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ABSTRACT

The practice of utilizing mathematics instruction, in which only practical engineering applications and not the pure mathematical proofs are stressed, is questioned. Three cases are presented in which students made erroneous conclusions concerning an engineering topic. In each situation, the student violated a mathematical principle which was not evident to him because his instruction had dwelled only on the practical applications of the formulae. (CP)

MATHEMATICAL FORMALITIES AND ENGINEERING STUDIES

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MATHEMATICAL FORMALITIES AND ENGINEERING STUDIES

Many engineering schools have incorporated a mathematics department in their units with the explicit objective to teach "engineering mathematics", as opposed to pure or rigorous mathematics. The subjects in mathematics are taught for direct application and "unnecessary or tedious" proofs are omitted or at best presented as unimportant. As a consequence, engineering students do not have much opportunity to acquaint themselves with carefully presented mathematical developments.

Since the engineering applications presented in class are already well developed and tacitly assumed to be corect, it may appear that no need exists to proceed carefully through mathematical proofs. The instructor omits such proofs in his lectures, often unintentionally "to save time", or sometimes intentionally because he considers them "irrelevant" mathematical formalities. For example, no explicit requirement may be stated that before dividing by a factor, this factor should not be zero, or no mention may be made that a series expansion may be convergent only in a certain range.

The objective of this paper is to discuss personal teaching experiences in which the lack of a basic understanding of mathematical principles precluded a clear comprehension of some engineering concepts.

<u>The consequence of too much confidence!</u>

Too much confidence in "proven" engineering methods may lead to inadmissible errors as demonstrated in this first case presented. In the finite element method for the dynamic analysis of framed structures, the axial, torsional, and flexural actions are expressed in terms of the elastic and inertial forces constituting respectively the stiffness and mass matrices. These matrices relate the forces to the displacements or to the accelerations of a basic beam element and are derived from assumed displacement functions. Specifically, it is assumed that the static deflection curve of a beam element also represents the deflection for a dynamic situation. This approach permits discretizing of the continuous member by means of its elastic properties as given in the stiffness matrix, and its inertial characteristics as given in the consistent mass matrix[1]. A more crude and simple method of discretizing the inertial properties of a beam element is to simply allocate half of its total mass to each end as a concentrated lump mass. The discretizing process has also been carried out mathematically by means of the appropiate differential equation. From this is derived the dynamic stiffness matrix which expresses the exact relation between harmonic forces and displacements in terms of transcendental trigonometric and hyperbolic functions.

Some time ago I proposed as a topic for a Master's Thesis, a parametric comparative study of the three methods of discretizing a beam element. The student working on this subject wrote a computer program and proceeded to test the program selecting simple numerical values. Specifically, he gave unit values to the modulus of elasticity, moment of inertia, and other parameters. It happened that all these three methods which normally produce relatively close results, for the simple unit values gave widely different answers. This unexpected result certainly added more interest to the subject of this thesis and resulted in a technical paper[2] in which it was shown that so-called stiffness and mass matrices were nothing other than the first two terms of the power series expansion of the exact dynamic stiffness matrix. The mathematical approach used to obtain these matrices, in contradistinction to the engineering approach, allowed the specification of the range convergence of the resulting series. It was found that the unitary numerical values used in testing the computer program were outside the range of convergence of the series. Consequently, the fact that these values led to divergent results with large errors was no longer surprising or unexpected.

Make the determinant equal zero!

The experience described above led to the following second case. As has already been stated the use of the exact equation for the discretization of framed structures yields the dynamic stiffness matrix containing trigonometric and hyperbolic functions. The common procedure to find the natural frequencies of such structures is to set the determinant of the dynamic stiffness matrix equal to zero. In the search for zeros, the determinant is evaluated for a series of values of frequency in the region of interest. A change in the sign of the value of the determinant indicates the presence of one or more natural frequencies. In other words, to find the natural frequencies, the analyst sets the determinant equal to zero and searches for its roots. As it happens, the searching process is not possible for certain values of the argument; for these values the determinantal function is not defined. This difficulty occurs in the study of flexural vibration as well as for torsional or axial vibration. The situation can be illustrated for the simple case of an axially-loaded rod. The governing equation in this case, as given in reference[3], is the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{m}{AE}\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

where m is the mass per unit of length, A the cross-sectional area, E the modulus of elasticity, and u the displacement at coordinate x and time t.

The solution of equation (1) yields a harmonic motion of amplitude

$$u = C_1 \sin \beta x + C_2 \cos \beta x \qquad (2)$$

where

$$\beta = \left(\frac{m \omega^2}{AE} \right)^{\frac{1}{2}} \qquad (3)$$

ω is the natural frequency and C_1, C_2 are constants of integration.

To obtain the dynamic stiffness matrix for the axially vibrating element, boundary conditions indicated by equations (4) and (5) are imposed.

$$u(x = 0) = \delta_A \qquad u(x = L) = \delta_B \qquad (4)$$

$$\frac{du(x = 0)}{dx} = - \frac{P_A}{AE} \qquad \frac{du(x = L)}{dx} = \frac{P_B}{AE}$$

where δ_A, δ_B and P_A, P_B are respectively the displacements and forces in the beam element as shown in Fig. 1.

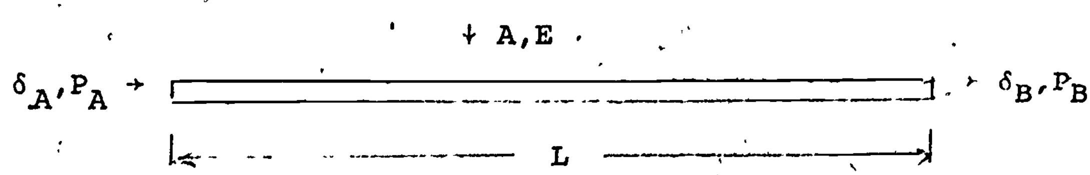

Fig. 1. Displacement and forces for axially loaded beam

Substituting the boundary conditions of equations (4) and (5) into equation (2) results in

$$\begin{bmatrix} \delta_A \\ \delta_B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin\beta L & \cos\beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \qquad (6)$$

and

$$\begin{bmatrix} P_A \\ P_B \end{bmatrix} = AE\beta \begin{bmatrix} -1 & 0 \\ \cos\beta L & -\sin\beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \qquad (7)$$

Equation (6) is solved for the constant of integration, namely

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\cot\beta L & \operatorname{cosec}\beta L \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_A \\ \delta_B \end{bmatrix} \qquad (8)$$

subjected to the condition

$$\sin\beta L \neq 0 \qquad (9)$$

Subtitution of equation (8) into equation (7) gives equation (10) which relates the forces to the displacements at the coordinates through the dynamic matrix.

$$\begin{bmatrix} P_A \\ P_B \end{bmatrix} = EA\beta \begin{bmatrix} \cot\beta L & -\operatorname{cosec}\beta L \\ -\operatorname{cosec}\beta L & \cot\beta L \end{bmatrix} \begin{bmatrix} \delta_A \\ \delta_B \end{bmatrix} \qquad (10)$$

It is interesting to observe that it is not possible to find the natural frequencies for an axially loaded beam by simply letting the determinant of the dynamic stiffness matrix in equation (10) be equal to zero. Subjected to the condition given by equation (9), the determinant of this matrix has a constant value, minus one. Furthermore, the condition given by equation (9), which is required to obtain equation (10), precludes the determination of the natural frequencies from the determinantal function. The zeros of sin βL are precisely the natural frequencies of a beam element undergoing axial vibrations. For such an element, the natural frequencies may be found by equating to zero the determinant of the 2 x 2 matrix in equation (7), but not by using the dynamic stiffness matrix as has been demonstrated. This difficulty arises because the natural frequencies of the system formed by only one member or element are obviously the same as the natural frequencies of that member. In an actual problem where the dynamic stiffness matrix for the system is obtained from several elements, the determinant of this matrix will not be defined for those values of frequency equal to any of the natural frequencies of the component elements. This fact which has been illustrated for an axially loaded beam is equally valid for torsional or flexural modes of vibration. Furthermore, it should be emphasized that any of these critical values (the natural frequencies for isolated members) actually may or may not be a natural frequency for the structure as a whole.

The nature of the difficulty in finding the natural frequencies by equating the determinant of the dynamic stiffness matrix to zero lies in the mathematical condition (equation (9) for axial vibration) required to obtain the dynamic stiffness matrix of the element. Failure to give due mathematical regard to conditions of the nature of equation (9) may produce either a spurious solution or a paradoxical situation such as the one described in this article.

Now make the determinant equal to infinity!

Another situation in which lack of a correct and careful mathematical treatment leads at best to embarrassing results is given by the following case described in detail elsewhere[4].

This case deals with the dynamic analysis of continuous beams presented in various texts on structural dynamics (Timoshenko (3), Biggs (5), Fertis (6)). The analysis consists of establishing the compatibility conditions at the intermediate supports of the beam. These conditions lead to an equation relating the redundant moments at three consecutive supports, of the same form as the well-known Equation of Three Moments in statics. Using this procedure, one equation may be written for each internal support of the continuous beam; the result is a system of linear homogeneous equations. In order for any free vibration of the beam to be possible, the determinant of the coefficients of the system of equations must be equal to zero.

To illustrate this method, the particular case of a continuous beam of two equal spans invariably is presented in texts on structural dynamics. For this case there is one redundant moment and only one equation. The corresponding coefficient of this equation is set equal to zero, resulting in the equation

$$2(\coth \lambda_n L - \cot \lambda_n L) = 0 \tag{11}$$

where

$$\lambda_n = \sqrt[4]{\frac{m\,\omega_n^2}{EI}} \tag{12}$$

In this equation, L is the length of one span of the beam, λ_n is the natural frequency corresponding to the n-mode, m the mass of the beam per unit of length and EI its stiffness. The roots of equation (11) are then determined numerically, or graphically by plotting $y_1 = \coth\lambda_n$ and $y_2 = \cot\lambda_n L$ as functions of the argument $\lambda_n L$. The first few of these roots are thus found to be:

$$\lambda_n L = 3.92,\ 7.06,\ 10.2,\ \ldots\ldots\ldots\ldots\ldots\ldots$$

Up to this point the procedure is a straightforward method for finding the eigenvalues of the system and the results appear to be satisfactory. However, as it is correctly realized by the authors of the texts on this subject, a whole series of eigenvalues are not included among the roots listed above. According to these authors, the missing values are then determined by letting the expression on the left side of equation (11) be equal to infinity, that is

$$\coth \lambda_n L - \cot \lambda_n L = \pm \infty \tag{13}$$

which then gives the missing roots as

$$\lambda_n L = \pi,\ 2\pi,\ 3\pi,\ \ldots\ldots\ldots\ldots\ldots\ldots$$

The natural frequencies may then be found by substituting into equation (12) values of λ_n and solving for ω_n. Although the final results give the correct numerical values for the natural frequencies of the two spans beam, it leaves the reader in a quandary with no explanation forthcoming for the formulation of equation (13), in which the left side of equation (11) was set equal to $\pm\infty$.

The elucidation of the answer to this question requires a careful examination of the mathematical development to the classic Equation of Three Moments for vibration of continuous beams. In this derivation it is necessary to divide by the factor $\sin \lambda_n L$, which for this purpose should be assumed as not equal to zero, thus,

$$\sin\lambda_n L \neq 0 \tag{14}$$

Consequently, this assumption precludes, as possible eigenvalues, the roots of $\sin\lambda_n L = 0$, which for the particular case of a continuous beam of two equal spans, are also eigenvalues of the problem. The troublesome, unexplained use of equation (13) can be eliminated if the analysis is carried out mathematically for the special case of the beam of two equal spans. For this case, the reduction and expansion of the determinantal equation is given by

$$\sin^3 \lambda_n L \sinh \lambda_n L (\cos \lambda_n L \sinh \lambda_n L - \sin \lambda_n L \cosh \lambda_n L) = 0 \quad (15)$$

from which the characterictic roots are found by setting the factors in equation (15) equal to zero, namely,

$$\sinh \lambda_n L = 0 \quad (16)$$

$$\sin \lambda_n L = 0 \quad (17)$$

$$\cos \lambda_n L \sinh \lambda_n L - \cosh \lambda_n L \sin \lambda_n L = 0 \quad (18)$$

Equation (16) gives only the trivial solution $\lambda_n L = 0$, while equation (17) gives correctly the series of eigenvalues $\pi, 2\pi, 3\pi$.......... and equation (18) which may be written as equation (11) for $\sin \lambda_n L \neq 0$ and $\sinh \lambda_n L \neq 0$ gives, as before, the series of eigenvalues 3.92, 7.06, 10.2 ....

Conclusions.

Three cases have been presented to illustrate that the lack of sound mathematical treatment of an engineering problem could result in an unacceptable or erroneous analysis. Many other examples from the classroom or professional practice may be added to the cases presented. Without taking an extreme position, I am inclined to believe that engineering students should not be exposed exclusively to applied or "practical mathematics" but also to rigorous presentations by competent professors of mathematics. In this way the student would develop an appreciation and respect for mathematical formulations, conditions and proofs.

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