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ABSTRACT

This resource paper on spatial allocation analysis is part of a series designed to supplement undergraduate geography courses. Spatial allocation analysis is the study of the distribution of economic flows and transactions over geographical space. This account represents a general introduction to the process and its application in geographical inquiry. It proceeds by examining a series of models of spatial allocation systems, where the term "model" signifies an idealized representation. Under examination is a transportation problem model that includes (1) a set of geographically distinct points or regions which produce some commodity, (2) a set of geographically distinct points or regions which consume the same commodity, and (3) a given unit cost for transportation of the commodity from any producer to any consumer. In examining the flow of commodities, constraints are observed so that no supplier's total productive capacity is exceeded and all consumers' demands are met. Attention is focused first on the purely computational properties of the simple transportation model, then on the theoretical underpinnings of the model, and lastly on a variety of important, formal generalizations from the model. (Author/DE)

AN INTRODUCTION TO SPATIAL ALLOCATION ANALYSIS

Allen J. Scott

COMMISSION ON COLLEGE GEOGRAPHY

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FOREWORD

The Resource Papers have been developed as expository documents for the use of both the student and instructor. They are experimental in that they are designed to supplement existing texts and to fill a gap between significant research in geography and readily accessible materials. The papers are concerned with important concepts in modern geography and focus on three general themes: geography theory; policy implications; and contemporary social relevance. They are designed as supplements to a variety of undergraduate college geography courses at the introductory and advanced level. These Resource Papers are developed, printed, and distributed by the Commission on College Geography under the auspices of the Association of American Geographers with National Science Foundation support. The ideas presented in these papers do not necessarily imply endorsement by the AAG. Single copies are mailed free of charge to all AAG members.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

AN INTRODUCTION TO SPATIAL ALLOCATION ANALYSIS

PREFATORY NOTE

In the following report use is made of some elementary mathematics, if only in the sense that certain of the expository material contained in the report is presented in terms of formal symbols. This may cause consternation to some students. However, there is in this respect no real justification for any diffidence on the part of the student. Every effort has been made to ensure that this material is fully self-explanatory and self-contained. Moreover, those mathematical processes which have been employed are reduced to their most basic and simple form. Any conscientious student should therefore be fully capable of pursuing the main argument which is developed in succeeding pages. For the rest, some knowledge of elementary economics is assumed, as well as familiarity with the most common conventions of mathematical notation. In addition, students are strongly encouraged to work with pencil and paper through the exemplary problem which is discussed in Section II.

Several individuals have read and criticized various early drafts of this report. In particular, thanks are due to Drs. L. J. King, F. Kenneth Hare, and Ross MacKinnon for their many helpful suggestions. Most of these suggestions have been incorporated in some way in the main text, and, as a consequence, the report has been much strengthened and improved.

A. J. S.

I. GENERAL CHARACTERISTICS OF SPATIAL ALLOCATION MODELS

Introduction

The term *spatial allocation analysis* is here defined to mean the study of the distribution of economic flows and transactions over geographical space. Here, attention is focussed on those particular flows and transactions which can be identified as producing simple patterns of trade in specified economic commodities. These patterns are thus assumed to possess a concrete character and to be susceptible to cartographic representation. In addition, consideration is largely limited in this report to those special kinds of patterns whose geographical conformation is determined by the operation of various sorts of *optimizing processes.* The full meaning of this latter term will become apparent in due course.

Spatial allocation processes are of central importance in economic geography, spatial economics, and regional science. Their operation may be discerned in a great variety of different kinds of transactions which occur over geographical space, and they are of particular relevance to the study not only of simple commodity flow systems but also of those production, transformation, and consumption processes which are usually associated with such systems. In addition, spatial allocation processes are readily amenable to generalization, so that they can be applied to an exceptionally wide assortment and range of problems.

The account which follows represents a general introduction to spatial allocation analysis and to its applications in geographical enquiry. In particular, the account largely proceeds by examination of a series of *models* of spatial allocation systems, where the term *model* signifies simply an idealized representation. This account moreover is especially concerned with the basic properties of a fundamental spatial allocation model designated the *transportation problem,* and with the various major extensions of this particular problem.

This transportation problem is an especially versatile mechanism. In its most simple and essential form it involves basically the definition of a simple economic system composed of the following ingredients: a) A set of geographically distinct points or regions which produce some commodity, b) a set of geographically distinct points or regions which consume the same commodity, and c) a given unit cost for transportation of the commodity from any producer to any consumer. The transportation model then animates this system by applying to it a kind of vital principle which optimizes the entire system. In short, the model designates an assignment of flows of the commodity from producers to consumers so that the total costs of transportation within the system are minimized. In doing this the constraints are observed that no supplier's total productive capacity must be exceeded and that all consumers' demands must be met. The generalizations of this kind of model are now discernible, and they extend into a series of models of increasing complexity involving transshipment processes, transformation processes, multicommodity flows, and the like.

Basic Assumptions for Spatial Allocation Analysis

A variety of basic assumptions must usually be satisfied before the transportation model can be said to be in any sense a satisfactory model of some economic system, however simple that system might be. These assumptions relate to the nature of the operating principles which govern the economic system represented by the model. Such principles are of two main types, and they correspond on the one hand to a system of complete centralization of decision-making, and on the other to complete decentralization of decision-making.

In the first place, then, the transportation model is an appropriate tool of analysis where the economic system under consideration is characterized by pure centralized control (or monopoly). Such a system might be controlled by a planning agency concerned with public or social benefits, or by a private organization concerned with purely private profit. In any case, it is in the nature of such systems to attempt to seek out cost minimizing solutions, (equivalent to benefit maximizing solutions), just as is done by the transportation model.

In the second place, the model is appropriate for the analysis of systems characterized by perfect competition in the classical sense. The reasons for this are fairly complex, and discussion of these reasons forms the major substance of Section III of this report. For the moment the assertion must be taken on faith. Nevertheless, it is an easy matter to specify the conditions for a regime of perfect competition in the context of the transportation problem. These conditions are: a) There must be many suppliers of the given commodity so that no supplier can gain control of the market, b) similarly, there must be many consumers, c) information about the market must be perfectly and freely available to all, d) sellers must be perfect profit maximizers, e) buyers must always buy at the lowest available price, and lastly f) the commodity in question must be sufficiently homogeneous that (price considerations apart) consumers are indifferent as to their source of supply.

These two major types of principles governing the validity and applicability of the transportation model in any given situation are in practice fairly general. Needless to say they are rarely completely satisfied in actuality. However, most real systems are sufficiently close to one or other of these two sets of conditions to make the transportation model reasonably valid and workable with respect to those systems. Only the most extreme cases of the development of special interest groups (as in international trade or oligopoly) would seem to be entirely resistant to analysis by the method of the transportation model. In any case, the model is a purely *normative* mechanism. That is, the model seeks not so much to describe real systems, as to represent an optimized ideal state to which real systems (whether centralized or decentralized) aspire. To the extent that any real system is less than purely monopolistically centralized or less than purely competitively decentralized, then to that extent may the system be expected to depart from the normative structure suggested by the transportation model.

The Transportation Model: Symbolic Presentation

Definition of System Elements

Suppose that there exists an economic system involving the production and consumption of a single homogeneous good as described above. Let there be n sources of this good, with the individual sources designated $i = 1, 2, ..., n$. Let there be m demand points or destinations for the same good, with the individual destinations designated $j = 1, 2, ..., m$. Now, the commodity will flow from sources to destinations. Let the magnitude of total flow from i to j be designated x_{ij}. In addition let the *unit* cost of transportation of the commodity from i to j be designated t_{ij}. Therefore, the *total* transport cost incurred by the flow x_{ij} will be $t_{ij}x_{ij}$. Additionally, let S_i denote the supply capacity of the i^{th} source, and let D_j denote the total demand at the j^{th} destination. These capacity and demand values are, for present purposes, taken to be perfectly inelastic; that is, they will never vary under any circumstance (such as a change in prices or costs).

Bearing these assumptions and definitions in mind, it is now possible to specify in a rigorous manner the general structure of the transportation model.

Basic Structures

Recall that the basic operating principle of the transportation model is to minimize the total costs of commodity flow. This principle can be written symbolically as an *objective function* as follows:

Minimize:

$$1) \qquad Z = \sum_{i=1}^{n} \sum_{j=1}^{m} t_{ij}x_{ij}$$

This objective function is now subject to a set of *constraints*, as follows:

$$2) \qquad \sum_{j=1}^{m} x_{ij} \leqslant S_i$$

which states that the total shipments out of the i^{th} source must always be less than or equal to the supply capacity of that source; and

$$3) \qquad \sum_{j=1}^{n} x_{ij} = D_j$$

which states that total shipments into the j^{th} destination must exactly equal the total demand at that destination. An additional constraint, or *side-condition,* also applies, namely,

$$4) \qquad x_{ij} \geqslant 0$$

for it is clearly not permissible to specify the magnitude of any flow as being equal to a negative number. The expressions (1)–(4) collectively are designated a *program.* More specifically they are also designated a *linear program* since the algebraic relations within each expression are all perfectly linear. This particular linear program is of course also the transportation problem.

Note that the program contained in the expressions (1)–(4) is written in extremely compact notation. The system can be re-expressed by writing out each supply and demand condition in full and this makes explicit the entire structure of the program. Thus, for simplicity, assume that $n = 4$ and that $m = 5$. For such a system a complete program may be written out as in Table 1. All variables are appropriately labelled along the top row of Table 1, and all

Table 1. Matrix representation of the transportation problem

Variables	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	Constraint values
Constraining conditions	1	1	1	1	1																$\leqslant S_1$
						1	1	1	1	1											$\leqslant S_2$
											1	1	1	1	1						$\leqslant S_3$
																1	1	1	1	1	$\leqslant S_4$
	1					1					1					1					$=D_1$
		1					1					1					1				$=D_2$
			1					1					1					1			$=D_3$
				1					1					1					1		$=D_4$
					1					1					1					1	$=D_5$
Transport costs	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	t_{21}	t_{22}	t_{23}	t_{24}	t_{25}	t_{31}	t_{32}	t_{33}	t_{34}	t_{35}	t_{41}	t_{42}	t_{43}	t_{44}	t_{45}	

constraint constants are shown in the right-hand stub of the Table. Then, the full system is expressed in matrix form in the main body of the Table, where the occurrence of the digit, one, in any column or row indicates that the corresponding variable occurs in the corresponding constraint. Any column-row intersection in the Table which is left blank is considered to be occupied by a zero.

In the matrix form typified by Table 1, the transportation problem is suitable for numerical solution by the somewhat involved computational procedure known as the *simplex method.* This method is a general solution procedure for all types of linear programming problems. However, the transportation model is, in addition, soluble by a simpler procedure which is especially designed to take advantage of the specific structural properties of this model. This procedure or *algorithm* is examined at length in Section II of this report.

Applications and Generalizations of the Transportation Model

Over the last twenty years there have been many successful applications of the transportation model and derivative models to the study of geographical problems. These applications have been concerned largely with the analysis and planning of commodity flow systems. However, there have also been significant applications of these models to such questions as the provision of public housing, the location of industry, and even to the scheduling of disaster relief operations.

An outstanding series of early studies concerned with the application of the transportation model was accomplished by Henderson (1955a, 1955b, and 1958). In these studies Henderson applied the transportation model to the analysis of coal production and distribution in the United States. By means of the transportation model, Henderson computed a sequence of normative representations of the United States coal industry for different periods of time. Then, by comparison of the actual coal trade in the United States against these normative solutions, he was able to isolate significant factors influencing the dynamic structure of the industry. In a similar study, Land (1957) has applied the transportation model to the study of coking coal movements over the British railway system.

The transportation problem has been applied further in the study of the organization and spatial structure of public facility systems. Yeates (1963), for example, has examined the spatial allocation of students to high schools in a part of Wisconsin. By means of the transportation problem, Yeates was able to draw up a set of optimal school district boundaries such that the total costs of transporting the students between home and school were minimized. Gould and Leinbach (1966) have applied the transportation model to the optimal organization of hospital service areas, and Garrison *et al.* (1959) have used the model to determine least cost allocations of patients to physicians.

One of the most important theoretical generalizations which emerges out of the transportation problem is the whole question of the equilibrium pricing of commodities and the spatial variations of these equilibrium prices. These theoretical notions have been most successfully applied in empirical cases to the study of agricultural commodities.

Thus Fox and Taeuber (1955) have studied the spatial structure of the livestock-feed economy of the United States and have shown how this spatial structure relates to the interdependencies in an interregional context among prices, supplies, and demands. Similar studies have been undertaken by Fox (1953) and Judge and Wallace (1958). In particular, in many of these studies of agricultural systems exhaustive and searching sensitivity analyses have been undertaken. Thus Morrill and Garrison (1960) have considered the problem of the regional stability of trade in wheat and flour across the United States, and they have shown for example what effects a hypothetical drought in the Great Plains would have upon this system.

A further important set of generalizations extending from the transportation model concerns the role of transshipment and commodity transformation processes in interregional trade. Here, Casetti (1966) has contributed an empirical study of great interest and significance. Casetti's problem was to study the geographical organization of the steel industry of Quebec and southern Ontario. This part of eastern Canada is linked together by the Great Lakes–St. Lawrence waterway system. The problem as formulated by Casetti is then to schedule a cost-minimizing program of shipping movements over this waterway system so that all steel plants are adequately supplied with coal and iron ore, and so that all finished steel is then taken to final markets. In addition, Casetti's model takes into account the input-output processes governing the transformation of the coal and ore into steel, and it also takes fully into account any costs on the movement of empty ships from one location to another.

These transshipment problems have been even further extended into an enormous variety of geographically-structured network problems. Here, the applications have been multifarious, though perhaps two such applications might be mentioned briefly for their relevance to geographical analysis. First, Gauthier (1968) has studied commodity flows in Brazil in relation to the carrying capacity of the Brazilian transportation system. In this study, Gauthier has applied the so-called "capacitated transportation problem," which is essentially a combination of the transportation problem with a simple network system. Second, Ridley (1969) has studied the problem of traffic flow through road networks and has devised a program for the optimal assignment of new investments to traffic-bearing networks, taking into account their flow characteristics.

This brief discussion of some of the applications and applied generalizations of the transportation problem provides a backdrop for the discussion which follows. This discussion now largely neglects the problems of direct empirical analysis and related questions of policy, and concentration is focussed upon computational procedures and formal theory. However, by occasional reconsideration of the material discussed in the preceding paragraphs this discussion of procedures and theory may be illuminated and made concrete. It should be especially kept in mind throughout all that follows that spatial allocation models are indeed normative rather than descriptive mechanisms.

II. THE TRANSPORTATION PROBLEM: COMPUTATIONAL PROCEDURES

In this section an *algorithm* (or set of repetitive computational rules) for solution of the transportation problem is developed. This algorithm proceeds by working in an iterative fashion from some arbitrary initial solution of the transportation problem through a series of intermediate solutions to the final optimal solution. The algorithm converges progressively on this optimal solution and the objective function value of the problem always diminishes as the solution process continues. At all stages of the computational process, feasibility of the program is rigidly maintained. Here *feasibility* is defined as that state where all the constraining conditions which apply to the problem are fully satisfied.

A Sample Problem

Rather than attempt to describe the transportation problem algorithm as a set of abstract principles, in this account the algorithm is described and developed with respect to a specific sample problem. This helps to make the computational process more immediately meaningful.

Suppose that a problem as shown in Figure 1 is given. In this Figure there are four points (designated by squares) which supply some commodity, and five other points (designated by circles) which consume that commodity. Transportation of the commodity from any source to any destination incurs a transportation cost, and a set of hypothetical transportation costs for this sample problem is shown in Table 2. These transportation costs are very roughly proportional to the actual distance separating source and destination points in Figure 1. Table 2 indicates the supply capacity of each source point and the demand requirement of each destination point. Thus supply capacities (shown in the right-hand stub of the Table) are, for S_1, S_2, S_3, and S_4 respectively, 7, 3, 5 and 15. In the same way demand requirements (shown in the bottom stub of the Table) are, for D_1, D_2, D_3, D_4, and D_5 respectively, 1, 8, 8,

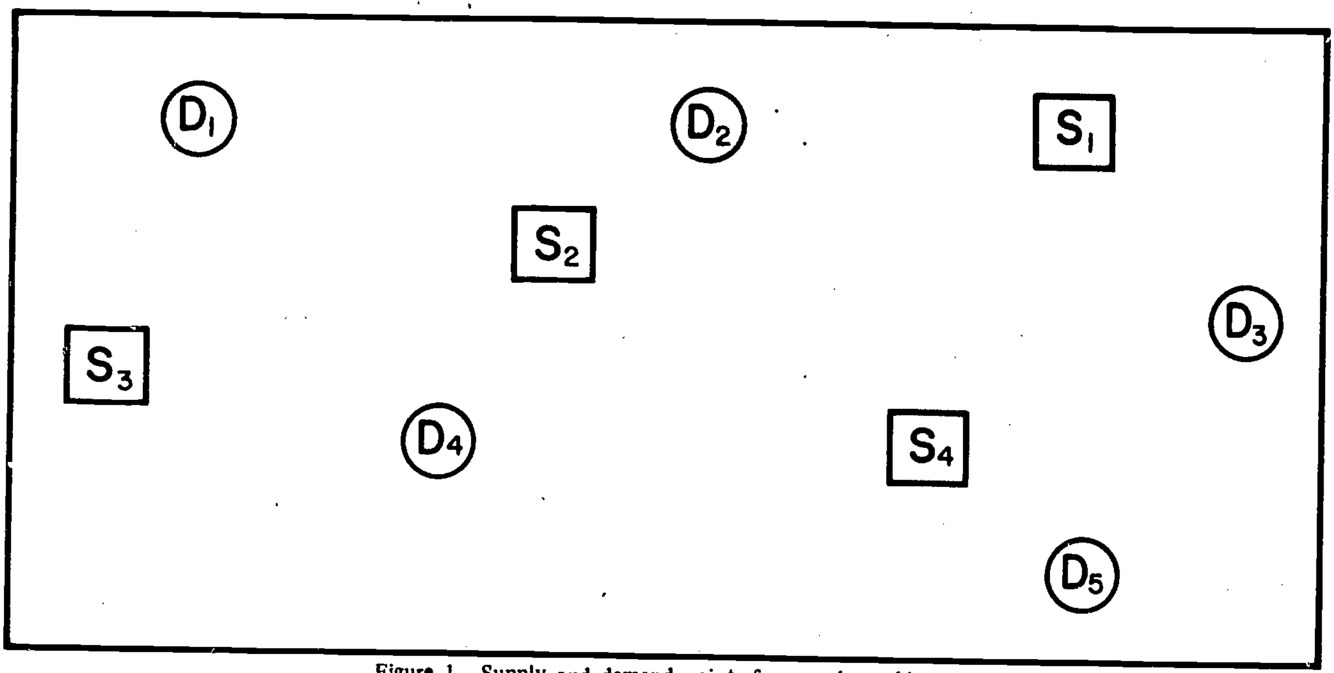

Figure 1. Supply and demand points for sample problem

Table 2. Transport costs, supply capacities, demands

To / From	1	2	3	4	5	Supply
1	9	4	3	7	5	7
2	4	2	7	2	5	3
3	3	7	12	4	9	5
4	9	4	3	5	1	15
Demand	1	8	8	9	4	30

Table 3. First feasible solution

To / From	1	2	3	4	5	Supply
1	9 1	4 6	3	7	5	7
2	4	2 2	7 1	2	5	3
3	3	7	12 5	4	9	5
4	9	4	3 2	5 9	1 4	15
Demand	1	8	8	9	4	30

$$Z = (9 \times 1) + (4 \times 6) + (2 \times 2) + (7 \times 1) + (12 \times 5) + (3 \times 2) + (5 \times 9) + (1 \times 4) = 159$$

9, and 4. Observe that global supply is made to equal global demand (30 units in each case) so that there is in this sample problem no slack capacity and no excess demand.

In addition, Table 2 provides a kind of tableau representation of the transportation problem, and this representation of the problem is notably more compact and easy to interpret than the matrix representation as given in Table 1. Indeed the computational procedure which is described below takes this tableau as a basic working statement of the problem. It is now required to find feasible numerical values, x_{ij}, for all flows from sources to destinations and such that all transport costs are a minimum.

The First Feasible Solution

Initiation of the transportation algorithm is dependent upon finding a first feasible solution. Usually, this solution will be quite far from optimality. Various methods are available for deriving some initial feasible solution, but an especially simple method will be demonstrated here. This method makes use of a basic operating principle known as the *north-west corner rule.* Essentially this rule involves assigning shipments to source-destination pairs by working through a tableau representation of the transportation problem, starting in the upper left-hand (or northwest) corner of the tableau and working progressively towards the lower right-hand corner. At each stage in this process as large a commodity shipment as possible is assigned.

For example, consider Table 3, (note that for ease of reference all transportation costs have been relegated to the corners of the cells of Table 3). At the outset take the upper left-hand cell of this Table. This is the cell [1, 1] representing a shipment from the first source to the first destination. Assign as large a value as possible to this shipment. Clearly this value must be 1 unit of the commodity, for whereas the supply point 1 can supply up to 7 units of the commodity, the destination point 1 requires only 1 unit, and this demand should not be exceeded. The supply point 1 now has a surplus of 6 units, while the demand of the destination point 1 is entirely satisfied. Therefore continue consideration of supply point 1, but now take cell [1, 2] which lies immediately to the right of cell [1, 1]. Destination point 2 has a total demand of 8 units. But supply point 1 can meet this demand only to the extent of 6 units. Therefore assign all of these 6 units to destination point 2. All of the commodity produced at the first supply point is now assigned to consumers, though destination point 2 has a demand deficit of 2 units. Therefore continue consideration of destination point 2, but move on to the cell [2, 2] which lies immediately below the cell [1, 2]. Clearly supply point 2 can now make up the entire deficit in the demand of destination point 2. Thus, assign 2 units to the cell [2, 2]. This still leaves supply point 2 with a surplus of 1 unit. Thus, consider cell [2, 3] to which that 1 unit can now be assigned. Now proceed on in this fashion, moving in an orderly fashion to the right and downwards through the Table, and always assigning as much as possible to each cell which is considered, while taking care not to contravene any supply or demand condition. Hence, continuing the allocation process, the quantities 5, 2, 9, and 4 are assigned in sequence to the cells [3, 3], [4, 3], [4, 4], and [4, 5].

These operations, then, lead to a complete first feasible solution of the sample problem, and this solution is shown in Table 3. That this solution is indeed feasible can easily be verified by summing up shipment values in any row or any column and observing that they do in fact conform to the appropriate constraint values. Note that the total cost of this first feasible solution is

5) $$Z = t_{11}x_{11} + t_{12}x_{12} + t_{22}x_{22} + t_{23}x_{23} + t_{33}x_{33} + t_{43}x_{43} + t_{44}x_{44} + t_{45}x_{45}$$

and this value of Z is 159, which, as will emerge in due course, is very far from optimal.

The First Iteration

With a first feasible solution of the problem now available it is possible to begin the iterative solution algorithm. This iterative algorithm begins by taking the first feasible solution as a starting point and then working out a set of successive program improvements. Each program improvement is itself accomplished after a series of computational steps, and these steps are now described in order.

Computation of shadow prices

The first stage in securing a program improvement involves the computation of a set of what for want of a better term will be designated *shadow prices.* In Section III of this report the physical meaning of these shadow prices will be made clear. For the moment, it is sufficient to accept them as purely computational devices.

A shadow price is computed for each source and each destination. This is the same as saying that there is a shadow price on each row and each column of the transportation problem tableau. Moreover, the shadow prices are computed only in relation to those given source-destination shipments which occur at some positive, non-zero level. Such shipments (from the first feasible solution) are designated symbolically in Table 4a by large solid circles. Let U_i be the shadow price on the i^{th} source or row, and let V_j be the shadow price on the j^{th} destination or column. Now for any intersection of row i and column j in Table 4a which contains a solid circle, the following conditions on the prices, U_i and V_j, must be made to hold:

6) $V_j - U_i = t_{ij}$, or

7) $V_j = t_{ij} + U_i$, or

8) $U_i = V_j - t_{ij}$

where the conditions (6), (7), and (8) are of course all algebraically equivalent. These conditions may or may not hold for any row and column intersection which does *not* contain a solid circle. In addition, some one shadow price is always arbitrarily assigned a value of zero, and for convenience of reference it will here be assumed that this rule applies to the shadow price U_1. Therefore, by definition, the following condition also holds,

9) $U_1 = 0.$

Tables 4a - 4c. First iteration

Table 4a.

Shadow prices →		9	4	9	11	7	
↓	To / From	1	2	3	4	5	Supply
0	1	9 ●	4 ●	3 9	7 11	5 7	7
2	2	4 7	2 ●	7 ●	2 9	5 5	3
-3	3	3 12	7 7	12 ●	4 ⑭	9 10	5
6	4	9 3	4 -1	3 ●	5 ●	1 ●	15
	Demand	1	8	8	9	4	30

Bearing these various conditions in mind a set of shadow prices relating to Table 4a can at this point be numerically determined. Hence, in conformity with (9) first set U_1 equal to zero. This permits calculation of V_1 which from (7) is equal to 9 + 0 =9. In the same way, the value of V_2 is computed as 4 + 0 =4. With this value of V_2 it is now possible to find U_2, which, from (8), is equal to $V_2 - t_{22}$ or 4 − 2 = 2. Recall that the operations generating these shadow prices apply only to row and column intersections which coincide with cells containing a non-zero shipment. Thus it is not permissible for example to compute U_2 by the operation $V_1 - t_{21}$, for the cell [2, 1] contains no positive shipment. By repeated application of the principles for deriving shadow prices on rows and columns it is possible to arrive at a fully identified set of U_i and V_j. For the case of Table 4a, these shadow prices are, in complete order, 0, 2, −3, and 6 for the U_i, and 9, 4, 9, 11, and 7 for the V_j.

The shadow prices are now used to compute an *opportunity cost* for every cell which lacks a positive non-zero shipment, and the properties of these opportunity costs are now considered.

Opportunity costs

Again, as was the case with the shadow prices, the notion of an opportunity cost will be defined here in terms of its purely computational role within the transportation algorithm, and its physical economic interpretation will be deferred until Section III.

Every cell of the transportation tableau which does not contain a positive shipment has associated with it an opportunity cost, denoted $\bar{c}_{ij}$. This opportunity cost is defined as

10) $\bar{c}_{ij} = V_j - U_i$

and with the strict proviso that there is no shipment from the source labelled i to the destination labelled j. These opportunity costs are now computed for all cells which lack a positive shipment, and the results are shown in Table 4a, where the opportunity costs are shown numerically in the main body of the Table.

Improvement of the basic solution

At this stage, Table 4a is scrutinized to determine whether any opportunity cost, $\bar{c}_{ij}$, is greater than the corresponding transport cost, t_{ij}. If no such $\bar{c}_{ij}$ is found then the existing program of shipments is optimal, and the computational algorithm is ended. However, in Table 4a, there are several instances of cells where $\bar{c}_{ij}$ exceeds t_{ij}. This being so, isolate that one cell where the difference between $\bar{c}_{ij}$ and t_{ij} is the greatest. This is the cell [3, 4] where the difference is equal to 14 – 4 = 10.

The cell [3, 4] is now used as a pivot or point of reference with respect to which improvements in the existing program of shipments are made. It is now required, in short, to assign some positive shipment to the cell [3, 4], and in doing this, the values of other shipments are readjusted so as to maintain program feasibility. This re-adjustment is accomplished in two stages: first by setting up a general pattern of shipment increments and decrements, and second, by numerical resolution of these increments and decrements.

Search for a generalized pattern of improvements. The objective of the present computational stage of the first iteration is to assign as large a shipment as possible from source 3 to destination 4. At the outset, then, locate a plus sign in the cell [3, 4], (see Table 4b), thereby indicating that this increment is eventually to be numerically effected. Obviously, however, if this shipment is made, then part if not all of the commodity which is shipped from source 3 to other destinations will have to be diverted so as not to exceed the supply capacity of source 3. Thus, search along row 3 for any cell which contains a positive shipment. There is only one such cell, and this is the cell [3, 3]. Place a minus sign in this cell, signifying that the corresponding shipment must be reduced. However, if this shipment is reduced then all of the demands of destination 3 will not now be sufficiently met. Therefore, search along column 3 for some other cell containing a positive shipment. Here there are two possible candidates for selection, either cell [2, 3] or cell [4, 3]. Suppose that cell [4, 3] is selected. Place a plus sign in this cell to indicate that the deficit in the demand at destination 3 is to be made up by increasing this shipment. An increase in the value of the shipment in the cell [4, 3] necessitates reducing the magnitude of other flows emanating out of source 4. Thus, search along row 4 for a cell containing a positive shipment. Such a cell is found at [4, 4]. Now insert a minus sign in the cell [4, 4] to indicate shipment reduction. As it happens this minus sign is located in the same column as the original plus sign in cell [3, 4]. This is a desired result, for it signifies that the search for a general pattern of shipment improvement is brought to a close. This search process may now be recapitulated in general as the following set of rules and procedures:

Table 4b.

To / From	1	2	3	4	5	Supply
1	9 ●	4 ●	3	7	5	7
2	4	2 ●	7 ●	2	5	3
3	3	7	12 ● –	4 +	9	5
4	9	4	3 ● +	5 ● –	1 ●	15
Demand	1	8	8	9	4	30

1) Insert a plus sign in that (empty) cell where the opportunity cost positively and maximally exceeds the transport cost.
2) Initiate an orthogonal pattern of search based upon this pivot cell. Here, to use an analogy with chess, the term *orthogonal* means that the pattern can be traced out through a tableau by the movements of a rook (which must move horizontally or vertically) but not by the movements of a bishop (which must move diagonally).
3) Let the search pattern extend in alternation over a row, a column, a row, a column, and so on.
4) The articulations in the orthogonal pattern of search are allowed to coincide only with cells which contain some positive non-zero shipment, (with the exception of the pivot cell which is empty).
5) At each articulation in the search pattern, insert either a plus sign or a minus sign so that the plus and minus signs form an alternating series around the search circuit.
6) The pattern of search is brought to a close whenever a minus sign is successfully located in the same column as the pivot cell.

Notice that during the search procedure it is often possible to make a decision which leads to an impasse. For example, in the case of the problem shown in Table 4b, the orthogonal search pattern could have proceeded after consideration of cell [3, 3] to cell [2, 3] (in which a plus sign would have been located) then on to cell [2, 2] (a minus sign), then to cell [1, 2] (a plus sign), and finally to cell [1, 1] (a minus sign). But from cell [1, 1] it is impossible to continue the search according to the rules laid out above. In this case, it would be necessary to re-trace the pattern back to some cell (in fact cell [3, 3]) from which the search can be continued and pursued to a successful close.

Implementation of the designated improvements. With the aid of the pattern of increments and decrements designated in Table 4b it is finally possible to improve upon the structure of the program as established in the first feasible solution. Remark that it is required to assign as large a shipment as possible from source 3 to destination 4. This means that the total shipment in one of the cells which contains a minus sign must be driven to zero, though not to less than zero, of course, because of the non-negativity side-conditions on the problem at large. In fact, the magnitude of that shipment which is driven to zero identifies the value of a constant which is then added to or subtracted from each cell according as that cell contains a plus sign or a minus sign. Denote this constant value by Δx. The specific numerical value of this constant, then, is set at the value of the *smallest* shipment from among those cells which contain a minus sign. In the present case (cf. Tables 3 and 4b) this value is 5 units. Now, actually carry out the series of additions and subtractions designated in Table 4b, i.e.,

$$\begin{aligned}
x_{34} &\leftarrow \Delta x, & &\text{or } x_{34} \leftarrow 5\\
x_{33} &\leftarrow x_{33} - \Delta x, & &\text{or } x_{33} \leftarrow 0\\
x_{43} &\leftarrow x_{43} + \Delta x, & &\text{or } x_{43} \leftarrow 7\\
x_{44} &\leftarrow x_{44} - \Delta x, & &\text{or } x_{44} \leftarrow 4
\end{aligned}$$

where the backward-pointing arrow, ←, denotes the operation "is changed to." Note that in this process the shipment in the cell [3, 3] is driven to zero, but that the non-negativity condition on all variables is preserved.

This is the final stage of the first iteration. Now take those new shipments as determined above together with those original shipments in Table 3 which remain unchanged, and construct a new tableau around them. This operation gives Table 4c which is a new and improved and feasible solution. The total cost of this new program is 109 units of money which is an improvement of 50 units of money over the total cost of the first feasible solution. This new program is used as the basic input to the second iteration which is now initiated and which seeks also to secure systematic program improvements. The computational procedures followed in the second iteration, as well as in all subsequent iterations, are identical to those followed in the first iteration.

Table 4c.

To From	1	2	3	4	5	Supply
1	9 1	4 6	3	7	5	7
2	4	2 2	7 1	2	5	3
3	3	7	12	4 5	9	5
4	9	4	3 7	5 4	1 4	15
Demand	1	8	8	9	4	30

Z = (9 X 1) + (4 X 6) + (2 X 2) + (7 X 1) + (3 X 7) +
(4 X 5) + (5 X 4) + (1 X 4) = 109

Continuation of the Iterative Algorithm

The second iteration: Résumé of the main computational steps

The second iteration may be expressed as a résumé of the main steps in the transportation algorithm:

1) Compute the shadow prices U_i and V_j, (Table 5a). U_1 is as usual and by definition set equal to zero. All other shadow prices are computed only in relation to shipments which have a non-zero value.
2) Compute all opportunity costs, $\bar{c}_{ij} = V_j - U_i$, for cells which have no positive shipment.
3) Out of those cells (if any) which have an opportunity cost greater than the corresponding transport cost isolate that one cell (which is now designated the pivot cell) where the difference between $\bar{c}_{ij}$ and t_{ij} is greatest. Observe that if no $\bar{c}_{ij}$ is found to be greater than the corresponding t_{ij} then the program is optimal and the iterative algorithm is brought to a close.
4) Take the pivot cell isolated in the previous step and locate a plus sign in it to denote that the cell is to be assigned a positive shipment. In the present instance (Table 5b) the pivot cell is cell [2, 4].
5) Trace out from the pivot cell an orthogonal circuit of alternating plus and minus signs and such that a) the plus and minus signs are located only in cells which have non-zero shipments (with exception of the pivot cell), and b) the circuit is always finally closed by locating a minus sign in the same column as the pivot cell.
6) Of all those cells in which a minus sign occurs, take that one such cell with the smallest shipment and set the

constant value Δx equal to that shipment. In Table 5a, this cell is [2, 3], and its shipment value is 1.

7) Increase shipments by the amount Δx in all those cells where a plus sign is located, and decrease shipments by the amount Δx in all those cells where a minus sign is located.

This ends the iteration, and the program currently developed is the new improved solution. For the second iteration of the sample problem under consideration here the new improved solution is shown in Table 5c, and this program is associated with an objective function value of 102 units of money.

Tables 5a - 5c. Second iteration

Table 5a.

Shadow prices		9	4	9	11	7	
	From \ To	1	2	3	4	5	Supply
0	1	9 ●	4 ●	3 9	7 11	5 7	7
2	2	4 7	2 ●	7 ●	2 ⑨	5 5	3
7	3	3 2	7 -3	12 2	4 ●	9 0	5
6	4	9 3	4 -2	3 ●	5 ●	1 ●	15
	Demand	1	8	8	9	4	30

Table 5b.

From \ To	1	2	3	4	5	Supply
1	9 ●	4 ●	3	7	5	7
2	4	2 ●	7 ● -	2 +	5	3
3	3	7	12	4 ●	9	5
4	9	4	3 ● +	5 ● -	1 ●	15
Demand	1	8	8	9	4	30

Table 5c.

From \ To	1	2	3	4	5	Supply
1	9 1	4 6	3	7	5	7
2	4	2 2	7	2 1	5	3
3	3	7	12	4 5	9	5
4	9	4	3 8	5 3	1 4	15
Demand	1	8	8	9	4	30

$$Z = (9 \times 1) + (4 \times 6) + (2 \times 2) + (3 \times 8) + (2 \times 1) + (4 \times 5) + (5 \times 3) + (1 \times 4) = 102$$

The third to the final iterations

Succeeding iterations are now computed in exactly the same manner. Thus the algorithm proceeds through a third iteration (Tables 6a to 6c), a fourth iteration (Tables 7a to 7c), and finally on to the beginning of a fifth iteration (Table 8) at which point the solution as shown in Table 7c is found to be optimal. The optimality of the program in Table 7c is indicated by the fact that the derivative opportunity costs (Table 8) are all less than their corresponding transportation costs. This final optimal solution has an objective function value of 95. The geographical structure of the solution is depicted in Figure 2 where all optimal source-destination flows are appropriately shown.

This completes the description of the principal features of the iterative algorithm for solution of the transportation problem.

Additional Computational Considerations

Number of non-zero variables in the solution

It may have been observed during the computational process discussed above that the number of non-zero variables (or positive shipments) in any program stage remained constant at eight. This feature in fact represents a kind of system stability and parsimony which is highly characteristic of the transportation problem. In more general terms, it can be shown mathematically that the number of non-zero variables in any solution of the transportation problem will usually be equal to n + m – 1, where n and m are defined as above. In particular, the

Tables 6a - 6c. Third iteration

Table 6a.

Shadow prices →		9	4	2	4	0	
↓	To From	1	2	3	4	5	Supply
0	1	9 ●	4 ●	3 2	7 4	5 0	7
2	2	4 7	2 ●	7 0	2 ●	5 -2	3
0	3	3 ⑨	7 4	12 2	4 ●	9 0	5
-1	4	9 10	4 5	3 ●	5 ●	1 ●	15
	Demand	1	8	8	9	4	30

Table 6b.

To From	1	2	3	4	5	Supply
1	9 ● −	4 ● +	3	7	5	7
2	4	2 ● −	7	2 ● +	5	3
3	3 +	7	12	4 ● −	9	5
4	9	4	3 ●	5 ●	1 ●	15
Demand	1	8	8	9	4	30

Table 6c.

To From	1	2	3	4	5	Supply
1	9	4 7	3	7	5	7
2	4	2 1	7	2 2	5	3
3	3 1	7	12	4 4	9	5
4	9	4	3 8	5 3	1 4	15
Demand	1	8	8	9	4	30

Z = (3 X 1) + (4 X 7) + (2 X 1) + (3 X 8) + (2 X 2) + (4 X 4) + (5 X 3) + (1 X 4) = 96

Tables 7a - 7c. Fourth iteration

Table 7a.

Shadow prices →		3	4	2	4	0	
↓	To From	1	2	3	4	5	Supply
0	1	9 3	4 ●	3 2	7 4	5 0	7
2	2	4 1	2 ●	7 0	2 ●	5 -2	3
0	3	3 ●	7 4	12 2	4 ●	9 0	5
-1	4	9 4	4 ⑤	3 ●	5 ●	1 ●	15
	Demand	1	8	8	9	4	30

Table 7b.

To From	1	2	3	4	5	Supply
1	9	4 ●	3	7	5	7
2	4	2 ● −	7	2 ● +	5	3
3	3 ●	7	12	4 ●	9	5
4	9	4 +	3 ●	5 ● −	1 ●	15
Demand	1	8	8	9	4	30

Table 7c.

To From	1	2	3	4	5	Supply
1	9	4 7	3	7	5	7
2	4	2	7	2 3	5	3
3	3 1	7	12	4 4	9	5
4	9	4 1	3 8	5 2	1 4	15
Demand	1	8	8	9	4	30

Z = (3 X 1) + (4 X 7) + (4 X 1) + (3 X 8) + (2 X 3) + (4 X 4) + (5 X 2) + (1 X 4) = 95

Table 8. Fifth and final iteration

Shadow prices		4	4	3	5	1	
	From \ To	1	2	3	4	5	Supply
0	1	9 4	4 ●	3 3	7 5	5 1	7
3	2	4 1	2 1	7 0	2 ●	5 -2	3
1	3	3 ●	7 3	12 2	4 ●	9 0	5
0	4	9 4	4 ●	3 ●	5 ●	1 ●	15
	Demand	1	8	8	9	4	30

number of non-zero variables will *never* exceed n + m − 1, though it may occasionally be less than n + m − 1. This latter condition is known as a state of *degeneracy*, and degeneracy can lead to minor computational difficulties.

Degeneracy problems and their resolution

Degeneracy as defined above leads to apparent program breakdown so that it appears to be impossible either to compute a full set of shadow prices or, often, to make any systematic improvements in the program structure. Moreover, any given program may be degenerate to the extent that it lacks only one positive variable, or two, or three, and so on up. In the following remarks the first, and simplest, case only will be explicitly dealt with. However, these remarks are easily generalizable to more complex cases.

The degeneracy problem is countered by selecting some cell which is entirely blank and treating it *as if* it contained a positive shipment. Designate that blank cell which is to be treated as though it contained a positive shipment a *θ-cell.* As the computational algorithm proceeds this θ-cell plays a role analogous to any cell containing a positive shipment. Thus row and column shadow prices are computed in the usual way in relation to this cell. The only complication occurs during the stage of program re-adjustment. Suppose in the first place that the θ-cell receives a plus sign during the orthogonal shipment re-allocation process. Then, the cell is simply assigned a value equal to the re-allocation constant Δx, and the degeneracy problem is automatically overcome. However, suppose in the second place that the θ-cell receives a minus sign during the re-allocation process. In this case there would be no physical re-allocation of shipments. Rather the θ-cell would now be relegated to its former status as simply a blank cell while the pivot cell would now be designated as the θ-cell.

These assertions may be exemplified by examination of Tables 9a to 9d. For simplicity neither transport costs nor supplies nor demands are shown in these Tables. In Table 9a a non-degenerate solution is shown together with a suggested re-adjustment pattern. In Table 9b these re-adjustments have been effected leaving a degenerate solution in which the cell [3, 3] is designated a θ-cell. Also in

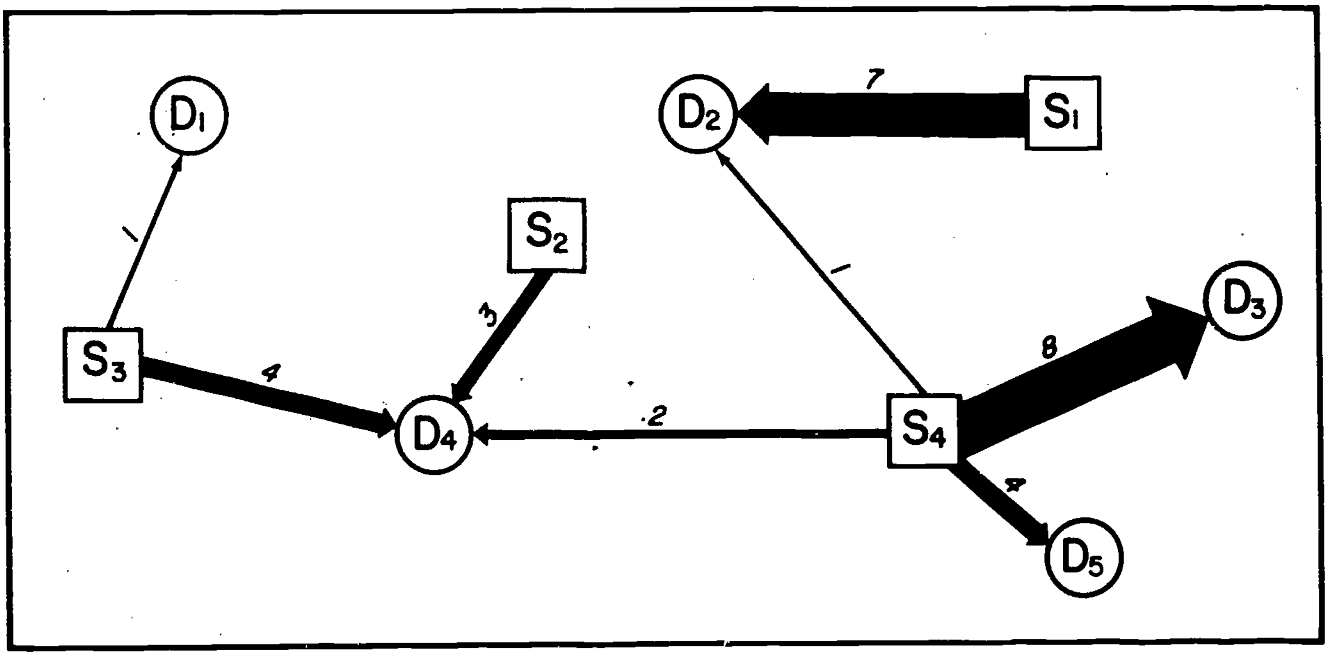

Figure 2. Optimal solution

Tables 9a - 9d. Illustrations of various degeneracy problems

Table 9a.

4	3	
2 -		6 +
+		2 -

Table 9b.

4 -	3	+
		8
2 +		θ -

Table 9c.

4	3 -	θ +
	+	8 -
2		

Table 9d.

4		3
	3	5
2		

Table 9b a further re-adjustment pattern is shown where a minus sign falls in the θ-cell. In effecting the re-adjustment, the cell [3, 3] becomes simply a blank cell while the pivot cell, [1, 3], now becomes the new θ-cell, (Table 9c). A final set of program re-adjustments called for in Table 9c causes a plus sign to be placed in the θ-cell. Thus, in Table 9d the re-adjusted program is once more non-degenerate.

Extension of the Elementary Tableau Format

Introduction of more complex cost functions

In real spatial allocation problems it would generally be rather unrealistic to minimize simply the costs of transportation. Producers at different locations will usually have different unit production costs, and these differences will be apparent in delivered commodity prices. Thus consumers may buy from a relatively distant producer providing that the additional transport cost incurred is offset by a concomitantly lower production cost. For this reason it is often desirable to minimize over a joint function of production costs and transport costs. The objective function corresponding to this joint cost structure would be

11) $$Z = \sum_{i=1}^{n} \sum_{j=1}^{m} (c_i + t_{ij})x_{ij}$$

which obviously should be a minimum, and where the term c_i is the unit production cost at source i. The usual supply, demand, and non-negativity conditions would apply to this problem.

Solution of this extended problem can be readily accommodated within the frame of reference of the transportation algorithm. This is accomplished simply by treating the sum $c_i + t_{ij}$ as an ordinary transportation cost. The problem is then manipulated in tableau format in the normal fashion.

Use of slack variables

The transportation model may further be significantly and realistically extended by defining slack variables which then identify levels of unused capacity at various sources. A corollary of this extension is that global supply capacity would now be greater than global demand.

Suppose in fact that the sample problem considered in Table 2 is extended to include a slack capacity variable for every source. In particular, let capacity at all four sources be increased by two units. Thus, in this extended problem,

12) $$S_1 = 9,\ S_2 = 5,\ S_3 = 7,\ S_4 = 17.$$

The complete problem is represented in tableau format in Table 10. Note that a new column containing the slack variables has been added to the tableau. The cells in this column represent purely fictitious shipments which in turn indicate the total quantity of unused capacity at any source. In addition, the cost attached to every cell in the slack column is zero, for it would normally be assumed that it costs nothing not to produce some commodity. Since the total slack capacity throughout the entire system is 8 units, the total demand indicated for the slack column is also 8 units.

The system is now ready for solution by the usual

Table 10. Transport problem with slack variables

To From	1	2	3	4	5	slack	Supply
1	9	4	3 8	7	5	0 1	9
2	4	2 2	7	2 3	5	0	5
3	3 1	7	12	4 6	9	0	7
4	9	4 6	3	5	1 4	0 7	17
Demand	1	8	8	9	4	8	38

Z = (8 X 3) + (0 X 1) + (2 X 2) + (2 X 3) + (3 X 1) + (4 X 6) + (4 X 6) + (1 X 4) + (0 X 7) = 89

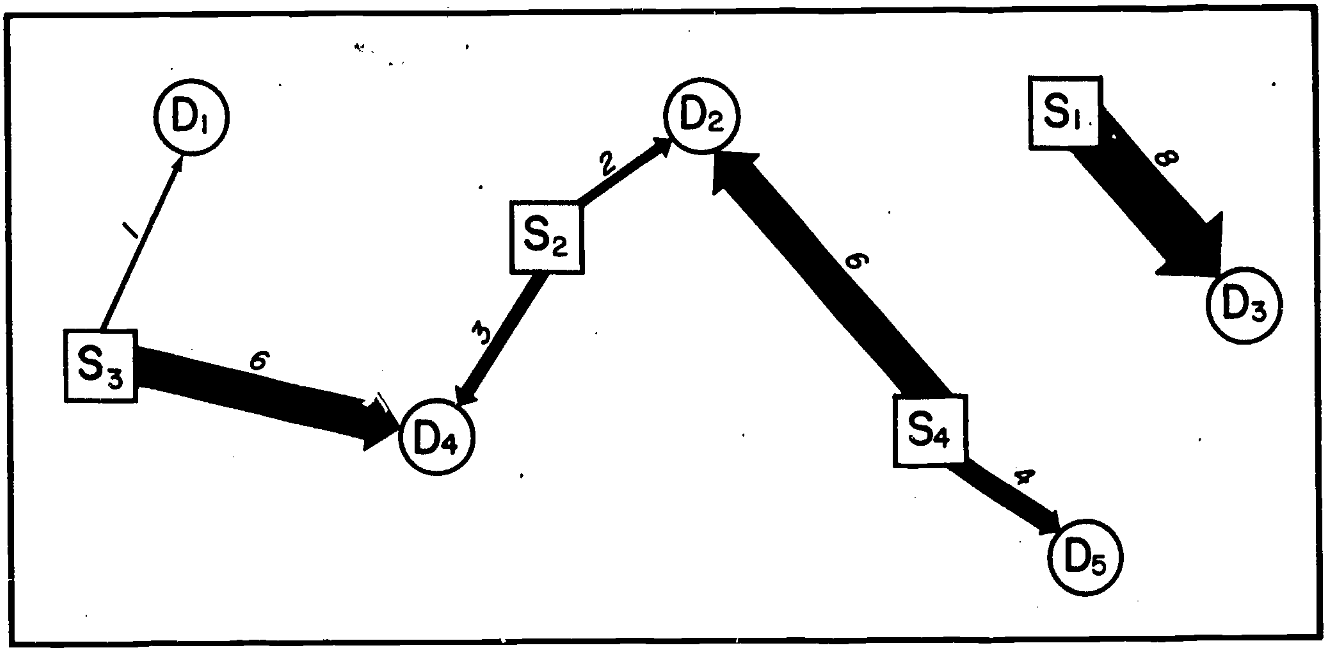

Figure 3. Optimal solution with slack variables

procedures of the transportation algorithm. An optimal solution for this specific problem is indicated in Table 10, and its graphical representation is shown in Figure 3. In this optimal solution sources 1 and 4 are seen to have slack capacity of 1 and 7 units respectively. This condition reflects the economically marginal position of these sources with respect to the total system. On the other hand sources 2 and 3 are left without any slack capacity whatever. The value of the objective function associated with this solution is 89. The wider economic meaning of these program changes due to capacity increases is of considerable interest. This question is dealt with in the succeeding Section where the various economic and theoretical ramifications underlying the transportation model are explored.

III. THE DUAL PROBLEM AND SPATIAL PRICE EQUILIBRIUM

The Dual Problem: Initial Definitions and Derivation

All linear programming problems are associated with a so-called *dual* problem, and the dual problem always bears a constant mathematical relationship to the original or *primal* problem. Examination of the general nature of this relationship is beyond the scope of the present report, but is dealt with at length in many standard texts on linear programming (e.g. Dantzig (1963), Dorfman *et al.* (1958), Hillier and Lieberman (1967)). Of immediate interest for present purposes however are the formal structure and theoretical implications of the particular dual program which is associated with the transportation problem.

At the outset, this dual can be specified in programmatic form as follows:

Maximize:

$$Z = \sum_{j=1}^{m} D_j V_j - \sum_{i=1}^{n} S_i U_i \tag{13}$$

subject to:

$$V_j - U_i \leqslant t_{ij} \tag{14}$$

$$V_j \geqslant 0, \; U_i \geqslant 0 \tag{15}$$

where the terms S_i, D_j and t_{ij} are defined as in Section II, and where U_i and V_j are dual variables whose physical interpretation forms the major substance of the present section. Indeed, the dual variables U_i and V_j will be immediately recognized as the shadow prices discussed in Section II. In addition, the constraint (14) will readily be recognized as no more than an explicit statement of the criterion for optimality of the primal problem; for, recall, first, that by definition $V_j - U_i = t_{ij}$ for any i → j shipment which is made at some positive level, and second, that for the program as a whole to be optimal then the condition $V_j - U_i \; (= \bar{c}_{ij}) \leqslant t_{ij}$ must obtain for all i → j source-destination pairs where no shipment is in fact made. Notice that since the primal problem is a minimization problem then the dual problem is therefore typically a maximization problem. In the succeeding account no attempt will be made to interpret directly the meaning of the dual objective function (13), though the constraining condition (14) will be considered at length.

The U_i and V_j are at this point defined explicitly and then interpreted at length below. Thus, assume that there is given some solution to a primal transportation problem in which a given commodity is shipped from a set of sources to a set of destinations. Then, any V_j represents the *equilibrium market price* of the commodity at destination j. Any U_i represents a *location rent* incurred at source i. Here the term *rent* is used in a special sense whose meaning will become apparent shortly. These specific definitions of the shadow prices will be illustrated and illuminated in the first instance by means of a simple example consisting of one destination or buyer and four sources or sellers. This simple system will then be generalized. The argument which follows is taken almost exclusively from Stevens (1961).

Economic Interpretation of the Dual

Case of one buyer and four sellers

Consider the spatial distribution of one buyer and four sellers as shown in Figure 4. Suppose that the buyer has a perfectly inelastic demand for Q units of some commodity. Suppose further that the four sellers can, collectively, supply more than Q units of the commodity so that after all of the buyer's demand has been met there will still remain some overall capacity. In addition, let it be supposed that all unit slack pre-transportation supply costs, c_1, c_2, c_3, c_4 (which would include a normal profit) are equal. Therefore, without any loss of generality, these costs can be neglected in the following analysis. This latter supposition also means that the only variations in delivered prices will be produced by differences in the costs of transportation, and these differences are in turn directly a function of locational variations among the four sellers.

A simple allocation process. In seeking to acquire the quantity Q of the given commodity the buyer would first address himself to seller 1, for, being closer to the buyer than any other seller, seller 1 can supply at the cheapest

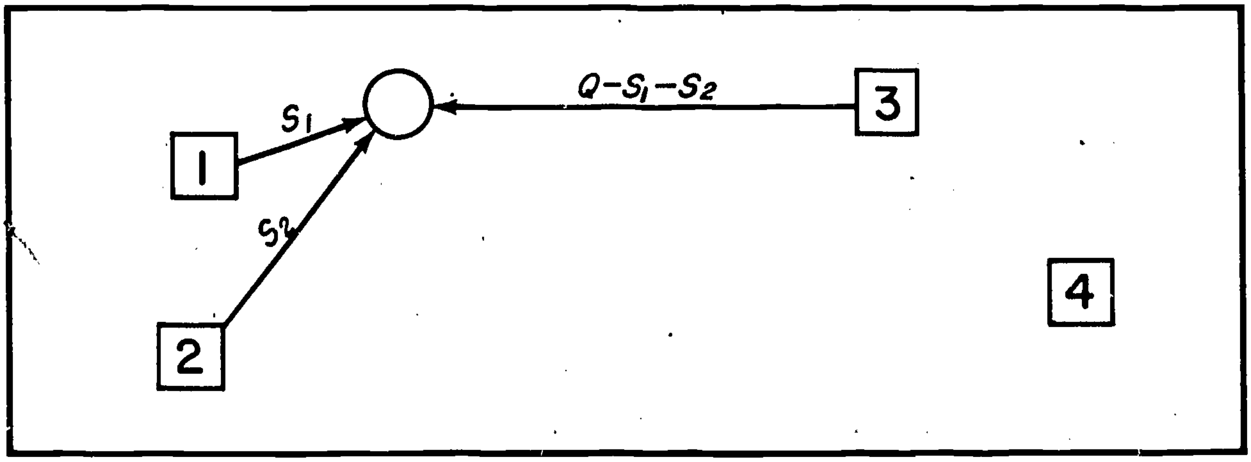

Figure 4. A hypothetical problem with one buyer and four sellers

price, t_1, (equating all c_i to zero). But suppose that seller 1's maximum supply capacity, S_1, is less than Q, (cf. Figure 5). The buyer will therefore attempt to make up the deficit, $Q - S_1$, in his total demand by purchasing from the next least expensive seller, seller 2, who can supply at the delivered price t_2. Again, however, let it be supposed that supplies are limited and that seller 2 can supply only up to $S_2 < Q - S_1$ units of the commodity. Continuing the same process, then, of always trying to make the best buy, the buyer will now turn to seller 3, even though seller 3's

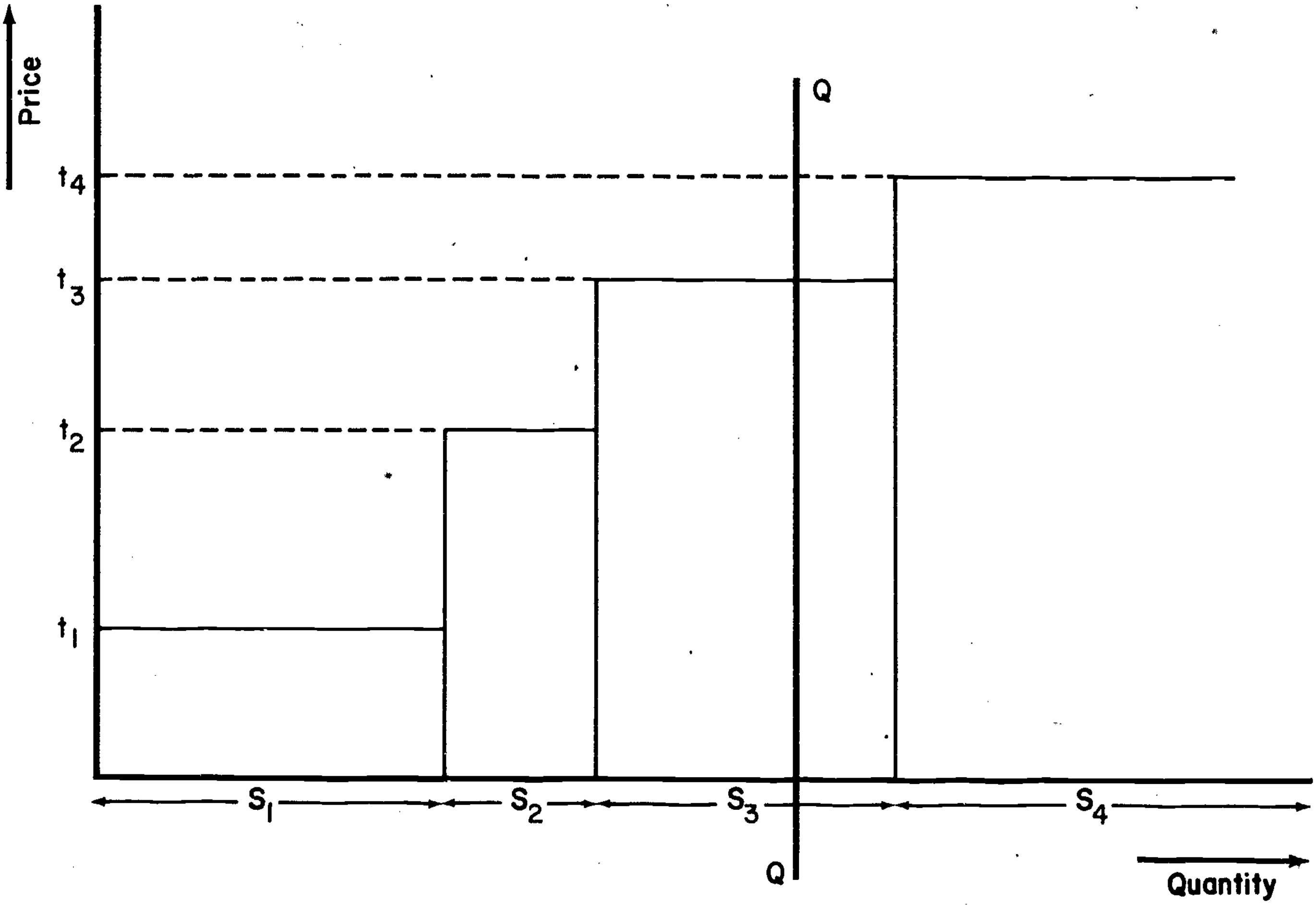

Figure 5. A pricing system for one buyer and four sellers

delivered price, t_3, is comparatively high. Suppose that seller 3 can more than meet the remaining deficit in the buyer's demand. Thus seller 3 delivers $Q \quad S_1 - S_2$ units of the commodity to the buyer, and is left with $S_3 - (Q - S_1 - S_2)$ units of slack capacity. Seller 4 is forced from the system entirely, and produces nothing.

Development of a market system. The problem at this point is what price or prices would now prevail at the market represented by the buyer, for it is by no means certain that once sellers have transported the commodity to the market they will be content to sell it simply at the prices t_1, t_2, and t_3, respectively, for sellers 1, 2, and 3. In fact, there will always be a tendency for any seller to try to increase his price to the greatest extent possible. Moreover this tendency leads to an equilibrium in which the commodity, whatever its source, is sold at a constant market price. The difference between this equilibrium price, say V, and the pre-equilibrium price, t_i, is precisely the location rent U_i.

Take, for example, the case of seller 3. Seller 3 can sell the commodity at a delivered price of t_3 and just make a normal profit. Suppose seller 3 experimentally raises his price by some arbitrarily small amount, say to the level $t_3 + U_3$. Providing the price $t_3 + U_3$ is not more than the minimum price, t_4, of the idle seller 4, seller 3 will still manage to sell the amount $Q - S_1 - S_2$ (since demand is completely inelastic) but with correspondingly increased total revenue. However, if the magnitude $t_3 + U_3$ exceeds the magnitude t_4 then seller 4 will immediately enter the market and undercut seller 3. For this reason, seller 3 will sell just at the level t_4, and this guarantees seller 3 a level of sales equal to $Q - S_1 - S_2$ at the maximum unit price of $t_3 + U_3 = t_4$, (cf. Figure 5). By the same arguments, seller 2 will raise his price to $t_2 + U_2 = t_4$, and seller 1 will raise his price to $t_1 + U_1 = t_4$. At this point it may appear that seller 3 would now lower his price just slightly, thereby undercutting sellers 1 and 2 and thus simultaneously increasing his sales and bringing some of his excess capacity into productive use. However, in any price war, sellers 1 and 2 can always ultimately win out over seller 3 (just as seller 1 can over seller 2) for, if necessary, sellers 1 and 2 can always sell at below seller 3's minimum delivered price, t_3. Therefore, the condition where sellers 1, 2, and 3, respectively, sell the quantities S_1, S_2, and $Q - S_1 - S_2$ at the single market price, t_4, is a steady and sustainable equilibrium.

The meaning of the shadow price, V, should now be somewhat more clear. V is the equilibrium market price, in this case equal to t_4. In addition, the meaning of the shadow prices U_1, U_2, and U_3 is now entirely clear. The U_i are excess profits or rents on every unit of commodity sold and they are generated by the process, equivalent to the conditions (8) and (14):

16) $U_1 = V - t_1$

17) $U_2 = V - t_2$

18) $U_3 = V - t_3$.

And these values are specially designated *location rents* because they result entirely from differences in sellers' locations.

The marginal seller. In any problem of the type discussed above, there is always a *marginal* seller. In the present instance the marginal seller is seller 3. This particular seller is marginal in the sense that given any reduction in total consumption, seller 3 would always be the first to suffer any loss of sales. Now, in most theoretical expositions of the structure of spatial market systems it is normally directly assumed that the marginal seller always earns a location rent of zero. But in the exemplary problem considered above, the marginal seller is shown to earn the non-zero rent, $U_3 = V - t_3$. Nevertheless, it is possible to reconcile this apparent discrepancy, and this can be done by demonstrating that the case where the marginal seller earns a zero rent is a kind of final limiting case. Thus, in any perfectly competitive system there would be not only a large number of actual sellers but also a large number of potential sellers, the latter ready to enter the market at any time when prices should be favourable to entry. If this is so, then the marginal seller's rent will always be effectively zero, for obvious reasons. This, then represents the limiting case of the simple (essentially oligopolistic) problem described in the preceding paragraph. In the succeeding account it will now be consistently assumed that the conditions of perfect competition are met, and that the marginal seller always earns a location rent of zero.

This assumption is, in addition, of direct relevance to the computational algorithm discussed in Section II. Thus, in this algorithm, instead of always arbitrarily setting U_1 equal to zero, it would be more meaningful from an economic point of view to equate to zero that U_i which relates to the marginal seller. If this is done then all other U_i (as well as the V_j) will always be greater than or equal to zero, as they should be given their physical interpretation. This, then accounts for the non-negativity condition (15) in the dual program. On the other hand, for purely computational purposes, the shadow prices U_i and V_j may be of any magnitude, and hence the condition (15) is not enforced in the primal computational algorithm. Note however that the U_i and V_j are always in constant proportional relationship to one another whatever their base of reference.

Effect of capacity changes on rents. One further important point about the location rents is in order. It should be remarked that if there is any change in any seller's supply capacity then there is also some likelihood that the set of U_i will also be changed. Indeed, these rents should properly be seen not simply as functions of location but rather as functions of location *in relation to productive capacity*. This point can be easily illustrated by reference to Figure 5. Suppose seller 2 invests in new capacity to the extent where he can entirely supply the buyer's demand deficit after seller 1 has supplied the amount S_1. Seller 2 will therefore force seller 3 from market. The new equilibrium price is now reduced to $V = t_3$, and the values U_1 and U_2 are also reduced accordingly. *Mutatis mutandis,* decreases in capacity will tend to cause an increase in system location rents.

These various assertions may now be generalized, and established with more solidity.

Case of many buyers and many sellers

Imagine that there is some general system consisting of n sellers and m buyers or markets, and characterized by some actual program of trade, which need not for the moment be optimal. On each market, j, there will be an equilibrium price V_j. By minor extension of the arguments given above, the price V_j can be seen as a function of the delivered price of the marginal or least competitive seller with access to that market. Every seller, i, will earn a location rent U_i. In addition, that seller who is marginal with respect to the entire system will earn a location rent which is equal to zero.

General criterion of equilibium. Let it be assumed that the system is not in fact optimal, so that commodity allocations are in a process of re-adjustment. This means that there will be at least one seller, i, and at least one market, j, for which $x_{ij} = 0$, and

19) $$V_j - U_i > t_{ij}, (\text{or } \bar{c}_{ij} > t_{ij}).$$

If this condition holds then it will always seem to be advantageous for seller i to sell at the market j (thus apparently improving his location rent), just as it will seem to be advantageous for the buyer or buyers at j to buy from i (thus apparently reducing delivered prices). The real end result however may not be as anticipated. The equilibrium delivered price, V_j, may indeed be reduced but in this case the seller will lose the anticipated increment to his rent. Or, the value of V_j may remain as it was so that the buyer is in the end no better off. In terms of the general system-wide program of trade however the net result is always beneficial, namely, reduction in the total real costs of the system by increasing the amount of flow over some comparatively cheap shipment route, and compensating for this by a reduction in flow over some comparatively expensive route.

The meaning of the opportunity cost, $\bar{c}_{ij}$ is especially well illustrated by the condition (19). Obviously, the opportunity cost $\bar{c}_{ij}$ is a measure of what the seller i apparently would lose by not selling at the market j. Whenever the condition $\bar{c}_{ij} - t_{ij} > 0$ is found, then, there is always some positive advantage to increasing shipments from i to j. By contrast whenever the condition $\bar{c}_{ij} - t_{ij} \leqslant 0$ is found then there is no advantage whatever in making a shipment from i to j. Moreover, this latter condition is, as has already been indicated, the criterion of final program optimality. Thus it is that the computational procedures already established in the description of the transportation algorithm correspond to elementary economic notions of the structure of a spatial trading system.

An interlocking system of rents and prices. With n sellers and m buyers, there always emerges in a competitive trading system an inter-connected and interdependent system of rents and prices. This system is such that if any seller earns a rent U_i in any market, j, then he always passes this rent on to all other markets in which he sells. As in the case of the simpler problem with one buyer considered above, this complete system of rents and prices ultimately relates back to the marginal seller who of course earns a rent of zero. In addition, in this more complex problem with many buyers and many sellers, *the* marginal seller is marginal with respect to the entire system.

These remarks may once more be clarified by means of a simple illustrative example. Thus, consider a simple market system extended along a line as shown in Figure 6. In this system there are four sellers and three buyers. Seller 2 is taken to be the marginal seller who therefore earns a rent of zero. This marginal seller sells to buyer 1 at a price of t_{21}, and he sells to buyer 2 at a price of t_{22}. Thus, the equilibrium prices which prevail at buyers 1 and 2 are $V_1 = t_{21}$ and $V_2 = t_{22}$ respectively. As it happens, seller 1 also sells to buyer 1. Moreover, since t_{11} is less than V_1 this means that seller 1 earns a location rent U_1 which is equal to $V_1 - t_{11}$. In the same way, seller 3 who sells to buyer 2 earns a rent of $U_3 = V_2 - t_{32}$. Since seller 3 also sells to buyer 3, the rent U_3 is therefore passed on to buyer 3. The equilibrium price at buyer 3 is then $V_3 = t_{33} + U_3$. Finally, seller 4 sells to buyer 3 and thus seller 4 earns a rent of $U_4 = V_3 - t_{43}$. Alternatively, this last relationship may be expressed,

20) $$\begin{aligned} U_4 &= U_3 + t_{33} - t_{43}, \text{or,} \\ U_4 &= V_2 - t_{32} + t_{33} - t_{43}, \text{or,} \\ U_4 &= t_{22} - t_{32} + t_{33} - t_{43}. \end{aligned}$$

From this last expression, which is no more than an exercise in simple algebraic substitution, it can readily be determined that all rents and equilibrium prices form an inter-connected network which finally can be traced back

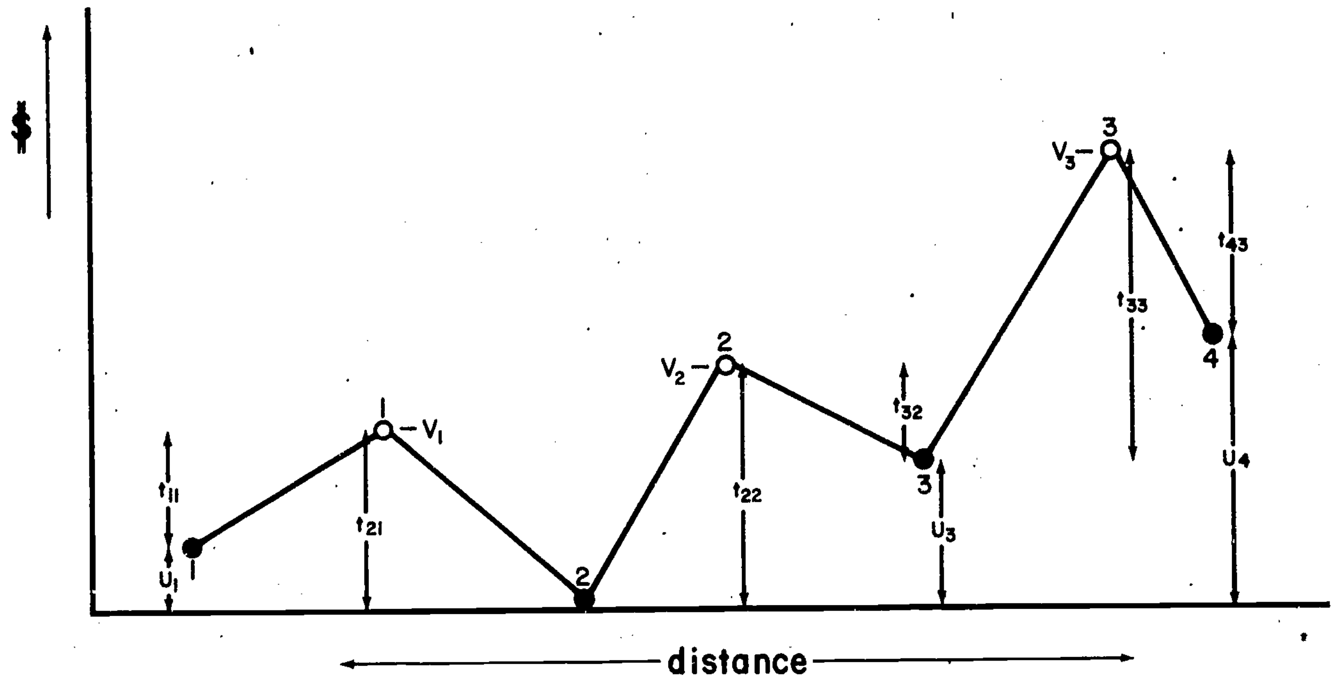

Figure 6. Rents and prices in a linear spatial system

to the marginal seller, whatever his location with respect to the system at large.

Spatial Price Equilibrium

The exceedingly simplified spatial economic theory developed above can, without too much difficulty, be extended to certain more general and realistic cases. These cases are centred upon the notion of *spatial price equilibrium*, whose relations to the transportation problem were first described by Samuelson (1952). The following account is very largely drawn from Samuelson, and reference must be made to his original paper for a more complete treatment of problem of spatial price equilibrium.

Briefly, the notion of spatial price equilibrium involves generalization of the transportation problem by assuming that supply and demand behaviour is governed by prices, just as prices themselves will in turn be affected by the level of supply and demand. Consider, for example a simple two-region case. This can be done without any loss of generality while, at the same time, the properties of the problem can be elucidated by graphical methods, (Figures 7 and 8). Each of the two regions, then, may be assumed both to supply and to consume some commodity. Suppose for the moment that the two regions are entirely isolated from one another so that no trade may occur between them. The internal supply and demand situation in each region will therefore be as in Figure 7. This Figure has been constructed in such a way that the supply and demand diagram for the first region has been placed back-to-back with that for the second region. Note that region 1 consumes the quantity Q_1 at the price p_1, and region 2 consumes the quantity Q_2 at the price p_2.

Now let the former isolation of the two regions be broken. In fact, suppose that a transport link is constructed between the two regions thus permitting interregional trade. Let t_{ij} denote as usual the unit transport cost from region i to region j. Observe from Figure 7 that

21) $p_2 > p_1 + t_{12}$, or, $p_2 - p_1 > t_{12}$.

This relation is precisely the relation (19) though in slightly different terms. In short, this is the prime condition for commodity flow *from* region 1 *to* region 2, for it signifies that even after discounting the cost of transportation, producers in region 1 can increase their total revenue by selling in region 2. Two immediate corollaries of this flow are, first, higher prices leading to lower consumption and higher supply in region 1, and second, lower prices leading to higher consumption and lower supply in region 2. Moreover flow will occur from region 1 to region 2 until that point of equilibrium where the excess of supply over demand (ES) in region 1 just exactly equals the excess of demand over supply (ED) in region 2. This point of equilibrium also coincides with the establishment of the identity

22) $\pi_1^* = \pi_2^* - t_{12}$

where π_1^* is the equilibrium after trade price in region 1 and π_2^* is the equilibrium after trade price in region 2.

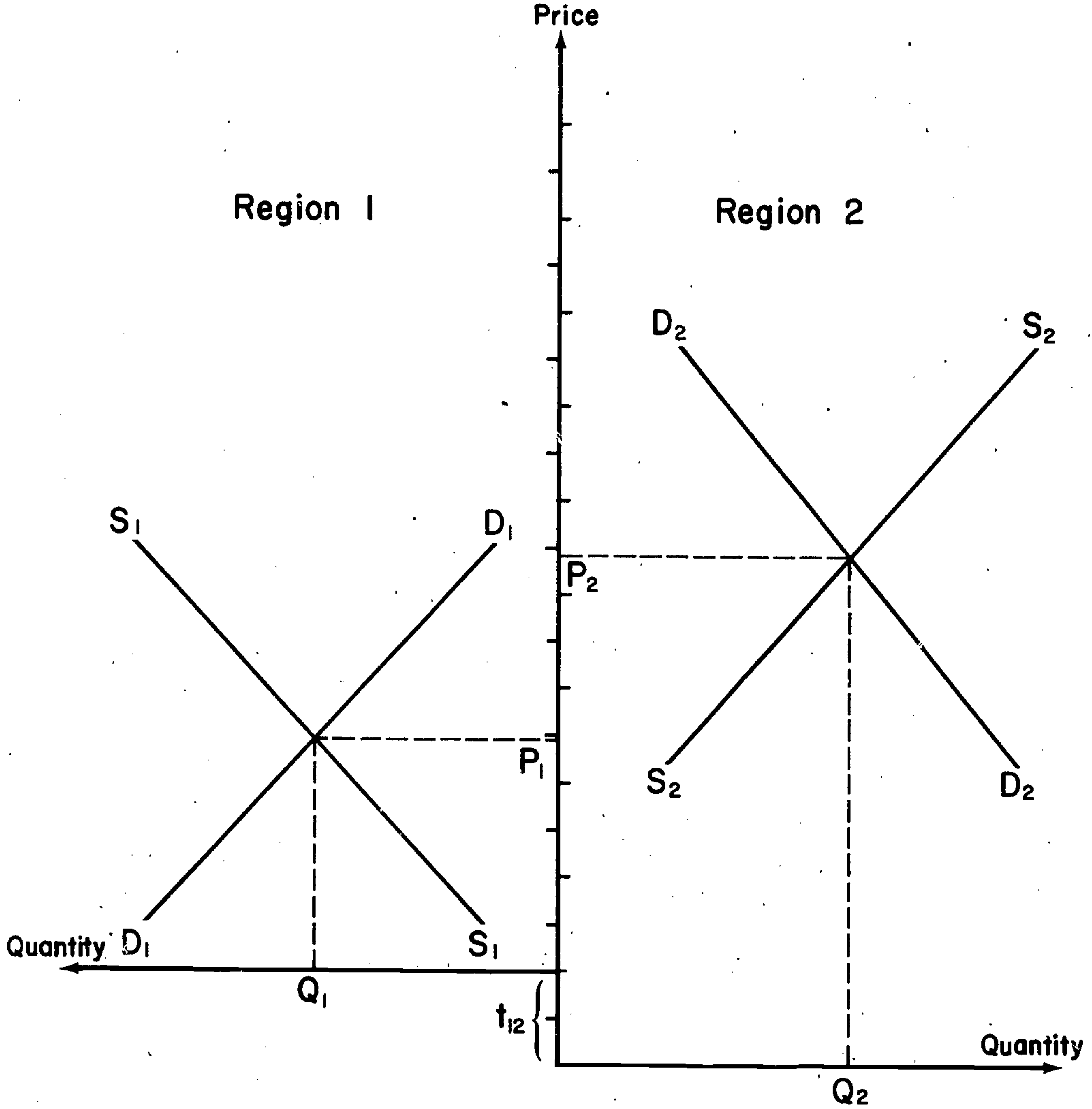

Figure 7. Before trade situation

These relations are shown graphically in Figure 8. In this Figure, the displacement of the supply-demand diagram for region 1 by t_{12} units up the price axis in relation to the supply-demand diagram for region 2, permits graphical correlation of the entire after trade situation. Notice in particular from Figure 8 that ES = ED, as it must for an equilibrium to exist. Now, consumption in region 1 has contracted to Q_1^* by comparison with the before trade situation, while demand in region 2 has increased to Q_2^*. In the same way, total supply in region 1 has now increased to S_1^* units, while supply in region 2 has contracted to S_2^* units. There is at this stage no further incentive for

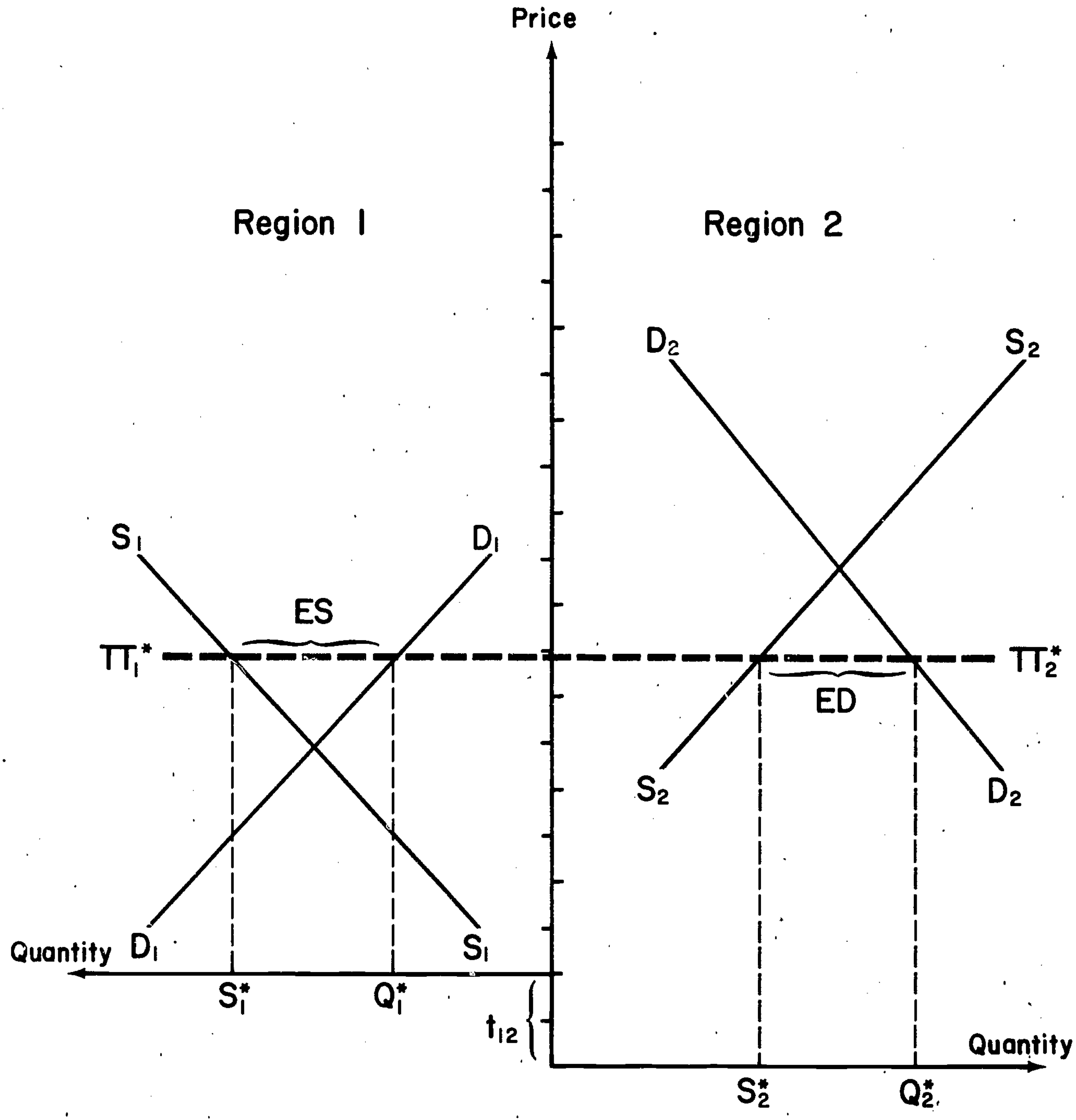

Figure 8. After trade situation

additional trade between the two regions, and the system as shown in Figure 8 is in stationary equilibrium.

This analysis can easily be generalized conceptually to the case of many regions, though no attempt to do this will be made here. In addition, it is possible to think of such a multi-region system as a special case of the transportation problem, but where the boundary supply and demand conditions are no longer constant but are subject to price-elastic schedules. In turn, prices will vary depending on the particular configuration of the trading pattern established in any solution.

In the next Section of the report a further and successive series of extensions and generalizations of the transportation problem will be considered.

IV. EXTENSIONS OF THE PRIMAL TRANSPORTATION PROBLEM

The total number and variety of extensions of the primal transportation problem are exceedingly great, and these extensions ramify throughout the whole of economic geography. Here an attempt will be made only to characterize a few of the most important of these extensions, and this will be done by consideration of a limited number of generic models. In particular the following three major extensions are considered in turn: a) The transshipment problem, b) the so-called Beckmann-Marschak problem, and c) a series of closely interrelated network problems.

The Transshipment Problem

The transshipment problem was developed originally by Orden (1956). Just like the transportation problem, the transshipment problem is concerned with the optimal pattern of flow of a single homogeneous commodity from n sources to m destinations. However, the transshipment problem further assumes the existence of a given number of transshipment points. These transshipment points function as purely intermediate nodes through which the commodity is passed on its way from the set of sources to the set of destinations. In practical terms, the transshipment points might for example be warehouses through which some commodity is shipped on its way from producers to final consumers.

Now, any degree of overlap between the set of sources, the set of destinations, and the set of transshipment points can in practice be accommodated without the slightest difficulty by the transshipment problem. For present expository purposes, however, it is convenient to permit the m destinations (and only the m destinations) to function as transshipment points. Then, the following *conservation condition* should always be preserved at these consumption-transshipment points: That total commodity flow into any such point must always equal total flow out of the same point plus whatever is retained at that point for purely local consumption. Symbolically, this condition may be expressed,

$$\text{23)}\qquad \sum_{i=1}^{n+m} x_{ij} = \sum_{k=1}^{m} x_{jk} + D_j$$

where the summation over n + m in the first set of terms signifies that the commodity may enter any point, j, either from one of the original n sources, or from one of the m transshipment points.

Symbolic representation of the transshipment problem

The transshipment problem can at this stage be written in symbolic representation as an objective function and a set of constraining conditions analogous to those already defined for the transportation problem. Thus, the objective function for the transshipment problem is

Minimize:

$$\text{24)}\qquad Z = \sum_{i=1}^{n+m} \sum_{j=1}^{m} t_{ij}x_{ij}$$

subject to:

$$\text{25)}\qquad \sum_{j=1}^{m} x_{ij} \leqslant S_i$$

which states simply that total exports out of source i may not exceed the supply capacity of that source;

$$\text{26)}\qquad \sum_{i=1}^{n+m} x_{ij} - \sum_{k=1}^{m} x_{jk} = D_j$$

which is a re-statement of the conservation condition (23), and which ensures that all demands at destination j are met while all transshipment inflows and outflows at that point are perfectly balanced. Lastly, the usual non-negativity side-condition applies,

$$\text{27)}\qquad x_{ij} \geqslant 0.$$

In the programmatic form as given by the expressions (24) – (27), the transshipment problem may be solved by the simplex method of linear programming. In addition, however, the transshipment problem can be solved by the method of the transportation algorithm, which provides a simple and compact means of numerical analysis of the problem. This is accomplished by initially setting up the transshipment problem in standard tableau format.

Table 11. Tableau representation of the transshipment problem

To From	A	B	C	D	E	A*	B*	C*	D*	E*	
1	8	12	6 2	9	10	8	12	6	9	10	2
2	7	8	3 2	1 2	3	7	8 θ	3 6	1	3 θ	10
3	15	17	8	4 3	7	15	17	8	4 4	7	7
A*	0	3 3	2	6	5	0 97	3	2	6	5	100
B*	3	0	6	7	6	3	0 100	6	7	6	100
C*	2 3	6	0	3	4	2 3	6	0 94	3	4	100
D*	6	7	3	0	2 4	6	7	3	0 96	2	100
E*	5	6	4	2	0	5	6	4	2	0 100	100
	3	3	4	5	4	100	100	100	100	100	519

Tableau representation and solution of the transshipment problem

Consider some specific sample problem such as that whose principal statistical and cartographic features are shown in Table 11 and in Figure 9, respectively. In this sample problem there are assumed to be three sources labelled, in turn, 1, 2, and 3. In addition, there are five destination-transshipment points, labelled A, B, C, D, and E. When these latter points are to be designated purely as transshipment points they are further differentiated as, respectively, A*, B*, C*, D*, and E*.

Consider now Table 11 which will be seen to have exactly the same general structure as any ordinary tableau for representation of the simple transportation problem. Transportation costs between any row and any column element are shown in the top left-hand corner of the cell corresponding to the intersection of that row and column. In addition, supply capacities are shown as usual in the right-hand stub of the Table while demands are shown as usual in the bottom stub of the Table. The detailed structure of the main body of the Table, however, is characterized by certain special features. In particular, the structure of the tableau can be divided off into four main quadrants describable as follows: First, the upper left-hand quadrant of Table 11; this quadrant represents a 3 X 5 commodity shipment matrix indicating connections between sources 1, 2, and 3, and direct destinations A, B, C, D, and E. Second, the upper right-hand quadrant which again is of dimensions 3 X 5 and which represents a matrix of shipments from the three sources to the points A*, B*, C*, D*, and E*, specifically for transshipment. Third, a lower right-hand quadrant of dimensions 5 X 5 which indicates shipment connections between transshipment

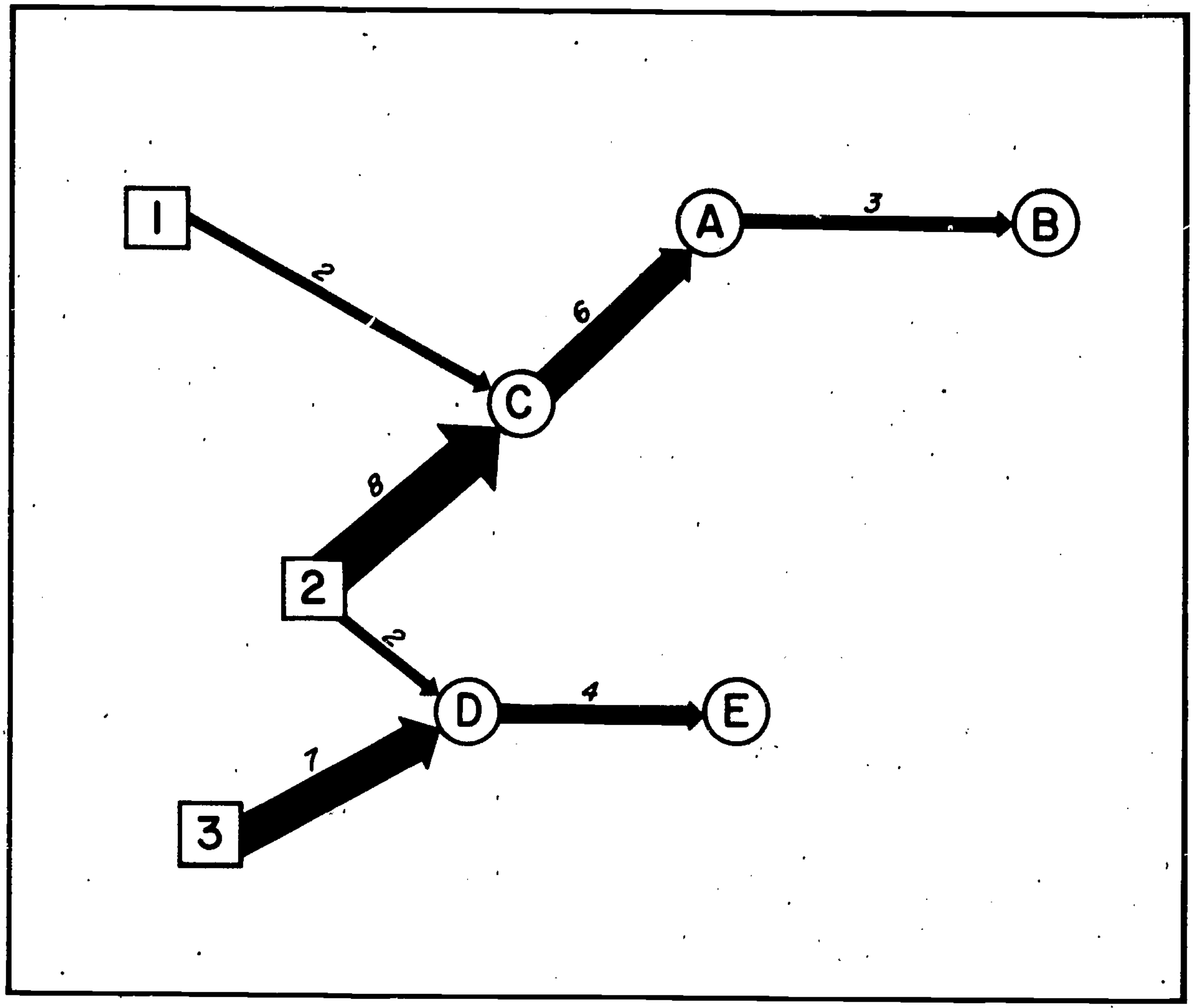

Figure 9. A transshipment problem

points and other transshipment points. Fourth, a lower left-hand quadrant of dimensions 5 X 5 which indicates shipments from transshipment points to final destinations.

In addition to these special features it should also be noted that in Table 11 a set of supply capacities and a set of demand requirements are specified for transshipment points. In all cases these elements are set equal to 100 units. These transshipment point capacities and requirements are of course entirely artificial since transshipment points are assumed neither to have upper limits on their activities, nor to have consumption requirements. However, these artificial quantities provide a basic mechanism which drives the transshipment system forward. They give rise to the development of purely fictitious stockpiles into which inflows for transshipment are added, and from which corresponding outflows are substracted. This also explains why the relatively high value of 100 has been assigned to these artificial capacities and requirements, for they must be such as to give rise to apparent stockpiles which will always exceed any conceivable transshipment flow through any point. These stockpiles may be thought of as being located in the cells [A*, A*], [B*, B*], [C*, C*], [D*, D*], and [E*, E*] of Table 11. Moreover, because it obviously costs nothing to maintain a purely fictitious stockpile, the transport costs in these particular cells are automatically equated to zero.

A numerical example. A solution for the sample problem discussed above can be obtained directly by now treating Table 11 exactly as though it represented an ordinary transportation problem. Table 11 in fact designates the optimal solution of this problem, and Figure 9 shows the main graphical features of the solution. The role of the apparent stockpiles in the transshipment process can now readily be demonstrated. For example, in Table 11 it will be observed that a shipment of 4 units is made from source 3 to transshipment point D*. This forces the stockpile at

[D*, D*] down to a level of 96 units so as to maintain the sum of the elements in column D* at 100. In the same way, the elements in row D* must also sum to 100, so that 4 units must make their appearance in some cell in row D* other than [D*, D*]. These units finally appear in the cell [D*, E]. In summary, this entire sequence of events represents a shipment of 4 units from source 3 to final destination E via the transshipment point D*.

Observe, in addition, that the solution shown in Table 11 is degenerate. Degeneracy is rather a common phenomenon in transshipment problems, though it can be overcome in precisely the usual way. Thus, the cells [2, B*] and [2, E*] are in the present case identified as θ-cells.

Lastly, it is evident from the structure of the solution of this sample problem that transshipment chains of any length can make an appearance in the transshipment problem. Thus, the chain 2 → C* → A* → B, which contains two transshipment elements, can be identified in Table 11.

The transshipment problem in its turn has many applications and generalizations, one of the more important of these being the Beckmann-Marschak problem which is now considered directly.

The Beckmann-Marschak Problem

The particular spatial allocation problem considered here was initially described (and with eponymous results) by Beckmann and Marschak (1955). This problem is essentially a transshipment problem; however, the transshipment points in this particular expression of the problem also represent transformation or manufacturing centres where some commodity passes through an industrial process before being sent on to final consumers. For example, the set of sources might represent production points for a raw material such as iron ore; this would then be shipped to various industrial centres where it would be manufactured into steel; the finished steel would then be shipped out from the industrial centres to final consumers. A graphical representation of such a system is shown in Figure 10. The main point to note here is that two entirely new elements have been brought into the discussion, namely, in the first place, a transformation or input-output process and in the second place, a two-commodity spatial allocation process. As a result of these new elements, the Beckmann-Marschak problem is not susceptible to numerical solution by the conventional transportation algorithm. Rather, except in certain very special cases, the problem can only be solved by the simplex method of linear programming. Thus, in the present case, all computational aspects of the problem will be neglected, and the discussion will concentrate instead on the formal and structural properties of the problem.

Formal structure of the Beckmann-Marschak problem

Let there be n sources of some given raw material, and m destinations for some derivative commodity which is manufactured out of that raw material. Let there be s centres of manufacturing or transformation. Let c_i be the unit cost of production of the raw material at i, and let c_j^* be the cost of transforming one unit of that raw material into the derivative commodity at transformation centre j. The term t_{ij} is the cost of transporting one unit of the raw material from source i to transformation centre j, and the term t_{jk}^* is the cost of transporting one unit of the finished commodity from transformation centre j to final consumers at k. Finally, let x_{ij} represent the magnitude of flow of the raw material from i to j, and let x_{ij}^* represent the magnitude of flow of the finished commodity from j to k.

The Beckmann-Marschak problem now represents a normative program which extends over the following cost elements: a) The cost of producing the raw material, b) the cost of transporting the raw material to manufacturing centres, c) the cost of transforming the raw material into the finished product, d) the cost of transporting the finished product from manufacturing centres to final destinations. The objective function corresponding to such a program is

$$28) \quad Z = \sum_{i=1}^{n} \sum_{j=1}^{s} (c_i + t_{ij}) x_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{s} c_j^* x_{ij} + \sum_{j=1}^{s} \sum_{k=1}^{m} t_{jk}^* x_{jk}^* .$$

This objective function is subject to a series of constraining conditions. These conditions are

$$29) \quad \sum_{j=1}^{s} x_{ij} \leqslant S_i$$

which is the familiar supply capacity condition;

$$30) \quad \sum_{j=1}^{s} x_{jk}^* = D_k$$

which is the familiar demand condition. In addition, a balance or conservation condition applies to the problem ensuring that all outputs at the transformation point j are exactly matched by a corresponding set of inputs. This conservation condition is

$$31) \quad \alpha_j \sum_{i=1}^{n} x_{ij} - \sum_{k=1}^{m} x_{jk}^* = 0$$

where the term α_j is an input-output coefficient which indicates how many units of output of the finished product

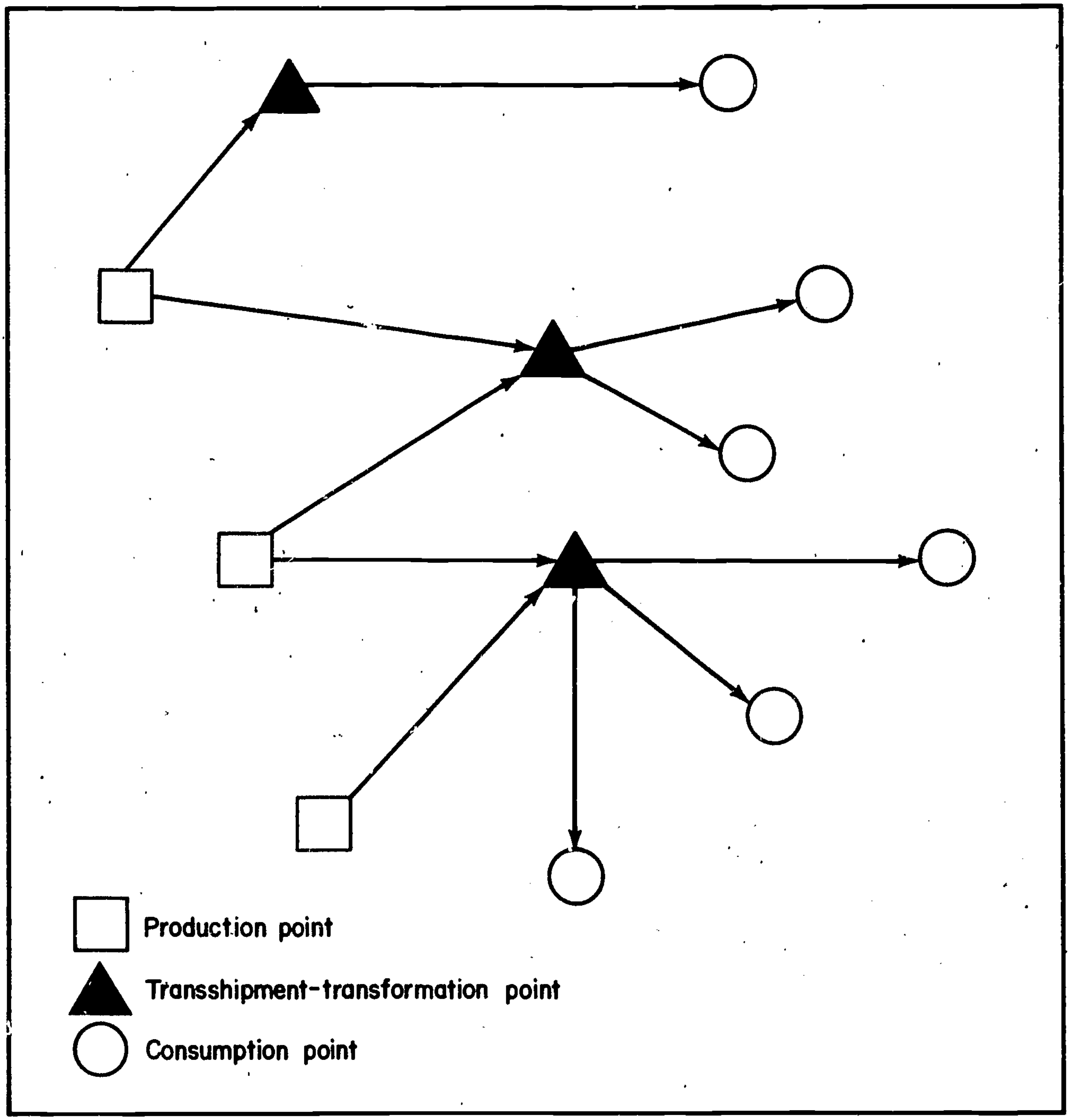

Figure 10. A Beckmann-Marschak problem

are obtained at the j^{th} transformation centre for each input of one unit of the raw material. A capacity constraint on any transformation centre may also be appropriately appended to the Beckmann-Marschak problem, giving,

32) $$\sum_{i=1}^{n} x_{ij} \leqslant K_j$$

where K_j is the total transformation capacity (in terms of raw material inputs) available at j. The non-negativity conditions on all variables necessarily apply:

33) $$x_{ij} \geqslant 0, x^*_{jk} \geqslant 0.$$

The output of such a program is an optimal plan of production, transformation, and trade. In addition, there would be available from the dual program for the

Beckmann-Marschak problem a set of equilibrium rents and prices extending over the entire system. In the case of the Beckmann-Marschak problem, these rents are in part a function of locational variations both among the set of raw material sources and the set of transformation centres; but they are also in part a function of the differential cost structures at the sources and transformation points.

Network Problems

In their most basic form, network problems are generally representable as either transportation or transshipment problems with the special characteristic that source-destination flows are explicitly aligned along a set of links or arcs. Of this class of spatial allocation problems there are two major types: Network flow problems on the one hand, and network structure problems on the other hand, though there is also a very great variety of additional network problems which do not fall readily into this categorization. All of these problems, however, have the almost invariant characteristic that they are structured around a kernel comprising either the transportation problem or the transshipment problem. For ease of exposition, the network systems to be discussed below will be built up entirely around the transportation problem. It should be borne in mind however that it is always possible and usually more realistic to embed a pure transshipment problem into these network systems.

A simple network problem: The capacitated transportation problem

As soon as the characteristic flows of the transportation problem are thought of as being restricted to the arcs of a network, the question of the carrying capacity of those arcs is brought into prominence. This question is dealt with explicitly by a linear programming problem known as the *capacitated transportation problem.* This linear program may be written immediately:

Minimize:

34) $$Z = \sum_{i=1}^{n} \sum_{j=1}^{m} t_{ij}x_{ij}$$

subject to:

35) $$\sum_{j=1}^{m} x_{ij} \leqslant S_i$$

36) $$\sum_{i=1}^{n} x_{ij} = D_j$$

Table 12a. Optimal solution uncapacitated network

To From	1	2	3	4	5	Supply
1	▥	4 7	3	▥	▥	7
2	4	2 1	▥	2 2	▥	3
3	3 1	▥	▥	4 4	▥	5
4	▥	▥	3 8	5 3	1 4	15
Demand	1	8	8	9	4	30

$$Z = (4 \times 7) + (2 \times 1) + (2 \times 2) + (3 \times 1) + (4 \times 4) + (3 \times 8) + (5 \times 3) + (1 \times 4) = 96$$

37) $$x_{ij} \leqslant K_{ij}$$

38) $$x_{ij} \geqslant 0$$

where K_{ij} is the maximum carrying capacity of the arc $\overline{i,j}$. Thus, all flows in this special version of the transportation problem are limited by upper bounds which are expressions of the capacity of the network over which those flows are carried.

Some of the features of this particular linear program are demonstrated by a sample problem as shown in Tables 12a and 12b and in Figure 11. The Figure 11 (which is derived

Table 12b. Optimal solution capacitated network

To From	1	2	3	4	5	Supply
1	▥	4 6	3 1	▥	▥	7
2	4 1	2 2	▥	2	▥	3
3	3	▥	▥	4 5	▥	5
4	▥	▥	3 7	5 4	1 4	15
Demand	1	8	8	9	4	30

$$Z = (4 \times 6) + (3 \times 1) + (4 \times 1) + (2 \times 2) + (4 \times 5) + (3 \times 7) + (5 \times 4) + (1 \times 4) = 100$$

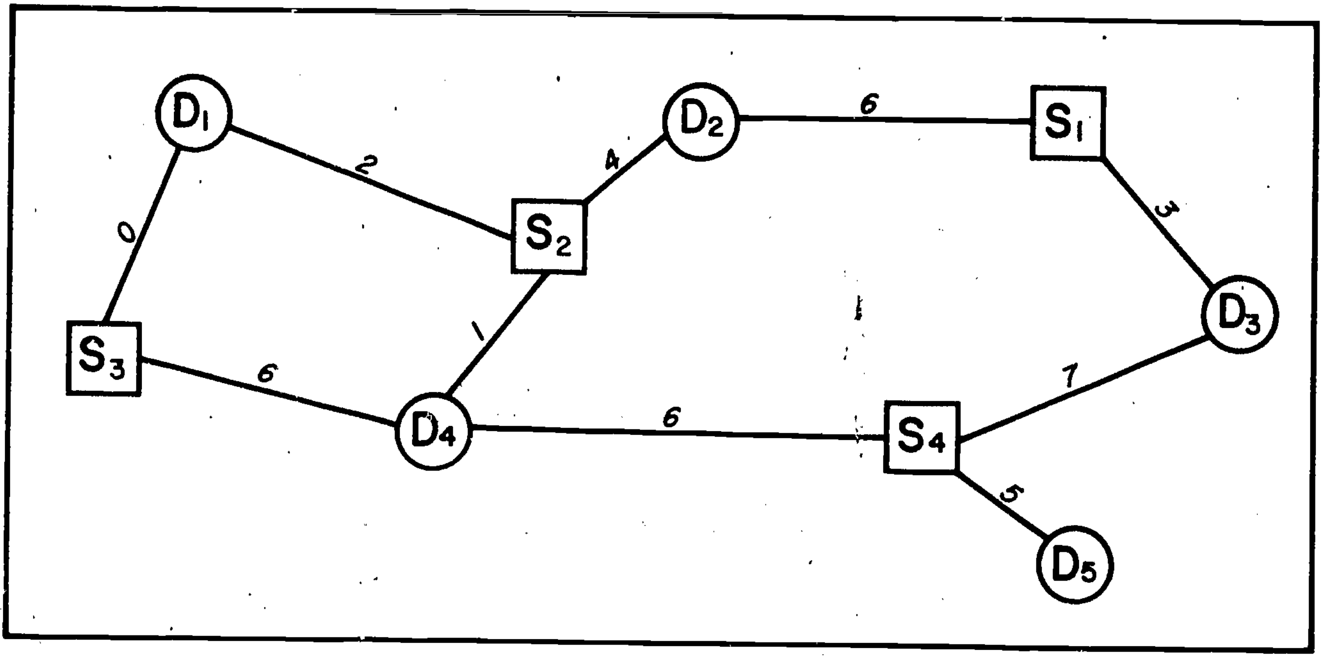

Figure 11. Network and arc capacities

from the exemplary problem discussed in Section II of this report) shows an elementary network with associated arc capacities. If any source-destination pair in Figure 11 is not directly connected by an arc then the corresponding cell in both Tables 12a and 12b is blocked out. Suppose now that some commodity is required to be sent from source nodes to destination nodes, but with the restriction that flow is rigidly confined to existing network arcs. Let all supply capacities, demand requirements, and transport costs for allowable source-destination connections be as they were in the sample problem discussed in Section II. Then, neglecting for the moment all arc capacity limitations, a first optimal solution for this system may be computed by the method of the simple transportation algorithm. During the computational process, shipments are kept off routes which lack a direct network connection by simply not considering tableau cells which correspond to those routes. This first solution is shown in Table 12a. A second optimal solution is now computed for the case where all arc capacity contraints are rigidly enforced. This new solution is shown in Table 12b, and it betrays a perceptible deterioration in the value of the objective function as compared with the solution shown in Table 12a. A special variation of the transportation algorithm may be used to compute solutions to this second, capacitated problem (cf. Wagner (1959)), though the simplex method would usually be more reliable.

From this sample problem, it is immediately evident that the arc capacity constraints greatly limit the total efficiency of any flow program. It is therefore of considerable interest to speculate as to the possibilities of extending the capacitated transportation problem in such a way that investment in new capacity on binding arcs may be possible, thus securing improved flow efficiencies. Some simple extensions of this type are now considered.

Some elementary network flow-investment models

Two main strategies are always available in any kind of investment problem. The first of these strategies is to assign a fixed budget to the problem and then to search out the best possible improvements in the structure of the problem while not exceeding the total available budget. This kind of strategy seems often to be adopted by public agencies, especially in cases where there has been a prior appropriation of public funds to some program. The second of these strategies is to attempt to trade off capital investment costs against concomitant savings in system operating costs produced by efficiency improvements. This second strategy, then, attempts to strike an economical balance between capital costs and operating costs. In economic terms, this strategy is therefore generally preferable to the former, though it presupposes great flexibility of the budget and it also leads to certain problems of definition, as will appear below. Both of these strategies will be exemplified in two parallel network flow-investment problems considered below.

Case of an inflexible budget. Suppose that some capacitated network is given, together with a set of sources and destinations. Further suppose that investments are to be made in new network capacity thus improving the total performance of the system. In cases where the total investment budget is given exogenously and is then to be applied to the problem of gaining the maximum improvement possible in system operating costs, the appropriate corresponding linear program is:

Minimize:

39) $$Z = \sum_{i=1}^{n} \sum_{j=1}^{m} t_{ij}x_{ij}$$

subject to:

40) $$\sum_{j=1}^{m} x_{ij} \leqslant S_i$$

41) $$\sum_{i=1}^{n} x_{ij} = D_j$$

42) $$x_{ij} - y_{ij} \leqslant K_{ij}$$

43) $$c_{ij}y_{ij} \leqslant B$$

44) $$x_{ij} \geqslant 0, \; y_{ij} \geqslant 0$$

where y_{ij} is a solution variable specifying the total increase in the capacity of the arc $\overline{i,j}$, c_{ij} is the cost of increasing the capacity of the arc $\overline{i,j}$ by one unit, and B is the total budget available for investment in the network. After solution of this system, the total capacity of the arc $\overline{i,j}$ will be $K_{ij} + y_{ij}$, and the constraint (42) effectively prevents x_{ij} from becoming larger than this amount. In addition, the constraint (43) prevents total new investments from exceeding the available budget.

The total result of the program (39)-(44) is both an improved network structure and a new improved pattern of commodity shipments.

Case of a flexible budget and a joint cost minimizing program. In the case where the investment budget is flexible and where the only criterion limiting new investment is that it should be such as to minimize joint operating and capital costs, an appropriate linear program may be expressed,

Minimize:

45) $$Z = \sum_{i=1}^{n} \sum_{j=1}^{m} t_{ij}x_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}y_{ij}$$

subject to:

46) $$\sum_{j=1}^{m} x_{ij} \leqslant S_i$$

47) $$\sum_{i=1}^{n} x_{ij} = D_j$$

48) $$x_{ij} - y_{ij} \leqslant K_{ij}$$

49) $$x_{ij} \geqslant 0, \; y_{ij} \geqslant 0$$

where all terms are defined as before.

Quandt (1960) has examined the economic properties of this particular network investment problem in some detail. The important point to notice for the moment is that the appearance of both the operating costs, t_{ij}, and the capital costs, c_{ij}, in the same objective function produces a number of problems of definition and accounting. Normally, the two costs, t_{ij} and c_{ij}, would be considered to belong to quite different dimensions of time. The former would usually be thought of as recurrent costs extending over some entirely short-run period, whereas the latter would usually be thought of as strictly non-recurrent, and extending over a reasonably long-run period. Thus, in order to make the objective function (45) realistic and meaningful it is essential that these two types of costs be reduced to a similar order of magnitude. This, at least in part, would involve re-definition of the capital costs into constant costs per unit of time, where this unit of time would correspond to the period of recurrence of the operating costs. To accomplish this difficult accounting problem it would be necessary to take into consideration such other time-dependent elements of the capital costs as interest rates, discounts, inflation, and the like.

Additional miscellaneous network problems

In addition to the simplified prototype problems considered in the preceding paragraphs, there is a further miscellaneous group of network problems with relevance to the general problem of spatial allocation. These additional problems are merely mentioned here in passing (with appropriate references), for they tend to represent specialized issues and to require somewhat esoteric mathematical techniques for their proper solution. Thus, among the more important of these problems are the following: The maximal network flow problem (Ford and Fulkerson (1962)), the minimal spanning tree problem (Scott (1969)), the shortest path problem (Dantzig (1963)), the travelling salesman problem (Scott (1969)), and a variety of quite complex network flow-investment problems (e.g. Garrison and Marble (1958), Quandt (1960), and Ridley (1969)). In particular, Quandt has considered at length certain dual problems in relation to network flow and investment programs, and his work should be consulted for an analysis of many of the basic theoretical economic issues which underlie these problems.

V. CONCLUSION

In the preceding account an attempt has been made to develop a few introductory notions about the basic workings of spatial allocation systems. This account has stressed the elementary methods, theory, and geographical applications of spatial allocation analysis. Thus, attention was focussed first on the purely computational properties of the simple transportation model, then on the theoretical underpinnings of the model, and lastly, on a variety of important formal generalizations of the transportation model. Throughout this account it has been evident, both by direct assertion and by implicit connotation, that the topic of spatial allocation has enormous ramifications, extending as it does throughout economic geography and the cognate disciplines of spatial economics and regional science. For this reason, the entire account above should be seen as no more than an elementary initiation into a small number of central issues.

Nevertheless, in spite of its apparent range and heterogeneity, spatial allocation analysis represents in fact a remarkably coherent frame of reference. This results in part from the purely formal similarities among spatial allocation models, and it results in part from the normative operating principle of those models. In particular, in spatial allocation analysis, geographical space is seen as a limiting factor in the teleological tendency to efficiency in real allocation systems. The problem then is to define final normative states for those systems so that the dissonant effects of geographical space are minimized.

Of course, spatial allocation analysis is not without certain limitations, and these limitations can often be serious. In the entire preceding discussion two limitations in particular have been consistently glossed over, and it now seems apposite to deal with these directiy. These limitations concern the implicit assumptions of divisibility and linearity which underlie much of spatial allocation analysis. In the first place, most spatial allocation models take it for granted that all variables are infinitely divisible. For certain special reasons this happens not to be the case so far as the transportation model is concerned, though it is largely the case in most extensions of the transportation model. But this condition of infinite divisibility is often not valid in empirical cases. To take just one example, increments to network capacity are surely not arbitrarily divisible as was assumed in the preceding section, but rather are composed of integral and irreducible elements, (see Scott (1971)). In the second place, and probably even more serious, the linearity assumption underlying most spatial allocation models is rarely satisfied in practice. Thus, to exemplify again, the Beckmann-Marschak model assumes complete linearity of all finished product outputs with respect to all raw material inputs at manufacturing centres. This is patently an infringement on the nature of real manufacturing processes where technical economies of scale are almost always obtainable. In addition, it may be noted that spatial allocation analysis as described above is virtually useless as a description of international trading situations for the geography of international trade is as much a political process as it is a normative spatial process in the narrow sense.

Doubtless, however, the advantage of spatial allocation analysis will frequently be found to outweigh the limitations. Above all, there are two outstanding advantages to spatial allocation models: as applied policy mechanisms, and as tools for theoretical analysis. These are substantial qualities. For policy purposes, these models have been found to be of prime pragmatic importance in the organization of planned economies. They have in particular been widely applied by Soviet economic geographers and regional scientists to the problems of industrial organization and integration over the vast geographical extent of the U.S.S.R. Elsewhere, these models have been applied to the planning of public facility systems, such as schools, hospitals, and road transport networks.

In addition, spatial allocation models yield significant theoretical insights into the workings of spatial economic systems. These insights emanate especially from the dual problem with its offshoots into classical rent and price theory. In turn, these issues lead on to the consideration of questions of geographic distribution and location à la Von Thünen and Lösch, (see Garrison (1959, 1960), and Isard (1960)). Moreover, spatial allocation analysis clearly demonstrates Pareto's subtle principle, which is at first apparently self-contradictory and then finally entirely reasonable, that competitive allocation systems tend to produce short run patterns of allocation which are similar to those produced by monopolistic or general welfare systems. Of course, this assertion does not mean also that long-run goals will be the same in each case. Moreover, in welfare situations there is always a natural tendency to make special provision for special cases. However, the assertion has a very general validity, for from all that has gone before, it is clear that each of these types of systems tends to produce cost minimizing (or, equivalently, benefit maximizing) solutions. At least, Pareto's principle applies to the kinds of simple systems typified by the transportation problem. The principle does not always apply in certain more complex cases, though it is a remarkable result even so, and spatial allocation analysis clarifies it entirely.

Finally, the generalizations of spatial allocation analysis are such as to incorporate virtually the whole of economic geography. Indeed, significant conceptual generalizations of this order of magnitude have been accomplished, (cf. Isard (1958, 1960), Moses (1960), and Stevens (1958)). These generalizations have succeeded in extending the kinds of simple models already considered to large interregional systems producing and consuming many different commodities, and where the commodities are subject to multifarious processes of exchange and transformation. These generalized models represent, in short, virtually complete economic systems. It is doubtful if at this point in time there can be much possibility of applying such models. Their importance, however, is that they provide an intellectually efficient way of thinking about an enormously complicated and important set of processes and phenomena. Indeed, the elemental components of these models lie at the basis of all economic life, and hence, ultimately, impinge upon all geographical problems.

The present account may be taken as a preface to the study and applied analysis of these wider issues. Concomitantly, spatial allocation analysis should be seen as one of the key conceptual frameworks in modern geographical enquiry.

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