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ABSTRACT

This course is part of an engineering certification program for men inspecting construction projects. It presumes some high school training in mathematics and science. The course begins with examples showing the importance of mathematics on the job. Following that is a section on algebra, one on geometry, and one on trigonometry. (KM)

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ENGINEERING CERTIFICATION PROGRAM SELF-STUDY COURSE

BASIC MATHEMATICS

- Algebra
- Geometry
- Trigonometry

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(Reprinted from R-5 Self-study Course)

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ENGINEERING MANAGEMENT
EM-7110-1

ENGINEERING CERTIFICATION PROGRAM
Self-Study Course

Basic Mathematics

(Reprinted from R-5 Self-Study Course)
May 1974

FOREWORD

The Engineering Certification Program -- Self-Study Course will be published in four separate issues. Pages are numbered consecutively throughout the four issues. Some pages have been omitted and a page inserted to indicate the omissions. This publication is numbered EM-7110-1. The three following issues will contain the same number followed by "a", "b", and "c".

These specific publications do not follow our standard format for Engineering publications and they are not of the printing quality we prefer for publishing. But, due to the timeliness and value of the excellent information compiled by Region 5, we are reprinting the Self-Study Course without retyping so that it may reach you as soon as possible.

Fran Owsley
Editor

INTRODUCTION

This course is designed for men inspecting construction projects in Region 5. It presumes that you have some, but not extensive, high school training in mathematics and science. It also presumes that you are interested in construction work and want to increase your knowledge and ability to do a better job. If you have this knowledge and ability, obtaining certification should not be difficult.

While we have all had formal education, many are unable to see the need or benefit of this education until after the opportunity for school has passed. We will try to show you how subjects studied in this course will be used and then concentrate on learning the basic information.

Study the material assigned thoroughly, and then do the homework problems. The problems will show you whether or not you have learned the study material. If you have trouble with the problems, go back and restudy the lesson material. Don't just try to get the answers. Understand what you are doing. If you have questions that aren't answered in the study material, make a note of them and ask your supervisor.

Vertical curves in a highway profile are parabolic, rather than circular. The formula for the correction from the tangent grade line to the curve is

$$d = m \left(\frac{D}{\frac{L}{2}}\right)^2$$

Here we have parenthesis, a complex fraction, and a term raised to a power, in this case squared.

The formula for the quantity of water flowing in a roller gutter is

$$Q = \frac{1.486}{n}(0.375Z)d^{\frac{8}{3}}s^{\frac{1}{2}}$$

The formula for flow in a triangular channel is

$$Q = 0.56\left(\frac{Z}{N}\right)S^{\frac{1}{2}}Y^{\frac{8}{3}}$$

And the grand-daddy formula from which these are derived is the Manning formula

$$V = \frac{1.486}{n}R^{\frac{2}{3}}S^{\frac{1}{2}}$$

If you are assigned as a plant man on a paving contract, you will need to compute required weights and percentages of aggregate to obtain a specified grading.

How many pounds of sand with x grading, and how many pounds of rock with a y and z grading are needed to get the end product. This involves ratio and proportion--again, algebra

The work of the inspector requires many use for algebra similar to those above. So emphasis on algebra is not just something to trip you up in an exam--it's a branch of mathematics you will use all through you career.

Section 1 - Algebra

INDEX

SECTION 1

Algebra

I - Basic Information

A - General

Generally, algebra continues the study of numbers and use of numbers started in arithmetic. Addition, subtraction, multiplication, and division are the same in algebra. The main differences are that the system of numbers is extended, letters are used as numbers, and relations of numbers are shown as equations.

B - Symbols

<u>Equation</u> - The equation symbol =, is shorthand for the expression "is equal to." Thus 2 + 2 =, or is equal to 4. All the terms to the left of the equation sign are equal to all the terms to the right of the sign.

The $\neq$ symbol means not equal to.

The $>$ symbol means is greater than; for instance, $2 > 1$.

The $<$ symbol means is less than; for instance, $1 < 2$.

<u>Signs</u> - The functions of addition, subtraction, and division are shown using the same signs used in arithmetic; +, -, and ÷. Multiplication is somewhat different. The product of two numbers, let's say a and b, are written $a \cdot b$, a x b, (a)(b), or just ab. The cross (x) is usually omitted because it may be confused with the letter x.

C - Definitions

<u>Negative Number</u> - Any number with a minus sign in front of it. For instance, temperature below zero, a downgrade.

<u>Positive Number</u> - Any number that is not negative. Since most numbers are positive, the plus sign is seldom shown in front of a positive number.

Expression - The showing of relation by the use of signs and symbols of algebra. Instances of expressions are 2d, 6c - 3, 5d ÷ t.

Term - If an expression consists of parts connected by plus and minus signs, each part is called a term. In the expression 6x - 4, 6x is a term and -4 is a term. The expression 6t - 4x + 5y + 7 has four terms.

Monomial - An expression of one term.

Polynomial - An expression of more than one term. A binomial has two terms, and a trinomial has three terms.

Factor - A factor of a number is any of the numbers that are multiplied to make the product.

For example, the factors of 3xy are 3, x, and y. Another example is 40ab, for which the factors are 2, 2, 2, 5, a, and b.

Prime Factor - A number that cannot be factored any further other than by itself and 1. Has 40ab (above) been factored into prime factors?

For the most part, when expressions are factored they should be shown as prime factors.

Coefficient - In products, any factor is called the coefficient of the other factor.

In 7cw the coefficient of w is 7c, 7 is the coëfficient of cw, and c is the coefficient of 7w.

Many times the numerical coefficient is said to be the coefficient of the term. If no coefficient is shown, it is understood to be one.

> Example: The numerical coefficient of such terms as x, wxy, and abe is assumed to be one. These terms are 1x, 1wxy, and 1abe.

Exponent - Where a quantity is to be multiplied by itself two or more times, the number of times it is to be multiplied by itself is written as a number above and to the right. Thus, $3 \cdot 3$ is 3^2, or $x \cdot x \cdot x \cdot x$ is x^4. The 2 and the 4 are exponents. The answer is called a power. Thus x^4 is the fourth power of x.

Root - A root is the opposite of a power. The number whose root is to be found is written under the radical sign $\sqrt{}$. If the square root is wanted, no other indication is given. If some other root is wanted, such as 3 (cube) or 7 (seventh), the root wanted is written in the notch of the radical sign. For instance, the cube root of x is written $\sqrt[3]{x}$.

The relationship of power and root can be demonstrated.

$$3^2 \text{ is } 9; \qquad \sqrt{9} \text{ is } 3$$

$$4^4 \text{ is } 256; \qquad \sqrt[4]{256} \text{ is } 4$$

II - Operating with Signs and Symbols

A - General

As stated under "Basic Information", it is necessary to extend the number system in algebra.

Numbers of arithmetic are such that we can apply every day quantities; for instance, in writing $8 - 5 = 3$ we can think of taking 5 peaches from 8 peaches. This gets fouled up when you try to take 8 peaches from 5 peaches.

But numbers are abstract in nature and do not need to be applied to any specific quantity.

The extension of the number system is done, in part, by bringing in new numbers called <u>negative numbers</u>.

The plus and minus signs used in arithmetic <u>only</u> to show addition and subtraction now are also used as integral parts of numbers. This now brings in double use of the addition and subtraction symbols. This double use of symbols will not be confusing if care is taken and the rules relative to the signs are observed.

If a number is written without a sign, the sign is assumed to be +.

B - Adding Signed Numbers

1. Numbers with like signs

$3 + 3 = 6,\ 4 + 4 = 8$

$-3 + (-3) = -6,\ -4 + (-4) = -8$

2. Numbers with unlike signs

A positive number added to a negative number gives the arithmetical difference and has the sign of the larger term.

$+4 + (-2) = 2$

$+2 + (-4) = -2$

C - Adding Algebraic Terms

Terms with literal factors the same are called similar, or like terms. 6cd and 3cd are like terms because they have the same literal factors c and d. It can be seen that 4ad and 7by are not like terms.

An example of adding like terms is:

$$\begin{aligned} 4xy + 7xy &= (4 + 7)xy \\ &= 11xy \end{aligned}$$

D - Subtracting Signed Numbers

To subtract signed numbers the sign of the number to be subtracted (the subtrahend) is changed and the numbers are added.

Example 1: Subtract -30 from -60.

$$\begin{array}{r} -60 \\ \underline{-30} \end{array}$$

Change sign in subtrahend and add.

$$\begin{array}{rl} -60 & \\ -30 & \\ \underline{(+)} & \\ -30 & \text{(difference)} \end{array}$$

Example 2: Subtract $-3ab$ from $4ab$.

$$\begin{array}{r} 4ab \\ \underline{-3ab} \end{array}$$

Change sign in subtrahend and add.

$$\begin{array}{rl} 4ab & \\ -3ab & \\ \underline{(+)} & \\ 7ab & \text{(difference)} \end{array}$$

Note: When subtracting, leave the original sign on the subtrahend then change the sign and put parenthesis around it. This will make checking easier when the question "Did I change the sign" arises.

E - Adding Polynomials

To add polynomials, arrange the terms in order, like terms under like terms, and then add.

Example: Add $5x^2 + 1 - 3x$, $2x + 7 - 4x^2$, and $5 - 4x^2 + 3x$

Solution:

$$\begin{array}{r} 5x^2 - 3x + 1 \\ -4x^2 + 2x + 7 \\ \underline{-4x^2 + 3x + 5} \\ -3x^2 + 2x + 13 \end{array}$$

Let $x = 2$ and substitute to check the solution.

Problems

Add: 1. $\begin{array}{r} 2x^2 \\ +6x^2 \\ -5x^2 \\ \underline{8x^2} \end{array}$ 2. $\begin{array}{r} -4abc \\ -9abc \\ +2abc \\ \underline{+5abc} \end{array}$

3. $\begin{array}{r} 5x - 7y - 3z + 12 \\ -3x + 5y + 9z + 15 \\ 5x + 4y - 7z - 17 \\ \underline{2x - 8y + 4z - 11} \end{array}$

F - Subtracting Polynomials

To subtract one polynomial from another, arrange the terms in order, then subtract each term in the subtrahend from the like term in the minuend.

Example: Subtract $5x - 4y - 5z$ from $6x - 3y + 2z$.

$$\begin{array}{rrrl} & 6x - & 3y + & 2z \\ & +\,5x - & 4y - & 5z \\ & (-) & (+) & (+) \\ \hline & x + & y + & 7z \text{ (difference)} \end{array}$$

Note that changing the sign in each term of the subtrahend and then adding all the terms gives the solution.

Problems

Subtract:

1. $$\begin{array}{r} 3x + 2y + 9 \\ \underline{2x + \ \ y + 5} \end{array}$$

2. $$\begin{array}{r} 6a^2 - 3a - 7 \\ \underline{a^2 + 4a + 9} \end{array}$$

3. $$\begin{array}{r} 8a^2 - 11 \\ \underline{-4a^2 + \ \ 1} \end{array}$$

4. $$\begin{array}{r} c - 2d \qquad \\ \underline{4c + \ \ d - 3} \end{array}$$

G - Multiplying Signed Numbers

A positive number times a positive number gives a positive product.

$+4 \times +5 = +20$; $3 \times 6 = 18$, etc.

A positive number times a negative number gives a negative product.

Just as 4 x 5 is actually 5+5+5+5 or +20, so is 4 x (-5) actually -5-5-5-5 or -20, and -4 x 5 = -20, etc.

A negative times a negative gives a positive.

Just as 4 x 5 = +20 and 4 x (-5) = -20, so does (-4) x (-5) = +20.

H - Dividing Signed Numbers

Since division is the reverse of multiplication, from the above, we determine that if

$$(+4) \times (+5) = +20 \text{ then } +20 \div +4 = +5$$

$$\text{also } +4 \times -5 = -20 \text{ then } -20 \div +4 = -5$$
$$\text{and } -20 \div -5 = +4$$

$$\text{also } -4 \times -5 = +20 \text{ then } +20 \div -4 = -5$$

From these we get the general rule for division:

The quotient of any two numbers having the same sign is positive, while that of two numbers of opposite sign is negative.

Examples: $\frac{-6}{-2} = 3$; $\frac{6}{2} = 3$;

$\frac{-6}{2} = -3$; $\frac{6}{-2} = -3$

III - Algebraic Operations

A - Grouping

Parenthesis ( ), brackets [ ], and braces { } are symbols used to show that terms within the symbols are to be considered a single quantity. These symbols are handy to show the sum or difference of two or more expressions.

Example: $(8a + 2y + 3z) + (7a + 3y + 2z)$ or

$(7b - 3x + 2d) - (7b + 3x - d)$.

In multiplication the symbols can be used to show what expressions are to be multiplied.

Example: $(c + d)(r + s)$

This means the sum of c and d is to be multiplied by the sum of r and s.

Example: $c + d(w + e) = c + dw + de$

Following are rules relative to use of parenthesis and other grouping symbols:

To remove parenthesis preceded by a + sign, write the enclosed terms with their given signs. To remove parenthesis preceded by a - sign, write the enclosed terms with their signs changed.

Removing parenthesis that are enclosed by other parenthesis:

<u>Rule</u>: When one set of parenthesis is enclosed by another, remove one set at a time, beginning with the innermost set.

Example: Remove parenthesis and simplify: $2x - [x - (3x + 4) - 1]$

Solution:

$$\begin{aligned} & 2x - [x - 3x - 4 - 1] \\ &= 2x - x + 3x + 4 + 1 \\ &= 4x + 5 \end{aligned}$$

<u>Problems</u>

<u>Remove parenthesis and simplify</u>:

1. $[a - (3a - 2)]$

2. $-\{3x - (4y - 5x) + 2y\}$

3. $\{[(3x - 5) - 8] + 2x\}$

B - Powers and Exponents

Powers - A power of a number is the product obtained by using the number as a factor one or more times.

Examples: The third power (cubed) of $3x$ is the product obtained by multiplying $3x \cdot 3x \cdot 3x = 27x^3$

The fifth power of $2m$ is

$2m \cdot 2m \cdot 2m \cdot 2m \cdot 2m = 32m^5$

The number of like factors to be used is indicated by means of an exponent.

Exponents - An exponent is a positive integer written to the right and slightly above another number to indicate how many times the number occurs as a factor

Thus 5^2 means 5×5 or 25

3^4 means $3 \times 3 \times 3 \times 3$ or 81

m^3 means $m \cdot m \cdot m$

$7m^2$ means $7 \cdot m \cdot m$

$n^2 + n^4 + 3n^3 - 5n^2$ is read n squared, plus n to the fourth power, plus 3n cubed, minus 5n squared.

If the exponent is 1, it is usually omitted but understood to be present. Thus x means x^1 and $3n$ means $3n^1$.

Multiplication with Exponents - The exponent of any letter in a product is equal to the sum of the exponents of that letter in the factors.

Therefore: $a^2 \cdot a^3 = (a \cdot a) \times (a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5$

$$a^2 \cdot a^3 = a^{2+3} = a^5$$

Similarly: $x \cdot x^4 \cdot x^3 = x \cdot (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x) = x^{1+4+3} = x^8$

Also: $3 \cdot 3^3 \cdot 3^5 = 3 \cdot (3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3^{1+3+5} = 3^9$

In general terms: $n^a \times n^b = n^{a+b}$

When like exponentials are multiplied, the exponents are added.

Additional Examples:

$$hx^3 \cdot x^2 = hx^{3+2} = hx^5$$

$$2d \cdot d^3 = 2d^{1+3} = 2d^4$$

$$5xy^2 \cdot x^3y^4 = 5x^{1+3} \cdot y^{2+4} = 5x^4y^6$$

$$(+7x^2) \cdot (-3x^3) = -21x^5$$

<u>Division with Exponents</u> - The exponent of any letter in a quotient is equal to the exponent in the dividend minus the exponent in the divisor.

Therefore: $n^5 \div n^3 = \frac{n^5}{n^3} = \frac{\cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdot n \cdot n}{\cancel{n} \cdot \cancel{n} \cdot \cancel{n}} = n^2$

But: $n^2 = n^{5-3}$ or $\frac{n^5}{n^3} = n^{5-3}$

Similarly: $2^5 \div 2^2 = \frac{2^5}{2^2} = \frac{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 2 \cdot 2}{\cancel{2} \cdot \cancel{2}} = 2^{5-2} = 2^3$

Also: $\frac{2b^3}{b^2} = \frac{2 \cdot \cancel{b} \cdot \cancel{b} \cdot b}{\cancel{b} \cdot \cancel{b}} = 2b^{3-2} = 2b$

Again: $\frac{cd^5}{d^4} = \frac{c \cdot \cancel{d} \cdot \cancel{d} \cdot \cancel{d} \cdot \cancel{d} \cdot d}{\cancel{d} \cdot \cancel{d} \cdot \cancel{d} \cdot \cancel{d}} = cd^{5-4} = cd$

In general terms: $n^a \div n^b = n^{a-b}$

When like exponentials are divided, the exponents are subtracted.

When a letter appears with a higher exponent in the divisor than the dividend, the quotient will be most conveniently expressed in fractional form with the excess factors of the letter appearing in the denominator of the fraction.

Thus $12x^3 \div 6x^7 = \frac{12x^3}{6x^7} = \frac{2}{x^4}$

It often happens that a letter appears with the same exponent in both dividend and divisor.

Since $x^2 \div x^2 = 1$

and $x^2 \div x^2 = x^{2-2} = x^o$

then $x^2 \div x^2 = x^o = 1$ (Note: Any power having a zero exponent equals 1)

Also $a^5b^2 \div ab^2 = \frac{a^5}{a} \cdot \frac{b^2}{b^2} = a^4 \times 1 = a^4$

Therefore, a letter or number with the same exponent in both dividend and divisor should not appear in the quotient.

Additional Examples:

$$\frac{n^4bc^3}{n^2c^3} = n^2b$$

$$2^2m^3n^5 \div 2^2n^5 = m^3$$

$$\frac{4^3x^2y^3z}{4x^2y^2z^4} = \frac{4^2y}{z^3}$$

<u>Negative Exponents</u> - Any quantity with a negative exponent is equal to the reciprocal of that quantity with the corresponding positive exponent.

Example: $\frac{x^3}{x^5} = \frac{x^3}{x^3x^2} = \frac{1}{x^2}$

But: $\frac{x^3}{x^5} = x^{3-5} = x^{-2}$

Hence: $x^{-2} = \frac{1}{x^2}$

In general: $a^{-n} = a^{o-n} = \frac{a^o}{a^n} = \frac{1}{a^n}$

In like manner: $\frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = a^n$

<u>Square Roots</u> - The square root of any number is one of the two equal factors whose product gives the monomial. To take the root of an exponent, divide the exponent by the root. $\sqrt{x^{10}} = x^5$

Examples:

$\sqrt{3^2} = \sqrt{9} = 3$	$3^2 = 3 \cdot 3 = 9$
$\sqrt{3^2 \cdot 4^2} = \sqrt{144} = 12$	$5^2 = 5 \cdot 5 = 25$
$\sqrt{4m^2} = 2m$	$(2m)^2 = 2m \cdot 2m = 4m^2$
$\sqrt{9m^6} = 3m^3$	$(3m^3)^2 = 3m^3 \cdot 3m^3 = 9m^6$
$\sqrt{25a^2m^4} = 5am^2$	$(5am^2)^2 = 5am^2 \cdot 5am^2 = 25a^2m^4$
$\sqrt{64m^4x^2} = 8m^2x$	$(8m^2x)^2 = 8m^2x \cdot 8m^2x = 64m^4x^2$

PROBLEMS

Multiplication with Exponents

1. $6x \cdot 3x$
2. $t^3 \cdot t$ (Ans. t^4)
3. $(-3n)(4n^3)$
4. $-y^2(y^3)$ (Ans. $-y^5$)
5. $(-3x)(-5x^3)$
6. $(6n^3y)(-2y^2)$ (Ans. $-12n^3y^3$)
7. $(-2n)^2(3nk)$
8. $(5x^2y)^2$ (Ans. $25x^4y^2$)
9. $2s^3(3ds^2)$
10. $(2p)(-3p^2)(5p)$ (Ans. $-30p^4$
11. $(-2x^2m)(3m^2)^3$
12. $(7x^2y^2)^2$ (Ans. $49x^4y^4$)
13. $(-3m^2p^2)(-3m^2p^2y)$
14. $5mn(-3m^2n)$ (Ans. $-15m^3n^2$)
15. $2n^2p(-3np^2)^2$

Division with Exponents

1. $y^7 \div y^3$

2. $-d^{12} \div d^4$ (Ans. $-d^8$)

3. $-5y^{12} \div (-2y^3)$

4. $39a^3d^7 \div 3a^4d^4$ (Ans. $\frac{13d^3}{a}$)

5. $36x^4p^3 \div 6x^5p^4$

6. $49x^5y^7 \div 7$ (Ans. $7x^5y^7$)

7. $75d^9y^5n \div (-25d^6n)$

8. $-56m^4n^2 \div (-4m^4n^2)$ (Ans. 14)

9. $a^5d \div a^{15}d^2$

10. $2x^2 \div x^{-2}$ (Ans. $2x^4$)

11. $d^3 \div d^{-2}$

12. $4n^5p^4 \div n^{-3}$ (Ans. $4n^8p^4$)

13. $42d^{12}b^{18}c^{15} \div 7d^{10}c^{12}$

14. $12x^5y^2 \div 3x^7y^2$ (Ans. $\frac{4}{x^2}$)

15. $50x^7yz \div 100x^2y^7z$

Find the value of the positive square roots of:

1. $\sqrt{4m^2}$

2. $\sqrt{9x^2}$ (Ans. $3x$)

3. $\sqrt{16y^4}$

4. $\sqrt{25a^2d^4}$ (Ans. $5ad^2$)

5. $\sqrt{36d^6m^2}$

6. $\sqrt{49m^2x^6}$ (Ans. $7mx^3$)

7. $\sqrt{81x^4n^6}$

8. $\sqrt{3^2 \cdot 9}$ (Ans. 9)

9. $\sqrt{2^2 \cdot 3^2 \cdot 5^2}$

10. $\sqrt{9 \cdot 16 \cdot 25}$ (Ans. 60)

C - Multiplying Polynomials by Monomials

To multiply a polynomial by a monomial, multiply each term of polynomial by the monomial.

Example: Multiply $6x^2 - 3xy + 4y$ by $-2xy$

Solution:

$$\begin{array}{r} 6x^2 - 3xy + 4y^2 \\ -\ 2xy \\ \hline -12x^3y + 6x^2y^2 - 8xy^3 \end{array}$$

Using parentheses, try solving the following by observation:

$$-2a(a^2 - 3a + 1) = -2a^3 + 6a^2 - 2a$$

Problems

Multiply

1. $2(x^2 - 3x + 1)$

2. $3 \bullet (8t^3 + 6t^2 + 1)$

3. $-2ab(b^3 - b + a)$

4. $-2y(-x - y)$

D - Multiplying Polynomials by Polynomials

Rules for multiplying polynomials by polynomials:

1. Arrange terms.
2. Multiply each term in the multiplicand by each term in the multiplier, writing like terms under like terms.
3. Add like terms.

Example: Multiply $2x^2 - 3x + 1$ by $3x - 2$

Solution:

$$\begin{array}{r} 2x^2 - 3x + 1 \\ \underline{3x \quad - 2} \\ 6x^3 - 9x^2 + 3x \\ \underline{- 4x^2 + 6x - 2} \\ 6x^3 - 13x^2 + 9x - 2 \end{array}$$

Problems

Multiply

1. $a + 5$ by $a + 11$

2. $3c + 4d$ by $3c + 4d$

3. $2a - 5$ by $3a + 5$

4. $7c^2 + 2c - 5$ by $-c - 3$

E - Dividing Polynomials by Monomials

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example: $\dfrac{4a^2x^2 - 6a^3x^3 + 14a^4x^5}{-2ax^2}$

$= \dfrac{4a^2x^2}{-2ax^2} - \dfrac{6a^3x^3}{-2ax^2} + \dfrac{14a^4x^5}{-2ax^2}$

$= -2a + 3a^2x - 7a^3x^3.$

The student should be able to omit the middle step above.

Problems

Divide

1. $\dfrac{17v^3 - 34v^2 - 51v}{-17v}$

2. $\dfrac{3Nn^{12} - 2N}{N}$

3. $\dfrac{-9t^4 - 18t^3 - 27t^2}{9t}$

F - Dividing Polynomials by Polynomials

To divide one polynomial by another

1. Arrange dividend and divisor in descending order of any letter.

2. Divide the first term in the dividend by the first term in the divisor and write the result as the first term in the quotient.

3. Multiply each term in the divisor by the first term in the quotient and write the product under the dividend, writing like terms under like terms.

4. Subtract like terms.

5. Bring down the next term of the dividend, or bring down more than one if needed.

6. Divide the first term in this remainder by the first term of the divisor and write the result as the second term in the quotient.

7. Continue multiplying, dividing, bringing down, dividing, etc.

Examples: Divide $6a^2 - 6x^2 - 5ax$ by $2a - 3x$

Solution:

$$\begin{array}{r|l} & \underline{3a + 2x\qquad\qquad} \\ 2a - 3x & 6a^2 - 5ax - 6x^2 \\ & \underline{6a^2 - 9ax\qquad\quad} \\ & \qquad\quad 4ax - 6x^2 \\ & \qquad\quad 4ax - 6x^2 \end{array}$$

Divide $v^2 + 2vw - 16w^2$ by $v - 3w$

Solution:

$$
\begin{array}{r}
v + 5w \\
v - 3w \enclose{longdiv}{v^2 + 2vw - 16w^2} \\
\underline{v^2 - 3vw} \\
+ 5vw - 16w^2 \\
\underline{+ 5vw - 15w^2} \\
- w^2
\end{array}
$$

Answer: $= v + 5w$, remainder $-w^2$

<u>Problems</u>

1. $\dfrac{6r^2 + 9r - 42}{2r + 7}$

2. $\dfrac{x^2 + xy - 2y^2}{x - y}$

3. $\dfrac{s^2 - 5s - 50}{s + 5}$

4. $\dfrac{13r^2 + 144rt + 11t^2}{r + 11t}$

<u>IV - Simple Equations</u>

In this part of algebra we will study linear equations of the first degree. The degree of an equation is the degree or power of the highest term.

An <u>equation</u> is defined as a statement of equality between two equal numbers or number symbols.

Examples: $5 + 12 = 21 - 4$

$3x + 5 = 7x - 9 - 4x + 14$

$3y - 7 = 8$

$m^2 - 2m - 8 = 0$

The part of the equation on the left of the equality sign is called the <u>first</u> or <u>left member</u> of the equation; that on the right, the <u>second</u>, or <u>right member</u>.

In an equation the letter whose value is sought is called the unknown letter or the unknown.

The process of finding the value of the unknown in an equation is called solving the equation.

If the value found for the unknown is substituted for the unknown in the given equation and if, when the result is simplified, the left member of the equation becomes identical with the right member, then the value found for the unknown is said to satisfy the equation.

A number or number symbol which satisfies an equation is called a root of the equation and the process of substituting the root for the unknown is called checking the solution.

Examples: $5x = 20$

$$x = \frac{20}{5}$$

$$x = 4$$

The value $x = 4$ satisfies the equation and consequently is the root of the equation.

$$7x - 5 = 16$$

$$7x = 21$$

$$x = \frac{21}{7}$$

$$x = 3$$

Axioms - An axiom is a statement whose truth is accepted without proof.

In solving equations constant use is made of five Axioms.

Axiom I - If the same number is added to each member of an equation, the result is an equation equivalent to the original one.

Example: Adding 5 to each member of the equation $x - 5 = 2$ gives the equivalent equation $x = 7$.

$$(x - 5) + 5 = 2 + 5$$

$$x - 5 + 5 = 2 + 5$$

$$x = 7$$

<u>Axiom II</u> - If the same number is subtracted from each member of an equation, the result is an equation equivalent to the original one.

Example: Subtracting 8 from the equation $x + 8 = 18$ gives the equivalent equation $x = 10$.

Or: $x + 8 = 18$

$$(x + 8) - 8 = 18 - 8$$

$$x + 8 - 8 = 18 - 8$$

$$x = 10$$

<u>Axiom III</u> - If each member of an equation is multiplied by the same number, the result is an equation equivalent to the original one.

Example: Multiplying each member of the equation $1/2x = 3$ by 2 gives the equivalent equation $x = 6$.

Or: $1/2x = 3$

$$1/2x(2) = 3(2)$$

$$2/2x = 6$$

$$x = 6$$

<u>Axiom IV</u> - If each member of an equation is divided by the same number, the result is an equation equivalent to the original one.

Example: Dividing each member of the equation $2x = 6$ by 2 gives the equivalent equation $x = 3$.

Or: $2x = 6$

$$2x/2 = \frac{6}{2}$$

$$x = 3$$

Axiom V - A term may be divided by a number as long as the term is multiplied by the same number and the value will not be changed.

$6yx \qquad \frac{6yx(2x)}{2x} = 6yx$ no change

Remember this: What you do to one side of an equation you must do to the other.

Example of solving and checking equations:

Solve	$5x - 2 = 3x + 6$
Adding 2 to each member	$5x = 3x + 8$
Subtracting 3x from each member	$2x = 8$
Dividing each member by 2	$x = 4$
Check: Substituting 4 for x	$(5)(4) - 2 = (3)(4) +$
	$20 - 2 = 12 + 6$
	$18 = 18$

Hence 4 is the root of the original equation.

Oral Exercise

Solve the following equations and state Axiom used.

$x - 3 = 8$	$k - 5 = -3$	$y - 2 = -2$	$-5 + x = 7$
$x + 5 = -4$	$k + 5 = 5$	$8 + y = 0$	$x + 1 = -2$
$1/2x = -5$	$1/3y = -2$	$1/4m = 3$	$1/2x = 0$
$5x = 75$	$7y = -21$	$3k = 9$	$16 = 4y$

$z - 1 = 6$

$4 + x = -6$

$1/3k = 2.5$

$-30 = 5x$

Example of problem solving:

A farmer has 380 chickens which are divided into three flocks. The second flock contains twice as many chickens as the first, and the third flock 20 more than three times as many as the first. How many chickens are in each flock.

Solution: The verbal statement of the equation is number of chickens in first flock + number in second + number in third = 380

Let x = number of chickens in first flock

then $2x$ = number of chickens in second flock

and $3x + 20$ = number of chickens in third flock

Substituting in the verbal equation:

$$x + 2x + 3x + 20 = 380$$

Collecting: $6x + 20 = 380$

Subtracting 20: $6x + 20 = 380,\ 6x + 20 - 20 = 380 - 20$

$$6x = 360$$

Dividing by 6: $\frac{\cancel{6}x}{\cancel{6}} = \frac{360}{6}$ or $x = 60$

Number in first flock = 60

Number in second flock = $2x$ = 120

Number in third flock = $3x + 20$ = 200

Rules to follow in problem solving:

1. Read the problem carefully, noting the known and the unknown numbers and the statement which will later be expressed as an equation.

2. State the implied equation briefly in words, using the symbols of operation +, −, x, ÷, and the equality sign.

3. Represent one unknown number by some letter such as x, y, etc., and the other unknown by means of algebraic expressions involving this letter.

4. Substitute these expressions in the brief statement of the implied equation.

5. Solve the resulting equation.

6. Check by substituting in the statement of the problem, the values found for the unknown.

PROBLEMS

Solve the following equations and check the results:

1. $2 + 3x = 2x - 3$

2. $3y - 5 = 3 + y$ (Ans. $y = 4$)

3. $3k + 8 = 10 - k$

4. $2x - 15 = x - 15$ (Ans. $x = 0$)

5. $7y + 12 = 2y + 12$

6. $2k - 7 = 5k + 8$ (Ans. $k = -5$)

7. $5y - 4 = 10 + 12y$

8. $2x - 2 = 4 + 3x$ (Ans. $x = -6$)

9. $7 + 3n - 6 = n - 1$

10. $x - 3 - 4x = 7 - 5x$ (Ans. $x = 5$)

11. The perimeter of a rectangle is 72 feet. Its length is 4 feet greater than its width. What are its dimensions?

12. One number is four times another. Their sum is 45. What are the numbers? (~~Ans. x = 9 y = 36~~)
9 and 36

13. A rectangular lot is three times as deep as it is wide. The fence around it is 400 feet. What are its dimensions?

14. A flagpole is half as tall as the building on which it stands. The top of the pole is 135 feet above the level of the ground. How long is the pole? (Ans. 45 feet)

15. The perimeter of a triangle is 42 yards. The first side is 5 yards less than the second, and the third is 2 yards less than the first. What are the lengths of each side?

16. The sum of three numbers is 75. The second is twice the first and the third is 5 more than four times the first. What are the numbers? (Ans. first = 10, second = 20, third = 45)

17. One number is seven times another and their difference is 216. What are the numbers?

18. A swimming pool is three times as long as it is wide. The difference between the length and width is 40 feet. What are the dimensions? (Ans. 20 x 60)

19. The second of four numbers is 3 less than the first, the third is 4 more than the first, and the fourth is 2 more than the third. Their sum is 35. What are the numbers?

20. Three groups of men are employed to clean streets. Group 1 is twice as large as group 2 and group 3 is as large as groups 1 and 2 combined. There are 72 miles of streets. How many blocks should each clean if there are 10 blocks to the mile? (Ans. Group 1 = 240 blocks, Group 2 = 120 blocks, Group 3 = 360 blocks)

<u>Transposition</u> - A term may be omitted from one member of an equation if the same term, with its sign changed from - to + or from + to -, is written in the other member. This process is called transposition. (Based on Axioms I and II)

<u>Example No. 1</u>

Solve $4x - 5 = 19$ (1)

Adding 5 to each member, (Axiom I), we get

$$4x - 5 + 5 = 19 + 5$$

In the first member $- 5 + 5 = 0$. Hence these two numbers may be omitted, and the equation becomes

$$4x = 19 + 5 \qquad (1)$$

$$x = 6$$

Check: Substituting 6 for x in (1)

$$24 - 5 = 19$$

$$19 = 19$$

Comparing (2) with (1), we see that -5 has vanished from the first member of the original equation and that +5 has appeared in the second member of the new equation. The number +5 has really been added to both members of the equation, but the effect is the same as if the -5 had been omitted from the one member of the equation and written in the other member with its sign changed from - to +.

<u>Example No. 2</u>

Solve $7x = 6x + 16 \qquad (1)$

If we apply Axiom II and subtract 6x from both members we get $7x - 6x = 16$. (2)

Combining $\qquad x = 16$

Check: Substituting 16 for x in (1)

$$7(16) = 6(16) + 16$$

$$112 = 96 + 16$$

$$112 = 112$$

Hereafter, instead of going through the details of adding a number to both members of an equation, (or subtracting a number from both members), we shall use transposition. We should remember, however, that the transposition of a term is really the addition of that term to (or the subtraction of it from) each member of the equation.

Like terms in the same member of an equation should be combined before transposing any terms.

Cross Multiplication

Application of Axioms III and IV.

To eliminate the coefficient of an unknown term follow Axioms IV and V. If the coefficient is on top, put it on the bottom; if the coefficient is on the bottom, put it on the top. (i.e. on other side of equation.)

$2x = 4$ $\quad x = \frac{4}{2}$ Axiom IV $\quad \frac{x}{2} = 4$ $\quad x = 4(2)$ Axiom III

$x = 2$ $\quad x = 8$

Example No. 3 (Using transposition and cross multiplication)

Solve $14x - 11 + 8x + 9 - 3x = 5x + 2 + 12x - 20$ (1)

Combining $\quad 19x - 2 = 17x - 18$

Transposing $\quad 19x - 17x = -18 + 2$ (Axioms I and II)

$2x = -16$

Cross Multiplication $\quad x = -8$ (Axioms III & IV

Check: Substituting -8 for x in (1)

$-112 - 11 - 64 + 9 + 24 = -40 + 2 - 96 - 20$

Combining $\quad -154 = -154$

Hence -8 is a root of the equation, for upon substituting -8 for x the equation reduces to an identity.

PROBLEMS

1. $5x + 4 - 2x = 25$

2. $11 + 5y + 1 = y + 12$ (Ans. $y = 0$)

3. $5x - 0 = 19 + 3x - 5$

4. $5n + 13 - 4n + 2 = 8 - 5n - 5 + 8n$ (Ans. $n = 6$)

5. $-2x + 1 + 5x - 6 + 3x + 18 - 25 = 0$

6. $4p - 9p + 9 + 5p = 7p - 12$ (Ans. $p = 3$)

7. $0 = 17n - 18 - 5n + 27 + 6 + 45$

8. $2x - 8 - 4x - 7 = 2x + 5 - 7x + 7$ (Ans. $x = 9$)

9. $2n + 13 - 3n + 3 = 7n + 16 - 4n - 8$

10. $4x - 17 + 15x - 31 - 10x + 3 = 0$ (Ans. $x = 5$)

11. $5p - 13 - p = -27 - 7p - 8$

12. $0 = 2y - 2 - 6y + 14$ (Ans. $y = 3$)

13. $5x - 4 + 6x - 40 = 0$

14. $5 + 2p - 3p + 3 = p - 2$ (Ans. $p = 5$)

15. $n + 1 - 3n + 3 = 4n + 22$

16. The sum of two numbers is 73 and their difference is 19. What are the numbers? (Ans. $n = 27$, $n + 19 = 46$)

17. The sum of the three angles of a triangle is always 180°. In a certain triangle the second angle is 15° greater than the first and the third is 10° less than three times the first. Find each angle.

18. The sum of three numbers is 60. The first is 7 less than the second and the third is 3 more than twice the second. What are the numbers? (Ans. 9, 16, and 35)

19. There are two numbers such that the first is five times the second and the first is also 64 more than the second. What are the numbers?

20. There are two numbers such that the larger is three times the smaller. If 8 is added to the larger, it will be five times as large as the smaller. Find the numbers. (Ans. 4 and 12)

V - Formulas

Formulas are literal equations which express relationships between certain quantities as applied to the sciences; such as in physics and engineering.

For instance, the formula for the distance traveled by a falling object acted on only by the force of gravity is: $s = 1/2gt^2$

Where: s = distance traveled
g = acceleration due to gravity
t = time object has traveled

Many such formulas are used in engineering and sciences. It is necessary for the student to be able to solve a given formula for the other values or variables.

For instance, if we solve $s = 1/2gt^2$ for g, we get $g = \frac{2s}{t^2}$

This solution was completed in accordance with the rules set out under "IV - Simple Equations".

PROBLEMS

1. In $E = IZ$, solve for I

$$I = \frac{E}{Z}$$

2. In $I = prt$, solve for t

$$t = \frac{I}{pr}$$

3. In $X = 2\pi fL$ solve for L

4. In $S = 1/2(a + b + c)$, solve for b

5. In $W = \frac{F2\pi R}{P}$, solve for R

6. In $m_1v_1 + m_2v_2 = (m_1 + m_2)v$, solve for m_2

7. In $C = \frac{5}{9}(F - 32)$, solve for F

8. In $Q = WS(T_1 - T_2)$, solve for T_2

9. In $K = \frac{e}{L(t_1 - t_2)}$, solve for t_2

VI - Simultaneous Solution of Equations

A - General

By now we have studied and learned to solve monomial equations, equations with one unknown, by various methods.

Example: $2x + 4 = 20$

when $x = 8$

Also we have studied polynomial equations with more than one unknown. Now we will learn to solve polynomial equations by various methods. Independent equations with more than one unknown and which have common solutions can be solved by the following methods:

(a) Elimination

1. By addition or subtraction
2. By substitution
3. By comparison

(b) Graphically

1. Plot the equations

Rule of Thumb:

There must be as many equations as there are unknowns before you can solve for the unknowns.

Example: One unknown needs one equation.
Two unknowns need two equations.
Three unknowns need three equations.
n unknowns need n equations.

B - Elimination

By elimination we try to eliminate all the unknowns but one; this will give us a monomial equation which we have learned to solve.

(a) - Addition or Subtraction

Rule: If necessary, multiply the given equations by such numbers as will make the coefficients of one of the unknown quantities in the resulting equations of equal absolute value. Add or subtract the resulting equations accordingly as the coefficients of equal absolute value are of unlike or like sign.

Given:

(1) $2x + 9y = 71$

(2) $12x - 9y = -15$

Solve for x and y

Solve for x by addition.

Note: We need not alter these equations to add; the y term will drop out.

Solution:

Add
$$\begin{array}{r} 2x + 9y = 71 \\ \underline{12x - 9y = -15} \\ 14x + 0 = 56 \end{array}$$

$$14x = 56$$

$$\frac{\cancel{14}x}{\cancel{14}} = \frac{56}{14}$$ Divide both sides by 14

$$x = \frac{56}{14}$$

$$x = 4$$

Now let us find the value of y this time by subtraction.

Given:

(1) $2x + 9y = 71$

(2) $12x - 9y = -15$

Note: If we subtract these two equations as they are now, we will not eliminate the unknown x term. Therefore, we must do something to one of the equations in order to make the x terms equal, so when we subtract the x term will be eliminated.

Let us multiply equation 1 by 6.

(1) $6(2x + 9y = 71)$

(1-R) $12x + 54y = 426$

Now the x terms of equation (1-R) and (2) are equal and we can proceed to subtract equation (2) from equation (1-R).

$$\begin{array}{lrcrcr} (1\text{-R}) & 12x & + & 54y & = & 426 \\ (2) & 12x & - & 9y & = & -15 \\ (-) & & (+) & & & (+) \\ \hline & 0 & + & 63y & = & 441 \end{array}$$

To subtract, change the sign and add

$63y = 441$

$\frac{\cancel{63}y}{\cancel{63}} = \frac{441}{63}$ Divide both sides by 63

$y = \frac{441}{63}$

$y = 7$

Now check the answer x = 4, y = 7.

Substitute into either equation 1 or 2.

$x = 4$ $\quad$ (1) $\quad 2x + 9y = 71$
$y = 7$ $\quad\quad\quad 2(4) + 9(7) = 71$

$$8 + 63 = 71$$
$$71 = 71$$ Answer checks

(b) - Substitution

Rule: From one of the given equations find the value of one of the unknown quantities in in terms of the other, and substitute this value in place of that quantity in the other equations.

Given:

(1) $2x + 9y = 71$

(2) $12x - 9y = -15$

First let us solve equation (1) for x in terms of y.

(1) $2x + 9y = 71$

$2x + \cancel{9y} - \cancel{9y} = 71 - 9y$ Add -9y to both sides

$2x = 71 - 9y$

$\frac{\cancel{2}x}{\cancel{2}} = \frac{71 - 9y}{2}$ Divide both sides by 2

$x = \frac{71 - 9y}{2}$

Now we have x in terms of y.

Next substitute the value of x in terms of y into equation (2) and solve for y.

(2) $12x - 9y = -15$ $\quad\quad x = \frac{71 - 9y}{2}$

$12\left(\frac{71 - 9y}{2}\right) - 9y = -15$ Substitute for x

$\frac{12(71)}{2} - \frac{12(9y)}{2} - 9y = -15$

$426 - 54y - 9y = -15$

$$-\cancel{426} + \cancel{426} - 54y - 9y = -15 - 426$$ Add -426 to both sides

$$-63y = -441$$

$$(-1)(-63y) = (-1)(-441)$$ Multiply both sides by (-1)

$$63y = 441$$

$$\frac{\cancel{63}y}{\cancel{63}} = \frac{441}{63}$$ Divide both sides by 63

$$y = \frac{441}{63}$$

$$y = 7$$

Now that we know the value of y we can substitute it into equation (2) and solve for x.

(2) $12x - 9y = -15$ $\qquad y = 7$

$$12x - 9(7) = -15$$

$$12x - 63 = -15$$

$$12x - \cancel{63} + \cancel{63} = -15 + 63$$ Add 63 to both sides

$$12x = -15 + 63$$

$$12x = 48$$

$$\frac{\cancel{12}x}{\cancel{12}} = \frac{48}{12}$$ Divide both sides by 12

$$x = \frac{48}{12}$$

$$x = 4$$

Now let us check our answer by substituting our values of x and y into equation (2) this time.

(2) $12x - 9y = -15$

$$12(4) - 9(7) = -15$$

$$48 - 63 = -15$$

$$-15 = -15$$ Answer checks

(c) - Comparison

Rule: From each of the given equations find the value of the same unknown quantity in terms of the other, and place these values equal to each other.

Given:

(1) $2x + 9y = 71$

(2) $12x - 9y = -15$

First solve equation (1) for x in terms of y.

(1) $2x + 9y = 71$

$$2x + \cancel{9y} - \cancel{9y} = 71 - 9y$$

$$2x = 71 - 9y$$

$$\frac{\cancel{2}x}{\cancel{2}} = \frac{71 - 9y}{2}$$

$$x = \frac{71 - 9y}{2}$$

Second, solve equation (2) for x in terms of y.

(2) $12x - 9y = -15$

$$12x - \cancel{9y} + \cancel{9y} = -15 + 9y$$

$$12x = 9y - 15$$

$$\frac{\cancel{12}x}{\cancel{12}} = \frac{9y - 15}{12}$$

$$x = \frac{9y - 15}{12}$$

Now equate the two values of x.

From equation (1) $x = \frac{71 - 9y}{2}$

From equation (2) $x = \frac{9y - 15}{12}$

$$\frac{71 - 9y}{2} = \frac{9y - 15}{12}$$ Cross multiply

$12(71 - 9y) = 2(9y - 15)$

$852 - 108y = 18y - 30$ Multiply out

$-108y - 18y = -30 - 852$

$-126y = -882$

$(-1)-126y = (-1)-882$ Multiply by (-1)

$126y = 882$

$$\frac{\cancel{126}y}{\cancel{126}} = \frac{882}{126}$$

$$y = \frac{882}{126}$$

$y = 7$

Now we have the value of y.

Substitute this value into either equation (1) or (2) and solve for x.

(1) $2x + 9y = 71$

$2x + 9(7) = 71$ $y = 7$

$2x + 63 = 71$

$2x + \cancel{63} - \cancel{63} = 71 - 63$

$2x = 8$

$$\frac{\cancel{2}x}{\cancel{2}} = \frac{8}{2}$$

$$x = \frac{8}{2}$$

$x = 4$

Now let's check our answer.

Equation (1) $2x + 9y = 71$ $\quad x = 4 \quad y = 7$

$2(4) + 9(7) = 71$

$8 + 63 = 71$

$71 = 71$ Answer checks

C - Graphically

Rule: The common point or points of intersection are the values sought.

1. Plot the equations.

Given:

(1) $2x + 9y = 71$

(2) $12x - 9y = -15$

Step One: By inspection we see both of these equations fit the general type of a straight line equation. Find the x and y intercepts of each equation by first, setting $x = 0$, and find the corresponding value of y. Then set $y = 0$ and find the corresponding value of x.

(1) $2x + 9y = 71$

$x = 0 \qquad 2(0) + 9y = 71$

$9y = 71$

$y = 71/9$

$y = 7.9$

$y = 0 \qquad 2x + 9(0) = 71$

$2x = 71$

$x = 71/2$

$x = 35.5$

Now the x and y intercepts of equation (1) are $y = 7.9$, $x = 35.5$

$$(2) \quad 12x - 9y = -15$$

$$x = 0 \qquad 12(0) - 9y = -15$$

$$-9y = -15$$

$$(-1) - 9y = (-1) -15$$

$$9y = 15$$

$$y = \frac{15}{9}$$

$$y = 1.67$$

$$y = 0 \qquad 12x - 9(0) = -15$$

$$12x = -15$$

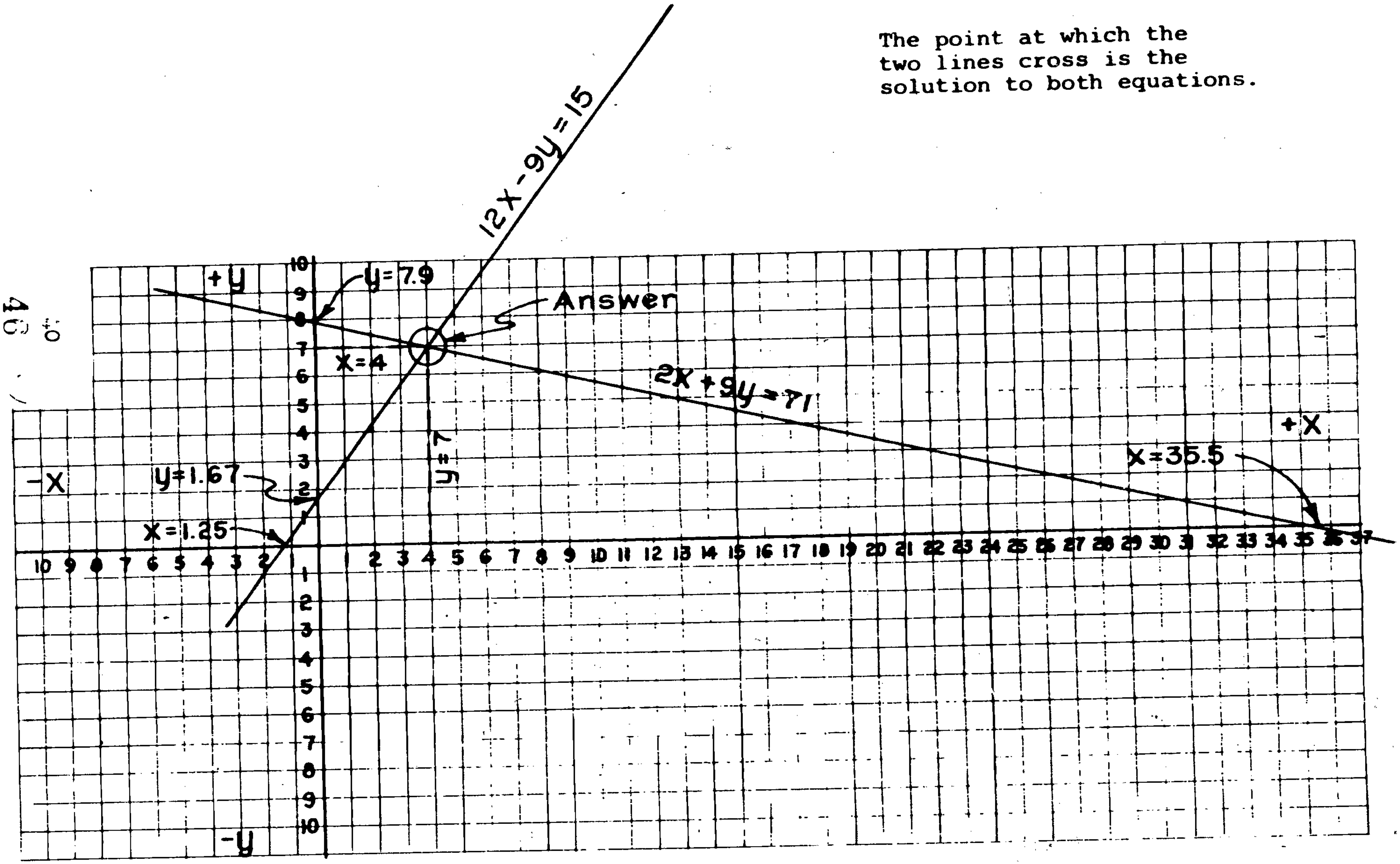
$$x = \frac{-15}{12}$$

$$x = -1.25$$

Now the x and y intercepts of equation (2) are $y = 1.67$, $x = 1.25$

Next, plot the two equations. The point at which the two lines cross is the solution to both equations.

The point at which the two lines cross is the solution to both equations.



PROBLEMS

Solve by both addition and subtraction:

1. $3x - 4y = -2$
   $2x + 4y = 12$

   Answer $x = 2,\ y = 2$

2. $c + 8h = 35$
   $2c - h = 2$

3. $9x - 15s = -27$
   $6x - 7s = -9$

   Answer $x = 2,\ s = 3$

4. $2a + 7b = 5$
   $5a - 9 = -3b$

5. $3/2t + 4/3v = -1$
   $2/3t + 1/4v - 7/12 = 0$

   Answer $t = 2,\ v = -3$

6. $r + s + t = 8$
   $5r + 2s + 2t = 10$
   $6r - 4s + 5t = 11$

   Answer $r = -2$
   $s = 3$
   $t = 7$

Solve by substitution:

7. $2x - y = 1$
   $x + y = 5$

8. $a - 4b = 1$
   $2a - 9b = 3$

   Answer $a = -3,\ b = -1$

9. $3a + 2b = 1$
   $2a - 3b = 5$

10. $x + y = 1$
    $2x + 3(1 - 2x) = 5$

    Answer $x = -1/2,\ y = 3/2$

11. $4a - b = 0$
$3b + 2a = 42$

12. $4y + 6x = -9$
$6y + 4x = -1$

Answer $x = -2.5$, $y = 1.5$

Solve the following by any method:

13. The sum of two numbers is 30 and their difference is 5. Find the numbers.

14. Two men borrowed a total of $90. One man borrowed $5 more than the other. How much did each borrow?

15. The sum of 7 times a smaller number and 3 times a larger number is 46. If 2 times the larger is subtracted from 9 times the smaller, the difference is 24. Find the numbers. Answer 6, 4.

VII - Products and Factoring

"Term", "factor", and "prime factor", have been defined in the first portion of this course. It would be well for the student to review the definition of these words.

In arithmetic we learned the multiplication table to make the solution of everyday problems less complicated. Everyone can see why it is necessary to remember that $4 \times 3 = 12$. Similarly, it is somewhat necessary to remember that the factors of 15 are 3 and 5.

In order to remember these products and factors we had to memorize them.

For students of algebra it is convenient to remember certain special algebraic products.

A - Special Products

1. A Binomial Squared - "Squaring a binomial" means to obtain a product such as $(x + y)(x + y)$ or $(x + y)^2 = (x + y)(x + y)$.

The product is

$$(x + y)^2 = x^2 + 2xy + y^2$$

X and y are any two numbers, and this product should be remembered as a formula.

Stated in words the formula is: The square of a binomial is the sum of the squares of the two terms plus two times their product.

PROBLEMS

By inspection, write the products of the following:

1. $(a + b)^2$ Answer $a^2 + 2ab + b^2$

2. $(2x + 3c)^2$

3. $(w^2y - 1)^2$ Answer $w^4y^2 - 2w^2y + 1$

4. $(6d^3h - 2)^2$

5. 32^2

6. 18^2

2. Product of the Sum and Difference of Two Terms

The product of $(x + y)(x - y) = x^2 - y^2$

Stated in words the formula is: The product of the sum and difference of two terms is equal to the square of the first minus the square of the second.

Multiplied out $(x + y)(x - y) =$

$$\begin{array}{rrr}
 & x & + y \\
 & x & - y \\
\hline
 & -xy & - y^2 \\
x^2 & + xy & \\
\hline
x^2 & & - y^2
\end{array}$$

By inspection, write the products for the following:

1. $(a - b)(a + b)$ Answer $a^2 - b^2$

2. $(x + 4)(x - 4)$

3. $(b^3 - 5)(b^3 + 5)$ Answer $b^6 - 25$

4. $(6 - y)(6 + y)$

5. $(21)(19)$

3. Product of Two Binomials With a Common Term

Consider the product of the two binomials $(x + y)(x + z)$. This product follows the formula "The product of two binomials with one common term equals the square of the common term, plus the product of the common term and the sum of the other two terms, plus the product of the different terms."

Following the formula we have:

Given: Find product of $(x + y)(x + z)$

Square of first term $= x^2$

Product of common term and sum of other two terms $= x(y + z)$

Product of other two terms $= yz$

Answer $= x^2 + x(y + z) + yz$

PROBLEMS

Using the formula, write the following products by inspection:

1. $(z - 2)(z + 3)$ Answer $z^2 + z - 6$

2. $(y + 7)(y - 9)$

3. $(2 - t)(3 - t)$ Answer $6 - 5t + t^2$

4. $(4n + c)(4n + 2c)$

5. $(ab - 5)(ab + 6)$ Answer $a^2b^2 + ab + 30$

6. $(7 + r)(7 + 3r)$

4. Product of Two Binomials With Like Terms

The following product gives a formula for this type:

$$(by + c)(dy + e)$$

$$= bdy^2 + (be + cd)y + ce$$

Example: Multiply $(2y + a)(3y - 2a)$

First Step: Find products of like terms,

$$(2y + a)(3y - 2a)$$

$$(2y)(3y) = 6y^2$$

and $(2y + a)(3y - 2a)$

$$= -2a^2$$

Second Step: Find the middle or cross product term.

$$(2y + a)(3y - 2a)$$

$$(2y)(-2a) + (a)(3y)$$

$$= -4ay + 3ay$$

$$= -ay$$

Third Step: The product is the algegraic sum of the products obtained above:

$$(2y + a)(3y - 2a)$$

$$= 6y^2 - ay - 2a^2$$

PROBLEMS

Using the formula write the following products by inspection:

1. $(2a + 3)(3a - 2)$ Answer $6a^2 + 5a - 6$
2. $(3b + 5)(2b + 7)$
3. $(3x - 5)(2x - 7)$ Answer $6x^2 - 31x + 35$
4. $(5d^2 + 3)(2d^2 - 5)$
5. $(9d + 8r)(2d + 5r)$ Answer $18d^2 + 61dr + 40r^2$
6. $(3by + 8)(4by + 10)$

5. Other Type Products

(a) The formulas for the cube of a binomial are:

1. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

by virtue of

$$\begin{array}{r} x + y \\ \underline{x + y} \\ x^2 + xy \\ \underline{+ xy + y^2} \\ x^2 + 2xy + y^2 \end{array} \quad \text{and} \quad \begin{array}{r} x^2 + 2xy + y^2 \\ \underline{x + y} \\ x^3 + 2x^2y + xy^2 \\ \underline{x^2y + 2xy^2 + y^3} \\ x^3 + 3x^2y + 3xy^2 + y^3 \end{array}$$

2. And Similarly

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

In words the formula is: The cube of a binomial is equal to the sum of the cubes of the two terms, plus three times the second term times the square of the first, plus three times the first term times the square of the second term.

(b) Two other type products are:

$(a + b)(a^2 - ab + b^2) = a^3 + b^3$

$(a - b)(a^2 - ab + b^2) = a^3 - b^3$

PROBLEMS

Write the following products by inspection:

1. $(d - 3)^3$ Answer $d^3 - 9d^2 + 27d - 27$

2. $(2c + d)^3$

3. $(2x^2y - 3d^4)^3$ Answer $8x^6y^3 - 36x^4y^2d^4 + 54x^2yd^8 - 27d^{12}$

4. $(6 - 5t^2)^3$

5. $(x - 4)(x^2 + 4x + 16)$ Answer $x^3 - 64$

6. $(2y^4 + 3d^3)(4y^8 - 6y^4d^3 + 9d^6)$

B. Factoring

1. General

In Part A of this section we found the product of various given factors.

Now we reverse the process and find the factors whose product is a given expression.

Learning and remembering how to factor will depend on the ability to recognize forms of certain type products.

In the following paragraphs under "Factoring" we will say that an expression is completely factored (1) if the factors are prime factors and (2) if no irrational* terms are introduced in the factors.

2. Factoring Common Monomials

It is very common to find that each term of an expression contains the same factor.

When this occurs the expression should be written as a product of the common factor and some other factor. For example:

$$bx + by = b(x + y)$$

*An irrational number or term is one that is not equal to the quotient of any entire two quantities. Examples of irrational numbers are $\sqrt{2}$, $\sqrt{17}$.

In the example it can be seen that b is a common factor, then the expression bx + by can be divided by b. This gives us a handy method for obtaining the other factor when the common factor is known.

The rules for factoring an expression that contains a common monomial factor are:

(a) Write the prime factors of each term and find by observation the common factor.

(b) Divide the original expression by the common factor to get the second factor.

(c) The product of the divisor and quotient (in Step 2) is the factored form.

Example: Factor $24dy - 18xy + 12y^2$

1. Factor each term.

$$24dy = 2\cdot2\cdot2\cdot3\cdot d\cdot y$$
$$18xy = 2\cdot3\cdot3\cdot x\cdot y$$
$$12xy = 2\cdot2\cdot3\cdot y\cdot y$$

The common factor is $2\cdot3\cdot y = 6y$

2. Divide the original expression by the common factor.

$$6y\ \overline{\big|\,24dy - 18xy + 12y^2}$$
$$\quad\ 4d - 3x + 2y$$

3. The product of divisor and quotient in (2) above is the factored form or

$$24dy - 18xy + 12y^2$$
$$= 6y(4d - 3x + 2y)$$

Example: (Where it is possible to factor by inspection) Factor $-12a^3b + 15a^2b^2 - 9a^2b$

1. By observation it is known that $3a^2b$ is the common factor. When the first term of an expression is negative, it is best to show the first term in the parenthesis as positive.

2. Thus the factored form of $-12a^3b + 15a^2b^2 - 9a^2b$ is $-3a^2b(4a - 5b + 3)$.

Factor the following completely:

1. $4x + 4y$ Answer $4(x + y)$

2. $6s - 6t$

3. $4t - 6$ Answer $2(2t - 3)$

4. $-3z^2 + 2z$

5. $x^3y^2 - x^2b^3$ Answer $x^2(xy^2 - b^3)$

6. $2r^2 + 3rd - 5r$

7. $12x^3a - 24x^2a + 42xa^3$ Answer $6xa(2x^2 - 4x + 7a^2)$

3. Factoring Common Binomial Factors

Expressions containing several terms (4 or more) at times can be factored by grouping the various terms that have common monomial factors. It may be found, after grouping, each group in the parentheses has the same expression. This expression then will become a common factor.

Following are two examples of factoring such expressions:

(a) Factor $2dy - ds + 4ry - 2rs$

The first two terms have the common factor d and the last two the common factor $2r$.

The expression becomes:

$$d(2y - s) + 2r(2y - s)$$

Dividing by $2y - s$

$$\begin{array}{r|l} & \quad d + 2r \\ \hline 2y - s & d(2y - s) + 2r(2y - s) \end{array}$$

and since the factors are the product of the divisor and quotient then:

$$2dy - ds + 4ry - 2rs = (2y - s)(d + 2r)$$

(b) Factor $6by^2 + 4cy^2 - 18bdy - 12cdy$

By observation you can see that there is a common factor which is 2y; therefore the expression can be set out as

$$2y(3by + 2cy + 9bd - 6cd)$$

Considering the four terms in the parenthesis, it can be seen that the first two have the common factor y and the second two contain the common factor 3d. Regrouping, the expression becomes

$$2y\left[y(3b + 2c) + 3d(3b - 2c)\right]$$

(c) Factor $3c(r + s)^2 + (r + s)$

Solution: No grouping is necessary, since it can be seen that $r + s$ is the common factor.

Dividing:

$$\begin{array}{r|l} & 3c(r + s) + 1 \\ \hline r + s & 3c(r + s)^2 + (r + s) \end{array}$$

Therefore:

$$3c(r + s)^2 + (r + s) = (r + s)\left[3c(r + s) + 1\right]$$

Or $\quad (r + s)(3cr + 3cs + 1)$

Factor the following:

1. $dr + ds + cr + cs$ Answer $(d + c)(r + s)$
2. $xy + xw - 4x - 4w$
3. $dn - 4n - 2d + 8$ Answer $(d - 4)(n - 2)$
4. $x^3 - x^2y + 3xy^2 - 3y^3$
5. $18a - 12a^2 + 24a^3 - 16a^4$ Answer $2a(4a^2 + 3)(3 - 2a)$

4. Factoring the Difference of Two Squares

In the study of "Special Products" we learned the difference of two squares appears in the form

$$x^2 - y^2 = (x + y)(x - y)$$

or the difference of two square factors is the product of the sum and the difference of the numbers.

Examples

1. Factor $9a^2 - a^4$

Solution: $\sqrt{9a^2} = 3a$

and $\sqrt{a^4} = a^2$

Therefore $9a^2 - a^4 = (3a + a^2)(3a - a^2)$

2. Factor $16r^2 - 36r^2s^2$

a. Take out common factor $4r^2$

$16r^2 - 36r^2s^2 = 4r^2(4 - 9s^2)$

b. The second factor is the difference of two squares and we get,

$16r^2 - 36r^2s^2 = 4r^2(2 + 3s)(2 - 3s)$

3. Factor $4x^4 - 64$

a. Take out common factor 4

$4x^4 - 64 = 4(x^4 - 16)$

and the second factor is the difference of two squares, therefore:

$4x^4 - 64 = 4(x^2 + 4)(x^2 - 4)$

and again the last factor is the difference of two squares so the expression completely factored is,

$4x^4 - 64 = 4(x^2 + 4)(x + 2)(x - 2)$

Factor the following:

1. $x^2 - 1$ Answer $(x - 1)(x + 1)$
2. $5 - 125x^2w^4$
3. $x^4a^6 - 9s^{10}$ Answer $(x^2a^3 - 3s^5)(x^2a^3 + 3s^5)$
4. $(x + y)^2 - 4$

5. $(a - b)^2 - w^2$

6. $(25)^2 - (5)^2$

5. Factoring the Perfect Square Trinomial

In memorizing the special product $(x + y)^2 = x^2 + 2xy + y^2$ we were drilling on the perfect square trinomial.

The student should learn that an expression is of this type if:

a. Two of the terms are perfect squares.

b. The third term is plus or minus twice the product of the square roots of the other two.

To apply the above rules we will inspect and factor an expression.

Example: Is $4a^2 - 12ab + 9b^2$ of this type?

First:

$$\sqrt{4a^2} = 2a, \text{ or } -2a$$

and

$$\sqrt{9b^2} = 3b, \text{ or } -3b$$

Second:

$(2)(2a)(3b) = 12ab$ which is the second term excepting the - sign.

Conclusion: The expression fits the rules and therefore is a perfect square trinomial.

To factor a perfect square trinomial:

a. Obtain the square root of the perfect square terms.

b. Write the square roots, with the second having the sign of the other term (non-perfect square) in the original expression.

c. The factored form then will be the square of the binomial in (b) above.

First Example: Factor $9a^2 + 16b^4 - 24ab^2$

a. (Above)

$$\sqrt{9a^2} = 3a$$

and

$$\sqrt{16b^4} = 4b^2$$

b. The sign of the other term is - so we write

$$(3a - 4b^2)$$

c. The factored form of the original expression

$$9a^2 + 16b^4 - 24ab^2 = (3a - 4b^2)^2$$

Second Example: Factor $4r^2 - 20r + 25$

a. Get square roots;

$$\sqrt{4r^2} = 2r$$

and

$$\sqrt{25} = 5$$

b. Use - sign on second term and factor is $2r - 5$.

Therefore;

$$4r^2 - 20r + 25 = (2r - 5)^2$$

Factor the following expressions:

1. $x^2 + 6x + 9$ Answer $(x + 3)^2$
2. $9y^2 - 12xy + 4x^2$
3. $9n^2 - 18n + 9$ Answer $(3n - 3)^2$

4. $-1 + 12x^2 - 36x^4$

5. $25y^2 + 50y + 100$ Answer $25(y + 2)^2$

6. $4a^2 - c^2 + 4c - 4$

6. Factoring the Sum or Difference of Two Cubes

It should be remembered that

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

and

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

a. Factoring the sum of two cubes:

Example: Factor $a^3 + 8b^3$

First: Find the cube of each term.

$$\sqrt[3]{a^3} = a$$

and

$$\sqrt[3]{8b^3} = 2b$$

The first root is $a + 2b$.

Second: The other factor is the square of the cube root of the first term, minus the product of cube roots plus the square of the second cube root.

Conclusion: The factors are $(a + 2b)$ $(a^2 - 2ab + 4b^2)$

or $a^3 + 3b^3 =$ $(a + 2b)(a^2 - 2ab + 4b^2)$

Example: Factor $81x^3 - 24y^6$

Solution:

First: Remove common monomial factor.

$$3(27x^3 - 8y^6)$$

Second: Find cube roots.

$$\sqrt[3]{27x^3} = 3x$$

and

$$\sqrt{8y^6} = 2y^2$$

Third: The first factor is

$$(3x - 2y^2)$$

Fourth: The second factor is

$$(3x)^2 + (3x)(2y^2) + (2y^2)^2$$

$$= 9x^2 + 6xy^2 + 4y^4$$

Conclusion:

$$27x^3 - 8y^6 = (3x - 2y^2)(9x^2 + 6xy^2 + 4y^4)$$

Or $81x^3 - 24y^6$

$$= 3(3x - 2y^2)(9x^2 + 6xy^2 + 4y^4)$$

Factor the following:

1. $x^3 - 1$ Answer $(x - 1)(x^2 + x + 1)$
2. $27a^3 - 8$
3. $a^3 - 8x^3$ Answer $(a - 2x)(a^2 + 2ax + 4x^4)$
4. $x^6 - 8$

VIII - Fractions and Fractional Equations

A - General

The rules for operation are the same for algebraic fractions as for arithmetic; however, operations with algebraic fractions are more involved. This is because many operations can only be indicated in algebra and not because of different rules.

From arithmetic we have learned that the fraction 4/8 can be reduced to 1/2 by dividing the numerator and denominator by 4, and this does not change the value of the fraction. We also have learned that we can multiply the numerator and the denominator by any number except 0 and not change the value of the fraction.

Example:

Given 4/8, multiply by 2/2

$$\frac{4 \cdot 2}{8 \cdot 2} = \frac{8}{16}$$

If we multiply the numerator and the denominator by the same number, we are in effect multiplying the fraction by 1 and this does not change the value of the fraction. From the above we can state a rule; <u>the value of a fraction is unchanged if the numerator and the denominator are multiplied or divided by the same quantity as long as the quantity is not zero</u>.

<u>B - Simplification</u>

When the numerator and denominator of a fraction are factored and divided by a common factor, it is possible to reduce the fraction to a simpler and more convenient form. When there is no common factor for the numerator and denominator we say the fraction is in its simplest form.

Example 1

Simplify: $\dfrac{2x^2 - x - 3}{2x^2 - 5x + 3}$

Factor the numerator: $2x^2 - x - 3 = (2x - 3)(x + 1)$

Factor the denominator: $2x^2 - 5x + 3 = (2x - 3)(x - 1)$

Rewrite: $\dfrac{(2x - 3)(x + 1)}{(2x - 3)(x - 1)}$

Divide numerator and denominator by the common factor $(2x - 3)$:

$$\dfrac{\overset{1}{\cancel{(2x - 3)}}(x + 1)}{\underset{1}{\cancel{(2x - 3)}}(x - 1)}$$

This is called cancellation. The answer in simplest form is:

$$\frac{x + 1}{x - 1}$$

Example 2

Given: $\dfrac{2a + 2b}{4a^2 - 4b^2}$

Factor: $\dfrac{2(a + b)}{4(a^2 - b^2)} = \dfrac{2(a + b)}{4(a + b)(a - b)}$

Simplify by cancelling common factors.

$$\frac{\overset{1}{\cancel{2}}\overset{1}{\cancel{(a + b)}}}{\underset{2}{\cancel{4}}\underset{1}{\cancel{(a + b)}}(a - b)}$$

Answer: $\frac{1}{2(a - b)}$

PROBLEMS

1. $\frac{35r^3t^2}{7rt} =$

2. $\frac{36ab^3c^2}{60a^4bc^2} = \frac{(3b^2)}{(5a^3)}$

3. $\frac{6a}{2a - 4b} =$

4. $\frac{18x^2}{12x^2 - 24x} = \frac{3x}{2(x - 2)}$

5. $\frac{x^2 - x}{x^2 + x} =$

6. $\frac{2x - 6}{4x - 12} = (1/2)$

7. $\frac{r + 2t}{2r^2 + 4rt} =$

8. $\frac{24a^2b^2 - 54b^4}{8a^2b^2 - 18b^4} = (3)$

9. $\frac{x^2 - 2x - 3}{x^2 + 4x + 3} =$

10. $\frac{4a^3 + 8a^2b}{4a^4 - 16a^2b^2} = \left(\frac{1}{(a - 2b)}\right)$

C - Signs

There are three signs that a fraction may have; one, the sign of the numerator; two, the sign of the denominator; and three, the sign of the fraction. If there is no sign shown it is assumed to be positive.

Rule: Any two signs of a fraction may be changed without changing the value of the fraction.

Example:

$$-\frac{+16}{+8} = -(16 \div 8) = -2$$

Change sign of the fraction and denominator.

$$+\frac{+16}{-8} = +(16 \div -8) = -2$$

Change sign of the fraction and numerator.

$$+\frac{-16}{+8} = +(-16 \div 8) = -2$$

Change sign of the numerator and denominator.

$$- \frac{-16}{-8} = -(-16 \div -8) = -2$$

Therefore:

$$\frac{-16}{8} = \frac{+16}{-8} = + \frac{-16}{8} = - \frac{-16}{-8} = -2$$

Example:

Simplify: $\frac{3x^2 - 9x - 12}{8 + 2x - x^2}$

Rewrite: $\frac{3x^2 - 9x - 12}{-x^2 + 2x + 8}$

Make coefficient of x^2 positive:

$$\frac{3x^2 - 9x - 12}{-(x^2 - 2x - 8)}$$

Factor numerator and denominator:

$$\frac{3(x^2 - 3x - 4)}{-(x^2 - 2x - 8)}$$

$$\frac{3(x - 4)(x + 1)}{-(x - 4)(x + 2)}$$

Cancel common factors:

$$\frac{3\overset{1}{\cancel{(x - 4)}}(x + 1)}{-\underset{1}{\cancel{(x - 4)}}(x + 2)}$$

Answer: $\frac{-3(x + 1)}{(x + 2)}$

Complete the following by inserting the proper value in the parenthesis:

1. $\frac{-6}{7} = - \frac{-6}{(\ )} = (\ )\frac{-6}{(\ )} = (\ )\frac{6}{(\ )} = (\ )\frac{-6}{-7}$

2. $\frac{2}{-7} = \frac{-2}{(\ )} = (\ )\frac{2}{7} = (\ )\frac{-2}{7} = -\frac{(\ )}{-7}$ Answer (+7, -, +, -2)

3. $\dfrac{a}{a-b} = (\ )\dfrac{-a}{a-b} = (\ )\dfrac{a}{b-a} = (\ )\dfrac{-a}{b-a}$

Simplify:

4. $\dfrac{5-5x}{x-1}$ = Answer -5

5. $-\dfrac{2-a}{7a-14}$

6. $\dfrac{1-a}{a^2-1}$ = Answer $-\dfrac{1}{a+1}$

7. $-\dfrac{m-t}{at-am}$

8. $\dfrac{t-2}{(2-t)^3}$ = Answer $\dfrac{-1}{(t-2)^2}$

9. $\dfrac{b^2+ab-6a^2}{2a^2+ab-b^2} =$

10. $\dfrac{b^2+ab-6a^2}{2a^2+ab-b^2}$ = Answer $-\dfrac{3a+b}{a+b}$

D - Multiplication

The product of two or more fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.

Example:

$$\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f} = \frac{ace}{bdf}$$

Or $\dfrac{2}{5}\cdot\dfrac{4}{6}\cdot\dfrac{7}{3} = \dfrac{56}{90} = \dfrac{28}{45}$

Example:

$$\frac{12a-4}{3a^2-10a+3}\cdot\frac{a^2-2a-3}{12a-6}$$

Factor: $\dfrac{4(3a-1)}{(3a-1)(a-3)}\cdot\dfrac{(a-3)(a+1)}{6(2a-1)}$

Multiply: $\dfrac{4(3a-1)(a-3)(a+1)}{6(3a-1)(a-3)(2a-1)}$

Divide: $\dfrac{\overset{2}{\cancel{4}}(\overset{1}{\cancel{3a-1}})(a-\overset{1}{\cancel{3}})(a+1)}{\underset{3}{\cancel{6}}(\underset{1}{\cancel{3a-1}})(\underset{1}{a-3})(2a-1)}$

Leave answer in factored form:

$$\frac{2(a+1)}{3(2a-1)}$$

Find the products:

1. $\dfrac{21}{48} \cdot \dfrac{32}{35} =$

2. $\dfrac{15x}{8y} \cdot \dfrac{4y}{5x} =$

3. $\dfrac{7}{22} \cdot \dfrac{143}{91} =$ Answer 1/2

4. $\dfrac{8bx}{15ay} \cdot \dfrac{5by}{4ax} =$ Answer $\dfrac{2b^2}{3a^2}$

5. $\dfrac{6a}{7c} \cdot \dfrac{5c^3}{8b} \cdot \dfrac{14b^2}{3a^2} =$

6. $\dfrac{34m^4n^3}{39ab^2} \cdot \dfrac{91a^2b^5}{119m^2} =$ Answer $\dfrac{2ab^3m^2n^3}{3}$

7. $\dfrac{6a-3}{10} \cdot \dfrac{15}{2-4a} =$

8. $\dfrac{x^2-y^2}{x^3} \cdot \dfrac{3x}{x+y} =$ Answer $\dfrac{3(x-y)}{x^2}$

9. $\dfrac{4r^2-t^2}{r^2-4t^2} \cdot \dfrac{r-2t}{t-2r} =$

10. $\dfrac{a^2-2a-8}{a^2+a-2} \div \dfrac{a^2-3a+2}{2a^2-5a+2} =$ Answer $\dfrac{a-4}{2a-1}$

E - Division

Find quotient of $\dfrac{a}{b} \div \dfrac{c}{d} = Q$

Or

$$Q = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \quad \begin{array}{l}\text{Numerator}\\ \text{Denominator}\end{array}$$

Multiply the numerator and denominator by the same thing or $\frac{d}{c}$

$$Q = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}}$$

We have not changed the value of the fraction because

$$\frac{\frac{d}{c}}{\frac{d}{c}} = 1$$

Now we reduce

$$Q = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{\cancel{c}}{\cancel{d}} \cdot \frac{\cancel{d}}{\cancel{c}}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Now again with numbers:

Find the quotient of $\frac{4}{8} \div \frac{1}{2}$

$$Q = \frac{\frac{4}{8}}{\frac{1}{2}}$$

Multiply numerator and denominator by $\frac{2}{1}$

$$Q = \frac{\frac{4}{8} \cdot \frac{2}{1}}{\frac{1}{2} \cdot \frac{2}{1}}$$

We know $\frac{2}{1} = 2$ so

$$\frac{\frac{2}{1}}{\frac{2}{1}} = \frac{2}{2} = 1$$

Therefore we have not changed the value.

Simplify:

$$Q = \frac{\frac{4}{8} \cdot \frac{2}{1}}{\cancel{\frac{1}{2}} \quad \cancel{\frac{2}{1}}} = \frac{4}{8} \cdot \frac{2}{1} = \frac{8}{8} = 1$$

<u>Rule of Thumb</u>: Invert the divisor and multiply.

Now a more complex example:

$$Q = \frac{8a^2 + 8ab}{2a - b} \div \frac{4a^2 - 4b^2}{4a^2 - b^2}$$

Rewrite:

$$Q = \frac{\frac{8a^2 + 8ab}{2a - b}}{\frac{4a^2 - 4b^2}{4a^2 - b^2}}$$

Invert and multiply:

$$Q = \frac{\frac{8a^2 + 8ab}{2a - b} \cdot \frac{4a^2 - b^2}{4a^2 - 4b^2}}{\cancel{\frac{4a^2 - 4b^2}{4a^2 - b^2}} \cdot \cancel{\frac{4a^2 - b^2}{4a^2 - 4b^2}}}$$

$$Q = \frac{8a^2 + 8ab}{2a - b} \cdot \frac{4a^2 - b^2}{4a^2 - 4b^2}$$

Factor:

$$Q = \frac{8a(a + b)}{2a - b} \cdot \frac{(2a + b)(2a - b)}{4(a + b)(a - b)}$$

Cancel common factors:

$$Q = \frac{\overset{2}{\cancel{8}}a\cancel{(a + b)}(2a + b)\cancel{(2a - b)}}{\cancel{4}\cancel{(2a - b)}(a + b)(a - b)}$$

Answer: $Q = \frac{2a(2a + b)}{(a - b)}$

Find the quotient:

1. $\frac{9}{16} \div \frac{3}{8} =$

2. $15 \div \frac{25}{18} =$ Answer $\frac{54}{5}$

3. $\frac{21a^5}{10b^3c} \div \frac{28a^2}{5bc^2} =$

4. $12m^3 \div \frac{24m}{33t^2} =$ Answer $\frac{33m^2t^2}{2}$

5. $\dfrac{36m^2}{7t} \div \dfrac{1}{3m} =$

6. $\dfrac{3x + 3}{4x - 12} \div \dfrac{9x + 9}{8x - 24} =$ Answer $\dfrac{2}{3}$

7. $(15x - 10) \div \dfrac{4x - 6}{6} =$

8. $\dfrac{a^2 - 16}{a - 2} \div \dfrac{a + 4}{4 - a^2}$ = Answer $-(a + 2)(a - 4)$

9. $\dfrac{x^2 + 6x + 8}{x^2 + x - 6} \div \dfrac{x^2 + 3x - 4}{x^2 - 5x + 6} =$

10. $\dfrac{m^2 + 6m + 9}{m^2 - 4m - 5} \div \dfrac{m^2 + 2m - 3}{m^2 - 2m - 15}$ = Answer $\dfrac{(m + 3)^2}{(m + 1)(m - 1)}$

F - Addition and Subtraction

We have learned from arithmetic how to add and subtract like fractions.

Example: $\dfrac{1}{2} + \dfrac{1}{2} = \dfrac{2}{2} = 1$

$\dfrac{3}{4} - \dfrac{1}{4} = \dfrac{2}{4} = \dfrac{1}{2}$

We also learned to add and subtract unlike fractions. To do this we must get the lowest common denominator. (L.C.D.)

$$\frac{3}{2} + \frac{1}{7} = ?$$

The L.C.D. is $2 \cdot 7 = 14$

Therefore, the first term is $\dfrac{21}{14}$

The second term will be $\dfrac{2}{14}$

So: $\dfrac{21}{14} + \dfrac{2}{14} = \dfrac{23}{14}$

Now let us apply this to algebraic terms.

Given: $\dfrac{3}{2x - 1} - \dfrac{2}{3x - 4} = ?$

In this case the L.C.D. will be

$$(2x - 1)(3x - 4)$$

The first term has (2x - 1), it needs (3x - 4). Now multiply the numerator and the denominator by (3x - 4). This does not change its value.

$$\frac{(3x - 4)}{3x - 4} \cdot \frac{3}{2x - 1} = \frac{3(3x - 4)}{(3x - 4)(2x - 1)}$$

Now the second term has (3x - 4), it needs (2x - 1). Again multiply the numerator and the denominator by (2x - 1) and the value is unchanged.

$$\frac{2x - 1}{2x - 1} \cdot \frac{2}{3x - 4} = \frac{2(2x - 1)}{(3x - 4)(2x - 1)}$$

Rewrite:

$$\frac{3(3x - 4)}{(3x - 4)(2x - 1)} - \frac{2(2x - 1)}{(3x - 4)(2x - 1)} = ?$$

Now we have like fractions so we can proceed as in arithmetic.

$$\frac{3(3x - 4) - 2(2x - 1)}{(3x - 4)(2x - 1)} = ?$$

Simplify:

$$\frac{9x - 12 - 4x + 2}{(3x - 4)(2x - 1)} = \frac{5x - 10}{(3x - 4)(2x - 1)} =$$

$$\frac{5(x - 2)}{(3x - 4)(2x - 1)}$$

PROBLEMS

Addition and Subtraction

Combine and Simplify:

1. $\frac{3}{4a} + \frac{5}{12a} =$

2. $\frac{x}{y} + \frac{y}{x} =$ Answer $\frac{x^2 + y^2}{xy}$

3. $\frac{3a - 2}{3} + \frac{2a - 1}{6} =$

4. $\frac{2x - 7}{25} + \frac{5 - x}{45} =$ Answer $\frac{13x - 38}{225}$

Note: Any number may be considered as having a denominator of 1 that is $7 = \frac{7}{1}$ and $x = \frac{x}{1}$.

5. $2a + 1 - \frac{5}{a} =$

6. $a - 1 - \frac{a^2}{a + 1} =$ Answer $-\frac{1}{a + 1}$

7. $\frac{-b^2}{x^2 - b^2} - \frac{b}{x - b} =$

8. $\frac{3b}{3a + 2b} - \frac{2a}{3a - 2b} =$ Answer $\frac{5ab - 6a^2 - 6b^2}{(3a + 2b)(3a - 2b)}$

9. $\frac{5}{6y - 3x} - \frac{3}{2x - 4y} =$

10. $\frac{2 - 11x}{2 - x} + \frac{x - 4}{2x - 4} =$ Answer $\frac{23x - 8}{2(x - 2)}$

G - Complex Fractions

Now we come to more complex problems where we apply what we have learned in addition and subtraction and multiplication and division.

Example:

Given: $$\frac{\frac{a - b}{b} - \frac{b - a}{a}}{\frac{a^2 - b^2}{b^2} - \frac{b^2 - a^2}{a^2}} = Q$$

Combine factors in the numerator.

$$\frac{a - b}{b} - \frac{b - a}{a} = \frac{a(a - b)}{ab} - \frac{b(b - a)}{ab} = \frac{a^2 - ab - b^2 + ab}{ab} = \frac{a^2 - b^2}{ab}$$

Combine factors in the denominator.

$$\frac{a^2 - b^2}{b^2} - \frac{b^2 - a^2}{a^2} = \frac{a^2(a^2 - b^2)}{a^2b^2} - \frac{b^2(b^2 - a^2)}{a^2b^2} =$$

$$\frac{a^4 - a^2b^2 - b^4 + a^2b^2}{a^2b^2} = \frac{a^4 - b^4}{a^2b^2}$$

Now rewrite: $$\frac{\frac{a^2 - b^2}{ab}}{\frac{a^4 - b^4}{a^2b^2}} = Q$$

Invert and multiply:

$$\frac{a^2 - b^2}{ab} \cdot \frac{a^2b^2}{a^4 - b^4} = \frac{\cancel{a^2 - b^2}}{\cancel{ab}} \cdot \frac{\overset{a\,b}{\cancel{a^2b^2}}}{(a^2 + b^2)\cancel{(a^2 - b^2)}}$$

Answer $= \frac{ab}{(a^2 + b^2)}$

PROBLEMS

Complex Fractions

1. $\dfrac{5 - \dfrac{3}{5}}{4 - \dfrac{7}{10}} =$

2. $\dfrac{2 + \dfrac{1}{12}}{4 - \dfrac{2}{3}} =$ Answer $\dfrac{5}{8}$

3. $\dfrac{\dfrac{1}{a} + \dfrac{1}{b}}{\dfrac{1}{a} - \dfrac{1}{b}} =$

4. $\dfrac{c + \dfrac{4}{c} - 4}{\dfrac{1}{3} - \dfrac{2}{3c}} =$ Answer $3(c - 2)$

5. $\dfrac{1 - \dfrac{1 - a}{1 + a}}{a} =$

6. $\dfrac{r^2 - 4t^2}{1 + \dfrac{3t}{r - t}} =$ Answer $(r - t)(r - 2t)$

H - Equations with Fractions

The first step in solving equations with fractions is to multiply all the terms by the L.C.D. This will give an equation (called the derived equation) which will contain no fractions. If the equation is cleared of fractions by multiplying both sides by the L.C.D., the solutions of the derived equation are also the solutions of the original equation provided the solution does not make the denominator zero.

Example 1:

Given: $\dfrac{1}{x} + \dfrac{3}{2x} - \dfrac{5}{6} = \dfrac{35}{12x}$

The L.C.D. is 12x.

Multiply both sides by the L.C.D.

$$12x\left(\frac{1}{x} + \frac{3}{2x} - \frac{5}{6}\right) = 12x\left(\frac{35}{12x}\right)$$

$$\frac{12x}{x} + \frac{36x}{2x} - \frac{60x}{6} = \frac{12x(35)}{12x}$$

Simplify:

$$\frac{12\cancel{x}}{\cancel{x}} + \frac{\overset{18}{\cancel{36x}}}{\cancel{2x}} - \frac{\overset{10}{\cancel{60}}x}{\cancel{6}} = \frac{\cancel{12x}(35)}{\cancel{12x}}$$

Rewrite: $12 + 18 - 10x = 35$

Rewrite to leave the x term on one side and the numbers on the other.

$$-10x = 35 - 12 - 18$$

$$-10x = 35 - 30$$

$$-10x = 5$$

$$\frac{-\cancel{10}x}{-\cancel{10}} = \frac{5}{-10}$$

$$x = -\frac{1}{2}$$

Check. Does $x = -\frac{1}{2}$?

$$\frac{1}{(-\frac{1}{2})} + \frac{3}{2(-\frac{1}{2})} - \frac{5}{6} = \frac{35}{12(-\frac{1}{2})}$$

$$-2 - 3 - \frac{5}{6} = \frac{-35}{6}$$

Use the L.C.D. of 6.

$$\frac{-12}{6} - \frac{18}{6} - \frac{5}{6} = \frac{-35}{6}$$

$\frac{-35}{6} = \frac{-35}{6}$ Checks. Yes, $x = -\frac{1}{2}$

Example 2:

Given: $\frac{x - 3}{x - 1} = \frac{x^2 - 9x + 20}{x^2 + x - 2}$ [(Note that) $x^2 + x - 2 = (x - 1)(x + 2)$]

Solution: L.C.D. $= (x - 1)(x + 2)$

Multiply both sides by the L.C.D.

$$(x - 1)(x + 2)\frac{(x - 3)}{(x - 1)} = (x - 1)(x + 2)\frac{(x^2 - 9x + 20)}{(x - 1)(x + 2)}$$

Simplify:

$$(x - 1)(x + 2)\frac{(x - 3)}{\cancel{(x - 1)}} = (x - 1)(x + 2)\frac{(x^2 - 9x + 20)}{(x - 1)(x + 2)}$$

$$(x + 2)(x - 3) = x^2 - 9x + 20$$

$$x^2 - x - 6 = x^2 - 9x + 20$$

$$x^2 - x^2 - x + 9x = 20 + 6$$

$$8x = 26$$

$$x = \frac{26}{8} = \frac{13}{4}$$

PROBLEMS

Equations with Fractions

1. $\frac{x}{3} + \frac{x}{6} = 18$

2. $\frac{y}{4} - \frac{3}{2} = \frac{y}{6}$ (Answer $y = 18$)

3. $\frac{10 - m}{2} - \frac{5 - m}{3} = \frac{5}{2}$

4. $\frac{1}{3x} = 3$ (Answer $x = \frac{1}{9}$)

5. $\frac{v - 7}{v + 2} = \frac{1}{4}$

6. $\frac{9}{x - 5} = \frac{10}{x - 3}$ (Answer $x = 23$)

7. $\frac{x + 3}{x - 3} = \frac{x - 3}{x + 3}$

8. $\frac{w - 2}{w + 3} = \frac{w - 1}{w + 5}$ (Answer $w = 7$)

Statement Problems

Note: (Rate)(time) = distance

9. A small plane averages 120 mph on a trip from Indianapolis to Kansas City. With a tail wind on the return trip the plane averaged 160 mph. If the round trip took 7 hours, what is the distance from Indianapolis to Kansas City?

10. A man estimates that a tractor could plow a field in 2 days and a team could do it in 6 days. How long would it take to plow the field if the tractor and team were both used? (Answer = 1½ days) (Hint: find rate.)

11. A 2-inch pipe can fill a tank in 6 hours, and a 4-inch pipe can fill the tank in 1.5 hours. (a) How long will it take for both to fill the tank? (b) For both to fill it two-thirds full?

12. On a river which flows at the rate of 3 miles per hour, a motor boat set at a given rate takes as long to go 4 miles upstream as it does to go 6 miles downstream. What is the rate of the motor boat in still water? (Answer 15 mph)

13. One pipe can fill a tank in 10 minutes. It will fill the tank in 35 minutes when the drain is open. If the tank if half full, how long would it take to drain it?

IX - Radicals

A - Changing the Form of Radicals

By definition $\sqrt[n]{a} = a^{\frac{1}{n}}$, so it is evident that any changes in radicals must follow the laws of exponents.

1. The radical may be changed by removing a perfect power from under the radical.

(A perfect power factor is a factor whose indicated root may be expressed without a radical.)

	No. 1	No. 2
Example:		
Simplify:	$\sqrt{12}$	$\sqrt[3]{24x^3y^8}$

Factor the radical so one factor is a perfect power factor.

$\sqrt{4 \cdot 3}$ $\qquad$ $\sqrt[3]{8x^3y^6 \cdot 3y^2}$

Extract the root of the perfect power factor.

$2\sqrt{3}$ $\qquad$ $2xy^2\sqrt[3]{3y^2}$

2. The radical may be changed by introducing a factor under the radical.

	No. 1	No. 2
Example:		
Given:	$3\sqrt{2}$	$2a\sqrt[3]{2ab^2}$

Raise the coefficient of the radical to the power of the index.

Index is 2 | Index is 3

so $3^2 = 9$ | so $(2a)^3 = 8a^3$

Write the quantity as a factor of the radicand.

$3\sqrt{2} = \sqrt{3^2 \cdot 2}$

$2a\sqrt[3]{2ab^2} = \sqrt[3]{8a^3 \cdot 2ab^2}$

or $\sqrt{9 \cdot 2} = \sqrt{18}$ or $\sqrt[3]{16a^4b^2}$

3. The radical may be changed by reducing the index of the radical.

Given: $\sqrt[6]{16x^4}$

Express the radical in exponential form.

$\sqrt[6]{16x^4} = (16x^4)^{\frac{1}{6}} = \left[(2x)^4\right]^{\frac{1}{6}}$

Rewrite:

$(2x)^{\frac{4}{6}}$ *

Reduce exponent: $(2x)^{\frac{2}{3}}$

Rewrite radical: $\sqrt[3]{(2x)^2} = \sqrt[3]{4x^2}$

*Stated in words this means we must take the number to the fourth power then take the sixth root of the expanded number.

PROBLEMS

Changing the Form of Radicals

Remove all perfect factors from the radical.

1. $\sqrt{27} =$
2. $\sqrt{40} =$ (Answer $2\sqrt{10}$)
3. $\sqrt[3]{54} =$
4. $\sqrt[3]{-256} =$ (Answer $-4\sqrt[3]{4}$)
5. $\sqrt{18x} =$

Write the coefficient of the radical as part of the radicand.

6. $2\sqrt{3} =$

7. $3\sqrt[3]{4} =$ (Answer $\sqrt[3]{108}$)

8. $c\sqrt{2} =$

9. $2m\sqrt{5m} =$ (Answer $\sqrt{20m^3}$)

10. $2a^2b\sqrt[3]{4ab^2} =$

Reduce the index.

11. $\sqrt[4]{4x^2} =$ (Answer $\sqrt{2x}$)

12. $\sqrt[6]{9x^2} =$

13. $\sqrt[8]{16m^4} =$ (Answer $\sqrt{2m}$)

14. $\sqrt[6]{(a^2 - b^2)^3} =$

B - Rationalizing the Denominator

To rationalize the denominator, apply the rules learned from fractions.

Remember: $\sqrt{2} \cdot \sqrt{2} = 2$

(The square root times the square root gives the number.)

Example:

Rationalize: $\dfrac{6}{\sqrt{21}}$

Multiply numerator and denominator by $\sqrt{21}$.

$$\frac{6}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{\overset{2}{\cancel{6}}\sqrt{21}}{\underset{7}{\cancel{21}}} = \frac{2\sqrt{21}}{7}$$

Divide by 3

PROBLEMS

Simplify:

1. $\dfrac{1}{\sqrt{2}} =$

2. $\dfrac{2}{\sqrt{6}} =$ (Answer $\dfrac{\sqrt{6}}{3}$)

3. $\frac{4a}{\sqrt{2a}} =$

4. $\sqrt{\frac{1}{5}} =$ (Answer $\frac{\sqrt{5}}{5}$)

5. $\sqrt{\frac{3}{5}} =$

6. $\sqrt{\frac{2c}{3x}} =$ (Answer $\frac{\sqrt{6cx}}{3x}$)

7. $\sqrt{3x^{-1}} =$

8. $\sqrt{8c^{-2}} =$ (Answer $\frac{2\sqrt{2}}{c}$)

X - Quadratic Equations

We have studied linear or first degree equations, e.g. $ax + b = 0$; this equation has but one root, $x = -\frac{b}{a}$.

Now we shall study quadratic or second degree equations, e.g. $ax^2 + bx + c = 0$. This equation has two roots, both real or both imaginary, which may be found by the following methods.

1. Factoring

2. Completing the square

3. Formula

1. Factoring

Given: $2x^2 - x - 3 = 0$

(a) What two terms multiplied together will give $2x^2$?

Try x and 2x

Check $(x)(2x) = 2x^2$. Now we have the first terms $(x \pm ?)$ and $(2x \pm ?)$

(b) What two numbers multiplied together will give -3?

Try +3 - 1

Check $(+3)(-1) = -3$. Now we have a trial second term $(x + 3)$ and $(2x - 1)$

Check: $\begin{array}{r} 2x - 1 \\ \underline{x + 3} \\ 2x^2 - x \\ \underline{+ 6x - 3} \\ 2x^2 + 5x - 3 \end{array} \neq 2x^2 - x - 3$

NO GOOD

Now let us try -3 and +1.

From inspection we know we have more of a problem than + or -. We must change our arrangement.

Try $(2x - 3)$ and $(x + 1)$

Now check: $\begin{array}{r} 2x - 3 \\ \underline{x + 1} \\ 2x^2 - 3x \\ \underline{+ 2x - 3} \\ 2x^2 - x - 3 \end{array} = 2x^2 - x - 3$

OK

Answer $x = \frac{3}{2}$, $x = -1$

PROBLEMS

Solve by factoring:

1. $x^2 - 12 = x$ Answer $x = 4, -3$
2. $8x^2 = 6x$
3. $6x^2 = 216$ Answer $x = \pm 6$
4. $4x^2 = 25$
5. $9x^2 - 5 = 0$ Answer $x = \pm \frac{\sqrt{5}}{3}$
6. $3x^2 = 4$
7. $\frac{x}{2} - \frac{10}{9x} = 0$ Answer $x = \pm \frac{2\sqrt{5}}{3}$
8. $25x^2 = 4(5x - 1)$

9. $A = \pi r^2$, solve for r   Answer $r = \sqrt{\frac{A}{\pi}}$

10. $V = \frac{1}{3} \pi r^2 h$, solve for h

2. Completing the square

Given: $2x^2 - x - 3 = 0$

(a) Simplify and transpose terms.

$$2x^2 - x = 3$$

(b) Divide by coefficient of x^2

$$\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x}{2} = \frac{3}{2}$$

$$x^2 - \frac{1}{2}x = \frac{3}{2}$$

(c) Complete the square by adding the square of one half the coefficient of x to both sides.

$$x^2 = \frac{1}{2}x + \left(\frac{1}{2} \cdot -\frac{1}{2}\right)^2 = \frac{3}{2} + \left(\frac{1}{2} \cdot -\frac{1}{2}\right)^2$$

Simplify: $x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 = \frac{3}{2} + \left(-\frac{1}{4}\right)^2$

Simplify: $x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{3}{2} + \frac{1}{16}$

$$x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{24}{16} + \frac{1}{16}$$

$$x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{25}{16}$$

We have created a perfect square for the left hand term.

Factor: $x^2 - \frac{1}{2}x + \frac{1}{16} = 0; \left(x - \frac{1}{4}\right)\left(x - \frac{1}{4}\right) = \left(x - \frac{1}{4}\right)^2$

Rewrite: $\left(x - \frac{1}{4}\right)^2 = \frac{25}{16}$

Extract $\sqrt{\ }$: $\sqrt{\left(x - \frac{1}{4}\right)^2} = \sqrt{\frac{25}{16}} = \frac{\sqrt{25}}{\sqrt{16}} = \pm\frac{5}{4}$

$$x - \frac{1}{4} = \pm\frac{5}{4}$$

$$x = +\frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{2}{3}$$

$$x = -\frac{5}{4} + \frac{1}{4} = \frac{-4}{4} = -1$$

Answer $x = \frac{3}{2}$ and $x = -1$

PROBLEMS

Determine the constant term which should be added to the expression to make a perfect square trinomial.

1. $x^2 + 4x + (\ )$ Answer $= 4$
2. $x^2 - x + (\ )$
3. $x^2 + 10x + (\ )$ Answer $= 25$
4. $x^2 - 9x + (\ )$
5. $x^2 - \frac{6}{5} + (\ )$ Answer $\frac{9}{25}$
6. $a^2 + 16a + (\ )$

Solve for the unknown by completing the square.

7. $x^2 - 2x - 8 = 0$ Answer $x = 4, -2$
8. $x^2 + 4x - 21 = 0$
9. $x^2 + x = 12$ Answer $x = 3, -4$
10. $2y^2 + 5y = 12$
11. $6a^2 = 11a + 10$ Answer $a = \frac{5}{2}, -\frac{2}{3}$
12. $w^2 - 8w + 4 = 0$

3. Formula

Derivation:

By completing the square.

Given $ax^2 + bx + c = 0$

(a) Simplify and transpose terms.

$$ax^2 + bx = -c$$

(b) Divide by coefficient of x^2

$$\frac{\cancel{a}x^2}{\cancel{a}} + \frac{bx}{a} = \frac{-c}{a}$$

(c) Complete the square by adding the square of one half the coefficient of the x term to both sides.

$$x^2 + \frac{b}{a}x + \left(\frac{1}{2}\frac{b}{a}\right)^2 = -\frac{c}{a} + \left(\frac{1}{2}\frac{b}{a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

(d) Rewrite:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-c4a}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

(e) Extract the square root of both sides.

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

(f) Simplify:

$$x = \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

PROBLEMS

Solve the following using the quadratic formula:

1. $x^2 + 4x - 21 = 0$ Answer = -7, 3
2. $2x^2 - 10x + 7 = 0$
3. $3a^2 + 20a + 12 = 0$ Answer $-\frac{2}{3}$, -6
4. $5a^2 - 28a + 32 = 0$
5. $9n^2 + 22n + 13 = 0$ Answer $-\frac{13}{9}$, -1

XI - Ratio, Proportion, and Variation

A - Ratio

When we say the ratio of a to b we mean the quotient $a \div b$ or $\frac{a}{b}$. Thus a ratio can be expressed as a fraction and written $\frac{a}{b}$, a/b, or a:b. Because a ratio can be written as a fraction, the rules for fractions apply to ratios.

Fundamentally, every measurement is a ratio. When we say the length of a room is 16 feet we mean the ratio of the length of the room to the unit of measure, 1 foot is 16:1. This is the ratio of units of the same kind. There are ratios of different units also, such as distance to time (this ratio is called average speed.) If a car travels 80 miles in 2 hours, the ratio 80 miles:2 hours, or 40 miles:1 hour, (usually written 40 mi/hr) is the average speed.

The ratios of certain line segments are so important in trigonometry that they are given names (sin, cos, and tan).

Ratios are used to solve problems. In particular, they offer a convenient method for converting dimensions from one kind of unit to another, provided we consider the units as algebraic quantities.

Example 1: Convert 8 yards to inches.

Solution: We know 3 feet = 1 yard. Now divide both sides of the equation by 1 yard.

$$3 \text{ ft} = 1 \text{ yd}$$

$$\frac{3 \text{ ft}}{1 \text{ yd}} = \frac{1 \text{ yd}}{1 \text{ yd}} = 1$$

Now, if we multiply any expression by $\frac{3 \text{ ft}}{1 \text{ yd}}$ we are multiplying the expression by 1 so its value does not change.

$$8 \text{ yd} = 8 \cancel{\text{yd}} \times \frac{3 \text{ ft}}{1 \cancel{\text{yd}}} = 24 \text{ ft}$$

We can cancel yards just as we can cancel a's in the expression.

$$8\cancel{a} \cdot \frac{3b}{\cancel{a}} = 24b$$

Thus we consider the dimensions as algebraic symbols and operate accordingly. Since 12 inches = 1 foot, we can perform in a similar manner.

$$8 \text{ yd} = 24 \cancel{\text{ft}} \times \frac{12 \text{ in}}{1 \cancel{\text{ft}}} = 288 \text{ in}$$

It should be noted that in addition to the numerical computations, we included all the units. We then considered these units as algebraic factors and canceled the same units in the numerator and denominator. Thus we see that when we wish to change an expression from one kind of unit to another, we set up ratios which are equal to one and multiply the given expression by these ratios. Each ratio is so determined that the dimension we wish to eliminate will cancel. The preceding example can be solved in one step.

$$y \text{ in} = 8 \text{ yd} = 8 \cancel{\text{yd}} \times \frac{3 \cancel{\text{ft}}}{1 \cancel{\text{yd}}} \times \frac{12 \text{ in}}{1 \cancel{\text{ft}}} = 288 \text{ in}$$

RATIO PROBLEMS

1. Change 891 inches to feet.

2. Convert 3 miles to meters. (1" = 2.54 centimeters and 1 meter = 100 centimeters) Answer = 4827 meters

3. A man runs 90 yards in 9 seconds. What was his average speed in (1) yards per second; (2) in miles per hour; (3) in feet per second?

4. A cubic foot of water weighs 62.4 pounds. How much does a cubic inch weigh? Answer = 0.0361 pound

B - Proportion

A proportion is an equation which states that two ratios are equal. Thus it is a relation in which two fractions are equal.

Example: $\frac{a}{b} = \frac{c}{d}$

This is also written a:b = c:d or a:b::c:d and is read "a is to b as c is to d". Because a proportion is an equation, the rules of equations apply to proportions.

Example:

Solve: 15:8::5:x

Rewrite: $\frac{15}{8} = \frac{5}{x}$

Simplify:

$(\cancel{8})(x)\ \frac{15}{\cancel{8}} = \frac{5}{\cancel{x}}\ (8)(\cancel{x})$ Multiply both sides by (8)(x)

Rewrite: $15x = 40$

Simplify:

$\frac{\cancel{15}x}{\cancel{15}} = \frac{40}{15}$ Divide both sides by 15

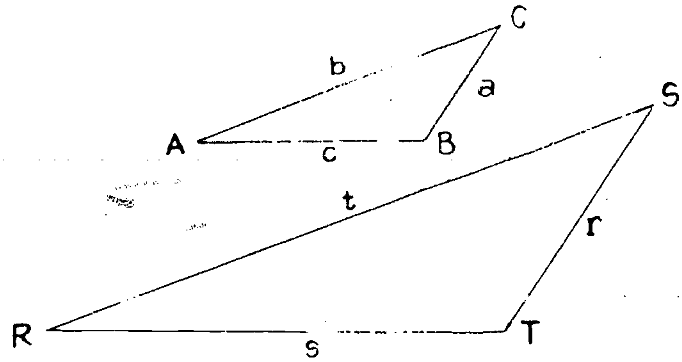
$x = \frac{40}{15}$

Reduce: $x = \frac{8}{3}$

Applications

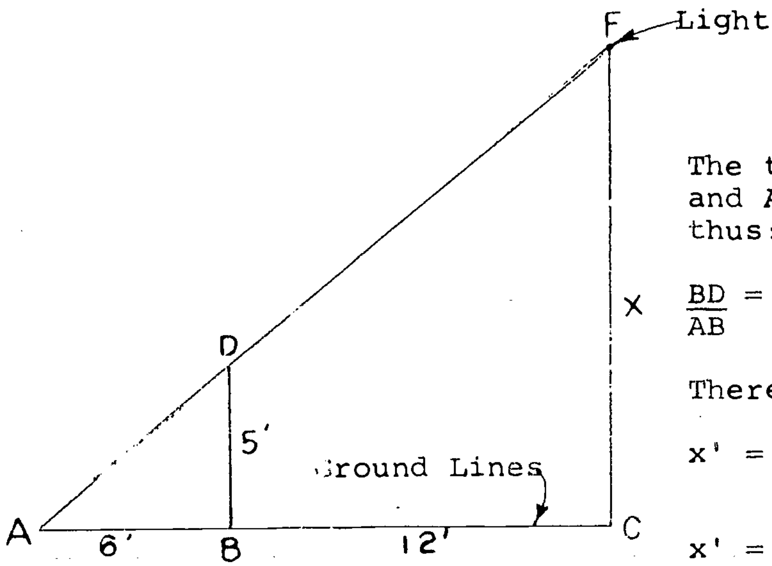
1. Similar geometric figures - If two figures are similar, the corresponding sides are proportional.

Example: Triangles ABC and RST are similar.



$\frac{a}{c} = \frac{r}{s}$; $\frac{a}{b} = \frac{r}{t}$; $\frac{b}{t} = \frac{c}{s}$

Example: A post 5 feet high is located 12 feet from a point directly below a street light. If the post casts a shadow 6 feet long. What is the height of the light above the ground?



The triangles ABD and ACF are similar thus:

$\frac{BD}{AB} = \frac{CF}{AC}$ or $\frac{5'}{6'} = \frac{x'}{18'}$

Therefore

$x' = \frac{5'}{6'} (18')$

$x' = 15$ feet

Example: The diameter of one sphere is 2 inches, and that of another is 4 inches. Compare their volumes.

Solution: (Volume of sphere $= \frac{1}{6}\pi d^3$)

Let v = volume of sphere with 2-inch diameter.

Let V = volume of sphere with 4-inch diameter.

Thus: $\frac{V}{v} = \frac{(4)^3}{(2)^3} = \frac{64}{8} = 8$

Therefore, the volume of the larger sphere is eight times that of the smaller sphere.

PROBLEMS

Solve for y.

1. 10:7 = 5:y
2. 5:y = 10:3 Answer y = 1.5
3. 7:3 = y:6
4. 6:y = 12:5 Answer y = 2.5
5. 12:5 = 3:y
6. 10:y = 12:$\frac{1}{4}$ Answer y = $\frac{5}{24}$
7. (2y - 3):4 = 3:(3y - 5)
8. 15:0.4::y:1.2 Answer y = 45
9. 27:y = y:3
10. (y + 4):(y - 2) = (2y - 1):3(y - 4) Answer y = -10, y = 5

Application problems.

1. Given two similar triangles. The sides of one triangle are 6, 8, and 10. The shortest side of the other triangle is 21. Find the other two sides.

2. A tree casts a shadow of 18 feet when the length of a shadow of a yardstick is 2 feet. What is the height of the tree?

3. The diagonal of a square is 12 inches. Find the length of the side of a square whose area is 4 times the area of the given square.

4. The altitude of a triangle is 3 feet and its base is 4 feet. Find the dimensions of a similar triangle with twice the area.

C - Variation

1. Direct Variation

When it is said y varies as x, y varies directly as x, or y is proportional to x, we can write

$$y = kx \text{ (k is a constant)}$$

We also can write $\frac{y}{x} = k$

Therefore we see in direct variation the variables are so related that their ratio is always constant. The constant k is called the constant of variation, or constant of proportionality. We can find k if we know y which corresponds to a value of x ($x \neq 0$).

Example: The distance traveled by a body falling from rest varies as the square of the time. If a body falls 64 feet in 2 seconds, how far will it fall in 5 seconds?

Solution:

Let s = distance the object falls

t = time the object is falling

We know: $s = kt^2$

Given s = 64 when t = 2

Substitute into formula:

$$64 = k(2)^2$$

$$64 = k4$$

$$k = \frac{64}{4}$$

$$k = 16$$

Thus: $s = 16t^2$

To find s when $t = 5$ we have:

$$s = 16t^2$$

$$s = 16(5)^2$$

$$s = 16(25)$$

$$s = 400 \text{ feet}$$

<u>Problems - Direct Variation</u>

Write each of the following as an equation:

1. The area of a semicircle A varies as the square of its radius r.

2. The kinetic energy, KE, of a particle varies as the square of its velocity, V.

Solve for the unknown:

3. It is known that y varies as x. If $y = 15$ when $x = 3$, find y when $x = 30$. (Find k before solving for unknown) Answer: $k = 5$, $y = 150$

4. Given that y varies as the square root of x. If $y = \frac{1}{2}$ when $x = 4$, determine y when $x = 36$.

State each of the following equations in the language of variation:

5. $C = 2\pi r$

6. $P = I^2R$ (P is power, I is current, R is resistance)

7. $v = \sqrt{2gh}$ (g is a constant)

2. Inverse Variation

When it is said y varies inversely as x, or y is inversely proportional to x, we can write

$$y = \frac{k}{x} \text{ (k is the constant of proportion.)}$$

From the above formula we see that as x increases y decreases, and as x decreases y increases.

Example: The volume of a given quantity of gas, at constant temperature, varies inversely as the pressure. A certain amount of gas occupies 20 cubic feet when the pressure is 16 pounds per square inch. What is the pressure when the volume is reduced to 5 cubic feet if the temperature is constant?

Let p = pressure

v = volume

Express as a formula:

$$v = \frac{k}{p}$$

Determine k.

Given $v = 20\ ft^3$ when $p = 16\ lb/in^2$

$$v = \frac{k}{p}$$

$$20\ ft^3 = \frac{k}{16\ lb/in^2}$$

$$k = 20\ ft^3\ (16\ lb/in^2)$$

$$k = 320\ \frac{ft^3 lb}{in^2}$$

Rewrite formula:

$$v = \frac{320\ \frac{ft^3 lb}{in^2}}{p}$$

Solve for p when $v = 5\ ft^3$

$$5\ ft^3 = \frac{320\ ft^3 lb}{p\ \ in^2}$$

$$p = \frac{\overset{64}{\cancel{320}}\ \cancel{ft^3} lb}{\cancel{5}\ \cancel{ft^3}\ in^2}$$

$$p = 64\ lb/in^2$$

Problems - Inverse Variation

Write the following statement as an equation:

1. For a given area the base of a triangle varies inversely as the altitude.

Solve for the unknowns:

2. Given that y varies inversely with the square of s. If $y = 25$ when $s = 2$, determine y when $s = 5$. (Before solving for unknown, find k.) (Answer $k = 100$, $y = 4$)

3. The quantity x is inversely proportional to the cube of t. If $x = 1$ when $t = 6$, determine x when $t = 2$.

State each of the following equations in the language of variation:

4. $I = \frac{k}{d^2}$

5. $p = \frac{k}{v}$

6. $f = \frac{k}{\sqrt{c}}$

3. Joint Variation

When we say s varies jointly as x and y, we may write

$$s = kxy$$

Example: The force of the wind on a surface, at right angles to the directions of the wind, varies jointly as the area of the surface and the square of the wind velocity. If it was found that a wind velocity of 10 miles an hour exerts a force of 8.4 pounds on a rectangular door whose dimensions are 3 by 7 feet, what will be the force on the door when the wind velocity is 30 miles an hour.

Solution:

Let f = force on surface
v = wind velocity
a = area of surface

Write the formula:

$$f = kav^2$$

Solve for k.

Given $f = 8.4\ lb$
$a = 21\ ft^2$
$v = 10\ mi/hr$

Substitute:

$$8.4\ lb = k(21\ ft^2)(10\ \frac{mi}{hr})^2$$

$$8.4\ lb = k(21\ ft^2)(100\ \frac{mi^2}{hr^2})$$

$$k = 0.004\ \frac{lb\ hr^2}{ft^2\ mi^2}$$

Rewrite the formula:

$$f = (0.004\ \frac{lb\ hr^2}{ft^2\ mi^2})av^2$$

Solve for f: When $a = 21\ ft^2$, $v = 30\ mi/hr$

$$f = (0.004\ \frac{lb\ hr^2}{ft^2\ mi^2})(21\ ft^2)(30\ \frac{mi}{hr})^2$$

$$f = (0.004\ \frac{lb\ \cancel{hr^2}}{\cancel{ft^2}\ \cancel{mi^2}})(21\ \cancel{ft^2})(900\ \frac{\cancel{mi^2}}{\cancel{hr^2}})$$

$$f = (0.004\ lb)(21)(900)$$

$$f = 75.6\ lb$$

Problems - Joint Variation

Write each of the following statements as an equation:

1. The area of a regular pyramid (a) varies jointly as the perimeter of the base (p), and the slant height (s).

2. The area of a spherical segment varies jointly as the radius and altitude.

Before solving for the unknown in the following problems, determine k.

3. The quantity x varies jointly as w and t. If $x = 1$ when $w = 10$ and $t = 100$, find t when $x = 50$ and $w = 3000$. (Answer $k = 0.001$, $t = 16\frac{2}{3}$)

4. The quantity s varies jointly as b^2 and h^2. If $s = 30,000$ when $b = 2$ and $h = 10$, find s when $b = 4$ and $h = 5$.

Make statements of the following in the language of variation:

5. $t = kvp$

6. $r = kx^2y^3$

4. Combined Variation

In many of the applications the relations actually involve several different types of variation. Thus, Newton's Law of Gravitation states that the force of attraction F, between two bodies varies jointly as their masses M and m, and inversely as the square of the distance, d, between their centers of gravity. This Law is an example of a combination of joint and inverse variation which can be written symbolically.

$$F = \frac{kMm}{d^2}$$

Example: The approximate crushing load for a square pillar varies directly as the fourth power of the thickness and inversely as the square of the height. If a 2-inch square wooden post 5 feet high is crushed by a weight of 16 tons, what weight will crush a pillar 10 feet high and 4 inches square?

Solution:

Let l = crushing load
d = length of sides (thickness)
h = height of pillar

Write the formula:

$$l = \frac{kd^4}{h^2}$$

Solve for k. Given $l = 16$, $h = 5$, and $d = 2$

$$16 = \frac{k(2)^4}{(5)^2}$$

$$16 = \frac{k(16)}{(25)}$$

$$k = \frac{25(\cancel{16})}{\cancel{16}}$$

$$k = 25$$

Therefore $l = \dfrac{25d^4}{h^2}$

Find l when $d = 4$ in and $h = 10$ ft

$$l = \frac{25(4)^4}{(10)^2}$$

$$l = \frac{25(256)}{100}$$

$$l = \frac{6400}{100}$$

$$l = 64 \text{ tons}$$

Problems - Combined Variation

Write each of the following statements as an equation:

1. T varies jointly as x and y and inversely as s.

2. The effective resistance R of two resistances, r_1 and r_2 in parallel, varies jointly as their product and inversely as their sum.

After determining k, solve for the unknowns:

3. The quantity Q varies directly as t and inversely as v^2. If Q = 18 when t = 10 and v = 2, find Q when t = 15 and v = 3. (Answer k = 7.2, Q = 12)

4. The quantity H varies jointly as m and t and inversely as the cube of L. If H = 32 when m = 4, t = 6, and H = 3, find l when H = 4, m = 8, and t = 3.

Write the following in the language of variation:

5. $v = \frac{kt}{p}$

6. $T = \frac{k\sqrt{I}}{v}$

Section 2 - Geometry

INDEX

GEOMETRY

I - Introduction

What is Geometry?

Geometry is that branch of mathematics which investigates the relations, properties, and measurement of solids, surfaces, lines, and angles. It deals with the theory of space and of figures in space.

Why is Geometry Important?

Where does geometry fit into the picture?

After the student learns to count and use arithmetic, then algebra and geometry form the foundation of mathematics.

Algebra supplies the language, rules, and methods for calculations; geometry develops the forms and methods of reasoning in mathematics and gives the rules and methods for measurement.

Geometry for our work is divided into two parts: 1 - Plane, and 2 - Solid.

(1) Plane Geometry

All the lines and figures which can be drawn on a plane, or flat surface, are called plane figures, and that portion of geometry which treats of their construction, relations, descriptions, and measurements is called Plane Geometry.

(2) Solid Geometry

All the lines, figures, and objects which can be drawn or formed in space without restriction to a plane are called solids, and that portion of geometry which treats of the construction, relations, description, and measurement of solids is called Solid Geometry.

II - BASIC GEOMETRICAL DEFINITIONS

Before beginning the study of geometry, fundamental definitions of certain terms must be known. These are:

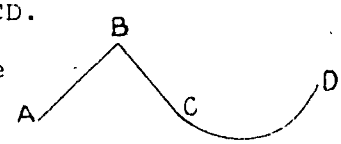
1. Point - A position which has no measurement other than its reference location; i.e., it has no length, width, or depth.

2. Line - The connecting link between separated points. It has no width or depth, but length only.

(a) A straight line is the shortest line which can join two points. The term "line", in general, infers a straight line.

(b) A curved line, or curve, is a line no portion of which is straight.

(c) Lines are usually designated by letters at the termini (ends) of their individual straight or curved segments such as the line ABCD. The first segment of this would be line $\overline{AB}$; the second segment would be line $\overline{BC}$; the third segment would be line $\overline{CD}$.



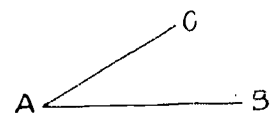
3. Plane - A plane surface is such a surface that the straight line joining any two points in it lies entirely in the surface. In ordinary terms a plane surface is a "flat" surface and has no thickness.

4. Curved Surface - A surface no portion of which is a plane.

5. Plane Figure - A figure formed by points and straight or curved lines, all of which lie in a plane.

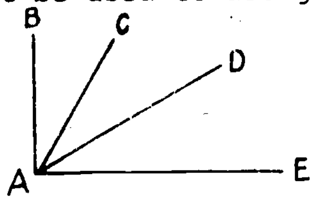
6. Angle - The figure formed by two straight lines meeting at or being drawn from a point. The two lines are called the sides and the point of intersection the vertex.

"AC" and "AB" are the sides. They meet at the vertex "A". The whole figure is the angle "CAB" or "BAC". For brevity this is written as $\measuredangle$ CAB or $\measuredangle$ BAC.



If there is but one angle at the point it may also be referred to as $\angle$ A. If three or more lines meet at a point, the three letters must be used to designate the various angles.

In this case we could have
$\angle$ EAD, $\angle$ DAC, $\angle$ CAB,
$\angle$ BAD, $\angle$ CAE, etc.



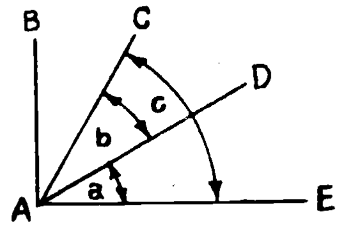
Or, the angles can be shown with small letters between the sides and an arc joining the sides used, such as:

$\angle$ a = $\angle$ EAD

$\angle$ b = $\angle$ DAC

$\angle$ c = $\angle$ CAE

etc.

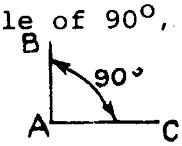


7. <u>Degree (°)</u> - The unit of measurement of angular magnitude. There are 360 degrees in a circle or complete angle.

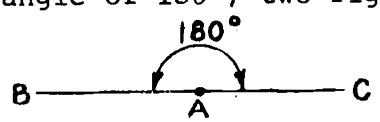
<u>Minute (')</u> - 1/60th of a degree

<u>Second (")</u> - 1/60th of a minute

8. <u>Right Angle</u> - An angle of 90°, one-fourth of a circle.

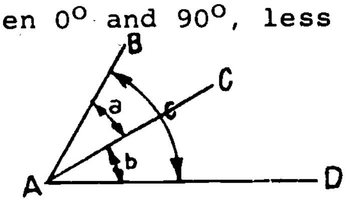


9. <u>Straight Angle</u> - An angle of 180°; two right angles; a straight line.



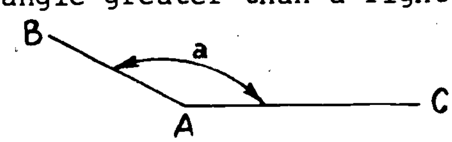
10. <u>Acute Angle</u> - An angle between 0° and 90°, less than a right angle.

All these (a, b, c) are acute angles.



11. <u>Obtuse Angle</u> - An angle greater than a right angle.

Angle "a" is obtuse.



12. Complementary Angle - When the sum of two angles is 90°, or a right angle, they are said to be complementary angles and each is the complement of the other.

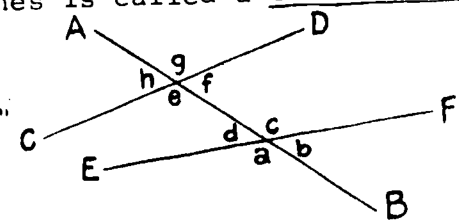
13. Supplementary Angle - When the sum of two angles is 180° they are said to be supplementary and each is the supplement of the other.

14. Perpendicular - A line drawn from a point to another line, in such a manner that the intersection forms two right angles, is said to be a perpendicular to the second line. A perpendicular is used only with reference to something. That is, a building can be vertical by itself, but it is perpendicular to the ground only if the ground is level or horizontal. The sign $\perp$ is used for the word "perpendicular".

15. Bisector - If a line divides an angle into two equal parts it is said to bisect the angle and is called the bisector. Or, if one line crosses another at its middle point the first is said to be the bisector of the second.

16. Parallel Lines or Parallels - Straight lines which are in the same plane and never meet, however far they are extended, are called "parallel lines" or "parallels". The symbol $\parallel$ is used for the words "parallel" or "is parallel to".

17. Transversal - A straight line which cuts any two, or more, other straight lines is called a transversal. Thus, AB is a transversal of CD and EF. In the eight angles formed the angles between the two lines cut are the "interior" angles (c,d,e,f); those lying outside are "exterior" angles (a,b,h,g).



These are related and distinguished in pairs as follows: "d" and "f" or "c" and "e" are alternate-interior angles; "a" and "g" or "b" and "h" are alternate-exterior angles; "a" and "e", "d" and "h", "b" and "f", or "c" and "g" are corresponding angles.

18. Curvilinear - A closed plane figure composed of a curved line or lines is sometimes called a curvilinear figure.

19. Rectilinear - A closed figure composed of straight lines is sometimes called a rectilinear figure.

The straight lines forming a rectilinear figure are called its sides, the angles formed by the interceding pairs of sides are the angles of the figure, with those inside the closed figure being the interior angles. The sum of the lengths of the sides is the perimeter.

20. Triangle - A rectilinear figure of three sides.

21. Quadrilateral or Quadrangle - A rectilinear figure of four sides.

22. Polygon - A rectilinear figure of "several" sides.

A polygon with equal angles and equal sides is known as a "regular" polygon.

23. Equilateral Triangle - An equal-sided triangle.

24. Isosceles Triangle - A triangle with two equal sides.

25. Scalene Triangle - A triangle with no two sides equal.

26. Right Triangle - A triangle with one of its angles being a right angle. The side opposite the Rt. ∡ is the hypotenuse.

27. Acute Triangle - A triangle with all of its angles acute.

28. Obtuse Triangle - A triangle with an obtuse angle.

29. Congruent Triangle - Triangles that are equal side for side and angle for angle.

The two sides of a triangle forming an angle are said to <u>include</u> that angle; each of these two sides is said to be <u>adjacent</u> to the angle; the third side is said to be <u>opposite</u> the angle.

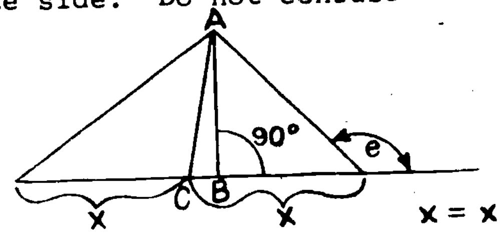
30. <u>Similar Triangles</u> - Triangles with angles equal each to each.

31. <u>Altitude</u> - The perpendicular drawn from a vertex to the side opposite, or to its extension.

32. <u>Median</u> - A line drawn from a vertex to the mid-point of the opposite side. Do not confuse with the altitude, thus:

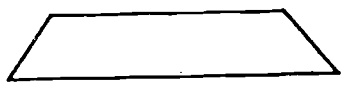
AB is an altitude

AC is a median



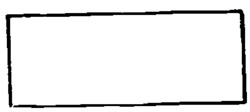
33. <u>Exterior Angle</u> - The angle formed by extending one of the sides through a vertex. (Angle "e" above.)

34. <u>Trapezoid</u> - A quadrilateral with two, and only two, of its sides parallel.



35. <u>Parallelogram</u> - A quadrilateral with its opposite sides parallel in pairs.

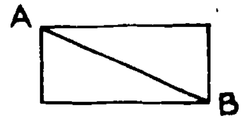
36. <u>Rectangle</u> - A parallelogram with all its angles being right angles.



37. <u>Square</u> - An equal-sided rectangle.

38. Diagonal - A line drawn between opposite vertices of a quadrilateral.

(i.e. line AB)



39. Altitude - A line perpendicular to the parallel sides of a trapezoid.

40. Circle - A curvilinear figure formed by a line every point of which is equally distant from one and the same point inside the figure, called the center.

41. Circumference - The length of the curved line forming a circle.

42. Radius - A straight line joining the center and any point on the circle, also used to mean the length of the radius.

43. Diameter - A straight line having its two ends on the circle and passing through the center, also used to mean the length of the diameter.

44. Chord - A straight line segment inside the circle with its ends on the circle. (A diameter is a chord.)

45. Secant - Any straight line which intersects a circle at two points.

46. Arc - A portion of a circle, such as the portion between the two ends of a chord, or the two points of intersection of a secant.

47. Tangent - A straight line touching a circle at one and only one point (the point of tangency.)

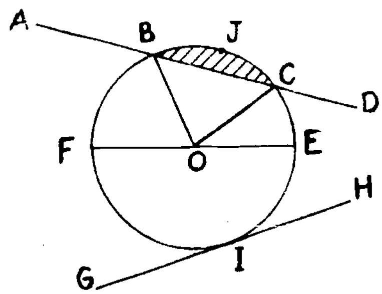
$\overline{FO}$, $\overline{OE}$, $\overline{OB}$ and $\overline{OC}$ are radii

$\overline{EF}$ is a diameter

$\overline{AD}$ is a secant

$\overline{BC}$ and $\overline{FE}$ are chords

$\overset{\frown}{BJC}$ is an arc



$\overline{GH}$ is a tangent, I is the point of tangency.

48. Segment - The figure formed by an arc and its subtended chord (shaded figure page 2-7).

49. Central Angle - The angle formed by two radii ($\angle$ BOC)

50. Sector - The figure formed by an arc of a circle and the central angle it subtends (the segment $\overset{\frown}{BC}$ plus the triangle BOC form a segment on page 2-7) or the figure COE is a segment.

51. Unit Circle - A circle whose radius R = 1 unit. The type of unit (foot, yard, or mile), is inconsequential when considering formula for the circle.

52. π (Pi) - The ratio of the circumference of a circle to its diameter.

Solid Geometry Definitions

53. Rectilinear Solid - A solid formed by plane surfaces.

54. Rectangular Solid - A solid with rectangular faces (the plane surfaces forming a solid).

55. Cube - A rectangular solid with square faces.

56. Properties of a Solid - The characteristic feature and the relation between dimensions, areas, and volumes.

57. Pyramid - A solid with a polygonal base and triangular faces all meeting at a common vertex.

58. Prism - A rectilinear solid, two of whose faces are polygons and the rest trapezoids or parallelograms.

59. Cylinder - A solid formed by rotating a rectangle about one of its axes.

60. Cone - A solid formed by rotating a right triangle about one of its legs.

61. Sphere - A solid formed by rotating a circle about its diameter.

62. Frustum, of a Cone - Formed by a cone with the top or apex cut off by a plane parallel to the base.

III - Axioms and Propositions

Axioms and propositions are next in the study of geometry because they establish principles and put forth certain truths to be demonstrated and certain operations to be performed.

(A) stands for Axiom which is a statement admitted to be true without proof.

(P) stands for Proposition which is a statement that is proposed for proof, construction, or discussion.

Following is a list of axioms and propositions that the student should become familiar with:

1(A) Things equal to the same thing or to equal things are equal to each other.

Example: $a = 2$, and $b = 2$, so $2 = 2$

therefore $\therefore$ $a = b$

2(A) If equals are added to equals the sums are equal.

Example: $a = 2$, $b = 2$ $\therefore$ $a + b = 2 + 2 = 4$

$x = y$, $a = b$, and $y = a$ $\therefore$ $x + a = y + b$

3(A) If equals are taken from equals the remainders are equal.

Example: Similar to (2).

4(A) If equals are added to or taken from unequals, the results are unequal in the same order as at first.

Example: $a = b$, $3 \neq 1$ and is larger

$a + 3 \neq b + 1$ and is larger

5(A) The doubles or any equal multipliers of equals are equal, and those of unequals are unequal in the same order.

Example: $x = y$, $2x = 2y$ $\quad$ $x = 3$, $y = 1$

$(2)(3) \neq (2)(1)$

6(A) The halves or any equal parts of equals are equal and those of unequals are unequal in the same order as at first.

Example: Similar to 5(A).

7(A) Equal powers and roots of equals are equal.

$$x = y \therefore x^2 = y^2 \text{ and } \sqrt{x} = \sqrt{y}$$

8(A) The whole is greater than any of its parts.

Example: $6 = 3+2+1 \therefore 6 > 3;\ 6 > 2;\ 6 > 1$

9(A) The whole is equal to the sum of its parts.

10(A) In any mathematical operation, anything may be substituted in the place of its equal.

Example: If $xy = 4$ and $x = y$ then $(x)(x) = 4$

or $(y)(y) = 4$

Geometrical

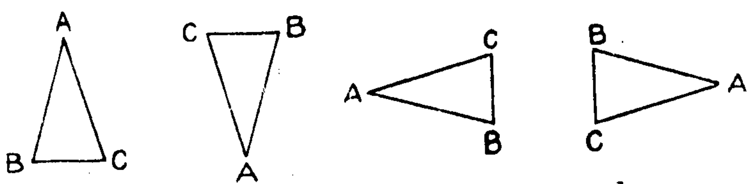
11(A) Through any two points in space there can be one and only one straight line - see Definitions.

12(A) A straight line may be produced to any length.

13(A) A circle may be described with any given point as center and with any given radius.

14(A) Motion of a geometrical figure does not change its size or shape.

Example:



All the above triangles are exactly equal in size, shape, and area.

15(A) All right angles are equal. See Definitions.

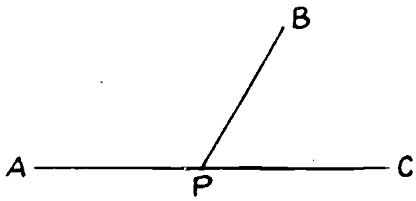
16(A) One and only one parallel to a given straight line can be drawn through a point outside that line. By definition.

17(A) At a given point in a line there can be but one perpendicular to that line. By definition.

18(P) If two adjacent angles have their exterior sides in a straight line, the angles are supplementary.

Example:

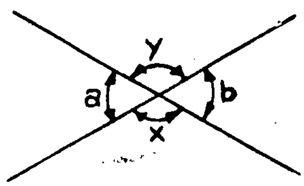
Since $\angle$ APB and $\angle$ BPC forma straight line they are supplementary by definition.



19(P) If two straight lines intersect, the OPPOSITE [redacted] angles are equal.

Example:

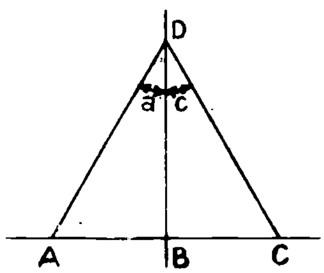
$\angle a = \angle b$ and $\angle x = \angle y$



20(P) Two straight lines drawn from a point in a perpendicular to a given straight line, and cutting off on the given line equal segments from the foot of the perpendicular, are equal in length and make equal angles with the perpendicular.

Example:

$\overline{DB}$ is $\perp$ to $\overline{AC}$ and $\overline{AB} = \overline{BC}$
then $\overline{AD} = \overline{DC}$ and $\angle a = \angle c$



21(P) Any point on the perpendicular bisector of a line is equal distance from the two ends of the line.

Example: Use figure above.

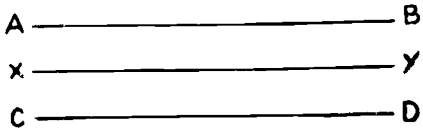
22(A) Only one perpendicular to a line can be drawn from a point outside that line and it is the shortest line that can be drawn from the point to the given line.

Example: By definitions.

23(A) Two straight lines in the same plane parallel to a third line are parallel to each other.

Example:

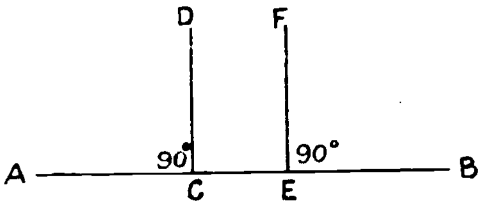
If AB is || to XY and CD is || to XY, then AB must be || to CD by definition.



24(A) Two straight lines in the same plane perpendicular to the same line are parallel.

Example:

$\overline{CD}$ is || to $\overline{EF}$



25(P) If two parallel lines are cut by a transverse line, (a) the alternate interior angles are equal; (b) the corresponding angles are equal; and (c) the two interior or exterior angles on the same side of the transverse line are supplementary.

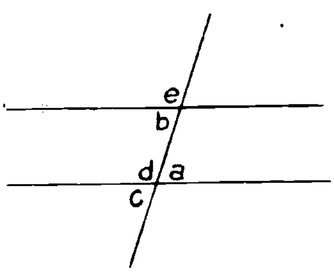
Example:

(a) $\measuredangle a = \measuredangle b$

(b) $\measuredangle c = \measuredangle b$

(c) $\measuredangle b + \measuredangle d = 180^\circ$,

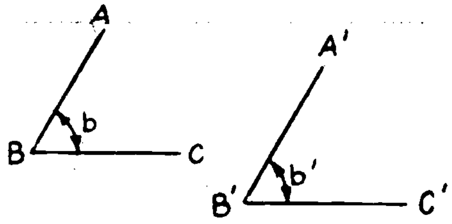
$\measuredangle c + \measuredangle e = 180^\circ$



26(P) If the sides of two angles are parallel, each to each, the angles are equal.

Example:

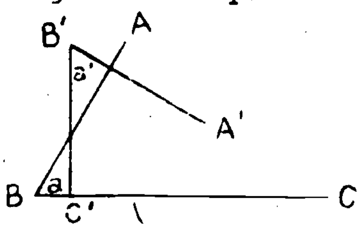
With $AB \parallel A'B'$
and $BC \parallel B'C'$
Then $\angle b = \angle b'$



27(P) If the sides of two angles are perpendicular, each to each, the angles are equal.

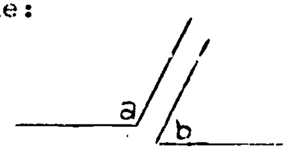
Example:

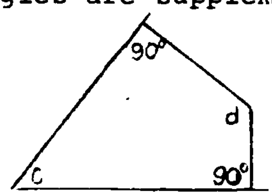
$A'B' \perp AB$
$B'C' \perp BC$
Then $\angle a = \angle a'$



28(P) If the sides of an acute angle are parallel or perpendicular, each to each, to those of an obtuse angle, the angles are supplementary.

Example:



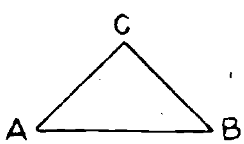


$\angle a + \angle b = 180^\circ$ $\angle c + \angle d = 180^\circ$

29(A) The sum of the angles of a triangle equals a straight angle.

Example:

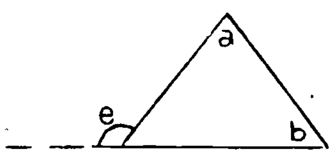
$\angle A + \angle B + \angle C = 180^\circ$



30(P) An exterior angle of a triangle equals the sum of the two opposite interior angles.

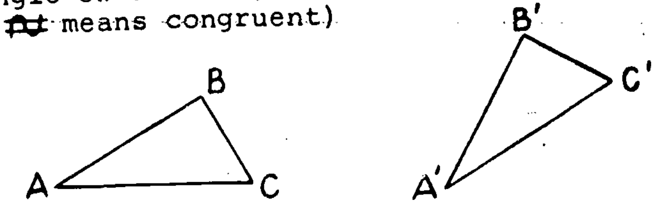
Example:

$\angle e = \angle a + \angle b$



31(P) If two sides and the included angle of one triangle are equal to two sides and the included angle of another, the triangles are congruent. ($\cong$ means congruent)

Example:



$\overline{AB} = \overline{A'B'}$; $\overline{AC} = \overline{A'C'}$ $\quad$ $\angle A = \angle A'$ $\therefore$ $\triangle$s are $\cong$

32(P) If two angles and the included side of one triangle equal two angles and the included sides of another, the triangles are congruent.

Example: Above figures $\angle A = \angle A'$; $\angle B = \angle B'$;

$\overline{AB} = \overline{A'B'}$ $\therefore$ $\triangle$s are $\cong$

33(P) If a side and any two angles of one triangle are equal to the corresponding side and any two angles of another, the triangles are congruent.

Example: Same figures. $\overline{AB} = \overline{A'B'}$ $\angle B = \angle B'$

$\angle c = \angle C'$ $\therefore$ $\triangle$s are $\cong$

34(P) If three sides of one triangle are equal to the three sides of another, the triangles are congruent.

Example: Same figures. $\overline{AB} = \overline{A'B'}$; $\overline{BC} = \overline{B'C'}$;

$\overline{CA} = \overline{C'A'}$ $\therefore$ $\triangle$s are $\cong$

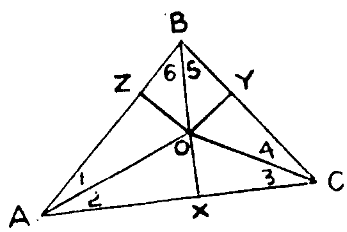
35(P) The three bisectors of the angles of a triangle meet in one point which is equidistant from the three sides.

Example: $\angle 1 = \angle 2$;

$\angle 3 = \angle 4$; $\angle 5 = \angle 6$

Pt. 0 is common to all bisectors

$\overline{OX} = \overline{OY} = \overline{OZ}$



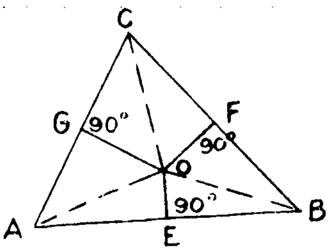
36(P) The three perpendicular bisectors of the sides of a triangle meet in one point which is equidistant from the three vertices.

Example:

$\overline{AE} = \overline{EB}$; $\overline{BF} = \overline{FC}$; $\overline{CG} = \overline{GA}$

Pt. "O" is common

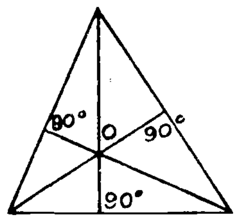
Then $\overline{OA} = \overline{OB} = \overline{OC}$



37(P) The three altitudes of a triangle meet at one point.

Example:

Pt. "O" is common



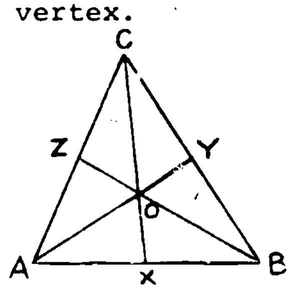
38(P) The three medians of a triangle meet in one point whose distance from each vertex is two thirds the median from that vertex.

Example:

$\overline{AX} = \overline{XB}$; $\overline{BY} = \overline{YC}$; $\overline{CZ} = \overline{ZA}$

Then Pt. "O" is common

$\overline{CO} = \frac{2}{3}\ \overline{CX}$; $\overline{AO} = \frac{2}{3}\ \overline{AY}$; $\overline{BO} = \frac{2}{3}\ \overline{BZ}$

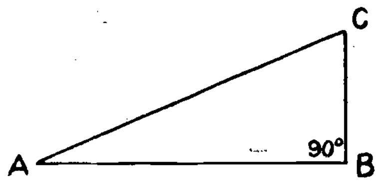


39(P) In an equilateral triangle the altitudes, medians, angle - bisectors and perpendicular - bisectors of the sides all coincide and concur at a point two-thirds the distance from each vertex to the opposite side.

Example: (Combination of last three propositions applied to an equilateral triangle.)

40(P) <u>The Theorem of Pythagoras</u> - The square of the hypotenuse of a right triangle equals the sum of the squares of the legs.

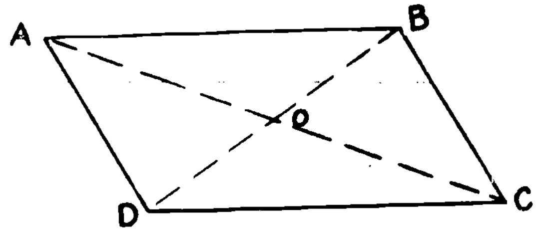
Example: $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$



41(P) The opposite sides of a parallelogram are equal.

Example: $\overline{AB} = \overline{CD}$

$\overline{AD} = \overline{BC}$



42(P) A diagonal divides a parallelogram into two congruent triangles.

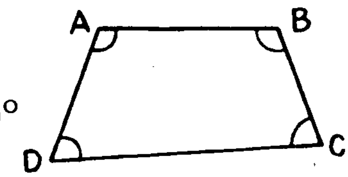
Example: Above figure:

$\triangle ACD \cong \triangle ABC$

43(P) The diagonals of a parallelogram bisect each other.

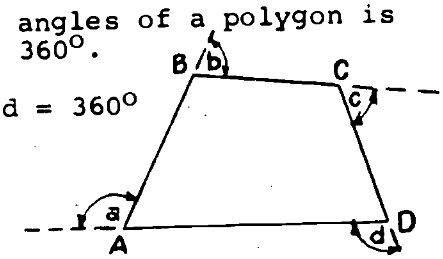
Example: Above figure: $\overline{AO} = \overline{OC}$; $\overline{BO} = \overline{OD}$

44(P) The sum of the interior angles of any polygon is (n - 2) straight angles where "n" is the number of sides.



Example: In ABCD there are 4 sides. Therefore the sum of the $\angle$s is $(4 - 2)\ 180^\circ = 360^\circ$
$[(n - 2) \times 180^\circ]$

45(P) The sum of the exterior angles of a polygon is two straight angles, or 360°.



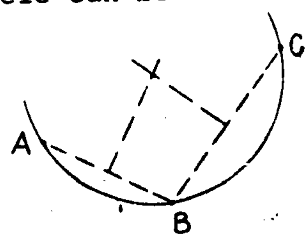
Example: $\angle a + \angle b + \angle c + \angle d = 360^\circ$

$\Sigma \angle = 360^\circ$

46(A) A straight line and a circle cannot intersect in more than two points.

47(P) Through three points not in the same straight line, one and only one circle can be drawn.

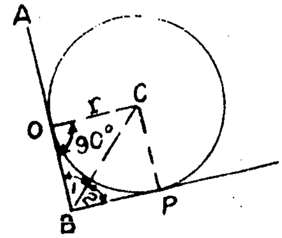
Example: Given ABC - Use propositions of perpendicular bisectors and distances from ends of lines.



48(P) Two circles can intersect in only two points.

49(P) A tangent to a circle is perpendicular to the radius drawn to the point of contact.

Example: the tan $\overline{AB}$ is $\perp$ to radius "r" at O.

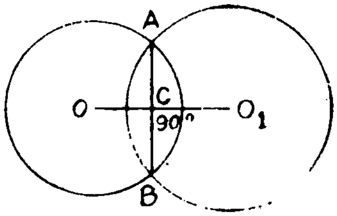


50(P) The tangents from an external point to a circle are equal and make equal angles with the line joining the point and the center of the circle.

Example: See above figure. $\overline{OB} = \overline{BP}$; $\measuredangle 1 = \measuredangle 2$

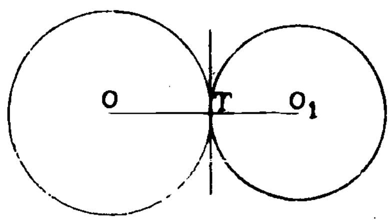
51(P) The line of centers of two intersecting circles is the perpendicular bisector of their common chord.

Example: $\overline{AC} = \overline{CB}$



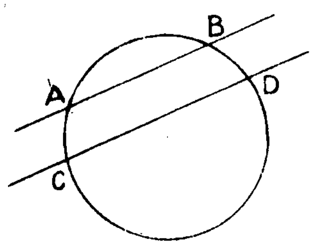
52(P) The line of centers of two tangent circles passes through the point of tangency.

Example: $\overline{O\text{-}O_1}$ passes through Pt "T"



53(P) Parallel lines intercept equal arcs on a circle.

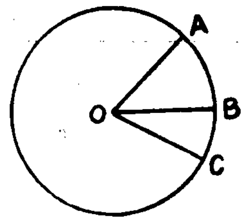
Example: $\overset{\frown}{AC} = \overset{\frown}{BD}$



54(P) In the same or equal circles two central angles have the same ratio as their intercepted arcs.

Example:

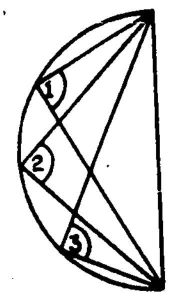
$\measuredangle AOB : \measuredangle AOC = \widehat{AB} : \widehat{AC}$



55(P) Any angle inscribed in a semi-circle is a right angle.

Example:

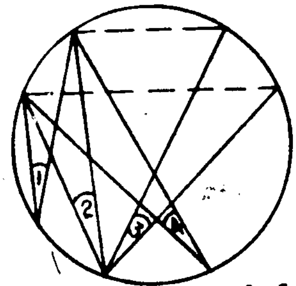
$\measuredangle 1$, $\measuredangle 2$ and $\measuredangle 3$ are all rt angles.



56(P) Angles inscribed in the same or equal segments are equal.

Example:

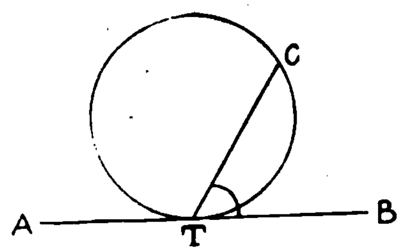
$\measuredangle 1 = \measuredangle 2 = \measuredangle 3 = \measuredangle 4$



57(P) An angle formed by a tangent and a chord from the point of tangency is measured by half the arc intercepted by the chord.

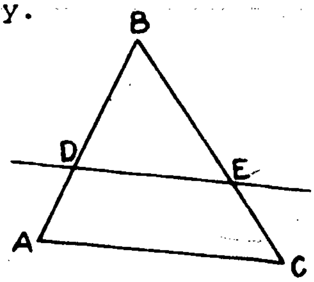
Example:

$\measuredangle CTB = 1/2$ arc $\widehat{CT}$



58(P) A line drawn across two sides of a triangle parallel to the third side, divides the two sides proportionally.

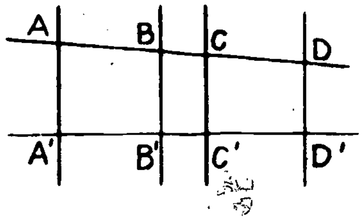
Example: $\frac{\overline{AD}}{\overline{DB}} = \frac{\overline{CE}}{\overline{EB}}$



59(P) If any number of parallels are cut by two transversals, the corresponding segments on the two are proportional.

Example:

$\frac{\overline{AB}}{\overline{A'B'}} = \frac{\overline{BC}}{\overline{B'C'}} = \frac{\overline{CD}}{\overline{C'D'}} = \frac{\overline{AC}}{\overline{A'C'}}$ etc.



60(P) If the altitude is drawn on the hypotenuse of a right triangle, then:

(a) The two right triangles formed are similar to the original triangle and to each other.

(b) The altitude is the mean proportional between the segments of the hypotenuse.

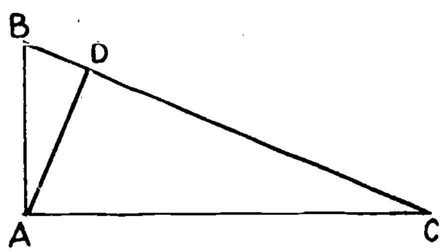
(c) Each leg of the original triangle is the mean proportional between its adjacent segment and the hypotenuse.

Example:

(a) $\triangle ABD \sim \triangle ADC \sim \triangle ABC$

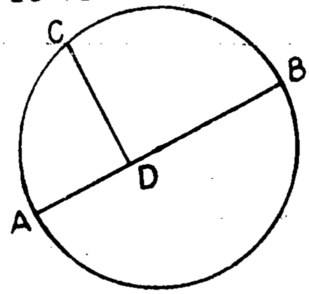
(b) $\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{DA}}{\overline{DC}}$

(c) $\frac{\overline{BD}}{\overline{AB}} = \frac{\overline{AB}}{\overline{BC}}$ and $\frac{\overline{DC}}{\overline{AC}} = \frac{\overline{AC}}{\overline{BC}}$



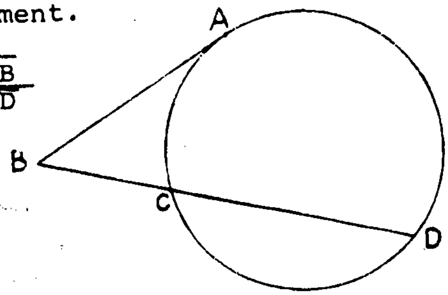
61(P) The perpendicular to the diameter from any point on a circle is the mean proportional between the segments into which it divides the diameter.

Example: $\frac{\overline{AD}}{\overline{DC}} = \frac{\overline{DC}}{\overline{DB}}$



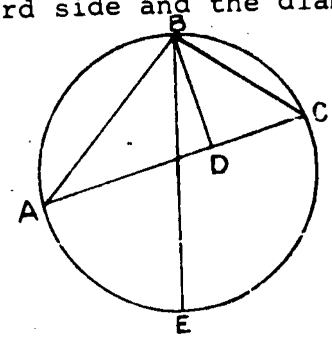
62(P) If, from a point outside a circle, a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.

Example: $\frac{\overline{BC}}{\overline{AB}} = \frac{\overline{AB}}{\overline{BD}}$



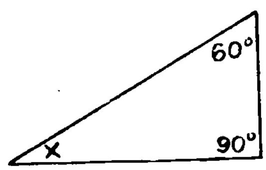
63(P) If a triangle is inscribed in a circle the product of any two sides equals the product of the altitude on the third side and the diameter.

Example: $\overline{AB} \cdot \overline{BC} = \overline{BD} \cdot \overline{BE}$

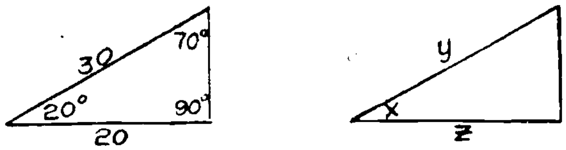


Problems that Apply Axioms and Propositions

1. What is the value of angle x in the following triangle? (Axiom 29)

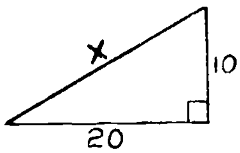


2. The following two triangles are congruent. What is the value of angle x, line y, and line z of Triangle No. 2? (Proposition 31)

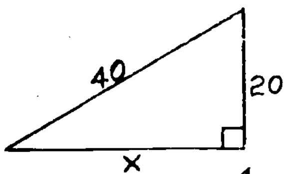


3. Using the Theorem of Pythagoras solve the following triangles for the unknown side:

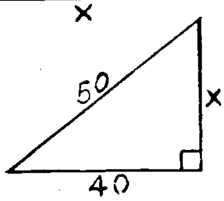
(1)



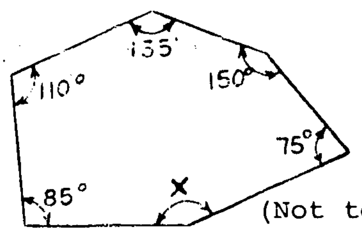
(2)



(3)



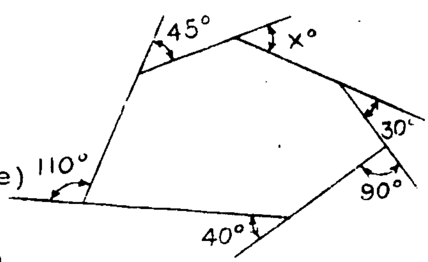
4. What is the value of angle x in the following figure? (Proposition 44)

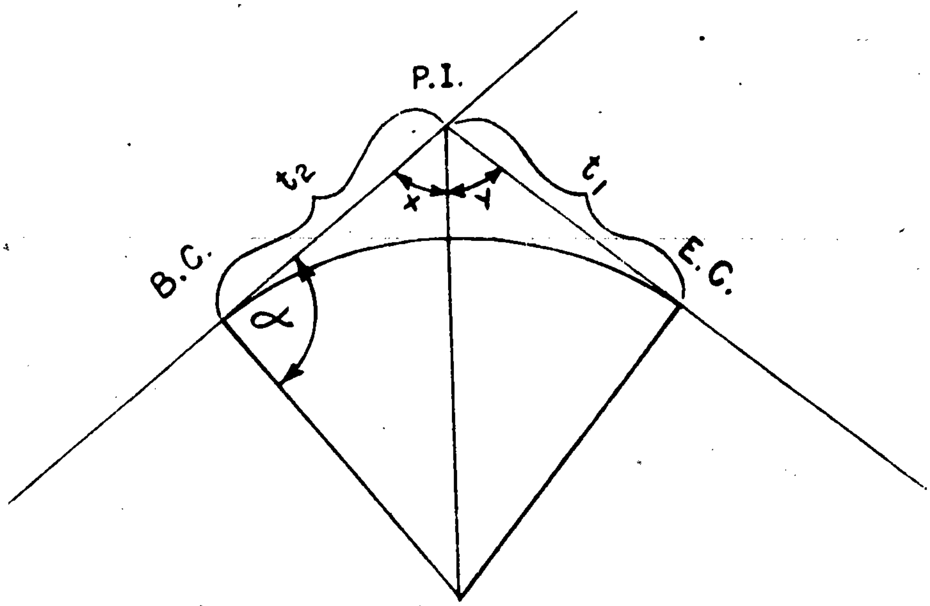


(Not to Scale)

5. What is the value of angle x in the following figure? (Proposition 45)

(Not to Scale)





6. What is the value of angle α ? State the Axiom.

7. Is t_2 equal to t_1? If so, state the Axiom.

8. Is x equal to y? If so, state the Axiom.

IV - Mensuration

Mensuration is defined as that branch of applied geometry concerned with finding lengths of lines, areas of surfaces, and volumes of solids.

In many engineering and mathematic handbooks lists of formulas are given. These formulas are a big aid and convenience in the solution of problems involving lengths, areas, and volumes.

The user of the formulas should have a fairly good knowledge of algebra, geometry, and trigonometry.

Up to this point in the course you have learned algebra and geometry. You will be able to use several of these formulas at present. Some of the formulas involve the relations to be learned later in trigonometry.

The following pages cover (1) common units of measure which are found under conversion factors in engineering handbooks, and (2) formulas for various lengths, areas, and volumes found under mensuration.

UNITS OF MEASURE

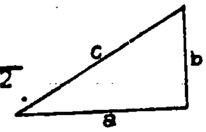
Acre	0.0015625 square mile; 10 square chains; 160 square rods; 4,840 square yards; 43,560 square feet.
Circumference	$2\pi R$ or 6.28319 radians.
Cubic Foot	1/27 cubic yard; 7.481 gallons (U.S.); 1728 cubic inches; 62.4 pounds of water.
Cubic Yard	27 cubic feet; 202.0 gallons (U.S.).
Gallon (U.S.)	@ 62^{o} F. weight 8.337 pounds; 0.13368 cubic foot; 231 cubic inches.
Chain	0.0125 mile; 4 rods; 66 feet; 100 links.
Mile	80 chains; 320 rods; 5,280 feet.
Radian	$\frac{1}{2\pi}$ or 0.159155 circumference or revolution.
Rod	0.25 chain; 16.5 feet.
Section	1 square mile; 640 acres.
Ton	2,000 pounds.
Township	36 square miles; 23,040 acres.
One Square Mile	640 acres.
One Square Yard	9 square feet.
One Square Foot	144 square inches.
One Station	100 feet.

MENSURATION: LENGTHS, AREAS, VOLUMES

Notation: a, b, c, d, s are lengths, A is area, V is volume.

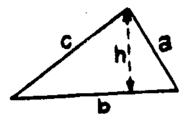
<u>Right Triangle</u> $A = 1/2\ ab.$

$c = \sqrt{a^2 + b^2}$, $a = \sqrt{c^2 - b^2}$, $b = \sqrt{c^2 - a^2}$.



<u>Oblique Triangle</u>

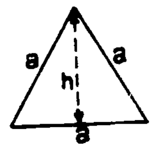
$A = 1/2\ bh.$



<u>Equilateral Triangle</u>

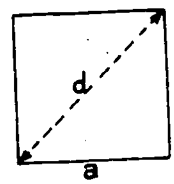
$A = 1/2\ ah = 1/4\ a^2 \sqrt{3}.$

$h = 1/2\ a \sqrt{3}.$



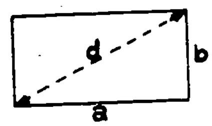
<u>Square</u>

$A = a^2$; $d = a\sqrt{2}.$



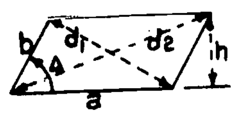
<u>Rectangle</u>

$A = ab$; $d = \sqrt{a^2 + b^2}.$



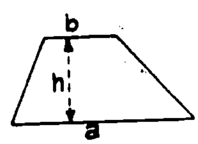
<u>Parallelogram</u> (opposite sides parallel)

$A = ah = ab \sin \Delta.$
$d_1 = \sqrt{a^2 + b^2 - 2ab \cos \Delta}$;
$d_2 = \sqrt{a^2 + b^2 + 2ab \cos \Delta}$.



<u>Trapezoid</u> (one pair of opposite sides parallel)

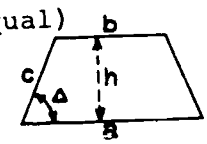
$A = 1/2\ h\ (a + b).$



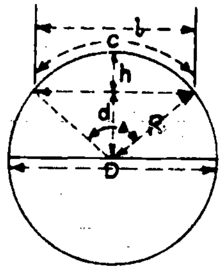
<u>Isosceles Trapezoid</u> (non-parallel sides equal)

$A = 1/2\ h\ (a + b) = 1/2\ c \sin \Delta\ (a + b)$

$= c \sin \Delta\ (a - c \cos \Delta) = c \sin \Delta\ (b + c \cos \Delta).$



Circle $\begin{Bmatrix} C = \text{circumference} \\ \Delta = \text{central angle in radians} \end{Bmatrix}$



$C = \pi D = 2\pi R.$

$c = R\Delta = 1/2\ D\Delta = D \cos^{-1} \frac{d}{R} = D \tan^{-1} \frac{l}{2d}.$

$l = 2\sqrt{R^2 - d^2} = 2\ R \sin \frac{\Delta}{2} = 2\ d \tan \frac{\Delta}{2} = 2\ d \tan \frac{c}{D}.$

$d = 1/2 \sqrt{4R^2 - l^2} = 1/2 \sqrt{D^2 - l^2} = R \cos \frac{\Delta}{2} = 1/2\ l \cot \frac{\Delta}{2} = 1/2\ l \cot \frac{c}{D}.$

$h = R - d.$

$\Delta = \frac{c}{R} = \frac{2c}{D} = 2 \cos^{-1} \frac{d}{R} = 2 \tan^{-1} \frac{l}{2d} = 2 \sin^{-1} \frac{l}{D}.$

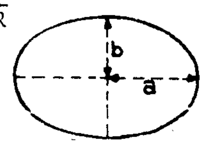
$A\ (\text{circle}) = \pi R^2 = 1/4\ \pi D^2 = 1/2\ RC = 1/4\ DC.$

$A\ (\text{sector}) = 1/2\ Rc = 1/2\ R^2 \Delta = 1/2\ D^2 \Delta$

$A\ (\text{segment}) = A\ (\text{sector}) - A\ (\text{triangle}) = 1/2\ R^2 (\Delta - \sin \Delta) = 1/2\ R \left(c - R \sin \frac{c}{R}\right)$

$= R^2 \sin^{-1} \frac{l}{2R} - 1/4\ l \sqrt{4R^2 - l^2} = R^2 \cos^{-1} \frac{d}{R} - d \sqrt{R^2 - d^2}$

$= R^2 \cos^{-1} \frac{R - h}{R} - (R - h) \sqrt{2Rh - h^2}.$



Ellipse

$A = \pi ab.$
Perimeter (s) =

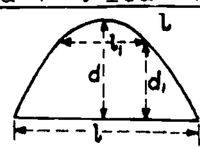
$$\pi(a + b) \left[1 + \frac{1}{4} \left\{\frac{a - b}{a + b}\right\}^2 + \frac{1}{64} \left\{\frac{a - b}{a + b}\right\}^4 + \frac{1}{256} \left\{\frac{a - b}{a + b}\right\}^6 + \ldots\right].$$

Parabola

$A = 2/3\ ld.$

Length of arc (s) $= \frac{1}{2}\sqrt{16\ d^2 + l^2} + \frac{l^2}{8d} \ln \left\{\frac{4d + \sqrt{16d^2 + l^2}}{l}\right\}$

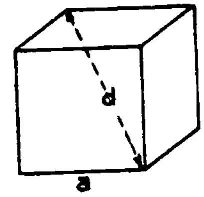
$= l \left[1 + \frac{2}{3} \left\{\frac{2d}{l}\right\}^2 - \frac{2}{5} \left\{\frac{2d}{l}\right\}^4 + \ldots\right].$



Height of segment $(d_1) = \frac{d}{l^2}(l^2 - l_1^2).$

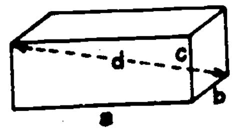
Width of segment $(l_1) = l \sqrt{\frac{d - d_1}{d}}.$

Cube



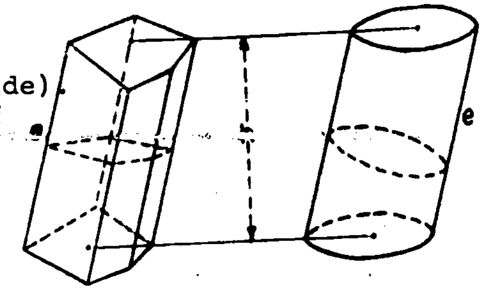
$V = a^3$; $d = a\sqrt{3}$.
Total surface $= 6a^2$.

Rectangular Parallelopiped



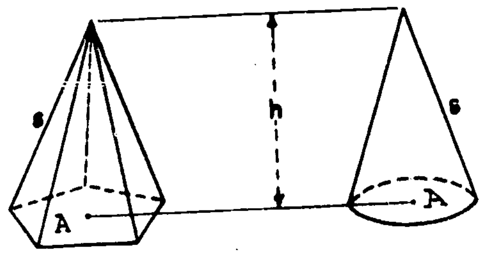
$V = abc$; $d = \sqrt{a^2 + b^2 + c^2}$.
Total surface $= 2(ab + bc + ca)$.

Prism or Cylinder



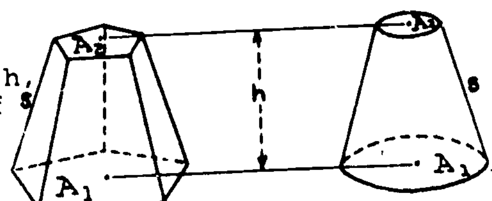
V = (area of base) x (altitude).
Lateral area = (perimeter of right section) x (lateral edge).

Pyramid or Cone



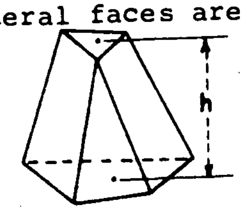
V = 1/3 (area of base) x (altitude).
Lateral area of regular figure = 1/2 (perimeter of base) x (slant height).

Frustum of Pyramid or Cone



$V = 1/3\ (A_1 + A_2 + \sqrt{A_1 \times A_2})h$,
where A_1 and A_2 are areas of bases, and h is altitude.
Lateral area of regular figure = 1/2 (sum of perimeters of bases) x (slant height).

Prismatoid (bases are in parallel plans, lateral faces are triangles or trapezoids)



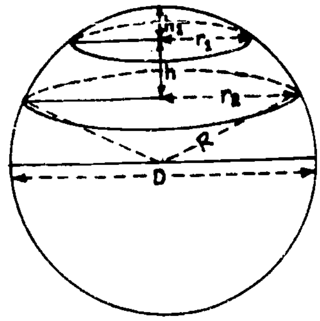
$V = 1/6\ (A_1 + A_2 + 4\ Am)h$,
where A_1, A_2 are areas of bases, Am is area of mid-section, and h is altitude.

Sphere

$$A(\text{sphere}) = 4\pi R^2 = \pi D^2.$$
$$A(\text{zone}) = 2\pi Rh = \pi Dh.$$
$$V(\text{sphere}) = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3.$$
$$V(\text{spherical sector}) = \frac{2}{3}\pi R^2 h = \frac{1}{6}\pi D^2 h.$$

V(spherical segment of one base)
$$= \frac{1}{6}\pi h_1(3r_1^2 + h_1^2) = \frac{1}{3}\pi h_1^2(3R - h_1).$$

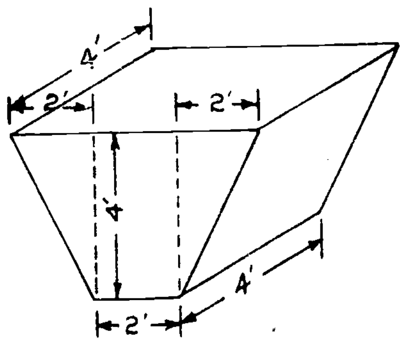
V(spherical segment of two bases)
$$= \frac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2).$$



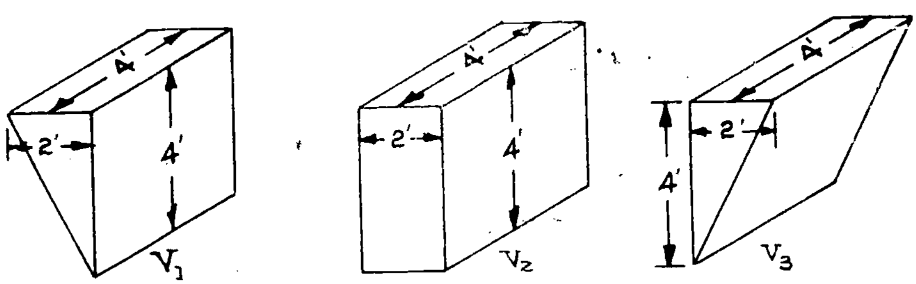
MENSURATION: PROBLEMS

1. Find the volume in cubic feet in Figure 1.

Figure 1



Solution: This problem can be solved in various ways. For example, it can be broken into simple sections, it can be computed by the trapezoidal formula, or by end areas. These three solutions are shown below.



$V = V_1 + V_2 + V_3$
$V = 1/2\ bhw + lwh + 1/2\ bhw$
$= \quad 16 + 32 + 16 = 64$ cu. ft.
$V = 1/2(a + b)hw = 1/2(2 + 6)\ 4 \times 4 = 64$ cu. ft.
$V = 1/2\ [(4 \times 6)+(2 \times 4)]\ 4 = 64$ cu. ft.

2. Find the volume in cubic feet of Figure 2.

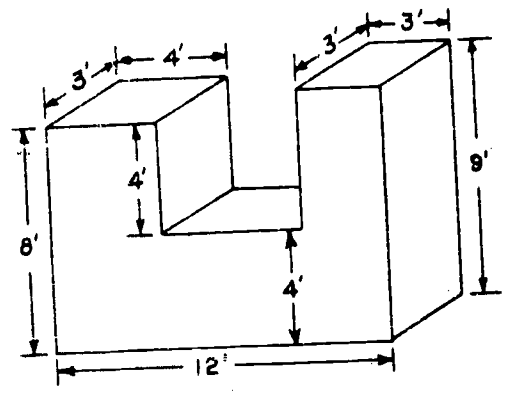


Figure 2

Answer - 237 c.f.

3. Find the cubic yards of concrete required to build the section shown in Figure 3.

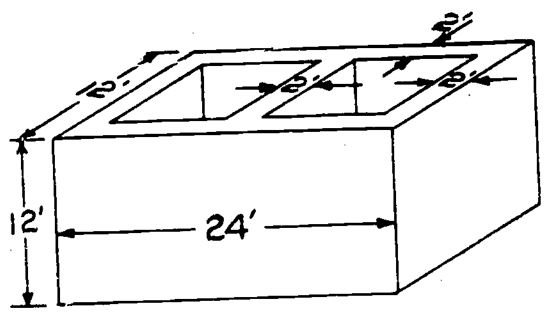


Figure 3

4. What is the amount of backfill needed in cubic yards to fill a trench that has an average cut of 6 feet, 3 feet wide, and 24" x 40' CMP is to be placed therein?

Answer - 22 c.y.

5. How many cubic yards would be used in constructing a wing wall one foot thick of the following dimensions.

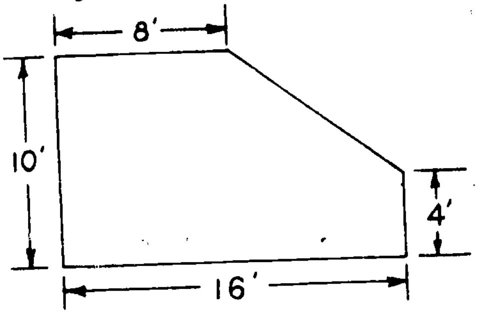


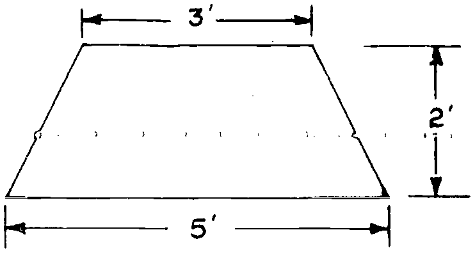
Figure 5

6. How many cubic yards of concrete are needed to build a concrete pipe 60" interior diameter, 6" thick and 8' long?

Answer - 2.56 c.y.

Area and Volume Problems

7. A spreader box deposits aggregate base material on the roadbed in a windrow as shown below:



At the weight of 150 lbs. per cubic foot, how many tons of material will be deposited in one mile?

8. The tank on a water truck is 20 feet long and 10 feet in diameter. Find the volume of the tank in cubic feet.

Answer - 1570 c.f.

9. Find the surface area in square yards on a one-mile length of 24-foot wide concrete pavement.

10. Portland cement concrete pavement is to be placed 0.67-foot thick and 24-feet wide. How many cubic yards of concrete will be required for 1 mile of pavement?

Answer - 3144.53 c.y.

11. Aggregate for road-mixed asphalt concrete surfacing is deposited on the roadbed in a triangular windrow 6 feet wide and 3 feet high. After mixing and spreading uniformly on a 24-foot wide roadbed, what will be the final thickness of surfacing?

12. The trench for an 8" perforated metal pipe is 18 inches wide, 4 feet deep, and 500 feet long. After the pipe is in place, how many cubic yards of permeable material will be required to backfill the trench?

Answer - 104.59 c.y.

13. If permeable material weights 150 lbs. per cubic foot, how many tons will be required in the above problem?

14. A 1-inch blanket of asphalt concrete is to be placed on a 24-foot wide highway for 6.4 miles. At the approximate weight of 150 lbs. per cubic foot for asphalt concrete, how many tons will be required?

Answer - 5068.80 tons

15. An asphaltic emulsion fog seal coat is to be applied to an existing 40-foot wide highway at the rate of 0.05 gallon per square yard. If 240 gallons of emulsion weighs 1 ton, how many tons will be required to seal 28.7 miles of highway?

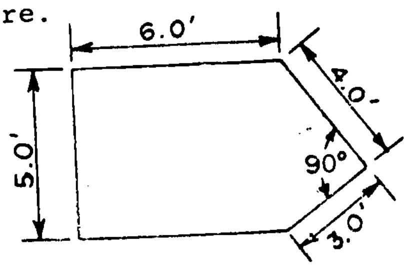
16. Curing seal is applied to portland cement concrete pavement at the rate of 1 gallon per 150 square feet. How many gallons will be required to seal 8.3 miles of 24-foot wide concrete pavement?

Answer - 7,011.84 gallons

17. A corrugated metal pipe is to be placed in a trench 3 feet wide, 5 feet deep, and 100 feet in length. How many cubic yards of structure excavation will be required?

Geometry Problems

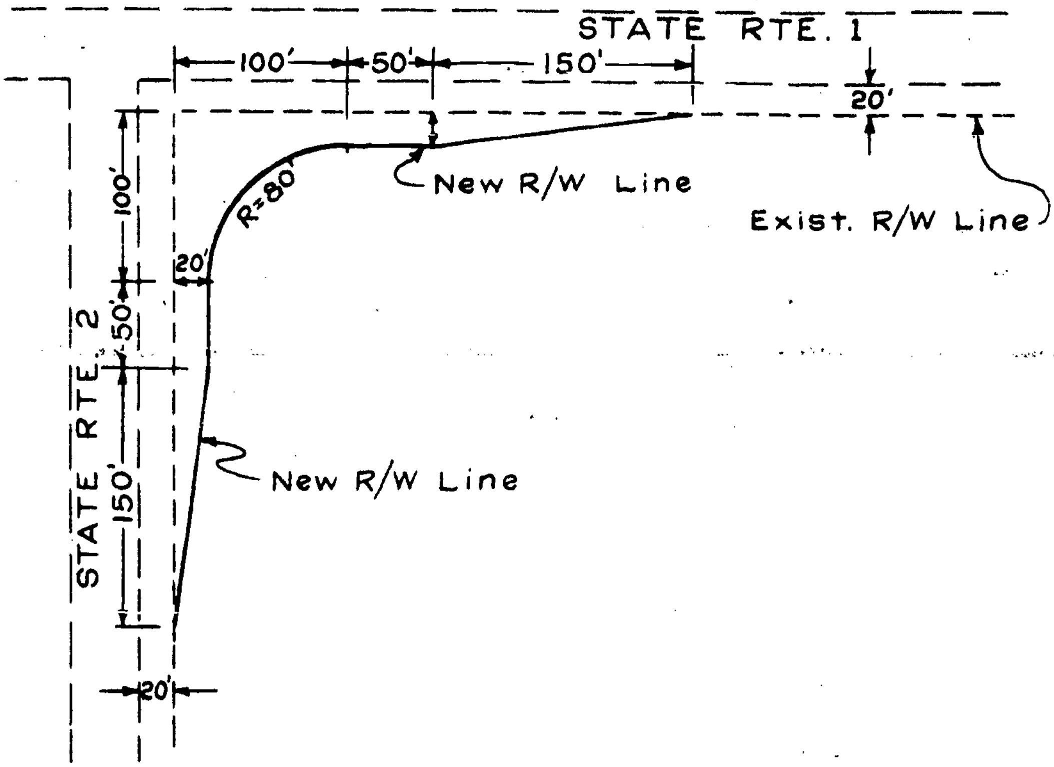
18. How many gallons of water will a tank 10' deep and shaped as follows hold? (1 c.f. = 7.481 gal.) See Units of Measure.



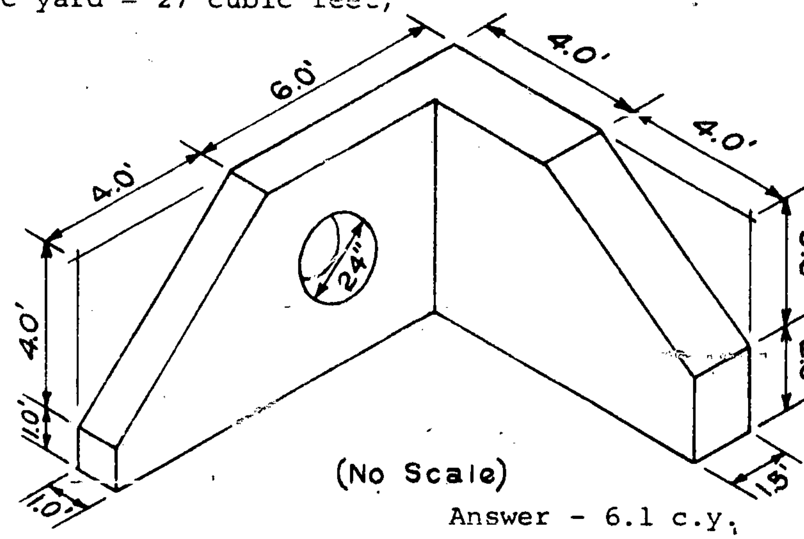
Answer - 2,693.16 gallons

19. In the following sketch how many additional acres of land will be required to make the indicated improvement? The construction of a free right-turn lane.

(One acre = 43,560 ft.^{2})

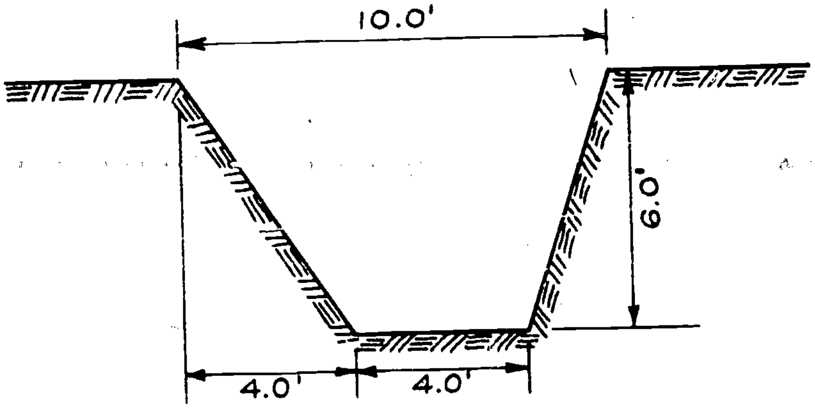


20. How many cubic yards of concrete must a contractor order to pour two head walls of the following shape. (One cubic yard = 27 cubic feet)



Answer - 6.1 c.y.

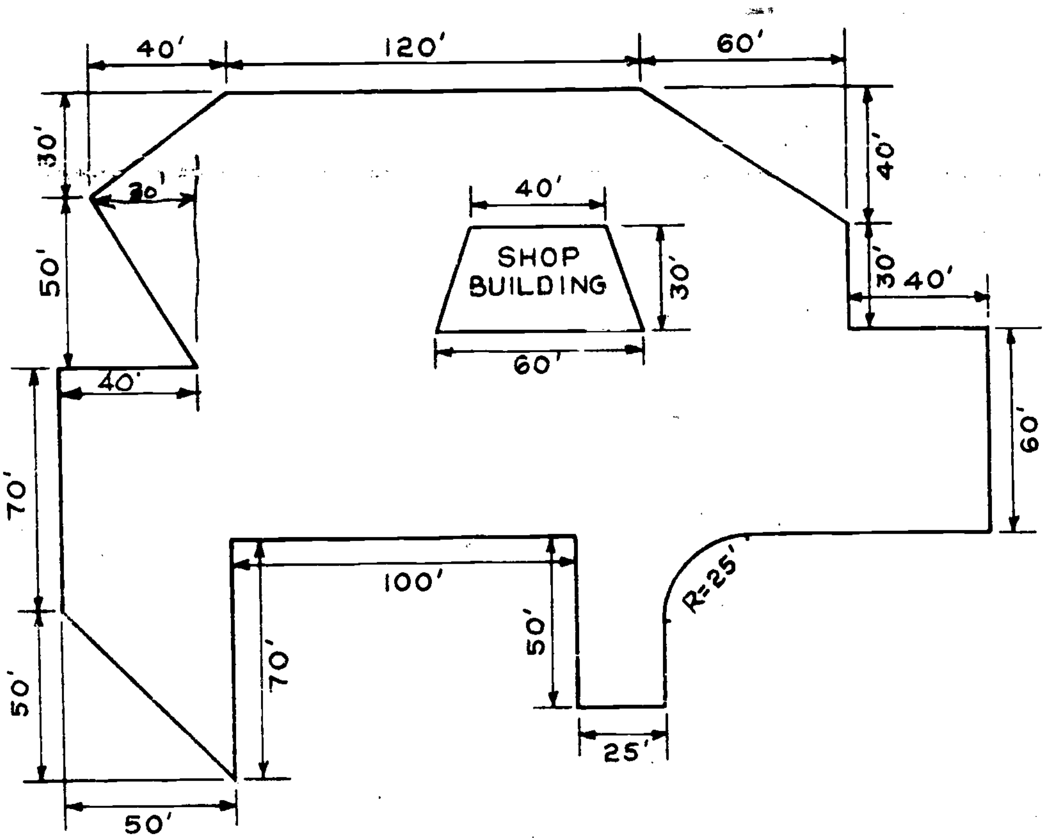
21. How many cubic yards of excavation is required to build a drainage ditch one mile long with the following cross section?



22. A contract calls for the installation of a 48-inch CMP to drain a low spot between two roadways. The pipe will drain the low spot into a creek 4,892 feet away. Flow line of the pipe shall be 8 feet below the existing ground and the excavation must be 2 feet wider than the diameter of the pipe. How many cubic yards of structure excavation and structure backfill are involved?

Answers - Structure Excavation = 8,697 cubic yards
Structure Backfill = 6,419 cubic yards

23. The area described below is under study for use as a parking lot. How many acres of parking will this area provide? If this area is used for a parking lot it will be paved with asphalt concrete 0.5 feet thick. How many tons of asphalt concrete will be required to pave this lot? (Asphalt concrete weighs approximately 150 lb/ft^3)



Answer - Area = 0.714 acre;
Tons A.C. = 1,166 tons

V - Common Prismoid Problems

A prismoid is a solid bounded by planes, the end faces being parallel to each other and having the same number of sides.

Solving for the volume of a prismoid is one of the most frequent problems encountered by the highway engineer. For estimates the designer must calculate volumes of earthwork, ditch and channel excavation, structure excavation and backfill, concrete, masonry, and many other items. Likewise the construction engineer must calculate the final volumes of these items for payments due to the contractor.

The methods used, by and large, for determination of volumes of such items are (1) the average end area method, and (2) the prismoidal formula method. The formula used will depend on the accuracy necessary. As a rule the end area formula will give somewhat larger volumes than the prismoidal formula.

An outline of the two methods follows:

(1) Average End Area Method

The simplest method of computing the volume of a prismoidal solid is to average the areas of the two bases (or ends) and multiply by the perpendicular distance between them, or by formula,

$$V = \frac{(A_1 + A_2)}{2} \times L$$

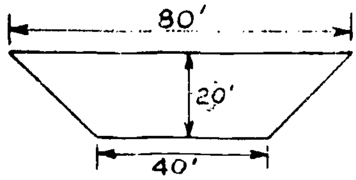
which is known as the End-Area Formula.

In California and throughout the remainder of the United States this method is used to a very great extent for earthwork computations. It does not give sufficiently accurate results for classes of work such as masonry or concrete.

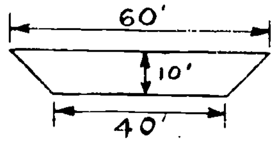
The Standard Specifications in use by the Division of Highways specify that the average end area method be used for computing quantities of roadway excavation except in special cases.

Example (using End-Area Formula)

Problem: Two cross-sections 100 feet apart are as follows:



First Area = A_1



Second Area = A_2

What is the volume of dirt removed from between the cross-sections?

Solution:

By the area of a trapezoid formula

$$A_1 = \left(\frac{40' + 80'}{2}\right)(20') = 1200 \text{ square feet}$$

$$A_2 = \left(\frac{40' + 60'}{2}\right)(10') = 500 \text{ square feet}$$

and from the statement of the problem $L = 100'$.

Substituting these values in the end-area formula

$$V = \left(\frac{A_1 + A_2}{2}\right) L$$

$$V = \left(\frac{1200 \text{ sq. ft.} + 500 \text{ sq. ft.}}{2}\right)(100')$$

$V = 85{,}000$ cubic feet, and 1 cubic foot $= \frac{1}{27}$ cubic yard

$V = 85{,}000$ cubic feet $\left(\frac{1}{27}\right) = 3148$ cubic yards.

(2) Prismoidal Formula Method

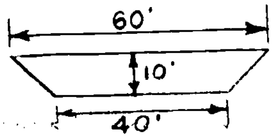
The correct volume of a prismoid is expressed by the "Prismoidal Formula":

$$V = \frac{L}{6}(A_1 + 4A_m + A_2)$$

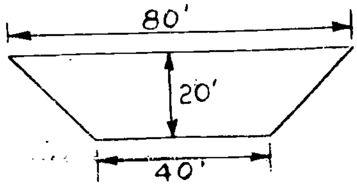
in which L is the perpendicular distance between the two bases (or ends), A_1 and A_2; and A_m is the "middle area", that is, the area half-way between the two bases. The dimensions of the middle section are found by averaging the corresponding dimensions of the end sections. The middle area is computed from the dimensions of this middle section and is <u>not</u> the mean of A_1 and A_2.

<u>Example (using Prismoidal Formula)</u>

Consider the two cross-sections to be ends of a prismoid of roadway excavation:



Area No. 1
$= A_1$



Area No. 2
$= A_2$

Problem: Find volume of roadway excavation:

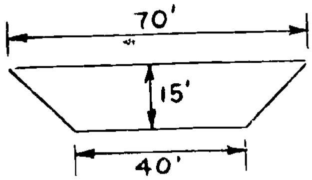
Upper measurement $A_m = \dfrac{60' + 80'}{2} = 70'$

Lower measurement $A_m = \dfrac{40' + 40'}{2} = 40'$

Height of $A_m = \dfrac{10' + 20'}{2} = 15'$

Or

A_m



Area No. 1 or $A_1 = \left(\dfrac{60' + 40'}{2}\right)(10') = 500$ sq. ft.

Area No. 2 or $A_2 = \left(\dfrac{80' + 40'}{2}\right)(20') = 1200$ sq. ft.

Middle Area or $A_m = \left(\dfrac{70' + 40'}{2}\right)(15') = 825$ sq. ft.

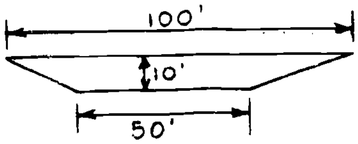
Substituting in the prismoidal formula:

$$V = \frac{L}{6}(A_1 + 4A_m + A_2)$$
$$V = \frac{100}{6}\left[500 + 4(825) + 1200\right]$$
$$V = 83{,}33\overline{3} \text{ cu. ft.} = 308\overline{6} \text{ cu. yds.}$$

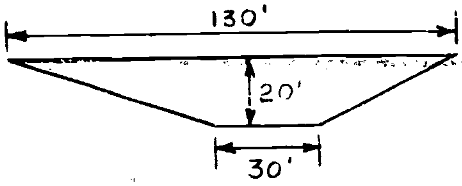
Compare this volume with volume shown under End-Area illustration.

Problems

1.



Area No. 1
or A_1

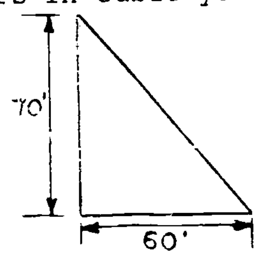


Area No. 2
or A_2

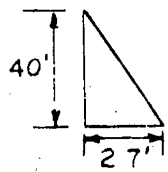
The two areas shown above are neat lines of a proposed roadway excavation section on a highway project. The distance between the two sections is 140'.

Find the volume of material that will be excavated between the two sections by the end area method and by the prismoidal formula. State your answers in cubic yards.

2.



Area No. 1
or A_1



Area No. 2
or A_2

The areas shown above are the bases (or ends) of a prismoidal solid. The solid is 300 feet long. Find the volume by both the end area method and prismoidal formula.

Answer: $V_A = 14{,}66\underline{7}$ cubic yards
$V_P = 13{,}74\underline{0}$ cubic yards

c. By determinants

(1) The value of each unknown in a system of linear equations is the quotient of two determinants.

(a) The determinant of the denominator consists of the coefficients of the unknowns written down in order, and is the same for all unknowns.

(b) The determinant of the numerator, for any unknown, is found by inserting in the denominator the constant terms in place of the coefficients of that unknown.

Example:

Given

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}\begin{matrix} d_1 & b_1 \\ d_2 & b_2 \\ d_3 & b_3 \end{matrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}} = \frac{d_1b_2c_3 + b_1c_2d_3 + c_1d_2b_3 - d_3b_2c_1 - b_3c_2d_1 - c_3d_2b_1}{a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1}$$

Each set of numbers multiplied together as indicated by the arrows. The product is given a positive sign for lines running down to the right and a negative sigh for lines running up to the right.

(3) The value of the determinant is:

(a) Zero, if each element of any row or column is zero.

(b) Zero, if two rows or columns have proportional or identical elements.

(c) Unchanged, if the rows and columns are interchanged.

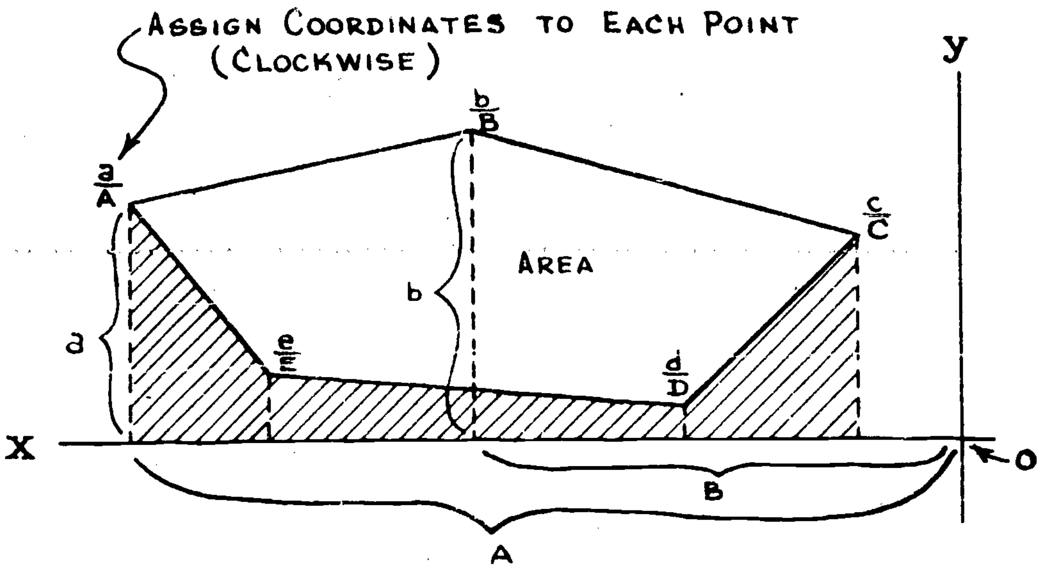
(d) Unchanged, if a constant multiple of each element of a column or row is added to the corresponding element of another column or row.

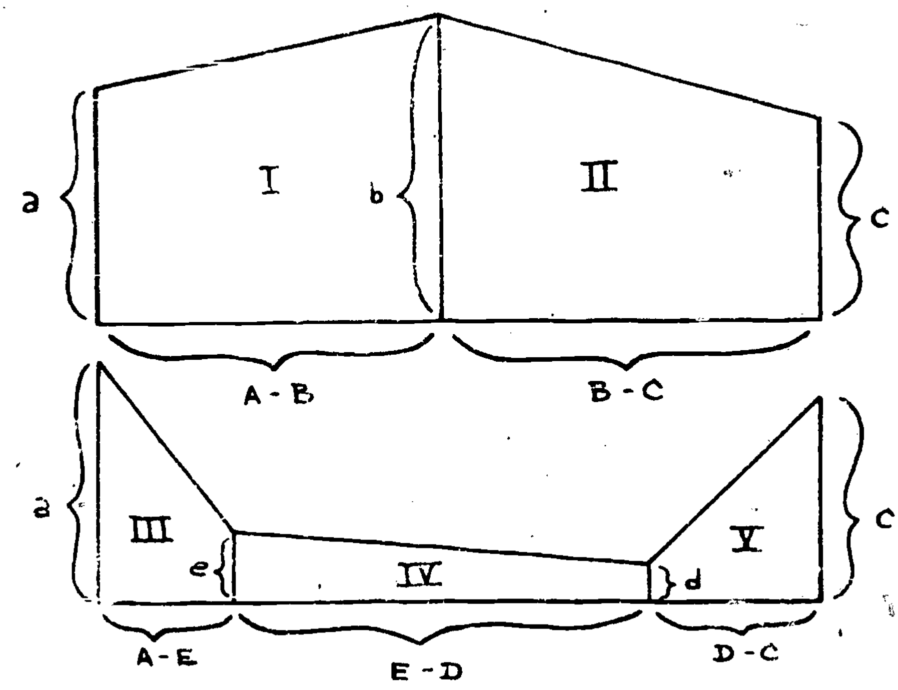
(e) Changed in sign, if any two rows or columns are interchanged.

(f) Multiplied by k, if the elements of any row or column are multiplied by k.

DOUBLE END AREA BY D.M.D.

(SUMMATION OF TRAPEZOIDS)





$$\text{AREA} = \left(\frac{a+b}{2}\right) h$$

PROCEDURE: CALCULATE AREA OF TRAPEZOIDS.
SUBTRACT TRAPEZOIDS III, IV, & V FROM
TRAPEZOIDS I & II.

$$A = \overset{\text{I}}{\left(\frac{a+b}{2}\right)(A-B)} + \overset{\text{II}}{\left(\frac{b+c}{2}\right)(B-C)} - \overset{\text{III}}{\left(\frac{a+c}{2}\right)(A-E)} - \overset{\text{IV}}{\left(\frac{e+d}{2}\right)(E-D)} - \overset{\text{V}}{\left(\frac{d+c}{2}\right)(D-C)}$$

MULTIPLY TERMS

$$A = \tfrac{1}{2}(aA + bA - aB - bB) + \tfrac{1}{2}(bB + cB - bC - cC) - \tfrac{1}{2}(aA + eA - aE - eE)$$
$$- \tfrac{1}{2}(eE + dE - eD - dD) - \tfrac{1}{2}(dD + cD - dC - cC)$$

MULTIPLY BOTH SIDES BY 2 AND CANCEL LIKE TERMS.

$$2A = Ab + Bc + Cd + De + Ea - aB - bC - cD - dE - eA$$

ALTERNATE METHOD

WRITE COORDINATES IN CLOCKWISE ORDER AND CROSSMULTIPLY

$$2A = \frac{a}{A}\ominus\oplus\ \frac{b}{B}\ominus\oplus\ \frac{c}{C}\ominus\oplus\ \frac{d}{D}\ominus\oplus\ \frac{e}{E}\ominus\oplus\ \frac{a}{A}$$

REMEMBER THIS IS TWICE THE AREA

Section 3 - Trigonometry

INDEX

SECTION 3

Trigonometry

I - General

Trigonometry (trig) is the detailed study of the properties of one geometric figure - the triangle. Highway engineers use the triangle in many ways--slope chaining, horizontal curves, latitudes, departures; in fact, our highest form of surveying is called triangulation.

In geometry, the triangle was studied. The Theorem of Pythagoras - Hypotenuse ^{2} = sum of the squares of the other two sides. Geometric study requires that in one way or another, the lengths of all 3 sides must be known.

Trigonometry considers the properties of the other part of the triangle - the angle formed where two sides intersect. Knowledge of these properties is invaluable. It is often much easier to merely measure the length of one side, and the angles at each end, than it is to measure two or three sides. Many times measurement of one side is impossible. It is much easier and more precise to measure angles with a transit than it is to measure distance with a tape.

To use this knowledge, certain properties must be known. These are called Trigonometric Functions. The three major ones are sine, cosine, and tangent, and they are explained in the next section.

Each angle has a definite value of sine, consine, and tangent. These have been computed accurate to the eighth decimal place for every degree, minute, and second of angle. These books are large and expensive. Numerous condensed tables are available. Surveying books often have them and we recommend the textbook "Surveying" by Bouchard and Moffit, published by International Text Book Company, Scranton, Pa. Approximate functions of angles can also be read from most slide rules. These are not precise enough for many calculations and should not be used in surveying.

II - The Right Triangle

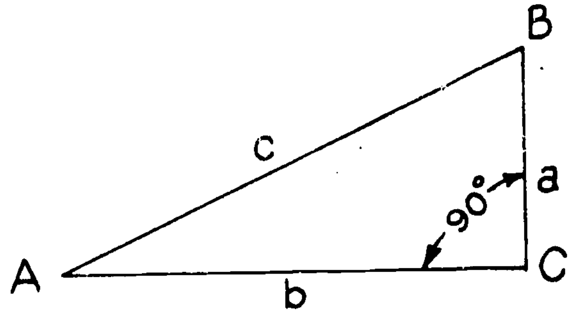


Figure I

The functions of the angles of a right triangle are the decimal equivalents of the ratio of the lengths of the various sides. Using Figure I, the <u>functions</u> of the angle at A are the ratios a/c, b/c, a/b, b/a, c/b and c/a, where a, b, and c are the lengths of the respective sides of triangle ABC. The functions of angle A are defined as follows:

(A) $\frac{a}{c}$ is called the Sine of A, written Sin A

$\frac{b}{c}$ is called the Cosine of A, written Cos A

$\frac{a}{b}$ is called the Tangent of A, written Tan A

$\frac{c}{a}$ is called the Cosecant of A, written Csc A

$\frac{c}{b}$ is called the Secant of A, written Sec A

$\frac{b}{a}$ is called the Cotangent of A, written Cot A

(B) That is:

$\text{Sin A} = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$ $\text{Csc A} = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$

$\text{Cos A} = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}$ $\text{Sec A} = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}}$

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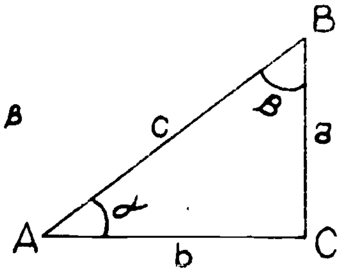
Tan A = $\frac{a}{b}$ = $\frac{\text{opposite side}}{\text{adjacent side}}$ Cot A = $\frac{b}{a}$ = $\frac{\text{adjacent side}}{\text{opposite side}}$

Pythagorean Theorem $a^2 + b^2 = c^2$

(C) Functions of Complementary Angles



Sin α = $\frac{a}{c}$ = Cos β Cos α = $\frac{b}{c}$ = Sin β

Tan α = Cot β Cot α = Tan β

Sec α = Csc β Csc α = Sec β

<u>The definitions of Sections B and C above should be thoroughly learned.</u>

There are also two other functions used in some engineering calculations. Versed Sine α = 1 - Cos α, and the coversed Sine α = 1 - Sin α. These are not used frequently, but should be at least kept in mind.

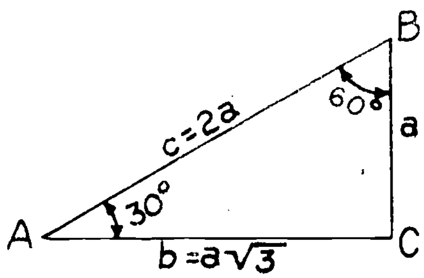
<u>Trigonometric Functions</u>

In general, any value of a trigonometric function is an endless decimal. By advance methods, the functions of any angle can be computed to as many decimal places as desired.

The trigonometric functions are the ratio of two sides of a triangle.

The functions used most frequently are the sine, cosine, and tangent.

Two common right triangles warrant special consideration: the 30° - 60° and 45° triangles. For these triangles the following sketches show their geometric relationships:



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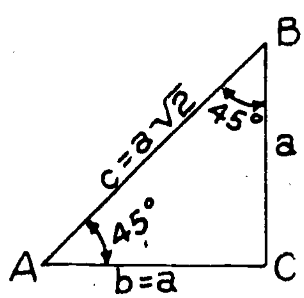


$\text{Sin } 30^\circ = \frac{a}{c} = \frac{a}{2a} = 1/2$

$\text{Cos } 30^\circ = \frac{b}{c} = \frac{\sqrt{3}a}{2a} = 1/2\sqrt{3}$

$\text{Tan } 30^\circ = \frac{a}{b} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = 1/3\sqrt{3}$

$\text{Sin } 45^\circ = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = 1/2\sqrt{2}$

$\text{Cos } 45^\circ = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = 1/2\sqrt{2}$

$\text{Tan } 45^\circ = \frac{a}{a} = 1$

CLASS WORK USING THE SIX FUNCTIONS OF A RIGHT TRIANGLE

1. Using Figure I, write the six functions of Angle B.

2. State which is the greater - (a) Sin A or Tan A

   (b) Cos A or Cot A.

3. Find the six functions of A when a = 2b.

4. Find the length of side a if Sin A = $\frac{3}{5}$, and c = 20.5.

5. Show that each function of an acute angle of a right triangle is equal to the co-named function of the complementary angle.

6. Which is larger, 2 Cos 15° or Cos (2 x 15°), if Cos 15° = .9659 and Cos 30° = .8660.

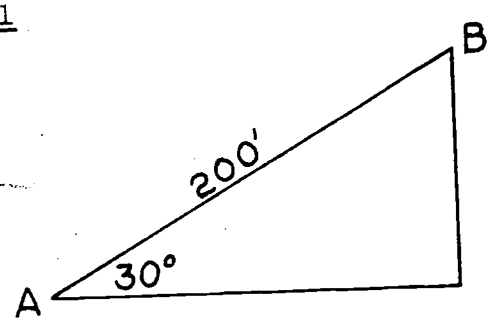
7. Prove that the Cos 60° = 1/2.

Use of Trigonometry

Now we have learned that if we have one angle and one side of a right triangle we can solve the triangle for all its unknown parts.

Example No. 1

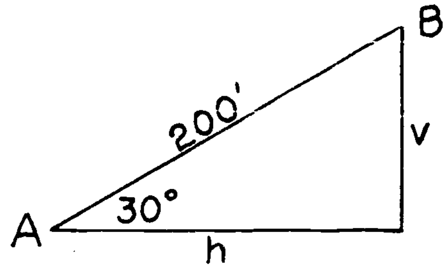
Given:



We are given the slope distance and the slope angle of a line from A to B. Wanted: the horizontal and vertical distance of B from A.

Solution:

Let the horizontal distance be h.
Let the vertical distance be v.



Write the functions of a 30° angle.

1. $\text{Sin } 30^\circ = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{v}{200'}$

2. $\text{Cos } 30^\circ = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{h}{200'}$

3. $\text{Tan } 30^\circ = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{v}{h}$

By observation we see that the solution of No. 1 will give the value of v.

$$\text{Sin } 30^\circ = \frac{v}{200'}$$

The value of the Sin 30° = 1/2. (From page 3-4.) Now substitute 0.5 for Sin 30° and solve the equation for v.

$$0.5 = \frac{v}{200'}$$

$$v = 200' (0.5)$$

$$v = 100'$$

Now we see the solution of No. 2 will give the value of h.

$$\text{Cos } 30^\circ = \frac{h}{200'}$$

The value of the Cos $30^o = 1/2\sqrt{3}$. (From page 3-4.) Now substitute 0.866 for the Cos 30^o and solve the equation for h.

$$0.866 = \frac{h}{200'}$$

$$h = 200' (0.866)$$

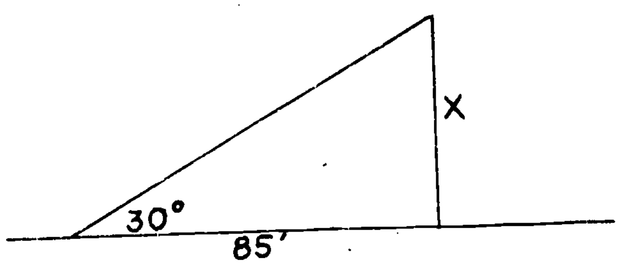
$$h = 173.2'$$

Answer: $v = 100'$, $h = 173.2'$

Example No. 2

The angle from the ground to the top of a flagpole is 30^o. The distance from the point of angle to the base of the pole is 85 feet. How high is the flagpole?

Given:



Wanted: Height x

Solution:

By observation we see that the given distance and the required height (adjacent side and opposite side respectively) involve the Tan of the given angle.

$$\text{Tan } 30^o = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{x}{85'}$$

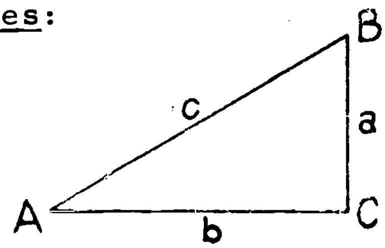
The value of the Tan $30^o = 1/3\sqrt{3}$. (From page 3-4.) Now substitute 0.577 for Tan 30^o and solve the equation.

$$0.577 = \frac{x}{85'}$$

$$x = 85' (0.577)$$

Answer: $x = 49.04'$

Problems involving right triangles:



1. $b = 20.4$ $\measuredangle A = 30^\circ$, find a.

2. $a = 78$ $\measuredangle B = 40^\circ$, find c.

3. $c = 345$ $\measuredangle A = 70^\circ$, find b.

4. Find the length of the horizontal shadow of a man six feet tall when the angle of elevation, from the horizon to the sun, is $75^\circ\ 36'$. Answer: 1.54 feet

5. A guy wire 35 feet long is stretched from level ground to the top of a vertical pole 25 feet high. Find the angle between the pole and the wire.

6. How high does an airplane rise in flying 4000 feet upward along a straight path inclined $28^\circ\ 47'$ from the horizontal? Answer: 1926 feet

7. Find the height of the Empire State Building in New York City if the angle of elevation of its top is $61^\circ\ 37'$ when seen from a point on the street level 675.4 feet from the building.

8. The largest tree in California is the General Sherman tree in the Sequoia National Park. At a point 185 feet from the tree, at the same elevation as its foot, the angle of elevation of the top of the tree is $55^\circ\ 49'$. How tall is the tree? Answer: 272 feet

9. A spy arranges to show a light at an oil tank 250 feet above the water level near an ocean beach. An observer 10 feet above the water level on an enemy submarine finds that the angle of elevation of the light is $3^\circ\ 27'$. Find the line-of-sight distance from the submarine's guns to the tank.

10. From the top row in a football stadium, 85 feet above the ground, the angle of depression of the center of the field is $32^\circ\ 10'$. Find the line-of-sight distance from the top row to the center of the field. Answer: 160 feet

Trigonometric Functions

∡	Sin	Cos	Tan
30°	0.50000	0.86603	0.57735
40°	0.64279	0.76604	0.83910
70°	0.93969	0.34202	2.7475
75° 36'	0.96858	0.24869	3.8947
44° 25'	0.69987	0.71427	0.97984
28° 47'	0.48150	0.87645	0.54938
61° 37'	0.87979	0.47537	1.8507
55° 49'	0.82724	0.56184	1.4724
3° 27'	0.06018	0.99819	0.06029
32° 10'	0.53238	0.84650	0.62892

III - Trigonometric Functions of Any Angle

In the first section, we only considered right triangles or triangles which could be cut into right triangles for purposes of solution. There are, however, oblique triangles which cannot conveniently be solved by merely separating them into right triangles.

Using Figure II, we consider OX as positive and OX' as negative. We also consider OY as positive and OY' as negative. With respect to angles, any angle is considered positive if the rotating line which describes it moves counterclockwise.

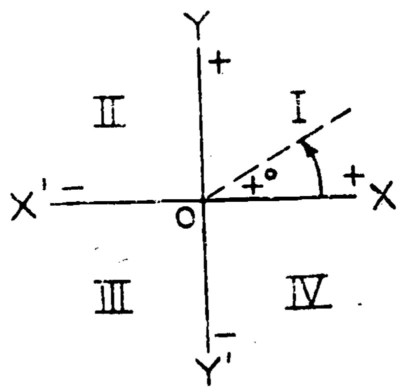


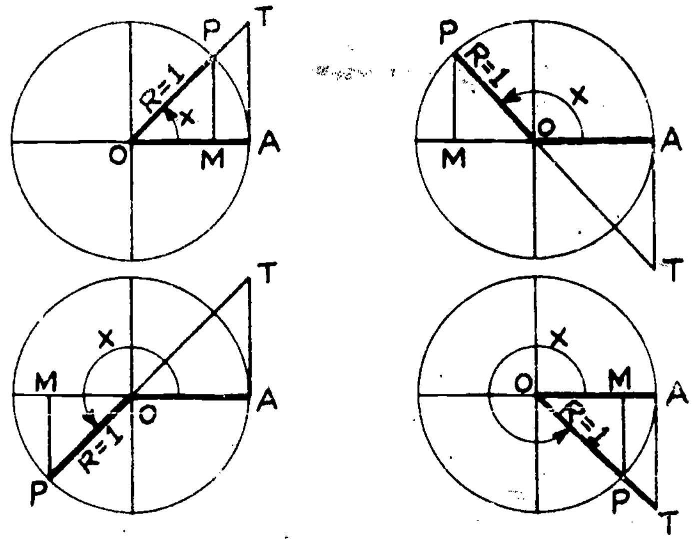
Figure II

By using a circle with a radius of unity, (R = 1), each of the 3 main functions will be as follows (neglecting signs).

$$\text{Sin } x = \frac{MP}{1} = MP$$

$$\text{Cos } x = \frac{OM}{1} = OM$$

$$\text{Tan } x = \frac{AT}{1} = AT$$



By examining the figures, we see that in Quadrant I, all the functions are positive. In Quadrant II, Sin only is positive. In Quadrant III, Tan only is positive. In Quadrant IV, Cos only is positive.

Also from examination of the figure and the unit circles, the following relationships are obtained:

Quadrant II

$\text{Sin } x = \text{Sin } (180^\circ - x)$

$\text{Cos } x = - \text{Cos } (180^\circ - x)$

$\text{Tan } x = - \text{Tan } (180^\circ - x)$

or, if $x = 170^\circ$

$\text{Sin } 170^\circ = \text{Sin } 10^\circ$

$\text{Cos } 170^\circ = - \text{Cos } 10^\circ$

$\text{Tan } 170^\circ = - \text{Tan } 10^\circ$

Quadrant III

$\text{Sin } x = - \text{Sin } (\cancel{180^\circ + x}) \; (x - 180^\circ)$

$\text{Cos } x = - \text{Cos } (\cancel{180^\circ + x}) \; (x - 180)^\circ$

$\text{Tan } x = \text{Tan } (\cancel{180^\circ + x}) \; (x - 180^\circ)$

or, if $x = 245^\circ$

$\text{Sin } 245^\circ = - \text{Sin } 65^\circ$

$\text{Cos } 245^\circ = - \text{Cos } 65^\circ$

$\text{Tan } 245^\circ = \text{Tan } 65^\circ$

Quadrant IV

$\text{Sin } x = - \text{Sin } (360^\circ - x)$

$\text{Cos } x = \text{Cos } (360^\circ - x)$

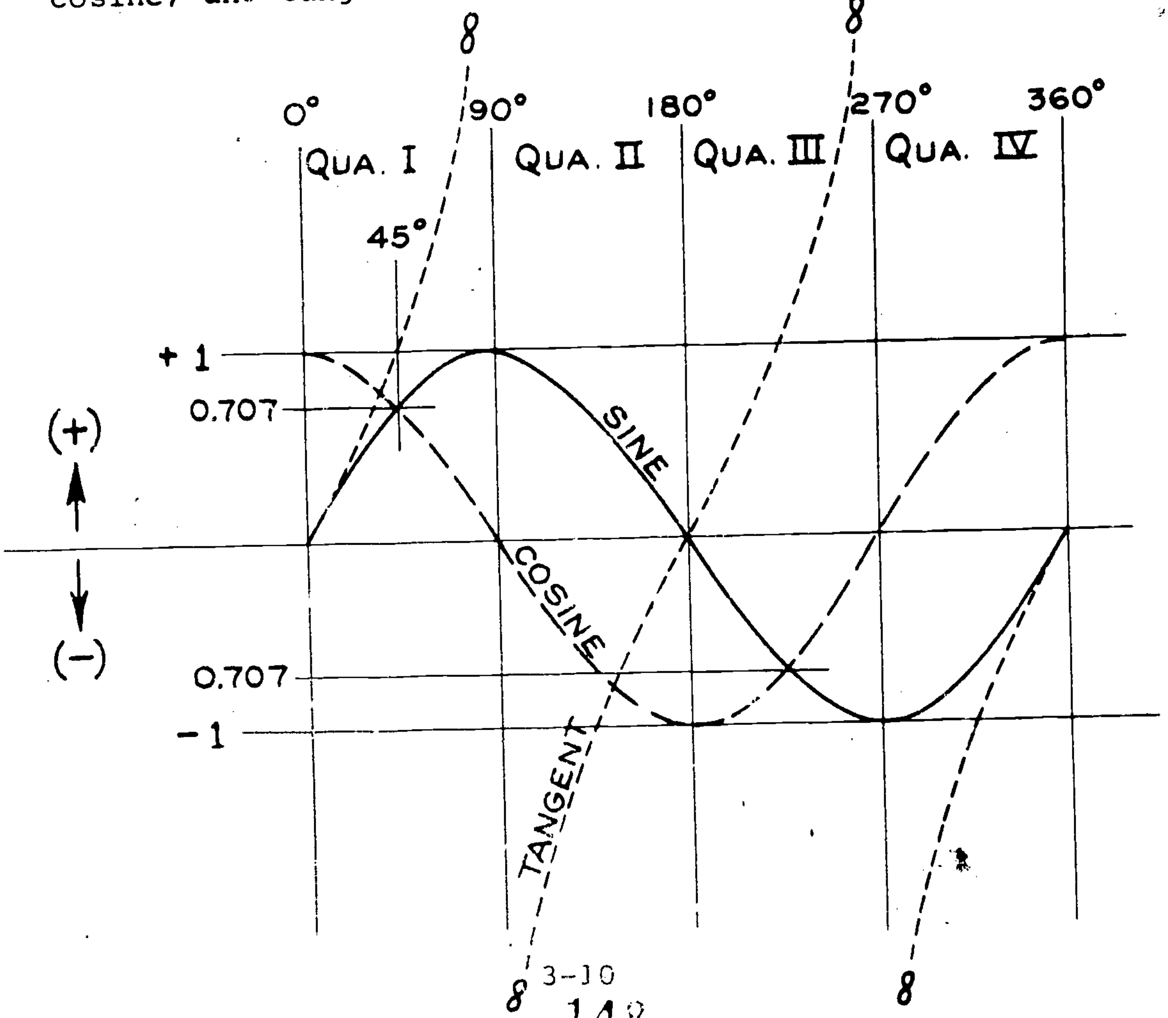
$\text{Tan } x = - \text{Tan } (360^\circ - x)$
or, if $x = 310^\circ$

$\text{Sin } 310^\circ = - \text{Sin } 50^\circ$

$\text{Cos } 310^\circ = \text{Cos } 50^\circ$

$\text{Tan } 310^\circ = - \text{Tan } 50^\circ$

Now, if we were to: First, lay the unit circle out into a straight line and call anything above the line positive and anything below the line negative, second, lay off four equal parts of 90° each and call these parts Quadrant I, II, III, IV, respectively, and third, plot the three trigonometric functions; sine, cosine, and tangent, we would obtain the following plot:



From this plot we can observe many things;

First:

(a) All functions are positive in Quadrant I.

(b) The Sine is the only function positive in Quadrant II.

(c) The Tangent is the only function positive in Quadrant III.

(d) The Cosine is the only function postive in Quadrant IV.

Second:

(a) The Sine of 0^{o} equals 0.

(b) The Cosine of 0^{o} equals +1.

(c) The Tangent of 0^{o} equals 0.

(d) The Sine of 90^{o} equals +1.

(e) The Cosine of 90^{o} equals 0.

(f) The Tangent 90^{o} equals ∞ (∞ = Infinity) and so on.

Third:

(a) The value of the Sine and Cosine are less than one (1) for all angles.

(b) The Tangent of 45^{o} equals 1; the value for the Tangent of angles < 45 is < 1.0. The value for the Tangent of angles > 45 is > 1.0.

Fourth:

The Sine and Cosine have equal values at 45^{o}.

Fifth:

The value of the Sine and the Tangent is almost equal for small angles.

Sixth:

What can we say about the relationship between the Sine and Cosine? What else can we learn about the Sine, Cosine, and Tangent from this plot?

Since the solution to a trigonometric problem may show a negative answer, this portion was included to show that the direction of rotation and the position of the positive and negative quadrants are the only way you have of knowing where an angle is in the solution of a complex trigonometric problem.

IV - Oblique Triangles

There are three main solutions to an oblique triangle when three independent parts are given. These solutions are:

1. Law of Sines
2. Law of Cosines
3. Law of Tangents

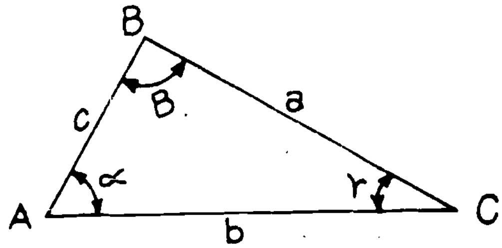
The first two of these, (Law of Sines and the Law of Cosines), are the most used.

1. Law of Sines

In any triangle, the lengths of the sides are proportional to the Sines of the opposite angles; that is:

(A) $a:b:c = \sin\alpha : \sin B : \sin\gamma$ or

(B) $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin B} = \dfrac{c}{\sin\gamma}$



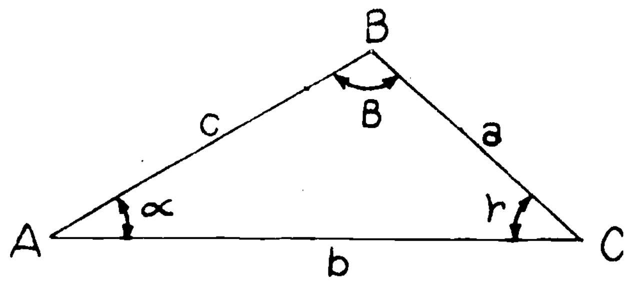
(C) In (B) we are abbreviating three equations:

$$\frac{a}{\text{Sin}\,\alpha} = \frac{b}{\text{Sin B}};\ \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin}\,\gamma};\ \frac{c}{\text{Sin}\,\gamma} = \frac{a}{\text{Sin}\,\alpha}$$

Example:

Given:

$\measuredangle\alpha = 25^\circ\ 10'$
$\measuredangle B = 110^\circ\ 20'$
$a = 130.6'$
$b = ?$



Solution:

With two angles and one opposite side we can solve for side b by the Law of Sines.

$$\frac{a}{\text{Sin}\,\alpha} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin}\,\gamma}$$

We are given a, $\measuredangle\alpha$, and $\measuredangle$ B, and we want side b. So we shall use:

$$\frac{a}{\text{Sin}\,\alpha} = \frac{b}{\text{Sin B}}$$

Solve for b: $b = \dfrac{a(\text{Sin B})}{\text{Sin}\,\alpha}$

Look up the functions of the given angles.

$\text{Sin } 25^\circ\ 10' = 0.42525\ (\text{Sin}\,\alpha)$

$\text{Sin } 110^\circ\ 20' = \text{Sin } (180^\circ - 110^\circ\ 20') = \text{Sin } 69^\circ\ 40'$
and $\text{Sin } 69^\circ\ 40' = 0.93769\ (\text{Sin B})$

Substituting these values in the above equation:

$$b = \frac{130.6\ (0.93769)}{0.42525}$$

$$b = 288.0'$$

2. Law of Cosines

In any triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice their product times the Cosine of their included angle; that is:

(A) $a^2 = b^2 + c^2 - 2bc \cos \alpha$

(B) $b^2 = a^2 + c^2 - 2ac \cos B$

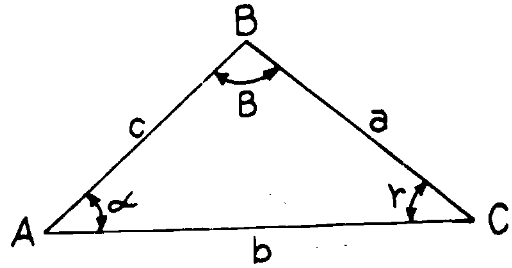
(C) $c^2 = a^2 + b^2 - 2ab \cos \gamma$

By solving A, B, and C for the Cosine we get:

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

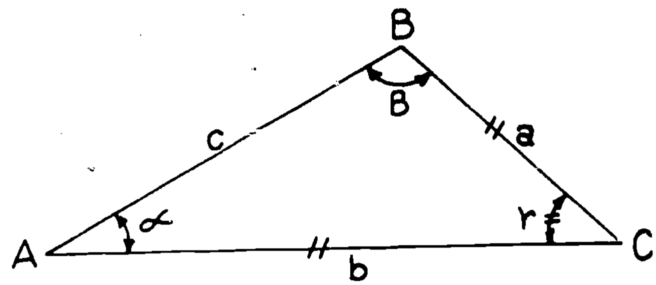
$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}$$



<u>Example</u>:

Given:

$\measuredangle \gamma = 41^\circ\ 21'$
$a = 140$ feet
$b = 620$ feet
$c = ?$



Solution:

With two sides and the included angle, we can solve for side C by the Law of Cosines.

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$a^2 = (140)^2 = 19{,}600$$

$$b^2 = (620)^2 = 384{,}400$$

$$2ab = 2(140)(620) = 173{,}600$$

$$\cos \gamma = \cos 41^\circ\ 21' = 0.75069$$

$$c^2 = (19{,}600) + (384{,}400) - (173{,}600)(0.75069)$$

$$c^2 = 404{,}000 - 130{,}320$$

$$c^2 = 273{,}680$$

$$c = \sqrt{273{,}680}$$

$$c = 523.15 \text{ feet}$$

Answer: $c = 523$ feet

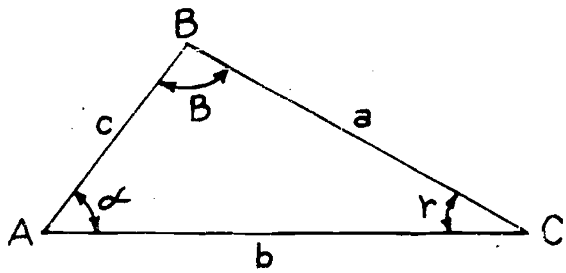
3. Law of Tangents

In a triangle, the difference of any two sides, divided by their sum, equals the Tangent of one-half of the difference of the opposite angles divided by the Tangent of one-half their sum.

$$\frac{a - b}{a + b} = \frac{\text{Tan } 1/2\ (\alpha - B)}{\text{Tan } 1/2\ (\alpha + B)}$$

$$\frac{c - a}{c + a} = \frac{\text{Tan } 1/2\ (\gamma - \alpha)}{\text{Tan } 1/2\ (\gamma + \alpha)}$$

$$\frac{b - c}{b + c} = \frac{\text{Tan } 1/2\ (B - \gamma)}{\text{Tan } 1/2\ (B + \gamma)}$$



We can use the Law of Tangents of we have two sides and an angle opposite, or two angles and a side opposite.

The Law of Tangents is the least used of all the laws due to its complex nature.

Example:

Solve the Δ ABC if $\angle \alpha = 78^\circ\ 48'$, $b = 726'$, $c = 938'$

Solution by Law of Tangents:

$$\frac{c - b}{c + b} = \frac{\text{Tan } 1/2\ (\gamma - B)}{\text{Tan } 1/2\ (\gamma + B)}$$

$$\text{Tan } 1/2\ (\gamma - B) = \frac{(c - b)\ \text{Tan } 1/2\ (\gamma + B)}{c + b}$$

Given: $\angle \alpha = 78^\circ\ 48'$

$b = 726'$

$c = 938'$

$\frac{1}{2}(\gamma + B) = \frac{1}{2}(180^\circ - \alpha)$. [Total of the interior $\angle$ of a triangle $= 180^\circ$]

$$
\begin{aligned}
\tfrac{1}{2}(\gamma + B) &= \tfrac{1}{2}(180^\circ - 78^\circ\ 48') \\
&= \tfrac{1}{2}(101^\circ\ 12') \\
&= 50^\circ\ 36'
\end{aligned}
$$

$$
\begin{aligned}
\tan \tfrac{1}{2}(\gamma - B) &= \frac{(938' - 726')\ \tan 50^\circ\ 36'}{(938' + 726')} \\
&= \frac{(212')\ \tan 50^\circ\ 36'}{1664'}
\end{aligned}
$$

The function for the Tan $50^\circ\ 36' = 1.2174$

$$\tan \tfrac{1}{2}(\gamma - B) = \frac{(212')(1.2174)}{1664'}$$

$$
\begin{aligned}
\tan \tfrac{1}{2}(\gamma - B) &= 0.1551014 \\
\tan \tfrac{1}{2}(\gamma - B) &= 8^\circ\ 49'
\end{aligned}
$$

Now we know:

$$
\begin{aligned}
\tfrac{1}{2}(\gamma + B) &= 50^\circ\ 36' \\
\tfrac{1}{2}(\gamma - B) &= 8^\circ\ 49'
\end{aligned}
$$

Simplify:

$$
\begin{array}{lrcl}
 & \tfrac{1}{2}\gamma + \tfrac{1}{2}B & = & 50^\circ\ 36' \\
\text{Add} & \tfrac{1}{2}\gamma - \tfrac{1}{2}B & = & 8^\circ\ 49' \\
\hline
 & \gamma & = & 58^\circ\ 85' \\
 & \gamma & = & 59^\circ\ 25'
\end{array}
$$

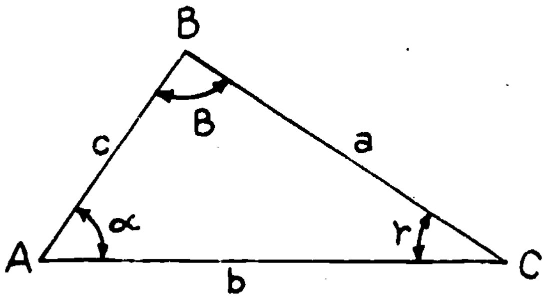
[Now we have two equations with two unknowns. Add to solve for γ. Subtract to solve for B.]

$$
\begin{array}{lrcl}
 & & & 49^\circ\ 96' \\
 & \tfrac{1}{2}\gamma + \tfrac{1}{2}B & = & \cancel{50}^\circ\ \cancel{36}' \\
\text{Subtract} & \tfrac{1}{2}\gamma - \tfrac{1}{2}B & = & 8^\circ\ 49' \\
(-) & (+) & & (-) \\
\hline
 & B & = & 41^\circ\ 47'
\end{array}
$$

Check angles:

$$
\begin{aligned}
\alpha &= 78^\circ\ 48' \\
B &= 41^\circ\ 47' \\
\gamma &= \underline{59^\circ\ 25'} \\
\Sigma &= 180^\circ\ 00'
\end{aligned}
$$

Now we have all three angles and two sides of the triangle. We can solve for the remaining side by the Law of Sines.



$\measuredangle \alpha = 78^\circ\ 48'$
$\measuredangle B = 41^\circ\ 47'$
$\measuredangle \gamma = 59^\circ\ 25'$

$c = 938'$
$b = 726'$
$a = ?$

$$\frac{a}{\text{Sin } \alpha} = \frac{b}{\text{Sin } B}$$

$$a = \frac{b \text{ Sin } \alpha}{\text{Sin } B}$$

$$a = \frac{726' \text{ Sin } 78^\circ\ 48'}{\text{Sin } 41^\circ\ 47'}$$

The function of the: Sin $78^\circ\ 48' = 0.98096$
Sin $41^\circ\ 47' = 0.66632$

$$a = \frac{726'\,(0.98096)}{(0.66632)}$$

$$a = \underline{1069'}$$

Answer: $a = 1069'$, $\gamma = 59^\circ\ 25'$, $B = 41^\circ\ 47'$

PROBLEMS - Oblique Triangles

Solve by Law of Sines:

1. $b = 5'$ $\measuredangle \alpha = 75^\circ$ $\measuredangle B = 30^\circ$

   find a, c, and $\measuredangle \gamma$ (Answer $a = c = 9.659'$, $\measuredangle \gamma = 75^\circ\ 0'$)

2. $a = 50'$ $\measuredangle \alpha = 37^\circ\ 30'$ $\measuredangle B = 71^\circ\ 10'$

   find b, c, and $\measuredangle \gamma$

Solve by Law of Cosines:

3. $a = 6''$ $c = 10''$ $\measuredangle B = 155^\circ\ 30'$

   find b. (Answer = $15.7''$)

4. $b = 4'$ $c = 11'$ $\measuredangle \alpha = 63^\circ\ 28'$

   find a.

Solve by Law of Tangents:

5. $b = 30'$ $c = 25'$ $\angle\alpha = 50^\circ$

find a, $\angle$ B, and $\angle\gamma$ (Answer: $a = 23.7'$
$\angle B = 76^\circ\ 02'$
$\angle\gamma = 53^\circ\ 58'$)

Solve by Any Method:

6. A battery commander "B" is order to shoot at a target "T" from a position "G", from which "T" is visible. To check on the range $\overline{G\text{-}T}$ as found by a range finder, "B" locates an observation point "H" from which "T" and "G" are visible. The bearing of "T" from "G" is N. 12° 48' E., of "T" from "H" is N. 6° 23' W., and of "H" from "G" is S. 82° 53' E. From a map it is found that $\overline{GH}$ = 3250 yards. Find the range of the target "T" from "G".

Trigonometric Functions

$\angle$	Sin	Cos	Tan
19° 11'	0.32859	0.94447	0.34791
24° 30'	0.41469	0.90996	0.45573
30° 00'	0.50000	0.86603	0.57735
37° 30'	0.60876	0.79335	0.76733
50° 00'	0.76604	0.64279	1.1918
53° 58'	0.80867	0.58826	1.3747
63° 28'	0.89467	0.44672	2.0028
71° 10'	0.94646	0.32282	2.9319
71° 20'	0.94740	0.32006	2.9600
75° 00'	0.96593	0.25882	3.7321
76° 02'	0.97044	0.24136	4.0207
76° 30'	0.97237	0.23345	4.1653
84° 19'	0.99508	0.09903	10.048

TRIGONOMETRIC FORMULAS

Radius, $1 = \sin^2 A + \cos^2 A$

$= \sin A \operatorname{cosec} A = \cos A \sec A = \tan A \cot A$

Sine $A = \dfrac{\cos A}{\cot A} = \dfrac{1}{\operatorname{cosec} A} = \cos A \tan A = \sqrt{1 - \cos^2 A}$

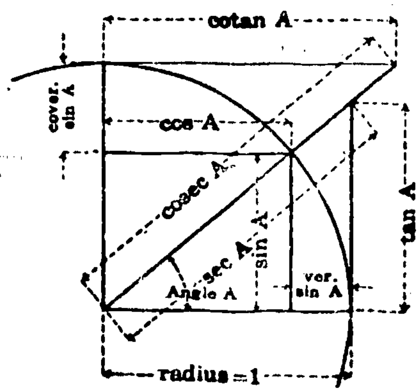
Cosine $A = \dfrac{\sin A}{\tan A} = \dfrac{1}{\sec A} = \sin A \cot A = \sqrt{1 - \sin^2 A}$

Tangent $A = \dfrac{\sin A}{\cos A} = \dfrac{1}{\cot A} = \sin A \sec A$

Cotangent $A = \dfrac{\cos A}{\sin A} = \dfrac{1}{\tan A} = \cos A \operatorname{cosec} A$

Secant $A = \dfrac{\tan A}{\sin A} = \dfrac{1}{\cos A}$

Cosecant $A = \dfrac{\cot A}{\cos A} = \dfrac{1}{\sin A}$



$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

$$\sin A + \sin B = 2 \sin 1/2\,(A + B) \cos 1/2\,(A - B)$$

$$\sin A - \sin B = 2 \cos 1/2\,(A + B) \sin 1/2\,(A - B)$$

$$\tan A + \tan B = \frac{\sin(A + B)}{\cos A \cos B}$$

$$\tan A - \tan B = \frac{\sin(A - B)}{\cos A \cos B}$$

$$\cos A + \cos B = 2 \cos 1/2\,(A + B) \cos 1/2\,(A - B)$$

$$\cos B - \cos A = 2 \sin 1/2\,(A + B) \sin 1/2\,(A - B)$$

$$\cot A + \cot B = \frac{\sin(B + A)}{\sin A \sin B}$$

$$\cot A - \cot B = \frac{\sin(B - A)}{\sin A \sin B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}$$

$$\sin 1/2\,A = \sqrt{\frac{1 - \cos A}{2}} \qquad \cos 1/2\,A = \sqrt{\frac{1 + \cos A}{2}}$$

$$\sin^2 A = \frac{1 - \cos 2A}{2} \qquad \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\tan 1/2\,A = \frac{\sin A}{1 + \cos A} \qquad \cot 1/2\,A = \frac{\sin A}{1 - \cos A}$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A} \qquad \cot^2 A = \frac{1 + \cos 2A}{1 - \cos 2A}$$

$$\sin^2 A - \sin^2 B = \sin (A + B) \sin (A - B)$$

$$\frac{\sin A \pm \sin B}{\cos A + \cos B} = \tan 1/2 \ (A \pm B)$$

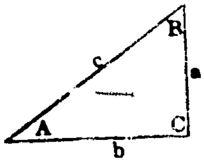
$$\cos^2 A - \sin^2 B = \cos (A + B) \cos (A - B)$$

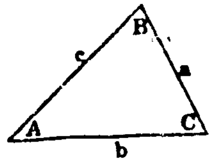
$$\frac{\sin A \pm \sin B}{\cos B - \cos A} = \cot 1/2 \ (A \mp B)$$

Quadrant	I	II	III	IV	Angle		
Angles	0° to 90°	90° to 180°	180° to 270°	270° to 360°	30°	45°	60°
Functions	Values vary from				Equivalent values		
sin	+0 to +1	+1 to +0	−0 to −1	−1 to −0	½	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
cos	+1 to +0	−0 to −1	−1 to −0	+0 to +1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	½
tan	+0 to +∞	−∞ to −0	+0 to +∞	−∞ to −0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$
cot	+∞ to +0	−0 to −∞	+∞ to +0	−0 to −∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$

Angle a < 90°				
Angle	sin	cos	tan	cot
$\phi°$	$\phi°$	$\phi°$	$\phi°$	$\phi°$
0° ± a	± sin a	+ cos a	± tan a	± cot a
90° ± a	cos a	∓ sin a	∓ cot a	∓ tan a
180° ± a	∓ sin a	− cos a	± tan a	± cot a
270° ± a	− cos a	± sin a	∓ cot a	∓ tan a

TRIGONOMETRIC SOLUTION OF TRIANGLES





$s = \frac{a+b+c}{2}$

Given	Sought	Formulae
		Right-Angled Triangles
a, c	A, B, b	$\sin A = \frac{a}{c}$, $\cos B = \frac{a}{c}$, $b = \sqrt{c^2 - a^2}$
	Area	$\text{Area} = \frac{a}{2}\sqrt{c^2-a^2}$
a, b	A, B, c	$\tan A = \frac{a}{b}$, $\tan B = \frac{b}{a}$, $c = \sqrt{a^2 + b^2}$
	Area	$\text{Area} = \frac{ab}{2}$
A, a	B, b, c	$B = 90° - A$, $b = a \cot A$, $c = \frac{a}{\sin A}$
	Area	$\text{Area} = \frac{a^2 \cot A}{2}$
A, b	B, a, c	$B = 90° - A$, $a = b \tan A$, $c = \frac{b}{\cos A}$
	Area	$\text{Area} = \frac{b^2 \tan A}{2}$
A, c	B, a, b	$B = 90° - A$, $a = c \sin A$, $b = c \cos A$
	Area	$\text{Area} = \frac{c^2 \sin A \cos A}{2} = \frac{c^2 \sin 2A}{4}$
		Oblique-Angled Triangles
a, b, c	A	$\sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}$, $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
	B	$\sin \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}}$, $\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}$, $\tan \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$
	C	$\sin \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}$, $\cos \frac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}$, $\tan \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$
	Area	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$
a, A, B	b, c	$b = \frac{a \sin B}{\sin A}$, $c = \frac{a \sin C}{\sin A} = \frac{a \sin (A+B)}{\sin A}$
	Area	$\text{Area} = \frac{1}{2} a b \sin C = \frac{a^2 \sin B \sin C}{2 \sin A}$
a, b, A	B	$\sin B = \frac{b \sin A}{a}$
	c	$c = \frac{a \sin C}{\sin A} = \frac{b \sin C}{\sin B} = \sqrt{a^2 + b^2 - 2ab \cos C}$
	Area	$\text{Area} = \frac{1}{2} a b \sin C$
a, b, C	A	$\tan A = \frac{a \sin C}{b - a \cos C}$, $\tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2} C$
	c	$c = \sqrt{a^2 + b^2 - 2ab \cos C} = \frac{a \sin C}{\sin A}$
	Area	$\text{Area} = \frac{1}{2} ab \sin C$

$a^2 = b^2 + c^2 - 2bc \cos A$, $b^2 = a^2 + c^2 - 2ac \cos B$, $c^2 = a^2 + b^2 - 2ab \cos C$