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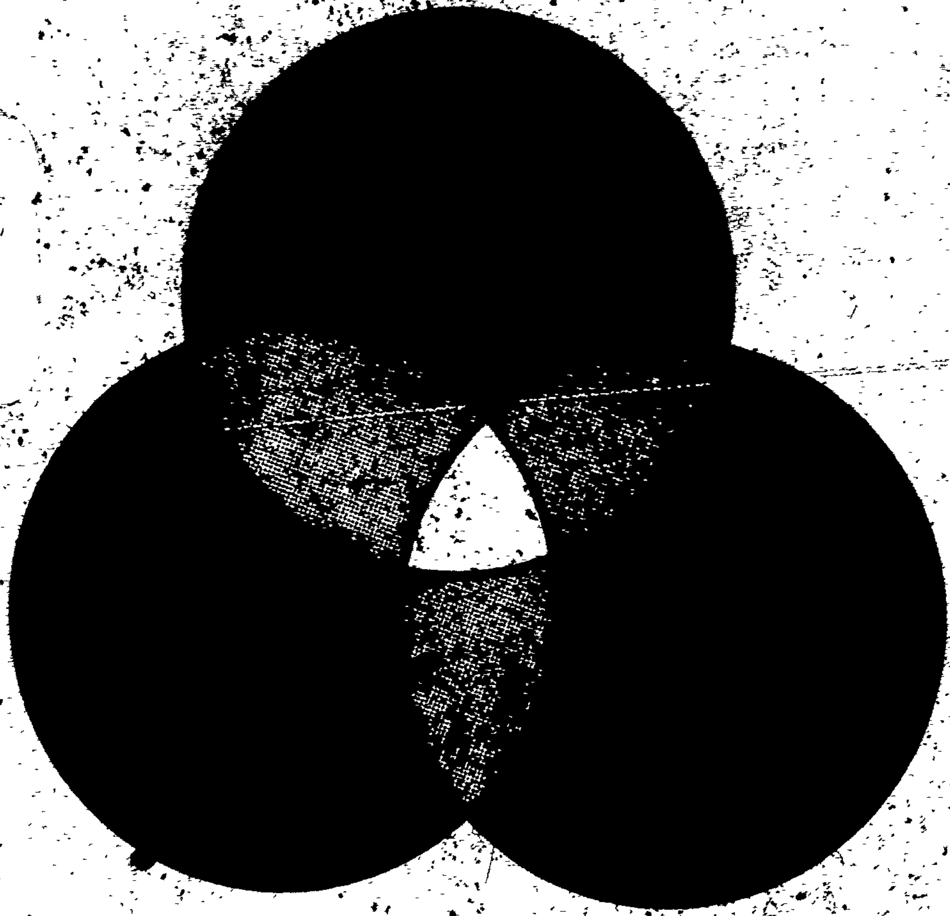
ABSTRACT

This self-instructive workbook focuses on a set of propositional abilities related to learning. It utilizes written materials, manipulative materials, videotapes, and students, and is supplemented by a program and support system which includes instruction, teaching experiences for practice, discussions, and individual conferences. The three major goals of this workbook are to enable the teacher to: (1) use a clinical method to administer, at any grade level, diagnostic tasks related to propositional abilities; (2) identify the developmental level each student in the class has attained in terms of propositional abilities; and (3) select, from resources available, subject matter appropriate to identified levels of development. Propositional tasks are included which deal with various types of reasoning (probablistic, disjunctive, combinational, propositional, deductive, and proportional). The workbook contains outlines for eight self-directed workshops some of which concern (1) developing the ability to administer propositional tasks, (2) administering propositional tasks to a student, (3) solving problems involving propositional abilities, and (4) reading about propositional abilities. There is also a section dealing with a review of some research on propositional abilities, a bibliography, and an appendix which gives examples of various types of problems related to propositional abilities. (BD)



learning about learning:

propositional abilities

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LEARNING ABOUT ..... SERIES

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learning about learning: propositional abilities

A PERSONAL WORKSHOP

by

Lawrence F. Lowery

This self-directed, personal workshop focuses upon the identification of a set of abilities related to learning. The set, described as propositional abilities, is not independent of but is distinct from other sets of abilities described in other books in the *"Learning About ... Series."* Although you will learn many ideas and develop various skills from experiencing this workshop, three major goals are identified below.

As a result of this workshop, you will be able to:

1. use a clinical method to administer, at any grade level, diagnostic tasks related to propositional abilities.
2. identify the developmental level each student in your class has attained in terms of propositional abilities.
3. select, from resources available, subject matter (content and manipulative materials) appropriate to identified levels of development.

Hopefully you will become able to determine levels of development by observing students during their normal activities without relying upon the administration of the tasks.

This workshop is self-instructive. It utilizes written materials, manipulative materials, videotapes, and students in your classroom. It is supplemented by a program and support system which includes instruction, teaching experiences for practice, discussions, and individual conferences.

This book on Propositional Abilities is one of several being developed under the general heading *"Learning About ... Series"* by the University of California Cooperative Teacher Preparation Project (UCCTPP).

The project, a new instructional model for teacher education, is being implemented through cooperation of the School of Education of the University of California at Berkeley, the Lawrence Hall of Science, and the Mt. Diablo and Vallejo Unified School Districts. Approximately 45 beginning teachers, both elementary and secondary, spend one year of pre-service preparation and their first two years of classroom teaching within the program. The project represents an effort to manage a much larger portion of the beginning teacher's experience than has hitherto been attempted. One major objective of the project is to assist new teachers and the experienced teachers working with them in becoming regular evaluators of their own instruction. Participants in the program are taught to use materials and techniques that help assess stages of intellectual development in students, to identify and prescribe learning activities commensurate with their students' intellectual development, and to use techniques for assessing their interactions with students. The research objective of the project is to determine the effectiveness of the model program in developing the instructional style of beginning and experienced teachers.

The *"Learning About ... Series"* is being developed through the resources of the School of Education and Lawrence Hall of Science, University of California, Berkeley. The research dimension is being funded through a National Science Foundation grant.

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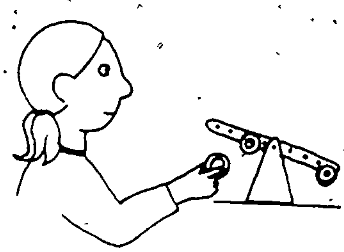
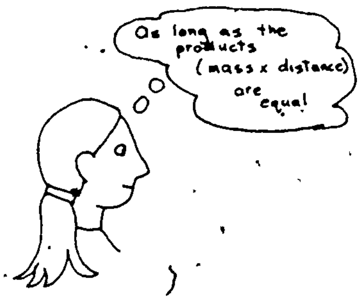
INTRODUCTION

The term propositional is used to describe the integration of a particular set of cognitive abilities and operations. These abilities and operations are revealed when a student can think systematically in purely abstract terms. Such a student has developed the capacity to deal with symbols and ideas verbally without the necessity for an intervening manipulation of or a recent experience with physical objects (reality becomes secondary to possibility). The individual can see given facts (reality) as the actually attained part of a complete set of possible transformations and can find the entire set of these transformations by hypothesizing.





For example, if a student attempts to predict which objects in a collection of objects will float and which will sink, he can generate a set of possibilities or hypotheses such as *"sinking is caused by the object's weight, or by its size, or by its weight in relation to its size, or by some other variables,"* then develop a plan for systematically testing the hypotheses in a way that will confirm or disprove each conclusively. This capacity to reason about possibilities or hypotheses, and not just about specific objects at hand, is the most distinctive feature characterizing propositional abilities.

Another important characteristic of propositional abilities is that cognitive operations are performed on other cognitive operations as opposed to concrete operations that are performed directly on objects and events. For example, if a student is to understand the formula for specific gravity,

$$\textit{specific gravity} = \frac{\textit{density of a substance}}{\textit{density of water}}$$

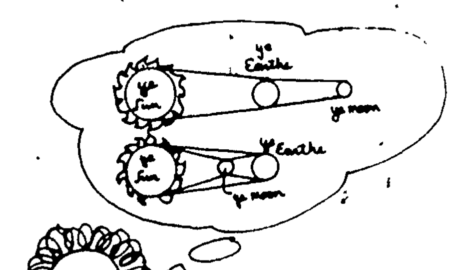
the student must first understand the formula for density,

$$\textit{density} = \frac{\textit{weight}}{\textit{volume}}$$

and to understand the formula for density, the student must utilize two reversible operations: If, according to the formula, the density of an object can be increased by adding weight while holding volume constant, it follows that this increase in density can be nullified either by removal of weight (inverse operation) or by a compensating increase in volume (reciprocity operation) or by certain combinations of both. Thus, understanding the specific gravity formula requires the ability to relate the two densities or the use of operations upon operations.

Propositional abilities and the operations one performs with them, emerge from age 12 to 18 and constitute the heart of Jean Piaget's Formal Operational Stage of cognitive development. Piaget separates this stage into two substages, which he labels IIIa and IIIb.

The first substage (IIIa) is distinguished by the appearance of a kind of reasoning based upon the logic of all possible combinations. For example, whether experimental or verbal problems are under consideration, combinatorial analysis (regard for all possible combinations) becomes an integral part of their solution. The student can be called upon to manipulate physically all combinations, as in the chemical substances task of Inhelder and Piaget (See Combinatorial Reasoning Task in this book), or he can be asked to identify mentally all combinations from verbal problems such as those developed by Morf (See Propositional Reasoning Task). The basic power of propositional abilities rests in being able to generate a set of hypotheses that are compatible with what one already knows about the problem and the implications of which can be tested by experiment. Through testing the implications of the hypotheses, one learns new information about the problem.



Making its appearance during the transition from substage IIIa to substage IIIb is the ability to separate variables by exclusion (the systematic testing of each variable individually while all others are held constant in order to determine which are relevant and which have no effect). Instead of being closely bound to immediate experience, the student can suggest, *"Maybe if I changed this, such and such would happen"* and then perform the necessary actions to confirm or disprove his suppositions. Inhelder and Piaget's pendulum problem is an example of this exclusion principle (See Controlling and Manipulating Variables Task).

The second substage (IIIb) is distinguished by the development of an individual's ability to organize his reasoning along the direction of particular logical operations such as conjunction (*"It must be A and B"*), disjunction (*"It must be either A or B"*), negation (*"It is neither A nor B"*), and implication (*"If it is A, then C will be true"*). As the student learns to think in increasingly abstract terms, he learns to combine such cognitive operations simultaneously.

Attainment of Piaget's final stage does not mean that intellectual development is at its peak. The mind continues to develop beyond adolescence, however, later changes in cognition are simply refinements of the kind of thinking which becomes more or less established by about age 15. There are no more major qualitative changes in the nature of cognition such as those which occur during childhood.

PROBABLE SEQUENCE IN WHICH

PROPOSITIONAL ABILITIES APPEAR

TYPE OF PROPOSITIONAL ABILITY	USUAL AGE AT WHICH ABILITY APPEARS									
	10	11	12	13	14	15	16	17	18	19
Probabilistic Reasoning		.....	——	——	...					
Disjunctive Reasoning			...	——	...					
Combinatorial Reasoning				...	——	...				
Propositional Reasoning				...	——	...				
Controlling and Manipulating Variables				...	——	——	...			
Deductive Reasoning						...	——	——	.....	
Proportional Reasoning							...	——	——	.....

Summary

With the attainment of propositional abilities, the student can utilize certain basic strategies of scientific thinking, such as the scheme of *"all other things being equal."* The student is able to generate hypothetical possibilities and to reason about them with completely coordinated reversibility, such as performing operations on other operations.

With a friend, take turns trying each of the propositional ability tasks in this section.

At first, try to use the suggested dialogue that accompanies each task, then gradually incorporate your own style.

If any questions or problems come to mind, discuss them with a UCCTPP staff member.

probabilistic reasoning

Principle

Probability is the relative frequency with which an event occurs or is likely to occur. Probabilistic reasoning involves mentally separating what is possible, what is real, and what is deductively necessary. At the Formal Operational Stage, this begins with the identification of possible events and ends with reality conceived as a realized part of the total number of possible events. For example, a student with this ability, given data that the number *three* is one of six numbers on a die which are equally likely to occur when the die is rolled, will predict in advance of rolling that the number *three* will appear on the average once in every six or ten in every sixty rolls of the die. Most students attain this ability by age 12 or 13.

Materials

2 opaque medium-sized paper bags.

14 black (or other color) poker chips.
14 white (or other color) poker chips.
} Other objects can be substituted.

Task

The student (S) is sitting opposite the experimenter (E).

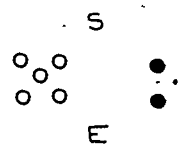
E places the black and white chips in two separate piles of twelve chips each, off to the side, but within reach of E and S.

Procedure

This task includes four separate parts. After each part, E may elect to discontinue the experiences depending on the performance of S. Guidelines for continuing or discontinuing are provided at the end of each part.

Part I

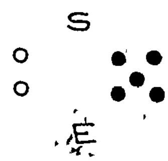
E arranges two separate groups of poker chips, one containing 5 white chips, the other 2 black chips, between E and S.



E begins: *"Here is a set of five white poker chips and two black poker chips."*

1. E places all 7 chips in a paper bag and shakes them up. As this is done, E states: *"I am putting the chips in this bag and shaking it. Now, if I were to take out 1 chip without first looking at its color, would I be more likely to get a white chip or a black chip?"* E allows S to respond, then asks: *"How do you know?"* E again allows S to respond, then returns all chips to the piles.

2. E now constructs two separate groups of chips, one containing 2 white chips, the other 5 black chips, between E and S. E says: *"Now I have two white chips and five black ones. If I put these chips in the bag as before and shake them up, would I be more likely to get a black chip or a white?"* E allows S to respond, then explores S's reasoning.



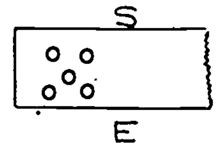
Guidelines for Proceeding

An acceptable response by S in either of the above experiences would be that the likelihood of drawing out a chip of a given color is greater if the bag contains more chips of that color than of the other. If S does not respond in an acceptable manner in either experience, E discontinues the task. Otherwise, E continues with Part II.

Part II

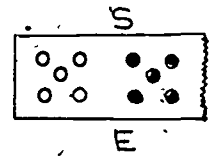
1. E places 5 white chips in a bag and sets the bag in front of S.

   E: *"There are five white chips in the bag. I want you to add black chips to the bag so that if we shook them in the bag and drew one out, we would be* _just as likely_ *to get a black chip as a white."* E allows S to respond, then asks: *"Why did you do what you did?"*



2. E next adjusts the number of chips in the bag so there are 5 black and 5 white chips.

   E continues: *"There are now five white chips and five black chips in the bag. Can you change the black chips so that you are* _more likely_ *to get a white than a black,* _without_ *changing the whites you have?"* E allows S to respond, then asks: *"Why did you do what you did?"*



Guidelines for Proceeding

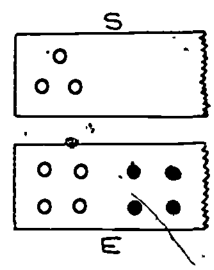
S passes the first experience if he adds 5 black chips to the bag and states that because there are now 5 chips of each color, the likelihood of drawing either color is the same. S passes the second experience if he removes any number of black chips from the bag and states that because there are now more white chips than black chips, the likelihood of drawing a white chip is now greater. If S passes at least one of the experiences, E continues with Part III. If S does not pass, discontinue the task.

Part III

E places one empty bag before S and one before E.

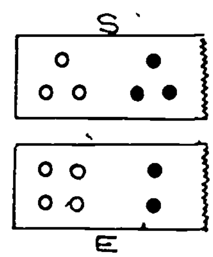
E: *"Here is a bag for you and a bag for me. We will each have a set of chips* _on_ *our bags. We will change our sets in different ways, but we will make it a rule that we always have at least one black chip and one white chip in each set. Do you understand?"*

E allows S to respond, then places two separate sets of chips, one containing 4 whites, the other 4 blacks, on E's bag. E then places one set of 3 white chips on S's bag.



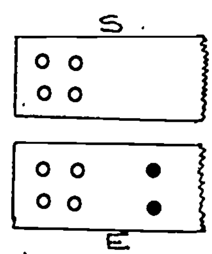
1. E: *"This time I have made a set of four whites and four blacks on my bag. You are to start with three whites on your bag. Don't change your whites. Now, using black chips from the pile of extras, make a set of black chips so that you are* _just as likely_ *to get a black as I am."* E allows S to respond, then asks: *"Why did you do what you did?"*

   After S responds, E adjusts S's chips, if necessary, so that S has two separate sets of 3 chips each, one white and one black, on S's bag. E then adjusts E's chips so that E has one set of 4 white chips and one set of 2 black chips.



2. E continues: *"I have changed my set to have four whites and two blacks. Your set has three whites and three blacks. Now, change your set by adding whites so that you are* _just as likely_ *to get a white as I am."* E allows S to respond, then asks: *"Why did you do what you did?"*

   E allows S to respond, then adjusts E's chips so that E has one set of 4 white chips and one set of 2 black chips. E adjusts S's chips so that S has one set of 4 white chips.



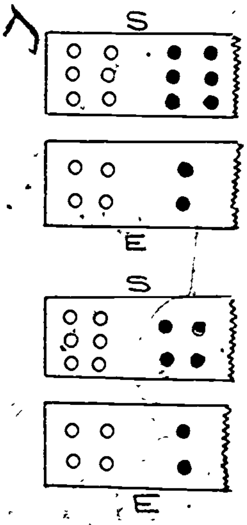
3. E continues: *"Now you have four white chips only. Add as few black chips as needed to give you a greater chance than I have of drawing a black chip. Don't change the whites."* E allows S to respond, then asks: *"Why did you do what you did?"*

E allows S to respond. E then retains one set of 4 white chips and another set of 2 black chips, but now adjusts S's chips so that S has one set of 6 white chips and another set of 6 black chips.

4. When this is done, E continues: *"I have left my set the same. Your set now has six whites and six blacks. Change it, without changing the whites, so that you are just as likely as I am to draw a black."* E allows S to respond, then asks: *"Explain why you did what you did."*

E allows S to respond. E then retains E's chips as before, but adjusts S's chips so that S has one set of 6 white chips and one set of 4 black chips.

5. E continues: *"This time I want you to change your set so that you are more likely than I am to draw a white. You can change either whites or blacks but not both. Don't change any more chips than you need to, and remember the rule that there must be at least one chip of each color."* E allows S to respond, then asks: *"Why did you do what you did?"*



Guidelines for Proceeding

S passes the first experience if he makes a separate set of three black chips and explains that since each of S's sets now contain an equal number of chips, the likelihood of drawing from one set is equal to that of drawing from the other.

S's response in the second experience is passing if S adds three whites to S's set of whites and explains that since the number of whites is now double the number of blacks in both S's and E's sets of chips, the likelihood of drawing whites is twice that of drawing blacks for both S and E.

S's response to the third experience is passing if S makes a set of three black chips and explains that since the number of blacks S now has is more than half of the number of whites S has, the likelihood of S drawing a black is more than it is for E (The number of blacks E has is still exactly one-half the number of whites E has).

S's response to the fourth experience is passing if S removes three black chips from S's set and explains that since the number of blacks S now has is exactly one-half the number of whites S has, the likelihood of drawing a black is the same as it is for E (E's blacks still number exactly one-half that of E's whites).

S's response to the fifth experience is passing if S changes S's chips so that the number of whites is at least one more than twice the number of blacks when S has completed the task and explains that since the whites now outnumber the blacks by more than two-to-one, S is more likely to draw a white than E (E's whites still outnumber E's blacks by exactly two-to-one). If S responds incorrectly to this last experience, E discontinues the tasks. Otherwise, E leaves E's and S's chips unchanged and continues with Part IV.

Part IV

E: *"You did the last experience by changing whites (or blacks). Now I would like you to do it by changing the other color (whites, blacks). Remember, you are to be more likely than I am to draw a white, and you are to change as few chips as necessary to do this."* E allows S to respond, then asks: *"Explain why you did what you did."*

Guidelines for Proceeding

A passing response is the same as for the last experience in Part III, except that S uses a different color.

Analysis

If S is at the Formal Operational Stage of development, S will estimate probability (or likelihood) as a relationship between reality (or what is given) and what is possible relative to the task presented. Consequently, S can predict the likelihood that an event will occur at some frequency relative to the occurrence of another event by comparing real instances of each, and then projecting or predicting their relative likelihood of occurrence in other possible situations. For example, a student at the Formal Operational Stage, given a real sample of five black poker chips and ten white chips from a universe of 100 mixed white and black poker chips, will predict that if 60 poker chips are drawn (a possible situation), then on the average, twenty will be black while forty will be white (an estimate of probability or likelihood).

If S is at the Concrete Operational Stage of development, S will not be able to estimate probability in the absence of a concrete model. For example, a student at this level, given a real sample of five black and ten white poker chips from the same universe of 100 mixed black and white poker chips mentioned above, might require not only the information that 60 chips will be drawn, but that the 60 chips be drawn in fact, and the black chips counted before the student will be able to determine the number of white chips in the sample.

In essence, the concrete operational student will revert to a simple additive model, rather than using a combinatorial one.

This task was adapted by permission from the research of Read Tuddenham, Professor of Psychology, University of California, Berkeley, California.

disjunctive reasoning

Principle

Solving problems based on factors that are disconnected or detached, require the ability to recognize the disjunctive aspects and order them in a way that reveals connections not directly apparent. Most students attain this ability by age 13 or 14.

Materials

2 story problems* (described below).

Task

The student (S) is sitting alongside or opposite the experimenter (E).

Procedure

E tells Story I to S, then seeks information concerning S's solution to the problem raised in the story. S is allowed to take notes if he wishes and can ask for information to be repeated.

Story I

I hiked with my brother in the mountains. When we arrived at a cabin, we found that we had forgotten our provisions. We looked in the cabin and found some remainders of food, but the food was not fresh.

We found a little condensed milk, some soup, and some old preserved meat. My brother ate the meat and some soup. I ate some soup and the condensed milk. An hour later we were both sick to the stomach. One of the three foods made both of us sick. Which could it have been? Explain your answer.

Checking Procedure

E tells Story II to S, then seeks additional information concerning S's explanations.

Story II

Mr. Smith is the manager of an apartment house that has three garages in it. The first garage belongs to Mr. Smith, the second to Mr. Moss, and the third to Mr. Williams. There are two keys to each garage. For the first garage, one key belongs to Mr. Smith and the other to Mr. Williams. For the second garage, one key belongs to Mr. Moss and the other to Mr. Williams. For the third garage, one key belongs to Mr. Williams and one to Mr. Smith.

One day it was found that some things were stolen from each of the garages. The police inspector said that whoever stole the items had keys to all the garages. Who did the police inspector arrest? Explain your answer.

Analysis

Careful listening to S's explanations will reveal that the type of justification given by an S who is at the Pre-formal Operational Stage of development, is limited to simple intuitive and causal solutions. An S at the Formal Operational Stage will take disjunction into consideration.

For example, a justification for Story I might be *"the milk"* or *"the meat made them sick because such foods are known to spoil and can cause sickness."* Even though a pre-formal S may be capable of telling who ate what and may retain all the facts perfectly, he will seek and give concrete causal solutions and ignore one part of the information. In this example, the fact that the milk or the meat was eaten by only one person was neglected.

Similarly, a justification given for Story II might be *"Mr. Smith because he is the manager."* In this justification, S focuses upon the occupation and intuitively feels that the manager must be the one who has access to all the garages. The pre-formal S never realizes that the manager does not have all the keys.

It is clear that pre-formal students will not make an attempt to work out solutions to these problems from the text. An intuitive or concrete situation in the story will dominate their explanations. Students in transition will often combine or confuse the causal and formal arguments in the same task.

E may also find that in Story I the statement *"one of these three foods made both of us sick"* is suppressed by pre-formal students to either *"one of"* or *"both of us."* This particular difficulty for the student seems to be in the combination or multiplication of three hypotheses with the phrase *"both of us."* The pre-formal student's reasoning cannot encompass all the conditions.

*Stories are adapted from: Morf, Albert. Les relations entre la logique et le langage lors du passage du raisonnement concret au raisonnement formel. In Apostol, L., B. Mandelbrot, and A. Morf. Logique, langage et theorie de l'information. Etudes D'epistemologie Genetique, Vol. 3. Paris: Presses Univ., 1957, 173-204.

combinatorial reasoning

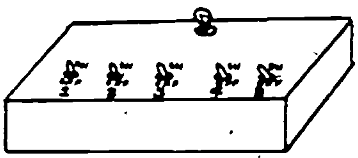
Principle

Solving problems that contain a number of possible solutions requires thinking that involves a complete combinatorial analysis. For example, combinatorial reasoning is necessary in order to determine all the possible five-card combinations that might be dealt from a standard deck of playing cards. Most students attain the ability to do this type of reasoning by age 14 or 15.

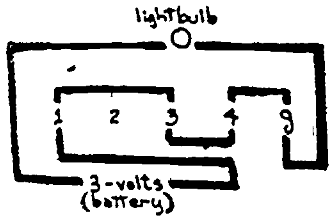
Materials

A small screen to keep five toggle switches from view.

A box made up of five toggle switches and a lightbulb.[1]



box



wiring diagram

Task

The student (S) is sitting opposite or alongside the experimenter (E).

Procedure

E shows S the box and tells S the light can be turned on by some combination of the switches in *on* and *off* positions.

E hides all the switches behind a screen and says: *"I am moving switches on and off behind this screen so as not to give you clues concerning how many of these switches, if any, one needs to move to make the light go on."*

After clicking a number of switches, E says: *"When I move the last switch, labeled "g", the light will go on."* E switches *"g"*, and the light goes on.

E again clicks a number of switches behind the screen, then returns each switch to its *off* position. NOTE: Be sure the number of clicks does not provide a clue.

E: *"Now I want you to make the light go on by using switches 1, 2, 3, 4, and g as you wish. For each trial, decide ahead of time which switches you are going to use, and tell me what they are."*

After S's first trial, E says: *"Return the switches you have just tried to the off position, making sure that switch g is the first one turned off."* E allows S to proceed with other trials but makes sure that before another combination is tried, the switches are returned to the *off* position by always returning the *g* switch first.

E keeps track of all the combinations that S tries.

If S does not proceed in a systematic way or becomes frustrated, discontinue the task.

If S is successful, E asks a series of questions:

a. *"Which switches make the light go on?"*

b. *"What will happen if the switches are turned on in this order: 2, 1, 3, g?; 1, 3, g, 2?; 4, 1, 3, g?; 1, 3, g, 4?"*

c. *"What do switches 2 and 4 do?"*

d. *"Is there any other way you can make the light go on?"* (There are two possible combinations)

Analysis

In order to turn on the light, other than by chance, the student must be equipped with a mental combinatorial system that enables him to keep the various permutations of possibilities in mind.

The student at Piaget's IIIa level of Formal Operational Thought will use a systematic approach to testing all possible combinations of the switches but will have some difficulty in answering the series of questions posed by the experimenter. The IIIb level student will have no trouble in answering the questions.

The pre-formal level student will not be able to systematically test all the combinations that are possible and will tend to explore possibilities in a partially systematic way or in a completely trial and error fashion.

Alternative Task

Piaget's chemical substances interview[2] can be used to examine the extent of a student's ability to subject all the variables of a problem to combinatorial analysis. This interview follows the procedures and questions described above but uses five colorless, odorless liquids in place of the box with switches. The liquids used are: 1) diluted sulphuric acid; 2) water; 3) oxygenated water; 4) thiosulphate; g) potassium iodide.

This alternative task requires the student to produce the color *yellow* by using some combination of 1; 2, 3, 4, and g. When 1 + 3 + g or 1 + 2 + 3 + g are combined, the color appears. The water (2) has no effect when added to 1 + 3 + g. Thiosulphate (4) bleaches the mixture 1 + 3 + g or 1 + 2 + 3 + g.

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[1]This task was adapted by permission from: Murray, Kenneth R. Chemical yellow - an electronic version. Paper presented at the University of California, Berkeley, 1971.

[2]Inhelder, B. and J. Piaget. The growth of logical thinking from childhood to adolescence. New York: Basic Books, 1958, 108-109.

propositional reasoning

Principle

Solving problems which contain a finite number of possibilities, each of which might be the solution, requires thinking that can systematically sort out the total number of possibilities, then combine the appropriate ones to formulate a solution. Most students attain this ability by age 14 or 15.

Materials

2 story problems* (described below).

Task

The student (S) is sitting alongside or opposite the experimenter (E).

Procedure

E tells Story I to S, then seeks information concerning S's solution to the problem raised in the story. S can take notes if he wishes and can ask for information to be repeated.

Story I

Complaints were received at a watch factory because there had been so many watches that had something wrong with them. The owner of the factory sent a man to study what was going on. The man examined the machinery that was used to make the watches and talked to the workers. He then telephoned the owner and said, "I have found that all the watches that were made in the month of September are faulty."

Later, the owner looked at some watches on his desk. He picked up the first watch and said, "This one was made in September; it has, therefore, a fault." Could the owner say that or not? Explain why.

The owner then looked at the second watch and said, "This was made in July, so I am sure that it doesn't have a fault." Could the owner say that or not? Explain your answer.

The owner next looked at the third watch, which he knew had a fault, and said, "This one has a fault, so I know it was made in September." Could he say that or not? Explain why.

Finally he picked up the fourth watch and said, "I know this one has not a single fault, so it was not made during the month of September." Could he say that or not? Explain.

Checking Procedure

E tells Story II to S, then seeks additional information concerning S's explanation.

Story II

A group of male students from all over California were given a trip to New York City. When they were about to leave California, the students who came from San Francisco decided that they would each wear a blue tie. While on the trip, two of the students from San Francisco were walking along a New York street and saw another student they knew was from California, but they couldn't see his tie because he was walking away from them. One of the students asked, "What color do you think his tie is?" What do you think his friend said? Explain.

Later they saw another student coming toward them wearing a blue tie. His friend asked, "Do you believe he is from San Francisco?" What was the reply? Explain.

*Stories are adapted from: Morf, Albert. Les relations entre la logique et le langage lors du passage du reasonnement concret au raisonnement formel. In Apostol, L., B. Mandelbrot, and A. Morf. Logique, langage et theorie de l'information. Etudes D'epistemologie Genetique, Vol. 3. Paris: Presses Univ., 1957, 173-204.

Analysis

When the solution to problems involving propositional reasoning is faulty, the relationship between given facts is always neglected in part or inverted. Such distortions in thinking are typical of the pre-formal operational level student. The distortions affect not only the answer given, but also the student's perception of the wording.

In the problem of the defective watches, the phrase *"all the watches that were made in September are faulty"* is the key to distinguishing formal operational thinking from pre-formal thinking. The phrase is the central part of a system of propositions: (a) made in September, (b) not made in September, (c) defective, and (d) not defective. Although pre-formal students will accept the phrase in its proper sense, they will at the same time convert it to *"all the defective watches were made in September."* Thus the implication of the phrase is treated as mutual. Formal operational level students will not treat the implication as an equivalence.

Essentially the problem involves two basic propositions about watches along with the negation of each: (1) either a particular watch was made in September, or it was not (i.e., was made in some other month); and (2) either the watch is defective, or it is not. The two propositions, or their negations, can be combined in four possible ways as illustrated by the following table.

		Was the watch made in September?	
		Yes	No
Is the watch defective?	Yes		
	No		

The statement of the problem gives some information as to which of the four possibilities actually exist. When the investigator reports that all watches made in September are defective, it means that the left-cells of the table can be filled in like this.

	Yes	No
Yes	+	
No	0	

Here a + means that there exist watches about which a particular combination of propositions is true. A o means that there are no such watches.

Suppose that a watch is found that is *not* defective. The information given would enable one to conclude that this watch was *not* made in September. Thus another cell can be filled in on the table.

	Yes	No
Yes	+	
No	0	+

But suppose another watch is found that *is* defective. Can one conclude that it was made in September? Certainly not. It might be a watch that belongs in the upper right-cell of the table.

The moral of the story is that, just because *"made in September"* implies <u>defective</u>, it does not automatically follow that <u>defective</u> implies *"made in September."* One has to consider <u>all</u> of the possibilities.

controlling and manipulating variables

Principle

Solving problems that involve the identification and systematic testing of each variable requires the ability to separate variables by exclusion. For example, factors that influence the rate of swing of a pendulum are determined by isolating possible factors and testing them one at a time while all others are held constant. Many students and some adults are never able to identify and isolate the possible relationships involved in this and similar problems. Most students attain this ability by age 14 - 16.

Materials

3 feet of thread.

1 sturdy yardstick with a small eyelet or U-nail driven in one end.

1 clothespin

1 paperclip

1 protractor

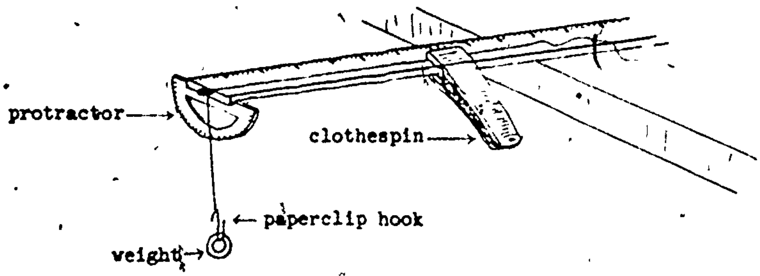
At least 6 identical washers or fishing-weights.

Tape

Task

The student (S) is sitting alongside or opposite the experimenter (E).

E attaches the paperclip to the thread, suspends the thread from the yardstick, hangs two weights on the paperclip, then sets the weights in motion.



Establishing the Variables

NOTE: In this problem there are at least four possible variables which can be tested: the length of thread, the weight, the angle at which it is released, the amount of force imparted when released.

E states: *"This is a pendulum and these are a number of different weights which can be attached to or removed from the thread."* E demonstrates by removing one weight.

E: *"Notice that the thread can be made shorter or longer."* E demonstrates how to adjust the length of the thread, then asks S to adjust it so that it is two-inches long.

E: *"Notice that the pendulum can be released from different positions."* E demonstrates how to release the weight using the protractor as a guide, then allows S to also release it.

E: *"Notice that I can release the weight with more or less force."* E demonstrates, then allows S to try.

Procedure

E says: *"See if you can find out what might make the pendulum swing faster or slower."*

S tries something.

E probes to obtain insight into whether or not S is following a systematic plan: *"Tell me what you are doing."*

After S has experimented with the pendulum and made some statement about its operation, E probes to determine the relationship or lack of relationship between the action of S, his observations, and inferences: *"Tell me why you think that is so."*

Since it is important for E to establish the status of such factors as string length, weight, release height, and release force in S's thinking, E might ask S about the role of any factors that S has not mentioned. For example, if S does not mention the possible effect of different weights, E might ask: *"If we make a pendulum using more (fewer) weights, will the pendulum swing faster or slower or the same?"* E might also arrange an untried experimental situation and ask S to make a prediction.

Analysis

If S is at the Formal Operational Stage of development, he will reveal concern about the possible interrelationships of variables and therefore will attempt to control as many as possible while varying only one. If S is not at the Formal Operational Stage, he might not consider the importance of controlling variables (e.g., S might explore the effect of increasing the weight of the pendulum but not consider it important to keep the length of the thread or the height of the first swing constant during his experiments).

The Formal Operational Stage student will be one who is able to perform controlled experiments where single variables are isolated in turn to study their effects. The student will also be able to report systematically all of the combinations that are possible. Furthermore, he will demonstrate his capacity for making hypotheses before he experiments. Instead of being closely bound to immediate experience, he can suggest *"What might happen if..."* and then perform the actions necessary to confirm or disprove his suppositions.

If S is at the Concrete Operational Stage of development, he will be able to make accurate observations, can investigate potential determinates, and might conclude that weight has an effect, however, he cannot design a controlled experiment and will not provide a correct conclusion.

If S is at the Pre-operational Stage, he will try to solve the task in a haphazard, random way. He will not observe accurately, and his conclusions will be unrelated to the evidence.

deductive reasoning

Principle

Deductive reasoning moves from the more general to the less general. A deductive argument or inference is one that is logically conclusive (i.e., its validity or consistency can be certified by logical considerations alone, usually through a transitivity of implications). For example:

All women are mortal.

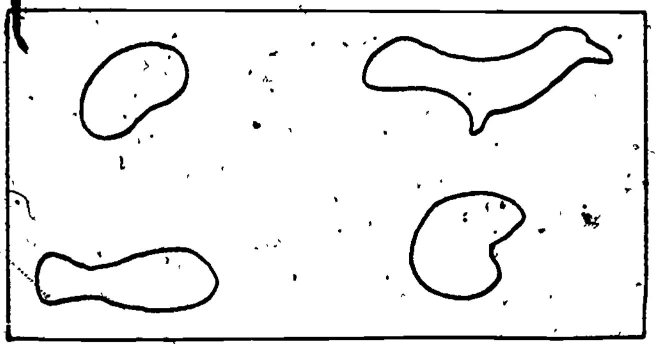
All queens are women.

Thus all queens are mortal.

The conclusion of a deductive argument is simply an explicit statement of something that is implicit in the premises. Many students (50-60%) attain the first substage level (IIIa) of this ability by age 15 - 17. Only a few (10-15%) reach the second substage (IIIb).

Materials

Consumable map showing four islands:



Task

The student (S) is sitting opposite or alongside the experimenter (E).

E places the map on the table in front of S and asks S to examine it for a moment.

Procedure

E provides information: *"This is a map of four islands in the ocean. The islands are called Bean Island, Bird Island, Fish Island, and Snail Island. You may make notes or marks on your map to help you remember the names and other information. You may ask questions about the information at any time."*

E continues: *"People have been traveling among these islands by boat for many years, but recently an inter-island airline started in business. Listen carefully to the clues I give you about the possible airplane trips. The trips may be non-stop between islands or they may include stops at one or more islands on the way. When I say a trip is possible, it can be made in* <u>*both*</u> *directions between the islands."*

E might indicate that the islands are so names because of their distinctive shapes. E should be sure S is listening and is ready to hear the clues.

E: *"First Clue: People can go by airplane between Bean and Fish Islands."*

E: *"Second Clue: People cannot go by airplane between Bird and Snail Islands."*

E: *"Use the two clues I've given to answer this question: Can people go by plane between Bean and Bird Islands?"*

After S responds, E asks S to explain his/her answer.

E: *"Third Clue: People can go by plane between Bean and Bird Islands."*

E: *"Use all three clues to answer these questions: 1) Can people go by plane between Fish and Bird Islands? 2) Can people go by plane between Fish and Snail Islands?"*

After each question, E allows S to respond and asks S to give a rationale. NOTE: E may repeat clues as often as requested by S.

Analysis

A student is said to be at the Pre-operational Stage of development if his explanation makes no reference to the clues and/or introduces new information. A pre-operational student might answer question 2: *"Yes, because there are flights"* or *"No, because the plane can run out of gas and go down in the water."*

A student is said to be at the Concrete Operational Stage of development if his explanation uses the clues to construct a concrete (symbolic) model or models that are then used to make predictions. A concrete operational student, using as a reference a model(s) he has drawn, might answer question 1: *"Can't tell because Bean Island has an airport, but Bird Island might not have an airport"* or question 3: *"No, Snail Island must be the one with no airport, so people from Fish Island can't get there."*

NOTE: The model-based approach, when correctly used, leads to correct answers; but within the framework of concrete operations, the use of the airport model assumes information not given in the clues and will not generalize to solve different puzzles with different data.

This task was adapted by permission from: Karplus, Elizabeth F. and Robert Karplus. Intellectual development beyond elementary school I: deductive logic. School Science and Mathematics, 70, 5, 1970, 398-406.

proportional reasoning

Principle

Proportional reasoning is the comparative relation of one thing to another in part or whole and is expressed in terms of magnitude, quantity, or degree. For example, ratio can be applied to solve a problem of comparing two different units of length using two objects of different lengths. A student with the ability to operate with ratio, using one-inch units of length to measure lengths of one-inch and three-inches respectively, can predict mathematically that if it takes two one-half-inch units to equal the shorter length (one-inch), then it will take six one-half-inch units (three times as many) to equal the longer length. Most students attain this ability by age 16 - 18.

Materials

1 chain of #1 Gem paperclips (10 clips long).

1 chain of jumbo-sized Gem paperclips (10 clips long).

1 worksheet depicting a drawing and statements (See Figure of Mr. Short).

1 display chart depicting Mr. Short.

several straight pins.

1 chalkboard or other writing surface including writing implement.

1 pencil for student.

Task

The student (S) is sitting opposite or alongside the experimenter (E).

E gives S the worksheet and chain of jumbo-sized paperclips.

E has a chain of #1 (regular-size) paperclips.

Procedure

E shows S the display chart depicting Mr. Short.

E: *"This problem is about Mr. Short, who is just like the man on your paper, and Mr. Tall, who is similar to Mr. Short but larger. I have his picture back at my office so I can't show him to you."*

E displays chart illustrating Mr. Short.

E: *"I will measure how high Mr. Short is with my chain of small paperclips - we'll call these clips 'smallies'."*

E displays the chain of clips.

E: *"We'll write down the results of the measurements on the board."* E writes *"Mr. Short"* and *"Mr. Tall"* on the board.

E: *"How high is Mr. Short?"*

E hangs his paperclip chain from a pin at the top of Mr. Short's head. S counts the number of clips and responds verbally. E writes the number on the chalkboard.

E: *"When I measured Mr. Tall in the same way, I found he was six smallies high."* E writes the number on the board.

E sets down the chart and puts away the small paperclips. These are not used again.

E: *"Now I would like you to do three things using your worksheet. First, measure Mr. Short using your big paperclips - we'll call the clips 'biggies'."*

After S measures, E says: *"Second, I want you to predict the height of Mr. Tall if you could measure him with biggies. Third I want you to explain how you made your prediction. Explain as best you can how you decided the number of biggies in your prediction."*

NOTE: S's questions or requests for more information are to be referred to data on the chalkboard. The measurements are not repeated after the display chart is put aside, nor is S given access to the small paperclips. E tells S that different people answer the questions differently, that there is no one correct way to solve the problem, that S can measure in his own way, and that great accuracy is not important. About 10 - 15 minutes are required to complete this task.

Analysis

A student is said to be at the Pre-operational Stage of development if his explanation refers to estimates, guesses, appearance, or extraneous factors without using the data or if he uses the data in an illogical way. A pre-operational student might answer: *"Six because that is what I think it is."*

A student is said to be at the Concrete Operational Stage if his explanation uses proportionality but is an estimation of length ratio instead of a computation of it from the data. A concrete operational student might answer: *"Six. I think that 1 biggie equals 1 1/2 smallies, so I took 4 biggies and added them up and got 6."*

A student is said to be at the Formal Operational Stage if he uses proportionality and makes clear how the ratio is derived from the measurements on the two figures. He may or may not use the word *ratio*. A formal operational student might answer: *"Six. The ratio of the smallies is 2:3 (two to three) so you figure the big paperclips would also have the ratio 2:3."*

NOTE: The fact that the numerical result given by each of the above students is correct, is not as important in determining the developmental level as the substance of the student's explanation of what he did.

This task was adapted by permission from: Karplus, Robert and Rita Peterson. Intellectual development beyond elementary school II: ratio, survey. School Science and Mathematics, 70, 9, 1970, 813-820; Karplus, E. F., R. Karplus, and W. Wollman. Intellectual development beyond elementary school IV: ratio, the influence of cognitive style. AESOP, Lawrence Hall of Science, University of California, Berkeley, 1973.

DIRECTIONS FOR PREPARING THE WORKSHEET AND DISPLAY CHART FOR PROPORTIONAL REASONING TASK

1. Make worksheet copies of the figure and the accompanying statements on the next page. The figure is Mr. Short. Be sure your copies of Mr. Short are exactly the same height as the figure (six chained #1 Gem paperclips high or four chained jumbo-sized Gem paperclips high). Use a worksheet with each student interviewed.

2. Copy the figure without the statements. Place the copy on the front of a display chart.

FIGURE
Mr. Short

How tall is Mr. Short when measured with big paperclips?

Predict the height of Mr. Tall when measured with big paperclips.

Explain how you figured out your prediction.

The following pages contain several self-directed workshops.

Except for the first workshop, the order in which they are done, the number that are done, and the time it takes to do them are up to you.

WORKSHOP **A**

developing the ability to administer propositional thinking tasks

The purpose of this personal workshop is to guide you in practicing the tasks shown in the previous section. After finishing this workshop, you should be able to administer the tasks and record, in a variety of ways, the responses of students.

Study the Guidelines on the following page, then practice administering any two of the propositional tasks with a friend. Consciously incorporate some of the ideas from the Guidelines as you practice.

Because the responses of students may be very complex, it might be helpful to use some type of a record keeping system. The two pages following the Guidelines show a few ways in which information can be easily recorded in chart forms.

Use the space below to jot down personal notes about your practice experiences.

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GUIDELINES FOR ADMINISTERING PROPOSITIONAL TASKS

The Cognitive Operations

Propositional tasks involve combinations of cognitive abilities. For each task it is important that you seek whether or not the student has the ability to perform the abilities two at a time, three at a time, and so forth. 16 cognitive abilities, identified by Piaget, are listed on page Most of these are included within the tasks presented in the previous section.

The Clinical Method

The clinical method is a consistent and powerful assessment tool. Essentially, it is the careful observation of a student's reaction to an interaction with some aspects of the physical world such as the swing of a pendulum, the balance of a scale, or the motion of an object.

When you use the method, it is important to remember:

a. that the purpose is to determine how a student approaches or goes about solving a task, rather than whether he "gets the right answer."

b. to interact with a student, manipulate the materials with him, but let the student's responses and actions guide the discussion.

c. to approach the student where he is, observe what he does and says, and note how he responds to the situation. Do this independently of any preconceived expectations you might have concerning how he should act or respond.

d. that the method will not arm you with a set of predetermined questions.

e. that the results will not produce standardized responses to adult questions. The results will simply indicate something about each student's interaction with his or her view of the world.

Use of the clinical method will point out that there are very few incorrect responses when one asks students questions, for responses usually are correct in terms of the intellectual framework in which they are given. Once this concept is understood and accepted, the whole approach to educating students can become more effective.

SAMPLE RECORD FORM
FOR PROBABILISTIC REASONING TASK

Directions: Record action and verbal response.

Procedure	Student's Actions black	Student's Actions white	Student's Verbal Responses	Comments
I_1		✓	Because there are more whites	
2	✓		Because there are more blacks	
II_1	+5		Makes the amounts even	
2	− 5		✓	
III_1	+ 3			
2	✓	+ 3		
3				
4				
5				
IV_1				

SAMPLE RECORD FORM
FOR COMBINATORIAL REASONING TASK

Directions: Record order of switch turned on.

Trials	Switches					Comments
	1	2	3	4	g	
1	1	2	3	4	5	
2	5	1	2	3	4	
3	4	5	1	2	3	
4						
5						
6						
etc.						

WORKSHOP **B**

assessing the administration of propositional tasks

The purpose of this workshop is to help you assess your ability to administer a task using the clinical method.

Prepare an audio- or videotape of your administering a propositional task to a student.

At your leisure, use the form on the next page to guide an assessment of your use of the clinical method. You might wish to discuss various points with colleagues.

SELF-ASSESSMENT: USING THE CLINICAL METHOD	
Assessment Considerations	Personal Notes
a. What aspects of the interview indicated that I was non-judgmental and accepting of the student's ideas?	
b. What examples in the interview indicated that I avoided leading the student toward an idea I had?	
c. What evidence is there that I let the student's responses and actions guide the discussion?	
d. What evidence indicates that I approached the student where he/she was and responded to the situation independently of adult expectations of how he/she should act?	
e. What evidence indicates I formulated hypotheses concerning the student's strategies and tested the hypotheses through manipulations of the materials?	
f. In what ways did I check for justifications?	

WORKSHOP **C**

administering propositional tasks to a student

The purpose of this workshop is to guide you in assessing the ability of one student. After doing this workshop, you should be able to identify a student's level of ability.

Select a student in your classroom about whom you want to learn more. Administer one or more propositional tasks to him/her.

At your leisure, study the student's responses in terms of the analysis given for each propositional task you used. Keep a record, diary, or profile of the assessment. Decide what kinds of experiences would be suitable in a learning experience for this student.

You may wish to audio- or videotape this experience so that you can discuss your assessment with a colleague.

WORKSHOP **D**

administering propositional tasks to students at different levels

The purpose of this workshop is to let you compare different levels of responses to the same experience. After doing this workshop, you should be able to better identify the level of ability of different students.

Select one task and administer it to three different students: one primary, one intermediate, and one high school.

At your leisure, compare the students' responses in terms of the analysis given for the task you selected.

You may wish to audio- or videotape this experience so that you can discuss the information with a colleague.

WORKSHOP **E**

observing the administration of propositional tasks

The purpose of this workshop is to provide additional practice in assessing levels of development and in analyzing interviewing procedures. After finishing this workshop, you should be able to better identify levels of abilities.

View the videotape on Propositional Abilities with several other teachers. The tape will show students at different levels of cognitive development responding to a propositional ability task.

When you finish viewing the tape, discuss the following with your colleagues:

1. What clues were observed that indicated the stage of development for each student interviewed?
2. Compare the responses of the students on this tape with responses of those you have interviewed (e.g., in Self-workshop B).
3. Did the interviewer use the clinical method throughout each interview? (You might use the guide given in Self-workshop B to check on the interviewer's procedures).

WORKSHOP **F**

solving problems involving propositional abilities

The purpose of this workshop is to give you an opportunity to do tasks which involve propositional abilities and to let you compare your thinking with an analysis of the logic involved. After this workshop, you should have an appreciation for the complexity of the propositional abilities.

Listed below are two problems. Solution to these problems requires several cognitive operations at the propositional thinking level.

Try solving one or both of the problems. Stick to the problem(s) long enough to give it a good try. If you have trouble, put it aside for a while, then return to it and try again.

If you solve the problem(s), list the steps one would go through to show another person how to solve it. This procedure will reveal to you the combinatorial aspect of propositional thinking.

If you do not solve the problem(s) (they are really tough!), do your best, then follow the analysis on the following pages. See the Appendix for additional propositional thinking problems.

1. The Five Suspects. Five suspects were rounded up in connection with a murder. Their statements were as follows:

   A: *"C and D are lying."*
   B: *"A and E are lying."*
   C: *"B and D are lying."*
   D: *"C and E are lying."*
   E: *"A and B are lying."*

   Who is lying?

2. The Twelve Coins. Someone gives you twelve coins (e.g., all quarters, all dimes). One coin in the set is counterfeit, the other eleven are identical in every characteristic. The counterfeit coin is different only in the characteristic of weight -- it is either heavier or lighter than the other coins.

   Using an equal-arm balance scale, determine in three weighings, which of the twelve coins is counterfeit and tell whether it is heavier or lighter than the other coins.

   Good luck!

ANALYSIS OF THE FIVE SUSPECTS PROBLEM

For clarity, the following matrix describes the statements and claims made by the five suspects.

STATEMENT \ CLAIM	A	B	C	D	E
A			✓	✓	
B	✓				✓
C		✓		✓	
D			✓		✓
E		✓			

In the explanation, the following notation is used:

- A : Person A is telling the truth.
- $\bar{A}$: Person A is lying.
- $\rightarrow$: Implies.
- $\wedge$: And.
- $\vee$: Either one or the other or both.
- $\Leftrightarrow$: Equivalent form.
- $\therefore$: Therefore.

To solve the five suspects problem let us assume that A is telling the truth and see where this assumption leads.

(1.0) $A \rightarrow \bar{C} \wedge \bar{D}$ $(\bar{C} \wedge \bar{D} \Leftrightarrow \bar{C}\,\bar{D})$

(1.1) $\bar{C} \rightarrow B \vee D$ $(B \vee D \Leftrightarrow B\,\bar{D} \vee \bar{B}\,D \vee B\,D)$

(1.2) $B \rightarrow \bar{A} \wedge \bar{E}$ but this contradicts (1.0) $\therefore$ $\boxed{\bar{B}}$, and since $\bar{B}$ is true, then D must also be true to satisfy (1.1). But if D is true, this contradicts (1.0) which requires $\bar{D}$ to be the case $\therefore$ our original assumption of A was wrong.

Now we can start again using our new knowledge: $\boxed{\bar{A} \text{ is the case}}$.

(2.0) $\bar{A} \rightarrow C \vee D$ $(C \vee D \Leftrightarrow C\bar{D} \vee \bar{C}D \vee CD)$

(2.1) $C \rightarrow \bar{B} \wedge \bar{D}$

(2.2) $\bar{B} \rightarrow A \vee E$ but A contradicts (2.0). Now $\bar{A} \rightarrow E \therefore E$ must be true.

(2.3) $E \rightarrow \bar{A} \wedge \bar{B}$ and we know $\bar{A}$, but $\bar{B}$ takes us back to (2.2). So far, so good.

(2.4) $\bar{D} \rightarrow C \vee E$ C leads us back to (2.1).
E leads us back to (2.3).

Since there are no contradictions and all of the possibilities have been analyzed, the problem is solved:

$\bar{A}$:	A is lying.
$\bar{B}$:	B is lying.
C :	C is not lying (i.e., is telling the truth).
$\bar{D}$:	D is lying.
E :	E is not lying (i.e., is telling the truth).

To check:

$\bar{A} \rightarrow C\bar{D} \vee \bar{C}D \vee CD$: $C\bar{D}$ is the case.
$\bar{B} \rightarrow A\bar{E} \vee \bar{A}E \vee \bar{A}\bar{E}$: $\bar{A}E$ is the case.
$C \rightarrow \bar{B}\bar{D}$ which is the case.
$\bar{D} \rightarrow C\bar{E} \vee \bar{C}E \vee CE$: CE is the case.
$E \rightarrow \bar{A}\bar{B}$ which is the case.

ANALYSIS OF THE TWELVE COIN PROBLEM

In the analysis, the following notation is used:

I, II, III	=	The three weighings.
1 through 12	=	The twelve coins.
L and R	=	The sides of the scale on which the coins are placed (Left and Right).
l, r, b	=	The results of a given weighing (l = left side goes down; r = right side goes down; b = the scale balances).

Weighing I

Place 1, 2, 3, 4 L and 5, 6, 7, 8 R.

If Ib (Then the counterfeit coin is either 9, 10, 11, or 12). Go to Weighing Ib II.

If Ir (Then either 1, 2, 3, 4 is light or 5, 6, 7, 8 is heavy). Go to Weighing Ir II.

If Il (Then follow the same steps outlined for Ir above).

Weighing Ir II

Place 1, 2, 5 L and 3, 6, 9 R.

If Ir IIb (Then either 4 is light or 7 is heavy or 8 is heavy). Go to Weighing Ir IIb III.

If Ir IIr (Then either 1 is light or 2 is light or 6 is heavy). Go to Weighing Ir IIr III.

If Ir IIl (Then either 5 is heavy or 3 is light). Go to Weighing Ir IIl III.

Weighing Ir IIb III

Place 4, 8 L and 9, 10 R.

If Ir IIb IIIb (Then 7 is it and it is heavy).

If Ir IIb IIIr (Then 4 is it and it is light).

If Ir IIb IIIl (Then 8 is it and it is heavy).

Weighing Ir IIr III

Place 1, 6 L and 10, 11 R.

If Ir IIr IIIb (Then 2 is it and it is light).
If Ir IIr IIIr (Then 1 is it and it is light).
If Ir IIr IIIl (Then 6 is it and it is light).

Weighing Ir IIl III

Place 5 L and 9 R.

If Ir IIl IIIb (Then 3 is it and it is light).
If Ir IIl IIIr (Not a possible result).
If Ir IIl IIIl (Then 5 is it and it is heavy).

Weighing Ib II

Place 9, 10 L and 11, 1 R.

If Ib IIb (Then 12 is the counterfeit coin).
If Ib IIr (Then either 11 is heavy or 9 is light or 10 is light). Go to Weighing Ib IIr III.
If Ib IIl (Then either 11 is light or 9 is heavy or 10 is heavy). Go to Weighing Ib IIl III.

Weighing Ib IIb III

Place 12 L and 1 R.

If Ib IIb IIIb (Not a possible result).
If Ib IIb IIIr (Then 12 is light).
If Ib IIb IIIl (Then 12 is heavy).

Weighing Ib IIr III

Place 9, 11 L and 1, 2 R.

If Ib IIr IIIb (Then 10 is it and it is light).
If Ib IIr IIIr (Then 9 is it and it is light).
If Ib IIr IIIl (Then 11 is it and it is heavy).

Weighing Ib IIl III

Place 9, 11 L and 1, 2 R.

If Ib IIl IIIb (Then 10 is it and it is heavy).
If Ib IIl IIIr (Then 11 is it and it is light).
If Ib IIl IIIl (Then 9 is it and it is heavy).

WORKSHOP **G**

recognizing propositional abilities in English

> *The purpose of this workshop is to have you transfer some of what you have learned to another field - English. After the workshop, you should be able to use your assessment skills more broadly.*

Being able to *read between the lines* and to comprehend *higher* levels of meaning beyond the printed words are indicators of an advanced level of thought.

The following is a listing of several literary sayings that a teacher might give to students and ask them to explain what each saying means. Below each saying is a possible statement given by a student. After you read them, decide whether or not the student's statement is at the Formal Operational Stage of development. Discuss your analysis with UCCTPP staff members.

1. KITES RISE HIGHEST AGAINST THE WIND - NOT WITH IT.

   What this really means to me is that it flies in the sky with the wind behind it.

2. IT OFTEN SHOWS A FINE COMMAND OF LANGUAGE TO SAY NOTHING.

   This means you talk too much.

3. IF THERE IS NO WIND - ROW!

   Once upon a time there was a man, and the man had a sailboat. One day the man thought to himself and said, "What a beautiful day for sailing." So he went and got his gear and went sailing. So here he was going along just fine until the wind stopped blowing, and he said to himself, "If there is no wind - row!" So he rowed back to his home and lived happily ever after.

4. CLOUDS - THE ONLY BIRDS THAT NEVER SLEEP.

   They are like birds in the sky, flying. They give you water and take it away. Sometimes they are black and sometimes white.

5. A FRIEND IS A PRESENT YOU GIVE YOURSELF.

   This means a lot to me. It reminds me of Bonnie. I like Bonnie a lot. And it means that it isn't a birthday present. It's a present of love and happiness.

WORKSHOP **H**

reading about propositional abilities

> *The purpose of this personal workshop is to provide for your study, a research topic that is related to the subject of this book.*

At your leisure, read the following chapter from one of Inhelder and Piaget's books (The Growth of Logical Thinking From Childhood to Adolescence. Basic Books Inc., New York, 1958). The chapter will add to your understanding of propositional abilities.

1

The Equality of Angles of Incidence and Reflection and the Operations of Reciprocal Implication[1]

OUR AIM in this chapter, and in the remainder of Part I, is not a systematic study of the concept of the equality of two angles. Actually, we already know how the concept is constructed: that it is first acquired at the level of concrete operations.[2] But it is precisely the fact that the concept is already so well known by the time the formal level (stage III) is reached that makes the reasoning process involved in the discovery of the equality between the angles of incidence and reflection so instructive. One of the aims of this study, then, is to isolate the operational mechanisms involved in the formal reasoning process itself, when this reasoning rests on notions already constructed at the concrete level.

The experimental apparatus consists of a kind of billiard game. Balls are launched with a tubular spring device that can be pivoted and aimed in various directions around a fixed point. The

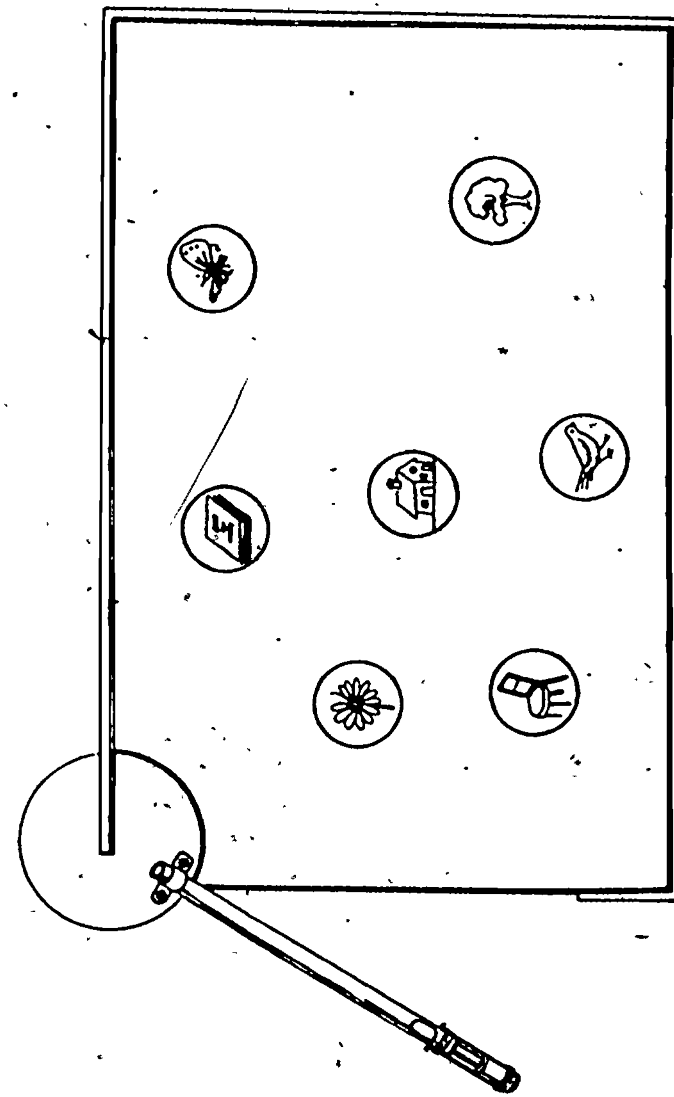


FIG. 1. The principle of the billiard game is used to demonstrate the angles of incidence and reflection. The tubular spring plunger can be pivoted and aimed. Balls are launched from this plunger against the projection wall and rebound to the interior of the apparatus. The circled drawings represent targets which are placed successively at different points.

[1] With the collaboration of H. Aebli, former research assistant, Laboratory of Psychology, Science Faculty, University of Geneva, professor, École normale supérieure, Zurich; L. Müller, former research assistant, Institut des Sciences de l'Éducation, University of Geneva; and M. Golay-Barraud, student, Institut des Sciences de l'Éducation.

[2] See Piaget and Inhelder, *The Child's Conception of Space* (Routledge & Kegan Paul, 1956), Chap. XII, and Piaget, Inhelder, and Szeminska, *La Géométrie spontanée de l'enfant*, Chap. VIII. (Not transl.)

ball is shot against a projection wall[a] and rebounds to the interior of the apparatus. A target is placed successively at different points, and subjects are simply asked to aim at it. Afterwards, they report what they observed.

But the equality between the angles of incidence and reflection is discovered only at stage III-A (11–12 to 14 years) and is often not formulated until stage III-B (14–15 years). Our problem is to understand why a concept as familiar after 7–9 years as that of the equality of two angles is utilized in the induction of an elementary law only at this late date and, especially, why formal operations are necessary for its use. We shall try to answer this question by retracing briefly the ground covered by the child before his arrival at the formal level, then by examining the latter more closely.

§ *Stage I*

In the course of stage I (up to about 7–8 years) subjects are most concerned with their practical success or failure, without consideration of means; often even the role of rebounds is overlooked. The result is that, except toward the end of the stage, the trajectories are not generally conceived of as formed of rectilinear segments but rather as describing a sort of curve:

DAN (5 ; 2)[b] succeeds at first: *"I think it works because it's in the same direction."* He adjusts the plunger by himself, but proceeds by empirical trial-and-error. Then he asks spontaneously: *"Why do you have to turn the plunger sometimes? . . . No, you have to put it there* [he fails]. *If it could be pushed a little further"* [he does this and succeeds]. But, although he knows how to control the rebounds successfully, DAN has no idea that they are made up of angles: the curve he describes with his finger is not tangent to the wall; he takes into account the starting point and the goal but not the rebound points.

WIRT (5 ; 5): *"It came out here and it went over there. . . . I'm sure to make it,"* etc. He succeeds occasionally but describes the trajectories with his finger only in the form of curves not touching the walls of the apparatus; he considers only the goal as if there were no rebounds.

NAN (5 ; 5), on the other hand, is astonished by the detour made by the ball which first touches the walls. *"It always goes over there."* But he does not succeed in adjusting his aim: *"Oh, it always goes there. . . . it will work later."*

PIT (5 ; 5) notes about one of his tries [a failure]: *"It was straight* [as if this were an exception].—*Why didn't it hit it?—I thought I hit it"* [no comprehension].

ANT (6 ; 6) becomes aware of the existence of rebounds at the same time that he notices the rectilinear character of the trajectory segments: *"It* [the ball] *hits there, then goes over there"* [his gesture indicates straight lines].

FER (6 ; 6), in contrast, in spite of his age, resorts to the curvilinear model: *"It goes there and it turns the other way"* [gesture indicating a curve].

The reactions of this stage are extremely interesting, for although the children demonstrate by their behavior that they know how to act in the experimental situation, sometimes successfully, they never *internalize their actions as operations*, even as concrete operations. In a general sense, by *concrete operations* we mean actions which are not only *internalized* but are also *integrated* with other actions to form general *reversible systems.* Secondly, as a result of their internalized and integrated nature, concrete operations are actions accompanied by an awareness on the part of the subject of the techniques and coordinations of his own behavior. These characteristics distinguish operations from simple goal-directed behavior, *and* they are precisely those characteristics not found at this first stage: the subject acts *only* with a view toward achieving the goal; he does not ask himself why he succeeds. In the experiment under consideration he is not aware of either the rectilinear nature of the trajectory segments or the existence of rebounds except toward the end of the stage (toward 6 or 6–7 years); consequently he cannot take note of the presence of angles at the rebound point.

[a] With a rubber buffer.
[b] Figures within parentheses indicate age in years and months—*i.e.*, five years; two months.

§ Stage II (Substages II-A and II-B)

Substage II-A is distinguished by the appearance of concrete operations in the sense just defined:

VIR (7 ; 7) succeeds after several attempts. He points out and then draws trajectories with two distinct rectilinear segments, saying: *"To aim more to the left, you have to turn* [the plunger] *to the left."*

TRUF (7 ; 10): *"I know about where it will go"*; in fact, he shows by his gestures that he realizes that the angle of rebound is extremely acute when the plunger is raised and extremely obtuse when it is lowered. Thus, he shows us that he has a vague global intuition of the equality between the angles of incidence and reflection. But he does not make it explicit, since he fails to divide the total angle indicated by his gesture into two equal angles.

BEND (8 ; 0): *"It's the corner* [the angle of rebound] *that makes it turn; you change the contour* [the size of the angle] *when you change the plunger"* [inclination of the plunger]. He demonstrates as did the preceding subject that the angle is extremely acute when the plunger is slightly inclined and extremely obtuse when it is sharply inclined. We ask him what he means by the contour, and he points to the opening of the angle with a gesture indicating that he is thinking of the very generation of the angle by the progressive rotation of the plunger and of the rebound of increasing amplitude which results.

DESI (8 ; 2): *"The ball always goes higher when the plunger is higher."* Then: *"The ball will go there* [further] *because the plunger is tilted more; I put my eyes high up* [= I pinpoint the rebound point] *and from the rubber* [= the rubber band attached to the wall on which the ball rebounds] *I look at the round pieces"* [= the disks serving as goals].

At substage II-B the preceding operations, which give rise to a model that includes straight lines and angles, are complemented by an increasingly more accurate formulation of the relations between the inclination of the plunger and that of the line of reflection:

NIC (9 ; 4): *"You have to move the plunger according to the location of the target; the ball has to make a slanting line with the target."*

KAR (9 ; 6): *"The more I move the plunger this way* [to the left—*i.e.*, oriented upwards], *the more the ball will go like that* [extremely acute angle], *and the more I put it like this* [inclined to the right], *the more the ball will go like that"* [increasingly obtuse angle]. KAR reaches the point of discovering that the ball returns to the starting point when the plunger is *"straight"*—*i.e.*, perpendicular to the rebound wall.

BAER (9 ; 6): "How do you explain it?"—*"It has to be at the same distance as the target"* [he points out the angle increasing with the withdrawal of the target and not the length of the line between plunger and rebound point or between the latter and the target].

ULM (9 ; 8): *"As you push the plunger up, the ball goes more and more like that* [acute angle], *and the more I put it like that* [inclined to the right], *the more the ball will go like that"* [obtuse angle].—"But, tell us more about what you are looking at."—*"I am still looking at that* [the goal], *and that's all, because it turns with the plunger"* [—because the direction of the path between the rebound and the goal changes with the inclination of the plunger].

DOM (9 ; 9): *"It hits there, then it goes there"* [he points out the equal angles, repeating his phrase for different inclinations of the plunger].

Thus we see that the subjects succeed in isolating all of the elements needed to discover the law of the equality of the angles of incidence and reflection, yet they can neither construct the law *a fortiori* nor formulate it verbally. They proceed with simple concrete operations of serial ordering and correspondences between the inclinations of two trajectory segments (before and after the rebound), but they do not look for the reasons for the relationships they have discovered. And they do not consider the segments except from the standpoint of the directions taken; thus the idea of dividing the total angle made up of the two segments into two equal angles (incidence and reflection) fails to occur to them.

However, in contrast to the stage I subjects, substage II-A and II-B subjects no longer limit themselves to overt performance but internalize their actions in the form of operations of placing or displacing: thus they have become aware of the facts that the plunger can be adjusted to specific slopes, that the trajectory of the ball is composed of two rectilinear segments, and, above all, of the fact that these two segments form an angle (whose peak coincides with the rebound point) whose size varies according to these slopes. They manage to order serially these latter inclinations, distinguishing between *"sharper"* or *"more to the left,"* *"higher,"* etc., and *"less sharp,"* etc., which amounts simply to a translation of the more or less well-ordered operations that they know how to execute beginning with stage I into coordinated operations of serial ordering. Similarly, they succeed in ranking the degrees of incline or the directions of the trajectory segments included between the rebound point and the goal ("the ball keeps going higher" or lower, it "will go here" or there, etc.). Finally, and particularly important, they establish a correspondence between the slope or direction of the plunger (and consequently of the first segment of the ball's trajectory) and the inclinations or directions of the second segment: "The more the plunger is (inclined, etc.), the more the ball will go (downwards, etc.)."

If the increasing inclinations of the plunger (and of the first segment of the trajectory for the ball leaving the plunger) are symbolized by the letters α, β, γ, etc. and the inclinations of the second segment (between the rebound point and the goal) by the signs α', β', γ', etc., the serial ordering and correspondence operations which subjects of this second stage can perform are as follows:

$$\alpha < \beta < \ldots \text{ or } \alpha' < \beta' < \gamma' \ldots \text{ and} \quad (1)$$

$$\alpha \leftrightarrow \alpha', \beta \leftrightarrow \beta', \gamma \leftrightarrow \gamma', \text{ etc.} \quad (2)$$

(where the sign $\leftrightarrow$ stands for the correspondence).

Why, then, does the correspondence between the two rank orderings fail to lead to the discovery of the law of the equality of the angles of incidence and reflection? It is because the subjects stick to the concrete rank ordering and correspondences without looking for the reasons behind this correspondence, just as the subjects at stage I knew what to do to attain their goal but did not look for the reasons behind their reactions (displacement of the plunger, etc.). The stage II subjects stick to dealing with facts whose accuracy is due to serial ordering and correspondence operations, but they do not seek to explain these facts further in terms of the formal operations of implication, etc., which are the conditions of hypothetico-deductive thought.

Since they do not seek an explanation of the observed facts, they must remain at the level of rough, global observation, certainly a great advance over that of stage I but still too global to lead to an analytic breakdown of the observed angles. Thus, because they are content to point out slopes of directions and to deal with the total angle composed of the two segments of the trajectory (BEND: the "contour"; ULMS "It turns"; etc.), they do not divide this total angle into the two equal parts that would give us the angle of incidence and the angle of reflection. That is why, although the subjects are very close to the discovery of the law and already possess all of its elements, it is not yet discovered; the formal operations needed for the quest for an explanatory hypothesis are lacking.

§ *The Formal Stage (Substages III-A and III-B)*

At this last stage the subjects finally discover the law of the equality of angles. At first the discovery is slow and partial, including verification or rejection of several specific hypotheses, then complete and rapid because subjects are oriented by the hypothesis that there is a necessary equivalence between two successive segments of the trajectory.

First, let us look at a case typical of substage III-A:

BON (14; 8) first invokes the launching force, then realizes that the trajectories are the same whether the balls are shot hard or soft. Next he invokes the role of the *"distances, how you have to place the rod."* Then he establishes concrete correspondences in the same way as the stage II subjects: *"It's the position of the lever* [of the plunger]: *the more you raise the target, the more you raise it here"* [the lever]. He uses a ruler to mark the trajectory of the ball between the rebound point and the target in such a way as to verify its correspondence with the orientation of the plunger. Then he hypothesizes that the angle is always a right angle: *"It has to make a right angle with the lever."* But after several trials he concludes: *"No, above* [= when the plunger is straightened] *it won't work."*—"It isn't ever a right angle?"—*"Yes, that's correct for one position."*—"And without that?"—*"When you turn, one should be smaller, the other larger. Ah! They are equal"* [he points out the angles of incidence and reflection].

Thus, in the later stages there is a search for a general hypothesis which can account for the concrete correspondences between the inclinations as soon as they are found. Subject BON thinks first of the right angle, then ascertains that the total angle is sometimes acute, sometimes obtuse; then he breaks it down to form two equal angles (incidence and reflection).

But the hypotheses found at substage III-A are still very close to concrete correspondences in that they attempt only to express the general factor which the correspondences contain. Substage III-B, on the other hand, is distinguished by a new exigency which is absent at substage II-B and still implicit at III-A: the need to find a factor which is not only general but also necessary —*i.e.*, which will serve to express beyond the constant relations the very reason for these relations.

In other words, the subjects at substage III-B are not completely satisfied with the establishment of a correspondence between the inclinations of the plunger and the line included between the buffer and the target, as are those of II-B. Nor are they satisfied with the search for a single constant factor which translates these correspondences, as are those of III-A. Initially they ask themselves why a certain difference in inclination x_r of the plunger necessarily corresponds to a difference x_2 in the buffer-target line: This pursuit of a necessary reason, in certain cases going as far as an immediate appeal to the concept of necessity, is what distinguishes formal thinking, with its operations of implication or equivalence (= reciprocal implication), from concrete thinking, with its simple statements of constancy. This is demonstrated by the following subject, who begins, like BON, with the hypothesis of the right angle but soon afterwards turns to a search for necessity:

DEF (14 ; 8) imagines at first that the two trajectory segments always form a right angle. But after three trials he says: *"The more the target approaches the plunger, the more the plunger must* [necessity] *also approach the target"* [which signifies evidently that the two inclinations of the plunger and of the line between the target and the buffer imply each other reciprocally].—"What do you mean by 'must also approach the target?' "—*"For example, if there were a line here* [he indicates a line perpendicular to the buffer], *the ball would come back exactly the same way."* Then he puts the plunger at 45°: *"That makes a right angle here and you have about the same distance as there"* [= the two angle openings]. Then he continues with several angles chosen at random and again verifies his law of equality. We object that the law may not be a very general one: *"It depends on the buffer too; it has to be good and straight—and also on the plane—it has to be completely horizontal. But if the buffer were oblique, you would have to trace a perpendicular to the buffer and you would still have to take the same distance from the plunger* [to the line and from it] *up to the target: the law would be the same."* The buffer is turned around, replacing the rubber by wood: *"Perhaps the wood is less elastic: the ball would be sent back with less force."*—"Then what about your law?"—*"The law doesn't vary."*

Even in this first case we see several new factors appear which psychologically distinguish formal from concrete thinking: the requirement of necessity ("the plunger *must* also, . . ." "it would *have to be* the same width here and there," "you still would *have to* take the same distance again," etc.); the ability to formulate hypotheses or hypothetical constructions not given by direct observation (trace an ideal perpendicular from the buffer, etc.); confidence in the generality of the law because it is conceived of as necessary, thus as holding true even if conditions are modified ("the law doesn't vary," etc.).

It is clear that new operations appear at this level (after a preparatory period beginning with substage III-A) which are superimposed on the concrete operations. Specifically, of what do these new operations consist? This should be made clear in the course of the examination of the following protocols, to be considered jointly with the preceding one.

GUG (14 ; 4), after several trials, says: *"The more you go toward the right angle* [*i.e.*, the more the plunger approaches a position perpendicular to the buffer] *the closer to the starting point the ball comes back."*—"Is that always true?"—*"Yes, or at least I think so: you'd have to check."* He continues his trials and, when there is chance dispersion due to deficiencies of the apparatus, he concludes: *"There must be something wrong."* After several new trials he concludes: *"You have to find the angle,"* and he looks first for equality in the complementary angles included between the walls of the apparatus and the plunger or the line buffer-goal. At last he discovers that: *"You have to trace the* *perpendicular"* [in relation to the buffer]. He then realizes the constant equality between the angles of incidence and reflection.

MUL (14; 3) begins with a series of correspondences: *"I was here and it went in this direction,"* etc.; *"You change the angle to see how it goes."* By systematically diminishing the total angle, he discovers the fundamental proposition: *"If I shoot it straight, at a right angle* [i.e., when the plunger is perpendicular to the buffer], *it will come right back."* Then he inclines the plunger progressively, according to the angles α_1, β_1, γ_1, etc., and ascertains that, as these angles increase, their complementaries α_1', β_1', γ_1' decrease [α', β' standing for the angles included between the plunger and the buffer]: *"The smaller you make the angle here* [α_1', β_1', etc.], *the larger the angle there"* [α_1, β_1, γ_1]. Then he perceives the equality which he had been seeking from the time he understood that, in the case where the plunger is perpendicular, the ball returns to its starting point. *"This angle* [α_1'] *is the same as that one* [α_2']; *you have to make it parallel to that one* [α_2']. *I am going to see* [he checks for several different angles]. *Yes, I think that's it. You have to carry over exactly that angle"* [the complementaries α_1', and α_2', etc.].

POM (15; 5) also begins by noting the correspondences between the angles: *"I look a bit at an angle. . . . The higher up you want to aim, the wider the angle has to be"* [he calculates on the complementary as did MUL]. In order to verify this hypothesis, he spontaneously places the plunger perpendicular to the buffer: *"If the lever is straight, the ball returns exactly."* Afterwards he adjusts the plunger in three different positions, but without moving the target and without firing, and concludes immediately: *"You have to have two angles: the inclination of the lever equals the angle that the trajectory of the ball makes"* [from the buffer to target].

LAM (15; 2): *"The rebound depends on the inclination* [of the plunger], *. . . Yes, it depends on the angle. I traced an imaginary line perpendicular* [to the buffer]: *the angle formed by the target and the angle formed by the plunger with the imaginary line will be the same."*

REV (15; 4): *"It's a right angle* [several trials]. *No, this slant has to be the same as that one."* When there are chance misses due to the apparatus, he says, *"I didn't move; the gadget isn't fair."*

COD (15; 9), after several fruitless trials: *"You would have to find the rebound angle."* First he indicates $\alpha_1' = \alpha_2'$ as did MUL. Then he traces the perpendicular and points out the angles of incidence and reflection: *"The two must be equal."*

FORT (16; 0) begins with several trials: *"You have to move the lever according to the target position and vice versa* [reciprocity]. *You must have an angle there, but it isn't always the same* [he continues his trials]. *It's obvious that everything changes."* Then: *"You have to think in straight lines. To the extent that the lever is displaced, you find the same distance in the other direction: you have to displace it according to the mean* [= the perpendicular from the rebound point to the buffer which he has spontaneously designed]. *The two distances* [the angle openings], *the two sides, always indicate the angles"* [of incidence and reflection].

JAN (16; 4): *"You have to find the corresponding angle: the more acute the target angle, the more the lever goes towards the middle and vice versa."*—"Can you measure it?"—*"It's more or less a right angle. No, it varies here the same way as there"* [same design as FORT].

BERG (16; 6), after analogous explanations, is shown a wooden buffer rather than the rubber one: *"I think that it's the same law. Yes, I'm sure of it. I take the perpendicular and I focus on the distances. Yes, now the angles have to be equal."*

Although they differ from each other in a number of respects, these examples of reasoning have in common several essential elements which must be differentiated before we can describe the differences between formal and concrete thinking as they relate to the experimental problem under consideration.

In the discovery of the law, the general starting point for these subjects seems to be the fact that the establishment of the concrete correspondences between the inclines of the plunger and the path of the ball after it strikes the buffer seems to lead automatically to the idea of a necessary reciprocity—*i.e.*, each incline implies the other and vice versa. For example, this is expressed by FORT: "You have to move the lever according to the target and vice versa." But this reciprocity, which adds the idea of mutual implication to that of one-to-one reciprocal correspondence, does not in itself entail the realization that the two angles are equal (as FORT, who is at first struck by the variation of the angles, demonstrates).

The bridge from the idea of reciprocity to that of equality—and this is the second point common to all of the answers—is actually furnished by the assertion (explicit, or in certain cases purely mental) that the ball returns to the starting point when the plunger is perpendicular to the buffer. Then it follows that if the null incline of the plunger implies the null incline of the ball's return course, any inclination of either implies an equal inclination of the other.

Once in possession of this double assertion (mutual implication of inclines and return of the ball to its starting point in the case of null incline), the subject will either imagine a perpendicular to the buffer from the rebound point, which leads him to discover the equality of the angles of incidence and reflection; or he will look for the complementary angles (located between the plunger and the buffer or between the former and the trajectory of the ball after the rebound), which step also leads him to the idea of equality.

In either case, the construction of the law is due to the quest for a necessary explanation of the observed inclinations; the serial orders and correspondences established prior to this point are not in themselves sufficient for the subject to discover the relationship between the angles, or even for him to break up the total angle included between the two successive segments of the trajectory into two partial angles.

Conclusion: The Transition from (Concrete) Correspondence to (Formal) Reciprocal Implication

In spite of what we have just said about the discoveries of our stage II subjects, we have yet to understand just what formal thought adds to concrete operations in the specific case, since subjects at stage II seem *a posteriori* so close to the formulation of the law. What is the contribution of formal operations to the solution of a problem that at first glance seems to require nothing more than correspondences and equalization? Actually, the context of stage III reactions is quite different from that of preceding stages: reasoning by hypothesis and a need for demonstration have replaced the simple stating of relations. In other words, henceforth thought proceeds from a combination of ***possibility, hypothesis,*** and ***deductive reasoning,*** instead of being limited to deductions from the actual immediate situation.

The distinction between the one-to-one correspondence of the angles of incline (at stage II) and the reciprocity leading to the idea of the equality of angles (discovered at stage III) is extremely fine as long as we are not in a position to state exactly what the differences are between the operations used at these two stages. Nevertheless, there is a difference. And though it is slight in this first case, it does give us an example of the general opposition of concrete and formal operations that we shall encounter again in increasingly clearer form in the following chapters.

The difference can be stated as follows: Although concrete operations consist of organized systems (classifications, serial ordering, correspondences, etc.), they proceed from one partial link to the next in step-by-step fashion, without relating each partial link to all the others. Formal operations differ in that all of the possible combinations are considered in each case. Consequently, each partial link is grouped in relation to the whole; in other words, reasoning moves continually as a function of a "structured whole."

Stated in symbolic terms, when two classes, A_1 and A_2, with their complementaries, A'_1 and A'_2, are taken, concrete class logic furnishes only four elementary products ($A_1A_2 + A_1A'_2 + A'_1A_2 + A'_1A'_2$). On the other hand, formal logic, taking the two propositions p and q with their negations $\bar{p}$ and $\bar{q}$, furnishes sixteen possible combinations derived from the four elementary propositional conjunctions $(p.q) \vee (p.\bar{q}) \vee (\bar{p}.q) \vee (\bar{p}.\bar{q})$, which define respectively relations of implication, disjunction, etc., depending on whether the conjunctions are taken one-by-one, two-by-two, three-by-three, the four together, or none at all. The implication of q by p, for example, corresponds to the sum of three conjunctions, $(p.q) \vee (\bar{p}.q) \vee (\bar{p}.\bar{q})$; the implication of p by q corresponds to the sum of $(p.q) \vee (p.\bar{q}) \vee (\bar{p}.\bar{q})$; and the equivalence of p and q (or reciprocal implication) corresponds to the sum of the two conjunctions $(p.q) \vee (\bar{p}.\bar{q})$. But, in order to affirm the truth of one of these three links, $p \supset q$ or $q \supset p$ or $p = q$, one also has to establish the respective falsehood of $(p.\bar{q})$ for $p \supset q$, of $(\bar{p}.q)$ for $q \supset p$, and of $(p.\bar{q})$ as well as of $(\bar{p}.q)$ for $p \doteq q$.

In other words, the difference between the concrete level subjects (who do not go beyond the formulation of term-by-term correspondences between the inclinations of the plunger and the course of the ball from the buffer to the target) and the formal level subjects (who look for necessary reciprocity immediately) can be wholly accounted for by distinguishing the step-by-step operations based on simple correlations found in class and relational groupings from the combinatorial operations based on the "structured whole" which constitute propositional logic. Thus, subjects at stage II are limited to stating successively the correspondences in question and to constructing from the resulting table that the more the plunger is inclined, the more the course of the ball between buffer and target is inclined. Certainly this could be called a law, but it is a law which is a simple summary of formulations made one by one.

In contrast, stage III subjects view the experiment from the start both in terms of the total number of possibilities and in terms of necessary relations, since they possess operations which both are combinatorial and contain the potential assurance of deductive necessity. In their first correspondence operations they do not merely take note of the empirical relationships but immediately proceed to search for an explanation—*i.e.*, they consider the correspondences as implications. Of course, in a sense the implication $p \supset q$ is still a statement of fact, equivalent to establishing that the case $(p.\bar{q})$ never occurs. Still, in order to establish this it is necessary to consider the four possibilities $(p.q) \vee (p.\bar{q}) \vee (\bar{p}.q) \vee (\bar{p}.\bar{q})$; in any case, the implication is nothing more than the addition of three possibilities (the first, the third, and the fourth) combined by the operation (v) which signifies "or"—*i.e.*, it is an addition of what is possible and not of "realities."

Actually, when faced with a correspondence $p.q$ (let p be the term for a certain angle of incline of the plunger and q the term for the corresponding angle of incline of the course of the ball between the buffer and the target), the stage III subjects are not restricted to pointing out the existence of the conjunction, as are those of stage II, who are satisfied at this point. They exclude the possibility $(p.\bar{q})$—*i.e.*, they introduce by hypothesis an implied link between p and q; but they also exclude $(\bar{p}.q)$—*i.e.*, they also introduce by hypothesis an implied link between q and p. Thus they proceed immediately from stating the conjunction $p.q.$ to stating the hypothesis of a reciprocal implication $p \gtreqqless q$, with the assumption that this reciprocity $p \gtreqqless q$ or $p \doteq q$ (which is not in itself an equality of content but a simple equivalence from the point of view of the truth of the propositions) covers the equality of some real factor.

At this point, the reasoning process of the 14–16-year-old subjects, based from the start on the twofold consideration of possible combinations and necessary links, is elaborated into a true hypothetico-deductive construction. Unlike stage II subjects, who are limited to noting the occurrence of various correspondences,[4] the adolescents at stage III sooner or later (and often very early) try to uncover the general principle underlying the special case of null inclination. Having established that the ball returns to its starting point, they immediately draw the conclusion that the corresponding inclinations must be equal and consequently the angles which determine them must also be equal; after verification with one or two they generalize the conclusion to all cases.[5]

In symbolic terms, the subject's reasoning at substage III-B is approximately the following (see as an example the extremely clear case of DEF):

$$p \gtreqqless q, \text{ because } (p.q) \vee (\bar{p}.\bar{q}) \text{ are true and} \quad (1)$$

$(p.\bar{q}) \vee (\bar{p}.q)$ false where p and q state corresponding inclinations having the respective values x and y. But

$$(x = 0) \gtreqqless (y = 0), \text{ and} \quad (2)$$
$$(x = a) \gtreqqless (y = a) \quad (3)$$

where a is a determinate inclination > 0. Therefore,

$$x \gtreqqless y, \text{ and} \quad (4)$$
$$\wedge x \gtreqqless \wedge y \quad (5)$$

where $\wedge x$ and $\wedge y$ are the angles of incidence and reflection (or their complementaries).

[4] Which may include the case in which the plunger is not inclined and the ball returns to the starting point, but from which they do not abstract the general principle.

[5] Note that the elementary reasoning by recurrence is itself accessible at the concrete level (see *La Géométrie spontanée de l'enfant*, Chap. IX, no. 4). It appears so late in this case because all of the subject's deduction is directed by preliminary reciprocal implications.

In sum, the discovery of the equality of the angles is the result of the reciprocal implication between the corresponding inclinations postulated from the start and not the inverse; this reciprocal implication differs from simple concrete correspondence by the fact that it results from a calculation of possibilities and not merely from an account of the empirical situation.

REVIEW OF SOME RESEARCH ON PROPOSITIONAL ABILITIES

Prolog

Propositional abilities constitute the basis for the most mature or advanced level of cognitive development. This level, described by Piaget as the Formal Operational Stage of cognitive development, generally develops throughout adolesence.

As the abilities develop, thinking begins to deal with statements or ideas about raw data, rather than with the raw data itself. The student might first use concrete operations of an earlier level of development to organize reality into classes or to order it in some way, but he proceeds to generate propositons that use the organizations and to operate on the propositions via such abilities as conjunction, disjunction, implication, negation, compensation, and equivalence. Because it employs operations upon operations and the resultant statements about statements, propositional thinking is considered to be second-level thinking (Flavel 1963).

To explain his notions of this stage of thinking, Piaget has developed two models that, when combined, constitute an integrated cognitive structure. The first is a general model that sets forth the possible propositional abilities that underly this level of thinking. The second is a specific model that describes the cognitive operations derived from the abilities.

a. The General Model

Theoretically, in approaching a problem, a student at the Formal Operational Stage of development can scan the realm of the possible, begin to test the validity of the possibilities empirically, and can reach some conclusions about the results. The following analysis of a problem in which multiple variables must be considered will show the abilities that underly the structure of this general model.

Suppose the solution of a problem required isolating the variables that cause a pendulum to swing rapidly. Considering only the variables of (1) the weight attached to the pendulum (w = heavy; $\bar{w}$ = light) and (2) the length of the pendulum (l = long; $\bar{l}$ = short), the following four combinations of variables are possible:

$(w \cdot l)$ heavy and long	$(\bar{w} \cdot l)$ light and long
$(w \cdot \bar{l})$ heavy and short	$(\bar{w} \cdot \bar{l})$ light and short

The student at the Formal Operational Stage can recognize these possible combinations before experimentation and can use them in generating propositions. The pre-formal level student becomes aware of the possible combinations only by manipulating the materials. There are 16 binary propositions that can be derived from the above four combinations of variables. The propositions are the possible outcomes of the experiment. They are derived by taking each possibility alone, then by taking the combinations of two possibilities, then taking combinations of three possibilities, and finally by taking all four possibilities together (Inhelder and Piaget 1958, p. 277).

THE 16 BINARY PROPOSITIONS*

Pendulum Experiment	Four Possible Combinations Of Two Variables In An Experiment			
	1	2	3	4
weight and length variables	$w \cdot l$	$w \cdot \bar{l}$	$\bar{w} \cdot l$	$\bar{w} \cdot \bar{l}$
Type Of Proposition	**All Ways In Which Outcomes Of The Combinations Of Two Variables Can Be Observed**			
1. Conjunction	$w \cdot l$	—	—	—
2. Non-implication ($l \not\to w$)	—	$w \cdot \bar{l}$	—	—
3. Non-implication ($w \not\to l$)	—	—	$\bar{w} \cdot l$	—
4. Conjunctive Negation	—	—	—	$\bar{w} \cdot \bar{l}$
5 Independence of w to l	$w \cdot l \lor$	$w \cdot \bar{l}$	—	—
6. Independence of l to w	$w \cdot l \lor$	—	$\bar{w} \cdot l$	—
7. Equivalence ($w = l$)	$w \cdot l \lor$	—	—	$\bar{w} \cdot \bar{l}$
8. Reciprocal Exclusion	—	$w \cdot \bar{l} \lor$	$\bar{w} \cdot l$	—
9. Inverse of Independence of l to w	—	$w \cdot \bar{l} \lor$	—	$\bar{w} \cdot \bar{l}$
10. Inverse of Independence of w to l	—	—	$\bar{w} \cdot l \lor$	$\bar{w} \cdot \bar{l}$
11. Disjunction	$w \cdot l \lor$	$w \cdot \bar{l} \lor$	$\bar{w} \cdot l$	—
12. Converse Implication ($w \subset l$)	$w \cdot l \lor$	$w \cdot \bar{l} \lor$	—	$\bar{w} \cdot \bar{l}$
13. Implication ($w \supset l$)	$w \cdot l \lor$	—	$\bar{w} \cdot l \lor$	$\bar{w} \cdot \bar{l}$
14. Incompatibility	—	$w \cdot \bar{l} \lor$	$\bar{w} \cdot l \lor$	$\bar{w} \cdot \bar{l}$
15. Tautology	$w \cdot l \lor$	$w \cdot \bar{l} \lor$	$\bar{w} \cdot l \lor$	$\bar{w} \cdot \bar{l}$
16. Negation	$\overline{w \cdot l \lor w \cdot \bar{l} \lor \bar{w} \cdot l \lor \bar{w} \cdot \bar{l}}$			

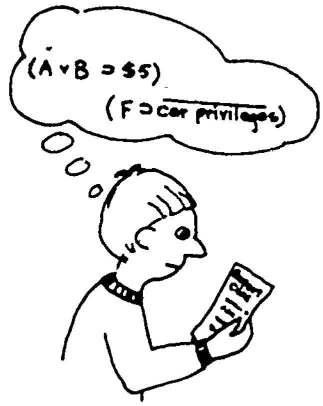
*Self-workshop H provides additional information about these propositions.

Although the propositions in the example might never be made explicit by the student at the Formal Operational Stage, it is clear that as he carries out an experiment, such propositions guide his experimentation. Generally the student will begin with an hypothesis, devise several experiments to test the hypothesis, experiment systematically holding all the variables constant except one so that the importance of each variable can be evaluated, use information gained to frame new hypotheses, and ultimately draw a conclusion based on all the experimental data.

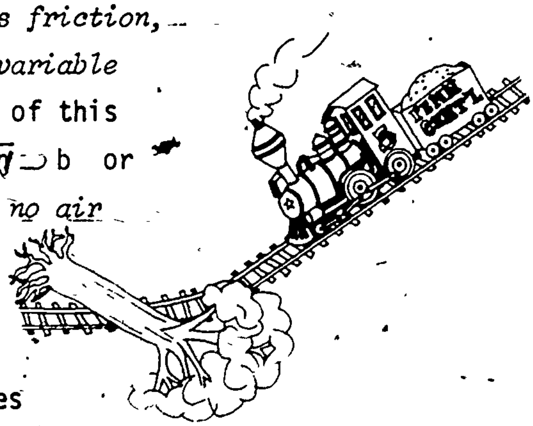
b. The Specific Model

Theoretically there are four cognitive operations that Formal Operational Stage students can use when dealing with information. In describing these operations, the following notation is used: $\supset$ = implies; v = and/or.

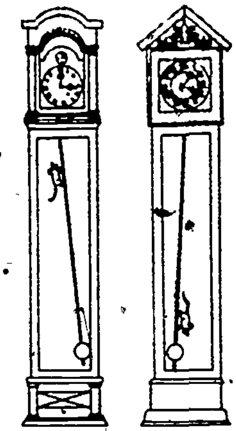
1. Identity Operation. This operation is a null operation that changes nothing. The identify operation of *"if x, then y"* is *"if x, then y."* Thus $I(x \supset y) = (x \supset y)$.



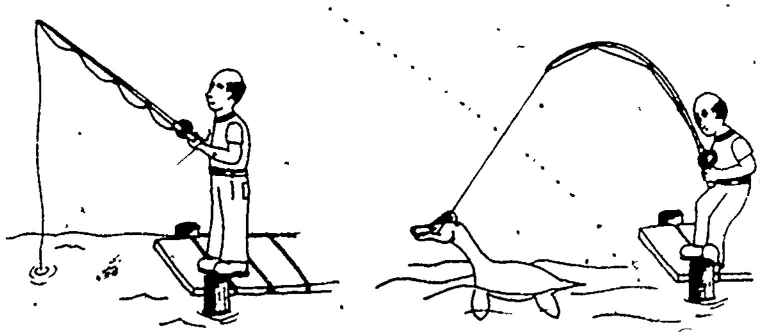
2. Negation (Inversion) Operation. This operation negates all aspects of a proposition. Suppose a student has already discovered a series of facts about a problem involving inertia — that a freely rolling train stops because of friction, air resistance, or other variables. This, translated into logical notation is $b \supset f \vee r \vee n$ or *"If the train stops, then there is friction, air resistance, or other variable operating."* The negation of this proposition is $\bar{f} \vee \bar{r} \vee \bar{n} \supset b$ or *"If there is no friction, no air resistance, or any other variable operating, then there will be no stopping of the train."* This states the principle of inertia, which cannot be observed in any concrete form but which can be derived by negating the initial proposition. Thus, through the negation operation all conjunctive propositions can become disjunctive propositions and all assertions can become negations.



3. Compensation (Reciprocity) Operation. This operation generates the reciprocal of a proposition. Suppose the variables in the initial proposition were weight and length, then the proposition $w \cdot l$ (heavy and long) can be transformed to the reciprocal $\bar{w} \cdot \bar{l}$ (light and short). The reciprocal operation does not change conjunctive or disjunctive propositions.



4. Correlation Operation. The correlation operation alters conjunctive and disjunctive propositions. In a problem about the flexibility of a series of rods, the following disjunctive proposition might be drawn: *"If bending occurs, then either*



length or the material or both are causing the bending" or $b \supset l \vee m$. The correlate of this proposition would be $b \supset l \cdot m$.

These two models (The General Model and The Specific Model) form the core of Piaget's theory about the nature of thought at the Formal Operational Stage of cognitive development. The final equilibrated state is based on a completely integrated structure of propositional abilities combined with a completely integrated set of the four operations.

Log

Researchers have carried out fewer studies on the propositional abilities and operations related to the Formal Operational Stage of development than they have on the abilities and operations related to other stages described by Piaget. Research dealing with thinking has increased tremendously in the past few decades; however, the efforts have been primarily directed toward the preschool and elementary levels. And although there is a large number of works on the affective and social life of the adolescent, there is little available on the adolescent's thinking.

a. Replication Studies

Carpenter (1955), Lovell and Ogilivie (1960), and Elkind (1961) were among the earliest researchers to replicate systematically Piaget's work on Formal Operational Stage thinking. They found evidence that supported the sequencing of stages and substages proposed by Piaget.

In England, Lovell (1961) repeated ten of the fifteen tasks originally described by Inhelder and Piaget (1958). 200 subjects, aged 8 to 32, were individually interviewed. The ten tasks used were: flexibility of rods, oscillation of the pendulum, falling bodies on an inclined plane, effects of invisible magnetization, combinations of colorless chemical substances, conservation of motion on a horizontal plane, equilibrium in the hydraulic press, equilibrium in the balance, projection of shadows, and correlations. Although the English subjects found the tasks more difficult than Piaget's Swiss subjects found them, Lovell's conclusions confirmed the sequence that Piaget observed in the development of propositional abilities. Lovell also found that many college students did not perform at the Formal Operational Stage.

Noelting (1961) conducted a longitudinal study using the same students and tasks used by Inhelder and Piaget. His findings corroborated the earlier findings. In addition he theorized that the structures of thought and those of mathematics were homologous.

The works of Hotyat (1961) and Lee (1968) also confirmed Piaget's sequential developmental pattern. Huttenlocher (1964) compared the thinking of adolescents with that of younger students and found, as Piaget did, qualitative differences between them.

Jackson (1965) investigated students of average IQ (90-110) on a variety of propositional ability tasks: oscillation of the pendulum, falling bodies on an inclined plane, conservation of motion on a horizontal plane, equilibrium in the balance, communicating vessels, and the law of floating bodies. His students ranged from 5 to 15 years of age. Half of the students (n = 24) at age 15 was found to be still at the pre-formal level of development. Although the other half was found to be at the Formal Operational Stage, only 5 of the students had reached the most equilibrated substage (IIIb).

Lovell (1968) conducted a study of gifted students in England to see if they achieved the Formal Operational Stage earlier than students of average IQ. His subjects, all with IQ's above 140, ranged in age from 8 to 11.5 years. The three tasks used were: oscillation of the pendulum, equilibrium in the balance, and combinations of colorless chemical substances. On these tasks, 15 of the 140 students were found to be at the Formal Operational Stage. Of these, 11 were only at the earliest substage (IIIa). Lovell concluded that gifted students do not acquire formal level abilities until at least age 12.

Keasey (1970) used five of the tasks with females who were college age and beyond. The five tasks selected were: flexibility of rods, oscillation of the pendulum, falling bodies on an inclined plane, combinations of colorless chemical substances, and equilibrium in the balance. She found that attainment of the highest substage of formal operations was rare. College coeds and older women showed some structural characteristics of propositional abilities, but these characteristics were not stable. The subjects in this study found the tasks more difficult to perform than did the subjects observed by Inhelder and Piaget.

McKinnon and Renner (1971) presented five tasks to college freshmen: flexibility of rods, oscillation of the pendulum, conservation of quantity, angles of incidence and reflection, and the relationship of weight and volume of floating and sinking objects. The researchers found that only the conservation of quantity task was successfully performed by more than 50% of the freshmen.

Lengell and Buell (1972) used the pendulum task and developed a hierarchy of levels that students follow as they solve the problem. The researchers claimed that the steps paralleled Piaget's developmental stages. Although the Identification Of Factors level corresponded to the earliest stage, it was believed that the Quantification level was more advanced than the IIIb substage. In the study, students moved progressively along the hierarchy from grade 7 to grade 12 (80% reached substage IIIb by grade 12).

HIERARCHY

Quantification
↑
Verification
↑
Tentative Conclusions With Exclusion
↑
Identification Of Dependent Variables
↑
Identification Of Independent Variables
↑
Identification Of Factors

b. Other Studies

Several investigators have designed their own tasks to examine the Formal Operational Stage of development.

Morf (1957) developed a series of brief stories, each posing a problem that required advanced reasoning for its solution. His procedures included sets of questions and the review of all details of a story until it was completely understood by the student. The most difficult stories were solved by only a few students at 15 years of age.

Case and Collinson (1962) asked students questions concerning selected passages from history, geography, and literature. They then categorized the responses as to the use of intuitive, concrete, or formal operational thought. They reported that *"between 13 and 15 years of age there is a considerable increase in incidence of formal thought, with a corresponding decrease in concrete and intuitive level answers"* (p. 107).

Yudin and Kates (1963) investigated adolescents' attainment of concepts. They found that at age 14 - 16, there were fewer perceptual errors and significantly more consistent and integrated strategies than at age 12. The findings were interpreted as being consistent with Piaget's assertion that formal thought begins at about age 12 and that the propositional abilities are generally well developed by ages 14 - 16.

Using a sample of students whose IQ's ranged from 80 to 130, Yudin (1966) reaffirmed his previous conclusions. He reasserted the constancy of Piaget's developmental sequence but pointed out variation in the rate at which the stages develop. For example, students in the lowest IQ range did not begin to employ propositional abilities until the ages of 14 - 16, but they did show significant gains during these two years.

Lunzer (1965) analyzed Inhelder and Piaget's original tasks and classified them into two sets. One set required considering one variable while holding all others constant; the other set required an understanding of proportionality. He then constructed other tasks to check on Inhelder and Piaget's conclusions. Among the tasks were verbal and numerical analogies. For example, a multiple choice statement required the student to finish the phrase *"leather is to shoe as wool is to ..."* with the word *"cardigan."* To Lunzer, such an analogy depends upon the abstraction of several relationships (Leather and wool might be united in the class of raw materials; shoe and cardigan might be in the class of wearing apparel) and the finding of second order relationships or conceptual intersections between the first two concepts and the last two. After testing 153 English students between 9 and 17 years of age, he concluded that both verbal and numerical analogies require the application of formal reasoning and that formal reasoning is qualitatively distinct from concrete reasoning because it clearly requires second-level relationships.

Gyr and Fleisher (1967) devised an apparatus which helped them examine college students' ability to solve a problem. Their data clearly distinguished levels similar to Piaget's stages. They found that 84% of the sample preferred to solve the problem at a concrete, pre-formal level and only 14% solved it using Formal Operational Level thought; however, when students were asked to solve the problem in as few trials as possible, 50% exhibited Formal Operational Level thinking.

In his work on adolescent thinking, Peel (1960) found that even capable students were likely to use concrete and less abstract methods because they required less cognitive effort.

Kessen (1960) indicates that, although the student may have the capacity to use propositional abilities, he is not compelled to do so. He may revert to any of the earlier levels of abilities, for earlier levels are not eradicated but instead are integrated into later stages.

Wason (1968) devised a task which involved the following conditional statement: *"If there is a D on one side of any card, then there is a three (3) on the other."* When represented in logical terms $(d \supset t)$, the following statements are possible: $(d \cdot t)$, $(\bar{d} \cdot t)$, $(d \cdot \bar{t})$, $(\bar{d} \cdot \bar{t})$. Only the third possibility is logically false given that $(d \supset t)$. The subjects, all university students, were then asked to prove whether the statement was true or false for a set of cards provided by the experimenter. Since this task involves a statement of the form *"if d, then t,"* the solution requires selection of variables and combinatorial reasoning. Wason found that the students performed far below the level expected. Students generally were not able to isolate variables and subject them to combinatorial reasoning. Wason further raised the question: *"Could it then be that the state of formal operations is not completely achieved at adolescence, even among intelligent individuals?"*

Stone and Ausubel (1969) attempted to determine if the transition from the concrete to the formal stage occurs simultaneously in all content areas. To find out, they asked students to read three difficult learning passages and to answer multiple choice questions. Tenth grade students performed at significantly higher levels than did 7th grade students with comparable IQ's. The researchers concluded that formal thought in any subject is not possible until sufficient concrete background experience has been attained in each content area.

Bart (1970) presented a set of tasks designed to be administered to groups of adolescents. The tasks involved readings and questionnaires covering three subject matter areas — biology, history, and literature. He administered the group tasks to students who were 13, 16, and 19 years of age and supplemented this by administering four of the Piagetian formal operational tasks to each student individually. The correlation coefficients between the Piagetian tasks and each of the subject area questionnaires were *"moderate"* according to Bart. He described his work as *"somewhat successful"* and stated that no valid group testing tool for the Formal Operational Stage was in existance at that time.

Buell and Bradley (1972) presented high school chemistry students with a graph of solubilities and requested that they interpret the graph and tell how they reached their conclusions. The researchers applied the 16 binary propositions in analyzing responses and found 4% of the students to be at the concrete stage and 14% at the formal stage. The remainder were in transition. Following two weeks of instruction on solubilities, the researchers found little change in the percentages. Students could perform less complex binary operations (e.g., conjunction, reciprocity) but not the more complex ones (e.g., implication, tautology).

Lynch (1973) developed a technique for determining levels of large groups of individuals. The group-administered technique obtained more information in a shorter period of time than the usual individually-administered tasks. Lynch found that the technique, a film and questionnaire, was equivalent to Piaget's pendulum task.

Karplus and various collaborators (1970, 1970, 1972, 1973) developed and tested tasks dealing with deductive reasoning and proportional reasoning. The deductive reasoning task required a kind of thinking primarily concerned with the logical coherence of a group of propositions — inferences were to be drawn only after having made an hypothesis as regards the truth of a proposition in question, given that another set of propositions (clues) were true. The ability to do this type of reasoning was present in only 10% of a large sample of teachers of science (n = 152), a sample composed entirely of persons who teach the principles embodied in many of Inhelder and Piaget's formal reasoning tasks. The proportional reasoning task involved direct proportions and revealed that consistent and correct use of this ability is not yet attained by the majority of above average 15 year old students. Since varying the task contexts is no problem for formal thinkers, having attained levels of abstraction that allow sensitivity to form regardless of content, it must be concluded from the students' lack of consistency on a variety of tasks that they were not at the Formal Operational Stage. In fact, these researchers have the impression that a stage of formal reasoning, as characterized by Piaget, may not exist. Their longitudinal research indicates that students' responses improve with age, but that there is no evidence for the existence of an invariant sequence of response categories.

c. Teaching Studies

Aebli (1951) devised an experiment to test the idea that if teaching imitated the operational nature of thought, learning would take place. Students were taught a sequence of lessons on the perimeter and area of a rectangle by operational or by non-operational methods. Students experiencing the operational method explored and experimented with physical materials and progressed to discussion and exposition of ideas. NOTE: Aebli was not promoting simple activity, but activity that was mobile and reversible and that reproduced the operational nature of the concept. It was found that operational teaching produced more learning than non-operational teaching for average and below-average students.

Lovell (1961) attempted to ascertain the effect of instruction on the level of thinking. His results indicated that knowledge gained by instruction prior to the development of the appropriate level of thinking either disappears or, if it remains, does so as rote knowledge.

Boyer (1969) achieved success when he used symbols of logic with abstruse logical statements to develop logical thinking in students.

Keasey (1970) found that her training procedure was very effective in changing problem solving behavior between pretest and the immediate post-test, but that there were no gains on a delayed post-test.

McKenzie (1972) was successful when he attempted to develop logical thinking in social studies by giving quizzes that required inferential thinking.

Marek and Renner (1972) used some of Inhelder and Piaget's tasks in a statewide investigation of thinking abilities of 588 seventh- through twelfth-grade students in Oklahoma schools. They found that the capability to think at the formal level improves through the grades. They also found that Oklahoma's current science program in biology, which requires the student to separate variables, see implications, use the principle of exclusion, and reason with the "*if ... then ... therefore ...*" construct, was appropriate for only 27% of the students enrolled in the program. They recommended that teachers must better select the curriculum on the basis of the students' learning characteristics.

Epilog

Although Piaget's work suggests that the student moves from the Concrete Stage to the Formal Stage in the period between the ages of 11 and 16, other research suggests that the level may not be reached until college age, and even then it may never become the preferred mode of cognitive operation for some individuals. In spite of the differences in ages at which the stage and substages appear, the sequence or order of appearance of the abilities seems to have been well corroborated.

Research has not shown clearly whether the Formal Stage can be accelerated by instruction. There is a general indication, however, that the selection of what is to be taught should be better matched to the student's level of development.

We need to learn much more about adolescent thinking.

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APPENDIX

Motivation, interest, and persistence often contribute somewhat to the attainment of solutions to propositional problems such as those given in Workshop F.

The following are examples of various types of problems related to propositional abilities. If you like thinking through possibilities or combining deduction with analysis, and if you enjoy tasks which require persistence, you and your students might try the following problems.

Although no solutions or analyses are provided for the problems, some references to solutions are given, and staff members will gladly discuss with you whatever you work out.

Probabilistic Reasoning Problems

Flipping Coins

Find the probability of three pennies falling all heads when tossed simultaneously.

Rolling Dice

Find the probability of a 5 turning up in one toss of a pair of dice.

Combinatorial Reasoning Problems

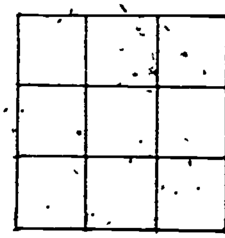
Choosing Socks

A drawer contains two red socks, two green socks, and two blue socks. What is the least number of socks you can take from the drawer, with your eyes closed, and be sure you have a pair that matches?

Making Change[1]

In how many different ways can a dollar be changed with an unlimited supply of halves, quarters, dimes, nickels, and pennies?

Making A Magic Square[2]



How can nine digits (1, 2, 3, 4, 5, 6, 7, 8, 9) be placed in a square array to form eight intersecting sets of three digits (rows, columns, and main diagonals), each summing to the same answer?

Printing License Plates

How many different license plates can be made if a letter of the alphabet is followed by a three-digit number on each?

Propositional Reasoning Problems

Traveling In The Mountains[3]

In the mountains there is a path where it is forbidden to pass when it is night and when you do not have a lantern. You can pass during the night when you have a lantern or without a lantern when it is daytime. May each of the following people pass or not pass along the path: 1) A tourist who arrives at night with a lantern? 2) A person who comes during the night and does not have a lantern? 3) Someone who has a lantern but it is daytime? 4) A tourist who comes during the day and has no lantern?

Identifying The Beads[4]

A king possessed a coffer filled with large and small beads of many different colors. In the interior of the coffer he placed a small golden box which contained the most precious of his beads and which were also of many different colors and sizes. One day he let his friend look at his beads and pick them up. Since he trusted his friend, he left him alone. But something unfortunate occurred. The friend let the golden box fall into the coffer, and the most precious beads fell among the others. It became no longer possible to distinguish them. The friend called the king's treasurer because he was afraid to tell the king what had happened. He asked the treasurer for a counsel. But the treasurer did not want to explain how the beads had been arranged and only said to him, *"The pink beads which were in the box were not little. There were also some small ones, but they were not pink."* The friend tried to manage with that information. He first took a large pink bead and asked himself if it could have fallen from the box. He did the same with a small white bead, then a small pink bead, and finally a bead which was neither large nor pink. What did the friend decide for each of these beads?

Controlling and Manipulating Variables Problems

Identifying Combinations of Variables[5]

A hamster is timed to see how fast he can find his way through a maze. At the start of the experiment, the first thing the animal must do is pass through either a blue door or a green door. When it has done this, it must pass through a door marked A, a door marked B, or a door marked C. Finally, it must go through either a door numbered 4 or one numbered 5. Once the hamster passes through any door, it cannot go back again. Identify all the different three-door paths that the hamster can take to get through the maze (Remember that each path has a colored door, a lettered door, and a numbered door).

Controlling Variables[6]

An agricultural researcher is planning to test three poisons labeled 404, 606, and 707 on two insect pests that are identified as type A and type B. If he can use each poison by itself and mix them together in any way, write the experiments that must be made to test every poison and poison combination on each pest.

Deductive Reasoning Problems

The Trainmen[7]

Smith, Jones, and Robinson are the engineer, brakeman, and fireman on a train, but not necessarily in that order. Riding the train are three passengers with the same three surnames, to be identified in the following premises by a "Mr." before their names.

1. Mr. Robinson lives in Los Angeles.
2. The brakeman lives in Omaha.
3. Mr. Jones long ago forgot all the algebra he learned in high school.
4. Smith beat the fireman at billiards.

5. The passenger whose name is the same as the brakeman's lives in Chicago.

6. The brakeman and one of the passengers, a distinguished mathematical physicist, attend the same church.

Who is the engineer?

The Homeowners[8]

Use the following facts to solve this problem.

1. There are five houses, each of a different color and inhabited by men of different nationalities, with different pets, drinks, and cigarettes.
2. The Englishman lives in the red house.
3. The Spaniard owns the dog.
4. Coffee is drunk in the green house.
5. The Ukrainian drinks tea.
6. The green house is immediately to the right (your right) of the ivory house.
7. The Old Gold smoker owns snails.
8. Kools are smoked in the yellow house.
9. Milk is drunk in the middle house.
10. The Norwegian lives in the first house on the left.
11. The man who smokes Chesterfields lives in the house where the horse is kept.
12. Kools are smoked in the house next to the house where the horse is kept.
13. The Lucky Strike smoker drinks orange juice.
14. The Japanese smokes Parliaments.
15. The Norwegian lives next to the blue house.

Now, who drinks water? And who owns the zebra?

The Ping Pong Contest

In a ping pong contest, Ken beat Rich and Bill; Rich beat Beth, Nancy, and Bill; Nancy beat Ken and Beth; Beth beat Ken and Bill; and Bill beat Nancy. Rank the players according to their winning ability.

Who's With Whom?[9]

A couple of times a year, in the province of Outer Suburbia, there's a get-together of five congenial families - the ALLISONS, the BAKERS, the CONNORS, the DAVIDSONS, and the ELLENDERS.

Naturally, all the adults are on a first-name basis. The wives are FRANCINE, GLORIA, HELEN, ISABEL, and JUDITH. Their husbands (not respectively!) are KENNETH, LEON, MICHAEL, NORMAN, and OSCAR.

As it happens, each of the adults is presently married for the second time; and what's more, each has been divorced from a member of the group. Each marriage produced one child, and each of the ten adults is the parent of one boy and one girl. The children whose parents are divorced live with their mothers. The get-togethers are attended by all ten children - PETER, QUENTIN, RUTHIE, SANDRA, TED, URSULA, VINCENT, WILMA, XAVIER, and YETTA.

Here are a few facts about the most recent bash:

1. Each couple arrived separately, accompanied by their child and the wife's child by her previous marriage.
   a. Helen and Xavier were among the occupants of the first car.
   b. Ursula and Vincent arrived in the second car.
   c. Leon and Sandra were among those in the third car.
   d. Mrs. Allison and Ted were in the fourth car.
   e. The party was held in the house where Norman and Wilma reside.
2. In the ladies' table tennis competition, the doubles match was won by Michael's wife and Xavier's mother, who defeated the team of Mrs. Connors and Judith.
   a. The two members of the winning team then opposed each other in a singles match in which Yetta's mother defeated Francine.
   b. The members of the losing doubles team also played singles against each other, and Kenneth's wife defeated Sandra's mother.
   c. The winners of the singles matches then opposed each other for the championship. After some brilliant volleying, the former Mrs. Baker finally edged out Gloria.

3. Later there was some really great singing by a quartet consisting of Mr. Davidson, Gloria's husband, Peter's father and Gloria's former husband.

   a. After their second number, Mr. Davidson and one other member of the group decided to take a break, so the remaining two members did a couple of selections with their daughters. Everyone agreed that Ursula, Michael, Ruthie and Mr. Ellender were pretty good.

   b. Then the two members of the original quartet who had taken a breather returned to the spotlight and did a number with their sons. When they finished, a round of applause went to Kenneth, Quentin, Vincent, and Helen's husband.

   c. Finally, the four youngsters who had sung with their dads did some rock 'n' roll improvisations, and the biggest hand of the day went to Quentin, Michael's daughter, Kenneth's son and Ruthie.

4. The only ones who didn't enjoy the day to the fullest were the Bakers.

   a. Mr. Baker had such an acute case of laryngitis that he had to borrow a pen from his son Xavier in order to communicate by written note.

   b. Mrs. Baker had sprained her ankle the day before, and even while leaning heavily on her son Quentin's arm, she was clearly in agony.

On the basis of these facts, you are now ready to figure out

(1) the full names of the five husbands

(2) who's presently married to whom

(3) whom they had previously been married to

(4) which child is the product of each marriage.

Proportional Reasoning Problems

Finding Values

If y varies directly as x and is 18 when $x = 6$, find the value of y if $x = 2$.

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Painting Fence Posts

The amount of paint needed to paint the sides of a cylindrically shaped fence post varies jointly as the radius and height of the post. Compare the amount of paint needed for a post 5 feet high of radius 1/2 foot with that needed for a post 6 feet high of radius 1/3 foot.

Taking Photographs

The exposure time necessary to obtain a good photographic negative varies directly as the square of the *f* number of the camera lens. If an exposure of 1/50 second produces a good negative at *f* 16, what exposure time would be necessary at a lens setting of *f* 8?

Bending Bars

The amount that a bar will bend under a given force varies inversely as the width of the bar when the thickness and length of the bar remain the same. If a bar 2.5 inches wide will bend 7° under a certain force, what is the width of a bar of the same material, length, and thickness that will bend 5° under the same force?

Measuring Distances

One of Kepler's laws states that the square of the time in days required by a planet to make one revolution about the sun varies directly as the cube of the average distance of the planet from the sun. If Mars is 1 1/2 times as far from the sun, on the average, as the earth is, find the approximate number of days required for it to make a revolution about the sun.

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