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ABSTRACT

This booklet consists of 11 bulletins designed to answer questions which teachers frequently ask about the learning and teaching of mathematics. (The bulletins are revisions of a set originally published in 1970.) Each bulletin is organized around a central topic, and presents questions related to that topic, summaries of research findings relevant to each question, and a selected bibliography. The titles of the bulletins are: (1) Attitudes and Interests, (2) Organizing for Instruction, (3) Promoting Effective Instruction, (4) Differentiating Instruction, (5) Instructional Materials and Media, (6) Addition and Subtraction with Whole Numbers, (7) Multiplication and Division with Whole Numbers, (8) Rational Numbers: Fractions and Decimals, (9) Geometry and Other Mathematical Topics, (10) Verbal Problem Solving, and (11) Planning for Research in Schools. The volume is indexed by the questions answered in the bulletins, for easy reference. (SD)

MATHEMATICS EDUCATION INFORMATION REPORT

Using Research: A Key to Elementary School Mathematics

Marilyn N. Suydam

J. Fred Weaver

THE ERIC SCIENCE, MATHEMATICS AND ENVIRONMENTAL EDUCATION CLEARINGHOUSE
in cooperation with
Center for Science and Mathematics Education
The Ohio State University

MATHEMATICS EDUCATION REPORTS

Revision of

Using Research:

A Key to Elementary School Mathematics

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Science, Mathematics and Environmental Education
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December 1975

Using Research:

A Key to Elementary School Mathematics

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Using Research:

A Key to Elementary School Mathematics

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This booklet consists of eleven bulletins which present answers to some questions about research on elementary-school mathematics. These are revisions of the bulletins which were originally prepared in 1970 as one facet of the "Interpretive Study of Research and Development in Elementary School Mathematics".[1] The questions are based on ones frequently asked by teachers about the teaching and learning of mathematics. The bulletins are organized by topic, and specific research findings are cited, with lists of selected references included for those who wish to explore a topic further. The intent is to provide a concise summary of specific ideas which may be applicable in a classroom.

The first five bulletins consider research findings which may apply across various age levels. The first bulletin involves the affective factors of "Attitudes and Interests". The second bulletin, "Organizing for Instruction", cites research findings on ways of organizing the classroom, while the third, "Promoting Effective Instruction", pertains to facets of learning which the teacher may directly

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control. "Differentiating Instruction" is the focus of the fourth bulletin, while some research on "Instructional Materials and Media" is contained in the fifth.

Bulletins 6 through 10 cite research findings on the content of elementary-school mathematics: "Addition and Subtraction with Whole Numbers", "Multiplication and Division with Whole Numbers", "Rational Numbers: Fractions and Decimals", "Geometry and Other Mathematical Topics", and "Verbal Problem Solving". The final bulletin, "Planning for Research in Schools", is designed to aid those teachers who want to become involved in doing research in their own schools. It can also aid readers of research reports, as it indicates the important factors to consider.

Following the last bulletin is an index of questions which can be of help in locating results of most interest.

In this revision, we have added sections on recent studies and rewritten some sections from the original bulletins to reflect more recent findings. But we have retained much from the original: there is still research from past decades cited, for it is important to recognize the contribution that such research has made to our present state of knowledge about the teaching and learning of mathematics.

As with the original, we have made a selection of studies, taking into consideration the quality of the research. We believe that the selection process has not distorted what research may have to say about a particular question. In most cases, however, there are more findings from a study than what we report, as well as other studies which could have been cited to affirm a point. It must also be recognized that there are times when we have generalized, and occasionally we have editorialized, especially when no research evidence is available, or to make a particular point.

The booklet is being distributed in loose-leaf form to facilitate use. Any of the materials may be reproduced to meet your local needs. We hope you will find them helpful.

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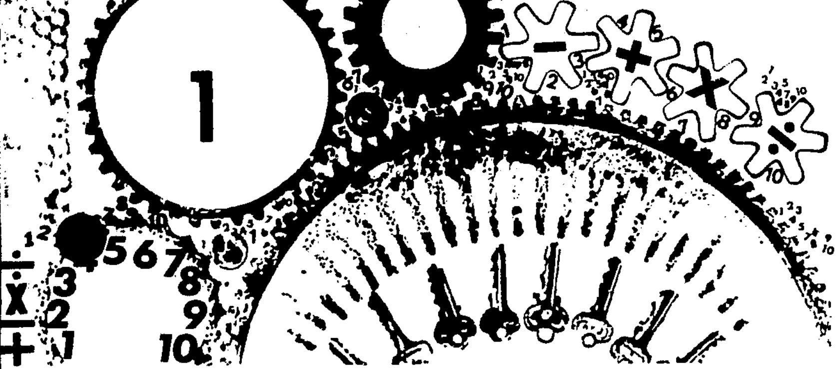

ATTITUDES AND INTERESTS

How are attitudes and interests investigated?

Attitudes and interests are affective things, having to do with feelings. Attitude toward mathematics involves many facets, ranging from awareness of the structural beauty of mathematics and of the important roles of mathematics, to feelings about the difficulty and challenge of learning mathematics, to interest in a particular type of mathematics or particular methods of being taught mathematics. Attitudes and interests are thought to exert a dynamic, directive influence on an individual's responses; thus attitudes and interests may be related to the teaching and learning of mathematics.

Attitudes and interests frequently have been investigated by the use of scales on which persons indicate agreement, or the degree of agreement or disagreement, with statements about mathematics. Sometimes various school subjects have been ranked by order of preference, or likes and dislikes have been indicated. Both methods obviously rely on the honesty of the individual in expressing his true feelings.

Do elementary school pupils like mathematics?

Many people believe that mathematics is disliked by most pupils--or that it is just about the least favorite subject in the elementary school. It is true that in some surveys a significant proportion of pupils rated mathematics as the least liked of their school subjects. But it is equally true that in these surveys approximately the same proportion of pupils (at least 20%) cited mathematics as the best liked or the second best liked school subject (Chase, 1949; Mosher, 1952; Chase and Wilson, 1958; Sister Josephina, 1959; Greenblatt, 1962; Curry, 1963; Faust, 1963; Rowland and Inskeep, 1963; Inskeep and Rowland, 1965).

More recently, Callahan (1971) reported that among eighth-grade students, 62% said they liked mathematics, while 20% expressed a dislike for mathematics. Levine (1972) noted that pupils in grades 3, 4, and 6 ranked mathematics highest, compared to English, science, and social studies, with respect to importance, enjoyment, best subject, and subject teacher taught best. Ernest and others (1975), in a study in which students in grades 2 through 12 were asked to rank the same four subjects, found that mathematics was liked best by 30% of the boys and 29% of the girls. It was liked least by 27% of the boys and 29% of the girls.

Such findings were supported by Dutton (1956, 1968) with evidence from answers given on scales of items. Similarly, Stright (1960) reported that:

(1) 9% felt that mathematics was a waste of time
(2) 20% thought mathematics uninteresting
(3) 58% said it was the best subject in school
(4) 66% wished they had more mathematics
(5) 80% said they really enjoyed mathematics.

In studies conducted 15 or more years ago, boys seemed to prefer mathematics slightly more than did girls, especially toward the upper elementary school grades (Chase and Wilson, 1958; Dutton, 1956; Stright, 1960). But in the study by Ernest and others (1975) mathematics was the only subject in which no sex difference in preference was observed.

The majority of evidence indicates that relatively definite attitudes about mathematics have been developed by the time children are in the intermediate grades. Generally, attitudes toward mathematics tend to become increasingly less positive as students progress through school. Evans (1971) and Neale and Proschek (1967) both found that attitude scores in grade 4 were significantly higher than in grade 6; Anttonen (1968) found that mean scores declined between grades 5-6 and grades 11-12, while Malcolm (1971) observed that attitudes became less positive between grades 3 and 7. Examining measures of attitude, self-concept, and anxiety as one phase of the National Longitudinal Study of Mathematical Abilities (NLSMA), Crosswhite (1972) reported that student attitude toward mathematics seemed to "peak" near the beginning of junior high school.

Does a more favorable attitude, or greater interest, lead to higher achievement?

There is no consistent body of research evidence to support the popular belief that there is a significant positive relationship between pupil attitudes toward mathematics and pupil achievement in mathematics. As Knaupp (1973) noted, we have little research basis for believing that these two things are causally related.

Lyda and Morse (1963) reported that among fourth-grade pupils, significant gains in mathematics achievement were associated with a combination of meaningful instruction and an increase in the favorableness of attitude toward mathematics. Nothing could be asserted, however, about the relation between achievement and attitude per se.

In investigations of the subject preferences of fifth-grade children, Chase (1949) reported no consistent pattern of relationship between pupils' relative preference for mathematics and their mathematics achievement level. Dean (1950), using some of the pupils involved in this study, found that pupils who did well in mathematics generally had indicated a preference toward it. However, preference for mathematics did not necessarily indicate that achievement would be better.

Intelligence, which cannot be separated from achievement, and its relationship to attitude was investigated by Rice (1963) and Greenblatt (1962), who noted that pupils with IQs above 110 had a greater interest in mathematics.

Faust (1963) and Shapiro (1962) found a low positive relationship existed between attitude and achievement. An analysis of recent studies indicates that no significant relationship was found in about half the studies (e.g., Abrego, 1966; Deighan, 1971; Keane, 1969), while low positive correlations (between .20 and .40) were reported by the other half (e.g., Anttonen, 1968; Burbank, 1970; Caezza, 1970).

Whether boys and girls differ on this factor was considered by many of the researchers. Greenblatt (1962) reported a significant relationship between relative preference for mathematics and mathematical achievement level on the part of girls in grades 3 through 5, but no such significant relationship existed for boys. At the sixth-grade level, Neale, Gill, and Tismer (1970) found attitudes and achievement to be significantly correlated for boys but not for girls. However, in other studies, no significant differences between girls and boys were found.

Evans (1971) investigated the nature of four attitude scales and concluded that a common construct was sampled; but Mastantuono (1971) analyzed data from administering four attitude scales and found that a significant correlation with achievement was obtained for only two of the four.

What is the relationship of teacher attitudes and interests to pupil attitudes and interests?

Teachers are often viewed as being prime determiners of a student's attitude and performance. There is some evidence to support this. Smith (1974), for instance, reported that students' perceptions of teachers were significantly correlated with mathematical growth in grades 4 through 6. Rosenbloom and others (1966) found that teaching effectiveness contributed significantly to the attitude and perceptions of pupils concerning their teachers and their methods, the school, text materials, and the class as a group.

Greenblatt (1962) reported a significant relationship between teacher preference for mathematics and pupil preference for mathematics in the case of children who had IQs above 110. No such significant relationship was found in the case of pupils in lower IQ groupings, however.

Chase (1949) reported a strong agreement between fifth-grade teachers' preference for mathematics and their respective pupils' preference for mathematics. A decade later, in a replication of this investigation, Chase and Wilson (1958) reported no consequential change: when teachers preferred mathematics, a majority of their pupils preferred it.

There is, however, some evidence on the other side. Caezza (1970), Deighan (1971), and Wess (1970) found no large or significant correlation between teacher attitudes and pupil attitudes or achievement. Similarly, Inskeep and Rowland (1965) reported a nonsignificant correlation between teacher preference for mathematics and pupil preference for mathematics. Van de Walle (1973) and Keane (1969) indicated that their data were inconclusive on this point.

Phillips (1970) reported evidence that the effect of teachers' attitudes may be cumulative. He found a significant relationship between most-recent-teacher attitude and student attitude at the seventh-grade level. He also observed that type of teacher attitude encountered by students for two and for three of their past three years was related significantly to their present attitude and achievement.

What affects attitudes?

Attitudes toward elementary school mathematics are probably formed and modified by many forces. The influence of other people could be named as one source: parents and other non-school-related adults, classmates and other children, and teachers in each of the grades.

The way in which the teacher teaches seems to be of importance--the methods and materials he or she uses, as well as his or her manner, probably affect pupils' attitudes.

The subject itself undoubtedly has an influence on a child's attitude: the precision of mathematics when compared with many other subjects; the need for thorough learning of facts and algorithms; the "building block" characteristic wherein many topics are built and often dependent on previous knowledge. Indeed, mathematics has traditionally been considered difficult, and its use as a mental discipline tool is still unfortunately being touted and abused by some persons.

The learning style of the child is also an important factor to consider. The orderliness which discourages some is the very aspect which attracts others.

Studies by Dutton (1956, 1968), Lyda and Morse (1963), and others have indicated that for some children the practical value and usefulness of mathematics in out-of-class situations contribute to the development of more positive attitudes toward mathematics.

Based on a survey of more than 1,000 pupils, Stright (1960) reported that 95% felt that mathematics would help them in their daily lives, while 86% classified mathematics as the most useful subject. Callahan (1971) reported that eighth-grade students gave the need for mathematics in life most frequently as the reason for liking it; not being good in mathematics was cited most often as the reason for disliking it.

Making pupils aware of the uses of mathematics seems related to developing more positive attitudes, yet newer programs have frequently tended to deemphasize this aspect. Dutton (1968), for example, noted that fewer students saw the practical uses of mathematics than did students surveyed ten years previously. However, Dutton also observed that fewer pupils were afraid of mathematics and more enjoyed the challenge of a mathematics problem at the time of his survey (1968) than pupils tended to ten years earlier.

How can attitudes be improved?

Among the reasons which children frequently give for disliking mathematics are lack of understanding, high level of difficulty, poor achievement, and lack of interest in certain aspects of mathematics.

On the other hand, children like mathematics primarily because they find it useful, interesting, challenging, and fun.

Certainly there are clues, in the reasons given above, for what to do in attempting to improve students' attitudes toward mathematics. And we have good reason to believe that interest and attitude can be improved if:

(1) the teacher likes mathematics and makes this evident to pupils,

(2) mathematics is an enjoyable experience, so that children develop a positive perception of mathematics and a positive perception of themselves in relation to mathematics,

(3) mathematics is shown to be useful, both in careers and in everyday life,

(4) instruction is adapted to students' interests,

(5) realistic, short-term goals are established--goals which pupils have a reasonable chance of attaining,

(6) pupils are made aware of success and can sense progress toward these recognized goals,

(7) provision is made for success experiences, to help the child to avoid failure and, in particular, repeated failure; diagnosis and immediate remedial help are imperative, and

(8) mathematics is shown to be understandable, through the use of meaningful methods of teaching.

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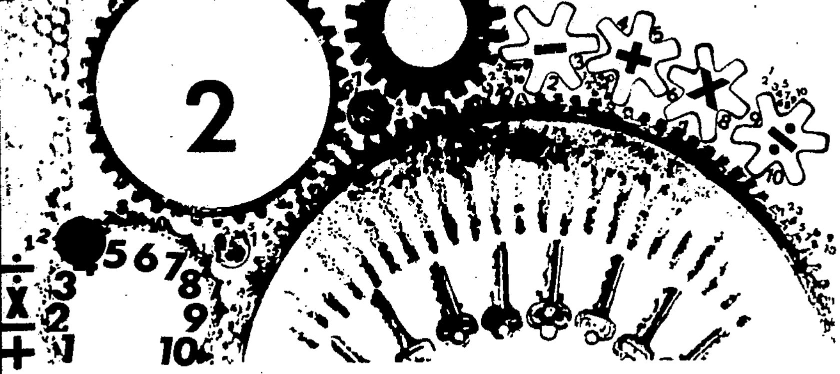

ORGANIZING FOR INSTRUCTION

Is there a "best way" to organize for mathematics instruction?

Educators have long searched for the "perfect" organizational pattern to meet individual pupil needs and increase achievement. A vast number of studies have been conducted to attempt to ascertain the efficacy or the superiority of departmentalization, team teaching, multi-graded, non-graded, or self-contained classrooms.

Recently there has been much interest in "open education." For instance, Earnshaw (1973) tried to determine what effect, if any, an informal learning climate had on pupil attitudes and motivation toward learning and school. One group had an open-education program in second grade, while the other had the "usual" program. The open-education program measurably influenced pupils in exhibiting more resourcefulness, creativity, initiative, self-reliance, and enjoyment of school. However, on standardized tests of mathematics, the pupils in the open-education program did not score as well as the pupils in the usual program.

Attempting to isolate and measure the effects of any organizational pattern is extremely difficult, since factors such as content organization and teacher background interact with the pattern. The definitions of the various patterns also tend to overlap--what one person labels team teaching another defines as departmentalization, etc.

It is apparent from a review of the research that no general conclusion can be drawn regarding the relative efficiency of any one pattern for mathematics instruction. There appears to be no one pattern which, per se, will increase pupil achievement in mathematics. A proponent of any pattern can find studies that verify his stand. Achievement differences are affected more by other variables such as the mathematical background of the teacher, than by the organizational pattern. Perhaps the most important implication of the various studies is that *good teachers are effective regardless* of the nature of classroom organization (Gibb and Matala, 1962).

How important is early or pre-first-grade instruction?

With a few exceptions, there is general agreement today that we will begin to teach mathematics systematically in grade 1, if not in kindergarten. Forty years ago, however, this was a matter of great debate. It was argued that formal study should be deferred "until the child could understand more and had a need for using mathematics." Therefore, until at least the third grade, mathematics should be learned "incidentally," through informal, unplanned contacts with number.

Opponents argued that such delay was a waste of time. Data to support this were collected; for instance, Washburne (1928) found that pupils who began mathematics in either grade 1 or 2 made better mathematics scores in grade 6 than did pupils who began mathematics in grade 3.

On the other hand, more recently Sax and Ottina (1958) found that by seventh grade, there was no significant difference in computation scores. Meaning scores were higher for pupils in a school in which formal instruction was deferred until fifth grade. However, with the emphasis today on teaching an increased amount of mathematics at any earlier age, the question of when to begin systematic instruction has not seriously been reopened.

A question of current great concern pertains to the effects of pre-first-grade mathematical experiences. This question has been of recurrent interest. In recent studies, Traywick (1972) reported no significant differences between the achievement of those who had had kindergarten experience and those who had not had such experience, when the children were in grades 2 to 6. Kristjansdottir (1972) reported that kindergarten was helpful in increasing children's achievement. These two reflect the evidence through the years. [Similarly, the research on pre-school experience (e.g., Yonally, 1972) indicates no clear-cut answer.] The effect appears to be highly school-related; that is, the experiences that have been provided to build on the foundation provided in the pre-school or kindergarten (in addition to the pre-school or kindergarten experience itself) have a vital effect on how much children achieve.

How should the sequence of content be determined?

For years the work of Washburne (1928) and the Committee of Seven strongly influenced the sequencing of topics in the curriculum. This group of superintendents and principals in the midwest surveyed pupils to find when topics were mastered, and then suggested the order and mental age or grade level in which each should be taught.

With the curriculum reform movement which began in the 1950's, much reorganization of content has been suggested. Generally, various topics and patterns have been "tried out" to see if they could be taught at a proposed level: research reflects many such trials.

Gagné has long been working on the development of hierarchies of learning tasks. Gagné and Bassler (1963) structured a hierarchy of "subordinate knowledge" which led to the development of a concept. They found that, in general, sixth grade pupils learned that concept when developed according to such a hierarchy. Although they did not retain all of the subordinate knowledge, they did continue to achieve well on the final task.

In a study with fifth and sixth graders, Buchanan (1972) examined instructional sequences to determine how prior experience with subordinate tasks affected mastery of a superordinate task, and the efficiency of performance within a sequence. The amount of prior experience with the introductory task had a significant effect on mastery of the superordinate task.

Phillips (1972) developed and evaluated procedures for validating a learning hierarchy from test data. A test to assess mastery at each of 11 levels of a hierarchy for computational skills of adding rational numbers with like denominators was administered and seven hierarchical orderings of the 11 subtasks were generated. One programmed instruction lesson was developed for each subtask. Fourth-grade pupils were assigned to seven groups defined by the hierarchical orderings. Results indicated that sequence, even if random, seemed to have little effect on immediate achievement and transfer to a similar task. However, longer term retention seemed quite susceptible to sequence manipulation.

"Task analyses" to establish learning hierarchies are much in vogue today and can be helpful if used sensibly. Not all persons, however, approach the sequencing issue in that way. For instance, Suppes (1969) is approaching the problem of organization and sequencing with the aid of computer-stored data on pupils' responses.

Is there research which identifies outcomes of programs of "meaningful" instruction?

Earlier in this century, it was doubted that children needed to understand what they learned. It was enough if they developed high degrees of skill. To take time to give explanations and develop understanding was deemed wasteful, besides being perplexing to the learners.

Then came the realization that certain things were to be gained if content made sense to the learner. When mathematics is taught according to the mathematical aim, learning becomes meaningful; when taught according to the social aim, significant. Children do not necessarily acquire meanings when they engage in social activities involving mathematics. Significant mathematical experiences need to be supplemented by meaningful mathematical experiences.

Dawson and Ruddell (1955) summarized studies, such as those by Swenson, Anderson, Howard, and Brownell and Moser, which were concerned with various aspects of meaning. They concluded that meaningful teaching generally leads to: (1) greater retention, (2) greater transfer, and (3) increased ability to solve independently. They also suggested that teachers should (1) use more materials, (2) spend more class time on development and discussion, and (3) provide short, specific practice periods.

Studies since that date have supported these findings. Greathouse (1966), for instance, found that groups taught by a group-oriented meaningful method achieved more than those taught by individually-oriented meaningful methods, but each achieved more than a group taught by a drill-computation method. Miller (1957) found that "meaning" methods were more effective for most computational areas and for understanding of the principles of mathematics. The "rule" method, however, seemed more effective for low IQ children.

To determine how the use of class time affects achievement, Shipp and Deer (1960) compared four groups, in which 75%, 60%, 40% or 25% of class time was spent on group developmental work while the remainder was spent on individual practice. Higher achievement in computation, problem solving and mathematical concepts was obtained when more than half of the time was spent on developmental activities.

In replications of this experiment, Shuster and Pigge (1965) and Zahn (1966) used other time allocations. They confirmed the finding that when the greater proportion of time is spent on developmental activities, achievement is higher.

Hopkins (1966) compared two fifth grade groups which spent 50% time on meaningful activities and 50% time (1) on practice or (2) in informal investigations of more advanced concepts. No significant differences between computation scores for the two groups were found, but significant differences on understanding measures occurred. Hopkins concluded that the amount of time spent on practice "can be reduced substantially and still retain equivalent proficiency in arithmetic computation." If activities are carefully selected, understanding can be increased.

How much relative guidance should be provided for learners?

In a pilot study with a small group of ten second-graders, Bassler (1968) provided groups with "intermediate guidance" in which pupils were led to a desired behavior through a "guided discovery" approach with directed questions by the teacher, or with "maximal guidance" in which teachers specifically told students what they were to do, followed by practice. The pattern of differences for posttest and retention achievement favored the "intermediate guidance" group. This group had higher transfer scores immediately following instruction, while the "maximal guidance" group had higher transfer scores on the retention test. Bassler, Hill, Ingle, and Sparks (1971) administered programmed mathematics units to students in grades 4, 6, and 8. No reliable differences were found between maximal and intermediate amounts of guidance in the materials.

Scandura (1964) conducted several studies concerned with "exposition" versus "discovery" in classification tasks. He found that pupils taught by "discovery" were (1) better able to handle problem tasks, (2) took longer to reach the desired level of facility, and (3) seemed more self-reliant.

Fleckman (1967) reported that classes of fifth and sixth graders taught division by a "guided-discovery" method learned more concepts than classes taught by conventional textbook procedures, while computation was equivalent. Scores of sixth graders who were taught geometry concepts with a discovery method increased over time, while scores of students taught with an expository method decreased, according to Scott (1971).

Barrish (1971) tested the hypothesis that "high-divergent" students in grades 4 through 6 would score higher on tests after instruction under an inductive-guided-discovery strategy than those encountering a deductive-reception strategy, while the opposite would be true for "low-divergent" students. Ten test problems called for "high cognitive" responses involving some degree of transfer, application in novel situations, or independent thinking. The remaining 25 problems were termed "low cognitive." They required recall and manipulations of algorithms in examples similar to those used in the lessons. It was found that levels of divergent production were not related to either initial learning or retention of the mathematical generalizations taught, regardless of the strategy presented. For the learning of low cognitive mathematical material, the deductive-reception strategy proved superior.

Olander and Robertson (1973) found that fourth-grade pupils who had seven months of expository instruction achieved significantly higher on computation tests, while those having discovery instruction scored significantly higher on the retention test on applications. Attitudes were significantly higher for the discovery group. The teachers were able to adapt to new techniques and procedures, and teacher behaviors in the discovery approach differed significantly from those in the expository approach. Robertson (1971) concluded that "it would appear that no one treatment or mode of instruction can be considered the best approach. The teacher who learns as many instructional modes as possible, identifies and diagnoses pupil needs and abilities, and uses this knowledge to individualize instruction may very well get the best results."

For kindergarteners, Anastasiow and others (1970) reported that the rule-example method was most efficient for mastery of simple classification tasks, while a guided discovery method appeared to be more efficient for mastery of more complex classification tasks. Since those with low scores on a picture vocabulary test learned best with the rule-example method, while others did well under either treatment, it might be possible to group children and teach with the method which would seem to promise greater success.

Armstrong (1968) studied the relative effects of two forms of spiral organization (area or topical) and two instructional modes of presentation (inductive or deductive). Sixth graders were assessed at each of six cognitive levels, within three areas (set theory, number theory, and geometry) and on four topics (terminology, relations, operations, and properties). The inductive mode of presentation fostered the learning of operations, while the deductive mode resulted in greater learning of mathematical properties. The interaction of curriculum organization and instructional presentation variables was not found to significantly affect mathematical learning.

Findings from Worthen's (1968) investigation appeared to support many of the claims made for discovery as opposed to expository instruction, but that conclusion had to be rescinded when Worthen and Collins (1971) re-examined the original data in terms of class means rather than individual pupil scores as units for data analysis.

There is no question that the equivocal or inconclusive nature of research on "discovery" and its role in instruction stems from the fact that the "discovery" label has been attached to methods or procedures that differ markedly in their distinguishing characteristics.

How have cognitive levels been considered?

Burron (1972) investigated the assumption that sixth-grade children of both low and high "success-potential" can profit from instruction at a variety of cognitive levels. Significant differences favored the high group at all but one cognitive level, but at least half of the low group attained "a respectable measure of success" at every cognitive level. Burron concluded that differences in the ability to function successfully at a variety of cognitive levels seemed more related to the level of complexity of a task than to cognitive level. Challenging all pupils to stretch their modes of thinking on a variety of cognitive levels seems to be a valid educational objective.

How effective are mathematics laboratories and activity-oriented approaches to instruction?

Bernard (1973) traced the historical development of the laboratory approach. Between 1966 and 1971, such an approach was used in more programs, discussed in more publications, and advocated by more educators than at any previous time. The literature was primarily devoted to philosophical discussions of the merits of the laboratory approach, and most writers spoke enthusiastically about their success with the laboratory approach.

But research has not conclusively indicated the "superiority" of the laboratory approach. Wilkinson (1971) compared the use of laboratory procedures with conventional instruction. One experimental treatment involved the use of laboratory units as a method of instruction. The laboratory units contained worksheets and manipulative materials; pupils were required to experiment with physical materials, collect data, and generalize the findings based on the data. In a second experimental treatment, cassette tapes were provided which contained a verbatim recording of all directions and questions on the laboratory worksheets. The control group was taught in a more conventional setting, using the textbook and teacher to provide the content and direction for the geometry lessons. No significant differences in achievement or attitude were found between the sixth-grade groups using conventional instruction or either of the two types of laboratory procedures.

The findings of a study by Ropes (1973) are similar to those of some other studies on the strategy. A group of sixth graders and a group of second graders each spent one 45-minute period per week for 14 weeks in a mathematics laboratory, working in small groups with a variety of manipulative materials and activity sheets. Compared with students not given a mathematics laboratory experience, these pupils had no significant change in overall attitude toward mathematics, although they did develop a greater awareness of the enjoyment to be derived from mathematics and an increased liking for it. On achievement tests, they scored as well as pupils in regular classes despite the 20% less time that laboratory students spent in regular mathematics lessons.

Vance and Kieren (1972) and Johnson (1971) also found non-significant differences pertaining to the effectiveness of laboratory and activity-oriented instruction. At best it can be said that pupils can learn from such instructional approaches (Vance and Kieren, 1971), which may be used to meet individual pupil needs (Brousseau, 1973).

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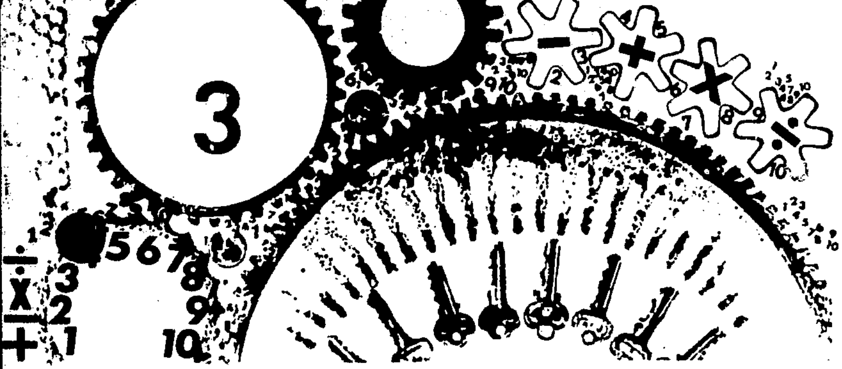

PROMOTING EFFECTIVE INSTRUCTION

Research to guide us in determining how we should teach and how children learn encompasses far more than one curriculum area. We have not attempted a broad survey of learning theory, but rather have selected that research which (1) is based on some phase of the elementary school mathematics curriculum and (2) provides specific suggestions to teachers of elementary school mathematics. Many of these findings have been substantiated not only in research across many phases of the curriculum, but also by practical use.

What factors associated with the learner influence achievement in mathematics?

Learning is not an "all or none" process. We generally acquire understanding progressively, in steps or stages. Perreault (1957) reported that the child's ability to count, to group, and to perceive the number of objects without counting appeared to reflect such developmental stages.

Brownell (1944) supplied interview data to support the conception of learning as a series of progressive reorganizations of processes and procedures.

Is there research to guide us in motivating learning?

Exactly what "motivation" is has been the subject of some debate. Let us assume that it includes what the teacher does to increase pupils' interest and achievement in learning mathematics. There are numerous reports about various games and materials which teachers have used successfully in increasing interest. The effect of teacher enthusiasm cannot be taken lightly.

What the teacher says--and how he says it--has been found to be particularly important. Not surprisingly, praise has been found to be a highly effective way to motivate.

Hollander (1968) recently studied the effect of different types of incentive on inner-city fifth and sixth graders following a test on addition and subtraction problems. He found that pupils worked faster when told they could earn a candy bar if they improved their own scores on a second test, and with greater accuracy when told they had performed exceptionally well. Those reproved by being told their scores were very low attempted fewer items and made more errors than were made under any of the other conditions.

The combined influence of teacher facilitative behavior and the effect of interpersonal compatibility between teacher and student were studied by Schultz (1972). Each of 20 tutors was assigned one student who appeared most compatible and one student least compatible to him, determined by responses to a test on interpersonal relationships. Student increases in achievement and in self-concept of arithmetic ability after nine tutoring sessions did not appear to be related to tutor predisposition for facilitative behavior and/or degree of interpersonal compatibility between tutor and student. However, when they were compatible, students rated their relationships with tutors as more facilitative.

Frumess (1973) reported that groups in which elementary-school pupils knew teacher-aims, or set their own aims, and charted their own scores made significantly greater gains on timed mathematics tests than did groups not knowing aims or progress. Having advanced organizers for transformational geometry materials helped fourth-grade students to score significantly higher than students having post-organizers or no organizers, according to Johnson (1973). Students given several models or applications achieved higher than those given only one model or application.

Rea and French (1972) reported on a small-scale research study with a class of sixth graders. One group used mental computation exercises; the other was given enrichment activities using the same content. For 24 days, both groups received their regular mathematics instruction plus 15 minutes daily of the special activities.

In both groups were individuals whose scores increased only slightly, and scores even decreased for a few. However, in both groups, the majority of the students gained rather dramatically; the average gain for the enrichment group on the achievement test was one full year, and for the mental computation group was eight months. There can be little doubt that the results were influenced by factors such as the halo effect, which often accompanies enthusiastic experimentation.

But why not capitalize on this in the classroom? Children do like variety--and children enjoy experimenting and being part of an experiment. Research is a way of motivating children.

Is there research to guide us in facilitating transfer?

Transfer infers that something learned from one experience can be applied to another experience. For instance, Olander (1931) found that pupils who studied 110 addition and subtraction combinations could give correct answers to the 90 untaught combinations. What facilitated this transfer best was instruction in generalizing, in teaching children to see patterns. Transfer increases as the similarity of problems and experiences increases. Much research has shown that meaningful instruction aids in transfer of learning. Recent studies also show that transfer is facilitated by discovery-oriented instruction.

In general, the older the child and the higher his ability level, the better he can transfer. However, Klausmeier and Check (1962) found that children of various IQ levels transfer problem solving skills to new situations when the children were given work at their own level of difficulty.

In one set of studies, Sawada (1972) studied a strategy for organizing a curriculum with explicit provision for transfer from lower- to higher-order objectives, with a system characterized by composition and reversibility. Eleven instructional sequences were presented via CAI. It was found that performance on an objective had little relationship with performance on the inverse objective. Pupils on their own apparently did not pick up the strategy of forming composites. In other words, pupils did not seem aware of reversibility inherent in the materials, nor of composition objectives. The need for explicit teaching, rather than expecting transfer to occur as a by-product, is indicated.

In most studies is the implication that transfer is facilitated when teachers plan and teach for transfer--and we must teach children how to transfer. Kolb (1967), for instance, carefully planned to have children transfer mathematical instruction to quantitative science behaviors, and achieved this transfer.

Is there research to guide us in facilitating retention?

Obviously, we want children to retain what we are teaching and they are learning. There is much research to show that when something has meaning to learners and is understood by them, they will be more likely to remember. Furthermore, Shuster and Pigge (1965) state that retention is better when at least 50% of class time is spent on meaningful, developmental activities. Klausmeier and Check (1962) reported that when a pupil solved problems at his own level of difficulty, retention was good regardless of IQ level.

Intensive, specific review will facilitate retention, according to Burns (1960). He prepared lessons which included not only practice exercises, but also review study questions which directed pupils' attention to relevant things to consider. Meddleton (1956) pointed out that such review should be systematic.

No significant differences in effectiveness for intermediate-grade pupils were found by Thomas (1974) when the mode of review was similar to or differed from the mode of initial instruction (semantic/didactic vs. developmental/figural). Pence (1974) also reported no significant differences in effectiveness for sixth-grade pupils when review was administered individually or in small groups.

Cummins (1975) found no significant differences in gain scores between groups given 15 minutes of review and 10 minutes of practice per day for eight weeks, and groups having the usual program in grade 6. Crawford (1970) reported on a study using a CAI drill-and-practice program in grade 7. Students who had 3 to 15 minutes of extra computational practice per day gained significantly, but scores were not significantly different from those of a group with no extra practice. In a review of seven studies which used CAI drill-and-practice programs, Vinsonhalter and Boss (1972) concluded that a substantial advantage for using CAI to augment traditional classroom instruction was indicated.

Many teachers have noted that children fail to retain well over the summer vacation. The amount of loss varies with the child's ability and age, but how long before the vacation material was presented is important. Practice during the summer and review concentrated on materials presented in the spring have been shown to be especially helpful in reducing retention loss.

Grenier (1975) investigated whether seventh-grade pupils showed a significant loss in arithmetic over the summer, and tried to determine the length of time required to return to the pre-summer level of achievement when mean losses did occur. She found that the students had a significant loss on the computation subtest; some gain was found after two weeks in school. On the concept subtest, the application subtest, and on a NLSMA test, gains were found between spring and fall testings.

Is there research to guide us in reinforcing learning?

One of the best ways of reinforcing learning is to give the child "knowledge of results"--by providing scores or by providing correct answers. Paige (1966) found that immediate reinforcement after a testing situation resulted in significantly higher achievement scores later. Having the student respond and then giving confirmation is more effective than prompting him with the correct answer before giving him a chance to respond (McNeil, 1965).

Kapos, Mech and Fox (1957) studied the effect of various amounts and patterns of reinforcement with third and fourth graders at several IQ levels. Different patterns of reinforcement produced differences in achievement. However, there was no clear indication of which quantity or pattern of reinforcement was best, nor was any relationship with IQ found.

The use of token reinforcements--plastic tokens which may be traded for candy, toys, or other desired items--has been reported to result in achievement gains in other curricular areas. Hillman (1970) reported that fifth graders given per-item knowledge of results, either with or without candy reinforcement, scored significantly higher in achievement with decimals than pupils given knowledge of results 24 hours later. He suggested that low achievers may profit more than high achievers. Heitzman (1970) studied pupils aged 6 to 9 in a summer arithmetic program. Those who were rewarded by tokens achieved significantly higher scores on a skills test than those who did not receive tokens (and also who may not have received knowledge of results). Immediate knowledge of results, rather than token reinforcement, may be the determining factor.

Masek (1970) reported significant increases in arithmetic performance and level of task orientation of underachieving first and second graders during periods when teachers emphasized reinforcement such as verbal praise, physical contact, and facial expression.

What type of homework is helpful?

Although some studies have reported an achievement gain when homework was used (Maertens and Johnston, 1972; Doane, 1973), evidence pertaining to the relative effectiveness of various types of homework is still rather nebulous. Grant (1971) found no significant differences in achievement between fifth-grade groups given differentiated homework on two levels of difficulty, textbook assignments, or no homework. Gray and Allison (1971) also reported that no significant differences were found when students were given three or no homework assignments per week in grade 6. Laing (1971) found no significant difference in achievement or retention between eighth-grade groups for which practice on a topic was massed in one assignment and those for which practice was distributed over several homework assignments.

Peterson (1970) found that an eighth-grade group receiving exploratory homework assigned for three days prior to the teaching of a topic, and a group receiving mathematical puzzles unrelated to the mathematics taught, each achieved better than a group receiving no homework. Those who completed at least 50% of the assignments in the first group retained and transferred more than the comparable portion of the puzzle group.

Is it effective to teach mental computation skills?

Relatively little recent research has been directed toward this increasingly important ability. Austin (1970) found that eighth-grade pupils who spent one period a week on mental computation scored significantly higher on standardized tests than students not given such instruction. Grumbling (1971) reported that fourth-grade pupils who were instructed in mental computation made a significant increase in arithmetic achievement and were better able to solve problems mentally than were pupils for whom mental computation was not stressed. Schall (1973) exposed fifth graders to short, frequent periods of oral practice administered in various modes. He found that the exercises resulted in increased ability to compute mentally and in a gain in attitude, although no significant differences were found between groups who used televised lessons, lessons on audio-tape, or programmed materials.

Rea and French (1972) noted that the majority of sixth-grade students who were given either mental computation exercises or enrichment activities gained "dramatically" in achievement scores.

How effective are pupil-tutors?

Evidence from several studies seems to indicate that use of pupil-tutors may be a valuable as well as inexpensive way of providing some remedial help for other pupils. Ackerman (1970) found that pairing low-achieving third graders with either high- or low-achieving sixth-grade tutors resulted in computation scores which were significantly higher than the scores of those third graders who only talked with sixth graders on non-arithmetic activities or who had no tutor-contact.

Burrow (1970) had high-achieving pupils in grades 6, 7, and 8 devise lesson plans organized according to diagnostic test results of low-achieving pupils in grades 3, 4, and 5. Tutored pupils achieved higher gain scores on computational skills than did untutored pupils, regardless of the achievement level of the tutors. Tutors themselves did not achieve significantly more than others who did not tutor.

Guarnaccia (1973) found that fourth graders having peer-tutors for four months gained significantly more in computation than did fourth graders not having tutors. No differences were found on tests of concepts or applications, nor were any differences found in grade 3. Tutors learned at least as well as tutees. On the other hand, Carlson (1973) found that tutoring or being tutored for six weeks did not increase the self-concept or achievement of six classes of fourth and sixth graders more than did working on individualized worksheets.

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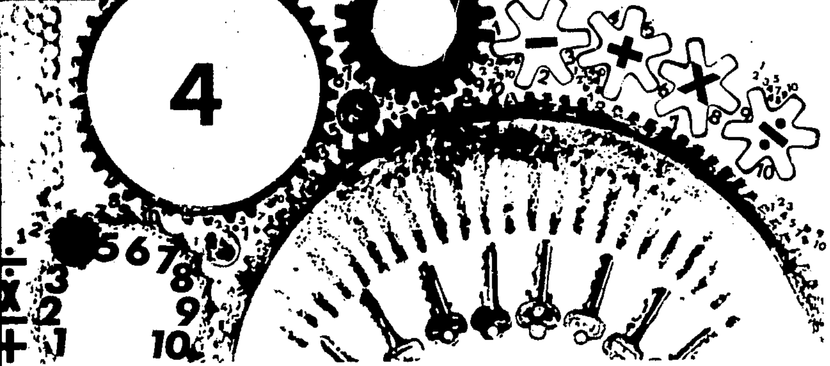

DIFFERENTIATING INSTRUCTION

By differentiating instruction we mean attempts to organize mathematics programs and instruction in relation to the unique needs and abilities of individual children. This includes, but is not restricted to, plans in which individual pupils work more or less completely independently. It seems apparent that there is no one plan which is best. Provision for differentiating is conditioned in part by school organization, in part by the particular teacher and pupils. The teacher must identify various factors related to pupils' achievement and interest in mathematics, and then decide on appropriate variations in content, materials, method, and time.

What factors are important to consider when differentiating instruction?

Wrigley (1958) was among those who studied the structure of mathematical ability. He concluded that high intelligence is the most important single factor for success in mathematics. He isolated a mathematical group factor which linked the different branches of mathematics, as well as specific verbal, numerical, and spatial factors which affect achievement. When the influence of intelligence was eliminated, verbal ability had little connection with mathematical ability.

Much additional research has shown that age and intelligence are highly related to ability to learn various specific mathematical ideas. Westbrook (1966), for instance, noted that the intellectual factors of reasoning and verbal meaning were related to achievement in mathematics in grades 4, 5, and 6. Meconi (1967) found that pupils with high ability were able to learn under any method that he investigated. Large variations in generalization ability, depending on the mathematical concept, intelligence level, and the visual pattern presented, were found on tests of varied mathematical content (Ebert, 1946).

Based on a review of 38 studies, Fennema (1974) concluded that pupils' sex was not a factor that influenced mathematics achievement during the early elementary years. In the upper grades, any observed achievement differences were apt to be in favor of boys on higher-level cognitive tasks and girls on lower-level ones. There appears to be no sound basis, however, for suggesting that mathematics instruction should be different for boys and girls.

It has been suggested that the most feasible way of coping with individual differences might be to alter instructional methods to fit the aptitude pattern of the learner. To ascertain whether students high in a given ability achieve better under one method of instruction than under another, King, Roberts, and Kropp (1969) tested 426 fifth and sixth graders after instruction with one of four sets of materials on elementary set concepts. There were significant interactions on inductive-deductive comparisons: it appeared that some students were identified who achieved better when taught inductively, while others achieved more when taught deductively.

At the eighth-grade level Gawronski (1972) found no significant achievement differences for inductive or deductive instructional approaches in relation to students having inductive or deductive learning styles. Branch (1974) found that for sixth-graders' work with integers, an inductive approach with a number line was more effective than a deductive approach, and that pupils with low-analytic cognitive styles were able to transfer better when taught inductively.

Cathcart and Liedtke (1969) suggested that pupils in grades 2 and 3 who were identified as having a "reflective" learning style took longer to consider their responses and achieved better than pupils with an "impulsive" style. Certainly learning style needs to be considered as we plan lessons and give directions.

Capps (1962) tentatively concluded from a comparison of "superior achievers" and "underachievers" that retardation in mathematics might be related to personal adjustment: perhaps emotional difficulties tend to foster difficulties, and vice versa. Other researchers have also suggested that personality factors may be more important than intelligence in promoting retardation.

There is evidence from research that children from low socioeconomic groups have less mathematical background when they enter school than do children from middle socioeconomic groups. Passy (1964) reported significant differences among third graders, with achievement level increasing as socioeconomic level of the parents increased. Unkel (1966) found that socioeconomic status had a significant effect on achievement in mathematics at all intelligence levels in grades 1 through 9.

Other studies have considered the effect of various personality factors; for instance, Poggio (1973) reported that grouping on the basis of personality characteristics appeared feasible for the sixth graders he studied, but the pertinent factors differed for boys and girls. Lawton (1971) reported that peer acceptance and acceptability were each significantly related to mathematics achievement in grades 5, 7, 9, and 11.

What types of differentiation are effective?

Intraclass grouping to facilitate individualization of reading instruction is a common practice in the elementary school. Evidence on the effectiveness of grouping for mathematics instruction is conflicting. Part of the conflict is due to grouping on different bases: ability and achievement.

When grouping is based on ability, some studies have shown that homogeneous grouping is especially effective for those with high IQs (e.g., Provus, 1960; Balow and Ruddell, 1963). Balow and Ruddell, however, found "decreased-range" grouping was more effective than either heterogeneous or homogeneous grouping for most pupils, while Savard (1960) found that such grouping tended to be effective for lower ability pupils and of less advantage for upper ability pupils. Balow and Curtin (1966) reported that grouping by ability did not significantly reduce the range of achievement.

Wallen and Vowles (1960) had each of four sixth-grade teachers use both ability and non-grouping methods for one year. No significant difference was found, though a significant interaction was found between teachers and the methods used. This was not tested in most other studies, and may be the most significant reason for differences in findings.

When grouping is based on achievement, Koontz (1961) found that fourth graders who were heterogeneously grouped achieved significantly higher scores than those homogeneously grouped. Eddleman (1971) found no significant differences in achievement between fifth-grade pupils grouped homogeneously and those grouped heterogeneously. Dewar (1963) concluded that providing three intraclass groups benefited high- and low-achieving groups more than did total-class instruction.

Holmes and Harvey (1956) found that there were no significant differences in achievement, attitude, or social structure within the classroom whether pupils were grouped permanently or flexibly (with the topic introduced to all, followed by grouping for further work).

Davis and Tracy (1963) reported that pupils in grades 4, 5, and 6 in self-contained classes scored significantly higher on factors such as verbal and quantitative ability, self-concept, anxiety, and attitude, than did those grouped by both ability and achievement across classrooms at each grade level.

Various other types of procedures to differentiate instruction have also been studied. Broussard (1971) found that fourth-grade students in inner-city schools given individually prescribed work through independent study, small-group discussion, large-group activities, and teacher-led discussions achieved significantly higher in skills and concepts than those taught by a traditional textbook, class-group method.

Bierden (1970) found that, for seventh graders, an intraclass grouping plan using group instruction followed by independent work for individualized objectives resulted in significant gains in computational skills, concept knowledge, and attitude, with a reduction in anxiety.

Lindgren (1968) reported no significant differences between team learning and learning through conventional teaching in grades 4 and 5, while Wolff (1969) found no significant differences in achievement among third-year pupils in individualized graded or nongraded classrooms.

Snyder (1967) found no significant differences in achievement between seventh and eighth graders who were allowed to select the mathematical topics they would study and those who could choose from a three-level assignment option. Both groups gained more on reasoning tests and less on skill tests than a third group receiving regular instruction.

McHugh (1959) reported on a two-year differentiated instruction program in grades 4, 5, and 6, in which extensive in-service help was provided to develop a program in which pupils would progress at their own rates, become self-directive and self-correcting, and give mutual help. Significant gains in problem solving were found in grades 5 and 6, and in computational skills in grade 5. The program produced gains "greater than normally expected for the IQ level" in all grades.

In a study of group size effects with 249 pupils in grade 4, Moody, Bausell, and Jenkins (1973) found that pupils in groups of 1, 2, or 5 displayed significantly greater attainment on a unit on exponents than did pupils in a class of 23. One-to-one instruction was significantly better than one-to-five instruction.

In general, acceleration has been reported to be effective for some children. Klausmeier (1963) reported no unfavorable academic, social, emotional or physical correlates of acceleration in fifth graders who had been accelerated from second to fourth grade. Ivey (1965) found that fifth graders who were given an accelerated and enriched program in grade 4 gained significantly more than those receiving regular mathematics instruction.

Jacobs, Berry, and Leinwohl (1965) reported that seventh graders who were in an accelerated program for either three or four years did significantly better on concepts tests than those who had been accelerated for only one year. There were no significant differences on problem solving tests.

How does diagnosis aid in differentiating instruction?

The purpose of diagnosis is to identify strengths as well as weaknesses, and, in the case of weakness, to identify the cause and provide appropriate remediation. As part of the process, there have been many studies which ascertained the errors pupils make. For instance, Cox (1975) reported on the systematic errors which children in grades 2 through 6 made on examples with each of the four operations with whole numbers. Roberts (1968) suggested that teachers must carefully analyze the child's method and give specific remedial help.

Most diagnostic tests have been concerned with skill development, but recently the focus has shifted to concept development. Paper-and-pencil tests such as those by Flournoy (1968) and Ashlock and Welch (1966) are not essentially diagnostic, but have implications for those attempting to diagnose pupil understanding. Harvey (1953) reported on diagnostic tests for each operation, and suggested the use of a testing-reteaching-retesting strategy to decrease errors.

Bernstein (1959), in a review of the research on remedial teaching of mathematics, noted that every cited experiment used lesson plans based on individual diagnosis as a basic teaching approach. Gray (1966), in reporting on the development of an inventory on multiplication, called attention to the individual-interview technique pioneered by Brownell: "facing a child with a problem, letting him find a solution, then challenging him to elicit his highest level of understanding."

The technique of skillful questioning and observing of pupils as they work was employed by Buswell and John (1926) in their diagnostic study on the four operations. They stress the need to analyze how the child works, which should lead to devising ways of teaching him better methods.

Scott (1970) matched 25 pairs of low-achieving seventh graders on computation, concepts, and applications. One-half then used programmed materials appropriate to meet diagnosed needs; they made significantly greater gain scores in computation than did students in the regular classroom, although differences on concepts and applications were not significant. Fennell (1973) compared small-group instruction with an approach using diagnostic, prescriptive, goal-referenced strategies for individual students. No significant differences were found between sixth-grade groups on achievement or attitude measures; the diagnostic strategy required less time for mastery, however.

Dunlap (1971) found no significant differences in achievement on the standardized test, between students in grade 4 given diagnostic activities or textbook materials. The activities group scored higher on the concept section of the experimenter's test, while the textbook group scored higher on computation.

What is the expected outcome of using the IPI or PLAN program?

Two programs make special provisions for instructional organization. In neither case is there evidence to show that the special provisions make for improved achievement. IPI (Individually Prescribed Instruction) and PLAN (Program for Learning in Accordance with Needs) both (1) involve the use of behavioral objectives, (2) identify activities and materials to meet those objectives, and (3) provide sets of test items for those objectives; in short, they provide a systems approach.

In a Progress Report on IPI (1969), it was concluded that "on standard achievement tests IPI pupils do as well as non-IPI pupils." No claim is made for higher achievement on the part of IPI pupils. For instance, at the third, fourth, and fifth grade levels Fisher (1968) found no significant achievement differences under three instructional treatments: (1) IPI, (2) "programmed learning instruction," and (3) "standard classroom instruction." Thomas (1972) found that IPI did not produce significant achievement gains over the conventional program, in either grade 5 or 6. In grade 6, attitude was more favorable toward the IPI program. Shumaker (1973) found no significant differences in mathematics achievement, study habits, or study attitudes between seventh-grade students who had had an IPI or a non-IPI program in elementary school.

In some studies, advantages were reported for conventional instruction. Fielder (1972) found that the non-IPI group generally achieved better than the IPI group for students in grades 3 through 6, and Verheul (1972) reported that pupils in sixth grade having conventional textbook instruction achieved higher than those using IPI.

A few investigations reported higher achievement, as well as a more positive attitude, for students using IPI (e.g., Clough, 1971).

Fewer studies have focused on PLAN. In one, Abate (1973) reported that in grades 1 through 3, but not grade 4, pupils using PLAN achieved as well on a mathematics test as students in a non-PLAN school. Ferney (1970) reported that fifth graders not using PLAN achieved significantly higher on arithmetic reasoning than the group using PLAN. Girls using PLAN achieved higher scores than did boys, and thus PLAN may be more appropriate to the learning styles of girls.

Schoen (1976) reviewed 36 studies in which elementary-school children in a self-paced mathematics program were compared to traditionally taught children. Twenty-one of the studies involved a teacher- or researcher-designed program, 12 tested IPI, and three included PLAN. He found that only five of the 18 studies for kindergarten through grade 4 favored any self-paced program, five favored the traditional program, while no significant differences were found in eight. In grades 5 through 8, three studies favored self-pacing, 12 favored the traditional program, and no significant differences were found in three. He concluded that, considering the additional cost and time needed when using a self-paced program, as well as the achievement data, "a teacher or principal should not feel he or she is necessarily failing to allow for individual differences [by deciding] not to implement a self-paced instructional program."

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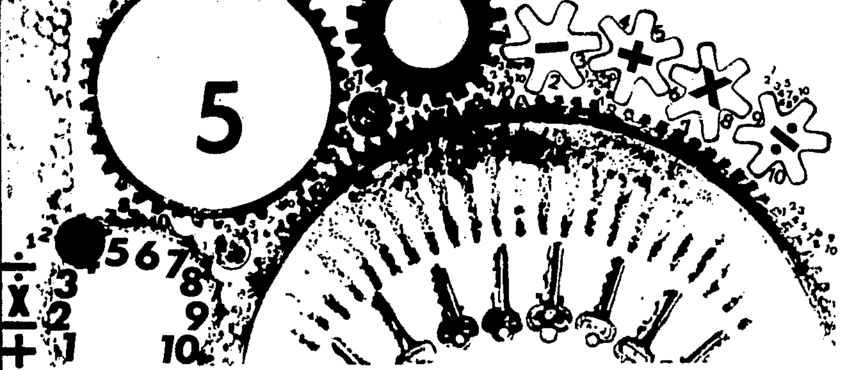

INSTRUCTIONAL MATERIALS AND MEDIA

What may we learn from analyses of mathematics textbooks?

Elementary mathematics textbooks have been analyzed for different purposes and from different bases. One of the most comprehensive analyses is that by Smith and Eaton (1942-43), which includes approximately 200 books used in this country between 1790 and 1940. Their purpose was to study "the basic characteristics and trends of textbooks of the past." Analysis was in terms of the social and economic life of the period, relative emphasis on various aspects of content, the psychological approach, purpose, and scope.

Dooley (1960) studied 153 series of elementary school mathematics textbooks published in the U.S. between 1900 and 1957, attempting to ascertain the effect of research on the content and methods suggested in textbooks. She found that when recommendations were "clear, concise and exact," they were incorporated in into many textbooks within five years.

Ten textbook series and accompanying workbooks and teacher's manuals were analyzed by Burns (1960). He presented specific information on the similar content included at each grade level, physical features, and points of emphasis. Folsom (1960) concentrated on manuals, using observations of classroom practice to determine how consistently teachers used suggestions about procedures, enrichment activities, and materials.

Sixteen textbooks for teacher education and texts for children were analyzed by Hicks (1968) to ascertain the similarities and differences in inclusion of content topics. Marksberry, McCarter, and Noyce (1969) checked cognitive objectives in textbooks with those from research committees and with questions and activities suggested in teacher's manuals.

McLaughlin (1970) compared two seventh-grade textbooks and measured the achievement of students on knowledge and understanding of those elements of mathematics which have been included in the curriculum as a result of experimental programs. Groups scored significantly higher using the textbook which (1) included more explanation and discussion of subject matter, (2) made greater use of symbolic notation, and (3) provided more examples with the explanations.

Text materials and test items often are analyzed in relation to a taxonomy of instructional objectives. Passi (1970); Callahan and Passi (1972) analyzed two recent textbook series and one older series. In all cases, low-level cognitive activities were more frequent than were high-level cognitive activities. The Manipulating (of symbols) level dominated the activities; the frequency of Translating, Analyzing, Synthesizing, and Evaluating levels was low.

Dahle (1970) used a grid of 120 objectives which ranged across five taxonomic levels. She found that a selected textbook series corresponded more closely to the distribution of objectives than did two standardized tests.

What do studies on the vocabulary of textbooks show?

Willmon (1971) found a total of 473 technical mathematics words in 24 textbooks for grades 1-3, with frequency of use ranging from 1 to 5,995. Seventeen words were repeated more than 1,000 times, but most were used less than 25 times. Stevenson (1971) reported that, of 396 technical and semi-technical words he found in third-grade mathematics textbooks and first- and second-grade readers, only 51 were used in both reading and mathematics books. However, 161 words were common to all four mathematics textbooks. Data from a study by Browning (1971) were less encouraging. She found a total of 743 mathematical terms in 15 textbooks used in grades 4, 5, and 6; only 10 words were common to all textbooks.

These studies indicate that every teacher of mathematics must consider the reading problem which a child may face. Smith (1971) added further evidence on this point. He found that the composite readability scores for sixth-grade textbooks ranged from 5.0 to 5.8; however, analysis of selections indicated a range of below grade 4 to grade 8. Tests ranged only from below grade 4 to grade 6 in reading level.

In a different type of vocabulary study, Olander and Ehmer (1971) administered a test from 1930 to pupils in 1968. On the test, 1968 pupils achieved higher scores on 74 of 100 items in grade 4, 59 items in grade 5, and only 48 items in grade 6 than did pupils who had taken the test in 1930. On a test of contemporary terms, mean scores were 49 for grade 4, 58 for grade 5, and 64 for grade 6 on the 100 items.

How useful are mathematics tests?

After analyzing a standardized mathematics test, Gridley (1971) reported that mathematics achievement in grades 2-5 as measured by that test appeared to consist of several empirically defined clusters of items. The clusters varied from grade to grade, and subtest headings did not represent distinct clusters. The meaningfulness of the total score, as well as the subtest scores, was questioned, since several skills or abilities were being measured.

Hoepfner (1974) found that norm-referenced tests were available for six of 11 educational objectives specific to mathematics. For "comprehension of numbers and sets," nine tests were available; for "operations with integers," 27; for four other objectives, one or two; for five objectives, none.

Does programmed instruction facilitate achievement?

Programmed instruction materials allow each pupil to progress at his own rate. Some studies ascertained the feasibility of using programmed instruction to teach specific content. For instance, Kalin (1962) used a two-week unit on equations and inequalities; Fincher and Fillmer (1965) used programmed materials on addition and subtraction with fractions. The use of programmed materials may result in a decrease in the time which most students must spend on a topic. It was also evident that "programmed materials are most effective when used to supplement the classroom teacher" (Banghart and others, 1963). Goebel (1966) noted a difference in the interaction pattern of teachers and pupils in a fourth-grade classroom. When arithmetic programmed materials were used, teachers devoted 68% of their time to work with individuals, while teachers using a conventional approach devoted only 3% of their time to individuals.

What types of materials have been found to be effective?

Harshman, Wells, and Payne (1962) reported on a study of first graders who were taught for one year by programs with varying content based on either (1) a collection of inexpensive, commercial materials, (2) a commercial set of expensive materials, or (3) materials provided by the teacher. Teachers in the first two instances received in-service training. When significant differences in achievement were observed, they were always in favor of the third program. It was concluded that high expenditure for manipulative materials does not seem justified.

Much research has been focused on the use of the Cuisenaire materials and program, in attempts to answer the question, "How effective is it?" Crowder (1966) reported that (1) a group of first graders using the Cuisenaire program learned more conventional subject matter and more mathematical concepts and skills than pupils taught by a conventional program; (2) average and above average pupils profited most from the Cuisenaire program; and (3) sex was not a significant factor in relation to achievement, while socioeconomic status was.

Working with first and second graders, Hollis (1965) compared the use of a Cuisenaire program with a conventional approach. He concluded that (1) children learned traditional subject matter with the Cuisenaire program as well as they did with the conventional method, and (2) pupils taught by the Cuisenaire program acquired additional concepts and skills beyond the ones taught in the conventional program.

Brownell (1968) used tests and extensive interviews in an analysis of the effect on underlying thought processes of three mathematics programs, with British children who had studied those programs for three years. He concluded that (1) in Scotland, the Cuisenaire program was in general much more effective than the conventional program in developing meaningful mathematical abstractions; and (2) in England, the conventional program had the highest overall ranking for effectiveness in promoting conceptual maturity, with the Dienes and the Cuisenaire programs ranked about equal to each other. Brownell inferred that the quality of teaching was decisive in determining the relative effectiveness of the programs.

Other studies have been concerned with the effect of use of the Cuisenaire program on a particular topic, for shorter periods of time. Lucow (1964) and Haynes (1964) studied use of the program to teach multiplication and division concepts for six weeks in third grade. Lucow attempted to control the effect of prior work in grades 1 and 2. He concluded that the Cuisenaire method was as effective as regular instruction in general, and seemed to operate better in a rural setting, especially with high and middle IQ levels, than in an urban setting. Haynes used pupils who were unfamiliar with the materials; no significant differences in achievement were found between pupils who used the Cuisenaire program and those who did not.

Prior background, length of time, and the specific topic may account for differences in the success of the Cuisenaire program. It has been suggested that it might be more effective in grades 1 and 2, with its effectiveness dissipating during third grade. No body of reported research is available about its effects beyond the third grade level.

What is the role of manipulative materials?

The role of manipulative materials in the learning of mathematics is being questioned by teachers at all levels today. Generally, we are bound philosophically to their use: but research increasingly indicates that we need to analyze <u>when</u> they are used, <u>with whom</u> they are used, <u>what types</u> should be used, and <u>how</u> they are used.

Throughout these bulletins, much evidence has been cited which indicates that the use of concrete materials appeared to be essential in providing a firm foundation for developing mathematical ideas, concepts, and skills. In many of these studies, it is concluded that the child must actively and individually handle the materials.

There is, however, an increasing body of evidence which indicates that perhaps having the children themselves manipulate materials may not be necessary for all topics--or for all children. For instance, <u>Bisio</u> (1971), conducting a study with 29 classes of fifth graders, compared three methods of teaching addition and subtraction of like fractions. In one treatment neither the teacher nor the students used manipulative materials. In the second treatment, the teacher used the manipulative materials as a demonstration for the students. And in the third treatment, both teacher and students manipulated materials. Children taught with materials, both <u>using</u> them <u>and</u> passively watching them being used, scored higher than those not using materials.

Knaupp (1971) also found that both teacher-demonstration and student-activity modes, using blocks and sticks in presenting addition and subtraction algorithms and ideas of base and place value to four second grade classes, resulted in significant gains in achievement.

Jamison (1964) compared instruction in counting in other numeration systems using (1) a large variable-base abacus, (2) a large abacus plus a small abacus for each pupil, and (3) only the chalkboard. There were no significant differences between mean gains.

Toney (1968) also found that a fourth grade group using individually manipulated materials for half a year was not significantly different in achievement from one seeing only teacher demonstrations. And Trueblood (1968) reported that fourth-grade pupils who watched the teacher manipulate materials scored higher than pupils who manipulated materials themselves during a unit on exponents and non-decimal bases.

Gilbert (1975) reported differing results from two schools. In one school, students manipulating materials individually scored significantly higher than students watching the teacher manipulate materials or handling materials in groups of four children. In a second school, no significant differences were found.

Many of the recent studies provide at least partial support for the long-held belief that the use of materials should proceed in stages--from concrete to semiconcrete (that is, pictorial) to abstract (or symbolic) (e.g., Olley, 1974). Investigations by Johnson (1971) and Portis (1973) in grades 4 through 6, Carmody (1971) in grade 6, and Punn (1974) in grade 3 indicate that use of either or both physical and pictorial aids result in significantly higher achievement than when only symbolic aids are used. Yet Fennema (1970, 1972a) found no significant difference in overall learning of a principle when second graders' learning was facilitated by either a meaningful concrete or a meaningful symbolic model, but use of the symbolic model resulted in significantly better transfer. It may be that the meaningfulness of the symbolic model was an important element. Fennema (1972b) concluded that research she reviewed appeared to indicate that the ratio of concrete-to-symbolic models used to convey mathematical ideas should reflect the developmental level of the learner. It might be that alternative models should be available so the learner can select the one most meaningful for him.

Another point of concern is whether the number of different materials--or "embodiments"--affect achievement. Wheeler (1972) tested the performance of 144 second graders on the use of the abacus, bundling sticks, the place value chart, and multi-base blocks, and then gave them two-digit and multi-digit addition and subtraction examples in written form. There were no significant differences between the means of the children at any of three levels of abstraction for regrouping, in solving two-digit examples in the symbolic mode. However, significant correlations were found between the number of embodiments children were able to regroup for two-digit examples and achievement on the multi-digit tests. It was concluded that children proficient in using three or more concrete embodiments had a significantly higher level of understanding of the regrouping concept than children without this proficiency with the concrete aids.

On the other hand, Gau (1973) found that fifth and sixth graders working with fractions with one, two, or three embodiments could operate with the symbolic embodiment essentially the same. Similarly, Beardslee (1973) found that one, two, or three embodiments had essentially the same effect on pupils' ability to generalize the concept. It is apparent that there is a need for more research on the question of how many varied materials are most appropriate--and for which children.

Sole (1957) concluded that (1) use of a variety of materials did not produce better results than use of only one material, and (2) the learning of mathematics depends more on the teacher than on the materials used.

When materials should be used is also of concern. For instance, Weber (1970) reported no significant differences between groups of first graders who used manipulative materials for follow-up activities and those who used paper-and-pencil activities at that point. Perhaps materials provide a foundation, but at some point they are no longer needed by children.

How are computers and calculators aiding in the instructional process?

Computer-assisted instruction is presently being used in some elementary school mathematics classes. Suppes (1969) has reported extensively on the use of both tutorial and drill-and-practice programs. He found that the drill-and-practice materials result in at least equivalent achievement in less time than it would take the classroom teacher using only conventional methods. The computer also readily collects data on how children are responding, thus facilitating diagnosis of their difficulties as well as increasing our knowledge of how they learn.

As the prices of hand-held calculators have decreased, they have appeared in many classrooms. While many teachers and researchers are exploring their use, little research has thus far been published. What has appeared indicates that it is feasible to use them and that certain aspects of mathematical achievement are facilitated by their use (e.g., Spencer, 1975), but precisely when and how they can be most effectively used, and what the long-range impact of their use is, have yet to be determined.

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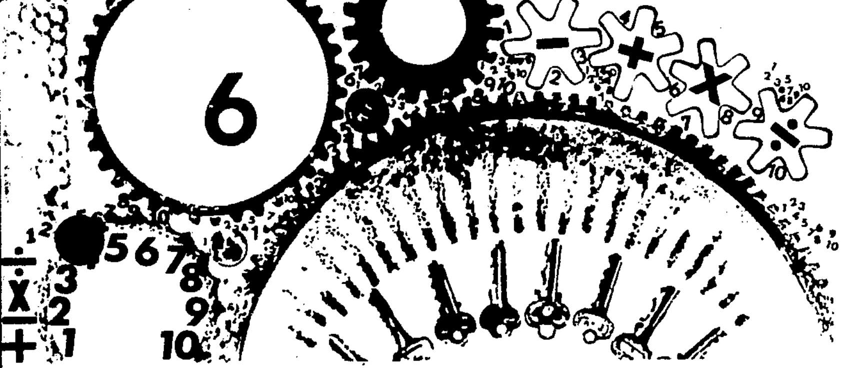

ADDITION AND SUBTRACTION WITH WHOLE NUMBERS

What foundation for addition and subtraction do children have upon entering school?

As teachers are well aware, some background for the development of skills in addition and subtraction is formed before children begin systematic study of these operations. The ability to count is of particular importance: children often use counting as a primary means of ascertaining and verifying addition and subtraction facts. The ability to recognize the number of a set without counting is also helpful.

Through the years, many investigations have been conducted to ascertain the counting skills and other mathematical abilities possessed by the pre-first-grade child (e.g., Douglass, 1925; Buckingham and MacLatchy, 1930; Woody, 1930; McLaughlin, 1935; Mott, 1945; Priore, 1957; McDowell, 1962; Holmes, 1963; Brace and Nelson, 1965; Williams, 1965; Schwartz, 1969; Rea and Reys, 1971). In some studies, it was found that many children could solve simple addition and subtraction examples in an oral or problem context. Across the studies, wide differences were found in children's ability to count. While some children could count to 100 or beyond, a few had difficulty counting to 10. Thus, the classroom teacher cannot assume that all children have the counting and other skills which appear necessary to work with addition and subtraction. Teachers must assess the attainment of the individual children in their classes.

Whether rote counting or rational counting should be taught first is a recurrent question, but has not been explicitly answered by research. Generally, the preschool child learns to say the number names and then begins to say them in order before associating the names with sets of objects.

Is conservation of number a necessary condition for understanding addition and subtraction?

It is not at all clear from research whether children's success in dealing with simple addition and subtraction situations is in fact dependent upon certain hypothesized prerequisites such as the ability to conserve number or numerousness (that is, to be able to specify that "if two sets are matched one-to-one, the number of objects in each is the same, regardless of the arrangement or rearrangement of the two sets"). For example, Steffe (1968) reported a direct relationship between children's level of number conservation and their success in solving addition problems, and LeBlanc (1968) reported a similar conclusion with respect to conservation level and subtraction problems. However, Mpiangu and Gentile (1975) concluded that number conservation was not a prerequisite to the development of mathematical understanding, and suggested that the two develop simultaneously.

What is the relative difficulty of addition and subtraction facts?

At one time, especially when stimulus-response theories of learning were prevalent, there was great interest in ascertaining whether some basic number facts or combinations -- e.g., 5 + 2 = 7, 9 + 6 = 15, 8 - 3 = 5, 17 - 9 = 8 -- were more difficult than others. Textbook writers as well as classroom teachers used the results of such research to determine the order in which facts would be presented. The assumption was that if the combinations were sequenced appropriately, the time needed to memorize them could be reduced.

Two common findings were evident which, despite the age of the studies (e.g., MacLatchy, 1933; Washburne and Vogel, 1928; Wheeler, 1939), may still be applicable:

3 + 4
4 + 3

(1) An addition combination and its "reverse" form tend to be of equal difficulty.

6 + 8
17 - 9

(2) Combinations with larger addends tend to be more difficult.

It should be noted, however, that Swenson (1939, 1944) questioned whether results on relative difficulty obtained under repetitive drill-oriented methods of learning are valid when applied in learning situations not so definitely drill-centered. When second graders were taught by drill, by generalization, and by a combined method, it was found that the order of difficulty seemed to be, at least in part, a function of teaching method. Thus research which aims at establishing the difficulty of arithmetic skills and processes should probably do so in terms of a clearly defined teaching and learning method.

Recently, Suppes and his associates (e.g., Suppes and Groen, 1967) have used the data-gathering potential of the computer to explore the relative difficulty of the basic facts. Drill-and-practice programs which present addition and subtraction combinations have been used as the vehicle to determine a suggested order of presentation and amount of practice.

Are open addition and subtraction sentences all of the same difficulty?

Various investigations pertaining to this question have been reported by Aims (1971), Engle and Lerch (1971), Weaver (1971, 1972, 1973), Grouws (1972, 1974), and Groen and Poll (1973). A synthesis of findings suggests that:

(1) open subtraction sentences are more difficult to solve than open addition sentences;

(2) sentences of the form $\square - b = c$ or $c = \square - b$ are clearly the most difficult of all types;

(3) sentences with the operation sign on the right-hand side of the equals sign are more difficult than those with the operation sign on the left-hand side;

(4) sentences with numbers between 20 and 100 are more difficult than those that are within the context of basic facts;

(5) children's methods of solving open sentences vary from type to type; and

(6) they also vary within each particular type.

Teachers should be careful of the order in which open-sentence types are introduced and studied. It is likely that within each column below, the types of open sentences are listed in order of increasing conceptual difficulty:

$a + b = \square$	$\square = a + b$	$a - b = \square$	$\square = a - b$
$a + \square = c$	$c = a + \square$	$a - \square = c$	$c = a - \square$
$\square + b = c$	$c = \square + b$	$\square - b = c$	$c = \square - b$

Teachers also may need to be careful of the pace at which open-sentence types are mixed.

Should addition and subtraction be taught at the same time?

It is somewhat surprising, considering how frequently this question is asked, to find that there has been little research on the topic. Early studies (such as Brownell, 1928) found that higher achievement resulted when addition and subtraction facts were taught together. Spencer (1968) reported that there may be some intertask interference, but emphasis on the relationship facilitates understanding.

Research has generally found that the subtraction combinations are harder for children to learn than those in addition, even when addition and subtraction are taught together.

Wiles, Romberg, and Moser (1973) studied the effects of two instructional sequences for teaching addition and subtraction of two-digit whole numbers to second-grade children. The traditional sequence of addition followed by subtraction was compared with an integrated approach in which addition and subtraction algorithms were developed concurrently, and in which carrying and borrowing were treated as a single entity, regrouping. The investigators concluded that the use of integrated sequences with this type of material was not supported for children of these ages. The addition algorithm was easier to learn than the subtraction algorithm; while regrouping was a major difficulty for both operations, it posed more of a problem for subtraction than for addition. It should be noted that expanded notation (e.g., 52 = 50 + 2) was used in conjunction with the development of algorithms, and this in itself may have posed greater difficulty within the context of subtraction than within the context of addition.

What type of problem situation should be used for introductory work with subtraction?

Gibb (1956) explored ways in which pupils think as they attempt to solve subtraction problems. In interviews with 36 second graders, she found that pupils did best on "take-away" problems and poorest on "comparative" problems. For instance, when the question was, "How many are left?", the problem was easier than when it was, "How many more does Tom have than Jeff?". "Additive" problems, in which the question might be, "How many more does he need?", were of medium difficulty and took more time. She reported that the children solved the problems in terms of the situation, rather than conceiving that one basic idea appeared in all applications.

Schell and Burns (1962) found no difference in performance on the three types of problems. However, "take-away" situations were considered by pupils to be easiest--thus they are generally considered first in introductory work with subtraction.

Coxford (1966) and Osborne (1967) found that an approach using set-partitioning, with emphasis on the relationship between addition and subtraction, resulted in greater understanding than the "take-away" approach.

What is the role of materials in developing understanding and skill in addition and subtraction?

Brownell (1928, 1941) and McConnell (1934) found that pupils use various ways of obtaining answers to combinations--guessing, counting, and solving from known combinations, as well as immediate recall. Brownell stated, "Children appear to attain 'mastery' only after a period during which they deal with procedures less advanced (but to them more meaningful) than automatic responses."

But before mastery is reached, experiences with concrete materials appear to provide an essential base for developing understanding of addition and subtraction concepts. Generally, researchers have concluded that understanding is best facilitated by the use of concrete materials, followed by semi-concrete materials such as pictures, and finally by the abstract presentation with numerals, symbols, and/or words. How crucial it is that every child proceed through all three phases is a question of some concern.

Gibb (1956), for instance, found that abstract contexts were poorest. She reported, however, that pupil performance was better on subtraction examples presented in a semi-concrete context, rather than with concrete materials. Nevertheless, she noted. "Children have less difficulty solving problems if they can manipulate objects or at least think in [the] presence of objects with which the problems are directly associated than when solving problems wholly on a verbal basis."

Ekman (1967) reported that when third graders manipulated materials before presentation of an addition algorithm, both understanding and ability to transfer increased. Use of materials was better than use of only pictures before introduction to the algorithm, or development of the algorithm without either aid.

How should subtraction with renaming be taught?

Over the years, researchers have been very interested in procedures for teaching subtraction involving renaming (once commonly called "borrowing"). The question of most concern has been whether to teach subtraction by equal additions or by decomposition.

How do you do this example?

$$\begin{array}{r} 91 \\ -\ 24 \\ \hline 67 \end{array}$$

You're using decomposition if you do it this way:

11 - 4 = 7 (ones); 8 - 2 = 6 (tens)

If you do it this way, you're using equal additions:

11 - 4 = 7 (ones); 9 - 3 = 6 (tens)

In a classic study, Brownell (1947; Brownell and Moser, 1949) investigated the comparative merits of two algorithms (decomposition and equal additions), in combination with two methods of instruction (meaningful and mechanical):

	meaningful	mechanical
decomposition	a	b
equal additions	c	d

He found that, at the time of initial instruction:

(1) Meaningful decomposition [a] was better than mechanical decomposition [b] on measures of understanding and accuracy.

(2) Meaningful equal additions [c] was significantly better than mechanical equal additions [d] on measures of understanding.

(3) Mechanical decomposition [b] was not as effective as either equal additions procedure [c or d].

(4) Meaningful decomposition [a] was superior to each equal additions procedure [c, d] on measures of understanding and accuracy.

It was concluded that whether to teach the equal additions or the decomposition algorithm depends on the desired outcome.

In recent years, the decomposition procedure has been used almost exclusively in the United States, since it was considered easier to explain in a meaningful way. However, some question has been raised about this: with increased emphasis in many programs on properties and on compensation in particular, the equal additions method can also be presented with meaning. For instance, pupils are learning that:

(a) $9 - 3 = \square$ means that $\square + 3 = 9$

or $3 + \square = 9$

They are learning that:

(b) $7 - 4 = 3$ is equivalent to

$(7 - 4) + 2 = 3 + 2$

Development of such ideas could facilitate the teaching of the equal additions procedure. There is no recent evidence that this is true. Evidence from earlier studies, however, indicates that use of the equal additions algorithm leads to greater accuracy.

What other aspects of teaching subtraction algorithms have been investigated?

Trafton (1971) investigated the effects on third-grade pupils of two initial approaches to two-digit subtraction. One approach consisted of the conventional decomposition algorithm. The second approach involved a more general method based on the main concepts of subtraction and using the number line as an aid to solution, before work with the decomposition algorithm. The "general" approach did not result in greater understanding of or performance with the decomposition algorithm than did prolonged development of the algorithm.

Brownell (1947) studied the use of a crutch such as

$$\begin{array}{r} {}^{4}\cancel{5}{}^{1}6 \\ -\ 39 \\ \hline 17 \end{array}$$

This seemed to facilitate understanding, but attempts to have pupils stop using the crutch were not wholly successful. Some persons suggest that this crutch should only be taught when it is needed.

The use of alternative addition and subtraction algorithms, involving less memory and more paper-and-pencil recording, is currently being investigated. "Reports of the research are being prepared for publication" (Hutchings, 1975).

Overman (1930) found that if pupils were taught to generalize about the renaming procedures in two-place addition and subtraction, they were able to do three-place examples. This was less time-consuming than having the teacher present two-place and then three-place examples separately.

Results from the first National Assessment of Educational Progress (NAEP), reported by Carpenter et al. (1975), provide some indication of how well children achieve on subtraction examples. At age 9, 55% of the nine-year-olds correctly found the difference between two two-digit numbers; only 27% were able to subtract a three-digit number from a four-digit number. There was a marked increase to 89% and 80% for 13-year-olds. Reversal errors were high among the nine-year-olds (15% to 20%), but were rarely made by older children.

What is the role of drill in teaching addition and subtraction?

Discussions on the teaching of mathematics in the primary grades once centered on whether programs should consist of isolated, repetitive drill or of an integrated approach involving the presentation of interrelated ideas. Prior to the 1930's, much research was done on the effectiveness of various types of drill. For instance, Knight (1927) reported on a successful program of drill in which the distribution of practice on basic facts was carefully planned--no facts were neglected, but more difficult combinations were emphasized.

Accuracy has been and is accepted as a goal in mathematics, and it is in an attempt to meet this goal that drill is stressed. In a series of articles, Wilson (1930) advocated no less than 100% mastery. He showed that, with a carefully planned set of materials, the goal was not as unattainable as some persons believed it to be.

Many other studies have shown that drill per se is not effective in developing mathematical concepts.

Programs stressing relationships and generalizations among the addition and subtraction combinations were found to be preferable for developing understanding and the ability to transfer (McConnell, 1934; Thiele, 1938). This has been supported by many studies since that time.

Brownell and Chazal (1935) summarized their research work with third graders by stating that drill must be preceded by meaningful instruction. The type of thinking which is developed and the child's facility with the process of thinking is of greater importance than mere recall. Drill in itself makes little contribution to growth in quantitative thinking, since it fails to supply more mature ways of dealing with numbers.

Pincus (1956) also found that whether drill did or did not incorporate an emphasis upon relationships was not significant, when drill followed meaningful instruction.

Should non-paper-and-pencil practice be provided?

Many mathematical problems which arise in everyday life must be solved without pencil and paper. Providing a planned program of non-paper-and-pencil practice on both examples and problems has been found to be effective in increasing achievement in addition and subtraction, as for other topics in the curriculum (Flournoy, 1954). Other researchers have suggested that certain "thought processes" which are especially suited to such practice should be taught. For instance, a left-to-right approach to finding the sum or difference is useful, rather than the right-to-left approach used in the written algorithm. "Rounding," using the principle of compensation, and renaming are also helpful. Increased understanding of the process may result.

Should children "check" their answers?

The answers which research has provided to this question are not in total agreement. We encourage children to check their work, since we believe that checking contributes to greater accuracy. There is some research evidence to support this belief.

However, Grossnickle (1938) reported data which should be considered as we teach. He analyzed the work of 174 third graders who used addition to check subtraction answers. He found that pupils frequently "forced the check," that is, made the sums agree without actually adding; in many cases, checking was perfunctory. Generally, there was only a chance difference between the mean accuracy of the group of pupils when they checked and their mean accuracy when they did not check.

What does this indicate to teachers? Obviously, children must understand the purpose of checking--and what they must do if the solution in the check does not agree with the original solution. With the increasing use of hand-held calculators, it is imperative that children attain this understanding.

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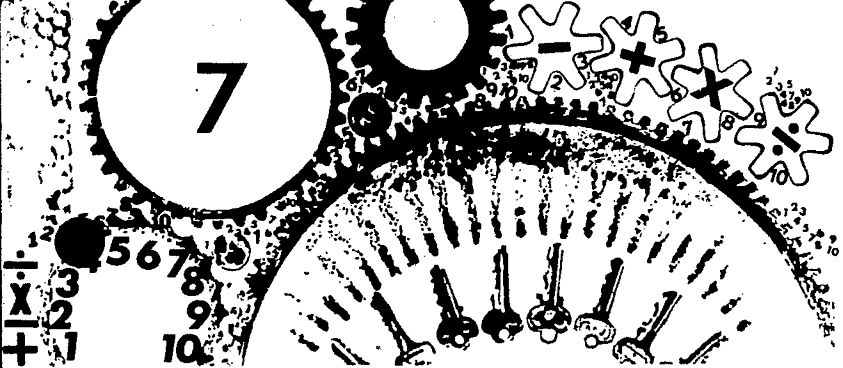


MULTIPLICATION AND DIVISION WITH WHOLE NUMBERS

Should children be encouraged to memorize basic multiplication facts?

At an appropriate time in the learning sequence it is desirable that children strive to achieve immediate recall of basic multiplication facts ($3 \times 5 = 15$, $6 \times 4 = 24$, $7 \times 8 = 56$, $9 \times 9 = 81$, etc.).

Findings from a comprehensive investigation with children in grades three to five by Brownell and Carper (1943) suggest that activities and experiences which contribute to pupils' understanding of the mathematical nature of multiplication should precede work which focuses on memorization of facts.

Teachers know that the number of specific basic facts to be memorized is reduced substantially if pupils are able to apply relevant properties of multiplication. Thus, learning that $4 \times 3 = 12$ should not be distinct from learning that $3 \times 4 = 12$; knowing that $\square \times 0 = 0$ and $\square \times 1 = \square$ makes it unnecessary to learn specific instances of those properties.

Brownell and Carper also suggested that development of the facts may lead to the organization of a multiplication table. This can aid in the identification of patterns and relationships; pupils can find answers to such questions as:

- -- If 1 is a factor, what pattern is true?
- -- If 5 is a factor, what digit will be in the units place in the product?
- -- If one factor is even, will the product be odd or even?

Ascertaining the relative difficulty of the multiplication facts was once a matter of great concern, based on the assumption that there is a fixed rank for each. Little commonality of levels of difficulty was evident among the studies, however, since this is apparently a function of (1) whether pupils are studied at the time of initial learning, or later; (2) the order and organization of the facts; and (3) the method of teaching, whether meaningful, with emphasis on relationships, or drill-oriented. Thus we need to ask, "Difficulty level for whom? at what age? under what method of instruction?"

In connection with analyzing data on difficulty levels, Jerman (1970) reported that students appeared to use different strategies for different multiplication combinations, and that the strategy used may be a function of the combination itself. Strategies used in grade 3 appeared to be the ones used for the same combinations in grade 6 in 72% of the cases.

Two findings that were frequently cited in the early studies (conducted under a drill approach) were that combinations involving zero presented difficulty, and that the size of the product was positively correlated to difficulty.

Investigations pertaining to the relative difficulty of open multiplication and division sentences have been conducted by Grouws and Good (1976) and by McMaster (1975). Findings parallel those from analogous investigations summarized earlier for open addition and subtraction sentences: difficulties associated with particular multiplication and division sentences types are similar to those for their respective addition and subtraction counterparts.

How should multiplication be conceptualized for children?

Traditionally multiplication of whole numbers has been conceptualized for children in terms of combining equal-sized groups and the addition of equal addends. For instance, "4 x 7" has been interpreted as "4 groups of 7" and "7 + 7 + 7 + 7." But there are logical difficulties inherent in this interpretation when the first factor in a multiplication example is 0 or 1. Some recent research has investigated the feasibility of using other conceptualizations of multiplication. One of these interpretations, which is independent of addition, is based upon the following relationship: if set A has a members and set B has b members, the Cartesian product of sets A and B has a x b members. Hervey (1966) reported that second-grade pupils had significantly greater success in solving, conceptualizing, and visually representing equal-addends problems than Cartesian-product problems. She was not able to determine the extent to which her findings may be influenced by the nature of prior instruction or by differences inherent in the mathematical nature of the two conceptualizations.

Another conceptualization of multiplication may be associated with rectangular arrays. At the third-grade level Schell (1964) investigated the achievement of pupils who used array representations exclusively for their introductory work with multiplication, as compared with pupils who used a variety of representations. He found no conclusive evidence of a difference in achievement levels.

Is attention to distributivity helpful in early work with multiplication?

We know, for example, that $3 \times (4 + 7) = (3 \times 4) + (3 \times 7)$. This is an instance of the distributive property of multiplication over addition which (in one form or another) is used to some extent in contemporary programs of mathematics instruction. Specific instances of this property often are illustrated with arrays.

Although Schell (1964) reported some findings regarding third-graders' ability to use distributivity, his observations were based upon a very limited amount of instruction: two introductory lessons. Such findings are tenuous at best.

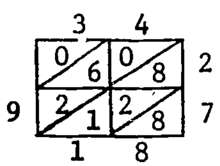
From a more comprehensive investigation with third-grade pupils and their beginning work with multiplication, Gray (1965) found that an emphasis upon distributivity led to "superior" results when compared with an approach that did not include work with this property. The superiority was statistically significant on three of four measures: posttest of transfer ability, retention test of multiplication achievement, and retention test of transfer. On the remaining measure--posttest of multiplication achievement--children who had worked with distributivity scored higher than those who had not, but the difference was not statistically significant.

Gray's findings add further support to a growing body of evidence on advantages to be expected from instruction which emphasizes mathematical meaning and understanding. The "pay-off" may not always be particularly evident in terms of skills-achievement immediately following instruction. Rather, the pay-off is much more clearly evident in relation to factors such as comprehension, transfer, and retention.

However, in a recent survey in grades 4 through 7, Weaver (1973) found that pupils exhibited very little sensitivity to the use of distributivity in solving examples varied in context, form, format, and number. Hobbs (1975) found a similar lack of sensitivity in his investigation, which was based on structured interviews with individual pupils in grade 5. This suggests that more emphasis must be placed on this property if we are to expect a "pay-off."

What multiplication algorithms have been found to be effective?

On the basis of multiple criteria, Schrankler (1967) evaluated the relative effectiveness of two algorithms for teaching multiplication with whole numbers to fourth grade pupils. As interacting factors, he considered (1) three intelligence levels and (2) two readiness backgrounds. From a variety of findings Schrankler concluded that methods using general ideas based on the structure of the number system are more successful than other methods investigated in achieving the objectives of increased computational skills, understanding of processes, and problem solving abilities associated with the multiplication of whole numbers between 9 and 100.



$$\begin{array}{r} 34 \\ \underline{\times\ 27} \end{array}$$

Hughes (1973) investigated the teaching of multidigit multiplication to fourth graders using the lattice method and the distributive method. The groups using the lattice method were able to compute multidigit multiplication exercises in significantly less time and more accurately than groups using the distributive method. (Whether or not the time to draw the lattices was included in the test time is unspecified.) No significant differences on tests of understanding of multiplication or attitude toward mathematics were found.

Other forms of a multiplication algorithm, involving less memorization and more paper-and-pencil recording, are reported effective with low achievers.

What is the difficulty level of division combinations?

Little research has been done on the difficulty level of the basic division facts, but great attention has been given to the difficulties inherent in the algorithm. Osburn (1946) noted 41 levels of difficulty for division examples with two-digit divisors and one-digit quotients. Pupils' ability to divide with two-figure divisors has been found to involve a considerable variety of skills varying widely in difficulty (Brownell, 1953; Brueckner and Melbye, 1940). Examples in which the apparent quotient is the true quotient (as in $43\overline{)92}$) are (of course) much easier than those requiring correcting (such as $43\overline{)81}$), with difficulty increasing as the number of digits in the quotient increases.

Which is it better to teach: the subtractive or the distributive form of the division algorithm?

During the 1940's and 1950's, the division algorithm typically taught in elementary school mathematics was:

```
    2
23)552        First think
   46         '2's in 5?'
    92
     etc.
```

(Some people refer to this as the distributive algorithm.)

A multiplicative and subtractive approach to the division algorithm has come back into use in recent years:

```
23)552
   230     10 x 23
   322
   230     10 x 23
    92
     etc.
```

In one investigation comparing use of the conventional (or distributive) and the subtractive forms, Van Engen and Gibb (1956) reported that there were some advantages for each. They evaluated pupil achievement in terms of understanding the process of division, transfer of learning, retention, and problem solving achievement. Among their conclusions were:

(1) Children taught the subtractive method had a better understanding of the process or idea of division in comparison with the conventional method used. Use of this algorithm was especially effective for children with low ability. Those with high ability used the two methods with equivalent effectiveness.

(2) Children taught the conventional (distributive) method achieved higher problem solving scores (for the type of problem in the study).

(3) Use of the subtractive method was more effective in enabling children to transfer to unfamiliar but similar situations.

(4) The two procedures appeared to be equally effective on measures of retention of skill and understanding. This seems to be more related to teaching procedures, regardless of the method of division.

Kratzer (1972; Kratzer and Willoughby, 1973) prepared two instructional units, both involving meaningful instruction. One used the distributive algorithm and the other used the subtractive algorithm, each as a method of keeping records while manipulating bundles of sticks. Twelve fourth-grade classes were taught one or the other of the division approaches. No significant differences in the approaches were found on achievement of familiar problems on immediate or retention tests. There was, however, a significant difference between the approaches on achievement of unfamiliar problems on both types of test: those using the distributive approach displayed a better understanding of the process.

Dilley (1970) also compared the teaching of division in grade 4 by using a distributive algorithm and a successive subtractions algorithm. Significant differences were found on an applications test favoring use of the successive subtractions algorithm, and on a retention test favoring the distributive algorithm.

Rousseau (1972) undertook an experiment with 12 fourth-grade classes to determine whether or not the foundations on which a division algorithm could be built affect children's ability to retain and transfer on tasks involving the algorithm. Four algorithms were developed on these foundations: (1) mathematical, based on the distributive property of division over addition; (2) real-world, based on the physical act of "quotitioning"; (3) real-world, based on the physical act of partitioning; and (4) rote, based on the memorization of routines. No significant differences in retention of algorithms were found. For extensions to cases of slightly greater difficulty, the rote algorithm was superior. For problems of greater difficulty, however, the quotitive and distributive algorithms were better than the rote and partitive algorithms.

What is the most effective method of teaching pupils to estimate quotient digits?

Inefficient algorithms need to be shortened to gain proficiency in division. Then pupils must be able to estimate quotient digits systematically. Several methods have been advocated: (1) the "apparent" or "round-down" method, in which the divisor is rounded to the next lower multiple of 10; and (2) the "increase-by-one" methods, in which the divisor is rounded to the next higher multiple of 10, (a) either "round-both-ways," depending on whether the digit in units' place is less or greater than 5, or (b) "round-up," no matter what. Which method do you use?

	apparent or round-down	increase-by-one	
		round-up	round-both ways
$42\overline{)216}$	$4\overline{)21}$	$5\overline{)21}$	$4\overline{)21}$
$47\overline{)216}$	$\overset{5}{4\overline{)21}}$	$\overset{4}{5\overline{)21}}$	$\overset{4}{5\overline{)21}}$

Efforts to resolve the issue of which method is best have focused on analysis and comparison of the success of each method on a specified population of division examples (e.g., Grossnickle, 1932; Osburn, 1946, 1950; Morton, 1947). Hartung (1957) critically reviewed these and other analytic studies. He concluded that "round-up" was the most useful method, because of the advantages of obtaining an estimate that is less than the true quotient (which decreased the need for erasing), and because of the relative simplicity of a "one-rule" method.

In one of the few experimental investigations on this topic, Grossnickle (1937) studied the achievement of groups taught by "round-down" and "round-both-ways." He concluded that there were no significant differences between the scores of the two groups.

How children apply the method was studied by Flournoy (1959), who found that "round-both-ways" was used as effectively as the "round-down" method. She stressed that perhaps not all children should be taught the "round-both-ways" method. Carter (1960) reported that pupils taught this method were not as accurate as those taught a one-rule method--nor did pupils always use the method taught.

What is the role of measurement and partition situations in teaching division?

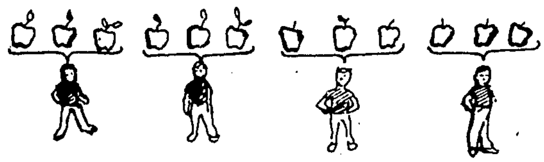
Measurement problems involve situations such as:

> If each boy is to receive 3 apples, how many boys can share 12 apples? (Find the number of equivalent subsets.)

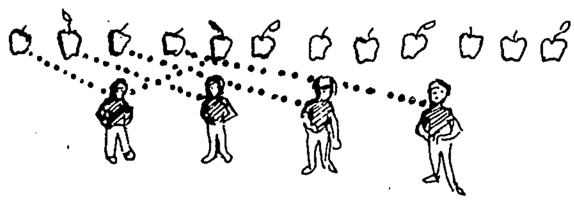
Partition problems involve situations like this:

> If there are 4 boys to share 12 apples equally, how many will each boy receive? (Find the number of elements in each equivalent subset.)

In a study with second graders (chosen since commonly children at this level have had little experience with division which would interact with the teaching in the research study), Gunderson (1953) reported that problems involving partition situations were more difficult than problems involving measurement situations. The ease of visualizing the measurement situation probably contributes to this. For instance, for the illustration above, a picture like this could be formed:



For the partition situation, the drawing might be:



and so on!

Zweng (1964) also found that partition problems were significantly more difficult for second graders than measurement problems. She further reported that problems in which two sets of tangible objects were specified, were easier than those in which only one set of tangible objects was specified. In an earlier study, Hill (1952) found that pupils in the intermediate grades indicated a preference for measurement situations, but performance was similar on both types.

In the study in which they compared two division algorithms, Van Engen and Gibb (1956) found that children who used the distributive algorithm had greater success with partition situations, while those who used the subtractive algorithm had greater success with measurement situations.

Scott (1963) used the subtractive algorithm for measurement situations and the distributive algorithm for partition situations. He suggested that: (1) use of the two algorithms was not too difficult for third grade children; (2) two algorithms demanded no more teaching time than only one algorithm; and (3) children taught both algorithms had a greater understanding of division.

Bechtel and Weaver (1975) used structured interviews with second-grade children to ascertain ways in which they manipulated objects to solve measurement and partitive problem situations prior to formal instruction on division. Findings confirmed that these situations are conceptually different for young children and suggest that systematic instruction should be designed accordingly.

Problem situations with non-zero remainders were found to be no more difficult for children to cope with than were problem situations with zero remainders, suggesting that no sharp dichotomy should be made between such instances when providing pre-division experiences in an instructional program.

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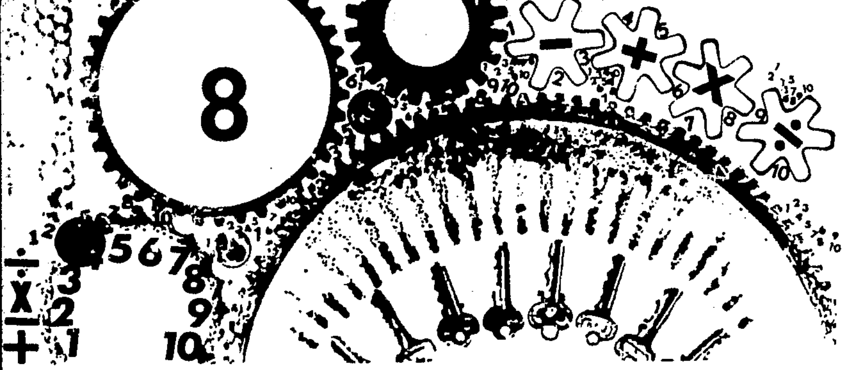

RATIONAL NUMBERS: FRACTIONS AND DECIMALS

Since several interpretations of the above words are possible, let's clarify how we're using them. We shall use the word fraction to refer to a number: a number that may be expressed in the form $\frac{a}{b}$, where a and b are whole numbers and $b \neq 0$. The word decimal will be used to refer to a particular kind of fraction: one that is expressed in our familiar positional place-value notation, with the implicit denominator being some power of 10.

Can young children learn fractional concepts?

We have found from surveys of what children know about mathematics upon entering school that at least 50% can recognize halves, fourths, and thirds, and have acquired some facility in using these fractions. Campbell (1975) surveyed five-, six-, and seven-year-olds on their understanding of the fractions one-half, one-third, and one-fourth, prior to formal instruction. The children consistently showed a higher level of understanding of "fraction of a whole" than of "fraction of a set" or "division" interpretations. More evidence of understanding was shown when concrete materials were used rather than semi-concrete representations. Gunderson and Gunderson (1957) interviewed 22 second graders following their initial experience with a lesson on fractional parts of circles. The investigators concluded that fractions could be introduced at this grade level, with the use of manipulative materials and through oral work with no symbols used.

Sension (1971) reported that area, set-subset, and combination representations for introducing rational number concepts appeared to be equally effective on tests containing items consistent with the experimental instruction. However, the combination treatment produced a higher level of generalization to a number-line model.

A planned, systematic program for developing fractional ideas seems essential as readiness for work with symbols. Use of manipulative materials appears vital in this preparation.

What sequence should be used in teaching fractions?

Investigators have focused on this question in varying ways. Novillis (1974), for instance, developed a hierarchy with 23 steps. Eighteen of these were found to be appropriate; that is, they depended on previously learned ideas. For example, associating fractions with part-whole and part-group models were prerequisite to associating a fraction with a point on a number line.

Bohan (1971) tried three approaches to teaching skills and concepts related to equivalent fractions in grade five. He found that approaches in which equivalent fractions were introduced with the aid of diagrams and sets of objects, followed by addition and then multiplication, resulted in higher achievement than an approach in which multiplication with fractions was taught first, then applied to equivalent fractions, followed by addition.

Four approaches to teaching comparison of fractions were investigated by Choate (1975): (1) pupils were taught a rule, without conceptual development; (2) the rule was developed meaningfully; (3) conceptual work for comparing fractions using diagrams preceded presentation of the rule; and (4) only the conceptual work was included, with no algorithmic work. He concluded, "The crucial consideration is the time of presentation of the algorithm in relation to the conceptual development." He suggested that the third approach, with conceptual work preceding presentation of the rule, would provide the strongest base.

What procedures are effective in work with addition and subtraction with fractions?

There is little evidence on the effectiveness of procedures for finding the common denominator in addition with fractions, and even less for subtraction with fractions. Anderson (1966) analyzed errors made by 26 fifth grade classes using two procedures for finding the least common denominator when adding two "unlike" fractions: by setting up rows of equivalent fractions, and by factoring the denominators. There were no significant differences between the two procedures on tests of four kinds of addition with fractions examples. Furthermore, Anderson reported that errors connected with (1) "reducing," (2) determining the numerator, and (3) addition, occurred most frequently, with the greatest frequency of error in examples in which the least common denominator was not apparent.

Bat-haee (1969) compared 112 fifth graders who were taught (1) the factoring method or (2) the "inspection" method of a current textbook series. Those taught by the factoring method scored significantly higher on the experimental posttests.

Howard (1950) reported on a study with 15 classes of pupils in grades 5 and 6 who were taught addition of fractions by three methods differing in the amount of emphasis on meaning, use of materials, and practice. Pupils retained better when they learned fractional work through extensive use of materials and with considerable emphasis on meaning, plus provision for practice. Feinstein (1952), Krich (1964), Sebold (1946), and Shuster and Pigge (1965) also support the importance of using meaningful methods for work with fractions.

Bisio (1971) conducted a study on addition and subtraction of like fractions with fifth graders. He found that the passive use of manipulative materials (that is, watching the teacher use them) was as effective as active use and better than non-use of materials.

Carney (1973) taught four classes of fourth graders to add and subtract fractions, using 30 lessons based on field postulates and other properties, and taught four other classes 30 lessons based on objects and the number line. The approach using the field postulates and other properties was more effective than the object-and-number-line approach.

In a study with fourth graders, Coburn (1974) reported that, while achievement on some concepts related to equivalent fractions was comparable for the two groups, students using the region approach achieved significantly better on adding and subtracting unlike fractions and on some retention and attitude measures.

How can we most effectively develop the algorithm for multiplication with fractions?

Green (1970) reported on a study using two different approaches in teaching multiplication of fractions in grade 5. An approach based on the area of a rectangular region was more effective than one based on finding a fractional part of a region or set. Each approach was studied in relation to two different modes of representation (diagrams and cardboard strips); the "area" approach taught with diagrams was most successful, the "fractional part" approach taught with cardboard strips was second, and the "fractional part" approach taught with diagrams was poorest.

Much of the other research on multiplication with fractions has involved the use of programmed instruction: the purpose of the investigation was to compare various programming strategies, while fractions served merely as the content vehicle. For instance, Kyte and Fornwalt (1967) used programmed materials on multiplication with fractions to ascertain the rate of mastery by pupils at two IQ levels. While they found that pupils with superior IQs were able to master identified types of examples more quickly than those with normal IQs, the study says nothing about what procedures were used to teach the operation with fractions.

Miller (1964) found that significantly higher gains in multiplication with fractions were made by pupils using programmed practice materials, which provided immediate knowledge of answers, than by pupils using conventional textbook materials. In another investigation, higher achievement on the experimental posttest resulted when multiplication with fractions was taught with multi-level materials rather than with single textbooks (Triplett, 1963).

What algorithm shall we use for division with fractions?

Bergen (1966) prepared booklets designed to teach pupils by complex fraction, common denominator, or inversion algorithms. No significant differences were found between complex fraction and inversion algorithms, but each was significantly superior to the common denominator algorithm on most types of examples.

common denominator

$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{2}{4} =$

$\frac{3 \div 2}{4 \div 4} = \frac{3 \div 2}{1} =$

$3 \div 2 = \frac{3}{2}$

inversion

$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} =$

$\frac{6}{4} = \frac{3}{2}$

Sluser (1963) compared teaching the common denominator and inversion algorithms with and without explanation of the reciprocal principle as the rationale behind inversion. The group given the explanation scored lower on tests of division with fractions than a group merely taught to invert and multiply. He suggested that only above average pupils could understand the principle. However, a large percentage of errors occurred because pupils performed the wrong operation. Krich (1964) reported no significant differences on immediate posttests for pupils taught why the inversion procedure works, as compared with those merely taught the rule. On retention tests requiring recall, however, the group taught with meaning scored significantly higher.

complex fraction

$$\frac{3}{4} \div \frac{1}{2} = \frac{\frac{3}{4}}{\frac{1}{2}} =$$

$$\frac{\frac{3}{4} \times \frac{2}{1}}{\frac{1}{2} \times \frac{2}{1}} = \frac{\frac{6}{4}}{1} =$$

$$\frac{6}{4} = \frac{3}{2}$$

In a study by Capps (1963) the effectiveness of the common denominator and inversion algorithms for division with fractions was compared. There were no significant differences in achievement on tests of addition, subtraction, and division with fractions, while pupils taught the inversion algorithm scored significantly higher on immediate posttests and on retention tests of multiplication with fractions than those taught the common denominator algorithm. This retroactive effect on multiplication was also reported by Bidwell (1968). He found that the inverse operation procedure was most effective, followed by complex fraction and common denominator procedures. The complex fraction procedure was better for retention, while the common denominator procedure was poorest.

Is it helpful to analyze errors pupils make with fractions?

Many earlier studies were concerned primarily with the specific errors children make. In general, it was found that, for all operations with fractions, the major errors were caused by (1) difficulty with "reducing," (2) lack of comprehension of the operation involved, and (3) computational errors (e.g., Brueckner, 1928a; Morton, 1924; Schane, 1938). Such findings frequently influenced the material included in textbooks.

Lankford (1972, 1974) reported the incorrect solutions given by seventh graders to various examples with fractions. This information could be very helpful to teachers in deciding what to stress as operations with fractions are taught.

Guiler (1936) was among those who reported success with a remedial program which provided practice on correcting errors which had been identified. Ramharter and Johnson (1949) had good and poor achievers think aloud while they attempted to correct errors in six examples involving subtraction with fractions. On subsequent tests, good achievers consistently corrected more errors, using a guidesheet effectively.

Aftreth (1958) had sixth grade pupils identify and correct errors imbedded in 19 completed sets of examples in addition and subtraction with fractions, while a control group worked the examples. No significant differences on either immediate or delayed recall tests were found for addition with fractions, while some significant differences favoring the group working the examples were found for subtraction with fractions. The author suggested that having pupils correct their own errors might be more effective than having them correct imbedded errors.

Fifth graders tested by Scott (1962) made more errors in subtraction with fractions involving regrouping than in subtraction with whole numbers involving regrouping. He suggested that current emphasis on the decimal system may reduce the "flexibility" which the child must have to deal successfully with subtraction with fractions when regrouping is necessary.

Romberg (1968) reported that among sixth graders who used a correct algorithm to multiply fractions, many pupils either did not express products in simplest form (as directed) or made errors in doing so. He attributed this difference to pupils' failure to "cancel," and suggested that the cancellation process is important--even essential--if efficiency in multiplication is one of the desired outcomes of instruction.

Is it helpful to relate decimals with fractions or place value?

Faires (1963) introduced some pupils to decimals through a sequence based on an orderly extension of place value, with no reference to common fraction equivalents, while others were taught fractions before decimals, as is usually done. Gains in computational achievement and at least as good an understanding of fraction concepts resulted. Faires indicated that "computation with decimals is [apparently] more nearly like computation with whole numbers than with fractions"; thus reinforcement of whole number computational skills is provided.

O'Brien (1968) reported that pupils taught decimals with an emphasis on the principles of numeration, with no mention of fractions, scored lower on tests of computation with decimals than those taught either (a) the relation between decimals and fractions, with secondary emphasis on principles of numeration, or (b) rules, with no mention of fractions or principles of numeration. On later retention measures, the numeration approach was significantly lower than use of the rules approach, but not significantly different from the fraction-numeration approach.

With fifth graders, Willson (1972) compared the fraction-then-decimal sequence with the decimal-then-fraction sequence. No significant differences in achievement were found, although greater raw-score gains were made by those having the decimal-then-fraction sequence.

With increased use of (1) the metric system and (2) hand-held calculators, both of which involve extensive use of decimals, the need to teach about decimals before fractions, or at least far earlier than is now done, will probably increase. Research is needed to ascertain more about the conditions under which it is possible to do so.

How should we teach children to place the decimal point in division with decimals?

Brueckner (1928b) and Grossnickle (1941) analyzed the difficulties with decimals which children have, citing misplacing of the decimal point in division as one of the major sources of error. Flournoy (1959) compared sixth grade classes taught to locate the decimal point in the quotient by (1) making the divisor a whole number by multiplying by a power of 10, and then multiplying the dividend by the same number, or (2) subtracting the number of decimal places in the divisor from the number of places in the dividend. Multiplying by a power of 10 resulted in greater accuracy, as Grossnickle (1941) had concluded earlier.

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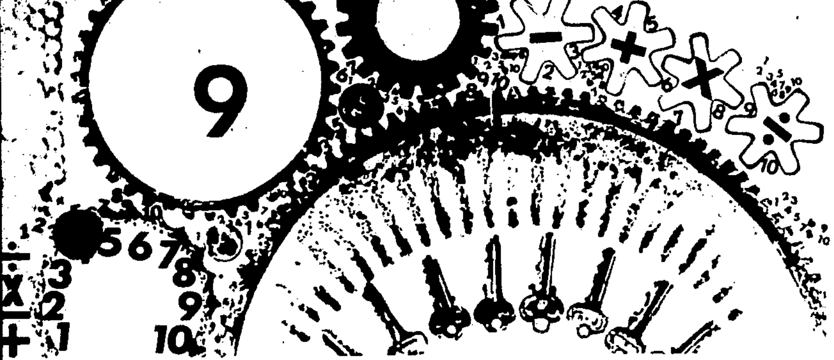

GEOMETRY AND OTHER MATHEMATICAL TOPICS

Is there common agreement on what geometry and measurement content will be presented?

During the past decade, geometry has been considered by curriculum developers as an important component in the elementary school mathematics curriculum. However, the importance of geometry is not always apparent in classroom practice. One reason for including geometry is that geometric ideas are used to facilitate some number ideas--for instance, in developing area representations and for work with a number line. No research has been conducted to identify other reasons; rather, scope-and-sequence questions have been emphasized.

Grimes (1971) identified more than 100 geometric topics included in elementary school textbooks. He pointed out that teachers need to know both the language of geometry and geometric concepts.

Neatrour (1969) analyzed 16 textbook series and surveyed 156 middle schools to determine the status of geometric content in their curricula. He found that while the amount of geometric content varied greatly, three times as much was included as in 1900, with an emphasis on informal geometry. Compartmentalization of geometric content into two- and three-dimensional ideas was common.

Paige and Jennings (1967) surveyed 39 textbook series, summarizing the measurement content. They noted that there were few experiences in which students created their own units of measure, too little emphasis on practical application, and too few problems requiring actual measuring.

What geometric ideas can children learn?

A number of studies were concerned with ascertaining the ability of children to recognize various geometric figures, to visualize plane sections of solid figures, or to perform various transformations.

For instance, McGlone (1974) found that understanding of each aspect of rotation increased as age increased from 6 to 8 years. Davis (1970) found that sixth graders scored significantly below those in grade 8 and 10 in their ability to perceive plane sections of selected solid figures. Palow (1970) confirmed that children appeared to be able to acquire this ability to visualize sections of solid figures at about age 12.

Rea and Reys (1971) identified the competencies of entering kindergarteners, and concluded that these children possessed many intuitive notions of geometry. Schnur and Callahan (1973) classified geometric concept areas into seven levels of difficulty.

Another large group of studies have been feasibility studies, to determine what can be taught. For instance, Williford (1971) checked the feasibility of a unit on transformational geometry with second and third graders.

From a set of tests administered after two weeks of teaching, Shah (1969) reported that children aged 7 to 11 learned concepts associated with plane figures, nets of figures, symmetry, reflection, rotation, translation, bending and stretching, and networks. In a pilot study, Denmark and Kalin (1964) found that fifth graders could satisfactorily (1) bisect an angle, (2) construct the perpendicular bisector of a line segment, (3) copy a triangle, (4) construct a perpendicular to a line through a point on the line, and (5) copy a quadrilateral. Lack of precision in the use of the compass accounted for many errors.

D'Augustine (1966) used programmed texts on topics such as paths and their properties, simple closed curves, and polygons with pupils in grades 5, 6, and 7. He reported that reading and mathematics achievement significantly affected success, but age, length of class period, grade, or sex did not.

In some studies, instructional variables related to geometric achievement were identified. Examples of such studies include Johnson (1973), who reported that students in grade 4 having advanced organizers for materials on transformational geometry scored significantly higher than those having post organizers or no organizers. Those given several concrete models achieved higher than those given only one model.

For intermediate-grade pupils working with geometric ideas, R. L. Johnson (1971) and Bring (1972) found that use of concrete materials resulted in higher achievement, while Prigge (1974) reported that the combination of demonstration and manipulation with materials was most effective.

Wilkinson (1971) found no significant differences between sixth-grade groups using conventional instruction or either of two types of laboratory materials on geometric concepts. Genkins (1971) reported that the paper-folding method was more effective than the mirror method for kindergarten children (though the mirror method was effective in helping second graders to discriminate more types of figures), while Cheatham (1970) found that gains in geometric concepts made by seventh graders were not significantly different from those who constructed models with compass and straightedge or with paper-folding techniques.

Burrows (1974) found that the groups of children in grades 4 and 5 taught by a mastery learning strategy achieved significantly better in geometry than those taught by programs including only objectives or test components.

Peck (1971) reported that groups using conventional or imaginative terminology did not differ significantly in achievement, but ability to transfer with certain concepts was significantly different.

What do children know about measurement?

What can they learn about measurement?

Four- and five-year-olds exhibit wide differences in familiarity with ideas of time, linear and liquid measures, and money, with little mastery evident (Davis, Carper, and Crigler, 1959). Rea and Reys (1971) found that over one-half of the kindergarteners they tested could differentiate between the characteristics of size and weight, and could identify the use of various measuring instruments. In another survey with first graders, Mascho (1961) reported that as age, socio-economic level, or mental ability increased, the children's familiarity with measurement increased. Familiarity was greater when the terms were used in context. It was suggested that (1) some ideas now considered appropriate for first grade should be considered part of the child's knowledge when he enters school, and (2) teachers need to study the composition of their groups in terms of age, socio-economic level, and mental ability when planning curricular activities with measurement. This may be especially important in view of Piaget's findings, which suggest that general concepts of linear measurement are not attainable for children until approximately age 8, when the child appreciates that a linear segment may be conserved even when subdivided.

Carpenter et al. (1975), reporting NAEP data, noted that only 7% of the nine-year-olds tested correctly computed the perimeter of a rectangle. The same percentage of 13-year-olds gave the correct answer to an area problem. Thus careful development of the two concepts is suggested.

Corle (1960) substantiated the need for experiences with measurement. He found that sixth graders could estimate weight, size, temperature, and time more accurately than fifth graders, but error was 45% for sixth grade and 61% for fifth grade. Sixth grade pupils measured with acceptable accuracy only about half the time; fifth graders, one-third of the time.

What approaches to teaching measurement are effective?

In a study with second and third graders, Montgomery (1973) reported no significant interactions between aptitude and treatment using an area or a unit-of-area approach to unit-of-length concepts. Significant main effects were found favoring the treatment emphasizing the unit-of-area approach.

Urbach (1973) compared two approaches (conventional and sweep, which incorporated the idea of the movement of a line segment over a given distance) and two methods (verbalization, in which formulas were introduced, and non-verbalization) for teaching area measure to fifth graders. The conventional approach was found to be better than the sweep approach on the final test, although no significant differences were found on retention and transfer tests. Non-verbalization was generally more effective than verbalization.

Ibe (1973) taught a five-day unit in which one group of sixth graders used estimation before direct measurement of angles while the other group measured without estimating first. The group taught to estimate had significantly higher scores for transfer, estimation, and achievement than the other group did.

Despite the fact that the United States is "going metric," there have been few recent studies on teaching measurement with the metric system. In one (Bargmann, 1973), a teaching unit on the metric system was developed and taught to ascertain the appropriate grade level at which to teach specified content. In several (e.g., Slobojan, 1975), it was determined that it was feasible to teach the metric system without reference to the English system. Since this is the way it must soon be taught, it is comforting to know that it can be done!

What aspects of graphing can be learned?

Dutton and Riggs (1969) used a programmed text to present pictographs and circle, bar, and line graphs to 393 fourth and fifth graders. The text was effective in improving skills on both a graph test and on graph interpretation items from a standardized test. There is some evidence from other research that, for third graders, pictographs and bar graphs are easier to interpret than line graphs.

How can we help pupils understand our numeration system?

Flournoy, Brandt, and McGregor (1963) found that the items missed very frequently by pupils in grades 4-7 on tests measuring understanding of our numeration system related to: (a) the additive principle; (b) making "relative" interpretations; (c) meaning of 1000 as 100 tens or 10 hundreds, etc.; (d) expressing powers of ten, such as 1000 = 10 x 10 x 10; and (e) the 10-1 place value relationship. Thus greater emphasis on these is necessary as we teach.

Following a task analysis of place-value skills, Smith (1973) constructed and administered a test on place value to second graders. He identified skills that were difficult: interpreting the value of each place in a two-place numeral; interpreting 10 ones as 1 ten and 1 ten as 10 ones; and exchanging tens for ones and ones for tens, or renaming the same number in several ways.

The study of non-decimal numeration systems was included in many mathematics programs because it was presumed that such work would strengthen understanding of the decimal numeration system. There is a limited amount of evidence which supports this presumption (e.g., Jackson, 1965). Many studies indicate that instruction with non-decimal bases does not have the transfer effect that was originally anticipated (e.g., Schlinsog, 1968; Scrivens, 1968; Smith, 1968; Kavett, 1969; Muckey, 1971; Higgins, 1972). Diedrich and Glennon (1970) found that study of the decimal system alone is as effective as study of several non-decimal systems in promoting understanding of the decimal system of numeration.

The effect of type of grouping (multibase or base ten) and the time at which base representations are introduced (initially or after counting, reading, and writing numerals) on achievement in numeration with 110 pupils in grade 1 was studied by Rathmell (1973). No significant differences were found between types of grouping. The pupils given representations late had consistently higher mean scores than the pupils given representations initially. Thus introducing two-digit symbols without regard to tens-and-ones meaning is apparently not detrimental to pupils. Diagnostic review instruction was found to be especially helpful.

What can pupils learn about integers?

There have been relatively few studies which provide an answer to this question. An exploratory study with six primary grade children showed that they could be taught some concepts about integers when the number line is used.

Burdick (1970) reported that grade 6 appeared to be the optimal level for teaching addition with integers, since there was the greatest increase in learning from instruction, attainment of group criterion performance, and non-significant loss on the retention test. However, fifth graders had the greatest increase in scores from pre- to retention test, among the groups tested in grades 5 through 8.

Coltharp (1969) reported no significant difference in achievement between sixth graders taught addition and subtraction of integers from an abstract, algebraic approach and those taught by means of a concrete, visual approach. According to Sawyer (1974), the group taught the related facts method achieved significantly higher on the concepts section of a standardized test than did the group taught the complement method. However, no significant differences were found for achievement on addition and subtraction of integers.

According to Tremel (1964), success in learning to add and multiply integers was not related to numerical and spatial abilities, but was related to verbal and problem solving abilities.

What set concepts facilitate achievement?

This is another example of a topic which has been included in many mathematics programs, yet evidence on its appropriateness is lacking. The ideas of sets are unavoidable in the introduction of number concepts and intuitive geometry, although the formal terms may not be used.

There has been some concern with how to picture groups of objects. In two older studies, Carper (1942) and Dawson (1953) concluded that the greater the complexity of the objects and the group configuration, the greater the difficulty children have in determining how many are in the group. Thus in the primary grades it seems important to picture relatively simple objects and groupings.

Suppes and McKnight (1961) found that concepts and operations with sets could be taught in grade 1, noting that "operations on sets are more meaningful to the student than operations on numbers," since the sets emphasized in the investigation were collections of concrete objects. As long as the notation introduced is explicit and precise and corresponds to simple concepts, no difficulties of comprehension seemed to arise. Holmes (1963), however, reported that first graders scored below the 50% level for tests on equality concepts, ordinal number, subsets, and number property of sets.

Harper, Steffe, and Van Engen (1969) reported success in teaching conservation of numerousness, including one-to-one correspondence and equivalent and non-equivalent sets, to children at the first grade level. They noted that "the teaching sequence used in these lessons, i.e., a progression from physical action of the children, to their manipulation of concrete materials, to their observation of semi-concrete illustrations, seems to be an effective approach to use in teaching early number concepts." [*Underlining added.*]

What can children learn about probability and statistics?

Mocciola (1975) found that only 29 of 105 districts (in the New York Metropolitan area) reported having curriculum guides that required the teaching of probability or statistics. Nevertheless, it has long been considered a mathematical topic with many applications.

Most of the research concerns what can be taught. Intermediate-grade children apparently have acquired considerable familiarity with probability from everyday experiences, and can apply knowledge about such topics as a finite sample space, the probability of a finite event and of mutually exclusive events, and the quantification of probabilities (Doherty, 1966; Leffin, 1969; Shepler, 1970; McLeod, 1971; Armstrong, 1972). Smith (1966) concluded that the following topics of probability and statistics seem to be appropriate for most seventh grade students: (1) possible outcomes of an experiment, (2) probability of events that are equally likely and events that are not equally likely, (3) mutually exclusive events, (4) Pascal's triangle, (5) histograms, (6) continuous and discrete data, (7) central tendency, and (8) measures of variation. There is some evidence from another study that the mode, the mean, and possibly the median can be introduced as early as grade 4.

Jones (1975) reported that children in grades 1 through 3 learned simple probability concepts better with some materials than with others.

What can children learn about logic?

If the child is to learn to think critically, it is important that he make logically correct inferences, recognize fallacies, and identify inconsistencies among statements. Hill (1961) concluded that children aged 6 through 8 are able to recognize valid conclusions derived from sets of given premises. There seems to be a "gradual, steady growth which is nearly uniform for all types of formal logic." Differences in difficulty were associated with type of inference, but these difficulties were specific to age. Difficulties associated with sex were not significant. Children can learn to recognize identical logical form in differing content. The addition of negation very significantly increased difficulty in recognizing validity. Roberge (1969) reported that negation in the major premise also had a marked influence on the development of logical ability in children in grades 4, 6, 8, and 10.

O'Brien and Shapiro (1968) confirmed Hill's findings, except that "little growth was detected between ages 7 and 8." Using a modification of Hill's test, they found that children experienced great difficulty in testing the logical necessity of a conclusion, and showed slow growth in this ability, which supports Piaget's theory that children reach the stage of ability to think logically later than age 8. They caution that Hill's research should be interpreted and applied with caution: hypothetical-deductive ability cannot be taken for granted in children of this age.

Fetzer (1972) gave 27 logic problems differing on content and validity to 206 students aged 8 through 15. In general, younger children appeared to base their judgments on the empirical conditions and did well on problems where the logical and empirical cues agreed, whereas older children were able to disregard the empirical content and base their judgments on the logical structure of the problem. Thus young children may appear to be responding to the logical structure of a problem when in fact they are responding merely to the truth of the empirical content.

A unit on logic developed by H. N. Johnson (1971) was found to be successful when used with sixth graders. Pupils were able to detect mathematical inconsistencies in a problem-solving situation better than those who did not have such a unit.

Does the use of negative instances or non-examples help?

Houde (1973) assigned sixth-graders to four instructional treatments on the geometric concept "similarity": all negative instances, all positive instances, alternating instances, or control. The all-negative series tended to confuse the low-IQ group; the all-positive series appeared to be much more helpful to them. The high-IQ group could use either positive or negative instances efficiently, but use of both positive and negative instances resulted in the best performance.

One aspect of the development of learning a concept about right triangles was studied by Sheppard (1972). Giving divergent examples was found to be better than giving convergent examples, while giving matched non-examples was better than giving non-matched non-examples. The combination of divergent examples and matched non-examples yielded predominantly correct classification behavior. Other combinations resulted in either over- or under-generalization--or confusion.

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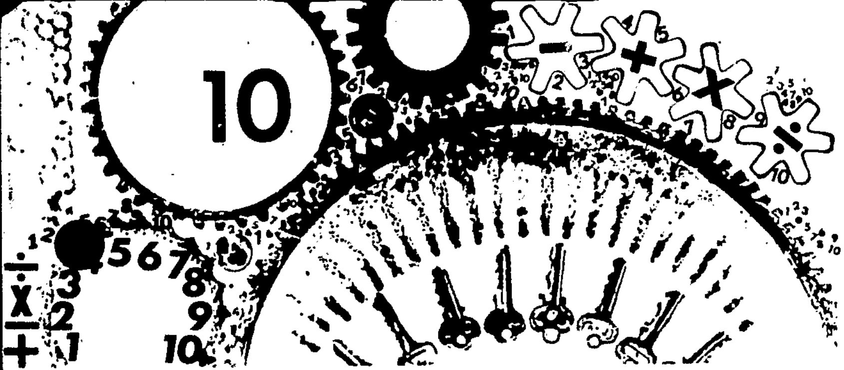

VERBAL PROBLEM SOLVING

Verbal problem solving has attracted more attention from researchers than any other topic in the mathematics curriculum. It is considered a plausible way to help children learn how to apply mathematical ideas and skills to the solving of real-life problems--and is a challenge to both pupils and teachers.

It should be noted that virtually all of the research on problem solving has been associated with whole numbers. We lack evidence about the extent to which the research can be generalized to other kinds of numbers. This is a topic for future research.

What factors are related to problem solving ability?

It is generally concluded that:

(1) IQ is significantly related to problem solving ability;
(2) sex differences do not appear to exist in the ability to solve verbal problems; and
(3) socio-economic status alone does not appear to be a significant factor.

What are the characteristics of good problem solvers?

Many researchers have proceeded on the assumption that if we can ascertain what problem solvers who are successful have in common, we may be able to help those who do not do as well. <u>Alexander</u> (1960) and <u>Hansen</u> (1944) compared pupils on selected factors thought to be related to problem solving ability. Among the factors which characterized high achievers were: (1) ability to note likenesses, differences, and analogies; (2) understanding of mathematical terms and concepts; (3) ability to visualize and interpret quantitative facts and relationships; (4) skill in computation; (5) ability to select correct procedures and data; and (6) comprehension of reading materials.

Related to these findings are the specific errors which <u>John</u> (1930) found that children in grades 4, 5, and 6 made in solving problems: errors in <u>reasoning</u>, in use of <u>fundamentals</u>, and in <u>reading</u> were found to be most frequent. Johnson (1944) noted that other researchers reported similar reasons why children do not succeed in solving problems: (1) ignorance of mathematical principles, rules or processes; (2) insufficient mastery of computational skills; and (3) inadequate understanding of vocabulary. In a somewhat more recent study, Chase (1960) reported test data collected from sixth graders showing that the three primary factors related to success in problem solving are computation, reading to note details, and knowledge of fundamental mathematical concepts.

Treacy (1944) and Alexander (1960) found that good and poor achievers in problem solving differed on many aspects of reading. Treacy concluded that reading should be regarded as a composite of specific skills rather than as a generalized ability. We may infer that reading and other interpretive skills should be specifically developed in the problem solving program.

Balow (1964) studied 468 sixth graders who had been classified by reading and computational levels. He reported that higher levels of problem solving ability were associated with higher levels of reading and computational ability, but that much of this relationship apparently was the result of the high correlation of these abilities with IQ.

Robinson (1973) administered a 16-item test on problem solving to 115 sixth graders; students scoring in the top third and the bottom third were then compared on several variables. She concluded that good problem solvers had significantly higher scores for IQ, reading comprehension, arithmetic concepts, arithmetic problem solving, and self-esteem, and significantly lower scores on test anxiety. More impulsive students were poor problem solvers, while more reflective students were good problem solvers.

In a study with 63 high and 63 low achievers in arithmetic in grade 4, Walek (1973) found that reflective students did significantly better than impulsive students on a one-step-problem-solving test designed to assess a student's ability to analyze a problem and select the appropriate operation needed to solve it. On a problem-solving test involving the ability to estimate, no significant differences were found.

Talton (1973) analyzed 38 mental, mathematical, reading, and personality scores for 56 sixth graders classified as high achievers in verbal problem solving and 56 classified as low achievers. Combinations of scores from which each group could be predicted with up to 95% accuracy were determined. She concluded several things that have been stressed frequently in the literature on problem solving:

(1) IQ is the greatest single contributor to high achievement in verbal problem solving.

(2) Activities stressing certain reading skills should improve problem-solving ability--selecting main ideas, making inferences, constructing sequences, following directions for simple and complex choices, and reading maps and graphs.

(3) Opportunities should be provided for children to determine the question to be answered, select specific facts necessary to solution, and choose the appropriate process for the solution.

(4) Children should have experience with problems which contain unnecessary data, insufficient data, and no numbers.

What factors associated with the presentation of verbal problems contribute to their ease or difficulty?

We know that many children have difficulty in deciding what operations to use to solve a given problem. It therefore has seemed evident to researchers that to make this decision without guessing or using trial and error procedures, pupils must understand both the <u>meanings</u> and the <u>effects</u> of the fundamental operations. <u>Pace</u> (1961) presented one group of fourth graders with systematic instruction in which children not only decided how to solve a problem, but <u>why</u> that process was appropriate, while another group merely solved the problems with no discussion. The first group made statistically significant gains on tests of problem solving. Interviews and other tests used to measure understanding showed that both groups improved, with greater gains for those who received specific instruction.

<u>Bolduc</u> (1970) reported that, for first graders to whom addition problems were read, those presented without a visual aid were significantly more difficult than those with a visual aid.

For fourth graders, <u>Linville</u> (1970) found that both vocabulary level and sentence structure were determiners of difficulty level of problems. In another study with fourth graders, <u>Swart</u> (1970) found that pupils who were directed to draw a picture "telling the entire story" did as well as those who were taught to write equations to solve the problems. For problems in area measure, those who drew pictures scored significantly higher.

Kellerhouse (1975) found that when the sets of objects identified in a verbal addition problem had different names (e.g., cats, dogs, animals), rather than the same name (e.g., only books), the presentation of a visual aid appeared to impede rather than facilitate the problem-solving performance of second-grade pupils.

Among those who experimented with the teaching of vocabulary was VanderLinde (1964), who reported that specific instruction on quantitative vocabulary was effective in increasing problem solving scores (for problems in which that vocabulary was used).

Harmon (1970), from a study with sixth graders, reported that an "expository" approach was more effective than an "inquiry" approach for problems involving ratio concepts, especially for children of average and high intellectual ability.

Heine (1972) found no significant differences in problem-solving performance between fifth-grade pupils who were or were not taught selected topics in elementary logic.

Steffe and Johnson (1971) investigated the relation of several variables to the performance of first-grade pupils on verbal problems involving these types of sentences: $a + b = n$, $a - b = n$, $a + n = b$, and $n + a = b$. The presence or absence of "described action" in the problem statements did not influence pupils' problem-solving performance. [This was not consistent with findings reported from earlier studies by Steffe (1967) and LeBlanc (1968) for problems involving sentence types $a + b = n$ and $a - b = n$.] Pupils' use or non-use of manipulable objects in connection with problem-solving performance was not related in a consistent way to problem types and to the presence or absence of "described action" in problem statements.

What problem settings are most effective?

Many persons have investigated whether children's success in solving problems is affected by the familiarity in the settings. Brownell and Stretch (1931) reported on the reactions of 256 fifth graders to carefully matched problems at four degrees of familiarity. They concluded that there is "no ground for reasonable belief that problems are made unduly difficult for children by being given unfamiliar settings."

While some other researchers confirmed this finding, there is conflicting evidence on this question. Washburne and Osborne (1926) concluded that unfamiliarity of setting has some influence on success in problem solving, although it is "not as large an element as might be supposed." On the other hand, Sutherland (1942) was among those who found that pupils were decidedly more successful on problems with familiar settings.

Kamins (1971) attempted to determine if the appearance of familiar settings, things, people, and subjects in the language of word problems would affect the success of black children from a lower socio-economic environment in solving word problems. For the 32 fifth graders involved, no significant difference in achievement was found between use of problems written by children and textbook problems.

It has been concluded by many researchers that children like a variety of problem settings. It seems important that children be interested in problems and in ways of solving them.

In studying a different aspect related to this question, Scott and Lighthall (1967) reported that no statistically significant relationship was found between "need content" in problems and degree of "disadvantage." ("Need content" was defined low if problems concerned food and shelter, and high if they concerned such factors as belongingness, education, travel, etc. "Disadvantage" was determined by whether or not pupils were assured of food and shelter.)

Does the order in which fundamental processes appear affect problem difficulty?

Citing data from 4,444 pupils in grades 4, 6, and 7, Berglund-Gray and Young (1940) said "yes." They reported that the easier order for each pair of operations with whole numbers in two-step problems was: addition before subtraction or division; subtraction before division; and multiplication before any of the three others. However, we should note that this study was conducted at a time when there was considered to be only one way of solving a problem.

Does the order of data affect problem difficulty?

Burns and Yonally (1964) reported that, when the data in each of ten multi-step problems were in the order required to solve them, significantly higher scores resulted than when data were not in the order in which they would be used. For the 95 fifth graders they studied, reasoning ability was positively related to pupil success with problems which presented numerical data in mixed order.

Four variables which significantly affected the difficulty of word problems were identified by Loftus (1971): number of operations, sequence of problems, complexity, and conversions. Verbal clues, order of operations, and number of steps had little effect on difficulty level. For the study, she used a computer-based teletype-presented program of 100 problems, and analyzed data from 16 sixth graders.

Should we place the question at the beginning or the end of a problem?

Williams and McCreight (1965) concluded that for fifth and sixth graders, there was "some advantage to the child when the question was placed first," though no significant difference between mean scores was found. Time to solve was less when the question was placed at the beginning.

Bolduc (1970) also noted no significant difference in first-grade pupils' problem-solving performance when the question was placed first vs. last.

What is the role of formal analysis in problem solving?

Research evidence does not show that formal analysis (that is, requiring pupils to answer a specific set of questions in order) is an effective procedure (e.g., Burch, 1953). Washburne and Osborne (1926) noted that "merely giving many problems . . . appears to be most effective." Pace (1961) also suggested that giving many opportunities to solve problems and letting children solve problems in a variety of ways were especially helpful.

What procedure for teaching problem solving is most effective?

In a well-controlled study, Wilson (1967) studied two problem solving procedures, one using equations which express the real or imagined actions in the problem (an "action-sequence" structure) and the other using equations which emphasize operations by which the problem may be solved directly (a "wanted-given" structure), and a third practice-only control treatment. He reported that differences for ability to choose the correct operation, accuracy, and speed favored those taught the "wanted-given" structure over those taught the "action-sequence" structure on tests given during instruction and after a nine-week retention period. The "wanted-given" structure was also significantly better than the practice-only treatment on the immediate posttest and the retention test. On the other hand, Lindstedt (1962) reported many differences favoring a group who used a text program in which equations are structured in terms of the action, over a group using a "traditional type of problem solving program."

Jerman (1973) reported no significant differences between fifth-grade groups using a general problem solving program, a wanted-given program, or the regular textbook. Some effect on strategies, especially for the wanted-given approach, was noted. Jerman pointed out the dependence of problem-solving skill on computational ability. Perhaps hand-held calculators can be used to advantage to minimize the extent to which computational facility influences the correctness of problem solutions.

Could it be that one of these procedures is better than the other for certain children?

What other techniques help in improving pupils' ability to solve problems?

Many specific techniques have been reported to be helpful, though how helpful has been impossible to determine from the structuring of the research studies. Among the techniques which researchers suggest are:

(1) Provide a differentiated program, with problems at appropriate levels of difficulty.

(2) Have pupils write the number question or mathematical sentence for a problem.

(3) Have pupils dramatize problem situations and their solutions.

(4) Have pupils make drawings and diagrams using them to solve problems or to verify solutions to problems.

(5) Have pupils formulate problems for given conditions.

(6) Present problems orally.

(7) Use problems without numbers.

(8) Have pupils designate the process to be used.

(9) Have pupils note the absence of essential data, or the presence of unnecessary data.

(10) Have pupils test the reasonableness of their answers.

(11) Use a tape recorder to aid poor readers.

Some evidence exists to support each of these. Keil (1965) found that pupils who wrote and solved problems of their own were superior in problem solving ability to pupils who had the "usual textbook experiences." Riedesel (1964) reported that sixth grade classes using specific procedures plus 30 sets of verbal problems at two levels of difficulty achieved higher mean gains on problem solving tests than did control groups who followed the regular textbook program. In another instance, Arnold (1969) reported evidence from sixth graders favoring the expression of problem relationships in number sentences. It should be noted that emphasis upon isolated word cues ("left," "in all," etc.) can be grossly misleading as a problem solving procedure. They may lead pupils away from recognition of the relationships inherent in the problem, which are crucial to its solution.

Is it helpful for pupils to work together in solving problems?

Evidence by investigators in other areas has indicated that children can learn more by working with partners or small groups than by working alone. In relation to verbal problem solving, however, this evidence has not been so clear.

Hudgins (1960) reported that fifth graders who worked on sets of verbal problems in groups of four solved significantly more problems than those who worked alone. When they then worked individually, no significant differences were found among their scores. In an extension of this study, Hudgins and Smith (1966) found that for pupils in groups of three, group solutions to problems were no better than the independent solutions of the most able member of the group, if he is perceived to be most able. (If he is not so perceived, the group will do better than he--or change their perception of him.)

Klugman (1944) found that two children working together at grades 4, 5, and 6 solved more problems correctly, but took a longer time than pupils working alone. In another study with fourth, fifth, and sixth graders, Dembo (1969) reported that there were no significant differences in the improvement of peer relations, attitude toward mathematics, or mathematical achievement between pupils working in small groups or independently.

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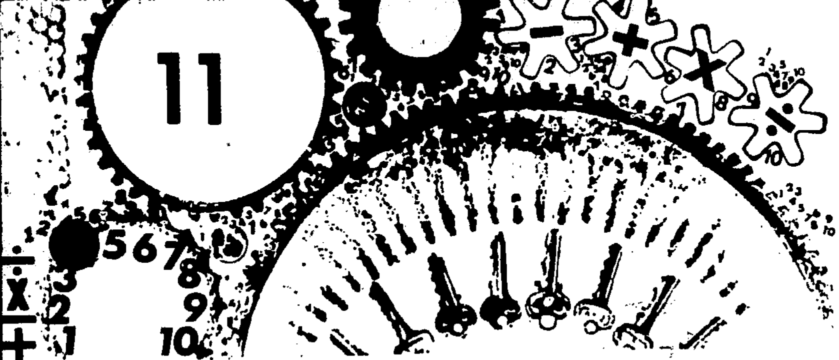


PLANNING FOR RESEARCH IN SCHOOLS

WHAT IS RESEARCH?

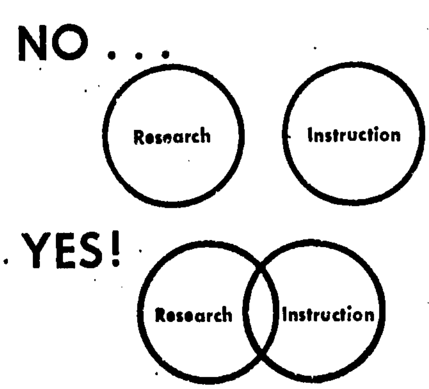
Research is controlled inquiry.

In these bulletins, we discuss research on the elementary school mathematics curriculum and research on the teaching and learning of mathematics. The vast majority of this research is product-oriented; there is, however, other research which is theory-oriented. The task of building a theory of the learning of mathematics concepts still lies before us, as Begle (1968), Glennon (1966), and Heimer and Lottes (1973) have noted.

Many of the studies we have cited have involved either experimental or survey research. By experimental we mean research in which the investigator has "manipulated" one or more specified variables, such as two methods of teaching, to measure their effect on another variable, such as achievement or attitude, thus testing a carefully formulated hypothesis or hypotheses. The variables which are manipulated are termed "independent," while those affected and measured are "dependent" variables. Experimental research is very difficult to conduct, because of the need to control the independent variable(s) and many other variables--which must be controlled since we want to interpret the results and generalize beyond the sample in the study. By survey we mean research which attempts to ascertain the characteristics of a population by studying a sample which answers a questionnaire or interview or test.

As we continue to discuss "research" in this bulletin, the focus is on experimental research. You should recognize, however, that certain of the things discussed are also applicable to other types of research. We should caution that, despite this focus on experimental studies, we are not thus implicitly stating that such investigations are the only ones which qualify as "true research." Other types of studies also contribute to the improvement of mathematics education.

Research is not independent of instruction. It is derived from and is applied to instruction. Actually, every teacher does a type of "action research" every day--whenever new ideas are tried out. You're constantly trying to find the methods and materials and procedures which will work best for you. You're assessing what pupils have learned, and using what you find out as you plan what to do next.



You're concerned with what will help you teach better, or help your pupils learn better. You've been using evaluation, and for some purposes--such as curriculum development--evaluation is vital.

For other purposes, however, research is essential. Research involves more precise controls. In experimental research, we are attempting to secure information which can be generalized to many other teachers and to many different situations. In survey research, we also maintain greater controls than in usual classroom testing--we want a more precise measurement of the status or level of learning.

WHY?

Research can provide a foundation on which to make curricular decisions and decisions about how to teach. Nothing has ever been proven by educational research--but it has provided guidelines to aid us in making decisions. It should be noted, however, that not all problems are amenable to research--some decisions must be made on the basis of your philosophy. For instance, research can provide an answer to "Can we teach logic to fourth graders?" but it cannot provide an answer to "Should we teach logic to fourth graders?"

Research has a valid role to play in assessing and improving the quality of instruction. In fact, merely being involved in research helps us to achieve this latter goal. As Pikaart and Berryman (1965) note, "Participating in research and contributing significant ideas was in itself motivating, and it contributed to self-esteem."

Local school systems may at times need to engage in their own research for other reasons. For instance, generalized findings may not be applicable when unique characteristics of the system are considered (e.g., ability level of the pupils).

HOW?

First of all, select a question which is important to answer. Then design the study: lay out an overall plan, delimiting the problem to make it researchable. This may be a long-term plan, but don't try to investigate everything at once: order your priorities logically.

You will need to identify and define or describe (1) the independent variable or variables and (2) the dependent variable or variables. You must also identify and control other relevant variables. Suydam (1967) reported that control of variables was one of the two most poorly handled facets of mathematics research studies (sampling was the other one). As Johnson (1966) noted, certain assumptions are made regarding what variables may affect the situation. During an experiment, the groups involved should have common experiences except for the treatment (independent) variables. Then significant differences at the end of the experiment can be attributed to the treatment. Johnson presents an example of an experiment in which many factors are controlled; Wilson (1967) and Worthen (1968) provide other excellent examples of research in which variables are well controlled.

Some pupil variables may be controlled in one of several ways (Kerlinger, 1964; Riedesel and Sparks, 1968): (1) eliminate the variable as a variable, by studying only a specified subset of the sample; (2) use the statistical procedure of analysis of covariance (but be careful not to "wash out" true differences, as may happen when you apply covariance to a factor of concern); (3) incorporate the factor as another independent variable; (4) match pupil for pupil (this may be difficult, depending on the number of factors on which pupils should be matched); (5) equate on the basis of group means. Or, you can use randomization, where you assume, since pupils are selected by chance, that variables are randomly distributed.

DeVault (1966) and Romberg and DeVault (1967) emphasize our need for realistic research that takes into account the complexity of the classroom setting. On the other hand, if you have a grandiose design that tries to take into account many, many factors, the study will become very complicated. Remember that there is a place to look at small (but not trivial) pieces (Van Engen, 1967).

Select or develop appropriate measuring instruments. Remember especially that "global" or standardized tests are not always appropriate. For example, if you're testing the effect of introducing multiplication in two ways, you'll find that a "global" test has a limited number of items which measure multiplication achievement. The study may result in no significant differences when in fact differences were present--but unmeasured. Instead of a "global" test, a test to measure achievement in multiplication must be constructed.

If two different treatments are to be evaluated, the test must be carefully constructed so it doesn't introduce a bias. Some research has been done in which the test contained a large number of items which only the experimental group would be able to answer (e.g., questions related to a story used to introduce the experimental treatment). Thus the findings of the research favor the experimental group--but not because the pupils did significantly better on the factor being studied.

After you've carefully outlined your research procedures, consider: could I replicate this study, that is, do it over again and expect to get the same results? If you can't answer "yes," replan! Then check your plans with someone who knows research--get professional assistance from your research department or from a university or college, whenever this is possible. This step often makes the difference between good research and a meaningless collection of data, between an answer to your question and no answer. This is the time to clarify questions like "What data should be collected?" and "How will the data be analyzed?" The procedures that are contemplated should not be independent of consideration of the way in which data are to be collected and analyzed. People have been known to collect data and then wander around trying to find a statistic to use. They don't always find one. In fact, one may not even exist!

Research is improved by being tried out first of all with a pilot study--problems are resolved before they affect your major study. For instance, in doing a survey, the questionnaire should be given to a small group before it is used in the study. A test should be administered to a small group, preferably one much like the group who will be involved in the research. You want to be sure each is valid and reliable, that is, that each measures what it's supposed to measure, consistently.

It is wise to consider the timing of research. Usually it's unwise to plan to begin a study on the first day of school. Beware of other things competing--such as a vacation or other projects which claim priority. Length of time should be appropriate to your problem--remember that most studies can't be done in one day. Also remember that the longer the study, the more problems you may have and the more difficult control becomes.

If you have only two classes at a grade level, the temptation is to have one teacher teach one treatment and the other teach the second treatment. Better yet, have both teachers try both--teaching some pupils by one method and some by the other. This eliminates some confounding, since data for each method can be pooled. The teachers must be doubly careful to not let biases interfere--they must do an honest job with each, despite a special preference for one. The way in which a teacher carries out the research plan is one of the most important factors.

Be sure your sample is appropriate for the population to which you want to generalize your results. There are times when it is reasonable to exclude data for a few children who are very different from the rest of the group, since they may bias the research. Better yet, analyze the data for them separately or differentially.

Whenever appropriate to the design, pupils should be randomly selected and assigned to a treatment. "How many children are needed?" cannot be answered in general: there's a number that will give each study sufficient "power." Remember that it may be wasteful of pupil time to use samples larger than necessary. On the other hand, too small a number raises questions about how representative they are, and how far the findings can be generalized.

There are instances in which it is feasible to conduct research only with intact classes. This situation presents certain problems of research design which need to be considered. Campbell and Stanley (1963) provide some help on this type of situation.

There is a time to pretest--when you think that pupils have some knowledge of the subject matter. But in other cases, when you can assume that pupils have no knowledge or equivalent knowledge (e.g., when non-decimal bases are introduced in grade 1), a pretest is not necessary. A pilot study using a pretest will indicate whether or not a pretest is necessary in the final study.

It is desirable for teachers involved to keep logs of what was done day by day, as well as anecdotal records of particular incidents and reactions. Then departures from the planned procedures can be noted; these are sometimes useful in interpreting findings. Also, there is a need for somebody to keep a finger on things as the study progresses, to make sure procedures are being followed.

Reporting and disseminating information about research should be carefully done. It is important that others know what you have done and found. Accuracy in reporting is essential, as well as readability. As Weaver (1967) noted, "We can go a long way toward extending the impact of research if each investigator accepts the obligation to report all significant aspects of this work as fully as is necessary to establish the integrity of his research and of the conclusions drawn." The interpretations must be derived from the data--and remember that there is a difference between findings and implications.

We have frequently cited differences which are "significant" or "statistically significant." By this we mean that there is a specified likelihood that such differences would not have occurred by chance. Usually, the level of significance is set at .05, or .01, or .001--thus the results might occur by chance only 5 times in 100, or only 1 time in 100, or only 1 time in 1000. That is, if the study were repeated by many investigators under the same conditions, you could expect to get the same results 95% or 99% or 99% of the time. "No significant differences" means that a specified level of significance was not reached--thus the results could occur more frequently by chance. Researchers set a level which seems appropriate to them in terms of the content and design of their study.

In summary, as you plan how to develop and implement your research, you may find these questions helpful:

(1) Is the problem practically and/or theoretically significant?

(2) Is the problem clearly defined?

(3) Is the design appropriate to answer the research question?

(4) Does the design control variables?

(5) Is the sample properly selected for the design and purpose of the research?

(6) Are the measuring instruments valid and reliable?

(7) Are the techniques of analysis of the data valid?

(8) Are the interpretations and generalizations appropriate to the data?

(9) Is the research adequately reported?

WHAT? We look only at mathematics research in these bulletins. Some findings from mathematics research, especially those cited in Set A, might be considered in regard to other phases of the curriculum--and research from other areas may be applicable to the teaching and learning of mathematics. One caution is necessary: don't take findings from one field and assume automatically that they are true for mathematics, or any other. It is important to recognize that conceptual learning is particularly important in mathematics. Mathematics may differ from other areas because there is a body of content (unlike language arts, but like science), which also is sequential; therefore there may be different problems for mathematics than for other areas.

Schools probably do not need to do research on those things on which there is "sound" research evidence already (for instance, on the benefits to be derived from meaningful instruction). There are large variations, however--what may be true in general for large groups may not be true for particular, unique groups. What may be true for one topic may not be true for another.

Teachers should test research findings in their own classrooms. Remember that just because research says that something was best for a group of teachers in a variety of classrooms, doesn't necessarily mean that it would be best for you as an individual teacher in your particular classroom. For instance, we're beginning to get evidence that there is an interaction between teaching and method. Thus research may show that an inductive approach is "good,"--yet some teachers may not be comfortable with it or can't manage it. An expository approach may be better for those teachers. Teachers have individual differences as well as pupils! In this same way, remember that learning modes of pupils differ, and that not all content lends itself to use of inductive strategies.

Teachers must be careful not to let prior judgments influence their willingness to try out and explore: open-mindedness is important in research. Be willing to investigate. But being open-minded doesn't mean you don't have beliefs about things--just that you don't let beliefs bias the conduct of research.

Often research may be generated by informal exploration that teachers make, which in itself is not research. Do this--but don't call it research; use it to generate hypotheses which can then be tested with research.

AND THEN . . . Research is not an end in itself--it should lead to some kind of action. You decide to change, or not to change; you will accept something, you will reject something. It may lead to other research. Do something as a result of research: incorporate the conclusions of research into your daily teaching.

Non-significant differences can be as important as significant differences--don't be disappointed or think automatically that research has "failed" when no significant differences result. There might in fact be no differences--and the decision is up to you!

In this bulletin, we have been able to give only a glimpse of some of the things which need to be considered as schools conduct research. You may wish to look further into the design and implementation of research as you plan for your own investigations.

Good luck!

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