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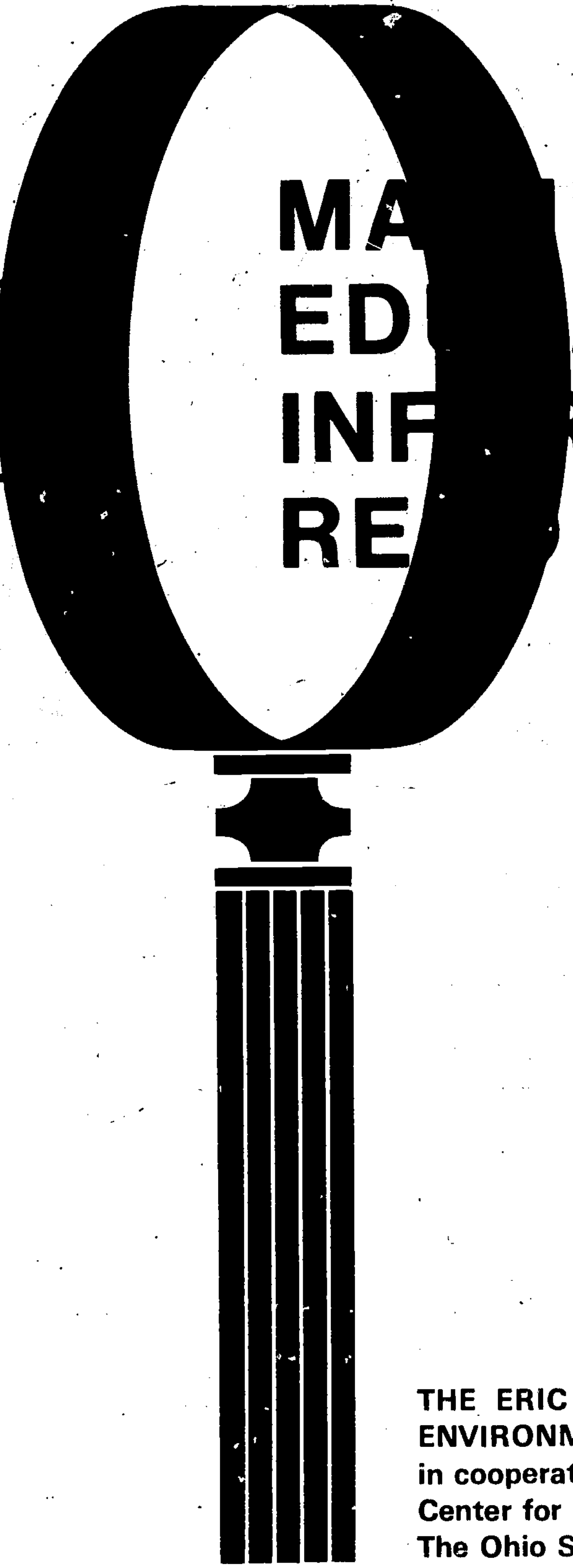
ABSTRACT

This volume contains a series of papers on algorithmic learning. Included are six reviews of research pertaining to various aspects of algorithmic learning, six reports of pilot experiments in this area, a theoretical discussion of "The Conditions for Algorithmic Imagination," and an annotated bibliography. All the papers assume a common definition of algorithmic learning as "the process of developing and/or applying methods or procedures, i.e., algorithms, with the goal of learning-how-to-learn." A common definition of algorithm is also used. Topics covered by literature reviews include algorithmic processes for cognition, algorithms and hierarchies, conceptual bases for the learning of algorithms, interference with the learning of algorithms, algorithmic problem solving, and algorithms and mental computations. Research papers report on studies related to algebra (3), arithmetic (2) and the use of desk calculators (1). The authors conclude that there are many open researchable questions in the area of algorithmic learning. (SD)



MATHEMATICS EDUCATION INFORMATION REPORT

THE ERIC SCIENCE, MATHEMATICS AND ENVIRONMENTAL EDUCATION CLEARINGHOUSE
in cooperation with
Center for Science and Mathematics Education
The Ohio State University

Algorithmic Learning

edited by

Marilyn N. Suydam
and
Alan R. Osborne

Mathematics Education Reports

Mathematics Education Reports are being developed to disseminate information concerning mathematics education documents analyzed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education. Reports in each of these categories may also be targeted for specific sub-populations of the mathematics education community. Priorities for the development of furture Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, the Conference Board of the Mathematical Sciences, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed.

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Algorithmic Learning

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I. Algorithmic Learning: Introduction

Algorithmic Learning: Introduction

Marilyn N. Suydam

To many people, "algorithmic learning" means "the learning of algorithms". They think of algorithms for addition, subtraction, multiplication, and division with whole numbers, such as:

	6		13
54	72	86	18) 234
+37	-45	x79	18
11	27	774	54
80		602	54
91		6794	

They think of algorithms for operations with fractions and decimals, of a square root algorithm, of procedures in the content of algebra and calculus and other mathematical areas.

But algorithmic learning involves more than just the learning of specific algorithms. It connotes having learners generalize from specific skills to broader process applications. It is related to learning-how-to-learn. As Simon (1975) pointed out, teaching the algorithm and teaching the characteristics of an algorithmic solution are two different things.

The importance of algorithmic learning is being increasingly recognized, across other content areas as well as within mathematics. In the past few years, it has been developed as the approach in at least one textbook. The research interest in artificial intelligence is built on a foundation of algorithmic learning. Several Russian psychologists, among others, have been very much concerned with the implications of algorithmic learning (Gerlach and Brecke, 1974; Landa, 1974).

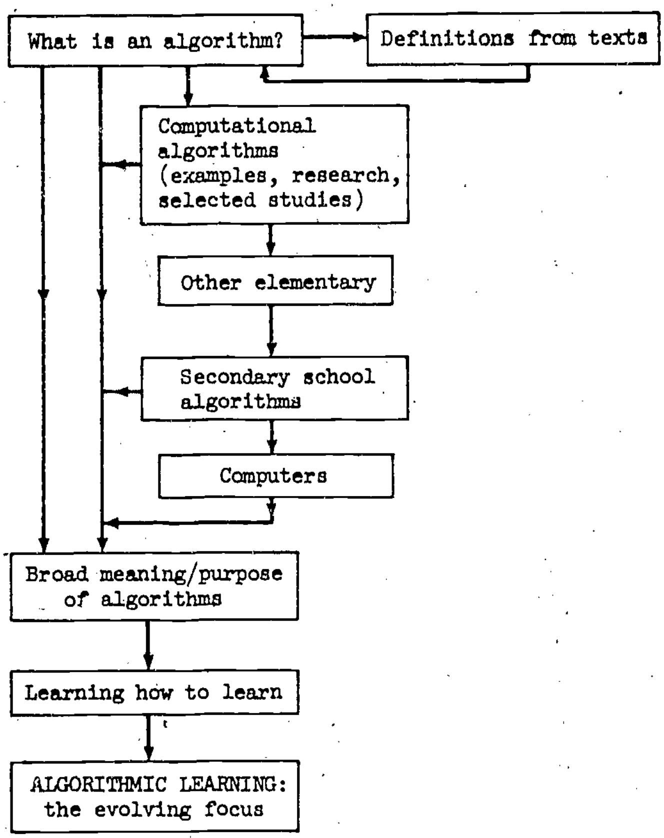
The focus of much current writing is still on algorithms, but the need to provide for algorithmic learning is becoming increasingly more evident. The use of hand-held calculators at all levels from the elementary years through life has raised new questions about algorithms--

and emphasizes the need to explore ways in which algorithmic learning can be promoted, as calculators decrease the need to focus so much of our attention on the algorithms for calculation.

Explanation

This document is not intended to be all-inclusive (although we had dreams of being comprehensive at one early point!). It is basically the report of a year of emphasis on algorithms and algorithmic learning in a seminar for mathematics education doctoral students at The Ohio State University. It doesn't include all that the seminar encompassed. But it does present some results, both in the form of research reviews and mini-research studies. It is hoped that it will serve to have others do more thinking about what is known about algorithmic learning, and, even more important, to think about what still needs to be explored and learned.

In proposing the seminar, it was noted that there is a tradition of concern for algorithms in the computational orientation of elementary school mathematics, but new information-processing models of learning seem to be stimulating a new body of research problems and studies. A more general interest is suggested, in broadly conceived algorithmic learning non-specific to the computational needs of young children. The relation of algorithmic learning to problem solving, logical ability, creativity, and the like have not been explored. And they should be. Our focus was indicated by this flow diagram for our initial discussions:



Thus, this document attempts to:

(1) review the status of some aspects of research related to algorithmic learning across the mathematics curriculum, and

(2) indicate a few of the directions which research on algorithmic learning and on computational algorithms has taken and might take.

It is not intended to be a state-of-the-art paper, but only another contribution to the increasing documentation on algorithms and algorithmic learning.

Definitions

We worked for many hours trying to find good definitions for "algorithm" and "algorithmic learning". In the course of this serarch, we found that algorithms have been defined in two ways:

(1) By example, especially at the elementary school level and in elementary school mathematics content and method textbooks for teachers.

(2) By simple definitions, such as:

(a) "A computational procedure, especially one that involves several steps, is often called an algorithm." (Bouwsma, Corle, and Clemson, 1967, p. 40)

(b) "Each arrangement of numbers for purposes of computation was called an algorism. . . .Many algorisms, or ways of setting down and arranging the figures, were tried for each of the four processes before those we now use finally prevailed." (Buckingham, 1947, p. 15)

(c) "The most natural algorism, or written record of the children's thinking, . . ." (Clark and Eads, 1954, p. 75)

(d) "An algorism is both the procedure for carrying out an operation and the arrangement of the numerals and operational symbols for computation." (Hollister and Gunderson, 1964, p. 29)

(e) "An algorithm is a set of procedures for performing a computation . . ." (Kelley and Richert, 1970, p. 47)

(f) ". . . a general procedure, called an algorithm," (Mueller, 1964, p. 71)

(g) ". . . the usual term algorithm will be used to refer to any computational device [where 'device' is a written procedure]." (Ohmer and Aucoin, 1966, p. 89)

(h) ". . . the advocates of the Hindu-Arabic system with its algorithms, or procedures, for computation." (Peterson and Hashisaki, 1963, p. 18)

(i) ". . . arithmetic based on the Hindu-Arabic numerals, more especially those that made use of the zero, came to be called algorism as distinct from the theoretical work with numbers which was still called arithmetic . . . we have the word loosely used to represent any work related to computation by modern numerals and also as synonymous with the fundamental operations themselves and even with that form of arithmetic which makes use of the abacus." (Smith, 1925, pp. 9, 10-11)

(j) "From a mathematical standpoint we may characterize an algorithm in terms of a finite alphabet (the digits 0 to 9 plus a few additional symbols in the case of arithmetic), an infinity of words made up of a sequence of elementary steps or rules that are required to handle any initial work in a unique way. The algorithm for column addition is a good example of such a scheme . . ."(Suppes, Jerman, and Brian, 1968, pp. 289-290)

We attempted to evolve a more inclusive definition, one not so specific to mathematics:

<u>algorithm</u>: a method (e.g., for computation) consisting of a finite number of steps, the steps being taken in a preassigned order and reproducible, that is specifically adapted to the solution of problems of a particular category.

And for

<u>algorithmic learning</u>: the process of developing and/or applying methods or procedures, i.e., algorithms, with the goal of learning-how-to-learn.

Beilin (1974) summarizes the problem in discussions and explorations of work on algorithms and algorithmic learning:

> The difficulty over the use of algorithmic methods stems in part from the lack of differentiation between conceptual algorithms and instructional algorithms. Instructional algorithms are devices, usually symbolic, that provide standardized ways of approaching the analysis or solution of problems and are essentially pedagogical instruments. . . .
>
> Although practical considerations are important in considering the value of algorithms, even more important is the need to determine what is essential for thought and problem solving to occur. . . .
>
> Algorithms, thus, are not simply arbitrary devices for solving school problems but enter into the very nature of the processes by which cognition develops. They may serve as instructional devices as well, but developments in computer simulation of thinking show that algorithms serve a much more serious and necessary function in reasoning and learning. . . . The task for mathematics education is to develop instructional algorithms whose structure and content will articulate most adequately with the structure and nature of conceptual algorithms. (pp. 129-130)

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II. Conditions for Algorithmic Imagination

There is nothing either good or bad but thinking makes it so.
(Hamlet--Act 2, Scene 2, Shakespeare)

Conditions for Algorithmic Imagination

Alan R. Osborne

Computers and small electronic calculators have recently become a part of our culture. What was a futuristic fantasy of science fiction (Asimov 1957) is now a portion of the reality requiring the thought and attention of educators. There is a reasonable expectation that calculators and computers will become more accessible and common in the immediately foreseeable future. Some would argue that this decreases the importance of teaching computation in the schools. Others would remark that the concern for "Why Johnny Can't Add" is misplaced and inopportune. Although such arguments may have more credence than they would have had even five short years ago, they are strawmen diverting the attention of designers of curricula and theoreticians of the instructional process from more pressing and vital questions about the experience of children and youth with arithmetic and mathematics.

The purpose of this paper is to raise some questions about the focus of mathematical experiences in the school given the fact of ready access to calculators and computers during the adult life of children presently in today's schools. The questions and issues raised by the community of scholars in mathematics education within the context of philosophizing about or considering needed research within the domains of computational proficiency and instruction for algorithms indicate some profound oversights in terms of the future needs of children.

A theme pervading Pirandello's plays is that reality is determined by the thinking and feeling of an individual. In Six Characters in Search of an Author (Pirandello, 1922), each character constructs his own reality. Historians of science hypothesize the same type of operational determination of reality for individuals contributing ideas to the evolution of science. Boring (1929) defines and documents the concept of zeitgeist operating within the field of psychology in his A History of Experimental Psychology. The prevailing philosophical orientation and spirit of the times, the zeitgeist, is a context that determines the categories of ideas to be prized and the questions and research important for psychologists of a given era to advance the state of knowledge. This provides limits to the imagination in much the same sense of T. S. Kuhn's concept of paradigm as explicated in his study of the historiography of science, The Structure of Scientific Revolutions (1962). Kuhn extends the concept of zeitgeist with the concept of paradigm to encompass the model of the science held or believed by practitioners in the field. This paradigm determines and is determined by what are considered legitimate questions and problems in that field, allowable research procedures, the philosophical orientation of the field, the type of apparatus used, and what is considered to be known with a degree of certainty. For both Boring and Kuhn, the limitations apply to the individual scholar and to the community of scholars as a whole. For the individual, this provides the matrix of beliefs, understandings and procedures from whence develops his sense of appropriateness for his own activities and the delimitations of his interests. Induced by his membership in the community of scholars in his field, it is a function of the nurture provided by that field which yields both the wellsprings of creativity and the limits on the imagination for an individual scholar.

Have the same sorts of factors operated within the field of mathematics? We would argue that this must be the case. Many creators of mathematics have demonstrated keen awareness of the legitimatizing character of the paradigm held by the community of scholars in mathematics. Consider Cardan's apologies in reporting his work with complex, non-real numbers or the hesitancy evidenced by the inventors of non-Euclidean geometry in publishing their studies. These two examples suggest a retarding effect on dissemination was operant if a creator of mathematics was (or is) aware of the existence of a paradigm within his discipline when his creation does not fit the paradigm. Many other examples can be found in the history of mathematics.

Of greater interest for our purposes is the set of ideas and approaches to mathematical problems and theories which were not created because of the existence of a paradigm. That is to say, have paradigms had a retarding effect (other than slowing dissemination and the spread of ideas) on the advance of the field of mathematics? No historical answer to this interesting question exists. One cannot provide historical evidence for the causes of a non-event; one must limit the arguments to supposition. Some examples of such arguments do exist. For example, Osborne (1968) argues that the Greeks' careful sense of closure concerning operations with lengths, areas and volumes prohibited their understanding and quantification of momentum even though the writings of Aristotle indicate that momentum was an important concept to the Greek scientists. Understanding of this rudimentary concept of science would await Galileo in an era in which the paradigm of Greeks' careful reasoning was relaxed and freed by the impact of the Dark Ages and the probable non-understanding of the niceties of Greek thought by the Arabs.

The history of modern algebra suggests the impelling force of mathematical paradigms or traditions. Stemming from a Greek tradition of geometrical algebra, it was the mid-fifteenth century before Bombelli would formulate algebraic arguments free of the hampering restriction of providing a magnitudinal base for numerical arguments. Vieta, approximately 25 years later, moved algebra somewhat in the direction of its own notation, yet it would be the turn of the eighteenth century before Peacock would attempt to free algebra completely from the need to provide 'real' referents for algebraic symbols. The traditions of providing real referents for the symbols of algebra suggest a hampered development of quaternion algebra by Hamilton and the more generalized description of a vector space by Grassman. Indeed, both Hamilton and Grassman were concerned with the question of whether a 'real' base for their algebras existed. One wonders what the retardation effect of the mathematical paradigm of needing real referents was on the field of modern algebra.

Paradigms provide limitations on the mathematical imagination and creativity of both an individual and for the community of users and doers of mathematics. On the one hand, it may be at the attitudinal level for specific individuals, forcing them into a construction of their own form of reality in the sense of a Pirandello character. On the other hand, it may be the more direct result of the traditions or appearance of traditions in the sense of the mathematical paradigms described above. In school mathematics at the elementary and secondary levels, the traditions and perceptions of what is legitimate mathematics is communicated through the experience of each individual child. The experiences of the child determine his zeitgeist or paradigm from whence his imagination and creativity will well. The modes of thought and processes that both limit and facilitate the child's productive use of mathematics are imprinted in much the same sense as the imprinting of intuition on the very young. The thesis of this paper is that if the child's experiences within the context of his school mathematics environment extablish and determine the paradigms of his thought, then mathematics educators need address the problem of whether an appropriate paradigm for our present and future ages in mathematics is being established.

We would argue that present school mathematics programs, and the associated supportive research concerning their effectiveness, does not address the problem of whether the goals and activities of the programs build paradigms and/or a zeitgeist fitting children's future adult needs in mathematics. The school mathematics program at the elementary and secondary school levels has been oriented by a need to produce students who are computationally proficient. Throughout our history this has been an important goal. Imagination and creativity, and the setting up of these attributes of individual performance, has been directed to the necessity of performing in the traditions of the existing mathematical thought and uses. The goal of computation has been quite appropriate. Individuals have needed to possess computational skills in order to participate fully in an adult life. Further, the very nature of the scientific and mathematical world has required computational skill. Note that by computational skill we mean much more than the capability of working with numbers but also are including the ability to work with higher-order mathematics even through the undergraduate level. Computational skills have been necessary to the individual in gaining a modicum of control over his personal environment beyond his application of mathematics to science or to mathematics per se. The housewife in coping with her budget, the golfer computing his score, and the home-improvement nut constructing a new patio each need a level of computational proficiency in order to fulfill expected roles in their personal life. In order to maximize participation in life, children needed to build computational competence.

Clearly computational proficiency is still important. A student of mathematics needs to know enough and be able to do enough computation so that teachers and other individuals can communicate with him. But it is an open question whether the operational proficiency of the past and present is sufficient to provide the zeitgeist or paradigm needed for the future adult life of today's children. Does the present treatment of school mathematics prepare a child for a world characterized by ready access to electronic calculators and to computers? Is the scope and sequence and approach of the school mathematics program sufficient to prepare individuals for intelligent application of devices capable of carrying out complex computations with the application of pressure on some buttons? Are we limiting the sort of problems which children can solve with the aid of machines?

The world of the future will be characterized by extensive use of the computer at many levels of our society. Individuals need to understand algorithmic processes if they are to take maximal advantage of computers. Although computer programmers are presently being trained on the base of present curricular orientation and content, is the efficiency of this training impaired because of a failure to stress the development of algorithmic thinking? Inadvertently are curriculum designers building limits on students' future creativity in the use and application of computers? Are habits of thinking or mind sets acquired during the early childhood experiences with mathematics that limit or retard algorithmic learning? Are students building appropriate intuitions?

The advent of the machine is changing the basic nature of mathematical endeavor. Algebra, number theory, and analysis are each evolving around new processes and styles of thinking which are directly attributable to the machine. Birkhoff's article, "Current Trends in Algebra" (1973), argues persuasively that the machine orientation of mathematical research in algebra is here to stay. Not only are new processes being used in modern algebra, but also a different style or type of problem is being considered as significant by the algebraist. The paradigm is shifting.

Finally, the student entering college today often encounters the use of the computer as an instructional device. We do not refer to computer-monitored instruction or computer-assisted instruction that uses the power of the machine as a means of teaching the usual mathematics by controlling individualization, administering drill and practice, or the general administration of instruction. Rather we are speaking of the use of the computer to exhibit and do mathematics that an individual with a pencil and paper could not accomplish. An example of this might be the examination of the limit of a function in a particular neighborhood. With rudimentary programming skill the student has access to mathematical examples unimagined in the instructional sense in the immediate past. An algorithmic sensibility would facilitate a student's perception of exactly what was happening in the example and perhaps make it real to him in a sense that is not available to many of our students today. We would argue that this entails more than the experience of programming; to know and be able to use a language is not sufficient. We would question whether the desired intuitions can be established through experience gained as late as junior high school and whether they can be acquired simply on the base of instruction in computers without attention to the mathematical orientation of algorithmic thought. The modes of thought necessary to successful use of the computer are essentially mathematical. A basic component of this mode of thought is algorithmic in character.

Bronowski (1965), whose field is the foundations of mathematics, argues persuasively that our mathematical imaginations are limited by what we know and do in mathematics. His supposition is that we cannot conceive readily of scientific and mathematical ideas that do not have a basis in the real number system. McLuhan (1964) hypothesizes that number concepts operating within the context of printing has channeled our imagination in directions accounting for the development of our scientific-technological society. Computational machines are going to have a comparable impact and influence on thinking. A new paradigm or a different zeitgeist will be established with both limits and facilitates our creativity in coping with our environment.

A significant question for curricular specialists is suggested by the shift to computational machines: Does the present curricular experience of the child facilitate use of the machine? That is to say, do present materials help establish a machine zeitgeist and creativity that will enchance the child's future work with computational devices? Or do presently designed mathematical experiences inadvertently establish inhibiting paradigms and modes of thought? We would argue that the latter is the case.

At the heart of productive use of computational devices is a capability for algorithmic thought. Whether the device is a low-level, hand-held calculator or the more sophisticated, programmable computer, effective power in using the machines depends upon developing a paradigm or zeitgeist facilitating rather than limiting algorithmic understanding in and or mathematics. But our thinking and research about algorithms has been limited for the most part to purely computational algorithms in terms of the elementary school arithmetic program. Even when algorithms are implicit in the content of the secondary mathematics sources, the algorithms are seldom treated as such but examined as a means to another content goal. At the elementary level, curricular development and related research has been limited almost exclusively to the establishment of computational competency rather than encompassing an understanding of algorithmic processes.

The phrases algorithmic thinking and algorithmic learning have been used above. A word of explanation is in order. Textbooks at the school level do not present algorithms as processes constructed by people which entail evaluative decisions. Within texts algorithms are defined explicitly as having a limited capacity of solving problems and are seldom considered as providing mathematical problems in and of themselves. Rather, a mathematical context is defined to which a specific, previously constructed algorithm applies. Now it may be the case that to this same contest more than one specific algorithm may apply, but the texts, if they present an alternative algorithm, reinforce the idea that no decisions are involved concerning the algorithm. For example, given an addition problem 238 + 95, the child is taught to use the regrouping or carrying algorithm:

$$\begin{array}{r} 238 \\ +\ 95 \\ \hline 333 \end{array}$$

The child may encounter an alternative algorithm such as

$$\begin{array}{r} 238 \\ +\ 95 \\ \hline 200 \\ 120 \\ 13 \\ \hline 300 \\ 30 \\ 3 \\ \hline 333 \end{array}$$

But this second algorithm is used with the intent of stregthening the student's understanding of place value and of the initial algorithm. The first algorithm is the favored technique for the addition problem. At no point, be it the context of addition at the early elementary school level or other computational contexts, is the learner let in on the fact that he has a choice of algorithms to apply. He is not allowed to make decisions concerning the efficacy and efficiency of algorithms. We would argue that choice decisions between alternative algorithms constitute an important component of algorithmic thinking.

The example considered above does not argue that the presentation of alternative algorithms is not an effective teaching device within the context of current curricular practices. (It should be remarked that researchers have amassed little firm evidence concerning how and when alternative algorithms should be presented or what outcomes may be predicted.) Rather it is to point out, through the use of an example from elementary school arithmetic, a characteristic of algorithmic thinking. Algorithmic thinking involves more than the application of a decision-free algorithm with the limited capability of only treating a single mathematical context. We argue that algorithmic thinking entails selection and decisions concerning alternate algorithms which apply to a single problem.

The most common strategy for instruction concerning an algorithm is a progression through three distinct steps:

1. The necessary, prerequisite mathematics for the conceptual base is developed carefully.

2. The algorithm is presented, typically with a rationale in terms of the conceptual base.

3. Opportunity for practice is provided.

Each of these steps is developed with the learner to establish an algorithm which has been constructed or borrowed for the learner by the author of the instructional materials. Students are not expected to construct or develop an algorithm themselves even though the necessary conceptual base has been established as the first step of the instructional strategy.

Most curricular reform of the past twenty years assumed a foundational precept of the learner needing to behave like a scientist if he were to understand the processes of science. The exhortation leveled at and by mathematics teachers was, "Mathematics is not a spectator sport." Students were expected to behave like mathematicians. But, curiously, this expectation did not extend to algorithms. Students were protected from behaving like mathemathicians with respect to algorithms. A mathematician does construct algorithms; this is a portion of the task of being a mathematician. For the curricular designers in mathematics of the late fifties and the sixties to proclaim that mathematics is not a spectator sport and then to design materials not allowing students to create their own algorithms is at the least ironic.

Mathematics educators have little or no experience in either allowing or expecting students to construct their own algorithms. The effect of this type of constructivist orientation on student achivement of computational proficiency is not known. The impact on attitudes and values may only be conjectured. It is not known if or how understanding would be extended beyound the traditional objectives which are considered important today. Would students display the confidence and sense of self-competence which contributes to being creative? Do maturity and experience factors contribute to the child's being able to construct algorithms? If younger children have limited capability for creating and evaluating algorithms, then what are the limiting constraints of their problem-solving ability which provide the interference? These questions are important if we are to extend participation in doing mathematics to algorithmic subject matter. A study in this vein is being conducted by Hatfield (1974). Preliminary results indicate that children have a capability for constructing algorithms as early as grade two, given an appropriate problem solving context.

Clearly some knowledge of how students cope with algorithmic learning exists in the literature of mathematics education. Some of this may be suggestive of questions and problems of import. Some of is may suggest hypotheses in need of testing. Perhaps the most comparable learning in mathematics which a child experiences is the idea of mathematical structure. This important unifying concept of mathematics is a set of ideas which taken together possess significance far beyond their significance taken separately. Research suggests learners need to acquire cognitive maturity and to have some experience with the separate ideas before they acquire the concept of a mathematical structure. If an algorithm is a fitting together of several processes into a complex decision network designed to solve each of a specific category of problems, then it is very similar to the concept of structure. Perhaps the learning of characteristics of algorithms and the consideration of algorithmic thinking as a process are subject to the same order of maturity factors. We do not presently have a research base which suggests when and what first experiences in constructing algorithms are most appropriate. We suspect that algorithmic learning is very similar to children acquiring a feel for mathematical structure. The child's preliminary experience with the important unifying concept of algorithm should be informal, intuitive and early. Formal expectations of students being able to construct algorithms probably should follow considerable experience in construction on an informal, exploratory basis. The task of the teacher in the early elementary grades may best be considered as providing foreshadowing experiences. But the precise nature of these early experiences has yet to be determined. It seems reasonable to expect the child's experiences to mirror the mathematical judgments to be made concerning algorithms. That is, students should begin early to compare algorithms as to their efficiency, to identify the types of problem contexts to which they apply, to assess their complexity, to note whether there are sub-algorithms within the primary algorithm, and the like. These are precisely the sorts of evaluative judgments that are needed when one shifts from one sort of electronic calculator to another or when one encounters a new programming language.

Another aspect of algorithmic thinking is identified with the word "awareness". A student should expect and be aware of the pervasiveness of algorithmic processes, particularly in mathematics but also in other fields. Many topics in mathematics at the secondary-school level are appropriately considered algorithms but are seldom treated as such in our curriculum. For example, a student typically encounters at least six different algorithms for solving simultaneous linear equations in the college-bound track of high school mathematics. But these approaches are seldom treated as algorithms and the algorithmic character of the approaches are not considered. The approaches are developed around a limited set of mathematical principles, namely substitution and the field properties. Students need to develop an awareness of the characteristics which suggest the application of each of the particular algorithms in order to become proficient in using each of these methods; this is precisely one of the characteristics of algorithms which needs to be highlighted. Indeed, one might argue that the entire set of processes for solving simultaneous linear equations should be collapsed into a single algorithm with the student making, for example, choices of a subroutine of determinants or substitution, depending upon the characteristics of the equations. This is to say, in order to make algorithmic decisions, the student needs an expectation of finding algorithms within the mathematics he or she is doing. An awareness of the pervasive character of algorithms in mathematics is an important first step to acquiring the zeitgeist facilitating creativity in using computational devices.

In summary, we characterize algorithmic thinking as requiring three components. First, we would expect the child to make decisions concerning the efficacy and efficiency of different algorithms. Thus, we expect the learner to acquire an ability and skill in evaluation of algorithms. Second, we would expect a learner to be able to construct algorithms. He should be able to decide whether a bit of mathematics is an algorithm or not. Finally, the learner must acquire an expectation of finding algorithms in the mathematics that he is doing.

Other attributes of algorithmic learning and thinking might well be described. For other topics in mathematics, mathematics educators are quick to label as limited and incomplete an instructional program which does not address the higher-order objectives of the Bloom taxonomy. The stress on evaluation, construction, and awareness is an attempt to examine the teaching of algorithms in the sense of providing a completeness to the set of objectives which are typically associated with algorithms. The curricular orientation advocated above is directed toward expanding the teaching of algorithms from the mechanistic limitations of tightly designed behavioristic hierarchial strategies. The prospect of a future characterized by ready access to machines built around use of algorithmic processes makes it incumbent on mathematics educators to direct the curriculum and curricular research to the more difficult levels of goals and objectives.

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III. Reviews of Previous Research

Algorithmic Processes for Cognition

Jesse D. Parete

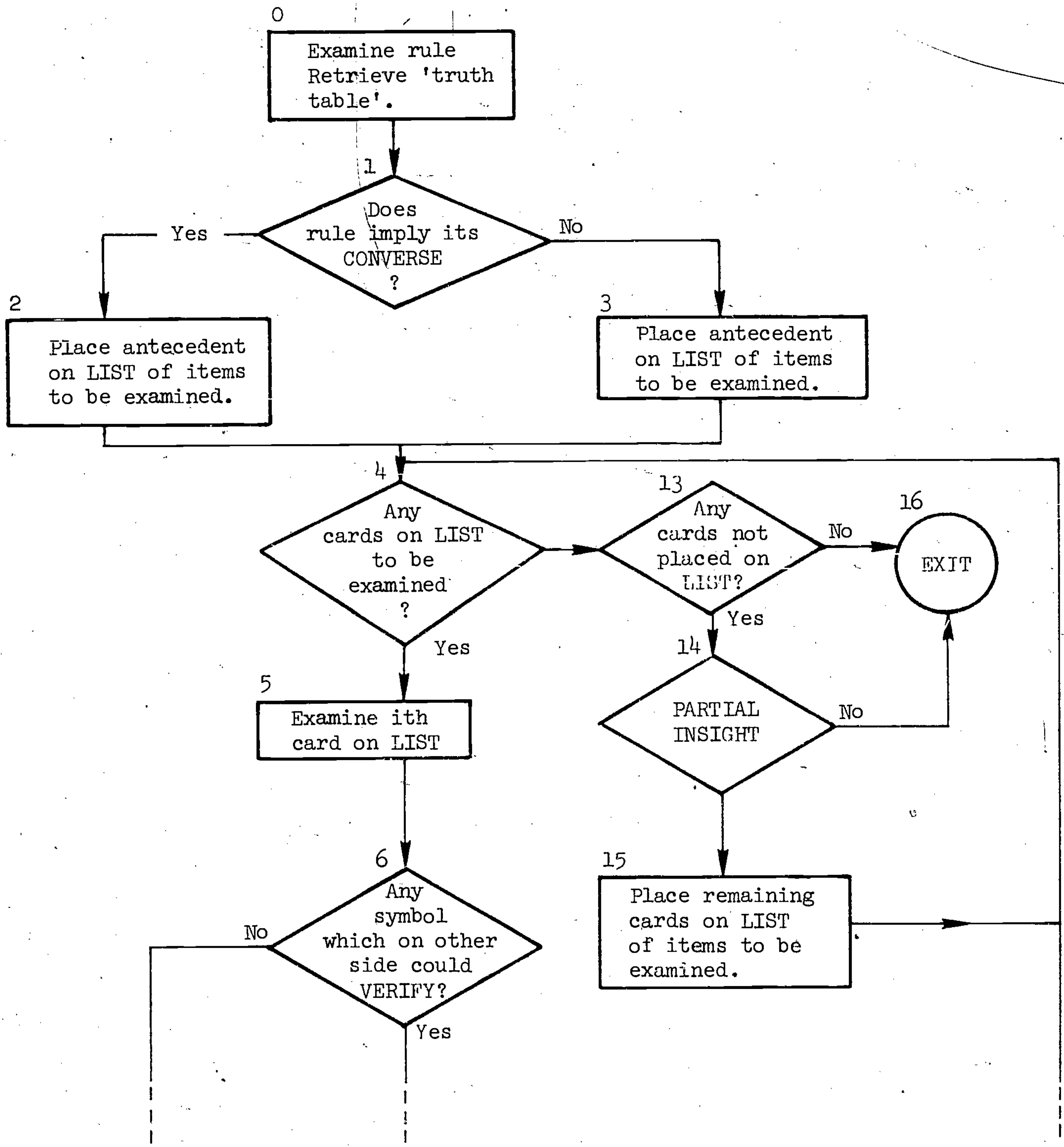
Learning theorists have contributed much to the study of strategies used by people when solving problem tasks. Often the subjects are unaware of the precise strategies they are applying. These strategies, therefore, are like internal or psychological algorithms. This paper reviews the literature which yields evidence that a great deal of human behavior in problem-solving or information processing tasks can be studied as applications of internal algorithms.

Such a point of view is compatible with Piaget's theory of development. Piaget theorizes schematic structures which originate out of our motor actions, building a lattice type pyramid of more elaborate and sophisticated behaviors (Piaget and Inhelder, 1969). This can easily be studied in terms of formulating algorithms and piecing together groups of simpler algorithms to form more sophisticated ones. Scandura (1971) has taken such an approach in his work. He has kept his theory much narrower than Piaget's and refers to it as a "partial theory". Scandura attempts to explain certain behaviors in problem solving as rule-governed behaviors. From simple rules are build more complex behaviors by development of a rule or rules to act on the simple rules. He calls such a rule a higher-order rule. For example, a person may have the rules for converting yards to feet and feet to inches. The combination of these two rules would enable the person to convert yards to inches. The combination of the two rules would be an application of the higher-order rule, composition of rules. He theorizes that whenever a problem requires a system of higher-order rules and associated simple rules for solution, a subject who possesses such a structure will apply it to the problem. To test his theory, Scandura and his associates taught a group of subjects, ages five to eight, how to use two simple rules comparable to those described above. Then each subject was tested to ascertain if he would solve a problem requiring for its solution the composite rule. Only one of the subjects was initially successful on this type of problem. Next, Scandura taught the subjects, using neutral materials, how to combine pairs of simple rules. "In short, we taught them a decision making capability for determining whether or not they had achieved the higher order goals" (p. 40). The subjects were then taught three new pairs of rules and given three corresponding problems which required the combination of the simple rules. All subjects who had successfully learned the skill of combining rules were successful on the three problems. Because the three problems were different with respect to all attributes except that they could be solved by the combination of a pair of simple rules, it could be argued that the subjects acquired an algorithm (higher-order rule) for this type of problem task. Scandura claims "that it has been possible to analyze a number of other, more complicated problem situations in very much the same way, including problems taken from Polya's pioneering yet atheoretical discussions of mathematical problem-solving" (Scandura, 1971, p. 40).

Scandura's approach to studying problem solving ignores many other facets of a very complex behavior. Most noticable is the role the heuristics play in helping people solve problems. Miller (1960), terming heuristics as insights which lead to plans about how to go about solving a problem, incorporates this aspect of problem-solving into a model which parallels Scandura's. He conceptualizes the process in terms of plans for solutions and plans for formulating or altering existing plans. This broader scope allows for insights or heuristics to enter a model for problem solving. The typical avenue for testing such a theory is to program a computer to act as if it possesses different types of insights associated with the observed human behavior. Miller encourages research in this area.

Johnson-Laird and Wason (1970a) present the flow-chart for the solutions to a reasoning task involving the conditional rule which incorporates insights such as Miller recommends that lead to solutions actually attained by individuals. The problem task was developed by Wason (1968). Subjects were presented with a set of four (or more) cards with a letter on one side and a whole number on the other. The task was to choose cards they wished to investigate (see what was on the face-down side) to determine whether the rule, "If there is a D on one side then there is a 3 on the other side", correctly described the lettering and enumeration of the cards. Showing would be a D, K, 3, and a 7. The correct choice was the D card and the 7 card. For ease of interpretation, think of the rule as "If P then Q", with D on the face of a card a P, and a 3 as Q. Any letter other than D, such as K, will be termed $\overline{P}$ (not P) and any number other than 3 will be termed $\overline{Q}$ (not Q). Thus, the correct choices are P and $\overline{Q}$. Subjects' choices in order of percent choosing it are, (1) P and Q, (2) just P, (3) P, Q, and $\overline{Q}$, and (4) P and $\overline{Q}$. The preference for the P and Q choice is attributed to a preference for searching for information to verify over searching for information to falsify. (The only way to verify this rule is to check all possible falsifying cases - P and $\overline{Q}$.) The choice of only P results if a subject does not assume the converse of the rule. The model to account for these choices is given in Figure 1 and incorporates three levels of insight. It also accounts for changes of insight which were observed by Wason (1969) attempting remedial procedures with subjects:

> All Ss will begin by placing either p and q (0,1,2) or only p (0,1,3) on their list of items to be tested. There are then three possible levels of insight. (In explaining the model, the numbers in parentheses refer to the different elements in the flow diagram and enable the reader to keep track of the behavior of a hypothetical subject.)


0
Examine rule Retrieve 'truth table'.
1
Does rule imply its CONVERSE ?
Yes
No
2
Place antecedent on LIST of items to be examined.
3
Place antecedent on LIST of items to be examined.
4
Any cards on LIST to be examined ?
Yes
13
Any cards not placed on LIST?
No
Yes
16
EXIT
14
PARTIAL INSIGHT
No
5
Examine ith card on LIST
15
Place remaining cards on LIST of items to be examined.
6
Any symbol which on other side could VERIFY?
No
Yes

(con't)

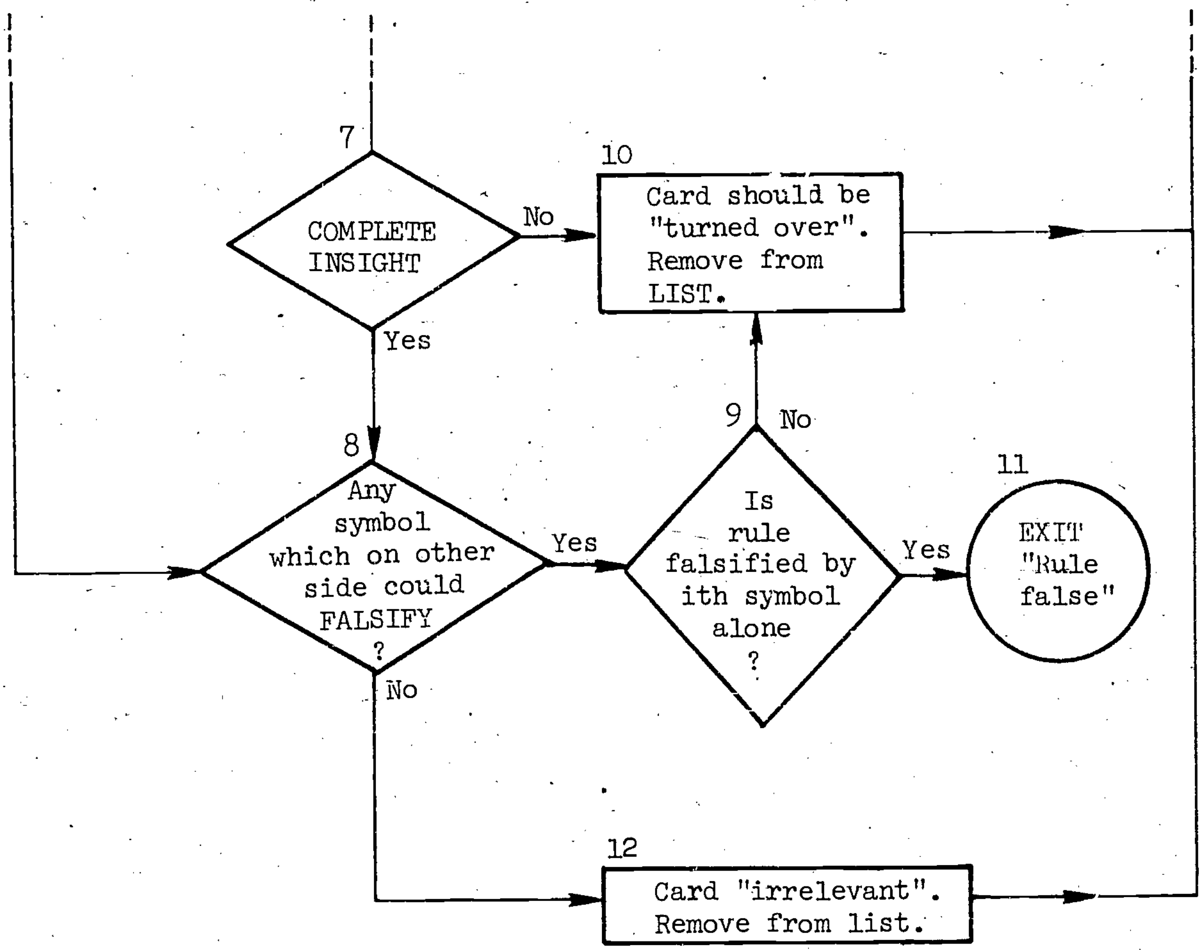


Figure 1. Insight to a reasoning task (Johnson-Laird and Wason, 1970, p. 143)

> *No insight*. Ss without any insight will select only these values because they alone could verify the rule (4, 5,6,7,10). They will test no further cards (4,13,14,16).
>
> *Partial insight*. Ss with partial insight will go on to place the remaining cards on the list of items to be tested (4,13,14,15). Regardless of the initial selection, $\bar{p}$ will be considered irrelevant because it could neither verify not falsify (4,5,6,7,8,9,10). An S, who did not initially place q on the list, will do so now and select it because it could verify. Thus an S with partial insight will ultimately select p, q, and $\bar{q}$.
>
> *Complete insight*. Ss with complete insight will select p and $\bar{q}$ and reject q because it could not falsify (4,5,6,7,8,12). Since the question of complete insight arises when S encounters a card which verifies the rule it can accur in two main ways. It may be gained during the initial tests. But if S initially rejected the converse, it may be gained after partial insight when S is testing q for the very first time. However, an S who initially accepts the converse and selects both p and q should be much less likely to gain complete insight after gaining partial insight. He would have no occasion to retest q and hence could not take the appropriate path in the flow diagram (from 6 to 7). (Johnson-Laird and Wason 1970a; p. 144)

As mentioned earlier the aspects of the model discussed in the last few sentences above compare favorably with the empirical data.

The subjects in the above study were college students assumed to be of high intelligence and well into the formal stages of cognitive development. This fact led Johnson-Laird and Wason to investigate the nature of their subjects' behavior further. They carried out an experiment (Johnson-Laird and Wason, 1970b) to test whether the bias for positive confirmation (verification of the rule) could be overcome by an instruction to falsify the rule as opposed to an instruction to verify it. This procedure was successful in bringing the group who were asked to falsify the rule, to the proper insight needed for solution. The instruction to falsify the rule apparently triggers a focus on the information value of the negated consequent.

If the subjects' behavior in the above experiment were interpreted as deliberate and conscious, it would seem that the discussion had strayed from the theme of "internal algorithmic processes." But Wason and Shapiro (1971) presented a similar sample of subjects with the same task using thematic rather than abstract materials. The rule was stated, "If I go to Manchester, then I travel by car." Cards were prepared with destinations on one side and modes of travel on the other. In this case subjects had little trouble realizing which cards to choose to verify the rule. Wason (1969, 1971) speculates that the subjects working with the abstract material exhibit behaviors characteristic of earlier stages of development. The implication is that the subjects rotely call upon old strategies (in this case inadequate ones) to deal with a problem they are unable to consciously and logically solve. Wason expands on this notion, "regression in reasoning", in his 1969 paper:

> The concept of cognitive regression is speculative. It is important to be clear about what it is intended to mean in the present context. It does not necessarily imply that the subject 'goes back in time' to a mode of thinking characteristic of an earlier stage of development. It implies only that certain salient features of earlier modes of functioning are still available, and are substituted for more sophisticated modes of functioning to cope with an unfamiliar problem (p. 480).

This inappropriate and regressive mode was maintained by the subjects as a reasonable strategy even in the face of contradicting evidence. These subjects were apparently applying an algorithm in a very rote manner.

Hans Furth and his associates (Furth, Youniss and Ross, 1970) demonstrate this very same phenomenon with young children in their experiments. In their initial experiment they analyzed school children's responses to six concepts; the children were in grades 1 to 6. The six concepts formed ". . . can be designated $S \cdot C$, $\bar{S} \cdot C$, $\bar{S} \cdot \bar{C}$ and $S \vee C$, $\bar{S} \vee C$, $\bar{S} \vee \bar{C}$; where S and C stand for affirmation of the two attribute classes shape and color, respectively; $\bar{\ }$ = negation; $\cdot$ conjunction; $\vee$ disjunction" (p. 39). The testing procedure consisted of the presentation of one concept together with a pictorial representation of the four possible instances; the instances depicted the presence or absence of the two attributes "shape" and "color".

On the basis of the children's responses, Furth was able to define three distinct behavior patterns. One group of subjects consistently answered "true" to cases where both attributes' positive values were present in the pictorial instance pattern and false when both were absent whether or not they were exemplars or nonexemplars of the concept being tested; they answered randomly with the other cases. This "level 1" group's behavior is dominated by an "attribute present factor."

The second group, "level 2", showed a consistent type behavior which implied that they dealt with a relation of logical truth. But in the cases where the attributes' truth values relative to the instance pattern and the concept were true-false (present-absent) or false-true (absent-present), these subjects answered randomly. Level 2 is much like a transitional stage; it leads to the total capacity to combine instance-presence or absence with a truth value consistent with the concept represented. Level 3 subjects exhibited this ability.

Furth's next experiment was performed with the same children and two new concepts, negation of conjunction ($S \bar{\cdot} C$) and negation of disjunction ($S \bar{\vee} C$). The results revealed a dramatic regression in performance. Subjects at level 3 now consistently performed at a level 2 type behavior and subjects previously at level 2 regressed to a level 1 style of he behavior. Thus Wason's speculation is once again demonstrated.

It is also quite interesting that the subjects could be sorted into three levels of behavior. These could quite easily be taken as a hierachy of the processes a subject must perform in dealing with such a task. Level 1 subjects can handle only the primative first level of this hierarchy while level 2 subjects are able to handle the next step in certain cases. If this were the case, manifested is a concise developmental pattern for the acquisition of a 'psychological algorithm'.

The "rule learning" experiments conducted by Bourne and his colleagues supports the above analysis of Furth's experimental results (Dodd, Kinsman, Klipp and Bourne, 1971; Bourne, 1970; Bourne and Guy, 1968a and 1968b; Haygood and Bourne, 1965). The four primary logical connectives, conjunction, disjunction, conditional and biconditional, form the rules in these experiments.

Bourne (1970) found sizable general positive intrarule and interrule transfer effects in subjects exposed to sequences of rule learning tasks. To explain these transfer effects Bourne presents the following analysis of the subjects' behavior:

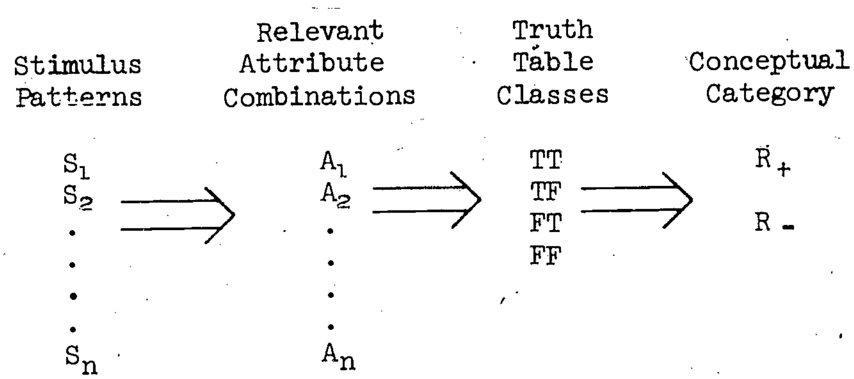


Figure 2. An analysis of the acquisition of a truth table strategy (Bourne, 1970, p. 552)

"In the course of multiple-rule learning, Ss acquire a mode of responding which is best described as an intuitive version of the logical truth table" (p. 552). The steps to achieving the the truth table strategy are identical to the levels Furth defined in his study. Bourne theorizes it is this model (Figure 2) which accounts for the intrarule learning transfer. The first step seems trivial; it is from stimulus patterns to recognition of the relevant attributes combination. In the Furth study, this step dominates the ultimate behavior of the level 1 children. Bower and King (1967) also demonstrate that this process cannot be taken for granted even with adult subjects.

The following model is presented to explain the additional interrule transfer effects found in Bourne's study (1970).

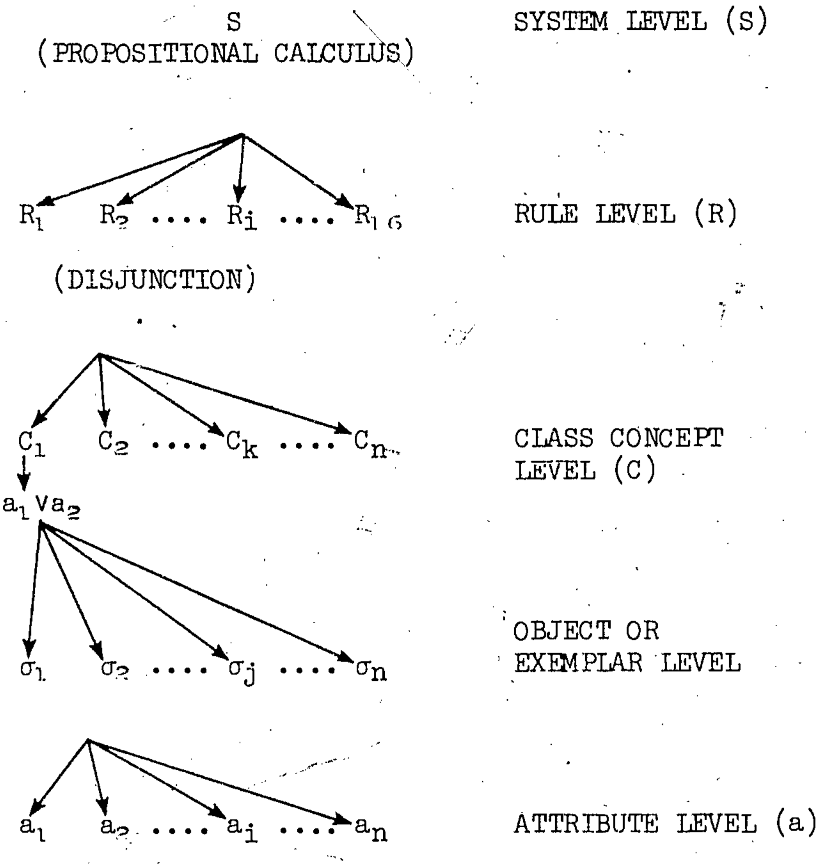


Figure 3. A structural, hierarchical model of concepts (Bourne, 1970, p. 555)

In this model, Bourne breaks down the structural heirarchy to which an individual in the rule learning task must react. The "System Level" is one step beyond the apparent mode of responding given in Figure 2. Given in Table 1 are the sixteen unique bidimensional partitions of a stimulus population forming the calculus of propositions and the basis for the "System Level".

Table 1. Sixteen unique bidimensional partitions of a stimulus population forming the Calculus of Propositions (Bourne, 1970, p. 554)

Truth-table	partitions															
class	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
TT	+	+	+	+	-	+	+	+	-	-	-	+	-	-	-	-
TF	+	+	+	-	+	+	-	-	-	+	+	-	+	-	-	-
FT	+	+	-	+	+	-	+	-	+	+	-	-	-	+	-	-
FF	+	-	+	+	+	-	-	+	+	-	+	-	-	-	+	-

The source of interrule transfer can be traced in part to the acquisition of the general strategy, the truth table, which is recognized (despite its simplicity) as a powerful deductive problem solving device for problems based on the primary bidimensional rules. But the truth table is more general than that. As seen in Table 1, it can be applied with equal utility to all 16 rules within the calculus of propositions. It yields a solution in the same number of steps and on the basis of the same information - one instance of each truth-table class - in all cases.

This suggests that in some sense S has learned not just the four primary rules (if he has learned them at all as specific individual cases), but the full conceptual system of rules - the entire calculus. He knows how to solve problems based on any rule within the system. He has encountered and solved a series of problems exemplifying a small set of rules, and from that experience he has learned a more general conceptual system. Just as the objects are positive instances of a class concept and class concepts are positive instances of a rule, the rules can be said to be positive instances of the system. (Bourne, 1970, p. 554)

The data Bourne has collected (Bourne, 1970 and Bourne and Guy, 1968a) support this theory. Performance on a new rule is a direct function of the number of different rules encountered during earlier tasks. The implication is that subjects acquire the simple yet powerful problem-solving strategy outlined above. The concise hierarchical structure of this strategy suggests that a large portion of this behavior is algorithmic. Subjects do not consciously formulate the calculus of propositions. It must be acquired through the acquisitions of the behaviors learned in accordance with the hierarchy model.

Much of the work in the rule learning stems from Bruner's study of concept attainment (Bruner, Goodnow and Austin, 1956). Bruner's experimental task allowed subjects to choose attribute cards to discover what attributes were used to form a specified conjunctive or disjunctive rule. He classified the subjects' behaviors into "focusing" type strategies for locating the relevant attributes. He also demonstrated that by manipulating variables such as time for solution, cognitive load and subject matter content, he could get the subjects to shift strategies depending on the situations. In his work there is abundant evidence for the contention that reasoning or problem-solving involves algorithmic strategies.

Laughlin and Jordan (1967) further studied the focusing strategy phenomenon discovered by Bruner. By varying the number of relevant attributes (2 or 4) in conjunctive, disjunctive, and biconditional concept-attainment problems, they were able to discern systematic shifts in strategies by the subjects. This then is additional evidence that these strategies exist as part of human psychological and mental processes.

The literature thus far concerns cognitive structures referred to as psychological algorithms which influence thought behaviors. The rest of this discussion will review that literature which gives evidence that there is a separate language processing mechanism which equally influences those behaviors in problem-solving tasks. It was Vygotsky (1962) who first proposed that two separate yet dependent systems of language and logical reasoning are developed in people. The recent work of the psycholinguists Chomsky (1965), Gough (1965), Clark (1969), and Sherman (1973) tend to support Vygotsky's theory.

It is with the negative operator that researchers have found considerable language-cognition interplay between development and use of negation. Eiffermann (1961) noted that the English word "not" has both a connotation of prohibition and denotation of negation: she took advantage of a double formulation of "not" in the Hebrew language. In Hebrew there are two forms of "not": (1) "lo", which is used in all contexts as the English "not" is used; and (2) "egno", which is restricted to use in all contexts except to express prohibition. One form, "lo", carries the full connotative and denotative impact of the English "not" while "egno" is similar to the negation operation. Eiffermann's study demonstrates that subjects processed information from sentences using "egno" more correctly than with sentences using "lo" to express negation. The processing of "lo" appears to be more complicated than that of "egno". One possible explanation is that the affirmative information in the sentences is processed separately from the negative operator (Gough, 1965) in both cases; however, in the "lo" case an additional process must take place to match the connotative or denotative interpretation to the context of the sentence. The point is that a language variable has added to the difficulty of using the negative operator.

Wason and Jones (1963) add support for the above analysis. Two groups of subjects were given the task of interpreting sentences using negation. The first group used sentences constructed with ordinary English using "not" for negation. The second group was trained to use two neutral signs (MED and DAX) which stand for assertion and denial of events. During a practice trial of the task the correct use of the symbols was taught by feedback, ("That's correct/incorrect."), to control any transference of connotations attached to "not" that might have resulted from verbal instruction.

The results were the same as those obtained by Eiffermann. The evidence supports intuitive claimers that linguistic usage influences the application of a logical operation; but, more importantly, the evidence can support the idea that there are structures which determine human reasoning behavior.

It has been established that negation with or without extraneous influences is a difficult operation (Bruner, 1956; Wason, 1959 and 1961; Wason and Jones, 1963; and Furth, Youniss and Ross, 1970). Sherman (1973) describes how this difficult-to-manage operation has influenced the development language; he also gives additional evidence for the theory of a linguistic processing mechanism separate from the logic structures. His study deals with the negative prefix "un", as in "unmanageable". Consider the following sentences: (1) He was not certain that she was not happy, and (2) He was uncertain that she was unhappy. The second sentence communicates the negative information with less strain than does the first. Sherman's results verify such a prediction. Reasons for the difference can be formulated by the linguistic theories presented by Gough (1965) and Chomsky (1965). The first and most easily interpreted semantic meaning processed by a "hearer" is the "base string" of the sentence. A base string is made of the syntactic variables of subject, verb, and object. There are two base strings in the first sentence: "he was certain" and "she was happy". A transformation must be effected to obtain the full semantic meaning. In the second sentence the negation is tied to the words of the base string: "uncertain" and "unhappy". The first sentence requires a transformation which reverses the meaning of a sentence, while the second sentence involves a word reversal meaning accomplished by the negative prefix "un". "The reversal of work meaning (caused by un-) is psychologically less complex than the reversal of sentence meaning (caused by not)" (p. 82). He speculates that the use of "un" was invented in language to deal with the cognitive strain of negation. This points up again the interrelationship of language and cognition. He also points out that "the results support the view that the language-comprehension mechanism is not a neutral device, responding with equal facility to all inputs, but, rather, that it is 'pre-set' to process certain inputs more quickly and accurately than others" (p. 81). Gough and Chomsky's theoretical formulations of this phenomenon were partially explored above. Other researchers involved with various other aspects of negation which lead to similar conclusions are Wason (1965), Green (1970), and Johnson-Laird and Tridgell (1972).

In an application of Chomsky's linguistic theory, Clark (1969, 1970) applies the "base string" information-processing idea to give insight into children's management of problems which involve the transitive relation. He sights Piaget's discussion of children's reactions to the following problem: "Edith is fairer than Suzanne, and Edith is darker than Lili." Responding to the question of which is the darkest, children ages 8 to 10 answered "Lili". Piaget's intuition helped him analyze the situation to come to the same conclusion as Clark using the linguistic theory to guide him. Both conclude that the children processed the base strings consisting of "Edith and Suzanne are fair" and "Edith and Lili are dark". Therefore, the children arrived at the solution that Suzanne is fair, Lili is dark and Edith is between the other two. Piaget (1928) states that, rather than tackle "the matter by means of judgments of relation, i.e., by making use of such expressions as 'fairer than' etc., the child deals simply in judgments of membership, and tries to find out with regard to the three girls whether they are fair or dark (speaking absolutely)." (p. 87). From this Piaget inferred that the children were deficient in relational thinking. The point is not whether they are deficient or not, for certainly they did not solve the problem successfully. But, rather, the question remains whether the children failed because of a lacking cognitive structure or because of the domination of a linguistic processing mechanism. Clark and other psycholinguists, as mentioned earlier, hold the opinion that the base string information is the easiest and the quickest semantic information processed by the brain (Clark, 1969; Gough, 1965; Chomsky, 1965). Perhaps the principal causes of the children's use of only the class membership information was the linguistic processing mechanism coupled with an over-load of their memory facility which inhabited further processing.

The theme throughout this discussion has been to demonstrate that there are systems of cognitive processes that act without the conscious deliberation of the individual. The last sections present research which shows that these systems interact with each other. This fact complicates the study of any one of these systems. Further work is certainly needed in investigating these systems, but there is even a greater need to investigate the consistent mappings from one system to the other. This latter approach may also yield valuable information about the individual systems which have been referred to as algorithmic processes.

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Algorithmic Learning and Hierarchies

Paul H. Wozniak

Introduction

Algorithmic learning may be thought of as hierarchical in nature in many aspects. In a report of a study on a mathematical topic, Gagné, (1962), perhaps the leading proponent of a "hierarchy of learning", refers to algorithmic learning as an ordered collection of specified intellectual capabilities. Subordinate skills are prerequisites for the final task. More specifically, some of the assumptions of his theory are given in an earlier article (Gagné, 1967b):

(1) Any human task may be analyzed into a set of component tasks which are quite distinct from each other in terms of the experimental operations needed to produce them.

(2) The task components are mediators of the final task performance, i.e., their presence insures positive transfer to a final performance and their absence reduces such transfer to near zero.

(3) The basic principle of design consists of (a) identifying the component tasks of a final performance, (b) insuring that each of these component tasks is fully achieved, and (c) arranging the total learning situation in a sequence which will insure optimal mediational effects from one component to another.

The learning of algorithms has been criticized for its dependence on memory and rote practice. In his theory, however, Gagné clearly makes the distinction between memory and mastery of subordinate competencies. Briggs (1968), in a review of the literature on hierarchies, gives an example of memorizing the Spanish equivalent of 100 different English words. He points out that the order in which the student memorizes the list may not matter. Whatever the sequence, the student will need several trials to master this task. In this case, it is not order of presentation which is important; learning depends on amount of practice (whole and partial list) and feedback. But if a student is to, say, solve linear equations, presentation and sequencing of instruction is different. Hopefully, when the student is to learn to solve linear equations (or many other kinds of algorithms), he is not merely presented with a number of completed equations to be mastered in the hope that he will learn how to solve them. Put another way, the student is not to memorize these equations and their solutions. Rather, he is to master first all the subordinate competencies it takes to be able to solve any equation of this type.

Gagné and Paradise (1961) looked at just this problem and concluded that subordinate competencies, unlike the memorizing example above, must be taught in a particular sequence (with options within layers of the hierarchical structure). They are taught not necessarily by direct presentation of parts of specific equations, but by supplying certain instructional events, materials, and exercises which lead to mastery of subordinate skills. This kind of learning is called by Gagné and Paradise "productive" learning to distinguish it from "reproductive" learning like the example of memorization.

Attempts at building hierarchies in different subject areas were the focus of much research. But as Briggs, (1968, p. 12) points out, in order to look at hierarchical structures we need to have objectives stated in behavioral form, not content form. Hence mathematics and science lend themselves to hierarchies, while subjects such as history and sociology do not.

From this it is clear that many of the important studies on hierarchical learning have led naturally into the study of algorithmic learning in mathematics. We will presently look at some of the factors involved with building hierarchies, with variations of the process, and with the efforts to substantiate their hierarchical nature.

Generating Hierarchies

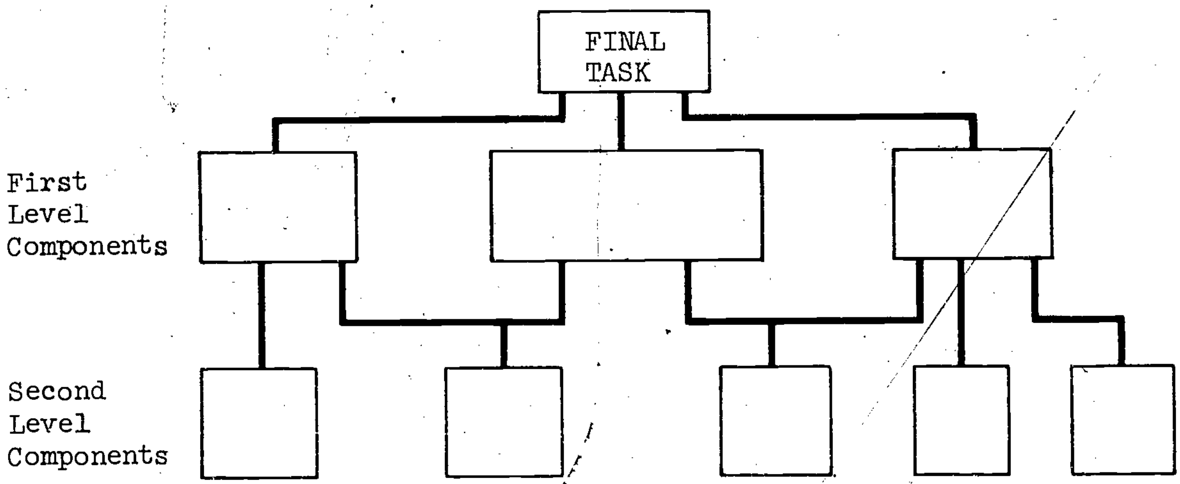
The generating of the hierarchy to achieve a final task can take different forms: teacher-generated, pupil-generated, or combinations of these. A common pattern is suggested by Mechner (1967):

(1) specification of behavioral objectives

(2) analysis of the subject matter in terms of component descrimination and generalization

(3) sequencing of those components for effective learning

The last two steps combine to form a procedure known as task analysis. According to Walbesser and Eisenberg (1972, p. 22), one of the best known forms of task analysis is one described by Smith (1964). To begin, the designer asks, "What must the learner by able to do if he has been told to perform a task, but has been given no specific training in the task?" This kind of questioning organizes the given task into components that resemble the following:



In effect then, the analysis starts at the "top" of the pyramid to determine what skill(s) are needed to perform the final task. These are the first-level components and are considered to be the final subordinate skills needed for attainment of the final task. The designer of the hierarchy must ask about these first-order components to ascertain what prerequisites are needed for their attainment. These prerequisites form the second-level components. This process continues until a point is reached at which the student may begin with his present skills.

As Walbesser and Eisenberg observe (1972, p. 23), the application of a task analysis procedure does not guarantee an "effective" sequence; it merely produces an hypothesized sequence that may succeed. Each task analysis generates a "best guess" sequence with respect to the author's experience. The extent to which learners are able to perform the final task after the sequence is one measure of the validity of the hierarchy.

To Smith (1964), an acquisition level of 85 per cent for the final task is considered desirable. In the work of Gagné and his associates, a 90 per cent acquisition level is usually sought.

The task analyses described above are developed by the instructor of the sequence. An alternative method which has been researched is in the area of student-generated hierarchies. This approach can lead to alternative algorithms appropriate for varied types of learners.

Mager (1961) wanted to see if student-generated hierarchies would be similar to instructor-generated sequences, and, if not, how would they differ. The experiment dealt with the learning of certain aspects of electronics. To generate the hierarchy the instructor only responded to those questions that the students asked. He did not initiate any instruction. The sample consisted of six adults. Two of the findings from the experiment were:

(1) learners start out asking questions on different topics than those followed in a general text.

(2) initial interest is on the concrete (how) rather than on the theoretical (why).

The results tended to indicate that the sequence generated by the student himself is more meaningful than the sequence given by the instructor or the text.

The findings of Mager were substantiated by a study by Kaplan (1964). The design was similar to Mager's. The subject matter was "vectors". The teacher again only responded to pupil-generated questions. Kaplan found that students moved from the concrete to the abstract, there was greater commonality of questions at the outset of instruction, and all students had some knowledge of the subject area regardless of how naive they claimed to be.

A fundamental question may be asked then: Do teachers, following textbooks, provide a sequence of instruction most meaningful to the student? As Briggs (1968, p. 30) points out, the experiments such as those cited above have something to say about motivation, interest, self-direction, and the importance of the student organizing information for himself. Hence in an algorithmic learning situation it may be more beneficial for the students to become more involved with the actual building of a hierarchical structure for a topic, be it a specific algorithm or other concept. In terms of a specific algorithm, the teacher may be better off to ask questions like "What would we need to know to do ________?" instead of, "This is what to do."

Another study in this area was one by Campbell (1964) using programmed instruction. He wanted to compare the effectiveness of programmed instruction (developed by the instructor) with student self-direction. He hypothesized that student self-direction was superior to programmed instruction. Two factors were stressed: (1) meaningfulness of materials to the learner and (2) motivation. Campbell believed that when problem-solving techniques are needed for highly structured material, small-step, fixed-sequence programs could interrupt the students line of thought. Also, he thought, the student is his own best judge of when an idea has been grasped, and this judgment is more easily exercised under self-direction than under programmed instruction.

Of several subject-matter areas sampled in a series of experiments, the only significant differences favoring self-direction over programmed instruction was for mathematics and that difference occurred only after coached practice in self-direction. The self-directed group was provided the following materials: (1) a short basic text, (2) supplementary examples and explanations, (3) self-testing questions, and (4) a two-page outline of the entire lesson. The programmed group used a linear program, self-paced. According to Briggs (1968, p. 29), since students benefited from coaching in the use of self-directing materials, it is possible that more prolonged use of self-direction methods without coaching would be needed for the superiority of the method to appear. Those who used self-direction with most benefit tended to be the better achievers among the students.

Parker (1973) also looked at, as part of a study, the problem of teacher-generated versus learner-generated task analyses in mathematics and science in terms of terminal objectives. Four programmed texts were developed from two different hierarchical arrangements of the subject matter: (1) a Gagné hierarchy and (2) a pupil hierarchy. These two hierarchies were then developed into two other texts which randomized the sequence of instructional units for the Gagné and child texts. Upon completion of the learning materials, subjects received an immediate posttest and weeks later a delayed posttest. The results showed no significant differential effects in learning final terminal objectives with different generation of hierarchies and sequencing of subordinate tasks.

Shriner (1970) and Seidl (1971) investigated the question of whether students of different ability levels would generate different learning sequences. The subjects for both studies were 24 early childhood elementary majors at the University of Maryland. Twelve high- and 12 low-ability students were determined by quality point average and rank. They were asked to build a hierarchy on the study of 2 x 2 matrices. One of the conclusions was that there were no significant differences between the learning sequences generated by high-and low-ability students.

The basic hypothesis using a hierarchical scheme of instruction is that subordinate skills are prerequisites for the attainment of the terminal task. According to Briggs (1968, p. 41-42), in one of his earlier works on this subject, Gagné (1967) theorized that lower-order skills serve as mediators of positive transfer from lower-level competencies and effects of instruction. At the very bottom of such hierarchies may be found either the entering relevant competencies brought to the course from prior learning, or very basic abilities identified as such.

In reference to the abilities at the bottom of such hierarchies, Gagné theorized that if learning programs were of perfect effectiveness, everyone would pass all the component tests in the hierarchy, the variance would be zero, and all correlations of tests on the various competencies with basic abilities would also be zero. But if learning programs are not perfectly effective, the probability that a person will acquire each competency will be increased to the extent of his score on a test of basic ability. To this critical hypothesis we now turn.

Subordinate and Final Tasks

A number of studies pertaining to the construction and testing of behavioral hierarchies have been conducted by the University of Maryland Mathematics Project in conjunction with Gagné. In one of these studies, Gagné and Paradise (1961) analyzed a final behavior represented by constructing solutions to linear algebraic equations. First a learning hierarchy was constructed by a task analysis procedure. The procedure identified three immediate subordinate behaviors. The analysis was then repeated on each of the three subordinate behaviors and yielded a collection of subordinate behaviors to each of the three successive iterations, producing a learning hierarchy of twenty-two behaviors subordinate to the terminal behavior and arranged in five levels. The study was designed to test the hypothesis that the acquisition of a terminal behavior depends upon the attainment of a hierarchy of subordinate behaviors which mediate positive transfer from one behavior to the next in the learning hierarchy and eventually to the terminal behavior.

A learning program was then constructed to teach students how to solve linear equations. The program was divided into eight booklets; students were given one booklet each day for eight days. Three performance measures were administered upon completion of the program: (1) 10 equations similar to those in the program, (2) 10 transfer type problems, and (3) attainment of each of the 22 behaviors in the hierarchy. There were a total of 118 subjects in four seventh-grade classes from two schools. The results showed validity estimates for the hierarchy ranging from .91 to 1.00, which supported the hypothesis that there was positive transfer to each behavior from relevant subordinate behaviors.

Briggs (1968, p. 44) points out that the authors recognized that other persons, especially proponents of "modern mathematics", might derive quite different hierarchies. It is not, however, a matter of there being only one "right" analysis; rather, the purpose if to find empirical "validation" for the method in terms of the hypothesis to be tested.

Gagné, Mayor, Garstein, and Paradise (1962) built a hierarchy around the addition algorithm and extended the previous study to look at another variable besides the one on acquisition of subordinate skills. Specifically, the purposes were (1) to find out if a final behavior (adding integers) depended upon the attainment of a hierarchy of subordinate behaviors, and (2) to investigate the variable of recallability of relevant subordinate behaviors and the integration of these behaviors into the solution of a new and different task.

The integration variable was studied by systematically varying the amount of guidance provided to the learner in leading his from one behavior to another. Repetition of previously developed behaviors was used to study the effects of the recallability of subordinate behaviors.

There were two tasks prescribed: (1) the addition of integers themselves and (2) formulating a definition of the addition of integers for specific numbers using the necessary properties. Analysis of the two tasks yielded a hierarchy of fourteen behaviors at six levels.

The study was conducted with 132 students in four seventh-grade classes from two schools. High- and low-ability students were identified by previous grades of the school year. Four combinations of instruction were formed: high guidance - high repetition, high guidance - low repetition, low guidance - high repetition, and low guidance - low repetition.

The instructional period was four days. There was a performance test on the addition of integers and a transfer test on subordinate skills of two questions on each skill. In order to pass on a particular skill, both questions had to be answered correctly.

Validity estimates ranged from .97 to 1.00 providing support for the initial hypothesis that acquisition of each behavior is dependent upon mastery of subordinate behaviors.

On the second purpose, there was no overall significance on the four combinations of high-low, guidance-repetition. The only significant difference was shown on the superiority of high guidance-high repetition over low guidance-low repetition on the task of stating a definition for the addition of integers. However, no significant difference was found for the task of adding the integers themselves. Commenting on this particular experiment, Briggs (1968, p. 45) suggested that these results may imply that if moderately good instruction is provided in the proper sequence, as compared to instruction not so ordered, the effects of this may overshadow other qualitative features in how material is programmed. This, he says, may account for the frequency of "no significant differences" findings in research designed to isolate "style" aspects of programming.

Still another task in mathematics was analyzed into a hierarchy by Gagné and the staff of the University of Maryland Mathematics Project (1965). The task in elementary geometry consisted of "specifying sets, intersection of sets, and separation of sets, using points, lines, and curves." In this study, the importance of sequencing of topic order was again noted in terms of the number of instances confirmed of higher competency acquisition dependent upon the acquisition of those lower in the hierarchy. According to Briggs (1968, p. 45), however, the variables of (1) variety of examples during learning and (2) passage of time between stages of learning, had no effect upon the learning effectiveness of the program.

Despite the above negative findings in regard to programming style variations (e.g., variety of examples) and their effects upon task acquisition, it was thought desirable to measure retention of knowledge of the same students. In a follow-up study, Gagné and Bassler (1963) measured the retention of students both on total task and on each component subskill nine weeks after the learning. The retention for the entire task was very high except for one group which had previously received a narrow variety of examples in the learning program. The level of retention, overall, ranged from 108 percent to 128 percent. In contrast, the level of retention for subordinate competencies ranged from 60 percent to 88 percent indicating that individual skills are much more susceptible to forgetting than the performance on a task as a whole. According to Briggs (1968, p. 45), this difference in retention of the part-skills need not have been learned in the process of learning the whole skill, because the contrary was shown to be the case in the original acquisition data.

From the practical point of view of maintaining ability to perform this terminal task, the forgetting of the subskills which originally aided in mastery of this task is of no importance, as these learners retained (or even gained) competency on the task as a whole. But if some of these same subskills are needed for new tasks to be learned later, this loss in retention of subskills in important and deserves efforts to prevent it. Hence remedial work on the subskills could improve learning of related tasks later.

Not all of the research findings are in agreement with Gagné's point of view. Studies by Anderson (1967), Merrill (1965), and Campbell (1963) are some that have challenged his contentions. Anderson stated that the notion of hierarchies as dealt with by Gagné and his associates cannot yet be said to be definitely tested. He cited two reasons for his statement:

(1) that the correlational type of analysis employed by Gagné is not sufficient evidence of the hierarchy notion, and

(2) an experiment by Merrill (1965) had resulted in findings contrary to Gagné's hypothesis concerning hierarchies.

Merrill tested the basic hypothesis that learning and retention of a hierarchical task are facilitated by mastering each successive components of the hierarchy before continuing in the instructional program. Merrill insured mastery by channeling a student who erred on any particular component into a two-stage correction/review procedure. The results of his study seem to indicate that it is not necessary to master one level before proceding to the next.

Despite studies such as Merrill's, most of the research supports Gagné's initial hypothesis on hierarchies.

There is also some question about sequencing order. According to Heimer (1969, p. 502), the research literature contains a number of studies about the effects of scrambling "ordered" sequences, but the purposes of these studies have not always been clear and their overall results have not been conclusive.

Roe, Case, and Roe (1962) reported a comparative study in which a 71-item program on elementary probability was presented to two groups; one group received the program in its normal ordered form, and one received a scrambled version of it. A criterion test was administered to each student immediately upon completion of the program. There were no significant differences reported on time required for learning, error score during learning, criterion test score, or time required for criterion test. However, in a subsequent study, Roe (1962) reported contrary results with an extended version of the probability program mentioned above, in which a random-sequence group performed significantly worse on learning time, errors made during learning, and on post learning test scores. Roe (1962, p. 409) concluded that "careful sequencing of items has a significant effect on student performance, at least for programs of some length and complexity."

Payne, Krathwohl, and Gordon (1967) hypothesized somewhat the same thing, i.e., the larger the size of the unit, the more detrimental scrambling will be. They further hypothesized that the more internal logical development a particular sequence had, the more detrimental a scrambled sequence would be. Their experiment consisted of three programs which varied in logical interrelatedness from low to fairly high. Both immediate and delayed retention tests were administered. The hypothesis was not confirmed by the results.

Pyatte (1969) argued that the lack of more information about the effects of sequence changes on variables such as achievement, retention, and transfer could be attributed in part to a neglect of clear specification of what an ordered sequence of materials is to be. This lack makes it impossible to decide whether a sequence purported to be ordered does meet this condition, and whether a scrambled version of the sequence fails to meet it. In an attempt to follow up on this idea, Pyatte (1969) conducted a study in which he defined an ordered sequence as structured or hierarchical. Assuming that in the hierarchy each level provides positive transfer to the next level, as Heimer argues (1969, p. 503), Pyatte considered the extent to which positive transfer was acting within a program as a measure of the extent to which the program was hierarchical, and hence ordered. His study was designed to provide a check on the effectiveness of the instructional materials, to provide a check on the ordered (structured) materials by examining the differences between these and unordered (unstructured) materials, and finally to test the hypothesis that no differences in achievement or transfer would be found between students taking the structured materials and those taking the unstructured materials.

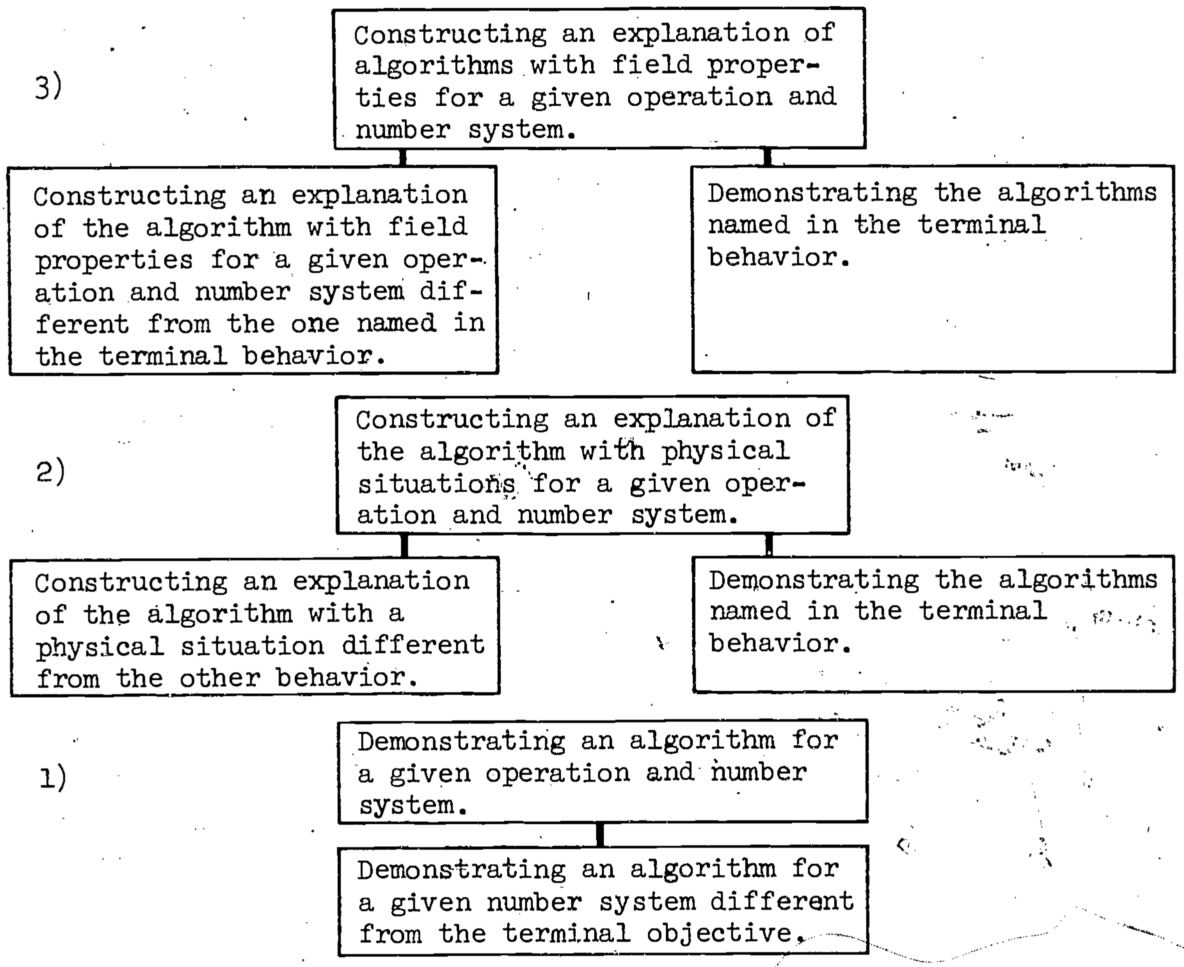
Pyatte reported that both versions of the instructional unit were judged effective - the structured unit was judged to have the defined structure, not; and the constructured unit was judged not to have it. No significant differences were found between the means on achievement or transfer for students taking the structured or unstructured unit. According to Heimer (1969, p. 504), among the statistically significant findings of the study was evidence that students of high basic ability reach higher levels of achievement and transfer knowledge than students of low basic ability, regardless of mode of program.

In concluding his analysis of the study, Pyatte (1969) stated that:

> the effects of sequence on learning measures should at this time be abandoned in favor of attempts to write programs which conform to a defined pattern and to develop the appropriate tools for testing these programs. . . . Having batteries of such well-defined programs, one would then be equipped with the requisite tools for answering questions about the type of program and its effects on such measures as achievement, retention, and transfer. (p. 260)

Alternative Hierarchy and Teacher Strategies

The University of Maryland Elementary Mathematics Inservice Program (1967, 1969) continued the series of hierarchy investigations with the analysis of an extensive learning hierarchy on arithmetic operations. Conventional task analysis was not employed in generating the hierarchy, according to Walbesser and Eisenberg (1972, p. 43). Rather, an ordering of clusters of three hypothesis of learning dependency were structured by number systems moving downward from rational numbers, to integers, then finally to whole numbers. Diagramatically it would look like:



The Demonstration Phase Report (1967, p. 4) suggests that the terminal task of the algorithm hierarchy is actually a triple of behaviors that the teacher will be able to exhibit after being exposed to the algorithm's instructional sequence. The three behaviors which constitute this terminal task represent the desired instructional output of the subordinate sequence.

As seen in the diagram, the first part of this triple (lower portion) describes a similar activity of elementary teachers--the literal demonstration of the procedures of an algorithm with no explanation of how or why it works. Unfortunately, as is well known, some instruction in algorithms at the elementary or even secondary level never proceeds beyond this mechanical level.

The second part of the triple (middle portion), describes the activity explaining how an algorithm works by relating the explanation of each procedure to observations of physical situations. This is another familiar activity of the elementary teacher when teaching an algorithm, according to the Report.

The third behavior (top portion), explaining the procedures of an algorithm by means of the rules of some "convincing game", represents those behaviors more characteristic of a contemporary mathematics curriculum with its appeal to the field properties and mathematical structure. This third behavior is one which the elementary teacher has most likely not acquired and yet, in many ways, it is the most critical to successful instruction in elementary mathematics today if learning is to go beyond rote memory of the algorithm presented.

The subordinate behaviors in the algorithm's hierarchy, as shown in the diagram, reflect this same triple of constructing and demonstrating behaviors, but are associated with a particular operation within a specified number system. The final task differs from the subordinate ones in that any algorithm could be presented to the teacher and he or she would be expected to be able to exhibit the specified behaviors without instruction.

Subordinate to the algorithm hierarchy behaviors are the convincing game rule behaviors. According to the Report, the behaviors associated with the identification and naming of the field properties are developed in the context of game rules for two reasons. First, games provide a vehicle for identifying the properties in a setting which promotes individual investigation and immediate application of the identified rules. And second, the departure from a formal mathematics presentation to a game presentation reduces the anxiety which frequently accompanies mathematics instruction for the elementary teacher.

Summary and Concluding Remarks

It is evident that much research and study has been done on hierarchies and their implications. In an algorithmic learning situation, hierarchies have been looked at, first of all, from the point of view of how best to construct the hierarchy. Does the task analysis that an instructor may construct differ from student-generated hierarchies? If it does, in what ways?

Second, what of the "validity" of such hierarchies? The basic tenant is that learning of lower level subskills will have a positive transfer effect on the learning of the terminal task. Sequencing of such sub-levels is also of interest and research on this has produced some conflicting conclusions. Studies on retention and transfer have also been researched from the point of view of the final task versus subordinate behaviors.

A hierarchy for teachers in the elementary school who deal with the teaching of algorithms, has been of interest. It has resulted in a slightly different hierarchical structure to describe different levels of teaching approaches in the classroom in regard to these basic algorithms.

The constructing of hierarchics for the teaching of algorithms, or for that matter, a larger class of algorithmic learning in general, has implications for what is happening in our schools. If they are constructed and used in a rote learning situation, they defeat the purpose and work of Gagné and others. If they are constructed carefully and used wisely, they can be of great value to both teachers and students alike.

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Investigations of Conceptual Bases Underlying the Learning of Algorithms

Diane Thomas

For the learning of a computational algorithm to be meaningful, there are prerequisite concepts that must have been mastered in order for the student to understand the rationale behind the algorithm. The algorithm will deal with some type of number--whole numbers, integers, rational numbers, irrationals, complex numbers--and the conceptual bases dealing with that particular type of number must have been developed beforehand. The algorithm will include at least one type of operation, probably more, and conceptual bases underlying the particular view of the operations used must have been established in relation to the type of number under concern. Properties such as distributivity of multiplication over addition, commutativity of addition and of multiplication but not of subtraction nor of division, and the role of the multiplicative and the additive identities should have been discussed previously if they are to be incorporated into the rationale for the algorithm. Further, concepts underlying the notation used in representing a number (both place value notation, e.g., 35 standing for 3 tens + 5 ones, and symbolism, e.g., $3\frac{1}{2}/14$ representing a fraction) will be involved in understanding a particular algorithm and these need to be provided for earlier in the child's learning process.

Defining "pure concepts" to be those dealing with numbers as such, with the properties of numbers, and with operations that can be performed on numbers, Dienes (1960) discusses the relationship between pure concepts and notational concepts:

> A child may have grasped the concept that to add two numbers you have to count on from the first number by as many steps as the second number. Yet he may be nowhere near realizing the complicated structure of the task 27 + 35 in which grouping and regrouping in tens must be achieved to perform the task economically. In other words, mathematical concepts and processes have to be learnt first in the pure form, followed by the same concepts and processes in the notational form, i.e., with the structure of the decimal system superimposed on them. (pp. 39-40)

Thus, if we are concerned with investigating and comparing the conceptual bases which underlie the learning of algorithms, it seems to follow that our concern must center around student learning of pure number concepts, of notational concepts, and of the tie-in between the two which leads to the understanding of computational algorithms.

Concept Learning

Looking first to educational psychology to provide some guidance in the general area of concept learning, it is apparent that much of the research of the educational psychologists has been in areas not directly applicable to our goal. Bruner has defined a concept to be "...a way of grouping an array of objects or events in terms of those characteristics that distinguish this array from other objects or events in the universe" (Bruner, Goodnow, and Austin, 1956, p.275); this idea of categorization has been used by many other psychologists as well in their investigations into concept learning. Much of this research, as a result, has been aimed at identifying optimal information processing strategies for concept attainment through manipulating task variables such as stimulus similarity, prompting procedures, sequence, and difficulty (Tennyson, 1972, p. 1). Another related approach is the one taken by Klausmeier, who postulated four stages involved in the attainment of the same concept at successively higher levels of inclusiveness and abstractness: concrete, identity, rudimentary classificatory, and formal. However, he restricted his model to those concepts "of the kind for which there are actual perceptible instances" and then noted that "not all concepts have perceptible instances; for example,...signed numbers" (Klausmeier, 1971, pp. 1-2).

Turning from this narrow view of concept learning, we approach the theories of mathematical concept learning as espoused by Skemp and by Dienes. According to Fehr (1966):

> For Skemp, the fundamental related ideas (concepts) are learned through intuitive methods through the use of well-chosen sensory activity situations, in proper sequence of presentation. In this way the fundamental concepts build up a schemata, which, acquired by the age when reflective activity of the mind has developed (age 12 years on), enable the child to appreciate and construct formal mathematical systems. Thus Skemp rejected, so far as the elementary school is concerned, any formal reflective procedures for the formation of basic mathematical concepts. He did accept perceptory-intuitive generalizations from sensory activity situations as the means of building the basic mathematical concepts. (p. 224)

Dienes has a similar theory of mathematical learning. In An Experimental Study of Mathematics Learning (1963) he described two of the principles he said should be followed in helping students attain a mathematical concept:

> The perceptual variability principle stated that to abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties. The mathematical variability

principle stated that as every mathematical concept involved essential variables, all these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved. The application of the perceptual variability principle ensures efficient abstraction; the application of the mathematical variability principle ensures efficient generalization. (p. 158)

These principles form the basis for the "multiple embodiment" approach to mathematics instruction, which demands that a variety of perceptually different materials be used by the teacher in helping students to develop a mathematical concept. However, based on reviews by Beougher, Kieren, and Suydam and Weaver that are summarized in general by Reys in the October 1972 issue of the Arithmetic Teacher, the research on multiple embodiments is reported to reveal inconclusive results. Reys found a wide range in the quality of the research, but concluded:

> Nevertheless, it is clear that the research does not consistently support or refute a multiple-embodiment approach to teaching mathematics. In fact, the one common thread among these studies is that learning mathematics depends more on the teacher than on the embodiment used. (p. 490)

Two research studies by Reys' students further confirm Reys' conclusions; however, the subjects used in both studies were preservice elementary school teachers and not youngsters at the elementary school level. Turek's study (1973) compared two instructional approaches, one based on Dienes' two principles and the other using a lecture approach, for teaching concepts about finite mathematical systems. The study was repeated twice--the first time significant differences were found favoring the Dienes-based approach, the second time no significant differences were found. Similarly, a study by Skipper (1973) compared three instructional methods for teaching concepts of positional numeration systems; one method used Dienes' Multibase Arithmetic Blocks along with variable base abaci, a second method made use of only the Multibase Blocks, and the third method was the lecture method. Two replications of the study yielded different results. In one of the replications there were no significant differences in performance by the three groups as measured by scores on an investigator-developed test. In the other replication, the students in the lecture method performed as well or better than those having materials presented through Dienes' approach, and those exposed to a variety of perceptual embodiments performed as well or better than those using only one embodiment.

The idea of meaningful learning must also be taken into account for a theory of mathematical concept learning. Brownell (1947) defined this aspect of instruction: "'Meaningful' arithmetic...refers to instruction which is deliberately planned to teach arithmetic meanings and to make arithmetic sensible to children through its mathematical relationships. Not all possible meanings are taught, nor are all meanings taught in the same degree of completeness" (p. 257). Brownell then suggested four categories under which the meanings of arithmetic can be roughly grouped:

> 1. One group consists of a large list of basic concepts. Here, for example, are the meanings of whole numbers, of common fractions, of decimal fractions, of percent, and...of ratio and proportion....Here, too, are the technical terms of arithmetic--addend, divisor, common denominator, and the like....
>
> 2. A second group of arithmetical meanings includes understanding of the fundamental operations. Children must know when to add, when to subtract, when to multiply, and when to divide. They must possess this knowledge, and they must also know what happens to the numbers used when a given operation is employed....
>
> 3. A third group of meanings is composed of the more important principles, relationships, and generalizations of arithmetic, of which the following are typical: When 0 is added to a number, the value of that number is unchanged. The product of two abstract factors remains the same regardless of which factor is used as multiplier. The numerator and denominator of a fraction may be divided by the same number without changing the value of the fraction.
>
> 4. A fourth group of meanings relates to the understanding of our decimal number system and its use in rationalizing our computational procedures and our algorisms. (pp. 257-258)

Besides meaningful learning, there are other dimensions to the teaching-learning situation that will affect student learning of pure and of notational number concepts. Weaver and Suydam (1972, p. 4) point out that the rote-meaningful dimension, the reception-discovery dimension, and the concrete-symbolic dimension may interact with each other in an instructional situation. Fennema (1969) was concerned primarily with the interaction between rote-meaningful instruction with material presented in a concrete-symbolic mode. Her study was an attempt to determine the relative effectiveness of a meaningful concrete model (Cuisenaire rods) and a meaningful symbolic model (a symbolic statement of repeated addition) in facilitating the learning of a mathematical principle (multiplication). Results showed that there were no significant differences between methods in the overall learning of the mathematical principle: "Second grade children were able to learn a mathematical principle by using only a symbolic or a concrete model when that model was related to knowledge the children had. This provides evidence that making the teaching of mathematical principles meaningful is as important as are the materials used to demonstrate that principle" (Fennema, 1969, p. xiii).

Pure Number Concepts

Some of the investigations centering around student learning of pure number concepts can be classified generally as studies looking at the different ideas incorporated in a specific operation on one type of number (usually whole numbers), perhaps also concerned with the symbolic-concrete mode of presentation, sometimes mentioning the meaningful-rote dimension, and not too often specifying whether the approach was inductive or deductive. The following five studies are some of those that fit the above criteria:

1. Gibb (1956) studied subtraction with whole numbers, identifying three types of applications for subtraction--take-away, additive-subtraction, and comparative-subtraction--at three levels of abstraction (abstract, semiconcrete, and concrete). Her results showed that second graders attained highest on take-away problems and lowest on comparative, that additive problems took a longer time, and that performance was better on problems in a semiconcrete mode than in a concrete mode and lowest in the abstract mode.

2. Van Engen and Steffe (1966) investigated first-grade children's concept of addition of natural numbers, when addition was defined in terms of the union of two sets. When student responses on a test of conservation of numerousness and on a paper-and-pencil test of addition facts were compared, findings showed that the student's ability to respond correctly to an addition combination seemed to have little or no relation to his ability to conserve numerousness. The authors concluded that the children had not abstracted the concept of the sum of two whole numbers from physical situations but rather had just memorized the addition combinations.

3. Hervey (1966) looked at multiplication of whole numbers represented by equal additions in contrast with multiplication as a Cartesian product, finding that equal additions multiplication problems were less difficult to solve and conceptualize for second-grade students than were Cartesian product problems.

   Tietze (1969) compared two methods of interpreting multiplication of whole numbers--the repeated-addition approach using an array as a physical referent and the ratio-to-one method using a coordinate system and ordered pairs of numbers as the physical referent;

lessons covered the basic facts from 1 x 1 through 9 x 9. No significant relationship was found between the method used and the acquisition, retention, and understanding of multiplication for the total group of fourth-grade subjects, but use of arrays with the repeated-addition method seemed better for average and low students.

4. Investigating the differences in difficulty between partitive and measurement division problems with whole numbers, Zweng (1964) found that partitive division problems were significantly more difficult than the measurement problems for second-grade students.

One study which investigated basic concepts about a type of number, rather than about an operation, was Sension's (1971). He compared three representations for introducing rational number concepts--through area, set-subset, and a combination of the two. Findings showed that all approaches seemed to be equally effective for second-grade students, when measured by student performance on a test using two types of pictorial models.

Since the rationale for an algorithm often is based on mathematical principles, we need to be concerned with how well students understand these principles, and will include "principles" as part of our look at pure number "concepts", even though authors often separate the two (see Higgins, 1973, p. 192). In a study comparing the use of the distributive property in understanding basic multiplication facts to the use of repeated additions and arrays, Gray (1965) found that instruction in the distributive property resulted in higher achievement for third-grade students. Knowledge of the distributive property appeared to help children proceed independently in the solution of untaught multiplication combinations; however, the children appeared not to develop an understanding of the distributive property unless it was specifically taught. Weaver (1973) reported a study on student performance on examples involving the distributive idea. Students in grades four through seven participated; findings showed that pupil performance level on an 8-item test was low--at each grade level at least 90% of the students gave criterion responses (applying distributivity without any computational error) on fewer than 3 of the 8 items. Flournoy (1964) gave seventh graders an 18-item test measuring ability to apply basic laws of arithmetic in each operation with whole numbers; an error rate of 30 percent or greater was found on 15 items and 50 percent or greater on ten items. The items most frequently missed were those related to the distributive property. Crawford (1965) found that the order of difficulty of field axioms (from easiest to most difficult) was commutativity, inverse, closure, identity, associativity, and distributivity, for students in traditional-content ninth- and tenth-grade classes.

Notational Concepts

Flournoy, Brandt, and McGregor (1963) found that on tests measuring understanding of our numeration system, students in grades four through seven very frequently missed items related to: (1) the additive principle (672 means 600 + 70 + 2); (2) making "relative" interpretations, which use varied ways of grouping rather than by individual places--for example, 2346 can be interpreted as 23 hundreds, 46 ones, or as 234 tens, 6 ones; (3) meaning of 1000 as 100 tens or 10 hundreds, and so on; (4) expressing powers of ten; and (5) the 10 to 1 relationship of each place in a numeral going to the left from the ones place, and the 1 to 10 or 1/10 relationship to the previous place in going to the right in the numeral.

Rathmell (1973) attempted to determine the effects of type of grouping (multibase or base ten) and the time that base representations were introduced (initially, or after counting and reading and writing numerals) on achievement in numeration in grade one. Results showed no significant differences between the multibase and the base-ten-only approach; however, low ability students achieved better in the base ten method. The group who had base representations introduced after counting and working with numerals had consistently higher means and adjusted means for the posttest and also had significantly more students with mastery on the retention test.

According to Diedrich and Glennon (1970), the evidence of previous studies (Brownell, 1964; Jackson, 1965; and Schlinsog, 1965) was not conclusive in showing that a study of nondecimal systems is more effective in enhancing student understanding of the decimal system than is a study of base ten alone. They further noted that the evidence did not tell which method is more effective in promoting increased understanding of the rationale of computation, in promoting increased understanding of a place value system in general, or in promoting retention of these understandings. In their own investigation, Diedrich and Glennon compared fourth-grade students studying five place value systems (bases 3, 5, 6, 10, and 12) with a group studying three different bases (3, 5, and 10), a group studying base ten only, and a control group receiving no instruction in numeration. Results showed that a study of the decimal system alone was just as effective as a corresponding study of nondecimal numeration in promoting understanding of the decimal system as shown on the posttest but that no single study was more effective than the others in promoting retention of achieved understandings. No differences among treatments were observed with respect to understanding computation with decimals on either the posttest or the retention test. With respect to understanding a place value system in general, a study of bases 3, 5, and 10 was as effective as a study of bases 3, 4, 6, 10, and 12; also, a study of nondecimal numeration was more effective than a study of the decimal system alone, as shown on the posttest. However, none of the studies was found to be more effective than the others in promoting retention. In discussing the implications of their findings, the authors stated:

> If one wishes to foster, at the fourth-grade level, understanding of the decimal system, the available evidence suggests that only the decimal system need be taught. Also, if one wishes to foster understanding of both decimal and nondecimal systems, the implication is that both decimal and nondecimal systems should be taught. (p. 171)

Understanding Computational Algorithms

Studies concerned with student understanding of computational algorithms for the most part seem to deal with comparing one algorithm with others, where different conceptual bases underlie each rationale. For example, the relative effectiveness of two algorithms for subtracting whole numbers was investigated by Brownell and Moser (1949). They compared the achievement of third graders taught to use the decomposition algorithm (which depends heavily on concepts of place value, grouping, and regrouping) to those taught to subtract using the equal additions algorithm (which is based on the concept, sometimes labeled as the "Law of Compensation", that increasing or decreasing each of two numbers by the same amount does not change the difference between them (Buckingham, 1953, p. 141)). Half of each group was taught meaningfully and half learned the procedure mechanically. Among the conclusions were that the equal additions algorithm appeared satisfactory for children with a background of meaningful arithmetic, but for children with a limited background the decomposition algorithm, taught with meaning, was better regardless of the criteria employed; that the equal additions algorithm was difficult to rationalize; and that some proficiency can be produced by mechanical instruction with either of the algorithms.

Like subtraction, division has more than one meaning. According to Buckingham (1953), there are essentially two kinds of division: "measurement, if you are to find the number of equal groups, knowing the size of each; partition, if you are to find the size of the equal groups, knowing how many groups there are" (p. 76). The following studies investigate the effectiveness of algorithms based on different meanings of division:

1. Van Engen and Gibb (1956) compared the use of the distributive algorithm for division to the subtractive form of the algorithm. Results showed that fourth-grade students taught the subtractive form had a better understanding of the process or idea of division than did those taught the distributive method, that use of the subtractive algorithm was especially effective with students of low ability, and that high ability students used the two methods with equal effectiveness. Use of the subtractive method was more effective in enabling children to transfer to unfamiliar but similar situations.

Children who used the distributive algorithm had greater success with partition situations, while those who used the subtractive algorithm had greater success with measurement situations.

2. Dilley (1970) also looked at the same two methods of teaching long division to fourth graders. The distributive algorithm was developed as a method of keeping records of a manipulation of bundles of sticks and the successive subtractions method was developed in a manner "closely paralleling the treatments given in popular elementary textbooks." Results showed significant differences on only two of the seven tests given to students. On the applications test the difference favored the method of successive subtractions. On the retention power test the difference favored the distributive method. It was concluded that there was little, if any, overall difference between the two methods of teaching long division.

3. In a similar study, also with fourth-grade students, Kratzer (1972) compared the Greenwood algorithm--the method of repetitive subtraction--to the distributive algorithm, both taught with the use of a manipulative aid (bundles of sticks again). No significant difference was found between methods on a test of familiar problems; however, the distributive group scored better on both immediate and retention tests of unfamiliar problems.

4. Rousseau (1972) defined four possible foundations of the division algorithm as (1) mathematical, based on the distributive law of division over addition; (2) real world, based on the physical act of partitioning; (3) real world, based on the physical act of quotitioning (measurement); and (4) rote, based upon the memorization of routines. Four different division algorithms were synthesized on these foundations and each taught to a different group of fourth graders. No significant differences were found for retention of the algorithm. For extensions of the algorithm to cases of slightly greater difficulty, the rote algorithm was found superior; as the degree of difficulty increased the ordering of quotitive (actually, the repeated subtraction algorithm), distributive, rote, and partitive (the conventional distributive algorithm) was established.

Algorithms for division of fractions, based on different underlying concepts, were of concern in the following studies:

1. Bidwell (1968), investigating three meaningful approaches to division of fractions taught to sixth-grade students, found that the inverse operation procedure was most effective, followed by the complex fraction method and the common denominator procedure. The complex fraction method was better for retention, while the common denominator method was poorest.

2. Comparing the same three approaches, also with sixth grades, Bergen (1966) found that there were no significant differences between complex fraction and inverse operation algorithms; but that each was significantly superior to the common denominator method.

3. In a study comparing the common denominator and the inversion methods, Capps (1963) found that sixth-grade students did not differ significantly in ability to divide fractions. The group taught by the inversion method scored significantly higher in ability to multiply fractions on the immediate posttest, but not on the retention test; analysis of gain or loss from posttest to retention test showed that the common denominator group gained significantly on multiplication of fractions.

Stenger (1972) compared two methods of teaching addition and subtraction of fractions to fifth graders: (1) a subset-ratio procedure based on the formal definitions of addition and subtraction of rational numbers, taught in a semiconcrete mode; and (2) the "traditional" approach based on the use of equivalence classes to find the least common denominator, taught in a symbolic mode. Results showed that the group taught with the subset-ratio approach did significantly better on both immediate and retention tests, but that the "traditional" group wrote significantly more correct answers in lowest terms.

Algorithms for operations on integers were investigated by Sherzer and by Sawyer. Sherzer (1973) studied the effects of two different methods of presenting instruction in adding integers to students in grades 3 through 6. One was the number line method, the other used the correspondence method which required matching positive and negative ones in the addends, then counting the unmatched numbers to get the answer (Sherzer, 1969, pp. 360-362). The following conclusions were reached: (1) students in grades as low as three could be successfully taught integer addition skills by the correspondence method, (2) the correspondence method was more effective than the number line approach overall for both proficiency skills and concept formation, (3) the correpsondence method appeared to work equally well with low and high achievement groups, (4) neither method appeared to be effective in imparting verbal skills (concept formation) to third graders, and

(5) the number line method appeared to be workable, although less effective, in the upper grades but not effective as low as grade three.

Sawyer (1973) compared achievement of seventh-grade students taught subtraction of integers by three methods: (1) complement method--a method of subtraction by adding the same number to both the minuend and the subtrahend, taught in a symbolic mode; (2) related number facts method, involving the relationship between subtraction and addition, introduced through the use of a number line; and (3) systems method, where a modular system is examined to show that $x - y = x + (-y)$ and then this result is generalized to the integers; introduction was in a semiconcrete mode. Results showed no consistent superiority of one method over another.

All of the studies discussed above deal with student understanding of computational algorithms by comparing one algorithm with others, where each of the algorithms in question springs from different conceptual bases. Another way that might be used to approach the problem would be to focus on just one particular computational algorithm and to determine the various conceptual bases underlying that algorithm. One way to define different conceptual bases for the same algorithm might be to consider the interaction of the concrete-symbolic dimension with the pure and notational concepts needed for understanding the algorithm. In this view, Wheeler's (1972) descriptive study could be classified as pertaining to appropriate and inappropriate conceptual bases. Wheeler analyzed the relationship of a child's performance in solving multi-digit addition and subtraction problems using concrete embodiments compared to his performance in the symbolic mode. It was found that second-grade children proficient in regrouping addition and subtraction examples on three or four embodiments scored significantly higher on the written tests of addition and subtraction than those children not proficient in using concrete materials, and significant correlations were found between the number of embodiments children were able to regroup and their performances on the written test. In an experimental study, care must be taken that all approaches are taught with the same degree of meaning. For example, Fennema (1969, pp. 21-24) cited a study by Ekman (1966) on teaching third-grade children addition and subtraction algorithms through symbolic, semiconcrete, and concrete modes; results showed that on a retention test, significant differences were found in favor of students taught using concrete materials. However, Fennema felt that the results of the study were confounded by the concrete approach being taught meaningfully while the symbolic approach was taught through a rote procedure.

A second possibility for defining different conceptual bases for the same algorithm would be to look at the different interpretations for the type of number involved, at different views of the operation involved, or at the various combinations that can be made between the two. For example, in Carney's (1973) study, fourth-grade students were expected to add and subtract rational numbers by changing each of the two given fractions to equivalent fractions both having the same denominator, combining the fractions, and then reducing when appropriate. Results from one group taught through the use of field properties (the identity element for multiplication, the commutative and associative principles, and the distributive law) were compared to those obtained by students taught by a standard method emphasizing equivalence classes and using objects (rods), number lines, and unit regions. Results showed the field property course to be more effective than the standard method; within each treatment group there were significant differences among mean gain performance of student subgroups based on achievement. Other studies that can be categorized as considering one algorithm and then looking at component parts are those by Green, Trafton, and Weinstein:

1. Green (1970) compared two approaches of teaching multiplication of fractional numbers, along with two types of instructional materials for each approach. One method was based on the area of a rectangular region and the other on finding a fractional part of a region or set; the instructional materials were diagrams and cardboard strips (called "materials"). Results showed that with fifth grade students, the area approach was more effective than the fractional-part-of approach; diagrams and materials seemed to be equally effective. The area/diagram combination was the most successful, followed by the part-or/materials approach, with part-or/diagram ranking last.

2. Trafton (1971) looked at two approaches to two-digit subtraction; one approach consisted of a prolonged development of the conventional decomposition algorithm, and the second was a more general method based on work with concepts of subtraction and use of the number line before the decomposition algorithm was taught. The more extensive development of the decomposition algorithm was found to be more effective than the second approach when used with third-grade classes.

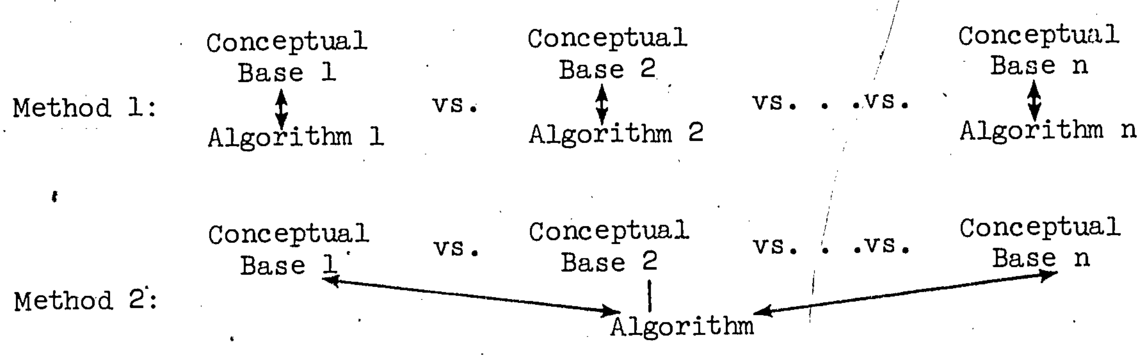
3. Weinstein (1973) compared the teaching of a mathematical algorithm by four types of justification methods: a pattern justification based on an analog to two-dimensional physical actions, an algebraic justification based on the algebraic principles for rational numbers as well as on the rules of logic, a pattern-algebraic sequence, and an algebraic-pattern sequence. Differences in performance among treatment groups were examined for each of four algorithms: multiplication of a fraction and a mixed number, comparison of fractions using the cross-product rule, conversion of a fraction to a decimal, and calculation of the square root of a fraction. The results showed

that, for fifth-grade classes, there were no significant differences between students taught by a strictly pattern approach and those taught by a strictly algebraic approach. However, evidence indicated that students taught by an algebraic approach, as a group, tended to do better on extension tests than their pattern-taught counterparts, and that students taught by a pattern method, as a group, tended to do better on simple algorithm computation tests than their algebraically-taught counterparts.

Summary

Studies which attempt to investigate and compare conceptual bases underlying the learning of algorithms have taken a variety of approaches. In the area of concept learning in general, the questions of using multiple embodiments and of attending to rote-meaningful, reception-discovery, and concrete-symbolic dimensions have been raised. The learning of pure number concepts has been the concern of several research studies which usually concentrated either on the different ideas incorporated in a specific operation on one type of number, or on student understanding of number properties such as the distributive law. Investigations of notational concepts generally have centered around comparing the study of decimal and nondecimal systems on student achievement in numeration and computation.

Research into student understanding of computational algorithms has been conducted along two lines. The majority of the studies reviewed deal with the comparison of two or more algorithms, where each of the algorithms under consideration stems from a different conceptual base. The second method used in setting up studies involves the selection of just one computational algorithm and the investigation of the various conceptual bases underlying that algorithm. The diagram below illustrates the difference between the two approaches:



Relatively few studies have been done which use Method 2 as a scheme for analyzing student understanding of computational algorithms; it would appear that more of the future research into learning algorithms might be profitably extended into this area.

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Algorithms: Interference, Facilitation, and Comparisons

Brady Shafer

Interference in the Laboratory: Retroactive and Proactive

Psychologists' interest in memory continues undiminished after ninety years. Serving as his own subject, Ebbinghaus (1913) performed a series of experiments in which he memorized and then attempted to recall strings of nonsense syllables. He found that the number of syllables remembered, after varying lengths of time, formed a decreasing function, with nearly all forgetting taking place in the first three hours.

Other investigators soon found that more was involved than mere passage of time. Equally important in determining forgetting rate was what the subject did during the time. Jenkins and Dallenbach (1924) found that when subjects slept during the retention interval, retention was better than when they were awake and going about their daily routines. Müller and Pilzecker (1900) noted that when subjects learned several lists of words, recall on the earlier lists was decreased; it was as if the later learnings dislodged or "interfered with" earlier learnings. The phenomenon has been replicated in widely varying contexts and is known as retrcactive inhibition or retroactive interference.

A converse phenomenon has also been noted. Prior learning sometimes makes the learning of new material more difficult. This is called proactive inhibition or proactive interference. For example, anyone attempting to learn the twenty-six letters of the alphabet in a new sequence would find the familiar a, b, c, ... intruding and making the new memory task more difficult than if he had not known the alphabet at all.

In a masterful bit of scientific detective work, Underwood (1957) showed that many researchers had inadvertently ballooned forgetting rates by keeping the same subjects for 6, 8, 16, and 20 lists; the larger the number of previous lists the higher the forgetting rate. Proactive interference had been, unnoticed, at work; the *n*th list had been more difficult for a subject precisely because of the *n-1* previous ones.

But frequently, one bit of learning aids in acquiring another. One always hopes that a child's first school experience with "two plus two" will aid in every future meeting. Proactive facilitation turns out to be the familiar transfer of training; review, formal or otherwise, is retroactive facilitation; and proactive interference amounts to negative transfer.

Retroactive and Proactive Interference in the Mathematics Classroom

In the typical retroactive/proactive interference study, the datum of interest is the number of items recalled. The subject is tested on memory alone. But in mathematics education two large differences appear. First, in most laboratory studies of PI and RI, the material is to be remembered over minutes or hours, whereas a mathematics item may be needed on a final exam after a lapse of months. Second, in mathematics class simply recalling the formula is usually not enough. One must cap the recall by using the formula to produce a correct solution. May retroactive and proactive interference be detected in this larger context? One obvious place to seek such interference is in a place where the student sees two or more methods for doing a given kind of calculation.

Often there is a single "best" algorithm. How many really different ways are there to differentiate a polynomial function? But there are at least five points in the school mathematics curriculum at which (a) two or more algorithms are widely taught and (b) research has attempted to measure the relative strengths and weaknesses of each. The five areas are:

- Division of fractions
- Long division (distributive vs. subtractive or Greenwood algorithm)
- Estimating quotient digits in long division
- Finding the lowest common denominator
- Placing the decimal point in division of decimals.[1]

The present paper brings together studies which involve the teaching of two alternate computation algorithms to students. Originally that last sentence ended, "to the same students." But one fascinating, frustrating result was noted again and again. Brownell (1938) hoped nearly forty years ago that "perhaps in teaching for understanding we shall some day depart from the well-nigh universal custom of offering children but one of several alternative forms for computation." This hope has not been realized in many mathematics curricula.

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[1] One topic in school mathematics distinguished by the fact that several algorithms are routinely presented to a given class is the solution of simultaneous linear equations. No study exists at present, however, which compares their effectiveness or which looks for retroactive/proactive phenomena. The writer is now at work on such a study. At this writing no data have been analyzed.

In a different but related area, Spencer (1968) attempted to teach addition and subtraction simultaneously, looking specifically for interference and facilitation.

To the present day the majority of studies comparing algorithms do so by presenting two or more groups of children with one algorithm each. The relative effectiveness of the algorithms is then inferred from a comparison of test scores, or gains, for the groups. This precludes the study of proactive and retroactive interference, as well as proactive and retroactive facilitation. Neither is observed, since neither has a chance to happen. The student does not learn two algorithms, so interference and facilitation are alike impossible.

In a small minority of experiments, however, students were taught more than one algorithm. These studies are noted in the review which follows. Of interest also from the interference point of view is that several experimenters, whose studies consisted of teaching the algorithms to different groups, nevertheless conclude by recommending the teaching of more than one algorithm to each child.

Studies on Division of Fractions

Bergen (1966) compared the complex fraction (reciprocal) method, the common denominator method, and the inversion method, on eight types of division problems involving mixed numbers, whole numbers, and simple fractions. On the first of four tests, the reciprocal method was found better than the inversion method ($p < .05$), while on all four tests the common-denominator method was found inferior to the other two methods ($p < .05$).

Bergen concluded by recommending that pupils begin their study of division of fractions by using the reciprocal method, since it is stronger at the outset. However, since this method is more involved than the inversion method, pupils should be taught the inversion method later as a shortcut.

Bidwell (1968), comparing the merits of the same three methods, came up with this ranking: inverse operation method best, followed by complex-fraction method and common-denominator method. He agrees with Bergen that the common-denominator method compares poorly with the others, but disagreed with her about the first-place finish. The difference may be that Bidwell included tests for four things which Bergen did not include: transfer between related concepts, integration of concepts, attainment of concepts, and the correlation between concept attainment and computational skill. He reported the inverse-operation method showed the lowest transfer error rate and the highest concept-attainment percent.

Another discrepancy appeared in the results obtained by Krich (1964) and Sluser (1963). In Krich's study, experimental-group pupils were given explanations of principles and were allowed, but not specifically asked, to develop the inversion algorithms for themselves. The control group was given rote learning and drill. For average students, a difference in favor of the experimental group developed on a two-month delayed test (though not at the end of the instruction). Comparing delayed test scores with pretest scores, Krich noted the control group actually lost ground, while the experimental group made a small gain (not significant at .05).

Sluser, on the other hand, reported that his experimental group got "an explanation" of the reciprocal principle while the control group did not. The experimental group fell behind the control group ($p < .02$). Analyzing results by IQ levels, Sluser reported that the brighter experimental-group children could understand the principle and were helped, but average and below-average students were confused.

As before, the discrepancy may be due to a difference in treatments, this time to the very large difference between a student's passively hearing an explanation and actively creating the explanation for himself. Krich used programmed instruction to reduce the effect of teacher variable, while Sluser presumably did not.

Looking at the same problem area from a somewhat different angle, Capps (1962) looked at the possibility of retroactive interference, induced by the algorithm for dividing fractions, with the algorithm for multiplication taught earlier. He compared the inversion method with the common-denominator method, giving a posttest which contained a multiplication-of-fractions subset. From this standpoint the inversion method was superior ($p < .01$). On a delayed test the difference was not as sharp, but still significant at the .05 level. It may be concluded that either the inversion method reinforces multiplication of fractions skills more than the common-denominator method, or it interferes less. The experiment did not include a base-line control group to determine which might be true.

The final study reviewed in this section, an older study, is of interest for this report chiefly because in it the same children were taught two algorithms. The study is reported by Brownell (1938) but was actually done by Thelma Tew. (Presumably Tew was Brownell's student; this writer can find no report of the study published by Miss Tew herself.) Details are few. More serious is the problem that the study was not well-controlled. Indeed Brownell's article includes no data whatever. But several observations are worthy of note.

Tew's sequence was: common denominator method first (since, the inversion method is particularly difficult to explain in a meaningful way), followed at length by inversion as a shortcut. It was found that pupils learned to divide by the common denominator algorithm "more easily and more intelligently than ever before when she had taught by the inversion method....Comparisons with children taught by the inversion method were consistently in favor of Miss Tew's group." This finding contradicts Bidwell and Bergen. How may it be explained? The Bergen and Bidwell studies appear to be more carefully done. This writer's conjecture is that in the Brownell-Tew study there could have been a Hawthorne effect strong enough to tip the scales the other way.

Brownell continues, "Promptly the more intelligent and capable children adopted the short-cut, while the slower children stayed with the more familiar common denominator method, -- which is precisely what they should have done at their stage of development." Presumably the second method took less than one week. Implicit in Brownell's "consistently" is that the children were doing at least as well as other children taught the textbook method (inversion). Brownell's report seems to imply that, for those children who used it, the shortcut method enhanced this superiority. But this is not clear, since no data were given.

What does seem clear, and germane for this report, is that (a) introducing a second method did not cause undue confusion in children's thinking, and (b) not all children used the second method. Brownell comments that this is what they should have done. Still unanswered is the larger question of whether children in similar situations will consistently do what they "should" do.

Studies in Long Division: Distributive vs. Greenwood (Subtractive) Algorithm

This group of studies compared the "distributive" and "subtractive" algorithms for long division. Three studies are reviewed.

Scott's (1963) experiment, like Brownell's above, is distinguished by the fact that in it a group of students was given more than one algorithm. Two groups of students were taught both methods; a third, the subtractive algorithm only; and a fourth, the distributive algorithm only. Scott's chief interest lay in comparing the two-algorithm classes with the one-algorithm classes. Among his conclusions were:

(1) The use of two algorithms for division computation neither confuses nor presents undue difficulty for young children. The two-algorithm groups proceeded at least as smoothly and efficiently as the one-algorithm groups.

(2) Teaching two algorithms takes no more teaching time than teaching only one.

(3) Children who use two algorithms are at least as efficient in solving division problems as those children who use only one.

(4) The two-algorithm children have a greater understanding of the division process than those who use only one.

Dawson and Ruddell (1955) compared the same two algorithms, using different groups of children. They reported that use of the subtractive algorithm was better than use of the distributive algorithm, but because of a design flaw (different visual devices, for instance, were used with the two groups) one cannot establish that the difference was related to the method.

Van Engen and Gibb (1956) noted that the distributive algorithm emphasizes the relation of division to multiplication, while the subtractive algorithm emphasizes the concept of division as repeated subtraction of multiples of the same divisor. Their chief interest lay in conceptual matters, but a subset of the study involved a comparison of the effectiveness of the two algorithms.

They found that the subtractive group attempted more problems than the conventional group. However, having started, the latter group has less difficulty with the processing.

> Disregarding any effects of classes, arithmetic achievement and intellectual ability, the conventional group achieved greater success in problem solving, although they did not attempt as many as did the subtractive group....Classes taught the conventional method were more successful in solving problems familiar to both groups. On the other hand, the subtractive group did better in solving problems unfamiliar to both groups. These findings suggest that the subtractive methods group had not reached a high level of skill, ... yet their understanding of the process was such that they were better able to transfer to new situations. (Van Engen and Gibb, 1956).

Van Engen and Gibb make no explicit sequencing recommendations (or indeed any recommendation) based upon this difference. Notwithstanding, fifteen years later Kratzer (1971) remarks that most textbooks, "following Van Engen and Gibb," present first the subtractive algorithm and later the distributive.

Kratzer's point is that if children are eventually to use the distributive algorithm, would it not be more efficient to begin and end with the one method rather than duplicating children's (and teachers') effort? He approached long division through "a partitioning distributive approach" using stacks of popsicle sticks as a visual-manipulative aid. He found his method at least as effective as the Greenwood method.

Estimating Quotient Digits in Long Division

An additional group of studies of division devoted attention to several competing methods for producing a quotient digit when dividing by a two-digit divisor.

Most of these studies were made at a time when the distributive method for long division was the method in general use. Many studies examine what Hartung (1957) called an "example population," cataloguing and counting the problems themselves as the data, while other studies analyze children's responses to problems. Grossnickle (1932) presented a study based on a problem population. He recommends the "always-round-down" method, not only because it offers good probability of giving the correct digit on first trial, but also because whenever it yields a wrong digit, correction procedures are easier.

Osborn (1946) came to a similar conclusion, but with more vigor. In the 1930s, Morton had published probabilities of getting the correct digit on first trial with the use of the "always-round-down," "always-round-up," and "round-both-ways"[2] method. The probabilities were based on a simple but lengthy count of several thousand division problems done by each of the three methods. Osborn chided Morton for omitting, by Osborn's count, some 20,700 problems. Osborn conceded that the both-ways rule gives success in 14,858 of these, but noted that his always-down rule works in 4,980 of them. More to Osborn's point, in the both-ways approach the student needs to be alert to the fact that "remainder larger than divisor" is a danger sign. In the always-down approach the error cue is a subtraction which cannot be done. He concluded, "Rule 1 has to be taught in any case, and the introduction of Rule 2 results in intolerable confusion."

Osborn said this, however, without actually talking to students and examining their work. Flournoy, who did, scolded him in turn for trying to settle the problem not on the basis of what children actually do but by what educators (looking at division problems) anticipate they might do.

Flournoy (1959a) found no evidence of "intolerable confusion." She agreed with Osborn, though, that with slow children one rule is probably enough. Her finding was that children use both methods equally well; she added the interesting fact that children tend to use both methods regardless of which was officially taught. She recommended teaching both methods.

On the other hand, Carter (1960) recommended only one method. Her treatments were: down, both ways, and down followed by both ways (after ten weeks of a twelve-week instruction period). She found the "two-rule" students, the third group, to be below either the round-down students or the both-ways students, with no significant difference between the latter two groups. On speed, an immediate posttest gave the same ranking to the round-down and both-ways groups, while an eighteen-week delayed test showed all three groups of equal speed. Carter noted, as did Flournoy, that children do not always use the method taught.

2 "Round-both-ways" is a short if inaccurate tag for the following rule: If the second digit in divisor is four or less, round the first digit down; if five or more, round up.

Placing the Point in Division of Decimals

Only one study has investigated the two methods for positioning the decimal point: Flournoy (1959b). The two competing algorithms are: multiplying both divisor and dividend by the appropriate power of 10, using a caret and making the divisor a whole number, *vs*. subtracting the number of decimal places in the divisor from the number of places in the dividend.

Her conclusions were:

(1) In general, the first method gave greater accuracy -- though with above-average pupils the second method was slightly superior.

(2) The nature of the subtractive method seemed to provide more opportunity for error.

(3) The caret method was the method of the textbook, so it is recognized that variations in presentation of the subtractive method were probably more widespread than for the caret method. But there was considerable indication what children taught the subtractive method understood the mathematical principle underlying their method as well as the children who were taught to use the caret method.

Flournoy remarked in summary that pupils will eventually attain a rather mechanical, but efficient, method for placing the decimal point, regardless of the method taught. Still unanswered is the question of whether presenting both methods to a student will produce facilitation or interference.

Finding Least Common Denominator

Again only one study was found which explored this topic. Bat-haee (1969) compared the methods of (a) factoring denominators and (b) finding LCD by inspection. The latter was the method of the adopted textbook. He found the inspection method much superior. Students saw only one of the two methods.

Two Operations at Once

Spencer's (1968) paper investigates a more ambitious proposal: not merely teaching simultaneously two algorithms for the same process, but teaching simultaneously two processes. The processes are whole-number addition and subtraction. He found some interference, but more facilitation and on the whole a gain over the usual segregated approach. Spencer ends by suggesting that instructional strategies may have been a factor in the interference.

How Important is Algorithm-Related Research and Teaching?

How frequent are errors involving algorithm, in comparison with other kinds of errors? On study gives an estimate. Roberts (1968) analyzed 148 papers of third-graders who took the computation section of the Stanford Achievement Test in 1966. He classified four kinds of errors: wrong operation, obvious computational error, defective algorithm (defined to be "correct operation but some other error than number-fact error") and random response.

Distributions of errors were analyzed from samples out of each quartile of achievement scores, with the following result. In the lowest quartile 29% of errors were classified as defective algorithm, 37% of the total number for students in Quartile II, 43% of errors in Quartile III, and 39% of the errors in the upper quartile; overall, 36%. These figures may be inflated a bit since "defective algorithm" seems to be in some measure a "none of the above" category. But there was a separate category for what were deemed to be random responses, so the "ballooning" was probably moderate. It seems safe to say that at least a fourth, and likely nearly a third, of the errors on computation problems in the third-grade SAT that year were algorithm-related -- the student knew whether to add, subtract, or whatever, but had trouble with choice and use of algorithm. It would seem, then that time spent in identifying and correcting algorithm-related errors is time well spent.

Summary

The majority of the research studies considered in this paper have made "side-by-side" comparisons of alternative algorithms. We may glean from them the following conclusions:

(1) Bergen concluded her study by recommending the use of two methods though her study did not actually do so.

(2) Tew, as reported by Brownell, did use two methods, with results judged successful.

(3) Spencer combined addition and subtraction, successfully.

(4) Scott attempted two algorithms for long division in sequence, successfully.

(5) Flournoy recommended the teaching of two methods for estimating quotient digits, though she did not actually do so.

(6) Kratzer indicated that the distributive method for long division is as good as the sequence sometimes taught, Greenwood method followed by distributive method.

(7) Carter stated that one rule is enough for estimating quotient digits. If a second is taught, it should be delayed to avoid confusion.

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Algorithmic Problem Solving

Richard W. Corner

One of the important goals for education is the development of problem solving ability. An educated person should be able to use his knowledge to create new ideas or to act to resolve practical problems. The highest order of problem solving is the solution of those problems which require invention by the solver. In Gagne's (1965) view, problem solving is the process in which two principles are combined to form a "higher-order" principle. This view of the role of problems in mathematics education has been well-stated by Dilworth (1966):

> ...problems should be formulated which present the student with an opportunity to perceive significant mathematical relationships capable of leading to a variety of significant non-obvious conclusions. (p. 83)

At the opposite end of the problem-solving spectrum is the rote application of rules to problems. Many students attempt to solve problems by chanting magic words over the problem. For example, instead of using cancellation to simplify rational expressions, some students will use the "cross out" rule. Thus, the expression $\frac{2x + 3}{3 + x}$ can be simplified: $\frac{2\not{x} + \not{3}}{\not{3} + \not{x}} = 2$. Pólya (1948) has criticized textbooks for only having problems of the "rule-under-your-nose" variety, which encourage mindless memorization.

Memorization can lead a problem solver into difficulty but at certain stages in learning it seems to be desirable. For example, for a child at the early elementary level "7 + 9 = ?" is a problem which the child can translate to concrete embodiments and solve. Later, as the child matures, the problem "7 + 9 = ?", asks only for the recall of a memorized fact; it is something the child "knows." The spirit of mathematics education today is based on the hypothesis that meaningful learning of facts and algorithms will result in the student being able to better use and transfer the facts. Of necessity some facts must be memorized.

It seems to me that there is an intermediate level of problem solving between memorization (meaningful or otherwise) and inventive problem solving which should be of concern to mathematics educators. Many students have the desire to use mathematics but have no desire (or ability) to be inventive problem solvers. These students need to use mathematics to solve problems in other fields such as economics, engineering, and biology. Much of the mathematics which has practical application is summarized by algorithms. Thus, the intelligent use of algorithms to solve problems seems to me to be a valid goal for mathematics education.

This paper will attempt to explore some of the facets of algorithmic problem solving. First the literature on problem solving will be reviewed. Based on the literature, a model for algorithmic problem solving will be proposed. This model points to some considerations for the future mathematics education.

What is a Problem?

The following defines "problem" in a general sense:

A problem is a set of stimuli and a goal, set in an environment. To solve a problem, a person must first perceive the goal. If the stimuli elicit behavior which results in achievement of the goal in a manner consistent with the environment, then we say that we say that the problem has been solved. A problem in mathematics must be solved consistently within the mathematical and logic system in which it is contained. Consider the following problem: Solve the quadratic equation $x^2 + 4 = 0$. The goal of this problem is finding the set of numbers which make the sentence, $x^2 + 4 = 0$, a true statement. If the environment in which we are operating only contains the real number system, then the set of numbers is empty. However, if we admit the complex numbers, then we have a non-empty set of numbers, $\{2i, -2i\}$. If it is claimed that the numbers 2 and -2 are solutions to the equation, then we have an inconsistency with the properties of the system in which we are operating.

A given problem may be classified as an inventive problem, an algorithmic problem, or a memorization type of problem, depending on the experience and knowledge of the problem solver. Merrill (1971) has extended Gagne's view of learned behavior and includes this view. Merrill classifies all learned behavior into 10 categories, including Gagne's eight and two additional categories (see Figure 1). As in Gagne's theory each lower level behavior is necessary for a high-level behavior. Furthermore, Merrill proposes that a person will display what he calls the "Push Down Principle." Since each succeeding level in the hierarchy increases the cognitive demand on a person, he will act in such a way as to reduce the cognitive load as much as possible. That is, a behavior acquired at one level will be pushed down to a lower level as soon as conditions have changed sufficiently so that the learner can respond to the stimulus situation using lower level behavior. In problem solving, as defined by Gagne, the learner evolves a new principle. In Merrill's scheme, on the second encounter with the same problem type, the learner only needs analysis behavior to apply the previously evolved principle. After several encounters with the same situation, the behavior required is reduced to the classification level. That is, the problem solver just needs to know if the problem is in the class which is solved by the previously evolved principle. This "Push Down Principle" makes it possible for persons to use previous facts to expand their knowledge.

Behavior Class	Behavior type		
Emotional	1. Emotional (Signal Learning)		
Psycho-motor	2. Topographic (Stimulus Response)	3. Chaining	4. Complex Skills
Memori-zation	5. Naming	6. Serial Memory (Verbal Association)	7. Discrete Memory (Multiple Discrimination)
Complex Cognitive	8. Classification (Concept Learning)	9. Analysis (Principle Learning)	10. Problem Solving

Figure 1. Types and classes of behavior.

Research on Problem Solving

In his review of research, Kilpatrick (1969) concluded that little research on problem solving was being done. He further stated that much of the research lacks direction and is of low quality. Two theoretical positions seem relevant to algorithmic problem solving, the behaviorist and the information processing theories.

1. The Behaviorist Approach

Skinner (1966) states in operant learning terms what may be considered the basic approach to the behaviorist theory of problem solving (see Figure 2). The problem acts as a discrimination stimulus, $S_{p,d}$; the response, $R_{p,d}$, is a "coding" by mediating processes into a secondary discrimination stimulus, $S_{p,d}$. $S^1_{p,d}$ elicts the response of selection of the appropriate rule (algorithm) for the problem ($R^1_{p,d}$). Then, the problem becomes the stimulus for application of the rule ($S_{p,r}$). The elicted response will be an S-R chain of length greater than or equal to one; the final response is the desired solution (R_n).

$S_{p,d} \longrightarrow R_{p,d} — s^1_{p,d} \longrightarrow R^1_{p,d} — S_{p,r} \rightarrow R_{p,e} — S_1 \longrightarrow R_1 \cdots S_n \rightarrow R_n$

The problem a discrimination stimulus	Mediated discrimination stimulus	Problem as stimulus for rule	Application of rule	Solution

Figure 2. The behaviorist theory of problem solving

The basic operant position has been modified by several others. Kendler and Kendler (Davis, 1966) view problem solving as a combination of horizontal and vertical processes. The horizontal processes are several ongoing S-R chains. Problem solving then is the vertical integration of two or more of these chains. Staats (1966) sees the process as a highly complex sequence of stimuli which may elicit multiple responses and responses which require multiple stimuli for their elicitation.

Davis (1966) summarizes the research on problem solving with the view that problem solving behavior is essentially the result of trial-and-error learning. If a person has prior experience with a given problem, then he has acquired the necessary S-R relations to apply a previously learned rule for solution. The research tasks usually associated with this type of problem solving are anagrams, water jug, and "insight" (e.g., matchstick or hat rack) problems. When a person does not associate the desire outcomes of a problem with a rule, he then operates in a trial-and-error manner. His trial-and-error behavior establishes the necessary S-R relations to allow the application of a rule for solution. The research tasks associated with overt trial-and-error are typically light-switch, classification, and probability learning tasks.

2. The Information Processing Theory

Newell and Simon (1972) have outlined the essential ideas of an information processing theory of problem solving. The essence of the theory is the assumption that a human acts as an information processing system in solving problems. The research done in this area has been designed to support this assumption.

An information processing system (IPS) has capability to solve problems in the form of a program. The program is written in a symbolic form, usually, but not necessarily, a computer programming language. The IPS has receptors which allow it to receive information from the environment. The IPS has a processor which connects the receptors with a long term memory. The long term memory of the IPS is capable of storing and retaining of symbolic structures such as programs or lists.

The processor of an IPS has a set of elementary processes which allow it to call from long term memory the structures needed to process inputs. Upon completion of the processing the results are communicated to the environment by effectors (see figure 3). This model of an IPS was stimulated by and has its embodiment in the modern digital computer.

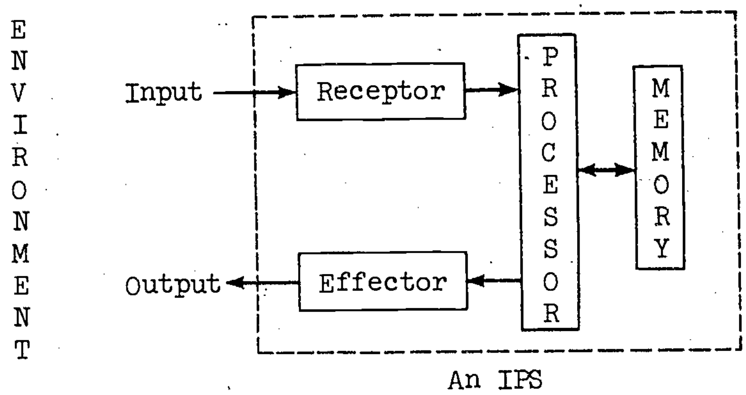

Figure 3. Representation of an information processing system

A simple IPS is a bimetallic thermostat. Its receptor is a bimetallic thermometer and its effector is a switch connected to a furnace. The processor is the bimetallic thermometer and the long term memory is fixed by the construction. A program might have the following steps:

1. Measure temperature, T.
2. If $T \geq 68°$, go to 1.
   If $T < 68°$, go to 3.
3. If $T \geq 66°$, go to 1.
   If $T < 66°$, go to 4.
4. Switch on furnace
5. Measure T.
6. If $T \geq 68°$, switch furnace off, go to 1.
   If $t < 68°$, go to 5.

Of course the IPS model is concerned with problems other that turning furnaces on and off. Paige and Simon (1966) have tested the theory as applied to algebra word problems. First a computer program was written to solve problems. Next, subjects are asked to solve a set of problems, talking aloud as they solve the problems. To test whether the program is a valid model of the problem solving procedure, the steps which the program executed are compared with the protocols of the subjects' solutions. Positive results have been obtained for algebra word problems, chess and symbolic logic problems.

As can be seen the theory is a non-statistical and highly content-specific at this time. Hallworth's (1969) comments point out the strengths and weaknesses of the theory. Because of the necessity of writing a program, the theory must be precisely stated and points out vagueness in other theories. He mentions that attempts to program what happens when a child passes from nonconservation to conservation in Piaget's theory, point to vagueness in Piaget's theory. The IPS theory is questionable in that its validity has been tested only in a few cases. Since the theory of solution of a specific problem is embodied in the computer program, the highly specific nature of a computer language limits the usefulness of theory.

The IPS theory brings many problems not usually thought of as algorithmic into the realm of algorithmic problem solving. If we assume that any programmable process is an algorithm, then the work in this area has greatly expanded the number of problems amenable to algorithmic solution. Algebra word problems are not usually thought of as being algorithmically solved. Certainly this area of research opens the possibility of finding new problem-solving techniques which may be easier to teach than those currently used.

A Model for Algorithmic Problem Solving

The model which is proposed for algorithmic problem solving is based partially on the behaviorist and the information processing theories of problem solving (see Figure 4). The problem provides a stimulus for the solver which causes him to select an algorithm from the set of all algorithms known to him. These algorithms can be thought of as stored in the solver's long-term memory. The selection may be based on previous instruction or non-previous trial-and-error learning. The algorithm then is tested as to its applicability to the problem. This test may be a simple multiple-discrimination task or involve some operating and testing (that is, there may be a testing algorithm). If the algorithm fits the problem, then it is applied to the problem and a solution is found. If the algorithm does not fit the problem, then it is applied to the problem and a solution is found. If the algorithm does not fit the problem, then another algorithm is selected and tested. The solver may fail to generate any algorithms for testing; in this case he then attempts to restructure the problem or to discover a new algorithm for the problem. If he is successful in changing the structure of the problem, he again selects and tests algorithms. A part of discovering a new algorithm would be the testing of its appropriateness. If he is unsuccessful in restructuring the problem or finding a new algorithm, he has failed to solve the problem. It should be noted that this model is not intended as a comprehensive model of problem solving, but it is intended as a model of algorithmic problem solving.

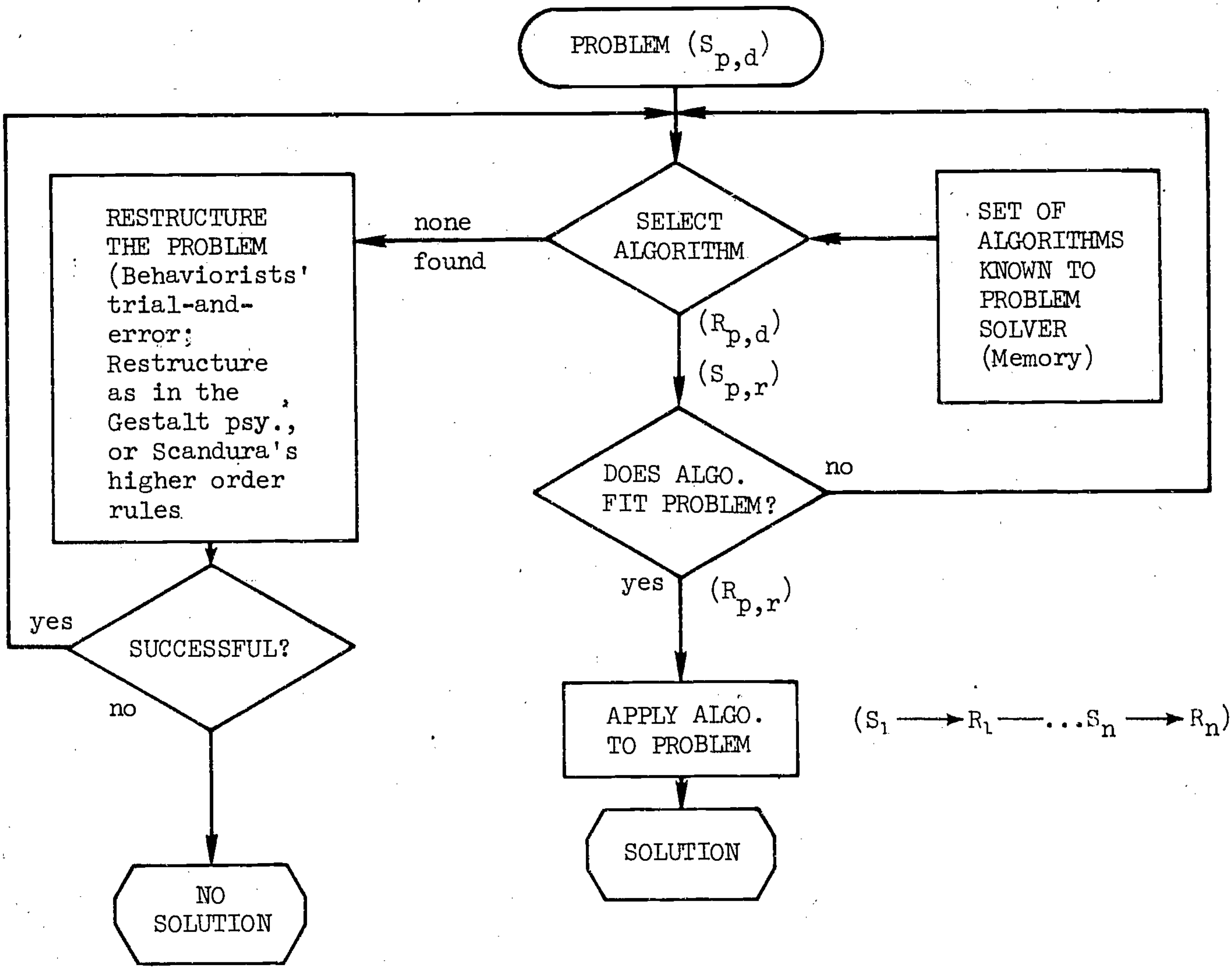

Figure 4. A proposed model for algorithm problem solving

Consider the following example of how a person reacts to a problem:

$$\text{Solve} \quad x^2 - 6x + 8 = 1.$$

1. Select algorithm.	None apply. Quadratic equations have form $ax^2 + bx + c = 0$
2. Restructure problem	$x^2 - 6x + 7 = 0$
3. Select algorithm.	Factor quadratic
4. Test algorithm.	Attempt to factor: $(x-7)(x-1) = x^2 - 8x + 7 \neq x^2 - 6x + 7$ No other seems to apply.

5. Select algorithm — Quadratic Formula

6. Test algorithm — Equation has form $ax^2 + bx = c = 0$ thus quadratic formula applies

7. Apply algorithm

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 - 6x + 7 = 0 \Rightarrow x = \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 7}}{2 \cdot 1}$$

$$= \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2}$$

8. Solution — $x = 3 \pm \sqrt{2}$

The restructuring of the problem could be thought of in several different psychological contexts. The Gestaltists as exemplified by Wertheimer (1959) view problem solving as restructuring. By insight (often sudden), the problem solver "sees" the structure of the problem and thus is able to solve the problem. The behaviorist view of this activity would be that if a problem does not elicit the behavior necessary for the solution, then trial-and-error behavior would occur. The trial-and-error behavior results in experience which relates the problem to a known rule and hence the solution.

Scandura's (1974) theory of behavior relates to what may occur during restructuring. His assumptions about problem solving are:

(1) If a person has a rule which allows achievement of the goal, then he will apply it.

(2) If a person does not have a rule for achieving a goal, then he will shift to a higher order goal of deriving a precedure which will satisfy the original goal.

(3) If the higher goal is satisfied, then the person returns to the original goal.

He feels that the ability to solve problems primarily depends on the presence or absence of higher order capabilities which make it possible to combine the parts of a problem into a whole which is adequate for solving the problem. It is necessary to have both simple rules (algorithms) and higher order rules which make it possible to modify simple rules. In Scandura's (1972) theory:

> ... rules are the basic building block of <u>all</u> mathematical knowledge, and that if looked at in the right way, <u>all</u> mathematical behavior is rule-governed. (p. 142)

Neither the Gestaltists nor the behaviorists theories suggest how instruction in restructuring might be done. Scandura (1972) suggests one way this might be done.

Some Questions for Mathematics Instruction

If the above model is valid, it points to certain areas of mathematics instruction which may be in need of revision. From elementary school through college the primary emphasis of the curriculum is the achievement of proficiency in carrying out the steps of algorithms. The result is that we see "solutions" for problems such as this:

Problem: Solve $(x+4)(x+3) > 0$

Solution: $(x+4)(x+3) > 0$

$\Rightarrow x + 4 > 0$ and/or (?) $x + 3 > 0$

The student often applies a familiar algorithm indiscriminantly without consideration of its appropriateness. We ask, does current instruction emphasize the importance of considering which algorithm is appropriate for a given problem?

Often several algorithms are presented for solution of a problem. For example, in algebra students will be taught to solve quadratic equations by factoring, completing the square, and the quadratic formula.

Consider the following problem:

Solve: $x^2 + 4x - 9 = 0$

Solution 1 (by the quadratic formula):

$$x = \frac{-4 \pm \sqrt{(4)^2 - (4)(1)(-9)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 + 36}}{2}$$

$$= \frac{-4 \pm \sqrt{52}}{2}$$

$$= \frac{-4 \pm \sqrt{4.13}}{2}$$

$$= \frac{-4 \pm 2\sqrt{13}}{2}$$

$$= -2 \pm \sqrt{13}$$

Solution 2 (by completing the square):

$$x^2 + 4x = 9$$

$$x^2 + 4x + 4 = 9 + 4$$

$$(x+2)^2 = 13$$

$$x + 2 = \pm \sqrt{13}$$

$$x = -2 \pm \sqrt{13}$$

Even though the efficiency of Solution 2 as compared with Solution 1 is obvious, when given the choice of algorithms students will usually use the one most recently studied. A similar example can be seen in the solution of systems of linear equations where we teach both substitution and elimination algorithms. In calculus we teach both the quotient rule and the product rule for differentiation; seldom, however, do we discuss when a quotient could be transformed into a product and more easily differentiated. How often do mathematics teachers give instruction in how to choose the most efficient algorithm for a problem?

Computing continues to have an increasing impact on mathematics. A recent IBM advertisement notes that 100,000 multiplications which cost $1.26 on a 1952 computer now cost one cent. The daily newspaper has advertisements for four-function hand-held electronic calculators costing $19.95 and "electronic slide rules" costing less than $100. These economic changes point to computing's continued growth in importance; it will be necessary for an accompanying change in mathematics instruction. The selection of the appropriate algorithm and the most efficient algorithm is important in computer applications. The need for human computers will continue to decline. At the same time the need for problem solvers who are able intelligently to apply algorithms will grow.

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Algorithms and Mental Computation

Raymond Zepp

There is no scarcity of journal articles which call for increased instruction on mental computation in the elementary school. These articles, for the most part, consist of high-minded but vague appeals to the necessity for grocery-store arithmetic and the like. A typical example of this kind of article is one by Koenker (1961) in the Arithmetic Teacher. His conclusions are:

> A ten minute daily period devoted to mental arithemtic would prove of great value in preparing a child for his present and adult out-of-school number experience. It would also help the child develop arithmetical understanding which cannot be taught by a pencil-and-paper type of arithmetic alone. (p. 296)

It is difficult to argue with the reasonableness of Koenker's statements. However, before we wholeheartedly launch programs of mental computation, we must answer precisely some specific questions. These questions fall into three categories as follows:

1. Is mental computation a well-defined topic? In a sense, all arithmetic is mental. Perhaps mental computation algorithms are essentially the same ones used with paper and pencil and involve precisely the same mental processes; then traditional written drill would be sufficient to increase nonwritten computational ability.

2. Can the algorithms involved in mental computation be taught at all? Perhaps it is an ability which, similar to IQ, cannot be cultivated beyond certain narrow limits. In other words, time devoted to mental computation may be totally wasted.

3. What are the effects of instruction in mental computation on the child's overall growth in arithmetic? Does knowledge of mental algorithms transfer to use of written algorithms? Would a student be more or less likely to gain the fundamental mathematical understanding and insight deemed so important in elementary education?

Let us examine these questions one-by-one.

1. Is mental computation a legitimate topic?

The least one can say is that there seems to be a concensus on the meaning of the term "mental computation" (or "mental arithmetic"). Hall (1954) noted that the term "mental arithmetic" appears as a separate listing both in the Education Index and in Webster's New International Dictionary, where it is defined as "the solution of arithmetical problems by mental processes, unassisted by written figures." Hall elaborates that the term should include "problems in which quick estimations are made which either may or may not be verified by a written response."

But the question of whether the algorithms of, along with the mental processes used in, mental computation are essentially different from written arithmetic, is more difficult to answer. Clearly, the two abilities will correlate highly, since many prerequisites, such as knowledge of multiplication tables, are common to both. There do, however, seem to be certain techniques in which the two differ. Flournoy (1959) attempted to differentiate between mental and written algorithms. For instance, to add 34 + 48, the algorithm which demands the adding of 8 + 4, carrying 1, etc. is much less amenable to mental computation than a procedure of adding 30 + 40 and then adding 12. Flournoy used her classification in a study which showed that many students, who were forced to use mental computation, automatically shifted to mental algorithms. Before and after a unit of instruction in mental computation, 150 pupils wrote explanations of the algorithms they used. For instance, in adding 43 and 28, a pupil stated, "I added 3 and 8. This is 11. I put down 1 and carried 1. This made 1 + 4 + 2, which is 7. I got 71 for an answer" (p. 137). This was classified as a written-type algorithm.

Before the instruction, 85 percent of the students used "paper-and pencil" thought. After the instruction, only 23 percent did. However, "there was very little change to the shorter or different ways of thinking when dividing whole numbers" (p. 138).

A summary of the algorithms used is the following:

> After instruction in how to add without using paper and pencil, pupils were using 11 different ways... In adding 34 and 48, the majority of pupils were using one of two methods:
> a) 30 + 40 = 70; 8 + 4 = 12; 70 + 12 = 82.
> b) 30 + 48 = 78; 78 + 4 = 82.
>
> After instruction in how to subtract without paper and pencil, pupils were using 10 different methods. In subtracting 24 from 62, the majority of pupils were using one method of thinking which was: 62 - 20 = 42; 42 - 4 = 38.
>
> After instruction on how to multiply without paper and pencil, pupils were using about 5 different ways of arriving at an answer. For the example, 16 x 11, the majority of pupils were using one of two methods. Almost half of the 150 pupils used this method:
> 16 x 10 = 160; 16 x 1 = 16; 160 + 16 = 176.
> And about half the pupils multiplied 11 by one-half the multiplier; then doubled the answer, as 8 x 11 = 88; 2 x 88 = 176.
>
> After instruction on how to divide without using paper and pencil, pupils generally used two ways of arriving at an answer

to a division example. While the majority of pupils used the regular pencil and paper procedure for dividing 174 by 3, a few pupils (about 10%) were using the following procedure:

$150 \div 3 = 50 \quad 24 \div 3 = 8 \quad 50 + 8 = 58.$

(Flournoy, 1959, p. 138)

Some correlational data have been collected on the relationship of ability in mental computation to other variables. Whimbey, et al. (1969) have tried to demonstrate an extremely close relationship to memory span. In two different experiments, college students took an ETS digit-span test along with a mental arithmetic test constructed by the author. In the first group, the tests "correlated .77, which, corrected for .87 reliability, gave .95 correlation." The precise nature of the correction was not stated. A similar result held for the second group of students. However, these results are only speciously convincing: if one takes a closer look at the tests, one finds mostly questions of the form "you have 8A, 3B, 2C, and 5D, and you add to this 2B and 5D, how many of each category do you now have?" on the mental test. It seems as though the so-called "mental arithmetic test" was constructed with the sole purpose of correlating with a digit-span test.

Better correlational data can be found. Perhaps the most far-reaching and experimentally rigorous research in the field is a study by Olander and Brown (1959). Seventeen-hundred students from grades 6 through 12 took a test of subtraction problems of 2 to 4 digits administered either orally or by flashcards. They also took a digit-span test, and scores on intelligence tests as well as Stanford-Achievement Tests were available. Olander and Brown noted:

(1) In relation to memory span--Before this study began it was assumed that ability in mental arithmetic was dependent to a considerable extent on a person's memory span. However, this expectation was not borne out by the results. The correlation between proficiency in mental arithmetic and memory span was found to be only .35.

(2) In relation to general arithmetic achievement--Based upon results in only grades 6 through 9, the correlation between ability in general arithmetic and mental arithmetic was .65. Compared with the correlation of .50 between intelligence and mental arithmetic, this is a high correlation, apparently indicating mental arithmetic is more dependent upon general arithmetic ability than it is upon intelligence.

(3) In relation to sex--Boys excelled girls in mental arithmetic. Girls showed superiority when paper and pencils could be used, though the difference was not significantly different.

It seems fair to say, then, that facility in mental arithmetic, although related to general arithmetic ability, is by no means the same thing.

2. <u>Can proficiency in mental arithmetic be effectively taught in the schools?</u>

Owing to the relationship between mental and general arithmetic mentioned above, it is natural to expect that some general arithmetic concepts must be mastered before mental computation can occur. Pigge (1967), in a study with 18 classes of fifth graders, compared three teaching methods: Method A-- 75% of instructional time was devoted to development and meaningful discussion versus 25% drill; Method B-- 50% development, 50% drill; and Method C-- 25% development, 75% drill. In the pretest and posttest and later recall test of addition and subtraction problems, nothing was said about the use of pencil and paper. On the recall test one month later, pupils displayed a partial reversion to written calculations. The conclusion was that: drill in written arithmetic seems to cause students to begin solving problems mentally. However, the experiment said nothing about the <u>accuracy</u> of the mental solutions. Furthermore, one might ask what would have happened if the students were <u>asked</u> not to use pencil and paper.

But what research has been done on the efficacy of direct instruction in mental computation? Quite convincing evidence has been submitted by Flournoy. In one study (Flournoy, 1959), a sixth-grade class spent 10 minutes of each arithmetic class for two months on mental exercises. A pretest and posttest in mental computation were given. The mean pretest score was 8.84, whereas the mean posttest score was 13.85, significant at the .01 level.

In the same study, classes of sixth graders were given three weeks of instruction in estimating and interpreting answers. A typical problem was to estimate or give an example of a distance of 250 miles. Practice in rounding numbers was also given. As compared with a control group, scores on a test of such problems were significantly higher. Out of 18 questions, the mean was 15.0 as compared with 9.2 for the control group. Flournoy's conclusion was that the skill of estimation not only can be taught, but "has to be taught, it isn't just caught."

Flournoy's previous study (1954) is similar, and even more striking. Five-hundred-fifty sixth graders were given 10 to 12 minutes per day of instruction in mental computation. Tests in both mental computation and problem-solving were administered before and after the treatment. All classes showed significant increases on both tests at levels from .05 to .001. Perhaps even more important was the fact that both fast and slow pupils showed increases. This would tend to dispel the thought that only bright students can learn mental computation skills. Dramatic results such as these appear to answer question 2 in the affirmative.

3. What effect does instruction in mental computation have on a child's growth in arithmetic?

A number of studies show positive results. In fact, this writer could find no studies with negative or even neutral results.

Flournoy's 1954 study cited above showed that in six of the classes drawn randomly from the 20, pretests and posttests on written computation and written problem-solving showed differences significant at the .01 level. Further, the Iowa Test of Basic Skills in Arithmetic Problem Solving revealed significantly more than average growth in arithmetic over the two months.

In a study by Hall (1942) 40 fifth- and sixth-grade students brought in their own practical oral problems to be solved mentally for 15 to 20 minutes each day. Unit Scales of Attainment were administered before and after the treatment. The results are listed on the next page. The figures are impressive for less than one year of instruction. Notice also that growth was relatively uniform over ability grouping.

Results of Hall (1942)

median IQ	Sept. 1941	Apr. 1942	Net Gain	
94	4-1	5-4	1-3	grade V
105	4-6	6-1	1-5	grade V
113	5-4	6-5	1-1	grade V
95	5-0	6-2	1-2	grade VI
104	5-4	6-9	1-5	grade VI
112	5-9	7-3	1-6	grade VI

A Unit Scale of Attainment score of 4-1 is to be interpreted as a mathematical growth level of 4th grade, 1st month.

In a study by Austin (1970), one teacher's seventh- and eighth-grade classes set aside one class period per week for mental computation problems made up by the students, for example, 8 + 4 x 2 - 3 x 6, etc. The scores of a random sample of 100 boys and 100 girls were compared to those of a control group on the SRA achievement test. A significant difference at the .01 level was found. The experimenter noted that the teacher variable was not controlled and may have been a factor. Another uncontrolled factor was the effect of a modern mathematics curriculum. No significant interactions of groups and IQ or of groups and sex were found.

Rea and French (1972) administered the SRA achievement test before and after a twenty-four-day period during which a sixth-grade class spent approximately 15 minutes per day using Kramer's Mental Computation Series. Although there was no rigorous statistical analysis, the class ($n = 13$) did show an average growth of eight months over the two-month period. Such striking results cannot be taken lightly.

Schall (1969) gave 399 fifth graders a pretest and a posttest to students who were given two weeks of instruction in mental arithmetic. The tests were in attitude, mental arithmetic and arithmetic achievement. Attitude improved. Mental arithmetic ability improved, but not too significantly ($p > .10$). No significant gains were found in paper-and-pencil computation, but gains were found in problem-solving. Schall concluded that pupils were able to transfer skills and concepts better after the two weeks.

The precise reason for the increases in arithmetic growth exhibited above is not known. Various explanations have been offered.

1) Pigge (1967) stated: "It has often been stated that reliance on paper and pencil solutions alone can lead to automatic computation without requiring much thinking. On the other hand, it is believed that the thought processes required in mental arithmetic enable the children to better understand the numerical relationships" (p. 589).

2) Flournoy (1954) appeared to concur with a statement she attributed to Spitzer: "Mental arithmetic tends to emphasize significant aspects of the number system" (p. 148).

3) Rea and French (1972) imply that success was due to increased motivation of students, i.e., fun with mental arithmetic serves primarily as a motivational device to get students to enjoy mathematics.

4) Hall's 1942 article seems to emphasize the fact that the students made up their own problems. The numbers, therefore, acquired a personal meaning and relevance to the students.

In only the first two of these explanations is there implication of direct transfer of skill in using mental algorithms to skill in using written algorithms. Whatever the reason the data are consistent and fairly conclusive that mental computation instruction produces good results in general arithmetic growth.

<u>Conclusions</u>

The literature on mental computation is fairly consistent in its proclamation of the value-of teaching mental computation. Mental arithmetic, while closely allied with written arithmetic, is a topic in itself which can be effectively taught to both slow and fast learners. Moreover, instruction in mental computation has been shown to be of significant value in enhancing students' overall growth in arithmetic.

If there is a set of algorithms unique to mental computation, and if knowledge of those algorithms is useful to students in learning mathematics, then it follows that research should be done as to the best method of teaching those algorithms. There has been much research on methods of teaching written algorithms, but almost none on methods of teaching mental algorithms.

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IV. Reports of Recent Exploratory Research

Algorithmic Processes in Arithmetic and Logic

Jesse D. Parete

Introduction

This study is designed to meet two objectives. The first is to investigate the relationship that may exist between (1) school students' ability to formulate and use the rule-governed behaviors necessary for success on the Furth-type task and (2) their ability ~~to use~~ arithmetic algorithms. For this study, the first variable is theorized to be a measure of the subject's innate ability to process information in an algorithmic fashion.

The second objective is to investigate the effectiveness of teaching division of fractions by two different algorithms. In this same context, two strategies for teaching division of fractions will be tested. They consist of presenting both algorithms to the students in two different sequential orders. The purpose of testing these two strategies is to find out if the learning of one algorithm influences the learning of the other.

Related Research

Hans Furth, a psychologist interested in the development of human intelligence, has studied students' mental behaviors as they worked with concepts formed with the logic rules of conjunction and disjunction and the logic operation of negation (Furth, Youniss, and Ross, 1970). His subjects were elementary school students (grades 1-6).

The paradigm he used in his study consists of presenting subjects with statements such as 'x and y' where x and y are values of two attribute dimensions. For example, 'house and red' would be presented as 'It is a house and it is red.' Together with this statement, the subjects would be presented a picture of some colored object. For such a pair, the subjects would respond to whether or not the statement and the picture matched, i.e., whether or not the picture was a positive exemplar of the concept being expressed. With a statement such as 'x and y', four task items are presented: one item for each possible truth value case (TT, TF, FT, and FF) of a bidimensional rule.

Furth tested his subjects using the following concepts: 'x and y', '$\overline{x}$ and y', '$\overline{x}$ and $\overline{y}$', 'x or y', '$\overline{x}$ or y', and '$\overline{x}$ or $\overline{y}$' (where $\overline{}$ means negation.) By analyzing consistent responses on certain item types, Furth sorted the subjects into three distinct groups.

To illustrate the item types, the four items formed from the concept '$\overline{x}$ and y' will be analyzed (see Table 1).

Table 1

Statement	Correct Response	Picture Instance	Truth Table Cases	Picture Instance Cases
1) Not a house or it is not red	match	red house	TT	s'c'
2) "	match	blue house	TF	s'c
3) "	match	red car	FT	s c'
4) "	not a match	blue car	FF	s c

Note: The symbol s indicates that the relevant value of the shape dimension (house in this case) is present in the picture and s' indicates that it is absent. Similarly for the color (red) c and c'.

The most primitive level of responding was found with responses to items like 1 and 4 in Table 1. One group of subjects (Level 1 subjects) consistently responds with a match for item 4 and not a match for item 1. This type of response was caused by an attribute present (or absent) factor and not the logical truth value of the instance or the concept it was to exemplify. In item 1 the relevant attributes, house and red, were absent (s' c') so the subjects in this group gave a negative response; in item 4 they were present (s c) and a positive response was elicited.

The second level of subjects (Level 2 subjects) demonstrated another consistent pattern of behavior which Furth interpreted as a transitional stage in the ability to deal with the relation of logical truth. These subjects could successfully answer item types 1 and 4 no matter which of the six concepts they were dealing with. Note that the truth value cases (TT and FF) are concordant for these two items while in items 2 and 3 they are discordant (TF and FT). Furth concluded that the subjects were beginning to deal with the relation of logical truth and that they were not able to apply their new skill in the discordant cases because the memory load interfered with information processing. The third level (Level 3 subjects) had little trouble with the task of dealing with the relation of logical truth.

The number of students in any level was related to the grade level. Because of this, Furth drew his conclusion that the three levels represented a developmental sequence.

These three levels also reflect the behavior of adult subjects applying a problem-solving strategy in "rule learning" tasks. Bourne (1970) analyzed this strategy as the application of three steps:

(1) identifying the truth value of the attribute dimensions, (2) placing the exemplar (or nonexemplar) instance into one of the four truth value cases, and (3) attaching a correct truth value to correspond to the rule. His experiments provide evidence for such a model by the behaviors that subjects exhibited. He also cited studies indicating that the subjects' application of this strategy can be enhanced by pre-training the subjects on the subtasks or steps (Haygood and Kiehlbauch, 1965; Bower and King, 1967; Bourne and Guy, 1968).

Bourne (1967) interprets this behavior as rule-governed. It may be applied to any one of the four bidimensional rules, conjunction, disjunction, biconditional, and conditional. It is like an algorithm in arithmetic. The subjects are not conscious of the reasoning behind their behaviors, but apply the rules to solve problems.

Capps (1963) and Bidwell (1968) report that different methods of teaching division of fractions influenced students' achievement on multiplication of fractions. The common denominator method caused interference. There does not seem to be any research on the effects of teaching multiple solutions for division of fractions and, thus, any research on how one algorithm may influence learning of another.

Hypotheses

The null hypotheses to be tested in the present study are:

H1. There are no differences among three groups of subjects formed by analysis of responses on a Furth-type logic test with regard to the likelihood that they will use both algorithms taught for solving division of fractions problems.

H2. There is no difference in achievement on a division of fractions test between students taught the inverse algorithm and students taught the complex fraction algorithm.

H3. For groups of students identified at each of the three levels of performance on the logic test, there is no difference in achievement on a division of fractions test between students taught the inverse algorithm and students taught the complex fraction algorithm.

H4. There is no difference in achievement on a division of fractions test between students taught the inverse algorithm followed by the complex fraction algorithm and those taught the same algorithm in reverse order.

H5. For groups of students identified at each of the three levels of performance on the logic test, there is no difference in achievement on the division of fractions test between students taught the inverse algorithms in reverse order.

Subjects

The subjects were 53 sixth-grade students in two classes in one elementary school in Columbus, Ohio. Their social background was predominantly of the lower socioeconomic level. There were nearly equal numbers of boys and girls, with ages ranging from 11.3 years to 13.2 years and IQ measures (for those available) ranging from 74 to 116.

Tests

A 24-item logic test was constructed with items similar to those used in the Furth paradigm. The same six concepts Furth worked with were used in the construction of items. For each concept, there were four items, one for each of the four possible truth value cases (TT, TF, FT, and FF). The two dimensions used were shape and color and the relevant attributes on all items were "house" for the shape and "red" for the color. A reliability of .59 (n = 37) was obtained for this test using the Hoyt Anova procedure. Factor analysis was used to validate the claim that this test could sort subjects into the three different types of behaviors as outlined in the introduction. On each of a three-, four-, five-, and six-factor analysis, one factor could be labeled as a "Level 1" factor and one a "Level 2" factor based on the dominance of item types associated with the given level. Since Level 3 subjects respond to almost all item types correctly, no factor was expected to reflect their behavior. Other factors obtained reflected differences between the conjunctive and disjunctive rules.

Two division tests were prepared to measure student achievement after instruction. The first test (D1) contained 15 problems all written in the following form: $a/b \div c/d$. The second test (D2) also contained 15 problems of which 11 were written in the same form as those on D1. Two problems were written in the complex fraction form and two were written as the equation which is used in the inverse algorithm solution. Hoyt Anova reliability coefficients for these two tests were .88 (n = 37) and .94 (n = 37), respectively.

Experimental Design

Subjects were grouped into one of three levels of performance (L1, L2, or L3) attained on the logic test. Within each level, subjects were randomly assigned to one of two treatment groups (T1 and T2). To test hypotheses H2 and H3, T1 consisted of instruction on the division of fractions with the inverse algorithm and T2 consisted of instruction with the complex fraction algorithm. To test hypotheses H4 and H5, T1 consisted of instruction on the division of fractions with the inverse algorithm followed by instruction on the complex fraction algorithm, plus the influence of test D1 given immediately after the instructional period for the first algorithm. Treatment T2 was the same except for the order of the algorithms. Both treatment groups were taught by the researcher. Table 2 presents the experimental design model.

Table 2. Experimental Design

Day 1	Day 2-5	Day 6	Day 7-10	Day 11
Logic Test	Instruction with Algorithm I	Test D1	Instruction with Algorithm C	Test D2
	Instruction with Algorithm C		Instruction with Algorithm I	

Algorithm I
(Inverse)

$1/3 \div 1/2 [\ ]$

$1/2 = [\ ] \times 1/2$

$1/3 = [1/3 \times 2/1] \times 1/2$

Ans. $1/3 \times 2/1 = 2/3$

Algorithm C
(Complex Fraction)

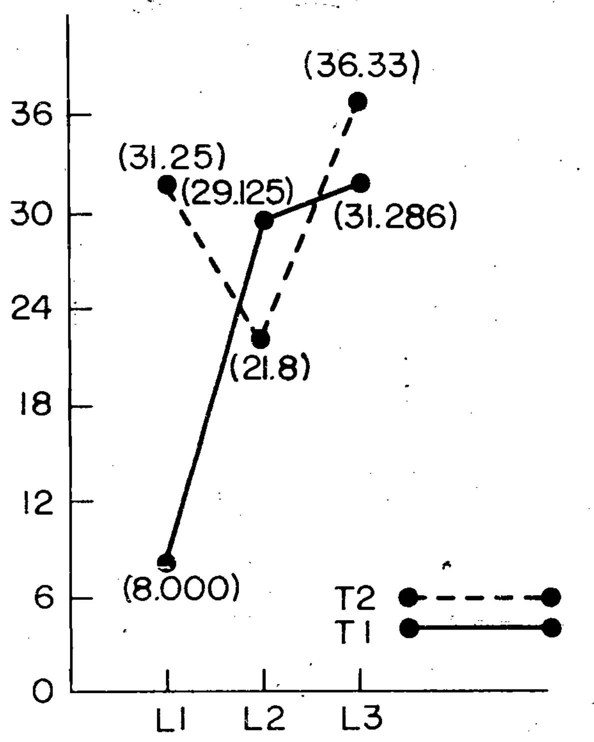
$1/3 \div 1/2 = [\ ]$

$\frac{1/3}{1/2} =$

$\frac{1/3}{1/2} \times \frac{2/1}{2/1} = \frac{2/3}{1} = 2/3$

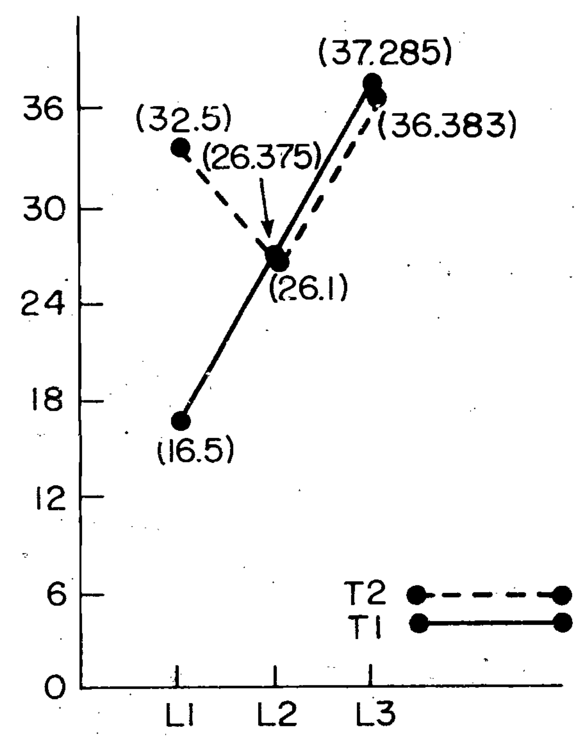
The random assignment of individuals to treatment groups was used in order that individuals rather than classes could be used as the experimental unit. Unfortunately, it was impossible to instruct subjects from both classes assigned to the same treatment group at a common time. Instead, four instructional groups were formed. Treatment T1 was administered twice; once to those subjects in one class assigned to T1 and once to those subjects assigned to T1 from the second class. Similarly, T2 was administered twice daily. The data were analyzed as if the random assignment into treatment groups had been achieved. This liberty with statistical assumptions was forced by the small sample size in this experiment. It was rationalized with two factors in mind. First, the researcher taught both classes for four weeks prior to administering the treatments and introduced all the prerequisite concepts for division with fractional numbers. Second, the researcher administered each treatment.

For the statistical design (see Table 3), the logic test was used as a blocking variable.



T1 - Inverse Algorithm
T2 - Complex Fraction Algorithm

Figure 1. Interaction on Test D1.



T1 - Inverse Followed by Complex Fraction Algorithm
T2 - Complex Fraction Followed by Inverse Algorithm

Figure 2. Interaction on Test D2.

Table 3. Statistical Design

	L1	L2	L3
T1	G_{11} (n = 2)	G_{12} (n = 8)	G_{13} (n = 7)
T2	G_{21} (n = 4)	G_{22} (n = 10)	G_{23} (n = 6)

A 2x3 factorial design using analysis of variance (SOUPAC)[1] was used to analyze the effects of the two treatment variables and the three levels obtained from the logic test. This design was run on each of the two division tests D1 and D2.

To test the first hypothesis H1, a Chi square test was run using the levels of performance on the logic test (L1, L2, and L3) as the independent variable and performance on four specially prepared items placed on the second division test D2 as the dependent variable. Two items were written in the form of the second step of the complex fraction algorithm and two were written in the form of the equation in the second step of the inverse algorithm (see Table 2). Success was considered to be achieved if the subject used the intended algorithm on all four problems.

Results

Results from the analysis of variance for the division tests D1 and D2 are given in Table 4.

Table 4. Analysis of Variance

		Test D1			Test D2		
Source of Variation	df	MS	F	P	MS	F	P
Treatments (T)	1	342.42	3.51	0.07	206.14	3.07	0.089
Logic Levels (L)	2	474.81	4.87	0.015	411.17	6.13	0.006
T x L	2	552.44	5.66	0.008	234.73	3.50	0.043
Within	31	97.55			67.05		
Total	36	1467.22			919.09		

In each analysis the interaction was significant beyond the .05 level and the graph of cell means indicated that the nature of the interaction was disordinal (see Figures 1 and 2). Therefore, only the simple effects at each of the three levels L1, L2, and L3 may be interpreted.

Critical values for differences in means at different levels of the logic measure were computed using the Dunn procedure for post hoc analysis. The critical value for the difference in means at L1 on test D1 was 30.68 at the .05 significance level; the observed difference was 23.125. At L2 on test D1, the critical differences were 16.829 and 11.243 at the .05 and .30 significance levels, respectively; the observed diffference was 7.325. At L3 the difference in means was very small, so no statistics were computed (since it was evident that there would be no statistical difference in the scores).

On the second division test, the only means that appeared different were at L1. The Dunn critical value at the .05 significance level for differences in means was 16.97; the observed difference was 17.0.

Tables 5 and 6 contain the test statistics for the division tests D1 and D2.

Table 5. Statistics for Test D1

	L1	L2	L3	
T1	M=8.00 (n=2)	M=29.13 (n=8)	M=31.29 (n=7)	M=27.53 (n=17)
	SD=7.06	SD=10.62	SD=6.55	SD=11.12
T2	M=31.25 (n=4)	M=21.8 (n=10)	M=36.33 (n=6)	M=28.05 (n=20)
	SD=4,11	SD=14.01	SD=4.72	SD=12.08
	M=23.50 (n=6)	M=25.06 (n=18)	M=33.62 (n=13)	M=27.81 (n=37)
	SD=12.82	SD=12.82	SD=6.13	SD=11.49

Table 6. Statistics for Test D2

	L1	L2	L3	
T1	M=16.5 (n=2)	M=36.83 (n=8)	M=37.29 (n=7)	M=29.71 (n=17)
	SD=0.71	SD=10.32	SD=4.42	SD=9.41
T2	M=33.5 (n=4)	M=26.1 (n=10)	M=36.83 (n=6)	M=30.8 (n=20)
	SD=1.29	SD=11.17	SD=4.17	SD=9.41
	M=27.83 (n=6)	M=26.22 (n=18)	M=37.08 (n=13)	M=30.29 (n=37)
	SD=8.84	SD=10.49	SD=4.13	SD=9.72

Levine's test for homogeneity of variance was run for both sets of data. This test indicated that the homogeneity condition was met ($p < .05$) to satisfy the analysis of variance model.

Table 7 presents the contingency table for subjects in groups L1, L2, and L3 who did or did not meet the criterion set for the four special problems from division test D2. A chi square of 4.23 (df = 2, $p < .125$) was calculated for this data. At the .125 significance level, group L3 was different from either L1 or L2, but L1 and L2 did not statistically differ from each other.

Table 7. Contingency Table for Special Problem Task on Division Test D2

	Meet Criterion	Failed Criterion	Total
L1	1	5	6
L2	4	14	18
L3	7	6	13
Total	12	25	37

Discussion

The biggest difference in mean scores appeared on division test D1 between the groups at the L1 level of the logic test (see Figure 1). These are the students who were judged to be least capable of processing information in an algorithmic fashion. They seemed to be quite successful using the complex fraction algorithm and quite unsuccessful with the inverse algorithm. The researcher served as the instructor throughout this experiment and it is his feeling that the subjects in the L1 group taught with the complex fraction algorithm were applying it rotely. The subjects in this level who were taught the inverse algorithm had trouble following the steps of this algorithm. The equation formed in the second step seemed to be a "trouble spot" for all students and therefore it was not as easily applied in the rote fashion in which the complex fraction algorithm appeared to be applied.

If the complex fraction algorithm is easier to perform, groups using it at each of the three levels should out-perform those using the inverse algorithm. While this appeared to be the case for subjects at L1 and L2, the reverse is true for level L2 subjects. The L2 subjects are those who were judged to be attempting to deal with the relationship of logical truth on the logic test. Most important is that, unlike the L1 subjects, they were attempting a meaningful solution to items on the logic test. It may be the case that the rationale for the inverse algorithm is easier to comprehend. For the inverse algorithm the subjects must understand the relationship between multiplication and division, i.e., that they are inverse operations. The complex fraction algorithm requires the students to deal with the formation of a complex fraction; this concept was new for them. There are no new concepts involved in the inverse algorithm; its rationale is built on concepts the students have already dealt with at one time or another.

If it can be assumed that the two algorithms differ in the manner described above, the differences in group means for test D1 at each level (L) reflect the characteristic measured by the logic test: the ability to perform with algorithmic processes. The L1 subjects were using an algorithm on the logic test as evidenced by their consistent responses to item types for which the instance patterns were sc or s'c'. But this is a rotely applied algorithm in that it lacked any external meaning. The level L2 subjects were attempting meaningful solutions. In so doing they became confused on items whose truth value case was discordant. The L1 subjects consistently answered some of these item-types correctly. For example, the statement, "Not a house and it is red" paired with a picture of a black house, is one such item-type. The truth value case for this item is TF, the pattern (picture) instance is s'c', and the correct response is "not a match." Thus those subjects attempting the meaningful solution on these types of items seemed to perform less well.

In the same fashion, subjects at level L1 using the complex fraction algorithm out-performed both groups at level L2 on division test D1.

The only group whose average fell after instruction with both algorithms (see Figure 2) was that group at L2 receiving instruction with the inverse followed by the complex fraction algorithms (T1). This tends to support the interpretations stated above concerning performance with the two algorithms. The fact that those subjects in the group L1 receiving instruction with the complex fraction algorithm followed by the inverse algorighm (T2) also gained on test D1 tends to detract from the interpretation of differences in the algorithm. Similarly, those in treatment group T1 at L1 did not gain as much as might be expected.

These events as well as those observed on the first test may be due to other factors such as individual differences not controlled for by the design of this experiment. The sample sizes for the two groups at L1 were 4 and 2, respectively— a small number of students from which to draw conclusions.

The results obtained for the special problems on test D2 were straightforward if not significant at the .05 level. The proportion of students meeting the criterion of success increased from L1 to L3; the proportions were .167, .222, and .538, respectively. These data add support to the theory that the logic test was measuring, in some way, subjects' ability to use algorithmic strategies.

The results of this small study warrant expansion to a larger scale. The use of classes as the statistical unit and IQ scores as a covariate could greatly increase the power of the statistics necessary to analyze data from a study of this nature. If there are strong ties between the logic test results and achievement in the arithmetic algorithmic setting studied that are accounted for by IQ, it could have significant implications for future educational practices. A construct of a higher-order skill more specific than general intelligence could be postulated. The construct of higher-order rules governing behavior in many domains has already been postulated and investigated by Scandura (1971). The higher-order skill postulated from the theory upon which this study is based is the facility to organize and process information in an efficient algorithmic manner. Again, assuming that this is a valid construct, instruction designed to develop this skill could help students improve in both the areas of logic and arithmetic and possibly other areas.

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A Comparison of Different Conceptual Bases for Teaching Subtraction of Integers

Diane Thomas

Purpose

There are a number of ways to interpret the operation of subtraction. In some situations, such as "If Kathy has 7 pieces of candy and eats 3 of them, how many does she have left?", subtraction becomes the process of taking away. In "How much more than 3 is 7?", subtraction is used to make a comparison between two numbers. Subtraction as a method of complementary addition is indicated in "What must be added to 3 in order to make 7?" When integers are considered, subtraction may be defined as an "adding the opposite" procedure. Each of these different interpretations can be thought of as forming a separate conceptual basis for the process of subtraction.

When students extend the number system they use from the whole numbers to the integers, and then consider subtraction as operating on these "new" numbers, the subtraction algorithm that they will use will be derived from one of these conceptual bases. The question of which conceptual base is most appropriate for the student's first introduction to subtracting integers was investigated in this study. Three algorithms, derived from three different conceptual bases, were compared in an attempt to ascertain which most facilitated student skill in computation.

Background

A variety of approaches are used in upper elementary-school textbooks for the first presentation of the topic of subtracting integers. In the 1972 Houghton Mifflin series, Modern School Mathematics--Structure and Use, subtraction of integers is introduced through the use of number patterns, eventually leading to the principle that subtracting an integer is the same as adding its opposite; the presentation is done in a completely symbolic mode. The 1968 Addison-Wesley series, Elementary School Mathematics, approaches subtraction through related addition facts ("Hence, in order to find a difference, we think of finding a missing addend." p. 368), and the presentation is in only the symbolic mode. In Modern Mathematics Through Discovery, the 1970 Silver Burdett series, the "finding a missing addend" approach also is used for an introduction to adding integers, but students are expected to use a number line in getting their answers; subtraction is presented in a like manner in the 1969 Ginn series, Essentials of Mathematics. The 1972 Laidlaw series, Progress in Mathematics, considers subtraction as adding the opposite and illustrates each problem with movement on the number line.

Similarly, methods texts and journal articles recommend varying approaches for the student's first encounter with subtracting integers. Butler and Wren, in The Teaching of Secondary Mathematics (1960), suggest that subtraction be defined as the process of finding a missing addend and that the number line be used as a vehicle for illustration. As another possible method, Butler and Wren include presenting subtraction of integers through the symbolic mode where number patterns are analyzed:

From	+8	+8	+8	+8	+8	+8	+8	etc.
Subtract	+3	+2	+1	0	-1	-2	-3	etc.
Difference	+5	+6	+7	+8	+9	+10	+11	etc.

(pp. 373-375)

Riedesel, in Guiding Discovery in Elementary School Mathematics (1967), opts for emphasizing that subtraction involves the idea of finding the difference between two numbers and suggests that word problems stressing the notion of distances above and below sea level, and of temperatures above and below zero, be used in conjunction with the number line for the student's first introduction to this concept (pp. 133-135). In the January 1973 issue of the Arithmetic Teacher, Werner discusses possible number line models of subtraction and concludes that the model involving finding the missing addend provides the smoothest transition from the system of whole numbers to the system of integers. Three suggestions for introducing subtraction of integers are made by Kennedy in his methods text, Guiding Children to Mathematical Discovery (1970): (1) using a number line with subtraction defined as finding the missing addend, (2) exploring the meaning of subtraction with integers by using Postman Stories that involve a mailman delivering and picking up bills and checks, and (3) employing David Page's method of using positive and negative money (pp. 381-383). Finally, Kennedy also recommends approaching the topic through a concrete mode by using pipe cleaner loops--a method first described by Fremont in a 1966 article in the Arithmetic Teacher. Fremont's method is summarized below:

> Pipe cleaners are used to represent positive and negative numbers; a pipe cleaner bent in this manner ⊃ represents +1 and one opening in the other direction represents -1. Subtraction is thought of as a take away process. The problem 4 - 3 would be worked as ⊅ ⊅ ⊅ ⊃ where slashes drawn through three of the loops indicate 3 have been taken away. In a problem such as 2 - 3, 2 is represented by ⊃ ⊃ . In order for three to be taken away, a zero is added to the 2, and the 2 is represented by ⊃ ⊃ ⊂ ⊂ . Now three are taken away-- ⊅ ⊅ ⊂ ⊄--leaving -1 as the result. (pp. 571-572)

Thus, many approaches to introducing subtraction of integers have been suggested to teachers and used in student textbooks; however, there has been little corresponding research into the relative merits of these different approaches. Two studies (Coltharp, 1969; Sawyer, 1973) investigated selected semiconcrete and abstract approaches to learning subtraction of integers; one study (Zelechoski, 1961) attempted to correlate learner characteristics with gain in knowledge of integers. Coltharp reported no significant differences in overall achievement between sixth graders taught addition and subtraction of integers from an abstract, algebraic approach through the use of ordered pairs of numbers and those taught by means of a visual approach through the use of the number line. However, only overall achievement was measured in Coltharp's work--there was no mention of the students' achievement in the specific area of subtraction. Sawyer compared achievement of seventh graders taught subtraction of integers by three different methods:

1. Complement method--method of subtraction by adding the same number to both the minuend and the subtrahend.

   Example: $(+5) - (-3) = ((+5) + (+3)) - ((-3) + (+3))$
   $= (+5) + (+3)$
   $= +8$

2. Related number facts method--method of subtraction involving the relationship between subtraction and addition.

   Example: $(+5) - (-3) = N$ iff $N + (-3) = (+5)$;
   therefore, $N = (+8)$
   (Number lines were used at the introduction of this method.)

3. Systems method--by examining a modular system, the student learns that $x - y = x + (-y)$. This is generalized to the integers.

   Example: $(+5) - (-3) = (+5) + (+3) = +8$

Results showed no consistent superiority of one method over another. Zelechoski found that for students in grades seven, eight, and nine, mental age correlated most highly with gain in knowledge of signed numbers, followed by algebra aptitude.

As pointed out by Sawyer in his study:

> ....There are many models for explaining subtraction of integers....(but) there seems to be no agreement as to which model is most easily used and retained by students. There does seem to be agreement that subtraction of integers is a troublesome area in mathematics as witnessed by the number of articles written on the subject. It seems that, because of the importance of subtraction of integers to the further study of mathematics and the concern of the people involved in the area, an investigation of the problem would be very important to the field of mathematics education. (p. 16)

Procedures

The present investigation attempted to compare three instructional treatments based on different conceptual bases for subtraction of integers to see how they affected student achievement on computation problems and to test for any interaction between instructional approach and student ability level. The three instructional treatments were defined as follows:

T1 -- Number line, subtraction as "adding the opposite"

A common algebraic definition of subtraction is given by:

In any ring R we define, for $a,b \in R$,

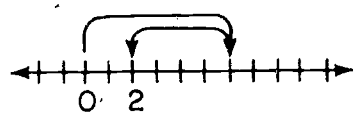
$$a - b = a + (-b)$$

(Introduction to Abstract Algebra, Dubisch, p. 41).

In T1, this definition was established by first using examples where the minuend and subtrahend were both whole numbers. The procedure used for adding integers on a number line served as the means for deriving the answer; for example:

$6 - 4 = 6 + (-4)$

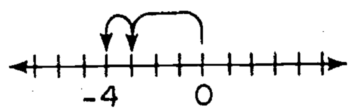
and $6 + (-4)$ is represented by



so $6 - 4 = 6 + (-4) = 2$

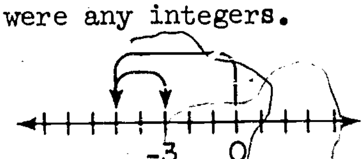
Then, the definition was used to obtain answers when the minuend was negative and the subtrahend was positive:

$$(-3) - 1 = (-3) + (-1) = (-4)$$



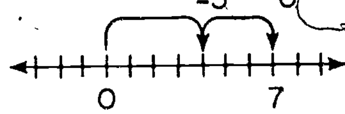
Finally, the definition was used to obtain answers when the minuend and subtrahend were any integers. Thus, for example,

$$(-5) - (-2) = (-5) + (+2) = (-3)$$



and

$$4 - (-3) = 4 + (+3) = 7$$

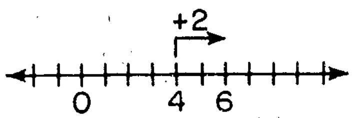


T2 -- Number line, subtraction as "finding the missing addend"

The rationale for this approach to subtraction of integers is based upon related addition facts and the commutative law:

$$6 - 4 = n \text{ may be rewritten as}$$
$$n + 4 = 6, \text{ or, equivalently,}$$
$$4 + n = 6.$$

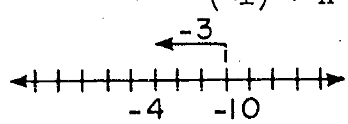
In this last form, the problem becomes one of locating the numbers 4 and 6 on the number line and determining the distance and the direction (the "missing addend") between the two points, starting from the subtrahend 4.



so $6 - 4 = 2$

The problem $(-4) - (-1)$ would be worked in the same manner:

$$(-4) - (-1) = n \text{ is equivalent to}$$
$$n + (-1) = 4$$
$$(-1) + n = 4$$



Since $(-1) + (-3) = (-4)$
we get that $n = (-3)$

T3 -- Number line with semiconcrete referent, subtraction as "take away"

Positive and negative integers are represented as follows:

⊂ = -1	+1 = ⊃
⊂⊂ = -2	+2 = ⊃⊃
⊂⊂⊂ = -3	+3 = ⊃⊃⊃
⋮ ⋮	⋮ ⋮

and the notions $x + 0 = x$ and $y + (-y) = 0$ are stressed.

Subtraction is viewed as a "take away" situation:

6 - 4 becomes ⊃⊃⊃⊃⊃⊃, take away ⊃⊃⊃⊃ or ⊅⊅⊅⊅⊃⊃, leaving ⊃⊃, or +2, as the result.

Whenever necessary, enough zeros are added to the picture of the minuend in order to facilitate the "take away" process:

1 - 4 means that we have $\supset$ and must take away $\supset \supset \supset \supset$; in order to do that, the 1 is represented as 1 + 0 + 0 + 0, or $\supset \subset \supset \subset \supset \subset \supset$. Then 1 - 4 becomes $\not\supset \subset \not\supset \subset \not\supset \subset \not\supset$ leaving $\supset \supset \supset$, or -3.

Similarly, (-3) - (-5) would be worked as

$$(-3) = (-3) + 0 + 0 = \subset \subset \subset \subset \supset \subset \supset$$

and $(-3) - (-5) = \not\subset \not\subset \not\subset \not\subset \supset \not\subset \supset = \supset \supset = 2.$

Two specific hypotheses were to be tested at the .05 level:

1) There is no significant differences between the three teaching approaches to subtracting integers, and

2) There is no significant interaction between student ability level and the instructional approach being used.

Three intact classes of sixth graders at one elementary school in the Columbus, Ohio public school system were used in the study. Students supposedly had been randomly assigned to classes before the study began. In order to control for the teacher variable, the investigator taught all three classes.

According to the regular classroom teachers, the topic of positive and negative numbers had not been previously discussed in their classes. The instructional unit on integers developed by the investigator lasted for one half-hour each day for each class, for a total of seven days. (See Appendix I for selected worksheets used in teaching the unit.) The activities for the first three days were the same for all three classes. On the first day, students were introduced to the concept of positive and negative numbers, they located integers on the number line, and they read coordinates of points already specified on the number line. A brief review was held on the second day, then a pretest on adding and subtracting integers was given to the students. The third day was devoted to adding integers, both with and without the use of a number line. Each class then was randomly assigned to one of three treatment groups for subtracting integers and on the fourth, fifth, and sixth days studied subtraction of integers according to the treatment specified. The choice of examples and problems worked during the introductory period on the fourth day by necessity was dictated by the treatment, so that students could begin with the easiest problems for that particular method. On the fifth and sixth days, all numerical examples used by the instructor during the class presentation, as well as all problems to be worked by each student, were the same for all three treatments. An attempt was made to teach all material meaningfully for the students in all of the groups. On the seventh day after a brief review of the subtraction procedures, students in each group were given the posttest on subtraction; upon completion of that portion of the test, they received the addition posttest.

Results of Analysis of Data

Since intact classes were assigned to the treatment groups, an attempt was made to determine whether the three classes were equivalent in ability to add and subtract integers before the treatment began. An investigator-constructed test of 12 items on adding integers (Kuder Richardson-20 estimate of reliability = .88) and of 12 items on subtracting integers (KR-20 estimate of reliability = .89) was given to all three classes. In analyzing the data, a blocking variable was used: student ability level as determined by scores on the California Test of Basic Skills--Arithmetic Level 2, given when the students were in the fifth grade. Raw scores ranging from 70 through 89 on the California Test were considered to indicate high ability, scores from 46 through 69 were classified as indicating average ability, and scores from 24 through 45 were considered to show low ability.

Table 1 shows the number of subjects per cell and per treatment level; Tables 2 and 3 show section means (weighted and unweighted) and standard deviations (unbiased) on the addition pretest and on the subtraction pretest.

Table 1. Number of Students Participating

		T1 Add the Opposite	T2 Missing Addend	T3 Take Away	Totals
Ability Level	High	7	3	7	17
	Average	8	6	9	23
	Low	5	6	6	17
	Totals	20	15	22	57

Table 2. Addition Pretest Section Means and Standard Deviations

Ability Level	T1	T2	T3	Level Means Level SDs	Unweighted Level Means
High	$\bar{X}_{11} = 5.43$ $\sigma = 1.99$	$\bar{X}_{12} = 6.33$ $\sigma = 3.52$	$\bar{X}_{13} = 8.57$ $\sigma = 1.90$	$\bar{X}_{11} = 6.88$ $\sigma = 2.57$	6.78
Average	$\bar{X}_{21} = 6.00$ $\sigma = 2.78$	$\bar{X}_{22} = 5.33$ $\sigma = 1.21$	$\bar{X}_{23} = 5.22$ $\sigma = .67$	$\bar{X}_{21} = 5.52$ $\sigma = 1.75$	5.52
Low	$\bar{X}_{31} = 4.20$ $\sigma = .84$	$\bar{X}_{32} = 4.50$ $\sigma = 1.64$	$\bar{X}_{33} = 6.00$ $\sigma = 2.61$	$X_{31} = 4.94$ $\sigma = 1.95$	4.90
Level Means SDs	$\bar{X}_{11} = 5.35$ $\sigma = 2.18$	$\bar{X}_{12} = 5.20$ $\sigma = 1.93$	$\bar{X}_{13} = 6.50$ $\sigma = 2.24$	$\bar{X}_{11} = 5.75$ $\sigma = 2.19$	
Unweighted Level Means	5.21	5.39	6.60	Unweighted Overall Mean	5.73

Table 3. Subtraction Pretest Section Means and Standard Deviations

Ability Level	T1	T2	T3	Level Means Level SDs	Unweighted Level Means
High	$\bar{X}_{11} = 4.14$ $\sigma = 1.07$	$\bar{X}_{12} = 4.67$ $\sigma = 1.53$	$\bar{X}_{13} = 4.71$ $\sigma = 1.70$	$\bar{X}_{11} = 4.47$ $\sigma = 1.37$	4.51
Average	$\bar{X}_{21} = 4.25$ $\sigma = 1.98$	$\bar{X}_{22} = 3.83$ $\sigma = 1.47$	$\bar{X}_{23} = 4.44$ $\sigma = 1.67$	$\bar{X}_{21} = 4.22$ $\sigma = 1.68$	4.18
Low	$\bar{X}_{31} = 4.60$ $\sigma = .89$	$\bar{X}_{32} = 4.50$ $\sigma = 1.52$	$\bar{X}_{33} = 3.33$ $\sigma = .82$	$\bar{X}_{31} = 4.12$ $\sigma = 1.22$	4.14
Level Means SDs	$\bar{X}_{11} = 4.30$ $\sigma = 1.42$	$\bar{X}_{12} = 4.27$ $\sigma = 1.44$	$\bar{X}_{13} = 4.23$ $\sigma = 1.54$	$\bar{X}_{11} = 4.26$ $\sigma = 1.45$	
Unweighted Level Means	4.33	4.33	4.16	Unweighted Overall Mean	4.28

A two-way analysis of variance was run on the addition pretest scores and on the subtraction pretest scores as a check to see if the three different classes possibly were not equivalent at the beginning of the instructional treatments. Unweighted means were used in the calculations. Summaries are given in Tables 4 and 5.

Table 4. Analysis of Variance of Addition Pretest Scores by Teaching Approach and by Student Ability Level

Source	df	SS	MS	F	Prob.
A (Teaching Approach)	2	19.8002	9.9001	2.5647	.087
B (Ability Level)	2	31.7968	15.8984	4.1187*	.022
AB (Method X Ability)	4	23.3085	5.8271	1.5096	.214
S/AB (Error)	48	185.2841	3.8601		
Total	56	260.1896			

*p < .05

Table 5. Analysis of Variance of Subtraction Pretest Scores by Teaching Approach and by Student Ability Level

Source	df	SS	MS	F	Prob.
A (Teaching Approach)	2	.3272	.1636	.0730	.930
B (Ability Level	2	1.4083	.7042	.3143	.732
AB (Method X Ability)	4	7.7047	1.9262	.8597	.495
S/AB (Error)	48	107.5413	2.2404		
Total	56	116.9815			

As shown in Table 4, no significant main effects were found for the teaching approach variable, nor were there any significant interaction effects. Similarly, the ANOVA for the subtraction pretest scores revealed no significant interaction effects. Thus, we have no evidence to say that the three treatment groups were not equivalent prior to instruction in adding and subtracting integers.

A two-way ANOVA then was used in analyzing student scores on the addition posttest and on the subtraction posttest. Table 6 shows means (weighted and unweighted) and standard deviations (unbiased) for scores on the addition posttest; Table 7 summarizes the analysis of variance performed on the addition posttest scores.

Table 6. Addition Posttest Section Means and Standard Deviations

Ability Level	T1	T2	T3	Level Means Level SDs	Unweighted Level Means
High	$\bar{X}_{11} = 9.14$, $\sigma = 2.85$	$\bar{X}_{12} = 9.00$, $\sigma = 4.36$	$\bar{X}_{13} = 8.56$, $\sigma = 2.91$	$\bar{X}_{11} = 9.00$, $\sigma = 2.94$	9.00
Average	$\bar{X}_{21} = 8.13$, $\sigma = 3.56$	$\bar{X}_{22} = 6.33$, $\sigma = 3.26$	$\bar{X}_{23} = 8.11$, $\sigma = 3.41$	$\bar{X}_{21} = 7.65$, $\sigma = 3.37$	7.52
Low	$\bar{X}_{31} = 4.00$, $\sigma = 4.30$	$\bar{X}_{32} = 5.83$, $\sigma = 4.58$	$\bar{X}_{33} = 7.33$, $\sigma = 3.67$	$\bar{X}_{31} = 5.82$, $\sigma = 4.16$	5.72
Level Means SDs	$\bar{X}_{11} = 7.45$, $\sigma = 3.94$	$\bar{X}_{12} = 6.67$, $\sigma = 3.94$	$\bar{X}_{13} = 8.14$, $\sigma = 3.23$	$\bar{X}_{11} = 7.51$, $\sigma = 3.66$	
Unweighted Level Means	7.09	7.06	8.10	Unweighted Overall Mean	7.42

Table 7. Analysis of Variance of Addition Posttest Scores by Teaching Approach and by Student Ability Level

Source	df	SS	MS	F	Prob.
A (Teaching Approach)	2	12.2441	6.1220	.4747	.625
B (Ability Level)	2	93.5688	46.7844	3.6280*	.034
AB (Method X Ability)	4	32.5400	8.1350	.6308	.643
S/AB (Error)	48	618.9782	12.8954		
Total	56	757.3311			

*$p < .05$

No significant main effects for teaching approach were found, nor were there any significant interaction effects, for the addition posttest scores. Although the main effects of the teaching approach were not found to be significant on the addition pretest ($p < .087$), because the probability was close to the .05 level an analysis of covariance was run on the addition posttest, with the addition pretest scores used as the covariate. Results of this test also showed no significant main effects for the teaching approach and no significant interaction effects.

Table 8 shows means and standard deviations for scores on the subtraction posttest, while Table 9 gives a summary of the two-way ANOVA performed on the subtraction posttest scores (unweighted means were used in the ANOVA calculations).

Table 8. Subtraction Posttest Section Means and Standard Deviations

Ability Level	T1	T2	T3	Level Means Level SDs	Unweighted Level Means
High	$\bar{X}_{11}$ = 7.57 σ = 3.31	$\bar{X}_{12}$ = 9.67 σ = 1.15	$\bar{X}_{13}$ = 10.14 σ = 2.12	$\bar{X}_{11}$ = 9.00 σ = 2.74	9.13
Average	$\bar{X}_{21}$ = 5.87 σ = 4.49	$\bar{X}_{22}$ = 9.33 σ = 1.21	$\bar{X}_{23}$ = 8.67 σ = 3.64	$\bar{X}_{21}$ = 7.87 σ = 3.72	7.96
Low	$\bar{X}_{31}$ = 3.40 σ = 4.88	$\bar{X}_{32}$ = 7.33 σ = 2.80	$\bar{X}$ = 3.67 σ = 3.39	$\bar{X}_{31}$ = 4.88 σ = 3.94	4.80
Level Means SDs	$\bar{X}_{11}$ = 5.85 σ = 4.31	$\bar{X}_{12}$ = 8.60 σ = 2.16	$\bar{X}_{13}$ = 7.77 σ = 4.01	$\bar{X}_{11}$ = 7.32 σ = 3.85	
Unweighted Level Means	5.62	8.78	7.49	Unweighted Overall Mean	7.30

Table 9. Analysis of Variance of Subtraction Posttest Scores by Teaching Approach and by Student Ability Level

Source	df	SS	MS	F	Prob.
A (Teaching Approach)	2	87.8194	43.9097	3.8936*	.027
B (Ability Level)	2	173.9838	86.9919	7.7139*	.001
AB (Method X Ability)	4	28.7223	7.1806	.6367	.639
S/AB (Error)	48	541.1313	11.2774		
Total	56	831.6568			

*$p < .05$

Table 9 shows that the main effect of the teaching approach variable was found to be significant at the .05 level, while the interaction effects were not significant. *Post-hoc* multiple comparisons of instructional approaches were performed using the Scheffé test to determine the specific nature of the differences. Results showed that students studying subtraction of integers by the "finding the missing addend" method scored significantly higher ($p < .05$) on the subtraction posttest than did students using the "adding the opposite" approach; other comparisons were not significantly different.

Limitations and Suggestions for Further Research

There were several limitations to this study. It was not possible to randomly assign students to treatments. Since intact classes were used, there might have been a teacher effect confounding the results, even though all instruction for the unit on integers was handled by the investigator. Further, it was learned that each class had been together for longer than just the sixth grade; test records for the California Test of Basic Skills revealed that the students had been in the same classroom units for the fifth grade also, so that a group effect could possibly be present. This study should be replicated using random assignment of students to treatments, or perhaps by randomly assigning more than one intact class to each treatment level.

The design of this study did not permit the investigator to ascertain the extent to which the subsequent instruction in subtraction either facilitated or hindered student ability in adding integers. A possibility for future studies would be to investigate the degree of interference taking place.

A further improvement for the study might be to measure student achievement not only on computation items, but also on items covering concepts and applications. Finally, including a retention test in the design of the study might yield useful information about the effectiveness of the various approaches to subtraction that would not be evident when only an immediate posttest was used.

Conclusions and Implications

Analysis of scores on the subtraction computation posttest showed that students taught to use the "finding the missing addend" method scored higher than those using the "take away" procedure, who in turn scored higher than those employing the "adding the opposite" approach. The difference which was significant at the .05 level favored the "finding the missing addend" group over the "adding the opposite" group. There were no significant interaction effects between the three instructional treatments and student ability level.

If replications of this study, correcting for the lack of random assignment of subjects to groups, would confirm the present results, implications could be made for classroom teaching. The results indicate that the conceptual basis which came from the definition of subtraction for integers, the "adding the opposite" approach, was more difficult for students to understand than were the other two methods which essentially extended the same procedures used when students learned about subtracting whole numbers. It appeared that students' previous experiences with using movement on a number line to illustrate subtraction of whole numbers led them to view subtraction as always meaning a motion to the left, a jump back. This prior learning, combined with the new ideas of directed numbers and of opposites, seemed to make it hard for the "adding the opposite" group to accept the generalized definition for subtraction of integers as being realistic. Part of the students' difficulties also appeared to stem from the time allotment for the study's instructional sequence. For all three treatment groups only one class period (a half hour) was allowed for instruction and practice in adding integers, and only part of that time was devoted to learning how to use a number line to illustrate the operation--too short a time for many of the students to become sufficiently competent with this technique. Yet, the "adding the opposite" algorithm depended heavily upon student ability to add integers using the number line, while the other two treatments did not. Certainly, a lack of mastery of this basic subskill would affect student understanding of the "adding the opposite" procedure. Thus, a teacher wishing to introduce subtraction of integers through the "adding the opposite" approach would be advised to be aware that its development will require more time than the "finding the missing addend" approach, and possibly than the "take away" method. Finally, if there is a limit on the instructional time available for providing students with a first introduction to the topic of subtracting integers, the teacher should consider that the "finding the missing addend" approach has been shown to facilitate student skill in computation to a greater degree than does the "adding the opposite" method.

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APPENDIX

Selected Worksheets

Fifth day
T1--"adding the opposite"

Name ____________________

PART I

State the opposite:

(1) (-7), ____ (2) 4, ____ (3) (-1), ____ (4) 2, ____

PART II

Name the number that is being subtracted (the subtrahend):

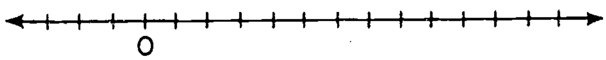
(1) 10 - 9 ____ (2) 6 - 4 ____ (3) 5 - 7 ____

(4) 2 - 8 ____ (5) (-1) - 3 ____ (6) (-8) - 1 ____

(7) 4 - (-3) ____ (8) 6 - (-2) ____ (9) (-3)-(-5) ____

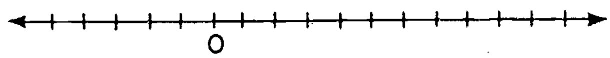
PART III

(1) 9 - 6 = n



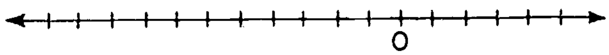
Answer: 9 - 6 =

(2) 4 - 6 = n



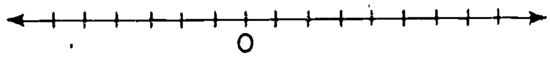
Answer: 4 - 6 =

(3) (-3)-(-7) = n



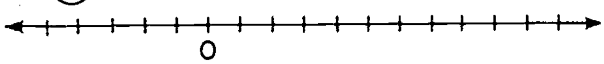
Answer: (-3)-(-7) =

(4) 1 - (-2) = n



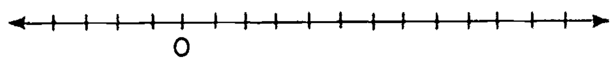
Answer: 1 - (-2) =

(5) 3 - 7 = n



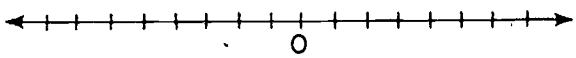
Answer: 3 - 7 =

(6) 4 - (-5) = n



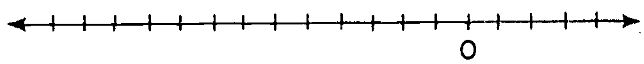
Answer: 4 - (-5) =

(7) (-3) - 2 = n



Answer: (-3) - 2 =

(8) (-6) - (-8) = n



Answer: (-6) - (-8) =

(9) 6 - 4 = n

(10) (-1) - 5 = n

Fifth day
T2--"find the missing addend"

Name ____________________

PART I

Write the related addition problem.

① 8 - 3 = n ____________________
② 4 - 1 = n ____________________
③ 9 - 5 = n ____________________
④ 6 - 8 = n ____________________
⑤ 2 - 7 = n ____________________
⑥ 2 - (-1) = n ____________________
⑦ 4 - (-6) = n ____________________
⑧ (-3) - 5 = n ____________________
⑨ (-1) - 8 = n ____________________
⑩ (-4)-(-2) = n ____________________
⑪ (-1)-(-7) = n ____________________
⑫ (-8)-(-3) = n ____________________

PART II

① 9 - 6 = n

0

Answer: 9 - 6 =

② 4 - 6 = n

0

Answer: 4 - 6 =

③ (-3)-(-7) = n

0

Answer: (-3)-(-7) =

④ 1-(-2) = n

0

Answer: 1 - (-2) =

⑤ 3 - 7 = n

0

Answer: 3 - 7 =

⑥ 4 - (-5) = n

0

Answer: 4 - (-5) =

⑦ (-3) - 2 = n

0

Answer: (-3) - 2 =

⑧ (-6) - (-8) = n

0

Answer: (-6)-(-8) =

⑨ 6 - 4 = n

0

Answer: 6 - 4 =

⑩ (-1) - 5 = n

0

Answer: (-1) - 5 =

Fifth day
T3--"take away"

Name ______________________

PART I Write the new symbols for:

(1) 5 = ______________________ (2) -3 = ______________________

(3) -1 = ______________________ (4) -5 = ______________________

PART II

(1) 4 + 0 + 0 = (2) (-3) + 0 + 0 + 0 + 0 = (3) 7 + (-1) + 1 =

(4) (-8) + (-1) + 1 + (-1) + 1 + (-1) + 1 = (5) 6 + (-2) + 2 =

PART III

(1) 9 - 6 =	(2) 4 - 6 =	(3) (-3) - (-7) =
(4) 1 = (-2) =	(5) 3 - 7 =	(6) 4 - (-5) =
(7) (-3) - 2 =	(8) (-6) - (-8) =	(9) 6 - 4 =
(10) (-1) - 5 =	(11)	(12)

Solving Quadratic Inequalities: More Than One Algorithm?

Brady Shafer

> To avoid the confusion that often arises in the minds of pupils from the presentation of a variety of methods, explanations, solutions, rules, remarks, etc., it has been the constant aim, in the preparation of this book, to present each subject in one form only...
>
> --Ray's Arithmetic, 1879

> Perhaps in teaching for understanding we shall one day depart from the well-nigh universal practice of offering children but one of several alternative forms of computation.
>
> --W. A. Brownell, 1938

Experiments at the secondary school level which compare students' learning of more than one algorithm for a given kind of problem are rare.[1] Typically two methods of problem solution are compared, but the question of whether the learning of one method facilitates or interferes with the learning of the other is not asked.

In elementary-school mathematics, a number of studies and discussions have examined alternative algorithms for certain arithmetic procedures. At least three elementary research studies have been characterized by the use of two algorithms with the same subjects. Scott (1963) concluded that teaching two algorithms in long division "does not confuse children, induces no undue difficulty, and takes no additional teaching time." In a study which dealt with estimating quotient digits, Carter (1970) found that a group given two rules (round divisor down if second digit is less than five, up if five or more) was both slower and less accurate than two groups which were given only one rule each. And as early as 1938 Brownell noted the "nearly universal practice" described at the beginning of this paper.

Brownell (1938) reported a study conducted by Tew which involved teaching two methods for dividing fractions, one for understanding and a second "as an efficient computation shortcut," with results he judged to be satisfactory. (No data were presented.)

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1 Suydam, Marilyn N. Annotated Compilation of Research on Secondary School Mathematics, 1930-1970, two volumes. U. S. Office of Education Final Report, February 1972. See also the annual research compilations by Marilyn N. Suydam and J. Fred Weaver in the Arithmetic Teacher and, since 1970, in Journal for Research in Mathematics Education.

The purpose of the research reported here was (1) to examine how second-year algebra students reacted to instruction on two algorithms for solving quadratic inequalities, and (2) to search for evidence of interference or facilitation by one algorithm with the learning of the other. Thus, in addition to comparing achievement on the two forms of the algorithm the study also investigated some of the consequences of presenting two algorithms in sequence. Did gains in achievement result after seeing a second method? Did students tend to "fix" upon the initial algorithm and ignore later ones? Did the procedure induce student confusion?

The Algorithms

One delimitation was necessary at the outset of the study. At the time of year when it was made (February), all students had considered quadratic expressions which could be factored. The quadratic formula had been introduced in some classes but not in all; therefore the study involved no problems for which the quadratic formula was necessary.

The two methods will be contrasted by use of examples.

METHOD A: VERBAL

The first method consists of examining possible cases, as follows:

Case 1: $x^2 + 5x < -4$

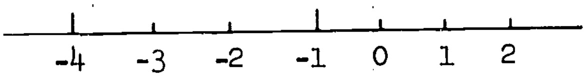
Case 2: $x^2 + 5x + 4 < 0$

Case 3: $(x + 1)(x + 4) < 0$

Hence a product is negative. One factor must then be negative, but not both. There are two possible cases. (1) The first factor might be negative, but not the second. In this case $x + 1 < 0$ and $x + 4 > 0$. Hence $x < -1$ and $x > -4$. The solution set for this sentence is the set of real numbers between -4 and -1, both endpoints omitted. (2) A second possibility is that the second factor might be negative but not the first. Thus $x + 1 > 0$ and $x + 4 < 0$, implying the $x > -1$ and $x < -4$. Since no number satisfies both statements simultaneously, the second case gives no additional solutions. Hence the solution set for the given inequality is $(x: -4 < x < -1)$.

METHOD B: VISUAL

The second method also begins by making one member of the inequality zero and factoring the polynomial which forms the other member, as in the previously cited Case 3. It then utilizes the number line in the following way. To know if the values of a continuous function are positive or negative, it is useful to know the zeroes of the function. The zeroes for the polynomial in Case 3 are of course -4 and -1. These are indicated on a number line:



The line is thus divided into three regions. For a given number x, in any of the regions, one can easily decide whether $(x + 1)(x + 4)$ is positive or negative.

If x is any number in the region right of -1, for instance, both $x + 4$ and $x + 1$ are positive; hence their product is positive. Thus no number larger than -1 can be a solution as the product is required to be negative. If x is between -4 and -1, one factor is negative and one positive, hence the product negative and the inequality satisfied. If $x < -4$, both factors are negative, the product is positive, and the inequality is not satisfied.

The x-values of -1 and -4 give 0 as a polynomial value. Hence they must be rejected since $0 < 0$ is a false statement. The solution set then consists of all real numbers between -4 and -1, with endpoints omitted.

Method B depends heavily upon continuity properties of the function involved. But since all polynomials are continuous, this involves no mathematical difficulty, nor did it seem to involve any pedagogical difficulty.

Procedure

The study was conducted with four Algebra II classes at Brookhaven High School, Columbus, Ohio. All classes met during morning hours. The study covered six days of instruction and four of tests. Daily activities are summarized as follows:

Day 1. A pretest was given to all four classes; the test was a ten-item instrument covering linear equations, quadratic equations, and linear inequalities.

Day 2. A review was given of linear equations, linear inequalities, and factoring.

Days 3 and 4. Two classes were shown Method A and the remaining two classes were shown Method B. Students were asked not to work or discuss homework with anyone except members of their own class.

Day 5. Test 1 was given to all classes. This test was abbreviated to only eight items since all class periods were shortened for an assembly.

Day 6. All classes were shown the method not previously taught. The original plan -- to spend two days teaching the second method -- was modified when students indicated they were ready for the test and were perhaps getting bored.

Day 7. Test 2, twelve items, was given to all classes, with instructions to work all problems by the new method.

Days 8 and 9. Transfer material was presented, consisting of such problems as

$$\frac{5x\,|x + 5|}{(x - 2)(x + 5)} > 0 \text{ and } \frac{x - 1}{x + 5} < 1.$$

Day 10. A twelve-item posttest was given, including four transfer items, with students given complete freedom in choice of algorithm to be used for all problems.

All classes were taught by the writer, to maintain some control over teacher variable. In addition, differences in ability were measured by the pretest, with no significant differences among classes noted in ability to do the kinds of problems on the pretest (see Table 1). To control the time-of-day variable (the possibility exists that early-morning classes might be fresher and therefore do better work regardless of treatment), the earliest and latest classes (1 and 4) were grouped together in assignment to treatment. The activities of Day 2 were an attempt to give a common background to the four classes through a review of the prerequisite skills.

Table 1. Means and Standard Deviations for the Four Tests Administered During the Study

	Pretest		Test 1		Test 2		Posttest	
GROUP	$\bar{y}$	s	$\bar{y}$	s	$\bar{y}$	s	$\bar{y}$	s
All students (N=71)*	3.86	2.10	5.79	2.12	8.24	3.43	7.45	2.68
Treatment AB: Verbal-Visual (N=37)	4.00	2.01	4.92	2.03	8.89	3.03	7.30	2.70
Class 1 (N=17)	3.94	1.86	4.82	1.89	7.65	3.51	6.88	2.56
Class 4 (N=20)	4.05	2.13	5.00	2.14	9.95	2.01	7.65	2.76
Treatment BA: Visual-Verbal (N=34)	3.71	2.18	6.74	1.77	7.53	3.68	7.62	2.65
Class 2 (N=16)	3.88	2.42	6.00	2.12	8.00	3.72	7.75	2.86
Class 3 (N=18)	3.56	1.92	7.39	1.01	7.11	3.59	7.50	2.43

*Data are reported only for students who took all four exams. Actual class sizes ranged from 22 to 28.

Results

For each of the four measures, a t-test was conducted for the difference of means between treatment groups AB and BA. No differences were significant at the .05 level.

The preceding comparisons do not take into account differences in pretest scores. To compare gains in performance, regression analyses were made. In three separate analyses, Test 1 scores were regressed against pretest scores, Test 2 scores against pretest, and posttest against pretest. Regression coefficients and correlations appear in Table 2.

At the time of Test 1, students had seen only one algorithm. In Test 2 (as already noted), they were asked to use the algorithm most recently taught, but were given free choice of algorithm on the posttest. Thus AB group used Method A in taking Test 1 and method B in taking Test 2. For groups using Method B (BA on Test 1, AB on Test 2) the difference in regression coefficients is not significant. But for groups using Method A, the difference in regression (0.37 vs. 1.22) is significant at the .001 level. Students who have seen Method B and then Method A achieve greater gains in performance with Method A than students who have seen Method A alone.

Analyses of variance and covariance were conducted using standard scores from the four tests. The difficulty level of the posttest was somewhat higher than that of Test 1 and Test 2 (see Table 3). By eliminating the two most difficult items[2] from the posttest, a ten-item subtest was obtained with a mean item-difficulty level of .311, which compares favorably with that of Tests 1 and 2. It was hoped by this means to adjust for the difference in test length forced on the study by school schedule. As Tables 4 and 5 show, the attempt was successful since in both analyses the F ratio for main effect due to tests is zero.

In the analysis of variance, which did not involve pretest scores, no other effects were significant. But in the analysis of covariance, in which scores are adjusted for pretest scores, three effects were significant at the .05 level: main effect for treatments, main effect for classes within treatments, and test-treatment interaction. These conclusions, as well as the earlier ones involving regression, must be qualified by noting that pretest reliability is only .65, lower than that of the other tests (see Table 3). However, some evidence is given to suggest that when gains in scores are considered, the sequences BA and AB affect student performance in a different fashion.

[2] The differences in error rate for the two treatment groups were not significant for either problem. Both problems were of transfer type.

Table 2. Regression Coefficients and Correlations between Pretest and Other Instruments

	Test 1 Against Pretest			Test 2 Against Pretest			Posttest Against Pretest		
GROUP	Regression Slope	Correlation	Method Used	Regression Slope	Correlation	Method Used	Regression Slope	Correlation	Method Used
AB	.37	.37	A	.67	.45	B	.68	.52	choice
BA	.43	.51	B	1.22	.70	A	.55	.43	choice

Table 3. Reliability and Difficulty Estimates For the Four Tests Administered During the Study

	Pretest	Test 1	Test 2	Posttest
KR20 reliability estimate	.652	.760	.870	.738
Mean item difficulty*	.614	.276	.313	.379

*Item difficulty is here defined as the percent of subjects who gave incorrect responses to the item; as it increases, the item becomes more difficult.

Table 4. Analysis of Variance of T-Scores from Test 1, Test 2, and a Ten-Item Subtest of Posttest

Source	Sum of Squares	df	Mean Square	F	P less than
Treatments	19680.934	1	19680.934	1.080	.375
Quizzes	0.006	2	0.003	.000	1.000
TxQ interaction	141688.813	2	70844.375	3.889	.147
Classes (within treatments)	25994.785	1	25994.785	1.427	.318
CxQ interaction	6423.230	2	3211.615	0.176	.846
Residual	54649.238	3	18216.410		

Table 5. Analysis of Covariance of T-Scores from Test 1, Test 2, and a Ten-Item Subtest of Posttest, with Pretest T-Score as Covariate

Source	Sum of Squares	df	Mean Square	F	P less than
Regression	2.876	1	2.876	0.004	.956
Treatments	17972.449	1	17972.449	24.382	.039
Classes (within treatments)	25970.672	1	25970.672	35.233	.027
Quizzes	0.005	2	0.003	0.000	1.000
CxQ interaction	6423.227	2	3211.613	4.357	.187
TxQ interaction	141688.688	2	70844.313	96.110	.010
Residual	1474.242	2	737.121		

Only on the posttest, it will be recalled, were students given a choice of algorithm. They were asked to show enough detail in written solutions so that the scorer could tell which method they were using. A frequency count was made of algorithm use, by class and by problem. Despite the instructions, 182 responses could not be judged as a consistent use of either method, while Method A was used 35 times and Method B, 498 times. Even if all "doubtful" responses were counted as instances of Method A (a highly unlikely occurrence), still Method B would be chosen by more than two to one over Method A. As the data stand, the margin of choice is more fourteen to one. The difference is significant at the .001 level. Differences between treatment groups were not significant at the .05 level on the entire set of problems, on those problems worked correctly, or on any individual problem.

Summary and Interpretation

No differences related to class means appeared, either in analysis by t-tests or in the analysis of variance. In the two analyses which took pretest scores into account, however, significant differences occurred.

One significant difference was the difference in regression coefficients between the treatment group which had seen Methods BA and the group which had seen Method A alone. Interpreting that difference is difficult. Informal evidence gleaned in conversations with students suggests one possible explanation. Students in the BA group were happy with Method B at the halfway point in the study. What may have happened -- despite a request to use Method A in Test 2 -- was that many BA students might have verified answers by using Method B. Of course test papers would give no evidence of the forbidden method if this were the case. At least two earlier researchers have noted discrepancies between the method "taught" and the method actually used by the children (Brownell, 1966; Flournoy, 1959).

There is also a second possible explanation. At the time the experiment ended, students in both treatment groups overwhelmingly chose Method B. As Table 1 shows, in both Test 1 and Test 2 there was a tendency toward better student performance with Method B, though not a statistically significant difference. Students were not confused by being confronted with two methods. There was some impatience manifested with the second method (in both groups) which resulted in the elimination of a second day with the new method. This may have been detrimental to Method A, which seems to require somewhat more time to explain. This in turn may account for finding (Table 2) that Method A alone was as good as BA.

Implications

Students liked Method B, the number-line method, and exhibited a tendency to better work with it. If only one method is to be taught, and if the teacher's goal is student performance at the close of the unit, it would appear that Method B is the better choice. Of note here is the fact that the textbook presented only the verbal method.

Nine problems in each class, on average, were solved by the verbal method. Would the needs of the students who used Method A have been well served if Method B had been taught alone? In this instance they were a decided minority; but with a different pair of alternative algorithms, the split might be more nearly fifty-fifty. Perhaps students should be presented both algorithms in such a case and then given a choice. The teacher can give guidance, of course, if it is evident that a student has made an unsound choice. But is not this the whole point of individualizing instruction?

No data were gathered to verify stability of results over a retention period, not did the study address the question whether spending more time on one algorithm might be as effective as teaching two. Further research is needed to answer these questions.

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A Comparison of Two Strategies for Teaching Algorithms for Finding Linear Equations

Richard W. Corner

An important part of teaching is the transmission of substantive content. The professor lecturing to the gathered students is the classic image of instruction. The lecture, still an important part of teaching, particularly in college, has been supplemented with instructional television, audio-tutorial systems, and programmed texts. The modes for the presentation of content would all be classified as direct communication strategies by Hough and Duncan (1970).

Fey (1969) points out that research on classroom behavior has been mostly of the nature of comparing "name" methods; e.g., "discovery" versus "expository" teaching. The problem with much of this research is that it lacked specification of what was meant by the particular name, thus leading to inconclusive or unreliable results.

To specify precisely the behavior of classroom teachers, Rosenshine (1970) suggests the use of category systems for the observation of classroom behavior. These systems are low inference systems; that is, the observed behaviors are categorized by the use of precise and narrow definitions. For example, the knowledge that a teacher responds to a student question with a question 42 percent of the time is much more informative than the statement, "The teacher usually asks leading questions." Rosenshine notes that there have been few studies relating teacher behavior and instructional outcome.

A number of category systems have been developed for the purpose of classroom observation. Among these are systems by Flanders; Smith, Meux, et al.; Hough and Duncan; and Henderson. The purpose of each of these systems is somewhat different; systems appropriate for the analysis of direct communication are of interest here.

Category Systems for Direct Communication Strategies

Henderson (1970) has formulated an instructional analysis system which is based on classroom teaching. His purpose was to analyze concept development through a taxonomy of the language used in talking about concepts. The categories or moves of his system can be partitioned into three classes. The first is connotative, talk about the concept. The second is denotative, which primarily involves giving examples of the concept. The final category is meta language or language about language, as seen in formal definitions. Henderson's system has been restricted to research on concept formation.

Henderson's system was derived from the more general work by Smith, Meux, et al. (1967). Smith et al. categorize behavior into "ventures" or episodes, which are complete subunits such as solving a problem, explaining a concept or proving a theorem. Based on classroom observation, Smith et al. identified seven types of ventures; each venture has a number of categories of behaviors. In a limited sample of mathematics classrooms Smith found three types of ventures; the three were the concept venture, the procedural venture, and the rule venture. The concept venture is Henderson's area of interest, previously discussed. In the procedural venture classroom discussion centers on the procedure for solving a problem. The rule venture involves the formation or justification of a rule. Smith's system has a problem in terms of size: there are a large number of highly specific categories in each of the seven ventures.

A system which seems highly appropriate for the analysis of direct communication strategies is the Content Analysis System (CAS) developed by Hill (1969). The system was developed for general classroom analysis. The ten categories of the system were based on the Gestalt ideas of figure and ground (see Figure 1). Hill's system has the advantage of a small number of categories as compared to the system of Smith et al. CAS can be used to analyze concept development as in Henderson's work or can be used for any other direct communication instruction. In his observation of 36 different junior high school classrooms, Hill found no identifiable strategies as analyzed by CAS. The categories of CAS seem highly appropriate for the specification and analysis of direct communication strategies.

Procedure and Results

The purpose of this study was to explore the use of CAS as a means of specifying different instructional strategies. In this initial endeavor the first objective was to develop materials which were appropriate for the students, specifying different strategies in terms of CAS. The second objective was to ask the question: For this topic and these subjects, is there any difference in the strategies?

Two different programmed lessons were written to introduce the algorithms for finding linear equations. The same number of examples and the same examples were used in each treatment. The only difference was the sequence of instruction (see Figures 2 and 3). One sequence was deductive in nature and the other sequence was inductive in nature.

The subjects were 26 students in a reduced-pace pre-calculus course at The Ohio State University. Students were randomly assigned to treatments. The students were given 35 minutes to complete the programmed lesson; all were able to complete the lesson in the allotted time. Because only 50 minutes were available for the experiment, a six-item test was used for evaluation. The students were given 15 minutes for the test and all were able to complete the test.

B - Background: Develops information or knowledge of the context or frame of reference within which the content idea, topic, or figure is set. May be a review of previously developed content.

D - Defining: Determines the precise significance or meaning of the content figure, idea or concept. Includes definition of terms used in the concept or figure.

N - Naming: Identifies or specifies the content figure by name, symbol, or image. Includes questions seeking identity.

E - General Example: Presentation or development of examples of a general or construct nature. May deal with the nature of many specific examples or the classes of a hierarchy. Includes derivation of formulas. Doubtful examples are classes as general.

Ea - Abstract Example: Communication which presents specific examples verbally or symbolically; presented in spoken or written form only. Includes charts, schematic drawings and graphs. No real or image form presented.

Ec - Concrete Example: Specific Examples which are presented in a real or image form, such as pictures or drawings. Example uses an object which represents the content figure. Includes any drawings representing three-dimensions.

En - Negative Example: Illustrates representations negative to the content figure. An example which is presented as a contrast or test of the figure.

A - Amplification: Content communication by which an enlargement or expansion of the focus of attention occurs. Two or more things are compared, contrasted or related.

An - Digression: Content communication which expands beyond the relevant content figure. Incorrect statements and accompanying corrective feedback are categorized here.

M - Miscellaneous: Non-content communication. Class management, procedures or control. Personal communication such as non-content opinion.

Figure 1. Categories of Hill's content analysis system.

1. Ea: Find equation of line through given points using similar triangles.

2. E: Find equation of line through general points using similar triangles a two-point formula.

3. Ea: Find equation of line through given points using two-point formula.

4. D: slope.

5. Ea: Find slope and sketch line.

6. Ea: Find slope and sketch line.

7. E: Use two-point formula to find point-slope form.

8. Ea: Find equation of line through two points. (parallel to X-axis)

9. D: line parallel to X-axis.

10. Ea: Find equation of line through two points (parallel to Y-axis).

11. D: line parallel to Y-axis.

12. E: Use point-slope form to find slope-intercept form.

13. Ea: Determine slope and Y-intercept.

14. Ea: Find equation of line through two points.

15. Ea: Find equation of line through two points.

Figure 2. Inductive sequence.

1. D: slope

2. Ea: Find slope and sketch line.

3. Ea: Find slope and sketch line.

4. Ea: Find slope.

5. D: Linear equation $(y = mx + b)$, y-intercept
Line parallel to Y-axis
Line parallel to X-axis

6. Ea: Given equation. Find slope and y-intercept.

7. Ea: Given 2 points. Sketch line, find slope, find y if $x = 0$.

8.(a) Ea: Given 2 points. Sketch line. Find equation. (parallel to X-axis)

8.(b) Ea: Given 2 points. Sketch line. Find equation. (parallel to Y-axis)

9. Ea: Given 2 points. Find equation.

10. E: Find equation of line through (x_1, y_1) and (x_1, y). (Point-slope form)

11. Ea: Sketch equation and verify point-slope form.

12. Ea: Given slope and point. Find equation.

13. Ea: Given two points. Find equation.

Figure 3. Deductive sequence.

The KR-21 test reliability was .43, which makes any conclusion about the results questionable. A t-test for the differences of means (H_o: $\mu_I = \mu_D$) and an F-test for equality of variances (H_o: $\sigma_I^2 = \sigma_D^2$) indicated no differences in the treatments (see Table 1).

Table 1

Data Analysis

Treatment	n	Sample Mean	Sample Variance
Inductive	12	4.17	1.61
Deductive	14	4.00	2.86

$t = .09$ (Not significant; $t_{.05,24} = 7.064$)

$F = 1.77$ (Not significant; $F_{.05,13,11} = 2.11$)

Discussion

The first objective, the development of materials using different CAS strategies which were appropriate for the students, seems to have been met. The students were able to complete the programmed lessons in the allotted time and scored reasonably well on the test. Because of the test reliability, any conclusions about the relative effectiveness of the strategies is highly tenuous; however, the results seem to imply that there was no difference in the student learning.

Several areas of further research are indicated. First, it would be interesting to include a transfer component in the evaluation of the results. Secondly, a long-term study comparing inductive and deductive strategies would be more likely to indicate differences. A treatment of 35 minutes duration is unlikely to reveal any difference.

An area for possible exploration would be the determination of an optimal instructional strategy for a given algorithm. It is possible that one algorithm may be best taught inductively while another is best taught deductively. The degree to which instructional strategies depend on content could be discovered.

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Some Computational Strategies of Students Using Desk Calculators

Raymond Zepp

Introduction

The calculator has recently come into prominence in American life with the development of small electronic calculators. It has naturally been considered as an educational aid in mathematics, particularly for students whose computational skills are weak. It is probably fair to say that calculators will play a tremendously increased role, both in school and society.

Although much has been said about the possible uses for calculators in the schools, very little has been said about the mental processes by which students operate the calculators. We need to examine and understand these processes in order to maximize the effectiveness of the use of calculators in education. Moreover, and perhaps more importantly, an understanding of the processes may provide insights into the nature of the learning and use of algorithms in general.

Lankford has been concerned with the nature and variety of "computational strategies" of students. He asks in what ways a student attacks computational problems. This question, applied to computation with calculators, is the focus of the present study. The study parallels some of the ideas and techniques and discussed by Lankford (1972, 1974).

Procedure

Twenty-three students from an inner-city high school in Columbus, Ohio were interviewed. These students, on the basis of low mathematics achievement, had been assigned to a classroom equipped with desk calculators. Their subject matter consisted of basic arithmetic, fractions, and decimals, with numerous practical story problems; programmed learning approach was used in the classroom.

The subjects were informed that the interviewer wished to learn how people think when they use calculators and that they were to work some examples, explaining or stating aloud (into a microphone of a tape recorder) all their thoughts concerning the examples. The interviews lasted twenty to thirty minutes apiece. The problems emphasized fractions, division, and combined operations.

Findings

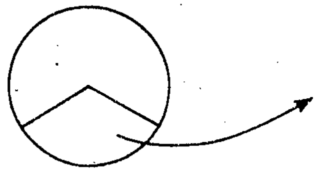
A. Division: The greatest single difficulty in division examples was with the sequencing of the numbers. Ten of the twenty-three students made at least one error in interpreting the order in which the numbers were considered in a division example. The confusion seemed to center around the verbalization of the problem, especially when using the words "divided by" and "goes into." The problem 384 ÷ 17 was read by nine students as "17 into 384," and by five students as "384 into 17." Of these five students, three worked the problem incorrectly, that is, by dividing in the wrong order. The other two, who had stated the problem incorrectly, proceeded to solve the problem correctly, that is, different from the way they had verbalized it. More strikingly, the problem 65/6 was read by four students as "6 into 65," but by eight students as "65 into 6." Of the eight, four worked the problem using the wrong order. In all cases, the way in which the problem was worded when the calculator was used was the same as when the student was asked to work the problem by hand, but only one student used the wrong order in actually working the problem by hand. Other wordings of division problems besides "384 divided by 17" were "384 divide 17," by several students (this would appear to be a direct consequence of using the calculators), and "384 divided through 17," by one student.

When asked what factors dictated the choice of which number to enter first in division, seven students replied that the bigger number is always entered first, and similarly, two students remarked that you can't divide the larger into the smaller. One student always entered the smaller number first. One girl entered the larger number in problems like 384 ÷ 17, but knew that in 65/6 the "top" number was to be entered first, whether larger or smaller. A few had much more difficulty with 65/6 than with 384 ÷ 17 because they had no idea that a division was called for. Finally, one boy solved 65/6 as follows: since 65 has one more place than 6, he wrote the problem as 65/6 x 1/10 : 1 5/60. The only reason that could be elicited was "that takes it to one more place."

Do students trained with calculators understand the concept of division, or are they merely punching buttons? A very rough estimate is that approximately one-third of the students were punching buttons. This estimate is based on the observation that approximately one-third could change a problem stated "384 + 17" into the form $17\overline{)384}$, but were totally at a loss as to how to begin to work with the algorithm. These same students seemed to have no "feeling" for the meaning of the answer. Of course, there is nothing to indicate that these students could have done any better had they not been trained on the calculator. The better students did quite well. Two even checked their answers using multiplication.

B. The Concept of Fraction: There seem to be three separate notions which many of the students interviewed could not connect, and therefore confused. These are the idea of fractional parts, the decimal representation of fractions, and the interpretation of the "-" or "/" as "divide," as in 4/2 = 2. A very clear example of this confusion is the case of one of the brighter girls, who had correctly divided 65/6, and who had a recollection that 2/3 was either 6.7 or .67, but refused to divide 2 by 3 on the calculator because "three just won't go into two, so it's impossible." She went on to explain that 65/6 meant divide, but 2/3 didn't, and she had no idea where the decimal representation came from. She knew the meaning of the fractional part 2/3, she knew that sometimes a slash or bar meant to divide, and she would compute with decimals, but she had no understanding of the interrelationship of the three notions.

Most students did understand what a fraction was and could give such examples as "2/3 of a pie," or "8/10 would be a collection of 10 objects with 8 taken away." One case, however, indicated that the "pie" notion may be rotely learned in school but not fully understood: one of the slower girls insisted that 2/3 was the smaller section of the pie,



while the larger segment was about 3/5. A slower boy thought that all fractions, including 5/3, were less than one. On further consideration, he stated, "There might be fractions bigger than 1, but I can't think of any." Two students could not give any explanation for fractions other than "2/3 is 2 divided by three;" this notion may possibly be traceable directly to their training on the calculators. But to reiterate, the students, by and large, did understand what a fractional part was, and could compute with decimals. The confusion seems to have been in transitions among the three notions stated above.

The question of the size of fractions is another matter entirely. It was here that students did poorest. They were asked which fraction is larger, 13/17 or 11/15, and why. Six quickly said that 13/17 was larger because 13 > 11 and 17 > 15. Upon further questioning, with examples such as 13/10 and 11/15, most of them were confused. One said that in a mixture, the numerator takes precedence, hence 13/10 > 11/15. Another said that the denominator takes precedence, hence 13/17 > 14/15. Other responses to the initial question were that the denominator <u>always</u> takes precedence (13/17 > 11/15) and that the numerator always takes precedence (13/17 > 11/15), while two students stated that the denominator takes reverse precedence, since 1/2 > 1/3. Thus 1/4 > 15/16 because only the denominator counts. By far the "best" answer, which two students arrived at separately, was that 1/17 is a smaller piece of pie than 1/15; 13/17 is four pieces away from a whole pie, and 11/15 is four of the larger pieces away from the whole pie; therefore 13/17 is closer to the whole pie, and thus larger. A third student, reasoning similarly, observed that both were 4 pieces away from a whole, so the two fractions were equal.

Five students immediately divided 13 by 17 and 11 by 15 using the calculator, and nine others first explained which was larger, perhaps incorrectly. But later, upon being asked, they were able to use the calculator to compare the two as decimals. However, of these nine, three reasoned incorrectly as follows: 13/17 = .76 11/15 = .73, hence 11/15 is larger because "decimals are just fractions to begin with, like 1/2 > 1/3." In other words, the fractional part .76, having higher digits, is actually the smaller fraction. This kind of statement is yet another example of the aforementioned confusion between fractions and their decimal representations.

C. Algorithms Involving Fractions: The common algorithms involving fractions are quite easily done on the calculator. For example 3/4 = 3/4 + 5/6 = (3÷4) + (5÷6) = .75 + .833 = 1.5683. The students could perform these operations rather well. On the other hand, they did extremely poorly in working the traditional written algorithms. For instance, only one girl of the 23 students knew that a common denominator was used in solving addition of fractions. It is impossible to say what effect the calculators have had here. One could perhaps argue that prolonged use of the calculators caused the students to forget the written methods. But another might argue that the fact that these students could not perform this kind of operation was the reason they had been placed in this class in the first place. Both arguments are probably correct to some extent. The lack of understanding of basic concepts which they exhibited certainly precludes further progress in mathematics. There was, however, some forgetting. Many students said that they "used to be able to do it," and there was evidence that students remembered bits and pieces of algorithms. For instance, in working with the fraction 3 2/9, three students wrote the fraction as 9 x 3 + 2, so 3 2/9 = 29. Another student said it was 3 x 2 + 9 = 15. Still, these students may not have been able to work the problems even if they were not enrolled in the calculator class.

The following table lists the number of pupils who solved various problems correctly by calculator techniques as compared to the number who merely used their written technique (correct or not) on the calculator. For example, the correct calculator technique for 3/4 + 5/6 is (3 ÷ 4) + (5 ÷ 6) = .75 + .833 = 1.583. An incorrect written technique would be 3 + 5/4 + 5 = 8/10, which could be "duplicated" on the calculator.

	3/4 + 5/6	4 - 1 2/9	4 1/3 x 2 1/4	2/3 x 75
calculator method	8	9	6	6
written method	2	4	6	7

Errors fell into two major categories as follows: 1) the misunderstanding of the relationship between fractions and their decimal equivalents, and 2) misunderstanding of placement and meaning of the decimal point.

Into the first category fall the following errors:

a) $3/4 + 5/6 = (3 \div 4) + (5 \div 6) = .75 + 1.20 = 1.95$

b) $3/4 + 5/6 = 3.4 + 5.6 = 9.0$, and similarly, $4 - 1\ 2/9 = 4 - 11/9 = 4 - 11.9$

c) $4\ 1/3 \times 2\ 1/4 = 4.3 \times 2.4 = 10.32$

d) $2/3 \times 75 = 2.6 \times 75 = 195$. The student went on to explain that she had remembered $2/3 = 2.6$ and $1/3 = 1.6$.

e) One girl observed that 3 won't go into 2, so you can't do 2/3 on a calculator. She was induced to try $2 \div 3$ on the calculator, but when she obtained .6667, she decided that was impossible, and gave up.

Into the second category fall the following errors:

a) $3/4 + 5/6 = 75 + 8333 = 8408$

b) $2/3 \times 75 = 6667 \times 75 = 499999$ (two students)

c) $1/3 + 1/4 = 33.5 + .25 = 33.75$

d) $5 - 3\ 2/9$: $2/9 = 9 - 2 = 4.50$, so $3\ 2/9 = 34.50$. $5 - 34.50 = 29.50$

D. Confusion of Operations: During the first three interviews, the students sometimes punched X instead of +, and vice versa, into the calculator. It was thought that the + written in the problem may have resembled a X to them. In subsequent interviews, the problems were read carefully to the students, but the mistake persisted. Most of the students immediately corrected their mistake, but three argued that although the problem said +, it was necessary to punch X in the addition of fractions. Two students also confused division with subtraction. An explanation is difficult to find. It may be that to a calculator user who merely punches buttons without understanding the underlying operations, one operation is as good as another. This showed up in statements like "I timesed 3 with 4," or I plussed 5 by 6." This explanation is unlikely, as not all operations were confused. A more plausible explanation is that when faced with a problem of some difficulty, the solver tries the strategy which is easiest to use or is most familiar. Only one student could add fractions without a calculator, while most could multiply fractions. So if multiplication of fractions is in fact easier, students might tend to substitute multiplication for addition. If one assumes that addition seems easier to students then division, then the following examples lend support to the above explanation.

The students had never encountered a problem of the form $\frac{12 + 71}{15 + 83}$. Four different students correctly added 12 and 71, as well as 15 and 83, but were puzzled over what to do with the resulting 83 and 98, so they added them. Upon questioning as to why they had added, none could explain, except by saying, "It seemed like a good thing to do." Another student worked the problem 2/3 x 75 as follows: 2 x 75 = 150. 3 x 75 = 225, then, after a long pause, 150 + 225 = 375. A final example is the student who, in working the same problem, found that 2/3 = .667, and then tried .6667 + 75. This particular student had previously stated that he was good at dividing.

E. Reasonableness of Answers: Does the use of calculators give students mny more insight into estimating the size of numbers? In the classes interviewed, particular stress had been given to the estimation of answers. A few students performed admirably. For instance, two students reasoned in this manner: 384 + 17 = 22.528 (on the calculator). "That's about right since 20 x 17 is three-hundred something." But by and large, the students had no conceptions about the plausible size of their answers. Most students did look at their answers in an attempt to check their calculations, as was shown by frequent winces and looks of dismay at answers on the calculator which they believed to be wrong. But it seems that the size was a rarely-used criterion by which answers were checked. Three or four students, for example, said that 384 + 17 = 22.528 was wrong because both 384 and 17 were whole numbers, so the answer should be a whole number.

The calculators can be set to read answers accurate to 2, 4, or 8 decimal places. When it was set to eight places, the answer 384 + 17 = 22.52848652 was usually a big surprise, and the usual response was that this answer was too big. Generally stated, the number of digits in the answers was the most important criterion used to determine the reasonableness of answers.

Four or five students said almost all their answers were reasonable for reasons such as, "I think I did it right," or "Machines don't lie." These answers suggested an absolute trust in the machines. But more often than not, students merely could not determine whether or not their answer was reasonable, even though they tried.

F. Attitude: The students were confident of their ability to use the calculators (but not necessarily of their mathematical ability). Almost all seemed to enjoy the interview, and many seemed proud to be explaining "math" to someone. It was thought prior to the interviews that many students with low self-concepts would fear to venture answers lest they manifest their "stupidity." These thoughts were not borne out; the students, in general, gave their answers with assurance.

The students also appeared to enjoy punching buttons and seeing the answer light up on the screen. They appeared to have a good feeling of having produced the number on the screen. One boy even punched the "clear" button after working problems when he was not using the calculator at all!

Attitude toward fractions was extremely negative. Many students uttered disparaging comments when presented with the first fraction problem. One girl could in no way be coaxed to attempt any fraction problems. When quizzed about their negative attitudes, most responded that they could never do fraction problems.

Conclusions

Very few conclusions can be drawn, as the students' mathematical achievement was very low before they used the calculators, and was still very low at the time of the interviews. Many can solve problems with the calculators which they would have little or no idea about how to solve otherwise, and they seem to enjoy doing so. The calculators have probably not added much to their mathematical understanding, but at this age, and with their long history of failure, it is doubtful that other methods would have any effect either. In general, only a few students were robot-like button pushers. Most made some attempt to understand the problem and to appreciate the numbers and issues involved.

Use of the calculator has changed a few concepts in some students. For instance, to a few the fraction 2/3 had no physical embodiment other than 2 divided by 3. Others have learned that all numbers are decimals and hence the need for operating with fractions is obviated. Most dangerous by far is the idea that if one punches numbers into a calculator, it will always be right. Any teacher who uses calculators must be sure to insist that students check the reasonableness of their answers, lest they lose contact with the feel for the size of the numbers which they are using.

References

Lankford, Francis G., Jr. Some Computational Strategies of Seventh Grade Pupils. Charlottesville, Virginia: School of Education. University of Virginia, 1972. (ERIC: ED 069 496). 96 pages.

Lankford, Francis, G., Jr. What can a Teacher Learn About a Pupil's Thinking through Oral Interviews? Arithmetic Teacher 21: 26-32, January 1974.

V. Annotated Listing of Selected Research References

Related to

Computational Algorithms

Annotated Listing of Selected Research References Related to Computational Algorithms*

Marilyn N. Suydam

Alessi, Galen James. Effects of Hutchings' "Low Fatigue" Algorithm on Children's Addition Scores Compared Under Varying Conditions of Token Economy Reinforcement and Problem Difficulty. (University of Maryland, 1974.) Dissertation Abstracts International 35A: 3502; December 1974.

Fourth graders who had high scores on basic addition facts were randomly assigned to two groups: one taught the "low fatigue" algorithm and one taught the conventional addition algorithm. Tests were at three levels of problem difficulty and administered under three conditions of token reinforcement. The "low fatigue" algorithm produced higher scores for both number of columns correct and columns attempted. Significant differences among means were also found for the difficulty level of columns correct. As the test forms increased in difficulty, the extent of superiority for the "low fatigue" algorithm decreased.

Bat-haee, Mohammad Ali. A Comparison of Two Methods of Finding the Least Common Denominator of Unlike Fractions at Fifth Grade Level in Relation to Sex, Arithmetric Achievement, and Intelligence. (Southern Illinois University, 1968.) Dissertation Abstracts 29A: 4365; June 1969.

Fifth graders (n = 112) were randomly selected from six classes, and assigned to be taught to find the LCD by the inspection method used in their textbook or by a set of six lessons on factoring of denominators. Pupils taught by the factoring method performed significantly better than those taught by the inspection method.

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*The reports which are listed have been included to illustrate various factors which are relevant to the study of computational algorithms. It should not be inferred that each study referenced is necessarily free of design flaws. An earlier version of this listing, prepared by Marilyn N. Suydam and J. F. Weaver, was used in the Postsession on Computational Algorithms sponsored by the Special Interest Group for Research in Mathematics Education at the Annual Meeting of the American Educational Research Association in 1973.

Baumann, Reemt Rikkelds. Children's Understanding of Selected Mathematical Concepts in Grades Two and Four. (The University of Wisconsin, 1965.) Dissertation Abstracts 26: 5219-5220; March 1966.

Forty randomly selected pupils from grades 2 and 4 were interviewed to ascertain their ability to attain and use concepts of commutativity, closure, and identity. Twenty-one tasks were presented and pupils were rated on definitions they gave before and after each task. It appeared that attainment of the concepts were difficult, and pupils did not generally evidence transfer of learning from pervious tasks. High-IQ fourth graders succeeded best, but even their "readiness" seemed questionable.

Baxter, Marion McComb. Prediction of Error and Error Type in Computation of Sixth Grade Mathematics Students. (The Pennsylvania State University, 1973.) Dissertation Abstracts International 35A: 251; July 1974.

Types of errors made by 96 sixth-grade pupils were identified and analyzed; effects of feedback, homework, and other factors were assessed. Algorithm errors appeared to be best predicted by mental age.

Bergen, Patricia M. Action Research on Division of Fractions. Arithmetic Teacher 13: 293-295; April 1966.

Booklets were designed to teach 63 pupils in three sixth-grade classes by the complex fraction, common denominator, or inversion algorithms. No significant differences were found between complex fraction and inversion algorithms, but each was significantly superior to the common denominator algorithm on most types of examples.

Bidwell, James King. A Comparative Study of the Learning Structures of Three Algorithms for the Division of Fractional Numbers. (University of Michigan, 1968.) Dissertation Abstracts 29A: 830; September 1968.

Three meaningful approaches were taught to 21 sixth-grade classes (n = 448) randomly assigned to treatment for eight days. The inverse operation procedure was most effective, followed by complex fraction and common denominator procedures. The complex fraction procedure was better for retention, while the common denominator procedure was poorest.

Brooke, George Milo. The Common Denominator Method in the Division of Fractions. (State University of Iowa, 1954.) Dissertation Abstracts 14: 2290-2291; 1954.

One group had division of fractions presented by the inversion method, and the other group used the common denominator method. Sixth-grade pupils in 28 classes (n = 772) were taught for four days. No significant difference between the two groups was found.

Brownell, William A. The Effects of Practicing a Complex Arithmetical Skill upon Proficency in Its Constituent Skills. Journal of Educational Psychology 44: 65-81; February 1953.

A test was administered to 17 fifth-grade classes (n = 367) before and after three weeks of instruction on division by two-place numbers. It was found that: (1) practice in dividing by two-place numbers (the complex skill) had no single, uniform, predictable results as far as proficiency in sub-skills was concerned; (2) in general, the oldest and best-established sub-skill (subtraction) seemed less subject to change than sub-skills recently taught, while the sub-skill (simple division) most like the complex skill seemed to be least stable; (3) loss in proficiency in sub-skills may be attributed to retroactive inhibition; (4) children with the lowest degree of proficiency in sub-skills made relatively little improvement on these while working on the complex skill.

Brownell, William A. and Moser, Harold E. Meaningful vs. Mechanical Learning: A Study in Grade III Subtraction. Duke University Research Studies in Education, No. 8. Durham, North Carolina: Duke University Press, 1949. 207 p.

In a study involving 1400 third grade pupils, half of the classes were taught to borrow using the decomposition algorithm; the other half using the equal additions algorithm. Each half was divided again, so that one group learned the procedure meaningfully and the other group, mechanically. Among the conclusions were: (1) the equal additions algorithm appears satisfactory for children who have a background of meaningful arithmetic, but for children with limited background the decomposition algorithm, taught with meaning, is better regardless of the criteria employed; (2) the equal additions algorithm is difficult to rationalize; (3) some proficiency can be produced by mechanical instruction with either the decomposition or equal additions algorithm; (4) crutches were needed, but were more helpful for the decomposition algorithm than for the equal additions algorithm.

Brueckner, Leo J. and Melbye, Harvey O. Relative Difficulty of Types of Examples in Division with Two-Figure Divisors. Journal of Educational Research 33: 401-414; February 1940.

Tests were administered to 474 pupils in grades 5 and 6. Long division was found to be not a single general ability but a process that consists of a considerable variety of skills found in combinations varying widely in difficulty. Examples in which the apparent quotient is the true quotient were much easier than those which required correcting. The mental ages at which less than 25 per cent error resulted ranged from 10 to 15 years.

Burdick, Charles Philip. A Study of the Effects of Academic Acceleration on Learning and on Retention of Learning Addition in the Set of Integers. (Syracuse University, 1969.) Dissertation Abstracts International 31A: 54-55; July 1970.

To determine the optimal grade level, a three-day unit on addition with integers was taught to 245 pupils in grades 5 through 8, with a retention test administered six weeks after the end of instruction. It appeared that grade 6 is the optimal level for teaching addition with integers, since there was the greatest increase in learning from instruction, attainment of group criterion performance, and nonsignificant loss on the retention test. However, the greatest increase from pre- to retention test was found in grade 5.

Burkhart, Lewis Leland. A Study of Two Modern Approaches to the Development of Understanding and Skills in Division of Whole Numbers. (Case Western Reserve University, 1967.) Dissertation Abstracts 28A: 3877; April 1968.

Fourth graders using the multiplicative approach had significantly greater mean achievement and retention than those using the subtractive approach, on measures of computational skills, understanding, and applications.

Capps, Lelon R. A Comparison of the Common Denominator and Inversion Method of Teaching Division of Fractions. Journal of Educational Research 56: 516-522; July-August 1963. (Also see Capps, Arithmetic Teacher 9: 10-16; January 1962.)

Sixth graders in 20 classes were randomly assigned for instruction on two methods of division of fractions. Groups did not differ significantly in ability to divide fractions, but the group taught by the inversion method scored significantly higher in ability to multiply fractions on the immediate posttest, though not on the retention test. Analysis of gain or loss from posttest to retention test revealed no difference between methods for addition, subtraction, or division of fractions, but the common denominator group gained significantly on multiplication of fractions.

Carney, Harold Francis. The Relative Effectiveness of Two Methods of Teaching the Addition and Subtraction of Rational Numbers. (New York University, 1973.) Dissertation Abstracts International 34A: 659-660; August 1973.

For eight fourth-grade classes, use of field postulates and other properties of whole numbers in teaching addition and subtraction with fractions was found to be more effective than use of objects and the number line.

Carter, Mary Katherine. A Comparative Study of Two Methods of Estimating Quotients When Learning to Divide by Two-Figure Divisors. (Boston University School of Education, 1959.) Dissertation Abstracts 20: 3317; February 1960.

For 12 weeks, 22 fifth-grade classes (n = 463) were taught (a) only the one-rule method, (b) only the two-rule method, or (c) first the one-rule method followed by the two-rule method as an alternative. Those taught one rule were more accurate than those taught by the two-rule method, and the combined method was also better than the two-rule method. Those taught the combined method did as well as those taught one-rule in both speed and accuracy. After a lapse of time, no significant differences in speed were found.

Coburn, Terrence Gordon. The Effect of a Ratio Approach and a Region Approach on Equivalent Fractions and Addition/Subtraction for Pupils in Grade Four. (The University of Michigan, 1973.) Dissertation Abstracts International 34A: 4688; February 1974.

Six classes of fourth graders were taught an instructional sequence for equivalent fractions based on an initial ratio thinking model, while six other fourth-grade classes were taught using a model which emphasized paper-folding activities. While achievement on some concepts was comparable for the two groups, students using the region approach achieved significantly better on adding and subtracting unlike fractions and on some retention and attitude measures.

Coltharp, Forrest Lee. A Comparison of the Effectiveness of an Abstract and a Concrete Approach in Teaching of Integers to Sixth Grade Students. (Oklahoma State University, 1968.) Dissertation Abstracts International 30A: 923-924; September 1969.

In a study with 79 pupils in four sixth-grade classes, addition and subtraction with integers was presented through a concrete procedure using the number line and other visual materials or with an abstract or algebraic procedure with ordered pairs. No significant differences in achievement were found.

Cosgrove, Gail Edmund. The Effect on Sixth-Grade Pupils' Skill in Compound Subtraction When They Experience a New Procedure for Performing This Skill. (Boston University School of Education, 1957.) Dissertation Abstracts 17: 2933-2934; December 1957.

It was found that sixth grade pupils who had learned the decomposition algorithm could change to the equal additions algorithm without significant interference effects. Hypothesized speed and accuracy advantages for equal additions were not observed.

Cox, L. S. Diagnosing and Remediating Systematic Errors in Addition and Subtraction Computations. Arithmetic Teacher 22: 151-157; February 1975.

Types of errors made by children were analyzed and categorized as systematic, random, or careless.

Coxford, Arthur Frank, Jr. The Effects of Two Instructional Approaches on the Learning of Addition and Subtraction Concepts in Grade One. (University of Michigan, 1965.) Dissertation Abstracts 26: 6543-6544; May 1966.

For the two higher-ability first grade classes in the control group, subtraction was based on the removal of a subset from a set, with no explicit use of the relationship between addition and subtraction. For the experimental group, which consisted of two lower- and two higher-ability first grade classes, subtraction was based on finding the missing part of a set when a set and one of its subsets was given, with extensive use of the relationship between addition and subtraction. Symbolism on addition and subtraction concepts was delayed six weeks in half of the classes in each treatment. Few significant differences were found between the two appraoches. For higher-ability groups, the control approach led to greater immediate proficiency in solving subtraction sentences, while the experimental approach tended to facilitate solutions of application of subtraction to a greater extent. Delayed symbolization led to greater transfer and applicability than did immediate symbolism when the experimental approach was used in the lower ability groups.

Crawford, Douglas Houston. An Investigation of Age-Grade Trends in Understanding the Field Axioms. (Syracuse University, 1964.) Dissertation Abstracts 25: 5728-5729; April 1965.

A 45-item test on field axioms was constructed and administered to 1000 non-randomly selected pupils in grades 4, 6, 8, 9, 10, and 12. Mean scores increased significantly from one even-numbered grade to the next. No significant differences were found for sex except between grades 8 and 9; IQ had an increasing effect as grade level increased. For traditional-content students in grades 9 and 10, the order of difficulty was commutativity (easiest), inverse, closure, identity, associativity, and distributivity.

Dawson, Dan T. and Ruddell, Arden K. An Experimental Approach to the Division Idea. Arithmetic Teacher 2: 6-9; February 1955.

Twelve fourth-grade classes were equated on seven variables and then taught for 22 days using the textbook approach or an approach in which division was presented as a special case of subtraction. Use of the subtractive concept resulted in significantly higher achievement on immediate and delayed recall tests. A greater understanding of division and its interrelationships with other operations was also found when the subtractive concept was used.

Crumley, Richard D. A Comparison of Different Methods of Teaching Subtraction in the Third Grade. (Unpublished doctoral dissertation, University of Chicago, 1956.)

Children in third grade tended to see subtraction as a take-away process regardless of the teaching procedure used.

Dilley, Clyde Alan. A Comparison of Two Methods of Teaching Long Division. (University of Illinois at Urbana-Champaign, 1970.) Dissertation Abstracts International 31A: 2248; November 1970.

Ten schools at three socio-economic levels were randomly selected, and one fourth grade from each school was randomly assigned to be taught division using either the successive subtractions method or the distributive method, taught meaningfully. On only two of seven tests was there a significant difference between treatments: on the application test the difference favored the successive subtractions method, while on the retention power test the difference favored the distributive method.

Ebeling, David George. The Ability of Sixth-Grade Students to Associate Mathematical Terms with Related Algorithms. (Indiana University, 1973.) Dissertation Abstracts International 34A: 7514-7515; June 1974.

From this study with 1094 sixth graders, it was concluded that: (1) the average sixth-grade student has the ability to associate fewer than half of the algorithms for operations with whole numbers with their mathematical terms; (2) writing an algorithm in horizontal or vertical form makes no difference in students' ability to associate the terms with the algorithms; (3) students are able to associate terms with algorithms when written in normal order significantly better than when written in inverse order.

Ellis, Leslie Clyde. A Diagnostic Study of Whole Number Computations of Certain Elementary Students. (The Louisiana State University and Agricultural and Mechanical College, 1972.) Dissertation Abstracts International 33A: 2234; November 1972.

A screening test on the four operations was followed by a diagnostic test used to tabulate errors and plan instruction for 690 pupils in grade 6. Division was found to be the most difficult operation, followed by subtraction, with addition least difficult.

Faires, Dano Miller. Computation with Decimal Fractions in the Sequence of Number Development. (Wayne State University, 1962.) Dissertation Abstracts 23: 4183; May 1963.

Two equated groups of eight fifth-grade classes were assigned to the two treatments. One group was introduced to decimals though a sequence based on an orderly extension of place value, with no reference to common fraction equivalents, while the other group was taught fractions before decimals, as is usually done. Gains in computational achievement and at least as good an understanding of fraction concepts resulted. It was concluded that computation with decimals is apparently more nearly like computation with whole numbers than with fractions; thus reinforcement of whole number computational skills is provided.

Flournoy, Frances. Children's Success with Two Methods of Estimating the Quotient Figure. Arithmetic Teacher 6: 100-104; March 1959.

Two fifth-grade classes (n = 61) were taught the one-rule method of rounding down and two classes (n = 63) were taught the two-rule method of rounding both ways. On a 10-item test, some children (including many low achievers) taught the two-rule procedure did not use it. However, the two-rule method appeared to result in greater accuracy.

Flournoy, Frances. A Consideration of Pupils' Success with Two Methods for Placing the Decimal Point in the Quotient. School Science and Mathematics 59: 445-455; June 1959.

Involved in this study were 137 pupils in six sixth-grade classes. Pupils taught to make the divisor a whole number by multiplying by a power of ten placed the decimal point in the quotient correctly more often than did pupils taught the subtractive method. Above-average achievers scored better with the subtractive method, but below-average achievers found it decidedly more difficult. Failure to place the necessary zeros in the dividend was common to those using either method.

Flournoy, Frances. Applying Basic Mathematical Ideas in Arithmetic. Arithmetic Teacher 11: 104-108; February 1964.

An 18-item test measuring ability to apply basic laws of arithmetic in each operation with whole numbers was administered to 106 students in four seventh-grade classes. An error of 30 per cent or greater was found on 15 items, and 50 per cent error or greater on ten times. Items related to the distributive property were most frequently missed.

Fuller, Kenneth Gary. An Experimental Study of Two Methods of Long Division. Teachers College Contributions to Education, No. 951. New York: Bureau of Publications, Teachers College, Columbia University, 1949.

Pupils in the experimental treatment were required to develop and use a table of multiples of the divisor, d, from 1 x d to 9 x d, to find quotient digits when working examples having two-digit divisors. Nonsignificant differences in achievement favored the experimental treatment over the control where pupils were taught the increase-by-one or two-rule procedure.

Gaslin, William Lee. A Comparison of Achievement and Attitudes of Students Using Conventional or Calculator Based Algorithms for Operations on Positive Rational Numbers in Ninth Grade General Mathematics. (University of Minnesota, 1972.) Dissertation Abstracts International 33A: 2217; November 1972.

Gaslin, William L. A Comparison of Achievement and Attitudes of Students Using Conventional or Calculator-based Algorithms for Operations on Positive Rational Numbers in Ninth-Grade General Mathematics. Journal for Research in Mathematics Education 6: 95-108; March 1975.

For six ninth-grade classes, use of units in which fractional numbers were converted to decimals and examples then solved on a calculator was found to be a "viable alternative" to use of conventional textbooks (including fractions) with or without a calculator, for low-ability or low-achieving students.

Gibb, E. Glenadine. Children's Thinking in the Process of Subtraction. Journal of Experimental Education 25: 71-80; September 1956.

Thirty-six randomly-selected pupils in grade 2 were interviewed about problems at three levels of abstraction and with three types of applications -- take-away, additive-subtraction, and comparative-subtraction. There were significant differences among applications for understanding, equation, solution, and time scores. Highest degree of attainment was on take-away problems and lowest level on comparative problems. Additive problems took a longer time. Significant differences were also found among contexts, with performance better on problems in semi-concrete context than in concrete context, and lowest in abstract context.

Gran, Eldon Edward. A Study to Determine Whether the Negative-Number Subtraction Method Can Be Learned and Used by Elementary Pupils. (University of South Dakota, 1966.) Dissertation Abstracts 27A: 4165-4166; June 1967.

Pupils in grades 3 through 6 learned the negative-number subtraction method with speed and accuracy superior to those taught by decomposition. Pupils demonstrated ability to apply the method to common and decimal fractions. However, they failed to continue to use the method as their habitual method of subtraction.

Gray, Roland F. An Experiment in the Teaching of Introductory Multiplication. Arithmetic Teacher 12: 199-203; March 1965.

Twenty-two third-grade classes were randomly assigned to instruction which introduced multiplication by stressing understanding of the distributive property or which explained multiplication in terms of repeated additions and arrays. The use of the distributive property resulted in higher achievement, and knowledge of the property appeared to help children proceed independently in the solution of untaught multiplication combinations. The children appeared not to develop an understanding of the distributive property unless it was specifically taught.

Green, Geraldine Ann. A Comparison of Two Approaches, Area and Finding a Part of, and Two Instructional Materials, Diagrams and Manipulative Aids, on Multiplication of Fractional Numbers in Grade Five. (The University of Michigan, 1969.) Dissertation Abstracts International 31A: 676-677; August 1970.

For a 12-day unit, 480 pupils in grade 5 were taught by treatments involving two approaches -- one based on area of a rectangular region and one on finding a fractional part of a region or set -- and by diagrams or materials. The area approach was more effective than the finding-a-part-of approach; diagrams and materials appeared to be equally effective. The area/diagram combination was most successful, with the part-of/materials approach second, and part-of/diagram ranking poorest.

Grossnickle, Foster E. An Experiment with Two Methods of Estimation of the Quotient. Elementary School Journal 37: 668-677; May 1937.

No significant differences were found between seven fourth-grade classes in one school who were taught the apparent method and seven fourth-grade classes in another school who were taught the increase-by-one method, on measures of accuracy, estimation scores, or mean number of errors.

Grossnickle, Foster E. Estimating the Quotient by Two Methods in Division with a Three-Figure Divisor. Elementary School Journal 39: 352-356; January 1939.

The result of division by the 810 three-figure divisors (which do not contain multiples of 10) were computed. Whether the apparent or the increase-by-one method of quotient estimation is used, in about 99 percent of the cases the true quotient is within a range of 2. Because of the difficulty of ascertaining what to increase, the apparent method was recommended.

Grossnickle, Foster E. Kinds of Errors in Division of Decimals and Their Constancy. Journal of Educational Research 37: 110-117; October 1943.

On tests from 400 pupils in grades 6 through 9, 21 different kinds of errors in division of decimals were found. Forty per cent of all errors resulted from improper usage of the decimal divisor. The average number of errors of each type was about the same at each grade level. The only constant error resulted from dividing an integer by a decimal.

Grouws, Douglas A. and Reys, Robert E. Division Involving Zero: An Experimental Study and Its Implications. Arithmetic Teacher 22: 74-80; January 1975.

Presenting division sentences involving zero before multiplication sentences involving zero was associated with significantly higher scores than the reverse sequence. Errors made when computation involved zero are noted.

Hall, Kenneth Dwight. An Experimental Study of Two Methods of Instruction for Mastering Multiplication Facts at the Third-Grade Level. (Duke University, 1967.) Dissertation Abstracts 28A: 390-391; August 1967.

Thirty classes (n = 701) of third graders were taught two sets of 36 lessons. No significant differences were found between groups taught by procedures emphasizing the commutative property and ordered pairs; with practice on uncommuted combinations, or by emphasis on the traditional approach, with practice on commuted combinations.

Hammond, Robert Lee. Ability with the Mathematical Principles Governing the Operations of Addition, Multiplication, Subtraction, and Division. (University of Southern California, 1962.) Dissertation Abstracts 23: 2372-2373; January 1963.

A test was developed and administered to 300 seventh graders to ascertain their understanding of mathematics principles and the relationship of this understanding to arithmetic and mental ability. Significant correlations were found between test scores and mental ability, arithmetic ability, and algebra aptitude scores. Mathematical ability factors were identified.

Hartung, Maurice L. Estimating the Quotient in Division (A Critical Analysis of Research). Arithmetic Teacher 4: 100-111; April 1957.

A critical analysis of significant research pertaining to the estimation of quotient digits when dividing by two-place divisors is presented. Advocated and defended is a preference for a one-rule "round-up" method of estimation instead of a one-rule "round-down" method or a two-rule "round-both-ways" method -- especially during the early stages of instruction.

Harutunian, Harold. Validation of a Learning Hierarchy Using Classroom Interaction. (Boston University School of Education, 1973.) Dissertation Abstracts International 34A: 5584-5585; March 1974.

Using Gagne's task analysis procedure, a learning hierarchy of thirteen subordinate skills was derived for adding fractional numbers. It was validated with a sample of five fifth-grade classes.

Hegstrom, William J. Construction and Clinical Testing of Programmed Instructional Units for Very Low Achievers in Junior High School Mathematics. (University of Miami, 1971.) Dissertation Abstracts International 32A: 3663-3664; January 1972.

Programmed instruction booklets on fractions and reduction of fractions appeared to be feasible for low-achieving junior high school students.

Hervey, Margaret A. Children's Responses to Two Types of Multiplication Problems. Arithmetic Teacher 13: 288-292; April 1966.

Sixty-four randomly selected second graders were administered one of two 10-item tests; they were asked to find the answer to multiplication problems and then select a representation, or they first selected a representation and then found an answer. Equal additions multiplication problems were less difficult to solve and conceptualize, and less difficult to select a "way to think about", than were Cartesian product problems. Cartesian product problems were more readily solved by high achievers in arithmetic than by low achievers, by boys than by girls, and by those with above average intelligence.

Hightower, H. W. Effect of Instructional Procedures on Achievement in Fundamental Operations in Arithmetic. Educational Administration and Supervision 40: 336-348; October 1954.

A critical review of 17 research studies on addition and subtraction led to the conclusion that additional variables and criteria must be used in research on method.

Hill, Edwin Henry. Study of Third, Fourth, Fifth, and Sixth Grade Children's Preferences and Performances on Partition and Measurement Division Problems. (State University of Iowa, 1952.) Dissertation Abstracts 12: 703; Issue No. 5, 1952.

Pupils in grades 3 through 6 (n = 844) were given a test on the two types of division problems, and asked to indicate their preference. Both boys and girls in grades 4-6 preferred measurement problems, while third graders indicated no preference for either type. Boys in grades 3-5 and girls in grades 3-6 scored equally well on both types of problems; boys in grade 6 scored significantly higher on measurement problems.

Hinkelman, Emmet A. A Study of the Principles Governing Fractions Known by the Fifth and Sixth Grade Children. Educational Administration and Supervision 42: 153-161; March 1956.

Thirty-one fifth- and sixth-grade pupils were tested by means of a 20-item true-false "principles of fractions" test (e.g., one item was: "Adding the same number to both the numerator and denominator of a fraction leaves the value of the fraction the same."). All ten principles were known to the pupils as a group, with a range of one to eight principles known by individuals. Means were 3.1 for grade 5 and 4.1 for grade 6.

Hostetler, Robert Paul. Toward a Theory of Sequencing: Study 2-1: An Exploration of the Effect of Selected Sequence Variables upon Student Choice in the Use of Algorithms. (The Pennsylvania State University, 1970.) Dissertation Abstracts International 31A: 4623; March 1971.

Using a CAI program on equivalent fractions with 24 fifth graders, evidence was found that (1) explicit instruction about the relative scopes of applicability of two algorithms did not significantly affect the algorithm preferences; (2) the order in which two algorithms are learned affected the algorithm preference of a student: strong support was obtained indicating that the preferred algorithm is the one learned last; and (3) the order in which two algorithms are learned exerted a significantly stronger influence on algorithm preference than did knowledge of the scope of applicability of the two algorithms under consideration.

Howlett, Kenneth Donn. A Study of the Relationship Between Piagetian Class Inclusion Tasks and the Ability of First Grade Children To Do Missing Addend Computation and Verbal Problems. (State University of New York at Buffalo, 1973.) Dissertation Abstracts International 34A: 6259-6260; April 1974.

First-grade pupils classified as Stage III on a class-inclusion test performed significantly better than Stage I pupils on both missing addend computation and verbal problems.

Hughes, Frank George. A Comparison of Two Methods of Teaching Multi-digit Multiplication. (The University of Tennessee, 1973.) Dissertation Abstracts International 34A: 2460-2461; November 1973.

The lattice method of multiplication was used with six classes of fourth graders, while six other fourth-grade classes used the distributive algorithm. Groups using the lattice method were able to compute in significantly less time and more accurately than groups using the distributive algorithm. No significant differences in understanding or attitude were found.

Hutchings, Barton. Low-stress Subtraction. Arithmetic Teacher 22: 226-232; March 1975.

A "low stress" algorithm, which involves regrouping before any computation is done, has been found to be effective with various types of learners.

Hutchings, Lloyd Benjamin. An Examination, Across a Wide Range of Socioeconomic Circumstance, of a Format for Field Research of Experimental Numerical Computation Algorithms, an Instrument for Measuring Computational Power Under Any Concise Numerical Addition Algorithm, and the Differential Effects of Short Term Instruction in Two Experimental Numerical Addition Algorithms and Equivalent Practice with the Conventional Addition Algorithm. (Syracuse University, 1972.) Dissertation Abstracts International 33A: 4678; March 1973.

The experimental rapid-acquisition algorithm produced "a quick, strong increase in computational power", conventional practice resulted in some improvement, non-treatment had little effect, and an alternative experimental algorithm was debilitating, for the fifth graders studied.

Ingersoll, Gary M. An Experimental Study of Two Methods of Presenting the Inversion Algorithm in Division of Fractions. California Journal of Educational Research 22: 17-25; January 1971.

In two experiments, 131 sixth-grade children from five classes were involved. After a program used by both groups on one day, pupils were randomly assigned to three different programs completed on the second day. The complex fraction method appeared to be more effective than a procedure using the associative property.

Jordan, Ralph James. Effects of Sequence of Presentation of Square Root Extraction Methods. (The University of Rochester, 1970.) Dissertation Abstracts International 31A: 3416; January 1971.

Over 200 eighth graders were present varied sequences of three pairs of square root methods. Immediately after presentation, the algorithm followed by the divide-and-average method was preferable to the reverse sequence. No significant differences were found between sequences for retention or transfer. The algorithm appeared to be the most preferred method.

Kansky, Robert James. An Analysis of Models Used in Australia, Canada, Europe, and the United States to Provide an Understanding of Addition and Multiplication Over the Natural Numbers. (University of Illinois, 1969.) Dissertation Abstracts International 30A: 1074-1075; September 1969.

Bases for meaningful instruction and the relationship of four classes of models of a number system to those bases were examined, to identify and analyze procedures and materials used with children in teaching addition and multiplication. Structural models used in textbooks were identified and classified, and the probable teaching effectiveness of each was analyzed with respect to mathematical and pedagogical criteria. Changes in the models now in use were suggested.

Kratzer, Richard Oren. A Comparison of Initially Teaching Division Employing the Distributive and Greenwood Algorithm with the Aid of a Manipulative Material. (New York University, 1971.) Dissertation Abstracts International 32A: 5672; April 1972.

Kratzer, Richard O. and Willoughby, Stephen S. A Comparison of Initially Teaching Division Employing the Distributive and Greenwood Algorithms with the Aid of a Manipulative Material. Journal for Research in Mathematics Education 4: 197-204; November 1973.

Six fourth-grade classes were taught division using the distributive algorithm as a method of keeping records of manipulating bundles of sticks; six other classes used the Greenwood algorithm, with sticks. No significant difference was found between methods on a test of familiar problems, but the distributive group scored better on transfer problems.

Lankford, Francis G., Jr. Some Computational Strategies of Seventh Grade Pupils. Final Report, USOE Grant No. OEG-3-72-0035. Charlottesville: The Center for Advanced Study, University of Virginia, October 1972. ERIC: ED 069 496.

Lankford, Francis G., Jr. What Can a Teacher Learn About a Pupil's Thinking Through Oral Interviews? Arithmetic Teacher 21: 26-32; January 1974.

The results of interviews with 176 pupils in grade 7 were presented. Frequency of right and wrong answers to examples for each operation, with whole numbers and with fractions; strategies frequently used; the nature of wrong answers; and some characteristics of good and poor computers were specified.

Leach, Mary Louise Moynihan. Primacy Effects Associated with Long Term Retention of Mathematical Algorithms. (University of Maryland, 1973.) Dissertation Abstracts International 34A: 7002-7003; May 1974.

Euclid's algorithm for finding the greatest common divisor of two numbers, the traditional square root algorithm, and the slide method of multiplication were arranged in six serial orders. Sixty elementary-education majors were randomly assigned to six groups, with each receiving one serial arrangement of the algorithms, presented via programmed booklets. No significant differences in retention were found.

Morton, R. L. Estimating Quotient Figures When Dividing by Two-Place Numbers. Elementary School Journal 48: 141-148; November 1947.

The results of estimating quotients by the apparent and the increase-by-one methods (on 40,014 examples) were presented: (1) the increase-by-one method is correct 79 per cent of the time when divisors end in 6, 7, 8, or 9; (2) the apparent method is correct 72 per cent of the time when divisors end in 1, 2, 3, or 4; (3) for any divisor ending in 1 through 9, the apparent method is correct 53 per cent of the time, and the increase-by-one method is correct 61 per cent of the time; (4) the apparent method is more successful with divisors ending in 5. It was concluded that pupils should be taught to round to the nearest multiple of tens.

O'Brien, Thomas C. An Experimental Investigation of a New Approach to the Teaching of Decimals. (New York University, 1967.) Dissertation Abstracts 28A: 4541-4542; May 1968.

Thirty-six sixth-grade classes were randomly assigned to the three treatments. Pupils taught decimals with an emphasis on the principles of numeration, with no mention of fractions, scored lower on tests of computation with decimals than those taught either (a) the relation between decimals and fractions, with secondary emphasis on principles of numeration, or (b) rules, with no mention of fractions or principles of numeration. On later retention measures, the numeration approach was significantly lower than use of the rules approach, but not significantly different from the fraction-numeration approach.

Osborne, Alan Reid. The Effects of Two Instructional Approaches on the Understanding of Subtraction by Grade Two Pupils. (The University of Michigan, 1966.) Dissertation Abstracts 28A: 158; July 1967.

The effects of continuing in grade 2 the instructional treatments used by Coxford (1966) in grade 1 were studied. The set-partitioning-without-removal approach resulted in significantly greater understanding of subtraction than did the take-away approach. Evidence concerning time for symbolism was inconclusive.

Osburn, Worth J. Levels of Difficulty in Long Division. Elementary School Journal 46: 441-447; April 1946.

Forty-one levels of difficulty for division with two-digit divisors and one-digit quotients were stated, with examples and the total number of possible exercises. The apparent method of estimating the quotient, with the instruction to try a quotient figure less by 1 when a subtrahend is too large, could enable the learner to handle all but five per cent of any long division he will ever be called upon to do.

Osburn, W. J. Division by Dichotomy as Applied to the Estimation of Quotient Figures. Elementary School Journal 50: 326-330; February 1950.

Analysis of division examples with divisors ending in 6, 7, 8, or 9, using a dichotomy, revealed that the apparent method is successful in 4,800 cases where the increase-by-one method is also successful. The apparent method fails in 9,846 cases where the increase-by-one method is successful, and is successful in 1,885 cases where the increase-by-one method fails. Both methods fail in 2,099 cases.

Pang, Paul Hau-lim. A Mathematical and Pedagogical Study of Square Root Extraction. (State University of New York at Buffalo, 1969.) Dissertation Abstracts International 30A: 1080; September 1969.

For students in grades 8 and 9, the direct-trial method was significantly better than the traditional algorithm and the average-and-divide method for finding the square root.

Phillips, Ernest Ray. Validating Learning Hierarchies for Sequencing Mathematical Tasks. (Purdue University, 1971.) Dissertation Abstracts International 32A: 4249; February 1972.

A hierarchy for the computational skills of adding rational numbers with like denominators was constructed using Gagne's task analysis. Sequence seemed to have little effect on immediate achievement and transfer to a similar task, but longer-term retention seemed susceptible to sequence manipulation, for the fourth graders studied.

Romberg, Thomas A. A Note on Multiplying Fractions. Arithmetic Teacher 15: 263-265; March 1968.

Analysis of tests from 691 sixth graders revealed that a larger percentage of students who had used "modern" programs were failing to cancel on problems dealing with multiplication of fractions, than were pupils who had had "traditional" programs.

Rousseau, Leon Antonio. The Relationship Between Selected Mathematical Concepts and Retention and Transfer Skills with Respect to Long Division Algorithms. (Washington State University, 1972.) Dissertation Abstracts International 32A: 6750; June 1972.

Twelve randomly-selected fourth-grade classes were randomly assigned to one of four treatments: (1) mathematical, based on the distributive law of division over addition; (2) real world, based on the physical act of quotitioning; (3) real world, based on the physical act of partitioning; and (4) rote, based on the memorization of routines. No significant differences were found in the retention of the division algorithms synthesized from these treatments. The rote algorithm was better for transfer to slightly more difficult problems, but for problems of greater difficulty, the quotitive and distributive algorithms were better than rote and partitive algorithms.

Ruch, G. M. and Mead, Cyrus D. A Review of Experiments on Subtraction. In Report of the Society's Committee on Arithmetic. Twenty-ninth Yearbook, National Society for the Study of Education. Bloomington, Illinois: Public-School Publishing Co., 1930. pp. 671-678.

Four methods of subtraction were presented and the experiments related to them described.

Sawyer, Ray Corwin. Evaluation of Alternative Methods of Teaching Subtraction of Integers in Two Junior High Schools. (University of Idaho, 1973.) Dissertation Abstracts International 34A: 6958; May 1974. (ERIC: ED 073 944)

The seventh-grade group taught the related facts method achieved significantly higher on the concepts section of a standardized test than did the group taught the complement method, but no significant differences were found for achievement on addition and subtraction of integers. In another district, two retention differences were noted.

Schell, Leo Mac. Two Aspects of Introductory Multiplication: The Array and the Distributive Property. (State University of Iowa, 1964.) Dissertation Abstracts 25: 5161-5162; March 1965.

Two nine-lesson sets of instructional materials were presented to nine third-grade classes. Five classes used arrays exclusively to illustrate multiplication; four classes used a variety of illustrations. The distributive property was used in three lessons. The Array group produced more correct drawings illustrating the commutative and distributive properties and multiplication word problems; the Variety group made more correct drawings for addition and subtraction word problems. Pupils in neither group adapted their illustrations to the "reality of the situation." Items dealing with the distributive property were more difficult for all pupils, and expecially for low-scoring pupils, than items dealing with other phases of multiplication tested.

Schell, Leo M. and Burns, Paul C. Pupil Performance with Three Types of Subtraction Situations. School Science and Mathematics 62: 208-214; March 1962.

Twenty-three pupils in grade 2 were asked to solve 36 subtraction problems. No significant differences were found in performance on the three problem types (take-away, how-many-more-needed, and comparison or difference). Take-away problems seemed to present fewest difficulties and were considered easiest by the pupils.

Schmidt, Mary Merle. Effects of Teaching the Commutative Laws, Associative Laws, and the Distributive Law of Arithmetic on Fundamental Skills of Fourth Grade Pupils. (The University of Mississippi, 1965.) Dissertation Abstracts 26: 4510-4511; February 1966.

Seven fourth-grade classes (n = 194) formed the control group, which used the Row-Peterson textbook during 1961-62. Seven fourth-grade classes (n = 215) formed the experimental group in 1962-63, for which the Row-Peterson textbook was supplemented with instruction on the five basic laws as applicable. At each of three ability levels, experimental classes made greater gains on the California Achievement Test than control classes did.

Schrankler, William Jean. A Study of the Effectiveness of Four Methods for Teaching Multiplication of Whole Numbers in Grade Four. (University of Minnesota, 1966.) Dissertation Abstracts 27A: 4055; June 1967.

Twenty-three fourth-grade classes were randomly selected and assigned to treatments. In a readiness phase, the 100 multiplication facts were emphasized for one group (n = 281); the commutative, associative, and distributive properties for multiplication were emphasized in the other group (n = 327). Then half the classes were taught the distributive algorithm using indentation, while half were taught the distributive algorithm using complete partial products. The properties-products group scored higher in understanding and problem solving, while the facts-indenting group was superior in computation directly after instruction. The properties-indenting group was superior in computation and problem solving on the retention test, while the facts-products group excelled in computational speed.

Scott, Lloyd. Children's Concept of Scale and the Subtraction of Fractions. Arithmetic Teacher 9: 115-118; March 1962.

Two 18-item tests were administered to 89 fifth graders after pupils had had several months of practice with the operations involving common fractions. Children made many more errors in subtracting common fractions involving regrouping than in subtracting whole numbers involving regrouping. Many regrouping errors in subtracting common fractions were related to children's tendency to relate this process to the decimal scale of our number system. Children involved in a contemporary arithmetic program made a greater proportion of total errors at the regrouping step in common fractions than did children in Brueckner's study of several decades ago.

Scott, Lloyd. A Study of Teaching Division Through the Use of Two Algorisms. School Science and Mathematics 63: 739-752; December 1963.

For a two-month period, four classes of third graders were taught division using one or two algorithms. The use of two algorithms neither confused nor presented undue difficulty; no more teaching time was needed than for teaching pupils to use only one algorithm. Those who used two algorithms were at least as efficient in solving division problems as were children who used one algorithm. Use of two algorithms resulted in greater understanding of the division operation; pupils were generally superior in their ability to set up a proper algorithm, distinguishing between partitive and measurement division, and defining division as a means for solving problems.

Sension, Donald Bruce. A Comparison of Two Conceptual Frameworks for Teaching the Basic Concepts of Rational Numbers. (University of Minnesota, 1971.) Dissertation Abstracts International 32A: 2408; November 1971.

For 162 pupils in grade 2 who were randomly assigned to treatments lasting 11 days, area, set-subset, and combination representations of introducing rational number concepts appeared to be equally effective on tests using two types of pictorial models.

Sluser, Theodore F. A Comparative Study of Division of Fractions in Which an Explanation of the Reciprocal Principle is the Experimental Factor. (University of Pittsburgh, 1962.) Dissertation Abstracts 23: 4624-4625; June 1963.

The teaching of the common denominator and inversion algorithms with and without explanation of the reciprocal principle as the rationale behind inversion were compared. A total of 299 sixth-grade pupils in 11 classes were involved for 20 days. The group given the explanation scored significantly lower on tests of division of fractions than the group merely taught to invert and multiply. A large percentage of errors occurred because pupils performed the wrong operation.

Smith, Charles Winston, Jr. A Study of Constant Errors in Subtraction and in the Application of Selected Principles of the Decimal Numeration System Made by Third and Fourth Grade Students. (Wayne State University, 1968.) Dissertation Abstracts International 30A: 1084; September 1969.

From each of two randomly-selected schools at each of two achievement levels, two third and two fourth-grade classes were selected. Errors made by 523 pupils on a diagnostic test and a place value test were analyzed. Pupils who correctly applied selected decimal numeration principles made few subtraction errors, and those proficient in renaming had less difficulty in subtracting. Errors committed most frequently by students who applied principles correctly were related to: basic subtraction combinations, subtracting the minuend from the subtrahend, and writing zero as an answer instead of borrowing.

Steffe, Leslie P. and Parr, Robert B. The Development of the Concepts of Ratio and Fraction in the Fourth, Fifth, and Sixth Years of the Elementary School. Technical Report No. 49. Madison: Wisconsin Research and Development Center for Cognitive Learning, University of Wisconsin, March 1968.

Six tests -- 4 pictorial, 2 symbolic -- were constructed and used to measure the performance of 4th-, 5th-, and 6th-grade pupils (in three different ability groups) on problems classified either as ratios or as fractions, where "reduction" to lower terms was involved and a missing numerator or denominator was to be found. Differential performance was observed with respect to grades, ability groups, and test types -- with a very low observed correlation between scores on symbolic and pictorial tests.

Stenger, Donald J. An Experimental Comparison of Two Methods of Teaching the Addition and Subtraction of Common Fractions in Grade Five. (University of Cincinnati, 1971.) Dissertation Abstracts International 32A: 3676; January 1972.

Eighty-one pupils from two fifth-grade classes were randomly assigned to two treatments for 16 days. The group taught with a subset ratio procedure achieved significantly better than the group taught by another (unspecified) procedure on both immediate and retention tests.

Stephens, Lois and Dutton, Wilbur. Retention of the Skill of Division of Fractions. Arithmetic Teacher 7: 28-31; January 1960.

For 74 sixth graders who had been taught the inversion method or the common denominator method, no significant differences were found on the retention test after three months.

Stocks, Sister Tina Marie. The Development of an Instructional System Which Incorporates the Use of an Electric Desk Calculator as an Aid to Teaching the Concept of Long Division to Educable Mentally Retarded Adolescents. (Columbia University, 1972.) Dissertation Abstracts International 33A: 1049-1050; September 1972.

The 15 secondary EMR students improved in skills with the division algorithm after instruction with the calculator.

Suydam, Marilyn N. and Weaver, J. Fred. Using Research: A Key to Elementary School Mathematics. University Park: The Pennsylvania State University, 1970.

This review of research on elementary school mathematics includes bulletins on addition and subtraction with whole numbers (B-1), multiplication and division with whole numbers (B-2), and rational numbers -- fractions and decimals (B-3).

Tietz, Naunda Meier. A Comparison of Two Methods of Teaching Multiplication: Repeated-Addition and Ratio-to-One. (Oklahoma State University, 1968.) Dissertation Abstracts International 30A: 1060; September 1969.

A random sample of 214 pupils in eight fourth-grade classes was randomly assigned to one of two treatments: (1) the repeated-addition approach using the array as the physical referent or (2) the ratio-to-one approach using a coördinate system and ordered pairs of numbers as the physical referent. No significant relationship was found between the method used and the acquisition, retention, and understanding of multiplication for the total group. However, use of arrays (with the repeated addition method) seemed better for average and low groups.

Trafton, Paul Ross. The Effects of Two Initial Instructional Sequences on the Learning of the Subtraction Algorithm in Grade Three. (The University of Michigan, 1970.) Dissertation Abstracts International 31A: 4049-4050; February 1971.

Eight third-grade classes were randomly assigned to two approaches to two-digit subtraction. More extensive development of the decomposition algorithm was found to be more effective than a procedure which included work with concepts and use of the number line before the algorithm was taught.

Tunis, Harry Brandriff. The Effects of Differential Rehearsal and Presentation Treatments on the Performance of a Mathematical Algorithm. (University of Maryland, 1973.) Dissertation Abstracts International 34A: 4093; January 1974.

A rehearsal strategy that did not involve grouping of algorithm steps (for finding the area of a triangle) was superior to strategies in which rehearsal steps were grouped, for 126 elementary-education majors.

Van Engen, Henry and Gibb, E. Glenadine. General Mental Functions Associated with Division. Educational Service Studies, No. 2. Cedar Falls: Iowa State Teachers College, 1956.

In this study with 12 fourth-grade classes, the use of the conventional, distributive algorithm was compared with the subtractive form. Some advantages were reported for each: (1) Children taught the subtractive form had a better understanding of the process or idea of division in comparison with the distributive method. Use of this algorithm was especially effective for children with low ability; those with high ability used the two methods with equivalent effectiveness. (2) Children taught the distributive algorithm achieved higher problem solving socres. (3) Use of the subtractive method was more effective in enabling children to transfer to unfamiliar but similar situations. (4) Children who used the distributive algorithm had greater success with partition situations, while those who used the subtractive algorithm had greater success with measurement situations.

Vest, Floyd Russell. Development of the "Model Construct" and Its Application to Elementary School Mathematics. (North Texas State University, 1968.) Dissertation Abstracts 29A: 3539; April 1969.

A system of theoretical concepts to be imposed on the area of teaching the operations with whole numbers and associated concepts was delineated. An organized catalog of models describing 20 families of models for addition and subtraction and 20 for multiplication and division was presented. Functions of models were determined and evaluated.

Weaver, J. F. and others. Some Factors Associated with Pupils' Performance on Examples Involving Selected Variations of the Distributive Idea. February 1973. ERIC: ED 075 199.

Weaver, J. Fred. Pupil Performance on Examples Involving Selected Variations of the Distributive Idea. Arithmetic Teacher 20: 697-704; December 1973.

Twelve 9-item tests were constructed and administered to pupils in grades 4-7 to ascertain whether there are differential achievement effects associated with context, form, format, and number variables. At all grade levels, pupils exhibited very little sensitivity to use of distributivity in solving the examples presented.

Weinstein, Marian Sue. An Investigation of Algorithm Justification in Elementary School Mathematics. (The University of British Columbia, Canada, 1973.) Dissertation Abstracts International 34A: 3045; December 1973.

No significant achievement differences were found between fifth-grade pupils taught fraction algorithms by a strictly pattern or a strictly algebric approach. Some evidence was found that teaching an algebraic approach followed by a pattern approach might be effective.

Wheeler, Larry Eugene. The Relationship of Multiple Embodiments of the Regrouping Concept to Children's Performance in Solving Multi-digit Addition and Subtraction Examples. (Indiana University, 1971.) Dissertation Abstracts International 32A: 4260; February 1972.

Second-grade pupils proficient in regrouping two-digit addition and subtraction examples on three or more concrete embodiments scored significantly higher on multi-digit tests than those not proficient in using concrete materials. A significant correlation was found between number of embodiments manipulated and achievement on multi-digit examples.

Wiles, Clyde A.; Romberg, Thomas A.; and Moser, James M. The Relative Effectiveness of Two Different Instructional Sequences Designed to Teach the Addition and Subtraction Algorithms. Technical Report No. 222. Madison: Wisconsin Research and Development Center for Cognitive Learning, The University of Wisconsin, June 1972.

Wiles, Clyde Allan. Comparisons of Three Instructional Sequences for the Addition and Subtraction Algorithms. (The University of Wisconsin, 1973.) Dissertation Abstracts International 34A: 6375; April 1974.

Investigated at the second grade level, a sequential and an integrated approach to the introduction of two algorithms for addition and subtraction examples involving renaming found no evidence to support any advantage of an integrated approach (introducing the two algorithms more or less simultaneously) over a sequential approach (introducing first the addition algorithm, then the subtraction algorithm).

Williamson, Bruce Merle. A Comparison of a Natural Algorithm with the Inversion Algorithm for Teaching the Division of Rational Numbers. (University of Minnesota, 1972.) Dissertation Abstracts International 33A: 150; July 1972.

Three classes of sixth graders used programs teaching an algorithm using equivalent fractions or the inversion algorithm. No significant difference was found between the two algorithms.

Willson, George Hayden. A Comparison of Decimal-Common Fraction Sequence with Conventional Sequence for Fifth Grade Arithmetic. (University of Arizona, 1969.) Dissertation Abstracts International 30A: 1762; November 1969.

Teachers of four fifth-grade classes (n = 112) were randomly assigned to use the usual textbook sequence of teaching common fractions followed by decimal fractions, or a re-ordered sequence using the same textbook. No significant differences were found on achievement, concept, computation, and problem solving tests. Greater raw-score gains were made by those using the decimal-common fraction sequence.

Wilson, Jean Alice. The Effect of Teaching the Rationale of the Reciprocal Principle in the Division of Fractions Through Programmed Instruction. (University of Pittsburgh, 1967.) Dissertation Abstracts 28A: 2926; February 1968.

The reciprocal principle was taught by programmed instruction, while the mechanical process of inversion was tuaght by the teacher. Sixth graders from one district were assigned to the inversion treatment, while sixth graders from two other districts comprised the reciprocal group (n = 630). Pupils using the inversion procedure scored significantly better on a computation test on division of fractions, while the retention test scores favored the reciprocal program group.

Zinn, Bennie Ardist, Jr. Extending the Teaching of Multiplication Facts at the Seventh Grade Level. (Texas A & M University, 1971.) Dissertation Abstracts International 32A: 4263; February 1972.

A set of nine lessons was developed which allowed students to use the concept of structure and to develop understanding of digit placement and expanded notation with two-digit multiplication examples. The unit was taught to three seventh-grade classes in three schools, while another class in each school had the regular program. The lessons appeared to be effective.

Zweng, Marilyn J. Division Problems and the Concept of Rate. *Arithmetic Teacher* 11: 547-556; December 1964.

Forty-eight second graders (randomly selected) were tested to ascertain differences in difficulty between partitive and measurement division problems and between basic and rate division problems. Partitive division problems were significantly more difficult than were measurement problems. Rate problems seemed to be easier than basic problems. Partitive basic problems were significantly more difficult than either basic measurement or rate measurement problems.

VI. Summary

Summary

So much is not considered in this publication on algorithmic learning:

--so much exploration needs to be conducted.

--so many variables need to be researched.

--so much thinking needs to be done.

--so many implications need to be drawn -- and tested.

There are implications for research questions and for research design. And even more important, there are implications for curriculum and instruction. . .