of the current?

    (c) Mary has typing to do which will take her at least 3 hours. If she starts at 1 P.M. and must finish by 6 P.M., how much time can she expect to spend on the job?

    (d) Jim receives $1.50 per hour for work which he does in his spare time, and is saving his money to buy a car. If the car will cost him at least $75, how many hours must he work?

Chapter 9

SUBTRACTION AND DIVISION FOR REAL NUMBERS

9-1. Definition of Subtraction

Suppose you make a purchase which amounts to 83 cents, and give the cashier one dollar. What does she do? She puts down two cents and says "85", one nickel and says "90", and one dime and says "one dollar". What has she been doing? She has been subtracting 83 from 100. How does she do it? - by finding what she has to add to 83 to obtain 100. The question "100 - 83 = what?" means the same as "83 + what = 100?". And how have we solved the equation

$$83 + x = 100$$

so far in this course? We add the opposite of 83, and find

$$x = 100 + (-83).$$

Thus "100 - 83" and "100 + (-83)" are names for the same number.

Try a few more examples:

$20 - 9 = 11$	$20 + (-9) = 11$
$8 - 6 = 2$	$8 + (\quad) = 2$
$5 - 2 = (\quad)$	$5 + (\quad) = 3$
$8.5 - (\quad) = 5.3$	$8.5 + (-3.2) = (\quad).$

From these examples you will agree that subtracting a positive number b from a larger positive number a, gives the same result as adding the opposite of b to a.

Since subtraction for positive numbers is already familiar to you, you probably wonder what we have accomplished. Our problem is to decide how to define subtraction for all real numbers. We have now described subtraction in the familiar case of the positive numbers in terms of operations we know how to do for all real numbers, namely adding and taking opposites. And so we define subtraction for all real numbers as adding the opposite. In this way, we extend subtraction to real numbers so that it still has the properties we know from arithmetic; and our definition has used only ideas with which we have previously become familiar.

To subtract the real number b from the real number a, add the opposite of b to a. Thus, for real numbers a and b,

$$a - b = a + (-b).$$

Examples:

$$2 - 5 = 2 + (-5) = -3$$

$$5 - 2 = 5 + (-2) = 3$$

$$(-2) - 5 = (-2) + (-5) = -7$$

$$2 - (-5) = 2 + 5 = 7$$

$$(-5) - 2 = \ ?$$

$$5 - (-2) = \ ?$$

$$(-2) - (-5) = \ ?$$

$$(-5) - (-2) = \ ?$$

Read the expression "5 - (-2)". Is the symbol "-" used in two different ways? What is the meaning of the first "-"? What is the meaning of the second "-"?

To help keep these uses of the symbol clear, we make the following parallel statements about them.

In "a - b",	In "a + (-b)"
"-" stands between two numerals and indicates the operation of subtraction. We read the above as "a minus b".	"-" is part of one numeral and indicates the opposite of. We read the above as "a plus the opposite of b".

We see that the operation of subtraction is closely related to that of addition. We may state this as

Theorem 9-1. For any real numbers a, b, c,

$$a = b + c \text{ if and only if } a - b = c .$$

Proof: Remember that in order to prove a theorem involving "if and only if" we really must prove two theorems.

Let us first prove: if $a = b + c$, then $a - b = c$.

$$
\begin{aligned}
a &= b + c \\
a + (-b) &= (b + c) + (-b) && \text{(Why?)} \\
a - b &= (b + (-b)) + c && \text{(Why?)} \\
a - b &= c && \text{(Why?)}
\end{aligned}
$$

Next we prove: if $a - b = c$, then $a = b + c$. To do this, note that "$a - b = c$" means "$a + (-b) = c$". The student may now complete the proof.

Problem Set 9-1

1. $(-5000) - (-2000)$
2. $\frac{3}{4} - (-\frac{1}{2})$
3. $(-\frac{9}{2}) - (-6)$
4. $(-0.631) - (0.631)$
5. $(-1.79) - 1.22$
6. $0 - (-5)$
7. $75 - (-85)$
8. $(-\frac{5}{9}) - \frac{13}{9}$
9. Subtract -8 from 15.
10. From -25, subtract -4.
11. What number is 6 less than -9?
12. -12 is how much greater than -17?
13. How much greater is 8 than -5?
14. Let R be the set of all real numbers, and S the set of all numbers obtained by performing the operation of subtraction on pairs of numbers of R. Is S a subset of R? Are the real numbers closed under subtraction? Are the numbers of arithmetic closed under subtraction?

15. Show why "$a - a = 0$" is true for all real numbers.

16. Find the truth set of each of the following equations:

(a) $y - 725 = 25$	(d) $3y - 2 = -14$
(b) $z - 34 = 76$	(e) $x + 23.6 = 7.2$
(c) $2x + 8 = -16$	(f) $z + (-\frac{3}{4}) = -\frac{1}{2}$

17. From a temperature of 3° below zero, the temperature dropped 10°. What was the new temperature? Show how this question is related to subtraction of real numbers.

18. Mrs. J. had a credit of \$7.23 in her account at a department store. She bought a dress for \$15.50 and charged it. What was the balance in her account?

19. Billy owed his brother 80 cents. He repaid 50 cents of the debt. How can this transaction be written as a subtraction of real numbers? (Represent the debt of 80 cents by (-80.))

20. The bottom of Death Valley is 282 feet below sea level. The top of Mt. Whitney, which is visible from Death Valley, has an altitude of 14,495 feet above sea level. How high above Death Valley is Mt. Whitney?

9-2. Properties of Subtraction

What are some of the properties of subtraction? Is

$$5 - 2 = 2 - 5?$$

What do you conclude about the commutativity of subtraction? Next, is

$$8 - (7 - 2) = (8 - 7) - 2?$$

Do you think subtraction is associative?

If subtraction does not have some of the properties to which we have become accustomed, we shall have to learn to subtract by going back to the definition in terms of adding the opposite. Addition, after all, does have the familiar properties.

For example, since subtraction is not associative, the expression

$$3 - 2 - 4$$

really is not a numeral because it does not name a specific number. Recall that subtraction is a binary operation, that is, involves two numbers. Then does "3 - 2 - 4" mean "3 - (2 - 4)" or does it mean "(3 - 2) - 4"? To make a decision, we convert subtraction to addition of opposite. Then

$$\begin{aligned} 3 - (2 - 4) &= 3 + \Big(-(2 + (-4))\Big) \\ &= 3 + \Big((-2) + 4\Big) \\ &= 3 + (-2) + 4 \end{aligned}$$

On the other hand,

$$\begin{aligned} (3 - 2) - 4 &= (3 + (-2)) + (-4) \\ &= 3 + (-2) + (-4) \quad . \end{aligned}$$

The second of these is the meaning we decide upon. We shall agree that

$$a - b - c \text{ means } (a - b) - c \ ,$$

that is

$$a - b - c = a + (-b) + (-c) \ .$$

Example 1. Find a common name for

$$(\frac{6}{5} + 2) - \frac{1}{5} \, .$$

We can think of this as $(\frac{6}{5} + 2) + (-\frac{1}{5})$, and then we know that we can write

$$\begin{aligned} (\frac{6}{5} + 2) - \frac{1}{5} &= (\frac{6}{5} + 2) + (-\frac{1}{5}) \\ &= \Big(\frac{6}{5} + (-\frac{1}{5})\Big) + 2 \\ &= 1 + 2 \\ &= 3 \ . \end{aligned}$$

Example 2. Use the properties of addition to write $-3x + 5x - 8x$ in simpler form.

$$-3x + 5x - 8x = (-3)x + 5x + (-8)x,$$

where we have used the theorem, $-ab = (-a)b$, for the first term, and the definition of subtraction and the same theorem for the last term. That is, we think of $-3x + 5x - 8x$ as the sum of $(-3)x$, $5x$, and $(-8)x$. Then

$$\begin{aligned} -3x + 5x - 8x &= \big((-3) + 5 + (-8)\big)x \quad \text{by the distributive property} \\ &= -6x. \end{aligned}$$

While it is not as precise, we use the commonly accepted word "simplify" for directions such as "find a common name for" and "use the properties of addition to write the following in simpler form". When there is no possibility of confusion, this term will appear henceforth.

Example 3. Simplify $(5y - 3) - (6y - 8)$.

$$\begin{aligned} (5y - 3) - (6y - 8) &= 5y + (-3) + \big(-(6y + (-8))\big) && \text{(Why?)} \\ &= 5y + (-3) + \big(-(6y)\big) + \big(-(-8)\big), \text{ since the opposite of the sum is the sum of the opposites.} \\ &= 5y + (-3) + (-6)y + 8 && \text{(Why?)} \\ &= (-1)y + 5 && \text{(Why?)} \\ &= -y + 5. \end{aligned}$$

Instead of the fact that the opposite of a sum is the sum of the opposites, we could also have used Theorem 7-2a which states that $-a = (-1)a$, and then the distributive property. Then our example would have proceeded as follows:

$$\begin{aligned} (5y - 3) - (6y - 8) &= 5y - 3 + \big(-(6y - 8)\big) \\ &= 5y - 3 + (-1)\big(6y + (-8)\big) \\ &= 5y - 3 + (-1)(6y) + (-1)(-8) \\ &= 5y - 3 - 6y + 8 \\ &= -y + 5 \end{aligned}$$

When you understand the steps involved, you can abbreviate the steps as

$$\begin{aligned}(5y - 3) - (6y - 8) &= 5y - 3 - 6y + 8\\ &= -y + 5\end{aligned}$$

You may be impressed by the way we are now doing a number of steps mentally. This ability to comprehend several steps without writing them all down is a sign of our mathematical growth. We must be careful, however, to be able at any time to pick out all the detailed steps and explain each one.

For instance, give the reason for each of the following steps:

$$\begin{aligned}(6a - 8b + c) - (4a - 2b + 7c)\\ &= \Big(6a + (-8b) + c\Big) + \Big(-\big(4a + (-2b) + 7c\big)\Big)\\ &= \Big(6a + (-8b) + c\Big) + \Big((-4a) + \big(-(-2b)\big) + (-7c)\Big)\\ &= \Big(6a + (-8b) + c\Big) + \Big((-4a) + 2b + (-7c)\Big)\\ &= \Big(6a + (-4a)\Big) + \Big((-8b) + 2b\Big) + \Big(c + (-7c)\Big)\\ &= \Big(6a + (-4)a\Big) + \Big((-8)\ b + 2b\Big) + \Big(1c + (-7)c\Big)\\ &= \Big(6 + (-4)\Big)\ a + \Big((-8) + 2\Big)\ b + \Big(1 + (-7)\Big)\ c\\ &= 2a + (-6)b + (-6)c\\ &= 2a + (-6b) + (-6c)\\ &= 2a - 6b - 6c\end{aligned}$$

Problem Set 9-2a

Simplify (In Problems 7 and 20 show and explain each step as in the first two parts of Example 3. In the remaining problems, use the abbreviated form of the third part of Example 3.)

1. (a) $3x - 4x$

   (b) $-5a - 3a$

   (c) $4x^2 - (-7x^2)$

   (d) $-4xz - xz$

2. (a) $-3y - 5y + y$

   (b) $-3c + 5c - \frac{1}{2}c$

   (c) $7x^2 - 4x^2 - 11x^2$

   (d) $3a^2 - 5a^2 + 6a$

3. (a) $-(3x - 4y)$

   (b) $-(-5a - 3y)$

   (c) $-(4a - 6a)$

   (d) $-(7 - x)$

4. $(3.6 - 1.7) + (2.7 - 8.6)$

5. $(\frac{3}{4} - \frac{1}{6}) - \frac{1}{4}$

6. $(\frac{3}{4} - \frac{1}{6}) + (\frac{1}{4} + \frac{1}{6})$

7. $(3\pi + 9) - (5\pi - 3)$

8. $(2\sqrt{5} + 8) + (2 - 7\sqrt{5})$

9. $-4y + 6y$

10. $-3c + 5c + \frac{1}{2}c$

11. $(3x - 6) + (7 - 4x)$

12. $(3x - 6) + (6 - 3x)$

13. $(5a - 17b) - (4a -$ [illegible]

14. $(2x + 7) + (4x^2 + 2 - x)$

15. $(3a + 2b - 4) - (5a - 3b + c)$

16. $(7x^2 - 3x) + (4x^2 - 7x - 8)$

17. $(7x^2 - 3x) - (4x^2 - 7x - 8)$

18. $(7xy - 4xz) - (8xy - 3yz)$

19. $(3n + 12p - 8a) - (5a - 7n - p)$

20. $(5x - 3y) - (2 + 5x) + (3y - 2)$

21. From $11a + 13b - 7c$ subtract $8a - 5b - 4c$.

22. What is the result of subtracting $-3x^2 + 5x - 7$ from $-3x + 12$?

23. What must be added to $3s - 4t + 7u$ to obtain $-9s - 3u$?

24. Prove: If $a > b$, then $a - b$ is positive.

25. If $(a - b)$ is a positive number, which of the statements, $a < b$, $a = b$, $a > b$, is true? What if $(a - b)$ is a negative number? What if $(a - b)$ is zero?

26. If a, b, and c are real numbers and $a > b$, what can we say about the order of $a - c$ and $b - c$? Prove your statement.

The definition of subtraction in terms of addition permits us to extend further our applications of the distributive property, and to describe in different language some of our steps in finding truth sets.

Example Simplify

$$(-3)(2x - 5)\ .$$

By applying the definition of subtraction, we have

$$\begin{aligned}(-3)(2x - 5) &= (-3)\big(2x + (-5)\big)\\ &= (-3)(2x) + (-3)(-5) \qquad \text{(Why?)}\\ &= \big((-3)(2)\big)x + 15 \qquad \text{What properties of multiplication have we used here?}\\ &= (-6)x + 15\\ &= -6x + 15.\end{aligned}$$

You would perhaps have done some of these steps mentally, and would have written directly:

$$(-3)(2x - 5) = -6x + 15,$$

thinking "(-3) times (2x) is (-6x)"
"(-3) times (-5) is 15."

Problem Set 9-2b

1. Perform indicated operations and simplify where possible:

(a) $4(3 - 5)$

(b) $2\big(-4 - (-a)\big)$

(c) $(-3)\big(4 - (-5)\big)$

(d) $(-x)(-2 + 7)$

(e) $(-4)(3 - x)$

(f) $5(3 - 2x)$

(g) $(-2)\big(3x - (-3)\big)$

(h) $2(-3x - 3)$

(i) $a(b - 2)$

(j) $(-y)(-x - 4)$

2. Perform indicated operations and simplify where possible:

   (a) $(-3)(-a + 2b - c)$

   (b) $(-3x + 2y) + 2(-2x - y)$

   (c) $(-2)(a - 2b) + 3(a - 2b)$

   (d) $4u(2u + 3) - 3(2u - 3)$

   (e) $x(x + y) - y(x + y)$

   (f) $2a(a - b) + b(a - b)$

   (g) $3(a - b + c) -(2a - b - 2c)$

   (h) $(-x)(4x - y) + 2x(-x - y)$

   (i) $a(b + c + 1) - 2a(2b + c - 1)$

   (j) $a(a + b + 3) + b(a - b + 3) + 3(a + b + 3)$

3. Solve:

   (a) $3x - 4 = 5$

   (b) $2a - 1 = 4a - 3$

   (c) $-3y = 2 - y - 6$

   (d) $-2 - 2y < - 1$

   (e) $4u + 3 > -5u - 3$

   (f) $0.7x + 1.3 = 3.2 + 1.4x - 0.3$

   (g) $-x - 1 < 4 - x$

   (h) $3a + 3 = 7a + 4 - 4a - 1$

   (i) $1.2 - 2.5c < - 3.3 - c$

4. (a) The width of a rectangle is 5 inches less than its length. What is its length if its perimeter is 38 inches?

   (b) If 17 is subtracted from a number, and the result is multiplied by 3, the product is 102. What is the number?

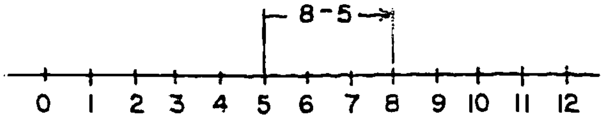
   (c) A teacher says, "If I had 3 times as many students in my class as I do have, I would have less than 46 more than I now have." How many students does he have in his class?

9-3. Subtraction in Terms of Distance

Suppose we ask: On the number line, how far is it from 5 to 8? If x represents the number of units in this distance, then

$$5 + x = 8.$$

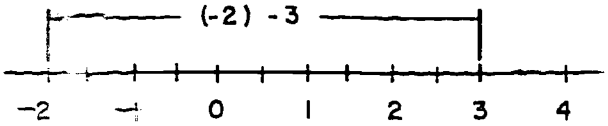
The solution of this equation, as we have seen, can be written as $x = 8 - 5$. Thus, $8 - 5$ can be interpreted as the distance from 5 to 8 on the number line.



Let us now ask how far it is from 3 to (-2). If y represents the number of units in this distance, then

$$3 + y = (-2)$$
$$y = (-2) - 3.$$

Thus $(-2) - 3$ can be interpreted as the distance from 3 to (-2).



The quantity $8 - 5 = 3$ is positive, while $(-2) - 3$ is negative. What does this distinction tell us? It tells us that the distance from 5 to 8 is to the right, while from 3 to (-2) is to the left. Therefore, $a - b$ really gives us the distance from b to a, that is, both the length and its direction.

Suppose we are not interested in the direction, but only in the distance between a and b . Then $a - b$ is the distance from b to a, and $b - a$ is the distance from a to b, and the distance between a and b is the positive of these two. From our earlier work, we know that this is $|a - b|$.

For example, the distance from 3 to (-2) is $(-2) - 3$, that is, -5; the distance between 3 and (-2) is $|(-2) - 3|$, that is, 5. In the same way, the distance from 2 to x is $x - 2$; the distance between 2 and x is $|x - 2|$.

Problem Set 9-3

1. What is the distance
   - (a) from -3 to 5?
   - (b) between -3 and 5?
   - (c) from 6 to -2?
   - (d) between 6 and -2?
   - (e) from 5 to 1?
   - (f) between 5 and 1?
   - (g) from -8 to -2?
   - (h) between -8 and -2?
   - (i) from 7 to 0?
   - (j) between 7 and 0?
2. What is the distance
   - (a) from x to 5?
   - (b) between x and 5?
   - (c) from -2 to x?
   - (d) between -2 and x?
   - (e) from -1 to -x?
   - (f) between -x and -1?
   - (g) from 0 to x?
   - (h) between 0 and x?
3. For each of the following pairs of expressions, fill in the symbols "<", "=", or ">", which will make a true sentence.
   - (a) $|9 - 2| \ ? \ |9| - |2|$
   - (b) $|2 - 9| \ ? \ |2| - |9|$
   - (c) $|9 - (-2)| \ ? \ |9| - |-2|$
   - (d) $|(-2) - 9| \ ? \ |-2| - |9|$

(e) $|(-9) - 2| \ ? \ |-9| - |2|$

(f) $|2 - (-9)| \ ? \ |2| - |-9|$

(g) $|(-9) - (-2)| \ ? \ |-9| - |-2|$

(h) $|(-2) - (-9)| \ ? \ |-2| - |-9|$

4. Write a symbol between $|a - b|$ and $|a| - |b|$ which will make a true sentence for all real numbers a and b. Do the same for $|a - b|$ and $|b| - |a|$. For $|a - b|$ and $||a| - |b||$.

5. Describe the resulting sentences in Problem 4 in terms of distance on the number line.

6. What are the two numbers x on the number line such that

$$|x - 4| = 1?$$

7. What is the truth set of the sentence

$$|x - 4| < 1,$$

Draw the graph of this set on the number line.

8. What is the truth set of the sentence

$$|x - 4| > 1 \ ?$$

9. Graph the truth set of

$$x > 3 \text{ and } x < 5$$

on the number line. Is this set the same as the truth set of $|x - 4| < 1$? (We usually write "$3 < x < 5$" for the sentence "$x > 3$ and $x < 5$".)

10. Find the truth set of each of the following equations; graph each of these sets:

(a) $|x - 6| = 8$

(b) $y + |-6| = 10$

(c) $|10 - a| = 2$

(d) $|x| < 3$

(e) $|v| > -3$

(f) $|y| + 12 = 13$

(g) $|y - 8| < 4$. (Read this: The distance between y and 8 is less than 4.)

(h) $|z| + 12 = 6$

(i) $|x - (-19)| = 3$

(j) $|y + 5| = 9$

11. For each sentence in the left column pick the sentence in the right column which has the same truth set:

$|x| = 3$
$|x| < 3$
$|x| \leq 3$
$|x| > 3$
$|x| \geq 3$
$|x| \not< 3$
$|x| \not> 3$

$x = -3$ and $x = 3$
$x = -3$ or $x = 3$
$x > -3$ and $x < 3$
$x > -3$ or $x < 3$
$x < -3$ and $x > 3$
$x < -3$ or $x > 3$
$x \leq -3$ or $x \geq 3$

*12. From a point marked 0 on a straight road, John and Rudy ride bicycles. John rides 10 miles per hour and Rudy rides 12 miles per hour. Find the distance between them after

(1) 3 hours, (2) $1\frac{1}{2}$ hours, (3) 20 minutes, if

(a) They start from the 0 mark at the same time and John goes east and Rudy goes west.

(b) John is 5 miles east and Rudy is 6 miles west of the 0 mark when they start and they both go east.

(c) John starts from the 0 mark and goes east. Rudy starts from the 0 mark 15 minutes later and goes west.

(d) Both start at the same time. John starts from the 0 mark and goes west and Rudy starts 6 miles west of the 0 mark and also goes west.

9-4. Division

You will recall that we defined subtraction of a number as addition of the opposite of the number:

$$a - b = a + (-b) .$$

In other words, we defined subtraction in terms of addition and the additive inverse.

Since division is related to multiplication in much the same way as subtraction is related to addition, we might expect to define division in terms of multiplication and the multiplicative inverse. This is exactly what we do.

> For any real numbers a and b $(b \neq 0)$, "a divided by b" means "a multiplied by the reciprocal of b".

We shall indicate "a divided by b" by the symbol "$\frac{a}{b}$". This symbol is not new. You have used it as a fraction indicating division. Then the definition of division is:

$$\frac{a}{b} = a \cdot \frac{1}{b} , \quad (b \neq 0).$$

As in arithmetic, we shall call "a" the <u>numerator</u> and "b" the <u>denominator</u> of the fraction "$\frac{a}{b}$". When there is no possibility of confusion, we shall also call the <u>number</u> named by "a" the <u>numerator</u>, and the <u>number</u> named by "b" the <u>denominator</u>.

Here are some examples of our definition. By $\frac{10}{2}$, we mean

$$10 \cdot \frac{1}{2}, \text{ or } 5; \quad \text{by } \frac{3}{\frac{1}{5}} \text{ we mean } 3\left(\frac{1}{\frac{1}{5}}\right), \text{ or } 3(5), \text{ or } 15.$$

Does this definition of division agree with the ideas about division which we already have in arithmetic? An elementary way to talk about $\frac{10}{2}$ is to ask "what times 2 gives 10?" Since $5 \cdot 2 = 10$, then $\frac{10}{2} = 5$.

Why in the definition of division did we make the restriction "$b \neq 0$"? Be on your guard against being forced into an impossible situation by inadvertently trying to divide by zero.

Problem Set 9-4a

Write common names for the following:

1. $\frac{75}{5}$
2. $\frac{570}{570}$
3. $\frac{2500}{1}$
4. $\frac{-30}{-5}$
5. $\frac{30}{-5}$
6. $\frac{-30}{5}$
7. $\frac{\frac{2}{3}}{\frac{1}{6}}$
8. $\frac{4}{\frac{3}{5}}$
9. $\frac{\frac{7}{8}}{2}$
10. $\frac{-\frac{2}{3}}{-\frac{1}{3}}$
11. $\frac{2b}{b}$
12. $\frac{x}{y}$

Mentally complete the following, and compare

$\frac{10}{2} = 5$ and $10 = 5 \cdot 2$,

$\frac{-18}{-3} = 6$ and $-18 = 6(\ )$,

$\frac{8y}{(\ )} = 2y$ and $8y = 2y(4)$.

What do these suggest about the relation between multiplication and division? Is the following theorem consistent with your experience in arithmetic?

Theorem 9-4. For $b \neq 0$, $a = cb$ if and only if $\frac{a}{b} = c$.

This amounts to saying that a divided by b is the number which multiplied by b gives a. Compare this with Theorem 9-1 which says that b subtracted from a is the number which added to b gives a.

Again, in order to prove a theorem involving "if and only if" we must prove two things. First, we must show that if $\frac{a}{b} = c$ $(b \neq 0)$, then $a = cb$. The fact that we want to obtain cb on the right suggests starting the proof by multiplying both members of "$\frac{a}{b} = c$" by b.

Proof: If $\frac{a}{b} = c$ $(b \neq 0)$, then $a \cdot \frac{1}{b} = c$,

$$\begin{aligned} (a \cdot \tfrac{1}{b})b &= cb, \\ a(\tfrac{1}{b} \cdot b) &= cb, \\ a \cdot 1 &= cb, \\ a &= cb. \end{aligned}$$

Second, we must show that if $a = cb$ $(b \neq 0)$, then $\frac{a}{b} = c$. This time, the fact that we do not want b on the right suggests starting the proof by multiplying both members of "$a = cb$" by $\frac{1}{b}$. This is possible, since $b \neq 0$.

Proof: If $a = cb$ $(b \neq 0)$, then
$$a \cdot \frac{1}{b} = (cb)\frac{1}{b},$$
$$a \cdot \frac{1}{b} = c(b \cdot \frac{1}{b}),$$
$$a \cdot \frac{1}{b} = c \cdot 1,$$
$$a \cdot \frac{1}{b} = c,$$
$$\frac{a}{b} = c.$$

Supply the reason for each step of the above proofs.

The second part of this theorem agrees with our customary method of checking division by multiplying the quotient by the divisor.

The multiplication property of 1 states that $a = a(1)$ for any real number a. If we apply Theorem 9-4 to this, we obtain two familiar special cases of division. For any real number a,

$$\frac{a}{1} = a,$$

and for any <u>non-zero</u> real number a,

$$\frac{a}{a} = 1.$$

<u>Problem Set 9-4b</u>

1. Prove that for any real number a,

$$\frac{a}{1} = a.$$

2. Prove that for any <u>non-zero</u> real number a,

$$\frac{a}{a} = 1$$

3. In the following problems perform the indicated divisions and check by multiplying the quotient by the divisor.

(a) $\frac{2500}{-2}$

(b) $\frac{-45}{5}$

(c) $\frac{-200}{-50}$

(d) $\frac{3\sqrt{5}}{3}$

(e) $\frac{35p}{7p}$

(f) $\frac{9\pi}{3}$

(g) $\frac{-\frac{2}{3}}{3}$

(h) $\frac{12}{\frac{1}{3}}$

(i) $\frac{-\frac{5}{8}}{-\frac{7}{8}}$

(j) $\frac{-976}{-976}$

(k) $\frac{0}{48}$

(l) $\frac{360}{2\pi}$

(m) $\frac{15a}{-3}$

(n) $\frac{-14}{0.1}$

(o) $\frac{93m}{-93}$

(p) $\frac{14\sqrt{2}}{2\sqrt{2}}$

4. Comment on $\frac{28}{0}$.

5. When dividing a positive number by a negative number, is the quotient positive or is it negative? What if we divide a negative number by a positive number? What if we divide a negative number by a negative number?

6. Find the truth set of each of the following equations:

(a) $6y = 42$

(b) $-6y = 42$

(c) $6y = -42$

(d) $-6y = -42$

(e) $42y = 6$

(f) $42y = 42$

(g) $6y = 43$

(h) $\frac{1}{5}x = 20$

(i) $\frac{x}{5} = 15$

(j) $\frac{3}{4}a = 9$

(k) $\frac{2}{3}b = 0$

(l) $5x = \frac{10}{3}$

7. Find the truth set of each of the following equations.

   (a) $5a - 8 = -53$

   (b) $\frac{3}{4}y + 13 = 25$

   (c) $x + .30x = 6.50$

   (d) $n + (n + 2) = 58$

   (e) $\frac{1}{3}a = \frac{1}{9}a + 4$

8. If six times a number is decreased by 5, the result is -37. Find the number.

9. If two-thirds of a number is added to 32, the result is 38. What is the number?

10. If it takes $\frac{2}{3}$ of a pound of sugar to make one cake, how many pounds of sugar are needed for 35 cakes for a banquet which 320 people will attend.

11. A rectangle is 7 times as long as it is wide. Its perimeter is 144 inches. How wide is the rectangle?

12. John is three times as old as Dick. Three years ago the sum of their ages was 22 years. How old is each now?

13. Find two consecutive even integers whose sum is 46.

14. Find two consecutive odd positive integers whose sum is less than or equal to 83.

15. On a 20%discount sale, a chair cost $30. What was the price of the chair before the sale?

16. Two trains leave Chicago at the same time: one travels north at 60 m.p.h. and the other south at 40 m.p.h. After how many hours will they be 125 miles apart?

17. One-half of a number is 3 more than one-sixth of the same number. What is the number?

18. Mary bought 15 three-cent stamps and some four-cent stamps. If she paid $1.80 for all the stamps, was she charged the correct amount?

19. John has 50 coins which are nickels, pennies, and dimes. He has four more dimes than pennies, and six more nickels than dimes. How many of each kind of coin has he? How much money does he have?

20. John, who is saving his money for a bicycle, said, "In five weeks I shall have one dollar more than three times the amount I now have. I shall then have enough money for my bicycle." If the bicycle costs $76, how much money does John have now?

21. A plane which flies at an average speed of 200 m.p.h. (when no wind is blowing) is held back by a head wind and takes $3\frac{1}{2}$ hours to complete a flight of 630 miles. What is the average speed of the wind?

22. The sum of three successive positive integers is 108. Find the integers.

23. The sum of two successive positive integers is <u>less</u> than 25. Find the integers.

*24. A syrup manufacturer made 160 gallons of syrup worth $608 by mixing maple syrup worth $2 per quart with corn syrup worth 60 cents per quart. How many gallons of each kind did he use?

25. Show that if the quotient of two real numbers is positive, the product of the numbers also is positive, and if the quotient is negative, the product is negative.

9-5. Common Names

In Chapter 2 we noted some special names for rational numbers which are in some sense the simplest names for these numbers, and which we called "common names". Two particular items of interest about indicated quotients were the following: We do not call $\frac{"20"}{5}$ a common name for "four", because "4" is simpler; similarly, $\frac{"14"}{21}$ is not a common name for "two-thirds" because $\frac{"2"}{3}$ is simpler. We obtain these common names by using the property of 1 and the theorem $\frac{a}{a} = 1$.

$$\frac{20}{5} = \frac{4 \cdot 5}{5} = 4\left(\frac{5}{5}\right) = 4(1) = 4$$

and

$$\frac{14}{21} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{2}{3}\left(\frac{7}{7}\right) = \frac{2}{3}(1) = \frac{2}{3}.$$

On the other hand, we cannot simplify "4" and $\frac{"2"}{3}$ any further.

In the above example, what permitted us to write $\frac{2 \cdot 7}{3 \cdot 7}$ as $\frac{2}{3}\left(\frac{7}{7}\right)$? This familiar practice from arithmetic is one which can be proved for all real numbers.

Theorem 9-5. For any real numbers a, b, c, d, if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Proof: $\frac{a}{b} \cdot \frac{c}{d} = \left(a \cdot \frac{1}{b}\right)\left(c \cdot \frac{1}{d}\right)$ (Why?)

$= (ac)\left(\frac{1}{b} \cdot \frac{1}{d}\right)$ (Why?)

$= (ac)\left(\frac{1}{bd}\right)$ Theorem 7-8d

$= \frac{ac}{bd}$ (Why?)

Example 1. Simplify $\frac{3a^2b}{5aby}$

$$\frac{3a^2b}{5aby} = \frac{3a}{5y}\left(\frac{ab}{ab}\right)$$, by associative and commutative properties,

$$= \frac{3a}{5y}\left(\frac{ab}{ab}\right)$$, by Theorem 9-5,

$$= \frac{3a}{5y}$$, by the property of 1.

<u>Example 2.</u> Simplify $\frac{3y - 3}{2(y - 1)}$. (Note: When we write this phrase, we assume automatically that the domain of y excludes 1. Why?)

$$\frac{3y - 3}{2(y - 1)} = \frac{3(y - 1)}{2(y - 1)}$$, by the distributive property,

$$= \frac{3}{2}\left(\frac{y - 1}{y - 1}\right)$$, by Theorem 9-5,

$$= \frac{3}{2}(1)$$, since $\frac{a}{a} = 1$, (here $a = y - 1$)

$$= \frac{3}{2}$$, by the multiplication property of 1.

After further experience, your mental agility will undoubtedly permit you to skip some of these steps.

<u>Example 3.</u> Simplify $\frac{(2x + 5) - (5 - 2x)}{-8}$

By the definition of subtraction,

$$\frac{(2x + 5) - (5 - 2x)}{-8} = \frac{2x + 5 - 5 + 2x}{-8}$$

$$= \frac{4x}{-8} ,$$

$$= \frac{x(4)}{-2(4)},$$

$$= \frac{x}{-2}$$ by the multiplication property of 1.

The numerals $\frac{x}{-2}$ and $\frac{-x}{2}$ and $-\frac{x}{2}$ all name the same number, and all look equally simple; the accepted common name is the last of these. Therefore,

$$\frac{2x + 5 - (5 - 2x)}{-8} = -\frac{x}{2} .$$

Problem Set 9-5

1. We have used the property of real numbers a and b that if $b \neq 0$, then

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} .$$

Prove this theorem.

In each of the following problems, simplify:

2. (a) $\frac{12}{9}$ (b) $\frac{12}{-9}$ (c) $\frac{-12}{10 - 1}$ (d) $\frac{-12}{-9}$

3. (a) $\frac{n \cdot n}{n}$ (b) $\frac{-n^2}{n}$ (c) $\frac{-n}{-n^2}$

4. (a) $\frac{2(x - 2)}{3(x - 2)}$ (c) $\frac{2(x - 2)}{-3(x - 2)}$ (e) $\frac{2x - 4}{6 - 3x}$

   (b) $\frac{2x - 4}{3x - 6}$ (d) $\frac{2(x - 2)}{3(2 - x)}$

5. (a) $\frac{xy + y}{x + 1}$ (b) $\frac{xy - y}{x - 1}$ (c) $\frac{xy + y}{y}$ (d) $\frac{xy - y}{y(x - 1)}$

6. (a) $\frac{8b - 10}{4b - 5}$ (c) $\frac{8(1 - b) + 2}{4b - 5}$

   (b) $\frac{8b - 10}{5 - 4b}$ (d) $-\frac{10 - 8b}{1 + 4(1 - b)}$

7. (a) $\frac{x + 2}{3}$ (b) $\frac{2x - 3}{2y - 3}$ (c) $\frac{2x + 1}{3}$ (d) $\frac{3x + 6}{3}$

8. (a) $\frac{2a - a^2}{a}$ (b) $\frac{2a - a^2}{-a}$ (c) $\frac{2a - a^2}{a - 2}$ (d) $\frac{2a - a^2}{a^2 - 2a}$

9. (a) $\frac{3}{3t - 6}$ (b) $\frac{3}{4t - 3 + 2t}$ (c) $\frac{(4t - 5) - (t + 1)}{3}$

10. (a) $\frac{6a^2b}{a}$ (b) $\frac{6a^2b}{3b}$ (c) $\frac{6a^2b}{2b^2a}$ (d) $\frac{3ab(2a + a)}{-15abc}$

11. (a) $\frac{(x + 1)(x - 1)}{x + 1}$ (c) $\frac{(x - 1)(2x - 3 + x)}{4x - 4}$

(b) $\frac{(x + 1)(x - 1)}{4x - 4}$ (d) $\frac{(-5x - 5)(2 - 2x)}{10x + 10}$

12. (a) $\frac{1 + 3x - 2y}{4y - 2 - 6x}$ (b) $\frac{1 - 3x - 2y}{4y - 2 - 6x}$

9-6 Fractions

At the beginning of Section 9-5, we recalled two conventions on common names which we have been using ever since Chapter 2: A common name contains no indicated division which can be performed, and if it contains an indicated division, the resulting fraction should be "in lowest terms". Then during Section 9-5, we stated another convention, this one about opposites: We prefer writing $-\frac{a}{b}$ to any of the other simple names for the same number, $\frac{-a}{b}$, $\frac{a}{-b}$.

Let us return to the conventions about fractions. In this course a "fraction" is a symbol which indicates the quotient of two numbers. Thus a fraction involves two numerals, a numerator and denominator. When there is no possibility of confusion, we shall use the word "fraction" to refer also to the <u>number</u> itself which is represented by the fraction. When there <u>is</u> a possibility of confusion we must go back to our strict meaning of fraction as a numeral.

In some applications of mathematics the number given by $\frac{a}{b}$ is called the <u>ratio</u> of a to b. Again we shall sometimes speak of the ratio when we mean the symbol indicating the quotient.

In the preceding section we used Theorem 9-5 to write a fraction as the indicated product of two fractions. For instance, we wrote

$$\frac{14}{21} = \frac{2\cdot 7}{3\cdot 7} = \frac{2}{3}\cdot\frac{7}{7} .$$

Let us now apply Theorem 9-5 in the "other direction" to write an indicated product of fractions as a single fraction.

Example 1: Simplify $\frac{x}{3}\cdot\frac{5}{6}$.

By Theorem 9-5, $\frac{x}{3}\cdot\frac{5}{6} = \frac{x\cdot 5}{3\cdot 6}$

$$= \frac{5x}{18} .$$

Sometimes we use Theorem 9-5 "both ways" in one problem.

Example 2: Simplify $(\frac{3}{2})(\frac{14}{9})$.

$$\begin{aligned}
(\tfrac{3}{2})(\tfrac{14}{9}) &= \frac{3\cdot 14}{2\cdot 9} && \text{(Why?)}\\
&= \frac{3\cdot(2\cdot 7)}{2\cdot(3\cdot 3)} && \text{because } \begin{matrix}14 = 2\cdot 7 & \text{and}\\ 9 = 3\cdot 3 & .\end{matrix}\\
&= \frac{7\cdot(2\cdot 3)}{3\cdot(2\cdot 3)} && \text{(Why?)}\\
&= \frac{7}{3}(\frac{2\cdot 3}{2\cdot 3}) && \text{by Theorem 9-5.}\\
&= \frac{7}{3} && \text{by the property of 1.}
\end{aligned}$$

Problem Set 9-6a

In Problems 1-10 simplify:

1. (a) $\frac{3}{8}\cdot\frac{7}{2}$ (b) $(-\frac{3}{8})\frac{7}{2}$ (c) $(-\frac{3}{8})(-\frac{7}{2})$ (d) $\frac{3}{2}(\frac{7}{8})$

2. (a) $\frac{4}{7}\cdot\frac{21}{10}$ (b) $\frac{4}{10}\cdot\frac{21}{7}$ (c) $\frac{4}{5+2}\cdot\frac{7+14}{7}$ (d) $\frac{1}{7}\cdot\frac{4\cdot 21}{10}$

3. (a) $(-2)\cdot\frac{5}{9}$ (b) $5\cdot(-\frac{2}{9})$ (c) $(\frac{-2}{-9})\cdot 5$ (d) $\frac{1}{9}\cdot\Big((-5)(-2)\Big)$

4. (a) $\frac{1}{n}\cdot\frac{1}{n}$ (b) $n\cdot\frac{1}{n}$ (c) $\frac{1}{n}\cdot\frac{1}{x}$ (d) $\frac{1}{n + n}$

5. (a) $\frac{x}{4}\cdot\frac{x}{3}$ (b) $\frac{-x}{4}\cdot\frac{-x}{3}$ (c) $\frac{x}{4}\cdot\frac{3}{x}$ (d) $\frac{-x}{-4}\cdot\frac{x}{-3}$

6. (a) $\frac{10}{3}\cdot\frac{3}{2}$ (b) $\frac{10}{3} + \frac{3}{2}$ (c) $\left(-\frac{10}{3}\right)\left(-\frac{3}{2}\right)$ (d) $\left(-\frac{10}{3}\right) + \left(-\frac{3}{2}\right)$

7. (a) $(4a^2)(\frac{a}{3})$ (b) $(4a^2)(\frac{3}{a})$ (c) $m(\frac{3}{a})$ (d) $\frac{a}{3}\cdot\frac{a}{3}$

8. (a) $\frac{3}{4}\cdot\frac{x + 2}{3}$ (b) $3\cdot\frac{(x + 2)}{4}$ (c) $\frac{3}{4}(x + 2)$

9. (a) $\frac{n + 3}{2}\cdot\frac{n + 2}{3}$ (b) $\frac{n + 3}{2}\cdot\frac{2}{n + 3}$ (c) $\frac{n + 3}{n + 2}\cdot\frac{2}{3}$

10. (a) $\frac{xy + y}{x + 1}\cdot\frac{xy - y}{x - 1}$ (b) $\frac{2a - a^2}{-a}\cdot\frac{2a}{a - 2}$

11. Can every rational number be represented by a fraction? Does every fraction represent a rational number?

12. The ratio of faculty to students in a college is $\frac{2}{19}$. If there are 1197 students, how many faculty members are there?

13. The profits from a student show are to be given to two scholarship funds in the ratio $\frac{2}{3}$. If the fund receiving the larger amount was given $387, how much was given to the other fund?

We can state what we have done so far in another way. A product of two indicated quotients can always be written as one indicated quotient. Thus, in certain kinds of phrases, which involve the product of several fractions, we can always simplify the phrase to just one fraction. If a phrase contains several fractions however, these fractions might be added or subtracted, or divided. We shall see in this section that in all these cases, we may always find another phrase for the same number which involves only one indicated division. We are thus able to state one more convention about indicated quotients: No common name for a number shall contain more than one indicated division. Thus the instruction "simplify" will always include the idea "use the properties of the real numbers to find another name which contains at most one indicated division."

The key to simplifying the sum of two fractions is using the property of one to make the denominators alike.

Example 3: Simplify $\frac{x}{3} + \frac{y}{5}$.

$$\frac{x}{3} + \frac{y}{5} = \frac{x}{3}(1) + \frac{y}{5}(1), \text{ by the property of 1,}$$

$$= \frac{x}{3}\left(\frac{5}{5}\right) + \frac{y}{5}\left(\frac{3}{3}\right), \text{ since } \frac{a}{a} = 1,$$

$$= \frac{5x}{15} + \frac{3y}{15}, \quad \text{by Theorem 9-5,}$$

$$= 5x\left(\frac{1}{15}\right) + 3y\left(\frac{1}{15}\right), \text{ by the def. of division,}$$

$$= (5x + 3y)\frac{1}{15}, \quad \text{by the distributive property,}$$

$$= \frac{5x + 3y}{15}, \quad \text{by the definition of division.}$$

Once again, you will soon learn to telescope these steps.

Problem Set 9-6b

In Problems 1-5 simplify:

1. (a) $\frac{5}{9} + \frac{2}{3}$ (b) $\frac{5}{9} - \frac{2}{3}$ (c) $\frac{-5}{9} - \frac{2}{-3}$
2. (a) $\frac{1}{3} + \frac{1}{4}$ (b) $\frac{1}{2} + \frac{1}{3}$ (c) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
3. (a) $\frac{4}{a} + \frac{5}{a}$ (b) $\frac{4}{a} + \frac{5}{2a}$ (c) $\frac{4}{a} + \frac{5}{a^2}$

4. (a) $\frac{x}{4} + \frac{x}{2}$ (b) $\frac{x}{4} \cdot \frac{x}{2}$ (c) $\frac{x}{4} - \frac{x}{2}$ (d) $\frac{x}{2} - \frac{x}{4}$

5. (a) $\frac{4a}{7} - \frac{a}{35}$ (b) $\frac{4}{7} - \frac{a}{35}$ (c) $\frac{4a}{7} - \frac{1}{35}$

6. (a) $\frac{x + 8}{10} + \frac{x - 4}{2}$ (b) $\frac{x + 10}{10} + \frac{x - 2}{2}$ (c) $\frac{x + 10}{x} + \frac{x - 2}{x}$

7. Prove that $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$ for real numbers a, b, and c $(c \neq 0)$.

*8. Prove that $\frac{a}{c} + \frac{b}{d} = \frac{ad + bc}{cd}$ for real numbers a, b, c, and d $(c \neq 0,\ d \neq 0)$.

9. Find the truth set of each of the following open sentences:

Example: $\frac{x}{3} - 2 = \frac{2}{9} x$

Two different procedures are possible.

$$\frac{x}{3} - \frac{2}{9} x = 2$$

$$\frac{3x}{9} - \frac{2x}{9} = 2$$

$$\frac{3x - 2x}{9} = 2$$

$$\frac{x}{9} = 2$$

$$x = 18$$

$$\left(\frac{x}{3} - 2\right) 9 = \left(\frac{2}{9} x\right) 9^{*}$$

$$3x - 18 = 2x$$

$$x = 18$$

* We multiplied by 9 because we could see that the resulting equation would contain no fractions.

(a) $\frac{1}{4} y + 3 = \frac{1}{2} y$

(b) $\frac{a}{3} + \frac{a}{6} = 1$

(c) $\frac{3 + x}{8} = \frac{15}{24}$

(d) $\frac{7}{9} x = \frac{1}{3} x + 8$

(e) $\frac{3}{4} x = 35 - x$

(f) $\frac{a}{2} - 3 = \frac{1}{3} - a$

(g) $3\ |w| + 8 = \frac{1}{2}\ |w| + \frac{41}{2}$

(h) $-\frac{3}{7} + |x-3| < \frac{22}{14}$

10. The sum of two numbers is 240, and one number is $\frac{3}{5}$ times the other. Find the two numbers.

11. The numerator of the fraction $\frac{4}{7}$ is increased by an amount x. The value of the resulting fraction is $\frac{27}{21}$. By what amount was the numerator increased?

12. $\frac{13}{24}$ of a number is 13 more than $\frac{1}{2}$ of the number. What is the number?

13. Joe is $\frac{1}{3}$ as old as his father. In 12 years he will be $\frac{1}{2}$ as old as his father then is. How old is Joe? His father?

14. The sum of two positive integers is 7 and their difference is 3. What are the numbers? What is the sum of the reciprocals of these numbers? What is the difference of the reciprocals?

15. In a shipment of 800 radios, $\frac{1}{20}$ of the radios were defective. What is the ratio of defective radios to non-defective radios in the shipment?

16. (a) If it takes Joe 7 days to paint his house, what part of the job will he do in one day? How much in d days?

(b) If it takes Bob 8 days to paint Joe's house, what part of the job would he do in one day? In d days?

(c) If Bob and Joe work together what portion of the job would they do in one day? What portion in d days?

(d) Referring to parts (a), (b), (c), translate the following into an English sentence:

$$\frac{d}{7} + \frac{d}{8} = 1.$$

Solve this open sentence for d. What does d represent?

(e) What portion of the painting will Joe and Bob, working together, do in one day?

*17. A ball team on August 1 had won 48 games and lost 52. They had 54 games left on their schedule. Let us suppose that to win the pennant they must finish with a standing of at least .600. How many of their remaining games must they win? What is the highest standing they can get? The lowest?

For simplifying the indicated product of two fractions, a key property was Theorem 9-5; for simplifying the indicated sum of two fractions, a key property was the property of 1. When handling the indicated quotient of two fractions, we have several alternative procedures involving these properties. Let us consider an example.

Example 4: Simplify $\dfrac{\frac{5}{3}}{\frac{1}{2}}$.

Method 1. Let us apply the property of 1, where we shall think of 1 as $\frac{6}{6}$. (You will see why we chose $\frac{6}{6}$ as the work proceeds.)

$$\frac{\frac{5}{3}}{\frac{1}{2}} = \frac{\frac{5}{3}}{\frac{1}{2}}\left(\frac{6}{6}\right)$$

$$= \frac{\frac{5}{3} \times 6}{\frac{1}{2} \times 6}$$

$$= \frac{10}{3}, \quad \text{by our previous work on multiplication.}$$

Method 2. Let us use the property of 1, where we shall think of 1 as $\frac{2}{2}$. We choose 2 because it is the reciprocal of $\frac{1}{2}$.

then $\frac{\frac{5}{3}}{\frac{1}{2}} = \frac{\frac{5}{3}}{\frac{1}{2}} (\frac{2}{2})$

$= \frac{\frac{5}{3} \times 2}{\frac{1}{2} \times 2}$

$= \frac{\frac{10}{3}}{1}$ numerator by previous work
denominator by choice of reciprocal of $\frac{1}{2}$

$= \frac{10}{3}$ because $\frac{a}{1} = a$ for any a.

Method 3: Let us apply the definition of division

$\frac{\frac{5}{3}}{\frac{1}{2}} = \frac{5}{3} \left(\frac{1}{\frac{1}{2}}\right)$

$= \frac{5}{3}(2)$ Since $\frac{1}{\frac{1}{a}} = a$

$= \frac{10}{3}$ by previous work on multiplication.

You may apply any one of these methods which appeals to you, <u>provided</u> that (1) you always understand what you are doing, and (2) you receive no instructions to the contrary.

<u>Problem Set 9-6c</u>

In each of the following, simplify, using the most appropriate method:

1. $\frac{\frac{3}{8}}{\frac{1}{4}}$

2. $\frac{\frac{a^2}{2}}{\frac{a}{4}}$

3. $\frac{\frac{ax}{y^2}}{\frac{3xy}{a^2}}$

4. $\frac{\frac{2}{3} + \frac{1}{6}}{\frac{5}{6}}$

5. $\dfrac{12}{\frac{3}{4} + \frac{1}{2}}$

6. $\dfrac{\frac{7}{8} - \frac{11}{12}}{2}$

7. $\dfrac{\frac{1}{3} + \frac{1}{4}}{\frac{1}{3} - \frac{1}{4}}$

8. $\dfrac{1 + \frac{1}{a}}{2 - \frac{2}{a}}$

9. $\dfrac{\frac{a - b}{2}}{\frac{a - b}{-4}}$

10. $\dfrac{a - b}{2} \cdot \dfrac{a - b}{-4}$

11. $\dfrac{\frac{x + 8}{9}}{\frac{3}{x + 2}}$

12. $\dfrac{\frac{xy + y}{x}}{3 + \frac{3}{x}}$

9-7. Summary

Definition of subtraction: To subtract the real number b from the real number a, add the opposite of b to a.

Theorem 9-1. For any real numbers a, b, c, $a = b + c$ if and only if $a - b = c$.

Agreement: $a - b - c = a + (-b) + (-c)$

On the number line,

$a - b$ is the distance from b to a

$b - a$ is the distance from a to b

$|a-b|$ is the distance between a and b

Definition of division: To divide the real number a by the non-zero real number b, multiply a by the reciprocal of b.

Theorem 9-4. For any real numbers a, b, c, where $b \neq 0$, $a = cb$ if and only if $\frac{a}{b} = c$.

Theorem 9-5. For any real numbers a, b, c, d, if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

The simplest name for a number:

(1) Should have no indicated operations which can be performed.

(2) Should in any indicated division have no common factors in the numerator and denominator.

(3) Should have the form $-\frac{a}{b}$ in preference to $\frac{-a}{b}$ or $\frac{a}{-b}$.

(4) Should have at most one indicated division.

Review Problems

1. Which of the following name real numbers? For each part write either the common name for the number or the reason why it is not a number.

(a) $3 \cdot \frac{1}{8}$

(b) $3 \cdot \frac{1}{0}$

(c) $3 \cdot 0$

(d) $\frac{0}{0}$

(e) $\frac{|0|}{7}$

(f) $5 \div 0$

(g) $\left|\frac{1}{0}\right|$

(h) $6 - 6 \cdot 7$

(i) $8 - 2 - 3$

2. Insert parentheses in each of the following expressions so that the resulting sentence is true

(a) $5 - 5 \cdot 7 = 0$

(b) $5 - 5 \cdot 7 = -30$

(c) $7 - 6 - 2 = 3$

(d) $7 - 6 - 2 = -1$

(e) $3 \cdot 2 - 2 \cdot 5 = 0$

(f) $3 \cdot 2 - 2 \cdot 5 = 20$

3. Find the value of the phrase, $b^2 - 4ac$, for each of the following:

(a) $a = 2$, $b = (-1)$, $c = 5$

(b) $a = 5$, $b = 6$, $c = (-3)$

(c) $a = 1$, $b = (-3)$, $c = (-2)$

(d) $a = 1{,}654$, $b = 2$, $c = 0$

(e) $a = 5$, $b = 0$, $c = -5$

(f) $a = \frac{1}{3}$, $b = \frac{1}{4}$, $c = -\frac{1}{5}$

4. Given the fraction $\frac{3x + 5}{2x - 7}$; what is the only value of x for which this is not a real number?

5. Use the distributive property to write the following as indicated sums.

   (a) $(-3)(2x + 1)$

   (b) $ab^2 (a - b)$

   (c) $m^2 (m + 1)$

   (d) $-(3x - 2y)$

   (e) $5a (a + 2b - 3c)$

   (f) $(x - 3) 7x^2$

   (g) $(2x - 3y)(x + 4y)$

   (h) $(2a - 3b)^2$

6. Solve the following sentences

   (a) $2a - 3 < a + 4$

   (b) $7x + 4 + (-x) = 3x - 8$

   (c) $6m \geq 135$

   (d) $\frac{x}{2} - \frac{3}{2} < \frac{x}{3}$

   (e) $\frac{1}{12}z + 1 = \frac{1}{3}z - 2$

   (f) $-3|x| \leq -6$

7. If $\frac{1}{4}$ of a number increased by $\frac{1}{8}$ of the number is less than the number diminished by 25, what is the number?

8. Find the average of the numbers

   $\frac{x + 3}{x}$, $\frac{x - 3}{x}$, $\frac{x + k}{x}$, $\frac{x - k}{x}$, where $x \neq 0$.

9. Show that $\frac{3}{8} < \frac{9}{20}$ and $\frac{9}{20} < \frac{7}{15}$ are true sentences. Then tell why you know immediately that $\frac{3}{8} < \frac{7}{15}$ is true.

*10. A haberdasher sold two shirts for $3.75 each. On the first he lost 25% of the cost and on the second he gained 25% of the cost. How much did he gain or lose, or did he break even on the two sales?

11. If $y = ax + b$ and $a \neq 0$, find an equivalent sentence for x in terms of a, b, and y.

12. Last year tennis balls cost d dollars a dozen. This year the price is c cents per dozen higher than last year. What will half a dozen balls cost at the present rate?

13. Find the truth set of each of the following equations.

    (a) $(x - 1)(x + 2) = 0$

    (b) $(y + 5)(y + 7) = 0$

    (c) $0 = z\,(z - 2)$

    (d) $(2m - 1)(m - 2) = 0$

    (e) $(x^2 + 1)\,3 = 0$

    (f) $(x - 3) + (x - 2) = 0$

14. If $\frac{a}{b} = \frac{c}{d}$ where a, b, c, and d are real numbers with $b \neq 0$, $d \neq 0$,

    (a) Prove that $ad = bc$

    (b) Prove that, if $c \neq 0$, then $\frac{a}{c} = \frac{b}{d}$

    (c) Prove that, if $a \neq 0$, $c \neq 0$, then $\frac{b}{a} = \frac{d}{c}$

    (d) Prove that $\frac{a + b}{b} = \frac{c + d}{d}$

15. Prove the theorem:

    If $b \neq 0$ and $c \neq 0$, then $\frac{ac}{bc} = \frac{a}{b}$.

16. Prove the theorem:

    If $a \neq 0$ and $b \neq 0$, then $\frac{1}{\frac{a}{b}} = \frac{b}{a}$.

*17. Given the set {1, -1, j, -j} and the following multiplication table.

Second Number

	x	1	-1	j	-j
First Number	1	1	-1	j	-j
	-1	-1	1	-j	j
	j	j	-j	-1	1
	-j	-j	j	1	-1

(a) Is the set closed under multiplication?

(b) Verify that this multiplication is commutative for the cases (-1, j), (j, -j) and (-1, -j).

(c) Verify that this multiplication is associative for the cases $(-1, j, -j)$ and $(1, -1, j)$.

(d) Is it true that $a \times 1 = a$, where a is any element of $\{1, -1, j, -j\}$.

(e) Find the reciprocal of each element in this set.

If x is an unspecified member of the set, find the truth sets of the following (make use of question (e)).

(f) $j \times x = 1$.

(g) $-j \times x = j$.

(h) $j^2 \times x = -1$.

(i) $j^3 \times x = -1$.

INDEX

The reference is to the page on which the term occurs.