DOCUMENT RESUME

ED 130 652 95 IR 004 192

AUTHOR Rapoport, Amnon
TITLE A Comparison of Two Tree Construction Methods for Obtaining Proximity Measures among Words. Number 47.
INSTITUTION North Carolina Univ., Chapel Hill. L.L. Thurstone Psychometric Lab.
SPONS AGENCY National Inst. of Mental Health (DHEW), Bethesda, Md.; National Science Foundation, Washington, D.C.
PUB DATE Jan 66
GRANT NIH-MH-10006; NSF-GS-82
NOTE 26p.; Archival document

EDRS PRICE MF-$0.83 HC-$2.06 Plus Postage.
DESCRIPTORS College Students; *Comparative Analysis; Higher Education; Males; *Paired Associate Learning; Structural Analysis; *Teaching Methods

ABSTRACT

The prediction that two different methods of constructing linear, tree graphs will yield the same formal structure of semantic space and measurement of word proximity was tested by comparing the distribution of node degree, the distribution of the number of pairs of nodes connected y times, and the distribution of adjective degree in trees constructed by the alternate methods. Fourteen male college students were each asked to construct two tree graphs from a list of 24 adjectives. For the first tree, the subject selected a pair of similar adjectives from the list and connected them with a line. He then sequentially connected the remaining adjectives to the pair according to their degree of similarity. Using the second method, subjects sequentially added single adjectives to the original pair, as in the first method, or started a new tree with a new similar pair or connected two trees together until all 24 adjectives were connected. Comparison of the formal structural distributions investigated showed no difference between the two methods. The methods did differ, however, with respect to their ability to yield the true ordering of the strength of association between paired adjectives. (KB)

Rothkopf

ED130652

A COMPARISON OF TWO TREE CONSTRUCTION METHODS FOR OBTAINING PROXIMITY MEASURES AMONG WORDS[1]

Amnon Rapoport

2

January 1966

Number 47

THE PSYCHOMETRIC LABORATORY
UNIVERSITY OF NORTH CAROLINA
Chapel Hill, N. C.

TR004 192

In order to determine the structure of a semantic space one needs to obtain proximity measures (measures of psychological similarity, nearness, or closeness) among the points (words) in this space. If n is the number of words, a symmetric n x n matrix of proximity measures among these words (a proximity matrix) can be subjected to one of the recently developed multidimensional scaling techniques (Guttman, 1965; Kruskal, 1964; Shepard, 1962). Each of these techniques will reduce the dimensionality of the matrix and provide a minimal semantic space, Euclidian in nature, where a monotone relationship exists between the proximity measures and the resulting distances among the words in this space.

There are several ways of obtaining a proximity matrix. An n-square confusion matrix can be constructed from the number of errors made in a paired-associate learning of a list of n words. An n-square proximity matrix can be obtained by rank ordering each set of n-1 words in terms of their similarity, association, or substitutability for a given target word, with ranking done separately for each of the n target words. Another way of obtaining a proximity matrix is by using the method of triads.

An alternative method has been suggested by Rapoport, Livant, and Rapoport (1965). When this method is used subjects are asked to construct ordinary linear graphs, in which the vertices are words and the edges are certain relations joining pairs of words. When the constructed graph is a "tree" (a finite connected undirected graph without cycles) the method determines uniquely all distances (inverses of the proximity measures) among all the n nodes (vertices) of the tree. The dimensionality of the resulting symmetric matrix of distances can then be determined by one of the multidimensional scaling techniques. Figure 1 portrays 3 different trees with 7 nodes each.

There may be several ways of constructing trees. The purpose of the present study is to compare two such methods used by the same Ss. The two methods differ psychologically from each other, but mathematically should yield the same structure. One of the methods has been used by Rapoport, Livant, and Rapoport (1965). Both methods have been used by Boyd and Livant (1964)

Method

Materials

A set of 24 scales was selected from the semantic differential study of the Thesaurus (Osgood, Suci, & Tannenbaum, 1957, p. 53). The scales were selected from the first 3 factors, i.e., Evaluation, Potency, and Oriented Activity. Eight scales were selected for each factor and one of the two polar adjectives defining each scale was then selected. (The only two exceptions, where both polar adjectives were selected, occurred on the third factor, which is defined by six scales only.) The resulting list

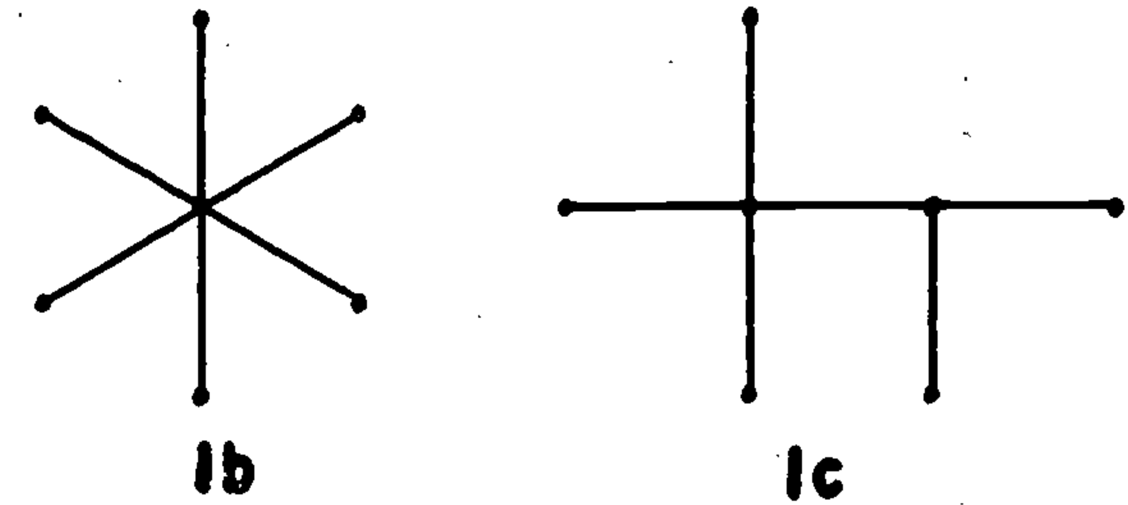

Fig. 1. Three different unlabeled trees with seven nodes each.

of 24 adjectives is given in Table 1.

Subjects

Ss were 14 male students enrolled at the University of Michigan (the same Ss used in the first study in Rapoport, Livant, & Rapoport, 1965). Each S was paid $1.50 per hour for participation in a language experiment.

Procedure

All Ss reported, as a group, to a lecture room. Each S was given a short printed introduction telling him that the experiment was concerned with obtaining similarity measures between adjectives. Each S was provided a list of the 24 adjectives, a blank sheet and a pencil. Ss were instructed as follows:

"Pick any two words from the list of 24 words which you think go together--that is to say--which you think are most similar to each other. Write the pair you have chosen on the blank paper and connect them with a line. Label the connecting line 1. Then go carefully over the remaining words in the list (which now includes 22 words) and pick the word which is the most similar to either of the two words you have already selected. Write this word down on your paper and connect it to the proper word already selected. Label the connecting line 2. Search carefully the remaining words (which now number 21) and select the word which is most similar to one of the three words already selected. Write this word down and onnect it with a line to the proper word.

Table 1

Adjective List Presented To Subjects

Bad	Negative
Beautiful	Painful
Calm	Passive
Cold	Reputable
Complete	Serious
Complex	Simple
Constrained	Slow
Excitable	Small
Feminine	Sociable
Foolish	Soft
Heavy	Spacious
Intentional	Strong

Label the connecting line 3. Please continue in this way until all 24 words have been exhausted."

Ss were asked to read the list of adjectives several times before starting the experiment. They were asked to work carefully, and when finished to wait for further instructions.

After all Ss finished the first part of the experiment, they were given another copy of the same list of adjectives and a new blank sheet. Ss were then instructed as follows:

"Start this part of the experiment in exactly the same way as before. I.e., pick any two words from the list of 24 words, which you think are most similar to each other. Write them down on a new blank sheet of paper and connect them with a line. Label the connecting line 1. Now you have two options: 1) You may look over the remaining 22 words on the list and decide that two of them are more similar to each other than any one of them is to either of the two words already selected and joined together. If so, you may select those two words and start a new tree, just as you did at the very beginning. Connect the two new words with a line and label it 2. 2) If you do not find two words which are more similar to each other than either one of them is to either of the two words already selected, proceed exactly in the same way as in the first part of the experiment. I.e., pick the word which is most similar to either of the two words you have already selected. Write this word down on your paper and connect it to the proper word already selected. Label the line 2.

As the experiment proceeds you may have a third choice. If you find that you made several trees you may connect any two of them together. Pick up any two words from any two separate trees which you feel are most similar to each other. Connect these words with a line and label the line according to the sequence already started.

In short, at any given time you have the following choices: 1) adding a new word to one of the trees you have already made, 2) starting a new tree with two new words, or 3) connecting two separate trees. Please continue in this way until all 24 words have been exhausted and until you connect all separate trees into one tree."

The end result of each of the two methods is a finite connected labeled tree without cycles. A distance between two adjectives (labeled nodes) is defined as the number of links connecting them. The distance between each pair of adjectives is, of course, unique. Note that the number of all possible trees with n labeled nodes is astronomical. It is given by n^{n-2} .

Each S constructed two trees, one by each method. Each tree consisted of n(n = 24) labeled nodes connected by n - 1 numbered links. The difference between the two methods is in the way a tree is constructed. In the first method S has to add new nodes to the old tree. In the second method S can either add new nodes to an old tree, start a new tree, or connect two trees together. We shall refer to the first method, where only one tree is constructed, as method C. The second method, which allows for the construction of many subtrees will be referred to as method H (after Boyd & Livant, 1964).

Results

The number of subtrees

Psychologically, the two methods differ from each other. Method H is less restrictive than method C as it allows S to separate the nodes into clusters, to deal with the clusters sequentially or in parallel, and to connect them together only in some later stage of the task. Thus in comparing the methods to each other the first thing that should be looked for is whether Ss used this option of constructing several disjoint subtrees. If they did not, then obviously method H would reduce to method C. The labeling of the links enable counting the number of subtrees constructed by each S under method H, and finding the stage at which two subtrees were linked together.

The number of disjoint subtrees, h, was counted separately for each S. The values of h ranged between 2 and 6, with a mean $\bar{h} = 4.357$, and variance $v_h = 1.374$. It is seen that Ss used the option given to them without exceptions. From the labeled links it is seen that the connecting of the ubtrees took place towards the end of the task. Ss tended to organize the nodes into separate connected clusters, and after exhaus.ing the whole list of 24 adjectives (but not the n - 1 links) connected these clusters together.

In spite of the psychological difference between method H and method C they are formally identical. Boyd and Livant (1964) have proved that if S has a metric on the words then the trees constructed under the two methods will have the same structure for a particular S. The prediction that the two methods generate the same formal structure is tested below by comparing several properties of the trees constructed by the two methods.

Distribution of node degree

Consider first the distribution of node degree (the distribution of number of links per node). The distribution of node degree in a random tree has been shown (Rapoport, Livant & Rapoport, 1965) to be well approximated for a large n by the Poisson distribution

$$(1) \quad P_n(i) = \frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!}, \qquad i = 1, 2, \ldots n - 1, \quad \lambda = \frac{n-1}{n},$$

where $P_n(i)$ is the probability of encountering a node of degree i in a random tree with n nodes.[2]

It should be noted that the only constraints on the number of links per node are 1) that a tree will contain at least two nodes with degree 1, 2) that the number of nodes with odd degree will be even, and 3) that there will be no node with degree higher than n - 1. Figure 1 portrays several trees exemplifying different distributions where x(i) denotes the number of nodes of degree i. Each tree has the same number of links, i.e., n - 1 = 6. It is seen that for the first tree (Fig. 1a) x(1) = 2, and x(2) = 5. For the second tree (Fig. 1b) x(1) = 6, and x(6) = 1. For the third tree (Fig. 1c) x(1) = 5, x(3) = 1, and x(4) = 1.

The distributions of x(i) were obtained for each S, and then summed over all Ss for each method. Table 2 compares the observed distributions obtained from method C and method H to the Poisson distribution predicted from (1). The fit of the Poisson distribution to each of the observed distributions is good ($p > .2$, using the one-sample Kolmogorov-Smirnov test). The difference between the two observed distributions, tested by the two-sample Kolmogorov-Smirnov test, is also nonsignificant ($p > .2$). Similar results were obtained for individual Ss.

Instead of comparing the predicted and the observed distributions of node degree a faster and simpler test is obtained by comparing the predicted and observed number of nodes with degree 1 (number of endpoints of the tree). Notice that this number is relatively large--at least one third of the nodes in any tree will have a degree 1. Let V_n denote the number of nodes with degree 1 in a random tree with n nodes. Let $M(V_n)$ and $D(V_n)$ denote the mean and standard deviation respectively of V_n. Rényi (1959) has derived the following formulas for $M(V_n)$ and $D(V_n)$:

(2) $$M(V_n) = n(1 - \frac{1}{n})^{n-2} ,$$

and

(3) $$D(V_n) = \sqrt{n(n - 1)(1 - \frac{2}{n})^{n-2} + n(1 - \frac{1}{n})^{n-2} - n^2(1 - \frac{1}{n})^{2n - 4}}$$

Note that

(4) $$\lim_{n\to\infty} M(V_n) = \frac{n}{e} .$$

Table 2

Observed And Predicted Frequency Distributions Of Node Degree

	Method C		Method H	
Degree	Observed	Predicted	Observed	Predicted
1	139	134	138	134
2	126	123	120	123
3	43	56	61	56
4	21	17	9	17
5	3	4	4	4
6	3	1	1	1
7	1	0	3	0

$\lambda = .917$

This asympototic result corresponds to what one gets from (1) with $i = 1$ and $\lambda = 1$.

For $n = 24$ equations (2) and (3) yield $M(V_{24}) = 9.40$ and $D(V_{24}) = 1.421$. The observed means of nodes with degree 1 are 9.929 and 9.857 for methods C and H respectively. The observed means do not differ significantly from the expected mean for the random tree ($\underline{t} = .372$, $p > .2$, and $\underline{t} = .322$, $p > .2$, for methods C and H respectively).

Distribution of number of pairs of nodes connected y times

Consider now the distribution of the number of pairs of adjacent nodes (connected by only one link) connected y times, $y = 0, 1, \ldots N$, where $N = 14$ is the number of Ss. There are altogether 276 $(\frac{n(n-1)}{2})$ possible pairs of nodes. As each pair may or may not be chosen by a particular S, the distribution of the number of Ss selecting a particular pair may be expected to be binomial with a parameter p ($p = 1/276$). The parameter p designates the probability of a specific pair to be selected. As the value of p is very small, the binomial distribution can be approximated by a Poisson distribution. The parameter λ, ($\lambda = pN$), the only parameter of the Poisson distribution, is interpreted as the "popularity" of a given pair of words relative to a set of words presented to a given population of Ss.

The popularity of a pair of words will generally depend on several factors. Prominent among them are the "semantic quality" of the two constituents of the pair, the specific experimental context, the specific set of words, and the particular population of Ss. Obviously, different pairs of nodes will have different popularities. Assuming that the popularity of different pairs of nodes is not constant as assumed above but is distributed in the population may lead to the distribution of the number of pairs chosen y times. Specifically, if we assume that λ is gamma distributed, it can be shown (Rapoport, Livant & Rapoport, 1965) that the distribution of the number of pairs chosen y times is given by the negative binomial distribution with parameters p and r

$$(5)\quad P(r,y) = \binom{r+y-1}{y} p^r q^y \;, \qquad 0 \leq p \leq 1,\ r > 0 \;.$$

Table 3 presents the distributions of the number of pairs of nodes chosen y times for method C and method H separately. The difference between the two observed distributions is nonsignificant ($p > .2$, by the two-sample Kolmogorov-Smirnov test). The negative binomial distributions, predicted from (5) for each of the two observed distributions, are also presented in Table 3. The parameters of the predicted distributions were estimated by the method of moments from the mean and variance of each of the observed distributions. The values of these parameters are presented in Table 3. The difference between the observed and predicted distribution was tested for each of the methods. In both cases the differences were nonsignificant ($p > .2$, using a chi-square test).

The good fit provided by the predicted negative binomial distribution to the observed distributions of pairs of nodes connected y times is consistent with the assumption about the distribution

Table 3

Observed And Predicted Frequency Distributions Of Number Of Pairs Of Nodes Chosen y Times

y	Method C		Method H	
	Observed	Predicted	Observed	Predicted
0	155	158	165	164
1	56	49	48	45
2	27	25	26	24
3	9	15	8	14
4	6	9	6	9
5	8	6	5	6
6	4	4	2	4
7	5	3	7	3
8	1	2	3	2
9	2	1	2	1
10	1	1	1	1
11	0	1	1	1
12	0	0	2	1
13	2	0	0	0
14	0	0	0	0
	276		276	

	Method C		Method H
mean =	1.167	mean =	1.163
variance =	4.414	variance =	4.890
$\hat{p}$ =	.264	$\hat{p}$ =	.238
$\hat{r}$ =	.419	$\hat{r}$ =	.363

of λ. The plausibility of this assumption asserting that λ is distributed over the population of pairs of nodes can be checked directly. The form of the distribution of λ is obtained by deriving the parameters of the assumed gamma distribution. The two parameters of the gamma distribution, r and u, are related to the parameters of the negative binomial distribution. The parameter r is identical in both distributions, and u is given by $u = p/q$, where $q = 1 - p$. Solving for r and u we get $r = .419$, $u = .359$ for method C, and $r = .363$, $u = .312$ for method H. The derived gamma distributions for both methods with both parameters smaller than 1 are monotonically decreasing over their ranges.

The form of the derived gamma distributions seems to be plausible. In considering the distribution of popularity of pairs of nodes, one would expect to find a high number of pairs with a very low popularity, and a small number of pairs with a high popularity. To put it differently, the distribution of λ would be expected to yield a monotonically decreasing function.

Distribution of adjective degree

So far we have investigated several statistical properties of the trees constructed by the two methods. We have not distinguished between unlabeled or free (topological) graphs and labeled graphs. Thus, in discussing the distribution of node degree the nodes were not labeled. For all purposes a node could be an adjective, a noun, or any sign whatsoever, without affecting the normal analysis which has been taken above.

If a complete comparison between the two methods is to be made, one might wish to compare the methods to each other with respect to the labeled graphs. Formally, two graphs whose nodes are labeled are the same if and only if for all i and j the same number of links are incident with the nodes labeled i and j in both graphs. Thus two labeled graphs may be considered distinct even though the two corresponding unlabeled graphs are isomorphic. For example, consider the graph portrayed in Fig. 1c. This is an unlabeled graph. The nodes of this graph may be labeled A, B, C, D, E, F, G, from left to right. Alternatively, the nodes of the same graph may be labeled by the same letters from right to left. The result is two distinct labeled trees even though their corresponding unlabeled trees are isomorphic.

Note that the formal identity of the two methods established thus far has no implications to the number of links incident with a particular adjective. In spite of the fact that the distributions of node degree do not differ from one method to the other, a particular adjective may have a large number of links affixed to it under one method, and a small number of links affixed to it under the other.

The number of links connected to a particular adjective (the adjective degree), relative to the group of adjectives selected, may be interpreted as a measure of associativity or a measure of specificity of meaning of this adjective. The smaller the number of links affixed to a given adjective the more specific is its meaning. If the equivalence between the two methods is not only formal, and if the same semantic space is used by Ss under both methods, one should expect the degree of specificity of each adjective to remain invariant under both methods.

The number of links connected to each adjective was summed over all Ss for each of the two methods. The frequency distributions of adjective degree for each method are presented in columns 2 and 3 of Table 4. Inspection of the table indicates that the distributions are very much alike. The product moment correlation between the two distributions is very high and significant ($r = .938$, $p < .001$).

The present measures of specificity of meaning can be compared with similar measures obtained in another study using different Ss and a different method for constructing graphs. In one of the studies reported by Rapoport, Livant, and Rapoport (1965) each of the 50 Ss was asked to construct a linear undirected graph using the same 24 adjectives given in Table 1. Each S was given a 24 x 24 matrix, where the rows and columns were labeled by the 24 adjectives. Each S was instructed to number the entry (ij) in the upper part of the triangular half-matrix if, in his judgment, adjectives i and j were associated. Number one was assigned to the most strongly associated pair, number two was assigned to the second most strongly associated pair, and so on up to 50. The end result was an undirected graph (with cycles permitted), not neces-

Table 4

Observed Frequency Distributions Of Adjective Degrees

Adjectives	Method C	Method H	Undirected graphs (N = 23)
Bad	30	29	133
Beautiful	27	25	104
Calm	37	34	129
Cold	27	27	100
Complete	22	26	70
Complex	29	29	83
Constrained	36	31	127
Excitable	23	22	61
Feminine	58	59	156
Foolish	25	29	101
Heavy	27	26	90
Intentional	26	24	74
Negative	23	22	105
Painful	21	26	103
Passive	24	28	124
Reputable	20	20	72
Serious	35	35	130
Simple	25	27	107
Slow	27	28	114
Small	26	18	54
Sociable	19	17	56
Soft	20	21	75
Spacious	14	17	51
Strong	23	24	81

sarily connected, with 24 nodes and 50 links.

The adjective degree was summed over all 50 Ss for each adjective for the first 23 labeled entries. The resulting distribution of adjective degree is given in the last column of Table 4. The product moment correlations between this distribution and the distributions obtained from method C and H are .755 and .803 respectively. Both correlations are significant ($p < .001$).

Discussion

The results suggest no difference between methods C and H with respect to the statistics investigated (distribution of node degree, distribution of number of pairs of nodes connected y times, and distribution of adjective degree). These results are relative, of course, to the specific group of adjectives sampled. Table 4 further suggests that the change in Ss and the construction of undirected graphs (where cycles are permitted and connectivity of the graph is not imposed by construction) instead of trees has no appreciable effect on the distribution of adjective degree.

The conclusion that the two methods do not differ from each other should be restricted to the formal structure of the graphs constructed under both methods. It may be argued that the statistics analyzed in the present study are too gross to yield a sufficiently detailed comparison between the two methods, and that a more refined analysis will expose differences between the methods. It is noted that the statistics reported above neglect an important aspect of the data--the numbering of the links. If the numbering of the links, which is supposed to reflect the strength of the association between every two paired adjectives, is considered, the two methods should yield different results.

The argument may be exemplified as follows: Supposing that for a given S the strongest association is between the adjectives Beautiful-Feminine and the second most strong association is between Bad-Negative. When method C is used S is forbidden by the method of construction from revealing his true ranking of the strength of his associations. Because if Beautiful-Feminine is put down as the first pair, the second pair should include either Beautiful or Feminine together with another adjective (which is unlikely to be either Bad or Negative). There is no way for S to indicate that Bad-Negative is his second most strongly associated pair. This constraint, however, is removed when trees are constructed under method H.

To assess the effect of this constraint the trees constructed under both methods were compared to one another with respect to the first four pairs of adjectives connected by S. The number of times each adjective appeared in the first four pairs was summed over all Ss for each method. No attempt is made here to provide a detailed comparison between the methods with respect to the frequency of appearance of each adjective, mainly because of the large individual differences. However, even a casual inspection of the results indicates differences between the methods. Thus, the adjectives, Bad, Negative, and Painful appeared 23 times when method H was used and not even once under method C. The adjectives Calm, Constrained, and Passive were used altogether 41 times under method C and only 19 times under method H. An inspection of the numbered trees indicates that under method C most Ss (10 from 14) started the task by connecting either Beautiful-Feminine, Calm-Passive, or Constrained-Slow, and thus could not "reach" one of the "negative" evaluation adjectives (Bad, Negative, Painful) within the first four connected pairs.

It may be concluded then that though the methods do not differ from each other with respect to the formal structure of the resulting graphs, they differ with respect to their ability to yield the true ordering of the strength of association between paired adjectives. As method H is more flexible than method C and does not impose the constraint discussed above it should be preferred.

The present results concerning the distribution of node degree seem to be at slight variance with results reported by Boyd and Livant (1964). They used similar instructions for both methods, a similar population of Ss, and nouns instead of adjectives. They found differences between the two methods with respect to the distribution of node degree. Method H showed less variability than method C. The discrepancy between the two studies, however, is only apparent. It can be attributed to two

Ss in Boyd and Livant (1964), who might have misunderstood the instructions.[3] Reanalysis of the data of the remaining Ss shows no difference between the two methods.

It was shown above that the hypothesis that links were affixed to nodes randomly could not be rejected. The correspondence between the distributions of node degree and the predicted Poisson distribution was very good for both methods. At the same time the correlation between the adjective degree in both methods was found to be high, suggesting that links are affixed to adjectives in a nonrandom fashion. The two findings are not incompatible with each other. They may be explained in terms of our distinction between labeled and unlabeled graphs. The task of constructing a labeled tree can be schematically decomposed into two parts, one dealing with the construction of the unlabeled tree, and the other dealing with the labeling of the nodes. The n - 1 links are affixed randomly to the n unlabeled nodes. However, the labeling or identification of the nodes is not done randomly. Adjectives which are close in meaning are put together so that nodes with a high degree are assigned to adjectives with general referents or multiple meanings, while nodes with a low degree are assigned to adjectives with a more specific meaning.

References

Boyd, J. & Livant, W. P. Some properties and implications of lexical trees. Unpublished manuscript, 1964.

Guttman, L. A general nonmetric technique for finding the smallest Euclidian space for a configuration of points. Psychometrika, 1965, (in press).

Kruskal, J. B. Multidimensional scaling by optimizing goodness of fit to a nonmetric hypothesis. Psychometrika, 1964, 29, 1-27.

Osgood, C. E., Suci, G. J., & Tannenbaum, P. H. The measurement of meaning. Urbana, Illinois: University of Illinois Press, 1957.

Rapoport, Anatol, Livant, W. P. & Rapoport, A. A study of semantic space. Unpublished manuscript, 1965.

Rényi, A. Some remakrs on the theory of trees. Publications of the Math. Inst. of the Hung. Acad. of Sci., 1959, 4, 73-85.

Shepard, R. N. The analysis of proximities: multidimensional scaling with an unknown distance function. I. Psychometrika, 1962, 27, 125-140.

Footnotes

1. The author is indebted to Lyle V. Jones for his assistance in the preparation of the manuscript. This study was supported by NSF grant no. GS-82 and by NIH grant no. MH-10006. A draft of this paper was written while the author was at the Mental Health Research Institute, University of Michigan. It was revised and prepared in its present form at the University of North Carolina.

2. If T_n denotes the number of all possible trees with n labeled nodes ($T_n = n^{n-2}$), then a random tree is one of the T_n possible trees chosen randomly with equal probability $\frac{1}{T_n}$.

3. The author wishes to thank J. Boyd and W. P. Livant for making their data available.