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ABSTRACT

This manual is a collection of materials and teaching strategies to motivate the development of mathematical ideas in secondary school mathematics programs or in beginning college mathematics programs. The unit is written for the instructor with step-by-step procedures including lists of needed materials. The exercises in this unit also appear in the separate publication, "Experiments in Experimental Mathematics." Contents include: geoboard activities in area and with the Pythagorean Theorem; exercises with arithmetic numerals; problems illustrating balance relationships; perfect number exercises, hidden combinations, and coordinate graphing; arrays, polynomials, and finite differences; physical problems that introduce convergent and divergent series; map coloring (Euler's Theorem); the analysis and prediction of patterns of motion with cycloids and area; and the Euler function. The unit concludes with more than 18 short investigations such as Tower of Hanoi Puzzle, box problem, Kohgisberg Bridges problem, limits of sequences, and games that employ mathematical analysis. (Author/JBW)

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Bishop College	Dallas, Texas
Clark College	Atlanta, Georgia
Florida A and M University	Tallahassee, Florida
Jackson State College	Jackson, Mississippi
Lincoln University	Lincoln University, Pennsylvania

Norfolk State College	Norfolk, Virginia
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Southern University	Baton Rouge, Louisiana
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CONTENTS

PREFACE

This unit is a collection of materials and teaching strategies which offer useful motivation for developing some important Mathematical ideas. The intuitive approach is stressed throughout. The procedures stimulate the imagination and help build self-confidence by the "doing" of mathematics. In using this approach, the teacher is urged to restrain himself from telling the formulas prematurely. Informal proofs are generally acceptable when they provide convincing arguments during class discussion.

This unit is written for the instructor with step by step procedures for developing many mathematical notions. The exercises included in this unit will also appear in a separate publication, "Exercises in Experimental Mathematics."

Special appreciation is due to Jack Alexander, Lee Evans and Charles Haynie, members of the Curriculum Resources Staff and to Roger Ingram, Kenneth Hoffman, James Kirkpatrick, William Nicholson, Newcomb Greenleaf, for their contributions to this unit.

For the tedius task of pulling together the materials of the contributors, special credit is due Johnsie Jo Posey and Carl Whitman.

GEO BOARDS

INTUTITVE DEVELOPMENT OF AREA AND THE PYTHAGOREAN RELATION

Provide each student with a geo board and a handful of rubber bands. A desirable board should have 1 inch spacing between pegs (small 3/4 nails) with a 13 x 13 array of the pegs (169 pegs) at the vertices of 1 inch squares covering the face of the board.

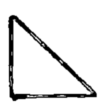
What can you do with this?

Allow students to explore the characteristics and potentials of the board. Move around among students encouraging their exploration, asking about the interesting things they have found out. When the class shows some restlessness with this activity, suggest that they consider some things you've found of interest.

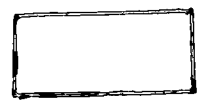
What do we mean by the word "Area"?

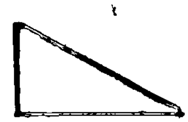
You are likely to hear a confusion of formulas such as Area = (e)(w)(h) or Area = $2\pi RH$.

After some discussion of what is me at by Area, it may be well to move to establish as a definition that  will be one unit of area and that the side is a unit length (not 2 pegs length).

What, then, would this area be? 

(Don't mention area of triangle formula, but clearly this area is half that of the unit square since two such triangles give the unit square.)

What is the area of this? 

What would the area of this be? 

Frequently we have to be satisfied with obtaining only a good guess of the area of geometrical figures.

Make up any closed figure and determine its area using only square units

The student should be led to the following possible solutions:

(a) A lower estimate of the area is found by counting the maximum number of squares which lie entirely within the figure. (A side or a corner of a square might be part of the boundary.)

(b) An upper estimate of the area is found by adding to the lower estimate the number of squares which meet the boundary. (Do not add those squares with only a vertex or a side in common with the boundary.)

If some students find the area by using triangles and note that the area is the average of these two estimates, stress that this is true only because of the limitation of using the geo board to construct figures. A discussion of areas of arbitrary plane figures would involve limits.

Area Formula

Form a 5 x 5 square by putting a large rubber band around such a square.

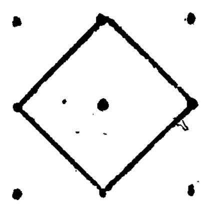
How many squares of different area can you find in this square?

Students will volunteer those with area 1, 4, 9, 16, and 25.

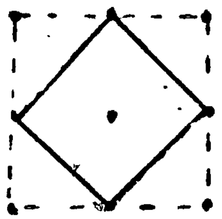
I can see more.

Eventually a student will come up with a square with sides rotated as

What is the area of this square?



Although there are many ways to determine this area, the following procedure is convenient.



Enclose the figure of unknown area as efficiently as possible inside a figure of known areas. As illustrated above, it is enclosed within a square of area 4. (However we also know the area of the four triangles which have been added, i. e., each has area 1/2.) Therefore the area of our given square is 4 - 4 (1/2) = 2.

Can you find a square with the areas of 5? 3? and 2 units?

Investigation of the length of the side for this and similar squares leads to the Pythagorean Relation. This is a rewarding excursion, but our illustrative development will proceed first to investigate a nice inductively available pattern related to area.

Now, using the whole geo board, make up any simple closed polygon* you like and determine its area.

When this has been explored at some length, the following exercise will point out the importance of the number of pegs on the boundary and the inside.

*To avoid the possibility of a student forming a "figure eight" type of polygon, you need to introduce the concept of "simple closed polygons." Otherwise the formula will not work.

Construct a figure on the geo board having the following properties and fill in the blanks. On graph paper copy the figures from the geo board.

B	I	A
3	0	___
4	0	___
5	0	___
6	0	___
3	1	___
4	1	___
5	1	___
6	1	___
3	2	___
4	2	___
5	2	___
6	2	___
___	___	___
___	___	___
___	___	___
___	___	___

B = Number of pegs on the boundary

I = Number of pegs in the interior

A = Area of the polygon

Although the students will begin to see a pattern taking shape, this will not be sufficient information for the students to generate the formula for area by themselves.

To generate fun with this procedure, you should arouse curiousity by your seemingly mysterious ability to get the answers instantly while the students take so long. Recall the previous exercise to reinforce the importance of the number of pegs, both inside and out.

Let students develop their own bookkeeping system for data relative to number of pegs — boundary and inside the polygons.

After considerable class work (and as possible homework) provide students with worksheets listing columns of pegs inside, given polygons as their boundaries, and their areas. The problem for students is to find a pattern to the data and formulate it in a simple expression. (At least half the class is likely to get the formula, eventually, they lose interest once it is established.)

Secret Formula

The formula which students will discover and will enable the teacher to verify polygon areas mentally is:

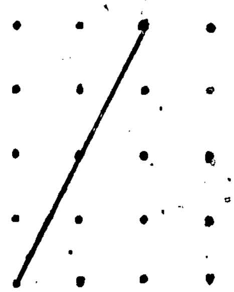
Area = 1/2 (No. pegs on Boundary) + No. pegs inside - 1

$$A = \frac{1}{2}B + I - 1$$

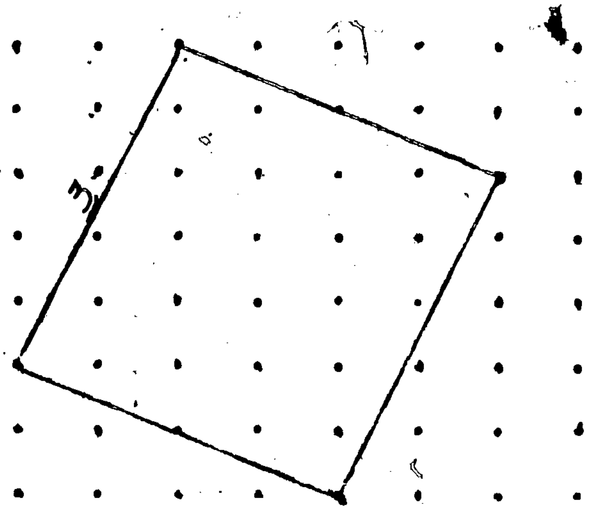
Pythagorean Theorem

We may now go back to the question, "What is the length of the side of a square"? If the area is 1, the square is of unit length but stress it as $\sqrt{1}$. If the area is 4, the length of each side is $\sqrt{4}$ or 2.

Now consider the length of any line which can be formed on the geo board.



The length of this line can be determined by finding the area of the square which has a side of this same length.



By previous experience the student will be able to determine that the area is 20, and therefore the length of the line is $\sqrt{20}$.

Proceed with this method several times.

To find the length of any given non-vertical, non-horizontal line on the geo board, form a right triangle with rubber bands using the given line as the hypotenuse; then form 3 squares using each side of the triangle as a side of one of the squares.

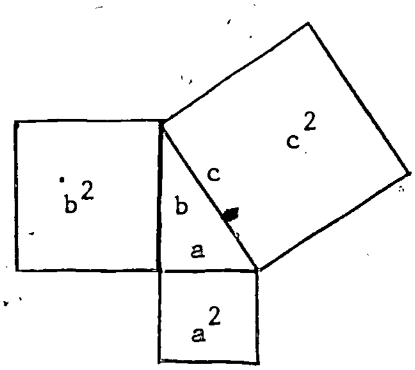
What is the length of a side, S?

Students may or may not volunteer the Pythagorean Relation. Many do not remember and understand, so encourage everyone to do the geo board configurations described below -- these are likely to be vividly remembered.

Put a band around any right triangle and then using 3 rubber bands form a square using each triangle side as a site of a square.

What special pattern do you notice about the areas of the squares?

$c^2 = a^2 + b^2$ in some form should be volunteered although the geometric descriptive is better for class discussion.

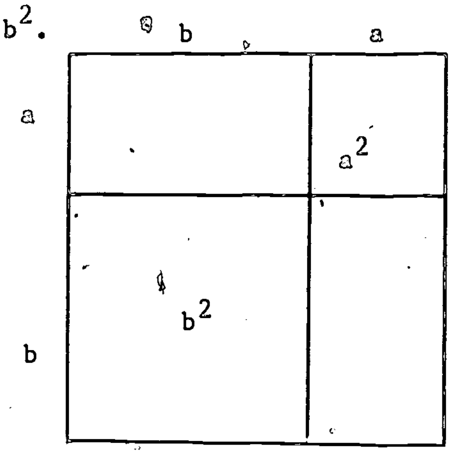


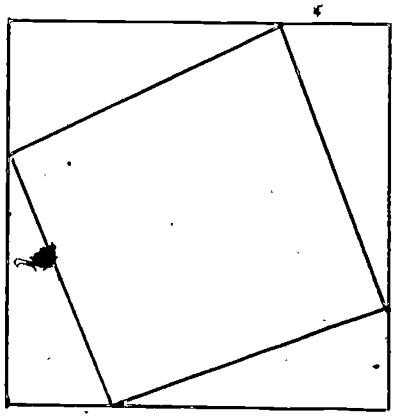
Do you think this pattern will always hold?

What does it mean?

How could you prove that it is always true?

Answers to this may lead to some of the dissection proofs of the Pythagorean Theorem such as for any right triangle with sides a, b, and c. Construct two squares with sides a + b. Subtract equals from equals to get $c^2 = a^2 + b^2$.





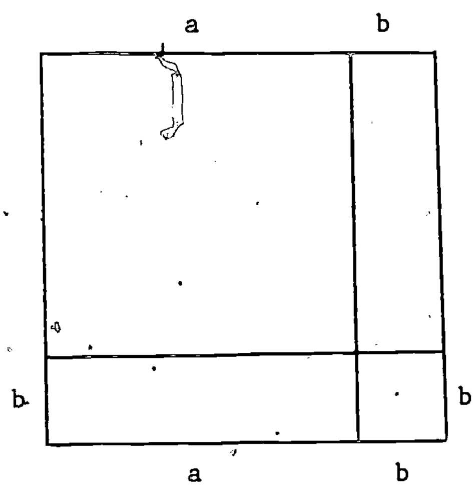
Exercises

Prove or find a proof of the Pythagorean Theorem. (A proof of the Generalized Pythagorean Theorem is given in this report.)

Using the geo board verify whether or not $(a + b)^2 = a^2 + b^2$.

If not what might the right side be?

The identity $(a + b)^2 = a^2 + 2ab + b^2$ is illustrated by the configuration at the right.

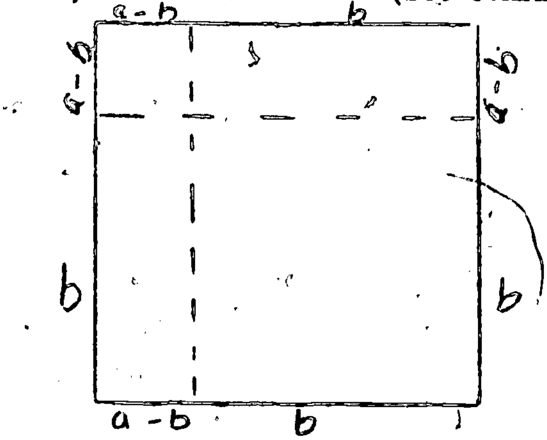


Demonstrate the identities with the geo board or using pencil and paper:

(a) $a(b + c) = ab + ac$

(b) $(a + b)(c + d) = ac + bc + ad + bd$

(c) $(a - b)^2 = a^2 - 2ab + b^2$ (See configuration below.)



(d) Generalized Pythagorean Theorem for all triangles.

The following is the standard form of the Cosine Law, but it can be reduced to a more geometric form in the following manner:

$$c^2 = a^2 + b^2 - 2ab \cdot \text{Cos } C$$

$c^2 = a^2 + b^2 \pm 2ab \frac{x}{b}$, where the + or - sign depends on the size of the angle.

Therefore $c^2 = a^2 + b^2 \pm 2ax$, where x is the length of the projection of b on a; the + sign is used if C is obtuse, the minus sign if C is acute.

The proof of these two statements depends upon the teacher having already taught the standard right angle form:

$$c^2 = a^2 + b^2$$

Case I

∠C is obtuse. (See diagram on page 12.)

On the geo board construct an obtuse triangle: $\angle C > 90^{\circ}$.

Using the geo board, construct the diagram on page 12.

Show that

$$c^2 = (a + x)^2 + y^2$$

$$c^2 = a^2 + 2ax + (x^2 + y^2)$$

$$c^2 = a^2 + 2ax + b^2 \quad (\text{since } b^2 = x^2 + y^2)$$

$$c^2 = a^2 + b^2 + 2ax$$

You should follow the geometric solution, rather than the algebraic.

Case II

∠C is acute

Using the geo board, construct △ABC, where ∠C is acute.

Using the geo board, construct the diagrams on pages 13 and 14.

Show that

$$c^2 = (b - x)^2 + y^2$$

$$(b - x)^2 = b^2 - x(b - x) - x(b)$$

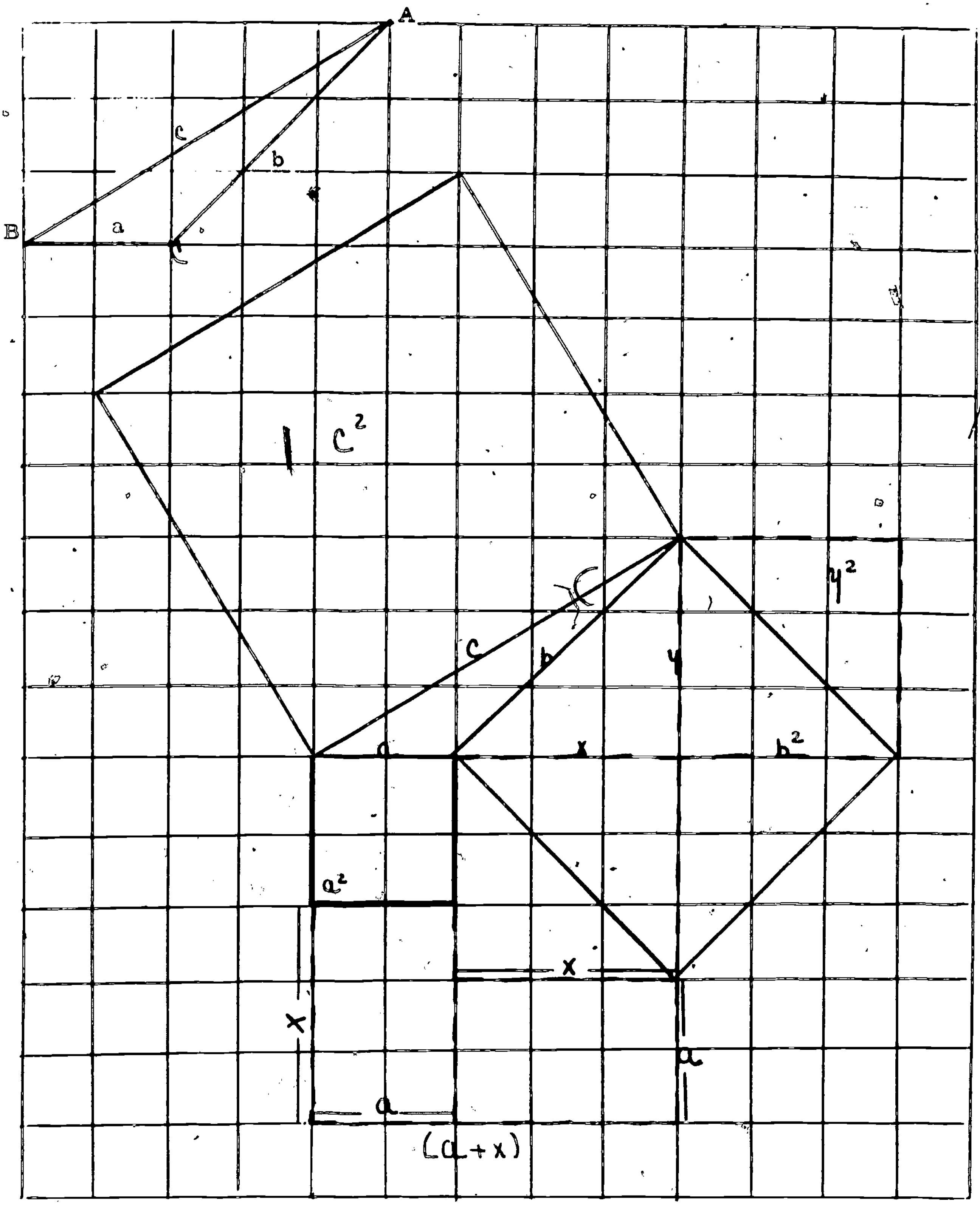
$$y^2 = a^2 - x^2$$

Rewriting: $c^2 = (b - x)^2 + y^2$

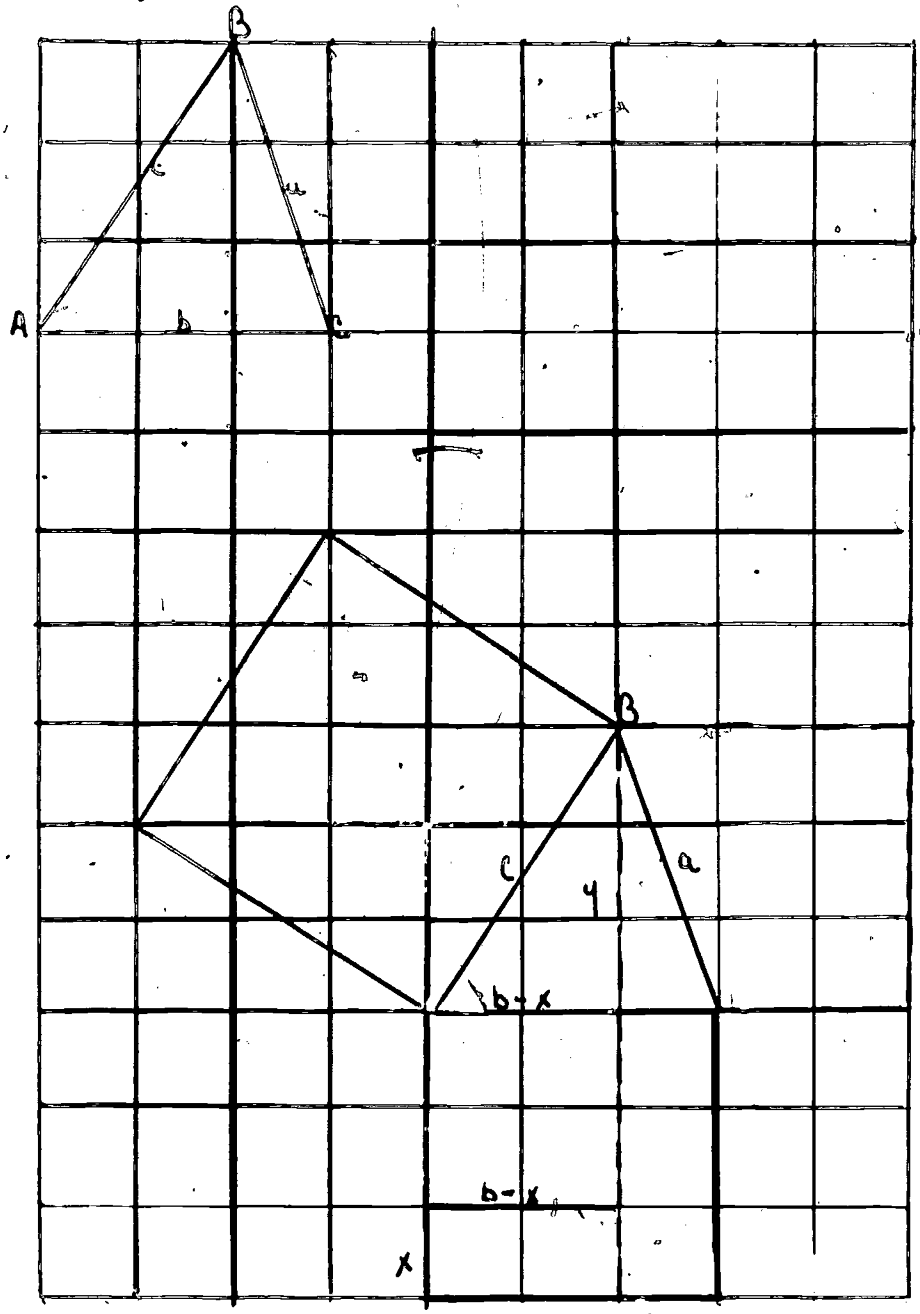
$$c^2 = (b - x)^2 + a^2 - x^2$$

$$c^2 = b^2 + a^2 - 2xb \quad (\text{See Exercise c})$$

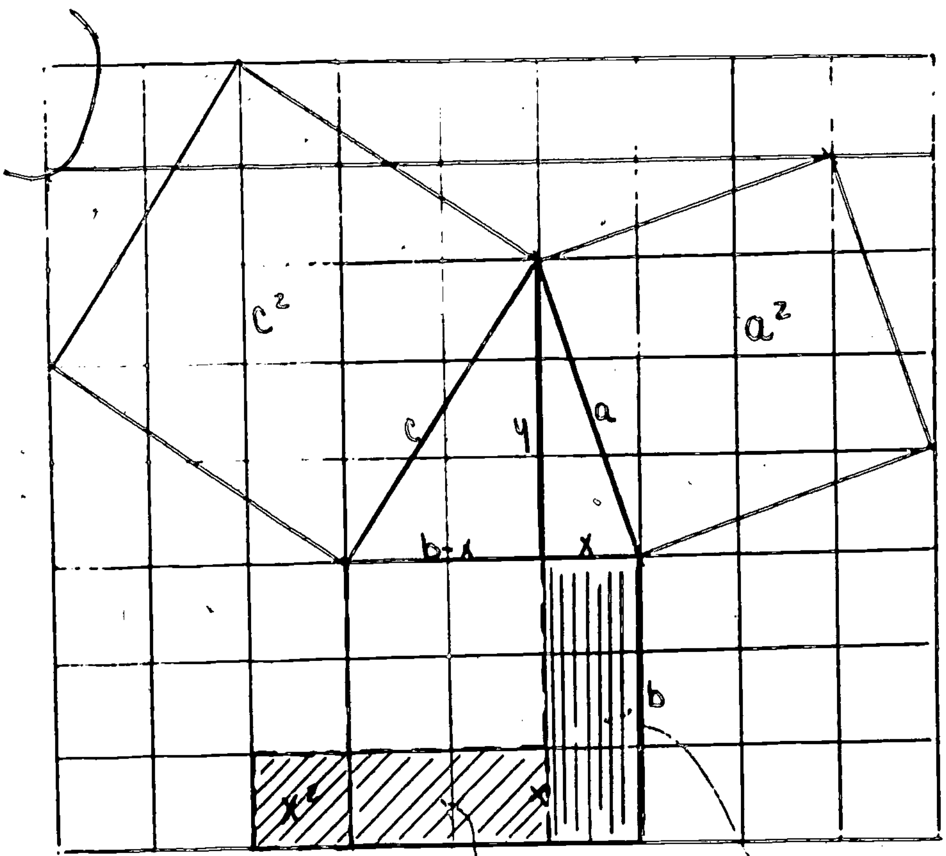
Case I: ∠ C is obtuse



Case II (a)



Case II (b)



Note: $(b - x)^2 = b^2 - xb - x(b - x)$

Other Uses Of The Geo Board

Other topics that may be developed using tne geo boards include:

Slope of a Line	Number Bases
Graphing Linear Inequalities	Figurate Numbers
Gaussian Integers	Computer Programming

ARITHMETIC NUMERALS

Originally suggested in a short monograph by B. H. Gundlach, this topic provides challenges in developing arithmetic expressions following special rules and operations. Many students find the problems attractive. In pursuing solutions, students review parts of arithmetic that often are troublesome to non-science students and they are introduced to both factorials and fractional exponents which may be new to them. Many variations can be introduced depending upon the ingenuity of the instructor.

Creating Arithmetic Numerals for the Counting Numbers 1 through 40

It is possible to construct arithmetic expressions for each of the numbers 1 through 20 using exactly four of the numerals 4 in each expression and using the four basic arithmetic operations, addition, subtraction, multiplication and division with two other arithmetic processes, grouping (using parentheses) and place notation (locating a decimal point in combination with digit numerals).

> Can you create an expression for each of the numbers 1 through 20? Try using pencil and paper and see how many you can find.

Provide several examples and then circulate among the students answering individual questions. If questions relating to the order of operations occur, have students show and defend their interpretations at the blackboard. Present the following for student responses:

(a) $2 + 3 \times 6 =$ 30 or $2 + 3 \times 6 =$ 20

(b) $45 + 3 + 2 =$ 17 or $45 \div 3 + 2 =$ 9

(c) $3 \times 5 + 6 \div 3 =$ 17 or $3 \times 5 + 6 \div 3 =$ 7

Establish the order of operations which requires simplication of arithmetic expressions to proceed in the following order:

(1) Perform operations inside a parentheses

(2) Raise to a power

(3) Multiply or divide taking operations in order left to right

(4) Add or subtract left to right

Many of the numbers may be represented by several arithmetic numerals or expressions within the given conditions. Instructor and class preferences can determine whether multiple expressions are desired.

When many students have many of the expressions, individuals may be invited, in rotation, to write beside a list of arabic numerals from 1 through 20 their expression which has not been previously shown. Certain expressions will prove difficult to obtain. Give no answers even if students haven't found, for instance, an expression for 19, but stimulate their curiosity and their pursuit of the question. If no one in the class has an expression for a number or numbers, remind the class of this at intervals of days or weeks.

When the class appears ready for a new challenge--

You can continue to work on the arithmetic numerals you haven't found yet. I'd like to see what you have when you get a difficult one, but let's change the rules a little and work on some more numbers.

Let's eliminate the place notation process and add s and "one-half" powering.* We are allowed to have 4^2 for 16 in expressions and it would be considered to use one "4". Also allowed are $4^{1/2}$ for 2 and $(4 \times 4)^{1/2}$ for 4 using one and two "4's" respectively.

Can you create arithmetic numerals for the numbers 21 through 30 using exactly four "4's", the four arithmetic operations (+, -, x, ÷,), the process of grouping (parentheses) and second and "one-half" powering?

* Note: The expression one-half power is used to introduce a fractional exponent as a new concept. this provides the motivation for writing $(4 \times 4)^{1/2} = 4$ rather than using the more traditional square root (and $\sqrt{4 \times 4} = 4$).

Procedures similar to those used when developing the arithmetic numerals for 1 through 20 may be used here. Remember to withhold answers to stimulate curiosity and to encourage the pursuit of productive questions. Then, when the class seems ready--

> Let's now have a new challenge for creating arithmetic numerals for the numbers 31 to 40. Let's restore place notations and add a new process, "factorializing," so that we have available 4! (=4·3·2·1) which uses one "4". Thus, the conditions are to create the numerals using exactly four "4's", the four operations (+, -, x, ÷) and the processes grouping, place noting (locating a decimal point among digits), raising to the second and half powers, and taking the factorial.

Exercises

(1) For homework find numerals for the numbers that are not obtained in class.

(2) For homework, students might like the following:

> Using exactly four "4's", the four arithmetic operations (+, -, x, ÷) and the two processes (parenthesis and decimal point) create arithmetic numerals for each of the fractions 1/2, 1/3, 1/4, 1/5, 1/6, 2/3, 4/5.

(3) For classwork or for homework the following challenge may be presented:

Given a collection of six numerals as follows "1/2", "1/3","1/4", "1/6", "1/6", "1/12" and the four arithmetic operations (+, -, x, ÷) with the process of grouping, use any or all of the given numerals in expressions equal to the counting numbers from 1 through 15. Example: (1/6 ÷ (1/6)) = 1.

(4) Students may write a short monograph describing their findings using different digits (three "3's", or four "6's" or something) and by experimentation find the conditions to impose (in terms of the numerals, and the operations and processes) for creating the expressions.

Note: Illustrative solutions for the requested arithmetic numerals are shown below:

(1) (4 + 4) ÷ (4 + 4)

(2) (4 x 4) ÷ (4 + 4)

(3) (4 + 4 + 4) ÷ 4

(4) 4 + (4 - 4) ÷ 4

(5) (4 x 4 + 4) ÷ 4

(4 x .4 +.4) ÷ .4

(6) 4 + (4 + 4) ÷ 4

(7) (44 ÷ 4) - 4

4 + 4 - 4 ÷ 4

(8) 4 + 4 + 4 - 4

4 x (4 + 4) ÷ 4

(9) 4 + 4 + (4 ÷ 4)

(10) $(4 \div .4) + 4 - 4$

$(44 - 4) \div 4$

(11) $(4 \div 4) + (4 \div 4)$

(12) $(44 + 4) \div 4$

(13) $(4 - .4) \div .4 + 4$

(14) $(4 + 4 \times .4) \div .4$

(15) $(44 \div 4) + 4$

$4 \times 4 - (4 \div 4)$

(16) $4 + 4 + 4 + 4$

$4 \times 4 + 4 - 4$

(17) $4 \times 4 + (4 \div 4)$

(18) $(4 \div .4) + 4 + 4$

(19) $(4 + 4 - .4) \div .4$

(20) $(4 + 4 \div 4) \times 4$

(21) $4^2 + 4 + (4 \div 4)$

(22) $4 \times 4 + 4 + 4^{1/2}$

(23) $(4 + 4 \div 4)^2 - 4^{1/2}$

(24) $4 \times 4 + 4 + 4$

(25) $(4 + 4 \div 4)^{4^{1/2}}$

(26) $4^2 + 4 + 4 + 4^{1/2}$

(27) $4^2 + 44 \div 4$

(28) $4^2 + 4 + 4 + 4$

(29) $(4 + 4 \div 4)^2 + 4$

(30) $44 - 4^2 + 4^{1/2}$

(31) $4^2 + 4^2 - 4 \div 4$

(32) $4! + 4 + 4^{1/2} + 4^{1/2}$

(33) $4^2 + 4^2 + 4 \div 4$

(34) $4! + 4 + 4 + 4^{1/2}$

(35) $(4 + 4^{1/2})^2 - 4 \div 4$

(36) $(4! - 4) \times 4^{1/2} - 4$

(37) $(4 + 4^{1/2})^2 + 4 \div 4$

(38) $44 - 4 - 4^{1/2}$

(39) $4! + 4^2 - 4 \div 4$

(40) $(4!) \times 4^{1/2} - 4 - 4$

Illustrative Solutions for Exercise 2

1/2: (4 + 4) ÷ (4 x r)

1/3: 4 ÷ (4 + 4 + 4)

1/4: 4 ÷ (4! − (4 + 4))

1/5: 4 ÷ (4 x 4 + 4)

1/6: 4 ÷ (4! + 4 − 4)

2/3: 4 ÷ (4 ÷ .4 − 4)

4/5: 4 ÷ (4 ÷ .4 + 4)

Illustrative Solutions for Exercise 3

1. (1/6) ÷ (1/6)
2. (1/3) ÷ (1/6)
3. (1/4) ÷ (1/12)
4. (1/3) ÷ (1/12)
5. (1/2 + 1/3) ÷ (1/6)
6. (1/2) ÷ (1/12)
7. (1/3 + 1/4) ÷ (1/12)
8. (1/3) ÷ (1/2) x (1/12)
9. (1/2) ÷ ((1/3) x (1/6))
10. (1/2 + 1/3) ÷ (1/2 + 1/3) ÷ (1/12)
11. (1/3 + 1/4 + 1/6 + 1/6) ÷ (1/12)
12. ((1/2) ÷ (1/12) x (1/3) x (1/6)

13. ((1/2) + (1/3) + (1/4) ÷ (1/12)

14. (1/6) ÷ ((1/6) x (1/12)) + (1/2) ÷ (1/4)

15. (1/4) ÷ ((1/6) x (1/6)) + (1/2) ÷ (1/2)

Balance Conditions for an Equal Arm Balance and Solutions to Linear Equations

Materials:

A. For demonstration

1. An equal arm balance - (low sensitivity from the physics lab or built for the occasion.)

2. An assortment with 6 each of 5 everyday items such as unused pencils, erasers, chalk sticks, washers, etc.

B. B. For students arranged in groups

1. Small colored chips or pebbles in 3 colors to provide approximately 20 of each color per group of students.

2. 1 centimeter cube - 2 or 3 handfuls within reach of each student.

3. "Balance Worksheets" - two per student marked off as shown.

Left Pan		Right Pan
	Λ	
	Λ	

This unit uses a physical demonstration to permit students to observe and state rules for manipulating unknown weights on a balance scale to determine the values of the unknowns. Students manipulate color chips which represent different weights so as to maintain simulated balance conditions. Permitted changes allow determination of the values of chips in a way that parallels permitted changes to given linear equations that lead to their roots. Interest can be killed by asking students to work equations by chips. In stead of using chips, students who need this unit most will attempt to work equations with x's and y's using half remembered rules from past experiences.

Student involvement comes from attractive challenges to manipulate colorful chips and to discover meaningful values and relationships. To sustain interest the teacher should promote investigation of the balance relationships for their own sake and avoid giving any hints to the mathematical payoff from the activity. A statement such as the following may help: As we proceed, "If you know, what we are doing mathematically, come to me and whisper your guess. If you want to, I will give you an individual project to work on."

Introduction

Introduce the unit as a study of balance and balancing operations.

Demonstrate to the class the ways that a balanced load may be changed.

For example, assume that one eraser balances four pencils. If you start with an eraser balancing four pencils, adding an eraser to each side will maintain balance and illustrate adding the same thing to each side. If you start with an eraser balancing four pencils, adding an eraser to the first side and four pencils to the other sides gives balance with two erasers balancing eight pencils. It illustrates adding equals to each side. Through similar demonstrations the following general rules for balance operations may be observed:

(1) For BALANCE - (these require essentially that what you do to one side you also do to the other)

(i) "Same Thing Operations"

Adding the same thing to each side

Taking away the same thing from each side

Multiplying each side by the same factor

Equally subdividing and balancing subunits from each side

(ii) "Balanced Things Operations"

Adding balanced (but different) things to one and the other sides

Subtracting balanced things

Multiplying by equal things

Equally subdividing - by equal things

(iii) "Interchange Operations" - Move each load to the opposite pan and maintain balance.

(2) For UNBALANCE - (these operations permit change to one side alone and adding unbalanced things to the sides of the balance)

(i) "One Side Change"

Adding to one side

Subtracting to one side

Multiplying to one side

Substituting a subdivision on one side only

(ii) "Unbalanced Things Operations"

Adding unbalanced things on to one side and other to other

Subtracting

Multiplying balanced sides by different factors

Subdividing into a different number of parts

(II)

A. Pass out the balance worksheets and state: "Assume that each of the centimeter cubes represents a unit weight and that colored chips are some multiple of this. Let chips of one color represent the same multiple." Then, ask, "Can you represent a series of balance combinations by putting chips and centimeter cubes on each side of the balance as represented on the exercise sheets?"

B. Circulate among the students; encourage them to make up any combination that pleases them. Ask individuals occasionally what meaning certain statements may have for them, without comment (except perhaps, to stimulate their thinking a little.)

C. Next establish with the class that a "simple balance statement" is one using only one colored piece balancing a number of unit cubes. Have the students set up on their worksheets several simple balances and then have them udertake a process which may be called "complicating" or "disguising" the simple statement. Complicating or disguising a simple balance statement is achieved by performing any of the balance operations on it. Allow an unlimited use of pieces of different colors and suggest that each student fill his worksheet with cimplications as wild as he can think of, and developed from some simple balance statement. It is probable that many students are not observing the conditions for "complicating" and so the instructor should circulate through the class, asking what the original or root statement was, and then checking that the complicated statement is consistent with the original simple statements. It is going to be difficult not to give oneself away by using the word "equation," but this term should be kept secret in favor of the "balance" terminology. Allow time for students to develop a facility, including a feeling of freedom, in setting up series of these complicated balance combinations rooted in original simple statements.

(III) "Seeing Through" Disguised Simple Statements

In this section students will challenge other class members to see through the disguises of simple balance statements which they got by "complicating" simple statements.

A. Have students use some one color chips with unit cubes to make a simple balance statement which they then "disguise" by complicating it. Remove the original simple statement from the worksheet and then invite other members of the class to exchange seats so that they can work back through the disguises to the original simple statement.

B. When students see that this activity using one color is rather trivial, suggest that they make two simple statements each using a different color and then disguise this pair of statements. Have students record their root statements on a slip of paper and then fill their worksheet with five or so disguised statements. When two students each have disguised sets, have them exchange seats and challenge each other to find the simple statements.

NOTE: It will be well to check the complicated sets at first, to insure that the intended simple statements are represented. However, some good discussions develop when students find that the statements are inconsistent.

This activity using two colors will probably hold the students' interest for one or several class periods. Have students use their second work sheet to put pieces to represent new balance statements derived from the disguised statements. After they are doing this easily, encourage students, probably one at a time as you circulate around the class, to record on paper in some manner the new statements derived from the disguised set. Allow students to devise their own recording system without imposing or even suggesting techniques. When students have developed a facility in seeing through two color disguises and can move easily from paper and pencil statement of balance to the blocks and cubes statements, they are ready to work with three colors.

C. Suggest to individual students first, and then announce to the class generally,

> "When you have mastered building balance statements on the second worksheet related to two color disguised statements on the first, and when you can record these new derived statements using pencil and paper, move on to disguising three simple statements."

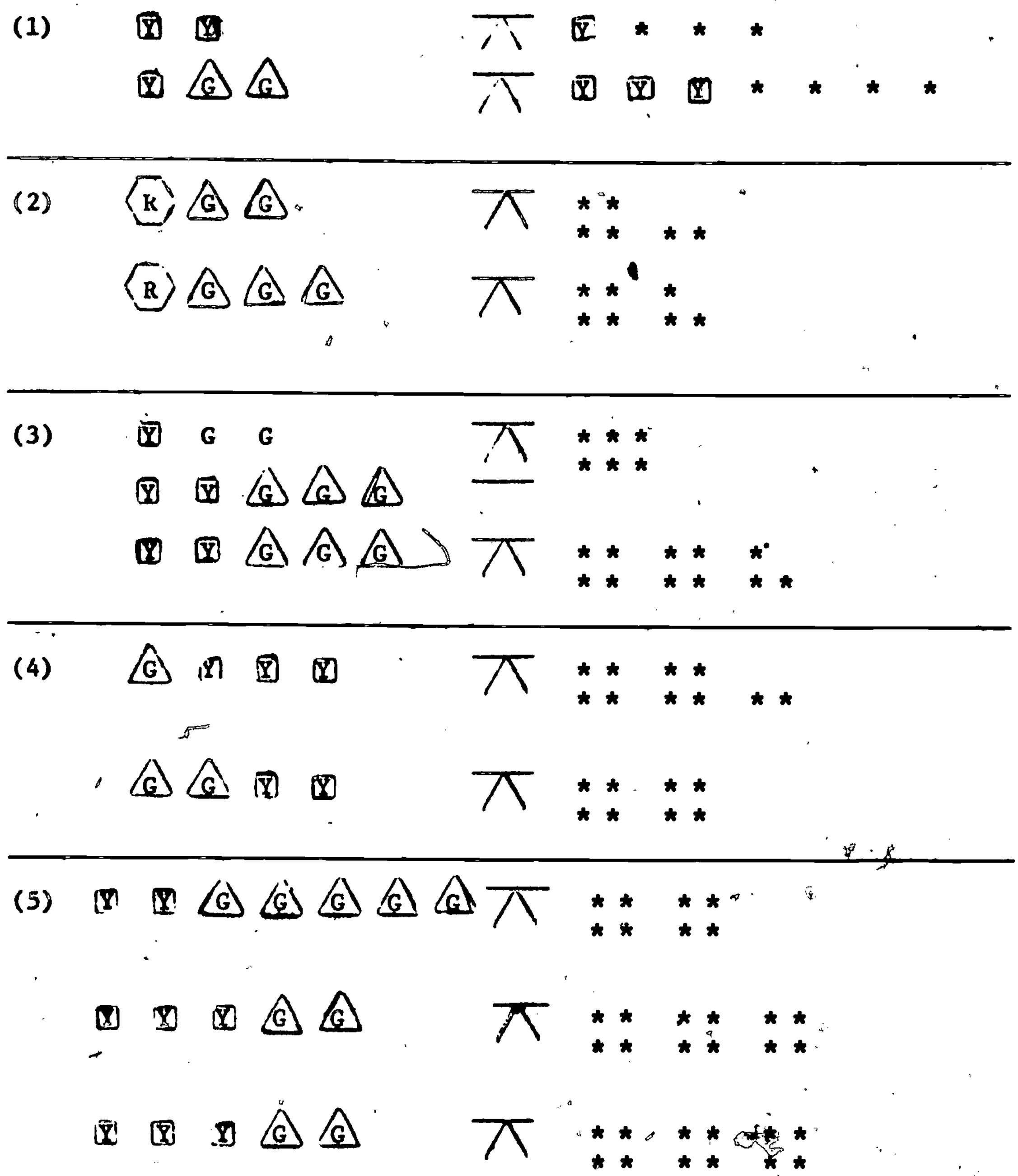
As before, have students fill a worksheet with five combinations before challenging a classmate to break the disguise. Students should record their simple statements on a slip of paper, and, in many cases, it is well to check disguised sets for consistency with the root statements.

It may now be appropriate for the instructor to set up one or several sets of disguised statements and invite students to find the root statements. Similarly, students may challenge the instructor.

Slower students will continue working on two color statements while quicker students may be working on involved situations with statements of three or more colors. When the class has at least a minimal proficiency with the two color sets, and seems ready, the instructor may wish to assign some of the exercises below for classwork or homework before moving on to Section IV, Guided Insights.

NOTE: These exercises are illustrative of many that may be used for classwork or homework. They progress in difficulty from two color simple statements wherein, for example, a green piece balances three cubes, to statements wherein a green piece balances an empty right side (zero cubes) and then to statements wherein a green piece and two cubes balance an empty right side (so that the value of the green piece is negative tow cubes). A few three color disguised statements are included. The level of difficulty is represented by little circles next to the number.

Exercises



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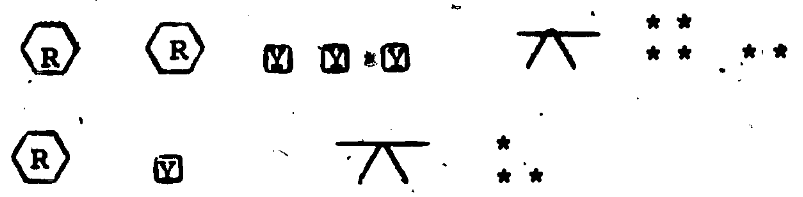

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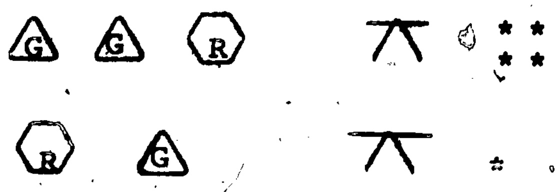

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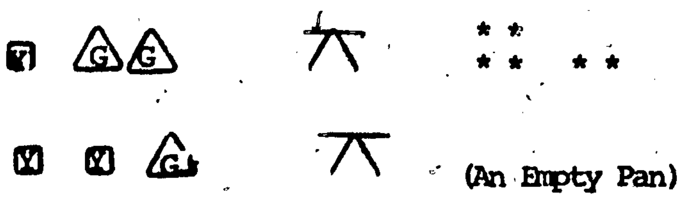

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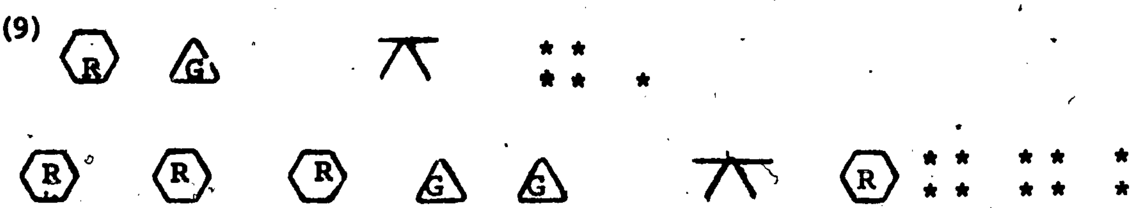

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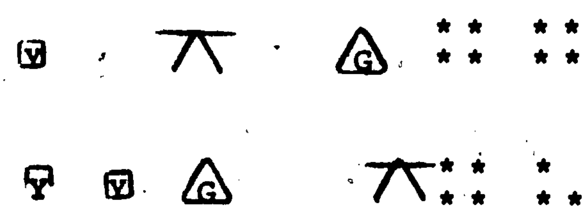

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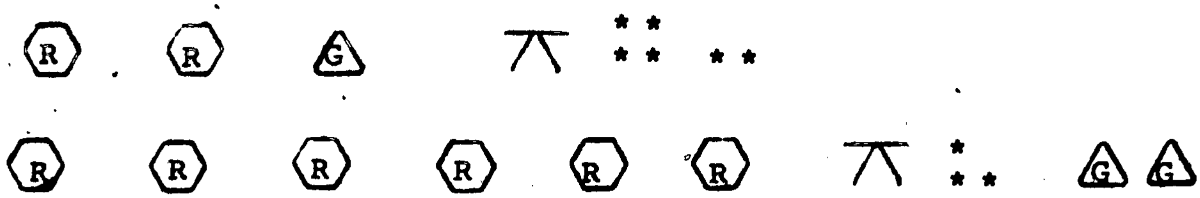

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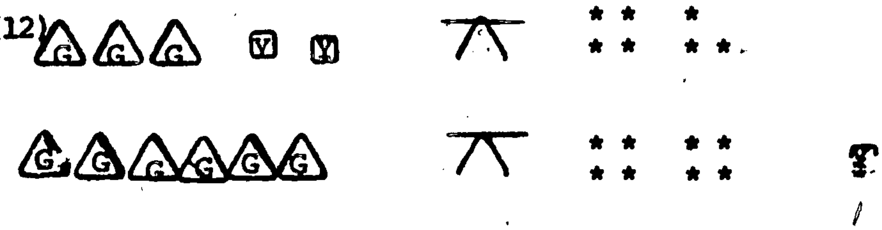

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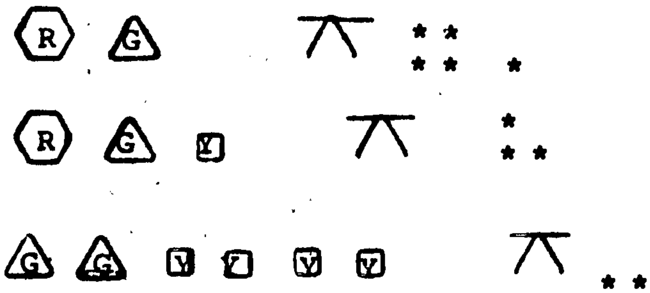

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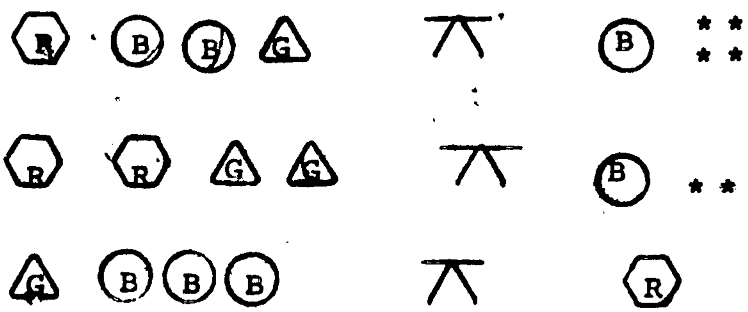


NOTE: The undisguised simple balance statements are:

(1) Y ⊼ 3 cubes G ⊼ 5 cubes	(7) Y ⊼ (-2) cubes G ⊼ 4 cubes
(2) Y ⊼ 4 cubes G ⊼ 1 cube	(8) R ⊼ 2 cubes G ⊼ 3 cubes
(3) R ⊼ 4 cubes G ⊼ 1 cube	(9) Y ⊼ 5 cubes G ⊼ (-3) cubes
(4) G ⊼ 1 cube Y ⊼ 3 cubes	(10) R ⊼ 2/3 cube G ⊼ 3 cubes
(5) Y ⊼ 4 cubes G ⊼ 0 cubes	(11) G ⊼ 5/3 cube Y ⊼ 2 cubes
(6) R ⊼ 3 cubes Y ⊼ 0 cubes	(12) G ⊼ 5 cubes R ⊼ 0 cubes Y ⊼ (-3) cubes

IV. Guided Insights

In the previous section we saw that a set of disguised statements could be unmasked to a set of simple statements. In this section we will show that one disguised statement leads to many pairs of simple statements. Thus, for example,

```
              * *  * *  *
G G G R ⊼     * *  * *  * *
```

```
may lead to  (1) G ⊼ * * *      (2) G ⊼ * *
                                        * *
                 R ⊼ * *            R ⊼ * * *

             (3) G ⊼ *                        etc.
                     * *   * *
                 R ⊼ * *   * *
```

To determine one simple statement pair you must be given two disguised statements that are essentially different (i.e. one is not a multiple or a constant value different from the other.)

Divide the blackboard into three sections. At the top of each section show a single simple statement. In the first, show a statement with one color, in the second with two colors and in the third with three colors. Then record in each section approximately four disguised statements derived from the simple statements of that section.

Keep only the disguised statements in each section, erasing the original simple statements.

Let us assume that we do not know the simple statements from which these disguised statements have come and that we know only the disguised balance statements.

Let us choose a disguised statement in one color and work back to its simple statement.

Record on the blackboard statements volunteered by students until the simple statement is reconstructed.

Let us choose a disguised statement in two colors and work back through its disguise, simplifying it.

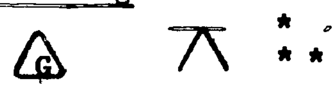
Again record statements on the blackboard. Students will find that any simplified statement they obtain still has two colors in it and many pairs of simple statements (each in one color only) satisfy it.

Let us choose a disguised balance statement in three colors and work back through its disguise to simplify it.

As before, any simplified statement will include three colors, which may be satisfied by several sets of simple statements. Students may be guided to the insight that to determine a single set of simple statements two disguised statements must be given for two colors, three disguised statements for three colors, four disguised statements for four colors, etc.

The examples following illustrate how one such class discussion might develop.

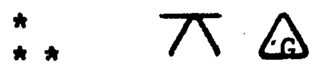
(A) Under the one color heading:

Original 

Disguising yields (1) 

(2) 

Simplifying yields (2) 

going to (2') 

or (2") 

(B) Under the two color heading

Original: 

Complicating yields (1) 

(2) 

(3) △G △G □Y □Y △G *

(4) □Y □Y □Y □Y □Y □Y □Y ⊼ * * * * △G △G

Simplifying (3) yields

(3) △G △G ⊼ □Y □Y △G *

(4) △G ⊼ □Y □Y *

For which pairs of simple statements in two colors might be:

△G ⊼ * * * * * or △G ⊼ * * * * * * *

□Y ⊼ * * □Y ⊼ * * *

or more conveniently in a talbe representing pairs of simple statements for △G and □Y we have:

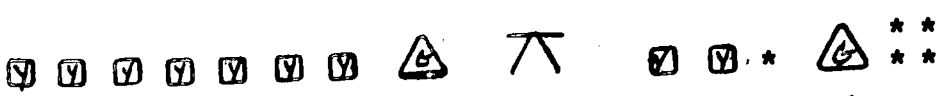
Possible Pairs

pair	△G	□Y
a	5	2
b	7	3
c	13	6
d	3	1
e	9	4
f	1	0

Simplifying (4) yields no simplification, but the statement accords with the pairs of simple statements included in the table:

Pair		
a	10	2
b	3	1
c	17	3
d	31	5
e	59	9

Simplifying by interacting two disguised statements leads by combining (3') and (4) to:



And to

(5) 

And by subdividing that

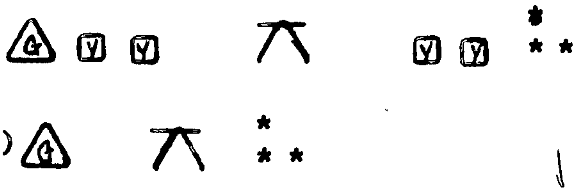
(5') 

Now (5') means that

(5') 

Combining (3') and (5") gives

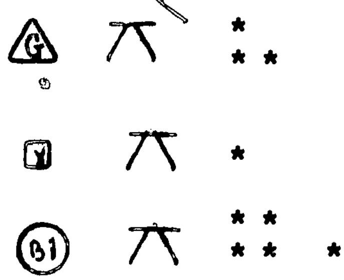
(6)
(6')



The pair (5') and (6') consititue for original simple balance statements for two colors.

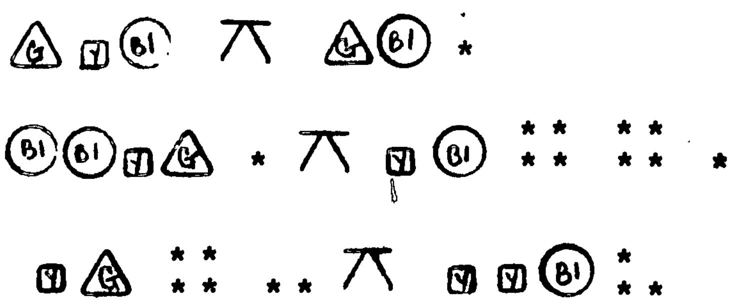
(C) <u>Under the three color heading</u>

<u>Original</u>:



<u>Complicating yields</u>:

(1)
(2)
(3)



Simplifying yields

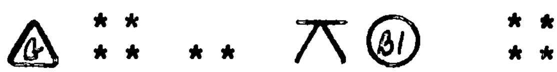
(1) 

(2) 

(3) 

Simplifying by interacting 1' with 3' gives

(4")



Adding (2') and (4") gives

(5")

So (5"') 

And by subdividing (5"") 

Adding (5"") interchanged to 2' we obtain

(6"') 

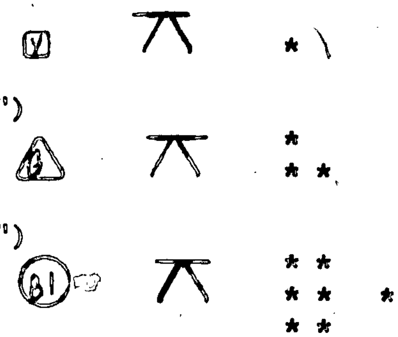
(6"") 

We have thus worked through the disguise to the original simple statement

(1)

(5'''')

(6'''')



V. The Set-up Exposed

Previous work has been discussed using only balancing vocabulary. Hopefully the students are thinking in balancing terms. Those who are really thinking this way may now be treated to one of the most pleasant experiences in mathematics - the insight that two abstract systems are in reality one. The instructor may now guide student thinking to enable them to see that the system of balance statements and operations parallel form an algebraic system involving linear equations.

A. <u>What are the balance operations that we may perform on a balance statement that will maintain balance?</u>

Record student responses on the blackboard.

Below is a set of statements that illustrate those that might be volunteered by students:

1) When you put a piece on the left, put one on the right side, also.
2) You can take the same piece from both sides.
3) You can switch the sides for the pieces on the balance.
4) You can add pieces together for two balance combinations.
5) You can take 2 or any number of times the number of of pieces on the left side to balance the same number of times the pieces on the right side.

B. You now know a lot about equations. Each of these balance statements can be written as an equation. How would you write equations for the disguised balance statements and their simplifications that we recorded on the blackboard in Section IV?

Record responses on the blackboard. These responses will probably include equality and the arithmetic operations although triangles, circles and squares are likely to persist for place-holders.

The final move to statements of a more traditional algebraic form may come when letters are used for placeholders. It may be interesting to ask the following questions:

> How do you recognize something in algebra when you look at it? Have you studied any algebra? What did you do?

The use of x's, y's and z's for placeholders or variables is one of the most visible features of algebra and is likely to be mentioned. If and when the use of letters as placeholders is mentioned, the instructor may explain their convenience. He may then solicit algebraic statements that represent the disguised and the simplifying balances and record the volunteered algebraic statements on the blackboard.

C. Since we are finding algebraic terms to use in place of the balancing terms, let us express the balance operations in algebraic terms.

The series of statements below may serve as an example of statements that may be volunteered and recorded on the blackboard.

New and equivalent equations result from:

1) Adding the same quantity to both sides of a given equation.

2) Adding equal quantities to both sides.

3) Subtracting the same or equal quantities from both sides.

4) Multiplying both parts by the same factor.

5) Dividing both parts by the same divisor.

6) Equations are symmetric so that if A = B, then B = A.

D. You now have a knowledge of a mathematical system in which the elements are linear equations. You may develop your skill in working a set of complicated equations back to the simple root equations by practice. Find some linear equations in an algebra book (starting with easier ones first) and work them out yourself.

Exercises: For classwork or homework students can work sets of linear equations from standard books to find the root equations as the students desire.

PERFECT NUMBERS

The factors of 6 are 1, 2, 3 and 6. Note: $1 + 2 + 3 + 6 = 2 \times 6$.
The factors of 28 are 1, 2, 4, 7 and 28. Note: $1 + 2 + 4 + 7 + 28 = 2 \times 28$.
What are the factors of 496? What is the sum of the factors of 496?

INTRODUCTION

Sometimes it is useful to have students generalize a well known theorem in order for them to become more familiar with the theorem itself as well as with related problems. The student is led to discover mathematics in such a way that it becomes fun for him. We shall demonstrate this approach by generalizing the concept of a perfect number to the concept of a number which is said to be $\frac{1}{a}$ - perfect where a is a natural number. The student is encouraged to try his hand at solving some famous unsolved problems.

(Part 1)

AN EXTENSION OF THE IDEA OF PERFECT NUMBERS

The Pythagoreans originated the problem about perfect numbers. We shall use this idea in the extension given here.

The Sum of Successive Powers of 2. The student should be led to prove the following well known theorem.

THEOREM 1.1: For any integer p, the sum of the first p powers of 2 is equal to $2^p - 1$, i.e.,

$$2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^{p-1} = 2^p - 1.$$

(a) Show that

$$2^0 = 2^1 - 1,$$

$$2^0 + 2^1 = 2^2 - 1,$$

$$2^0 + 2^1 + 2^2 = 2^3 - 1,$$

$$2^0 + 2^1 + 2^2 + 2^3 = 2^4 - 1.$$

(b) Suppose that for some natural number k, it is true that

$$2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^{k-1} = 2^k - 1.$$

Then multiply both sides of the above equation by 2 and add 1 to both sides to determine the sum.

$$2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^{k-1} + 2^k.$$

(c) What have you proved using Mathematicsl Induction?

(d) Is this true: $2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^{p-1} = 2^p - 1$?

PERFECT NUMBERS. If we exclude 6 from the divisors of 6, then 6 is the sum of its divisors, since 1 + 2 + 3 = 6. The same is true for the number 28. On account of this, they are considered to "perfect numbers." Euclid has given a formula for all the even perfect numbers. His formula says that $2^{p-1}(2^p - 1)$ is a perfect number if p and $(2^p - 1)$ are both primes.

Exercises:

(a) Try to find a natural number n which is not prime but for which $2^n - 1$ is a prime number. What conclusion do you reach?

After a while the students should conjecture that if $2^n - 1$ is a prime number, then n itself is a prime number.

(b) Show that, except for the number itself, the divisors of 2^p $2^{p-1} (2^p - 1)$ are

$$1, 2, 2^2, \ldots, 2^{p-1}, 2(2^p - 1), \ldots, 2^{p-2} (2^p - 1).$$

(c) Show that the sum of these divisors is given by

$$1 + 2 + 2^2 + \ldots + 2^{p-1} + (2^p - 1)(1 + 2 + \ldots + 2^{p-2}).$$

(d) Expand the above sum to show that it is equal to

$$2^{p-1} (2(1 + 2 + 4 + \ldots + 2^{p-2}) + 1).$$

(e) Use Theorem 1.1 to show that the above sum is equal to

$$2^{p-1} (2(2^{p-1} - 1) + 1).$$

(f) Expand the above result to get

$$2^{p-1}(2^p - 1).$$

(g) What theorem has just been proved about perfect numbers?

THEOREMS ABOUT PERFECT NUMBERS. An elementary number theory theorem states:

> Each even perfect number (Base Ten) ends in either a 6 or an 8.

The student is hereby encouraged to prove the counterparts to this theorem in other bases. For example,

THEOREM 1.2: All even perfect numbers (Base Nine), except the perfect number 6, end in 1.

(a) Make a (Base Nine) multiplication table.

(b) Fill in the following table:

i	3	5	7	10
2^i	8	35		
2^{i-1}	4			
$2^i - 1$	7			
$2^{i-1}(2^i - 1)$	31			

- TABLE 1 -

(c) Prove the following theorem.

THEOREM 1.3: Each odd power of 2 (Base Nine) ends in 2, 8 or 5.

(d) (d) Prove the following theorem.

THEOREM 1.4: Each even power of 2 ends in 4, 7 or 1.

(e) Prove the following theorem.

THEOREM 1.5: For any odd integer i, $2^i - 1$ ends in 1, 7 or 4.

(f) Prove the following theorem:

THEOREM 1.6: For any odd integer i,

1. If 2^i ends in 2, then 2^{i-1} ends in 1 and $2^i - 1$ ends in 1.
2. If 2^i ends in 8, then 2^{i-1} ends in 4 and $2^i - 1$ ends in 7, and
3. If 2^i ends in 5, then 2^{i-1} ends in 7 and $2^i - 1$ ends in 4.

(g) Prove the theorem.

THEOREM 1.7: For any odd integer i, $2^{i-1}(2^i - 1)$ ends in 1.

(h) Prove the following corollary to theorem 1.7. For any even perfect number n, except 6, n ends in 1 if n is given in Base Nine.

Exercises:

1. Show that in Base Eleven

   (a) Each even perfect number ends in a 6, 1 or 4.

   (b) If an even perfect number ends in 4, then it is of the form

   $$2^{p-1}(2^p - 1)$$

   where p is a prime of the form 10a + 9 (Base Ten).

   Hint: Fill in the following table.

i	2	3	5	7	9	10	12
2^i	4	8	2d				
2^{i-1}	2	4	15				
$2^i - 1$	3	7	29				
$2^{i-1}(2^i - 1)$	6	26					

2. Show that in Base Thirteen, except for the number 6, each even perfect number ends in a 1, 2, 3, or 8.

3. Show that in Base Fifteen in which the fifteen symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, d, e, x, y, and z, each even perfect number, except 6, ends either in 1 or y.

4. Show that in Base Seventeen, each even perfect number except 6 ends in 1, 2, 4, or e.

5. What are the possible endings for even perfect numbers if they are written in Base Nineteen?

6. Give endings for perfect numbers written in bases six, eight and twelve.

The Sum of Successive Powers of 3. The student should be led to prove the following theorem.

THEOREM 1.8: For any integer p, the sum of the first p powers of 3 is equal to $\frac{3^p - 1}{2}$, i.e.,

$$3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^{p-1} = \frac{3^p - 1}{2}$$

(a) Show that

$$3^0 = \frac{3^1 - 1}{2}$$

$$3^0 + 3^1 = \frac{3^2 - 1}{2}$$

$$3^0 + 3^1 + 3^2 = \frac{3^3 - 1}{2}$$

$$3^0 + 3^1 + 3^2 + 3^3 = \frac{3^4 - 1}{2}$$

(b) Suppose that for some natural number k, it is true that

$$3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^{k-1} = \frac{3^k - 1}{2}.$$

Then multiply both sides of the above equation by 3 and add 1 to both sides to determine the sum

$$3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^{k-1} + 3^k .$$

(c) What have you proved using Mathematical Induction?

(d) Is this true: $3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^{p-1} = \frac{3^p - 1}{2}$?

HALF-PERFECT NUMBERS. We would like to introduce a problem about half-perfect numbers. If we exclude 117 from the divisors of 117, then the sum of its divisors is 65 because 1 + 3 + 9 + 13 + 39 = 65.

Observe that

13 is prime and divides both 65 and 117.

65 is 5 x 13, and

117 is 9 x 13.

5 is approximately half of 9.

A simiiliar relationship holds for the number 796797, i.e., if we exclude 796797 from the divisors of 796797, then the sum of its divisors is 398945.

Note that

$$1 + 3 + 9 + 27 + 81 + 243 + 729 + (364)\ (1093) = 398945.$$

We find that

1093 is prime and divides both 398945 and 796797.

398945 is 365 x 1093 and

796797 is 729 x 1093.

365 is approximately half of 729.

Thus, the numbers 117 and 796797 both have the special property of each having itself and sum of its factors as multiples of the same non-composite number and having a ratio of approximately one-half. On this account, we name them "half-perfect numbers". In each case the non-composite number is given by the formula $\frac{(2 + 1)^p - 1}{2}$ for some prime p. We now give a formula for some half-perfect numbers. Our formula says: A number N is said to be half-perfect if there is a prime p such that

(a) $\frac{(2 + 1)^p - 1}{2}$ is prime,

(b) $\frac{(2 + 1)^p - 1}{2} \times (2 + 1)^{p-1} = N$, and

(c) the sum of the factors of N (excluding N itself) is

$$\frac{(2+1)^{p-1}+(2-1)}{2} \times \frac{(2+1)^p-1}{2}$$

Exercises:

If both p and $\frac{3^p-1}{2}$ are primes, then

$$3^{p-1}\left(\frac{3^p-1}{2}\right)$$

is a half-perfect number.

(a) Show that, except for the number itself, the divisors of

$3^{p-1}\left(\frac{3^{p-1}-1}{2}\right)$ are

$$1,\ 3,\ 3^2,\ \ldots,\ 3^{p-1}, \left(\frac{3^p-1}{2}\right),\ 3\left(\frac{3^p-1}{2}\right),\ \ldots,\ 3^{p-2}\left(\frac{3^p-1}{2}\right)$$

(b) Show that the sum of these divisors is given by

$$1+3+3^2+\ldots+3^{p-1}+\left(\frac{3^p-1}{2}\right)(1+3+\ldots+3^{p-2}).$$

(c) Change the above sum to show that it is equal to

$$\frac{3^p-1}{2}+\frac{3^p-1}{2}+\frac{3^{p-1}-1}{2}$$

Use Theorem 1.8.

(d) Expand the above result to get

$$\frac{(2+1)^{p-1}+(2-1)}{2} \times \frac{(2+1)^p-1}{2}.$$

Exercise: Show that the number 423,644,039,001 is a half-perfect number.

THEOREMS ABOUT HALF-PERFECT NUMBERS. Here are some other theorems about half-perfect numbers which the teacher might take up in class.

THEOREM 1.9: The larger the value of p the closer to one-half is the sum of the factors of a half-perfect number N to the number N.

Exercise:

1. Show that no number can be both perfect and half-perfect.
2. Show that the sum of the factors of a half-perfect number (disregarding the number itself) is always greater than 1/2.

THEOREM 1.10: There exists infinitely many numbers that are half-perfect.

THEOREM 1.11: There are no even half-perfect numbers.

Exercises:

1. Fill in the blanks.

If we exclude ____________ from the divisors of 423644039001 then the sum of its divisors is ___________. We observe

1 + ____ + ____ + ... + ____ = ____________________.

__________ is prime and divides both ______ and 423644039001.

__________ is ___________ x 797161 and

__________ is ___________ x 797161.

__________ is approximately ________ of ________.

Thus, the number ___________ has the property of having itself and the sum of its factors as multiples of the same ________ and having a ration of approximately ________. The number 423644039001 is one-half perfect since there exists the prime ________ such that

(a) ___________ is prime,

(b) 423644039001 + _______ x _____________, and

(c) the sum of the factors of 423644039001 (excluding 423644039001) is __________ x _____________.

Exercises:

1. Show that in Base Ten

   (a) Each half-perfect number ends in a 1 or a 7.

   (b) If a half-perfect number ends in 1, then it is of the form

   $$3^{p-1} (3^p - 1) /2$$

   where p is a prime of the form 4a + 3.

   Hint: Fill in the following table.

i	3	5	7	9	11
3^i	27	243			
3^{i-1}	9	81			
$3^i - 1$	13	121			
$3^{i-1} (3^i - 1)/2$	117				

2. Show that in Base Nine each half-perfect number has an odd number of face-values, the first one being equally divided between a string of 4's and a string of 0's. As an example, the smallest half-perfect number in Base Nine is 140. The second smallest half-perfect number in Base Nine is 1444000.

3. Show that the Base Nine number 1444444000000 is half-perfect.

4. Is the Base Nine number 1444444400000 half-perfect?

5. Is the Base Nine number 144444440000000 half-perfect?

6. Show that corollaries to the theorem of (2) above are:

   (a) Every half-perfect number is a multiple of nine.

   (b) Except the smallest half-perfect number, every half-perfect number is a multiple of seven hundred twenty-nine.

   (c) Except for the two smallest half-perfect numbers, every half-perfect number is a multiple of 3^{18}.

The Sum of Successive Powers of 4. The student should be led to prove the following theorem.

THEOREM 1.12: For any integer p, the sum of the first p powers of 4 is equal to $\frac{4^p - 1}{3}$, i.e.,

$$4^0 + 4^1 + 4^2 + 4^3 + \ldots + 4^{p-1} = \frac{4^p - 1}{3}$$

(a) Show that

$$4^0 = \frac{4^1 - 1}{3}$$

$$4^0 + 4^1 = \frac{4^2 - 1}{3}$$

$$4^0 + 4^1 + 4^2 = \frac{4^3 - 1}{3}$$

$$4^0 + 4^1 + 4^2 + 4^3 = \frac{4^4 - 1}{3}$$

(b) Suppose that for some natural number k, it is true that

$$4^0 + 4^1 + 4^2 + 4^3 + 4^4 + \ldots + 4^{k-1} = \frac{4^k - 1}{3}.$$

Then multiply both sides of the above equation by 4 and add 1 to both sides to determine the sum

$$4^0 + 4^1 + 4^2 + 4^3 + 4^4 + \ldots + 4^{k-1} + 4^k.$$

(c) What have you proved using Mathematical Induction?

(d) Is this true: $4^0 + 4^1 + 4^2 + 4^3 + \ldots + 4^{p-1} = \frac{4^p - 1}{3}$?

ONE-THIRD-PERFECT NUMBERS. A number N is said to be one-third-perfect provided that there is a prime p such that

ONE-THIRD-PERFECT NUMBERS. A number N is said to be one-third-perfect provided that there is a prime p such that

(a) $\frac{(3+1)^p - 1}{3}$ is prime,

(b) $N = \frac{(3+1)^p - 1}{3} \times (3+1)^{p-1}$, and

(c) The sum of the factors of N (excluding N itself) is

$$\frac{(3+1)^{p-1} + (3-1)}{3} \times \frac{(3+1)^p - 1}{3}$$

THEOREM 1.13: There exist primes p and $\frac{4^p - 1}{3}$ in which

$$4^{p-1}\left(\frac{4^p - 1}{3}\right)$$

is not one-third-perfect.

STEP 1. For the Prime p = 2, show that $\frac{4^p - 1}{3}$ is a prime.

STEP 2. Find the sum of the factors of $4^{p-1}\left(\frac{4^p - 1}{3}\right)$.

STEP 3. Show that the result found in STEP 2 does not satisfy (c) of the definition.

THEOREM 1.14: There are no numbers that are one-third-perfect.

The Sum of Successive Powers of 5. The student should be led to prove the following theorem.

THEOREM 1.15: For any integer p, the sum of the first p powers of 5 is equal to $\frac{5^p - 1}{4}$, i.e.,

$$5^0 + 5^1 + 5^2 + 5^3 + \ldots + 5^{p-1} = \frac{5^p - 1}{4}$$

STEP 1. Show that

$$5^0 = \frac{5^1 - 1}{4}$$

$$5^0 + 5^1 = \frac{5^2 - 1}{4}$$

$$5^0 + 5^1 + 5^2 = \frac{5^3 - 1}{4}$$

$$5^0 + 5^1 + 5^2 + 5^3 = \frac{5^4 - 1}{4}$$

STEP 2. Multiply both sides of the last result by 5; add 1 to both sides of this new equation to get your next equation to go in STEP 1.

STEP 3. Repeat STEP 2 until you are convinced that

$$5^0 + 5^1 + 5^2 + 5^3 + \ldots + 5^{p-1} = \frac{5^p - 1}{4}.$$

ONE-QUARTER-PERFECT NUMBERS. We would like to introduce a problem about quarter-perfect numbers. If we exclude 775 from the divisors of 775, then the sum of its divisors is 1 + 5 + 25 + 31 + 5 x 31.

Observe that

31	is prime and divides both 217 and 775.
217	is 7 x 31 and
775	is 25 x 31.
7	is approximately 1/4 of 25.

A similar relationship holds for the number 305171875, i.e., if we exclude 305171875 from the divisors of 305171875, then the sum of its divisors is 76307617.

Note that

$$1 + 5 + 25 + 125 + 625 + 3125 + 15625 + 3906 \times 19531 = 76307617.$$

We find that

19531	is prime and divides both 76307617 and 305171875.
76307617	is 3907 x 19531 and,
305171875	is 15625 x 19531.
76307617	is approximately 1/4 of 305171875.

Thus, the numbers 775, and 305171875 both have the special property of each having itself and the sum of its factors as multiples of the same non-composite number and having a ratio of approximately one-quarter. On this account, we name them "1/4-perfect numbers". In each case the non-composite number is given by the formula $\frac{(4+1)^p}{4}$ for some prime p. We now give a formula for some 1/4-perfect numbers. Our formula says: A number N is said to be 1/4-perfect if there is a prime p such that

(a) $\frac{(4+1)^p - 1}{4}$ is prime,

(b) $\frac{(4+1)^p - 1}{4} \times (4+1)^{p-1} = N$, and

(c) the sum of the factors of N (excluding N itself) is

$$\frac{(4+1)^{p-1} + (2-1)}{4} \times \frac{(4+1)^p - 1}{4}$$

Exercise:

If both p and $\frac{5^p - 1}{4}$ are primes, then

$$5^{p-1}\left(\frac{5^p - 1}{4}\right)$$

is a quarter-perfect number.

STEP 1. Show that, except for the number itself, the divisors of $5^{p-1}\left(\frac{5^p - 1}{4}\right)$ are

$$1,\ 5,\ 5^2,\ \ldots,\ 5^{p-1},\ \left(\frac{5^p - 1}{4}\right),\ 5\left(\frac{5^p - 1}{4}\right),\ \ldots,\ 5^{p-2}\left(\frac{5^p - 1}{4}\right).$$

STEP 2. Show that the sum of these divisors is given by

$$1 + 5 + 5^2 + \ldots + 5^{p-1} + \left(\frac{5^p - 1}{4}\right)(1 + 5 + \ldots + 5^{p-2}).$$

STEP 3. Change the above sum to show that it is equal to

$$\frac{5^p - 1}{4} + \frac{5^p - 1}{4} + \frac{5^{p-1} - 1}{4}$$

Use THEOREM 1.12.

STEP 4. Expand the above result to get

$$\frac{(4 + 1)^{p-1} + (4 - 1)}{4} \times \frac{(4 + 1)^p - 1}{4}$$

The student should now be able to score well on the following exercises.

Exercises:

1. Show that for any integer p, the sum of the first p powers of 6 is equal to $\frac{6^p - 1}{5}$.
2. Give a formula for finding the sum of the first p powers of 7.
3. How would you define a number to be one-fifth perfect?
4. With the definition given in (3) above, can you find any numbers that are one-fifth perfect?
5. Define numbers to be one-sixth-perfect in such a way that 6725201 is one-sixth-perfect number. Show that every one-sixth-perfect number either ends in 1 or 3 (Base Ten).
6. State and prove three theorems about quarter-perfect numbers. Do not include any of those mentioned here.
7. Show that there exists primes p and $\frac{6^p - 1}{5}$ such that $(\frac{6^p - 1}{5})$ is not a one-fifth-perfect number.
8. Define the set of one-tenth-perfect numbers and give an example.
9. Give a generalization in which you define for a natural number a, the set of numbers which are 1/a-perfect.
10. Show that in Base Ten each quarter-perfect number ends in 75.
11. State a corollary to the Theorem of (10) above for Base Eight quarter-perfect numbers.
12. State a corollary to the Theorem of (10) above for Base Four quarter-perfect numbers.
13. State a corollary for Base Five quarter-perfect numbers.
14. The Base Ten quarter-perfect number 775 and 305171875 are represented in Base Nine by 1051 and 7072101, respectively. Make a prediction about quarter-perfect numbers in Base Nine. Try to prove or disprove your conjecture.

(PART 2)

FURTHER GENERALIZATIONS WITH APPLICATIONS

We shall now generalize the concept of a perfect number to the concept of a number which is said to be $\frac{1}{a}$ - perfect where a is a natural number.

DEFINITION 2.1: A number N is said to be $\frac{1}{a}$ - perfect if there is a number p such that for some number a,

(a) both p and $\frac{(a+1)^p - 1}{a}$ are primes,

(b) $N = \frac{(a+1)^p - 1}{a} \times (a+1)^{p-1}$, and

(c) the sum of the factors of N (excluding N itself) is: $\frac{(a+1)^{p-1} + (a-1)}{a} \times \frac{(a+1)^p - 1}{a}$.

The following theorems follow immediately.

THEOREM 2.1: Any number which is $\frac{1}{1}$ - perfect is also perfect by Euchlid's definition.

THEOREM 2.2: There are exactly three numbers less than the number

129140162 x 43046721

which are half-perfect numbers.

THEOREM 2.3: There are exactly two numbers less than the number

12,207,031 x 9,765,625

which are one-fourth-perfect numbers.

THEOREM 2.4: There are no one-fifth-perfect numbers.

THEOREM 2.5: There is no number which is both a tenth-perfect number and smaller than: 74×10^{19}.

THEOREM 2.6:

(a) If a number N has 2 as a factor exactly once, and the sum of the factors of N is S, then $\frac{N}{2}$ does not have 2 as a factor and the sum of the factors of $\frac{N}{2}$ is $\frac{S}{1 + 2}$.

If a number

(b) If a number N has 2 as a factor exactly twice, and the sum of the factors of N is S, then $\frac{N}{4}$ does not have 2 as a factor and the sum of the factors of $\frac{N}{4}$ is $\frac{S}{1 + 2 + 4}$.

(c) If a number N has 2 as a factor n times, and the sum of the factors of N is S, then $N/2^n$ does not have 2 as a factor and the sum of the factors of $N/2^n$ is given by the formula:

$$\frac{S}{1 + 2 + 4 + \ldots + 2^n} = \frac{S}{2^{n+1} - 1}$$

THEOREM 2.7: If $N = f_1^{e_1} \cdot f_2^{e_2} \cdot f_3^{e_3} \cdots f_i^{e_i}$

where the f_i are distinct primes and the sum of the factors of N is S, then

$$\frac{N}{f_1^{e_1} \cdot f_2^{e_2} \cdot f_3^{e_3} \cdots f_i^{e_i}} =$$

has no factor other than 1 and the sum of the factors of N is given by

$$\frac{S}{\frac{f_1^{e_1+1}-1}{f_1-1}\cdot\frac{f_2^{e_2+1}-1}{f_2-1}\cdot\ldots\cdot\frac{f_i^{e_i+1}-1}{f_i-1}}.$$

or by

$$S \times \frac{(f_1-1)(f_2-1)\cdots(f_i-1)}{(f_1^{e_1+1}-1)(f_2^{e_2+1}-1)\cdots(f_i^{e_i+1}-1)}$$

THEOREM 2.8: If N is a perfect number, i.e., the sum of the factors of N is $2 \cdot N$, then

$$1 = 2 \cdot \frac{f_1^{e_2}(f_1-1)\cdot f_2^{e_2}(f_2-1)\cdots f_i^{e_i}(f_i-1)}{\left(f_1^{e_1+1}-1\right)\cdot\left(f_2^{e_2+1}-1\right)\cdot\left(f_i^{e_i+1}-1\right)}$$

The condition is also sufficient.

THE SEARCH FOR PERFECT NUMBERS. If one wishes to search for perfect numbers or numbers that are $\frac{1}{a}$ - perfect, then TABLE II should be filled in and primes should be selected satisfying the conditions of THEOREM 2.8. In particular, if an odd perfect number is desired, then the following two theorems are applicable.

THEOREM 2.9: If N is a perfect number, then $(f_n - 1)$ divides $\left(f_n^{e_n+1} - 1\right)$.

THEOREM 2.10: If N is an odd perfect number, then exactly one of the numbers $(f_n^{e_n+1} - 1) / (f_n - 1)$ is even (having only one 2-factor).

Hidden Combinations: A Thoughtful Game of Clues and Implications

This game stimulates analytical thinking, deductive thinking and skillful interpretation of language statements. It is related to David Page's "Hidden Numbers"* game and to several other question games.

Materials

(A) Decks of twenty, small (approx. 2" x 2") opaque cards one deck for each four students. The cards should be numbered on one side so that there are four cards for each of the numbers 1 through 5.

(B) A second deck (also one deck per four students) of 12 cards (possibly 3" x 5" file cards) on each of which one of the questions listed below is written.

List of questions to be typed on question cards.

1. Are there any numbers that are missing?
   If so, how many are missing?
2. How many numbers are there which appear less frequently than the other numbers?
3. How many 5's do you see?
4. How many times does the most frequently appearing number appear?

5. How many numbers appear more than two times?

6. How many numbers do you see exactly once?

7. What is the sum of the numbers that you can see?

8. How many triples can you see?

9. How many numbers are multiples of two?

10. Do you see more 4's and 5's or do you see more 1's or 2's?

11. Do you see more odd or more even numbers?

12. Shuffle the cards and pass them on.

Play

Allow students to arrange themselves in groups of four and then pass out a number deck and a question deck to each group. The number deck is shuffled and each player draws three cards, face down, which he then (without seeing his numbers) props up in front of him for only the other three players to see. When play begins each person can see the number combinations of all the players except his own. The player who begins draws the top card from the question deck, reads the question aloud for all players to hear clearly and answers the question in accordance with the three combinations which he can see, before replacing his question card to the bottom of the deck. Play, then moves clockwise to the next player who draws a question card, reads, answers and replaces it as before. Players who hear answers receive clues to their number combinations. (For instance, suppose the four players are A, B, C, and D. If player A answers that the sum of the numbers he can see is 26, he gives a clue to player D who can see the sum of combinations in the B and C hands. Suppose the sum of B's cards is 8 and of C's cards is 9, then their sum together is 17 and D concludes that his sum is 26 - 17 = 9) Subsequent answers provide additional clues and eventually a player believes he knows the numbers in his set. When he believes that he knows his cards, he says "I declare that my combination is ..." (and states the numbers). If he is correct, he scores 6 points, but if he gives the wrong combination he must throw in his hand and start anew by drawing three new number cards. A player may declare his hand at any time. His declaration does not affect the play of the others. They continue to deduce their combinations independently of other declarations until they decide individually to declare their combination. Play may continue until time runs out or until someone reaches a predetermined total score.

After students have become familiar with the game and are ready to move on from small group play, the teacher may then simulate play using a blackboard on which three hands are shown for players A, B, and C. Each student in the class assumes the role of D. The teacher, holding D's set hidden draws question cards successively for the players. (That is for A, B, C and D). In rotation the teacher assumes the role of the several players and answers the question drawn for each particular player to simulate the responses of the players in a real game. All the students in class have the same clues and try to deduce D's combination with minimal clues.

Exercises

1) Below is shown a sample of an exercise sheet that may be developed for classwork or homework. The sheet represents a record of play among players A, B, C and D with C's hand unknown to the reader. Answers of successive players are given as they would in actual play. The question is to find the implications of each answer and deduce C's hidden combination.

Note: Similar exercise sheets may be easily be developed to provide additional Exercises.

2) Create a similar game using a deck of letter cards instead of number cards, by writing appropriate questions for the question deck.

Note: Several potentials seem to exist for modifying the game by using:

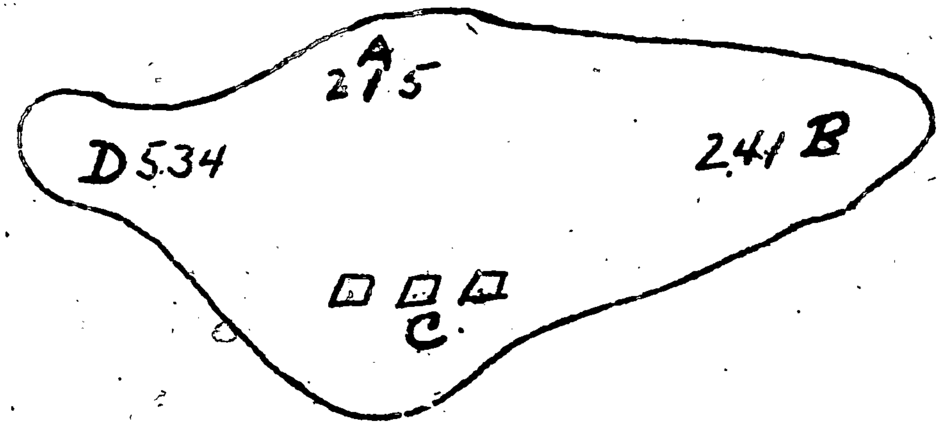
a) negative numbers

b) fractions;

c) multiples and/or divisors

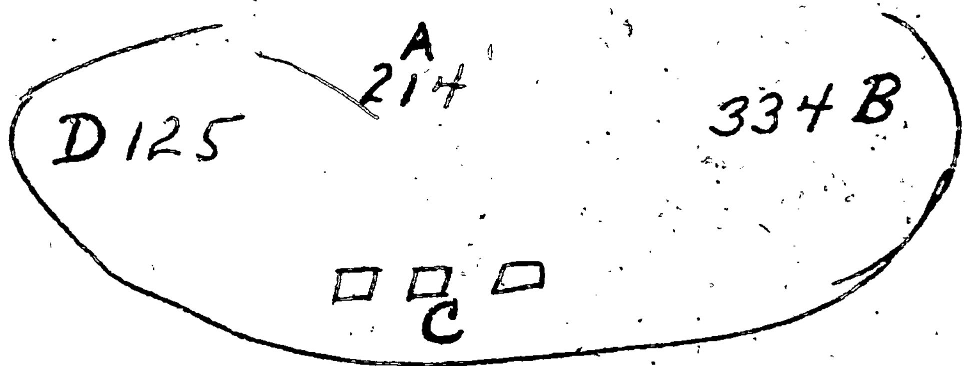
or by allowing players in turn to ask questions of their own choosing.

WHAT'S MY SET? GAME I



FROM THE ANSWERS GIVEN BY PLAYERS "D", "A" AND "B", CAN YOU DETERMINE WHAT THE CARDS ARE IN "C"'S SET?

QUESTION NO.	RESPONDENT	QUESTION	ANSWER	ANALYSIS AND SOLUTION
(i)	D	HOW MANY CARDS THAT HAVE NUMBERS THAT ARE MULTIPLES OF 2 CAN YOU SEE?	"THREE"	
(ii)	A	HOW MANY 5'S CAN YOU SEE?	"TWO"	
(iii)	B	DO YOU SEE MORE ODD OR EVEN NOS.?	"MORE ODD"	
(iv)	D	WHAT IS THE SUM OF THE NUMBERS YOU CAN SEE?	"TWENTY-SIX"	

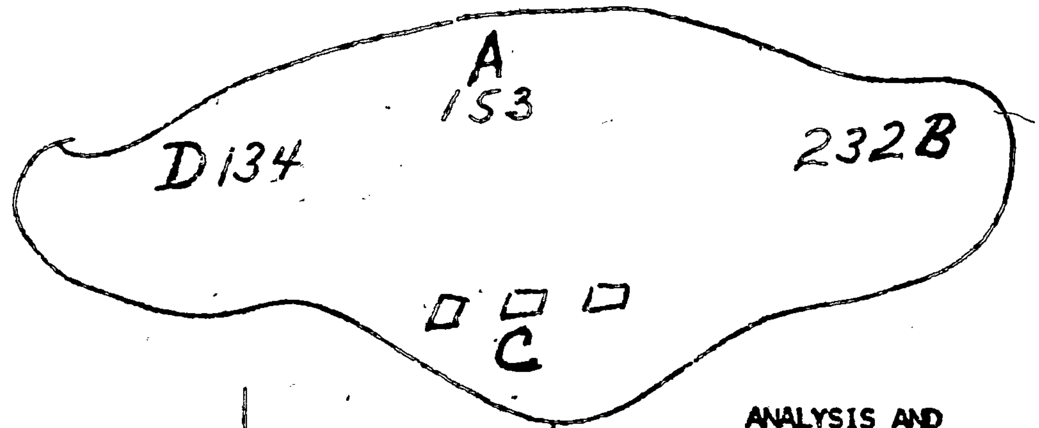


WHAT'S MY SET GAME II

FROM THE ANSWERS GIVEN BY PLAYERS "D", "A", AND "B", CAN YOU DETERMINE WHAT THE CARDS ARE IN "C"'S SET?

QUESTION NO.	RESPONDENT	QUESTION	ANSWER	ANALYSIS AND SOLUTION
(i)	D	HOW MANY CARDS ARE MULTIPLES OF 2?	"FOUR"	
(ii)	A	DO YOU SEE MORE 4'S AND 5'S OR 1'S AND 2'S?	"NEITHER"	
(iii)	B	DO YOU SEE MORE ODD "NUMBERS" OR MORE EVEN "NUMBERS"?	"MORE ODD"	
(iv)	D	HOW MANY 5'S DO YOU SEE?	"ONE"	

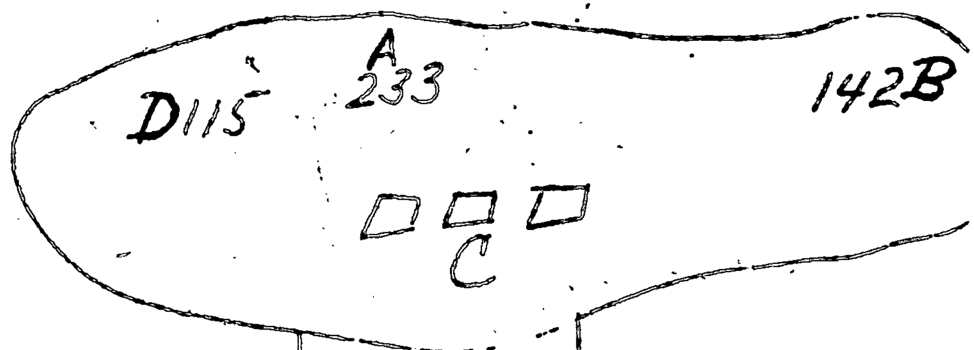
WHAT'S MY SET? **GAME III**



FROM THE ANSWERS GIVEN BY PLAYERS 'D' "A", AND "B" CAN YOU DETERMINE WHAT THE CARDS ARE IN 'C'S SET?

QUESTION NO.	RESPONDENT	QUESTION	ANSWER	ANALYSIS AND SOLUTION
(i)	D	ARE THERE ANY NUMBERS MISSING? IF SO, HOW MANY?	"NONE MISSING"	
(ii)	A	HOW MANY TRIPLES CAN YOU SEE?	"ONE"	
(iii)	B	HOW MANY NUMBERS DO YOU SEE EXACTLY ONCE?	"ONE"	
(iv)	D	WHAT IS THE SUME OF THE "NUMBERS" THAT YOU CAN SEE?	"TWENTY-SEVEN"	

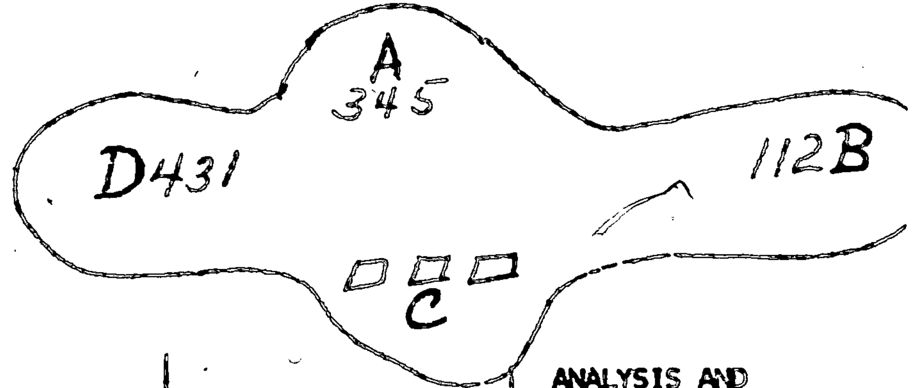
WHAT'S MY SET? GAME IV



FROM THE ANSWERS GIVEN BY PLAYERS "D", "A" AND "B", CAN YOU DETERMINE WHAT THE CARDS ARE IN "C"'S SET?

QUESTION NO.	RESPONDENT	QUESTION	ANSWER	ANALYSIS AND SOLUTION
(i)	D	HOW MANY NUMBERS APPEAR TWO TIMES EXACTLY?	"THREE"	
(ii)	A	DO YOU SEE MORE ODD OR MORE EVEN NUMBERS?	"MORE ODD"	
(iii)	B	DO YOU SEE MORE 4'S AND 5'S OR MORE 1'S AND 2'S?	"MORE 1'S AND 2'S"	
(iv)	D	HOW MANY NUMBERS ARE THERE WHICH APPEAR LESS FREQUENTLY THAN THE OTHER NUMBERS?	"THREE"	
(v)	A	HOW MANY CARDS HAVE NUMBERS THAT ARE MULTIPLES OF TWO?	"THREE"	
(vi)	B	WHAT IS THE SUM OF THE NUMBERS THAT YOU CAN SEE?	"TWENTY-THREE"	

WHAT'S MY SET? GAME V



FROM THE ANSWERS GIVEN BY PLAYERS "D", "A", "B", CAN YOU DETERMINE WHAT THE CARDS ARE IN "C"'S SET?

QUESTION NO.	RESPONDENT	QUESTION	ANSWER	ANALYSIS AND SOLUTION
(i)	D	HOW MANY NUMBERS APPEAR MORE THAN TWO TIMES?	"NONE"	
(ii)	A	DO YOU SEE MORE CARDS WITH ODD OR WITH EVEN NUMBERS?	"MORE ODD"	
(iii)	B	DO YOU SEE MORE 4'S AND 5'S THAN 1'S AND 2'S.	"MORE 4'S & 5'S"	
(iv)	D	HOW MANY CARDS HAVE NUMBERS THAT ARE MULTIPLES OF TWO?	"THREE"	
(v)	A	HOW MANY NUMBERS DO YOU SEE EXACTLY ONCE?	"TWO"	
(vi)	B	ARE THERE ANY NUMBERS MISSING? IF SO, HOW MANY ARE MISSING?	"NONE, MISSING"	
(vii)	D	HOW MANY 5'S DO YOU SEE?	"TWO"	

Two Students Walk: The Interpretation and Application of Coordinate Graphs

A primary objective of this study is to breakdown patterns in which many students require artifical classroom cues to call up and utilize their knowledge and skills in coordinate graphing and, thus, to establish more general responses. This topic, in effect, asks students, "What are the best ways to describe motion?" Word, pictorial, graphical and algebraic descriptions are considered before coordinate graphs emerge as convenient, information-packed descriptions.

Materials

1) 1/2 inch graph paper -- several sheets per student.
2) (Optional) A yard (or meter) stick and stop watch to measure data accurately.
3) (Optional) A cloth tape or knotted string about thirty feet long with one foot markers.

(I) Two Students Walk: Different Speeds Along A Straight Path Regular Type

While it is hoped that students will learn a good deal about coordinate graphing and its applications as a result of the investigations in this unit, the topic is not introduced as graphing per se. Frequently outcomes are not predictable when the instructor listens to students and allows them a role in developing the topic. Although the instructor may indirectly encourage a consideration of graphs, it is probable that graphs will emerge as the major concept toward the end of this unit. When the demonstrations are introduced avoid even using the terms "graph" or "graphing".

Regular Type Student Demonstration

Before class starts the instructor should call two students, let us call them Reggie and Sam, aside and instruct them to walk at a steady rate on a straight line toward a finish line. They should start shoulder to shoulder at a starting line or point, which we will call "A" and proceed to the finishing line or point, called "B". Have the students count to themselves at a regular tempo of about one count per second with Reggie taking one step per second and Sam taking one step every two seconds. (The students will benefit from 3 or 4 practice runs outside the classroom.) Then have them demonstrate before the class.

Let us see how accurately you can observe and describe the activity of Reggie and Sam who are going to give a demonstration for you.

Have the two students demonstrate their "walk" and then --

"How many things did you observe?"

"Suppose you had a friend in San Francisco and you wanted to describe Reggie and Sam's demonstration in a letter to him. Draw a sketch or some sort of diagram that would show, as many of your observations as possible."

"After you have done this, give a second description using word statements to list as many observations as possible."

Pass among the students raising questions to stimulate thinking and to demonstrate weaknesses without giving answers. Try to resist the temptation to tell what happended or how to represent it as this is likely to kill the challenge and student interest. If questions arise from conflicting observations have Reggie and Sam repeat their demonstration without verbalizing what they did. Students, thus, must observe and resolve their differences themselves.

As you find characteristically different sketches, diagrams and graphs ask individuals to show them on the blackboard. Next, have a student recorder list on the board, one at a time, the verbal observations. Again, if students disagree on what they observed, have Reggie and Sam repeat their demonstration. If students continue to disagree or if they wish greater accuracy than paces and seconds, the distances may be measured with a yard stick and timed with a stop watch.

Example

Examples of student responses to the above may include;

(A) For word statement observations:

(1) One student walked faster than the other.

(2) The distances walked by both students were the same.

(3) One student took longer than the other to reach the end point.

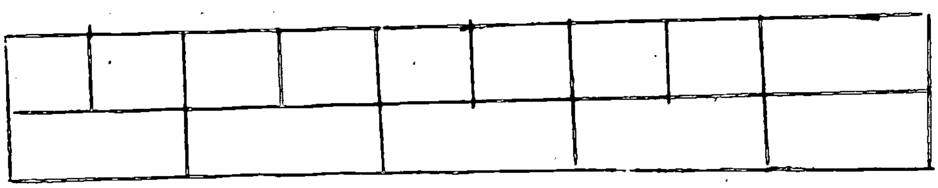
(4) They walked in a straight line.

(5) They started and ended at the same point.

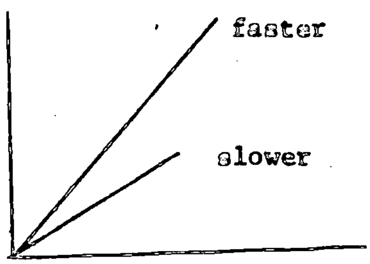
(B) For illustrations or diagrams:

(1)

(2)



(3)



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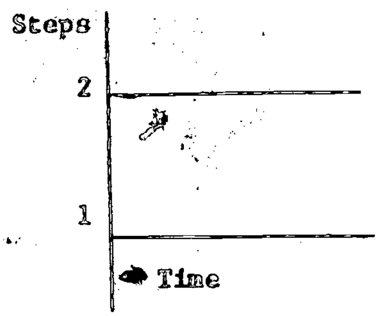


Do the diagrams shown on the board accurately represent the observations?" "Do the word statements and the diagrams agree?" "Which is the best diagram?

Compare diagrams using such criteria as the ones below to compare and evaluate them.

(1) Does it show where each person was a given time?

(2) Does it show where they started?

(3) Does it show where they ended?

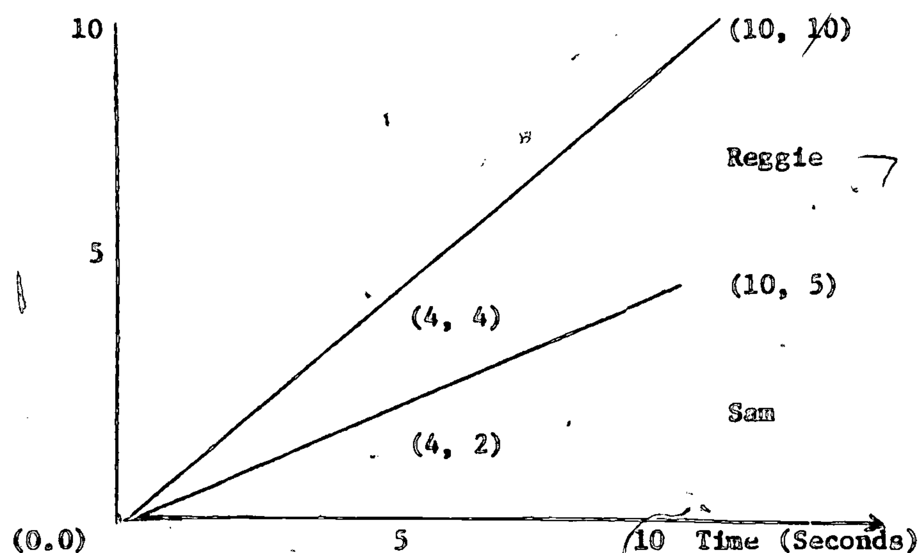
(4) Does it show how far they traveled?

(5) Does it show how long they took?

(6) Does it tell how fast they traveled?

etc.

Assuming that the distance from A to B is 30 feet and that Reggie paces 3 feet per second with Sam's paces, 3 feet every 2 seconds, one possible graph describing the motion is shown below:



To insure understanding of the displacement-time graph questions such as the following may be asked: Where is Reggie (or Sam) one (or two) seconds after the start? What is the velocity of Reggie (or Sam) at the end of one (or two) seconds? etc.

$$\left(\text{Average velocity} = \frac{\text{Displacement}}{\text{Time for that displacement}} \right)$$

Displacement from A is used in preference to distance from A since the direction of displacement will clarify the description of actual situations. For instance if Reggie started six feet further from the finish line, B, than the original starting line, A, his original displacement from "A" would be -6 feet, and after 2 seconds (at 3 feet per second) he would have displacement from "A" = 0 and after a total traveling time of 12 secinds his displacement from A (in the B direction) would be +30 feet.

Displacement and Velocity are vector quantities having both magnitude and direction. Distance and speed are scalar quantities with only magnitude. If a plane flies a distance of 500 miles from Washington one does not know whether the plane is over land or water, but if the plane has a displacement of 500 miles east from Washington it is clearly over water.

(II)/ Two Students Walk: Traveling At the Same Speed One Reverses His Direction At the Midpoint

Coach the students and have them practice as before. This time, however, have both Reggie and Sam walk at the rate of one pace (3 ft.) per second. Have Reggie walk directly from A to B covering the 30 ft. in 10 seconds while Sam goes from A to the midpoint, M, (between A and B) where he reverses his direction without loosing his pace so that he returns to A the same instant Reggie arrives at B. Sam travels from A to M (15 ft.) and M to A (15 ft.) for a total of 30 feet traveled in 10 seconds, also.

Suggest that the students list observations, sketch graphs, etc. to describe what they have observed. Compare and evaluate the different modes -- word statements, pictorial sketches and graphs -- of presenting the observations.

Exercises

For classwork or homework the following questions may be asked along with similar questions growing out of class discussions. Students may demonstrate many of the activities described.

(1) Sketch a displacement-time graph showing Reggie walking from start, A, to midpoint, M, at one pace per two seconds and from M to finish, B, at one pace per second.

(2) Sketch graphs showing:

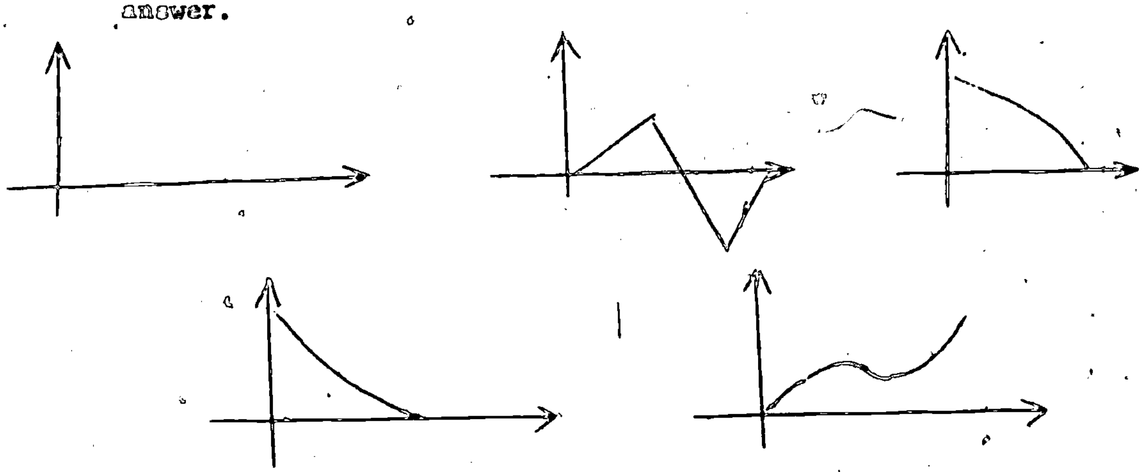
(a) Reggie and Sam start from A at the same time. Reggie walks twice as fast as Sam until he reaches the midpoint when he, Reggie, slows to Sam's rate. Both continue traveling until each is at the finish, B.

(b) Reggie and Sam start together from A to travel to destination B. Reggie walks twice as fast as Sam, but when Reggie has traveled 3/5 distance to B he stops and waits for Sam to catch him. Then, they continue to finish (B) at Sam's rate.

(3) Sketch a graph showing R starting at A and heading for B at two paces per second while S starts at B heading for A at one pace per second. If they walk head on where do they meet?

(4) Sketch two graphs with "interesting" breaking points and different rates of travel. In class, challenge neighbors to describe in words what each graph represents.

(5) Discuss whether or not a person can move according to the descriptions of the following graphs. Explain your answer.



(6) Write an algebraic formula describing Reggie's motion when he travels at 3 feet per second from A to B. What is Sam's formula if he moves 3 feet every two seconds?

(7) Write an algebraic formula (or formulas) for Reggie's motion in Exercise (1) above.

(8) Write algebraic formulas to describe the motions described in Exercise (2) above.

(9) What method of decription communicates characteristics of students' movement most satisfactorily (a) word statements, (b) pictorial sketches, (c) graphs or (d) algebraic formulas?

One Student Walks: Two Observers Are Located One At the Starting Line and the Other At the Finish Line

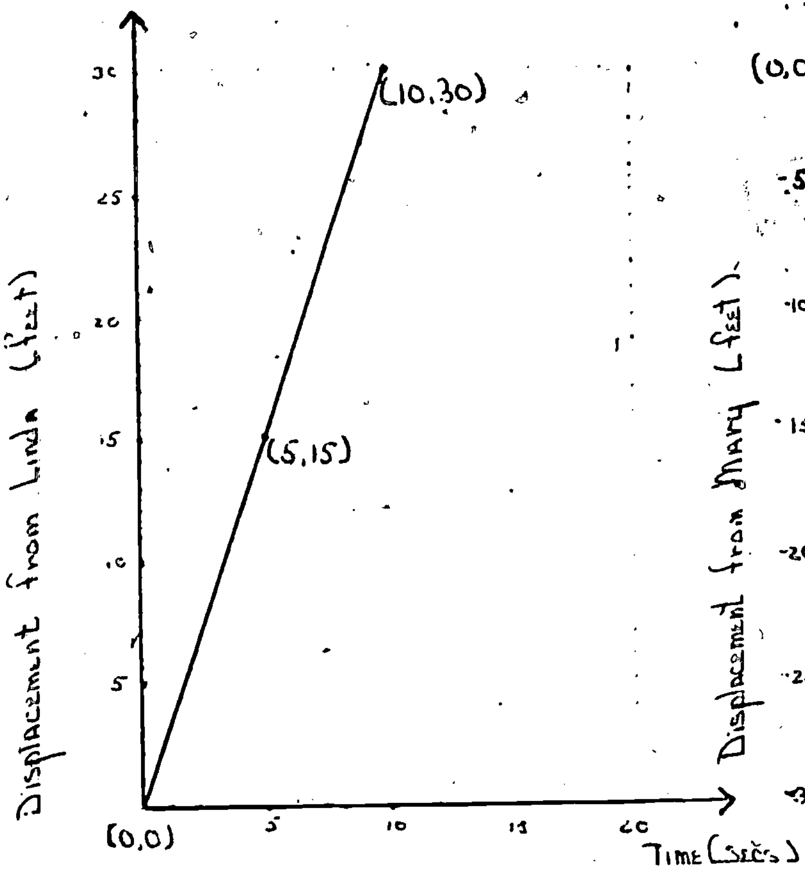
Have two students, let us call them Linda and Mary, act as observers at points A and B respectively while a student, Reggie, walks at one pace per second from A to B. Students may actually demonstrate the activity called for, and the other class members may, then, draw two graphs, one for Linda's observations and one for Mary's observations.

Linda and Mary are going to observe Reggie as he walks from A to B. On a sheet of paper tabulate observations of "displacement from observer" and "time from start" first as Linda (at A) and then second as Mary (at B) would record these observations. Include 4 or 5 tabulation pairs for each observer.

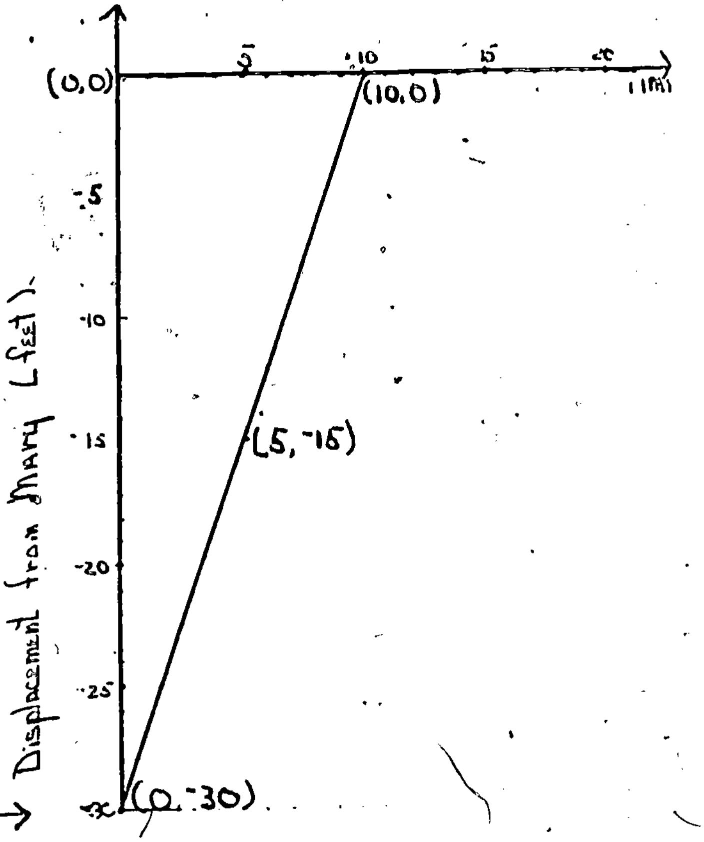
Then sketch two graphs, one for Linda's observations and one for Mary's.

Note: Ordered pairs associated with graph points and a clear description of the quantities in these pairs provide a key to understanding the graphs. Thus, displacement is not a sufficient description to use in the observations above, we need to know, for example, whether we have "displacement from Linda" or "displacement from Mary" before we can draw or interpret graphs.

If B is taken to be in the positive direction from A the girls' graphs might appear as below:

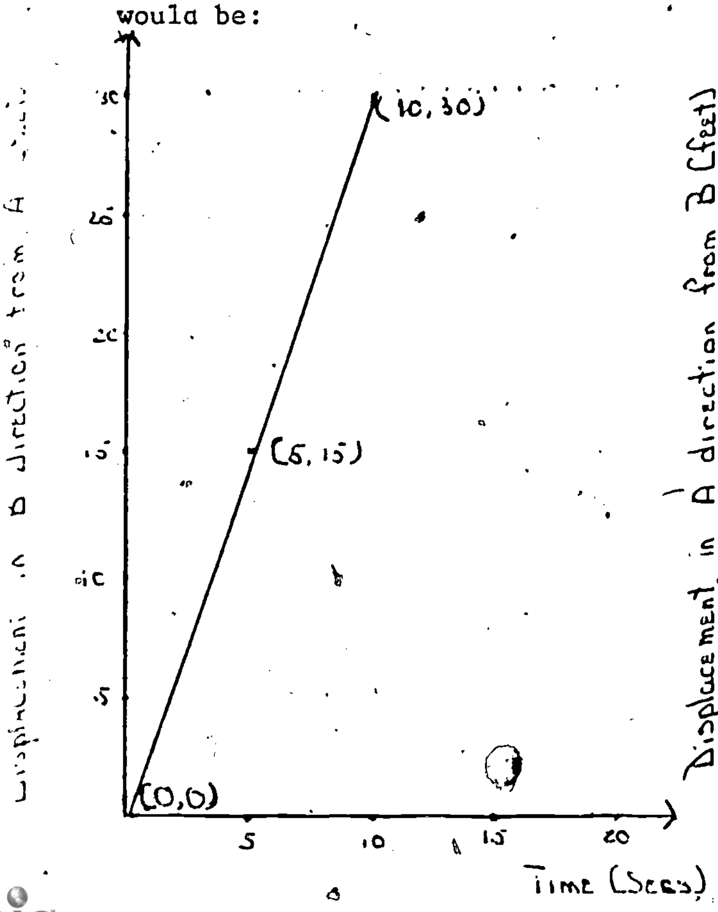


Linda's Graph

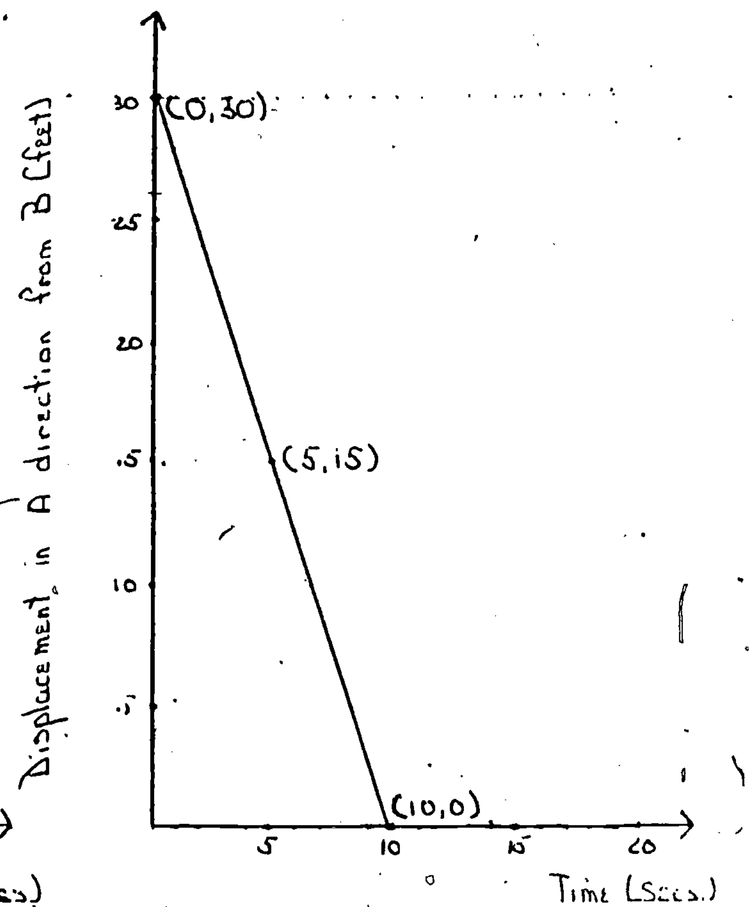


Mary's Graph

If, however, Linda and Mary face each other and B is +30 feet from Linda (at A) and A is +30 feet from Mary (at B) then their graphs would be:



Linda's Graph



Mary's Graph

The instructor may introduce as many variations as seems desirable in the light of class responses. Frequently the complexities that are introduced by moving the location of the observer are avoided by using special cases or by imposing a rather strict set of unexpressed conventions. When instructors avoid complexities in these ways in order to "cover topics" in a course, students, accept graphs in the special way that they are "given" to them and can give back mechanical answers but they lack the depth of understanding that would make the graphs functional for them.

It is helpful in developing student understanding to establish the following or similar sequence of steps in relating graphs to phenomena and vice versa.

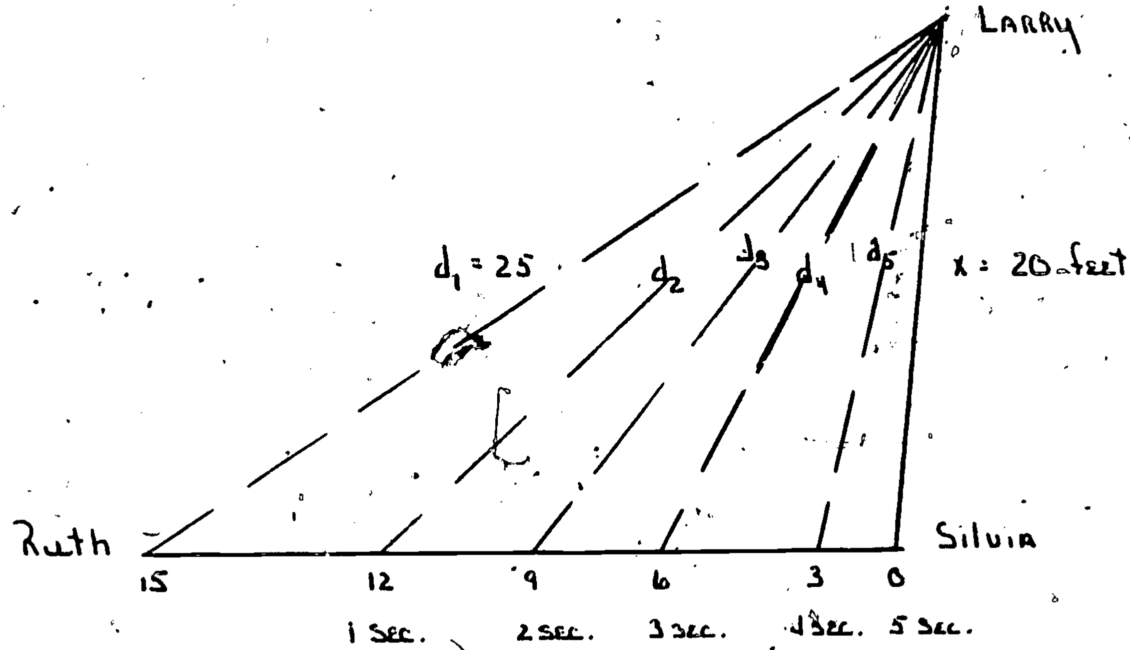
(a) A graph point → An ordered pair of numbers: (2, 6)

(b) An ordered pair → Things measured: (time from start, displacement from Linda)

(c) Things measured → Event observed: (Reggie is 6 feet from Linda after 2 seconds from start)

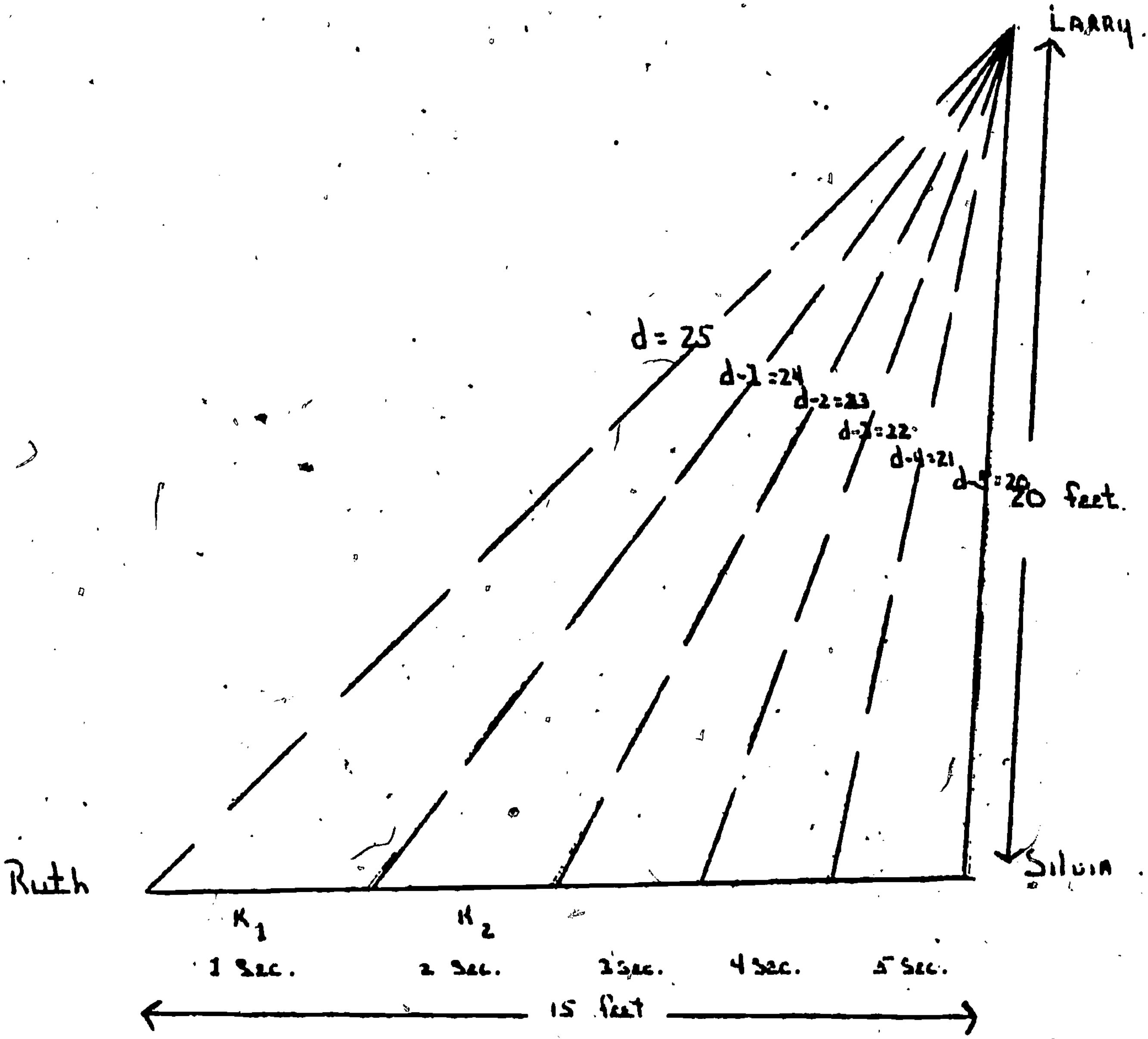
IV. One Student Walks: Observer 20 Feet From The Student's Path

The instructor may have students perform demonstrations and/or experimentations with the following situations as seems best. This particular investigation may challenge the more advanced and quicker students in class who may study the questions raised while other students finish up earlier work.



Ruth, Silvia and Larry are located at vertex points of a 15, 20, 25 right triangle. Ruth walks at a constant rate of 3 ft. per second along the 15 foot side directly toward Silvia who is located at the vertex of the right angle. Thus, as Ruth walks, she is at 12, 9, 6 ... feet from Silvia at the end of 1, 2, 3 ... seconds. At the end of 5 seconds Ruth is alongside Silvia and the separation distance is zero. Larry holds the knotted rope taut between himself and Ruth. He records the distances Ruth is from him at intervals of 1, 2, 3 ... 5 seconds (after Ruth starts to walk).

<u>Can you graph the distance, d, aganist the time, t? What is the average velocity? Can you measure Δd for a given Δt and find the ratio $\frac{\Delta d}{\Delta t}$ for several cases? Can you sketch the graph of $\frac{\Delta d}{\Delta t}$ against time?</u>


LARRY
d = 25
d-1 = 24
d-2 = 23
d-3 = 22
d-4 = 21
d-5 = 20
20 feet
Ruth
Silvia
K_1
K_2
1 Sec.
2 Sec.
3 Sec.
4 Sec.
5 Sec.
15 feet

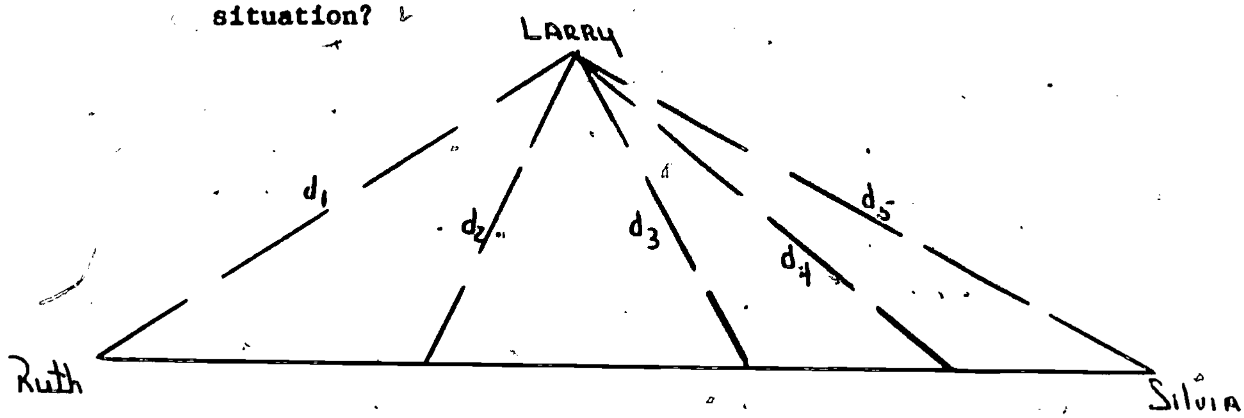
Exercises

1. (a) Could you change the situation to the following so that the uniform motion (i.e. the change in distance each second is the same as in each other second of time.) is now with respect to Larry?

   (b) Can you graph $(15 - K_n)$ against time? What is the graph?

   (c) Can you sketch $\frac{\Delta d}{\Delta t}$ against time? What is the graph?

2. What are the questions and what would you find with this situation?



3. Can you graph or draw the path of the moon as it proceeds with the earth around the sun?

Arrays, Polynomials and Finite Differences

The method of finding polynomials using finite differences in table of data is quite powerful and often convenient to use. The method may be introduced to students by "maneuvering" on special number lattices or arrays and investigating regularities in the arrays. The illustrative development below shows one possible way of developing the topic in a classroom.

On the blackboard show the following array:

10	3	16	73	198	415	748	1221
-7	13	57	125	217	333	473	
20	44	68	92	116	140		
24	24	24	24	24			

Can you figure out how this table was constructed? Can you extend the array keeping the same pattern? Find and write the necessary numbers to extend it to the right.

This is the easiest direction to extend the array and leads to significant interpretations.

Now, can you extend the array in the other direction such as to the left or down? Try it and put in some values. Can you extend the array upwards? Can you find the one hundredth number to the right of 10 in the first line?

Extending to the right requires the assumption that the next number in the bottom row will be 24. Then, moving up and to the right we can fill in 164, 637 and 1858. Successive diagonals to the right could be added similarly. Moving to the left can be done but becomes complicated because many of the operations for moving to the right are inverted. Moving down contributes little of value, but should be asked for completeness and understanding. Moving up is not possible without having one value either given or assumed on the higher level. (This relates to the necessity of finding a constant after integration in calculus.)

These arrays are related to equations. For instance, if we were given the equation $y = 3x^2 + 5x + 2$ we would have the following array:

2		10		24		44		70
	8		14		20		26	
		6		6		6		

and if we had the equation $y = x^2 + 2x - 5$, we would have the following:

-5   -2   3   10   19   30   43

3   5   7   9   11   13

2   2   2   2   2

and if we had $y = 2x^2 + 7x + 13$, we would have:

13   22   35   52   73   98

9   13   17   21   25

4   4   4   4

Can you predict the equation which has the following array?

1   9   27   55   93   141

8   18   28   38   48

10   10   10   10

The function for the above table is $y = 5x^2 + 3x + 1$. It was generated by tablulating values of y for x = 0, 1, 2, 3, 4, ... and taking differences between terms in one line to obtain terms in the next lower line. Thus, we have the table for $y = 5x^2 + 3x + 1$:

X	0	1	2	3	4	5
Y	1	9	27	55	93	144

and the first row of the array is the sequence of y values. Other arrays may be developed for class use or homework as may seem best.

Write down to explain to a friend in Tanzania the rules used to find the formula above.

Circulate among the students and raise questions to point out errors.

Now, can you dind an equation which would be similarly related to the array below?

A   B   C   D   E

P   Q   R   S

W   W   W

All the formulas or equations we have discussed so far had an x squared term. Let us consider some different situations.

What would be the formula for y in terms of x related to the following array?

1 4 7 10 13

3 3 3 3

And for this array?

1 0 5 22 57 116

-1 5 17 35 59

6 12 18 24

6 6 6

The first array is related to $y = 3x + 1$ and the second to $y = x^3 - 2x + 1$.

The general analysis for formulating the value of a second degree function in one variable is suggested below. Additional data relative to first, third and fourth degree functions is also provided.

Given $y = ax^2 + bx + c$; this implies the following table:

X	Y	1st dif. in val. of "Y"	2nd differences (dif. of dif.)
0	c		
		a + b	
1	a + b + c		2a
		3a + b	
2	4a + 2b + c		2a
		5a + b	
3	9a + 3b + c		2a
		7a + b	
4	16a + 4b + c		

The array derived from this would be:

(c) (a+b+c) (4a+2b+c) (9a+3b+c) (16a+4b+c)

(a+b) (3a+b) (5a+b) (7a+b)

(2a) (2a) (2a)

Let us stop our analysis now, and consider the inductive situation which we often face in practice. We will attempt to reconstruct the original equation. We note that,

1) We needed 2nd differences before obtaining unchanging difference terms. This implies a quadratic.
2) The coefficient of the squared term is 1/2 the value of the 2nd difference (which, as indicated, is constant).
3) The constant in the quadratic expression is the first term of the first line in the array.
4) The coefficient of the linear term in the expression is obtained by subtracting the quadratic coefficient (represented originally by "a") from the first term in the second line (which our analysis showed to equal "a+b").

<u>Example</u>: Given the array discussed above, namely

A B C D E

P Q R S

W W W

The observations above indicate that the related equation is

$y = (W/2)\, x^2 + (P - W)\, x + A.$

Similar analyses can be made based upon the following analyses.

1. $y = ax + b$

x	y	differences
0	b	
		a
1	$a + b$	
		a
2	$2a + b$	
		a
3	$3a + b$	

one difference

first degree

2. $y = ax^2 + bx + c$

x	y	first differences	second differences
0	c		
		$a + b$	
1	$a + b + c$		$2a$
		$3a + b$	
2	$4a + 2b + c$		$2a$
		$5a + b$	
3	$9a + 3b + c$		$2a$
		$7a + b$	
4	$16a + 4b + c$		

two differences

second degree

3. $y = ax^3 + bx^2 + cx + d$

x	y	first differences	second differences	third differences
0	d			
		$a + b + c$		
1	$a + b + c + d$		$6a + 2b$	
		$7a + 3b + c$		$6a$
2	$8a + 4b + 2c + d$		$12a + 2b$	
		$19a + 5b + c$		$6a$
3	$27a + 9b + 3c + d$		$18a + 2b$	
		$37a + 7b + c$		$6a$
4	$64 + 16b + 4c + d$		$24a + 2b$	
		$61a + 9b + c$		
5	$125a + 25b + 5c + d$			

three differences

third degree

4. $y = ax^4 + bx^3 + cx^2 + dx + c$

x	y				
0	e	a + b + c + d	14a + 6b + 2c	36a + 6b	24a
1	a + b + c + d + e	15a + 7b + 3c + d	50a + 12b + 2c	60a + 6b	24a
2	16a + 8b + 4c + 2d + e	65a + 19b + 5c + d	110a + 18b + 2c	84a + 6b	
3	81a + 27b + 9c + 3d + e	175a + 37b + 7c + d	194a + 24b + 2c		
4	256a + 64b + 25c + 5d + e	369a + 61b + 9c + d			
5	625a + 125b + 25c + 5d + e				

Exercises:

1. Find a formula that relates y to x in the following tables:

a)

x	y
0	1
1	4
2	7
3	10
4	13

b)

x	y
0	0
1	2
2	8
3	18
4	32

c)

x	y
0	0
1	2
2	6
3	12
4	20
5	30

2. Find a formula that relates y to x in the following tables:

a)

x	y
0	b
1	1 + b
2	2 + b
3	3 + b
4	4 + b

b)

x	y
0	0
1	2 + b
2	8 + 2b
3	18 + 3b
4	32 + 4b
5	50 + 5b

c)

x	y
0	0
1	a
2	2a
3	3a
4	4a
5	5a

3. Graph the following points, (0, -45), (1,0), (2,21), (3,24), (4,15).

   a) What kind of curve do these points form?

   b) Find a formula for the curve.

4. One number in each of these two sequences is misprinted. Determine which one and explain.

   a) 1, 2, 4, 8, 15, 27, 42, 64, 93

   b) 1, 3, 11, 31, 69, 123, 223, 351, 521, 739, 1011

5. A table of values showing the number of terms in a sum and the value of the sum can be developed to find a formula for the value of a sum, given the number of terms in the sum. Then a formula generalizing the pattern of the table can be developed using the method of finite differences. Find the sum of the first 93 odd counting numbers using this method.

6. Developing a table as suggested in (5), find a formula for the sum of the n odd counting numbers that follow the p-th odd counting number.

A Chain Loop Puzzle: An Analysis of the Number of Triangles of Given Perimeters That Can Be Formed With Integral Sides (Integral Inequalities, Modular Arithmetic and Polynomial Formulations)

Preparation of a Demonstration Model

Materials: A manilla file folder and some winged brass paper fasteners.

Cut from the folder about 20 strips of cardboard 4 inches long and one inch wide. Punch a hole near each end of each strip, one-half of an inch from the end, and centered with respect to the long edges. Using the winged fasteners, join ten or more strips end to end to form a chain. Join the ends of the chain to make a chain loop.

Statement of the Puzzle

A chain is made of links of equal length. The ends of the chain are joined to form a loop as in a bicycle chain. The loop can be deformed into a triangle by using as vertices three properly chosen points where links are joined and pulling taut between each pair of these vertices the part of the chain that joins them. How many different triangles can be formed in this way from a chain loop of given length? (This problem appeared as elementary problem E 1825, page 1020, American Mathematical Monthly, November, 1965.)

Procedure

Read the puzzle to the class, and demonstrate the formation of a triangle from the chain loop. Then raise the question, "Under what conditions should we consider two triangles formed in this way to be the same?" The discussion of this question will make clear the need for clarifying the formulation of the problem by defining what is meant by "different triangles." Ask the students to formulate the definition: Two triangles will be considered different if they are not congruent.

Now propose the following question: "How can we state the essence of this problem without referring to links and chains at all?" Develop through discussion this reformulation of the puzzle as a puzzle about numbers:

For a given positive integer n, find the number of noncongruent triangles with integral sides and perimeter n.

Then ask, "What is the lowest possible value of n we may use? After eliciting the fact that $n \geq 3$, ask the students to determine the answer to the puzzle for the specific cases where

$n = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$ and 14.

Ask for the results obtained by the students, and list them on the board. Invite the students to challenge any answers that they think are wrong. The students will identify a triangle with perimeter n by giving the lengths of its three sides. Record this information as an ordered triple. You may be given some ordered triples like (4,2,1). If they are not challenged by a student, ask that the triangle be made from a chain loop of appropriate length. Through such trials and discussion, lead the class to recognize that the numbers must satisfy the condition that the sum of two sides of a triangle is greater than the third side.

If you are given as separate solutions (4,2,1) and (4,1,2), these will have to be challenged. Ask for a convenient way of representing the triangles by ordered triples that will make it easy to avoid duplication. The discussion of this question should end with an agreement that the three members of an ordered triple that represent a triangle will be listed in descending order. Then ask, if (a,b,c) is an ordered triple that represents a triangle whose perimeter is n, where a, b, and c are positive integers, what assertions may we make about the numbers a, b, and c? That is, what conditions must the numbers satisfy?

Allow the students time to formulate the answers in their own way first. If necessary, however, step in with these more specific questions: What is the condition imposed by the fact that the perimeter of the triangle is n? How do we express the condition that we write the members of the ordered triple in descending order? What is the condition imposed by the fact that the sum of two sides of a triangle is greater than the third side? The final outcome of this discussion will be the listing of the three basic conditions that must be satisfied by an ordered triple (a,b,c) that represents a triangle whose perimeter is n:

1. $a + b + c = n$. (Perimeter condition)

2. $a \geq b \geq c$. (Descending order agreement made to eliminate duplication.)

3. $a < b + c$. (The sum of two sides of a triangle is greater than the third side.)

For any integer $n \geq 3$, denote by f(n) the number of non-congruent triangles with integral sides and perimeter n. Use specific examples to be sure that the notation is understood. For example, f(3) = 1, f(4) = 0, f(5) = 1, etc. You will now have recorded on the blackboard the triangles and the values of f(n) for values of n from 3 to 14, as follows:

n = 3.	(1,1,1).	f(n) = 1.
n = 4.	none	f(4) = 0.
n = 5.	(2,2,1).	f(5) = 1.
n = 6.	(2,2,2).	f(6) = 1.
n = 7.	(3,3,1) (3,2,2).	f(7) = 2.
n = 8.	(3,3,2).	f(8) = 1.
n = 9.	(4,4,1) (4,3,2).	f(9) = 2.
n = 10.	(4,4,2) (4,3,3).	f(10) = 2.
n = 11.	(5,5,1) (5,4,2) (5,3,3) (4,4,3).	f(11) = 4.
n = 12.	(5,5,2) (5,4,3) (4,4,4).	f(12) = 3.
n = 13.	(6,6,1) (6,5,2) (6,4,3) (5,5,3) (5,4,4).	f(13) = 5.
n = 14.	(6,6,2) (6,5,3) (6,4,4) (5,5,4).	f(14) = 4.

Our problem is to find a way of computing f(n) for any given value of n. We shall consider the problem solved if we discover a simple, systematic and foolproof procedure for listing easily all the ordered triples that represent triangles with integral sides and perimeter n. Then f(n) can be found by simply counting the triples. Once such a procedure is known, a second, more advanced type of solution can be found in the form of a formula for f(n).

If an ordered triple of positive integers satisfies the three conditions listed above, suggest that it be called an <u>acceptable</u> ordered triple. Ask the class to think of a natural way in which the job of listing all the acceptable ordered triples may be broken down into a sequence of separate steps. If the information is not supplied by the class, suggest these steps: First, list all possible values of a. Then, for each possible value of a, list all the possible values of b.

To determine possible values of a, for a given value of n, ask the class to see what conditions 2 and 3 tell about possible values of a. If necessary, suggest that a be added to both members of the inequality in condition 3. Then we have

$$a < b + c,$$

$$a + a < a + b + c,$$

$$2a < n,$$

$$a < \frac{n}{2}.$$

Condition 2 makes two assertions about a:

$$a \geq b,$$

$$a \geq c.$$

Adding these inequalities, we get $a + a \geq b + c$.

Adding a to both members, we get $a + a + a \geq a + b + c$,

$$3a \geq n,$$

$$a \geq \frac{n}{3}, \text{ or } \frac{n}{3} \leq a.$$

Thus, we know that $\frac{n}{3} \leq a < \frac{n}{2}$.

This information enables us to list all possible values of a for a given value of n.

For example, if n = 15, $\frac{15}{3} \leq a < \frac{15}{2}$.

Then the possible values of a are 5, 6, and 7.

If n = 16, $\frac{16}{3} \leq a < \frac{16}{2}$.

Then the possible values of a are 6 and 7.

If n = 17, $\frac{17}{3} \leq a < \frac{17}{2}$.

Then the possible values of a are 6, 7, and 8.

Now we proceed to the second step of finding a way of listing all the possible values of b for a given value of a, with n fixed. The clue is found in condition 2. Ask the class to uncover it. The clue is that b cannot be greater than a, and c cannot be greater than b. Moreover, once we have chosen a and b, we can calculate c by subtracting a + b from n, in view of condition 1.

For example, consider the case where n = 15, and a is taken to be 7. Then the highest possible value of b is 7. The corresponding value of c is 1. Now take lower and lower values of b. Each time b is decreased by 1, c is increased by 1. As soon as c becomes greater than b, we have an unacceptable triple, and the process stops. Thus, the acceptable triples for n = 15 and a = 7 are : (7,7,1) (7,6,2) (7,5,3) (7,4,4). The complete list of acceptable triples for n = 15 is as follows:

(5,5,5)	(6,6,3)	(7,7,1)
	(6,5,4)	(7,6,2)
		(7,5,3)
		(7,4,4)

Consequently f(15) = 7.

Have the class evaluate f(n) for n = 16, 17, 18, 19, 20, using the procedure just discovered above.

If the class wishes to go on to derive a formula for f(n), tell them that twelve separate cases have to be considered, and each leads to a separate formula. The class may seek the formula for each case, one at a time, as follows:

Classify all possible values of n by the remainder you get when you divide by 12. If the remainder is r, then n = 12 m + r. If this notation is not familiar to the students, develop it by asking the students how to check a long division example. This question should elicit the rule,

dividend = (divisor x quotient) + remainder.

Then, if m is the quotient and r is the remainder when you divide n by 12, n = 12 m + r. Thus there are twelve separate classes of numbers corresponding to the twelve possible values of r. The numbers in these classes have the following forms respectively: 12m, 12m + 1, 12m + 2, 12m + 3, 12m + 4, 12m + 5, 12m + 6, 12m + 7, 12m + 8, 12m + 9, 12m + 10, 12m + 11.

Ask the class to identify the first seven values of n for which n = 12m. Determine the corresponding values of f(n). Prepare a table of values with two columns, one for m, and the other for f(n). Ask the class to discover from the table a formula that expresses f(n) in terms of m. Next ask the class to identify the first seven values of n for which n = 12m + 1. Determine the corresponding values of f(n). Again, prepare a table of values with two columns, one for m, and the other for f(n). Ask the class to discover from the table a formula that expresses f(n) in terms of m. Repeat this procedure with each of the other classes of possible values of n.

The twelve formulas are given below. . After all the formulas have been found, ask the class to estimate mentally the value of f(n) for large values of m. Is there a single formula that gives an approximate value of f(n) for large values of m? (The single formula is $f(n) = 3m^2$.)

The twelve formulas for f(n) are given in the following table:

n	f(n)
$12m$	$3m^2$
$12m + 1$	$3m^2 + 2m$
$12m + 2$	$3m^2 + m$
$12m + 3$	$3m^2 + 3m + 1$
$12m + 4$	$3m^2 + 2m$
$12m + 5$	$3m^2 + 4m + 1$
$12m + 6$	$3m^2 + 3m + 1$
$12m + 7$	$3m^2 + 5m + 2$
$12m + 8$	$3m^2 + 4m + 1$
$12m + 9$	$3(m + 1)^2$
$12m + 10$	$3m^2 + 5m + 2$
$12m + 11$	$3m^2 + 7m + 4$

Over the Edge: A Physical Problem That Introduces Converging and Diverging Series

This topic presents a most interesting physical problem that is an application of the notion of sequences and series that could lead to converging and diverging series. The presentation below illustrates a possible development of the topic with a class and may be helpful although it should be modified and changed to suit the instructor's and student's preferences.

Materials: 8 to 10 uniform slats per student performing the experimentation. These uniform pieces may be rulers, meter sticks, 1 1/4 inch wooden lattice cut to uniform lengths of 18 to 24", etc.

Can you arrange a pile of slats at the edge of a table so that successively higher slats extend beyond the edge of the table until at least one is completely beyond the edge of the table?

Note: Students will set up many arrangements of slats that will collapse on them. Encourage them to experiment with different arrangements until some students are successful. Their success will spur others to greater effort.

Circulate among the students. When a student is successful ask him how he did it. When several students are successful they may be encouraged to make a record of their solution using pencil and paper.

> Write down how you would explain your solution to a friend in Kenya who might want to know how to do this.

Encourage students to make correct interpretations of their set up by asking questions that would show up inaccurracies or weaknesses in their diagrams or explanations.

> Using as many slats as you wish, how far out from the edge can a slat be supported?

Also - - -

> Apparently it makes a difference how many slats you have. What is the maximum distance to the end of a slat beyond the edge as the number of slats increases?

Notice that: (a) Students will readily find that with one slat the maximum distance end of slat to table, D, is 1/2 the length of a slat, L. With two slats D may be 3/4 L and that with six slats D may be greater than L.

(b) An example of a student's solution, in point shown below, had the sequence of gaps 1/2, 1/4, 1/5, 1/8, 1/10, 1/13.

Some sequence like this should be presented to the class for discussion. Ask if they can pick out any fractions that seem out of place and, if so, what they can substitute in their place. Students might suggest that 1/5 and 1/13 should be replaced by 1/6 and 1/12 respectively. They might further suggest that the next term in the sequence should be 1/14. This means that an additional slat could be added with 1/14 L gap. When another slat is added with a gap 1/14 L the arrangement extends further and will not fall. At this point, students in the class may divide in opinion as to whether this could go on forever or that sooner or later the slats will fall.

When students have all had an opportunity to experiment hanging over slats and have formed an opinion regarding the extent to which slats can be extended suggest that they hang over slats with gaps making the sequences 1/2, 1/4, 1/8, 1/16, 1/32 + ...

Students will determine by experimenting that no matter how many slats they use they will not be able to support a slat beyond the edge of the table.

We may investigate the difference in these two sequences by looking at the patterns of successive sums of terms (partial sums). Let us call these sums s_1, s_2, s_3, etc. for sums of the first 1, 2, 3, ... terms. What difference in patterns do you find for the successive sums using these two sequences? Does this help explain why one sequence allows "hanging over" and the other does not?

Note: The definition for adding rational numbers $\frac{a}{b} + \frac{1}{4} = \frac{4 + 2}{8}$

and

and $\frac{7}{8} + \frac{1}{10} = \frac{70 + 8}{80}$

Exercises

1. Write an explanation for a friend in California who is familiar with the "hanging over" problem using uniform slats that will convince your friend that the series of gaps 1/2, 1/4, 1/6, 1/8, 1/10, ..., etc. allows a slat to hang over, whereas the series of gaps 1/2, 1/4, 1/8, 1/16, ... does not.
2. Explain the possibilities of "hanging over" slats whose lengths are successively 1/2, 1/4, 1/6, 1/8, ..., or 1/2, 1/4, 1/8, 1/16, ... or some other sequence of lengths.

Research Notes on Over the Edge

1. Tim Barclay posed the following problem:

Arrange 8 rulers on a desk, and on top of the other, so that the uppermost ruler was hanging completely off the edge of the desk.

Few of us arranged them in the same way that he did. When asked to tell why he did things the way he did, he came up with mathematical argument.

Suppose that we had one ruler to work with, how far out could it stick:

Clearly, half way. (Let us assume from now on that we shall be using rulers of the same length and weight which is evenly distributed along the lengths; for the sake of discussion we shall assume they are all 1 unit long.)

Suppose you had two rulers; now how far out could you project the furthest one, stacking them on the desk?

You may be able to see other ways of maximizing the extended distance, but this diagram shows the best you can do with two unit length rulers. Looking at the bottoms of these rulers, we see two "gaps", one under the first ruler, until the second begins; the other gap underneath the second ruler ends with the desk top.

Now you have three rulers, again all one unit long, and you have to stack them to extend one ruler as far out as possible. What do you do? (Try it and see, before reading on!) At this point, many people guess on the basis of the two previous "gap" distances, 1/2, and 1/4, that the new gap will be 1/8. And they arrange the three rulers:

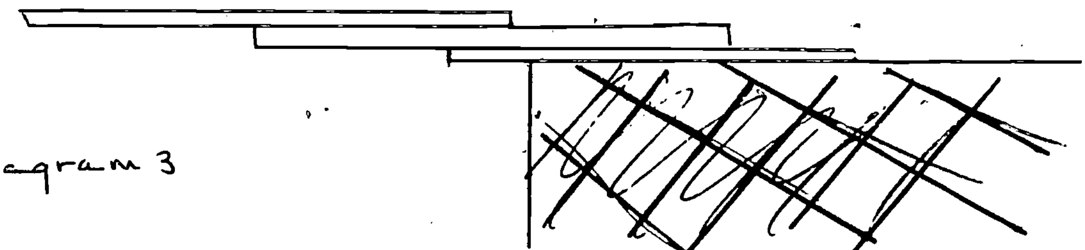


diagram 3

If you try this, you realize that it is possible to slide the bottom ruler further out than 1/8, in fact out to 1/6 for the third gap. In inches, if the rulers are all one foot, that would be the difference between 1 1/2 inches and 2 inches easily seen. (Of course, all these distances are somewhat off, because these gaps are never quite realized due to the necessity of making sure that the rulers don't fall over.)

Now the sequence of gaps is 1/2, 1/4, 1/6. Any guesses for the rest of this sequence? Try them out and see, before reading further.

2. There are some principles of physics that help one predict how these rulers can best be arranged to maximize the extension of the top ruler. What we seek, mathematically, is to slide out the set of rulers in such a way that if any one ruler were to be extended any further it would tip over the edge of the ruler right beneath it and fall to the ground. That means that the weight of all the rulers above a given ruler is just balanced at the edge of that given ruler. In the case of the top ruler, if it is extended 1/2 unit out above the next ruler, the balance point will be the center of mass of the first and second ruler, since the first rests on it. Looking at diagram 2, if

the top ruler is out 1/2 over the second ruler, then the weight is symmetrically distributed over a point 3/4 from either end, or 1/4 from the end of the first gap. The third ruler will have to end at that point, producing a gap of 1/4 between it and the end of the second ruler, to maximize the extension of the top ruler from the desk. (In diagram 2, that ruler is the top of the desk.) What about the third ruler? Can you calculate the center of mass of the three rulers (with gaps 1/2, 1/4) to see where the fourth ruler ought to end? Try it and see. (If the rulers have their weight uniformly distributed along the length, then you can assume that all the weight is concentrated at the center of the ruler , or 1/2 unit from either end.)

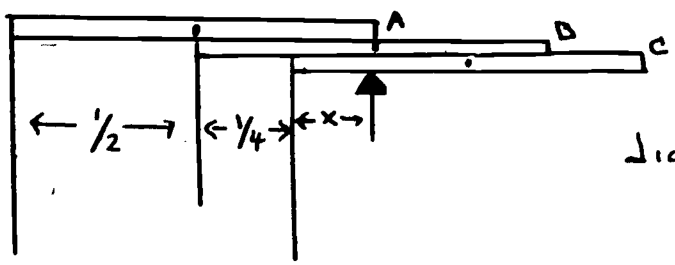


diagram 4:

The solid triangle under the bottom ruler represents a fulcrum, upon which the block of rulers are exactly balanced. How far in from the end of ruler three should that fulcrum be? We call if x, and proceed to balance the moments of the various rulers about that fulcrum, each moment being the product of the weight of the ruler (say they are all one ounce) and the horizontal distance from the vertical line through the fulcrum. Some moments counteract others, so that we shall have to call those tending the object to rotate clockwise to be + moments, and those tending the other rotation to be - moments. We get for the sum of all the moments (which should be zero if the object is balanced):

$$-1\left(\frac{1}{4} + x\right) + 1\left(\frac{1}{4} - x\right) + 1\left(\frac{1}{2} - x\right) = 0$$

Solving this equation, we get: $x = \frac{1}{6}$.

3. At this point, someone will suggest that the sequence is: 1/2, 1/4, 1/6, 1/8, 1/10, 1/12, 1/14, ... and that the nth term will be 1/2n ... Can we prove this? It might still be useful to check out the 1/8 for those who consider this too hazardous a guess.

Let us assume that we have stacked "n" rulers to maximize at each stage the extension and that we have gotten the gaps as predicted above. Let us see how we might best add one more ruler, the (n + 1)st ruler, at the bottom of the stack, and what the gap for that last ruler ought to be to maximize the new extension.

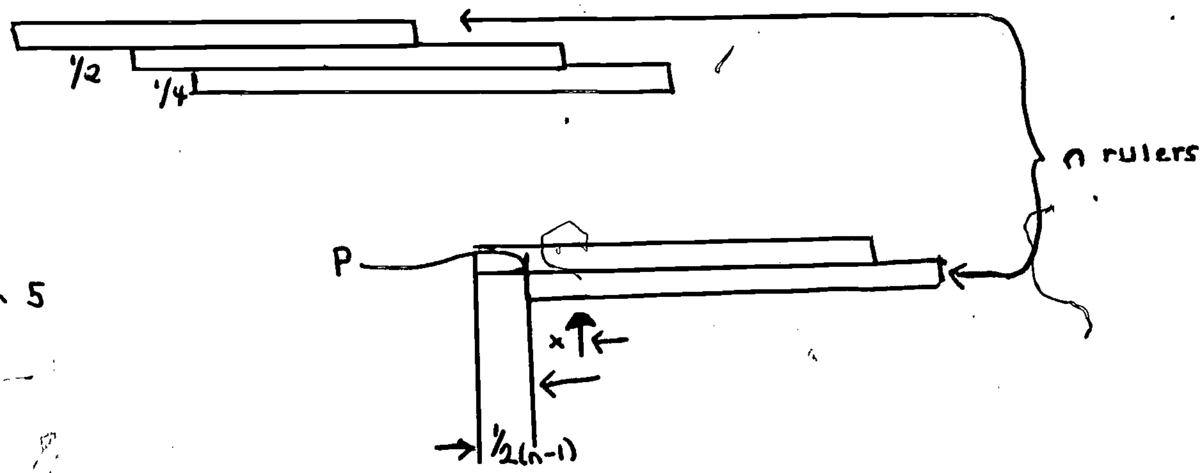


diagram 5

According to the predictions, the gap between the nth ruler and the next ruler should be 1/2n. This means that the top n rulers should balance at a point 1/2n to the right of the nth ruler's end. If a fulcrum were placed there, it should balance. Do you think it will? Let us place the fulcrum a distance x inside the nth ruler (see diagram 5) and calculate the distance x by balancing the positive and negative moments. (Remember, the previous (n-1) rulers must have just balanced at the end of the nth ruler, or a point P in diagram 5 — that is why that ruler was placed there.)

$$-(n-1)x + 1\left(\frac{1}{2} - x\right) = 0 \quad \text{or} \quad x = \frac{1}{2n}$$

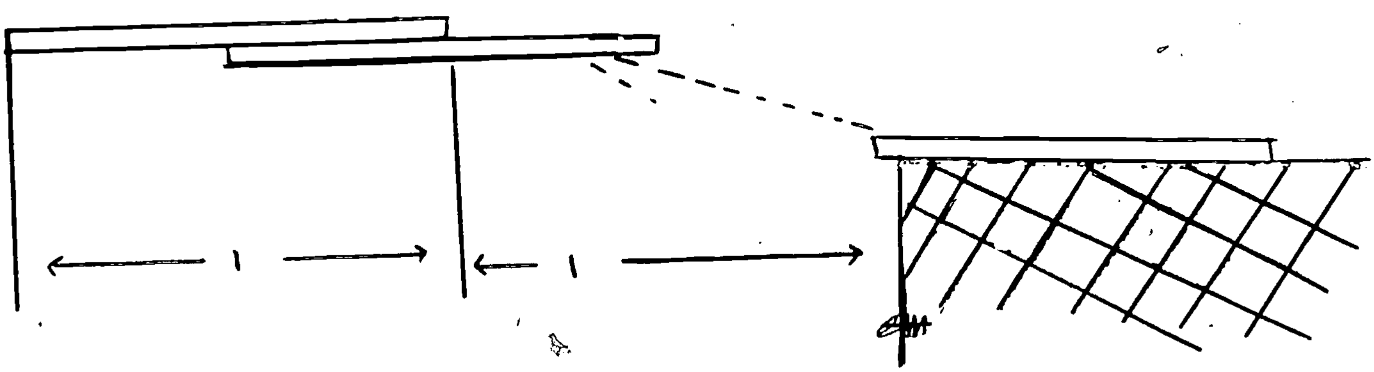
This is exactly what we predicted would be the gap under the nth ruler.

Those who recognize mathematical induction will see that what we have done in effect is to prove that our predictions for the sequence of gaps is correct. Those who are not familiar with induction proofs can see that what we have shown is Q: "If our prediction — our theory — is correct for the case of the (n-1)st gap, then it gives correct predictions for the nth gap." But n could be any number (bigger than 1). If n=4, what this theorem means is that "if our predictions are correct for the 3rd gap, then our predictions are correct for the 4th gap." But we based our predictions on what we knew about the first three gaps; so we know our 3rd gap prediction is correct. The statement in quotes then tells us that our prediction ($\frac{1}{2\cdot 4} = \frac{1}{8}$) is correct for the 4th gap. But going further and letting n=5, we can conclude; "If our prediction is correct for the 4th gap, then our prediction is correct for the 5th gap." Again we have just seen that our prediction, 1/8, was correct for the 4th gap, so this quoted statement says that our prediction ($\frac{1}{2\cdot 5} = \frac{1}{10}$) is correct for the 5th gap. This could go on and on. And it does. It is clear that proving statement Q which involves the indefinite letter n, means proving an infinite chain of statements that link together to prove that every one of the gaps that we predicted at the beginning of section 3 was correct. Proving statement Q is kind of like proving that "you can take a step up a ladder." You can deduce from this that you could rise to any step on the latter by repeated use of the statement in quotes. If you understand this argument, then you understand the essence of mathematical induction!

4. How far out could you go, if you had as many rulers as you wanted? Any guesses, before you read further?

If you have n rulers, the gaps between the rulers are: 1/2, 1/4, 1/6, 1/8, 1/10, ..., 1/2(n+1). But then you could slide the whole bunch of n rulers out past the desk, using the desk corner as the last fulcrum, to get another gap of 1/2n. So the question really boils down to:, what is the sum of 1/2 + 1/4 + 1/6 1/8 + 1/10 + ... + 1/2n? If you factor out 1/2 from each term, the sum then equals 1/2 of the sum of the series 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + ... + 1/n. What is that?

Those who have worked with series before may recognize this as a harmonic series. The first term is twice 1/2; the second term equals 1/2; the third and fourth terms are respectively greater than and equal to 1/4, and so their sum is greater than 1/2; the fifth through eighth terms are each either greater than or equal to 1/8 so they add up to greater than 1/2; the ninth through sixteenth terms for the same reasons add up to greater than 1/2, and so, always groups of terms adding up to more than 1/2. This sum can thus be made greater than any number you choose; this is a property of a certain category of divergent series. What does this say about what kind of extension of the top ruler is possible, using as many rulers as you want? Does it help to calculate how many rulers you should order so that you can extend the forward edge of the top ruler 2 units out from the desk top?



5. We still have before us the general question: how should we arrange a given number of rulers to produce the maximum extension? About all we have shown is this: if you build a stack of rulers, extending over the desk edge,

by the special process of stacking up all the rulers you have and first extending over, as far as you can, the top ruler, then extending the top three rulers until the balance point is reached, etc., ... you will get gaps between the rulers equal to 1/2, 1/4, 1/6, 1/8, ... 1/2n. What it doesn't tell us about is whether we could get a further total extension of many rulers, if we didn't proceed in exactly that way. Who knows, it might still be true that if we were more conservative at the beginning, for example, not extending the top ruler so far, or not extending the top two rulers a full 1/4, etc., we might be able with the remaining rulers to produce an even bigger gap. Who can say? For me, this is an unsolved problem. I have looked at what happens in some simple cases, in which I do not push the top ruler out as far as 1/2, but instead out a distance of $(\frac{1}{2} - e)$, where "e" stands for some small positive quantity, as yet undefined. Then with two rulers, one can see how big the next gap is.

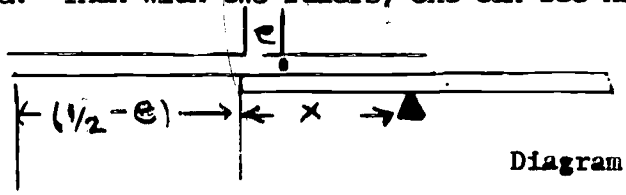


Diagram 7

Balancing the moments as before, $-1(x-e) + 1(\frac{1}{2} - x) = 0$

$$\text{or, } x = \frac{1}{4} + \frac{e}{2}$$

Now, it is true that this second gap has increased by e/2 over the usual case, but the total extension now is $(\frac{1}{2} - e) + (\frac{1}{4} + \frac{e}{2}) = \frac{3}{4} - \frac{e}{2}$, or $\frac{e}{2}$ less than before. Thus, it seems that no matter what the reduction from 1/2 is for the first gap, the second gap never gains enough to make up for that reduction. It might be useful to set up a big stack of rulers arranged according to the usual pattern; then move in any one of the rulers by some distance and see if the others can be extended so that:

1) at none of the edges do the rulers resting above that edge tend to fall over the edge, and

2) the total extension is greater than what the theory predicts for that many rulers.

6. Would it help to beat the "theory" if we allowed stacks with some rulers projecting above the ones below, as usual, and some rulers pulled back from those below?

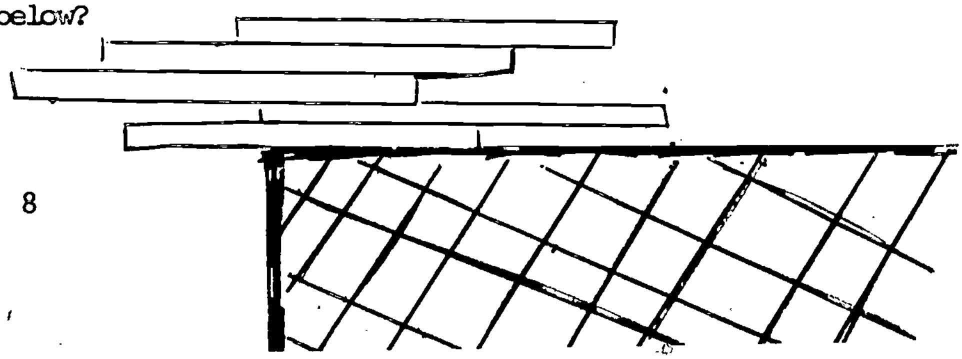


Diagram 8

Would it help to pull back the upper rulers to the extent that they tip backwards, like so?

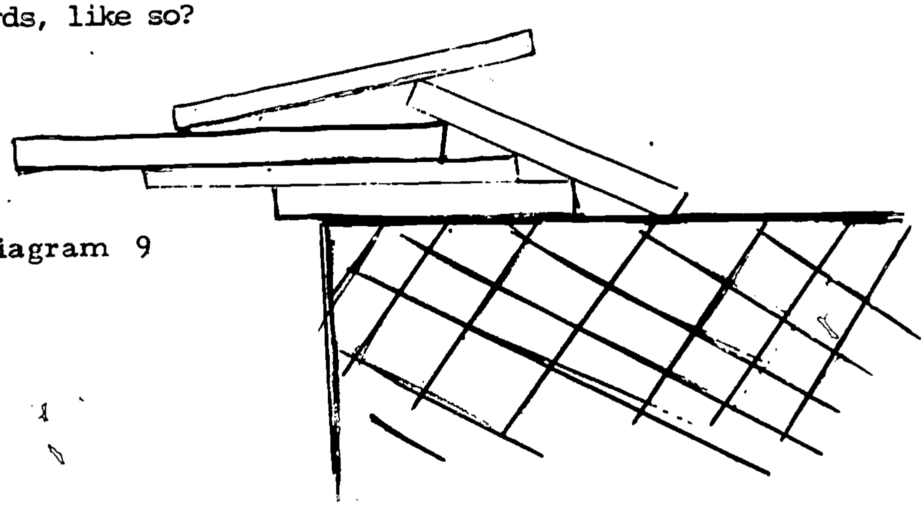


Diagram 9

7. **Another question suggested by this is:**

Given boards of lengths 1, 2, 3, and 4 units, what would be the best way to arrange them so that you get the maximum extension over the desk's edge? It is worthwhile to make boards of these lengths and to try to arrange them yourself to maximize the extension, it is not at all obvious what the best way is.

There are two methods which are suggested immediately, which produce unexpected results. One way is to stack the boards so that the largest one is on the bottom, and the smallest on the top. The other way is to stack them in the reverse order, that is, the largest on the top and the smallest on the bottom. (It looks weird when you do it). Which do you think will give the greatest extension?

It is possible, by the usual method, to calculate the gaps for the four boards, for each of these arrangements, maximizing the extensions.

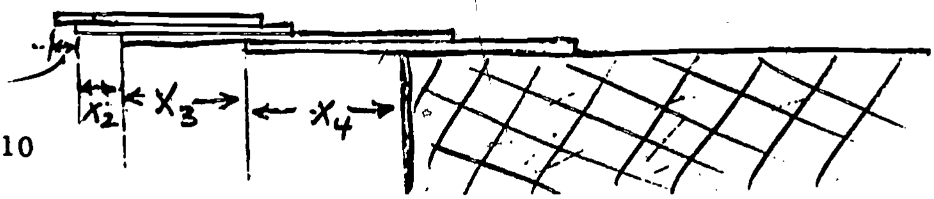
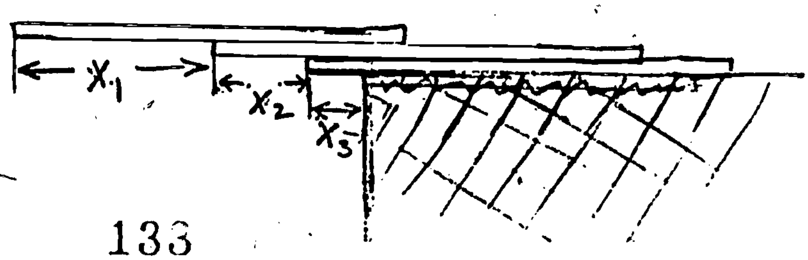


Diagram 10

x_1 in the diagram 10 is clearly 1/2 unit. To calculate x_2, we concentrate the weight of the length-one board at its center, 1/2 from either end, and the weight of the length-two board at its center, 1 unit from either end. The moment about the leading edge of the third board is: $-1(x_2) + 2(1 - x_2) = 0$, or $x_2 = 2/3$. To get x_3, a similar calculation for the moments about the leading edge of the fourth board: $-1(x_3 + 2/3) - 2(x_3 + 2/3 - 1) + 3(3/2 - x_3) = 0$, or $x_3 = 3/4$, and $x_4 = 4/5$. Are there guesses for what happens if we continue with boards of increasing dimensions? Will this sequence get bigger and bigger? Will it get bigger without bound? Or is there some number, past which it doesn't get? What is that number?

Sticking to the 4 board problem, what if the biggest boards were on top? Is it conceivable that the total extension is greater than $1/2 + 2/3 + 3/4 + 4/5 = 163/60$?

Diagram 11



Of course the x = 2, since the top board is 4 units long. To get the x_2 we sum the moments about the leading edge of the third board:

$$-4(x_2) + 3(3/2 - x_2) = 0 \text{ or } x_2 = 9/14$$

To get x_3:

$$-4(9/14 + x_3) + 3(3/2 - 9/14 - x_3) + 2(1 - x_3) = 0, \text{ or } x_3 = 2/9;$$

and to get x_4:

$$-4(9/14 + 2/9 + x_4) + 3(3/2 - 9/14 - 2/9 - x_4) + 2(1 - 2/9 - x_4) + 1(1/2 - x_4) = 0, \text{ or } x_4 = 1/20.$$

The total extension is: 2 + 9/14 + 2/9 + 1/20 = 3673/1260 = 2.91 (approx) whereas 163/60 = 2.72 (approx). Thus, as absurd as it looks, the second arrangement extends further. Do you think this will be true for only three of those boards: lengths 1, 2, and 3? What about five boards, lengths 1, 2, 3, 4 and 5? What about boards of lengths 10, 11, 12, 13?

This raises the general question of boards, of uniform density (weight per unit length), where they vary in total length.

What about boards of the same length but of different weights? Some heavier, some lighter? What about boards where all these characteristics are varying? What is the best scheme for extending them over the desk edge? Maybe some experimentation will suggest a general solution to this problem.

Switches and Batteries

The purpose of this unit is to involve the student in the discovery of a mathematical system suitable for describing a particular physical situation. The student also encounters a part of our technological environment. That the technology is in the form of a switch is not important. What is important is the idea that mechanical and electrical things are not too complicated to understand. We want to get the student to think, "I am smarter than that gadget and I can figure out how it works."

Materials required for a class of 20:

1. 50 Fahnestock clips #33-7102 — $ .45
2. 25 D cell batteries #6256 — $3.00
3. 1 roll vinyl plastic electrical tape #99-H-8015 — $ .54
4. 1 - 6 1/4 inch long nose pliers #13-H-5578 — $1.19

   Lafayette Industrial Electronics
   1400 Worcester Street (Route 9)
   Natick, Mass. 01760

5. 1 copy Lattices to Logic by Roy Dubisch

   Blaisdell Publishing Co.
   135 West 50th Street
   New York, N. Y. 10020

6. 1 copy each Batteries and Bulbs, Books 1, 2, 3, 4 — $6.27

    Science Service Desk,

    McGraw Hill Book Co.

    Webster Division

    Manchester Road

    Manchester, Missouri

7. 75 single pole single throw switches (slide switches): H.H. Smith type #515 — $9.50/100

8. 25 bulbs, G. E. #46 — $3.26

9. 25 miniature screw base sockets: H.H. Smith type #1934 — $11.00/100

    Terminal Hudson Electronics

    236 West 17th Street

    New York, N. Y. 10011

10. 1 roll 100 feet bare copper wire #20, 22, or 24

    (Alpha Wire Co. #297 is suitable and may be obtained from:

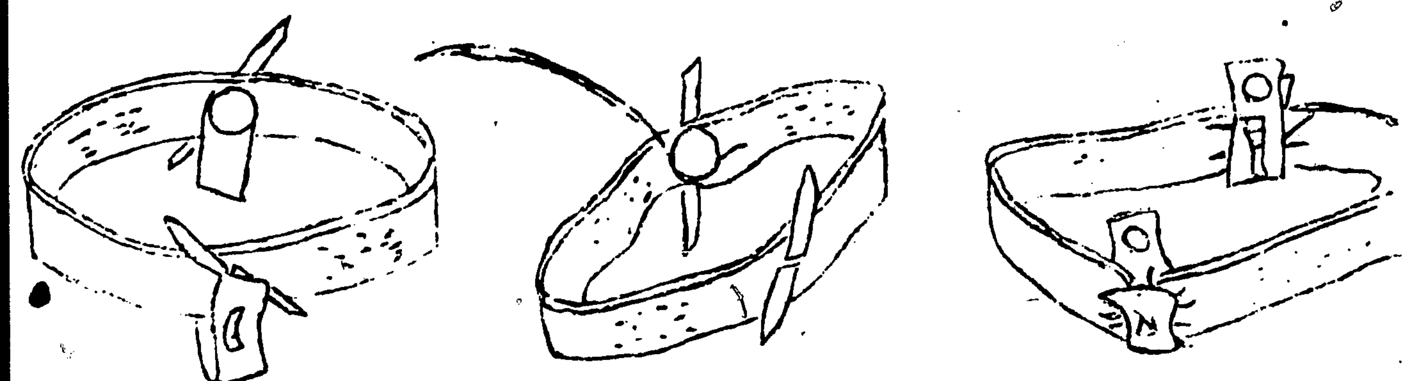
    DeMambro Electronics

    1095 Commonwealth Avenue

    Brighton, Mass.)

At the beginning of a class pass out a battery, bulb and socket, and some 5 or 6 inch pieces of bare wire but not any switches. If you wish, you can ask, "Can you light the bulb?" However, no one will be listening to you as they will all be trying to light their bulbs. (This is a good unit when you have laryngitis.) The number of students who have difficulty lighting a bulb is often large. Some may even be afraid to pick up a battery, so be prepared for anything. At this time the need for a continuous electrical path, i. e., a "closed circuit," may come up in the discussion. If so, good; if not, it will come up later when switches are part of the circuit.

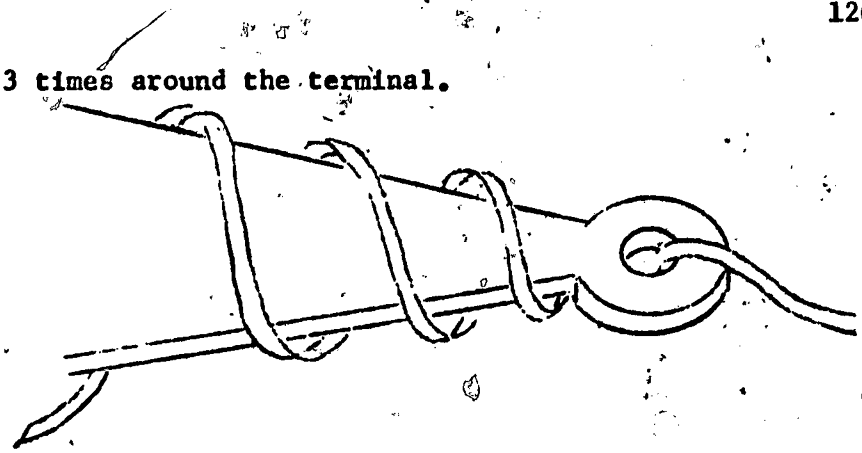
After a student has found how to light a bulb, pass out a switch and the makings of a battery holder.



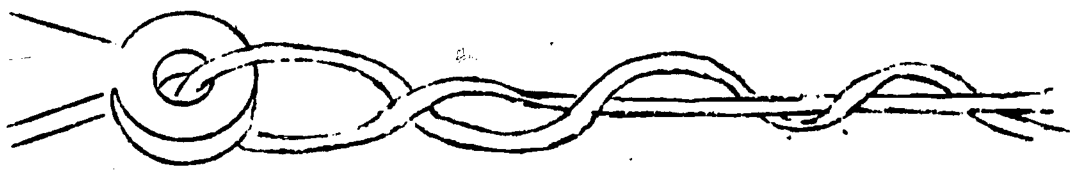
Three Ways to Make a Battery Holder

Using a Rubber Band, Paper Fasteners and/or Fahnestock Clips

Having a few battery assemblies ready to show the students is the best way to demonstrate their construction at this time. You may also want to give a general demonstration on how wire can best be wrapped around the various terminals. It really is very easy. One simply pokes one end of the wire through the connector hole and then, holding the long end, wraps the wire 2 or 3 times around the terminal.



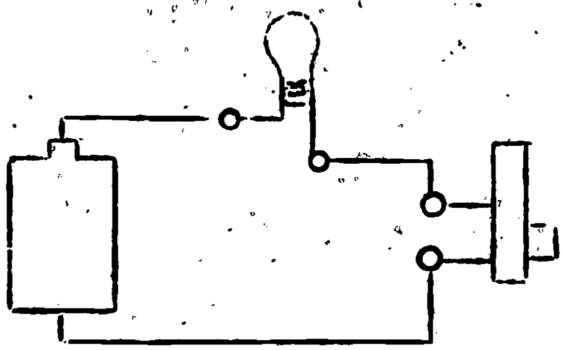
A connection like the following is not very satisfactory as both open and short circuits easily occur.



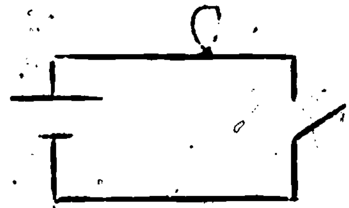
Moreover, with several switches connected as above, everything starts flopping around on the desk. Another possibility, and perhaps a more desirable one, is to give individual wrapping instructions to students needing help as you move about the room. It is also helpful to have pieces of black electrical tape available. These can be used to tape floppy circuits to the desk in dire circumstances. They can also serve as an alternative to the Fahnestock clips to hold wires to the battery.

Before the students get involved in putting their switches in a circuit you might ask if they have any ideas as to which is the "on" position and which is the "off". Many instinctively know that  is "on" and  is "off". However, most students tend to make statements in terms of right or left or some other irrelevant parameter. Few will look carefully at the switches, see how contacts are made, and give a clear description of the switch operation. A few students should be encouraged to take a switch apart, using pliers, so that everyone can see how it works.

When students hook their switch up with a battery and bulb, two circuits come up with about equal probability.

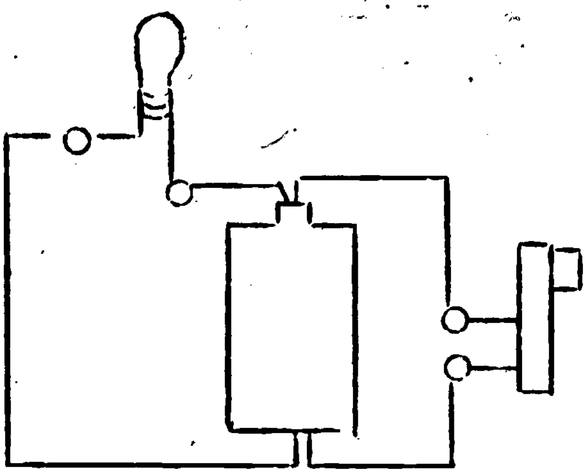


Pictorial Diagram

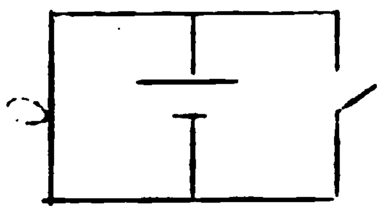


Equivalent Schematic Diagram

CIRCUIT I



Pictorial Diagram



Equivalent Schematic Diagram

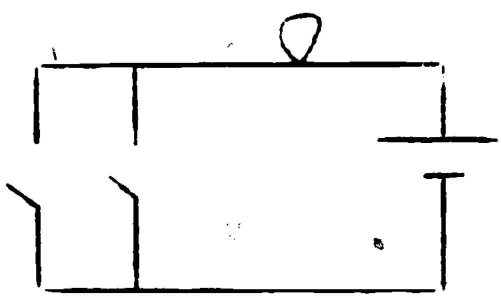
CIRCUIT 2

In Circuit 1 the bulb lights when the switch is  , i. e., "on".
In Circuit 2 the bulb lights when the switch is  , i. e. "off !
A good argument should now ensue as to which is really the "on" position. While the argument proceeds, everyone should keep the switches in the position that lights the bulb. Otherwise, there will be a number of dead batteries.

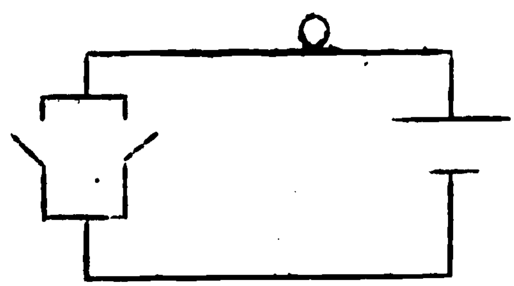
The problem of why the bulb lights in both these circuits implies the question, Why does the electricity flow in the wire rather than through the bulb? At this point, if the class is interested, it can pursue a study of electricity using the ideas in the Elementary Science Study unit, Batteries and Bulbs as a basis.

Going on to the Boolean Algebra aspect of this unit, a second switch can be passed out once the students agree that Circuit 1 represents all valid switching circuits composed of a single switch, battery and bulb. With the second switch, the question is again to look for various types of switching circuits.

When several circuits are drawn on the blackboard, discuss which are really the same (with respect to the flow of electricity) and which are really different. For example

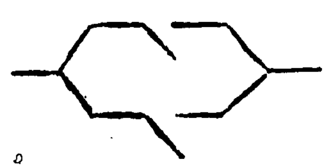


and

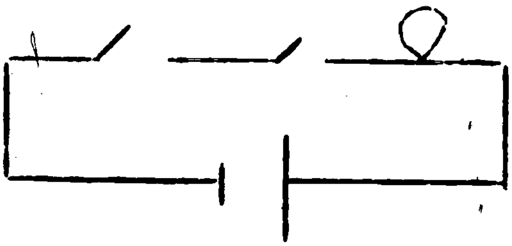


are really the same electrically but slightly different physically. There are only two possible circuits with two switches that are electrically different:

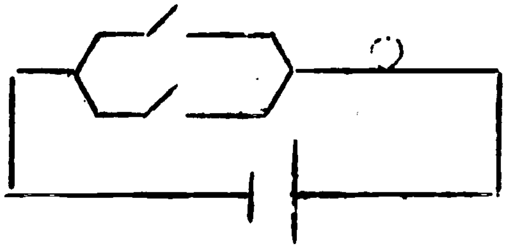
and



These switching diagrams are simplifications of diagrams of the whole circuit,



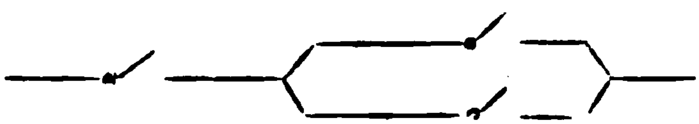
and



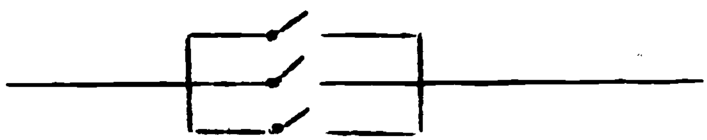
but the single bulb and battery remain constant throughout, thus making the switching arrangement the only varying component to diagram.

Also consider what circuits can be made with three switches that light the bulb. Diagrams and discussion. (There are three possible circuits.)

all on



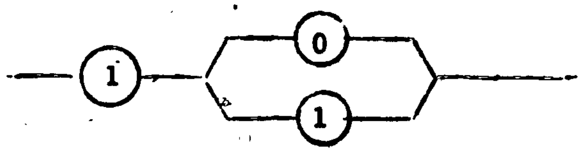
2 or 3 on



1, 2, or 3 on

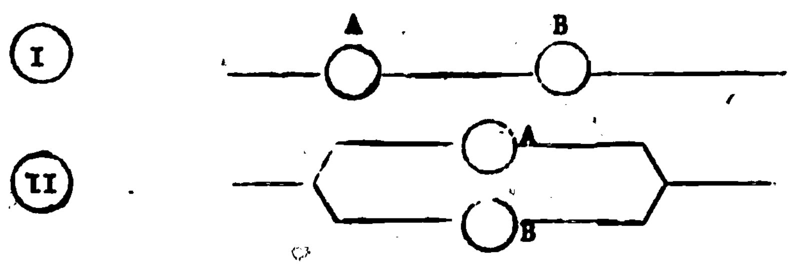
Encourage students to try for simple representations, even to a representation that does not involve drawing a switch as above.

After maybe one more switch question (light the bulb with four switches) ask, <u>Since this is a class in mathematics, can you represent the state of a switch (i. e., "on" or "off") with a number?</u> I think this question is crucial to get to a "0" and "1" representation. "0" and "1" are the two "simplest" numbers to use. "0" is a logical choice for "off" as nothing happens when the switch is off. It is really arbitrary whether off is "0" or "1", but it is much more convenient to use "0" when we go to the algebra. By this time a circuit might be represented as



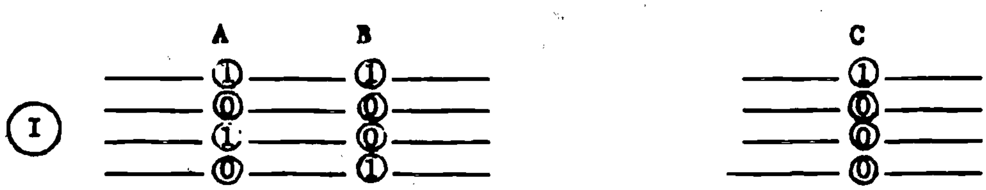
From now on most of the discussion can be abstract; however, the switches are always available for the doubters. At any point in what follows there is a direct correspondence between an abstraction and the switches. This is a particular beauty of two-valued Boolean Algebra.

Now consider the two switch circuits where A and B simply refer to a given switch.

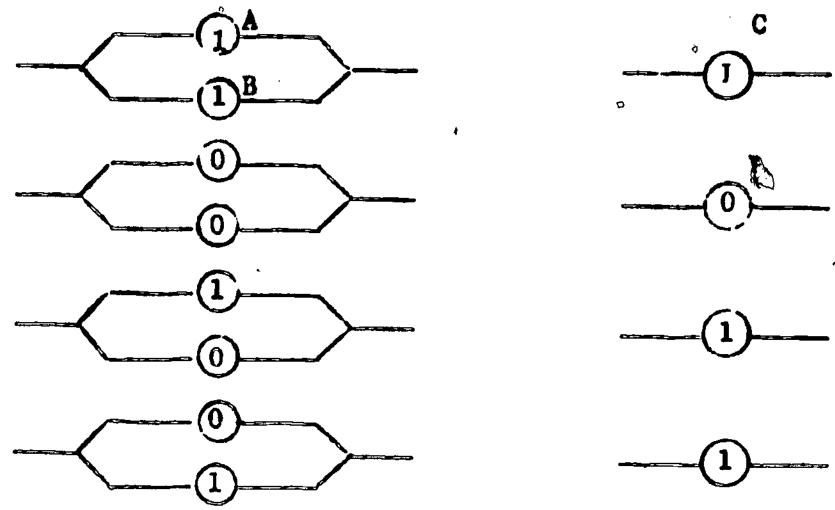


The fact that (I) is a series circuit and (II) is a parallel circuit might be mentioned and used. The words are not important but their use is convenient.

For these two-switch circuits, all possible on-off arrangements can be listed, and the further question posed, "Can you find a way to replace each of the 8 sets of two switches with a single switch?" (The switches labeled C below represent the single switch substitution.)



II



One can write down the tables for the above switches:

Series Circuit

A	B	C
1	1	1
0	0	0
1	0	0
0	1	0

Parallel Circuit

A	B	C
1	1	1
0	0	0
1	0	1
0	1	1

Ask if anyone sees a similarity between the switches and mathematics. Maybe write down all the tables for addition, subtraction, multiplication, and division with the numbers 0 and 1:

a + b = c		
1	1	2
0	0	0
1	0	1
0	1	1

a - b = c		
1	1	0
0	0	0
1	0	1
0	1	-1

a x b = c		
1	1	1
0	0	0
1	0	0
0	1	0

a ÷ b = c		
1	1	1
0	0	?
1	0	?
0	1	0

Now what similarities are there?

Let's see what happens if we think of a series circuit as being similar to multiplication and a parallel circuit as being equivalent to addition but with a funny rule for adding 1 and 1. You might want to use

$\oplus$ and $\otimes$ as the "addition" and "multiplication" symbols in this algebra. We have made a correspondence between switching circuits and the following tables:

A	$\oplus$	B	=	C
1	$\oplus$	1	=	1
0	$\oplus$	0	=	0
1	$\oplus$	0	=	1
0	$\oplus$	1	=	1

A	$\otimes$	B	=	C
1	$\otimes$	1	=	1
0	$\otimes$	0	=	0
1	$\otimes$	0	=	0
0	$\otimes$	1	=	0

From now on the letters A, B, C, etc. can be used as a variable (with only two values). It also represents a switch which can be in either one of two states.

While it shouldn't be discussed in class at this time it should be noted by the teacher that $\oplus$ is equivalent to "or" and $\otimes$ is equivalent to "and" in the context of such logical statements as:

If X is true or Y is true then Z is true.

If X is true and Y is true then Z is true.

This equivalence and a discussion of logic can serve later as a separate unit. One more equivalence not to be mentioned in class: $\oplus$ and $\otimes$ correspond to union and intersection respectively in the language of sets.

Now with this "multiplication" and "addition" table available let the students discover some general properties of the system:

(1) $A \oplus A = A$

(2) $A \oplus 1 = 1$

(3) $A \oplus 0 = A$

(4) $A \oplus B = B \oplus A$

Remember A is a variable or a switch which can be in one of two possible positions. I don't think it is worthwhile to call (4) the "commutative law for addition", but it might be called "John's Law", after a discoverer, to give it and John some importance. In the above and what follows the similarities between various properties of this system and the axioms of ordinary algebra and arithmetic should be discussed. It would be wise to refer occasionally to switches and what physically $A \oplus B$ and $B \oplus A$, for example, mean.

More generalities:

(5) $A \otimes 0 = 0$

(6) $A \otimes 1 = A$

(7) $A \otimes A = A$

(8) $A \otimes B = B \otimes A$

For three elements:

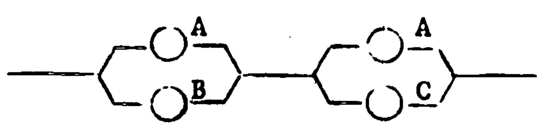
(9) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

(10) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

(11) $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$

What do the above mean in terms of switches? Is everything still the same as in ordinary algebra and arithmetic? After all we have a pretty funny addition table that we started from.

There are circuits in which we would want two switches to always be in the same position. For example, the motor and amplifier of a tape recorder should be on or off together. In the following circuit,



the two upper switches are either both open or both closed at the same time, hence we can designate them both as A. Now a tricky question:

What is $(A \oplus B) \otimes (A \oplus C)$? (This represents the above circuit.)

$$(A \oplus B) \otimes (A \oplus C) = \left[(A \oplus B) \otimes A\right] \oplus \left[(A \oplus B) \otimes C\right]$$

$$= \left[(A \otimes A) \oplus (B \otimes A)\right] \oplus \left[(A \otimes C) \oplus (B \otimes C)\right]$$

but $(A \otimes A) = A$

$$= A \oplus (A \otimes B) \oplus (A \otimes C) \oplus (B \otimes C)$$

$$= A \otimes (1 \oplus B \oplus C) \oplus (B \otimes C)$$

and since $(1 \oplus B) = 1$, $(1 \oplus C) = 1$, and $(A \otimes 1) = A$

$$= A \oplus (B \otimes C)$$

Compare the word statements about switches associated with the initial and final representations above and see that these both make sense.

There will also be occasions when you wish one switch to be off when another is on. For example, if a loudspeaker is used in conjunction with a tape recorder and a radio (and operates continuously) you would want the radio off when the tape recorder is on and vice versa. We can represent such a situation by calling one switch A and the other $\overline{A}$. $\overline{A}$ means that if A is 1, $\overline{A}$ is 0 and if A is 0, $\overline{A}$ is 1. It is equivalent to "not A".

What is $A \oplus \overline{A}$? (1)

$A \otimes \overline{A}$? (0)

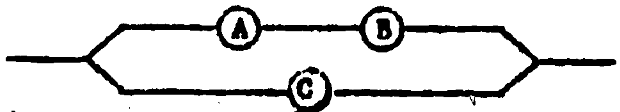
$\overline{\overline{A}}$? (A)

Some problems:

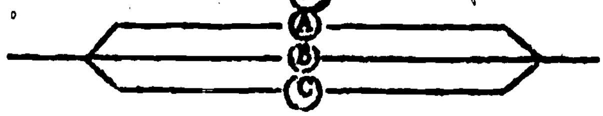
Express the following arrangements algebraically:

(1)

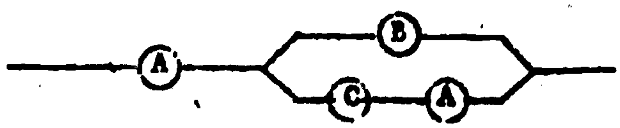
(2)



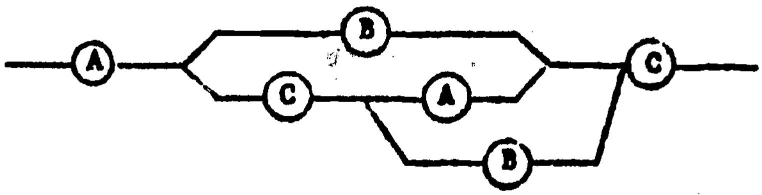
(3)



(4)



(5)



(6) Draw a wiring diagram to illustrate:

(a) $A \oplus B$

(b) $(A \oplus B) \otimes B$

(c) $(A \otimes B \otimes C) \oplus B$

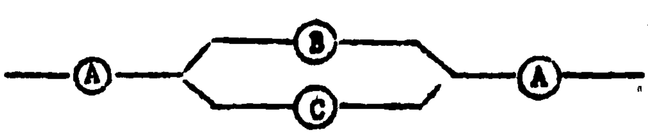
(d) $(A \otimes B) \otimes (C \oplus B)$

(e) $(A \otimes B) \oplus B \oplus C$

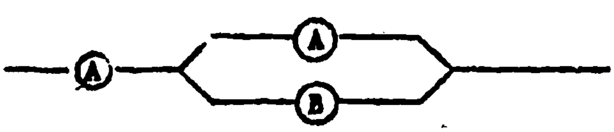
More problems:

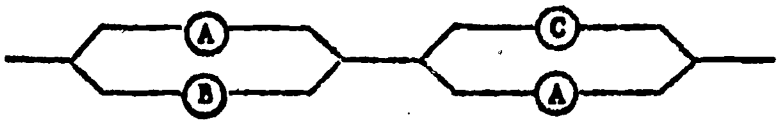
Design a new circuit with fewer switches that will do the same as the following:

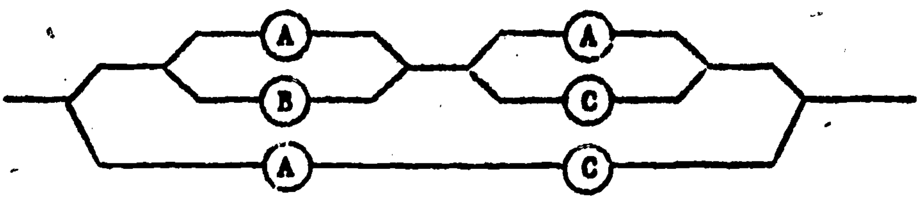
(7)

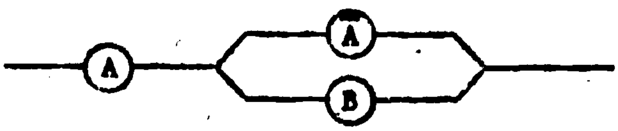


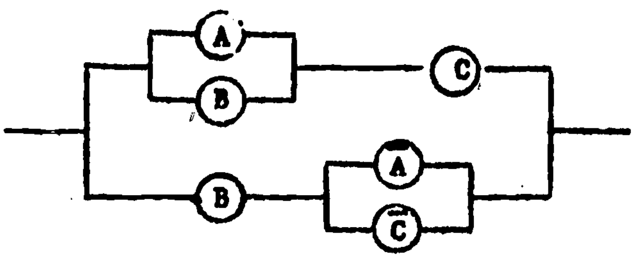
(8)



(9) 

(10) 

(11) 

(12) 

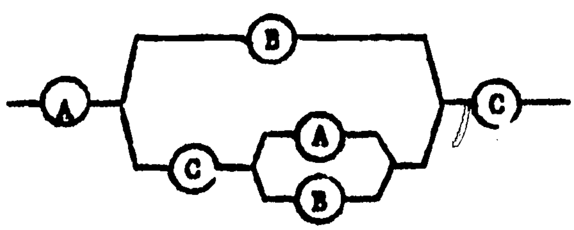
Solutions:

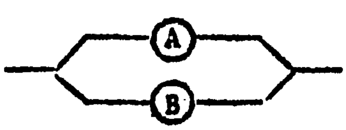
(1) A ⊗ B

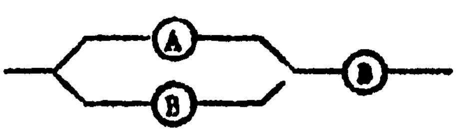
(2) (A ⊗ B) ⊕ C

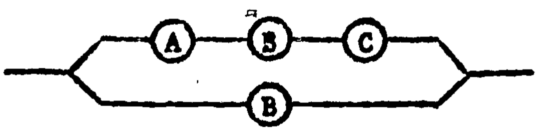
(3) A ⊕ B ⊕ C

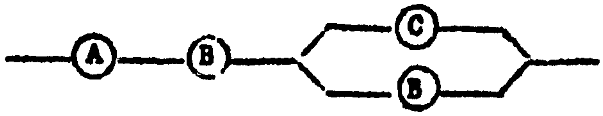
(4) A ⊗ [B ⊕ (C ⊗ A)]

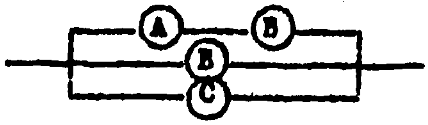
(5) A ⊗ [B ⊕ C ⊗ (A ⊕ B)] ⊗ C 

(6) a. 

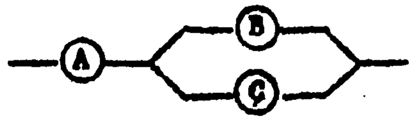
b. 

c. 

d. 

e. 

(7) $A \otimes (B \oplus C) \otimes A = A \otimes (B \oplus C)$

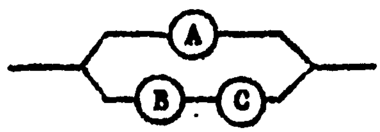


(8) $A \otimes (A \oplus B) = (A \otimes A) \oplus (A \otimes B) = A \otimes (1 \oplus B) = A$

(9) $(A \oplus B) \otimes (C \oplus A) = (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes A) \oplus (B \otimes A)$

$= A \otimes (1 \oplus B \oplus C) \oplus (B \otimes C)$

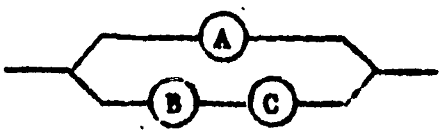
$= A \oplus (B \otimes C)$



(10) $(A \oplus B) \otimes (A \oplus C) \oplus (A \otimes C) = A \oplus (B \otimes C) \oplus (A \otimes C)$

(See problem 9) $= A \otimes (1 \oplus C) \oplus (B \otimes C)$

$= A \oplus (B \otimes C)$



(11) $A \otimes (\bar{A} \oplus B) = (A \otimes \bar{A}) \oplus (A \otimes B)$

$= 0 \oplus (A \otimes B)$

$= A \otimes B$

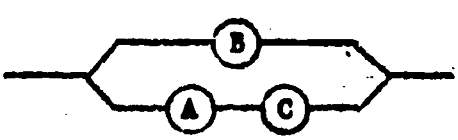
(12) $[(A \oplus B) \otimes \bar{C}] \oplus [B \otimes (\bar{A} \oplus \bar{C})] = (A \otimes C) \oplus (B \otimes C) \oplus (B \otimes \bar{A}) \oplus (B \otimes \bar{C})$

$= [B \otimes (C \oplus \bar{C})] \oplus (A \otimes C) \oplus (B \otimes \bar{A})$

$= B \oplus (B \otimes \bar{A}) \oplus (A \otimes C)$

$= B \otimes (1 \oplus \bar{A}) \oplus (A \otimes C)$

$= B \oplus (A \otimes C)$



Map Coloring: Activities, Conjectures and Proofs Concerning Maps On A Plane and On Solids, the Five Color Theorem and Euler's Theorem

This unit on map coloring is designed to achieve maximum student involvement in discovering problems and forming hypotheses. Hence, it is essential that the student be unaware of the vast literature on map-coloring until after he has worked his own way into the subject. After the class has formulated the four color conjecture, it would be a good idea to tell them of the long history of this conjecture.

This unit would work best with a class which has already studied the regular solids. If the class has not done so, it would be possible to skip the applications to regular solids, but it might be better to work it in with the rest of the unit.

Chapter 13 of the book, Mathematics, the Man-made Universe by Stein, contains a clear exposition of map-coloring. The pamphlet Multicolor Problems by Dynkin and Uspenskii, contains a large number of problems, applications, and examples.

The material of this unit has been divided into the five topics indicated below. The fourth and fifth topics could be interchanged, or either could be omitted. The topics have been built around the various materials needed. A possible schedule would be to cover the first three topics in 3 or 4 classes, while the last two might take 3 or so classes.

Classes may be conducted as laboratory sessions with the instructor circulating among the students giving new tasks and problems to those students who finish what they were doing, trying to help out those students who are stuck (without giving them too much information), and generally encouraging any kind of constructive activity. The unit should lend itself to encouraging a variety of student activity at many different levels. As the unit progresses the instructor may challenge students with completed maps to color them with fewer colors. Students may then attempt to use fewer colors or they may assert that it is not possible to use fewer colors. In the latter case students should give a convincing argument why it is not possible. Although students may find this difficult at first, they should soon be able to present a reasonable argument.

Unit Outline

I. The coloring of geographic maps; the drawing and coloring of maps.

II. Maps which can be colored with two colors.

III. Maps which require more colors; the four color conjecture; the history of the map-coloring problem.

IV. The equivalence between maps on the plane and on the sphere; coloring the regular solids; maps on the torus.

V. Proof of the five color theorem; Euler's theorem; applications.

Map-coloring - I

Materials: Geographical outline maps of various areas in the world (several for each student); broad, colored flow-pens (more fun to use than crayons) of various colors (about 10 different colors would be good, and there should be enough of them so that each student has three or four — they can share them to obtain greater variety of colors); plain paper, colored chalk.

The goal of the first topic is to introduce the idea of a proper coloring of a map, and the notion of an abstract (non-geographical) map. By actively participating in map-coloring and map construction the student should be able to arrive at these notions for himself. The teacher can then help the class to formulate precisely what it has discovered. We suggest the following outline for the topic.

It seems best to start with real (geographical) maps. The maps and flow pens should be distributed, with instructions probably kept to a minimum. The class might merely be told to color the maps in such a way as to make them more clear. After comparing efforts, the class should be encouraged to arrive at a set of "rules" for map coloring. The following ideas should be brought out in some form:

(1) All of a given country should be colored the same color.

(2) Every country should be colored (the white of the paper can count as a color, if desired);

(3) If two countries have a common border, they must be colored different colors.

It is, of course, rule (3) which is the key idea in map coloring. The following points may well arise. They can be left open, or the class may decide them as it wishes (they can be treated in later sessions).

(4) What must be done with countries which touch only at a point (Colorado-Arizona)? Can they be colored the same color?

(5) What is the role of the ocean? Must it be colored? Is it different from the other countries?

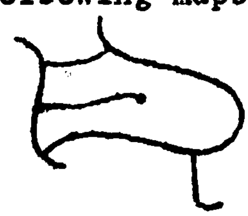
The term proper coloring may be introduced for a coloring which satisfies (1) - (3).

At some point the students should be encouraged to count the number of colors which they have used to color the various maps. If they have used a large number of colors, they should be encouraged to try to color the same map over again with fewer colors. The class might compare notes to see who has properly colored each of the maps with the smallest number of colors.

The class should now take blank paper and draw their own maps, and then color them. After all, this is a mathematics class, not a geography class, and the fact that the maps happen to correspond to geographical realities is of no mathematical interest. It is the idea of map-coloring which is of interest. There isn't enough variety in the geographical maps to illustrate adequately the mathematical ideas. In drawing maps, almost anything goes. The only rule is the following:

(a) Every boundary line must separate two different countries.

Hence the following maps are not legal:



The first maps drawn will probably tend to imitate geographical maps quite closely. The students should be encouraged to eliminate those aspects of their maps (wiggly borders, etc.) which are irrelevant to the map coloring problem. The following question may be raised, and answered, for the time being, as the class desires:

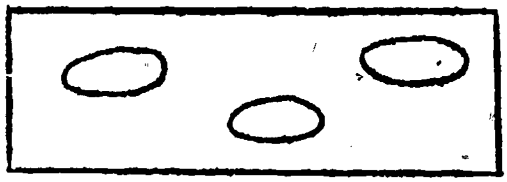
(b) Should every country consist of a single connected region, or may it have several parts (like East and West Pakistan)?

The idea of using as few colors as possible should be kept in mind throughout. One way to properly color a map is to use a different color for each country. But this is aesthetically unpleasing, economically unfeasible, and mathematically uninteresting. There are many "games" which can be built around this idea. A student could devise a map which he can properly color with, say, five colors. The class could then be challenged to color his map with five, or even with four colors. Maps can either be drawn on the board or can be quickly run off on ditto.

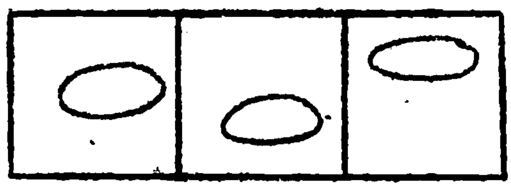
Map-coloring - II

Materials: Blank paper; red flow pens; red chalk.

The simplest maps to deal with are those which can be properly colored with two colors. We say that such a map can be 2-colored. A simple example of a map which can be 2-colored is a map of a collection of distinct "islands" in an ocean, i.e.



Another example occurs if we just draw a straight line across the map. We can combine these two types in a map of the form



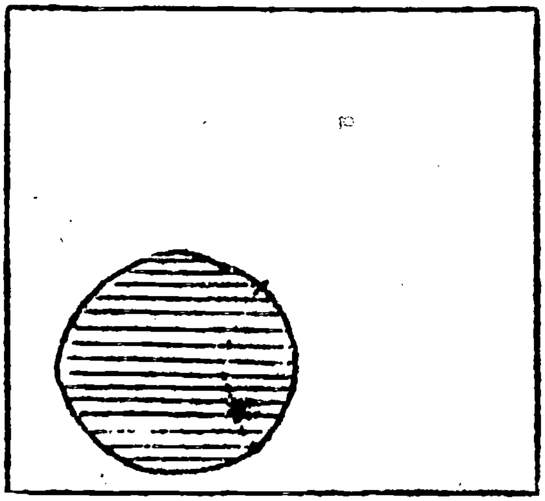
We must now return to question (4) in the first session. If countries which meet only at a point must be colored differently, then the above map is the most complicated type which can be 2-colored. Hence it is necessary to adopt the following rule (in order to arrive at any interesting maps which can be 2-colored):

> Two countries which meet only at a point can be colored the same color.

We can now construct a great variety of maps which can be 2-colored, by drawing straight lines and circles which may or may not intersect. At each stage in the construction we simply change all colors on one side of the line or inside the circle just drawn. The resulting map is then already properly colored with two colors.

Constructing a map which can be properly colored using two colors.

1.



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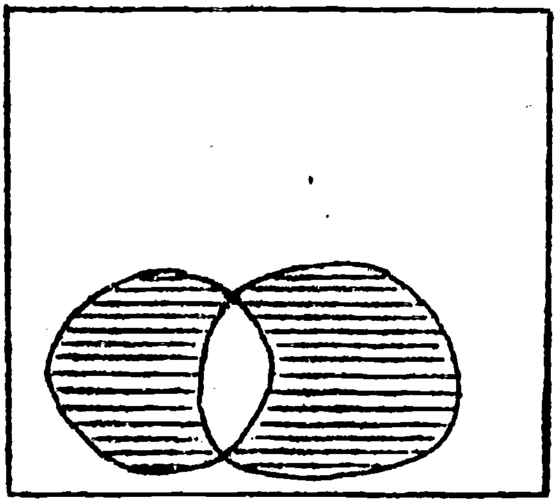
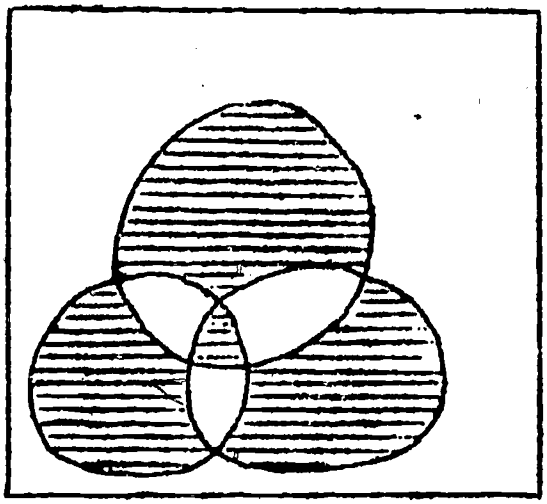


3.



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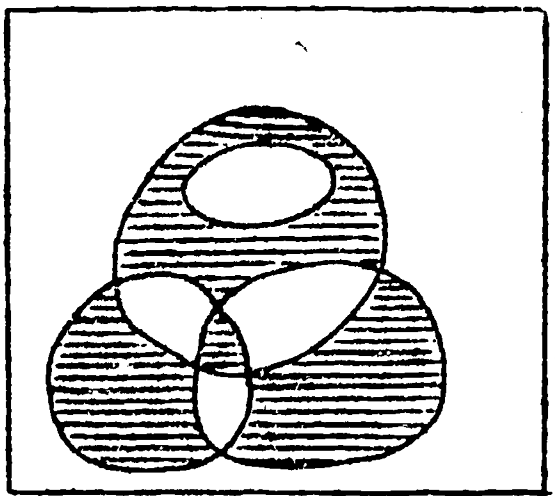

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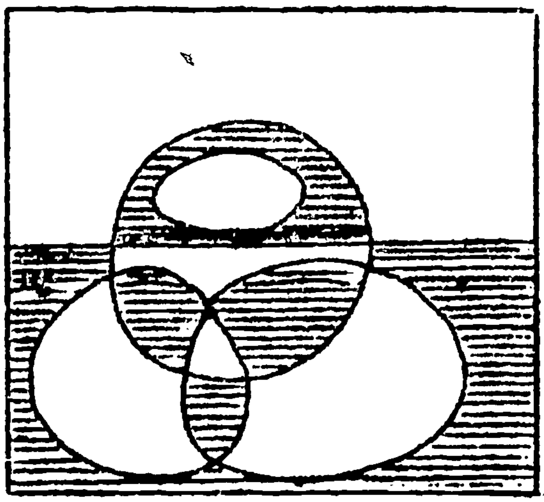

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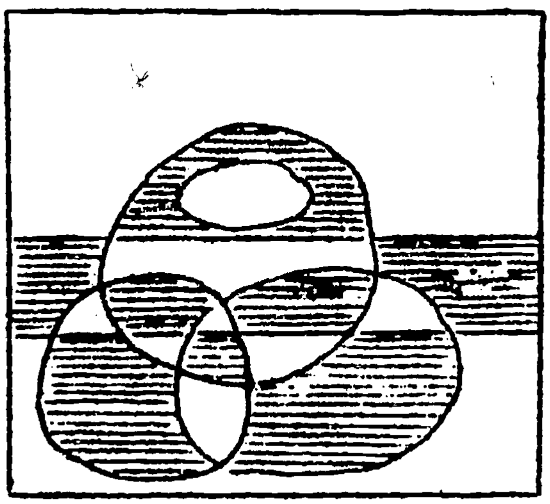


We also have to worry about the role of the "ocean". For instance, a checkerboard may be colored with two colors, provided that one does not wish to color the border of the board. If the border is colored, then three colors are necessary.

Once the class has established considerable facility in drawing maps which can be 2-colored, they can be faced with the converse problem; given a map, how can one tell whether or not it can be 2-colored. On way is trial and error. If you succeed, o.k., but if you fail, how do you know that it can't be done? The students should begin to see what is involved after some experimentation, but to help them, they should be given the definition of the degree of a point on a map (see the enclosed sheet for the definition). They should then be able to arrive at the following two conjectures.

I. If a map can be 2-colored, then every point on the map is of even degree.

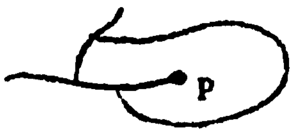
II. If a point on a map is of even degree, then the map can be 2-colored.

The statements I and II can, of course, be combined into one, but they should eventually be stated seperately, since I is almost trivial to verify, while a proof of II is considerably more subtle (see pp. 176-177 in Stein for a nice explanation).

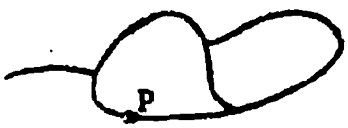
The Concept of the Degree of a Point on a Map

A point which does not lie on a boundary line has <u>degree zero</u>.

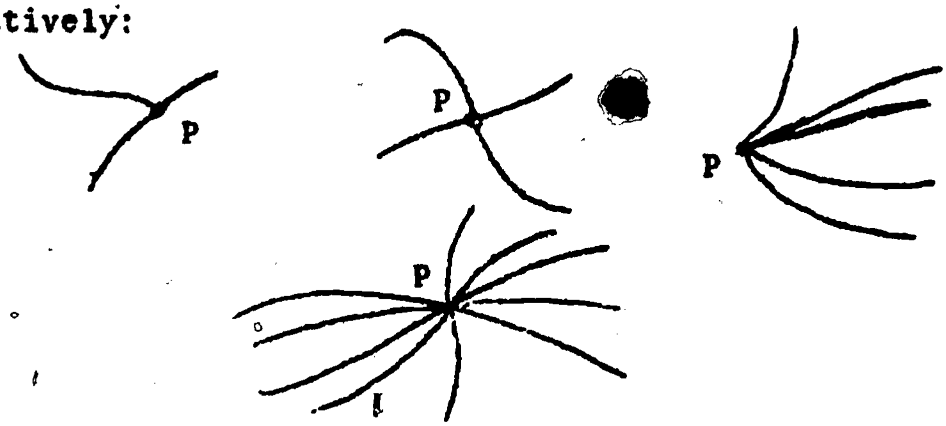
A point which lies at the end of a boundary line has <u>degree one</u>. (This never occurs on the maps which we study, since each boundary must separate two countries.)



A point in the middle of a boundary line has <u>degree two</u>.



The following points have degrees <u>three</u>, <u>four</u>, <u>five</u>, and <u>ten</u>, respectively:



Again, if we are coloring an "island" and not trying to color the "ocean", then the result is slightly different. Namely, we find that an island can be 2-colored provided that every point not on the boundary of the island has even degree. (Stein's terminology of <u>wet</u> and <u>dry</u> points may be used at this point.)

Map-coloring - III

Materials: Blank paper, five or six colors of flow pens and chalk.

The object of this topic is to study maps which require more than two colors, with the goal of arriving at the four color conjecture. After the class has arrived at the four color conjecture on their own, they should be told the history of the conjecture. (The sheet "The History of the Map-coloring Problem" may be handed out to them at this point.)

The class should first construct maps which can be 3-colored (but not 2-colored), then maps which can be 4-colored (but not 3-colored), and then should try to construct maps which cannot be 4-colored. After failing at this they should arrive at the four-color conecture (with suitable encouragement).

We note some pitfalls which may occur:

(1) While every known map can be colored with 4 colors, it is not always easy to find the proper coloring. Often, having arrived at a situation where a fifth color is necessary, one must start all over again to avoid this situation.

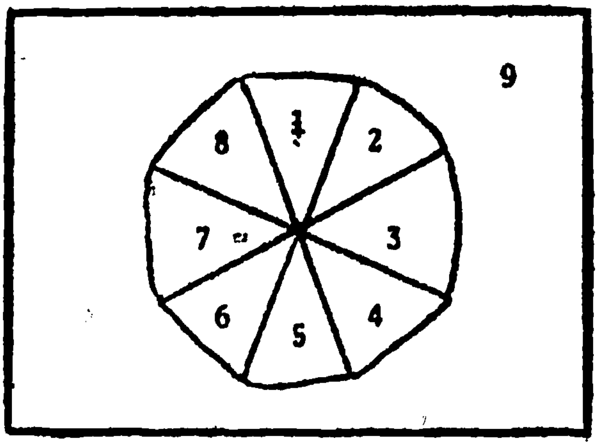
(2) If countries which meet only at a point must be colored differently, then it is possible to construct maps in which n colors are necessary, for any integer n. (Simply have n countries touch at a common point.) Hence, if the question of coloring countries which meet only at a point has not yet been resolved, it must be decided that they can be colored the same color. Otherwise the whole problem would lose all mathematical interest.

(3) If countries are allowed to consist of several parts (East and West Pakistan), then it is possible to construct maps which require n colors for any integer n (see the enclosed sheet for a picture). Hence, if this question is still undecided, such countries must be disallowed.

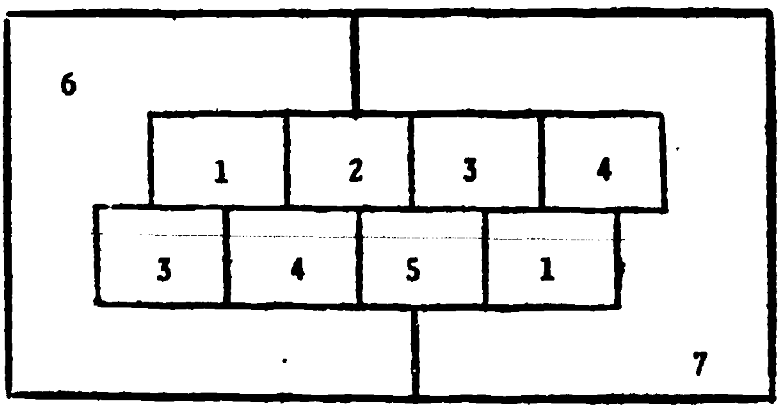
(If the class is not worried about (2) and (3), it would be a mistake to spend much time on them.)

Maps which require many colors

A. If countries which meet only at a point must be colored different colors, then this map requires 9 colors.



B. If countries need not be connected, then this map requires 7 colors.



(The two squares labelled 1 are parts of the same country, etc.

The History of the Map-coloring Problem

"As far as we know the problem was first mentioned by Mobius[1] in his lectures in 1840. Both Kempe[2] and Tait[3] published "proofs" that four colors are sufficient. Actually Tait merely proved that the four color problem could be solved if one could solve an equally difficult problem on the coloring of graphs. But needless to say, this graph problem is still unsolved to this day. The error in Kempe's proof is more delicate and in fact for ten years the error went undetected until Heawood[4] pointed out the mistake in Kempe's proof. In this very same paper Heawood proved that five colors are sufficient, and it is this proof that we give ...

[1]Alfred F. Mobius, 1790-1868

[2]A. B. Kempe, "On the Geographical Problem of the Four Colors," American Journal of Mathematics, vol. 2 (1879), pp. 193-200

[3]Peter Guthrie Tait, "Note on a Theorem of Position," Transactions of the Royal Society of Edinburgh, vol. 29 (1880), pp. 657-660.

[4]P. J. Heawood, "Map-colour Theorem," Quarterly Journal of Mathematics, vol. 24 (1890), pp. 332-338.

From The Pleasures of Math by A. W. Goodman, (Macmillan, 1965) pp. 93-94.

Further Heawood solved the map-coloring problem on the torus, and we give this solution .... This last result is remarkable, because the torus is a more complicated surface than the plane or sphere, and normally one would expect that the map-coloring problem would be more difficult on the torus.

The fact that this problem in the plane, now regarded as unsolved, was considered as solved during the years from 1880 to 1890, is rather disquieting. It suggests that perhaps there are today many theorems that we regard as proved, that really have not been proved, because the "proofs" offered contain errors, as yet unnoticed.

Each student of mathematics has a duty to himself to examine each proof as carefully as he can in order to convince himself that the proof is indeed correct."

"The problem of coloring maps with four colors has a long history. The experience of map makers indicated that any map on the globe could be colored with four (or fewer) colors. Mobius mentioned it in a lecture in 1840; de Morgan discussed it in 1850; and Cayley remarked, in 1878, that he could not prove it. In 1879 Kempe published an erroneous proof in a paper that contained these remarks:

Some inkling of the nature of the difficulty of the question, unless its weak point be discovered and attacked, may be derived from the fact that a very small alteration in one part of a map may render it necessary to recolor it throughout. After a somewhat arduous search, I have succeeded, suddenly, as might be expected, in hitting upon the weak point, which provided an easy one to attack. The result is, that the experience of the map makers has not deceived them, the maps they had to deal with, viz: those drawn on a sphere, can in every case be painted with four colors.

The flaw in Kempe's proof was exposed in 1890 in a paper by P. J. Heawood, which began:

The Descriptive-Geometry Theorem that any map whatsoever can have its divisions properly distinguished by the use of but four colors, from its generality and intangibility, seems to have aroused a good deal of interest a few years ago when the rigorous proof of it appeared to be difficult if not impossible, though no case of failure could be found. The present article does not profess to give a proof of this original Theorem; in fact its aims are so far rather destructive than constructive, for it will be shown that there is a defect in the now apparently recognized proof ....

In the same paper Heawood showed that Kempe's technique could be used to prove that every map on the sphere can be colored with five (or fewer) colors."

From Mathematics, the Man-made Universe by Sherman K. Stein. (Freeman, 1963) pp. 183-184.

While the four-color conjecture has never been proved, some progress has been made on it. For instance, it was shown by Reynolds in 1926 that any map with at most 27 countries can be colored with four colors, and in recent years this has been extended to maps with at most 37 countries. Hence, if there is a map which cannot be colored with four colors, it must be quite complicated, in particular it must have at least 38 countries. This makes the task of trying to construct such an example by trial and error rather difficult, though many people have spent a long time trying.

Map-coloring - IV

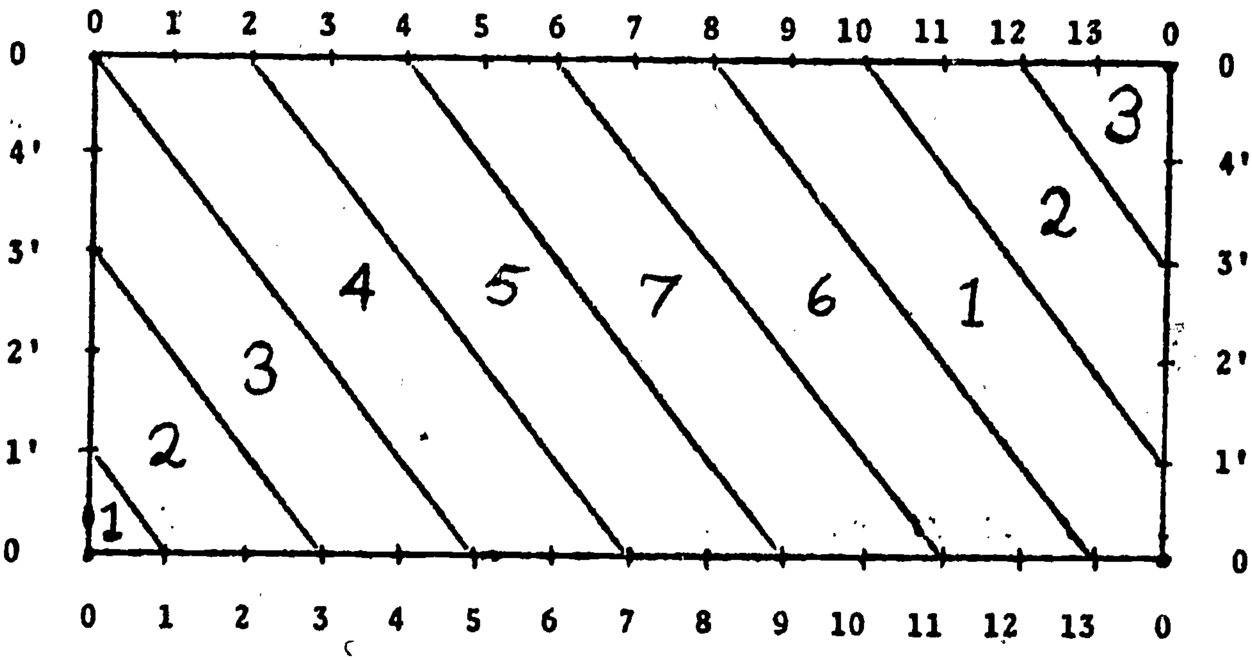
Materials: A small globe of the world (unmounted); the regular solids kit; several rigid models of regular solids (can be made of wood, clay, or plaster of paris, with the cardboard model possibly used as a mold); several sheets of thin, stretchable rubber; flow pens; two inner tubes from automobile tires (optional)

The purpose of this section is to show that maps in the plane are equivalent to maps on the sphere (at least as far as any properties relevant to map-coloring are concerned), to consider the regular solids as maps on the sphere, and possibly to show that maps on the torus (auto tire) have completely different properties.

The transition from flat maps to maps on the sphere may be hard for the students to visualize. The use of rubber sheets allows him to see the process in action, and the results can be quite striking. Wrap the globe with rubber, so that the rubber is fairly smooth over most of the globe, except for the region where the edges of the rubber are gathered together. This latter spot should be in some spot like the north pole. While the rubber is held in place, the outlines of the continents can be quickly sketched with a black flow pen or colored chalk. When the rubber is flattened out, a map of the world is obtained. There will be a great deal of distortion in this map. The area around the north pole will be greatly enlarged and will lie around the edges of the sheet of rubber. But this stretching does not change any properties of relevance to map-coloring problems. Another natural place to put the gather would be in the middle of an ocean. Then we would truly find the ocean around the edges of the map. But now the gather can be put in the center of a (fairly large) country, like the USA or USSR. The resulting map will have the USA or USSA spread out around the edge of the map, playing the role of "ocean". This should convince the students that in map-coloring problems the role of the ocean is not significantly different from that of the other countries. It might be a good idea to reverse the process; draw a map on a rubber sheet while it is flat, and then stretch it around a ball, making a spherical map out of it.

The regular solids can be viewed as spheres which have been somewhat flattened. If we regard the edges of the solids as the edges of a map, then we have map-coloring problems. It seems a very worthwhile exercise to draw the corresponding flat maps. The rubber sheet supplies a perfect tool for doing this. Stretch the rubber around the regular solid, making sure that the gather lies in the middle of a face. Then outline the edges on the rubber with black flow-pen. When spread out, a flat map of the regular solid will result. After this has been done for a couple of solids the class should be challenged to draw the corresponding flat maps for the remaining solids without use of a rubber sheet. The solids (or the corresponding maps) should then be colored, and the minimum number of colors necessary for each solid should be determined.

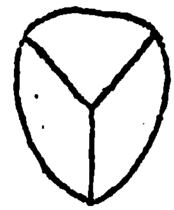
Attention can then be turned to map-coloring on the torus. If two inner tubes are available, one of them can be cut around both circumferences so that it can be flattened out. This should enable the student to see that maps on the torus correspond to maps on a rectangle where the top and bottom are identified and the left and right edges are identified. The class should then be challenged to draw a map on the torus consisting of n countries, each of which touches all of the other n - 1. The maximum such number is n = 7, so there exist maps on the torus which require seven colors. The remarkable thing is that it can be proved that any map on the torus can be colored with seven colors. Hence map-coloring on the torus is an easier problem than map-coloring on the sphere or in the plane. The following neat map of seven countries, each of which touches all of the other six, was discovered by the mathematician Peter Ungar in 1953. After they have discovered such a map, the students should transfer it to the inner tube.



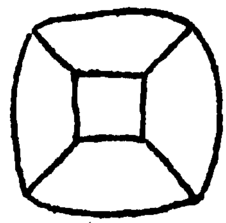
The Regular Solids

(The extra face is the "ocean".)

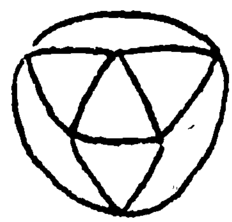
Tetrahedron - 4 colors



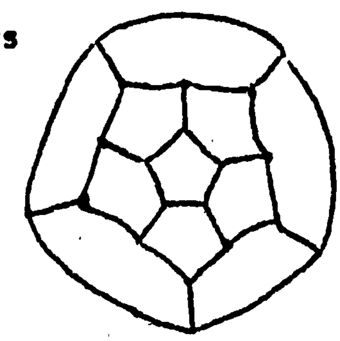
Cube - 3 colors



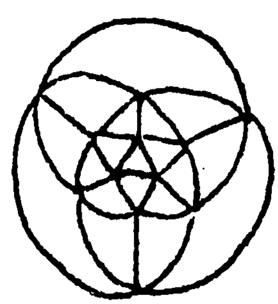
Octahedron - 2 colors



Dodecahedron - 4 colors



Icosahedron - 3 colors



Map-coloring - V

The fifth topic is the proof of the five color theorem. This will require a much more active participation on the part of the teacher than the previous ones. It would probably be best not to devote a class to the proof, but rather to encourage those students who are interested to read the proof on their own, or to organize a special outside class.

A quite readable account of the proof of the five color theorem is to be found in Stein's book, on pp. 184-191 (the proof is given there in great detail - it is not as complicated as one might think from the fact that it occupies eight pages). Hence we shall limit ourselves here to commenting on Stein's proof and indicating some alternatives.

(1) It is now necessary to introduce the word "vertex". Previously we have dealt only with the countries and the boundary curves of a map. But the proof of the five color theorem (particularly in Euler's theorem) depends heavily on a careful counting procedure, and to do this we must choose certain points on the boundary curves which we call the vertices of the map. Every point of degree greater than two must be a vertex, and some points of degree two may also be vertices. Each edge will start and end at a vertex, and will contain no vertices in its interior.

(2) On page 184 Stein remarks that the figures in (12) are not countries. This restriction, while convenient in his proof, is not essential. If a country X surrounds some other countries, then the surrounded countries can be colored completely independently from the rest of the map, with X treated as an ocean. [Fu]rther Lemma 3 (Euler's Theorem) is true for much more general

types of maps (see below).

(3) The proof of Lemma 6 is by mathematical induction. If the students are not familiar with mathematical induction, they might find an argument by contradiction more convincing. This would go as follows: "Assume that the statement of the lemma is false. Then there exists a regular map which cannot be colored with five colors. Among all such maps, pick one with the least number of countries. Then proceed as in Case 1 or Case 2." (Note that Lemmas 1 and 3 could also have been proved by mathematical induction.

(4) Case 2 in Lemma 6 is somewhat more complicated than Case 1. The method of Case 1 is easily seen to yield a proof that every map on the sphere can be colored with six colors.

(5) Lemma 3 (Euler's Theorem) may already have been seen by the students while studying the regular polyhedra. It would probably be a good idea to go over the proof again. The following treatment is perhaps more likely to engage the imagination of the student than the one given by Stein.

Consider a map on a sphere or on the plane as a network of dams. All of the countries are below the water level of the ocean (any country may be designated as the ocean). To visualize the rest of the proof it is easier to think of the countries as forming an island in the ocean. See topic IV. There are only two restrictions on this system of dams:

[i] Each dam starts and ends at a vertex and contains no vertex in its interior.

[ii] The system of dams is connected, so that it is possible to walk between any two points on the dams along the top of the dams.

We will now flood the entire world, but will do this by destroying as few dams as possible. Thus, we will always destroy a dam which is dry on one side and wet on the other. In the end we will have a network of dams completely surrounded by ocean (with no dry land). Each time we destroy a dam we flood one more country and hence reduce the number of dams by one, and do not change the number of vertices. Hence the quantity

$$V - E + C$$

remains unchanged throughout. Now consider the final system of dams, completely surrounded by water. We claim that there is one and only one path along the dams between any two vertices. If there were two routes between the vertices A and B, then these routes would surround some land, which would still be dry. If there were no route between A and B, then at some stage we would have destroyed a dam which had to be crossed to get from A to B. But if there was no way around this dam, there must have been water on both sides, and such dams were not destroyed. Hence there is precisely one route from A to B along the remaining dams. Now fix a vertex A. If B is any other vertex, we associate to B that dam which is crossed last in going from A to B. In this way we pair off the remaining dams with the vertices (except A). Hence in the "map" produced by the remaining dams, we have

$$C = 1$$

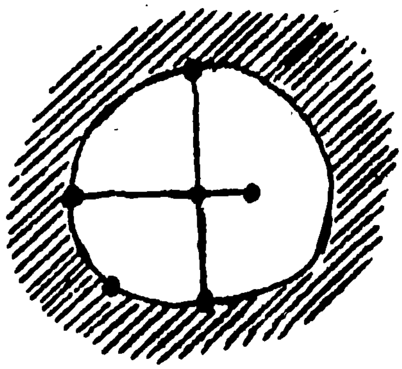
$$V = E + 1$$

so that

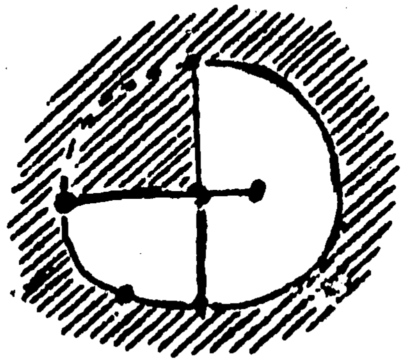
$$V - E + C = 2 .$$

Since, as we noted above, the quantity V - E + C does not change as the dams are destroyed, it must have been equal to 2 for the original map.

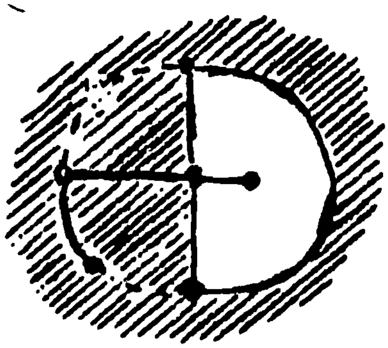
The Proof of Euler's Theorem (an example)



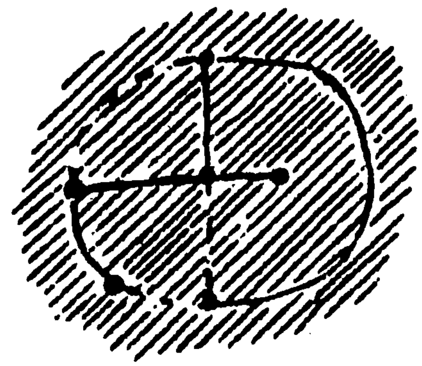
V = 6; E = 8; C = 4.



V = 6; E = 7; C = 3.



V = 6; E = 6; C = 2.



V = 6; E = 5; C = 1.

At each stage we have V - E + C = 2.

(6) Note that Lemma 3 (Euler's Theorem) is false if the map (or system of dams) is not connected. To see this, just compute a simple example.

(7) It can be shown by analogous reasoning that on the torus

$$V - E + C = 0 .$$

(However here we must make sure that none of the countries stretches all the way round the torus in either direction.) This result can then be used to show that any map on the torus can be colored with seven colors. Since we have constructed maps on the torus which cannot be colored with less than seven colors, the map-coloring problem on the torus is completely solved.

(8) Euler's formula has many other applications. For instance, it can be used to prove that there are at most five regular polyhedra (see below). A variety of puzzles can be invented to which Euler's Theorem applies, i.e. one shows that certain configurations are not possible in the plane. See problems 47-51 in Multicolor Problems (answers on pp. 59-62).

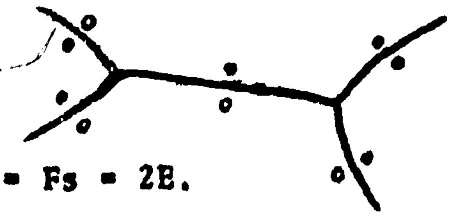
(9) Proof (by Euler's formula) that there are at most five regular solids.

Let a regular solid have F faces, E edges, V vertices, with each face having s edges (and s vertices); and t edges (and t faces) meeting at each vertex. Place a dot inside each vertex on each face:

Counting all of the dots, we have

$$\text{number of dots} = Fs = Vt.$$

Now erase the above dots and place one dot inside each edge on each face. Counting the dots:



$$\text{number of dots} = Fs = 2E.$$

Hence $$F = \frac{2E}{s}, \quad V = \frac{2E}{t}.$$

Substituting in Euler's formula:

$$V - E + F = \frac{2E}{t} - E + \frac{2E}{s} = 2$$

so

$$\frac{2E}{t} + \frac{2E}{s} = 2 + E$$

and

$$\frac{1}{t} + \frac{1}{s} = \frac{1}{E} + \frac{1}{2} > \frac{1}{2}$$

Noting that s and t must be at least 3, the only possibilities for s, t which give

$$\frac{1}{s} + \frac{1}{t} > \frac{1}{2}$$

are

(a) $s = 3, t = 3$

(b) $s = 3, t = 4$

(c) $s = 4, t = 3$

(d) $s = 5, t = 3$

(e) $s = 3, t = 5$.

Using

$$E = \frac{1}{\frac{1}{t} + \frac{1}{s} - \frac{1}{2}}$$

and

$$F = \frac{2E}{s}, \quad V = \frac{2E}{t}$$

it is easily seen that (a) - (e) correspond to the known regular solids.

This shows that we cannot obtain more regular solids even by allowing curved edges and faces.

Roll Along With Galileo: A Study of Cycloids and Area

This unit will hopefully develop some intuitive insights into geometry and can also serve as an introduction to integral calculus. Further, it is intended to give students practice in analyzing and predicting patterns of motion.

Materials: Plywood discs, squares, elipses, rectangles, and triangles. These plane figures should be between 6 and 10 inches in diameter. Pasteboards, scissors, and 1/4 inch graph paper will also be useful.

Procedure: The teacher may want to proceed as follows:

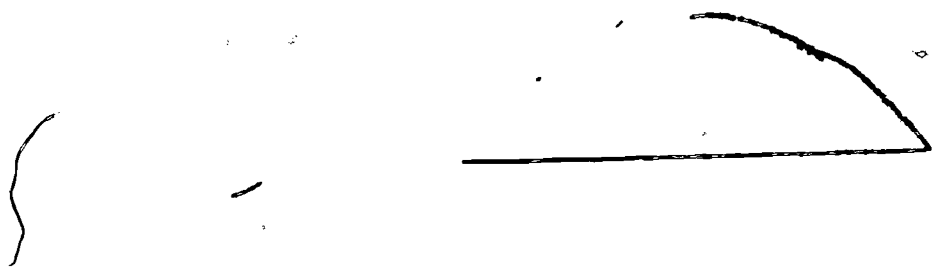
Galileo (1564 - 1642) was interested in the path of a point on the rim of a rolling wheel. (This path is called a cycloid.) To get an appreciation for this, drill a hole in a plywood disc near the edge, put a piece of chalk in this hole. Now roll the disc along the blackboard, using the ledge as a guide. What kind of path did Galileo and you discover?

There are some interesting questions to ask about the above figure. Suppose the radius of the circle is 3 inches?

1) How far is it from A to B?
2) How long is the path of the point (chalk)?
3) What is the area enclosed by the path?
4) Does the chalk repeat, if you continue rolling the circle?

With a little help students will probably see that the circle makes one complete revolution in going from A to B. Therefore, the distance AB is merely the circumference of the given circle ($2\pi r$).

The next two questions are not so easily answered. To get a clue to the answer of the second question one might try to use a cord or something flexible. A piece of copper wire works well. To estimate this area, cut out a paste board circle and the path and weigh the pieces on a sensitive scale.



You will probably see that the path length is about four times the diameter of the given circle, and the area is three times that of the circle. One could also use a past board circle and a pencil to trace the cycloid on a piece of 1/4 inch graph paper. You could then count squares to determine the area enclosed by the path.

All of these questions do not have to be cleared up at the beginning. The teacher could go on to other geometric figures. When you have explored the other figures to the extent that you desire, you may want to return to some of the unanswered questions.

Consider the square or rectangle. (The teacher should have plywood squares and rectangles on hand.) What kind of pattern would be formed if you rolled the rectangle or square along the chalk edge?

Does it make any difference where on the perimeter the chalk is placed?

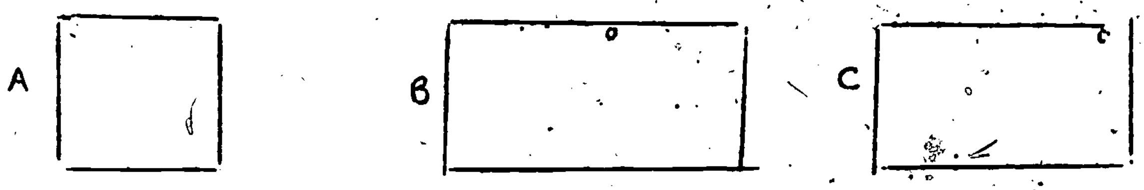
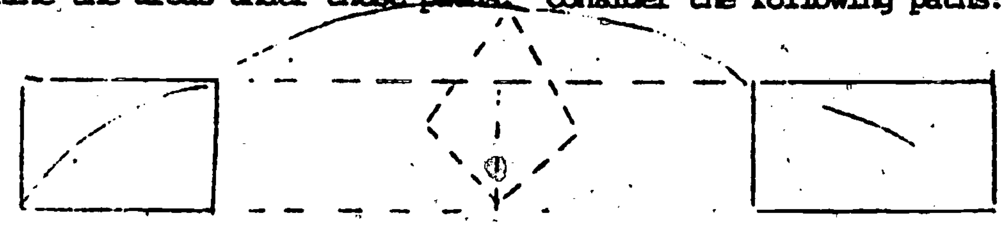


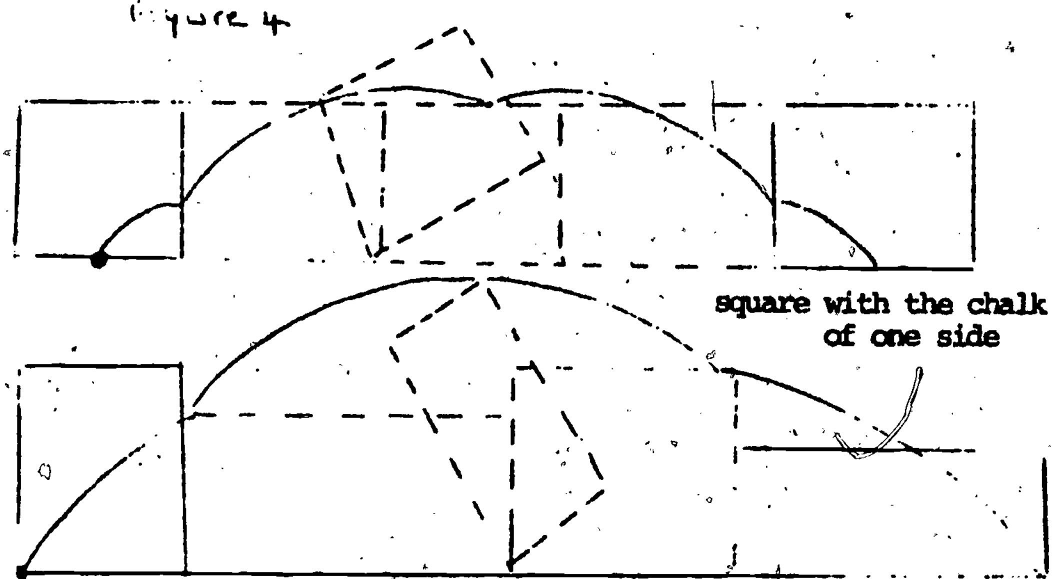
Figure 3

Once students have discovered some of the patterns that are formed, try to determine the areas under these paths. Consider the following paths:



square with the chalk in the corner

Figure 4



square with the chalk at the midpoi of one side

rectangle with the chalk in the corner

We are also interested in the length of these different paths.

Questions for further discussion:

1. Noting that we get different paths using the square, depending on where the chalk is placed, there are two questions to be asked:

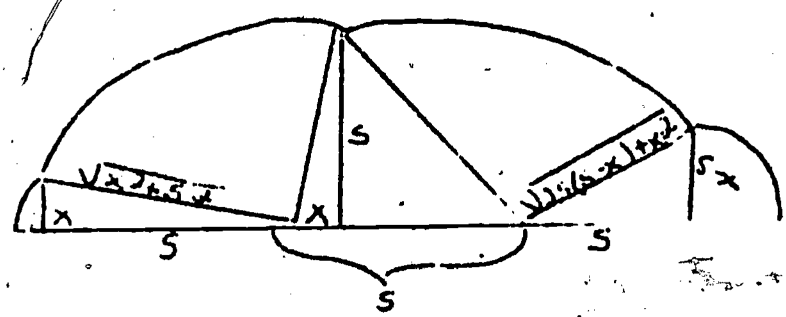
a) Where can the chalk be placed so that we obtain maximum area? Minimum area?

b) Where can the chalk be placed to get maximum or minimum path length?

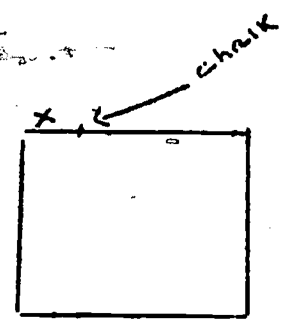
2. Obviously the larger the circle, square, rectangle, etc. are, the larger the path of points. Suppose that we have plane figures (circles, rectangles, triangles) all having the same area. Which figure produces the largest path? The smallest path?

If a square is used the area and path length can always be determined. In the general case for the area consider the discussion below:

figure 5



This path was generated by a square with the chalk at a point x distance from the corner.



It is not too difficult to see that the combined area of the triangles indicated in figure 5 is s^2. To determine the remaining area we note that each of the circular portions are 1/4 of a circle. Therefore we can write:

$$\pi/4\,(x^2 + x^2 + s^2 + 2s^2 - 2sx + x^2 + (s - x)^2)$$

which simplifies to: $\pi(x^2 - sx + s^2)$. If you substitute $x = 0$, this describes the case where the chalk is in the corner; the area is $s^2 + \pi s^2$.

It is also not too hard to determine the path length in the general case. If you refer to figure 5, it is clear that the following formula will give the path lengths: $2\pi/4\,(x + \sqrt{x^2 + s^2} + \sqrt{2s^2 - 2sx + x^2} + s - x) =$

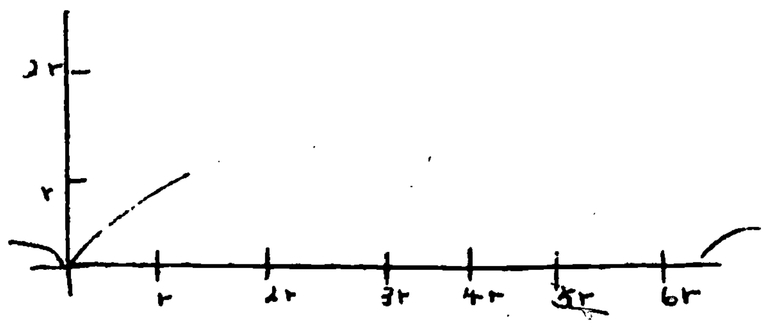
$$\pi/2(s + \sqrt{x^2 + s^2} + \sqrt{2s^2 - sx + x^2})$$

Again for the square with the chalk in the corner we have the path length given below:

$$\pi/2(s + \sqrt{s^2} + \sqrt{2s^2}) = \pi/2s(2 + \sqrt{2})$$

It is left to the reader to generate formulas for the other geometric configurations.

Returning the discussion to the cycloid or path of a point on the rim of a rolling wheel, it is interesting to note that there is no easy non-calculus way to determine the path length and the area enclosed by the path. To give the reader some idea of how nasty the calculus could be to determine just the area, consider the derivation below:

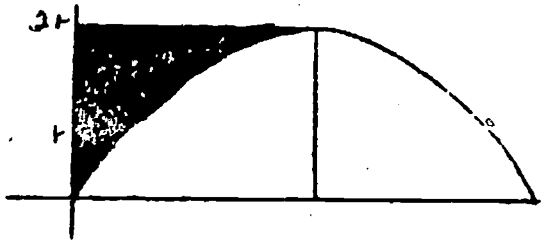


The curve can be described by the two equations $x = r(\theta -$si

and $y = r(1 - \cos\theta)$.

If we eliminate θ we can get one equation: $x = r\cos^{-1}\frac{r-y}{r} - \sqrt{2ry - y^2}$.

We can do the integration necessary on the y axis. Consider the picture below:



We know that the indicated rectangular area is $2\pi r^2$, since the x distance of the whole curve is $2\pi r$. By the integration we can find the area of the shaded portion. Now if we subtract the area of the shaded portion from the area of the indicated rectangle, and then multiply by 2 we would have the area under the cycloid.

Let's do the integration. Without loss of generality we can let $r = 1$. Therefore, we have, using the integration tables,

$$\int_{0}^{1} (-(\cos^{-1}(1-y)) - \sqrt{2y - y^2})dy =$$

$$[-((1-y)\cos^{-1}(1-y) - \sqrt{1-(1-y^2)})] - 1/2[(y-1)\sqrt{2y-y^2} + \sin^{-1}(y-1$$

$$= \pi/2$$

Since the area of the rectangle is 2π, the area of the cycloid is $(2\pi - \pi/2).2$ Therefore we can write area of cycloid = 3π.

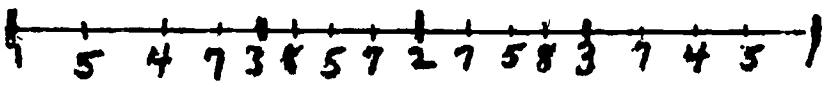
Q. E. D.

Charlie Haynie has some more elegant proofs for both the path length and the area. His derivation may also be straight forward enough for some of your good students to comprehend.

THE EULER ϕ FUNCTION

A SIMPLE INTRODUCTION

Take a sheet of paper and draw a line the length of it. Place the numeral 1 at each end of the line. Subdivide the line and put the numeral 2 at the midpoint. Subdivide each segment again, putting 3 at the midpoint of each new segment. Continue subdividing, placing at the midpoint of each new segment the sum of the endpoints of the segment. Thus after four subdivisions, the original line should look like



See next page for a more detailed subdivision.

<u>Question</u>: What do you notice? Possible comments might be:

a. The smallest new number is always next to the ends of the line.

b. The numbers keep getting bigger; after a while any given number will stop coming up.

c. There are more odds than evens.

(Each of these comments leads to interesting observations. For comment a., one might ask, "After three subdivisions, what is the smallest number added to the line? (4)? How about after five subdivisions? (6)." They will probably be able to generalize and say that after n subdivisions, the smallest number added is n + 1. Does this mean that in order to make sure we have, say, all the 17's, we have to perform 16 subdivisions? They should be able to convince themselves that this won't be necessary.

Comment c. will be treated briefly in an appendix.)

After a while, the comments may start getting more specific, such as:

c. There are a lot of 7's

d. And 11's and 13's.

They or the teacher might suggest tabulating the number of occurrences of the various integers. From the lines on the next page, we can get the following tables (filling in some of the spaces near the end points mentally by exploitation).

number	frequency
1	2
2	1
3	2
4	2
5	4
6	2
7	6
8	4
9	6

number	frequency
10	4
11	10
12	4
13	12
14	6
15	8
16	8
17	16
18	6

number	frequency
19	18
20	8
21	12
22	10
23	22
24	8
25	20

Once the student has this table, he can begin to be more specific in his comments:

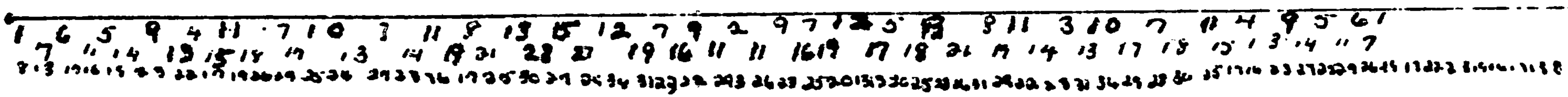
a. Every number occurrs an even number of times.

b. Except for 2

c. Every odd number occurs one less time than itself. 11 comes up 10 times, 13 comes up 12 times, 19 comes up 18 times.

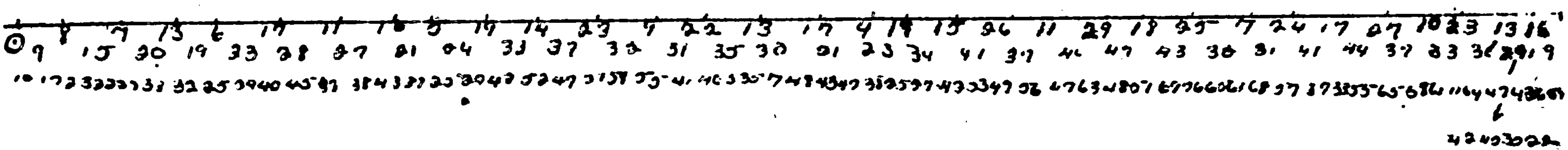
d. But 9 only comes up 6 times and 15 only comes up 8 times.

After worrying over this for a while, they may come up with something equivalent

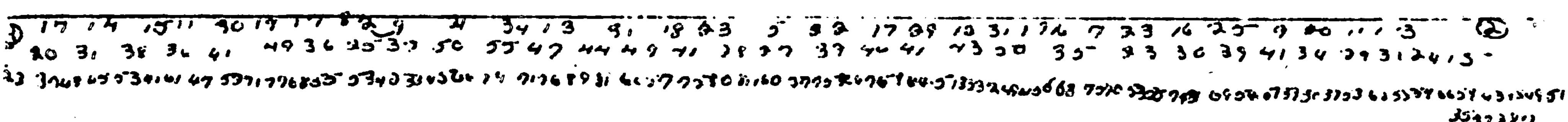


You might suggest, or the students might notice, that more numbers could be squeezed in if the lines were broken up into double pieces, each one stretched out.

thus we could consider.



And



This is the result of 9 subdivisions. The students should do nearly this many

etc.

to x

e. If a number is prime, it will occur one less time than itself.

Once they've made this conjecture, ask how many 29's they would expect, how many 37's, how many 103's.

Perhaps the following observations would come up:

f. 12 is equal to 3x4, and the number of 12's is equal to the number of 4's times the number of 3's.

g. 20 is equal to 4x5, and the number of 20's is equal to the number of 4's times the number of 5's.

h. 15 is equal to 3x5, and the number of 15's is equal to the number of 3's times the number of 5's.

Conjecture: If a number is the product of two other numbers, then the number of occurrence of the first number is equal to the product of the number of occurrences of the other two.

i. 12 is equal to 2x6, but the number of 12's is more than the number of 2's times the number of 6's.

j. 25 is equal to 5x5, but the number of 25's is more than the number of 5's times the number of 5's.

After some time, the notion of relative primeness might come up, and the following modified conjecture might be offered (phrased in different language, perhaps)

Conjecture: If c=ab, then the number of c's equals the number of a's times the number of b's, provided a and b have no common factor. Otherwise, this product will be less than the number of c's.

Questions: How many 35's do you think there will be? How many 42's? How many 70's? How many 63's.

Now what about numbers that are powers of other numbers?

k. There is one 2, two 4's, four 8's, eight 16's. I bet there will be sixteen 32's and thirty-two 64's.

i. But how does this help up figure out how many 9's or 25's there will be?

Now might be a good time to go back to the original line and look at the actual sums which occur to give any number. For instance, we see that 7 occurs six times, in the following combinations: 1+6, 2+5, 3+4, 4+3, 5+2, and 6+1. All possible ways of writing 7 as a sum of two numbers actually appeared. Now look at a number like 6. This only occurs as 1+5 and 5+1. The combinations 2+4, 3+3, and 4+2 do not appear. Let's look at some more numbers.

9	12		14		15		16	
1+8	1+11	5+7	1+13	5+9	1+14	5+10	1+15	5+11
2+7	2+10	6+6	2+12	6+8	2+13	6+9	2+14	6+10
3+6	3+9		3+11	7+7	3+12	7+8	3+13	7+9
4+5	4+8		4+10		4+11		4+12	8+8

The combinations which actually occur on the line are circled.

Question: Why do some combinations occur and some don't? What determines whether or not a combination will occur?

There will probably be a number of preliminary comments at this stage:

a. The two numbers can't be the same.

b. You can't have an even number.

c. That's not so, because the combination 2+5 occurs to give 7, and 2 is even.

d. Well, both numbers can't be even.

e. Both numbers can't be multiples of three, either.

Somewhere along the line, this will hopefully get generalized to the following:

Conjecture: Any combination in which the two numbers have a common divisor will not occur. Any combination in which the two numbers are relatively prime (or some such) will occur.

Armed with this conjecture, now, the students have a technique which will permit them to count the number of occurrences of any reasonably small number. Use this technique to calculate the number of occurrences in the table on page 1 as a check. Have them compute the number of, say, 30's and 42's by this conjecture and compare the results with the results of the preceding conjecture. Have them compute the number of 25's if they haven't made their table that far, and the number of 27's. This will give us a little more data with which to attach the questions posed at the bottom of the preceding page.

Returning to this question of powers of numbers, let's write down what we know:

1 2	4 5's	2 3's	and perhaps	6 7's
2 4's	20 25's	6 9's		42 49's
4 8's		18 27's	using the last conjecture to compute w.	
8 16's				
16 32's				

Questions: How many 81's do you think there will be? How many 125's? How many 343's ($=7^3$) do you think there will be? How many 121's? The pattern should be clear enough so that the class can answer these questions. They are probably ready now to make a conjecture, which, depending on their familiarity with exponential notation may go something like

Conjecture: If p is a prime, then the number p^n will occur

$(p-1)p^{n-1}$ times.

Question: What if p is not a prime, does the rule still work? For instance, $16=4^2$. Does (4-1)4 give us the number of 16's which occur?

Let's summarize the rules we have so far.

Rule 1: (Really a special case of rule 2) If p is a prime, then p occurs p-1 times

Rule 2: A number of the form p^n occurs $(p-1)p^{n-1}$ times.

Rule 3: If c=ab, where a and b have no divisors in common, then the number of occurrences of c equals the number of occurrences of a times the number of occurrences of b.

Using these rules, we can now predict the number of occurrences of just about any number.

Example: How many 10,000's will there be? Well, $10{,}000=2^4 \times 5^4$. Further, 2^4 and 5^4 are relatively prime. (an interesting digression might be to figure out why this is so.) By rule 2, we know that 2^4 will occur eight times, and that 5^4 will occur five hundred times. Therefore, using rule 3, there will be 8x500=4000 occurrences of the number 10,000.

Let us go back and examine the conjecture we made on page 4. There we said that the combination c=a+b will occur only if a and b have no common divisors. But if a and b do have a common divisor, then this number will also divide c (why?). Conversely, if a number divides c and a, then this number will also divide a and b. Our conjecture can then be rephrased in the following way.

Conjecture: Given a number c, to find all pairs a,b such that a+b=c will occur on our line, we need only find all numbers a less than c, which are relatively prime to c. For if a is less than c, set b=c-a. Then if a and c are relatively prime, so are a and b, and the combination a+b will occur.

Corollary: The number of times that c will occur on our line is equal to the number of numbers less than c which are realtively prime to c. This gives us an even more efficient technique for writing down all combinations giving c which will occur.

Remark: The number of integers less than a given integer and relatively prime to it is very important in number theory, and the associated function has a special name: the Euler φ-function, and is defined as follows; given an integer c,

$f(c)$= number of integers less than c and relatively prime to c.

Let's look at the conjecture on page 4 again. Can we prove it? We can break it down into the following two statements:

1. No matter how many subdivisions we make, any two adjacent numbers will always be relatively prime.
2. If c=a+b, and a and b are relatively prime, then after a suitable number of subdivisions, the numbers a and b will be adjacent somewhere on the line. Equivalently, we can phrase this in the following suprising fashion: given any two relatively prime numbers, they will occur side by side at some time (i.e., after a suitable number of subdivisions) somewhere on the line. (why is this equivalent?)

Consider statement 1 first: After two subdivisions we have the following situation:

Statement 1 is certainly true in this case.

After three subdivisions, we have:

and again statement 1 is true.

Can we prove that if statement 1 is true after so manu subdivisions, then it will be true after the next subdivision? Sure. Suppose after n subdivisions we have ...a b ... being two adjacent numbers somewhere on the line. Then after the next subdivision, the situation looks like ... a a+b b ... . If a and b are relatively prime, then so are a and a+b, as well as b and a+b; i.e., adjacent numbers are still relatively prime.

Statement 2 can be demonstrated by a similar inductive argument. The only way of writing 3 is 1+2. We see that the numbers 1 and 2 are adjacent after the first 1st subdivision. The only way of writing 4 is 1+3. The numbers 1 and 3 are adjacent after the second subdivision. 5 can be written as 1+4, 2+3, 3+2, 4+1. 3 and 2 are adjacent after the 2nd subdivision, 4 and 1 are adjacent after the third, etc.

Now suppose we have established in some fashion that statement 2 holds for c=2,3 ....out to some number n. Can we show that it must be true for c=n+1? For instance, suppose we know that the statement is true for c=2,3 .....15. Can we prove that it is true for c=16? Let's see. Take the case 16=13+3. Can we prove that after a suitable number of subdivisions that the numbers 13 and 3 will be adjacent? Sure. By our assumption, statement 2 holds for c=13. In particular, since 13=10+3, after a suitable number of subdivisions, the numbers 10 and 3 must have been adjacent, i.e., somewhere on the line it looked like ... 10 3 ... . After the next subdivision, then, this part of the line would look like ... 10 13 3 ...; i.e., 13 and 3 are adjacent, which we wanted to show. Statement 2 can be proved by rigorous induction with a little more work, but there is not much point in doing it in class. With suitable buildup, a class might well be able to produce and follow the reasoning sketched out.

APPENDIX

Comment c on page 1 leads into some interesting channels that have little to do with the rest of this unit, but which might be worked up into something interesting in their own right,

The comment is made that there are more odds than evens on the line. How many more ? A check reveals that after each subdivision there are almost exactly twice as many odds as evens. An even more careful check will reveal, as Tim Barclay commented, that the pattern of odds and evens is highly regular, then after every subdivision we seem to have the following pattern:

...OEOOEOOEOOEOOEOOEO...

A little checking will show that this pattern is indeed self-reproducing, and that if we put between every two letters the parity of their sum, we will get back exactly the same pattern. There will always be twice as many odds as evens.

What happens to other patterns if we do the same thing? Are there other reproducing patterns? Does every repetition of the interpolation process?

For instance, the pattern ...EE EEEEEEEEE... is obviously self-reproducing, while the pattern ... OOOOOOO... looks like ...OEOEOEOEOE ... after one interpolation, and like ... OOEOOEOOEOOEOOE ... after a second interpolation. We know that this pattern is self-reproducing. The pattern ... OOOOOOOOEEEEEEEEE ... crowds the E's to the right with successive interpolations and tends to look more and more like ... OOEOOEOOEOOE... Do all patterns tend toward this one?

Instead of interpolating the sums between the original symbols, what happens if we replace the original pattern by the pattern obtained from summing every two adjacent symbols? The pattern ...EEEEEEEEE.... is still

fixed, and so is ... OOEOOEOOEOOEOOE... . The pattern ...OOOOOOO.....
now goes to ... EEEEEEEE... . after one application of the rule. The pattern
...OOOOOOOOEEEEEEE... tends to look more and more like ...EEEEEEEEEEE...
with successive applications of the rule.

Can you think of other interesting patterns? Can you think of other interesting transformation rules to use? In the appendix to the unit on Fibonacci numbers handed out a couple of days ago, we were concerned with patterns like OOUOUOOUOUO, having no connection with odd or even integers. Can all this playing with patterns be generalized somehow?

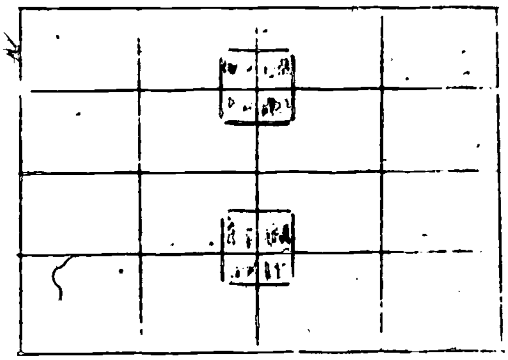
Short Investigations Presented With Discussions of Possible Outcomes

FOUR SHORT INVESTIGATIONS USING ARRAYS OF SQUARES AND CUBES:

A STUDY OF PATTERNS IN NUMBER SEQUENCES

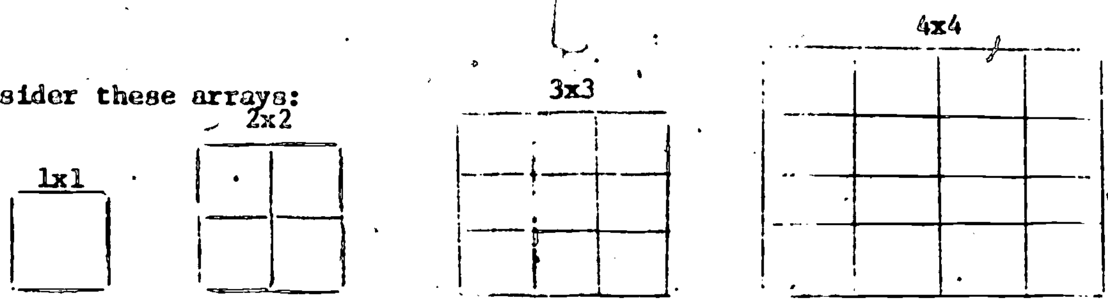
Note: The teachers' supplement which suggests some "mathematical outcomes" from these "non-mathematical beginnings" is illustrative of things which may be discovered during investigations of the problems. However, unexpected discoveries by students are especially to be encouraged and the teachers' supplement should be continually enlarged to include these new potential outcomes.

Short Investigation: 0 Squares in Arrays of Squares



(1) Given the figure at left, how many different squares* can you see? Do not look now, but at the bottom of the page the answer is given to check yourself.**

Consider these arrays:



(2) Can you tell the number of different squares* to expect in each?

(3) Suppose you have a 5 x 5 array, can you predict the number of squares in this? Check your prediction by an actual count.

(4) Now predict the number of squares in a 6 x 6 array. And for a 10 x 10 array? What is the pattern? Can you give a formula to describe this pattern?

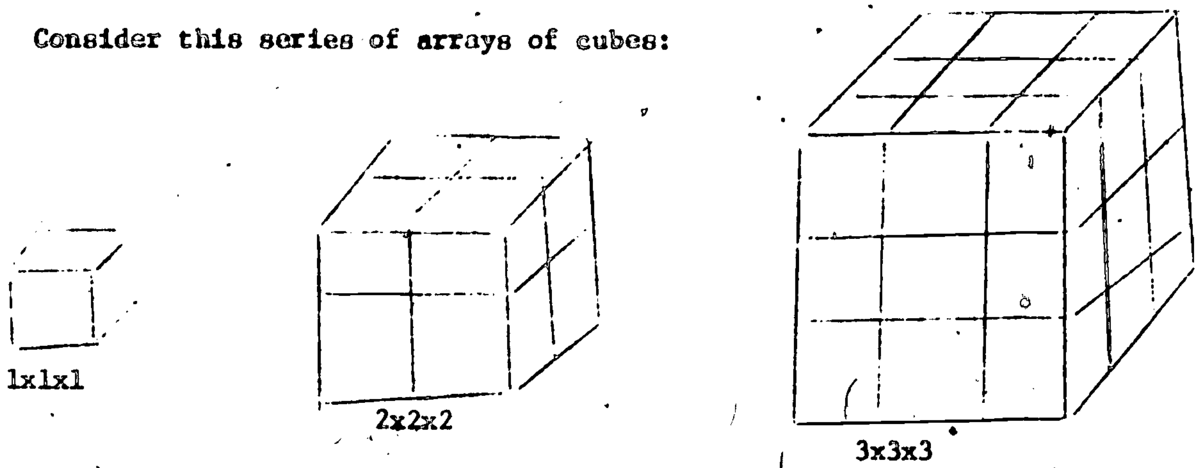
*Different squares may be interpreted to mean squares that include at least one square not in another square. Other interpretations of different squares will lead to different, but interesting results.

**It is possible to see forty different squares including ten which are shaded.

Short Investigation: 80 Cubes in Arrays of Cubes

Note: This investigation builds an insight developed in SI-1.

Consider this series of arrays of cubes:



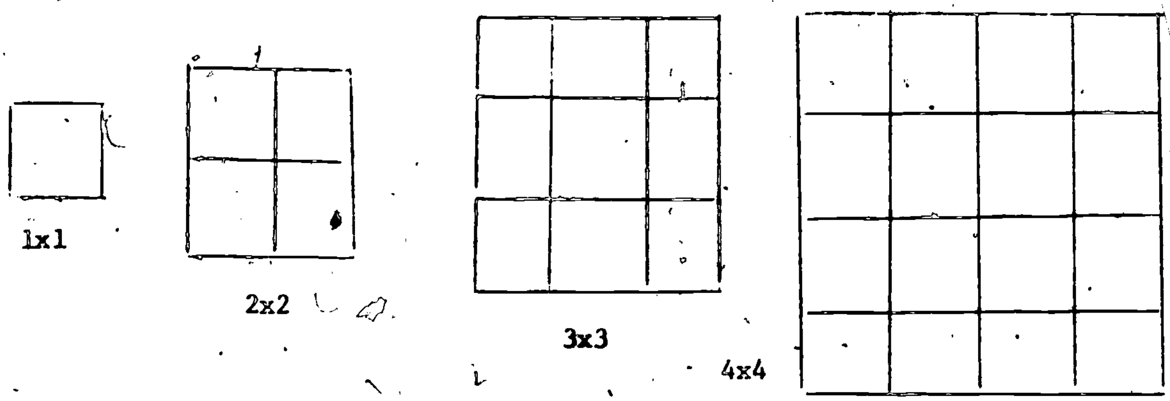
(1) Can you tell the number of different cubes that can be distinguished in each array?

(2) Suppose you have a 4 x 4 x 4 array, can you predict the number of cubes in it? Check your prediction and explain how you counted the number of cubes in the 4 x 4 x 4 array.

(3) Now predict the number of cubes in a 8 x 8 x 8 array. What is the pattern? Can you give a formula to describe this pattern?

Short Investigation: 00 Rectangles in Arrays of Squares

Consider these arrays of squares:



(1) Can you tell the number of rectangles (including squares) that you can see in each of the arrays?

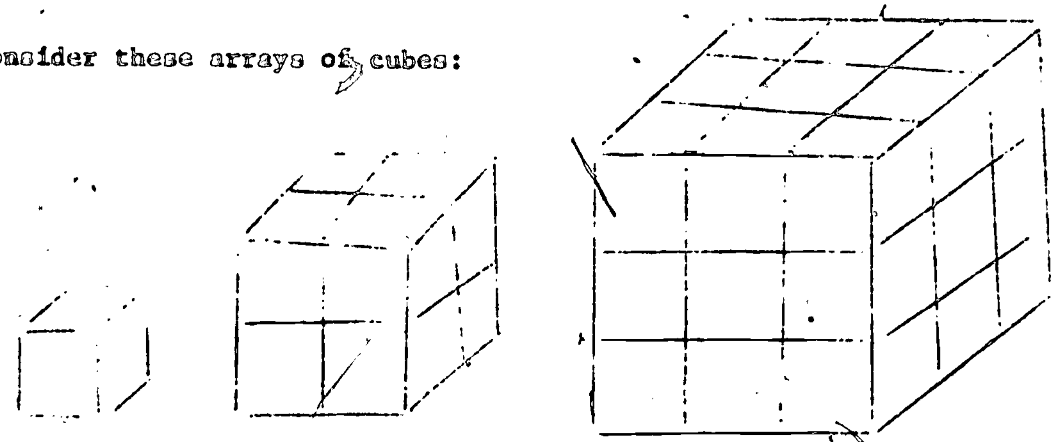
(2) Suppose you have a 5 x 5 array, can you predict the number of rectangles in this?

(3) Now predict the number of rectangles in a 7 x 7 array. What is the pattern? Can you give a formula for the number of rectangles in any nth array (an n x n square array of squares)?

Short Investigation: 00 Parallelopipeds in Arrays of Cubes

Note: A collection of one inch multi-colored cubes which may be used to physically construct cube arrays of cubes will be helpful.

Consider these arrays of cubes:



1x1x1 2x2x2 3x3x3

(1) Can you tell the number of different parallelopipeds, N, (including cubes) that can be distinguished in each array?

(2) How many parallelopipeds in a 4 x 4 x 4 array? What is the pattern? Can you give a formula to give the number parallelopipeds in any nth array?

* A parallelopiped is a kind of "Three dimensional rectangle". It is a three dimensional figure whose faces are rectangles.

Outcomes: Teacher's Supplement to Investigation of Arrays of Squares and Cubes

Note: The difficulty of the Short Investigations is suggested by the stars preceeding each title as follows: 0 A Challenge, 00 Difficult, and 000 A Brain Buster

Short Investigation Discussion: Squares in Arrays of Squares

(1) In the given array it is possible to see: 16 1 x 1 squares; 9 2 x 2 squares; 4 3 x 3 squares and 1 4 x 4 square plus 10 shaded squares for a total of 40 different squares.

(2) The total number of squares that can be seen, N, in each of the arrays is: 1 x 1, N = 1; 2 x 2, N = 5; 3 x 3, N = 14; and 4 x 4, N = 30.

(3) The sequence 1, 5, 14, 30 has the difference between successive pairs of terms 4, 9, 16. The next member of this difference sequence would apparently be 25 suggesting that the next member of the original sequence be 30 + 25 or 55.

An actual count of the squares in a 5 x 5 array verifies this.

(4) A prediction for a 6 x 6 array would be 55 + 36 = 91. A prediction for a 10 x 10 array is $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$

$= 1 + 4 + 9\ 16 + 25 + 36 + 49 + 64 + 81 + 100 = 385$

A generalized formula N = $\begin{pmatrix}\text{Number of Squares} \\ \text{can see in n x n array} \\ \text{of squares} \quad =\end{pmatrix}$ $1^2 + 2^2 + 3^2 + \ldots (n-1)^2 + n^2$

This is a "recursive formula" which essentially requires the answer to the (n-1) x (n-1) array before finding the answer to the n x n array. However, it is possible to express N in terms of n in a "discrusive formula" that gives the value of N directly when a value for N replaces N. This is: $N = \frac{(n)(n+1)(2n+1)}{6}$. This formula may be obtained from the sequence of values of N by using the method of finite differences.

Short Investigation Discussion: Cubes in Arrays of Cubes

(1) In the 1 x 1 x 1 array 1

In the 2 x 2 x 2 array 9

In the 3 x 3 x 3 array 36

(2) In a 4 x 4 x 4 array, there will be 64 1 x 1 x 1 cubes, 27 2 x 2 x 2 cubes, 8 3 x 3 x 3 cubes and 1 4 x 4 x 4 cube to yield a total of 100 different cubes.

(3) A prediction for the 8 x 8 x 8 array is:

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 = 1296.$

The pattern is that, when the length of a side of the array of cubes increases from n to (n+1), the number of cubes that can be seen increases by $(n+1)^3$.

Teacher's Supplement to Investigation of Arrays (Cont'd)

Thus N = Number of cubes that can be seen in a n x n x n array of cubes $= 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3$

This is a recursive formula. It is interesting to look at the sequence of N for which we have 1, 9, 36, 100 ... We may recognize that this is $1^2, 3^2, 6^2, 10^2$ ... which is sequence of squares of the triangle numbers 1,3,6, 15 ... (these are the successive partial sums of the counting numbers). Since the nth triangular number is given by the expression $\frac{(n)\ (n+1)}{2}$ we have the discursive formula for the number of cubes seen in an array, $N = \frac{n(n+1)}{2}^2$.

Short Investigation Discussion: Rectangles in Arrays of Squares

(1) The number of different rectangles that can be distinguished, N, in the sequence of arrays is 1 x 1, N = 1; 2 x 2, N = 9; 3 x 3, N = 36; 4 x 4, N = 100

As an example of the mode of analysis, let us consider the count of the number of rectangles in the 3 x 3 array of squares.

There would be the squares which number $1^2 + 2^2 + 3^2 = 14$ and in addition there would be

1 x 2 rectangles	6
1 x 3 rectangles	3
2 x 1 rectangles	6
3 x 1 rectangles	3
2 x 3 rectangles	2
3 x 2 rectangles	2
	36

(2) In a 5 x 5 array of squares we would predict the next in the sequence 1, 9, 36, 100 ... This sequence may be recognized as the sequence of squares of triangular numbers which occurred in SI-2 and thus the next term squared which is 225.

(3) The number of rectangles predicted in a 7 x 7 array of squares would be the seventh triangular number, 28 = (1 + 2 + 3 + 4 + 5 + 6 + 7), squared which is 784.

The pattern is that the number of rectangles, N, in any nth of square arrays of squares is the nth triangular number squared. And the formula for this:

$$N = \left(\frac{(n)\ (n+1)}{2} \right)$$

The Tower of Hanoi Puzzle and Variations: Questions

(1) Using a regular tower puzzle, what is the minimum number of moves to putting a larger disk on a smaller one?

(2) Starting with a pile on any number of disks alternating in color, say green and brown, what do you find if you restack the disks in two piles on two other posts so that the piles are all green or all brown?

(3) Part (1) again but with four posts.

(4) Try part (1) again but with six posts. How many moves for say 100 disks?

(5) Try part (2) again. Here just separate into two piles, one green, the other brown. (One color may be stacked on the peg on which the stack originated)

In each case find the minimum number of moves and write a formula that will yield this.

Tower of Hanoi -- and Interesting Variations: Discussion

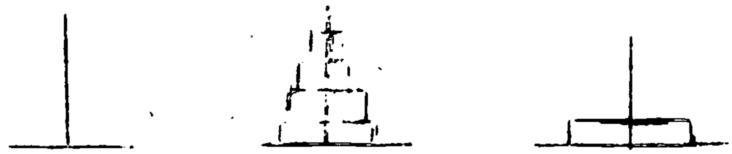
(1) Students find this Tower of Hanoi game quite interesting and, in fact, make as an additional feature the speed element of moving a given number of pegs. I recommend that six or more games be made available to the class several weeks before bringing up the following topics. This will enable the students to learn how the game is played before asking the mathematical question.

In the standard game of Tower of Hanoi, using the rules which accompany the game, how many moves are required to move the discs from one peg to another? Can you generalize to n pegs? In those classes where back ground is sufficient, the formula for the number of moves can be proved by mathematical induction. Many times students will arrive at the proof with no help from the teacher.

Shall we try it? How many moves if you start with

1 disc 1
2 discs 3
3 discs 7
4 discs 15
5 discs ?

Perhaps someone will consider the following.



To solve 5 I must first move four discs to a single peg (15 moves), then move the fifth disc to the correct peg, and now my problem is to move four again (15 moves again)

$$15 + 1 + 15 = 31$$

Any conjecture --

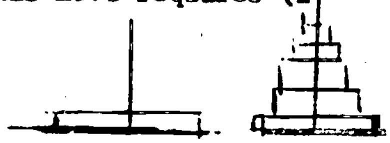
n discs require $2^n - 1$ moves or n discs require twice the number of moves required for (n-1) discs + 1, i.e., n discs requires $2(2^{n-1} - 1) + 1$ moves.

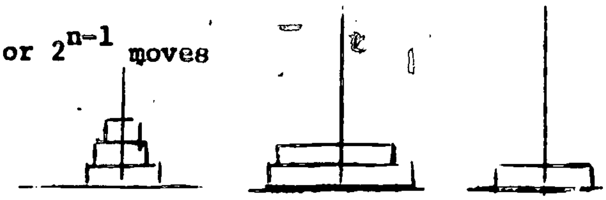
(2) Using alternating colors starting with any number of discs on one peg, what is the minimum number of moves required to separate the colors into two stacks of descending size on the other two pegs?

No. of discs	No. of moves
1	1
2	2
3	5
4	?
5	
10	731

The technique for getting the expression for n moves is to to use Part (1) for help in solving the new question. For example, for n discs

This move requires $(2^{n-1} - 1) + 1$ moves or 2^{n-1} moves





This move requires $2^{n-3} + 1$ or 2^{n-3} moves.

Conjecture: n moves require

$2^{n-1} + 2^{n-3} + 2^{n-4} + 2^{n-6} + 2^{n-7} + \ldots + (2^1 \text{ or } 2^0)$ depending on whether n is even or odd. If n is even the last term is 2^0; if n is odd, the last term is 2^1.

(3) Using the original rules, but adding one more peg (making four) what is the minimum number of moves required to stack n pegs?

The following bookkeeping might aid in this question.

No. of discs	No. moves 3 pegs	No. moves 4 pegs
1	1	1
2	3	3
3	7	5
4	15	9
5	31	13
6	63	17
7	?	?x
8	?	?

Conjecture:

x for 7 will be the minimum calculation from the following:

7 = 4 + 3;   2(9) + 7, or 2(5) + 15

7 = 5 + 2;   2(13) + 3, or 2(3) + 31

7 = 6 + 1;   2(17) + 3, or 2(1) + 63

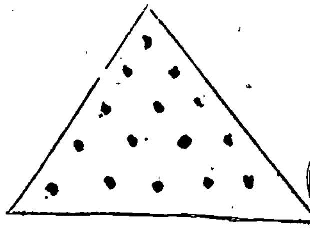
x =

(4) and (5) are not discussed in order to whet your own creativity; however, E is nice if worked before B and then used in the solution for (2).

Short Investigations: Peg Puzzles, Patterns and Equivalence Classes

Investigations starting with the Triangle and other Peg Type Puzzles

1. Using a triangle puzzle of the type that has pegs in a peg board --



a) Start with one empty space in one corner.

b) Make jumps without removing the peg jumped.

c) Get a vacancy in each of the places.

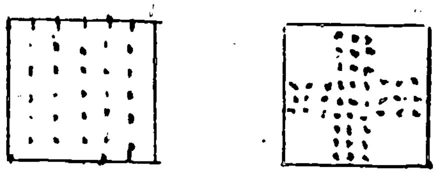
d) How many unique starting places are there?

e) How many vacancies are associated with each starting place?

f) What do you get for the next size triangle? etc.

g) Generalize for a triangle with n rows.

2. Using the square peg board, or any other peg board, such as the following:



a) Try the same problem with any of the peg boards.

b) Generalize to an n x m or n x n.

3)

a) Go on to cubes (it helps to use colored cube blocks here).

b) Generalize for n-dimensional cubes.

4. Try tetrahedrons.

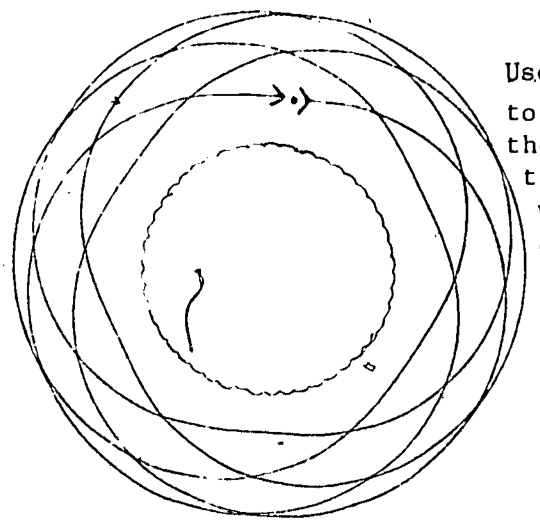
5. It might be interesting to have students make a random type peg board, and answer the same type of questions as above.

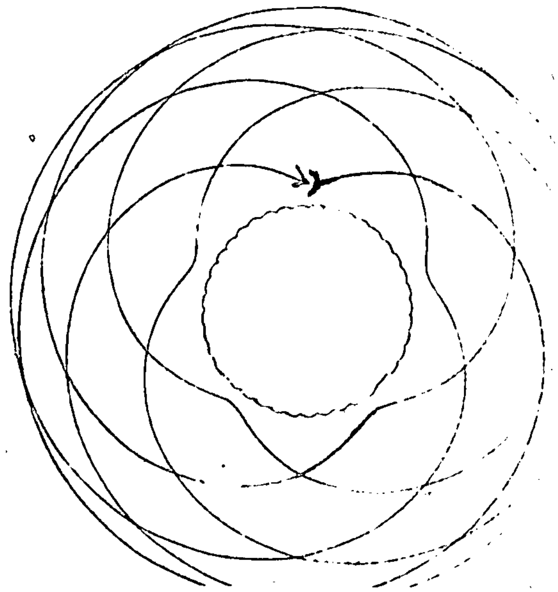
6. How many different jumps are possible on the triangle puzzle? (There might be several interpretations of this question.)

SPIROGRAPHS & GCD'S

Something to try with the Spirograph (kenner #401)

Take two plastic gear wheels from the Spirograph set, say the #40 and #48, pin one of them down on some paper on top of cardboard and move the other wheel around it with a pen in one of the holes. Carefully note your starting point. Then switch the wheels; see what happens!





Let x = the no. of times the no. 40 wheel goes around the no. 48 wheel. 40/x = ?

Let y = the no. of times the no. 48 wheel goes around the no. 40 wheel. 48/y = ?

Any conclusions?

Try this activity with some other Spirograph wheels. Pick any two; what do you think will happen? Can you predict beforehand how many times one wheel will go around the other?

How many times will the #24 wheel go around the #36 wheel?

How many times will the #36 wheel go around the #24 wheel?

How many times will #45 go around #75, and vice versa?

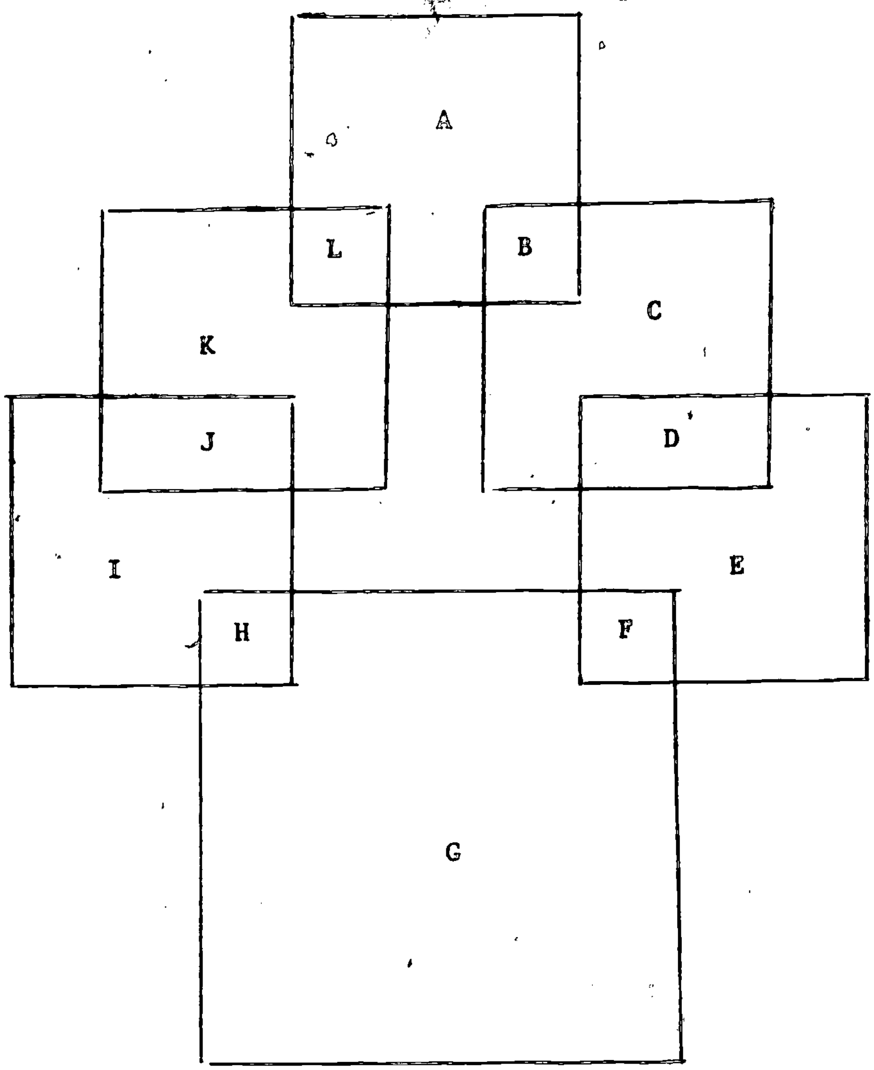
How many times wil #40 go around #80, and vice versa?

Not so easy: How many times will the #63 go around #64, etc.?

Do you think you can make up a theory or an explanation of how this works?

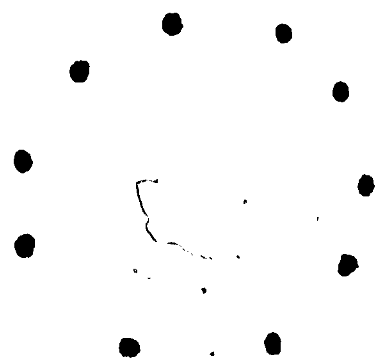
Short Investigation: The Box Problem

In place of the letters below place the numbers 1-12 in such a way that the sum of the numbers in each box is the same and greater than 15.



Short Investigation: One Hundred Dots

Look at the circle of dots, below, and imagine that you had to draw all the possible lines (straight) between them. How many lines would that be?



Problem: Suppose the circle had 100 dots on it, instead of the ten above, how many lines would that be connecting all the dots?

Problem: Suppose, instead of looking at the straight lines drawn between the dots, you counted all the different triangles you could make, using these dots as vertices, how many would that be? (You can first look at the case where there are ten dots; and then if you think you are really good at this, look at the case where there are one hundred dots on the circle.)

Problem: Same idea as above, but now you want to count all the quadrilaterals (four sided figures) with vertices among the dots.

Problem: Would it make any difference if the dots were not on a perfect circle; maybe an ellipse (a slightly squashed circle)? Try putting the dots (ten, say) on different kinds of boundaries, and see what happens to the answer.

Short Investigation: Squares, Cubes and Averages

Patti Griffith, a teenager friend of mine that has been in our class on several occasions, once had a problem in math to do. It involved squaring various numbers:

$5^2 = 25$, $6^2 = 36$, $7^2 = 49$, etc.

I tried to confuse her, after she had gotten 5^2 and 7^2, and was about to get 6^2, by reminding her that 6 was the "average" of 5 and 7. And that maybe that would give her a quick way to get the square of six. She leaped at the "clue". Then, all I have to do is average 25 and 49, and I get 6^2?" 25 + 49 = 74, 74/2 = 37. Whoops! What happened?

Why don't you pick several examples and see what happens, if you see this "short cut method". Given: $8^2 = 64$ and $10^2 = 100$, what is 9^2? Given: $11^2 = 121$, $13^2 = 169$, what is 12^2? Make up some examples of your own.

What happens when the first two numbers -- the ones you will square -- are not two units apart? Suppose you were given: $9^2 = 81$ and $13^2 = 169$, what is 11^2? (I have picked them further than two apart, and so that the average is still a whole number. It gets more confusing with fractions. Right?)

Suppose you were given: $7^2 = 49$ and $15^2 = 225$, what is 11^2? OR: $2^2 = 4$, and $14^2 = 196$, what is 8^2?

Is there any pattern to this squaring and averaging business? Can you formulate any rule?

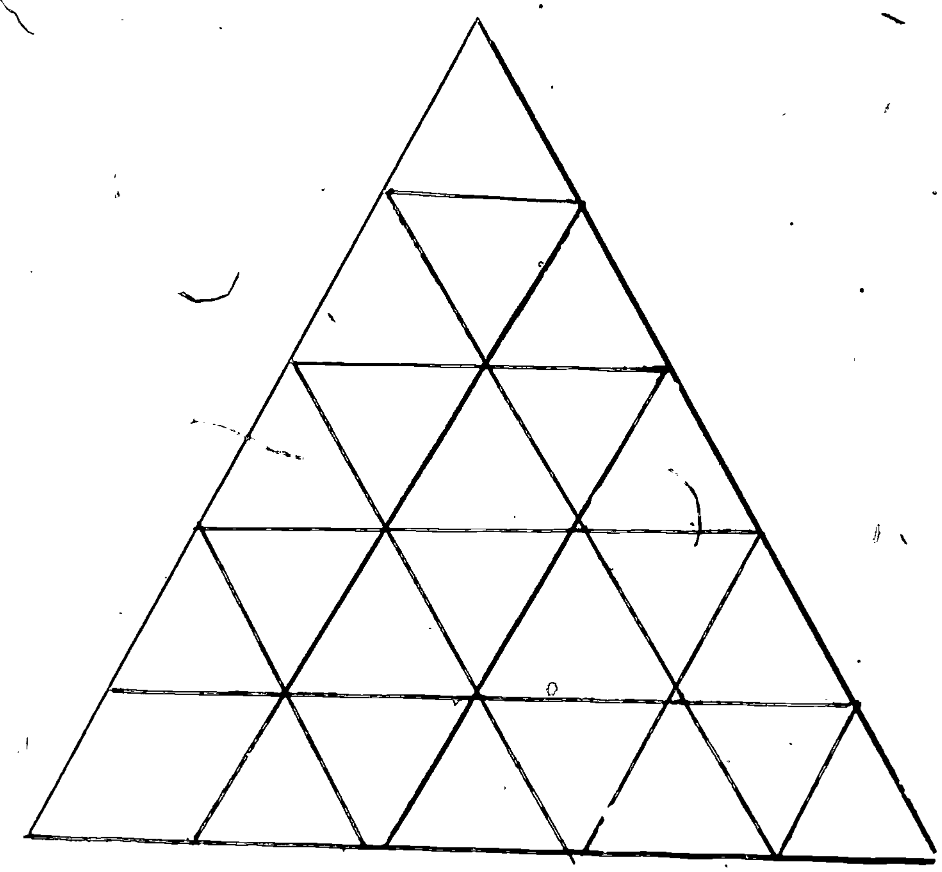
Problem: What about cubing? Does the averaging method work for cubing? Try it and see. Make up some system for cubing.

Problem: What about square roots, does the averaging method work there?

Problem: What about the problem of multiplying a number by seven: does the averaging method work there? Given: 17 x 7 = 119; and 15 x 7 = 105, what is 16 x 7 = ???? (Or a similar problem.)

Short Investigations: An Array of Triangles

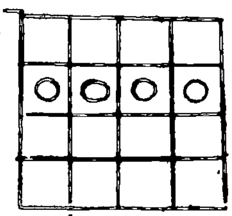
1. How many triangles can you see in the array of triangles below?
2. How many trapezoids can you see?
3. How many line segments can you see?
4. How many parallelograms can you see?
5. How many dots can you see
6. How many regular hexagons?
7. Can you predict how many of each type of geometric figure you would have for different size arrays?



Investigations with 3 Dimensional Tic-Tac-Toe Games

1. How to play 3-D Tic-Tac-Toe

The best commercial game is Parker Brothers' Qubic. It consists of four platforms, one above the other, each of which has four rows of four spaces.



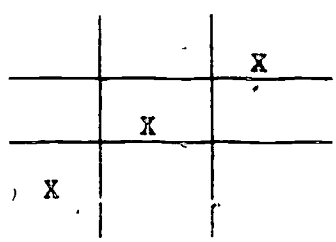
The object of the game is to get four chips in a row just as players in regular (2-D) tic-tac-toe try to get three in a row. There are many ways to win. Four in a row, 4 in a diagonal, through all four plat forms 4 in a column, a main diagonal. Try them out and see. Play the game a few times.

2. Investigation. How many different ways are there to win? Try finding them on the game.

3. Investigation. What is the greatest number of chips of a single color you can place in the game without getting four in a line? It must be greater than or equal to 27; do you know why? Can you find a number which it is clearly less than or equal to?

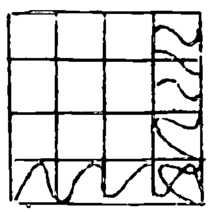
4. Why is the game 4 x 4 x 4? Why is it not 3 x 3 x 3?

We might try a different question. Remember ordinary tic-tac-toe; it is 3 x 3



and you must get 3 in a line in order to win. Try playing a 2 x 2 game of two dimensional tic-tac-toe.

How many in a line would be required to win? Who wins? Can you work out a strategy? Which is better the 2 x 2 or 3 x 3 game. Why? Now try playing the 3 dimensional game as a 3 x 3 x 3 instead of 4 x 4 x 4. Revove one of the platforms and lay paper strips to cover over an outside row and column on each platform.

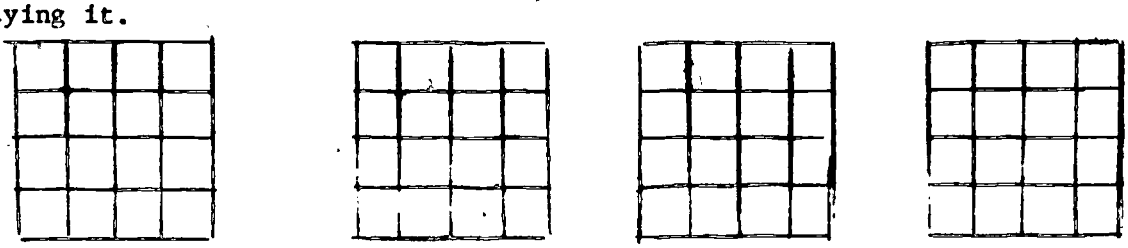


Now try playing the game with someone. How many in a row are needed to win? Who wins? Can you work out a strategy? Which do you think is better the 3 x 3 x 3 or 4 x 4 x 4 game?

What can you infer about the length of a side of a game with respect to the dimension of the game? Could there be a one-dimensional game? Why?

5. Look at The Top Platform. It forms a plane. On that plane you have columns, rows, and diagonals of 4 spaces. How many more planes equivalent to that plane can you find?

6. Could you play a 3 dimensional tic-tac-toe in 2 dimensions, i.e., on a flat piece of paper? Try projecting the game on to the paper and then playing it.

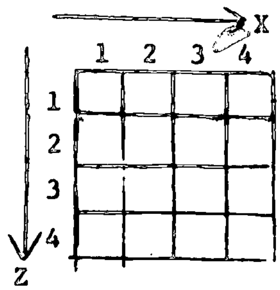


bottom or 1st platform | 2nd | 3rd | 4th or top

Is it easier or harder to play this way? Can you tell if someone has won?

7a) Suppose now we describe the game with a coordinate system.

There are four platforms in the game. Let us look at the bottom or first platform.



Starting in the upper left corner, let us label the spaces horizontally 1, 2, 3, 4 just as we might on the x-axis in Cartesian coordinates. Then label the spaces down 1, 2, 3, 4. Each platform, too, would have a number counting from the bottom upward 1, 2, 3, 4 (y-axis). Then every point would have a coordinate (x, y, z).

There is no need to impose "x, y, z" on the students. A better approach would be simply to ask them to make up a coordinate system of their own so that they could describe the position of any chip just by giving just 3 numbers. Maybe they would come up with A, B, C or more likely R,C,L; Row, Column, Level.

7 b) Once some kind of descriptive coordinates can be agreed upon, the interesting mathematics begins.

Practice putting chips at varying places in the game and letting students determine the coordinates.

Now give them the coordinates for four chips and ask if they can decide whether or not it is a winning combination.

For example in the R, C, L System:

1) Is (1, 1, 1,), (1, 1, 2), (1, 1, 3), (1, 1, 4) a winner?

2) How about (1, 1, 4), (2, 2, 3), (3, 3, 2), (4, 4, 1) ?

3) Is (1, 2, 3), (2, 3, 4), (4, 3, 2), (3, 2, 1) a winner?

4) (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4)
What about this?

Some interesting generalizations can be made about winning combinations and the patterns of the numbers in the coordinates. A good way to analyze these is to "stack" the coordinates., i.e.,

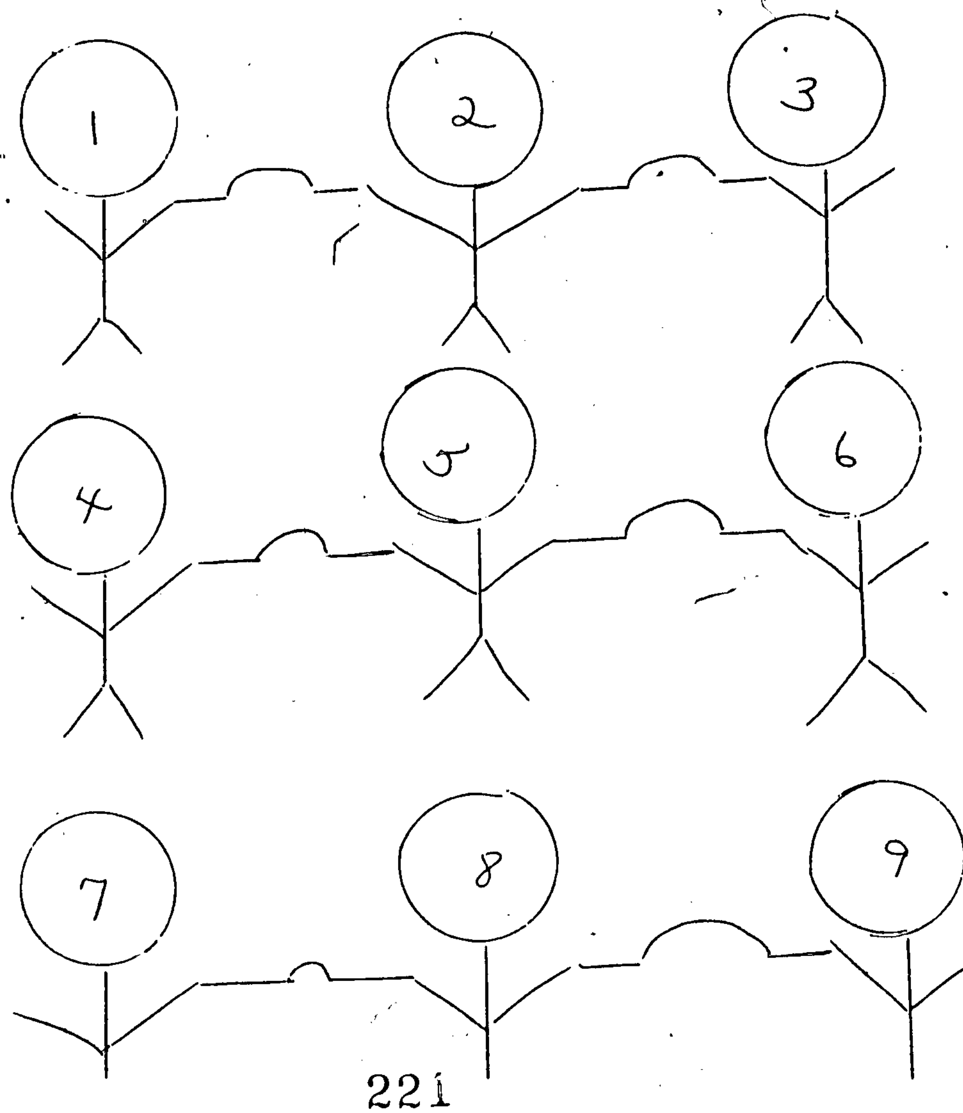
1) 1, 1, 1,	2) 1, 1, 4	3) 1, 2, 3	4) 1, 1, 1
1, 1, 2	2, 2, 3	2, 3, 4	2, 2, 2
1, 1, 3	3, 3, 2	4, 3, 2	3, 3, 3
1, 1, 4	4, 4, 1	3, 2, 1	4, 4, 4

Try writing down as many of the winning combinations in this way as you can. (Note: not all of the sets of coordinates given above are winners). Keep doing this until you see some patterns. If you can generalize and make up some theorems, do so. Then test them and see if you can find any counter examples. The proofs for some of the theorems you can make up from this exercise are not hard. Try proving your theorems.

Short Investigations: The Handcuffed Prisoners

Once upon a time there were nine prisoners of particularly dangerous character who had to be carefully watched. Every weekday they were taken out for exercise, handcuffed together, as shown in the sketch below. On no one day in any one week were the same two men to be handcuffed together. It will be seen below how they were sent out on Monday. Can you arrange the men in three's for the remaining five days?

It will be seen that number 1 cannot be handcuffed to number 2 (on either side), nor number 2 with number 3, but, of course, number 1 and number 3 can be put together.



Short Investigations: Bees, Rabbits, and One-Way Streets

A. B-Biol-o-Gee:

It is biologically true that a drone has only one parent, a queen, whereas a queen bee has two parents, a queen and a drone. Given a particular drone, how many ancestors did he have in the seventh generation back? What is the number of ancestors in the n^{th} generation back? What is the total number of ancestors for the previous n generations.

B. The Fibonacci Rabbit Problem:

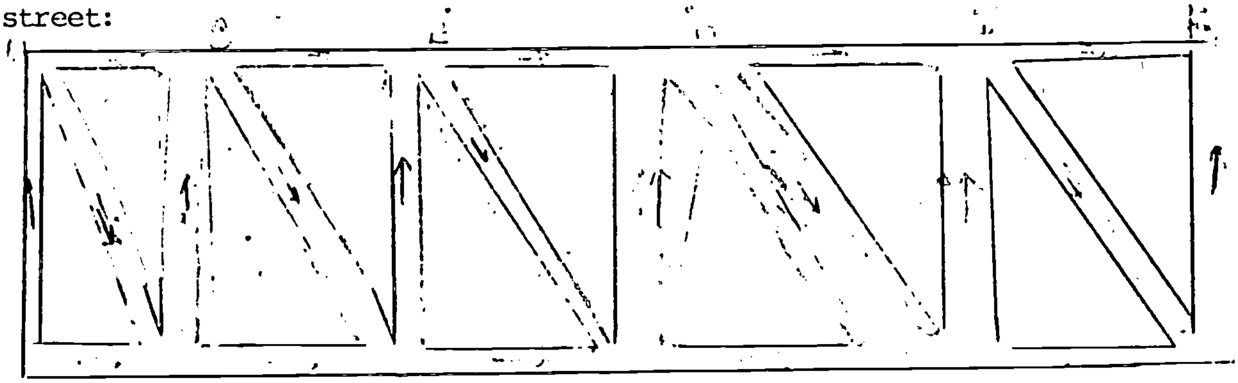
You put a mated pair of newly born rabbits into a cage and leave them -- feeding them, of course -- for a year. When you return, how many rabbits are there?

The facts about rabbits are: It takes two months from the birth of a mated pair of rabbits to the time that they produce their first litter. Every litter is exactly one mated pair. After the first litter, parent rabbits produce a litter every month.

Can you now compute how many rabbits there will be in the cage at the end of one year ------365 days, to be precise?

C. A One-Way Street Problem:

Suppose that a City has the following streets, where the arrows along each street represent the single direction you must follow when driving along that street:



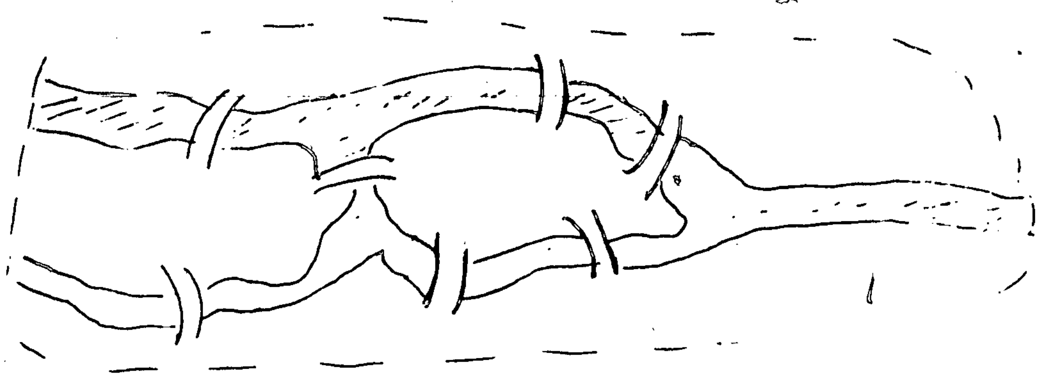
One path you might take to get from start "S" to final position "K" might be: S-A-C-E-F-G-I-J-K. And, there are many others. How many?

Problem: How many different routes are there from S to K, consistent with the street directions?

Suppose the map was extended so that five more square blocks were added to the city's boundaries — roughly doubling the size of the city — what then?

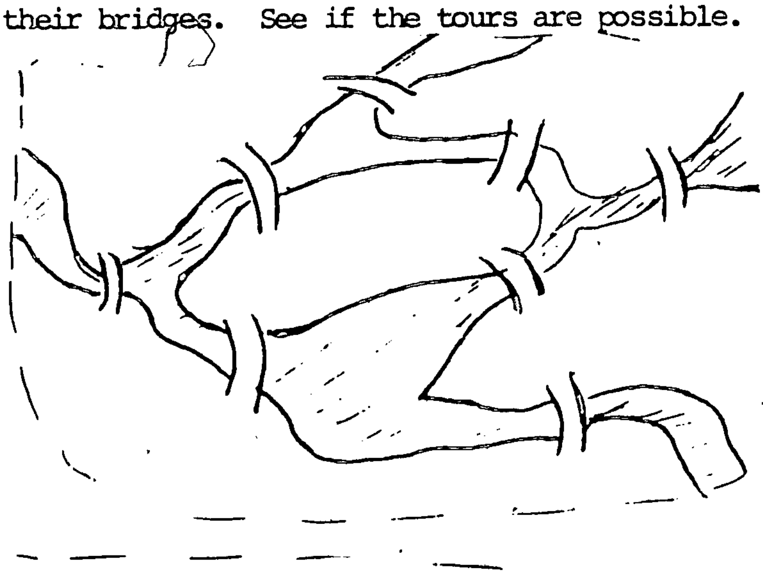
Short Investigation: Kongisberg Bridges

There is a famous "Konigsberg Bridge" problem that set mathematicians off creating the important field of mathematics called "Topology". Here it is:

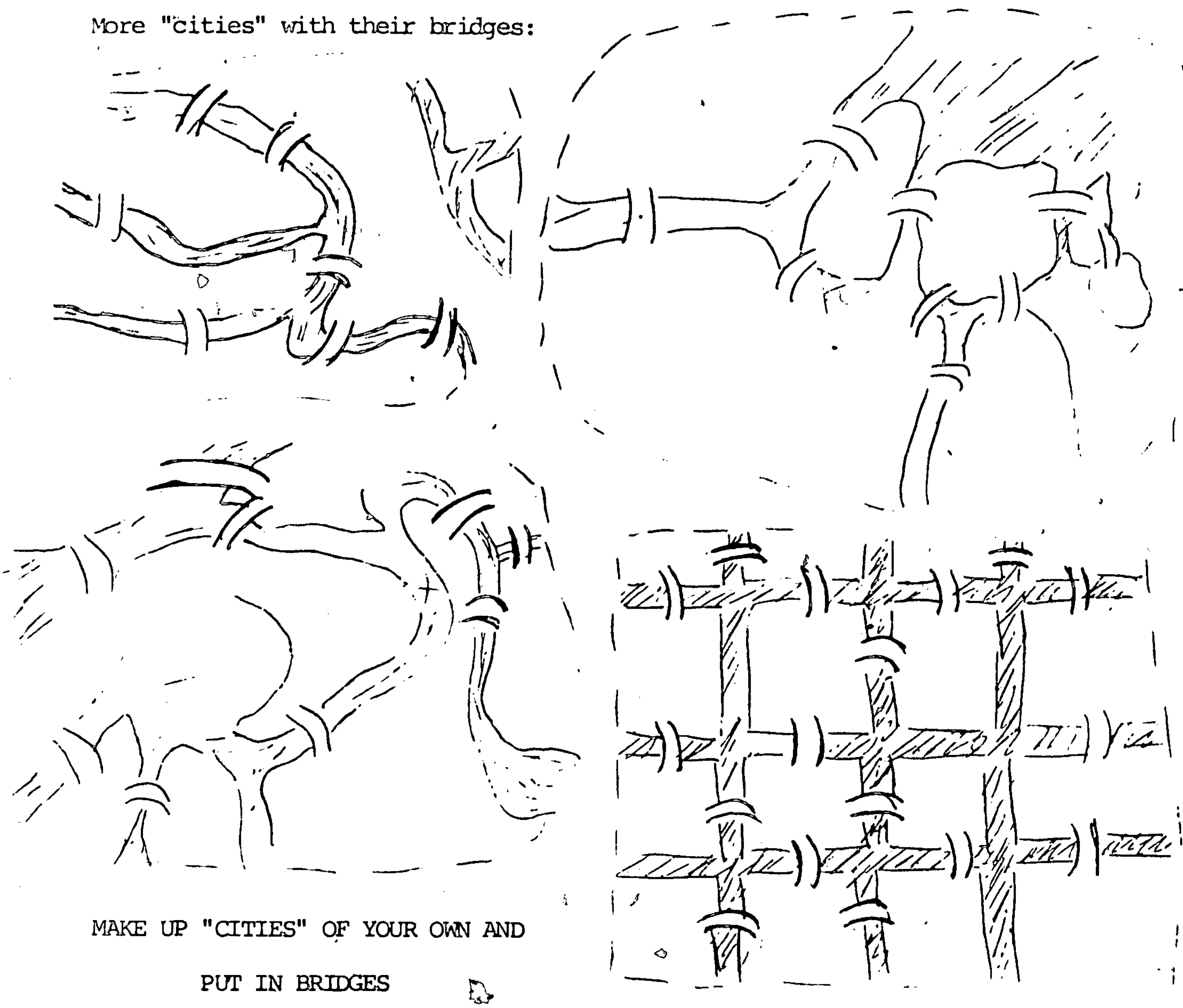


Problem: If you lived in Konigsberg, could you take a wlaking tour of the city and cross each of the seven bridges exactly once? Try it.

Here are some other "cities" with their bridges. See if the tours are possible.



More "cities" with their bridges:



MAKE UP "CITIES" OF YOUR OWN AND

PUT IN BRIDGES

Problem: Can you explain why, for some cities, there are tours, and for other cities there are not? Do you have any theory?

Problem: Are some of these maps really "the same"?

Problem: What is there about a city map that is essential, and what is really not important (as to whether there is or is not a tour)?

Short Investigations: Tree Graphs

Note: Tinker Toy or similar knob and dowel toys may provide helpful models in the investigations below.

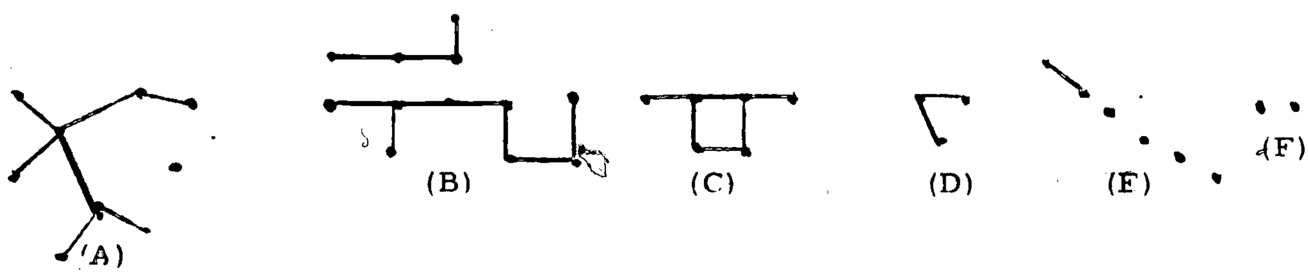
There are some special "graphs" called "tree graphs". First, some definitions:

A Graph: is a picture containing some points, and where some of the points are joined by lines, (not necessarily straight; and they can cross and it does not matter.)

A Tree Graph: is a graph in which there are no cycles, that is connected.

A Cycle: in a graph, is a series of points and lines that connect in a loop.

Some examples:



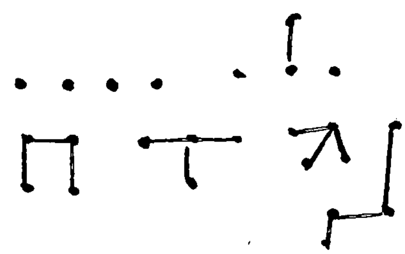
Which ones of the above graphs contain cycles?

(Connected: means that any two points are connected by a series of lines.) Are all these pictures graphs? How many of them are tree graphs? Can you find cycles in any of them? Is "F" a graph? Make up some pictures that are graphs, of your own and see whether they are actually tree graphs?

Tree Graphs:

Mathematicians study tree graphs as a special kind of graph. They look at all the "different" tree graphs one can make with a specific number of points in each graph. For example, here are all the tree graphs you can make with just four points:

Are these different?



It all hangs on how you want to define the word "Different". How?

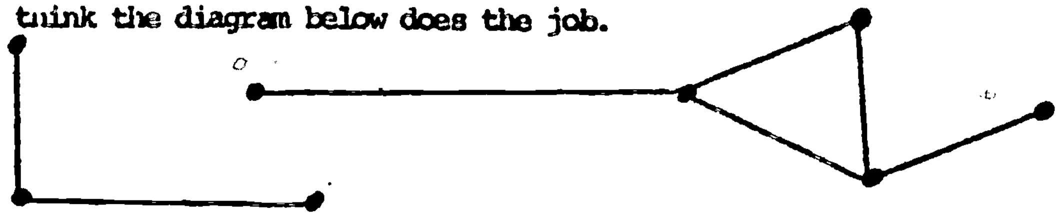
Problem: How many "different" tree graphs can you make with 6 points? ten points? etc.

Problem: How many graphs can you make with three points? Five points? 100 points?

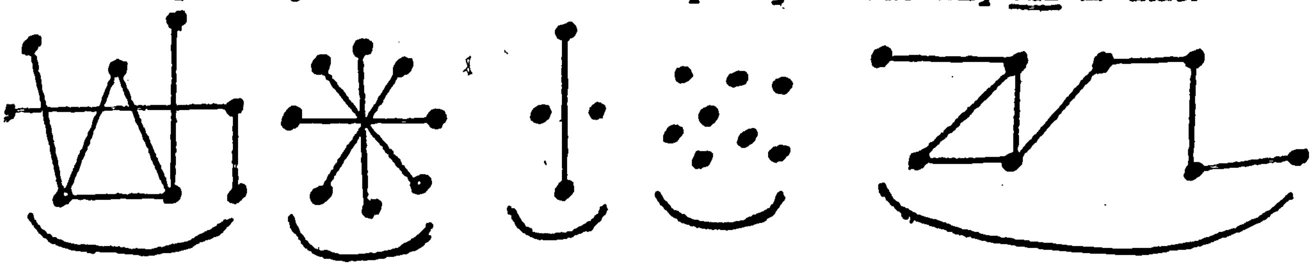
Short Investigations: Friendship Diagrams

I have a group of friends, (let's call them, Mr. A; Mr. B; Mr. C; etc. for the sake of argument), and I can relate the various friendships that exist among them. A knows B; B knows C and D — who know each other, as well; and D knows E; G knows F and H. Got that?

How much more straightforward and easily comprehended would it be if I could find a way to make a diagram that exhibits these friendships visually! See if you think the diagram below does the job.

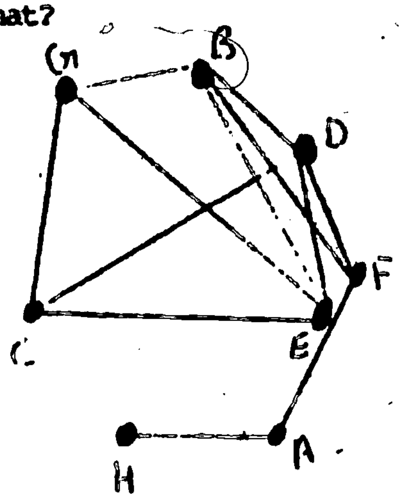


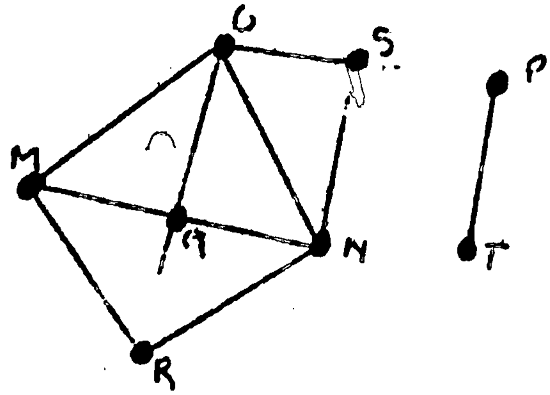
(Oops, I forgot to tell which points refer to which people: Can you do that?) Of course, some people make diagrams one way, some another: Here are several other "friendship diagrams" that are supposed to represent the same set of friendships among Mr. A, thru Mr. H. Do you agree that they all do that?



Can you explain why they do, or why they don't?

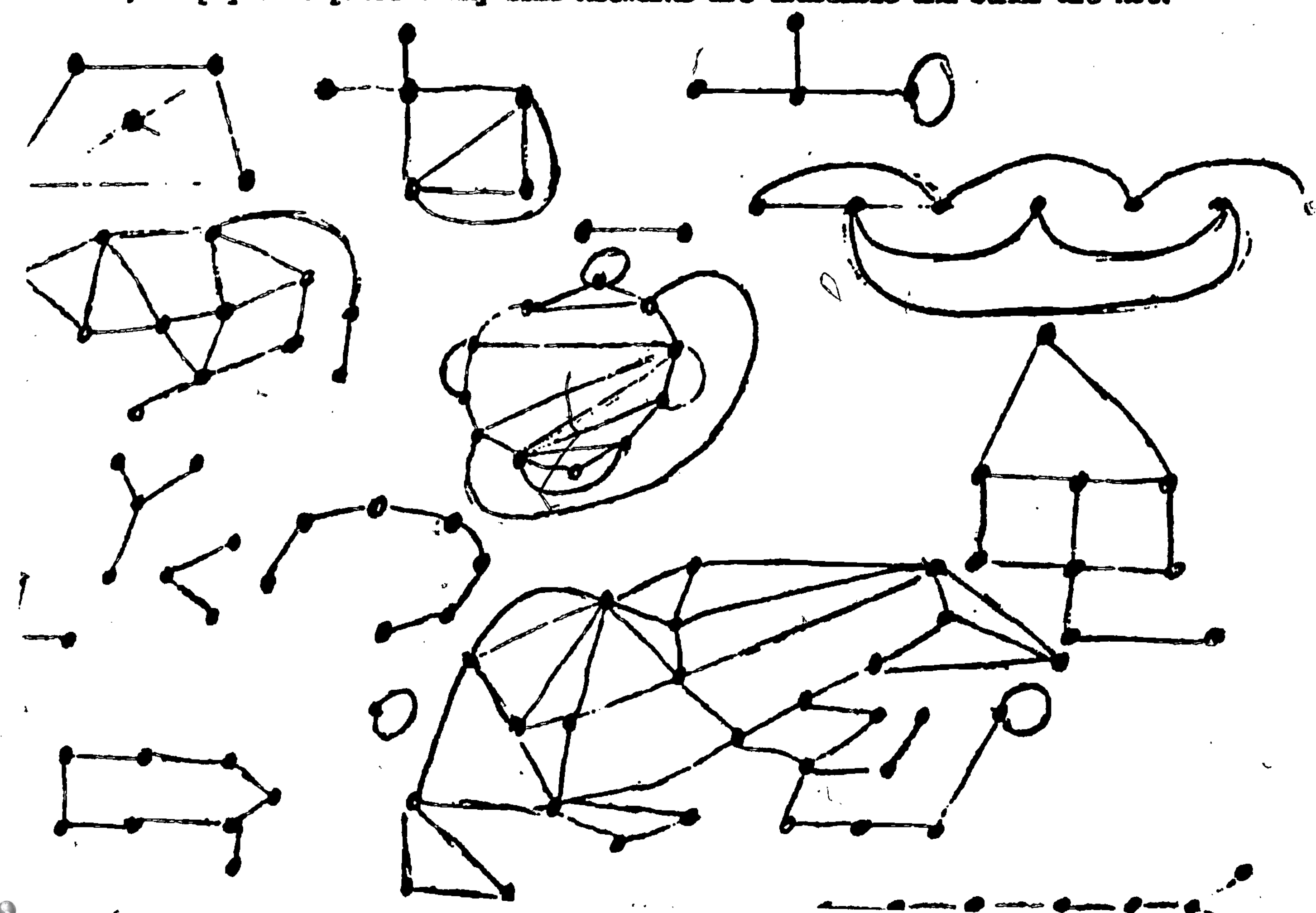
Problem: I am going to give a friendship diagram for a group of men (Mr. A thru Mr. H), and then a friendship diagram for a group of women (Miss M thru Miss T) and your job will be to marry them off! But, with the proviso that if two men know each other, their wives would too, and vice versa. Can you do tnat?





Short Investigations: Networks

(This is a sort of follow-up to a Number four; or it can stand alone.) At the bottom of this sheet are many "networks," composed of lines and points. For each of the networks, see if you can trace over the network in such a way that you cover each line exactly once. (no more, no less.) Let us call such networks "traceable"; the others, "non-traceable". See which of these networks are traceable and which are non-traceable. Can you see any reason why some are and some are not traceable? It may be useful to divide the traceable networks into those for which the pencil path can begin at any point in the network, and those for which there are a restricted group of points at which you must start (and end). See if that, or other ideas you may have, help you to predict why some networks are traceable and other are not.



INTERESTING QUESTIONS FOR EXPLANATION

1. On a 3-D Tic-Tac-Toe measuring 3 x 3 x 3, what is the most number of counters that you can put down without having any two lie on the same straight line? Is there a unique solution?

I have found two different ways of placing 6 counters. I don't know if these are examples of the maximum possible or if more than 2 solutions exist. I have not looked yet at a 4 x 4 x 4.

2. Given a 3 x 3 lattice of points, it is possible to draw four straight lines without lifting your pencil, and pass through each point once and only once. What is the minimum number of lines needed to pass through all the points in a 3 x 3 x 3 cubical point lattice?

I am still working on 2-D cases. I can pass through all the points of a 4 x 4 lattice using 6 lines. I don't know if this is the minimum number, or if there is more than one way to do this.

3. Two Ways to Cut a Pie if You Don't Care Whether all the Pieces Are the Same Size and Shape.

a) With each cut, which must be a straight line, get the most pieces possible. What is the maximum number of pieces possible with 10 cuts?

b) Join all pairs of points that are on the edge of the pie. What is the maximum number of pieces with 10 points?

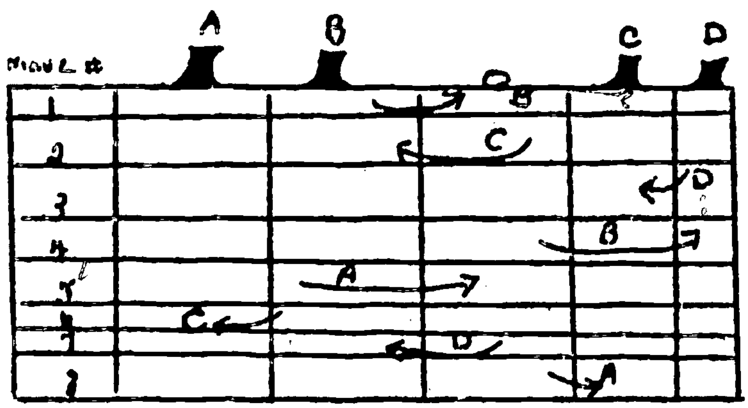
I have found geometrical patterns that come up as you draw successive cuts and have also developed a general formula for case a) and one that doesn't check for case b).

COMMENTS AND QUESTIONS ABOUT A PEG GAME*

(Called "Brain Buster" commercially; also the same as Carolyn Chesson's heads and tails penny game.)

In this game the rules provide that a piece can move ahead either by moving one slot or by jumping a piece of the opposite color. Jumping a piece of the same color is not permitted.

Let us try to imagine how a student might make a record of the developments in the two red — two white peg game:



Specifically, this procedure of recording keeps a complete record of how the game progressed. (It could be reproduced.) One could get students to invent such a schematic method of keeping this record by asking them to explain on paper, without words, in some diagram, how they did the two peg problem. There are 15 moves in the three peg problem or else I would exhibit a diagram for it. Actually, I don't know what the students would do; but I imagine something like this. It would be fun to see what they would do.

* NOTE: The game is called "Brain Buster" commercially and is equivalent to a heads and tails game suggested by Carolyn Chesson.

One of the things that comes out of such an explicit record is the total number of moves that each piece makes. It isn't at all obvious.

Piece	Number of Moves it Makes
A	2
B	2
C	2
D	2

A number of things appear here. The symmetry between the red and the white comes out in that the forward red piece moves just as often as the forward white piece, etc. In fact, all pieces move the same number of times.

What would be the piece by piece breakdown for a three - three peg game? Here are summarized the results of some fiddling around with the 2, 3, 4, and 5 peg games, counting each piece.

2	2	3	4
2	3	3	3
	2	3	4
2		3	3
2	3		4
	2	3	
	3	3	3
		3	4
		3	3
			4
			3

Does this help you guess what will happen for larger and larger numbers of pieces? Can you guess what the total number of moves for one hundred red and white pieces will be?

INTERESTING QUESTIONS REGARDING NETWORKS

A couple of days ago in the topology session, someone observed that if you take a point out of the middle of a line segment it is divided into two pieces. They then conjectured that this would happen with any figures made out of lines and curves. At this point someone else observed that you could remove one point from a circle and the remainder would still be in one piece. This leads to a whole series of fairly intriguing questions:

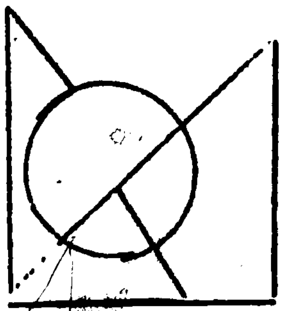
Question 1: Can you remove two points from a circle and leave the remainder in one piece?

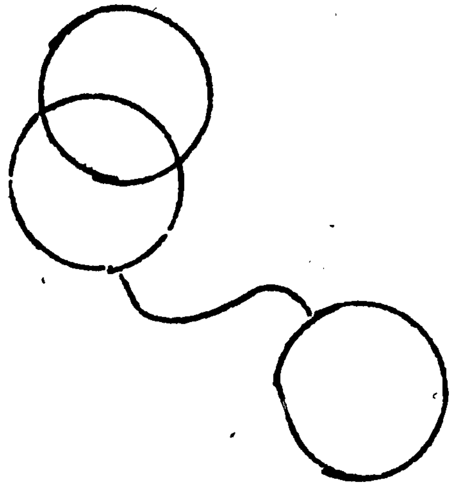
Question 2: Can you draw a figure from which you can remove two points and still leave the remainder in one piece?

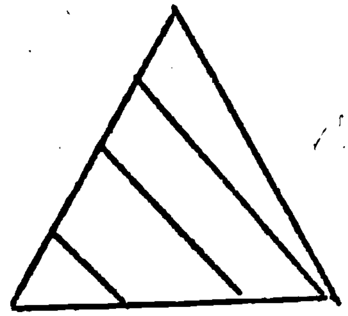
Question 3: Can you draw a figure from which you can remove exactly two points and still leave the remainder in one piece?

Question 4: Do all figures that have this property (that you can remove exactly two points and leave the remainder in one piece) look alike? How are they similar? How different can they be from one another?

Question 5: If you generalize the preceding three questions to removing 3, 4, 5, 6, ...points, what can you say? Try lots of examples at this stage. Sit down and draw a variety of figures and investigate what happens when you remove various points from these figures. For example, consider the following figures. In each case determine the maximum number of points that can be removed, leaving the remainder in one piece?



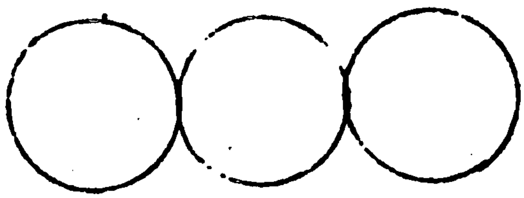


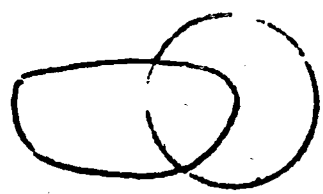


(All figures are linear — they contain no points inside the lines)

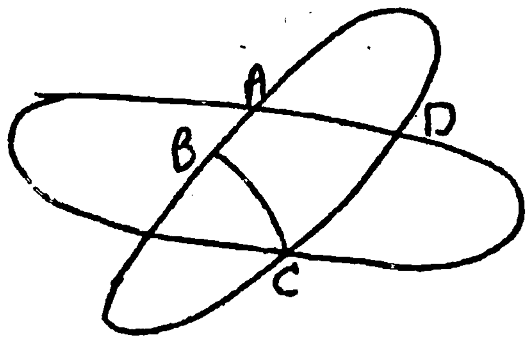
<u>Question 6:</u> What's going on here? Can you derive some general criterion for determining at a glance the maximum number of points that can be removed from a given figure?

A number of interesting related questions arise from looking at the "special" points where three or more lines come together. Consider the following two figures:





They are alike in that the maximum number of points you can remove from either one is three. Yet in the first figure it is impossible to remove either of the two special points and still have a connected figure, while in the second figure either special point can be removed and the result will still be connected so long as you don't remove anything else. Try lots of other examples of this nature. For instance look at the following figure:



If you remove point A, what is the maximum number of other points which can be removed and still leave a connected figure?

Same question for point B.

Same question for point C.

What happens if you remove B and D? A and C?

A PROBLEM INVOLVING SEQUENCES AND LIMITS OF SEQUENCES

Consider the

x	y
1	1
2	3
5	7
12	17
29	41
70	99
.	.
.	.
.	.

In general, each x value is the sum of the preceding x and y values while each y value is the sum of the two preceding x-values.

The question is, what is the value of the limit of ratios y/x? My friend was certain he knew the answer (as you will be too if you calculate this ratio for the first few values of x and y) but he couldn't prove it. Can you?

Some other questions of this sort that might be asked are:

1. What would happen if we had started off with values other than 1's for x and y?
2. Can you construct other sequences sort of like this one and evaluate the associated limits?
3. In the original sequence above, what is the value of the limit of the ratio of

successive values of x's? Of y's? What is the value of the limit of the ratio of y to the next (instead of the preceding) x?

GAMES THAT EMPLOY MATHEMATICAL ANALYSIS

I. Hoffman's Game Number One

A. Given sixteen squares in a row, players make alternate moves by cancelling out two adjacent boxes - these can be anywhere as long as those two have not already been cancelled. The player that cannot make a move loses. Thus

Player no. 1		X	X													
Player no. 2		X	X					X	X							
no. 1		X	X					X	X		X	X				
no. 2		X	X			X	X	X	X		X	X				
no. 1		X	X			X	X	X	X		X	X			X	X
no. 2		X	X			X	X	X	X		X	X	X	X	X	X
no. 1		X	X	X	X	X	X	X	X		X	X	X	X	X	X

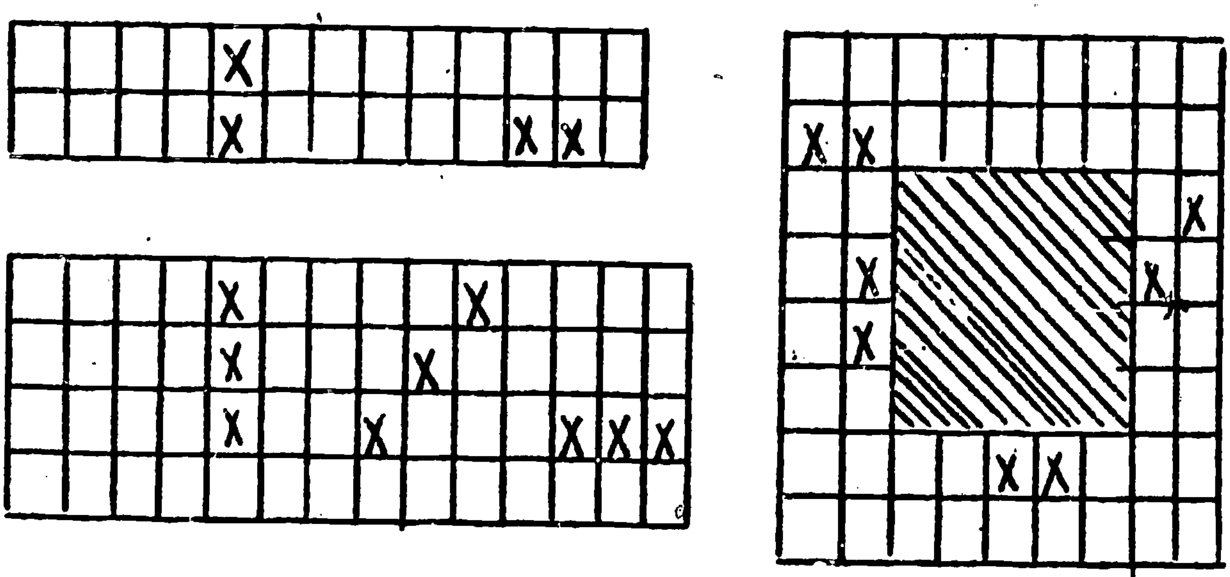
The first player wins, because the second player was unable to make a move in the last diagram. Students are encouraged to invent winning strategies for this game. This might include: do you want to go first?

B. Variations of this game that have been thought up so far include:

1. You could change the number of squares from 16 to some other number: would it make any difference what that number is? If you could go first all the time, what lengths would you prefer?

2. Instead of changing the length, one might allow the players to cancel three adjacent squares? Or four? Or one, for that matter? Which of these new games is interesting?

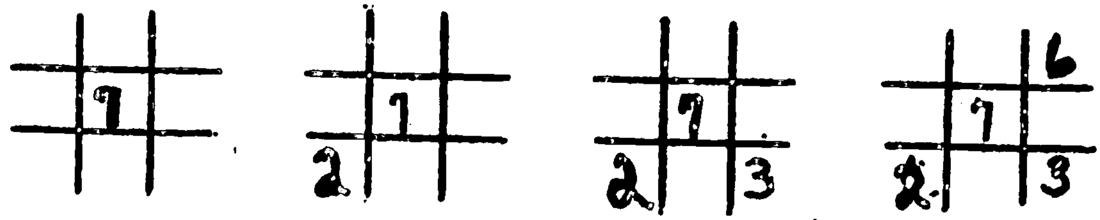
3. Or, one could arrange a different starting board:



In some of these it might be useful, and interesting, to allow not merely the cancellation of two squares adjacent and in one row, but perhaps two squares, one above the other (as shown): or maybe even two squares along a diagonal and still adjacent (as shown).

II. Hoffman's Game Number Two

The first player uses the numbers: 1, 3, 5, 7, 9; while the second player uses: 2, 4, 6, 8. The players fill in the tic-tac-toe board, 3 x 3, alternately putting in their numbers: the first player to complete a row, column or diagonal with the sum 15, wins.



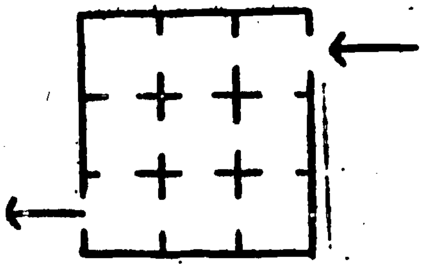
The second players wins. What is the winning strategy? Do you want to go first? Why? Why not?

Variations:

1. Could you arrange a different set of numbers? Would that affect the winning strategy? What set of nine numbers?
2. Should we make it a different winning sum? What sum? Would there always be a winning player? Why? Why not?
3. Suppose we merely said that the first player had to use only 1's and the second player only 2's; and that the player who completed an odd column, row, or diagonal wins. (Another game: you try to get an even column, row, or diagonal.)
4. What would a four by four version of this game look like?

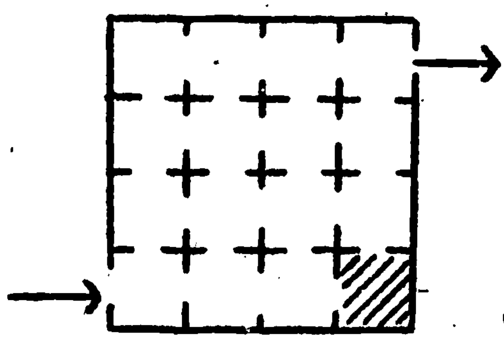
III. <u>Mr. Simplex Saves the Aspidistra</u> (see movie of the same name)

Given a 3 by 3 square floor plan for a house, where all the straight walls have doors in them as indicated, and where the two doors to the outside are indicated also, can you find a path starting at the front door going through each room only once?



How many different paths are there?

Does this work on a 4 by 4 house, similarly constructed:



Office

Suppose one of the rooms is closed off as an office, is it possible to make a "tour" going through each room other than that office, exactly once? Does it make a difference *where* the office is? Where could it be put? Where could it not be put?

Suppose the arrangement of rooms is changed to a 5 by 3? A 4 by 3? A 6 by 4? For what combinations can a tour be made, without an office? For what combinations can a tour be made with an office?

Suppose you are given a 5 by 5 arrangement: how many distinctly different tours are there starting at the front door and ending at the rear door?

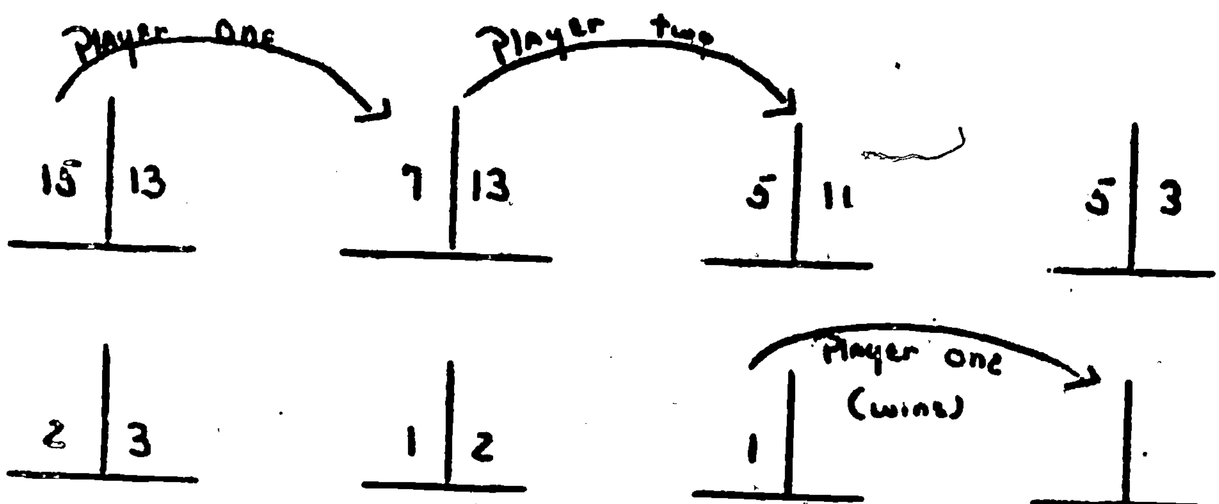
Suppose there were two floors, and there is a trap door connecting every room on the first floor to the one directly above it, along with the usual doors between rooms on both floors, and we shall put the front and rear doors on the first floor as before. Is a tour possible? Do you need an office? Just one? How many tours, if there are any?

Suppose there are more than two floors, with all the connecting doors? Then what? Can you construct a complicated house with many floors and many rooms on different floors, and see what happens?

IV. Wythoff's Game

Start with two numbers, or two piles of things, or two piles of anything. Players move alternately. Moves can be either (1) removing as many chips as the player wants from the first pile, or (2) removing as many chips as he wants from the second pile, or (3) removing an equal number of chips from both piles. The player that cannot move, without removing all chips, loses.

Here is how one game progressed:



Are there winning strategies:

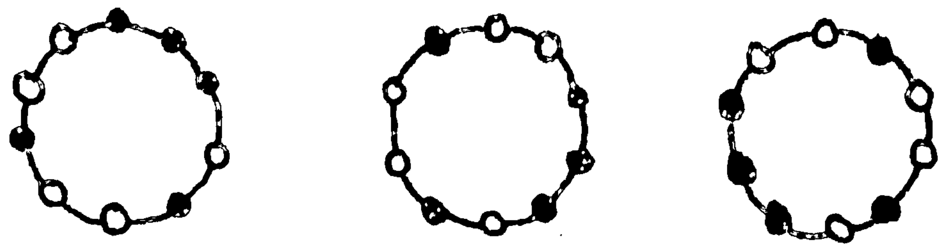
Variations include, naturally, varying the two numbers that begin the game. Suppose that you knew what the two numbers were to begin with; would you choose to go first or wait and go second? On what basis?

Other variations include:

1. Eliminate the third kind of move. Is the game trivial, then?
2. Suppose you can move, in the third move, an equal number of chips from both piles, up to five chips, but no more?
3. Suppose you can move a fourth kind of move as well: remove as many chips as you want from one pile and that number plus one more from the other pile. Then what happens to the strategy?
4. Suppose you start with three piles of chips, of some numbers, and make up some appropriate rules like the ones we began with for the allowable move, then what consitutes a winning strategy?

V. A Problem About Necklaces

Suppose you have a factory that produces with ten beads on each necklace; how many different necklaces must you produce, using beads of two colors, if you want to be able to satisfy any request?



NOTE: All three of the above are the same, and you only have to produce one of them to satisfy any of the three requests. The second can be obtained from the first by rotating the necklace two beads in a clockwise direction; the third can be obtained from the second by flipping the whole necklace about an axis that is vertical through the middle of the necklace, turning it over. How many other arrangements, that might appear to look different, can really be obtained from the first necklace by some sort of motion as described? Does that help the plant manager of the factory to organise his factory.

Obvious variations:

1. Change the number of beads in the necklace from ten to twenty. or any number you want.
2. Suppose there are three different colors of beads to choose from, then what? What about four? What about just one color?
3. Suppose you were producing beads where there was a clasp, like most necklaces have, located between two of the beads, and you were asked to solve the original problem with this feature?

4. Suppose the necklace is really a "figure eight" composed of five beads in one loop and five more beads in the second loop and the loops were connected, as shown? (in two colors)

5. Same as four, except that the two loops are not connected; when one buys a necklace one gets a package with the five-bead loops inside. Now what?

6. Make up your own variations.