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ABSTRACT

This publication contains the substance of all papers submitted at the fall 1974 Association of Teachers of Mathematics in New England (ATMNE) conference. As the title indicates, the focus of the publication is directed toward mathematical applications. The 25 papers cover topics ranging from applications at the elementary school level to those involving college level mathematics. Subjects of the papers include: function concepts in intermediate grades; mathematical modeling for election decisions; mathematics applications to music, art, grocery store arithmetic, vocational-technical school, economics, communications, ecology, and traffic flow; research and applications; Euclidian geometry; audio-tutorial instruction; computing pi; tanagrams; UISCM materials; the Developing Mathematical Processes program; and computer approximations. Some of the articles include bibliographies and/or references. (JBW)

Mathematics: Perspectives on Applications

ASSOCIATION OF TEACHERS OF MATHEMATICS IN NEW ENGLAND

NCTM

PROCEEDINGS OF THE ANNUAL FALL CONFERENCE

November 1974
Manchester, New Hampshire

Normand H. Côté, Editor

FOREWORD

Concordia, res parvae crescunt a posse ad esse.

Summer is ending now; the local farmers are gathering the seeds in preparation for next years growth.

This publication terminates the activities of the Fall 1974 ATMNE Conference. It is the first time in ATMNE's history that such a publication project has been attempted; comments addressed to your ATMNE representatives concerning your interest or lack of it for a continuation of this procedure will strongly influence future decisions.

At a regional conference of the size of the ATMNE conference there is always such a multitude of speakers and of workshops and such frenetic activity that the participant senses he may have missed at least as much as he has gained. These proceedings open an opportunity to reflect upon the thoughts of speakers you may have heard or of speakers whom you missed. By this instrument, the seeds of their thoughts are broadcast; what is reaped depends, as always, upon the quality of the seed and of the earth receiving them as well as on the climatic conditions which prevail. This is a non-juried publication containing the substance of all papers which were submitted.

I would like to express my sincere thanks to all who worked to make the 1974 Conference a success: every speaker and presider at the conference, and especially the contributors to these proceedings; the Program Committee members, Richard Evans, chairman, Russell J. Call, William Driscoll, Richard O. Kratzer, Norton Levy, Dorothy Meserve, Patricia Nolan, Edward Roth, James Swenson; the Registration Committee members, Donna Hurley, chairman, Jane Brandt, George Chase, Ronald Clark, William Faulkner, Bess Goodwin, Bev Guinessd, Warren Hulser, Roberta Kieronski, Gene Ladlev, George Smith, Mary Vachon; for Hospitality and Exhibits affairs, Malcolm Murray; for program distribution, Thomas Armstrong; for sale of NCTM Materials, Sara McNeil, chairman, Geraldine Phelps, Kenetha Marshall; Stanley Brown, president of ATMNE for his efforts in publishing the proceedings; Tom Goulart and Wally Stuart of the Plymouth State College AV Department for their invaluable assistance during the conference and during the publication of the proceedings; Janis MacDonald, Maude Stiles for their help in typing; Andrea Kroll, for typing the entire manuscript of the proceedings; Betsy Cheney, Jerry Deneau, for their kind advise, the editors of the CLOCK for use of their equipment; the many who gave a hand when it was most needed. Special mention should be made for the help and encouragement received by the administration of Plymouth State College of the University System of New Hampshire.

Normand H. Côté
Plymouth State College
General Chairman - Editor

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APPLICATIONS: A HISTORICAL PERSPECTIVE

By Morris Kline, Courant Institute, N.Y.U., and Visiting Distinguished Professor Brooklyn College of C.U.N.Y.

It is always neat and satisfying if one can present the essence of what he is trying to explain in a one-sentence definition. One that has been offered is that applied mathematics is that science of which pure mathematics is a minor branch. A more sober definition is that applied mathematics is the science which attempts to study some phenomenon or class of phenomena in the real world with the tools provided by mathematics. It is a sustained attempt to discover the structures that underlie our preceptions of the universe. It strives to lay bare the hidden structures behind classes of phenomena. Historically the phenomena have been almost entirely those of the physical sciences but there is no reason to limit the class of phenomena. The domain of applied mathematics is the real world.

One could ask, how can we be sure that there are patterns in nature's behavior? Perhaps the effort to discover them is a vain one. Fortunately the Greeks believed that nature is mathematically designed. This belief is epitomized in the Pythagorean saying that "everything is number" and Plato's statement that "God eternally geometrizes." Because the Greeks acting on their beliefs did create some mathematical structures that describe nature's ways, the chief ones were Euclidean geometry as the science of physical space and Ptolemaic astronomy, the successors of the Greeks continued the search for patterns and enjoyed further successes. Today the successes have been so numerus and so valuable to mankind that we are convinced of the applicability of mathematics. Unfortunately mathematics is not emblazoned on the face of nature and so the task of applying mathematics is a formidable one. I should like to describe the stages in the work of applying mathematics.

The first task is that which I call *idealization*. I can illustrate this readily. When one seeks the area of a field and he decides on the basis of observation or measurement that the shape is a rectangle, he has idealized. No actual field is a mathematical rectangle but one ignores minor irregularities in the shape of the boundaries and possibly in the rightness of the right angles and he proceeds on the basis that he has a rectangle. A slightly more complicated task of idealization is presented in the following problem. A man wishes to calculate the height of a flagpole and he measures the angle of elevation of the top and the distance from his feet to the base of the pole with the intention of applying the trigonometry of right triangles. Before applying the trigonometry he regards the flagpole as a line segment and he may even regard himself as a point on the ground some number of feet from the base of the pole. Having thus idealized he is prepared to apply trigonometry. He may idealize in a slightly different way. He may take his own height into account and replace himself by a line segment as long as the distance from the ground to his eye-level. The latter idealization will give a more accurate result than the former one because he measures the angle of elevation of the top of the pole at his eye-level. Treating the earth as a sphere is another idealization.

Beyond idealizing a problem the applied mathematician often deliberately simplifies a problem by omitting factors that ought to be taken into account but which complicate the problem so much that he takes the chance that the simplification will not introduce significant error. Thus in determining the motion of a planet around the sun, many simplifications are introduced. The planet and sun are treated as point masses because their sizes are small compared to the distances involved. The effect of the other planets on the one whose motion is being studied is ignored (at least in the basic problem). And one assumes that the planet moves in a vacuum so that there is no drag arising from motion in a medium. The use of point masses would not do in studying the motion of the moon around the earth, in studying the effect of the moon on the tides on the earth, or in studying eclipses of the earth. Just how far one can go in simplifying a problem is a major question and I shall come back to it. One can see that some insight into the physical phenomenon is necessary.

The third task of the applied mathematician is to *make a mathematical model*. The efforts of Newton in dynamics, of Euler in acoustics, of Euler, Lagrange and others in fluid dynamics, of Cauchy, Poisson, Navier and Stokes in elasticity and of Maxwell in electromagnetic theory, produced the fine models for the respective fields. However new models are constantly required as new phenomena are investigated. The newer fields of quantum theory, relativity, magnetohydrodynamics, statistical mechanics and information theory have acquired models only recently.Solid state physical and molecular chemistry are struggling with their models. The social sciences are still seeking significant models.

In this third task, too, the applied mathematician must be deeply informed in the physical field and he may even have to create the mathematical concepts and technique just as the entire subject of the calculus and differential equations had to be created to provide mathematical models.

The fourth task of the applied mathematician and the one which most people think of in connection with such work is the *manipulation of the mathematics* itself to produce the mathematical conclusion. What mathematics is involved? Of course all of the elementary branches and especially the calculus are used. But the heart of the mathematics used in this phase of applied mathematics is differential equations, ordinary and partial, and the reason for the concentration on this field is simply that the laws of nature seem to be most amenable to differential equations. One need only think of Newton's second law, the formulations of Lagrange and Hamilton, the linear and non-linear equations of acoustics, the equations of fluid dynamics, The Navier-Stokes equations of elasticity, Maxwell's equations, and Schrodinger's equation to appreciate how much of physics is embodied in differential equations.

However the mathematics proper which is used in applied problems extends far beyond differential equations. Prominent today are the calculus of variations - one need only recall the many variational principles (Least time, Least Action, etc.) to see the uses that subject - differential geometry including Riemannian geometry, complex function theory (conformal mapping in the solution of Laplace's equation and evaluation of integrals), integral equations (as a reformulation of differential equations and even as the natural starting point for many problems as Hilbert pointed out), matrix theory (for transformations and solution of equations), probability theory (for quantum mechanics, statistical mechanics and information theory), special functions (for solution of all sorts of differential equations), spectral theory (eigenvalues and eigenfunctions), transform theory (Laplace, Fourier, Hankel and others) for solution of differential equations, and other branches.

There is a fifth major task, *to get useful answers*. Since very few applied problems can be solved exactly it is the task of the applied mathematician to get results by hook or crook. One necessary measure is the use of approximations. Ignoring a minor term in a differential equation, perturbation methods (Picard method of solving $\dot{y} = f(x,y)$ is a prototype), series, asymptotic expansions, and the numerical solution of differential and integral equations (finite difference methods) which lean squarely today on the use of computers are approximation methods. In making the right approximations physical knowledge of what the quantities neglected mean is indispensable.

After his flight into the abstract the applied mathematician must return to earth to face a sixth task, *the interpretation of the results*. What does the mathematics say about the physical phenomenon? Interpretation is a major task and a crucial one. Idealization, simplification, the erection of a model, and approximation are all sources of error and the test of whether all the mathematical work is significant is whether it tells us something useful and reasonably accurate about the phenomenon. In fact even experimental check on the results is often called for.

There is one more characteristic of applied mathematics which to some extent distinguishes it from pure mathematics and this is the matter of *rigor*. There is of course no such thing as absolute rigor. This is one of the sad facts of our times. But beyond this the applied mathematician is somewhat more indifferent to rigor than the pure mathematician. I have pointed out that even after the applied mathematician sets up his model he makes approximations to get useful answers. It is usually very difficult and sometimes impossible to estimate the error introduced by these approximations. When the applied mathematician uses perturbation methods or series and stops short with a few steps or a few terms he usually cannot estimate the error. He may use integrals which are known to be divergent as in the case of quantum field theory. He uses series whose convergence is not even provable. (This was always the case in astronomy and is still true today.) When the applied mathematician is quite certain of his steps on intuitive or physical grounds he does not bother with mathematical rigor. The pure mathematician would insist on estimation of error, on knowing that his series converges, that his integrals converge, and so forth.

Insofar as rigor is concerned the applied mathematician has employed concepts and techniques which horrify the pure mathematician. In the eighteenth century before a satisfactory theory of convergence was made available by Bolzano and Cauchy, great debates took place on the use of series. In our own time, Heaviside introduced and used infinite series and fractional powers of d/dx and $\partial/\partial x$ with no logical basis at all and Dirac used the δ-function and its derivatives even though these are not legitimate functions. Of course the pure mathematicians declaim against such work and Heaviside, for example, was subject to personal attack. Some of his answers are famous. He said "Shall I refuse my dinner because I do not fully understand the process of digestion?" "Logic can be patient for it is eternal." "Ha, the series diverges. Now we can do something with it." He became contemptuous of the logic-choppers, as he called them. The proofs mathematicians demanded, he called whimsical fancies. With respect to rigor the applied mathematician boasts that he can find the solution to any difficulty whereas the pure mathematician seeks only to find the difficulty in any solution.

It may be clear from this description of the work of the applied mathematician that he faces formidable tasks. He must be a highly skilled mathematician in numerous branches of mathematics and he must be deeply versed in the branches of physics to which he applies mathematics. He might also have to know some engineering. The problems of aerodynamics, for example, were initiated by engineers. The applied mathematician's role lies somewhere between those of the pure mathematician and the theoretical physicist. The theoretical physicist is largely concerned with the inductive process of generalizing to laws from observations, experiments, intuitive arguments, and even pure guess as to what might or must happen. The applied mathematician is more concerned with the deductive process of obtaining conclusions about real phenomena by employing the basic laws produced by the theoretical physicist and applying to them all the methods and techniques produced by mathematicians. Of course, there is no sharp dividing line at either end. The applied mathematician may create the mathematics himself and often has to and he may also discover fundamental physical principles as did Newton, Euler, Lagrange, Hamilton, Gauss and in fact most great mathematicians of the past.

Though applied mathematics, to use modern terminology, has been the major concern of mathematicians of all centuries through the nineteenth, a sharp split has arisen between those who would continue to devote themselves to the study of real phenomenon and thus who seek sustenance in problems arising within mathematics itself. These "pure" mathematicians have concentrated in fields such as topology, functional analysis, abstract algebra and mathematical logic. They raise and answer questions which have no connection with the use of mathematics in real problems. What is the motivation of these people and why do they concentrate in the fields just mentioned?

As to motivation one possible answer is intellectual challenge. But this challenge is surely present in problems arising from real phenomena and indeed more so because, as I indicated above, there are several difficult tasks in the work over and above mathematics proper. Moreover the challenge of applied work and of questions bearing on applied work is no idle one.

Another possible motivation is beauty. A man states that he finds such and such a topic beautiful and does not find another equally beautiful. There can be no argument about tastes. But I would venture an opinion. Very little mathematics is really beautiful and in my opinion the search for beauty in mathematics is much overrated. I worked among the greatest mathematicians when I spent two years at the Institute for Advanced Study and I have worked among some very good ones for the past twenty-five years at N.Y.U. I would say that beauty plays very little role. Challenge, personal gratification in accomplishment, and satisfaction in doing some good for society are certainly there but the term beautiful is only rarely applied to a theorem or a proof. As a matter of fact I have heard almost continually dissatisfaction with ugly proofs that must be accepted because they do the job. Some of the ugliest proofs I have seen are in topology and I dare anyone to tell me that the proofs in Russell and Whitehead's *Principia Mathematica* are beautiful. Even when a mathematician sincerely asserts that something *he* has done is beautiful - the other fellow's work never is - my suspicion is that the man is an egotist (most mathematicians are anyway) and so he thinks that what he produces must be beautiful. Moreover today the pure fields are so heavily burdened with terminology and definitions that even expert research workers in these fields are complaining of their inability to keep up with the language. One must read these works with a dictionary in hand but unfortunately the dictionary does not exist.

Despite the absence of any pervasive, sound motivation for the creation of the so-called pure mathematics, the mathematics having no connection with reality, the fact is that most mathematicians are concocting pure mathematics and in the fields mentioned above. Why? May I give you my own analysis and answer? This country has become research conscious in the last thirty years or so and mathematicians wish to do research in order to earn their spurs. They choose the easy way out. To do applied mathematics calls for an extensive background in mathematics because this brand of mathematics is now over 200 years old and has been explored by the greatest mathematicians we have had. Applied mathematics also calls for an extensive knowledge of at least one major domain of science and each of these is 100 to 200 years old. In addition to the task of combining the two in the manner applied work calls for mastering major difficulties of other sorts as indicated above. It is far easier to concentrate on some isolated topic which requires no knowledge of science, raise questions of one's own choosing that one may have some chance of answering, rather than questions about nature, and then write papers. The pure mathematician can confine himself to one neat limited field and prove what he can. The applied mathematician must call upon numerous branches of mathematics and science and his results must yield knowledge of the physical phenomenon he is studying. Applied mathematics is pure mathematics with many more difficulties, requirements and obligations.

This analysis is borne out by the choice of subjects in which the pure mathematicians work. They have selected new fields which are already abstract and in which seemingly the only requisite is the knowledge of a few axioms and a few definitions. This accounts for the concentration on topology, abstract algebra, mathematical logic, and functional analysis. Of course to do sound work in these fields one should know the intimate connections with the solid core of mathematics, know what the real problems are, and treat those in the abstract domains. But merely to write papers one can pick any old problems. As a matter of fact the question of why the problems are tackled is rarely raised.

In further support of my analysis, I ask you to note that there is no concentration of pure mathematicians in say partial differential equations. This is a difficult field and one must have considerable background and considerable ability to make progress in it.

Let me give another example. Between the first and second world wars, the Polish mathematicians decided to build up mathematical research. What did they select? They chose point set theory for which no background was needed, at least to write papers. Of course there were good mathematicians among them and since point set theory is a legitimate field some good work was done.

It seems to me that the distinction with which the mathematical world should be concerned is not that between pure and applied mathematics, because this distinction cannot be made solely on the basis of subjects involved, but the distinction between mathematics that is undertaken with sound objectives and mathematics that is undertaken to satisfy personal whims and goals, between pointed mathematics and pointless mathematics, between significant and insignificant mathematics, between vital mathematics and inert, bloodless mathematics.

The pure mathematicians, emboldened by the number of their cohorts, no longer hide the fact that they work on problems that merely satisfy their own tastes and goals. Quite aware of their abandonment of the true obligation of mathematicians, they have tried to defend their activities by giving their own interpretation of history. They claim, for example, that non-Euclidean geometry was created by men who were engaging in an intellectual pastime and yet 75 years later non-Euclidean geometry proved useful in the theory of relativity. Their conclusion is then that purely intellectual investigations having no ties to the real world prove as useful as those which do start from real problems.

But the pure mathematicians do not know their history. The centuries of effort on the parallel axiom were made by men who were deeply concerned as to whether the axioms of geometry fit physical space. Euclidean geometry was always meant to be an accurate representation of physical space and the mathematicians sought to insure it. It was in fact an overconcern if anything with the physical correctness of Euclidean geometry which brought about non-Euclidean geometry. There is an oft-repeated assertion that Gauss, the creator of non-Euclidean geometry, went as far as to measure the sum of the angles of a triangle in order to decide whether Euclidean geometry is the proper description of physical space. (This story is not substantiated by history but it does represent Gauss's concern.)

I do agree that good mathematicians have often carried a development far beyond the immediate physical needs and that sometimes these extensions have found new applications. But they found new applications because they were sound developments to start with. There is no example in the history of mathematics of a major purely speculative development which has later proved useful.

Since history does not recommend or justify their work, the pure mathematicians have adopted another maneuver. They have tried to take the bull by the horns and brazen out their willfulness. Let us listen to Marshall Stone in his American Mathematical Monthly article of October 1961 entitled "The Revolution in Mathematics".

"While several important changes have taken place since 1900 in our conception of mathematics or in our points of view concerning it, the one which truly involves a revolution in ideas is the discovery that mathematics is entirely independent of the physical world...but we should also not fail to observe how closely this development has been involved with an emphasis on abstraction and an increasing concern with the perception and analysis of broad mathematical patterns. Indeed, upon closer examination we see that this new orientation, made possible only by the divorce of mathematics from its applications, has been the true source of its tremendous vitality and growth during the present century." Vitality and growth there has been; the flood of papers has swamped the journals but the worth of this material is another matter. Stone continues, "A modern Mathematician would prefer the positive characterization of his subject as the study of general abstract systems, each one of which is an edifice built of specified abstract elements and structured by the presence or arbitrary but unambiguously specified relations among them." This statement, like the abstract mathematics he defends, is so vague that it is hard to know what he means. Further, "For it is only to the extent that mathematics is freed from the bonds which have attached it in the past to particular aspects of reality that it can become the extremely flexible and powerful instrument we need to break into areas beyond our ken. The examples which buttress this argument are already numerous..." And then Stone mentions genetics, game theory, and the mathematical theory of communication. But these advances have come about by applying good sound classical mathematics and not from homological algebra or the study of abstract structures.

Thus Stone dares to proclaim a thesis which is contradicted by the entire history of mathematics. One can say, as the pure mathematicians confidently do, that their work will prove valuable 50 years hence but I would say that we need some evidence beyond the confidence of these men in the value of their work.

Unfortunately Stone does reflect the work and the position of the great majority of the mathematicians. Mathematics is now dominated by abstraction and formalism. It is form at the expense of substance and reminiscent of medieval Scholasticism. The professional mathematician, generally speaking, is a specialist in logical systems and rigor. His narrowness and lack of flexibility make him incapable of exercising the essential functions of mathematics in science and engineering, which is to separate the relevant from the irrelevant, to simplify the formulation of complex phenomena, to synthesize and to unify substances rather than form. Rigor and abstract formalism are purely technical contributions and impede invention.

Of course there is some value in abstraction and structural insight. Mathematical ideas such as complex numbers, matrices and operations are not descriptions of physical substance. But abstractions and abstract structures are valuable only insofar as they shed light on mathematics of proven worth. On the other hand, to emphasize just abstraction and independence of physical relationships is a misinterpretation of what is important in mathematics. The life blood of mathematics rises through roots which are deeply imbedded in reality. The nourishing soil is the physical sciences and engineering. Moreover abstraction and generalization are not more vital than individual phenomena or than induction from intuitive situations. It is the interplay of the mathematization of real problems and abstraction and generalization that keeps mathematics alive. Mathematics must not be allowed to split into a pure and applied variety. It must remain a unified vital strand in the broad stream of science and not a brittle film that is all gloss and no substance. Abstract structures created in and for themselves are empty shells, peanut shells.

Topologists who do not know the connection of their subject with analysis or know it but ignore the contribution which topology should make to analysis will produce trivia. Group theorists who do not know Galois theory or the use of group theory in quantum mechanics will not produce material worth the paper on which it is printed.

The idea that there can be a pure mathematics distinct from applied mathematics is a threat to the life of mathematics. Many wise men have seen this danger and have warned against it. As far back as 1890 when Cantor declared in good faith that "the essence of mathematics is its freedom", Felix Klein rejoined that the privilege of freedom carries with it the obligation of responsibility. More recently John Von Neumann in his essay "The Mathematician" warned "As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from 'reality', it is beset with grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude on insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much 'abstract' inbreeding, a mathematical subject is in danger of degeneration".

A still more dismal prophecy was made by Richard Courant in the Mathematische Annalen of 1957. He said, "There exists the danger that the 'applied', [he meant all real] mathematics of the future will be developed by physicists and engineers and professional mathematicians of rank will have no connection with this new development." There is no question of the future of the mathematics designed for the sciences but there is considerable question about the future of purely speculative creations.

The attitude of the pure mathematicians is reflected in the teaching of mathematics on the high school and college levels and is poisoning the atmosphere there. Even if there were some reason to create mathematics which is independent of reality, this type of mathematics should not be presented to novitiates.

Mathematics presented as a subject which exists in and for itself has no motivation, no meaning and no purpose. The possible values such as beauty and intellectual challenge do not serve as motivation. The beauty does not serve because very little of what we teach on the high school and college levels is beautiful. Witness the quadratic formula. Intellectual challenge does motivate a very small percentage of the students but even many of these ask after a while, what good is all this?

The meaning of mathematics is not found in mathematics *per se* until one has reached a rather high level of ability of grasp abstractions. All the elementary branches arose in response to physical problems and the concepts adopted have meaning only in terms of the physical objects or relationships they represent. The properties of numbers and the axioms of geometry were adopted because they fit reality and enable us to work with real phenomena. The meaning of all this mathematics lies outside of mathematics.

And as far as purpose is concerned, the one purpose that will attract most students is to see how mathematics helps in the understanding and mastery of nature.

Moreover, who constitute the audience in high school and college classes? Those who will use mathematics later are mostly engineers and physical scientists. Surely they should be taught how mathematics is used in physical studies and what mathematics means in physical terms.

This is, of course, what should be the case. But what have the curriculum makers, who are dominated by the pure mathematicians, done? They offer a sophisticated, abstract, rigorous mathematics which stresses deductive structure and axiomatics. They have eroded the substance; they present uninspiring and pointless abstractions; they have isolated the subject from other bodies of knowledge; and they offer dogmatic presentation of final versions of branches of mathematics. Formalism, whether of the present axiomatic variety or of the older manipulative variety, can lead only to a decline in vitality and to authoritarianism.

The pure mathematicians, aware of their neglect of science, defend what they have incorporated in the curriculum by saying that they are teaching mathematics as the language of science. But they have invented a totally new vocabulary which certainly has not been used in science and just as surely will not be. They have wallowed in vocabulary as though new words will solve problems. They also talk about the need to teach students *abstract* structure, such as groups and fields, as if structure will teach them how to make models of scientific phenomena. Beyond that, what they really teach are some of the most artificial structures mathematics possesses, the development of the real number system from Peano's axioms and Hilbert's foundation of Euclidean geometry. In fact all logical structures are artificial reorganizations of real mathematics, reorganizations that strip the subjects of any indication of how they arose, why anybody wanted them, and what one does with them. These canned highly artificial structures have about as little to do with true model building as child's play has to do with real engineering. Moreover what mathematicians really stress in these structures is hair-splitting rigor.

There is, I believe, but one road to pedagogy in mathematics and this is the road which mathematics itself took during the centuries and the road which applied mathematicians necessarily take in their work. The mathematics must arise out of and be built up for the sake of real problems. Moreover, the students must create the mathematics that is needed and their very participation in the creative act will enable them to perceive the life and spirit which true mathematics possesses. The students must introduce the concepts and methods and apply the mathematics, of course, with the guidance of the teacher. The students must learn to think like physicists while concentrating on the role which mathematics plays in the study of nature. What I am saying, in short, is that insofar as pedagogy is concerned, the approach of the applied mathematicians is the only one that should be considered.

If I may summarize, the function of mathematics, the life of mathematics, the needs of society and government, and wise pedagogy point to only one kind of mathematics, the mathematics that is clearly and unmistakeable devoted to the study of reality and the mathematics that operates in partnership with the sciences. The glory of mathematics, the true appreciation of its power, and its claim to being a significant body of human knowledge all rest squarely on the value of the subject for the study of nature and real problems generally.

I congratulate the Association of Teachers of Mathematics in New England for recognizing the importance of applications of mathematics and for devoting this meeting to furthering the teaching of applications.

Postscript

The idea that mathematics should be taught in . lose conjunction with applications is not new. It was advocated by the British engineer John Perry and by the American mathematician E. H. Moore as far back as 1902. Unfortunately the educational leaders of this country were not sufficiently prepared at that time to appreciate Perry's and Moore's ideas. Instead, especially in the recent so-called reform, the new mathematics, the country turned in the precisely opposite direction of isolating mathematics from reality. It is because of this turn of events that I ventured to reopen an old theme and to encourage the members of this Association.

THE WORLD OF BUCKMINSTER FULLER

By Ernest R. Ranucci, SUNY at Albany andO.A.S. Fellow at Universidad de Costa Rica

The difficulty in writing a definitive statement about the work of Bucky Fuller is that you never know when to stop. The man is eighty years old. He still tosses off brilliant insights, still circles the globe on an apparently non-stop basis, still numbs his listeners and disciples with the length, breadth and height of his visions, the scope of his global concerns and indestructible fundamental optimism about man and his sorry lot on this earth. When you try to write about this man - a living legend in his day - he refuses to stay put. He's quicksilver in motion. You just can't toss off blithely some pat statement about the Leonardo da Vinci of the twentieth century.

The man is known to most teachers of mathematics as the inventor of the geodesic dome. This, alone, would be insufficient grounds for the statement made yb more than one writer: it's not *whether* or not he'll get the Nobel Prize, but *when* he'll be so recognized. In a world preoccupied with problems: overpopulation, insufficient food, droughts, shortage of energy-producing sources, his was a preoccupation which goes back forty years or more. Perhaps it would be better to merely enumerate reasons why teachers of mathematics ought to know something about the man. Statements made by two men are worth noting with regard to the monumental achievements of Buckminster Fuller. Robert Marks, in his book, "The Dymaxion World of Buckminster Fuller", has this to say:

> It is a difficult matter to interpret Bucky. He has the genius' constant onrush of dream flow and dream logic. And he is graced with the quality now known, in cybernetic circles, as positive feedback - mirror multiplication of the information communicated. Each thought that Bucky expresses feeds back into his mind, there to generate families of fresher thoughts, broader in scope and more intense.
>
> Bucky has never been easy to understand - even by those best equipped to grasp his meanings, and those who know him best and love him most. The reason is both psychological and semantic. He overloads the channels of communication. He is ever ready to give too much of himself too spontaneously, too richly, and too quickly. The simplest question evokes a torrent of insights. And these he expresses in an incisive, private argot, resplendent with word coinages, hyphenated Latinisms and tropes.
>
> Although his cardinal ideas have about them the skeletal simplicity

> we associate with the best Greek thought, they sometimes come through to the casual listener as a cascade of ambiguities. And this only because there is too much. You would not expect to take in the first six books of Euclid at a single hearing, nor without a reduction of text language to conversional level. Yet, with Bucky, the equivalent of this technical richness is offered untranslated, at each meeting. His conversation, thus, is always a subtle form of flattery. It implies that he believes you are at ease in all the areas of his talk, and that you can with equal agility go "second powering", "tetrahedroning", "inwardly-outwardly-to-and-froing", or go bouncing on a four-dimensional pogo stick down the slopes of Parnassus.

While the Marks book takes the reader up to the year 1960 and no farther, it is the best single source I know of for Fuller fans. The pictures and drawings are stunning and the writing style is graceful and lucid.

The comments of world-reknowned architect Frank Lloyd Wright, on the occasion of the emergence of Fuller's book: "Nine Chains to the Moon" (1938) throw additional light on our man Fuller:

> Buckminster Fuller - you are the most sensible man in New York - truly sensitive. Nature gave you antennae, longrange finders you have learned to use. I find almost all your prognosticating nearly right - much of it dead right, and I love you for the way you prognosticate. To address you directly will be a hell of a way of reviewing your book - I know. I should write all around you, take you apart, and put you together again to show - between the lines - how much bigger my own mind is than yours and how much smarter than you I can be with it and leave the essence of your thought untouched.
>
> But I couldn't do it if I would and I wouldn't if I could. To say that you have now a good style of your own in saying very important things is only admitting something unexpected. To say you are the most sensible man in New York isn't saying much for you - in that pack of caged fools. And everybody who knows you knows you are extraordinarily sensitive...
>
> Faithfully, your admirer and friend, more power to you - you valuable 'unit'.
>
> FRANK LLOYD WRIGHT
>
> Taliesen
> Spring Green, Wisconsin
> August 8th, 1938.

Fundamental to an appreciation of the overall Fuller Philosophy is his Dymaxion Principle. It permeates much of what he did and is still doing today:

> In its simplest form, Fuller's Dymaxion concept is that rational action in a rational world, in every social and industrial operation, demands

the most efficient overall performance per units of imput. A Dymaxion structure, thus, would be one whose performance yielded the greatest possible efficiency in terms of the available technology (Marks).

The importance of the principle lies in the universality of its application to diverse problems. Fuller used it to advantage in such apparently unrelated structures as: a house that hung from a pole, a new type of automobile, a new system of map projection and, of course, geodesic domes.

Certain basic principles permeate the work of Fuller. It is here that the teacher of mathematics can latch on to intriguing ideas and materials suited to everyday teaching.

Regular solids constantly emerge in the work of Buckminster Fuller. This intrigues me greatly since the Greeks studied much of their geometry in order to cast light on the five regular polyhedrons. The virtual demise of the teaching of solid geometry in the secondary schools, a minor crime - in the opinion of one teacher who has been at it for forty-two years - has resulted in a generation of young teachers who are, at times, less than literate in this field. The regular solid permeates the structures of such diverse fields as bacteriology, atomic structures and architecture. With respect to just one of the applications of regular solids, Drs. Klug and Finck of Birbeck College, London, England wrote to Fuller in June of 1959 confirming the fact that the polio virus had a structure quite like that of the regular icosahedron.

The altered regular icosahedron and other of the regular and semi-regular solids, are basic to an understanding of the structure of geodesic domes. The term "altered" is used advisedly, and needs explanation. A geodesic dome in the form of a complete regular icosahedron (as in a radar dome) would have, for its vital statistics: 30 equal edges, 12 vertices and 20 congruent equilateral triangles as faces. At each of the vertices would emanate five equal segments, forming five equal face angles of the solid angle encountered. The sum of these face angles would be 300° (60×5); this merely affirms the fact that the sum of the face angles of a polyhedral angle is always less than 360°. In actual practice, geodesic domes are usually convenient fractional parts of the complete geodesic sphere. If large-scale domes were to be constructed according to the basic statistics quoted, sheer dead weight would snap the long struts required for support. In actual practice the basic triangle involved - the essential face of the regular icosahedron - is subdivided into smaller units. In a two frequency basic structure, figure 1, triangle ABC represents one of the original faces of the icosahedron. On

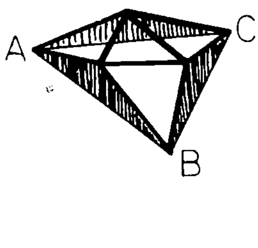

FIG. 1.

each of the equal edges project two equal chords. Triangulation is carried out as indicated. The resulting skeletal structure will appear on each of the twenty faces. In the eventual geodesic dome which would emerge, AB, BC and AC would not even appear. Each of the substructures would mesh with its three neighbors. In a three-frequency dome, figure 2, AB and the other original edges would be replaced by three chords. It is difficult to general-

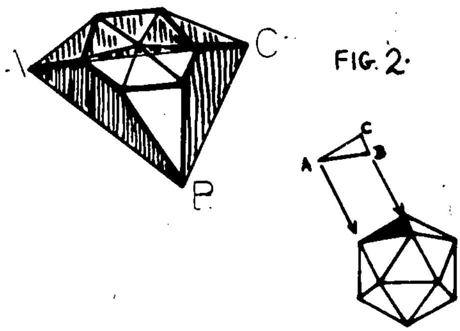

FIG. 2.

ize. In a dome such as the one which appeared at EXPO in Canada, you might find that a sixteen-frequency dome might be involved. In essence, Fuller was the pioneer in inventing a triangulation device for constructing a structure which used linear elements to approximate the curvilinear features of a sphere. Phenomenal strength results. What is more, the elements which go together are quite short. Yet tremendous structures evolve. To cite one, not the largest in the world, the Union Tank Car Company had a dome 384 feet in diameter and 116 feet high constructed in Baton Rouge Louisiana. It was, at least in 1958, the largest structure in the world of a clear-span type. It was built, fundamentally, as a roundhouse for the re-building of railroad cars. The lack of internal supports was, obviously, its strongest feature.

Before I got interested in the construction of geodesic domes I was quite confused by one particular element. Every teacher of mathematics knows that if six congruent equilateral triangles have one vertex in common, a regular hexagon emerges. Teachers also know that the elements of a regular hexagon are coplanar. I would look at some of the pictures of complex geodesic structures and see what appeared to be congruent regular hexagons. If this were so, how could a three-dimensionality (Fuller is not the only one who can invent new words) result? If we place three congruent regular hexagons in juxtaposition with each other, the resulting structure is still coplanar. Further ratiocination, however, revealed that you seldom see a pure regular hexagon in geodesic structures. The hexagons are always subtle relatives of the regular hexagon, adjusted in such a manner that three-dimensionality results. The original calculations for these elements were carried out *manually* by Fuller and an assistant early in the inventive stages. Now, complicated tables are available. These resulted from programming of the problem, by computer.

Before we move on to other Fullerisms, I wish to interject a personal note. Buckminster Fuller is, at present, the God of the grass-roots aficionados in the United States. The geodesic dome has sprung up as a personal dwelling in many parts of the United States, particularly in arid parts of the United States. There may be a reason for this (my source is that indispensible compendium of man's knowledge - The Last Whole Earth Catalog). Geodesic domes, when constructed with machine-made elements, is one thing. When smaller ones are constructed by the hammer-saw method, errors, small as they may be at the outset, have a tendency to aggravate themselves. When it rains ... you might do well if your name were Noah. Curiously, the old post-beam construction, verticals with horizontal roof supports, is still one of the easiest ways of achieving some protection from the elements. The moral, if there is one, may well be: it's not necessarily bad if it's old and it certainly is not necessarily good if it's new. Reformers of mathematics education, kindly note ...

One of the vital features of even a surface-knowledge of Fuller demands some understanding of what Fuller calls: tetrahedronizing. In the two-dimensional realm, a set of discrete points can be joined in such a manner that nothing but triangles emerge. With such a structure, rigidity would be assured since the triangle is the fundamental building block of plane geometry. A quadrilateral with four components for edges has no rigidity. It wiggles and wobbles unless diagonals are added for triangulation. If both diagonals are added, then bolted at their intersection, strength results. *Eight* triangles contribute to the strength of the basic quadrilateral. In a similar manner, the tetrahedron is the fundamental building block in three-dimensions. The use of modern machinery makes the reproduction of identical components in a system ridiculously simple. Thus, in two-dimensions, repeated equilateral triangles would be an ideal way of covering a plane. It used to bother me back in my work in high school mathematics (it was during this century) that repeated, congruent, regular tetrahedrons could not be used to completely fill space. We know that repeated cubes - as in the case of baby's blocks - will fill three-dimensional space, but not regular tetrahedrons. The angle formed by adjoining faces of a regular tetrahedron contains $70^{\circ}31'44''$ (I just *happen* to know it). Since this value is *not* an integral divisor of 360°, space-filling in three dimensions is an impossibility with repeated regular tetrahedrons. When the regular tetrahedron is combined with the regular octahedron, intriguing results ensue - specifically the Fuller Octet Truss. In actuality, the geometric basis for the F.O.T. has been known for centuries; nevertheless, give Fuller credit for making it an integral part of certain of his structures. The arithmetic basis for the marriage of the regular tetrahedron-regular octahedron rests in the values of the plane angles of adjacent faces of the tetrahedron-octahedron. The plane angle for the regular octahedron is: $109^{\circ}28'16''$. A little Euclidean bird told me so. The plane angle is the basic measure which indicates the angle at which two planes meet. It is formed by taking a point on the edge common to two intersecting plane and constructing perpendiculars, one lying in each plane. The measure of this angle is the angle at which these planes meet(figure 3).Study figure 4; the sum of: 109/28/16, 109/28/16, 70/31/44 and 70/31/44 is 360°. Actually this calculation is nothing but a parallel to

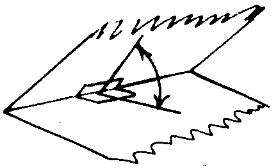

FIG. 3

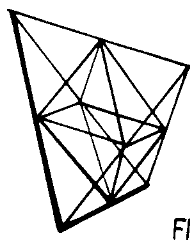

FIG. 4

the well-known homily: how come the seashore is always so close to the ocean? It is the *spatial* fact that repetitions of regular octahedrons and regular tetrahedrons will fill three-space that results in this specious bit of arithmetic. When equal struts are joined in the manner suggested by figure 4, a fantastically strong structure emerges. Fuller used this basis idea in the construction of the Ford Rotunda building in Dearborn, Michigan (1953). The construction of this particular dome seems to have given Bucky Fuller a particular pleasure - it just about fulfilled his prediction, made in 1927, that it would take a quarter of a century before certain of his principles would make it. He refers to his first customer as: "Mr. Industry himself". The Octet-Truss caused quite a furore when it was used in an exhibition of certain of the Fuller structures at the Museum of Modern Art in New York in 1959. This particular construction was 100 feet long, 35 feet wide and 4 feet deep. The one-layered Octet-Truss (Marks p. 282 for a picture) can be extended above and below by a judicious addition of struts. In essence, when a regular tetrahedron is affixed to each of the faces of a regular octahedron - they share certain of the linear components - new openings emerge for the addition of new layers of regular octahedrons. This is difficult to visualize and to draw but quite simple if construction sets like D-Stix or Geodestix are used. You'll have to combine rubber connectors to accommodate the complex nodes which arise - some of which require twelve struts. Remarkable constructions can be designed with the Octet-Truss principle. The horizontal coverage suggested by the drawing (Marks p. 282) would make it possible to cover a tremendous flat space. What is not at all obvious - it requires that a different orientation of the Octet-Truss be observed - is the fact that proper repetitions of the unit will result in the formation of a regular tetrahedron. I can't do justice to this concept in the space alloted to me. All I can say is: look at pages 232 and 233 of the Marks-Fuller book.

No discussion of Fuller, however brief, could fail to mention his forays into cartography. He finds that the Mercator Projection, which still defaces the front walls of many of our schools, quite unsuited to the needs of 20th century navigation. There are many versions of Fuller's Dymaxion Map (how he loves the world). The original map, since superseded by other versions, was invented in 1943. It caused quite a stir when It appeared in the now defunct magazine LIFE in the issue of March the 22nd, 1943. This issue is a collector's item. If you can get hold of it, treasure it. If you can get two, get one for me. The original Dymaxion Map, since superseded by other versions, is drawn on p. 152 of Mark's book. If the corners of a cube are lopped off as shown in figure 5, a cuboctahedron results. Fuller imagined that a sphere was inscribed within the solid. Boundaries of earth masses were projected on to the faces of the solid - either equilateral triangles of squares. Judicious arrangements of the fourteen faces resulted in a fairly true picture of the relations of the earth land and sea measures. This picture is certainly more amenable to the use of maps at a time when plane travel is such a common means of transportation. The Dymaxion Map: "... provides global information with negligible distortion of magnitudes" and "it is the only flatsurface plot of the earth which presents all the true geographic scale areas in a single, comprehensive picture without any breaks in any of the continental contours, or any visible distortion of the relative shapes or sizes of these whole land masses." (Marks) On the original Dymaxion Map it is possible to locate a Dymaxion Equator - a great circle - which passes through Cape Kennedy, Florida, across the United States, through Cape Mendocino, California thence completely around the globe and back again. This great circle has the unusual distinction of intersecting no

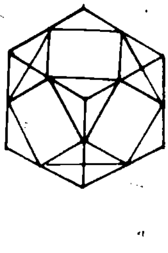

FIG. 5

other land mass than North America. Thus, such a path violates no air space of any other power. Fuller also points out that the 50-50 point on the earth - Lat. 50^{O}No., Long. 50^{O}E. is in Russia, at the foot of the Ural Mountains where Russia, at the time of Marks' book, maintained launching stations. This is the pole of a hemisphere which contains 93% of the earth's population. Thus you can sense its importance in this era of rocketry and its intimations of world domination.

The Buckminster Fuller I have cited briefly, takes us up to the year 1960. During the years 1960-1975, Fuller hasn't been exactly dormant. Peripatetic Fuller, a truly global figure (he used to carry three wristwatches; one which carried the local time, one which carried the time where he was yesterday and one where he carried the time as of the place where he would be tomorrow) is recognized for the genius he is, all over the world. It was not always this way.

P.S. Fuller's most ambitious project has been, and still is, the accumulation of global information on a truly staggering scale. It all flows into a central repository in the United States and is part of a Fuller Foundation activity. He would hope that global concerns - man caring for man - could best be handled when *total* information is available.

P.P.S. I am writing this article in San Jose, Costa Rica and lack a blow-by-blow description of the details of the project. I do know that the New York Times ran a comprehensive article about it. The best of the Fuller information pertinent to the now of 1975 comes from the periodicals. I suggest use of the Reader's Guide.

P.P.P.S. The following story, which I take from Marks, is too delicious and I repeat it in detail:

> He learned at an early age that the teachers lacked satisfactory answers to all the questions he had to ask. One day, for example, the geometry teacher attempted to explain the basic definitions. She put a point on the blackboard, then rubbed it out. "A point", she said, "does not exist - it has no dimensions." She then drew a line. "A line," she continued, "is made up of points but there are no lines." Bucky looked at her wide-eyed as she defined a plane in terms of parallel lines. His eyes opened wider when she announced that no planes exist. The final blow was her presentation of the cube. "A cube," she said, "is a solid stack of square planes whose edges are equal."
>
> "I have some questions," Fuller said, raising his hand. "How long has the cube been there? How long is it going to be there? How much does it weigh? And what is its temperature?"

... All of which suggests the Buckminster Fuller who was to emerge and which also suggests the beautiful little poem of J. A. Linden: POINTS have parts or joints, How then can they combine to form a line?

DEVELOPING THE FUNCTION CONCEPT IN THE INTERMEDIATE GRADES

by Bruce A. Allen, University of Maine

Many authors of elementary school mathematics textbooks have included lessons on developing the concept of a function in their current editions. The rationale for a more explicit treatment of the function concept is probably different for each author, but the following reasons might be cited to support the introduction of what was formerly thought of as a secondary school concept: 1) The idea of a function is a major concept in mathematics 2) Discovery, pattern hunting, and organizing data are all learning strategies currently being promoted in the elementary grades, and functions can be easily developed using these strategies, and 3) lessons promoting function type thinking can be readily designed in mathematics laboratory settings where the pupils are actively involved in their own learning. The elementary school teacher should become more familiar with the function concept, and should explore techniques to present this idea to children in an interesting way.

Arithmetic is essentially a study of number systems which may be thought of as an investigation of different sets of numbers, operations on sets of numbers, rules governing these operations, and relations that exist within sets of numbers. It is in the study of relations that the idea of a function emerges.

Relations may be thought of in many ways. Usually children think of family associations such as mother, father, brother, or sister, when thinking about relations. Later, the idea of a relation as an association may have more general meaning. Children feel the warmth that usually exists between the appearance of the sun and the temperature of the air. There is a quantity association that exists between the amount of candy that they can buy and the amount of money they have. There is a distance association that is experienced between the force used in throwing a ball and how far the ball will travel. A relation, then, may be thought of as an association between two objects or ideas.

A mathematical concept of a relation is developed in school when the learner grasps the meaning of equality in dealing with numbers or of congruence when studying geometry. As children refine their understanding of a mathematical relation they can begin to perceive special types of relations.

The particular characteristic that distinguishes a function as a special type of relation is the uniqueness of the outcome in an association between two mathematical ideas. For example, whenever any number is multiplied by four, a single-valued product is the outcome. Many compound associations may also yield single-valued outcomes. For example, multiplying any number by three and then adding seven to the product will result in a single-valued solution. The following example illustrates a numerical association found in elementary school textbooks.

n	0	1	2	3	...
2n + 4	4	6	8	10	...

Example A

The association that exists between each pair of numbers is indicated below the line in the example. Since there is only one value that is paired with any number in this particular example the association can be thought of as a function.

The numerical associations found in the pupils textbooks may be thought of as descriptions, rules, or formulas. These associations, or functions as they are mathematically called, may be represented with a picture of a function machine, a table of number pairs, or a graph. In a classroom what functions are called and how they are represented, of course, depends on the mathematical sophistication of the pupils.

A function machine, or magic box as they are called in the primary grades, is simply a picture of a device which has been "programmed" to do special arithmetical manipulations. The machine in Figure 1 has been "programmed" to add six to any number dropped into the machine.

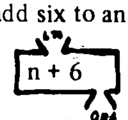

Figure 1

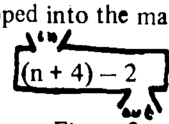

Figure 2

A function machine can be "programmed" by the teacher or a pupil to perform arithmetical operation or any combination of these operations such as the function machine in Figure 2.

A table of number pairs is simply a way of organizing data to aid in exposing any pattern that may exist in the data. Table 1 is an example organizing data to expose the relation that exists between the length of the side of a square and the area of that same square.

Length of side of square	1	2	3	4	5	...	n
Area of Square	1	4	9	16	25	...	(n)(n)

Table 1

The graphing technique for representing functions is more appropriate for the upper elementary grades. With this technique, a geometric interpretation of the data can be developed on graph paper, the pupils then can be guided to gather more data from the given information (by interpolating and extrapolating).

There are many activities that children enjoy doing that will promote hunting for patterns and trying to discover the relationship (function) that exists between sets of numbers. The following "experiments" have proven to be good introductory activities with intermediate grade children.

Patterns in Polygons

This activity involves partitioning polygons into triangles. Have the pupils draw several polygons using a straight edge; 3-sided, 4-sided, 5-sided, ... 8-sided polygons that are convex (corners pointing outward) would be ideal. After the polygons are drawn, pose the question, "What is the least number of triangles into which a polygon can be divided?" When all the polygons have been partitioned into triangles, the pupils can be directed to record their results in a table like the one drawn below.

Summary of Angle Measure of Polygons

Number of sides	Minimum No. of △s in polygon	Sum of angle measures of △	Sum of angle measures of polygon
3			
4			
5			
⋮			

A few questions related to the activity might also be presented to the pupils. "What drawing technique will ensure the production of the minimum number of triangles?" (Always start from the same vertex.) "Why are the numbers 1 and 2 omitted from the first column of the table?" (The minimum number of sides of a polygon is three.) "If the table were extended to include a 14-sided polygon, what would be the minimum number of triangles into which the polygon could be partitioned?" "How about a 39-sided polygon?" "A 100-sided polygon?"

The last two columns of the table should be used only if the pupils have had considerable experience with a protractor. Completing these two columns is another activity in and of itself and is probably more appropriately used in the junior high school.

Building Pyramids

A bag of marbles and a small amount of clay are all the materials needed to generate interesting number patterns. The clay can be shaped into a platform into which marbles can then be stacked on different shaped bases in a pyramidal shape.

Consider building pyramids with triangular bases as described above. How many marbles in a pyramid with a triangular base that is two marbles wide? Three marbles wide? Four marbles wide? Ten marbles wide?

Marbles on side of triangular base	1	2	3	4	5	6	7	8	9	10	...
Total number of marbles											

A similar activity can be done with pyramids built with square bases.

Marbles on side of square base	1	2	3	4	5	6	7	8	9	10	...
Total number of marbles											

How Many Squares?

A. How many squares are enclosed in a checkerboard?
Consider those squares that may overlap with other squares.

B. You may want to simplify the problem by beginning with first a 1 x 1 square, then a 2 x 2 square, then a 3 x 3 square, etc.

C. Organizing your information in the following chart may be helpful.

Length of side	1	2	3	4	5	6	7	8
Number of squares	1			30				

The generalization of the numerical data into a formula has been purposely avoided because the prime purpose of the activities in the elementary grades is to involve the pupils in organizing information, exploring for patterns, and discovering any association that may exist between sets of numbers. If the pupils are able to predict outcomes based on their observations the lessons should be considered successful. Later, as they become more sophisticated in their mathematical knowledge, techniques for generalizing to the symbolic stage may be developed. When lessons are pursued to the stage of abstract thinking, and the children are not capable of working at that level, very often an element of frustration may be introduced into the class that would diminish the enthusiasm which may have been created with "hands-on" mathematics.

There are many activities which will generate interesting number patterns. Mathematics enrichment books and laboratory manuals written specifically for mathematics classes are excellent sources for these activities. Children enjoy "doing" mathematics, and their interest in thinking quantitatively will be enhanced if they are provided with activities similar to those described in this paper.

MATHEMATICAL MODELING FOR ELECTION DECISIONS

by Earl M. L. Beard, University of Maine

What I intend to do is to illustrate, with some examples, how mathematics can be used to construct a model for an election process. Before I do so, I should point out (as all speakers do) the advantages or payoffs that the mathematics teacher might expect from introducing this topic to the class.

First: This is relevent! Elections and ways to rig or manipulate the decision process appeal to both liberal and conservative students.

Second: The rating of candidates by individuals is a relation. It has all the properties of a mathematical relation and the instructor can investigate properties of relations in a "real life" context.

Now let's look at the basic problems involved in an election - namely deciding who has won the election. When there are only two candidates the decision procedure is easy. One of the candidates must have a majority of the votes cast. Look at figure 1. In this case candidate A has a majority of the votes. In case the candidates are tied with 50% of the votes for each candidate any decision will not elect a candidate opposed by a majority of the voters.

# of votes	candidate
51	A
49	B

Figure 1

# of votes	candidate
40	A
30	B
30	C

Figure 2

The decision process becomes a problem when there are more than two candidates. In all my examples the total number of voters will be 100. Even though this is unrealistic it makes the total number of votes for any one candidate equal to his percentage of the vote. I also limit my examples to three candidates. This is the smallest number of candidates that present decision problems. The same problems presented by three candidates exist for four, five, six or more candidates. The models for election decisions involving three candidates are easily modified for four or more candidates. In the classroom situation, I would assign the problems for four or more candidates as exercises.

In figure 2 we see an election result where candidate A receives more votes than either of his opponents, but does not receive a majority of the votes cast. The question is: Is A the most popular (or desired) candidate or do a majority of the voters prefer either B or C? That is, has A received a plurality because two more favored candidates divided a majority of the vote. To answer this question we must look at who the individual voters would choose were their first choice of candidate not available. It is ap-

# of votes	choice of voters	
	1^{st}	2^{nd}
40	A	B
30	B	C
30	C	B

60% of the voters prefer either B or C to A.

The plurality winner - A is opposed by 60% of the voters.

Figure 3

parent that in this election the plurality winner, A, is opposed by 60% of the voters. That is, if a runoff election were held between A and B or between A and C, A would be soundly beaten 40%-60%.

In order to make such a decision it is necessary that the decision maker have more information from the voter than just the voter's first choice of candidates. Thus in modeling our voting procedure we require the voter to rate all the candidates in order of preference. Requiring the voter to rank all the candidates enables the decision procedure to hold run-off election without recasting the ballots.

Now we must model the voter, i.e. we must model his rating possibilities.

SOME POSSIBLE VOTER PREFERENCES

Voter Schedules

Voter rating	S_1	S_2	S_3	S_4	S_5	S_6
1^{st}	A	A	B	B	C	C
2^{nd}	B	C	C	A	A	B
3^{rd}	C	B	A	C	B	A

Voter preference assumptions

1) Voter preference is a transitive relation.

2) If a candidate were removed, voters would note their preferences as indicated.

Figure 4

Figure 4 shows some of the possible ways in which the voter can indicate his preference of candidates. A voter whose first choice is A, second choice is B and third choice is C would vote schedule S_1. If his choices were B, then A, and then C he would vote schedule S_4.

These are certain things to note about figure 4.

First: The number of voter schedules are simply the number of orderings of three objects or 3!

Second: The voter schedules of figure 4 are not the whole story. The ordering of candidates is a non-transitive relation. That is, a voter may prefer candidate A to candidate B and may prefer candidate B to candidate C, but also prefer candidate C to candidate A.

To illustrate this last remark consider the problem of a young lady who has three suitors. In order to decide which would make the best husband she rates them according to wealth, good looks and personality. (See figure 5). Since A ranks higher than B on two of the three criteria, it is clear

Rating of suitor

Criteria		1	2	3
	Wealth	A	B	C
	Looks	B	C	A
	Personality	C	A	B

A rates higher than B on 2 of 3 criteria

B rates higher than C on 2 of 3 criteria

C rates higher than A on 2 of 3 criteria

Figure 5

that A is "better" than B. But B ranks higher than C on two of the three criteria so that the young lady should choose B before C. When she makes the comparison between A and C she sees that C is "better" than A on two of three counts. On these three criteria there is no best suitor.

In evaluating candidates on a number of criteria, voters often run into exactly the same problem. Thus whenever a voter must rate candidates based on several criteria, he may be faced with a non-transitive relation.

Thus a complete listing of voter schedules should also contain schedules like:

1st	A	A-B	A-B-C	B	
2nd	B-C	C		A-C	...
3rd					

From the above, for time (and space) considerations, we will make two postulates concerning the voter.

Postulate 1: Voter preference is a transitive relation, i.e. we limit ourselves to the schedules of figure 4.

Postulate 2: If a candidate were removed, voters would vote their preference as indicated.

Now we are ready to model the election decision procedure. We saw earlier (figure 1 and 2) an election where the plurality winner might have been the least popular candidate. Let us look at an election where complete voter preference is known. (See figure 6). In this election A is the obvious

VOTER PREFERENCE SCHEDULES

	A	A	B	B	C
	C	B	C	A	B
	B	C	A	C	A
# of votes	25	19	9	21	26

Plurality: A; 25 + 19 = 44 first place voter

B; 9 + 21 = 30 first place voter

C; 26 first place voter

Run off between A and B

A; 25 + 19 = 44 votes

B; 9 + 21 = 56 votes

Figure 6

VOTERS PREFERENCE SCHEDULES

1^{st}	A	B	C	C
2^{nd}	C	C	A	B
3^{rd}	B	A	B	A
# of votes	38	33	10	19

Plurality: A-38, B-33, C-24

A wins.

Run of A-48, B-52

B wins.

Condorcet: A vs. C - C wins

A-38, C-62

B vs. C - C wins

B-33, C-67

C will beat either A or B in any 2 way race.

Figure 7

plurality winner with 44% of the votes. However, in a run-off election (omitting C because he had the fewest number of votes) we see that B wins. Thus we see a clear voter preference for B over A.

On the other hand, since we have complete voter schedules why be satisfied with just a run-off between the two candidates with the highest plurality counts? Why not look at all possible run-offs? i.e. A vs. B, A vs. C, B vs. C. This method of looking at all possible two way elections to choose the winner is called the Condorcet count. According to Condorcet the winner of an election should be the one who would win against every other candidate. Although we might expect the Condorcet count winner to be the same as the winner using the run-off system, figure 7 shows that this is not always true.

In this election we see that while A is the plurality winner and B is the run-off winner, candidate C will beat either A or B with ease in any two man elections. Thus it seems we now have a good decision process. Namely, if there is no majority winner, look at the Condorcet count.

Before we become too confident, let's look at one more example. (See figure 8). In this election A is the plurality winner with 43% of the votes. B

VOTER PREFERENCE SCHEDULES

	A	B	B	C
	C	C	A	B
	B	A	C	A
# of votes	43	12	19	26

Condorcet count:

B vs. A:	B-12 + 19 + 26	= 57
B wins	A-	43
A vs. C:	A-43 + 19	= 62
A wins	C-12 + 26	= 38
C vs. B:	C-43 + 12 + 26	= 89
C wins	B-12 + 19	= 31

Non-transitivity of rating even though voter preference is transitive.

Figure 8

VOTER PREFERENCE SCHEDULES

1st	A	B	B	C
2nd	C	C	A	B
3rd	B	A	C	A
# of votes	43	12	19	26

Borda count: one point for every person with lower preference

A;	43 × 2 + 19 × 1	= 105
B;	12 × 2 + 19 × 2 + 26	= 88
C;	43 × 1 + 12 × 1 + 26 × 2	= 107

Figure 9

is the run-off winner, 57% to 43% for A. But look at the Condorcet winner. We see that B beats A, A beats C and C beats B.

Thus our decision procedure has non-transitivity of choice even though we postulated transitivity of individual voter preference. This non-transitivity was first discovered by Condorcet in 1785. Lewis Carroll rediscovered it. It is worth nothing that it was virtually ignored by politicians and political theorists until the 1940's when Duncan Black brought it to light for the third time.

From this somewhat unhappy state of affairs there is little that we can salvage except to assign weights to the voter ratings. In this procedure one point is given to a candidate for every candidate rated lower on a voter's schedule. The winner is the candidate with the highest point count. The number of points that each candidate receives is called the Borda count. Figure 9 shows the determination of the Borda count for the election of the last example. The Borda count winner is C with 107 points, the Condorcet winner is B and the plurality winner is A.

Any student who wishes to "rig" an election will enjoy using the Borda count because the relative weighs assigned determine the rating of candidates. If we assign weight w_1 to first choice, w_2 to second choice, w_3 to third choice on a rating schedule we have the general Borda count as illustrated in figure 10.

GENERAL BORDA COUNT

$1^{st}w_1$	A;	$43w_1 + 12w_3 + 19w_2 + 26w_3$
$2^{nd}w_2$	B;	$43w_3 + 12w_1 + 19w_1 + 26w_2$
$3^{rd}w_3$	C;	$43w_2 + 12w_2 + 19w_3 + 26w_1$

Figure 10

The problems of voting are very relevant to any student of the political process - they occur in committees in legislatures - especially when both are voted on in pairs. This method of voting compares the bills pairwise until all but one are eliminated. From the non-transitivity that occurs when the Condorcet method is used it should be apparent that the later a bill is presented, the better is its chance of being chosen.

The problem of formulating an election procedure as outlined here have been neatly summarized by an economist named Kenneth Arrow. Arrow set four criteria for any decision system S. That is, we would like a decision procedure to satisfy the following:

1. S should work, i.e. S should determine a ranking of candidates for any collection of schedules submitted by the voters.
2. If every voter prefers A to B, then S should determine a ranking in which A is prefered to B.
3. There is no dictator. This means that there is no outside decision maker.
4. The relative ranking of A and B depends only upon the relative ranking of A and B on the voter schedules.

This last criteria might need a little more elucidation. Look at figures 11a and 11b.

Voter Preference Schedule

A	B	E	C
B	C	D	A
C	A	A	B
D	D	C	D
E	E	B	E
35	20	15	30

Figure 11a

A	B	C	E
B	D	E	A
D	A	A	B
C	E	D	C
E	C	B	D
35	20	15	30

Figure 11b

In both of these elections, the same number of voters rated A and B the same relative to each other. Criteria 4 demands that the decision system, S, rate A relative to B the same in both these elections. That is, the rating of candidate C, D and E should not affect the relative ranking of A and B by S.

As reasonable as the 4 criteria for S may seem, Arrow showed that there is no system S that will satisfy all these criteria. This work of Arrow's not only won him a nobel prize, but initiated a new field of study called *Theory of Social Choice*. What this means to the student is that given any decision procedure he can discover an election in which at least one of these criteria is violated.

MEASUREMENT AND THE METRIC SYSTEM

by James E. Bierden, Rhode Island College

There are obviously many reasons for talking about the metric system. The fact that it represents an application of mathematics - the theme of this conference - is one of the best reasons. However, I would like to take a somewhat different view of the metric system and its relation to mathematics and the teaching of mathematics. This paper will focus on the topic: the United States conversion to the metric system is a good thing because it gives teachers an opportunity to look at *measurement* in a fresh light.

A fresh look at measurement certainly will not hurt any of us. In many instances today, the study of measurement in school has become a series of paper and pencil exercises dealing with conversions and formulas. Very little of this gets at the important concepts of measurement which students should be learning.

The overall theme of this paper is directed toward the need for any study of measurement to be *activity-based*. The paper will consider two specific questions:

1. What are the important measurement concepts we should be teaching?
2. What are appropriate activities associated with these concepts?

The answers to these questions will be illustrated using concepts from the metric system of measurement. To help carry out the activity - based theme, you will be given some tasks to help you review measurement and understand the metric system.

Measurement is Comparing an Object to a Unit

This initial measurement concept is seen in most elementary school programs today. Children are introduced to measurement using such non-standard units as books, fingers, paper clips, and their own bodies. Using these units, they measure a variety of objects, from desks to corridors. In this way the childrens' measurement experiences are active and related to their environment.

Measurement activities using non-standard units have a dual purpose. First, they give students many experiences with the comparison aspect of measuring. Second, the lack of any standardization or system ensures exposure to the problems of utility and communication which create the need for standard units of measure.

Choice of Appropriate Units

Before children will be able to choose appropriate units for measuring, they have to have experiences which give them a thorough understanding of the units most often used. The "size" of these units, as well as the types of units used, are both very necessary.

This important concept is often overlooked by many adults. They have become such sophisticated measurers that this concept is no longer operational for them. For example, an appropriate unit of measure for finding the area of a rectangular region - such as the floor of a room - is an *area* unit, usually a square unit. Adult measurers often use a *linear* unit to measure the length and width of the region and then convert these linear measures to an area measure using the formula

area = length x width.

All of this is pretty sophisticated for children, especially if it has not been preceded by many activities which acquaint them with units of length, area, and volume (perhaps even weight and temperature), with the use of these units, and with the relationships between them.

The metric system provides many opportunities for developing this concept with children. The appropriate metric unit of length for initial activities with children is the centimeter. A 20 centimeter or 30 centimeter ruler is just the right size for children to measure many objects they are familiar with. The list below illustrates appropriate experiences in developing an understanding of linear measure. Try them.

MEASURE (to the nearest cm):

1. The length of your middle finger.
2. The width of your thumb nail.
3. The length of this paper.
4. The length of your shoe.
5. The length of your smile.

NOW ESTIMATE (to the nearest cm):

6. The length of the diagonal of this paper.
7. The length of an unsharpened pencil.
8. The width of a telephone book.

For initial activities with the meter, a flexible meter-stick is best. Children can easily construct one out of heavy cloth or by taping together 10 cm strips of tagboard, and then use it to measure a variety of lengths. The flexibility comes in handy for measuring waists, chests or other circumferences, as well as usual linear distances.

To gain experience with other measurement units, children should use square centimeters, square decimeters and square meters to measure regions. The latter can be made from newsprint. Again, try it.

Somewhere in their metric education, children should be introduced to a model of a cubic meter - easily made with twelve lath strips and masking tape. An ambitious class might fill their cubic meter with cubic decimeters - but remember, it takes 1000 of them! Other uses include a "quiet place" for reading, a storage area or, with paper sides, a "computer" for input - output games. If you have never seen a cubic meter model, make one of your own.

All of the above activities with units are designed to develop the child's ability to choose the appropriate unit (by size or by type) when confronted with a measurement problem.

All Measurement is Approximate

We use approximation in most of the measurement we do. Children should have experiences which help them understand why measurement is approximate and how approximations are used. This understanding can only come if the child is an active measurer.

Along with approximation, students should be given a feel for estimation. Measurement activities should regularly begin with children estimating the measurements they are about to make.

It's not a bad idea for adults to follow this procedure too, as they learn the metric system. Using common references such as

meter: a little longer than a yard,
liter: a little larger than a quart,
Kilogram: a little more than two pounds,

we should take the opportunity to estimate lengths, distances, amounts, sizes, and weights whenever and wherever we can. After all, we too are going to have to live in a metric world.

Need for a Standard Unit.

The realization that standard measurement units are necessary for such things as reference, duplication and communication can be achieved in children through a variety of activities. One favorite method uses parts of the body (hands, arms, feet) from different children making the same measurement.

Work on this concept should also include some exposure to the history of the metric system, since it is the history of man's most extensive attempt to standardize measurement. This history has been treated extensively in recent articles in *The Arithmetic Teacher*, as well as many other professional journals.

While many of the facts related to the history of metrication are interesting - such as the fact that the metric system was legally adopted as the official United States system of measure in 1893 - what should be of most interest is the history of the process we have gone through to arrive at the metric system. It is a fascinating story.

Relationships Between Units

This major part of measurement is not one concept but a series of concepts, all of which are important for the child's understanding of the measurement process. The metric system provides a golden opportunity for emphasizing these concepts. Indeed, many people say that the systematic relationship between metric units is the most important aspect in its favor.

There are two main relationships between units in the metric system. The first concerns the decimal base of the system. Multiples and subdivisions of a given metric unit are related to each other the same way as the place values of our base-ten numeration system. This is illustrated by the units of length listed below.

kilometer = 1000 meters
hectometer = 100 meters
decameter = 10 meters
METER
decimeter = .1 meter
centimeter = .01 meter
millimeter = .001 meter

An activity-based approach to measurement should also be in evidence in the development of these concepts. The units, their relative sizes and their relationships are assimilated through varied use over a long period of time. For a change of pace from actual measuring, these relationships can also be reinforced using games and puzzles, such as the tic-tac-toe game shown in Figure 1.

METRIC TIC-TAC-TOE

10 dm	1000 m	10 hm
1 mm	1 m	1 dam
10 m	10 cm	100 cm

1000 l	10 hl	100 dal
10 l	10 dal	100 l
100 dl	100 cl	1 kl

000 cg	1 kg	1000 mg
100 dg	100 g	1000 g
10 g	1 hg	100 dag

Figure 1

There is another relationship between the meter - basic unit of length, the liter - basic unit of capacity, and the kilogram - basic unit of mass or weight. The relationship is easily shown. Construct a cubic decimeter (.1 meter). This box has a capacity of 1 liter. When the box is filled with very cold water (4°C), you have a good model of a kilogram. Using cardboard, tape and plastic bags, students can construct these measuring standard themselves giving them opportunity for some insights into the relationships involved. The standards can then be used for other measuring activities.

Conclusion

We have discussed five major concepts related to measurement and examples of these concepts in the metric system. A further discussion of these areas can be found in the article "Metric, not *if*, but how" by the *NCTM Metric Implementation Committee* in the *May, 1974* issue of *The Arithmetic Teacher.*

One final thought for teachers who are hesitant about learning and teaching the metric system. You will find it the same as other topics you teach: the metric system is easy if you understand it, hard if you do not. Meaning and understand - through activities - must come before formalism and structure. It will be best understood and used if it spiralled throughout the curriculum. So, be of good heart, take that first step, and always - THINK METRIC.

APPLICATIONS OF MATHEMATICS TO MUSIC AND ART

by Clifford Boatner, Quincy (Mass.) High School

Having taught mathematics at various levels for over ten years, I have found what I consider a serious problem common to these various levels -- namely, retention of skills. Students seem to be able to manipulate arithmetic skills while taking arithmetic, but as soon as they go into algebra, they forget arithmetic. Students seem to be able to handle algebra while taking the course but as soon as they hit a geometry course, algebra fades away.

This should not be the case because algebra is arithmetic using letters and practically all algebra can be expressed geometrically.

For the most part I am referring to the so-called "standard" mathematics student -- the student who is college bound but is not particularly fond of mathematics but is taking the courses only because he has to. Naturally, being dis-interested in mathematics makes it very easy for him to cast it out of his mind after the final examination is over.

Though the honor and advanced placement students seem to be more self motivated, they too have problems with retention. These "gifted" students are sometimes neglected as far as making various subjects fit into their own individual talents, hobbies, interests, etc.

So, as I see it, two important things should be investigated: the effectiveness of *integrated or parallel learning* within the structure of mathematics and a more concerted effort to relate, as much as possible, required mathematics to the interests of students. I have already experienced some success with students by constantly seeking ways to present mathematics as a living, useful, beautiful subject.

Some of the methods that I have found useful to me in the teaching of mathematics have led me to develop materials for classroom instruction.

Geometry And Art

Formal plane geometry is one of the most hated mathematics courses in the high school college course curriculum. Thus the teaching of it is a great challenge for any teacher desiring every student in his classes to get something pleasurable out of the course.

Unfortunately, many of the teachers care little for the subject and constantly refer back to the time when they took geometry and because of the dull, drab, approach along with their lack of understanding of the subject, ended up memorizing axiom after axiom, postulate after postulate, theorem after theorem, in agony. Some teachers have even admitted that geometry almost stopped them from becoming mathematics majors. Naturally, this attitude is bound to affect the attitude of the pupils. Heads of mathematics departments aware of this attitude, have jokingly threatened to give teachers geometry to teach, if they don't "behave".

There is the mathematics teacher who positively adores geometry and cannot understand why some students in his class do not "dig" geometry. "How can anyone not see the beauty of abstract thinking, inductive and deductive reasoning?" So they continue shoving one concept after another down the throats of students who would prefer to be anywhere else but in the geometry class.

It is my desire to introduce geometry in such a way that every student in my classes can leave the course with not only enough background to tackle the next mathematics course and the College Boards, but with a feeling that there was something in the course worth remembering. One way is to relate geometrical problems and concepts to everyday items, having the students make drawings in which geometrical forms are dominant.

Musimatics And Mathemusic

Musimatics and Mathemusic (the use of music symbols in place of fractions in the study of arithmetic, algebra, geometry, and trigonometry.) can be used in both music and mathematics classes with students who have some background with rational numbers (fractions), or who are in the process of learning fractions. The most ideal situation, however, would be to have a mathematics teacher working together with a music teacher, using classes that they both have (if this is junior high school). In an elementary school where a music specialist comes to teach a class, the same interdisciplinary technique can be used. At the high school level, it can be used on an individualized basis.

Many musicians are not fully aware of the reliance they have on mathematics while starting, continuing, and perfecting their art.

Mathematics is related in some way, to almost everything in life. Everyone needs mathematics. Some can get by with a little mathematical skill, while others need more, and still others much more.

It is the belief of the author that if musically inclined students, or students who dislike mathematics in general, are given a mathematical system based on symbols that they are familiar with and feel at home with, they can not only develop a greater skill in mathematics but a greater understanding of music, and perhaps an appreciation for mathematics, as well.

It must be noted that though there is a parallel between the fractional value of music notation and common fractions of arithmetic. Certain aspects of music meter, from a musician's standpoint, must be considered. For example: Three-fourths time is not the same as six-eighths time: six-eighths time is a compound time, two beats to a measure. Also, from the mathematician's point of view, three-fourths is not equal to six-eighths, but rather three-fourths is proportional to six-eighths in the strictest sense.

Because there are many fine methods of teaching mathematics and because there is no one way to teach or learn mathematics, these exercises are to be done under the direction of the instructor or instructors involved, or, if permitted, done the way the student understands best. The main thing to realize is that mathematics and music are both abstract, and there are many concepts that we learn but may not immediately see their usefulness.

All of this has results in helping students to "take a second look" at mathematics, and they have found pleasure in the subject that many of them hated previously.

GROCERY STORE ARITHMETIC

by Russell J. Call, Northeastern University

The title I have chosen, "Grocery Store Arithmetic," may evoke different mental images from each of you. A "grocery store" to some may be a "super market" to others. The intent is both to refer to "grocery stores" as well as to so-called "applications" that are indeed "under our noses".

What I hope to do is to suggest to you and to share with you some points of view. You should take what you want with you and discard the rest. What you take should, of course, be worked into your own respective life styles.

The assumption used is that you all accept the fact that mathematics is very important but that there are also other things that are important too! In short, while mathematics is a discipline in its own right, in the spirit of the fall, 1974 NCTM-ATMNE conference, mathematics must have some "... perspectives on applications ..."

Before any meaningful discussion can take place on any topic, all of the participants of the discussion must have a common framework. Even though we are all acquainted with mathematics we are still a heterogeneous group.

First of all we need to have a common framework about the word "applications." We need to look at Benjamin Bloom's Taxonomy of the "Cognitive Domain": Evaluation, Synthesis, Analysis, Application, Comprehension, Knowledge.

Most of you knew already that "application" is a middle-level element in this six element list.

Consider now the eight types of learning as developed by Robert Gagne: 1. Signal Learning, 2. Stimulus-Response Learning, 3. Chaining, 4. Verbal

Association, 5. Multiple Discrimination 6. Concept Learning, 7. Principle Learning, 8. Problem Solving.

Clearly, "problem solving" is the goal of all education, and it surely involves many "applications," but perhaps beginning with "concept learning" comes some type of "application," if we use the meaning of "application" similar to that suggested by Bloom.

While "grocery store arithmetic" may sound rather innocent and/or childish, it is neither! The complaints of society indicate that what we once called "consumer" math has once again become important. In the mid-1970's when the "energy crisis" and what some have called "inflation," others "depression," and others "recession," the topic to "applications" has, in the words of Piaget, found a "readiness" on the part of students and teachers alike.

If we were to search for a common experience among students and teachers, perhaps the grocery store is it! To survive, one must visit the grocery store.

Consider the simple operation of the collection of the money for the groceries purchased. Each of the prices of the items would be recorded on the adding machine and that total would be collected. The cash register would be used just for the total amount of the items. Let's say that the groceries totaled $11.37 on the adding machine. The purchaser might hand the grocer a ten dollar bill and a five dollar bill. What does the grocer do? Assuming that the cash register does not do the subtracting, the grocer probably counts as follows:

Removes from Cash Drawer	Says something like this
3 pennies	$11.37 and 3 are $11.40
1 dime	and a dime is $11.50
1 fifty-cent piece	and fifty is $12.00
3 one-dollar bills	and three dollars is $15.00

Notice the built-in review of elementary arithmetic and how we "count" to the next denomination. Does it not remind you of the addition process of 8 + 7 where we count to 10 with 5 more than ten?

There are some cash registers which even do the subtracting. In the earlier illustration, the total would have shown $11.37 and when the shopper handed the register operator a ten dollar bill and a five dollar bill, the register operator would have recorded a $15.00 as the amount received and upon depressing another button the amount of change would have appeared as $3.63. In fact, some registers would actually have returned the 63 cents automatically.

These illustrations may sound far too basic. Yet, there are some profound mathematic principles illustrated. Also, the more complex the cash register the more practice the operator needs. But the skills he needs are of a different sort. If the register does the computing, the operator is left only to return the amount of change directed by the cash register. But a higher level operation is needed - not "knowledge" but perhaps "evaluation" for the ability to judge whether or not the register is correct.

While we are still visualizing the cash register operation, consider the situation whereby three cans of soup were purchased butwhen the groceries were placed on the cash register counter the three cans got separated from one another. Thus, the cash register operator saw a can with the price "3 for 25" labeled on it. The decision made at that time probably was to punch 9 cents for the single can. A while later another can appeared and 9 cents was again punched. Finally, the third can appeared and if the cash register operator remembered the other two, he may have explained to the customer that since the three cans were purchased and were to be sold at "three for twenty-five" and since eighteen cents had already been rung-up, he would only ring up 7 cents for the third can.

But there is some important mathematics involved here! The concepts of "error" and "correction" are but illustrative. There are also some fundamental ideas we need to explore related to "learning theory." In the elementary school and later we teach "rounding" and the general "rule" is to round to the nearest whole number, or when it is in the middle, to round to the even whole number. Using this "rule," the cost of one item in a 3 for 25 situation would be 8 cents, since 8 1/3 rounds to 8. But, alas, we need to use the "grocery store" "rule" which requires that anything over the whole number goes to the next highest whole number. Thus 8 1/3 rounds to 9. Note also that we are able to develop the concept of "error" here. For if we round to the nearest whole number we see that the error is only 1. Thus, 8 1/3 rounded to 8 and 3 times 8 = 24, and 24 is 1 away from 25. If we round up to the next highest, we get 8 1/3 = 9 and 3 times 9 = 27 or 2 away from the 25.

If we carried that simple example - 3 for 25 - to a further sophistication, we could carry it into the field of "logic" and maintain that a single item could be sold for any amount the grocer decided. A listing of "3 for 25" means exactly that and nothing more! That three will be sold for a certain price does not mean that one or two will be sold for any specific price. Principles of "logic" would say that the information "3 for 25" in no way suggests what "one" would cost. In fact, if you go to Fenway Park in Boston the peanut man advertises his peanuts as "10 cents a bag - three for a quarter.

When I referred to "learning theory," I meant the concept of "interference." For there surely is a chance for "overteaching" the concept of "rounding." If so, some students may have difficulty in using the "rounding up" concept in grocery store arithmetic for it conflicts with the usual "rule." Then, you know you need to cope with these inconsistencies which are forms of "interference."

The determination of the cost of more than one but for less than the multiple cost is computed in more than one way. For example, consider a multiple price as follows: "7 for 64." The price of a single item is computed then to be 9.14.

For five items, one way is to multiply the computed unit price by five. Thus, the unit price when rounded is ten cents and five items would be fifty cents.

Another approach is to multiply the computed unit price before rounding. Thus 5 x 9.14 = 45.70. 45.70 would be rounded to 46 cents.

Think about the conçept of "spoilage" in grocery store arithmetic. When a store purchases perishables it must build in a cost for spoilage. Unlike soap powder, for example, many items of produce and of meat must carry a greater margin of profit to offset the loss caused by spoilage. There is an enormous number of math topics here - profit, markup, etc.

Have you ever noticed how coupons are redeemed? Sometimes, you must pay the full grocery bill and then you will receive the cash in exchange for the coupons. In other stores, the amount of the coupons is subtracted from the item to which the specific coupons were related. In still other stores the amount of the coupons is deducted from the grocery store bill. Notice that the next amount of money spent for groceries is the same regardless of which of these procedures is used.

One of the most fascinating things in a grocery store is the "express" line. The purpose of an "express" line is to prevent the necessity for a customer with only a few items to stand in a line behind some persons with huge orders. The "variable" (a good math word) is the "number of items" and this is said to affect the thing we are interested in saving - "time!"

Listen to the following statements that appear at check-out counters: Express line 8 or fewer items, Express line less than 8 items, Express line 12 or fewer items, Express line less than 12 items, Express line 10 items or less.

From a mathematical standpoint, "8 or fewer items' does not mean the same as "less than 8 items."

The terms "continuous" and "discrete" are involved here. With respect to the topic of "items," we would have to admit that "item" suggests "discreteness." In other words, there is a next item to item "twelve" and it is called item "thirteen."

On the other hand, "continuous" when referring to something mathematical and relating to grocery store arithmetic could be "weight." In other words, weight does not immediately jump from three pounds to four pounds - there is increasing weight between three and four pounds.

"Eight or fewer items," since "items" are "discrete," would suggest the following: 1, 2, 3, 4, 5, 6, 7 or 8 items; while "less than 8 items" refers to: 1, 2, 3, 4, 5, 6 or 7 items. "Less than 8 items" would identify the same thing as would "7 or fewer items," but the condition is that we treat "items" as "discrete."

The rather interesting thing is how 8 items, 10 items, or 12 items is arrived at. Quite obviously, the purpose is to make it possible for people to be served well. Would, for example, the use of 18 items as the cutoff ite up the express line to the detriment of those with 10 items? Perhaps 18 items belong in the regular line. But the question is a mathematical one and can be a very enjoyable one.

What is being considered here is mathematics in its most practical form. We are attempting to define a cut-off point which separates the fast-moving express orders from the slower-moving large grocery orders.

That suggests, of course, that the human being actually running the cash register is a "function" of the degree of "express." With that in mind and knowing that the person running the register is a member of a set of "variables," it is often faster to go through a non-express line!

Still another consideration is the fact that the grocer needs to serve all his customers. He cannot tie up a register for express orders if in fact all of his regular lines are already too long. Thus, the mathematics of economics comes into play. If a steady stream of "express customers is using the express line, that is one thing - but if the very existence of the express line is too expensive because of too little use, that is something else again!

If we were to talk in terms of "calculus," we would consider what things need to be "maximized" and what needs to be "minimized." For example, one item we desire to be "maximized" is the speed through which customers can go through the check-out lines. An example of an item needing to be "minimized" is the cost of operating the check-out line of the express type.

Let's change to another topic for a moment. Consider the topic of determining how much help the grocer needs and when. He wants enough help at any one time - but not too much! He wants the maximum help when the maximum business is taking place.

It is, initially, often not possible to compute these facts accurately. Experience is necessary! That experience does not have to be the experience of that grocer - he can use data collected by other grocers. There is plenty of applicable mathematics here!

Another maxima/minima application involves packaging. The producers of cereals, for example, want to maximize the volume of boxes while minimizing the cardboard needed to construct the boxes. The grocer, on the other hand, wants to maximize the number of types of cereal he can display which means minimizing the width of the boxes.

Still another topic in grocery store arithmetic can be named "the best bargain" which would be essentially the collecting of data and performing some divisions. Here is an example: A 49-ounce package of soap powder was listed at 81 cents. The same brand of soap powder was listed at $4.69 for a 20-pound box. The price per ounce in the first instance turns out to be 1.653 cents while in the second instance it turns out to be 1.465 cents an ounce.

I'm sure you already know that there is ample opportunity for applications in the grocery story. I have but reminded you of a few! I do want to touch briefly on the concept of "application" as it relates to teaching. Clearly, we live in a pragmatic world - but then again, we always have! So we cannot appropriately say that our theme of this conference is "new." It is "timely," however! And I want to indicate very briefly how good teachers have always behaved and all the literature written by the "romanticists" does not change what we know to be so about good teaching.

One of the speakers at this conference, Dr. Vincent Glennon, has written and spoken about good teaching and I have adapted it. He maintains that good teachers need to know their content, to have a good cultural foundation of their field, and to make use of the knowledge of learning and teaching theories as they relate to the characteristics of the students of the particular age we are teaching. Good teaching comes about during the times when these three factors "intersect."

I also want to refresh your memories of what good teaching was in the nineteenth century according to Herbart. His five teaching steps were as follows: 1) Preparation, 2) Presentation, 3) Association or Comparison, 4) Generalization, or Abstraction, 5) Application. Thus, the fifth Herbartian step made use of what we know about learning theory. While we do not necessarily need an immediate use for our knowledge, if we practice a skill by applying the knowledge, it tends to stick with us better.

Now, it is very important that we make certain that we bring our teaching skills into the twentieth century. Therefore, we look to the 1970 list (Marks, Purdy, and Kinney): 1) Preparation, 2) Exploration and Discovery (we no longer present; we allow our students to discover), 3) Abstraction and Organization, 4) Fixing Skills, 5) Application. We find "application" still there - holding an important place in the field of "good teaching."

Hopefully, some questions have been raised in your minds. ' Application" is not a basis, low-level activity. It is vital to the process of teaching for it "puts it all together." It helps in the process of synthesis.

"Application: does not answer the question of your student, "How do I use it?" Rather it should provide the appropriate experience for your students to utilize knowledge and comprehension in new and different situations. And that is an important characteristic of a well-educated person!

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MATHEMATICS AND FLUID POWER TECHNOLOGY

by Richard H. Carter, University of Maine

I. Introduction

The conference theme "Mathematics: Perspectives on Applications" really has much appeal and is most appropriate in today's quest for relevant education. The theme reminds me of my initial impression of Algebra, which I regret was negative. I am sure my instructor felt that he motivated everyone by proving algebraically that the weight of a mouse equalled the weight of an elephant. While seeing is believing, the approach and justification for such a course left me cold as I was looking for more practical applications of mathematics.

The topic "Mathematics and Fluid Power Technology" assumes that this audience has a limited knowledge of technology, and in particular of fluid power technology, which will compensate for the speaker's limited knowledge of mathematics. Hopefully it will prove that Technology needs Mathematics and that Mathematics needs Technology.

II. Fluid Power Defined

Fluid power -- the science and technology of power transmission by means of potential energy changes in a fluid medium. Fluid Power may be defined as the transmission of power, energy or force by means of a fluid, like air or oil under pressure.

III. History and Growth of Fluid Power

Fluid power is one of three major means of transmitting power (other means being mechanical and electrical) and has been known for many years. In fact the first known use was by the Egyptians in building the pyramids and it was used to split stone. Its refinement and application evolved very slowly, however, and it was not until World War I, with its demand for increased production that fluid power came into its own.

In the past two decades, however, its growth and development, because of its many inherent advantages and flexibility, has been rapid and significant. In fact, its growth rate in sales dollars has been over 1000%.

IV. Fluid Power Systems

Since the Systems Concept in education and technology is the in-thing it might be helpful to review a basic systems model and then apply it to a basic fluid power system.

With the systems concept in mind a fluid power system may be developed to meet a practical requirement. It will identify parameters for design criteria and allow for the optimum selection of sub-systems and their related components.

For this purpose, let's assume:

1. You have a work requirement that you must accomplish (this is an output of the system). The requirements are to lift a load (weight) of 1000 pounds a distance of 12". You must have the capability of stopping and holding this load at any point in the 12" lift. You must be able to control the speed at which the load is returned or lowered.

An examination of the system requirements will be invaluable in the selection of components to complete the system and to determine mathematical considerations. The mathematical requirements and considerations for this system will include:

Output -- Force (load), Speed (work rate), Direction.

Process and Control -- Pressure requirements, Flow requirements (velocity) Reynolds Numbers, Flow direction.

Input -- Flow (volume and capacity) G.P.M., Fluid conditioning (temperature and quality), Prime mover -- type and size (horsepower), Tank storage -- capacity.

Total System -- Efficiency INPUT ⟷ OUTPUT

V. Efficiency and Economics

Today like never before in our history we are demanding technology to become more efficient. Proper design, maintenance and operation of fluid power systems can and will make this possible.

"Technology needs mathematics to achieve this goal. Mathematics needs technology to be relevant." *Let's get together.*

ENERGY: AN INTERDISCIPLINARY APPROACH TO MATHEMATICAL APPLICATIONS

by Paul L. Estes, Department of Mathematics and
William J. Taffe, Department of Natural Science
Plymouth State College

Introduction

Mathematics teaching may be approached from several viewpoints. One instructor might present mathematics as symbolic logic, rigorous thinking, proof, emphasizing relationship and language. Another might concentrate on structures or systems. The theme of these proceedings is yet another approach to mathematics, a route via applications. Within the application-oriented route, however, there are different paths. One is to introduce new mathematics by means of example, developing the mathematics while concurrently showing its use in a particular physical problem; new problems are chosen to illustrate each new bit of mathematics. But, as Morris Kline stated, "Mathematics is not an isolated, self-sufficient body of knowledge. It exists primarily to help man understand and master the physical, the economic and the social worlds."[1] So, a second applied approach might go one step further than the first. An extended, broad-scope problem might be examined and as the problem is unfolded, the mathematics needed for each new aspect is introduced. The goal is the solution of a human problem; mathematics becomes an interdisciplinary part of a problem-oriented study, not seen in naked isolation but as part of human learning.

We opted for this latter approach and designed an experimental course to test the idea. Mathematics became "objective-oriented rather than subject-oriented".[2] It was encountered while analyzing and solving a real problem. Understanding the problem, its solutions, their meanings and implications, was the primary objective. But it was often necessary, and therefore a secondary objective, to find and understand methods to reach the goal, to learn the mathematics necessary to analyze the problem and reach a solution.

The problem we chose was energy. We examined man's energy demand and nature's energy supply. We studied the ways man taps the storehouse of energy and the transformations it must undergo before it is in directly useable form. We unraveled the pathways by which energy flows from source to user. As we sought to comprehend energy supply and demand, students learned and utilized some mathematics, physics, chemistry, biology, geology, meteorology, engineering, social science and tapped any other body of knowledge which aided an attack on the problem.

But how does mathematics fit into this problem? Briefly stated, it is because the problem is quantitative. How *many* people will require how *much* energy from a finite source variable in space and time? Quantitative formulations are needed to answer quantitative questions. So we built mathematical models of populations and their growth, of percapita energy demand and its growth, of energy flow networks and other aspects of the problem. The goal was to understand the energy problem. Mathematics was an indispensible element in the fully interdisciplinary formulation of this applied problem.

An Overview of the Program

The National Science Foundation offered support for such an approach through a Student Science Training Program grant. We accepted a group of twenty-seven students from several states with cities, suburbs and rural regions all represented. We intentionally sought students who had high potential and above average motivation, but whose local schools, through lack of facilities, staff, etc., were unable to allow them to unleash those abilities.

The program fully occupied the middle six weeks of the summer, an intensive experience yet within the capability of a motivated student. The daily format varied as the program proceeded. Lecture, seminar, discussion, problem sessions, each was used when it fit the immediate objective. The first weeks leaned toward lecture, but as the students developed greater understanding of basic principles, discussion assumed greater importance.

As discussed earlier, the energy problem is a most interdisciplinary topic. Most of the physical sciences, some biological science, mathematics, engineering, computer usage, and the social sciences of economics, sociology and politics must be utilized to offer viable solutions. In an approach of this nature, with such wide scope, the problem must be intentionally limited. So, while recognizing that any solution is required to be economically competitive, socially acceptable and politically feasible (high school students are wonderfully optimistic on this last point), we concentrated our efforts on the scientific and mathematical analyses of the problem.

The studies began with a consideration of man's energy demand due both to population increases and to the growth in per capita energy use. This was pursued mathematically as the students wereintroduced to graphical methods of data representation, exponential growth of populations, logarithmic graphs, etc. The basic concepts of energy were examined from a physical, chemical, biological, and ecological approach. Energy transfer mechanisms and the balance of energy on the globe and in the biosphere followed.

The sources of society's present energy supply, both continuous and depletable were then examined. The geology of coal and petroleum, their chemistry and combustion, led to the analysis of the present energy supply for various economic sectors and energy uses. The finitude of these depletable supplies led to estimates of the time scale available before new energy sources are needed.

Simultaneously, the mathematical projections of energy demand developed as algorithmic thinking, flowcharting, and BASIC programming were learned and applied to the extrapolation of present into future conditions. The models of various authors (for example, John Fisher in "Energy Crises in Perspective"[3]) were examined for underlying assumptions.

The discussion of energy use generated two additional mathematical needs. The consideration of rates led to the concept of a derivative. Thus, a brief introduction to differential calculus was begun. But since the rate of energy use has varied in time, to understand the total consumption, integral calculus was also needed. Both topics were introduced in a manner such that the student could grasp their physical (geometric) interpretation and were developed in response to a problem. Differential calculus was also used to develop regression line techniques which were used to analyze both population growth and energy per capita growth rates. Integral calculus was further extended to energy problems such as hydrostatics problems presented by hydroelectric power stations.

The mathematical techniques allowed students to make their own computer-based projections of the total energy demand and to examine the effects of various alternative assumptions in their own models.

While the students were developing their energy demand models, they began a series of seminars which were designed to elucidate some of the newer techniques for meeting that demand.

Student groups prepared reports on topics such as breeder reactors, fusion, geothermal power, solar power, MHD, and others. The discussions considered the basic methods of each source, its advantages, disadvantages, side effects, economic potential and other features. When necessary, the instructors preceeded the seminars by introducing basic science concepts. A guest lecturer presented the effects of the various pollutants (particulate, thermal, radioactive) on the biosphere.

The energy demand made by transportation systems was then considered. The students recognized that, as fossil fuels diminish, an oil economy will be replaced by an alternate, and the possibilities of the hydrogen and electric economies were examined.

A last topic was the analysis of energy transmission. Electric grids, such as New England Power Exchange, as well as the need for new transmission capabilities (superconducting transmission lines, etc.), and other technology were presented in student seminars. The basic mathematics of network analysis was developed and applied to the problem, such as laying a minimal-cost network of natural gas pipelines from the Gulf of Mexico to the consumer.

The final week was devoted to one goal. In teams of four, the students were asked to develop and present a proposal for the energy demand and supply for the United States for the next 50 years. The proposal was to be quantitative. It was to consider the demand for energy (on a regional basis) and what methods were feasible for meeting that demand. Students considered the source types most available to a region, the stresses each source would place on the environment, and the distribution of energy. Under consideration were such topics as the technological possibility of introducing a new source at an economic par with other sources, the social alterations involved, and the political difficulties raised in developing new tehcnologies. The students were not expected to face the energy problem in its totality but to be as detailed as time permitted. They experienced the frustration of quantitative planning based on uncertain assumptions, the vagaries presented by developing but yet unproved technology, the interplay between scientific-technological and social factors encountered wherever the problem solver is involved in issues with social implications, and the realization that simplistic solutions usually need to be rethought in a more critical light. Moreover, they learned something about how to attack a problem, how to use a systems approach to its solution, and how to employ quantitative reasoning, precise statements of relationship and mathematical models in the solution of a real problem.

Mathematics: Topics, Sequence, and Examples

Specifically, what Mathematical topics can be used and how can they be integrated into a study of the energy problem? The possibilities are unlimited; almost any branch of Mathematics can be integrated into a study of some aspect of the energy problem. The magnitude and complexity of the energy problem present us with tremendous quantities of raw data which must be organized, analyzed, and interpreted. This calls for both graphic and statistical techniques. Various types of graphs (for example linear, semi-logarithmic, full logarithmic) must be well understood so that each can be used in the appropriate circumstances. A complete arsenal of statistical techniques can be used to full advantage: regression and correlation, hypothesis testing, probabilistic analysis applied to governmental and business decision-making. One of the first major uses of linear programming was to the classical blending problem: in what quantities to mix various grades of gasoline to obtain an optimum blend. The following are some of the mathematical topics we drew on in our institute: exponential functions, least squares fitting of straight lines, calculus, and network analysis. In addition to these mathematical topics, the computer language BASIC, used on the college's time-sharing computer facilities, proved to be invaluable.

Exponential functions enter early into the discussion of the energy problem whenever one asks: How did we get into our present predicament? The answer, from a purely mathematical point of view, is extremely simple: energy use has been growing exponentially, not only in the United States, but worldwide.

EXPONENTIAL GROWTH OF ENERGY USE

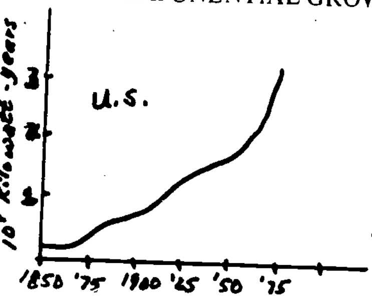


Figure 1

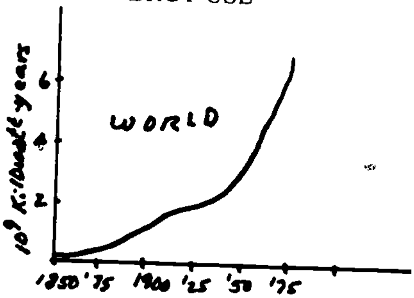


Figure 2

The annual growth rate of energy use, r, enables us to express the growth by an exponential function

$$A = A_0 e^{rt}$$

Where A is the amount after t years starting with $A = A_0$ when $t = o$. Using this general exponential function, it is easy to derive (after a discussion of the irrational number e, logarithms, and the concept of a limit) the rule of thumb for calculating doubling times: $t_d = \frac{70}{r}$. Thus, from an annual growth rate of 7% for electrical energy production, we know that production doubles every ten years.

Later in the program, in studying nuclear energy, we are confronted with the problem of radioactive wastes. Since these wastes decay exponentially, we have another use for the exponential function. But now the growth rate is negative, doubling times are replaced by half-lives, and the characteristic rising exponential growth curve is replaced by a declining decay curve as shown in figure 3.

One of the major objectives in studying the energy problem is to try to predict, as accurately as possible, the energy needs for the years ahead. In various references one sees different projections, many of which are constructed quite glibly. The most frequent type of projection is a simple extrapolation of past exponential growth. It is instructive to carry out such a projection just to see where it leads and as an exercise in preparation for more refined projections.

How should we carry out a more refined projection? One that might first come to mind is to project the demand for oil, for coal, for gas, for nuclear energy, etc. and then add up the separate projections. This method is full of uncertainties, especially when we consider that the separate demands are mutually dependent. For example, the demand for oil is dependent upon its cost relative to the cost of the alternatives.

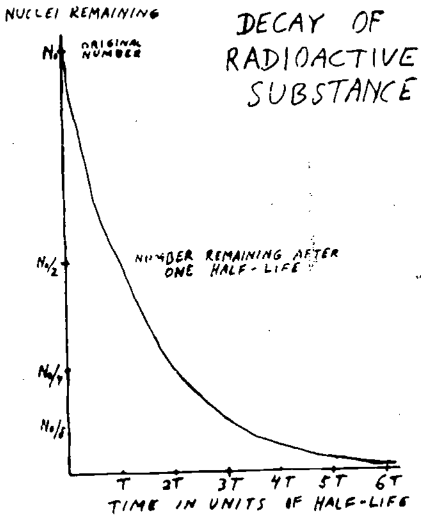


Figure 3

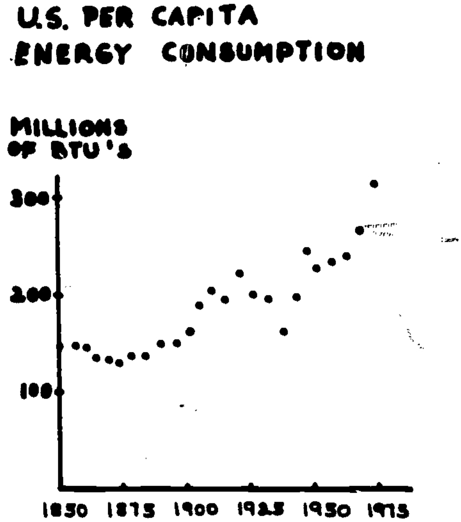


Figure 4

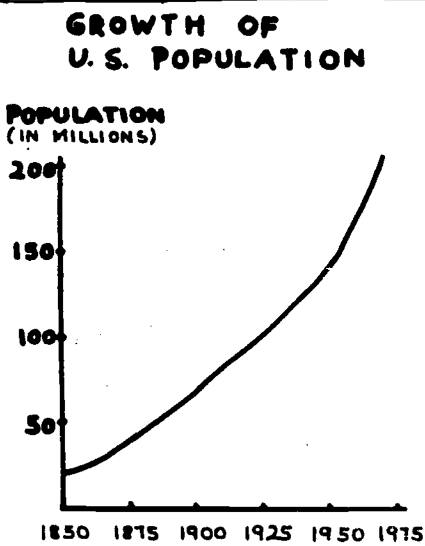


Figure 5

The method we employed for arriving at energy demand projections requires a further analysis of past energy consumption. Figure 4 shows how per capita energy consumption in the United States has increased since

1850. It has slightly more than doubled.

Figure 5 shows U.S. population growth during the same period. It has increased nine-fold. Thus the total growth in U.S. energy consumption has been the result of two factors: more people and more consumption per capita with the population increase clearly being the predominant factor. If we can project the percapita growth of energy consumption and the population growth, we can combine our figures and get a projection of total energy consumption for the years to come. This was our approach.

To project per capita consumption we fitted a regression line by the method of least squares. See Figure 6.

REGRESSION-LINE PROJECTION OF U.S. PER CAPITA ENERGY CONSUMPTION

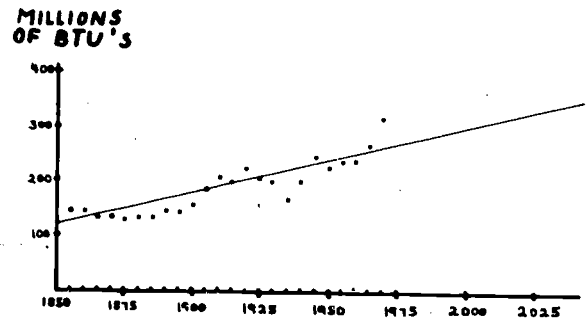


Figure 6

Then with a linear equation in hand, it was a simple matter to project the percapita consumption for the year 2000 or any other year we might choose.

The need for fitting a regression line motivates quite a bit of Mathematics. First, the student must be thoroughly experienced in ordinary graphing of straight lines. Then, the problem at hand raises the question of how to find a straight line which best fits a set of graphed raw data points. After the usually preliminary discussion of crude eyeballing techniques and a consideration of what is meant by a line of "best" fit, the student is guided to the least squares criterion: The line of best fit is defined to be that line for which the sum of the squared deviations of the data points from the line are a minimum.

To find the least squares formulas for the unknown slope and Y-intercept of the regression line, we must now apply differential calculus to solve the above minimization problem. In our program, we had at this point already introduced some differential calculus in connection with previous discussions of rates of change (of moving objects, of growing populations, etc.) Thus, the calculus foundation which had already been laid merely had to be built upon a little further to complete the task of deriving the method of least squares. As is often the case, the consideration of a single problem has taken us on an extended mathematical journey: from the problem of predicting future per capita energy consumption to straight line graphing to a search for a line of best fit to differential calculus to a derivation of the method of least squares and finally back to the solution of the problem that motivated it all.

The above journey in search of the solution to one problem has given the student tools with which to tackle additional problems. Described a few paragraphs below are our techniques for predicting population. The method of least squares is needed again but this time the student is already prepared. Before we delve into these population predictions however, let us first observe that other seeds have been planted which will later bear fruit.

The student has been unknowingly getting ready to learn some integral calculus and apply it to the solution of other energy-related problems.

He has been exposed to the concept of a limit; and while developing the machinery for regression analysis, has encountered summation notation. Thus, he is prepared to look at the limit of a Riemann sum. The need for evaluating such things has arisen in the discussion of energy usage. From a graph showing the rate of energy consumption as a function of time, we can find the total consumption over a period of years by computing the area under the graph.

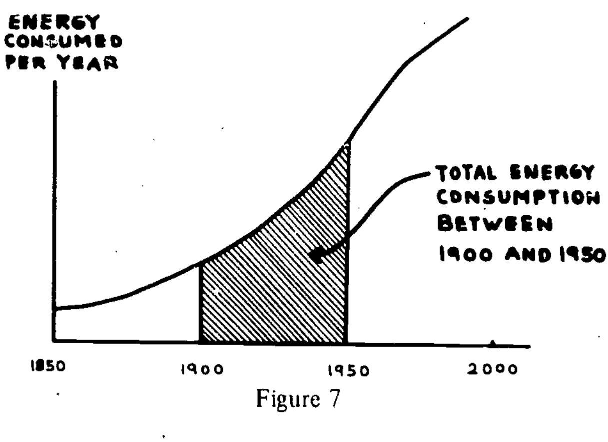


Figure 7

Whenever the graph is a continuous curve, we have a need to evaluate the limit of a sum of areas of rectangles, or in other words, the limit of a Riemann sum. Thus, we have discovered integral calculus as a necessary tool for proceeding with our investigation of the energy problem.

After working a bit with integrals, the student is equipped to solve some of the problems which arise in the consideration of hydroelectricity. For example, in designing the dam for a new hydroelectric plant, the engineer must know the total force on the face of the dam of the water which is backed up behind the dam. This force is calculated by evaluating a definite integral. As a second example, suppose a utility wants to store energy for periods of peak demand by pumping water up to a reservoir on a nearby hill. How much energy is required to do the pumping? This question too is answered by evaluating a definite integral.

Let us return to the task of predicting population growth. We first observe that although U.S. population has been growing, the growth *rate*, which is graphed in figure 8, has been declining.

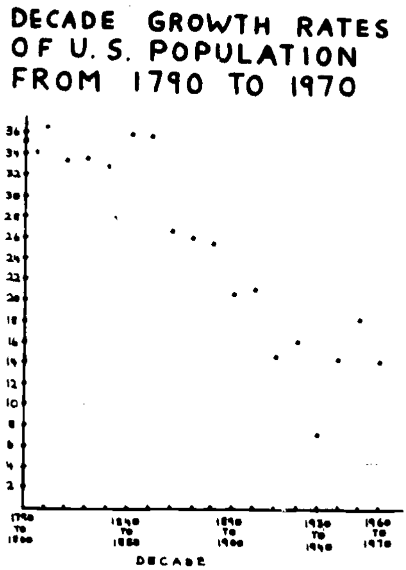


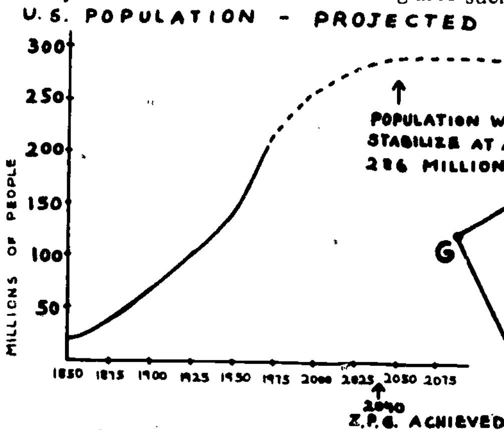
Figure 8

Since there is a growing consensus that the growth rate should decline to zero and in fact a conscious effort is underway to make this occur, it seems likely that the trend will continue. Hence we felt justified in fitting a regression line to this data to obtain an equation which could then be used to predict the growth rates and thus in turn the growth itself. From our regression equation, we estimate that the U.S. population growth rate will reach zero about 2040 A.D. with the population then stabilizing at around 286 million people. The projected growth curve is shown in figure 9.

We are finally ready to project total energy demand for any year we choose. To predict total demand in the year 2000, say, we multiply the projected population for that year times the previously calculated figure for per capita demand. To get an overall demand picture of the decades ahead, we performed these calculations for the years 1980, 1990, ..., 2040.

Our methods of projection are clearly not the only ones possible, but they seem to us to be at least as reasonable as the methods utilized by other authors. Furthermore, they require no high-level Mathematics and hence can be incorporated into the curriculum as early as the twelfth grade.

Another branch of Mathematics which we found useful in studying energy is network analysis. For transmission of energy, goods, and services in our society, we have networks of roads, transmission lines, pipelines, railroads, airline routes, television relay stations, routes of ocean-going vessels, etc. For purposes of constructing minimal-cost connecting networks, all of these possess the same, or similar, abstract structure. Each can be regarded as a system of vertices and connecting arcs such as the one in figure 10.



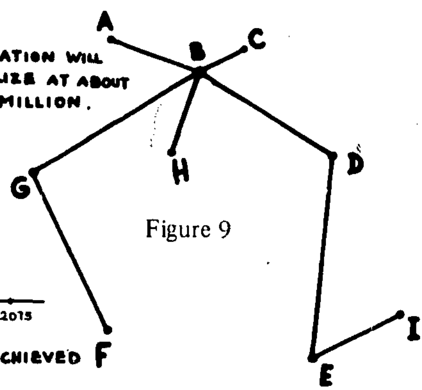


Figure 9

Figure 10

To construct a minimal-cost network, there is a very simple algorithm due to Kruskal by which one proceeds as follows: First, build a table of costs for all possible connecting links. Choose the cheapest link to begin the network. Then, at each successive step, add the cheapest link provided that the new link does not complete a circuit together with the previous links. The proof that this rule does in fact give a minimal-cost network is suitable for presentation to high school students. It requires no advanced Mathematics.

Kruskal's algorithm is a powerful one in that it can be used to solve a large class of network problems. However, many network problems are beyond its reach because of their extreme complexity. Frank and Frisch[4] discuss the use of network analysis in designing a minimal-cost system of pipelines to collect natural gas from wellheads in the Gulf of Mexico. Because of constraints on the allowable pressure of the gas, the pipes had varying diameters - seven different sizes in all; and changing the diameter of one pipe would affect the pressure of the gas elsewhere in the system. The complete solution of this particular problem is a little too advanced for presentation at the secondary level. However, the introduction of the problem, especially when one mentions the savings which were effected (ten million dollars!), provides excellent motivation for further study of network analysis.

To handle the extensive calculations involved in our investigations, especially in the population and energy demand projections, we used time-sharing computer facilities (teletype terminals at our Plymouth campus linked by telephone to the IBM 360 at Durham). Our students enjoyed working at these terminals and interacting with the computer.

The calculations required were extensive in the sense that they were long, tedious, and repetitive; they were not especially deep. For example, to extrapolate the exponential growth of a population, one uses the same exponential function repeatedly with only the time variable changing by taking on successively larger values. This is the sort of calculation which is ideally suited to a computer. The computer is performing the boring tasks quickly which the student's interest is heightened by an introduction to the exciting world of computers.

The use of teletype terminals has two advantages over entry of data via punched cards. First, the student can interact with the computer; he gets immediate answers or error messages. There is no need to wait hours or days for his results. Secondly, there is no need for a lengthy introduction to FORTRAN. The usual language for terminal use is BASIC (Beginner's All-purpose Symbolic Instruction Code). BASIC, being much closer to conversational English than FORTRAN, is very easy. Our students were quite proficient after only a few hours of instruction. Yet in spite of its simplicity, the BASIC language is sufficiently powerful for the programming of fairly complex problems. Hence, it is an easy and effective means of communicating with the computer.

All of the above mathematical topics arose naturally in the course of our analysis of the energy problem. We never were asked that question that students so frequently ask: "Why do we have to learn this?" Each topic was introduced because it was needed to help solve some energy-related problem. As a result, we had no problem in motivating students. They were able to see good reasons for learning what they were studying and therefore went about their work with a rare enthusiasm. Also, partly because of their high motivation, we were able to cover much more material than could normally be done in six weeks: the mathematics of exponential growth, a fair amount of statistics, a modest introduction to differential and integral calculus, a bit of network analysis, and an introduction to computer programming.

Summary

How far can this approach be pushed? What levels of mathematics can be taught in this manner? As we indicated previously, the possibilities seem unlimited and our investigations thus far have proven very rewarding and successful. We intend to pursue the energy problem further. But many other topics will also lend themselves to this approach, some more so than others. Precisely which problems and how much mathematics they require are "left as an exercise for the reader."

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THE MOVE TOWARD APPLICATIONS: WHERE DOES RESEARCH FIT OR DOES IT?

By William E. Geeslin, University of New Hampshire

At present there is evidence of concern over the mathematics curriculum. In particular, "modern" mathematics curricula are confronted with increasing criticisms and some proposed changes. One change, requested by Kline (1958) and others, is to include more applications and/or applied mathematics in curricula. Kline requests that we sequence mathematics curricula in a manner similar to the historical development of mathematics. He suggests that we use applications as both motivational devices and goals of mathematical learning. The purpose of this paper is to convince educators that opportunities for research exist in this area and that *now* is the time to conduct this research.

Consider a historical example of reform in mathematics education. Over fifteen years ago the Russians launched Sputnik. A great public outcry arose against the mathematics and science curricula. Mathematicians, mathematics educators and others claimed we need "modern mathematics." This new curricula (actually a new approach to the same mathematics) would produce more and better mathematicians, scientists, and technologists thus allowing us "to catch" the Russians. Although the effect of the new curricula is unclear, we did overtake the Russians and are presently over-supplied with mathematicians. Of course, some of these effects are due to variations in government spending, etc. A massive change of curricula occurred due to forces outside education. Little or no research was done on the effects of the new curricula. We are in much the same position today. That is, we have a call for reform, namely applications but we do not have empirical information on the effect or direction of the proposed reform.

An examination of the last reform is revealing. Although forces outside education allowed reform to take place by providing money, support, and demands for change, educators made many claims concerning the new curricula: knowledge of structure is required for full understanding of subject-matter (Begle, in preparation); enhances retention (Bruner, 1960); facilitates problem solving (Wilson, 1971); leads to transfer to similar and perhaps even new situations (Schwab, 1962); results in intellectual excitement (Schwab, 1962); leads to an aptitude for learning (Klopfer, 1971); allows the solution of problems not solvable by computational methods (Begle, 1971); contains something more than rote algorithms and skills (Bruner, 1960; Report of the Cambridge Conference, 1963); and is a prerequisite to problem solving and principle learning (Gagne, 1965). That is, modern mathematics with its emphasis on structure and understanding would solve all our teaching problems: it is exciting, motivating, produces better problem solvers (both pure and applied), produces better understanding, more advanced knowledge, etc. The difficulty is that these claims were suppositions and assumptions, i.e. not based on empirical fact. As will be pointed out later, these claims are very similar to claims made concerning applications.

The original evaluations of modern curricula, if any, were teacher opinions during tryouts of material already written. Effects on students were not considered except in a second-hand manner using information from persons convinced that the new curricula were inherently good. After adoption of new curricula was well underway, a few systematic evaluations were conducted. For example, in 1963-1968 SMSG (SLSMA) performed a five year longitudinal study comparing modern and conventional curricula. Among the results of this study were: students learned what they were taught, did not learn what they were not taught, students using modern texts fared better on understanding and problem solving, and students using conventional texts were better on computation (Begle, 1973). These results are nontrivial but they appeared in print over ten years after curricula changes were made. This was certainly too late for schools to use the information in selecting texts. Our move toward applications is in the stage of supposition. Thus we should proceed with research now if we are to accept or reject the inclusion of applications in mathematics curricula in a knowledgable manner.

Before proceeding with a discussion of necessary research concerning applications, we will examine a study that illustrates the type of research, difficulties, and methods which could provide useful information to teachers. The most publicized claims of the new mathematics concern its emphasis on structure. In fact, structure and its effect on student understanding is a major reason for the existence of modern mathematics. Yet, only recently has anyone attempted to examine learning of mathematical structure (cf. Branca & Kilpatrick, 1972; Geeslin, 1974a; Geeslin & Shavelson, 1975a, 1975b; Scandura, 1971). Consider briefly a study by Geeslin (1974b; cf. Geeslin & Shavelson 1975a) concerning the learning of mathematical structure. Although this study indicates the rather large difficulties in assessing empirically the claims made by curriculum makers, it indicates that progress can be made.

One of the first steps in determining whether students learn mathematical structure is to define mathematical structure. With the exception of Begle (in preparation), I was unable to find a definition of structure in the mathematics literature. That is, claims were being made without even defining major terms. For the purposes of Geeslin's study (Geeslin & Shavelson, 1975a), mathematical structure was defined as the relationships between concepts within a set of abstract systems. Content structure is the web of concepts and their interrelations in a body of instructional material (Shavelson, 1971, 1972). Cognitive structure is a "hypothetical construct referring to the organization (interrelationships) of concepts in long-term memory [Shavelson, 1971, p. 9]." The purpose of the study was to compare content structure and cognitive structures in students to determine if students learn structure and to see if this learning was correlated with attitudes or achievement (i.e. ability to solve problems). Since the details of the study are published elsewhere (Geeslin & Shavelson, 1975a), only a summary will be reprinted here.

The study investigated learning of mathematical structure. Eighth grade students (N = 87) were assigned randomly to read either a programmed text on probability (experimental group) or one on prime numbers (control group). The subject matter structure of the probability text was mapped with the method of directed graphs. Structure in students' memories, cognitive structures, was investigated using a word association technique. Cognitive structure and achievement data were gathered at pretest, posttest, and retention test. The directed graphs provided an interpretable map of subject matter structure. Experimental students learned and retained the content structure but the control students did not. A comparison of word association, achievement, and attitude data indicated that learning of structure may differ from learning measured by achievement tests.

Although the results of such a study are not definitive, they are useful. However, studies such as this are being done approximately fifteen years after the implementation of modern curricula and several future studies are necessary to ascertain the effects of learning structure on other desired student behaviors. This research is coming much too late. In fact, several curricula changes may occur prior to our knowing much about the effect of current curricula. Classroom teachers generally have not participated in such research and thus it has had little impact in the classroom.

Adoption of new curricula with more emphasis on applications may occur also with no empirical foundation. However, if we begin now we can gather enough information to make knowledgable curricula changes. We are presently beseiged by cries for new curricula. Distinguished persons such as Morris Kline are calling on mathematics educators to add more applications to the mathematics curriculum, to emphasize applications, and to present mathematics in a manner comparable to its historical development. Research probably will not be able to control this new movement, but it can be used to establish the validity (or lack of validity) of many of the claims. Although monetary support is necessary to change curricula, empirical studies can give us information necessary to concentrate on changes most likely to improve curricula. Now is the time to examine claims of those advocating reform, not after changes have already occurred.

A cursory examination of recent issues of *The Mathematics Teacher* alone produces a sizable list of claims concerning applications. Among the more common are: applications might provide motivation (Bell, 1971); build a student's intuition (Wilder, 1973); reinforce concepts (Mizrahi & Sullivan, 1973); make mathematics more enjoyable (Adler, 1972); make mathematics more interesting (Fremont, 1974); cause more efficient learning (Fremont, 1974); contribute to problem solving (Fremont, 1974); aid student reasoning even after the student is in the formal operations stage (Bell, 1971); and provide anchors for mathematical ideas (Fremont, 1974). In summary, applications will solve all our teaching problems (just like modern mathematics did).

Hopefully, few educators actually believe random introduction of applications into present curricula will correct all the problems in mathematics education. Nonetheless, proper use of applications may correct particular difficulties or significantly improve the present situation. The claims concerning applications definitely suggest important and practical research studies. It is not our intent to discourage the use of applications, but rather to encourage systematic investigations which will determine those applications of benefit to the student, and when and where to present these applications for the most effective results.

A rather simple study could determine whether inclusion of applications promotes student facility with word problems as opposed to instruction only on necessary skills for solving the problems. (One should make certain the included application is not simply a practice element for the set of problems.) A similar study could determine if use of applications increases performance on achievement tests (i.e., reinforces concepts) or retention tests. Attitude scales concerning motivation, interest, and enjoyment could be used to ascertain the effect of applications on these variables. (Such scales are in existence although improvements could be made.) These studies would be quick and easy to conduct, even for the classroom teacher, and would satisfy the "publish or perish" need as well as serve to refine our methods and intuitions concerning important variables.

Let us turn to some more difficult but more significant studies. We have carefully avoided defining application. Results of the above studies might change drastically according to: 1) what is considered an application; 2) which applications are used, i.e. what types of applications; 3) how many applications are presented; and 4) the type of student. An operational definition of application must be developed and consideration must be given to the important variants of applications. Given this, a series of systematic studies similar to those mentioned above could be conducted.

Having established what an application is and what the important characteristics are, we may proceed to another question. Where in the instructional sequence should an application(s) be placed? Does an application create a need/relevancy/motivation for a new topic and thus increase learning or efficiency by preceeding the mathematics? Perhaps applications function as advance organizers. Or, should applications be placed after mastery of prerequisite skills to show usefulness, provide practice, or as a connector between concepts? Maybe applications should be placed at both the beginning and end of an instructional sequence. One would describe several elements that can be manipulated in an instructional sequence (teaching model) and systematically vary these elements to ascertain the most effective strategy.

We are now better able to proceed with the investigation of the connection between the inclusion of applications in curricula and variables such as problem solving, insight, intuition, cognitive structure, and attitude. Studies with a design similar to the studies concerning structure would be appropriate. Naturally, or perhaps unfortunately it is necessary to define terms such as insight. In fact, these definitions alone would represent a significant contribution (and effort) to mathematics education.

Several other assertions could be investigated. Inclusion of applications increases the stress on thinking and decreases the stress on memory. Applications are a prerequisite (aid) to mathematical intuition. Applications aid discovery. Applications (models) increase achievement even after the student is in Piaget's formal operations stage. Applications provide concrete anchorage in cognitive structure, allowing students to organize mathematics better. Applications produce students who focus on techniques alone at the expense of their analytical abilities. (Conversely, modern curricula force students to analyze statements and assumptions and this transfers to other areas.) Rigor does not make mathematical ideas more clear and thus is not necessary, perhaps even harmful. Obviously, no one person can investigate all these assertions in a reasonable time span. Thus we should attempt to coordinate our efforts and disseminate our ideas. Classroom teachers who need assistance in conducting research should make use of their colleagues in colleges, universities, and state departments of education. Most researchers are happy to help in the conducting of experiments. In fact, the "publish or perish" syndrome requires they do some research and the classroom teacher can provide the "laboratory" for research. Some initiative and desire are probably the only prerequisites for beginning the examination of applications.

For those who are not overjoyed by the difficulties in acquiring empirical data on learning, I suggest some different activities which are significant also. Has the development of mathematics preceeded or followed physical problems? In looking at various articles, anecdotal evidence was presented for both cases. A serious historical analysis could prove quite valuable. What types of applications and/or mathematical models are used in ordinary life? What applications fit well in curricula and where? The previous two questions are concerned with locating applications that are "mathematically correct," that can be stated easily, that can be understood with a minimal background in the application discipline, and dissemination of these findings to classroom teachers. We need further examination of the relationship between mathematics and cultural development, including value judgements concerning "good experiences" for students to have in mathematics as it relates to our cultural development. Given various applications, one could producea task analysis (Gagne, 1965) for each application indicating the prerequisite skills and thus indicate where it is possible to place the application in the curricula. This type of analysis would be very beneficial to teachers.

In summary, we should attempt to benefit from lessons of the last reform in mathematics education. Claims concerning applications are similar to claims concerning modern mathematics. These claims are based on assumptions, prejudice, hunches, and hope for some magical answer to the difficulties encountered in the teaching of mathematics. A blind rush toward applications will do little to help mathematics education and may, in fact, harm it. On the other hand systematic experimentation with decisions based on empirical evidence concerning the learner would likely uncover many aspects of applications that would be beneficial. If we move now, future curricula changes could be based on hard evidence. This would probably save both time and money in the long run, and could lead to continuous systematic improvement of the curricula. In fact, it might increase the rate of change. If we pass up this opportunity to begin research now, then I suspect in another ten to fifteen years we will again become disatisfied, find discouraging results, and be in much the same position as we are today.

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MAXES, MINS, AND OTHER CRITICAL POINTS

by T. A. Giebutowski, Plymouth State College

Although most calculus teachers, in both college and senior high school, draw their motivating examples from classical physics, there are many useful examples from the "softer" sciences (economics, sociology, etc.), that are easily accessible to the student. I see by the program that a fair number of the speakers here will concern themselves with economics, using mathematics of the VonNeuman-Morgenstern persuasion, i.e., Game Theory and Linear Programming. In this room, however, let us bypass the modern superhighway of the new economics and take for a while the cracked and grass-encroached macadam, the classical approach, that the new way tries to keep in sight.

One purpose of this talk is to present to you a specific use of classical analysis in economics, and to give an example which can (with some explanation and a little hand-waving) be presented to high school students in an advanced math class. Another purpose is to show how, if our problem gets a little tougher, a little bit heavier analysis can be used. Lastly, perhaps you will leave with a little better insight into the interplay of calculus and (horrors!) linear algebra.

First, a quick review:

Recall that the derivative of a function of one variable at a point represents the slope of the tangent line to the graph at that point. You all remember this picture:

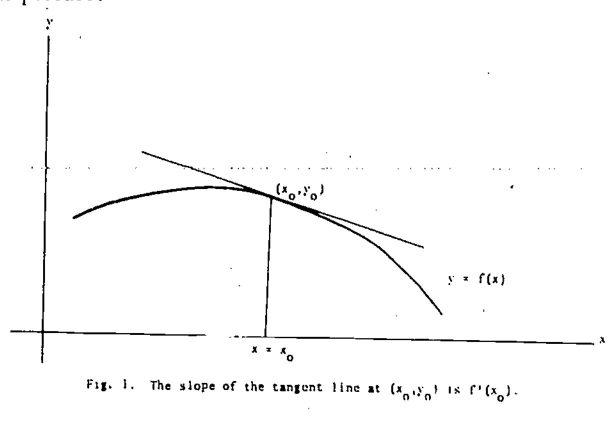

Fig. 1. The slope of the tangent line at (x_0, y_0) is $f'(x_0)$.

Figure 1

Now it is very little trouble to go to a function $z = f(x,y)$ of two variables being represented as a nice surface in 3-space:

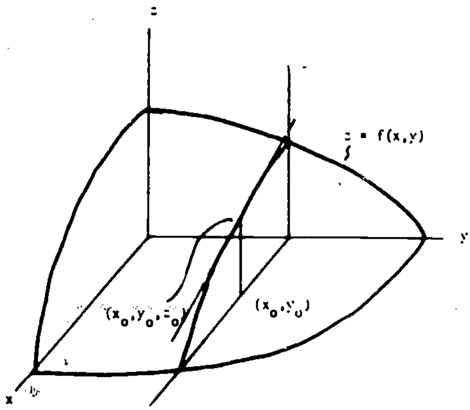

Fig. 2. Graph of a function $z = f(x,y)$.

Figure 2

Pick a point in the domain (x_0,y_0) and ask what happens when you consider only those points of the domain on the straight line $y = y_0$. If you now move your z-axis over to y_0 on the y-axis, this generates a plane which cuts the surface of our function $z = f(x,y)$. Now, if we lift that plane right out of there, complete with locus of points of its intersection with our surface $z = f(x,y)$, we get a picture like:

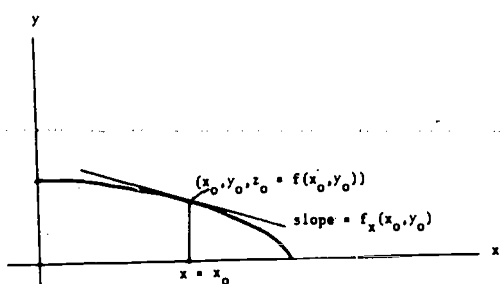

Fig. 3. Plane $y = y_0$ as it intersects surface of Fig 2 with tangent.

Figure 3

Now, the slope of *that* tangent line at that point is the partial derivative of our function with respect to x at (x_0, y_0). Recall the notation: $\frac{\partial f}{\partial x}(x_0,y_0)$ or $\frac{\partial z}{\partial x}(x_0,y_0)$ or $f_x(x_0,y_0)$.

You can illustrate the same idea at (x_0,y_0) using the line $x = x_0$, with the z-axis moved to x_0 on the x-axis in a parallel way. Here is that picture:

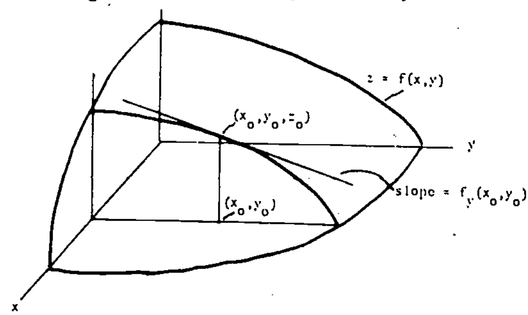

Fig. 4. $z = f(x,y)$ showing intersecting plane $x = x_0$ and embedded tangent.

Figure 4

It should be fairly evident, that working symbolically, this corresponds to:

If you want f_x, hold y constant and differentiate with respect to x, for f_y, hold x constant, and differentiate with respect to y, which gives you, since it can be done at *any* point (x,y) interior to the domain, two new *functions* $z = f_x(x,y)$, and $z = f_y(x,y)$. Of course, as with a function of one variable, you can go through these processes for f_x and f_y again, getting $f_{xx} = (f_x)_x$, $f_{xy} = (f_x)_y$, $f_{yy} = (f_y)_y$ and $f_{yx} = (f_y)_x$.

I think if we all gaze at this picture for a bit, we will recall that maximums, minimums, and horizontal points of inflection (a curious terminology at best!) occur mostly where $f'(x) = 0$.

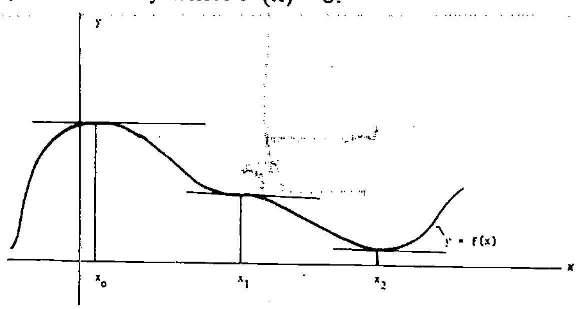

Fig. 5. $f(x)$ with (relative) maximum at x_0, a horizontal point of inflection at x_1, and a (relative) minimum at x_2.

Figure 5

Switching to functions of two variables, we can see that a maximum for the function (see figure 6) will ('nice' functions only!) have the property that at the point (x_0,y_0) in the domain where the function $f(x,y)$ has a maximum, both f_x and f_y are zero. A similar statement holds for minimums (turn figure 6 over).

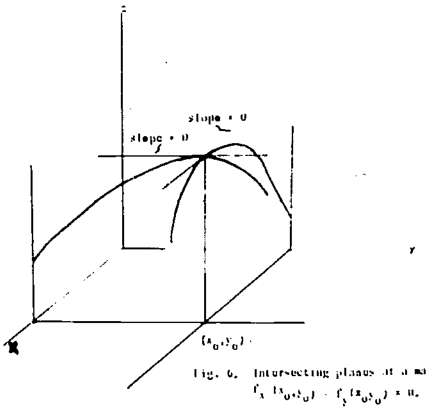

Fig. 6. Intersecting planes at a maximum, illustrating $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$.

Figure 6

Of course, as we see next, $f_x = 0$ and $f_y = 0$ can occur at what we call a saddle point:

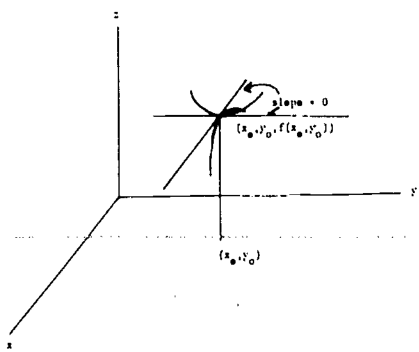

Fig. 7. Profile of tangent lines at the saddle point of a function $z = f(x,y)$.

Figure 7

All of the above can be presented to an advanced math class which has gotten past max-min problems of a single variable, without necessarily writing down the definitions of the partial derivatives (of course, if you have the time, it wouldn't hurt!). Now, we move to a theorem that they will just have to take on faith.

First, a critical point is a point of the domain of a function, where all partial derivatives are zero.

Theorem: Classification of critical points of $z=f(x,y)$:

I. If, at (x_0,y_0) (a critical point), $f_{xx} \cdot f_{yy} > (f_{xy})2$ then $f_{xx} > 0$ indicates a relative minimum; $f_{xx} < 0$ indicates a relative maximum.

II. If $f_{xx} \cdot f_{yy} < (f_{xy})^2$, then (x_0,y_0) is a saddle point.

Now to economics.

Suppose a firm sells two products, let x (measured in, say, 100,000 item lots) be the amount produced of one, y the amount of the other, and, suppose we have the following relation between the price P_1 charged for the first product, and x the amount sold: $P_1 = 12 - 2x$ (you see, the more they have to sell, the lower the price must be, the old supply-demand idea). And for the second product: $P_2 = 32 - 4y$.

Suppose also that it costs: $C(x,y) = x^2 + 2xy + y^2$ to produce these items.

Find the price and output which will maximize profits.

Now, before we begin the solution, it should be mentioned that the formulation of those functions for a particular industry is in itself a formidable task involving gathering data, market surveys, cost analyzing, curve fitting, etc., and that the functions we have here are greatly simplified as opposed to any actual situation. But from this point, the general ideas coincide.

We know: Profit = Revenue - Cost, and that Revenue $= P_1 \cdot X + P_2 \cdot Y =$ $12x - 2x^2 + 32y - 4y^2$, so profit:

$f(x,y) = 12x - 2x^2 + 32y - 4y^2 - (x^2 + 2xy + y^2) =$
$12x - 3x^2 + 32y - 5y^2 - 2xy$

is what we want to "maximize".

First, set: $f_x = 12 - 6x - 2y = 0$

$f_y = 32 - 10y - 2x = 0$

Solve for (x_0,y_0), get $(1,3)$.

Now $f_{xx}(1,3) = -6$, $f_{yy}(1,3) = -10$, $f_{xy}(1,3) = -2$, says, $(f_{xy})^2 = (-2)^2 < (-6)(-10) = f_{xx} \cdot f_{yy}$ at $(1,3)$, so we see, since this is true, and $f_{xx}(1,3) = -6 < 0$, by our criterion, 100,000 of the first item and 300,000 of the second will produce maximum profit:

$f(1,3) = 12 - 3 + 32(3) - 5(9) - 2(3) = 54$, and the price that should be charged: $P_1 = 12 - 2 = 10$, $P_2 = 32 - 4(3) = 20$.

I reiterate that this is an oversimplification of what an actual situation would produce, but it does give the idea, that the underlying structure of economics and business problems can be taken to be a highly mathematical model which assumes continuity, differentiability, etc.

But let's look at what's wrong with the formulation of this problem and see what might be involved in correcting it. First, the price functions (in actuality) will rarely be linear, and there probably would be more than two items being produced. In addition, there may be unmentioned constraints on production (in which case linear programming or the method of Lagrange multipliers might be employed). But let's expand the problem along the lines of our first objections, so that after tortuously figuring out cost functions and price functions for a company which produces some n different items, we arrive at a profit function $z = f(x_1, x_2, x_3, \ldots, x_n)$, a function of n variables. How do we handle this expanded problem?

First recall that a partial derivative of a function of more than two variables is found by holding *all* the variables other than the chosen one fixed and proceeding with a differentiation. Example,

If $f(x_1, x_2, x_3) = x_1x_2 - x_2^2 + x_1x_3$, and you want to find $\frac{\partial f}{\partial x_1} = f_{x_1}$

you would treat x_2, x_3 as constants and take the derivative with respect to x_1: $f_{x_1} = x_2 + x_3$. And a critical point is again where all the partials equal zero. To test these critical points for max-mins, we need the following matrix:

$$A = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ \vdots & \ddots & \vdots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{bmatrix}$$

each 2nd partial $f_{x_ix_j}$ is being evaluated at a critical point (x_0, y_0).

With that small bit of information, and relying on your fond remembrances of linear algebra and matrices, let me now simply quote some theorems necessary to our goal.

Theorem 1. If $f(x_1, x_2, \ldots, x_n)$ is twice continuously differentiable (i.e., all mixed 2nd partial derivatives are continuous functions), then $f_{x_ix_j} = f_{x_jx_i}$. (This means A is a symmetric matrix with like rows and columns.)

Theorem 2. (Canonical form) A a symmetric matrix implies there is a nonsingular matrix P such that P^TAP is a matrix with possibly ones, minus ones and zeros down the diagonal and zeros elsewhere.

Definition: Let $A' = P^TAP$ as above, then A is:
positive definite if A' has only ones down the diagonal
negative definite if A' has only minus ones down the diagonal,
and indefinite if A' has at least one one and one minus one,
but no zeros down the diagonal.

Theorem 3. For $f(x_1, x_2,\ldots, x_n)$ as above, $\vec{x_o} = (x_{o_1}, x_{o_2},\ldots,x_{o_n})$ a critical point of f,

if A is positive definite, f has a relative minimum at $\vec{x_o}$;
if A is negative definite, f has a relative maximum at $\vec{x_o}$.
and if A is indefinite, f has a saddle point at $\vec{x_o}$.

Now, in practice, the effect of P and P^T can be obtained by performing successive paired elementary row and column operations, as you will see in our example.

Suppose we wish to analyze the function:
$f(x_1,x_2, x_3) = x_1^2 + 3x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 8\,x_2x_3 - 2x_1 + 4x_2 - 2x_3.$
To find critical points, we set $f_{x_1} = f_{x_2} = f_{x_3} = 0$, as in:

$$f_{x_1} = 2x_1 - 4x_2 + 2x_3 - 2 = 0$$

$$f_{x_2} = 6x_2 - 4x_1 - 8x_3 + 4 = 0$$

$$f_{x_3} = 2x_3 + 2x_1 - 8x_2 - 2 = 0$$

If you solve this system of equations, you find $x_o = (2/3\ , 0, 1/3)$ as a critical point (again, things are not usually this easy!), then calculating *all* of the mixed second partial derivatives, we get our matrix:

$$A = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 6 & -8 \\ 2 & -8 & 2 \end{bmatrix}$$ (See, it's symmetric!)

Recall how if we wish to perform a column elementary operation on a matrix, we post multiply by the identity matrix adjusted by the operation and to perform a row elementary operation, we premultiply the row adjusted identity matrix, so that:

$$A_1 = P_1^TAP_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -4 & 2 \\ -4 & 6 & -8 \\ 2 & -8 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 6 & -5 \\ 0 & -5 & -9/2 \end{bmatrix}$$

and this equivalent matrix has zeros in the corners, now let's make $a_{12} = a_{21} = 0$: Form

$$A_2 = P_2^TA_1P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -4 & 0 \\ -4 & 6 & -5 \\ 0 & -5 & -9/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & -5 & -9/2 \end{bmatrix}$$

Now let

$$A_3 = P_3^T A_2 P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & -5 & -9/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 16/2 \end{bmatrix}$$

and finally,

$$A^1 = P_4^T A_3 P_4 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2/16} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 16/2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2/16} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, by Theorem 3, A^1 an indefinite matrix implies, the critical point is neither a max nor a min, but a saddle point, and our analysis of this (mathematical) max-min problem is complete.

I have tried in this hour to present to you an example of a "different" application (i.e., a non physics oriented one) of the calculus, which, if you have a little time, you could reasonably show your advanced math students in high school (or your business calculus freshman in college). Perhaps the second example, analyzing these critical points for functions of several variables, has shown you yet another powerful application of linear algebra to analysis. But the overlying idea here, is that we are sending our students out into an increasingly mathematicized society. Even in the world of business, the need for quite sophisticated mathematics is being felt. The M.B.A. Master's in Business Administration, program in many schools requires a course in calculus. Many graduate schools in economics, psychology, etc., actually prefer applicants who choose mathematics for an undergraduate major.

On the other hand, if remembering bits and pieces of this talk gives you one more answer to THAT question: "What's all this stuff good for?", I'll be happy.

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EUCLID MUST GO

By Jay Graening, University of Arkansas

During the past decade there has been renewed interest in the curriculum question "How should we teach high school geometry?" Unfortunately, the questions of "Why, what, and when should we teach geometry?" have not nearly received the same amount of attention.

In practice, the primary goal of high school geometry has been "to develop the entire body of traditional content as a unified, abstract mathematical system - a geometry of ideas - based on undefined terms and reasonable assumptions from which the remaining information is obtained by proper application of rules of inference." (Forbes, 336) This objective has been so dominating and influential that many other worthwhile goals for geometry have received little or no attention. Some of these less emphasized but desirable goals for geometry are to develop inductive and creative thinking, to transmit important information about space, to develop an understanding of the nature of a mathematical model, to develop the ability to think critically, to develop an understanding of an axiomatic structure, to develop skill in applying the several methods of geometric development to the solution of original problems, to develop within students an appreciation for both the intellectual strength and the intrinsic beauty of working with abstractions, to provide students with the opportunity for original investigation and the construction of valid arguments within the context of geometric content, to exhibit the unity of mathematical ideas through an integration of arithmetic, algebraic, and geometric concepts, to build the students' geometric intuition so that geometric models can be used in further mathematics instruction, and to introduce and extend those mainstream mathematical ideas that arise most naturally within the content of geometry. The attainment of such worthwhile goals for geometry is severely limited by the pervasive, current, primary goal of developing the entire body of traditional geometric content as a unified, abstract mathematical system.

Since an increased amount of geometry is being taught at the elementary and junior high school levels, much of the present tenth-grade geometric content is not new for students. (Forbes, 336)

Furthermore, the present day course in geometry has failed to take advantage of the fantastic advancements made in geometry since the time of Euclid. (Eccles, 103, 165)

Steven Szabo also has commented on this existing estrangement between geometry and algebra.

For the most part, the studies of algebra and geometry in the high school curriculum are not at all related. In fact, it is the case in many instances that the study of geometry turns out to be merely a strange interlude between the study of algebra in the ninth grade and the continued study of algebra in the eleventh grade. (Szabo, 218)

Even deductive reasoning, the sacred cow of high school geometry, has come under attack from many quarters during the last decade. Irving Adler is one who feels, as I do, that there has been too much emphasis given to formal, deductive reasoning. (Adler, 229)

Not only can we de-emphasize deductive reasoning profitable, but we can also eliminate many of the obvious proofs by assuming them. (Willoughby, 306)

Stewart Moredock has suggested that we need a careful balance between what we assume and what we prove. (Moredock, 221)

What then should be the content of high school geometry? Willoughby has suggested that a high school geometry course should "emphasize mathematical creativity and insight on the part of the pupil rather than formalism and form of proofs. It should give the pupil a feeling for deductive reasoning without so much rigor that *rigor mortis* sets in." (Willoughby, 306). Coxeter has argued that the intuitive 'interest' approach through problems significant to the student is more appropriate than the axiomatic approach with rules and definitions. "The systematic use of axioms in geometry is admissable only after the students have already had several years of experience with simple deductions. Actually, for exercises in deductive reasoning, algebra is probably more suitable than geometry. Geometry should be taught rather for its interesting results and as an exercise in informal reasoning." (Coxeter, 9)

Meserve has probably come closest to my point of view. He feels that informal approaches are appropriate first. Then, he says,

> The emphasis on informal approaches can now be shifted from providing a basis for the recognition of assumptions to providing a basis for the observations (conjectures) that are to be proved or disproved. For these proofs a wide variety of approaches will be sought - direct synthetic proofs, indirect proofs, disproofs by counterexample, coordinate proofs, vector proofs. Such a broadening of the discussion of proof strengthens the emphasis on proofs in a deductive system while removing much of the tedium of seemingly endless synthetic proofs. Synthetic proofs have a major role but become one approach rather than the approach to a proof. (Meserve, 178)

It is my thesis that high school geometry should be a blend of Euclidean (synthetic), coordinate, vector, and transformation approaches that emphasize proof and deductive reasoning but in a much more informal way than we have had traditionally.

Rather than beginning with the ultimate abstractions of points, lines, and planes, the point of departure for a tenth-grade geometry course should be the familiar behavior of rigid objects in the physical world. Three-dimensional objects and the geometric figures they suggest should be explored. Thenthe box or rectangular solid can be given special attention, since it is one of the shapes most familiar to students. This investigation can lead to such topics as cubes, rectangles, squares, line segments, parallel line segments, parallel line segments, vertices, nets of cubes and rectangular solids, etc.

The focus can then be switched to two-dimensional tiling patterns in order to begin intuitively the study of symmetry; some shapes have line symmetry while others have halfturn or point symmetry. Metric units should be used along with American (English) units of measurement throughout the course. Angles should be explored from several points of view: as the shape of a corner, as an amount of turning, and as the union of two concurrent rays. Coordinates should be introduced early and used as a unifying thread. Rectangular, polar, and other coordinate systems should be considered.

With these preliminary background experiences, students can begin a sequential investigation of shapes that simultaneously builds up some important notions of informal deductive arguments. Some four-sides shapes

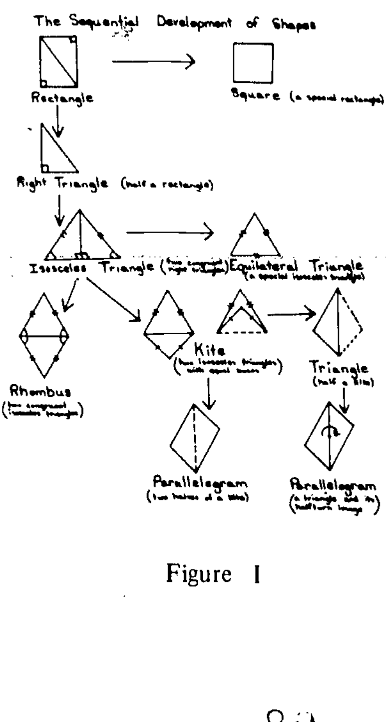

Figure I

fit their outlines in one way; others fit in two, three, four or as many as eight ways. The rectangle and the rhombus are the only quadrilaterals that fit their outlines in exactly four ways. Shape-fitting suggests one important distinction between a rhombus and a rectangle. A rectangle has every corner change positions from one-fitting to another; this is not true for a rhombus. Incidentally, investigating the rhombus provides a unifying focus for studying several basic constructions.

At this stage, students should be ready for a more detailed and more mathematical description of line reflections and rotations. Coordinate ideas can be particularly useful here in extending and reinforcing the concepts.

Some vector ideas can then be studied, beginning again informally, through the ideas of displacements and trips. This leads to a discussion of directed line segments and translations.

The course can continue with an investigation of similarity and circles. Then more direct attention should be focused on deductive reasoning and proof by examining statements, counterexamples, implications, converses of implications, and equivalent sentences. Attention should also be given to theorems and their converses, minimum conditions, direct proof, indirect proof, and contrapositives.

Finally, the previous work can be expanded to include vector geometry and its related proofs, dilations (similarity transformations), and the composition of various types of transformations.

In summary, the content of tenth-grade high school geometry should be a blend of Euclidean (synthetic), coordinate, vector, and transformation geometries. The study of symmetry, motion, and shapes should be fundamental in the development. Coordinates should be a unifying thread running throughout the entire course. The content should meaningfully relate to everyday experiences and the physical world. There should be a balance between theoretical and applied geometry. There should be strong interplay between geometric and algebraic ideas throughout the course. Proof and deductive reasoning should play a central role but be less formal than in the past.

The starting point for the development of concepts in high school geometry should be physical models and real-world situations. Students should be encouraged to use their intuition, to manipulate models, to make conjectures, and to explore and study shapes. Obvious properties such as betweenness and plane separation should be assumed. Proof, like other concepts, should be developed in stages which increase in difficulty and complexity. Initially, proofs should be informal and intuitive within simple axiom systems. In short, students should learn geometry through student-centered activities that encourage them to think mathematically about their experiences in the real world. This approach should provide students with greater motivation and meaning in their study of geometry.

The blend is the best approach for revitalizing the high school geometry course and for bringing geometry back into the mainstream of mathematics. Euclid may stay, but the present, formal, tenth-grade geometry course must go.

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MUSICAL MATHEMATICS

by Henry P. Guillotte, Rhode Island College

The title "Musical Mathematics". is the title of a workshop the author conducted at Salve Regina College in Newport, R.I. during the summer of 1974. The title is used only because it sounds better than "Mathematical Music."

In order to put the subject under discussion in the proper perspective consider the following:

There are composers whose works can be distinguished even by the non-music student, for instance many people recognize works by Bach even though they might not know the names of the compositions. Other familiar composers - Beethoven, Mozart - also have a quality in their works, a property, and here we are getting mathematical, about their works that make them identifiable. The identification however is done by listening to the composition. This consideration leads quite naturally to the following question: Is it possible that a composer's works might have an identifiable *mathematical* structure so that his compositions would be recognized mathematically rather than musically? Although that question was not answered in the workshop, a foundation was laid for the investigation of the question.

To look at how mathematical structures might be imposed on musical compositions, one needs to look first at the fundamental elements of music. Only three of these will be discussed in this paper:

1. Melody: a horizontal sequence of pitches sounded successively
2. Harmony: a vertical sequence of pitches sounded together
3. Rhythm: a duration of pitches, silence, meter, tempo.

One of the building blocks of melody is the scale. The rule of a major scale is that there be 1 full tone between the first note and the second; 1 full tone between the second and third notes; 1/2 tone between the third and fourth notes; 1 full tone between the fourth and fifth, fifth and sixth, and sixth and seventh; and 1/2 tone between the seventh and eighth. This rule applies no matter what note is used as a starting note.

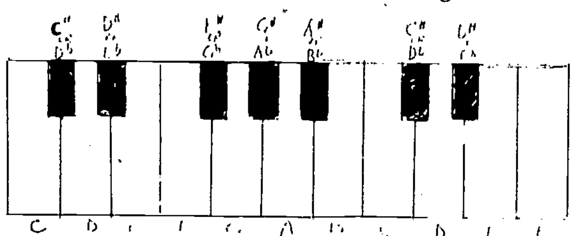


Figure 1

Figure 1 is a sketch of part of a piano keyboard. There is a full tone interval between adjacent white notes except E, F and B, C which are 1/2 tone intervals. There is a 1/2 tone interval between adjacent white and black notes; for example between C and C♯ and between F♯ and G.

Using the rule and the keyboard one can develop the C Major Scale, so named because its starting note is C: C,D,E,F,G,A,B,C. Figure 2 illustrates the C Major Scale written on a musical staff.

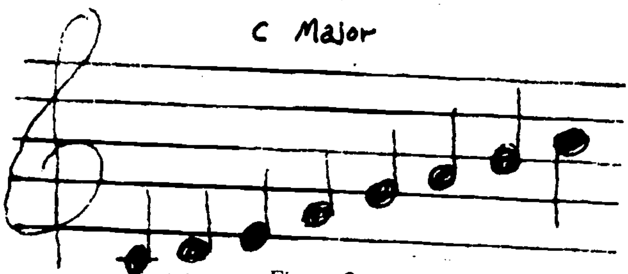


Figure 2

The task of finding the notes of the F Major Scale is not difficult. By the rule these are: F,G,A,A♯ or B♭, C,D,E,F. However, placing these notes on the staff poses somewhat of a problem. If one chooses A♯ as the fourth note then one is faced with representing two different notes, A and A♯, in the same space. The problem can be avoided by choosing B♭ instead of A♯ and in order to avoid constant use of the "♭", this symbol is placed at the beginning of the staff to indicate that the B is flatted. Thus the key signature of F major is '♭' placed on the middle line. See figure 3.

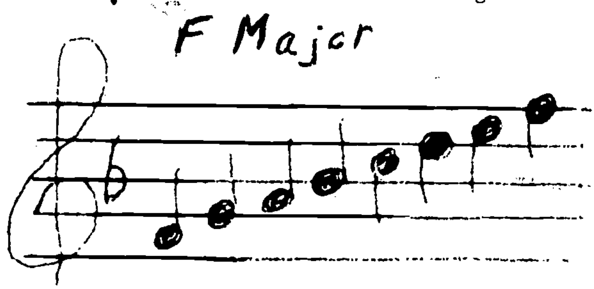


Figure 3

Consider the problem of finding the key signature of the scale which begins with the note D♯ or E♭. The rule generates the notes: D♯ or E♭, F, G, G♯ or A♭, A♯ or B♭, C, D, D♯ or E♭. The staff can be made 'clean' by choosing E♭, F, G, A♭, B♭, C, D, and E♭, and placing the ♭ symbols on the third line and in the first and second spaces to indicate that B, E, and

A, respectively are flatted. Thus the E♭ Major Scale leads to the key signature of three flats. See figure 4.

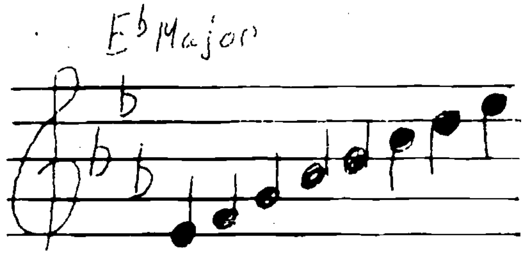



Figure 4

There is a pattern for finding the various key signatures. Start at C, see figure 5, and count clockwise five notes. F, the fifth note has one flat. Continue clockwise. The next note, has two flats so B♭ is chosen rather than A♯. This pattern gives F-1♭, B♭-2♭, E♭-3♭, A♭-4♭, D♭-5♭, G♭-6♭, C♭-7♭. A counterclockwise movement produces in turn the keys which are sharped: G-1♯, D-2♯, A-3♯, E-4♯, B-5♯, F♯-6♯, C♯-7♯.

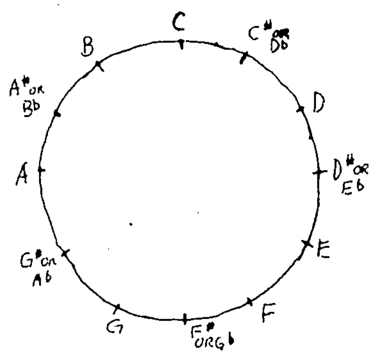


Figure 5

The pattern of tone intervals for the major scales, 1, 1, 1/2, 1, 1, 1, 1/2, is not the only pattern used. Various minor scales have different patterns. The minor natural has the rule: 1, 1/2, 1, 1, 1/2, 1, 1. The minor harmonic has the rule: 1, 1/2, 1, 1, 1/2, 1, 1/2, 1/2. The minor melodic has the rule: 1, 1/2, 1, 1, 1, 1, 1/2.

One natural kind of mathematical structure which can be imposed on a piece of music is a sequence of numbers. The very simplest of these is the sequence of natural numbers 1 to 8 with the eight tones of a scale. The first few bars of the melody of *Old McDonald* by the scheme above would produce the sequence: 8,8,8,5,6,6,5,10,10,9,9,8,5,8,8,8,5,6,6,5,10,10,9,9,8. This numbering system works only in cases where all the melody is played with the major tones of the given scale. A numbering system to avoid this restriction will be shown below.

The vertical aspect of music, namely harmony, involves many different kinds of chords. A major chord involves the 1st, 3rd, and 5th notes of the major scale. The minor, $(1, 3\flat, 5)$; the major 6th, $(1, 3, 5, 6)$; the minor 6th, $(1, 3\flat, 5, 6)$; and the 7th, $(1, 3, 5, 7\flat)$ are examples of some of the different sounds which can be sounded together either by one instrument such as the piano or a combination of instruments.

An obvious mathematical structure which can be imposed on the notes of a chord is an n-tuple and on a sequence of chords such as the first bar of Chopin's *Polonaise*, see figure 6, the beginnings of a matrix. Observation of the range of notes indicates that a simple 1 to 8 is no longer appropriate. Rather, one imposes the numbers 1 to 88 on the notes of the complete keyboard. Such an allocation gives for the first eight notes of the *Polonaise* the 8 rows of the matrix:

(13,25,37,41,49)
(25,32,37,41,44)
(25,32,37,41,44)
(25,32,37,41,44)
(25,32,39,42,46)
(25,32,39,42,48)
(25,32,39,42,49)
(25,32,39,42,51)

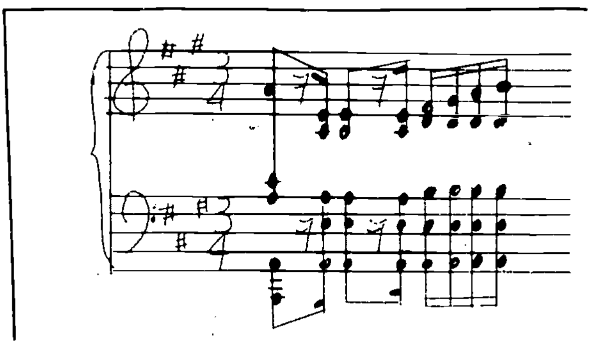



Figure 6

The last aspect of the elements of music which will be discussed is rhythm. One aspect of rhythm is time duration of the notes. Choosing one kind of note, say the whole note as the unit, then some of the combinations of other notes induce the numbers 1/2, 1/4, 1/8, 3/8, 3/4, 1/16. This allocation of numbers to the melody of *Old McDonald* gives the following sequence: 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/2, 1/4, 1/4, 1/4, 1/4, 1/2, 1/2, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/2, 1/4, 1/4, 1/4, 1/4, 1.

Although there are other mathematical structures which can be imposed on a musical composition, some of which will be discussed in a subsequent paper, it might be well at this point to look at the other side of the coin, a mathematics which is musical.

If we take part of the expansion for $\sqrt{2} \approx 1.41421356237237184425$ and associate with 1 the note C, with 2 the note D, with 3 the note E, etc., one is able to write and listen to a simple composition *Square Root of Two.*

See figure 7. However if one wishes to rewrite $\sqrt{2}$ in base twelve one gets the following sequence: 1.4e79170t08t26879t24e5, which translated to music by assigning the note C to 1, C♯ to 2, the note D to 3, etc., one gets the different composition illustrated in figure 8. The striking thing

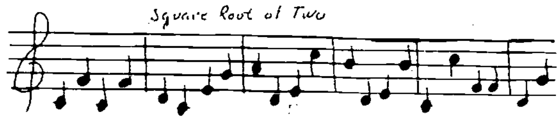



Figure 7

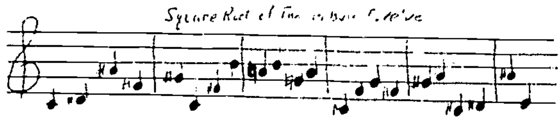


Figure 8

about the two compositions is that not only can one see the difference between the two different numerals, one is now able to hear the difference. To indicate how limitless are the possibilities in this direction, figure 9 is the composition of *Square Root of Two in Three-quarter Time*. If a composer were to concentrate upon irrational numbers set to music, then the 'unfinished symphony' would really apply.

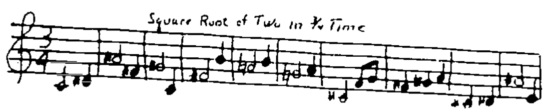


Figure 9

As a final example of the kind of analysis which can be made, consider Beethoven's *Minuet in G* which was analyzed for the frequency of 1/2-tone intervals for the successive notes of the melody. The meaning here is that the amounts of change between the first note and the second, between the second and third, between the third and fourth, etc., were recorded. The graph in figure 10 is the result of this analysis. As an indication of how one is to interpret the graph, note that there were 75 occurrences of moving up 1/2 tone, for example from E to F or A to A♯.

The reader is left with the following questions: Would other compositions by Beethoven produce similar graphs or would one have to look at triples rather than pairs before one noted similarities? Is searching for similarities like searching for a needle in a haystack? Needle anyone?

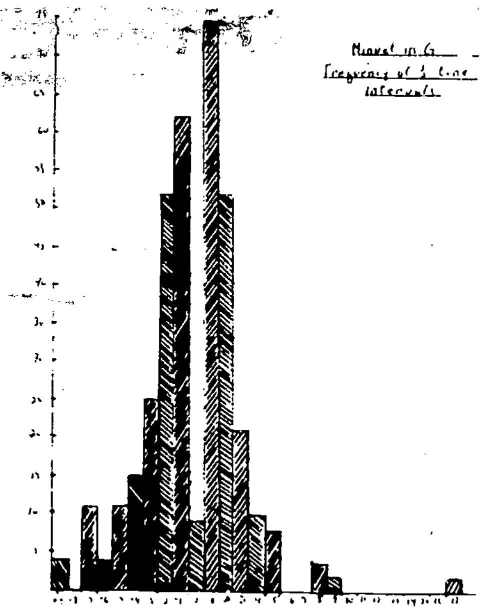


Figure 10

DESIGNING AUDIO-TUTORIAL INSTRUCTION: A CASE STUDY IN MATHEMATICS

by Harry O. Haakonsen and Robert M. Washburn, Southern Connecticut State College

Why Individualize?

What is Audio-Tutorial instruc*t*ion? That's not an easy question to answer. For starters, it is a way of individualizing instruction. Perhaps that is where we should begin, with a brief discussion of individualized instruction itself. Why would anyone want to individualize instruction by any means? What have we learned about the learning process that suggests the desirability of individualized instruction? If several people were asked to make lists of reasons for individualizing instruction, it is highly likely that many of these lists would contain three very fundamental ideas.

One fact which has been well supported by research studies concerns itself with readiness. We are told that the learning of most concepts requires the prior acquisition of a set of skills and related concepts. For example, it would seem senseless to teach the concept of addition of whole numbers before one has acquired the concept of whole numbers themselves and is able to perform the related skill of counting. It would be equally silly to attempt to teach the concept of addition of fractions before one has learned how to add whole numbers. If one plans a lesson on the addition of fractions, he bases the lesson plan on the assumption that his students already know how to add whole numbers. These concepts upon which the teaching of new concepts are based are called the pre-concepts of the lesson. When one exposes a class of students to a particular lesson, he is assuming that every student has the necessary pre-concepts for that lesson; that is, he is assuming that all students in the class have arrived at the same degree of readiness. As we all know, this frequently is not a valid assumption. Seldom, if ever, are all students ready to learn the same concepts at the same time. We may have tried to remedy this through homogeneous grouping, but this does not really solve the problem of readiness. Even if one assumes that the grouping has succeeded in placing students according to their degree of readiness according to one concept, what is the chance that they will then possess the same degree of readiness for some other concept? Readiness, then, is one important reason for individualizing instruction, to permit the learner to learn a new concept when he is ready to do so.

When one watches any kind of race, whether it is a foot race or a horse race, no one expects all the participants to reach the finish line at the same time - in fact, such an occurrence would seem incredible. It would be hard to believe that all of the runners could run at the same rate. In the same sense, it is illogical that all people learn the same concept at the same rate, even though they may have been exposed to the same set of learning experiences. Most of us are aware that no matter how well we teach a lesson. there will be some who become bored because we go too slow and others who become lost because we go too fast. Again, it has been proposed that homogeneous grouping would also solve this problem but it was no more successful at this than it was with the problem of readiness. Thus, it is desirable to adjust one's teaching pace to each student's learning rate. Only through a mode of individualized instruction can we do this.

One would not attempt to teach a blind person using pictures nor would one ask a deaf person to listen to a tape recording. In these extreme cases, we are aware that a person who loses one of his senses learns to use his other senses more acutely. Thus a blind person relies more heavily on his senses of hearing and touch and a deaf person becomes able to learn through his senses of sight and touch. What research has revealed is that even persons who possess all five of their senses in a so-called "normal" range differ from one another in the degree to which they rely on their senses to learn concepts and skills. Some people rely very heavily on their sense of hearing to learn new ideas. Such persons have an audio strength. Others rely more heavily on their sense of sight. Such people are said to have a good visual strength. Such familiar phrases as "seeing is believing" or "a picture is worth a thousand words" express the need of people who rely very much on their visual strengths. Research indicates that each of us requires an interplay between our five senses to maximize our learning efficiency. Such interplay varies with the individual. Thus, what one must see to learn, another must hear and yet another must see and touch. To further illustrate this point, Figure 1 summarizes a study in which the same concept was taught in three different ways. One presentation was entirely visual, one audio, and the third was a combination of audio and visual. The amount of retention after three hours and three days was dramatically higher for the group who learns through the audio-visual presentation.

Methods of Instruction	Recall	
	3 Hours	3 Days
A. Audio	70%	10%
B. Visual	72%	20%
C. Audio and Visual	85%	65%

Figure 1

Thus, through individualization of instruction one can attempt to communicate to the learner utilizing as many of the learner's senses as seems feasible.

Among others, there are three very fundamental reasons why we should consider the possibility of individualizing instruction - first, in order to allow students to begin a particular lesson when they possess the desired degree of readiness in terms of the pre-concepts and skills required to maximize learning; second, in order to allow students to progress through a learning sequence at their own rates; and, third, in order to match more closely instruction with the sensory requirements of the learner. The degree to which an instructional technique accommodates these three requirements can be a measure of the success of that technique towards the achievement of individualization.

Why A-T?

Now that we have explored the major reasons for individualizing instruction, it seems appropriate to ask why one should consider individualization using audio-tutorial techniques. The reasons center on two points - students enjoy audio-tutorial instruction and find it an efficient and effective approach to learning; furthermore, audio-tutorial instruction systems are generally built upon sound principles of learning espoused by educational theorists and psychologists such as Bruner, Gagne, Skinner, Ausubal, Cronbach and Mager.

In analyzing the literature on instructional systems, there appears to be six major areas of concern. It would seem that if learning-teaching systems are to be effective, they must:

1. Clearly state the goals of instruction in behavioral terms so they may be evaluated [11];
2. Structure the learning environment so that students are exposed to fundamental concepts that can be used in future learning and problem solving [1, 4, 9];
3. Match effective educational media with a specific learning event [3];
4. Facilitate the development of a proper attitude toward learning and make students increasingly responsible for their own intellectual development [7];
5. Provide for evaluation that keeps the learner informed of his progress and helps the teacher judge the adequacy of his teaching methods [4, 5];
6. Make allowance for aptitudinal, attitudinal, and personality variability among students [6, 10].

Audio-tutorial systems are designed with these parameters in mind. In an A-T system, careful attention is given to the statement of objectives, the selection of modes of instruction, and the kinds of learning involved in achieving stated objectives. Ideas are presented in an orderly sequence and students are able to proceed at their own rate of speed. If the sequence is right and the learner is informed about the goals of his learning, the motivation for learning will be built in [8].

Cronbach [6] has suggested that it is the task of educators to devise or select instructional methods that will interact with differences in learners so that the achievement of all students working toward a given educational objective will be significantly better than it would be if only a single "best" method of instruction were used. To be effective, we as teachers must become designers and managers of dynamic, diversified instructional materials. We must remember that it is the interaction of the learner and his aptitudes with the instructional environment and its stimulus materials which will determine what is learned. Students in A-T courses set their own pace for learning. They are free to repeat materials until they are confident of their understanding. Then, when the student is ready, he can proceed to the evaluation stage, where he measures his success in terms of behaviorally-stated objectives. If instructional materials are creatively designed and comply with learning theory recommendations, they stand an excellent chance of maximizing student potential for learning.

A Procedure for A-T Unit Development

Audio-Tutorial units are designed to assist students in mastering specific cognate materials or developing specific psychomotor skills while simultaneously developing a positive attitude toward learning. The materials are designed following a flow chart outlined in Figure II.

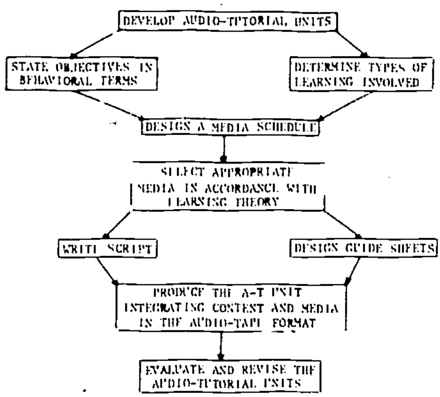


Figure II

In order to properly design the material the goals of instruction must be clearly stated, preferably in behavioral terms. The behavioral objectives are studied to determine what type of learning they involve. If psychomotor skills are to be developed, then the instructional program should include activities that foster psychomotor skill development. If the objectives fall into the cognative domain, then they should be scrutinized to see if they represent mastery of knowledge, comprehension, application, analysis, synthesis, or evaluation. If the objectives all deal with the lower items in the hierarchy in the cognitive domain, a reevaluation of the objectives of instruction may be appropriate.

Once the objectives have been written out and evaluated, a media schedule is developed.

In general, a media schedule consists of a list of instructional goals and all the possible instructional media that can be utilized in helping students master the goal. Brainstorming sessions with colleagues often uncover new innovative and effective approaches to mastering specific behavioral objectives. When a thorough media schedule has been developed, it is time to begin writing script.

There are many alternative scripting formats that may prove effective. For our purposes, we develop script using the format in Figure III.

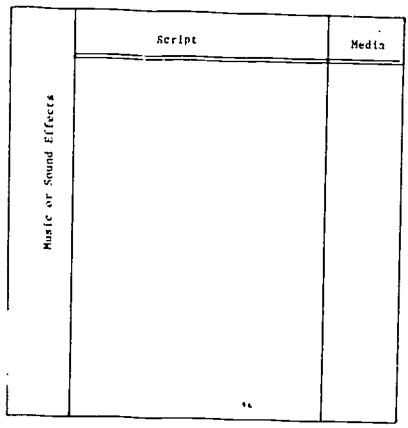


Figure III

The script is developed as a tutorial narrative. As the writing progresses, imagine that you are tutoring an individual through an instructional sequence. In the left margin jot down information on music or sound effects that might be appropriate. In the right column sketch out the guide sheets, slides or other materials which you will be integrating into the sequence. As the scripting progresses, integrate slides, guide sheets, laboratories, readings and pauses where they become important in the instructional flow. Select the best media from your media schedule and integrate them into the script where they are most appropriate.

When the script and guide sheets are completed, read through the script and make necessary editorial modifications. Then record the script and assemble the A-T instructional package integrating all the instruction media, including the guide sheets.

When the package has been assembled, have a few students listen to the unit and evaluate it for you. The scripts, guide sheets, laboratories, demonstrations, readings, and supplementary materials should be constantly revised on the basis of student evaluation.

An Example

To illustrate the procedures for developing an A-T lesson which have been described, let us suppose that we wish to design a lesson for eighth grade students concerning the area of a circle.

Preparing Behavioral Objectives

The first questions to consider are "What are the objectives of the lesson?" and "What is to be accomplished?" Perhaps an initial response to these questions might be that we want the student to understand the formula for the area of a circle. But what does one mean by "understanding the formula for the area of a circle"? This phrase is too vague, general and ambiguous. It is subject to too many interpretations. To avoid this problem, we behavioralize the objectives. There could be several behavioral objectives which relate to understanding the formula for the area of a circle. For the purposes of this paper, let us accept the following:

Objectives

After completion of all activities related to this lesson, you should be able to:

1. Recall the formula for the area of a circle;
2. Sketch a diagram to demonstrate how a circle may be rearranged to approximate the shape of a parallelogram;
3. With the aid of the diagram above, explain how the formula for the area of a circle can be derived from the formula for the area of a parallelogram;
4. Calculate the area of a circle given the measure of its radius of diameter at a performance level of 80%.

Media Scheduling

Now that we have identified the objectives for the lesson, we can begin to consider the appropriate media for the lesson. We could begin to investigate the availability of various types of media. From this, we begin to select the specific media which seems to be most appropriate for the students being taught and for the objectives to be accomplished. The media which was selected for this particular lesson was: a cassette recording, a series of slides, a wooden model of a circle, a set of worksheets, a record with the song "All the World's a Circle."

Script Writing

We now begin to coordinate the various media selected and determine a sequence of learning activities. The final outcome is a script. A portion of the script for this lesson follows below.

Script	Media
In this lesson, you are going to do some experimentation which may help you to discover	A portion of the song, "All the

and understand how to find the area of a circle. In order to do this lesson, it is necessary that you already know certain ideas about area and about circles. To see if you are ready to begin this lesson, take the worksheet which is on the table behind the recorder.

Try to answer the questions and then check your answers with those on the second page. If you are not able to answer all of these questions correctly, you probably should not yet try this lesson. On the other hand, if you can answer these questions, you are ready to continue with this lesson. Turn off the recorder now and answer the questions. When you are ready to continue, turn the recorder back on.

OK, now are you ready to learn how to find the area of a circle? (pause) In front of you, you should see a model of a circle. Pick it up and examine it. (10 second pause) Notice that it is cut in half. We call each half a semi-circle. Each semi-circle is cut into several pieces, which are joined by a leather strap around the outside of the circle. If you wish, you may take some time now to experiment with this model. For example, you may want to take it apart and put it back together a different way to make different shapes and designs. Turn off the recorder while you do this and when you are ready to continue, turn it back on.

[More of the script continues here but is left out for this article. Here begins another excerpt of the script.]

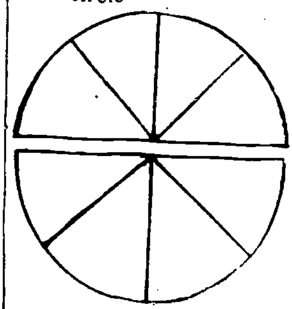
... Now look at the second picture. Do you see that when you arrange the pieces this way the figure looks like a parallelogram? If your pieces are not already arranged to look like the figure in the picture, do it now.

Now obviously the figure is not really a parallelogram because the sides are not straight. But what would happen if we were to cut the circle into more and more pieces? What

World's a Circle" is played - gradually fades out.

Worksheet with a short pre-test to determine readiness (Figure IV)

A wooden model of a circle



Student is asked to look at slide 1 and manipulate model.

Slide 2, picture of model assembled to resemble a parallelogram. (See Figure V) would happen to these sides? Look at picture number 3 to help you ... notice that as the number of pieces increases, the sides seem to appear straighter and the figure looks more and more like a parallelogram.

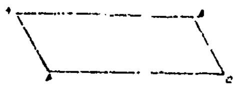
Suppose we pretend that this figure is a parallelogram. If we find its area, we will also be finding the area of the circle because the circle is simply rearranged to form this parallelogram.

[The script continues in a similar fashion but is not included in this article.]

Slide 3 [See Figure V.]

[illegible]

. Suppose that the figure at the right is a parallelogram. Draw in a line between bases AB and CD whose length is the height of the parallelogram.



1. The formula for the area of a parallelogram is

   A = ______________________

3. The formula for the circumference of a circle is

   C = ______________________.

The answers to these questions are on the next page. If your answers are correct, you are ready to continue with the rest of this lesson. Please turn the recorder back on when you are ready.

Figure IV

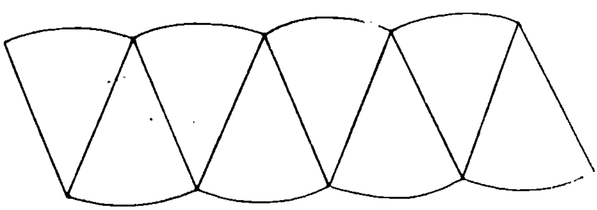


Figure V

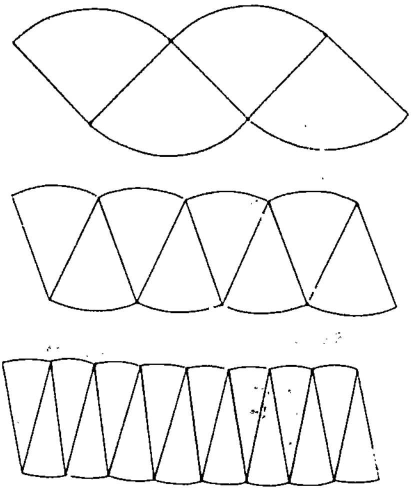


Figure VI

Once the script has been prepared it is often a good idea to design a guide sheet which is available for the student. It summarizes the purpose of the lesson, the materials if there seems to be an arrangement which would optimize the effectiveness of the instruction. The guide sheet for this particular lesson follows.

GUIDE SHEET

Topic: Area of a Circle

Materials: one cassette tape recorder and headset
one cassette tape labelled "Area of Circle"
one slide viewer
one envelope containing seven slides
one set of worksheets
one wooden model of a circle

Objectives:

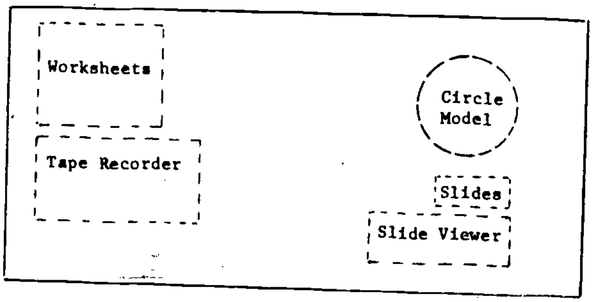
After completion of all activities related to this lesson, you should be able to:

1. Recall the formula for the area of a circle;
2. Sketch a diagram to demonstrate how a circle may be rearranged to approximate the shape of a parallelogram;

3. With the aid of the diagram above, explain how the formula for the area of a circle can be derived from the formula for the area of a parallelogram;
4. Calculate the area of a circle given the measure of its radius or diameter at a performance level of 80%.

Carrel Arrangement:

Please arrange your carrel as indicated by the diagram below.



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ON THE COMPUTATION OF PI

By Warren H. Hill, Westfield State College

One of the common misconceptions that students bring to a high school Euclidean Geometry class is the belief that a value for PI must be known before the circumference or area of a circle can be computed. In reality this belief places the cart before the horse. Actually the geometric definition of the value of PI is expressed in terms of a ratio between the circumference of a circle and its diameter. Consequently, to avoid a circular argument (no pun intended!) the teacher should have at his disposal some procedure for determining the circumference of a circle prior to the introduction of the constant PI. Once some method for approximating the circumference of a circle has been employed, an approximation for PI can then be easily computed. One viable approach to this predicament is through the computation of the perimeters of certain inscribed and circumscribed polygons relative to a given circle.

The proceeding sequence of activities which are employed to determine an approximation of PI will use the following inscribed and circumscribed regular polygons: square, hexagon, and octagon. After completing these activities, the students (and teacher) may desire to attempt a similar activity with a regular dodecagon and a regular 16-gon.

It should be mentioned, parenthetically, that in addition to the valuable experience of placing PI in a proper perspective, several other positive by-products are derived from these exercises. In particular, the activities which are employed in the following exercises require the application of certain elementary geometry constructions, use of congruence theorems of triangles, properties of the equilateral triangle and $30^{\circ} - 60^{\circ} - 90^{\circ}$ triangle, and the Pythagorean Theorem.

The Square

Beginning with a circle having a radius of one unit, a square is inscribed by first drawing an arbitrary diameter and constructing the perpendicular bisector of that diameter. The endpoints of the diameters are then joined to form the square (figure 1).

To circumscribe a square about the circle perpendicular diameters are again constructed and perpendiculars are then constructed to the endpoints of the diameters. Note that, by a geometric theorem, these perpendiculars are tangent to the circle (figure 2).

Turning again to the inscribed square it can be observed that since the diameters are perpendicular and the measures of the radii of a circle are equal, then the resulting four triangles are right isosceles triangles. An ap-

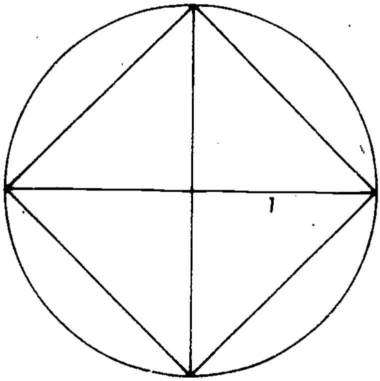


Figure 1

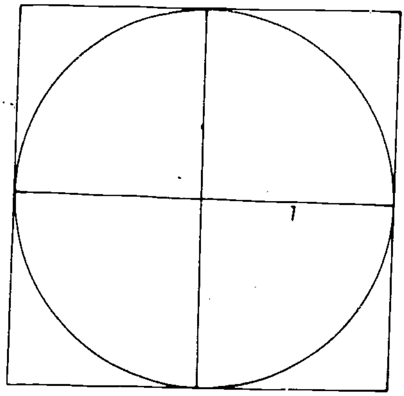


Figure 2

plication of the Pythagorean Theorem yields a hypotenuse of length $\sqrt{2}$ and, consequently, the perimeter of the inscribed square is $4\sqrt{2} \approx 5.656$. Finally, it can be seen that if C is the circumference of the circle then $5.656 < C$.

Returning to the circumscribed square, these diameters are also perpencicular and the tangents to the circles are perpendicular to these diameters. The resulting four squares have sides of length 1 and hence the perimeter of the circumscribed square is 8 units. The conclusion relative to the circumference, C, of the circle is: $C < 8$.

Combining the two bits of information obtained thus far, the conclusion is:

$$5.656 < C < 8.00.$$

Since the diameter of the circle is 2 units the resulting ratio C/D yields:

$$2.828 < C/D < 4.00.$$

If an average is computed (realizing the hazard that the perimeter of one square might be a better approximation of the circumference of the circle than the other) the resulting approximation of PI is 3.414.

It should, however, be obvious that the perimeters of the squares are not a close approximation of the circumference. To obtain a better approximation a polygon with a greater number of sides should be used.

The Hexagon

To inscribe a hexagon in a circle, begin by constructing a circle with radius of 1 unit. Maintaining the length of the radius between the endpoints of the compass, the compass point is then placed on any arbitrary point on the circle and an arc is swung. The procedure is repeated until the circle has been divided into six equal arcs. The resulting six points are joined consecutively to form an inscribed regular hexagon. Finally, to complete the construction, three diameters are drawn (figure 3).

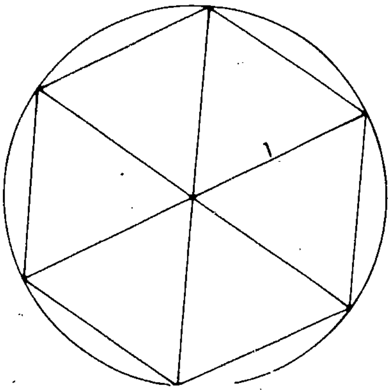


Figure 3

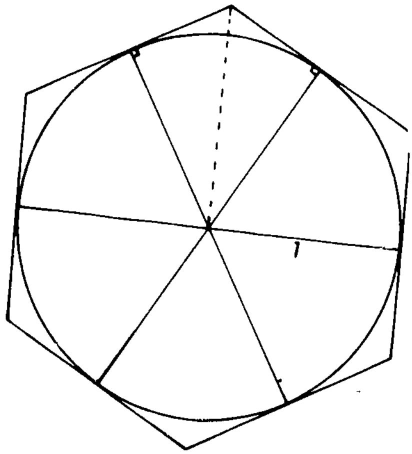


Figure 4

The construction of a circumscribed hexagon begins in an analogous manner. After the six points have been located, three diameters are constructed and tangents to the circles are constructed at the endpoints of these diameters (figure 4).

The computation of the perimeter of the inscribed hexagon is straightforward. The six triangles formed by the diameters are equilateral (chords are equal in length to the radius of the circle by construction) and congruent (s.s.s.) and hence the perimeter of the hexagon is 6 units. It is, therefore, established that $6 < C$, where C is the circumference of the circle.

The computation of the perimeter of the circumscribed hexagon requires an additional construction. A line is drawn from the center of the circle to a vertex of the hexagon creating two right triangles. It can be shown that the two triangles are congruent (hypotenuse-leg) and consequently each triangle is a 30° - 60° - 90° triangle.

If the length of the hypotenuse is defined as x then the length of the side opposite the 30° angle is $x/2$. Applying the Pythagorean Theorem

$$x^2 = \left(\frac{x}{2}\right)^2 + (1)^2 \quad \text{or} \quad x = \frac{2\sqrt{3}}{3}$$

It can also be shown that x is also equal to the length of one side of the hexagon and hence the perimeter of the hexagon is equal to $6(2\sqrt{3}/3)$ or $4\sqrt{3} \approx 6.928$.

Thus $C < 6.928$. Combining the information concerning the inscribed and circumscribed hexagon:

$$6 < C < 6.928$$
$$3 < C/D < 3.464.$$

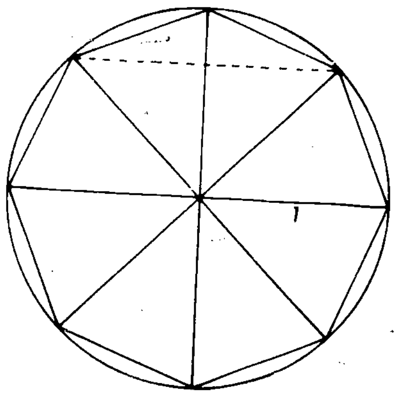


Figure 5

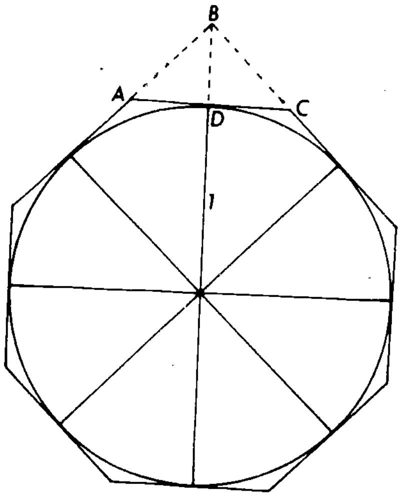


Figure 6

The construction of a circumscribed octagon begins in a similar manner. After the four diameters have been drawn, tangents to the circle are constructed to the endpoints of these diameters (figure 6). Using an averaging technique again with its inherent dangers, the following approximation is obtained: PI $\approx$ 3.232. The observation can again be made that, although the perimeter of a hexagon is a better approximation of the circumference of a circle than the perimeter of a square, a polygon with a greater number of sides will yield a closer approximation.

The Octagon

After constructing a circle with radius of one unit, an octagon is inscribed in the circle by first constructing perpendicular diameter and then bisecting the resulting right angles. The endpoints of four diameters are then joined to form an inscribed regular octagon (figure 5).

As in the computation of the perimeter of the circumscribed hexagon, an additional construction is needed in order to compute the perimeter of the inscribed octagon. A line is first drawn connecting the endpoints of two adjacent sides of the octagon as seen in figure 6.

Upon examining an enlargement of this construction (figure 7), it is apparent that $\triangle AOD \cong \triangle BOD$ (S.A.S.) and, consequently, $\triangle ADC \cong \triangle BDC$ (S.A.S.).

Since $\triangle BOD$ is an isosceles right triangle with a hypotenuse of length 1, the sides OD and BD must both have lengths equal to the $\sqrt{2}/2$. Furthermore OC is a radius of the given circle and hence $\overline{DC} = 1 - \sqrt{2}/2$. Realizing

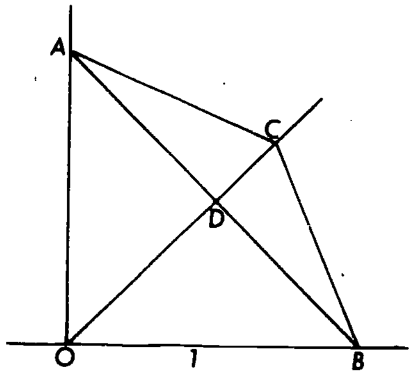


Figure 7

that $\triangle$BDC is a right triangle an application of the Pythagorean Theorem is in order.

$$\left(1 - \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = x^2$$ where x is the hypotenuse of $\triangle$BDC.

Solving this equality yields: $x^2 \approx .586$ or $x \approx .766$.
The value .766 represents the approximate length of one side of the octagon and the perimeter is, therefore, 8(.766) or 6.128. Thus $6.128 < C$ where C is the circumference of the given circle.

Turning to the circumscribed octagon, the solution for the perimeter becomes somewhat easier. With the additional construction in Figure 6 it is apparent that a square is formed with sides of length 1 and a diagonal of length $\sqrt{2}$.

Using the information that the diagonal bisects the angles of the square, it is discovered that not only is $\triangle ADB \cong \triangle CDB$ (A.S.A.) but also that the two triangles are right isosceles triangles. Furthermore since $\overline{BD} = \sqrt{2} - 1$ then $\overline{AD} = \sqrt{2} - 1$ and $AC = 2(\sqrt{2} - 1)$.

Hence the perimeter of the octagon is $16(\sqrt{2} - 1) \approx 6.624$. The conclusion is that $C < 6.624$. Combining information the following inequalities are obtained:

$$6.128 < C < 6.624$$
$$3.064 < C/D < 3.312.$$

If the midpoint between the two values is computed then 3.188 is obtained as an approximation of PI.

Conclusions

Using the preceding activities which admittedly employed polygons which were coarse approximations of a circle, it is possible to derive an approximation of PI which is accurate to one decimal place. In addition several important mathematical ideas can be presented at an intuitive level by making observations about the sequence of values which are obtained.

Compilation of Results

$5.656 < C < 8.00$	$2.828 < C/D < 4.000$	$PI \approx 3.414$
$6.000 < C < 6.928$	$3.000 < C/D < 3.464$	$PI \not\approx 3.232$
$6.126 < C < 6.624$	$3.064 < C/D < 3.312$	$PI \not\approx 3.188$

First note that the sequence of perimeters of inscribed polygons form a montonic increasing sequence and the sequence of perimeters of circumscribed polygons form a monotonic decreasing sequence. A justification of this assertation can be offered based upon a geometric interpretation of what occurs as the number of sides of the polygons is increased. In particular consider the relationship between the inscribed square and the inscribed octagon.

Furthermore, upon examining the two sequences, it should also be, at least intuitively, apparent that the value of PI is a Least Upper Bound for the ratio of inscribed perimeter to diameters and a Greatest Lower Bound for the ratio of circumscribed polygons to diameters.

A final observation reveals that several additional avenues of investigations are apparent. For example:

1. Are the perimeters of the inscribed polygons a better approximation of the circumference of the circle than the perimeters of the circumscribed polygons (or vice versa)?
2. Is it possible to compute the area of the preceding inscribed and circumscribed polygons? If so, do these ultimately yield a closer approximation of PI than the perimeters?
3. Is it possible to compute the perimeters of inscribed (and circumscribed) dodecagons or 16-gons within the realm of high school Euclidean Geometry? (Is it necessary to resort to trigonometric functions to determine these perimeters?)

MATHEMATICS: ITS APPLICATION IN THE VOC.-TECH. SCHOOL

by Frank J. Levanti

Aims - Mathematics has always been an integral part of the curriculum in Connecticut Vocational-Technical Schools. Even though the curricula and methodology of mathematics have varied through the years there has always been agreement as to its need in a program of training skilled tradesmen. As a result of advances in industrial and scientific knowledge, this conviction is stronger than ever. Therefore, it is imperative that a program of mathematics be so designed and administered as to fulfill the broad aims of vocational education and the following specific objectives:

1. To meet the needs of the specific trade area;
2. To develop in students an understanding of fundamental mathematical principles and an ability to use them;
3. To develop in students the ability to think and organize effectively in the changing technology and society of our times; and
4. Develop each student for the fullness of his capabilities.

Preparation of Outline - To ascertain the content of mathematics in the Connecticut Vocational-Technical Schools trade instructors in specific areas working with their school directors prepared a list of mathematical skills and knowledge which is essential for a student to possess at a specific time if the student is to progress satisfactorily through the established training program.

The Mathematics Curriculum Committee prepared a list of minimum skills and knowledge in mathematics which should be required of all graduates of Connecticut Vocational-Technical Schools.

This lists the following knowledge and skills as essential:

1. Fundamental operations of whole numbers;
2. Any arithmetic process involving decimals, fractions, percentage, ratio proportion or denominate numbers;
3. Square root;
4. Mensuration including angle measurement, areas and volumes and similar triangles following solution of simple literary equations and formulae;
5. The techniques of problem solving;
6. Basic geometrical constructions;
7. Definitions of geometric terms;
8. Facts concerningrelationshipsexisting between geometric figures;
9. Elementary, right triangle, trigonometry; and
10. Logarithms.

Furthermore, programs of mathematics of various academic and technical schools were surveyed and evaluated.

As a result of these reports and investigations the content of the program was determined.

General Plan - Since it is recognized that the ninth grade program is that of exploration the mathematics in this grade is of general nature and so named - *General Mathematics.*

It cannot be over-emphasized that this phase of the mathematics program must be presented most constructively. It is not intended to be a mere review or exercise type of program, teaching for understanding and accuracy must be stressed constantly, in fact, it is probable that a great deal of time may be spent on the remedial aspects of mathematics. The subject matter for general mathematics approximates very nearly the standard ninth grade junior high school mathematics program.

The program for mathematics for grades ten through twelve - Related Mathematics - is an integrated program based on the elements of algebra, geometry and trigonometry essential to trade mastery. It is arranged in a sequence of logical teaching order and also more accurately and immediately to meet the needs of both trade training and related science education. It is hoped that this arrangement of the content will fulfill our specific objectives.

Basic Assumptions - It must be assumed that students in the technical schools have the ability to learn mathematics. Since it has been agreed by shop and related instructors and the supervisors of instruction that this basic program is either needed by the student to successfully complete his trade training or as a basic need for a secondary school education, it follows that every student should successfully complete the mathematics program.

Teaching Procedure - A student to be able to relate his experience in mathematics to his trade program must have a thorough understanding of the fundamentals involved. He must be able to analyze and solve problems and to think effectively. Therefore, it is important in teaching mathematics that considerable time be spent on *teaching for understanding and not just the manipulative skills of mathematics.* It is very evident that to encourage, stimulate and challenge the student applications of mathematics to the trade plays a very important part. Therefore, wherever possible these applications should be made evident to the pupil by the instructor.

Furthermore, it is felt that the sequence is so arranged as to afford the best possible avenue of instruction. However, if for some reason an instructor needs to change the order or add to it, he should feel free to do so, but it must be remembered that the achievement level set forth at each year should be attained. It is expected that group instruction will prevail in all classes with individual attention given to students in keeping with sound teaching procedures.

Related Mathematics - Grade Nine - In the mathematics department, we are preparing students to enter their chosen trade with a background of mathematics to enable them to function adequately. By this we mean, the student should be able to figure\for specifics. These specifics may be measuring or estimating in the trade of his choice.

For years we have been getting students who are not prepared in the fundamentals of arithmetic. For this reason the ninth grade mathematics is of a general nature to insure a familiarity with fundamentals. These fundamentals must be taught to insure accuracy and understanding. In many cases there is much work to be done for the student who just doesn't understand figures. Where time allows remedial work has to be done.

The technical school ninth grade parallels somewhat the ninth grade general mathematics in high school. The only difference may be that we use problems which are of a vocational nature. The problems are related to things he is doing or that he shall do in the future. Information that is given with preliminaries of a lesson should refer to the usefulness of the problem in question. This is something that may be lacking in his previous educational experiences.

The student should be made conscious that the mathematics he is learning is a tool, and that it has practical application in his trade work. He should be brought face to face with the need to think out the processes so that he understands what he is doing. This is more important than a particular answer to a specific problem. Understanding principles, and their aims are the important factors. There is more than the "how" of a problem to learn. He must learn "where", "when" and "why" these general principles are important. He must learn "what" the general benefit of the whole subject is to him. Learning by rote gives the students no reason to learn. If a student knows where he is going he is more apt to go. To accomplish this, the instructor should use every means to illustrate vividly to this end. The result should be a student who will learn to think by means of the orderly mathematical process.

To get down to specifics, the ninth grade should teach the students whole numbers, fractions, decimals, addition, subtraction, multiplication of these numbers, graphs, mensuration, ratio and proportion and the fundamentals of algebra. The algebra depends upon the ability of the particular group. These basic principles should be instilled. We should work to have the students see the need to acquire these skills, and we should use every means available to reach this objective.

Related Mathematics - Grade Ten - At the beginning of the school year the student (according to the approved mathematics outline) reviews denominate numbers, ratio and proportion and measurement of line and angle to be followed by the elements of algebra.

Grade ten classes begin the school year studying the elements of algebra, so the students will be acquainted with the processes involved in solving a simple equation or formula. This skill is needed in grade ten science (physics) classes and in grade ten electrical trade theory.

At this point the algebra books are put aside and the class applies these principles to ratio and proportion. In studying ratio and proportion, denominate numbers (English and Metric) are used. This section is related to various shop problems and each class member, regardless of shop, participates in each trade area. For example;

Auto Shop	Gearing versus speed
Carpentry	Paint versus Square Foot
Electrical	Transformers Wire Length versus Resistance
Machine & Machine Drafting	Gearing-belting, Wire Area versus Load
Plumbing	Pipe Diameter versus Capacity
Sheetmetal	Area versus Air Flow

The principles or elements of algebra are continued until the student is able to transpose, solve and primarily understand a formula or equation. This is the principle goal of the tenth grade program, and also the basic achievement level, although some advanced groups study quadratic and simultaneous equations.

It should be understood that in this area some algebraic principles must be treated from an academic viewpoint, although wherever possible we use and point out trade applications. Presently I am working on a project to expand our present mathematics outline, taking each topic and listing possible teaching activities, assignments, references, trade applications (Delmar workbooks) and for example -- teaching aids.

In all cases we try to teach for an understanding of the topic so "Thinkers" will result, not "Parrots".

Related Mathematics - Grade Eleven - The beginning of the eleventh grade is given over to the review of fundamentals of algebra. This is done both to aid retention of these fundamentals and to insure that the student has the necessary base for the year's work. For the most part, this review is devoted to formula and equation solving.

After the review a short unit in numerical trigonometry is studied. This unit is so placed not because it is necessary in the development of the mathematics program, but rather as an aid to the machine area departments. The purpose of this unit is threefold:

to acquaint the student with the trigonometric functions;

to enable him to read a table; and

to develop an ability to solve simple problems.

This unit is designed to be related to the needs of the trade. This does not mean that it is necessary to demonstrate numerous trade applications. Time does not permit such depth; furthermore, it can be more effectively handled during the twelfth grade.

This brings us to the beginning of the geometry course. The first few weeks are devoted to terminology and constructions. To bring in trade applications of constructions seem artificial and not to the best interests of the student. First of all the student does not understand why the construction work...this can come only after a study of geometry. The main advantage of construction work at this point is not actual knowledge of the constructions for this is soon forgotten, rather it is the terminology involved. After a week's work with construction problems, such terms: arc,bisect, perpendicular and parallel have become firmly established in the vocabulary of the student.

In order to better understand our geometry course, it might be well to take a brief look at the history and development of geometry and then to contrast our course to that of the academic high school.

Practically all the geometry taught in the secondary school today is contained in Euclid's Elements, written about 300 B.C.

Nothing could be further from our needs and objectives than to teach geometry in the traditional methods(with which so many of us are familiar from our own student days). This method consisted of memorizing the proof of theorem after theorem and, it seemed, never getting to any practical applications. We must bear in mind that this logical development of a system of geometry which was given to us by Euclid was not considered as an exercise in practical mathematics. Euclid was not a mathematician, but a philosopher. Geometry, as he developed it, was intended to serve as an introduction to logic and general philosophical studies; therefore, he emphasized logical structure rather than mathematical insight, neglected practical applications completely and carefully selected the subject matter in the light of his philosophical orientation. "He wrote for scholars, not for schoolboys" -- and especially not for tradesmen. And yet if we take almost any academic geometry textbook written more than ten years ago and follow it religioulsy, we would be following the course outline set up by Euclid. Very recently there have been sweeping changes in the academic geometry course, but these are designed for the college preparatory student and would not be of great benefit to us.

How do we teach geometry? What makes our course related mathematics? Both the method and the content differ greatly from the academic course.

METHOD: Emphasis is not placed upon the formal deductive proof. The theorems are arrived at, for the most part, by informal discussion and experimentation. In cases where this is not possible, the theorems are simply postulated. The main emphasis of the course is placed upon problem solving. The student learns the mathematical approach to a problem not by strict formula rules, but rather by being confronted with numerous practical problems. Until this point in the mathematics program, there have been set rules to follow and usually only one way to arrive at the correct answer. It is very gratifying to watch them discover that there may be many different ways to solve the same problem. Different approaches are encouraged, and the student soon learns that as long as he has a legitimate reason for each step, he will arrive at the correct answer. Such an approach provides a real opportunity for creative thinking. The student also learns that a large part of the mathematical approach is trial and error.

CONTENT: Time does not allow us to cover as much material as in the academic geometry course; therefore, we concentrate on those theorems which have the greatest practical value. For instance, theorems dealing with congruent triangles, so necessary in the development of Euclidean geometry, serve little purpose in our course. On the other hand, the theorems dealing with tangents to circles receive great emphasis because of the numerous practical applications involving measurement of dovetails and angle-cuts. The main criterion used in selecting the theorems to be covered is the practicability of the theorem for trade applications. Thus, the geometry course is truly related mathematics.

This completes the basic requirements for the eleventh grade as defined by the outline. We then spend a few weeks working on the algebra which was not completed in the advanced section of grade ten.

The next topic is the slide rule. We cover only multiplication, division, squares, square roots and proportions. While this unit has no specific trade applications, we try to point out both the advantages and the limitations of the slide rule for the tradesman.

If time allows we attempt to combine the entire year's work in one unit. Using handbooks and slide rules, we cover the sections on mensuration in workbooks. This unit involves the knowledge of geometrical facts, the manipulation of algebraic formulas and the ability to use a slide rule. It also provides an opportunity, in the machine trades, to become more familiar with the handbook. The greatest benefit of this unit is that it shows the student why he studied each of the topics involved, and it allows him to take these new mathematical tools and use them in solving a practical trade problem.

Related Mathematics - Grade Twelve - The seniors begin the year with a review of the Pythagorean Theorem and square root. Writing and learning the rules accompanied by practice work is followed by such topics as work methods, arrangement, accuracy, use of diagrams, tables, formulas, approximate numbers, significant digits, rounding off, errors in computation, short methods and checking. The use of square root and cube root tables is included.

The course then continues with a review of examples. At the board simple work with triangles is illustrated to show the concept of constant ratios, using angles of 30^{o}, 45^{o}, and 60^{o} in triangles where the length of sides change, but not the angle. This is really an introduction to the use of the tables of natural functions. Herein the students are shown how to obtain the function when the angle is given, and the reverse. Some experience much difficulty learning to interpolate and anti-interpolate; thus, plenty of practice is required. The students are now ready for the solution of right triangles.

Thus far, the students have been working with triangles, each having two values given. They have had no practice with word problems requiring interpretation and visualization of the trig situation. Such applications are now explored. As complete a variety as possible, representative of all trades, is presented for practice. This is the end of the basic achievement level. Continuing, beyond the achievement level with the better groups, treatment of oblique triangles is similar to that of right triangles, i.e., explanation and illustration of the laws of sines and cosines, practice examples in each of the four cases. Areas of triangles by means of trigonometry are included. Again, application to the trade situations by means of word problems is employed.

Graphs and logarithms follow the trigonometry section, and if time permits, a general review is conducted.

Some Deviations - Although vocational-technical schools have many similarities there are some recognizable differences Some of the basic differences are the three and four year programs.

As a result of these program differences there are some deviations in the mathematics curriculum. Thus, at the technical schools which conduct a three year program, grades ten through twelve, the four year mathematics outline is covered in three years.

Teaching Technique - The school director should observe whether or not a variety of teaching techniques is employed by the instructor. Specifically, students should be encouraged to participate in discussions, demonstrate their achievements at the board, be free to question the instructor, be encouraged to experiment and be shown concrete practical applications of the basic principles under study.

Major Problem - "Related" Mathematics - Many people feel the term related mathematics simply means that the student in a given trade shop should solve a number of typical problems found in the trade. This was being done approximately fifteen to twenty years ago when the mathematics instructor passed out the blocks of practical mathematics problems and from that point on everyone was on his own.

No doubt, everyone can see the many shortcomings in that type of approach. Needless to say, there were some few benefits.

Today, we find isolated, and to some extent obstinate, instructors using this approach. On the other hand, many instructors are being rightly criticized for using a college preparatory approach.

It is hoped that the basic course in the vocational-technical schools will be found somewhere between these two extremes. This course will provide the student with mathematical skills and understandings which he needs in order to make satisfactory progress in his trade training program.

The troublesome task comes in implementation. Thus, in observing the mathematics instructor one will find these various means of implementation being employed:

1. The specific trade application made after understanding has been established.
2. Illustrating the principle by using problems taken from a variety of trades.
3. Devoting one period per week to solving specific trade problems.
4. Devoting a marking period to trade problems.
5. Devoting part of the senior year to trade problems.
6. Use of trade problems from a variety of trades and the use of trade problems in the theory program.

From informal and standardized tests no program proved better than the other. What should our futureapproach be?

Perhaps this could only be answered in the hope that more classroom or action research be undertaken to help solve this problem.

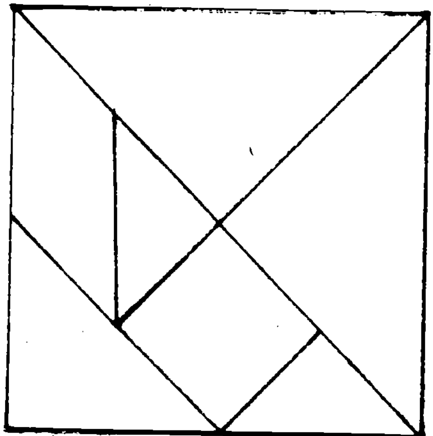
Center for the Improvement of Mathematics Education
In-service Training Workshop
TANGRAMS

by Samuel M. Lipman

The tangram puzzle is cut from a square as illustrated below. Use this as a pattern (but don't peek when you are trying to put it back together). Please, please ask children to cut their own. Then it truly is their puzzle - and at least they should be convinced that it actually does go back into a square. The suggestions listed below develop some mathematical concepts for which the tangram puzzle serves as a model.

1. Put all 7 pieces back together to make a square. If this gets too frustrating, don't sneak a glance at your neighbor's. Skip this and come back later, or hope in the meantime the pieces will fall together when you're trying something else. Don't rob yourself or your students of the joy of discovering the answer.

2. Put all 7 pieces together to form a rectangle. (I think there are two dif-



ferent ways of doing this.) Now, by moving only 1 piece from the rectangle, you can change it to be a triangle, a parallelogram, and a trapezoid.

3. Can you make a square from just 1 piece of your puzzle? from 2 pieces? from 3? 4? 5? 6? 7? Do the same investigation for triangles, rectangles, parallelograms, and trapezoids.

4. One way to organize this as a class activity is to explain the problem with the square, and then put a large chart on the board.

When a child discovers one, he can go up and mark yes or no on the chart, and put his name also so others can either challenge him or see if their results are the same. Encourage dialogue among the students - they often seem to explain things to each other as well as or better than teachers do.

5. Take the square, the parallelogram, and the middle-sized triangle. Compare their areas. Can you convince your friends why what you think is true? Which of these three has the smallest perimeter? (Or ask the question in another way: If I were to plant a garden, and needed a fence to keep the dog out, which of the three shapes needs the shortest fence?) How do you know?

6. Suppose I say the square piece has an area of 1 square unit. How much is the area of each of the other pieces? What is the area of the big square made of all 7 pieces?

Suppose I change my mind and say the big square made of all 7 pieces has the value of 1 square unit. Now figure the area of all the other pieces.

Choose any number to represent the number of square units in the big square (like 5 or 11 or your age or your favorite number) and then figure the area of the others. Children need some facility with fractions to do this.

AND NICEST OF ALL -- whenever someone puts all 7 pieces together to make an interesting shape, he should trace it on tagboard, name it, sign it, and put it in the class tangram box so others can try to fit their pieces in the shape.

APPLICATIONS OF LINEAR PROGRAMMING IN ECONOMICS

by Ken McCaffrey, Brattleboro Union High School, Vt.

Prologue:

The Simplex Method of linear programming was originally developed in 1947 by George Danzig who first recognized the widespread applicability of a process of maximizing a linear function. The Simplex Tableau was later refined by, among others, Professor A.W. Tucker.

After developing the arithmetical basis of the Simplex Method, we present the application of this method to a specific problem of maximizing a profit function which is constrained by a system of linear inequalities.

Section 1, The Simplex Tableau and Pivot:

Consider the following system of three equations in five variables:

$$\left.\begin{array}{l} x + y + s_1 + s_2 + (-1)s_3 + (-22) = 0 \\ (-7)x + (-1)y + 2s_1 + (-1)s_2 + (-2)s_3 + 46 = 0 \\ 3x + 2y + (-3)s_1 + (-2)s_2 + 4s_3 + (-12) = 0 \end{array}\right\} (1)$$

and an equivalent system:

$$\left.\begin{array}{l} (-1)x + (-4)y + 40 = s_1 \\ (-3)x + (-1)y + 30 = s_2 \\ (-3)x + (-4)y + 48 = s_3 \end{array}\right\} (2)$$

In this form s_1, s_2 and s_3 are dependent variables while x and y are independent variables. A system which has been expressed in terms of dependent and independent variables is said to be in canonical form. If we set the independent variables equal to zero, and if the system then has a unique solution, this solution is called a basic solution. In the above case let $x = y = 0$ and $s_1 = 40$, $s_2 = 30$ and $s_3 = 48$ is a basic solution.

Clearly, (2) is not the only canonical form. We could solve for x, substitute, and arrive at a canonical form in which s_1, x and s_3 are dependent while s_2 and y are independent; as follows:

$$(1/3)s_2 + (1/3)y + (-10) + (-4)y + 40 = (1/3)s_2 + (-11/3)y + 30 = s_1$$

$$2\uparrow$$

$$\begin{array}{l} (-1)x + (4)y + 40 = s_1 \\ (-3)x + (-1)y + 30 = s_2 \xrightarrow{1} (-1/3)s_2 + (-1/3)y + 10 = x \qquad (3) \\ (-3)x + (-4)y + 48 = s_3 \end{array}$$

$$2\downarrow$$

$$s_2 + y + (-30) + (-4)y + 48 = s_2 + (-3)y + 18 = s_3$$

System (3), also in canonical form, was derived from system (2) and differs from it in that x and s_2 have exchanged roles: x is now a dependent and s_2 an independent variable. This operation, in Tableau format is the Simplex Pivot.

To write system (2) in Simplex Tableau format we first rewrite it as:

$1(-x) + 4(-y) + 40(1) = s_1$

$3(-x) + 1(-y) + 30(1) = s_2$

$3(-x) + 4(-y) + 48(1) = s_3$

Now writing the independent variables and '1' as column multipliers we have:

-x	-y	1	
1	4	40	$= s_1$
3*	1	30	$= s_2$
3	4	48	$= s_3$

(2)

To interchange x and s_2 we pivot on the 3* which is in the column under (-x) and the row of s_2. Details of the pivot operation follow:

-w	-v	
⋮	⋮	
.... p*	r	= u
⋮	⋮	
.... c	e	= z
⋮	⋮	

p = pivot element
r = element in row or pivot
c = element in column of pivot
e = any other 'off axis' element.

In the resulting Tableau the w and the u will exchange places. Each of the elements noted above will be replaced with:

-u	-v	
⋮	⋮	
.... 1/p	r/p	= w
⋮	⋮	
.......c/-p	E	= z
⋮	⋮	

where $E = \frac{pe - cr}{p}$

The following are the calculations for Tableau (2) above:

element	replacement	result
the pivot	1/p	
3*	1/3	1/3
row elements	r/p	
1	1/3	1/3
30	30/3	10
column elements	c/-p	
1	1/-3	-1/3
3	3/-3	-1
other elements	$\frac{pe - cr}{p}$	
4	$\frac{4(3) - 1(1)}{3}$	11/3
40	$\frac{40(3) - 1(30)}{3}$	30
4	$\frac{4(3) - 3(1)}{3}$	3
48	$\frac{48(3) - 3(30)}{3}$	18

System (2) has been transformed into system (3):

$-s_2$	$-y$	1		
-1/3	11/3	30	$= s_1$	
1/3	1/3	10	$= x$	(3)
-1	3	18	$= s_3$	

The following sequence of pivots illustrates the original system (1) can be transformed into equivalent systems and finally to a canonical form, system (2). The zeros are interchanged in the same way as variables, however, when they become column multipliers the column can be eliminated since multiplication by zero gives zero.

$-x$	$-y$	$-s_1$	$-s_2$	$-s_3$	1		
-1	-1	-1*	-1	1	-22	= 0	
7	1	-2	1	2	46	= 0	(1)
-3	-2	3	2	-4	-12	= 0	

-x	-y	0	$-s_2$	$-s_3$	1	
1	1		1	-1	22	$= s_1$
9	3	2	3*	0	90	= 0
-6	-5	3	-1	-1	-78	= 0

-x	-y	0	$-s_3$	1	
-2	0		-1	-8	$= s_1$
3	1		0	30	$= s_2$
-3	-4		-1*	-48	= 0

-x	-y	0	1	
1	4		40	$= s_1$
3	1		30	$= s_2$
3	4		48	$= s_3$

(2)

Section 2: A System of Inequalities:

The following system of inequalities in two variables and its graphical representation will be referred to throughout the remainder of this paper. The graph is intended to aid the reader in visualizing the concepts presented in the next four sections, however, the numerical methods discussed are not restricted to two dimensional systems.

1. $x + 4y \leq 40$
2. $3x + y \leq 30$
3. $3x + 4y \leq 48$
4. $x \geq 0$
5. $y \geq 0$

In the following diagram the arrows indicate the direction of each inequality. That set of points which satisfies all five inequalities is called the feasible region. The coordinates of the points of intersection have also been noted.

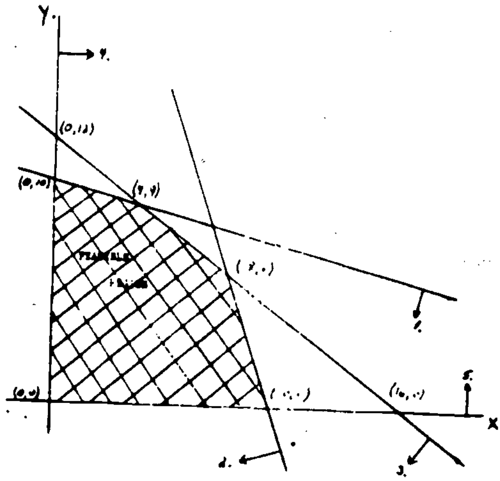


Figure 1

Section 3: Rewriting an Inequality as an Equality: Using Slack Variables:

Consider the first inequality of Section 2: $x + 4y \leq 40$. Since the expression $(x + 4y)$ is constrained to be less than or equal to 40, we can add a non-negative variable, s_1, to $(x + 4y)$ so that it is equal to 40. Thus:

$$x + 4y + s_1 = 40 \qquad s_1 \geq 0.$$

Now solving for s_1 gives:

$$(-1)x + (-4)y + 40 = s_1.$$

By doing this to the first three inequalities we obtain system (2), a system of equalities, subject to the constraint that all variables be non-negative. Thus:

$$\begin{aligned} (-1)x + (-4)y + 40 &= s_1 & s_1 &\geq 0 \\ (-3)x + (-1)y + 30 &= s_2 & s_2 &\geq 0 \\ (-3)x + (-4)y + 48 &= s_3 & s_3 &\geq 0 \\ & & x &\geq 0 \\ & & y &\geq 0 \end{aligned}$$

Section 4: Pivoting Through the Canonical Forms:

As was noted above, there are several canonical forms ($_5C_3 = 10$), which we will now examine more closely. Consider the following:

1. In canonical form, if the independent variables of a system (the two at the top of the Tableau) are set equal to zero, the right hand column contains the values of the dependent variables for a basic solution;

2. Each variable, when equal to zero, represents the boundary line of a constraining inequality;

3. The non-slack variables, x and y, are the axes of the graph (see fig. I) and whether independent or dependent their values are the coordinates of a point;

4. Graphically, the solution set of two linear equations is the point of intersection.

In each canonical form, the values of x and y in the basic solution are the coordinates of the point of intersection of the two boundary lines represented by the independent variables.

Please refer to fig. I while considering the following sequence of pivots:

-x	-y	1		(x,y)
1	4	40	$= s_1$	
3*	1	30	$= s_2$	(0,0)
3	4	48	$= s_3$	

$-s_2$	-y	1		
-1/3	11/3	30	$= s_1$	
1/3	1/3	10	= x	(10,0)
-1	3*	18	$= s_3$	

$-s_2$	$-s_3$	1		
8/9*	-11/9	8	$= s_1$	
4/9	-1/9	8	= x	(8,6)
-1/3	1/3	6	= y	

$-s_1$	$-s_3$	1		
9/8	-11/8	9	$= s_2$	
-1/2	1/2	4	= x	(4,9)
3/8	-1/8	9	= y	

Pivoting through the canonical forms produces points of intersection.

Section 5: Maintaining Feasibility:

I have perhaps mislead the reader, in the preceeding section, into thinking that the points of intersection denoted by the canonical forms are corner points of the feasible region. Without the consideration of two rules to be developed in this section, they are not necessarily such points. Consider the following pivot:

A

$-s_1$	$-s_3$	1		
9/8	-11/8	9	$= s_2$	
-1/2*	1/2	4	$= x$	(4,9)
3/8	-1/8	9	$= y$	

B

$-x$	$-s_3$	1		
9/4	1/4	18	$= s_2$	
-2	-1	-8	$= s_1$	(0,12)
3/4	1/4	12	$= y$	

Referring to fig. 1, note that this point (0,12) is not an element of the feasible region; also that this point is on the wrong side of inequality no. 1, into which the slack variable s_1 was introduced, in Tableau B notice that a **basic** solution gives $s_1 = (-8)$; but, all variables were constrained to be non-negative. To realize how we could have avoided this loss of a feasible canonical form, notice that in Tableau A we pivoted on (-1/2). The algorithm of the Simplex Pivot requires that the elements in the row of the pivot (except the pivot itself) be divided by the pivot. Dividing a positive 4 by a negative 1/2 will surely give a negative result. In this paper we are restricting the discussion to canonical forms which are already feasible. That is, the column under the '1' in any tableau will contain only non-negative elements. Therefore, to maintain feasibility: *pivot on a positive element.*

But this is not enough. Referring to Tableau A again pivot now on 3/8 in the first column to obtain:

$-y$	$-s_3$	1		
-3	-1	-18	$= s_2$	
4/3	1/3	16	$= x$	(16,0)
8/3	-1/3	24	$= s_1$	

The point (16,0) is on the wrong side of inequality no. 2, and $s_2 = (-18)$. Since the (-18) appeared as an 'off axis'' element of the pivot, we consider the following skeletal tableau:

$$
\begin{array}{ll|l}
 & & 1 \\
p^* & \cdots\cdots\cdots\cdots & e_p \\
\cdot & & \cdot \\
c & \cdots\cdots\cdots\cdots & e_c
\end{array}
$$

We already have feasibility so e_p and e_c are non-negative. We must pivot on a positive element, p. Again recalling the algorithm of the Simplex Pivot, the element e_c will be replaced with:

$$\frac{pe_c - ce_p}{p}$$

which we desire to be non-negative. If c is either negative or zero, then the expression above will be positive. If c is positive we need: $pe_c - ce_p \geq 0$. Transforming this into:

$$\frac{e_c}{c} \geq \frac{e_p}{p} \quad ,$$

we see that feasibility will be maintained if, in a given column, we divide each positive element into the constant term on that row, and ***pivot on the element which produces the smallest quotient.***

When these two rules are followed, those canonical forms which are generated by successive pivots will yield basic solutions of which the x and y values are coordinates of the corner points of the feasible region.

It is shown in Glicksman [1], Kemeny [2] and many algebra textbooks that a linear function which is constrained by a system of inequalities takes on a maximum value at a corner point of the feasible region. It is for this reason that we are especially interested in the basic solutions of the canonical forms.

We have now covered the basics of the Simplex Pivot and are ready to solve a linear programming problem.

Section 6: Maximizing a Linear Function:

Zxy Musik Kompanie makes two instruments, a xylophone and an automatic yodeler. Three processes are involved in their manufacture: part production, polishing and final assembly. The union which represents the skilled production workers and the polishers demands a work week of no more than 40 hours. The polishers must spend 10 hours a week on machine maintenance so a maximum of 30 hours is available for the polishing process. The semi-skilled assembly workers can put in overtime, but up to no more than 48 hours a week.

It requires 1 hour to produce the parts for a xylophone and 4 hours for a yodeler. 3 hours of polishing are needed for a xylophone, only 1 hour for a yodeler. It takes 3 hours to assemble a xylophone, 4 hours to assemble a yodeler.

A xylophone sells for $1000 while a yodeler sells for $2000. How many of each should Zxy Musik Kompanie make to realize a maximum weekly income?

To summarize the problem:

let x = number of xylophones produced

y = number of yodelers produced

Thus:

	xylophone	yodeler	constraint
part prod	1x	4y	$\leq$ 40
polishing	3x	1y	$\leq$ 30
assembly	3x	4y	$\leq$ 48
selling			
price thousand $	1x	2y	= Income (I)

This is the system of inequalities noted in Section 2 with the additional function: 1x + 2y = I to be maximized. We now rewrite the inequalities as equalities by the addition of slack variables s_1, s_2 and s_3; include the function I to be maximized; write the problem in the format of the Simplex Tableau and pivot to its solution. Thus:

-x	-y	1		ratios
1	4	40	$= s_1$	40/1 = 40
3*	1	30	$= s_2$	30/3 = 10
3	4	48	$= s_3$	48/3 = 16
-1	-2	0	=I	

The basic solution in this canonical form is: x = y = 0 and I = 0. We are producing zero products and income is zero, but it is feasible. Noting that: 1x + 2y + 0 = I, especially that the coefficients of x and y are positive (negative in tableau format). Of course Zxy Musik Kompanie should produce more than zero instruments if it wishes to have some income; the tableau indicates that x and y should not be independent variables and set equal to zero. We must pivot x and y to be dependent variables. Since both x and y have negative coefficients in the tableau, let us arbitrarily pivot in the column of -x. Observe the calculations for the least ratio and pivot on

3*. Thus:

$-s_2$	$-y$	1		
-1/3	11/3	30	$= s_1$	30/(11/3) = 8^+
1/3	1/3	10	$= x$	10/(1/3) = 30
-1	3*	18	$= s_3$	18/3 = 6 ←
1/3	-5/3	10	$= I$	

We are now producing zero yodelers, 10 xylophones and realizing an income of 10 thousand dollars. In this canonical form we have: $(-1/3)s_2 + (5/3)y + 10 = I$. The solution I = 10, will be improved by pivoting so that y becomes a dependent variable. Again, examine the ratios and pivot on 3* gives:

$-s_2$	$-s_3$	1		
8/9*	-11/9	8	$= s_1$	8/(8/9) = 9 ←
4/9	-1/9	8	$= x$	8/(4/9) = 18
-1/3	1/3	6	$= y$	negative
-2/9	5/9	20	$= I$	

There is still a negative entry in the last row indicating that s_2 should be pivoted into the solution. Thus:

$-s_1$	$-s_3$	1		
9/8	-11/8	9	$= s_2$	
-1/2	1/2	4	$= x$	(4,9)
3/8	-1/8	9	$= y$	
1/4	1/4	22	$= I$	

Now we have: $(-1/4)s_1 + (-1/4)s_3 + 22 = I$ and can do no better. If either s_1 or s_3 is given a non-zero (of course positive) value, the 22 thousand dollars will be decreased Zxy Musik Kompanie will do best to manufacture 4 xylophones and 9 yodelers and will have a gross income of \$22,000 per week.

Notice that in the original function to be maximized:

$$1(x) + 2(y) + 0 = I$$

$$1(4) + 2(9) + 0 = I$$

$$22 = I$$

Part of the final solution is $s_2 = 9$. This was the slack variable originally introduced into the inequality constraining the total polishing time:

$$3(x) + y + s_2 = 30$$

$$3(4) + 9 + s_2 = 30$$

$$21 + s_2 = 30$$

$$s_2 = 9$$

The polishing department will use 21 hours per week leaving 9 hours as unused 'slack' time.

Section 7: Summary of the Simplex Algorithm for Maximizing a Linear Function:

1. Slack variables are introduced to rewrite the constraining inequalities as equalities. All variables are constrained to be non-negative.
2. The equations are written in Tableau format with the function to be maximized as the bottom row.
3. Feasibility is maintained by pivoting on a positive element. If there are more than one positive elements in a given column, pivot on the one which as a divisor gives the smallest quotient with the element in the right hand column.
4. The function is maximized when the bottom row contains only non-negative elements. If there are negative elements in the bottom row, pivot in one of these columns. Several pivots are usually necessary to complete a problem.

Final note: the procedures discussed in this paper apply to maximizing a linear function. Further considerations are necessary to solve a minimizing problem.

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NEW UICSM MATERIALS – INTRODUCTION TO ALGEBRA AND GEOMETRY AND PROBABILITY

by Robert C. McLean, Northeastern University

In 1958 I made a study of teachers' attitudes toward integrated algebra-geometry courses for college preparatory students who were not going on in mathematics. The almost universal reaction was "These subjects cannot be integrated. It is mixing oranges and apples." Since the simile is from "old math" perhaps teachers of the "new math" which can handle mixing sets of apples and sets of oranges, are ready to accept a mixture of algebra and geometry in the same course. Historically, algebra and geometry were mixed in a branch of mathematics called analytic geometry. Thus, mathematically the problem has been solved. Is there an educational solution to the problem?

The responses I received to my questionnaire reflected more than the respondents' lack of recognition of the existence of a mathematical solution to the problem. It reflected the educational solutions to the problem which were then being offered. At that time, integrated algebra geometry courses might be classified into two large categories. One consisted of teaching a chapter from an algebra book, a chapter from a geometry book, a second chapter from the algebra book, and so forth. These were a set of alternate units in geometry and algebra. Some of these mixtures probably still exist today. In fact, most books written specifically for integrated algebra and geometry courses consist of alternating chapters. The content of the chapters is more closely correlated than are chapters of different books, but the structure is very much the same.

The second method of constructing courses of study that mix algebra and geometry is to teach them on alternate days. They may even have two different teachers. Mr. A may teach algebra periods 1 and 2 on Monday for two classes while Miss B is teaching geometry periods 1 and 2 on Monday to two other classes. On Tuesday Mr. A's algebra students go to Miss B for for geometry, and Miss B's geometry students go to Mr. A for algebra. Some of you may be acquainted with this technique. I have heard of its still being tried in some school systems in this region.

Both of these methods of mixing algebra and geometry accomplish, to a greater or lesser degree one of the major purposes of such programs. It reduces the memory gap between first year algebra and second year algebra. Inserting a year of geometry between the two years of algebra has a bad effect on the student's ability to recall first year algebra concepts when they enter the second year algebra class.

A second, more important, argument for developing integrated algebra-geometry programs, however, is very poorly served by simple chronological mixing of the subjects. This argument says that it is a desirable end for the students to recognize the close relationship that exists between algebra and geometry and that they should be seen as branches of the general field of mathematics which are mutually supportive. At the lowest level, such support consists of using geometry as a source of "applications" for algebra, and of pointing out each time it is used how algebra is a tool for solving many geometric problems. In a specially designed integrated algebra-geometry textbook, this last function can be served, very well. A program which uses two different textbooks may not only fail to serve this function, it may be counterproductive. That is, the algebra textbook may chose geometry problems as applications before they appear in the geometry part of the course, and the geometry textbook may call upon the use of certain algebraic techniques before they have been developed in the algebra textbook. Neither method of designing a curriculum, however, seems to support the idea that algebra and geometry are different manifestations of the same discipline. Or, more correctly, are different views of the same manifestation, and that it is sometimes advantageous to take one view and sometimes advantageous to take the other. Thus, they are not competing subjects but complementary subjects.

Considerations of the educational problem, as roughly outlined above, leads to the proposal that a program which drew upon the mathematical solution to the problem might be the best one. If we accept this point of view, then our problem can be restated. How do you design a course in coordinate geometry for students who have not had a course in algebra or in geometry?

The designers of these materials, in fact, presented themselves with an even more challenging problem. They knew that most of high school mathematics was invented to solve real problems. It was not thought up in someone's head as an exercise in pure logic. Thus, they reasoned should it not be possible to present mathematics in a similar manner, that is, as an invention of man that grew out of his need to solve certain problems. You must carefully note the difference between this use of applications and the common use of them. In the common use, the applications are used to illustrate the mathematics. In this program, the applications are used as the reason for the very existence of the mathematical materials being studied. The problem is presented before the mathematical mechanism has been developed for solving it. The mathematical mechanism is developed in the process of solving the problem.

If we include this second purpose in our consideration of the design of the course, we will find that geometry is not the only source of problems.

In fact, it is the least appropriate source. Many of the problems come from mechanics, that area of physics that deals with motion, Newtonian mechanics, to be exact.

This source of problems does not mean this is a course in physics. It certainly is not. The methods of attacking problems is not that of the physicist, but of the mathematician. This is, truly, a mathematics course. Your friendly colleague in physics would not want to substitute it for his own course, but he would certainly encourage you to substitute it for yours. I will take an example from this program to illustrate why this last statement is valid.

Physics teachers are prone to ask this particularly annoying question of mathematics teachers, "Don't you teach these kids any algebra?" If you ask for it, he can even give you an illustration. For example, he may claim that his students cannot solve simple uniform motion problems using the formula d = rt, given distance and time or given distance and rate. They usually can solve it given the rate and time -- but not always.

One very probable reason for this failure on the part of your former (or current) students to recognize a "physics problem" as an "algebra problem" is because the physics teacher refuses to tell them it is. However, it may be because of our love of x and y in mathematics classes.

Is $d = rt$ the same equation as $y = ax$? Not exactly, but the method of solving it is the same. Can your students solve $ay = x$? How about $3x = 9$? All of these are equations in which one number is shown to be the product of two other numbers. It is one of the most common types of equations found in applications. $A = lw$ is another example. $C = \pi d$ is a third. $I = PR$ is still another.

Recognizing the frequency of such equations in applications, how much time do you give to solving them -- for any one of the three numbers?

Do you give your students equations such as $x = yz$ or $a = bc$, then assign values to different pairs of these variables and ask the students to find the value of the third one? This is what the physics teacher wants his students to be able to do with $d = rt$.

Of course, the physics teacher could help bridge the symbol-gap by trying to find out how his students read algebra or what kind of notation they can read. He could then recast his physical formulas in "algebraic" dress appropriate for the readiness of his students. Finally, he could lead them from that notation to the one with which he wants them to deal. Let me give you an example of such a change of appearance.

Problem: Using the formula d = rt, find the time necessary to travel 20 miles at a constant rate of 5 miles per hour.

Let x = time	$d = rt$
20 = distance	$20 = 5x$
5 = rate	

Somehow, doesn't your brain relax when it sees that familar form, $20 = 5x$? Isn't it easier for you to read than $20 = 5t$? Don't be hasty to say, "No!" You are letting your mind rule your heart. Deep, down inside of you, all of your mathematical training has prepared for you to handle equations stated in terms of x. You can handle T's, but you do not like to.

I want to take this illustration one step further to a point which may or may not be reached in a high-school physics class. $d = rt$ is not the most general form of this law for motion at a uniform speed. This is the more general one.

$$d = rt + d_o$$

What is d_o? The distance from the origin at time o. In terms of coordinate systems, what does this mean? It means that the occurrences of $d = o$ and of $t = o$ are not simultaneous. Let us take the local turnpike as an example. Distances on it are marked by mileposts which begin with o at the Massachusetts border. Let us say that the milepost out here reads 15. Let us say that we also begin from here and travel at a speed of 50 miles per hour. Can we write an equation to describe our location on the turnpike? How about this one?

$$d = 50t + 15$$

Look at this equation very carefully. It is, supposedly, pure physics. Do you have a geometric interpretation for it? Do you want to see it in x's and y's?

$$y = 50x + 15$$

It is a linear equation in two variables. It is in what form? The slope-intercept form. What is 50? The slope. What is 15? The y-intercept. What have we illustrated? Is it a translation between the distance formula in physics and the slope-intercept form of a linear equation from analytic geometry? In fact, can we emphasize this a little by interpreting the slope of the line as a rate of change? The rate of change of the y-coordinate with respect to the x-coordinate. In physics, it is the rate of change of the distance-coordinate with respect to the time-coordinate, is it not?

Notice how the algebra, the geometry, and the physics get intertwined in a natural manner. There is no forced marriage. By learning to think in terms of coordinate systems, many such relationships can be quickly identified.

This initial example from the new UICSM materials was chosen to illustrate this close relationship which is established among algebra, geometry, and motion. This is true integration of subject matter. It is not a mixture of discrete elements. It is, in the language of the new mathematics, a continuous set of elements too closely allied to be separable.

DEVELOPING MATHEMATICAL PROCESSES

by Nancy R. Martin, University of Maine

A mathematics program designed for children must include material which children find fun and exciting to use. It must provide the foundation necessary to understand most strands of mathematics. It must recognize and deal with each student according to his needs, ability, and learning style. One must do more than promote this as the ideal, accomplished on rare occasions by a "super teacher". A new program was needed which provided a useable format and the necessary tools which would enable all teachers to deal more adequately with the vast array of individual differences and group needs in our classrooms.

The developmental edition of such a program (see bibliography) was field tested beginning in 1971. During that time the team of investigators from the Wisconsin Research & Development Center for Cognitive Learning consistently asked three questions while children interracted with the materials: "How do children like to learn?", "What should children learn?", and "How do children learn?". When using the materials with children at the Madison and Chicago field-sights, reactions of teachers and students were carefully considered to insure that activities included in the final program would be fun and exciting for children. To be a significant component of a mathematics program, the activity should build on an overall sequential approach in which the learner develops the ability to use each of the basic strands as a valuable tool in problem solving. To be a pedigogically sound program, it must recognize that no one approach meets the needs of all children. Each child has his own special strengths and weaknesses and within a class many learning styles and developmental levels will be present. Therefore, a sound program includes the variety of approaches necessary when introducing any given concept.

All too often, teachers (and published programs as well) pay lip service to manipulative materials at best. The manipulatives are given casual attention in the teacher's guide and the teacher quickly uses them to demonstrate to the whole class or the children are invited to use the materials at a learning station during their spare time - assuming that the children who need the materials the most will even have the spare time. Observation and questioning of an individual student as he interracts with the materials are not utilized to discover the child's thought processes. This approach ignores the research of leading psychologists who state that the child will progress through various stages of development, beginning with the preoperational stage, then the concrete operational stage, and reaching the stage of formal operations at approximately twelve years of age. It is only then that a child can easily attach meaning to formal, abstract operations. If a child fails to acquire a specific skill, it is usually remediated by more practice via another dittoed work sheet. Unfortunately, there are many ills which the purple plague has failed to cure.

Many children do not readily attach meaning to abstract symbols. Even extensive use of manipulative materials can be of limited value if the teacher does not provide an approach utilizing a gradual transition from the use of objects to the use of symbols.

I was recently working with a fourth grade child whose achievement in math appeared to be significantly lower than his ability in other areas. John's class was learning to regroup in addition and subtraction using the following method:

$$\begin{array}{rcrcrcl} 339 & = & 300+30+9 & = & 200+130+9 & & \\ \underline{-165} & & \underline{100+60+5} & & \underline{100+\ \ 60+5} & & \\ & & & & 100+\ \ 70+4 & = & 174 \end{array}$$

At one of our first sessions, he proudly demonstrated his ability to solve the above correctly. Shortly after this I laid six blocks on the table. I then removed two of them and asked John to write a mathematical sentence which would tell us what had just happened. He immediately asked, "Is that plus, minus, times, or division?" Because of our constant tendency to question students only when they get the wrong answer and because of our tendency to give them formal approaches at a very early stage of their development, I fear that their are many other such students in our classroom who can easily go undiscovered. If one cannot picture what is happening and apply this to new situations, memorized formulas and facts are of little long term value.

The chart below (Figure 1) illustrates the mathematical content included in this program. Note that applied mathematics, i.e. problem, solving is the central strand. Problem solving and measurement will be interwoven with all other strands. Children will be encouraged to develop their own method to solve a problem - not to think always of there being only one correct way.

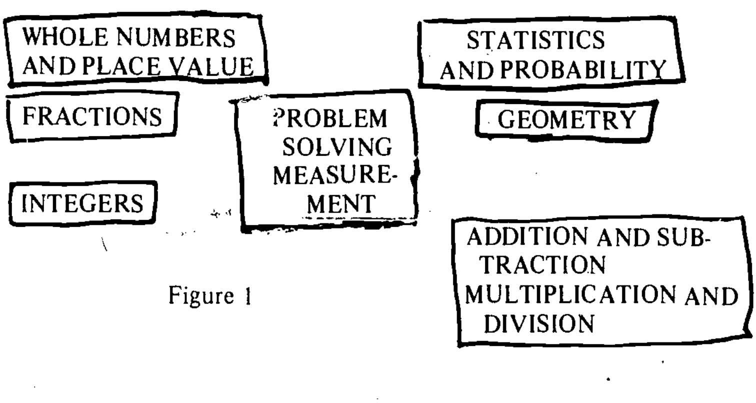


Figure 1

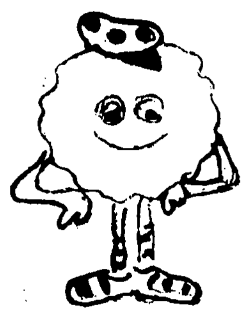
Activities in this program seek to develop the child's skills when working with the nine basic processes identified below (Figure 2). They must work with each of these processes at various levels of sophistication (super-processes) which include the *physical, pictorial,* and *symbolic representation.* This is followed by *validation* in which the student begins with the

BASIC PROCESSES	SUPER PROCESSES: REPRESENTING			VALIDATING		
	Physical	Pictorial	Symbolic	Physical	Pictorial	Symbolic
DESCRIBING & CLASSIFYING						
COMPARING & ORDERING						
EQUALIZING						
JOINING & SEPERATING						
GROUPING & PARTITIONING						

Figure 2

symbolic approach but then demonstrates that his thinking processes were appropriate and his solution accurate using physical or pictorial representation. Processes are best understood when experienced. Try some of the following activities to acquaint yourself with the materials and the nine basic processes.

This is Silly Sylvester, one of the DMP characters. How many attributes of Silly could you list when DESCRIBING him to your friend?



Some of the possibilities which you might have listed are thin legs, a spotted hat, a fuzzy body, or striped shoes.

When describing an object, the number of possible attributes is limited only by the child's imagination - there is always more than one correct answer.

Classifying is the process of sorting objects on the basis on one or more attributes. Look at the lines below and circle the one which you think belong together.

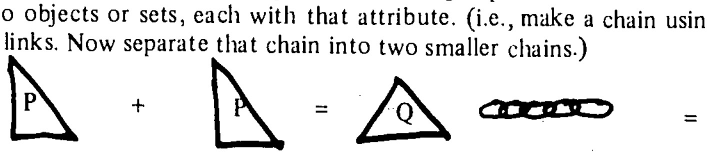
Some of you might have chosen to concentrate on "short lines", others on "wavy lines", and still others on "long, straight lines". Notice that once again there is no one correct answer! The emphasis is upon the child's ability to think independently and to find a logical solution. Children at one stage of development will focus on only one attribute while others might observe and sort according to two or more attributes simultaneously. Those attributes most commonly encouraged in the program are those related to shape, length, weight, numerousness, and capacity.

Closely related to the processes of describing and classifying are the processes *comparing* and *ordering*. When *comparing*, the child is asked to look at only one attribute and to determine if two objects or sets of objects are the same or different. (For example, are both of your shoes the same weight?) When *ordering* the signs $<$, $=$, and $>$ are introduced. Children first focus on "greater than" and "less than" using a character named Greedy Duck who is always hungry and therefore, always opens his bill, ready to eat the greater amount. A first grade teacher who used this program as a pilot program in Chicago reported that at the beginning of a second grade there was almost 100% retention of the meaning of $<$, $=$, and $>$ which to me is proof of Greedy Duck's effectiveness as an activity designed to give meaning to abstract symbols normally very confusing to most children.

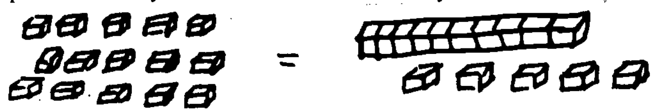
Equalizing is the process of making two objects the same on a given attribute. This is accomplished either through adding on or taking away. (Note that in equalizing you begin with two objects, do something to one of them, and end with two objects.)

For example, place the loose change which you have in your pocket on one side of the scale and the loose change which your friend has on the other. What would you have to do to make the weight of both sides the same?

Joining and separating are two processes closely related to equalizing. When *joining* you combine two objects on the basis of a common attribute (i.e., the area of the triangle formed when you combine triangle P with another triangle P will be equal to the area of triangle Q). *Separating* is the process of taking apart an object or set having a specified attribute to make two objects or sets, each with that attribute. (i.e., make a chain using lots of links. Now separate that chain into two smaller chains.)



Grouping is the process of arranging a set of objects into groups of a specified size. For example, given a collection of unifix cubes, how many groups of ten can you make? Will there be any cubes left over?



Grouping is used extensively to build the foundations needed for an understanding of place value and for conversion from one unit to another in measurement.

Partitioning exists when you know the number of groups which you must make from a set of objects but do not know the number of objects which will be in each group. When asked to deal a deck of cards to "x" players you are partitioning. Partitioning is used to build an understanding of division and fractions.

Overriding the development of the nine basic processes just described, are the processes of *representing* and *validating*. As John's disabilities have shown us, using physical materials does not always insure that the child will be capable of drawing the conclusions necessary to use meaningfully abstract symbols. As you follow a child through a given sequence of activities in the program you will find that he moves very gradually from using physical representations, pictorial representations, and finally, symbolic representations to help him comprehend the process. (See Figure 3.)

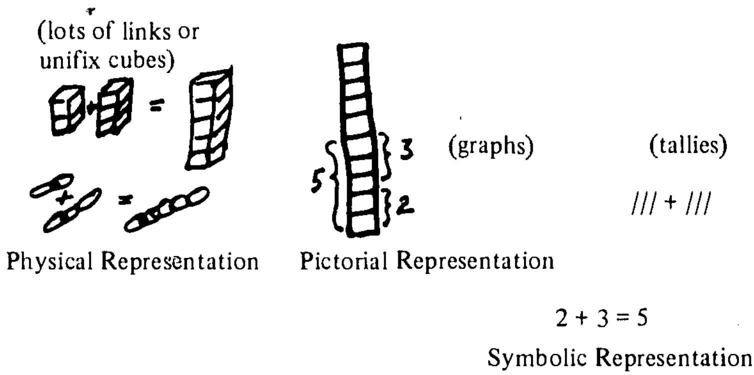


Figure 3

During the *validating* process the child begins with the symbolic representation, and then moves to the pictorial representation and/or the physical representation. (See Figure 4.)

```
  15        15
x 24   or  x24   or
  60        20
 300        40
 360       100
           200
           360
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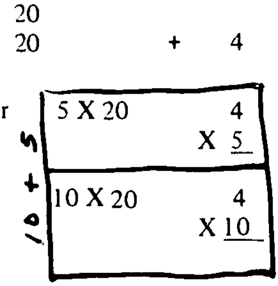

(The child validates two-digit multiplication by finding the area of sub-parts of a 24 X 14 unit rectangle.)

Figure 4

This is significant both because of the immediate feedback which it provides, and because when errors are made the child can begin to analyze the erroneous aspects of his own thinking and learn from his mistakes. Children *validate* all work when learning a concept. It is important that a teacher questions both the wrong answers and the right answers because a right answer doesn't always insure a thorough understanding of the processes involved in achieving the correct answer. Careful observation of the child's method of *validating* help you as a teacher understand the child's thinking.

Children must be viewed as individuals. They learn in many ways. They learn through interraction with the teacher, interraction with peers, and interraction with materials. Individualization of mathematics implies more than individualization of pacing. Materials have been provided which permit varied approaches to achieve the same objective. Some children will need to use all approaches but most children or groups of children will find that one of the alternatives will be sufficient and will master a given objective without needing to utilize the options available. Concepts are developed through the use of games, stories, inquiry-discovery activities at learning stations, and independent work.

Rarely is there only one way to solve legitimately a given problem. Each child is encouraged to choose a method which makes sense to him. Some children learn best when working alone. Others learn best when interracting with peers. Some provide their own structure and others must rely upon the teacher for the structure. Some need the physical representation far longer while others function easily using only pictorial representation and/or abstract representation. A teacher should be willing to accept this philosophy and should not only allow but encourage children to use their own initiative to solving problems.

When comparing this program with other primary mathematics programs currently in use teachers will observe that children reach the abstract level of many processes at a later age. It is the philosophy of this program that one should not look at what concepts a child can be taught, but at the age at which a concept is learned with ease. A spiral approach implies that concepts taught at one level are repeated at another level not for the purpose of reteaching but only for maintenance skills. Although a child is not hurried to use the abstract representation, when he reaches that point, he hits it hard, soon leaving the other approaches behind. This is possible only when the proper foundations have been laid.

Charlotte Junge states, "A slow learner is not helped by mere repetition, an enlarged view, carefully developed, is better for him than endless repetition." As objects are manipulated the pupil tells what he sees, what relationships he observes, and chooses a logical approach to solve a given problem. The teacher plays a key role in the selection of materials which are appropriate for the given objective and the student's learning style. The teacher must encourage the pupil to think for himself, but ask the questions which call critical points to the pupils' attention. Objects are manipulated and the students then use pictures or symbols to record what has happened. Pre- and post-assessment through informal observation and formal inventories combined with individual group record keeping permit each teacher to assess both the individual and group needs of her class and to provide for them. The use of a variety of objects and activities continually encourages the child to see how mathematics relates to his total world.

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GRAPH NETWORKS - APPLIED TO COMMUNICATIONS, ECOLOGY AND TRAFFIC FLOW

by Frances C. Pascale, Albertus Magnus College

Which one of us has escaped the following challenges: The object of each is to start at a corner and to trace the entire figure with one continuous line without lifting the pencil from the paper, without tracing over any part of a line segment more than once, and without crossing over any segment.

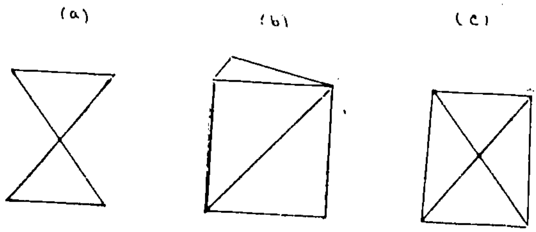


Figure 1

After "going through the motions" we conclude that (a) and (b) are possible and that (c) is not. Even with this conclusion safely in tow, we feel that there is also some difference between (a) and (b).

Students react most favorably when asked: "What if I were to say that there is a way to tell without ever tracing the diagram. This method would allow us to make up our own puzzles of this type as well. Would you be interested?" But first some history.

The Königsberg Bridge Problem

There was in Germany (actually East Prussia) a town called Königsberg. Now, the River Pregel passed through Königsberg in a rather peculiar way.

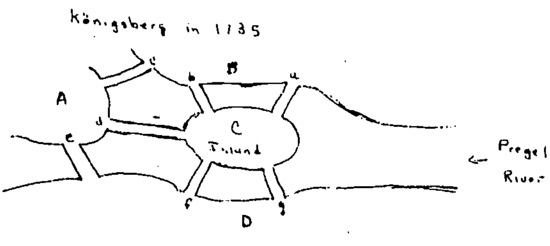


Figure 2

To get around town, the people of Königsberg built seven bridges. Thus began what is now commonly called "The Königsberg Bridge Problem." Each Sunday when the weather permitted, the people of Königsberg were accustomed to take an afternoon stroll. This, of course, was before autos, air pollution and gasoline shortages! They were challenged with the following problem: "Is it possible for a person to leave home, walk crossing all seven bridges continuously without recrossing any of them, and to end up back at home?"

In 1735, Leonhard Euler, a Swiss mathematician, settled the question and at the same time set the foundation for what has come to be called Graph

Theory (a branch of topology). Euler proceeded by replacing land areas with points (vertices) and bridges with lines (edges) connecting these points.

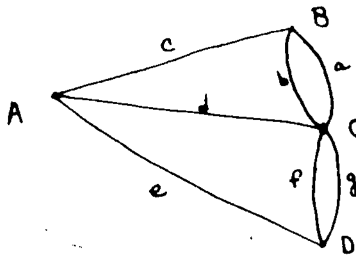


Figure 3

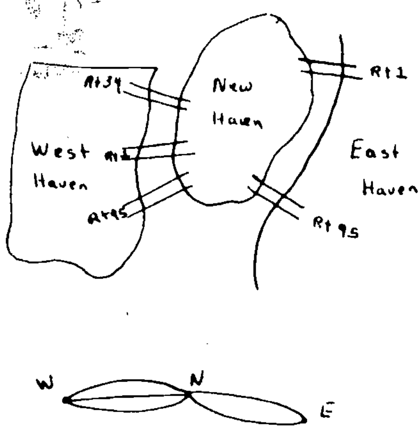


Figure 4

By such a conversion, the problem reduces to that of starting anywhere and traversing the "graph" with one continuous "sweep" of a pencil without lifting the pencil from the paper. Euler realized that this could only be done if all the vertices of the graph are what we shall call "even." He also showed that if a graph contained no more than two "odd" vertices, it may be traversed in one journey, but it is impossible to return to the starting point. But first let us consider some terms.

Basic Terms and Theorems (Cf. Roberts)

Graph: A graph is a figure in the plane or space which consists of a set of points called vertices, and a set of line segments or curves called edges. An edge cannot have any intersection with itself, except possibly at its two endpoints.

Path: A path is a sequence of different edges in a graph that can be traversed continuously without retracing any edge.

Even Vertex: A vertex is even, if the number of edges ending at it is even.

Odd Vertex: A vertex is odd, if the number of edges ending at it is odd.

We are interested in the number of even and number of odd vertices for each graph. It is here that I suggest construction of the following table where students may record results from many puzzles:

Puzzle	*Traceable*	*Number of Even Vertices*	*Number of Odd Vertices*
(a)	Yes	5	0
(b)	Yes	3	2
(c)	No	1	4
Kõnigsberg	No	0	4

Given a sufficient number of puzzles the students have little or no difficulty arriving at the following results:

Theorem 1. In any graph the total number of odd vertices is even.

Theorem 2. If a graph has more than two odd vertices, it can't be traversed by a single path.

Theorem 3. If a graph has no odd vertices then it can be traversed by a single path. Moreover, we can start from any vertex in our route.

Theorem 4. If a graph has exactly two odd vertices, it can be traversed by a path which starts from one odd vertex and ends at the other (but you can't get back to where you started.)

These theorems demonstrate Euler's results in solving the Königsberg Bridge Problem. In particular, from Theorem 2 we learn that the bridge problem cannot be solved by a single path since it contains 4 odd vertices. We can now handle all puzzles of the type originally posed. Students may find it interesting to take a map of their immediate locality and apply the above examination. For example, consider the New Haven area.
This graph has exactly two odd vertices, N and W. It can be traversed by a path which starts from one odd vertex and ends at the other.

Graph Theory Develops

It was here that graph theory remained for quite some time - just useful for solving puzzles. In 1847, Kirchhoff used it, however, in working with an electric networks problem. Ten years later, Cayley applied it to a chemical isomers problem. It was also applied by Sir William Hamilton to his "Around the World Game." Certainly many have applied it to the Four Color Conjecture. (See Harary for discussion of these applications.)

However, it was not until our own twentieth century that graph theory was extended and enthusiastically received to handle problems in such diverse areas as communications, ecology, transportation, sociology (a study of group dynamics and cliques), psychology (a study of preference), and game theory (in which category I include such studies as that of war strategies.)

General Examples: Before we can show some examples we need another new term:

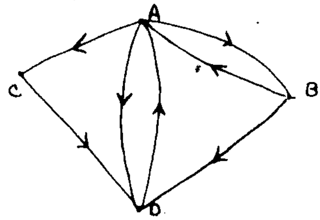
Directed Graph or *Digraph* (Cf. Roberts): A digraph is a graph in which a direction may be specified along a given edge. Thus A ⟶ B or A ⟵ B or A ⟷ B.

We can see that paths are going to be harder to find since we can't go "against" the given direction. Examples of digraphs are plentiful. If X, Y are vertices, X ⟶ Y, the digraph, we might have the following interpretation in: (a) Communications: "X can communicate with Y." (b) Tournaments: "X beats Y." (c) Psychology: a person "prefers alternative X to alternative Y." (d) Food Webs in Ecology: "X preys on Y." (e) Games: "It is a legal move to go from position X to position Y." (f) Transportation: "There is a direct link from location X to Y."

We have mentioned some areas where applications can be found. Let us look at some applications in greater detail.

Application in Communication Networks: Suppose we have four individuals: Al, Betty, Carol and Don who can communicate with one another in the way represented by digraph D.

D =



$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Figure 5 Figure 6

(note: uRv if u can communicate directly with v.) Now, these digraphs get rather cumbersome, so we introduce an algebraic device known as the adjacency matrix of the digraph.

Adjacency Matrix: Let D be a digraph. Suppose we list its vertices as $u_1, u_2, \ldots, u_n$. The adjacency matrix A associated with D is the matrix (a_{ij}) defined by $a_{ij} = 1$, if $u_i R u_j$, i.e. $u_i \longrightarrow u_j$ and $a_{ij} = 0$, otherwise.

Now the above digraph has the adjacency matrix A.

There is a very useful theorem in graph theory that would be a delight to any student who might be studying multiplication of matrices:

Theorem: If D is a digraph with adjacency matrix $A = (a_{ij})$, then the i, j entry of A^t gives the number of sequences of length t in D which lead from u_i to u_j. (Cf. Roberts)

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

Figure 7

Thus a simple matrix multiplication would answer, "In how many ways can Betty get a message to Carol in three steps." The answer is in just one way. One might also mention that as the adjacency matrices become large, we might prefer to have a computer do our computation for us.

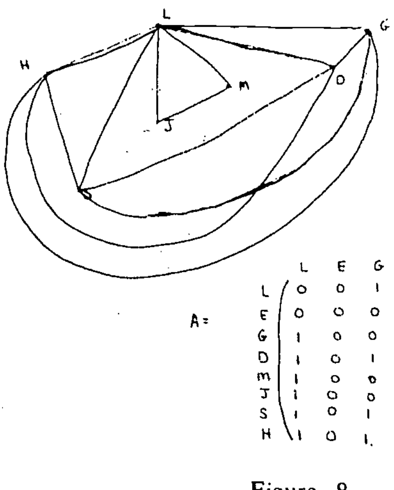
Application in Sociology: One aspect of a study of structure within a group of people is determination of all possible cliques.

Clique: A clique of a communication network is a subset C of individuals containing at least three members, with the following two properties:

(1) Every pair of members of the clique has two-way communication.

(2) The subset C is as large as possible with every pair of members having property (1). (Cf. Kemeny, Snell)

Consider the following: You are observing a nursery school play yard as part of a study of group interaction. During the first week, your records show that Laura played with everyone except Eileen, Gretschen has played with all the boys, David has played with all the boys, May and Jeannette have played only with each other and with Laura, and Steve has played with Harvey. (Cf. Harary et al.) Let G and A be the corresponding graph and adjacency matrix respectively.



A =

	L	E	G	D	M	J	S	H
L	0	0	1	1	1	1	1	1
E	0	0	0	0	0	0	0	0
G	1	0	0	1	0	0	1	1
D	1	0	1	0	0	0	1	1
M	1	0	0	0	0	1	0	0
J	1	0	0	0	1	0	0	0
S	1	0	1	1	0	0	0	1
H	1	0	1	1	0	0	1	0

Figure 8

Cliques are going to be those submatrices which are of largest size and which contain 1's everywhere except on the main diagonal. Thus our cliques are [L, M, J] and [L, G, D, S, H]

Traffic Planning Example

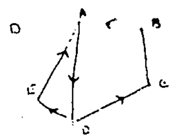
Reachability Matrix: Associated with a digraph D is its reachability matrix $R = (r_{ij})$ defined by $r_{ij} = 1$ if u_j is reachable from u_i (no limit to the number of steps) $r_{ij} = 0$, otherwise.

Strongly Connected Digraph: A digraph D is strongly connected if for every pair of vertices u and v, v is reachable from u and u is reachable from v. Thus R has 1 in each position.

We have the following very useful theorem. (It helps to have a computer for the computations.)

Theorem: Suppose D is a digraph of n vertices with adjacency matrix A and reachability matrix R. Then $R = B\,(I + A + A^2 + \ldots + A^{n-1})$ where $B(x) = 0$ if $x = 0$ and $B(x) = 1$ if $x \neq 0$.

I have been told that at times Boston has not been strongly connected. Let us consider an example. Suppose that there are 5 landmarks in Boston: A, B, C, D and E connected by a series of one-way streets as described in figure 9 by digraph D and adjacency matrix A.



$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Figure 9

We ask is D strongly connected? In order to answer this we find the reachability matrix R, which is shown in figure 10.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad A^4 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

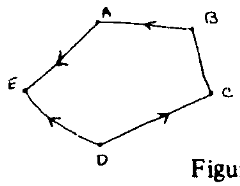
$$I + A + A^2 + A^3 + A^4 = \begin{pmatrix} 3 & 1 & 2 & 2 & 2 \\ 3 & 4 & 3 & 2 & 1 \\ 2 & 2 & 4 & 1 & 1 \\ 2 & 2 & 3 & 3 & 3 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$R = B\,(I + A + A^2 + A^3 + A^4) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Figure 10

Since R is composed entirely of 1's we conclude that D is strongly connected.

But what if we, or the Boston Traffic Authority, decide to change just one of these one-way streets, resulting in the situation depicted in figure 11.



$$R = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

Figure 11

Alas, D is no longer strongly connected. So, if our hapless tourist is located at point E when the change takes place, it will be impossible to leave!

Applications to Ecology and Food Webs

Another example of a use for digraphs can be seen in ecology, especially in the construction and investigation of *food webs*. A food web for an ecological community is a digraph whose vertex set is the set of all species being considered. An arc is drawn from species u to species v if u preys upon v. From the food web, we can define the corresponding *competition graph*. This graph has as vertex set the set of all species, and two species are joined by an edge if and only if they are in competition, i.e. they have a common prey. An example of a food web and its corresponding competition graph are shown in figure 12.

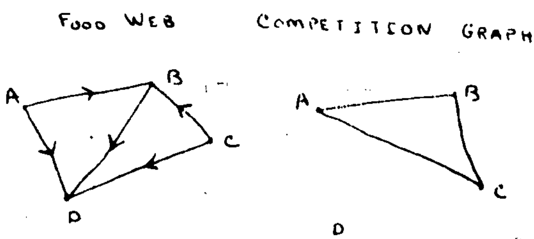


Figure 12

Many ecological questions are now being phrased, discussed and in some cases solved using these graph theoretic techniques.

Conclusion

Simple graph (or digraph) notions are being extended to allow for signs (positive or negative) on the edges. This leads to a discussion of balance in the graph or system described by the digraph. Another natural extension puts various weights or probabilities on the edges of our graphs.

I have attempted to give some flavor of applications in this rapidly growing field of graph theory. If I have succeeded in wheting your interest, may

I recommend those books mentioned in the bibliography. My experience has been that students are very receptive to these notions and that there is some graph theory that can be done at almost any level.

Footnotes:

[1]F. Roberts, Finite and Discrete Mathematics as Applied to the Social and Environmental Sciences, (title subject to change) Mimeographed notes to be published as a text.

[2]Ibid

[3]Ibid

[4]J. G. Kemeny and J.L. Snell, Mathematical Models in the Social Sciences, Blaisdell, New York, 1962.

[5]F. Harary, R.Z. Norman and D. Cartwright, Structural Models: An Introduction to the Theory of Directed Graphs, Wiley, New York, 1965.

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USEFUL COMPUTER APPROXIMATIONS IN ELEMENTARY FUNCTIONS

by Ann Waterhouse

I always enjoy teaching elementary analysis because the computer can be used effectively to build intuitive notions of limit, slope and extreme values without calculus. Also, for the first time in high school mathematics the computer becomes an indispensable tool; the programs, although quite simple, involve iterative processes which defy long-hand calculation.

I have chosen three different problems to illustrate ways in which the computer can enhance the study of functions. First, let's consider a typical polynomial in factored form,

$$f(x) = (x + 3)(x + 1)^2 (x - 1)(x - 2)^3 \qquad (A)$$

Sketching the graph is a simple exercise. As a polynomial of odd degree, it "enters" through quadrant III and "leaves" through quadrant I. The zeros, -3, -1, 1 and 2, are obvious. Furthermore, a quick check of the signs of the individual factors over the open intervals determined by the zeros enables us to conclude that

f(x) is negative over $(-\infty, -3)$;
f(x) is positive over $(-3, -1)$;
f(x) is positive over $(-1, 1)$;
f(x) is negative over $(1, 2)$; and
f(x) is positive over $(2, \infty)$.

Using these results one can sketch a graph which exhibits the general characteristics of the function. See Figure 1.

My initial homework assignment requires the student to make rough sketches of the graphs of six polynomials in factored form using the above outlined techniques. Sample polynomials might be

$f(x) = (x + 3)^3 (x + 2)^2 x^3 (x - 1)$,
$f(x) = (x + 2)^3 (x + 1)^2 (x - 1)(x - 3)^2$, and
$f(x) = (x + 2)^2 (x - 1)^2 (x - 4)$.

Once these sketches are completed the logical question is, "What can we do to make these graphs more precise?" Most students readily agree that we should determine the actual locations of the various turning points since nothing has been drawn to scale. Together we design a computer search for maximum and minimum values. A detailed flowchart of the procedure is given in Figure 2

The user has the option of finding either a maximum or minimum. His choice is stored as Y$. The initial search interval is [A,B]. Divide this interval into 10 subintervals of width H = (B - A)/10 as indicated in Figure 3.

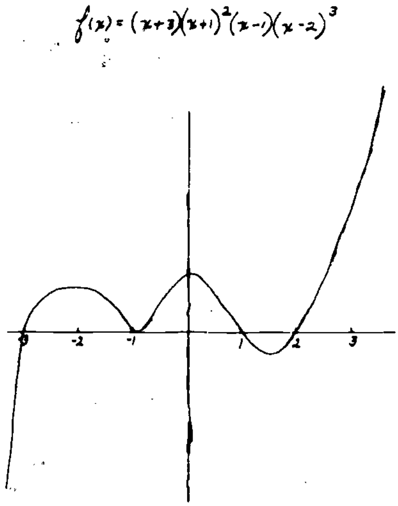


Figure 1

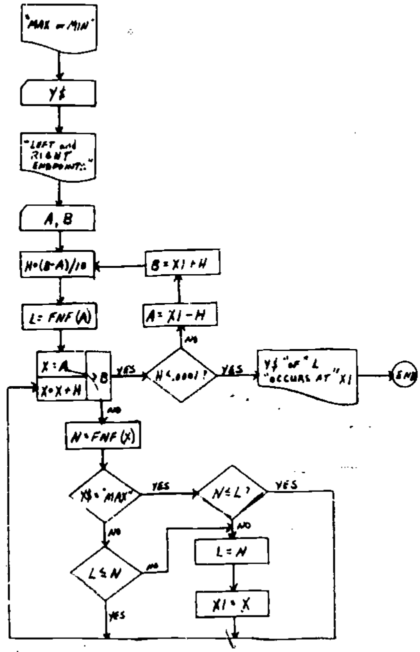


Figure 2

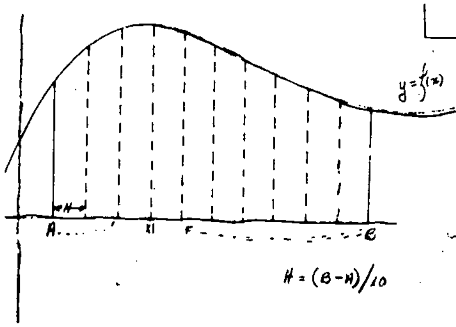


Figure 3

Assume temporarily that the largest (or smallest as the case may be) function value on the search interval is f(A). Store this value as L. Now compare the function value at each of the 10 abscissas greater than A with L. If the function value is larger (or smaller), save it as L and its abscissa as X1. At the end of this search X1 and L will contain the coordinates of the ordered pair with the largest (smallest) ordinate of the selected points on the interval [A,B]. To refine our search further, let A = X1 - H and B = X1 + H, subdivide the new closed interval [A,B] into 10 parts, set L equal to F(A), and repeat the comparisons. Continue this reduction process until the desired accuracy is obtained. Figure 4 contains a LIST and RUN of this program for the original polynomial function (A).

```
LIST
 100 DEF FNF(X)=(X+3)*(X+1)↑2*(X-1)*(X-2)↑3
 105 DEF FNR(X)=INT(X*10000+.5)/10000
 107 PRINT "MAX OR MIN";\INPUT Y$
 110 PRINT "LEFT AND RIGHT ENDPTS OF SEARCH INTERVAL";
 120 INPUT A,B
 130 LET H=(B-A)/10
 140 LET L=FNF(A)
 150 FOR X=A TO B STEP H
 160 LET N=FNF(X)
 165 IF Y$="MAX" GO TO 170
 167 IF L<N THEN 200
 168 GO TO 180
 170 IF N<L THEN 200
 180 LET L=N
 190 LET X1=X
 200 NEXT X
 210 IF H<.0001 THEN 250
 220 LET A=X1-H
 230 LET B=X1 + H
 240 GO TO 130
 250 PRINT Y$;" OF ";FNR(L);"OCCURS AT "; FNR(X1)
 260 END

READY

RUN

MAX OR MIN? MAX
LEFT AND RIGHT ENDPTS OF SEARCH INTERVAL? -3,-1
MAX OF  361.5274 OCCURS AT -2.5515

READY

RUN

MAX OR MIN? MAX
LEFT AND RIGHT ENDPTS OF SEARCH INTERVAL? -1,1
MAX OF  24.056 OCCURS AT -.2425

READY

RUN

MAX OR MIN? MIN
LEFT AND RIGHT ENDPTS OF SEARCH INTERVAL? 1,2
MIN OF -2.3422 OCCURS AT  1.3806

READY
```

Figure 4

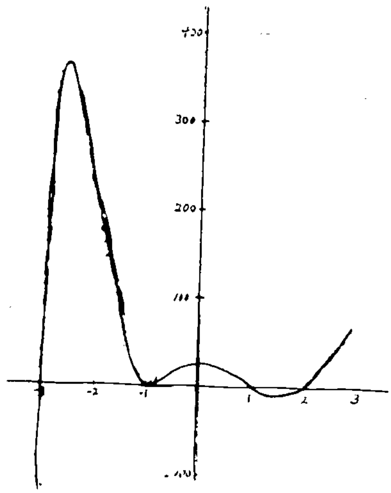


Figure 5

After the program MAXMIN has been written each student determines the proper maximum and minimum values for two of the original six graphs and makes new sketches with accurate turning points. Figure 5 shows the polynomial (A) drawn to scale.

Are there additional improvements which could be made to the sketches? Checking the graph of polynomial (A) we notice a tangency at -1, which corresponds to the only squared factor in (A). This suggests that it might be fruitful to investigate the behavior of the graph at each of the zeros, in particular to compute the slope of the curve at each zero by a sequence of approximations $\frac{f(x+h)-f(x)}{h}$ as h approaches 0. A sample program with RUNS for the zeros of polynomial (A) is shown in Figure 6. Note that for this exercise the slope is computed accurate to the nearest 1000th; slopes tend to be large for factored polynomials (2000 at -3) and round-off error can play havoc with the output if you retain too many decimal places.

For the final assignment the students determine the slopes of their particular curves at the zeros using SLOPE and modify the graphs once more to indicate the true behavior of the functions at these points. By comparing results they confirm the tangencies which occur for squared factors of the functions and discover those factors which give terrace points. In fact, our original polynomial (A) has a terrace point at x = 1 as indicated by the cubic term, and the graph in Figure 5 should be changed to show this.

```
OLD

OLD PROGRAM NAME--SLOPE

READY

LIST
 100 DEF FNF(X)=(X+3)*(X+1)↑2*(X-1)*(X-2)↑3
 105 PRINT "WHAT IS X COORDINATE";
 110 INPUT C
 120 LET H=.5
 130 LET A1=(FNF(C+H)-FNF(C))/H
 140 IF ABS(A1-A)<.001 GOTO 180
 150 LET A=A1
 160 LET H=H/2
 170 GOTO 130
 180 PRINT "THE SLOPE IS "; INT(A1*100+.5)/100
 190 END

READY

RUN

WHAT IS X COORDINATE? -3
THE SLOPE IS  2800

READY

RUN

WHAT IS X COORDINATE? -1
THE SLOPE IS  0

READY

RUN

WHAT IS X COORDINATE? 1
THE SLOPE IS -16

READY

RUN

WHAT IS X COORDINATE? 2
THE SLOPE IS  0

READY
```

Figure 6

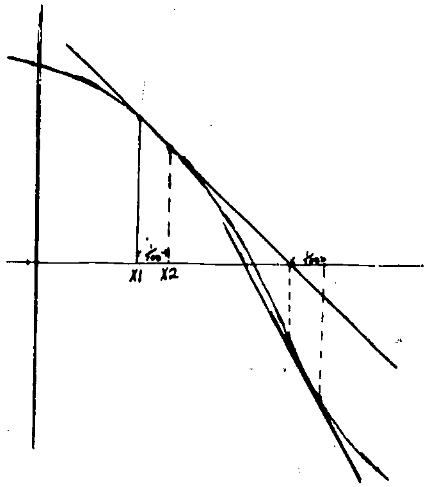


Figure 7

As a second application of the computer may I suggest Newton's method for approximating irrational zeros of polynomial functions. Perhaps you have been like me, teaching bisection techniques to avoid introducing derivatives. How unnecessary! If we must approximate the curve by a tangent line to use Newton's method, why not use a "good" secant line instead! In Figure 7, X1 is an approximate zero of y = f(x). Choose a second point, X2 on the graph, say 1/100 units from X1. The secant line through (X1, f(X1)) and (X2, f(X2)) is an excellent approximation for the actual tangent line to the curve at (X1, f(X1)); use its x-intercept as a second approximation for the zero and repeat the above procedure to obtain more accurate approximations.

A sample program is given in Figure 8. For demonstration purposes the user can select the distance between X1 and X2. With a difference as large as .1 only 6 interations are required to give the zero accurate to 6 decimal

```
OLD

OLD PROGRAM NAME--NEWTON

READY

LIST
 100 DEF FNF(X)=X↑3-4*X*X-3*X+2
 110 READ H
 120 DATA .1
 130 PRINT "FIRST APPROXIMATION";
 140 INPUT X1
 150 LET X2=X1+H
 160 LET M=(FNF(X2)-FNF(X1))/(X2-X1)
 170 LET X=X1-FNF(X1)/M
 180 PRINT X1
 190 LET X1=X
 200 GOTO 150
 210 END

READY

RUN

FIRST APPROXIMATION? 4.5
 4.5
 4.560546
 4.561512
 4.561551
 4.561553
 4.561553
 4.5
↑C

READY

RUN

FIRST APPROXIMATION? .4
 .4
 .4373957
 .4384037
 .4384454
 .4384471
 .4384472
 .4384472
 .4384472

↑C

READY
```

Figure 8

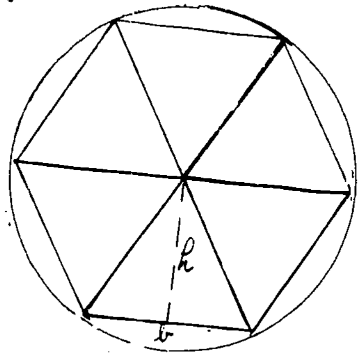


Figure 9

```
LIST
 100 LET P=3.14159
 110 INPUT R
 120 LET C=P*R↑2
 130 LET N=10
 140 LET X=2*P/N
 150 LET Y=(P-X)/2
 160 LET B=R*SIN(X)/SIN(Y)
 170 LET H=SQR(R↑2-(B/2)↑2)
 180 LET A=H*B*N/2
 190 PRINT N,A,C
 200 LET N=2*N
 210 GOTO 140
 220 END

READY

RUN

? 2
 10              11.7557         12.56636
 20              12.36067        12.56636
 40              12.51475        12.56636
 80              12.55345        12.56636
 160             12.56313        12.56636
 320             12.56555        12.56636
 640             12.56616        12.56636
 1280            12.56631        12.56636
 2560            12.56635        12.56636
 5120            12.56636        12.56636

↑C

READY
```

Figure 10

places. The standard number of interations using bisection techniques is about 20!

My final application concerns the epsilon-delta definition of limit. In recent years I have proceded the formal study of limits in AP Calculus with an "exploration assignment" using the computer. This assignment has been so successful in helping the students understand the definition of limit that I recommend it to you as a good pre-calculus project.

The notion of limit arose early in mathematics in the calculation of the area inside a circle. Suppose this region is cut into a large number (n) of congruent pie-shaped segments, as shown in Figure 9. Each segment is approximately a triangle with base b and height h. Since the number of pie-shaped pieces is n, the total area of the circle is approximately n x ½bh. Let's rewrite this as

$$A_n = \frac{h}{2}(bn) \qquad (1)$$

We recognize bn as the perimeter of the regular inscribed polygon of n sides and we feel intuitively that this perimeter should be very nearly equal to the circumference C of the circle. Also the height h should be close to the radius r of the circle. Thus,

$$A_n \sim \frac{r}{2}C \qquad (2)$$

The students' first assignment is to choose a radius r for a circle and write a computer program which will print n, A_n (equation 1) and the limiting value $\frac{r}{2}$ C for increasing values of n. Figure 10 shows a sample program and output. Note that the number of sides is doubled each time. Let the students choose their own formulas for increasing n. Someone always increments by 1; others use 10 or 50. The varied choices are especially important for the second part of this exploration exercise.

By comparing entries in the program print-out it is apparent to the student that A_n approaches $\frac{r}{2}$ C when n is large. Can we show this result in another way? Why not measure the difference between A_n and $\frac{r}{2}$ C. If A_n truly approaches $\frac{r}{2}$ C, the difference measured by

$$\left|A_n - \frac{r}{2}C\right| \qquad (3)$$

should approach 0. This observation is the basis of the second assignment which follows:

> Imagine that you are playing a game against an opponent who challenges you to make the difference in (3) less than .001, or .0002, or any small positive number E. Modify your first program so that it will print out the number of segments n necessary to make the difference less than E. Try a few runs. Given enough computer time do you think it is always possible to find a suitable n for any positive number E?
>
> A sample RUN is shown in Figure 11.

Since the students originally incremented n in various ways, Line 200 will vary from program to program and the value of n will not be unique for a given E as shown in Figure 12. By sharing their results, the students can observe that there is a smallest value of n which will work (obtained by the fellow who plods upward by 1!) but that any larger value is certainly suitable within the framework of the problem. Acceptance of this is so fundamental in understanding the formal definition of limit and its application in epsilon-delta proofs.

```
LIST
 100 LET P=3.14159
 110 LET R=2
 115 PRINT "WHAT IS E";
 116 INPUT E
 120 LET C=P*R↑2
 130 LET N=10
 140 LET X=2*P/N
 150 LET Y=(P-X)/2
 160 LET B=R*SIN(X)/SIN(Y)
 170 LET H=SQR(R↑2-(B/2)↑2)
 180 LET A=H*B*N/2
 185 IF ABS(A-C)<E THEN 215
 200 LET N=2*N
 210 GOTO 140
 215 PRINT "N=";N
 220 END

READY

RUN

WHAT IS E? .01
N= 160

READY

RUN

WHAT IS E? .001
N= 320

READY

RUN

WHAT IS E? .0005
N= 640

READY

RUN

WHAT IS E? 5
N= 10

READY
```

Figure 11

```
200 LET N=2*N

RUN

WHAT IS E? .001
N= 320

READY

200 LET N=5*N

RUN

WHAT IS E? .001
N= 1250

READY

200 LET N=N+50

RUN

WHAT IS E? .001
N= 310

READY

200 LET N=N+1

RUN

WHAT IS E? .001
N= 300

READY
```

Figure 12

A TASTE OF PI

by Eugene Wermer, Norwich University

The title of this paper is a bit misleading. In fact we want to talk about Monte Carlo methods, i.e. the method of statistical trials, in an effort to stimulate the use of our computers in similar applications. The material used herein was taken, in large part, from the subject matter of a course called "Computer Simulation of Systems" which was offered to NSF participants at the University of Massachusetts during the summer of 1967. Though the reference list appended cites the sources, Dr. Val Punga's Lectures were the foundation of this paper. We wish to introduce the subject by considering the number π. In Eves' Introduction to the History of Mathematics we learn that though there were early approximations of the value of π, the first real attempt to compute π was done by Archimedes about 240 B.C. He used the classical method of computing the perimeter of regular inscribed and circumscribed polygons and found $223/71 < \pi < 22/7$. Eves goes on to give a chronology of π and refers to a more complete list in The Mathematics Magazine, January - June 1950. We now turn to a probability method of finding a value for π.

The first example of a Monte Carlo method applied to computation was described by Buffon in a paper published in 1777. The Monte Carlo procedure consisted of dropping a needle onto a surface upon which are ruled parallel lines and counting the number of times the needle touched a line. As described below, if the lines are spaced properly, it is possible to obtain a numerical approximation of π. In 1850, Volser used Buffon's method and obtained a value of $\pi = 3.1596$ and in 1901 it is reported that Lazzerini got a value correct to six decimal places using 3408 tosses, (but a little computation indicates the result is contrived.)

Refer now to Figure 1, let D = distance between two adjacent lines. Let

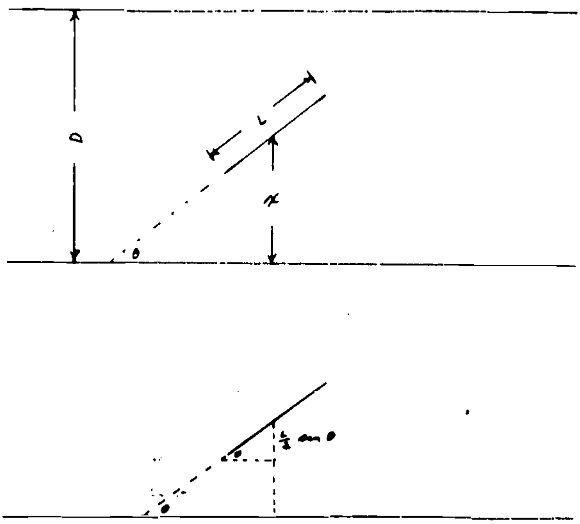


Figure 1

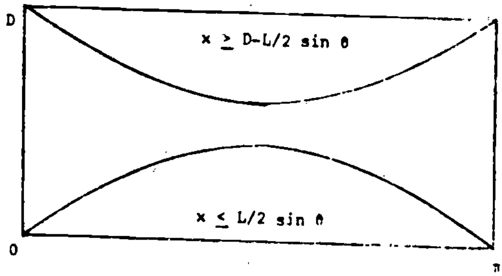
the needle have length $L < D$ so that $L/2 < D/2$. The needle will, of course have some inclination with respect to the lines. Let θ = angle of inclination. It is clear $0 \leq \theta < \pi$. Finally let x = distance from the center of the needle to the lower line shown. Thus $0 \leq x < D$. The projection of half the needle on a vertical line is given by $L/2 \sin \theta$. There are therefore two conditions that determine if the needle touches a line:

(1) $x \leq L/2 \sin \theta$

(2) $D\text{-}x \leq L/2 \sin \theta$ or $x \geq D\text{-}L/2 \sin \theta$

It appears that x and θ are uniformly distributed in their domains, hence we have a sample space of ordered pairs (θ, x). Those pairs that satisfy the inequalities (1) and (2) indicate the event, H, "The needle touches a line".

The probability of the event, H, is given by the ratio of the area of the event to the area of the sample space as shown in the diagram.



Area of sample space = πD

Area of event H = $2\int_0^{\pi} L/2 \sin\theta \, d\theta = L\int_0^{\pi} \sin\theta \, d\theta = 2L$

Probability of H = $2L/\pi D$

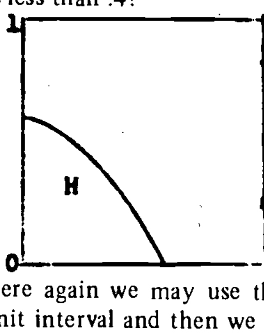
Hence if D =2L, P(H) = $1/\pi$ or $\pi = 1/P(H)$

We can now approximate P(H) by actually tossing a needle and counting the number of hits (on a line) we obtain. $P(H) \approx \frac{\text{Hits}}{\text{Trials}}$ or $\pi \approx \frac{\text{Trials}}{\text{Hits}}$

We can simulate the needle tossing process by using a computer. θ and x are chosen by a random process. The constraint inequalities (1) and (2) are tested and the number of hits is recorded. Figure 2 gives a program with results for 10,000 needle tosses. You will note D = 1, L/2 = .25 in the program.

Two runs of the program are shown in Figure 3, and an average of the results gives the value 3.1481.

A problem similar to the preceeding one is this: Suppose a square with side equal to 1 ft. is set up and darts are randomly thrown at it. What is the probability that the square of the distance of a dart from a specified corner is less than .4?



For the required event, H, to happen, a point (x,y) must satisfy (3): $x^2 + y^2 < .4$ P(H) = area of the quarter circle in the unit square. $P(H) = \frac{\pi r^2}{4} = \frac{\pi(.4)}{4} = .1\pi$

Here again we may use the computer to select x and y randomly in the unit interval and then we test the ordered pair in inequality (3). Figure 3

```
NEEDLE        12 MAR 72    20:15

10 PRINT "TRIALS", "PI"
20 LET N = 0
30 LET T = 0
31 RANDOMIZE
40 FOR I = 1 TO 10
50 FOR J = 1 TO 1000
60 LET T = T + 1
70 LET A = 3.14159265*RND
75 LET X = RND
80 LET W = .25*SIN(A)
85 IF (X - W)<=0  THEN 90
86  IF (1-X-W)>0  THEN 100
90 LET N = N + 1
100 NEXT J
105 LET P = T/N
110 PRINT
115 PRINT T,P
120 NEXT I
125 END
READY
```

```
PI-DIST        12 MAR 72     20:00

5  RANDOMIZE
10 PRINT "TRIALS", "PI"
15 LET N = 0
20 LET T = 0
25 FOR I = 1 TO 10
30 FOR J = 1 TO 1000
35 LET X = RND
40 LET Y = RND
45 LET D = X*X + Y*Y
50 LET T = T + 1
55 IF (D - 1)> 0 THEN 100
60 LET N = N + 1
100 NEXT J
105 LET P = 4*N/T
110 PRINT
115 PRINT T,P
120 NEXT I
125 END
```

NEEDLE 12 MAR 72 20:15 20:13

TRIALS	PI	PI
1000	3.16456	2.90658
2000	3.15956	3.01659
3000	3.10881	3.06122
4000	3.12734	3.13726
5000	3.11526	3.12305
6000	3.10396	3.13800
7000	3.09733	3.14465
8000	3.07182	3.16306
9000	3.07906	3.17019
10000	3.10655	3.18979

Figure 2

PI-DIST 12 MAR 72 20:00

TRIALS	PI
1000	3.24
2000	3.12
3000	3.1
4000	3.08
5000	3.104
6000	3.10667
7000	3.2
8000	3.10125
9000	3.10666
10000	3.171

Figure 3

shows the result of such an experiment.

A third application of the Monte Carlo method is in evaluating definite integrals. The only requirement on a function, f(x), is that it be bounded and measureable. Suppose we are to find $\int_a^b f(x)\,dx$. Let p(x) be the uniform probability density function so that $p(x) = \frac{1}{b-a}$ for $x \in [a,b]$ and p(x) = 0 otherwise. Then the expected value of f(x) is

$$E(f(x)) = \int_a^b f(x) \quad p(x)\,dx = \frac{1}{b-a} \int_a^b \quad f(x)\,dx$$

Now if values of x are selected randomly in [a,b] and f(x) is computed,

$E(f(x)) \approx \frac{\sum f(x_i)}{N}$, if the number of trials is large

Hence $\int_a^b f(x)\,dx \approx \frac{(b-a)}{N} \sum_{i=1}^{N} f(x_i)$.

A generally poorer procedure is one that uses the ratio of two areas in the probability sense of the dart game previously described. Here let $0 \leq f(x) \leq c$ for x in $[a,b]$. Then $\int_a^b f(x)\,dx$ is the area, A, under the curve bounded by $x = a$, $x = b$, $y = 0$. A is a portion of the rectangle with base b-a and height c. Hence points chosen randomly in the rectangle should fall in A with a relative frequency approximately the same as the ratio $\frac{A}{c(b-a)}$. If H counts the number of such points in N trials, we have $\frac{A}{c(b-a)} = \frac{H}{N}$ and $\int_a^b f(x)\,dx \approx c(b-a)\frac{H}{N}$. Programs for both of these cases are easily written by following the method of the three illustrated programs presented earlier.

A fourth application of the method of stochastic trials is in the solution of certain systems of linear equations. Though we wish to solve $A x = b$ where A is an $n \times n$ matrix, x is the column vector of the variables $x_1, x_2, \ldots, x_n$, and b is the column vector of constants $b_1, b_2, \ldots, b_n$, we shall instead show the theory for a 3×3 case so that the details are clear. The application to the $n \times n$ case is essentially the same.

Consider the equation $\sum_{j=1}^{3} a_{ij}\, x_j = b_i,\ i = 1, 2, 3.$

Solve for x_j: $x_1 = \frac{a_{12}x_2}{a_{11}} - \frac{a_{13}x_3}{a_{11}} + \frac{b_1}{a_{11}}$

Similarly for x_2, x_3. We might replace the coefficients with new symbols p_{ij}. Hence

$$
(10):\quad
\begin{aligned}
x_1 &= p_{12}\, x_2 + p_{13}\, x_3 + b_1\\
x_2 &= p_{21}\, x_1 + p_{23}\, x_3 + b_2\\
x_3 &= p_{31}\, x_1 + p_{32}\, x_2 + b_3
\end{aligned}
$$

This system can be solved by Monte Carlo methods if (1) $\sum_{\substack{j=1\\ i \neq j}}^{3} p_{ij} < 1$, $i = 1, 2, 3$ and (2) $p_{ij} > 0$.

First, split the b_i's into two factors $p_i\, B_i = b_i$ such that $p_i + \sum p_{ij} = 1$. That is:

$$
\begin{aligned}
p_1 + p_{12} + p_{13} &= 1 \qquad & b_1 &= p_{11}\, B_1\\
p_2 + p_{21} + p_{23} &= 1 \qquad & b_2 &= p_{22}\, B_2\\
p_3 + p_{31} + p_{32} &= 1 \qquad & b_3 &= p_{33}\, B_3
\end{aligned}
$$

The system (10) now reads

$$
(11):\quad
\begin{aligned}
x_1 &= p_{12}\, x_2 + p_{13}\, x_3 + p_{11}\, B_1 & & & x_1 &= p_{11}\, B_1 + p_{12}\, x_2 + p_{13}\, x_3\\
x_2 &= p_{21}\, x_1 + p_{23}\, x_3 + p_{22}\, B_2 & &\text{or} & x_2 &= p_{21}\, x_1 + p_{22}\, B_2 + p_{23}\, x_3\\
x_3 &= p_{31}\, x_1 + p_{32}\, x_2 + p_{33}\, B_3 & & & x_3 &= p_{31}\, x_1 + p_{32}\, x_2 + p_{33}\, B_3
\end{aligned}
$$

If we detach the p_{ij} to form their matrix, we see we have a Markov or stochastic matrix.

$$(12):\quad \begin{array}{c|ccc} & S_1 & S_2 & S_3 \\ \hline S_1 & p_{11} & p_{12} & p_{13} \\ S_2 & p_{21} & p_{22} & p_{23} \\ S_3 & p_{31} & p_{32} & p_{33} \end{array}$$

Each row can be thought of as describing the probability of some system changing from state S_i to state S_j. Thus p_{23} is the probability of S_2 changing to S_3. p_{11} is probability of S_1 staying in S_1. Let capital P_{ij} be the probability that the system moves from state S_i to state S_j in some number of steps and remains there. We are thinking, as an example, of a set of urns with certain balls in the urns marked STOP and other balls marked with numbers of other urns. One would pick balls and move from urn to urn until a STOP ball is selected. Then there are nine equations

$$(13):\quad \begin{aligned} P_{11} &= p_{11} + p_{12}P_{21} + p_{13}P_{31} \\ P_{12} &= p_{12}P_{22} + p_{13}P_{32} \\ P_{13} &= p_{12}P_{23} + p_{13}P_{33} \\ P_{21} &= p_{21}P_{11} + p_{23}P_{31} \\ P_{22} &= p_{21}P_{12} + p_{22} + p_{23}P_{32} \\ &\;\vdots \end{aligned}$$

Now suppose a payoff is made of amount B_j when the system stops at S_j. Thus if we start at S_1, N_1 times and let n_{11} be the number of times S_1 is finish point (i.e. payoff B_1 is had), n_{12} be the number of times B_2 is the payoff, and n_{13} the number of times B_3 is the payoff, then the average payoff $y_1 = \dfrac{n_{11}B_1 + n_{12}B_2 + n_{13}B_3}{N_1}$ or more generally $y_i = \sum_j \dfrac{n_{ij}}{N_i} B_j$ if we start at S_i.

However by the Law of Large Numbers, $\lim_{N_i \to \infty} \left(\dfrac{n_{ij}}{N_i}\right) = P_{ij}$

Therefore we may write $y_i = \sum_j P_{ij}B_j$ or in the illustration, $y_1 = P_{11}B_1 + P_{12}B_2 + P_{13}B_3$.

Now suppose in (13) we multiply by B_j:

$$(14): \quad \begin{array}{r} P_{11}B_1 + p_{12}P_{21}B_1 + p_{13}P_{31}B_1 \\ p_{12}P_{22}B_2 + p_{13}P_{32}B_2 \\ p_{13}P_{23}B_3 + p_{13}P_{33}B_3 \end{array}$$

Adding these nine equations in groups of three and factoring we have

$$(P_{11}B_1 + P_{12}B_2 + P_{13}B_3) = P_{11}B_1' + p_{12}(P_{21}B_1 + P_{22}B_2 + P_{23}B_3) + p_{13}(P_{31}B_1 + P_{32}B_2 + P_{33}B_3)$$

But the expressions in parentheses are y_1, y_2, y_3 respectively, so

$$y_1 = p_{12}y_2 + p_{13}y_2 + p_1B_1 \quad \text{and similarly for } y_2 \text{ and } y_3,$$

$$(15): \quad y_2 = p_{21}y_1 + p_{23}y_3 + p_2B_2$$

$$y_3 = p_{31}y_1 + p_{32}y_2 + p_3B_3$$

Note we have the same system as (11)! All we have to do then is find the average values y_i and we have the required solution!

Since we have the matrix (12) we simply program the computer to follow the procedure we have outlined. To be more concrete, let us take the equations

$$(16): \quad \begin{array}{l} 36x_1 - 18x_2 - 12x_3 = 23 \\ -18x_1 + 90x_2 - 60x_3 = 7 \\ 15x_1 + 12x_2 - 60x_3 = 1 \end{array}$$

We put these in the form (11)

$$x_1 = 1/2x_2 + 1/3x_3 + 23/36$$

$$x_2 = 1/5x_1 + 2/3x_3 + 7/90$$

$$x_3 = 1/4\,x_1 + 1/5x_2 - 1/60$$

Now we see we have a system that meets our conditions for solution. Then

$$23/36 = p_{11}B_1 = 1/6 \cdot 23/6 \quad \text{where } p_{11} = 1/6$$

$$7/90 = p_{22}B_2 = 2/15 \cdot 7/12 \qquad p_{22} = 2/15$$

$$-1/60 = p_{33}B_3 = 11/20 \cdot (-1/33) \qquad p_{33} = 11/20$$

and p_{11}, p_{22}, p_{33} have been chosen so

$$1/2 + 1/3 + 1/6 = 1; \; 1/5 + 2/3 + 2/15 = 1; \; 1/4 + 1/5 + 11/20 = 1$$

our matrix (12) now is

$$12^* \quad \begin{pmatrix} 1/6 & 1/2 & 1/3 \\ 1/5 & 2/15 & 2/3 \\ 1/4 & 1/5 & 11/20 \end{pmatrix}$$

Suppose we wish to find x_1.

We now set up the following scheme: For row 1, set up intervals on [0,1] to correspond with the values of p_{11}, p_{12}, p_{13}:

[0, .16667) [.16667, .66667) [.66667, 1]

Similarly for row 2: [0, .2) [.2, .33333) [.33333, 1]

row 3: [0, .25) [.25, .45) [.45, 1]

Now choose a random number on [0,1]. If it falls in the p_{11} interval [0,.166667)tally a count for B_1 since the system will remain in state S_1. If the number is in the p_{12} interval. [.16667, .66667) we proceed from state S_1 to state S_2, that is we move to row 2.

We draw a new random number, tally a count for B_2 if it is in the "stop" interval (i.e. the p_{22} interval). We then begin again with row 1 as at the very beginning. However, if we were not stopped, we proceed to the row indicated (for instance p_{21} would send us back to row 1) and continue the process. Whenever a "stop" is reached, we start anew at row 1. On the other hand if we were finding x_3, all new starts would be at row 3. After N trials we compute $x_1 = (n_1/N)(23/6) + (n_2/N)(7/12) - (n_3/N)(1/33)$. Attached is a computer program to solve this problem and we see the results are quite good. This method has an advantage in that one can solve for one unknown without being required to find the other unknowns.

Instead of the computer, we could have used three urns, as mentioned

```
MONTE        12 MAR 72    20:26

10 REM MONTE CARLO SOLUTION OF THREE EQUATIONS IN THREE
11 REM UNKNOWNS.  EXPECTED VALUES ARE:  X1 = 1, X2 = .5, X3 = .33
12 PRINT "HIT 1", "HIT 2", "HIT 3", "X"
15 RANDOMIZE
20 FOR I = 1 TO 3
25 LET N1 = 0
30 LET N2 = 0
35 LET N3 = 0
40 FOR J = 1 TO 20000
45 ON I GO TO 50,75,100
50 LET A = RND
55 IF (A - .16667)>0 THEN 70
60 LET N1 = N1 + 1
65 GO TO 120
70 IF (A - .66667)>0 THEN 100
75 LET A = RND
80 IF (A - .2)<0 THEN 50
85 IF (A - .33333)>0 THEN 100
90 LET N2 = N2 + 1
95 GO TO 120
100 LET A = RND
105 IF (A - .25)<0 THEN 50
110 IF (A - .45)<0 THEN 75
115 LET N3 = N3 + 1
120 NEXT J
121 PRINT
125 LET X = (23*N1/6000 +7*N2/12000 -N3/33000)*.05
130 PRINT N1,N2,N3,X
140 NEXT I
145 END
```

Figure 4

```
MONTE        12 MAR 72    20:27

HIT 1          HIT 2          HIT 3          X

4873           2614           12513          0.991276
2078           4186           13756          0.4[illegible]
1661           1505           16834          0.33674[illegible]
```

I. $\quad x_1 - 16x_2 - 17x_3 = \ldots$

$\quad -10x_1 + 90x_2 - 60x_3 = 7$

$\quad 15x_1 + 12x_2 - 60x_3 = \ldots$

II. $\quad x_1 = \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{23}{30} \qquad p_{11}B_1 = \frac{1}{\ldots} \; \frac{23}{6}$

$\quad x_2 = \frac{1}{6}x_1 + \frac{2}{3}x_3 + \frac{7}{90} \qquad p_{21}B_2 = \frac{2}{15} \; \frac{7}{12}$

$\quad x_3 = \frac{1}{4}x_1 + \frac{1}{5}x_2 + \frac{1}{60} \qquad p_{31}B_3 = \frac{11}{20} \left(\frac{1}{33} \right)$

III. $\begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & \frac{2}{15} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{5} & \frac{11}{20} \end{pmatrix}$

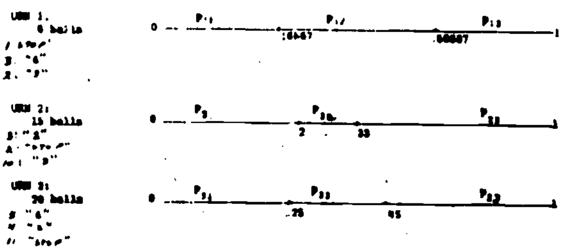


Figure 5

previously, to solve this problem. The attached sheet shows the number of balls in each urn. In urn 1, one ball is marked STOP, three balls are marked 2, and two balls are marked 3.They act respectively as p_{11}, p_{12}, p_{13} and dictate whether one tallies B_1 or moves to urn 2 or urn 3 to continue drawing balls, similarly for the other urns.

In closing it should be mentioned that the Monte Carlo method is sufficiently accurate for many practical applications. Improvements in results is related to N where N is the number of trials, so there are limitations to this method. Reference No. 2 discusses this and similar questions in great detail.

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TEACHING THE "OLD" DIVISION ALGORITHM USING OBJECT MANIPULATION

By Peter H. Williams, University of Maine at Farmington

A great deal has recently been written and spoken about the importance of object manipulation in teaching mathematical concepts. In this country the increased use of math lab materials and activity cards is but one example of a growing trend. Object manipulation is also an important component of the math material being developed in England and Canada. My experience in working with students from the lower elementary through the graduate level has convinced me that people of all ages who lack a concept (i.e. place value) are much more effectively taught the concept if they are provided with manipulative material as a starting point for a sequence of activities leading from objects to symbols.

For the past few years I have devoted a considerable amount of time to two questions: 1) How does one explain each of the whole number and fraction algorithms using object manipulation; and 2) What is an appropriate sequence of activities for moving from the objects to the symbols? The sequencing of activities is very important and, in my opinion, would be basically the same for any topic. Although the discussion of sequencing cannot be covered extensively here, I would like to develop it enough to provide more meaning to the material which will follow.

It seems there are at least four basic levels of communication involved in the instructional process. These four levels are object, verbal, picture and symbol, each of which can be used by both the teacher and the child. These levels are displayed in the following four-by-four communications model. It is a deductive model based on the assumptions: a) that the levels object, verbal, picture and symbol increase in their degree of abstractness; thus, the levels are progressively more difficult for the child to deal with when learning a concept; b) that the teacher should introduce and use each new level first and the child should progress from the object level, the child's easiest, to the point where the child uses the new medium.

The easiest communication is the situation where the teacher represents a concept in object form and the student does the same. This communication is denoted object-object, where the teacher is first and the child second. An example of another type of communication is symbol-picture: the teacher presents a concept in symbol form (using a worksheet or the blackboard) and the child represents the concept in picture form. The sequence of communications beginning with object-object communication (i.e. 1) and ending with symbol-symbol communication (i.e. 16) can be represented in a four-by-four table where the numbers in the cells indicate the order of the activities.

4 x 4 Communications Model

Child

Teacher		object	verbal	picture	symbol
	object	1	3	7	13
	verbal	2	4	8	14
	picture	5	6	9	15
	symbol	10	11	12	16

Figure 1

The data that I have gathered to date indicates that this model is significantly more effective than the usual textbook approach. My experience indicates that the omission of one of the levels, such as the picture level, greatly reduces the effectiveness of the model. For example, teachers who jump from using objects to using symbols experience much less success than do teachers who use most or all of the sixteen steps in the model.

Because of my belief in the model and the need to explain each of the traditional algorithms through the use of object manipulation, the following material on division was developed. Before being taught the material on division children should be given a review of the topics of place value, addition, and subtraction using the place value charts. The activities could involve such things as:

a) Show me 124 on your place value chart.

b) Near the top of your board put 257; near the bottom put 465; now combine the two sets starting at the right and regrouping where possible.

c) Show me 403 – that is what you have; you owe me 135 – how would you pay me?

Having done activities of this type one could progress to the following problem.

You have 433 dollars in the appropriate denominations (i.e. 4 hundreds, 3 tens and 3 ones). You are asked to distribute the money equally among *three* people. How would you do it? Most people did the following:

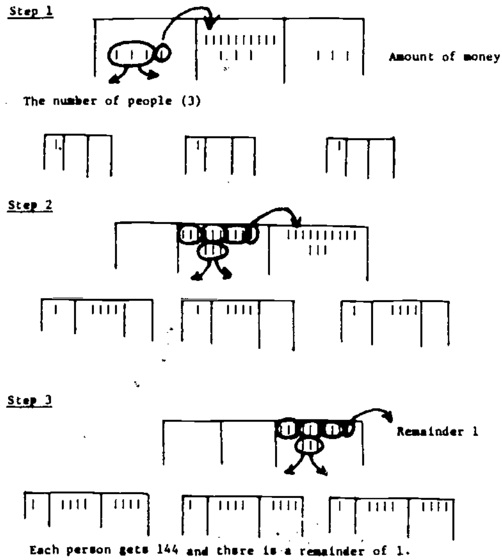


Figure 2

The way the participants worked the problem and others like it is probably the most "natural" way to solve the problem.

Each of the steps in Figure 2 can be described by using symbols:

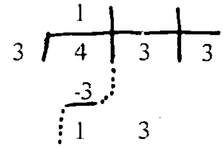
Amount each gets

No. of people 3 ⟌ 4 | 3 | 3 Denomination of the money

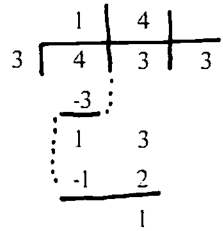
Step 1. There were four one hundred dollar bills and three people, so each person got one which used up three and left one.

$$\begin{array}{r|c|c|c} & 1 & & \\ \hline 3 & 4 & 3 & 3 \\ & -3 & & \\ \hline & 1 & & \end{array}$$

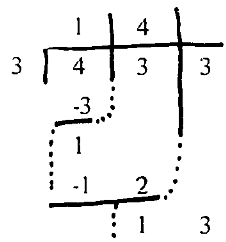
The one hundred dollar bill needs to be exchanged for ten tens which is denoted by "bringing down".



Step 2. There are now thirteen tens to distribute among the three people so each person gets four and one is left over.



The one ten is regrouped into the ones place by "bringing down".



There are now thirteen ones to distribute so each person gets four and one is left as a "remainder".

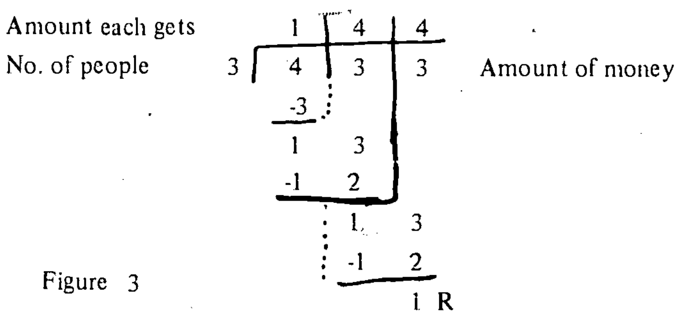


Figure 3

Thus, the numbers presented in Figure 3 describe the steps involved in solving the problem on the place value chart.

The sequence of activities can progress from the use of the place value charts to the picture sequence as presented in Figure 2 and then to the symbol sequence as presented in Figure 3.

In applying the model it is often helpful to list the four types of activities: object - place value chart activities; verbal - explaining the process of distributing the counters; picture - figure 2; symbol - figure 3. Then the sixteen steps can be implemented by matching the four types of activities in pairs in the order described in the model (Figure 1). In my opinion, the "old" division algorithm is still an efficient computational technique which can be taught to children in a meaningful way by beginning with object manipulation on the place value chart and progressing through the four-by-four communications model.

The activities presented here are but a sample of the activities done in the workshop session. A copy of the individualized module will be provided by writing the author.

The place value charts were constructed of a piece of poster board (9" x 22") divided into three sections. Each person was also provided about 30 counters (1" x 3").

It is interesting to note that in solving the problem with counters no one swapped *all* his counters to the ones place to start and then made piles of three with all 433 counters. This is the process suggested by the "guess method" algorithm which is being taught to many children today.

WHAT YOU HAVE ALWAYS WANTED TO KNOW ABOUT GRAPHING BUT WERE AFRAID TO ASK!

By Eldwin A. Wixson, Plymouth State College

Graph the statement; x = 3. This is really an unfair request because you could have a variety of solutions.

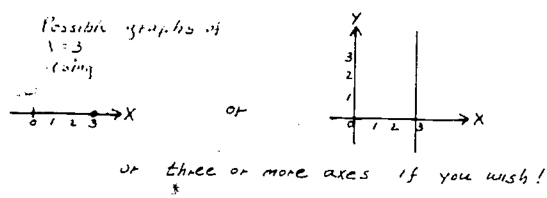


Figure 1

Thus, certain ground rules are followed.

G.R.1. A placeholder for the name of a thing is a *variable.*

G.R.2. A sentence containing at least one variable is an open sentence.

Thus, □ + □ = 12 is an open sentence with □ as the variable. Now, what can we put in the variable spots? G.R.3. The set of things that may be put in place of the variable in an open sentence to be tested as to whether or not the sentence is then true is called the *domain of the variable.*

G.R.4. The set of elements (from the domain) that make an open sentence true is the *solution set.*

Note that someone must establish the domain of the variable before testing of an open sentence may begin. If no domain is specified, it is assumed that the domain of the variable is the set of all the numbers on the number line, i.e., all real numbers.

Now consider again □ + □ = 12. If the domain is the set of real numbers, the solution set is {6}. If the domain is the set of odd counting numbers, the solution set is empty.

If the problem is restated as □ + △ = 12 and the domain of both variables is the counting numbers, then the solution set in table form is:

□	11	10	9	8	7	6	5	4	3	2	1
△	1	2	3	4	5	6	7	8	9	10	11

Notice the difference in results when we change the domain or when we change the sentence from a one variable sentence to a two variable sentence.

Finally, if we consider □ + △ = 12 with the domain of all real numbers for both variables, then the solution set is lots and lots of pairs of numbers! How, then, can we display this infinite solution set?

Before we pursue the solution to this question we need a few more ground rules.

G.R.5. An open sentence whose verb is = is an *equation.*

G.R.6. An open sentence whose verb is $<$ or $>$ or $\leq$ or $\geq$ or $\neq$ is an *inequality.*

G.R.7. A line consisting of points each of which corresponds to a directed number is a *number line.* Each point is the *graph* of the corresponding directed number and each directed number is the *coordinate* of the corresponding point.

G.R.8. The graph of a solution set of an equation or inequality is called the *locus* of the equation or inequality.

Thus, our notation says $\{x|...\}$ to mean "the set of all x such that...". This solves the unfair request at the onset. We should have written $\{x|x = 3\}$, if we wished to consider the one-axis problem and the first diagram in figure 1 is the appropriate graph. Graph the following:

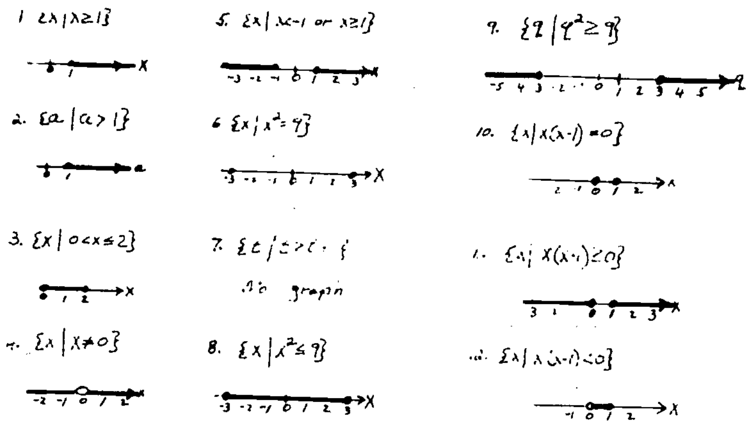


Figure 2

In order to answer our question of ▭ + △ = 12 in two variables, we will need to extend our ground rules.

G.R.9. The *Cartesian Product* of set A times set B, denoted A X B, means to form all possible ordered pairs in which the first element comes from set A and the second element comes from set B. In set notation - A X B = {(a,b)|a ∈ A and b ∈ B}.

For example, let A = {1, 2, 3}, B = {1}, and C = {2, 4, 6, 8}. Then A X B = {(1,1), (2,1), (3,1)}. B X A = {(1,1), (1,2), (1,3)}. A X C = {(1,2), (1,4), (1,6), (1,8), (2,2), (2,4), (2,6), (2,8), (3,2), (3,4), (3,6), (3,8)}. Notice that if the number of elements in each of A and B is finite, then A X B will have common ordered the product of these two numbers.

When plotting A X B on a graph, we almost always use two number lines as axes, placing them perpendicular to each other and intersecting them at the zero points. We then associate the first element of an ordered pair with a horizontal coordinate and the second element with a vertical coordinate. Hence, the "order" in ordered pair.

You, the reader, should consider what the graph of R X R would be like where R is the set of real numbers.

We are now ready to extend our ground rules again in order to graph solution sets with two variables.

G.R.10. If A is a set, then a subset of A X A is called a *relation in A*.

Let A = {-4, -3, -2, -1, 0, 1, 2, 3, 4}. Graph the following relations in A. (Remember, we seek the ordered pairs that make the statement true!).

Graph the following:

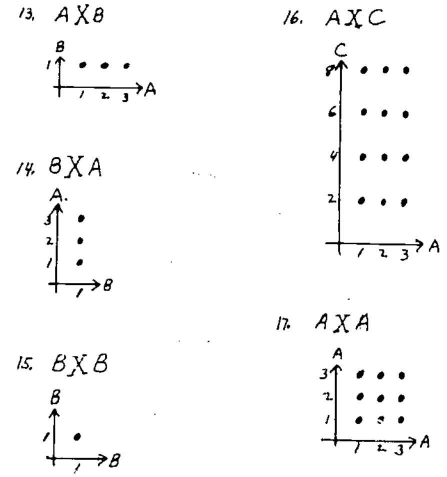


Figure 3

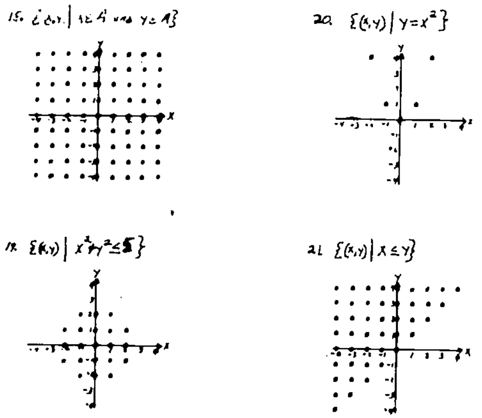


Figure 4

Again, let us assume that A = R, the set of real numbers. How would each of the graphs of the relations in Figure 4 be changed? With the answer to this question the reader is ready to graph □ + △ = 12, or if you would rather x + y = 12.

Now that we have completed the theory portion of the topic, let us consider its application.

Assume that a car starts from rest and accelerates smoothly to 45 m.p.h. in 11 seconds. Find the following:

1. The graph of velocity versus time
2. The graph of acceleration versus time
3. The graph of displacement versus time

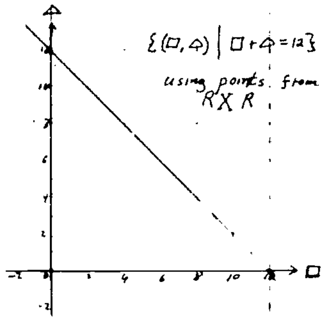


Figure 5

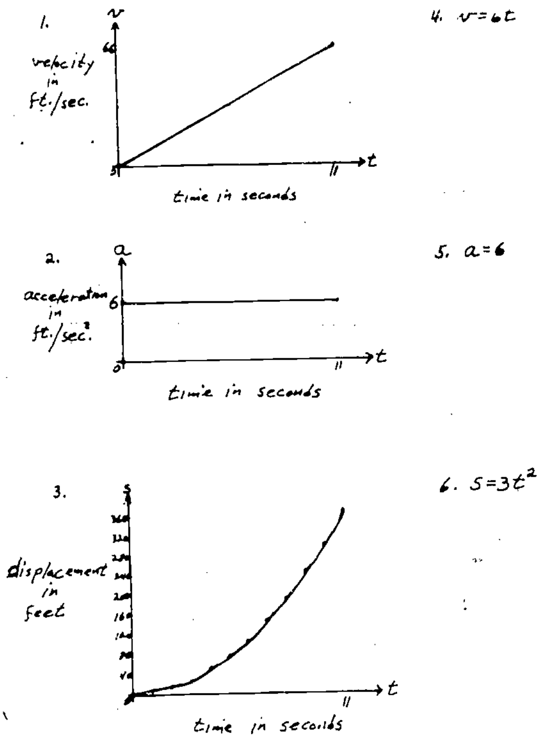


Figure 6

4. The algebraic statement showing the relation between velocity, v, and time, t.
5. The algebraic statement showing the relation between acceleration, a, and time.
6. The algebraic statement showing the relation between displacement, s, and time.

Once again we must extend our ground rules.

G.R.11. In a graph involving non-pure number quantities, each quantity may have its own scale and its own units associated with its number line.

G.R.12. Units used in an applied problem must be compatible.

The answers to the six questions are displayed in Figure 6. The reader may be interested in calculating the area between the velocity and the horizontal axis. What does this area represent?