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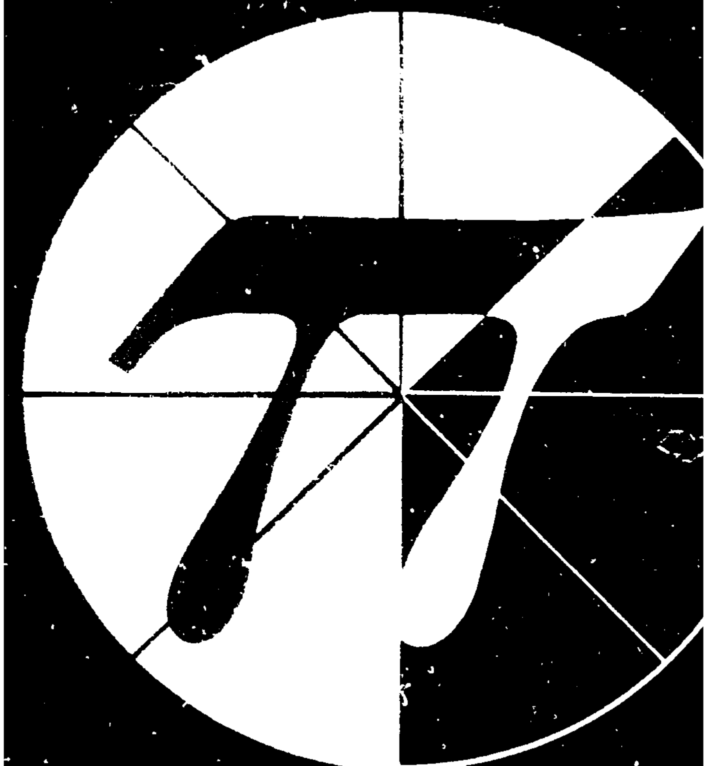
ABSTRACT

This curriculum guide covers Course I of a three-year sequence for high school mathematics in New York which was intended to provide an alternative to the regular Regents sequence of ninth-, tenth-, and eleventh-grade mathematics. A listing of scope and content for Course I is provided, along with a suggested time allotment. Four mathematics units are covered: logic; aspects of algebra and geometry; probability, permutations, and statistics; and rectangular coordinate systems. For each of the units, the general goal for that unit and the material to be covered are discussed. Some teaching suggestions are given. Appendix 1 summarizes items generally required in a traditional ninth-grade algebra course but which are not required in Course I of this program. Appendix 2 suggests other possible orderings for topics in the course outline. (DT)

SEQUENCE FOR HIGH SCHOOL

mathematic

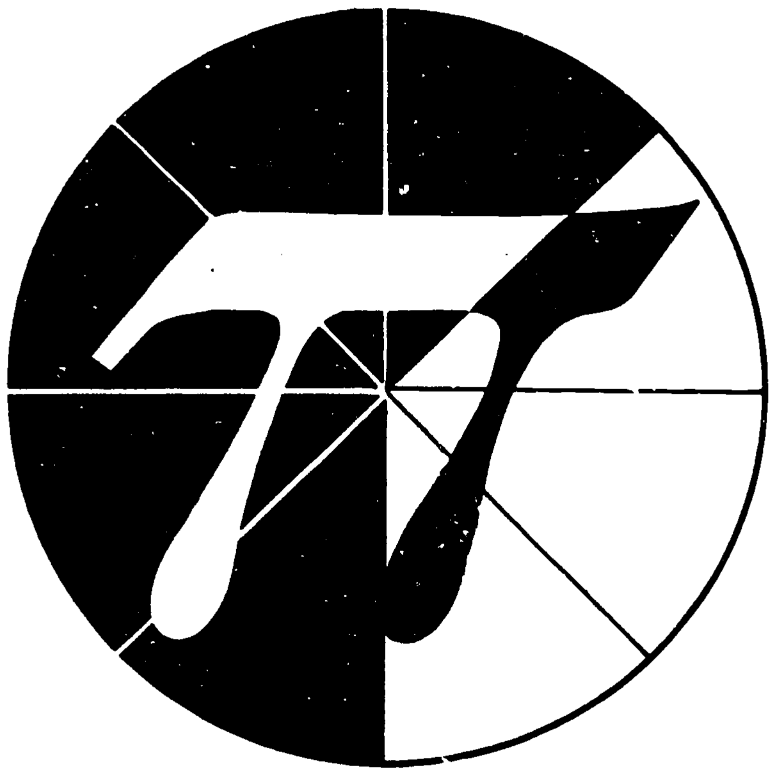


COURSE I

The University of the State of New
THE STATE EDUCATION DEPARTM
of General Education Curriculum
Albany, New York

THREE-YEAR SEQUENCE FOR HIGH SCHOOL

mathematics



COURSE I

The University of the State of New York
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Albany, New York

FOREWORD

This is the first in a series of three publications designed for the new *Three-Year Sequence for High School Mathematics* program consisting of three unified courses - *Course I, Course II, and Course III*. Currently, this sequence provides an alternate to the regular Regents sequence of ninth, tenth, and eleventh year mathematics. Examinations will be available each year for each course, and students who successfully complete the courses will be credited with a Regents sequence in mathematics.

In June 1972, an Advisory Committee was convened by Frank Hawthorne, then Chief of the Bureau of Mathematics Education, to consider the development of a three-year high school mathematics program which would bring together the various branches of mathematics that historically have been treated independently. Members of this committee included: Robert Atkins, John Adams High School, New York City; Donald Holquist, then of Mineola High School, now at Baldwin High School; Anthony Prindle, Linton High School, Schenectady; Ralph Roberts, Pelham Memorial High School; Harry Ruderman, Hunter College High School; George Salterelli, Amherst High School; Stephen Willoughby, New York University; and Sister Ann Xavier, Nazareth High School, Rochester.

Using the philosophical deliberations and tentative outline developed by the committee, *Course I* was organized and written by Sister Ann Xavier of the committee and Richard Klutch of Hunter College High School, working under the direction of Fredric Paul, then an associate and now Chief of the Bureau of Mathematics Education. In September 1974, 40 school districts began offering the course.

As a result of a series of meetings with classroom teachers during 2 years of teaching the course, together with their written evaluations at the end of that time, *Course I* was revised during the summer of 1976. Emrys Ellis of New Hartford High School, Alice Garr of Shelter Rock Junior High School, Herricks, and Alice [illegible] of North High School, Binghamton, analyzed the evaluations and made recommendations which were incorporated into the revision of the materials produced by the original writers.

The final draft was prepared and edited by Mr. Paul and Aaron Buchman, Associate in Mathematics Education. Robert F. Zimmerman, Associate in Secondary Curriculum, represented the Bureau of General Education Curriculum Development on the committee and served as the liaison with the Bureau of Mathematics Education. The final manuscript was prepared in the Bureau of General Education Curriculum Development by the undersigned.

Herbert Bothamley, *Chief*
Bureau of General Education Curriculum Development

Gordon E. Van Hooft, *Director*
Division for Curriculum Services

October 1976

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INTRODUCTION

This booklet presents an outline and notes for the first four sections of a proposed new program of mathematics study. One of the chief goals of this program is to integrate the various branches of mathematics (i.e., algebra, geometry, trigonometry) heretofore treated as independent year- or semester-long courses by most schools in the State. This move is long overdue; surely it is no longer desirable, as we approach the last quarter of the 20th century, to maintain the artificial separation of these subjects.

It is estimated that the first four sections presented here can be handled confortably by most classes in 150 school days (i.e., one school year), and it is assumed that this year will be the ninth. (Estimated time for the entire program is 3 years: 9, 10, 11). Though this course is written with ninth grade students in mind, it is possible that some students may begin the material in an earlier or later year, or complete it in less or more than one year. It is expected that individual teachers will consider their students' abilities and interests in making judgments concerning time allotments, aspects of the material to be stressed, or extra material to be presented. For teachers seeking additional material, there are some suggestions given. It should be noted that teachers need not present topics in the order given in this outline.

The inclusion of two topics, logic, and probability and statistics, will constitute a substantial departure from tradition for most students and teachers. Perhaps, then, a few words on these topics are in order.

Logic has been included here as a necessary, desirable, and, it is hoped, *enjoyable*, preclude to future work in mathematics work in other academic disciplines, and in everyday living. The teaching of logic is based on the conviction that the logical process of reasoning, as a human endeavor, is something that can be and should be *taught*. The students' mathematics course, even if they choose the ninth grade course to be the last, will help to prepare them to sort out many of the confusions and complexities of a world which is growing ever more confusing and complex. More "locally," a good logical background is essential for continued mathematical study, from elementary algebra and geometry to those delta-epsilon proofs of calculus.

The degree to which our world is shaped and described by statistical inference is apparent to everyone. Therefore, we have an obligation to expose students to the principles and terminology of this branch of mathematics; that is, of this way of ordering knowledge, and so prepare our students to understand and deal with the world as it is shaped today. The introduction to probability offered here is a basic one, and one upon which some sophisticated work can be based in succeeding years. The study of statistics, per se, is not carried too far at this stage, but students will be taken through at least the idea of standard deviation in grade 11.

It will be noticed that some of the work with algebraic expressions traditionally found in grade 9, Algebra I course has been omitted. Most of this work will appear, as needed, in suceeding years of the program. These topics have been postponed, not because this work in algebra is considered unimportant, but because its true importance is most clearly seen when it is placed in proximity with the problems it helps to clarify. Similarly, some "tenth grade geometry" appears here, at a time when it can illuminate and be illuminated by other mathematical concepts. At the end of this outline, there is a summary of traditional ninth grade topics not required in Course I of this curriculum.

There are no formal units on the computer or calculator in this outline. Nonetheless, the use of calculators, computers, and flow-charting is suggested in those areas where such devices can contribute to mathematical understanding, increase student participation and enjoyment, or broaden the area from which applications of mathematical ideas can be selected. Individual teachers may want to give lesser or greater emphasis to the role which these modern components can play in mathematics study.

One of the goals of this program is to give students a useful and enjoyable experience with mathematics which may encourage them to continue their mathematical studies. In this regard, it is suggested that teachers give priority to a classroom atmosphere where there is ample time for student investigation, discovery, and discussion rather than to the task of "covering the material." Given enough time for unpressured involvement in the various possibilities these topics have to offer (consideration of advertising claims, political speeches and editorials, historical arguments, and so forth, in the case of logic, and the entire range of statistical arguments in the social sciences), the students will participate in their first year of high school mathematics with a fuller appreciation of the fact that mathematics is a broader and more exciting field of study than many students often perceive it to be.

It is our belief that the program whose beginnings are outlined in this booklet is honest and coherent, and that it reflects what is happening in mathematics today. It is our hope that students who participate in it will be more likely to accept mathematics as an agreeable and relevant part of their total experience - and want to continue with it.

I. Logic

A. At this level the student should be able to appreciate the need for, and the appropriateness of, some form of reasoning in both mathematical and nonmathematical situations. A system of logic is concerned essentially with the investigation and analysis of the truth value of a given statement. Such a truth value can be assigned only to a declarative sentence free of any ambiguous or vague words. The truth value of a sentence such as:

It is hot today.

or

Spinach is delicious.

may be a matter of opinion, but sentences such as:

Tomorrow is Tuesday.
Albany is the capital of New York State.
$7 > 5$.

have objective truth values which are generally accepted. (The truth value of the first changes from day to day but the truth value of the others is constant.)

In addition to statements such as these which have a definite truth value at any given time, some sentences such as:

(1) He is a member of this class.
(2) X is divisible by three.

may be either true or false as the pronoun or variable is replaced. For example, sentence (2) is true when x is replaced by 12 and false when x is replaced by 11. Such a statement may be called an *open sentence*. The set from which replacements are taken is called the *universe, domain*, or *replacement set*, and can be any set whose elements, when used in place of the variables, make sensible sentences, whether true or false. The members of a replacement set of an open sentence which make the sentence true constitute the *solution set* of the open sentence. If every such replacement makes the open sentence false, the solution set is empty.

B. Notation is introduced into any area of mathematics to aid understanding and the communication of ideas, and should be used by the teacher insofar as it will do this for the students. The basic statements of logic will be designated p, q, r, etc. From these basic statements, the following new statements can be formed:

<u>Negation</u> - not p - (symbol: $\sim p$)

The negation of a statement always has the opposite truth value:

$\sim p$ is false when p is true.
$\sim p$ is true when p is false.

p: Today is Tuesday.

$\sim$p: Today is not Tuesday. (It is not so that today is Tuesday.)

q: Albany is the capital of New York State.

$\sim$q: Albany is not the capital of New York State.

r: $7 < 5$

$\sim$r: $7 \nless 5$

The negation of the statement which is true on Tuesday and false on all other days is false on Tuesday and true on all other days. The negation of the true statement is false. The negation of the false statement is true.

$\sim(\sim p)$ is p (The negation of the negation of p is p.)

Conjunction - p and q (symbol: $p \wedge q$)

The conjunction of two statements is true when both are true, and false when either or both are false. This definition may be summarized in the following table, called a truth table.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example:

Let p represent: *x is odd.*
Let q represent: *x is prime.*

Then the conjunction $p \wedge q$ represents: x is odd and x is prime.

When x is 3: p is true and q is true;
hence, the conjunction, $p \wedge q$, is true.

When x is 9: p is true and q is false;
hence, the conjunction, $p \wedge q$, is false.

When x is 2: p is false and q is true;
hence, the conjunction, $p \wedge q$, is false.

When x is 6: p is false and q is false;
hence, the conjunction, $p \wedge q$, is false.

Disjunction - p or q (symbol: $p \vee q$)

The disjunction of two statements is true when either or both are true, and false when both are false.

p	q	$p \vee q$
T	T	T
T	F	[illegible]
F	T	T
F	F	F

Truth Table for Disjunction

Example:

Let p represent: *x is divisible by 2.*
Let q represent: *x is divisible by 3.*

Then the disjunction $p \vee q$ represents: x is divisible by 2 or x is divisible by 3.

When x is 12: p is true and q is true;
hence, the disjunction, $p \vee q$, is true.

When x is 14: p is true and q is false;
hence, the disjunction, $p \vee q$, is true.

When x is 15: p is false and q is true;
hence, the disjunction, $p \vee q$, is true.

When x is 31: p is false and q is false;
hence, the disjunction, $p \vee q$, is false.

Examples such as the following can offer the student practice in working with negations, conjunctions, and disjunctions.

Let p represent: *Thomas Jefferson was the third president of the United States.*
Let q represent: *Hackensack is the capital of New Jersey.*
Let r represent: *The square of a negative number is negative.*

Then p is true and $\sim p$ is false.
Then q is false and $\sim q$ is true.
Then r is false and $\sim r$ is true.

Thus:

i $p \wedge q$ is false
ii $p \wedge \sim q$ is true
iii $\sim q \wedge \sim r$ is true
iv $p \vee q$ is true
v $p \vee \sim q$ is true
vi $\sim p \vee q$ is false
vii $p \wedge (\sim q \wedge \sim r)$ is true
viii $(\sim p \vee q) \vee (p \wedge \sim q)$ is true
ix $\sim (\sim p \vee q)$ is true (negation of vi).

Students should be able to work with expressions comparable to those in i through vi. Teachers may wish to introduce more complex problems similar to those in vii through ix to abler students.

<u>Conditional</u> - If p then q (symbol: $p \rightarrow q$)

The conditional, or *implication* as it is sometimes called, is the connective most frequently associated with the reasoning process.

The conditional, $p \rightarrow q$, is true when p is true and q is true, when p is false and q is true, and when p is false and q is false. The conditional is false when p is true and q is false. These facts may be summarized in the following table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In the conditional $p \rightarrow q$, p is called the *hypothesis* or *antecedent* and q is called the *conclusion* or *consequent*. Although the hypothesis always precedes the conclusion when the conditional is written in symbols, this is not necessarily the case when it is written in words.

Another formulation the conditional $p \rightarrow q$ may take is "p only if q." Students may not readily see the equivalence of "if p then q" and "p only if q"; the following examples may be of help.

If a person lives in Cleveland, then he lives in Ohio.
A person lives in Cleveland only if he lives in Ohio.
If x is a horse than x is an animal.
X is a horse only if x is an animal.

Example:

Let p represent: *x is divisible by 3*
Let q represent: *x is divisible by 2*

Then the conditional $p \rightarrow q$ represents:

If x is divisible by 3, then x is divisible by 2
or x is divisible by 2 if x is divisible by 3
or x is divisible by 3 only if x is divisible by 2

When x = 6: p is true and q is true;
hence, $p \rightarrow q$ is true.

When x = 9: p is true and q is false;
hence, $p \rightarrow q$ is false.

When x = 4: p is false and q is true;
hence, $p \rightarrow q$ is true.

When x = 5: p is false and q is false;
hence, $p \rightarrow q$ is true.

We can see from the truth table and these examples that in a true conditional:

i a true hypothesis is followed by a true conclusion
ii a false hypothesis can be followed by either a true or false conclusion
iii a true conclusion can be preceded by either a true or false hypothesis.

The student may have difficulty accepting the truth of a conditional with a false hypothesis since this convention may not correspond with common usage as he perceives it. An example such as this one may help:

Suppose Mr. Jones made the following promise to his daughter, Sally. "If you do the dishes tonight, then I will raise your allowance." The chart below illustrates the possible situations.

Sally does the dishes.	Mr. Jones raises allowance.	Mr. Jones kept his promise
Yes	Yes	Yes
Yes	No	No
No	Yes	?
No	No	?

Since no mention was made in Mr. Jones' promise of what would happen if Sally did *not* do the dishes, students may feel that they do not have sufficient information to decide whether or not Mr. Jones kept his promise in the last two cases. However, since such ambiguities are unacceptable in a mathematical system, it is agreed to assign the truth value "true" in all those cases where the antecedent is false. A justification for this choise, in the case of Mr. Jones for example, is that the promise was *not broken.*

Some authors distinguish between the terms "conditional" and "implication" by defining an implication as a true conditional, but it is suggested that this distinction not be probed at this time. Students may, however, confuse the truth of a conditional with the truth of its conclusion, and it is suggested that some attention be given to this possible difficulty.

C. Truth tables can be introduced as the work of a section progresses or can be used to summarize what has been developed in that section. Tables for statements such as $p \vee q$ or $[(p \rightarrow q) \wedge p] \rightarrow q$ can be used to present and analyze statements which are always true.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Tables can also be used to analyze statements like $p \wedge \sim p$, which are always false:

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

A compound statement which is always true regardless of the truth value of the basic statements of which it is composed is called a *tautology.*

A principle of logic which can be drawn from the second tautology given above (i.e. $[(p \to q) \wedge p] \to q$; "If we know that a conditional is true and also that its antecedent is true, we may conclude that its conclusion is true.") is one of the most important and frequently used rules of logical reasoning. From "If today is Thursday, then class 92 has chemistry lab 4th and 5th period" and "Today is Thursday" we conclude that "class 92 has chemistry lab 4th and 5th period." Also from "If $|x| > 5$ then $x^2 > 25$" and "$|x| > 5$" we conclude that "$x^2 > 25$." This rule of inference is often called the Law of Detachment.

It should be noted that the manipulation of truth tables is not an end in itself. The tables are introduced as an efficient means of summarizing and recording mathematical investigations.

D. Students should be familiar with the four related conditionals:

original: $p \to q$
converse: $q \to p$
inverse: $\sim p \to \sim q$
contrapositive: $\sim q \to \sim p$

How the truth values of these statements are related should be investigated through examples and truth tables. Some of these can be taken from political or historical sources, or from advertisements. One of the most common fallacies in reasoning is to assume that the converse or inverse of a given statement has the same truth value as the given statement, which is not necessarily the case.

Statements like "If a creature is a horse then it is an animal" about situations familiar to everyone can provide examples of statements which are clearly true and whose converses and inverses are clearly false.

Statements like "If a man lives in Trenton then he lives in the capital of New Jersey" can provide examples of true statements with true converses and inverses. In many "real life" situations, of course, the truth value is less obvious. For example, the advertisers and manufacturers of Woompies might like us to assume the truth of each of the following:

If you eat vitamin-enriched, whole wheat Woompies, you will have a satisfying and nutritious breakfast.

If you do not have a satisfying and nutritious breakfast, you do n-t eat vitamin-enriched, whole wheat Woompies. (contrapositive)

If you have a satisfying and nutritious breakfast, you eat vitamin-enriched-whole wheat Woompies. (converse)

If you do not eat vitamin-enriched, whole wheat Woompies, you do not have a satisfying and nutritious breakfast. (inverse)

Even if you assume the truth of the first statement, the truth of the last two cannot be determined from this assumption. Similar discussions can be built around mathematical statements such as, "All rectangles are parallelograms" (That is, if a figure is a rectangle, then it is a parallelogram.) or political statements such as, "Elect Hector Wrector for honest government," (That is, if you elect Hector Wector then you will have honest government.).

E. A biconditional (symbol $p \leftrightarrow q$) is true whenever p and q have the same truth value.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table for Biconditional

It can be seen by comparing the table below with the one above that the biconditional $p \leftrightarrow q$ is itself equivalent to the conjunction of $p \to q$ and $q \to p$, i.e., $(p \leftrightarrow q)$ will always have the same truth value $[(p \to q) \land (q \to p)]$.

p	q	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The biconditional $p \leftrightarrow q$ can be stated: (if p then q) and (if q then p). This can be restated: (p only if q) and (p if q). Finally, this can be rewritten: *p if and only if q*. The latter expression, perhaps the most common in expressing the biconditional, is often abbreviated *p iff q*.

Further examples of sentences involving the four connectives are:

$\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and negation, $\sim$:

Let p represent: The diameter of a circle is equal to twice the radius.
Let q represent: Tacoma is the largest city in the United States.
Let r represent: Every water molecule contains two atoms of hydrogen.

Then:

i p is true and $\sim p$ is false, q is false and $\sim q$ is true, r is true and $\sim r$ is false, ii) the conditional $q \rightarrow \sim p$ is true since the antecedent, q, is false.
iii $p \leftrightarrow r$ is true
iv $\sim r \leftrightarrow \sim q$ is false
v $(p \wedge q) \leftrightarrow r$ is false
vi $p \vee (q \vee r)$ is true
vii $p \rightarrow (q \wedge r)$ is false

Optional: Those tautologies known as the de Morgan Laws deal with the negation of the conjunction and the disjunction:

$\sim (p \wedge q) \leftrightarrow \sim p \vee \sim q$
$\sim (p \vee q) \leftrightarrow \sim p \wedge \sim q$

II. Aspects of Algebra and Geometry

The goal of this section is to help students become comfortable, even excited, with the language of algebra they will be using throughout their courses in mathematics. The individual topics are probably familiar to the teacher of a standard algebra course and are included because they represent the necessary algebraic skills all students must learn before moving on in mathematics. However, some topics from standard algebra courses do not appear here. In most cases, these topics will appear in outlines for the second and third courses of this program, where students will be able to learn them with more ease and at a time when their usefulness in the scheme of things can more readily be seen. It cannot be overstressed, especially in discussions of those aspects of a mathematics curriculum which have frequently appeared tedious or mechanical, that an essential ingredient for maximum success is that the teacher have sufficient time to do *whatever* is done in a classroom atmosphere of relaxation and unhurried progress.

Many of the topics of this section will be "old hat" to teachers of this course; what follows are some comments on the choice and order of the topics, suggestions for emphases within topics, transitions between topics, etc. It is recommended that wherever possible geometric ideas be a source of motivation and application in the development of algebraic concepts.

A. Obviously, students must master the fundamental elementary number operations before moving on to more advanced work. Probably this review will have been some part of classroom activity since the beginning of the term. This might be a good time to encourage a somewhat deeper discussion of the commutative, associative, and distributive properties. Such a discussion is genuinely fundamental, not just an exercise in vocabulary, and it is the properties themselves, rather than their names, which ought to be stressed. For, although the behavior of the rationals or integers may appear to be obvious or not worthy of special attention, there are many important systems that behave differently. It is suggested that the teacher try to underscore this point by discussing some cases where these properties fail to hold. For example:

(1) Subtraction of rationals is neither associative nor commutative.
(2) Multiplication is distributive over addition but addition is not distributive over multiplication or over itself.
(3) The binary operation of averaging is commutative for the set of integers but is not associative.

Teachers may wish to devise other systems which display or fail to display one or more of these properties.

B. Fundamental operations with monomial expressions are the building blocks for further work with equations and polynomial expressions and should be explored thoroughly. It is suggested that the operation of subtraction, which often poses a problem to beginning students, be considered in its relation to the operation of addition, perhaps even by definition: $a - b = a + (-b)$. There should be a special effort to eliminate such common errors as $2x - x = 2$ or $(3x)(3x) = 6x$ or $(2x)(3x) = 5x^2$ by helping students to understand the dependence of such computation on the distributive, associative and commutative properties.

Students should understand and be able to apply the rules for exponents:

$$a^m \bullet a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

They should also understand and be able to apply the definitions:

$$a^{-n} = \frac{1}{a^n}$$

$$a^0 = 1$$

It is suggested, however, that teachers avoid devoting a disproportionate amount of time on exercises of a purely mechanical nature.

C. Students should learn to solve linear equations with some degree of ease, including those whose solutions may require application of the distributive and other properties previously studied. (For example, $2x + \frac{1}{2}(x - 3) = \frac{1}{7}x + 2(\frac{1}{4}x + 1)$. Of course, the complexity of the equation ought to be increased gradually.

So-called "literal equations," ($ab + c = d$, solve for b), occur often in mathematics and the physical and social sciences, and should not be neglected. It is suggested that ample time be allowed for this type of equation. It can be introduced anywhere during the progress of this topic. Useful examples can be drawn from students' knowledge of geometry and the physical sciences.

Although primary emphasis should not be on rules, some time might be given to those properties of our number system which enable us to solve linear equations. For example,

$$a + c = b + c \leftrightarrow a = b$$
$$ac = bc,\ c \neq 0 \leftrightarrow a = b$$

Teachers may wish to identify these statements as examples of the Cancellation Law. Identities, inverses, and so on, should also be discussed.

Of course, "applications" of these equations are plentiful in any textbook and should be assigned, at the teacher's discretion, throughout this section. Some realistic and genuinely useful applications are available and should be sought out, but teachers may wish to take care not to give the appearance of believing that the majority of "verbal problems" found in textbooks, concerning such things as how many more fish Peter caught than Mary, are in this category. Some of these problems can be treated ironically, (or made deliberately absurd, and amusing to the class) with the promise (a true one generally) of more serious and realistic problems to come when students' mathematical skills are more sophisticated. It is stressed that the chief purpose of such "applications" is not to train students to produce "correct" answers to standarized problems - or to "fill in the boxes" correctly - but, rather, to enable students to deal mathematically with situations encountered in nonmathematical form.

It is unfortunate that inequalities are often deemphasized in algebra courses, considering their importance in higher mathematics; for one example, the basic definitions and arguments of the calculus. It is thus desirable that students become as comfortable with statements about quantities which are unequal as they are with statements about quantities that are equal. Simple linear inequalities should be taught (e.g., $2x + 1 > 7$) and the rules corresponding to those for solving equalities discussed.

$$(a < b) \rightarrow a + c < b + c$$
$$(a < b) \wedge (c > 0) \rightarrow (ac < bc)$$
$$(a < b) \wedge (c < 0) \rightarrow (ac > bc)$$

It would be good for students to meet problems with answers like "any price less than $1.25" rather than only ones with answers like "Marilyn paid $1.25 for the fish."

Example:

John wants to earn at least $10 this week. His father agrees to pay him $4.00 for cutting the lawn and $2.50 an hour for weeding the garden. If John cuts the lawn, what is the least number of hours he must spend weeding to earn at least $10.

D. The goal here is to insure that students will feel comfortable with the ideas and terminology which form the basis of the geometry they will be studying in future years. While many students will already be familiar with much of this material, their newly acquired algebraic skills will enable them to solve new types of related problems.

Examples:

(1) The measure of an angle is 5° less than 3 times its supplement. Find the measures of both angles.
(2) The perimeter of a rectangle is 39 cm. If its length is 3 cm more than twice its width, what are the dimensions of the rectangle?

Hopefully, this will increase the students' understanding of the geometric concepts as well as their appreciation of algebraic techniques. Teachers should extend the work with geometric relationship to include such topics as the measure of an angle inscribed in a circle, the measure of an angle formed by two chords intersecting within a circle. Formulas for these measures may be derived informally and may provide new applications of algebraic technique. There is also an opportunity to discuss the concept of an axiomatic system: defined and undefined terms, axioms, and theorems.

It is suggested that the concept of congruence be dealt with here on a more intuitive than formal level. Construction of triangles with rulers and protractors or with compasses and straightedge can be used to investigate the conditions sufficient for congruence of triangles (SSS, SAS, ASA) and insufficient (SSA).

Formulas for area and volume of plane figures and solids should be reviewed, and related problems investigated, including the effect of change of dimensions on area and volume.

Examples:

(1) The side of an equilateral triangle is tripled. How is its area affected?
(2) The length of a rectangle is doubled and its width multiplied by 5. How is its area affected?
(3) How is the volume of a cube affected if its side is halved?
(4) A crossword puzzle in the form of a square measures 5 inches on a side and has squares which measure $\frac{1}{3}$ inch on each side. If $\frac{1}{5}$ of these squares is blacked in, how many squares remain to be filled in?

E. The students should recognize the significance of ratio as a method of comparison, be familiar with the various notations for ratio and for the proportion formed when two ratios are equal, and understand percent as the ratio of a number to one hundred. These understandings, and previous work with equations, should facilitate students' ability to handle problems involving ratio, proportion, and/or percent.

Example:

The Apex Appliance Company advertises a color television set for \$509.15. This price represents a discount of 15 percent off the original price. For how much did the television set sell originally?

Solution:

Let x represent the original cost of the television set. Then, \$509.15 must represent 85% of x. Therefore, the ratio of 509.15 to x must equal the ratio of 85 to 100.

$$\frac{509.15}{x} = \frac{85}{100}$$

$$85x = 50915$$

$$x = \frac{50915}{85}$$

$$x = 599$$

The television set originally sold for \$599.00

As with congruence, it is suggested that the concept of similarity be dealt with on a more intuitive than formal level, and students should play a central role in the investigation of conditions required for the similarity of plane figures. Proportionality of sides of such figures may suggest useful and "everyday" problems of a nontrivial nature.

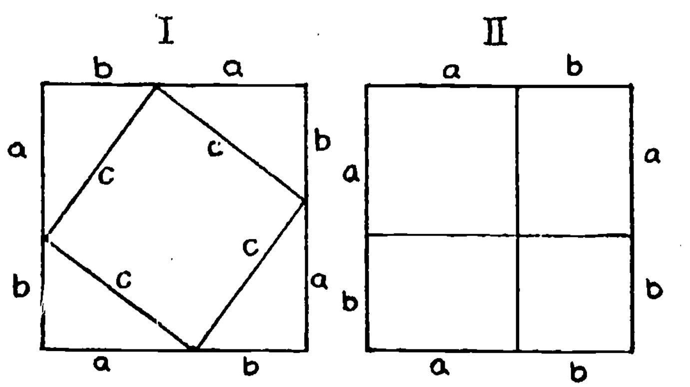
F. The rational numbers can be better understood as a special system if the students are familiar with some numbers which are not rational. The definition of *rational number* should be emphasized and examples given of numbers which satisfy and fail to satisfy this definition. Rational numbers which may not *appear* to be expressed in the required form ($-\frac{2}{5}$, 0.0021, $0.\overline{3}$, $0.\overline{45}$) can be shown to be so expressible. The notion of repeating and non-repeating decimal can be reviewed at this time. The fact that a number is rational if and only if it has a representation as a repeating decimal can be used to demonstrate the existence of numbers, besides the common radicals, which are not rational (e.g., .12112111211112..., .123456789101112...). In the case of more capable students, a proof that there exists no number whose square is two and which satisfies this definition. For example, a proof that $\sqrt{2}$ is irrational can be presented, if there is time. (There are several such proofs available.) The difference between an irrational number (say $\sqrt{3}$ and a rational approximation of it (1.732) should be stressed.

The Pythagorean Theorem is a topic of vital and continuing importance in the development of mathematics. A study of the theorem might begin with a brief discussion of triangle classification with special consideration given to the right triangle.

Students could be asked to draw several right triangles, label the hypotenuse *c*, and the legs *a* and *b*, measure these sides as carefully as possible, and fill in a chart similar to the one below.

Triangle	a	b	c	a^2	b^2	c^2
#1						
#2						
#3						

If necessary, an additional column headed "$a^2 + b^2$" may be added. Hopefully most students will be able to arrive at a fair statement of the Pythagorean Theorem with the aid of such a chart. In classes where this is not the case, teachers will have to direct students' attention to the appropriate selection. At some point this relation can be verified by a simple demonstration such as the one suggested below.



I		II
$c^2 + 4(\frac{1}{2}ab)$	$=$	$a^2 + b^2 + 2ab$
$c^2 + 2ab$	$=$	$a^2 + b^2 + 2ab$

$$c^2 = a^2 + b^2$$

There are many other simple proofs available. Many textbooks can provide applications of the Pythagorean Theorem. Teachers may wish to introduce problems similar to the following:

(1) A 26-foot ladder is placed against a wall with the foot resting 10 feet from the wall. How high up the wall does it reach?

(2) A baseball diamond is actually a square with sides 90 feet. To the nearest foot, how far is home plate from second base?

If we vary the size of the baseball diamond in this problem and use the rule $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ to simplify the result, we are led to the generalization that the diagonal of a square with side s is equal to $s\sqrt{2}$. Teachers may then wish to prove this generalization.

It is suggested that the number π be introduced, or reintroduced to those students who have used it before, as the ratio of the circumference of a circle to its diameter. A good starting point is to ask students to measure (with a tape measure or string and ruler) the circumferences and diameters of assorted circles - perhaps the tops of open tin cans that they have been requested to bring to class for this purpose. Each student would be asked to divide each circumference obtained by the corresponding diameter. A summary of these results will lead to the observation that the ratio of the circumference to the diameter of a circle is, apparently, some number a little bigger than 3. Then, π would be brought into the picture and its nature as an irrational number *approximately* equal to 3.14 or to $\frac{22}{7}$ discussed fully. Teachers should emphasize, for example, that the circumference of a circle of diameter 10 is not exactly equal to 31.4 or to $\frac{220}{7}$, but *about* equal to either of these *rational approximations*.

Example:

If the radius of a circle is 7, then the area of the circle is

(1) exactly 44
(2) exactly 153
(3) between 153 and 154
(4) exactly 154

Before the teacher presents the formula for the area of a circle, the students' observations and suggestions can be elicited by asking them to compare the area of a circle drawn on a sheet of graph or other paper with the area of a square whose side is the radius of the circle.

Sample Problem:

Luigi's Pizza Parlor displays a sign with the following information:

8" pizza - $1.50
12" pizza - $2.50
14" pizza - $3.00

John ordered an 8" pizza. Mary and Jane share a 12" pizza. Paul, Pete, and Peg shared a 14" pizza. Assuming that those who shared pizzas, shared equally:

- Who ate the most pizza? Who ate the least?
- Who got the best bargain (the most pizza for the money)?
- Who got the worst bargain?

It is important that students be allowed to study and analyze the figures and situations involved and be able to participate in the justification of the methods of finding volume. Although students may be required to know some formulas, it should be emphasized that students should not be dependent upon memorization to solve problems.

Sample Problems:

(1) The Yum Yum Biscuit Company contracts to provide the Ding Dong Ice Cream Company with ice cream cones which have a capacity of 100 cc. If the machines at Ding Dong Ice Cream Company require that the height of the cone be 12 cm, what must be the diameter of the base?

(2) The height of a right circular cylindrical tank is 5 ft. Express the relation between the capacity of the tank and the radius of its base. Approximate the capacity of the tank when the base has radius: 10 ft., 12.5 ft., 15.5 ft.

(3) The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ How is volume of the sphere affected if its radius is doubled?

G. It is recommended that the introduction of multiplication of binomials be related to the distributive property:

$$
\begin{aligned}
(x+2)^2 &= (x+2)\,(x+2) \\
&= x(x+2) + 2(x+2) \\
&= x^2 + 2x + 2x + 4 \\
&= x^2 + 4x + 4
\end{aligned}
$$

$$
\begin{aligned}
(2x + 3)\,(x^2 - 5) &= 2x(x^2 - 5) + 2\,(x^2 - 5) \\
&= 2x^3 - 10x + 3x^2 - 15 \\
&= 2x^3 + 3x^2 - 10x - 15
\end{aligned}
$$

Students should gain facility in finding common monomial factors, and in factoring trinomials and expressions involving the different of two squares. Students should use these results and the property of the reals, $(ab = 0) \rightarrow [(a = 0) \vee (b = 0)]$, to solve quadratic equations. Teachers may wish to limit work with quadratic equations to those cases in which the leading coefficient is 1.

III. Probability, Permutations, and Statistics

Although the idea of teaching probability theory, especially to students as young as ninth graders, may seem unusual to some, the basic ideas involved are fairly easy for students to visualize. Also, there is the interest generated by the idea of solving problems involving gambling, dice, cards, as well as the many other nonmathematical subjects which may be included at the discretion of the teacher. In any case, it is the students' understanding and appreciation of the concepts outlined in this section, rather than any particular symbolism or developments which is of major importance.

A. A central idea is that the probability of an event occurring is the ratio of the number of ways that event can occur to the total number of possible outcomes.

Examples:

(1) The probability of rolling a "3" in one roll of a fair die is $\frac{1}{6}$ since there is one face with the 3, and 6 faces altogether.

(2) The probability of rolling an odd number in one roll of a fair die is $\frac{3}{6}$ since there are 3 ways (1, 3, 5) of "success" and six outcomes altogether.

(3) The probability of drawing a red ace from a standard deck of 52 cards is $\frac{2}{52}$.

(4) The probability of drawing a seven or a red king is $\frac{6}{52}$ since there are 6 successes (the kings of hearts and diamonds, and the four sevens) out of 52 possibilities.

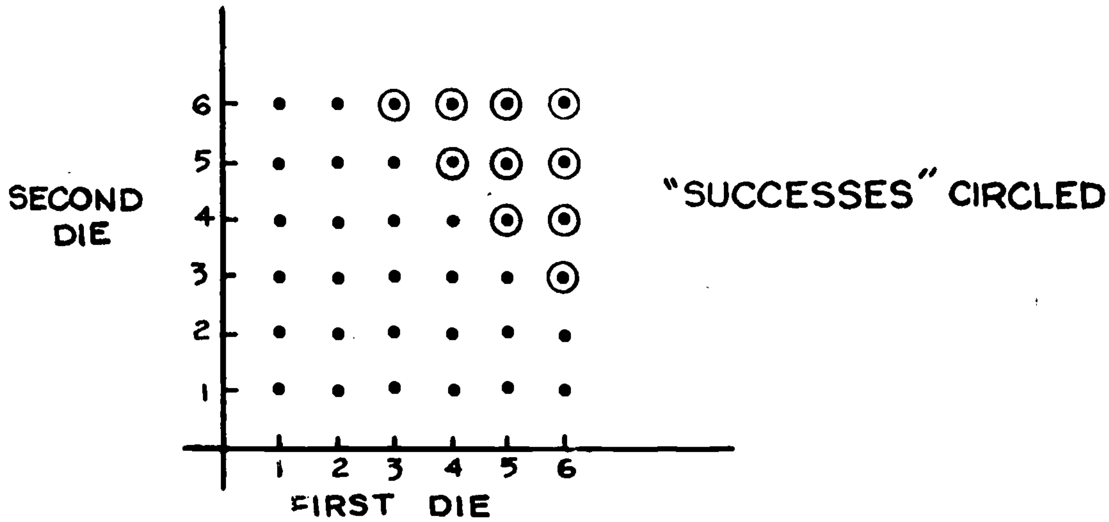
(5) The probability of drawing *a* card which is a red king *and* a seven is 0. (i.e., $\frac{0}{52}$ since there is *no way* a success can occur out of 52 possibilities). There are no cards which are red kings *and* sevens simultaneously.

(6) The probability of drawing a card which is red and a king (i.e., a red king) is equal to $\frac{2}{52}$. In this problem students familiar with set terminology may recognize the set of successful events as the intersection $E \cap F$ of the sets E = red cards and F = kings.

(7) The probability of drawing a red card *or* a king is somewhat more challenging to find. There are 26 red cards, and four kings, but two cards, the king of hearts and the king of diamonds, are members of *both* sets. These cards would thus be counted twice if we simply added 26 and 4 to arrive at the number of successful outcomes. The true number of successes may be found either by actually counting the cards which satisfy the given conditions for success, or by reasoning that the sum 26 + 4 must be reduced by the number of cards counted twice (2), or by appeal to the set-theoretical rule, $(A \cup B) = (A) + (B) - (A \cap B)$. A corollary for the case when A and B are mutually exclusive might also be introduced: if $A \cap B = \phi$, $(A \cup B) = A + B$.

Note: Fractions here have not been reduced to lowest terms. Teachers may prefer such reduction or not, but it is suggested that an issue not be made of it; reducing fractions may actually hide the true significance of the original fractions, or the method by which they are obtained.

(8) What is the probability of rolling a 9 or higher with a roll of two fair dice? Solution: There are 36 outcomes possible. Of these, ten - (3, 6); (6, 3); (4, 5); (5, 4); (6, 4); (4, 6); (5, 5); (6, 5); (5, 6); (6, 6) - are successes, hence P(9 or higher) = $\frac{10}{36}$. See diagram below.



Illustrations like this one are very useful in helping students to picture the situation under discussion.

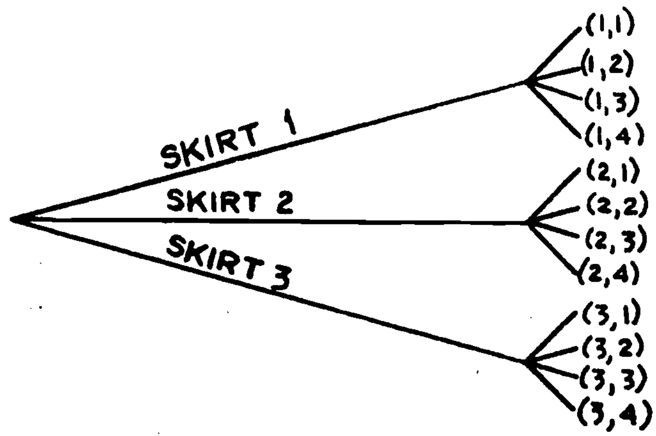
(9) If two fair dice are rolled, what is the probability that the resulting number will be divisible by 4? Answer: If we count the points which represent success (in the illustration above) we conclude that P (divisible by 4) = $\frac{9}{36}$.

It should be noted that each of the preceding problems deals with a situation where each element of the outcome set is equally likely to occur. For example, the dice in these problems were not "loaded." If the dice were reconstructed so that the probability of rolling a six was increased somewhat, with the probability of other outcomes correspondingly reduced, then the solution to these problems would, of course, be different. The idea that any probability will have to be equal to or larger than zero and equal to or smaller than 1 is easily derivable from the nature of the fractions: the numerator cannot be larger than the denominator since there cannot be more 'successes' then total possibilities. In classes using set notation, it can be pointed out that for any outcome set S and any set of successful outcomes E, the relation $E \subset S$ must hold. Of course, it might be the case that $E = S$ or $E = \phi$.

B. The following elementary idea is often called the "Counting Principle": If one activity can occur in any of m ways, and another in any of n ways, then the total number of ways both activities can occur is given by mn.

Example 1:

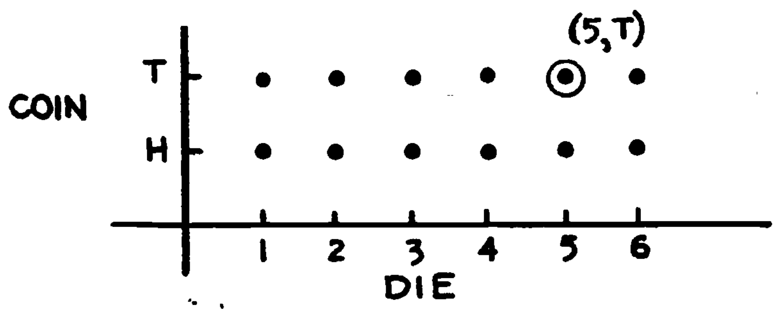
Sally has three skirts and four blouses ready for wear on a particular day. How many different outfits can Sally choose? Solution: Twelve, since, for every skirt there is a choice of four different blouses, as this diagram shows:



There are 12 different paths, each representing a different outfit. Each ordered pair at the right can be considered to represent an outfit.

Example 2:

A fair coin and a fair die are tossed. How many different pairs (outcomes) are possible? Solution: Twelve, since there are two ways for the coin to land and six for the die.



In this diagram each dot represents an outcome of this experiment; the dot labeled corresponds to *five on the die, tails on the coin.*

Example 3:

Suppose there are seven roads between Troy and Utica. In how many different ways can Mr. Smythe travel from Troy to Utica and return by a different route? Solution: He can go in any of seven ways, but once he has chosen a route, he has a choice of only six by which to return. Hence, he may make the round trip in any of 7 x 6 = 42 ways.

These results can be visualized by diagrams of various sorts, or by appeal to ordered pairs. Appeal to ordered pairs (or triples, quadruples, ...) is often an effective device in solving the kinds of problems given in the section, as the examples above have shown.

Following from the Counting Principle is the idea that if the probability of one event, E, is given by m ($0 \leq m \leq 1$) and the probability of another event, F, by n ($0 \leq n \leq 1$), then the probability of E and F jointly occurring is given by mn.

Example 1:

A card is drawn from a standard pack of 52, set aside and another card drawn. What is the probability both cards are aces? Solution: P(first card is an ace) = $\frac{4}{52}$. Assuming the success of an ace on the first draw, there now remain 51 cards in the pack of which three are aces, so p(ace on second draw) = $\frac{3}{51}$. Thus, P (both cards drawn are aces) = $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$.

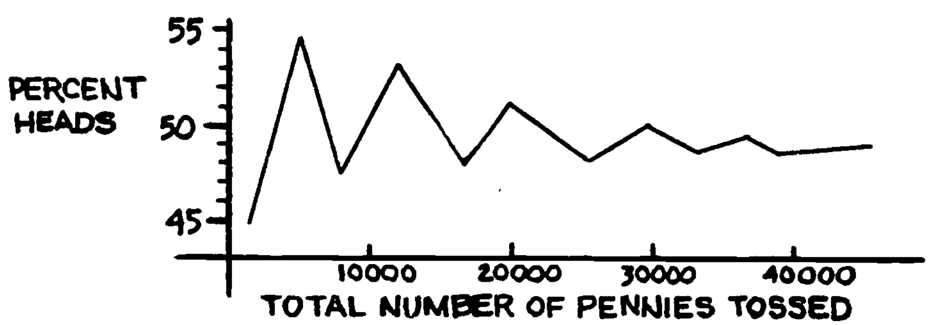
Example 2:

Assuming that the probability of a woman's giving birth to a boy is $\frac{1}{2}$, what is the probability that in a family of 5 children, all the children are boys? Solution: $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = (\frac{1}{2})^5 = \frac{1}{32}$. For a family of n children the solution is $(\frac{1}{2})^n$.

Example 3:

The probability that a person will be able to predict correctly every result when a fair die is rolled r times is $(\frac{1}{6})r$.

Is is suggested that students be asked, as an accompaniment to their theoretical results, (even some of the obvious ones like P (Heads) = $\frac{1}{2}$), to check these results empirically by actually performing a large number of trials of the experiment in question, e.g., tossing a fair coin. As a possible procedure, each student in the class could be asked to toss 10 coins 10 times each night, and report the results the following day. A running account should be kept of the percent of heads obtained as students complete this assignment. A tally could be kept which might resemble this one:



Such a procedure can help students grasp the idea of the *stabilization of relative frequency;* i.e., the tendency of the cumulative relative frequency to get closer to an ideal (theoretical) value as the number of trials increases. This is an important concept which will be used later in this section, and it is suggested that time be taken at this stage to study it.

C. The items in this section can be touched on lightly, or probed in some depth, depending on the skills and interests of the students; teachers can find information and exercises in any number of probability and/or statistics texts.

Combinatorics involves computing the number of ways r things can be chosen from a total of n things (all different or not) with either the stipulation that:

1. order, or arrangement, counts; i.e., Δ □ Q is a different arrangement from Δ Q □, and thus a different choice, or

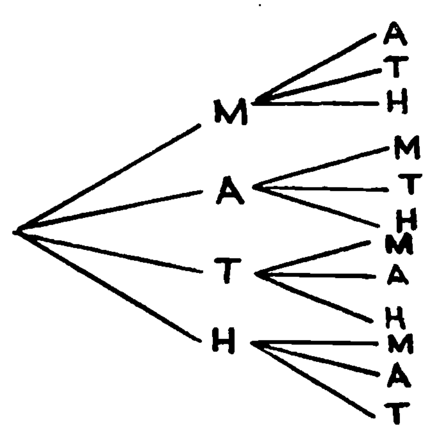
2. order does not count; i.e., Δ □ Q and Q Δ □ constitute the same choice.

Each choice in (1) is called a *permutation;* each choice in (2) is called a *combination.*

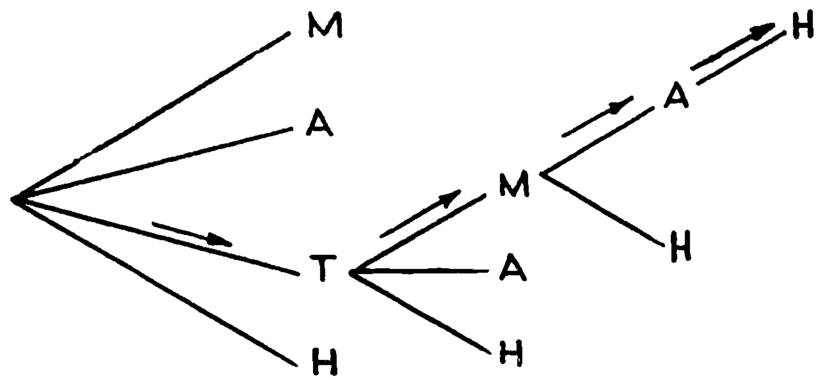
The major concern in this course will be with permutations of things which are *all different.*

Example 1:

How many different arrangements of the letters in MATH are possible? (That is, how many permutations are there of four different objects?) Solution: We may reason as follows: There are four spaces to fill: ----. The first may be filled with any of the four letters, M, A, T, H. Once we have filled the first space, we must fill the second, and there are three letters remaining with which this can be done. That is, for each choice of a letter for the first blank, we have three choices for the second, as shown in the diagram.

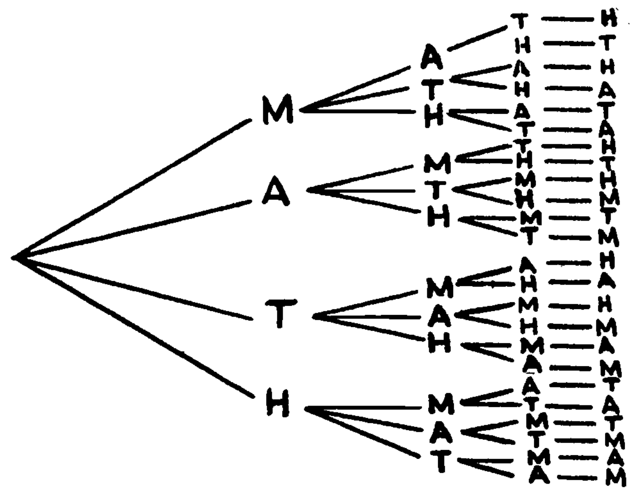


After we have filled the first two blanks, two choices remain for the third. (For example, if we had chosen T for the first, and M for the second blank) we would have a choice of A or H for the third. Finally, of course, we are forced to choose, for the fourth blank, the one letter remaining to us. The arrows in the diagram below correspond to the choice T M A H.



To find the total number of ways (permutations) the blanks could have been filled, we may count the number of different paths in our completed diagram, since each path corresponds to a different permutation. There are 24 such paths, or, more compactly, using the counting principle, $4 \times 3 \times 2 \times 1 = 24$.

Diagrams of the sort shown to the right are called *tree diagrams*. They are extremely useful for many problems in probability, and are an excellent visual aid in instruction.



In a like manner, the total number of permutations of the letters in THINK (total number of permutations of five things taken five at a time) is $5 \times 4 \times 3 \times 2 \times 1 = 120$. The number of permutations of six things taken all at a time is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$; of seven things $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ and so on. It is apparent that the number of permutations of n things (which are all different) taken n at a time is $n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots \times (2) \times (1)$. This quantity is symbolized $n!$ (read "n factorial" or "factorial n"); thus, $n! = n\ (n - 1)\ (n - 2) \ldots (2)\ (1)$ and $5! = 120$, and $(7 - 3)! = 4! = 24$.

Example 2:

How many "words" of three letters can be formed from the letters QXVBZTK? Solution: We have three blanks to fill. The first may be filled in any of seven ways (Q, X, V, B, Z, T, K), the second in six ways, and the third in five. Thus, we have $7 \times 6 \times 5 = 210$ different ways. (For example, there are 210 permutations of seven things taken three at a time). The number of permutations of n things taken r at a time is symbolized ${}_nP_r$ (read Enn-Pee-Ahr). Thus ${}_7P_3 = 210$ and ${}_nP_n = n!$ for all n, as we have seen.

Example 3:

How many ways can Sally and nine other students be seated in 10 fixed seats if Sally has to sit in a certain seat because of poor vision? Solution: The seat for Sally can be filled in only one way (with Sally). Once we have seated Sally, the second seat can be filled in any of nine ways, the third in eight ways, etc. The answer is 1 x 9 x 8 x 7 x ... x 2 x 1 = 1 x 9! = 9!

It is possible that classroom activities may lead to a discussion of problems involving combinations, and teachers may wish to consider this topic briefly. However, it is suggested that the overall time allotment be a major factor in such decisions; following years will see a continuation of work in the area of probability.

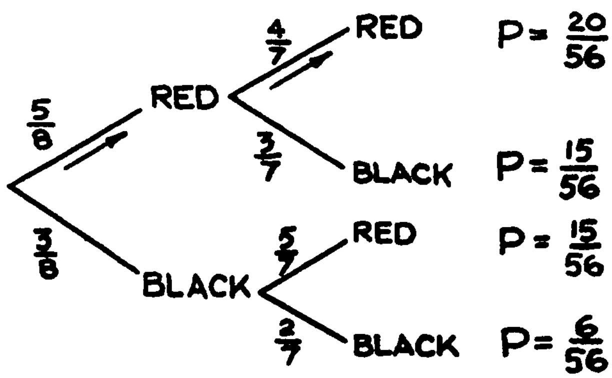
"Urn problems" are classic models of a certain type of probability consideration and are studied for that reason.

Example 1:

An urn contains five red and three black marbles. If one marble is drawn at random, what is the probability it is red? The answer is $\frac{5}{8}$, since there are 8 possible outcomes of which five are successes.

Example 2:

Same urn as problem 1. Suppose a marble is drawn, not replaced in the urn, and another marble drawn. What is the probability both are red? Solution: P (first is red) = $\frac{5}{8}$. Now, if the first marble drawn was red, there remains in the urn seven marbles of which four are red, so P (second marble is red) = $\frac{4}{7}$. Using the counting principle, we have $\frac{5}{8} \cdot \frac{4}{7} = \frac{20}{56}$. A tree diagram for this situation is given below.



The paths with the arrows show the 'successful' red-red combination. Note that the diagram gives us other information, including:

P (black, black) = $\frac{6}{56}$
P (1st black, 2nd red) = $\frac{15}{56}$
P (one red, the other black) = $\frac{15}{56} + \frac{15}{56} = \frac{30}{56}$

Note also that the sum of the probabilities at the right of the tree $\frac{20}{56} + \frac{15}{56} + \frac{15}{56} + \frac{6}{56}$ is equal to 1, as it should be, since the outcomes, of which these numbers give the corresponding probabilities, include all possibilities.

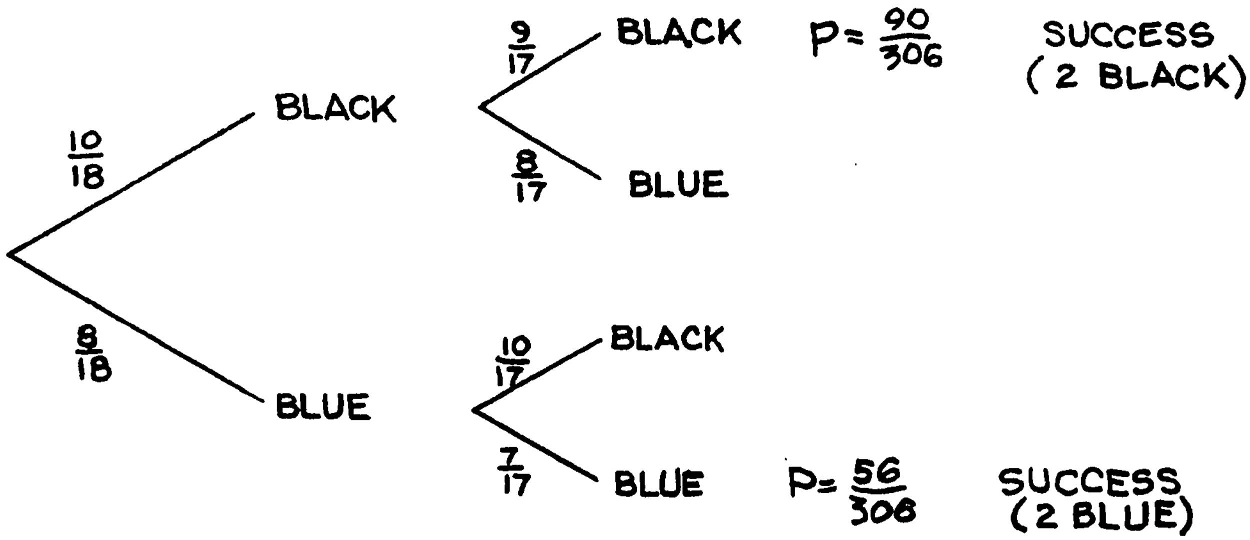
Example 2 was an urn problem without replacement. Example 3 is an urn problem with replacement.

Example 3:

An urn contains one white marble and five blue ones. A marble is drawn at random, its color noted, and then replaced in the urn. A marble is drawn again. What is the probability of drawing *blue* both times? Answer: $(\frac{5}{6})^2 = \frac{25}{36}$. In this case, since the marble is returned to the urn, P (blue) remains constant for every draw. For example, P (blue 5 times in a row) = $(\frac{5}{6})^5$.

Example 4:

Mr. Smythe groggily reaches into his sock-drawer one dim December morning and pulls out two socks, not looking at them very carefully. If the drawer contained 10 black socks and eight navy blue socks, what is the probability Mr. Smythe pulled out a matching pair?



P (success) = P (black, black) + P (blue, blue) = $\frac{90}{306} + \frac{56}{306} = \frac{146}{306}$.

D. Until this point, only situations involving events whose probabilities could be determined theoretically have been discussed. But there are situations involving events whose probabilities must be determined empirically (i.e., by observation and recording). An example of this type of event is the now pedagogically famous thumbtack: If a thumbtack (of a certain shape and weight distribution) is thrown in the air, what is the probability it will land point up? (i.e., as opposed to). There is no theoretically correct answer for this question, so to answer it as accurately as possible it is necessary to experiment. One hundred tosses will provide an approximate answer, and 1,000 tosses a better one.

The central aim here, as before, is to reinforce the idea of the stabilization of relative frequency, and an accurate chart should be kept of the progress of such experimentation.

Note: For various reasons, teachers may wish to use conically shaped paper drinking cups instead of tacks.

E. It is difficult to overestimate the effect that statistics, and their interpretation, have on all of us. Also, the philosophical, legal, and moral issues involved in the use of statistics may be at least as important, and may be of greater interest, to students than the purely mathematical consideration. It is suggested then, that the classes be allowed to discuss, if they wish, some of the nonmathematical ramifications of statistical inference before formal mathematical instruction begins.

Since the essential material of statistics consists of many bits of information, or *data,* and since the inferences made from the analysis of data are often of far-reaching significance, it is reasonable to investigate the adequacy of the methods by which they were collected. Some discussion of sampling techniques may be in order here.

Example:

Suppose it is required to determine the number of families in Gloversville who eat Diet Woompies for breakfast. How shall families be polled? Must *all* family members eat Diet Woompies to be counted? Will we telephone potential respondents, and if they do not answer at the time of day we call, will we call back? Will we assume that every respondent will answer truthfully? Finally, when we have finished our study, how can we be sure we have reached a fair sampling of the families in Gloversville, rather than a nonrepresentative subset of them?

The organization and presentation of data are, of course, prime factors in their usefulness. Bar graphs and circle graphs are among those with which students may already be familiar and the cooperation of social studies and business education departments might be solicited at this point to provide additional examples of the use and variety of graphs. Newspapers and magazines may also provide additional source material for this study. In this development we consider the frequency histogram and cumulative frequency histogram as two particularly useful methods of organizing data.

Example:

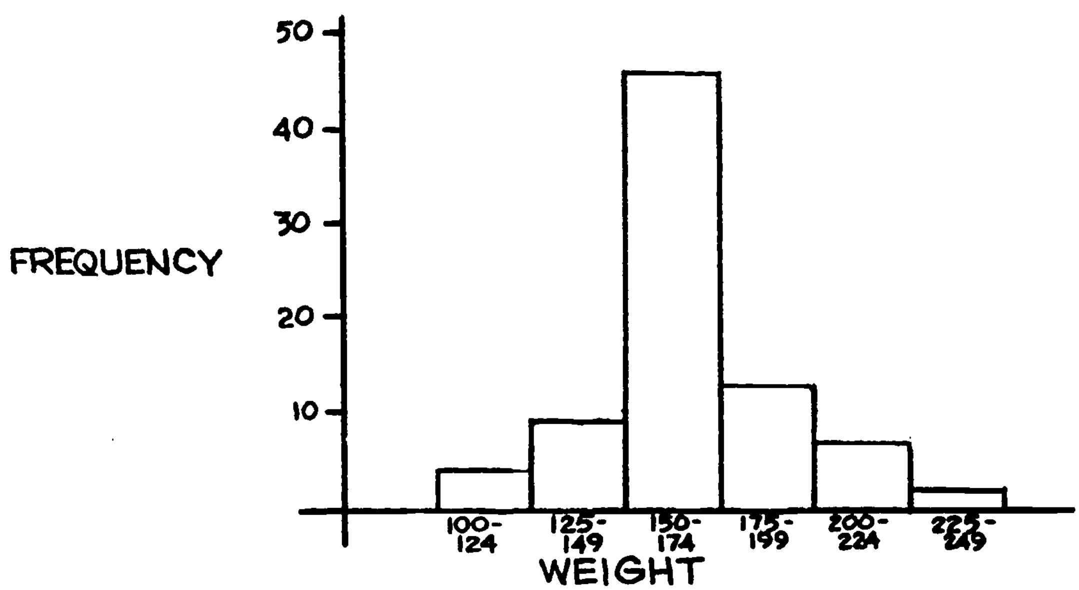
The medical office of Woompie State College reveals the following weights for the 80 male members of the entering freshmen class of 1976.

161	172	157	160	188	176	180	201
193	195	162	138	210	162	161	176
163	119	180	140	154	114	162	166
194	157	156	166	238	138	170	149
126	219	172	173	170	222	169	210
170	164	190	180	153	158	155	129
171	162	160	152	158	168	173	169
166	141	216	122	184	156	156	155
210	168	167	165	162	196	128	169
163	150	101	174	165	156	130	178

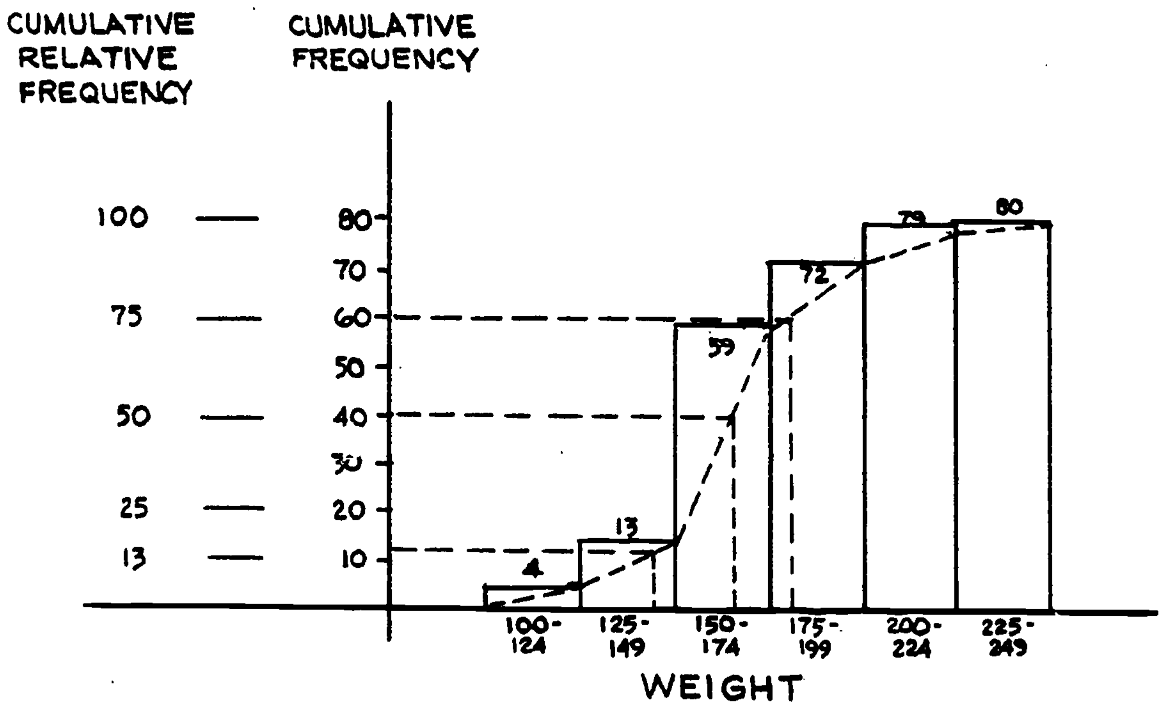
This data can be rearranged in the following table:

Interval	Tally	Number (frequency)
100-124	||||	4
125-149	𝍸 ||||	9
150-174	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 |	46
175-199	𝍸 𝍸 |||	13
200-224	𝍸 ||	7
225-249	|	1

From this tally, the following frequency histogram can be drawn.



A corresponding cumulative frequency histogram appears below:



F. Certain statistically significant measures can be 'read off' the graph given above (see dotted lines beginning at left of graph):

(a) The *median* (measure at or below which 50% of all the measures fall) weight of our group appears to be 165.
(b) The *upper quartile* (the measure at or below which 75% fall) appears to be 176.
(c) The *lower quartile* (the measure at or below which 25% fall) appears to be 151.

It is also possible to determine *percentile* values. (The nth percentile is the measure at or below which n% of the measures fall). For example, in our sample, the 13th percentile appears to be about 140. People often confuse the meaning of *percentile* and *quartile*, believing these terms to indicate ranges rather than specific measures, and it is suggested that attention be given to this problem.

Two other measures of central tendency are the *mean* and the *mode*. The mean, or *arithmetic mean*, is the most commonly used such measure, and students will know it as the familiar "average." Its importance is well known, but there are situations where knowledge only of the mean might not be helpful in drawing useful conclusions about the set of measures at hand. For example, the set of measures 2, 3, 2, 4, 3, 3, 3, 500 has as its mean 65, but this number is not a useful index of central tendency for these measures.

The *mode* is the measure which occurs most frequently. The mode in the previous set of numbers is 3, and in this case appears to be a better measure of central tendency than the mean. The median is also 3.

Depending on the skill of the class, individual teachers may wish to introduce, with varying degrees of stress, some of the axiomatic set-theoretical basis of modern probability theory, or to delve more deeply than suggested here into the area of combinatorics or statistics, and these teachers will find ample material for this purpose in any of the many probability and/or statistics textbooks available. However, teachers should be aware that it is planned to return to these topics in succeeding years of this program, and should not spend a disproportionate amount of time on supplementary topics.

IV. Rectangular Coordinate System

The visual, graphical experience of mathematics is the part of elementary algebra which many teachers have found appeals most directly to a large percentage of students they teach. It will be noted that in the outline given here, it is suggested that this graphical experience precede the corresponding analytical experience of the same topic. This approach allows the student to "see" or "play with" some of the notions (polynominal functions, step functions, absolute value functions, etc.) he will be asked to deal with in a more analytical or exact way later.

Besides leading to deeper understandings, the graphical approach also suggests some effective motivation for most analytical work. For example, the drawing of straight lines can lead to student discovery and discussion of the meaning of slope, the significance of the "m" in the equation $y = mx + c$ to $y = mx + b$. Also, the conjectures "Suppose the lines don't intersect at some nice point like (2, 3). Suppose they really intersect at 1.97162, 3.01111) and question, "How do we find that out?" can lead nicely to a discussion of the algebraic solution of simultaneous equations. In other, nonlinear, cases, through conjectures about graphs they have drawn, students may be able to reach conclusions of their own about the solution of quadratics or of simple higher-order polynomial equations. For additional material, functions or relations which produce "interesting" pictures should not be overlooked.

Example:

a "diamond-shape" ($|x| + |y| = 5$).

V. Optional Topics

As time permits, teachers may wish to explore the following topics.

A. Application of graphing to linear programming
B. Additional work with graphing - the quadratic, absolute value, and simple higher order polynomial functions
C. Scale drawings and practical examples (blue prints, etc.)
D. Aspects of transformation geometry

APPENDIX I

Below is a summary of the items generally required in a traditional ninth grade algebra course but which are not *required* in Course I of this new program.

(1) Students are not required to be able to refer to the associative, commutative, and distributive properties specifically by name. They are, however, required to understand the nature of the properties and to distinguish between systems displaying or not displaying one or another of them.

(2) "Type" problems (e.g., "investment" problems, "motion" problems) are not required as such. The emphasis instead is on general problem solving technique, chiefly the ability to translate from a given nonmathematical (or "verbal") statement of a situation to a mathematical (algebraic) one.

(3) Absolute value is not required. Many teachers may want to introduce the concept of absolute value as the year progresses, perhaps to facilitate the expression of some other idea, to provide new examples of equation solving, to provide a fairly easy example of a nonlinear graph, or for some other reason.

(4) Division of polynomials and work with polynomial fractions are not required in Course I. Most of the essential aspects of this material have been placed in Courses II or Course III where they are more relevant to other topics studied during those courses. Likewise, the process of rationalizing denominators is not required in Course I.

(5) There is no work in trigonometry in Course I. This material appears in Course III where it will be discussed from the point of view of circular functions.

APPENDIX 2

Some teachers may prefer an order other than the course outline. Possible suggestions for such changes of order are given below and may, as has been indicated in the introduction, be revised in any way.

A Introduction to Algebra (II - A, B)

B Logic (I)

C Solution of Equations (II - C)

D Rectangular Coordinate System (IV)

E Multiplication of binomials, factoring and application (II - G)

F Probability, Permutation, and Statistics (III)

G Aspects of Geometry (II D, E, F)

A Logic (I)

B Probability, Permutation, and Statistics (III)

C Introduction to algebraic operations (II - A, B, C, G)

D Rectangular Coordinate System (IV)

E Applications to Geometry (II - D, E, F)

A Logic (I)

B Introduction to Algebraic Operations (II - A, B, C, G)

C Rectangular Coordinate System (IV)

D Probability, Permutation, and Statistics (III)

E Aspects of Geometry (II - D, E, F)

A Logic (I)

B Introduction to Algebraic Operations (II - A, B, C)

C Rectangular Coordinate System (IV)

D Algebra and Geometry (II - D, E, F, G)

E Probability, Permutation, and Statistics (III)